DISCRETE-TIME STOCHASTIC ANALYSIS OF LAND COMBAT

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UĞUR ELİİYİ

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Prof. Dr. Canan Özgen
Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

Prof. Dr. Çağlar Güven
Head of Department

This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

Assoc. Prof. Dr. Levent Kandiller Co-Supervisor	Assoc. Prof. Dr. Nur Evin Özdemirel Supervisor

Examining Committee Members

Assist. Prof. Dr. Müge Avşar

Dr. Haluk Aygüneş

Assoc. Prof. Dr. Levent Kandiller

Prof. Dr. Ömer Kırca

Assoc. Prof. Dr. Nur Evin Özdemirel

ABSTRACT

DISCRETE-TIME STOCHASTIC ANALYSIS OF LAND COMBAT

Eliiyi, Uğur
M.S., Department of Industrial Engineering
Supervisor: Assoc. Prof. Dr. Nur Evin Özdemirel
Co-Supervisor: Assoc. Prof. Dr. Levent Kandiller

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In this study, we present the implementation and experimental analysis of a modeling approach for analyzing tactical level land combat to generate information for weapon and ammunition planning. The discrete-time stochastic model (DSM), which can handle small and moderately large force levels, is based on single shot kill probabilities. Forces are assumed to be heterogeneous on both sides, and both directed and area fire types are modeled by means of combinatorial analysis. DSM considers overkills and can handle noncombat loss and engagement processes, discrete reinforcements, force combinations and divisions. In addition to experimenting with DSM, we estimate attrition rate coefficients used in Lanchester combat models, such that the two models will yield similar figures for force levels throughout the combat.

Keywords: Discrete-Time Stochastic Model, Combat Modeling, Attrition Rate Estimation, Military Applications.

ÖZ

KARA MUHAREBESİNİN KESİK-ZAMANLI STOKASTİK ANALİZİ

Eliiyi, Uğur

Yüksek Lisans Tezi, Endüstri Mühendisliği Bölümü

Tez Yöneticisi: Doç. Dr. Nur Evin Özdemirel

Ortak Tez Yöneticisi: Doç. Dr. Levent Kandiller

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Bu çalışmada silah ve mühimmat planlamasında kullanılabilecek taktik seviyede bir kara muharebesi modelinin uygulaması ve deneysel analizi amaçlanmıştır. Küçük ve orta büyüklükte kuvvet seviyelerinde kullanılması amaçlanan söz konusu kesik-zamanlı stokastik model (DSM) tek atışta vuruş olasılıklarına dayanmaktadır. Muharebenin her iki tarafı için de kuvvetler heterojen varsayılmış, direkt ve alan atış tipleri kombinatoryel analiz teknikleri ile modellenmiştir. DSM, muharebe dışı kayıp ve angajman süreçlerini, kesikli takviye, kuvvet birleştirme ve bölme etkilerini de dikkate almaktadır. Modelin deneysel analizine ek olarak, deterministik ve stokastik Lanchester modellerinde kullanılan zayiat katsayılarının tahmini yapılırken DSM ve Lanchester modellerinin küçük ölçekli muharebe benzetimlerinde benzer sonuçlar vermesi amaçlanmıştır.

Anahtar Sözcükler: Kesik-Zamanlı Stokastik Model, Muharebe Modelleme, Zayiat Katsayı Tahmini, Askeri Uygulamalar.

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CHAPTER 1

INTRODUCTION

The governments all around the world try to save from funds allocated to military budget. Nevertheless, defense expenditures still constitute a major percentage of the overall budget for most countries. Therefore, scientific effort towards cost-efficient use of weapon systems and effective exploitation of ammunition on hand is exceptionally worthy, and this constitutes the major motivation of our study.

Combat is the term used for circumstances during which at least one *combatant* (or weapon system) employs lethal means against at least one other. All other situations are preludes or postludes to combat, which either set the initial and boundary conditions for the next combat, or simply end the combat (Ancker, 1995).

Lanchester (1916) presented his theory of battle attrition by a system of differential equations, which we refer to as the Lanchester model (LM). Lanchester's square law for directed fire and linear law for area fire are the two fundamental attrition equations in LM, which will be explained in detail in Chapter 2. LM makes use of *attrition rate coefficients* (ARC) for units, which is a measure of effectiveness and defines the rate at which a unit destroys the opposing unit. In other words, ARC is the number of targets killed by one combatant per unit time.

LM can be effective in representing combat dynamics when the number of combatants is sufficiently large. However, if the number of combatants is small (say less than 20), the randomness in engagement and killing should be taken into account in modeling the attrition process. In this respect, Snow (1948) studied stochastic Lanchester equations. Stochastic Lanchester model (SLM) is a general renewal model except that inter-fire times are negative exponentially distributed. However, later on, Gafarian and Ancker (1984) showed that neither the exponential model nor the deterministic LM is a satisfactory approximation of the stochastic combat model. Hence, there seems to be a need for further research on new combat models, especially those considering the random aspects.

Land combat is our primary subject in this study. This combat environment is the most involved one, and the most complicated to deal with (Ancker, 1995). Conventionally, *blue units* refer to ally forces, and *red units* to enemy forces. We assume that both forces control a number of *military units* such as infantry, artillery and tank, implying heterogeneous combat. Each military unit is composed of a number of identical combatants. In *homogeneous combat*, on the other hand, each force consists of only one type of combatant, such as only infantry or only artillery. LM and SLM are originally proposed for homogeneous combat.

The major distinguishing characteristic of a military unit is its single shot kill probability. *Single shot kill probability* (SSKP) is the probability that a combatant kills its target (an opposing combatant) at a single shot. SSKPs may be different for different military units, but remain constant throughout the battle.

In this study, we focus on the implementation of and experimentation with a recent model developed by Kandiller *et al*. (2002). The model is a discrete-time stochastic model (DSM) based on SSKPs, which can handle small and moderately large force levels. Forces are assumed to be heterogeneous on both sides, division and combination of units are allowed, and both directed and area fire types are modeled by means of combinatorial analysis. Targets are selected at random in each salvo, meaning there is no coordination between firers, and overkills are allowed.

DSM treats combat as a stochastic process, which is composed of *salvo*s. A salvo is a firing cycle of fixed duration, within which every military unit engages one or more opposing units, and fires simultaneously. In each salvo, a combatant may either utilize *directed fire* (or aimed fire) aiming at a particular opposing combatant, or *area fire* towards the opposing military unit. Attrition occurs only after all firing is over in a salvo; therefore, there is a possibility that a combatant can both kill and be killed in the same salvo. This, of course, may not be suitable for combats between certain types of weapon systems. It may be appropriate for artillery or missile exchanges, but not suitable for close-range directed fire engagements. The assumption of simultaneous salvos can be relaxed by means of the engagement process, which will be described in Chapter 3.

A unit can distribute its force and fire at a number of opposing units. In addition, a number of units may be pooled against a single opposing unit. These allocations may be specified in the combat scenario, or they may emerge from an optimization model such as Özdemirel and Kandiller (2001) have employed.

The major extensions of DSM are noncombat loss and stochastic engagement process, which handles different firing rates of military units. As for the minor extensions, discrete reinforcements and synergy effects due to force division and combination, which slightly modify SSKPs, are considered. Mean and variance of force levels can be estimated using respective survival state probabilities of the units, and risk analysis can be conducted based on these statistics.

In addition to experimentation with DSM, we try to estimate ARCs used in LM and SLM, such that DSM and LM will yield similar figures for force levels throughout the combat. This is attempted through comparison of force levels at the end of each DSM salvo with the respective force levels obtained by SLM using estimated ARCs.

The rest of the thesis is organized as follows. In the next chapter, a review of relevant combat simulation models is provided. Problem definition, DSM approach in modeling of land combat, and DSM details are given in Chapter 3. In Chapter 4, we present implementation design stages, experimentation with DSM, and results as to comparison with the SLM. A method for estimating the ARCs used in LM is given in Chapter 5 with force level comparisons. Our major findings, conclusions and future work directions follow in Chapter 6.

CHAPTER 2

A REVIEW OF COMBAT SIMULATION MODELS

The employment of scientific methods in military problems originated in early twentieth century with the development of deterministic combat models by Lanchester (1916). He worked on air combat in World War I, by applying ordinary differential equations to populations of fighter planes. Since then, many researchers have elaborated on these models with some additional features. These models represent attrition of opposing sides under different types of engagement. Initial studies, which involved deterministic models, were followed by advances leading to stochastic models.

A taxonomy of combat models available in recent literature is shown in Figure 2.1. Optimization models in the first branch are in general concerned with force allocation and deployment. Models in the third branch include game theoretic approaches, in which the combat is modeled as time sequential two-person zero-sum games. This chapter presents the summary of reviewed literature on a subset of the second branch of combat modeling in the figure, i.e. simulation modeling, since it is within the scope of our study. In particular, we review analytical deterministic and stochastic models, leaving out the large subset of the discrete-event simulation models.

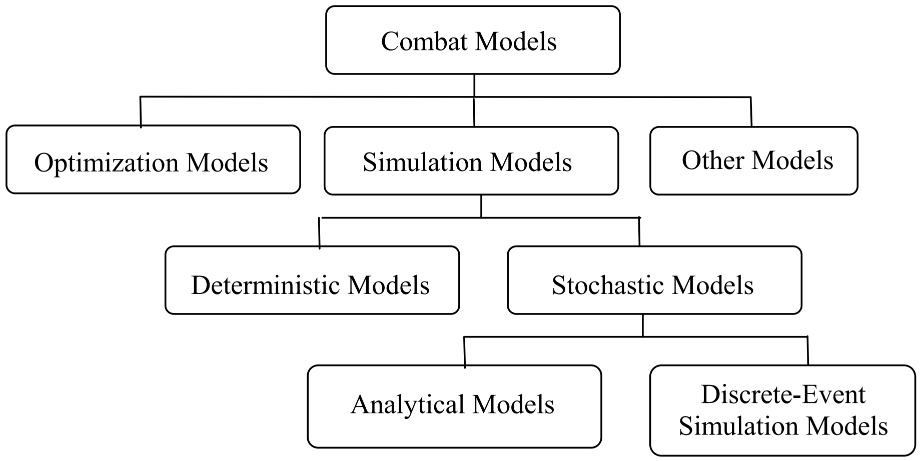


Figure 2.1 Taxonomy of combat models

Combat simulation models estimate attrition and number of survivors in an engagement. Therefore, these models provide predictions that will help in making decisions for upcoming stages of the combat. Analytical combat simulation models are classified as deterministic and stochastic models, which are reviewed in the following two sections.

2.1 Deterministic Combat Simulation Models

Lanchester (1916) proposed using systems of differential equations with static attrition rate coefficients to model combat. These models describe combat dynamics as change in force levels of opponents over time from a deterministic point of view. Lanchester models have been used for attaining information on the general behavior of units, and are applicable to aggregated units involving large numbers of combatants. In Lanchester model, combat is assumed to take place between two homogenous forces (each consisting of only one type of combatant or weapon), and the attrition effects are reflected with constant attrition rate coefficients (ARCs). These fundamental models are represented as a couple of differential equations. In directed fire, where the firer acquires and fires at a single target, the attrition of a force is only dependent on the size of the opposing force.

The directed fire model is as follows:

$$\frac{dB}{dt} = -aR \tag{2.1}$$

$$\frac{dR}{dt} = -bB \tag{2.2}$$

In the above equations, B=$B(t)$ and R=$R(t)$ denote the number of combatants (or weapons) of blue and red forces at time t, whereas a and b are the ARCs symbolizing the effectiveness of one combatant of red and blue forces, respectively. As an example, a represents the number of blue combatants killed by one red combatant per unit time. The initial number of red and blue combatants, B_0 and R_0, sets the initial conditions for these equations.

In the system of differential equations for area fire, where the targets cannot be detected individually but the region they are located is known, the attrition of a force is dependent on both its own size and the size of the opposing force.

$$\frac{dB}{dt} = -aBR \tag{2.3}$$

$$\frac{dR}{dt} = -bRB \tag{2.4}$$

These two models above represent Lanchester's square law for directed (or aimed) fire, and linear law for area fire, which are the two basic attrition equations in Lanchester modeling.

The third fundamental model by Brackney (1959) is called the mixed fire, where the blue force is subject to directed fire and the red is subject to area fire.

$$\frac{dB}{dt} = -aR \tag{2.5}$$

$$\frac{dR}{dt} = -bRB \tag{2.6}$$

It is easy to obtain closed form solutions for time-dependent force levels in directed and area fire models simply by solving the related differential equations with given initial conditions, when forces are homogenous and attrition rate coefficients are constant. Although it is not possible to attain closed form solutions for B and R separately in the mixed fire case, once a final value of either B or R is provided the force level of the other can easily be computed.

The fourth model is called the logarithmic law, which formulates the noncombat loss process and is added to directed fire equations below.

$$\frac{dB}{dt} = -aR - B\beta \tag{2.7}$$

$$\frac{dR}{dt} = -bB - R\alpha \tag{2.8}$$

Here, α and β are noncombat loss rates of blue and red, respectively.

The fifth and final model we review is proposed by Helmbold (1965).

$$\frac{dB}{dt} = -a\left(\frac{B}{R}\right)^{1-\omega} R \tag{2.9}$$

$$\frac{dR}{dt} = -b\left(\frac{R}{B}\right)^{1-\theta} B \tag{2.10}$$

In these equations, $\omega \in (0,1)$ is the fraction of the blue force that can be used effectively against the red, when initial force ratio B_0 / R_0 is high. The motivation behind this model is that when the blue force has too many combatants, they cannot all be used against red simultaneously. θ is defined similarly for the red unit. A symmetrical power function of surviving force ratios is added in this model.

Przemieniecki (1994) concentrated on determining ARCs, which are the basic components of Lanchester systems. The simplest relationship for ARC of a blue unit *i* when the target acquisition time is negligible can be expressed as

$$b_i = SSKP_i \times firing\ rate_i \tag{2.11}$$

where *firing rate* depends not only on the weapon's technical capability but also on the skills of the weapon operator and combat conditions. Finding firing rate during active combat is difficult, hence estimating ARCs is also difficult in practice. However, most of the studies assume that ARCs are available. Dupuy (1979) measured the combat potential of opposing forces by quantifying their total weapons firepower by developing the *Operational Lethality Index* (OLI) concept. This index is a composite measure of a number of factors. The major components of OLI are mission factor, spatial effectiveness measure, and casualty effectiveness measure.

SSKPs, on the other hand, can be obtained from technical and operational data of weapon systems, and from field exercises observing combatant skills. This is a major advantage of the models that use SSKPs instead of ARCs.

There is a literature on extensions and modifications of the above basic Lanchester models for heterogeneous forces. However, most researchers have dealt with homogenous forces due to practical or computational reasons.

Isbell and Marlow (1956) contributed to initial efforts by allowing heterogeneity of forces and investigated the distribution of fire over a number of targets.

Howes and Thrall (1973) developed a procedure that employs Perron-Frobenius theory of eigenvalues and eigenvectors to compute the overall weight (effectiveness) of a heterogeneous force. This overall weight is defined as the sum of weighted-averages of individual weapon effectiveness values, which are derived from inter-weapon effectiveness matrices. These matrices are assumed to be given in their study.

Taylor (1974a) utilized Lanchester equations in combination with optimal control theory in order to find the optimal fire distribution policy. Range-dependent attrition coefficients are used with the assumption that all weapons of a force have the same range capability. Variable attrition rate coefficients are defined as

$$b_i(t) = k_{bi}h(t) \tag{2.12}$$

For blue weapon i, k_{bi} is the constant portion and $h(t)$ is the variable portion of the ARC. Values are assigned per unit of surviving forces, and a two-on-one combat is analyzed. The analyzed combat is heterogeneous in the sense that different survival values and ARCs are assigned for each type of weapon. Formulation for distribution of fire over n targets is presented in the study, and the approach is demonstrated for n=2.

In another study, Taylor (1974b) reviewed common issues related to fire distribution. When the targets are subject to the square law (directed fire) process, the fire is concentrated on one target type, which is known as the 0-1 allocation rule. Besides, the allocation is not completely dependent upon force levels in this case. On the other hand, when the targets undergo the linear law (area fire) process, the fire may be divided between target types; hence, fractional policies other than the 0-1 allocation rule may be applied. The allocation is directly dependent upon force levels in area fire. Additional conclusions are also presented from a different perspective. When intelligence and command control systems are exceedingly efficient, the optimal tactic is to concentrate fire on a specific target type. However, the optimal tactic becomes proportional allocation of fire over target types, when capability for redirection of fire from destroyed targets is rather poor.

As another extension, Taylor (1975) handled the fire distribution problem including force-level constraints in time sequential allocation problems. He used measures of strategic value of firing at a target, indicating that optimal fire distribution policy depends on force levels but not on time. One major finding of this study is the motivation to value the targets directly proportional to their fire effectiveness.

Afterwards, Taylor and Brown (1978) made another improvement on fire distribution problems, which was verified on a two-on-two heterogeneous combat. With this combat, the authors explored the optimal allocation of supporting fires through the tactic that involves attacking infantry to contact enemy defensive positions.

Later on, Taylor (1983) characterized two types of target acquisition processes for Lanchester type combat models, namely serial and parallel acquisitions. In serial acquisition, a firer (or a weapon) cannot acquire targets while it is engaged to another target, while in parallel acquisition it can search targets uninterruptedly while engaging other targets.

Enhancements to Lanchester models cause the problem to be intractable; hence obtaining a closed form solution analytically is either difficult or impossible. Thus, numerical methods for solving these problems are to be investigated. Taylor (1983) illustrated the formulation of Lanchester equations for heterogeneous forces, and suggested the use of simplest methods, such as Euler-Cauchy method, since they are shown to work efficiently due to well behavior of Lanchester equations.

Protopopescu *et al*. (1989) later developed a combat model using partial differential equations featuring the effects of spatial dependence and nonlinearity, in an attempt to overcome some shortcomings of Lanchester equations. Their formulation introduces some realistic concepts that do not exist in classical Lanchester models. One of these, diffusion, is defined as the natural tendency of any force to lose its original configuration as it moves, fights, or simply as time goes by, due to fatigue, loss of concentration, loss of motivation etc. Another notion is advection, which is defined as large-scale, ordered flow of troops in battlefield. In addition, they employ state dependent attrition of forces; ARCs change as forces close on one another.

Hudges (1995) studied the measure of combat power's mental effect, which is the suppression of enemy actions. Based on the observation that the apparent effects of combat power are not only physical but also mental, he developed a quantitative approach using Lanchester square law to illustrate the suppression effect of enemy fire.

Taking into account the solution of Lanchester equations for heterogeneous forces, Jaiswal (1997) described a method that depends on the use of eigenvalues and eigenvectors, where some conditions are necessary for implementation. Fowler (1999) established two techniques for aggregating heterogeneous quadratic Lanchester systems into a homogeneous one. Özdemirel and Kandiller (2001) estimated the attrition of a force by summing up attritions generated by heterogeneous opposing units, and employing division and combination effects.

2.2 Stochastic Combat Simulation Models

When detailed observation of the behavior of each combatant is necessary in engagements between small units (as, for example, in a two-on-three combat of tanks), stochastic models become inevitable. Rather than the force sizes, randomness in engagement, shooting and killing plays an important role.

Snow (1948) was the first to contribute to the stochastic combat simulation literature by treating stochastic Lanchester equations thoroughly. Stochastic Lanchester model is a general renewal model except that inter-fire times are negative exponentially distributed. This assumption brings the memoryless property, which greatly simplifies the analysis but does not make it trivial. Stochastic Lanchester model will be examined in detail in Chapter 3.

Robertson (1956) modeled infantry fire as a sequence of simultaneous salvos forming a Markov chain, and applied this model for homogeneous units with a maximum size of 15. Helmbold (1968) built expressions for expected force levels in many-on-many duel using alternating volleys rather than simultaneous salvos.

Taylor (1983), Gafarian and Ancker (1984), Kress (1987) and Gafarian and Manion (1989) are the major early studies on stochastic combat simulation modeling. They handle stochastic duels (or small firefights) involving small number of combatants. Since then, researchers have contributed by modeling and analyzing different stochastic combat characteristics.

Taylor (1983) made use of classical continuous-time treatment, and utilized attrition rates to determine state probabilities. Since the attrition rates are taken from deterministic LM, his model is referred to as the stochastic Lanchester model (SLM), which will be explained in detail in Chapter 3.

Gafarian and Ancker (1984) studied the general two-on-one stochastic duel as an extension of one-on-one stochastic duel by Ancker (1982). There are two combatants on side A and one on side B, force compositions are homogenous and engagement type is directed fire for both sides. They modeled two stochastic processes, one with negative exponential and the other with Erlang-2 interfiring time distributions. Their main contribution is the computation of state probabilities for the first time for two-on-one stochastic duel. Utilizing these probabilities, they also derived probability of win, and mean and variance of the number of survivors.

Later on, Kress (1987) investigated the general many-on-one stochastic duel conditioned on the order in which the targets are attacked. He has utilized SLM with homogenous forces and directed fire. Other than exponentially distributed interfiring times for both sides, he also studied a special case where the distribution is gamma for the blue force. As results of his study, state probabilities are derived for five different cases for the number of red units, and relative firepower effectiveness of both sides is examined utilizing kill rates or reciprocal of the mean killing times as measures.

Gafarian and Manion (1989) considered two versions of two-on-two homogenous stochastic combat, with the motivation of developing more realistic firefight models. They utilized stochastic process with homogenous forces and directed fire. Interfiring times are assumed to follow a Gamma-2 distribution. Aiming configurations are defined, and states are decomposed with regard to initial aiming configurations. They computed state probabilities, derived probability of win, mean and variance of the number of survivors, and mean and variance of battle duration. They also compared their model with equivalent exponential and deterministic LM.

Yang and Gafarian (1995) introduced an algorithm based on solving a set of exact Kolmogorov equations and approximating the kill rate of one combatant in homogenous stochastic combat models. The kill rate is conditioned on the state of the system. They studied discrete-time many-on-many homogenous systems with directed fire, where they utilized counting process and Kolmogorov equations. They derived exact Kolmogorov equations for states, probabilities for interior and boundary transient states and state-dependent kill rates. They argued that huge amount of computation is necessary for battles larger than four-on-four.

Jaiswal *et al*. (1995) modeled the combat as a continuous-time discrete state space Markov process, and estimated some combat characteristics such as distribution, mean and variance of combat duration, probabilities of win, expected number of survivors at termination, etc. They also presented some numerical results for stochastic Lanchester equations of directed fire, area fire and warfare with smart weapons.

Speight (1995) employed a discrete-time Markov chain model as a stochastic correspondent of deterministic and continuous-time Lanchester formulation, and contrasted the results at the mini-battle level.

Anderson (1995) expressed attrition formulas for large-scale combat under a variety of conditions. He dealt with heterogeneous many-on-many combats, and treated area and directed fires separately. In his study, it is assumed for the area fire that, targets are uniformly distributed in the area, fires of weapons may overlap in each salvo, and a target is killed with a kill probability if it is in the fatal area. He devised directed fire and area fire attrition equations for uncoordinated, partially coordinated and coordinated fire cases.

Parkhideh and Gafarian (1996) studied development of general solutions to many-on-many heterogeneous stochastic combat. They modeled the system as a continuous-time stochastic process, where the combat is modeled as a sequence of aiming and killing events. Engagement type is directed fire for both sides, and time between consecutive kills is randomly distributed. A firer-dependent time-to-next-kill distribution, where target selection is random in aiming events is utilized. Combat ends when any side reaches its predetermined breakpoint. They computed state probabilities by enumerating all possible routes that the combat may go through via itemizing sequences of aiming and killing events, then finding probabilities of events that take the combat to a specific state. They assumed that heterogeneous combat involves only two opposing units, each consisting of different types of combatants utilizing directed fire.

Jaiswal *et al*. (1997) modeled homogenous combat with reinforcements as a continuous-time discrete state space Markov process. He analyzed the effect of reinforcements made at prespecified force levels on various combat characteristics.

McNaught (1999) investigated the effects of applying Exponential Stochastic Lanchester (ESL), which is the stochastic version of deterministic square law for directed fire, to battles that have been split into smaller engagements (mini-battles). He modeled the many-on-many combat as a Markov Chain where the forces are homogenous, and directed fire is employed for both sides. He computed probabilities of win and the expected number of survivors at each mini-battle. He also observed increase in the number of mini-battles in the first stage, change in the force ratio, and random (uneven) split of battles using Monte-Carlo simulation.

McNaught (2001) later solved two variants of homogenous one-on-one duels with directed fire for both sides. He modeled the combat as two continuous time Markov chains, where the distribution of inter-firing times follow a 2-phase Erlang distribution in the first model, and exponential distribution in the second.

Armstrong (2001) studied stochastic duel between two opposing units, in which both kills and suppression effects of firepower are possible, with the motivation of creating a more realistic model. He formulated the one-on-one homogenous combat with directed fire as a Markov Chain. Results are provided regarding probability of win, expected duration of the duel, expected proportion of time the red is suppressed, and expected number of rounds fired by red, which can be utilized in computation of expected consumption of ammunition and the effect of suppression.

Salim and Hamid (2001) used a Bayesian stochastic model in formulating homogenous many-on-many stochastic combat with directed fire, where beta distribution is chosen as a prior distribution for survivor probability. They have estimated the distribution of the number of survivors, and expected value of the attrition rate coefficient using their model.

Pettit *et al*. (2003) illustrated that Bayesian statistical methods may be used both to predict which side will win the combat and to choose between alternative stochastic fire type models, and utilized the results for comparing different weapon systems.

Very recently, Aygüneş (2003) has presented a modeling framework for combat with heterogeneous forces. He proposed an integrated system, consisting of three interacting models: An optimization model for force allocation, an attrition simulation model including a discrete-time stochastic model (DSM) that validates allocation results, and a weapon effectiveness index update model. DSM can handle heterogeneous forces of relatively larger size. Our study is mainly concerned with implementation of and experimentation with DSM. Therefore, an overview of DSM will be provided in Chapter 3.

2.3 Discussion

As it was stated previously, there are common critiques as to the deficiencies of Lanchester models in literature, such as difficulty in estimating ARCs, use of constant ARCs and homogenous forces only. Besides these, Protopopescu *et al*. (1989) indicated some additional shortcomings, such as ignoring movement of forces in battlefield, and not taking into account command and control mechanisms. Ancker (1995) reviewed combat theory and found some deficiencies. Based on two axioms and a theorem on combats, his study emphasizes the fact that analysis of combat as a hierarchical network of firefights is compulsory to better comprehend combat models, where a firefight is a terminating stochastic target attrition process on a discrete state space.

Reviewed literature clarifies some deficiencies or shortcomings in combat models. Combat simulation models can be more efficiently used in force and ammunition planning, as they distinguish more accurately the critical aspects of the problem in hand. It is crucial that the dynamic nature of combat is increasingly taken into consideration by recent developments. However, use of advanced models and techniques are limited, as opposed to the classical Lanchester models, due to computational restrictions.

Deterministic Lanchester models are usable basically for aggregate forces (size more than 20), and it has been revealed in many studies that their performance for predicting smaller size combats is rather unsatisfactory. Furthermore, there is no possibility of a risk analysis in a deterministic model. Besides, successful convergence for more accurate results occurs for very large force sizes only, which has limited relevance to today's war situations. Therefore, we need stochastic models for small forces that will allow us to conduct risk analysis. Although numeric solutions can be obtained by stochastic models available in the literature, most of which are continuous time, they require excessive computation time and can handle very small force sizes most of the time.

A discrete-time stochastic model (DSM) has been developed by Kandiller *et al.* (2002) that can handle relatively larger force levels. In this study, our main contribution is the implementation of and experimentation with this model. In addition, we try to estimate ARCs for Lanchester model, such that the use of these models will yield similar figures for force levels. This is attempted through comparison of force levels at the end of each DSM salvo with the respective force levels obtained by stochastic Lanchester model using estimated ARCs.

CHAPTER 3

DISCRETE-TIME STOCHASTIC MODEL

Our contribution in this study is the implementation of and experimentation with DSM developed by Kandiller et al. (2002). Therefore, DSM is described for the sake of completeness. As we have stated previously, there are many critiques as to the deficiencies of Lanchester models in literature, such as difficulty in estimating ARCs, use of constant ARCs and homogenous forces only. Deterministic Lanchester models are usable basically for aggregate forces (size more than 20), and it has been revealed in many studies that their performance for predicting smaller size combats is rather unsatisfactory. Furthermore, there is no possibility of a risk analysis in a deterministic model. Therefore, we need stochastic models for small forces that will allow us to conduct risk analysis.

DSM is based on SSKPs, which can handle relatively larger force levels, as compared to previous stochastic models in literature. Forces are assumed to be heterogeneous on both sides, division and combination of units are allowed, and both directed and area fire types are modeled by means of combinatorial analysis.

The main features and assumptions for this combat model can be summarized as follows:

- Time advancement is achieved by assuming the combat proceeds in salvos. Combatants of both forces fire simultaneously in a salvo.

- Both sides can consist of a number of units of different types (having different SSKPs), which implies heterogeneity of the forces.
- Directed and area fire are modeled by concepts of combinatorial analysis. Target selection is a random process, accordingly, overkills of a target are allowed, as no coordination among firers exists.
- Noncombat loss and stochastic engagement processes can be included as well as discrete reinforcements, division and combination effects.
- The mean and variance of the remaining force level at the end of each salvo are found, allowing for risk analysis.

In the following section, we will firstly describe SLM, which we implemented in our study for estimating ARCs and checking whether the use of these models yields similar figures for combat force levels. The subsequent two sections describe DSM for homogeneous and heterogeneous combat cases in sufficient detail. The last section is about the extensions of DSM, whose major processes are employed in our implementation.

3.1 Stochastic Lanchester Model (SLM)

The classical continuous-time treatment by Taylor (1983) makes use of attrition rates to determine the state probabilities. This model can be applied for homogeneous combat where only two opposing units are involved. Let the state definition be (t,i,j) where t indicates time, $i = 0,1,\ldots,m$ and $j = 0,1,\ldots,n$ are the number of combatants alive in blue and red units at time t. Let $P(t,i,j)$ be the probability of having i blue and j red combatants at time t. The initial condition is $P(0,m,n) = 1$. There are three possible state transitions in this model. These are no loss, one blue casualty and one red casualty. Let $A(t,i,j)$ be the attrition rate of the blue unit at time t when there are i blue and j red combatants. The attrition rate of the red unit is denoted by $B(t,i,j)$. Since the attrition rates are borrowed from deterministic LM, this model is referred to as the stochastic Lanchester model (SLM). The attrition process is modeled as $\frac{dB}{dt} = -A(t,i,j)$ with B_0=m and

$\frac{dR}{dt} = -B(t,i,j)$ with $R_0=n$. This is a Markov process since the probability of any particular future state is determined by the present state and not on how the state is reached. Assuming Poisson process for casualties, it follows that

$$P(\text{one blue casualty in time from } t \rightarrow t+\Delta t) = A(t,i,j)\Delta t \quad (3.1)$$

$$P(\text{one red casualty in time from } t \rightarrow t+\Delta t) = B(t,i,j)\Delta t \quad (3.2)$$

$$P(\text{more than one casualty in time from } t \rightarrow t+\Delta t) = 0 \quad (3.3)$$

Hence, the conditional probability for state $(t+\Delta t, i, j)$ is

$P(t+\Delta t, i, j) = P(t,i,j)\ P(\text{no casualties in } \Delta t) + P(t, i+1, j)\ P(\text{one blue casualty in } \Delta t) + P(t, i, j+1)\ P(\text{one red casualty in } \Delta t)$.

Applying Equations (3.1), (3.2) and (3.3) to the first term of the above equation, we see that $P(\text{no casualties occur}) = (1-A(t,i,j)\Delta t\ )\ (1-B(t,i,j)\Delta t) = 1 - (A(t,i,j) + B(t,i,j))\Delta t$. The term $A(t,i,j)B(t,i,j)\Delta t^2 = 0$, since we cannot have more than one casualty at any given time. Another motivation in ignoring this term is that the infinitesimal duration Δt is so small to define legitimate probabilities in (3.1), (3.2) and (3.3), yielding its square being close to zero. Substituting this into the equation above, with the suitable choice of infinitesimal time step Δt, the state transition probabilities are found as:

$$P(t + \Delta t, i, j) = [1 - (A(t, i, j) + B(t, i, j))\ \Delta t]\ P(t, i, j) + A(t, i+1, j)\ \Delta t\ P(t, i+1, j) + B(t, i, j+1))\ \Delta t\ P(t, i, j+1). \quad (3.4)$$

One can assume that Δt is the salvo length and define the salvo sequence as 0, Δt, $2\Delta t$, $3\Delta t$, and so on. Equation (3.4) can then be treated as a difference equation and it can be used to calculate discrete-time state probabilities. Time-dependent expected values and variances of remaining force levels can also be calculated.

SLM models the combat as a non-homogeneous Poisson process where $A(t,i,j)\Delta t$ is the expected number of blue casualties during Δt. Therefore, Δt should be determined such that the probability of one casualty ($A(t,i,j)\Delta t$ for blue or $B(t\ i,j)\Delta t$ for red), does not exceed one. If we employ Δt as the salvo length in a discrete-time approach, we would have to observe combat dynamics over a very large number of salvos, which will require significant computation time. In Chapter 4, we compare SLM and DSM by taking salvo length as an appropriate multiple of Δt.

3.2 DSM for Homogeneous Combat

The main features of DSM that distinguish it from SLM are advancing time in discrete steps as opposed to continuously, the manner in which SSKPs are used, and the relaxation of single casualty per salvo assumption. The discrete-time nature of the model requires focusing on binomial processes.

In DSM, combat between two units each having a small number of combatants is modeled as a two-dimensional death process. Let the state definition be (t,i,j) where t = 0, 1,… is the discrete time counter denoting the salvo number, i = 0, 1,…, m and j = 0, 1,…, n are the number of surviving combatants in blue and red units at the end of salvo t. Let $P(t,i,j)$ be the probability of having i blue and j red combatants at the end of salvo t. The initial condition is $P(0,m,n) = 1$.

3.2.1 Directed Fire in Homogeneous Combat

Directed fire is the situation where a firer detects and aims at a single target and fires. Consider a combat situation in which there are i identical firers shooting independently at j = 3 identical targets. Let $\{A,B,C\}$ be the pattern denoting the number of firers engaged with each target such that $A + B + C = i$. If, for example, i = 4, {3,1,0} means that three firers shoot at one target, and the remaining firer engages with one of the remaining targets. Note that overkills are possible with this pattern definition. When arrangement of targets is considered, the pattern

$\{A,B,C\}$ would repeat as $\{A,B,C\}$, $\{A,C,B\}$, $\{B,A,C\}$, $\{B,C,A\}$, $\{C,A,B\}$, $\{C,B,A\}$. The number of arrangements, $n_T(A,B,C)$, is $\frac{3!}{1!1!1!}$ if A, B, C are all different, $\frac{3!}{2!1!}$ if only two of them are the same, and $\frac{3!}{3!}$ if all three are the same. When we also consider arrangement of firers, there are additional repetitions. For example, with firers a, b, c, d and target arrangement (3,1,0), possible firer arrangements are ($abc,d,$-), ($abd,c,$-), ($acd,b,$-), ($bcd,a,$-). The number of these repetitions $n_F(A,B,C)$ is $\frac{i!}{A!\,B!\,C!}$ for each target arrangement. Hence the total number of arrangements for pattern $\{A,B,C\}$ is:

$$n(\{A,B,C\}) = n_T(A,B,C)\, n_F(A,B,C). \tag{3.5}$$

Let p_k be the SSKP of a single firer. Given a pattern $\{A,B,C\}$, the probability of having l = 0, 1, 2, 3 casualties is calculated considering whether or not the first target subject to A shots is killed and so on, i.e.:

$$P\left(l \text{ casualties} \mid \{A,B,C\}\right) = \sum_{\substack{l_1,l_2,l_3:0 \text{ or } 1 \\ l_1+l_2+l_3=l}} \begin{matrix} \left[1-(1-p_k)^A\right]^{l_1}\left[(1-p_k)^A\right]^{1-l_1} \\ \left[1-(1-p_k)^B\right]^{l_2}\left[(1-p_k)^B\right]^{1-l_2} \\ \left[1-(1-p_k)^C\right]^{l_3}\left[(1-p_k)^C\right]^{1-l_3} \end{matrix} \tag{3.6}$$

Hence, the probability of l casualties in the presence of i firers is calculated as:

$$P_i(l) = P(l \text{ casualties with } i \text{ firers}) = \sum_{\substack{A\geq B\geq C \\ A+B+C=i}} \frac{n(\{A,B,C\})}{j^i} P\left(l \text{ casualties} \mid \{A,B,C\}\right) \tag{3.7}$$

where j^i is the total number of arrangements over all patterns.

An example with i =4 firers, j =3 targets, and p_k = 0.2 is presented in Table 3.1.

Table 3.1 Directed fire casualty probabilities for i=4, j=3, p_k=0.2

$\{A,B,C\}$	$n_T(A,B,C)$	$n_F(A,B,C)$	$n(\{A,B,C\})$	$P_4(0)$	$P_4(1)$	$P_4(2)$	$P_4(3)$	Total
{4,0,0}	3	1	3	0.4096	0.5904	0.0000	0.0000	1.0000
{3,1,0}	6	4	24	0.4096	0.4928	0.0976	0.0000	1.0000
{2,2,0}	3	6	18	0.4096	0.4608	0.1296	0.0000	1.0000
{2,1,1}	3	12	36	0.4096	0.4352	0.1408	0.0144	1.0000
Total			81	0.4096	0.4637	0.1203	0.0064	1.0000

3.2.2 Area Fire in Homogeneous Combat

In area fire, a firer cannot identify targets individually, but it has information about the region in which the opposing unit is positioned. The targets are assumed uniformly distributed over an area of radius R_A as in the study by Anderson (1995). An area shot divides this region into two, where the first division becomes the fatal area (*FA*) of radius r, and the second becomes the non-fatal area (*NFA*) as in Figure 3.1.

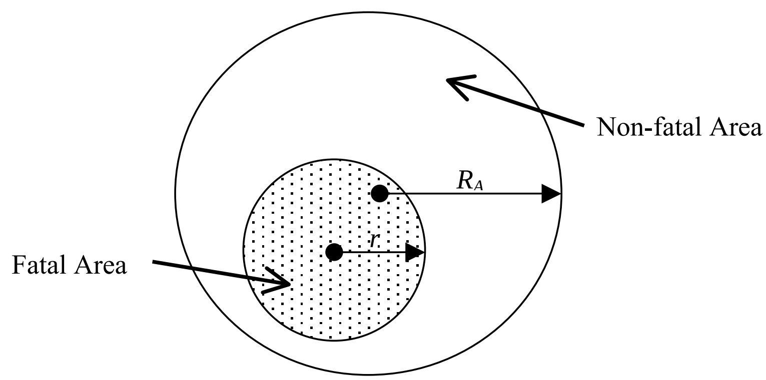


Figure 3.1 Regions of fatality in area fire

The SSKP in *NFA* is assumed to be zero, whereas it is p_k in *FA*. Let $\xi = \dfrac{r^2}{R_A^{\ 2}}$ be the probability that a target is in the *FA*. In the presence of a single firer, the probability of l casualties out of j targets depends on the condition that there are $f\left(l \le f \le j\right)$ targets in *FA* and l of them are killed. Therefore,

$$P_1(l) = P(l \text{ casualties with 1 firer}) = \sum_{f=l}^{j} \binom{j}{f} \; \xi^f (1-\xi)^{j-f} \; \binom{f}{l} p_k^l (1-p_k)^{f-l} \qquad (3.8)$$

Consider the case where there are i=3 firers. If the targets neutralized by each firer were mutually exclusive (i.e. there were no overkills), then the probability of having l casualties in a salvo would be calculated by the following 3-way convolution:

$$P_3\left(l\right) = P\left(l \text{ casualties with 3 firers}\right) = \sum_{\substack{l_1, l_2, l_3: \\ l_1+l_2+l_3=l}} P_1\left(l_1\right) P_1\left(l_2\right) P_1\left(l_3\right) \qquad (3.9)$$

In the presence of overkills, let the pattern (l_1, l_2, l_3| l_{12}, l_{13}, l_{23}| l_{123}) denote the number of casualties where l_1 is the number of kills only by the first firer, l_{12} is the number of kills only by the first and the second firer, and so on. For example, the pattern (1,2,0|1,2,0|1) indicates that one target is overkilled by all three firers, another is overkilled by the first and the second together, two targets are overkilled by the first and the third together, one is killed only by the first firer, and two only by the second. If there are i firers, we have patterns of dimension 2^i-1 in the form of ($\binom{i}{1}$entries|…|$\binom{i}{k}$entries | … |$\binom{i}{i}$ entry). When we consider arrangement of targets, the number of repetitions for pattern (l_1, l_2, l_3| l_{12}, l_{13}, l_{23}| l_{123}) is:

$$\begin{aligned} n\left(l_1, l_2, l_3 \mid l_{12}, l_{13}, l_{23} \mid l_{123}\right) &= \binom{j}{j-l}\binom{l}{l_{123}}\binom{l-l_{123}}{l_{12}}\binom{l-l_{123}-l_{12}}{l_{13}} \cdots \binom{l_1}{l_1} \\ &= \binom{j}{j-l, l_1, l_2, l_3, l_{12}, l_{13}, l_{23}, l_{123}} = \frac{j!}{(j-l)!\,l_1!\,l_2!\,l_3!\,l_{12}!\,l_{13}!\,l_{23}!\,l_{123}!} \end{aligned} \qquad (3.10)$$

where $l = l_1 + l_2 + l_3 + l_{12} + l_{13} + l_{23} + l_{123}$ is the total number of casualties.

Let l_A, l_B, l_C denote the number of kills (including overkills) by the first, second and third firers, i.e., $l_A = l_1 + l_{12} + l_{13} + l_{123}$, $l_B = l_2 + l_{12} + l_{23} + l_{123}$, $l_C = l_3 + l_{13} + l_{23} + l_{123}$. Given l_A, l_B, l_C, we face some size restrictions. For instance the two-way overkill value l_{12} = max{0, $l_A + l_B - j$},…, min{l_A, l_B}. That is, if we have j=3 targets, i=2 firers, l_A=3 and l_B=1, then we cannot have the pattern (1,0,0|2,0,0|0) since the number of targets overkilled by the first two firers (l_{12}=2) cannot exceed the overall number of casualties due to the second firer (l_B=1). The pattern (2,1,0|1,0,0|0) is also impossible when l_A=3, l_B=2, and j=3 since there cannot be more than j casualties. So, l_{12} should be at least $l_A + l_B - j = 2$ in which case l_{12}=2, l_1=1 and l_2=0 should be true. Let (*) denote such restrictions and $n(l_A,l_B,l_C)$ be the total number of pattern repetitions possible under these restrictions, i.e.

$$n(l_A, l_B, l_C) = \sum_{(l_1,l_2,l_3,l_{12},l_{13},l_{23},l_{123}) \in S_3} n(l_1, l_2, l_3 \mid l_{12}, l_{13}, l_{23} \mid l_{123}) \tag{3.11}$$

where

$S_3 = \{(l_1, l_2, l_3, l_{12}, l_{13}, l_{23}, l_{123}) \in Z_+^7 : (*) \text{ is satisfied}\}$,

$l_A = l_1 + l_{12} + l_{13} + l_{123}$,

$l_B = l_2 + l_{12} + l_{23} + l_{123}$,

$l_C = l_3 + l_{13} + l_{23} + l_{123}$.

Hence, the probability of l casualties in the presence of 3 firers is calculated as

$$\begin{aligned} P_3(l) &= P(l \text{ casualties with 3 firers}) \\ &= \sum_{\substack{(l_1,l_2,l_3,l_{12},l_{13},l_{23},l_{123}) \in S_3 \\ l_1+l_2+l_3+l_{12}+l_{13}+l_{23}+l_{123}=l}} \frac{n(l_1,l_2,l_3 \mid l_{12},l_{13},l_{23} \mid l_{123})}{n(l_A,l_B,l_C)} P_1(l_A)P_1(l_B)P_1(l_C) \end{aligned} \tag{3.12}$$

An example with i=2 firers and j=3 targets is illustrated in Table 3.2. Suppose that $P_1(l)$ values are found by Equation (3.8) as 0.4096, 0.4944, 0.0931, 0.0029 for l=0,1,2,3, and they are the same for the two firers. Consider the rows where l_A=2 and l_B=1, generating the patterns {1,0|1} and {2,1|0}. Using Equation (3.10), the number of repetitions is n(1,0|1)=6 for the first pattern and n(2,1|0)=3 for the second. Thus, n(2,1)=6+3=9. According to Equation (3.12), the joint probability $P_1(2)P_1(1)$=0.0460 is distributed between $P_2(2)$ and $P_2(3)$ with proportions 6/9 and 3/9, resulting in the values 0.0307 and 0.0153, respectively. After processing all patterns in this manner, $P_2(2)$ and $P_2(3)$ are found as 0.3035 and 0.0422. Note that, l_A=2, l_B=1 and l_A=1, l_B=2 are treated separately to account for firer arrangements.

Table 3.2 Area fire casualty probabilities for i=2, j=3 when FA locations are unknown

$(l_1, l_2 \mid l_{12})$	l_A	l_B	$n(l_1, l_2 \mid l_{12})$	$n(l_A, l_B)$	$P_1(l_A)P_1(l_B)$	$P_2(0)$	$P_2(1)$	$P_2(2)$	$P_2(3)$
(0, 0 | 3)	3	3	1	1	0.0000				0.0000
(1, 0 | 2)	3	2	3	3	0.0003				0.0003
(2, 0 | 1)	3	1	3	3	0.0014				0.0014
(3, 0 | 0)	3	0	1	1	0.0012				0.0012
(0, 1 | 2)	2	3	3	3	0.0003				0.0003
(0, 0 | 2)	2	2	3	9	0.0087			0.0029	
(1, 1 | 1)	2	2	6	9	0.0087				0.0058
(1, 0 | 1)	2	1	6	9	0.0460			0.0307	
(2, 1 | 0)	2	1	3	9	0.0460				0.0153
(2, 0 | 0)	2	0	3	3	0.0381			0.0381	
(0, 2 | 1)	1	3	3	3	0.0014				0.0014
(0, 1 | 1)	1	2	6	9	0.0460			0.0307	
(1, 2 | 0)	1	2	3	9	0.0460				0.0153
(0, 0 | 1)	1	1	3	9	0.0244		0.0815		
(1, 1 | 0)	1	1	6	9	0.0244			0.1630	
(1, 0 | 0)	1	0	3	3	0.2025		0.2025		
(0, 3 | 0)	0	3	1	1	0.0012				0.0012
(0, 2 | 0)	0	2	3	3	0.0381			0.0381	
(0, 1 | 0)	0	1	3	3	0.2025		0.2025		
(0, 0 | 0)	0	0	1	1	0.1678	0.1678			
Total			64			0.1678	0.4865	0.3035	0.0422

3.2.3 Salvo Treatment in Homogeneous Combat

State transition probabilities in DSM are calculated based on the binomial processes discussed in the previous two subsections. DSM allows multiple casualties in a salvo in both forces. Possible states that can be reached from state (t,i,j) in the case of directed fire are as follows:

$$(t,i,j)\rightarrow (t+1,i,j), (t+1,i-1,j), (t+1,i,j-1), (t+1,i-1,j-1), (t+1,i-2,j), (t+1,i,j-2),$$
$$(t+1,i-2,j-1), (t+1,i-1,j-2), (t+1,i-2,j-2),\ldots, (t+1,\max\{i-j,0\},\max\{j-i,0\})$$

If area fire is involved, all the states down to $(t+1,0,0)$ can also be reached.

The state transition $(t,i,j)\rightarrow (t+1,i-\Delta i,j-\Delta j)$ indicates that there are Δi blue casualties with probability $P_j^B(\Delta i)$, and Δj red casualties with probability $P_i^R(\Delta j)$ in a duel of i blue versus j red combatants. Regardless of the fire type, the corresponding state transition probability is found as

$$P((t,i,j)\rightarrow (t+1,i-\Delta i,j-\Delta j))= P_j^B(\Delta i)\ P_i^R(\Delta j) \tag{3.13}$$

As an example, Figure 3.2 illustrates all possible state transitions from the state (t,4,3) under directed fire. Casualty probabilities for the red unit with $p_{k:B,R}$=0.2 are taken from Table 3.1. Let casualty probabilities for the blue unit with $p_{k:R,B}$=0.3 be 0.3430, 0.4899, 0.1569, 0.0101, 0.0000 for 0,1,2,3 and 4 casualties, respectively. The probability of staying in the same state is the probability of no blue or red casualties, 0.3430 (0.4096) = 0.1405. The probability of transition to state $(t+1,2,2)$ is the probability of having two blue casualties and one red casualty, which is 0.1569 (0.4637) = 0.0728. Since we have only 3 red combatants, more than 3 casualties in blue are impossible as shown in the last row of Figure 3.2. Transitions from a lower state such as $(t,2,1)$ to an upper state such as $(t,3,2)$ would also be impossible.

			Red			
			P(0 cas.)	P(1 cas.)	P(2 cas.)	P(3 cas.)
			0.4096	0.4637	0.1203	0.0064
Blue			j=3	j=2	j=1	j=0
P(0 cas.)	0.3430	i=4	0.140493	0.159050	0.041262	0.002195
P(1 cas.)	0.4899	i=3	0.200678	0.227186	0.058938	0.003136
P(2 cas.)	0.1569	i=2	0.064282	0.072773	0.018879	0.001004
P(3 cas.)	0.0101	i=1	0.004147	0.004695	0.001218	0.000065
P(4 cas.)	0.0000	i=0	0.000000	0.000000	0.000000	0.000000

Figure 3.2 State transition probabilities from the state (*t*,4,3)

Given the initial condition $P(0,m,n) = 1$, the state probabilities for subsequent salvos when both blue and red units are subject to directed fire are calculated as

$$P(t+1,i,j) = \sum_{\Delta i=0}^{\min\{m-i,n\}} \sum_{\Delta j=0}^{\min\{n-j,m\}} \left[P_j^B(\Delta i)P_i^R(\Delta j)\right] P(t,i+\Delta i, j+\Delta j) \qquad (3.14)$$

The state probabilities under area fire are:

$$P(t+1,i,j) = \sum_{\Delta i=0}^{m-i} \sum_{\Delta j=0}^{n-j} \left[P_j^B(\Delta i)P_i^R(\Delta j)\right] P(t,i+\Delta i, j+\Delta j) \qquad (3.15)$$

Because of the initial condition $P(0,m,n) = 1$, the state probabilities $P(1,i,j)$, i=0,1,…, m=4, j=0,1,…, n=3 at the end of the first salvo are the same as the transition probabilities from the state (0,4,3) given in Figure 3.2. Marginal probabilities of having 4,3,2,1 blue combatants alive at the end of the first salvo are 0.343000, 0.489938, 0.156938, and 0.010125. Therefore, expected value and variance of the number of surviving blue combatants are found as 3.165813 and 0.512944. The same values for the red unit are 2.276504 and 0.4790442, respectively.

State probabilities $P(2,i,j)$ for the second salvo are given in Figure 3.3. The probability of transition from state (1,4,3) to (2,3,2) is 0.227186 in Figure 3.2. The contribution of this transition to $P(2,3,2)$ is 0.227186 $P(1,4,3)$ = 0.031918, as $P(1,4,3)$ = 0.140493. Similarly, contributions of transitions to state (2,3,2) from states (1,4,2), (1,3,3) and (1,3,2) are found as 0.028828, 0.028696, and 0.056996. In Figure 3.3, $P(2,3,2) = 0.146438$ is found by adding up all four contributions.

Red

Blue		j=3	j=2	j=1	j=0
	i=4	0.019738	0.054267	0.056049	0.027147
	i=3	0.063436	0.146438	0.122257	0.044006
	i=2	0.075235	0.139552	0.089591	0.021992
	i=1	0.038825	0.055087	0.025292	0.003748
	i=0	0.007517	0.007517	0.002165	0.000141

Figure 3.3 State probabilities at the end of the second salvo

Expected number of blue and red combatants for the first ten salvos and respective variances are given in Table 3.3. As the salvo number increases, the rate of change in the expected number of combatants decreases indicating the convergence. The variance increases as the diffusion from the initial state takes effect. It would start to decrease eventually as the absorbing states start getting higher probabilities. It is also possible to find the confidence intervals around the expected values for a given confidence level.

Table 3.3 Results for the first ten salvos

	Blue		Red	
Salvo	Expected	Variance	Expected	Variance
0	4.000000	0.000000	3.000000	0.000000
1	3.165813	0.512944	2.276504	0.479042
2	2.532910	0.913262	1.715330	0.807202
3	2.072989	1.254982	1.317807	0.958030
4	1.768699	1.497381	1.067736	0.997139
5	1.583666	1.651056	0.921509	1.004077
6	1.476729	1.746568	0.839240	1.007064
7	1.416499	1.805713	0.793784	1.010683
8	1.382942	1.842004	0.768834	1.014282
9	1.364299	1.863985	0.755151	1.017173
10	1.353935	1.877126	0.747631	1.019214

3.3 DSM for Heterogeneous Combat

Combat between more than two units each having a small number of combatants is modeled as a multi-dimensional death process. Let the state definition be (t, $i_1,\ldots, i_I, j_1,\ldots, j_J$) where $t = 0,1,\ldots$ is the discrete time counter denoting the salvo number, $i_1=0,1,\ldots,m_1,\ldots, i_I = 0,1,\ldots,m_I$ and $j_1=0,1,\ldots,n_1,\ldots, j_J = 0,1,\ldots,n_J$ be the number of combatants remaining in blue and red units at the end of salvo t, respectively. Let $P(t, i_1,\ldots, i_I, j_1,\ldots, j_J)$ be the probability of having $i_1,\ldots, i_I$ blue and $j_1,\ldots, j_J$ red combatants at the end of salvo t. The initial condition is $P(0,m_1,\ldots,m_I,n_1,\ldots, n_J)=1$.

3.3.1 Directed Fire in Heterogeneous Combat

If there are I blue units with $i_1, i_2,\ldots, i_I$ firers shooting at j targets in a certain red unit, the number of different arrangements is $j^{i_1} j^{i_2} \cdots j^{i_I}$.

In particular, consider a combat situation where there are i_1 and i_2 firers in two different units shooting independently at j=3 identical targets with SSKP values p_{k1} and p_{k2}. Let $\{A_1A_2|B_1B_2|C_1C_2\}$ be the pattern denoting the number of firers engaged with each of the j=3 targets such that $A_1+ B_1+ C_1=i_1$ and $A_2+ B_2+ C_2=i_2$. For i_1=4 and i_2=3, an example pattern is {12|21|10}. The total number of repetitions for a pattern $n(\{A_1A_2|B_1B_2|C_1C_2\})$ is found as in Section 3.2.1 for the homogeneous combat case by considering both target and firer engagements. Given a pattern $\{A_1A_2|B_1B_2|C_1C_2\}$, the probability of having l = 0,1,2,3 casualties depends on whether or not the first target subject to A_1 shots from the first unit and A_2 shots from the second unit is killed, and so on. Then, simplifying p_{k1} and p_{k2} as p_1 and p_2,

$$P\left(l \text{ casualties} \,|\, \{A_1A_2 \,|\, B_1B_2 \,|\, C_1C_2\}\right) = \sum_{\substack{l_A,l_B,l_C:0 \text{ or } 1 \\ l_A+l_B+l_C=l}} \left[1-(1-p_1)^{A_1}(1-p_2)^{A_2}\right]^{l_A}$$
$$\left[(1-p_1)^{A_1}(1-p_2)^{A_2}\right]^{1-l_A} \left[1-(1-p_1)^{B_1}(1-p_2)^{B_2}\right]^{l_B} \left[(1-p_1)^{B_1}(1-p_2)^{B_2}\right]^{1-l_B}$$
$$\left[1-(1-p_1)^{C_1}(1-p_2)^{C_2}\right]^{l_C} \left[(1-p_1)^{C_1}(1-p_2)^{C_2}\right]^{1-l_C} \qquad (3.16)$$

Hence, the probability of l casualties under fire from two units is

$$P_{i_1,i_2}(l) = \sum_{\substack{A_1+B_1+C_1=i_1 \\ A_2+B_2+C_2=i_2}} \frac{n\left(\{A_1A_2 \mid B_1B_2 \mid C_1C_2\}\right)}{j^{i_1} j^{i_2}} P\left(l \quad cas. \mid \{A_1A_2 \mid B_1B_2 \mid C_1C_2\}\right) \quad (3.17)$$

An example where i_1=4, i_2=3 and j=3 with p_1=0.2 and p_2=0.3 is illustrated in Table 3.4. The probability of two casualties for the pattern {21|11|11} is calculated as P(2 cas.|{21|11|11})=2[1-0.8^2(0.7)][1-0.8(0.7)][0.8(0.7)]+[0.8^2(0.7)][1-0.8(0.7)][1-0.8(0.7)]=0.358758. This pattern contributes to the overall 2 red casualties with 0.358758 (216/2187) = 0.035433.

The analysis so far in this subsection is carried out from the viewpoint of firers. The same phenomenon could also be analyzed from the viewpoint of targets. When we consider the above example from the targets' perspective, there are two enemy units creating casualties (casualty probabilities for the first blue unit are given in Table 3.1). The overall attrition process of targets is simply the convolution of the two attrition processes due to different firing units. This approach is similar to the one we use for the area fire. If we apply the same analysis presented in Section 3.2.2 and summarized by Equation (3.12), we obtain the results given in Table 3.5. The overall casualty probabilities calculated in Tables 3.4 and 3.5 are exactly the same. This means that we can use either viewpoint in combining multiple units.

Table 3.4 Casualty probabilities for j=3 when i_1=4, i_2=3, p_1=0.2, p_2=0.3

$\{A_1A_2\|B_1B_2\|C_1C\}$	$n(\{A_1A_2\|B_1B_2\|C_1C_2\})$	$P_{4,3}(0)$	$P_{4,3}(1)$	$P_{4,3}(2)$	$P_{4,3}(3)$
{43|00|00}	3	0.1405	0.8595	0.0000	0.0000
{40|03|00}	6	0.1405	0.4716	0.3879	0.0000
{42|01|00}	18	0.1405	0.6197	0.2398	0.0000
{41|02|00}	18	0.1405	0.4957	0.3638	0.0000
{40|02|01}	18	0.1405	0.4089	0.3602	0.0903
{41|01|01}	18	0.1405	0.4699	0.3254	0.0642
{33|10|00}	24	0.1405	0.6946	0.1649	0.0000
{30|13|00}	24	0.1405	0.5054	0.3541	0.0000
{30|10|03}	24	0.1405	0.4381	0.3572	0.0641
{32|11|00}	72	0.1405	0.5299	0.3296	0.0000
{32|10|01}	72	0.1405	0.5148	0.2997	0.0449
{31|12|00}	72	0.1405	0.4694	0.3901	0.0000
{31|10|02}	72	0.1405	0.4329	0.3612	0.0654
{30|12|01}	72	0.1405	0.4120	0.3585	0.0890
{30|11|02}	72	0.1405	0.3905	0.3595	0.1095
{31|11|01}	144	0.1405	0.4221	0.3527	0.0847
{23|20|00}	36	0.1405	0.5785	0.2810	0.0000
{20|20|03}	18	0.1405	0.4272	0.3472	0.0851
{22|21|00}	108	0.1405	0.4806	0.3789	0.0000
{22|20|01}	108	0.1405	0.4467	0.3386	0.0741
{21|20|02}	108	0.1405	0.3984	0.3598	0.1013
{21|21|01}	108	0.1405	0.4064	0.3617	0.0914
{23|10|10}	36	0.1405	0.5698	0.2585	0.0312
{20|13|10}	72	0.1405	0.4857	0.3216	0.0522
{22|11|10}	216	0.1405	0.4530	0.3461	0.0604
{20|12|11}	216	0.1405	0.4073	0.3559	0.0963
{21|12|10}	216	0.1405	0.4261	0.3662	0.0671
{21|11|11}	216	0.1405	0.3939	0.3588	0.1069
Total	2187	0.1405	0.4451	0.3479	0.0665

Table 3.5 Casualty probabilities for j=3 when i_1=4, i_2=3, p_1=0.2, p_2=0.3 using area fire approach (target viewpoint)

$(l_1, l_2 \| l_{12})$	l_A	l_B	$P_4(l_A)P_3(l_B)$	$n(l_1, l_2 \| l_{12})$	$n(l_A, l_B)$	$P_{4,3}(0)$	$P_{4,3}(1)$	$P_{4,3}(2)$	$P_{4,3}(3)$
(0,0|3)	3	3	0.0000	1	1				0.0000
(1,0|2)	3	2	0.0009	3	3				0.0009
(2,0|1)	3	1	0.0032	3	3				0.0032
(3,0|0)	3	0	0.0022	1	1				0.0022
(0,1|2)	2	3	0.0007	3	3				0.0007
(0,0|2)	2	2	0.0173	3	9			0.0058	
(1,1|1)	2	2	0.0173	6	9				0.0115
(1,0|1)	2	1	0.0610	6	9			0.0407	
(2,1|0)	2	1	0.0610	3	9				0.0203
(2,0|0)	2	0	0.0413	3	3				0.0413
(0,2|1)	1	3	0.0028	3	3				0.0028
(0,1|1)	1	2	0.0668	6	9			0.0445	
(1,2|0)	1	2	0.0668	3	9				0.0223
(0,0|1)	1	1	0.2351	3	9		0.0784		
(1,1|0)	1	1	0.2351	6	9			0.01567	
(1,0|0)	1	0	0.1591	3	3		0.1591		
(0,3|0)	0	3	0.0025	1	1				0.0025
(0,2|0)	0	2	0.0590	3	3			0.0590	
(0,1|0)	0	1	0.2077	3	3		0.2077		
(0,0|0)	0	0	0.1405	1	1	0.1405			
Total				64		0.1405	0.4451	0.3479	0.0665

3.3.2 Area Fire in Heterogeneous Combat

Consider the case where there are two different area firing blue units with i_1, i_2 firers, and there are j red targets. Suppose $P_k(l) = P(l$ casualties with k firers) have already been calculated independently for $k = i_1, i_2$ using Equation (3.12). When two units are combined, the pattern $(l_1, l_2 | l_{12})$ represents the number of casualties where l_1 is the number of kills only by the first unit, l_{12} denotes the number of overkills by the first and second units. Let $n(l_1, l_2 | l_{12})$ denote the number of casualty arrangements and $n(l_A, l_B)$ be the total number of repetitions as defined in Section 3.2.2. With the same definition of S_2, the probability of l casualties under the simultaneous fire of two units is calculated as

$$P_{i_1,i_2}(l) = \sum_{\substack{(l_1,l_2,l_{12}) \in S_2 \\ l_1+l_2+l_{12}=l}} \frac{n(l_1,l_2 \mid l_{12})}{n(l_A,l_B)} P_{i_1}(l_A) P_{i_2}(l_B) \tag{3.18}$$

Note that here the area fire computations are carried out twice in a hierarchical manner. First, Equation (3.12) is used to find $P_{i1}(l_A)$ by combining firers of the first unit, and this is repeated independently for the second unit. Then, the two units are combined to find overall casualty probabilities $P_{i1,i2}(l)$. An example is given in Table 3.5 as already mentioned in Section 3.3.1.

3.3.3 Mixed Fire in Heterogeneous Combat

Suppose that, of the blue units combined against a red unit, some employ directed fire and others area fire. The approach proposed in Section 3.3.2 for combining multiple area firing units can also be used for combining these mixed units. We owe this to the equivalence of the combined casualty probabilities under the two viewpoints, as explained at the end of Section 3.3.1. For example, if the first unit employs directed fire, then $P_{i1}(l_A) = P(l_A$ casualties with i_1 firers) must be calculated by Equation (3.7), otherwise by Equation (3.12). In other words, the fire type affects only the individual probabilities in the convolution $P_{i1}(l_A)$ $P_{i2}(l_B)$ but, given these, the combined casualty probabilities $P_{i1,i2}(l)$ are the same.

3.3.4 Salvo Treatment in Heterogeneous Combat

Let us assume that there are I blue units $B_1,\ldots, B_I$ with $m_1,\ldots, m_I$ combatants each, and J red units $R_1,\ldots, R_J$ with $n_1,\ldots, n_J$ combatants each. The state space is then denoted by $(t,i_1,\ldots, i_I, j_1,\ldots, j_J)$ resulting in $(m_1+1)\cdots(m_I+1)(n_1+1)\cdots(n_J+1)$ many states. The initial condition is $P(0, m_1,\ldots, m_I, n_1,\ldots, n_J) = 1$.

It is possible for a blue unit to divide its force among multiple red units. Let $x_{B1,R1},\ldots,x_{B1,RJ}$ denote the allocation fractions for the first blue unit. These fractions might be specified as part of the scenario. We know that $x_{B1,R1}+\ldots+ x_{B1,RJ} \leq 1$ meaning that B_1 can reserve a certain fraction of its force.

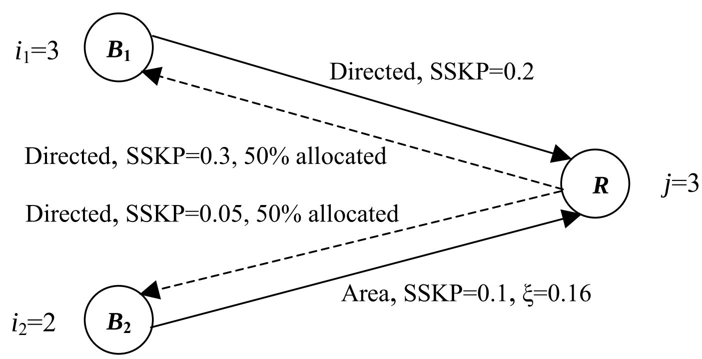


Figure 3.4 Example heterogeneous combat situation

An example heterogeneous combat situation is illustrated in Figure 3.4. There are two blue units, B_1 with 3 combatants and B_2 with 2 combatants, firing at a red unit R with 3 combatants. B_2 employs area fire, and all remaining fires are directed. R divides its force evenly between B_1 and B_2. The state space notation is (t,i_1, i_2, j) and we have 48 states in each salvo. The allocations are $x_{B1,R} = 1.0$, $x_{B2,R} = 1.0$, $x_{R,B1}=0.5$, and $x_{R,B2}=0.5$.

In salvo t, red unit R with j combatants is subject to attrition due to blue units allocated to R with fractions $x_{B1,R},\ldots, x_{BI,R}$. We have shown in the previous sections how to find the probabilities of Δj casualties in R under the fire of $[x_{B1,R} \cdot i_1]$ firers of B_1 and $[x_{B2,R} \cdot i_2]$ firers of B_2. However, the number of allocated firers given in brackets should be integers. Without loss of generality, let us assume that $[x_{B1,R} \cdot i_1]$ is not an integer. Let $\check{i}_1 = \lfloor x_{B1,R} \cdot i_1 \rfloor$ and $\hat{i}_1 = \lceil x_{B1,R} \cdot i_1 \rceil$. Let $w_{\hat{i}_1} = x_{B1,R} \cdot i_1 - \check{i}_1$, and $w_{\check{i}_1} = \hat{i}_1 - x_{B1,R} \cdot i_1$ be the interpolation weights. Then, the probability of R_1 having Δj casualties under attack by B_1 is calculated as

$$P^R{}_{i1}(\Delta j \mid [x_{B1,R} \cdot i_1]) = w_{\check{i}_1} P^R_{\check{i}_1}(\Delta j) + w_{\hat{i}_1} P^R_{\hat{i}_1}(\Delta j) \qquad (3.19)$$

Generalizing the above equation, we have

$$\begin{aligned} P^R{}_{i1,\ldots,iI}(\Delta j) = P^R{}_{i1,\ldots,iI}(\Delta j \mid [x_{B1,R} \cdot i_1],\ldots,[x_{BI,R} \cdot i_I]) = \\ w_{\check{i}_1} \cdots w_{\check{i}_I} P^R_{\check{i}_1,\ldots,\check{i}_I}(\Delta j) + \cdots + w_{\hat{i}_1} \cdots w_{\hat{i}_I} P^R_{\hat{i}_1,\ldots,\hat{i}_I}(\Delta j) \end{aligned} \qquad (3.20)$$

In the example of Figure 3.4, R divides its j=3 firers evenly between B_1 and B_2. Let us consider the attrition in B_1. We have $\check{j} = \lfloor 0.5(3) \rfloor = 1$ and $\hat{j} = \lceil 0.5(3) \rceil = 2$, and $w_{\check{j}} = 0.5 = w_{\hat{j}}$. Since the probability of one casualty in B_1 is 0.450000 when j=2, and 0.300000 when j=1, $P^{B_1}{}_3(1)$=(0.5)0.450000 + (0.5)0.300000 = 0.375000. Similarly, $P^{B_2}{}_3(1)$=(0.5)0.096250 + (0.5)0.050000 = 0.073125.

The state transition $(t, i_1,\ldots, i_I, j_1,\ldots, j_J) \rightarrow (t+1, i_1-\Delta i_1,\ldots, i_I - \Delta i_I, j_1-\Delta j_1,\ldots, j_J - \Delta j_J)$ indicates that there are Δi_1 casualties in B_1, Δi_2 casualties in B_2, and so on. The corresponding state transition probability is

$$\begin{aligned} P\big((t, i_1,\ldots, i_I, j_1,\ldots, j_J) \rightarrow (t+1, i_1-\Delta i_1,\ldots, i_I-\Delta i_I, j_1-\Delta j_1,\ldots, j_J-\Delta j_J)\big) = \\ P^{B_1}_{j_1,\ldots,j_J}(\Delta i_1) \cdots P^{B_I}_{j_1,\ldots,j_J}(\Delta i_I) P^{R_1}_{i_1,\ldots,i_I}(\Delta j_1) \cdots P^{R_J}_{i_1,\ldots,i_I}(\Delta j_J) \end{aligned} \qquad (3.21)$$

In our example, suppose an area firing combatant of B_2 has an effective radius of r=40, and the region containing j=3 red targets has radius R=100. The casualty probabilities for unit R are found by Equation (3.8) as $P^R{}_1(0)$=0.952764,

$P^R{}_1(1)$=0.046476, $P^R{}_1(2)$=0.000756, $P^R{}_1(3)$=0.000004. When we combine two such firers, the effect of B_2 alone yields $P^R{}_2(0)$=0.907759, $P^R{}_2(1)$=0.089282, $P^R{}_2(2)$=0.002927, $P^R{}_2(3)$=0.000032 according to Equation (3.12). The effect of B_1 individually with 3 firers is calculated by Equation (3.7) as $P^R{}_3(0)$=0.512000, $P^R{}_3(1)$=0.416889, $P^R{}_3(2)$=0.069333, $P^R{}_3(3)$=0.001778. If we combine the effects of B_1 and B_2 by Equation (3.18), the red casualty probabilities are $P^R{}_{3,2}(0)$=0.464773, $P^R{}_{3,2}(1)$=0.436554, $P^R{}_{3,2}(2)$=0.094258, $P^R{}_{3,2}(3)$=0.004415.

Let us now consider the state transition (t,3,2,3)→ (t+1,2,1,1), meaning that one B_1 casualty, one B_2 casualty and two R casualties occur in salvo t. Recall that, $P^{B1}{}_3(1)$=0.375000, $P^{B2}{}_3(1)$=0.073125, and $P^R{}_{3,2}(2)$=0.094258. Hence,

$P((t,3,2,3)\rightarrow (t+1,2,1,1)) = P^{B1}{}_3(1)\, P^{B2}{}_3(1)\, P^R{}_{3,2}(2) = 0.002585.$

The state probabilities for the first salvo, which can be seen in Figure 3.5, are the same as the transition probabilities from the initial state due to the initial condition $P(0, m_1,\ldots, m_I, n_1,\ldots, n_J) = 1$. State probabilities for subsequent salvos are determined as

$$P(t+1, i_1, \ldots, j_J) = \sum_{\Delta i_1 = 0}^{m_1 - i_1} \cdots \sum_{\Delta j_J = 0}^{n_J - j_J} P(t, i_1 + \Delta i_1, \ldots, j_J + \Delta j_J) P^{B_1}_{j_1, \ldots, j_J}(\Delta i_1) \cdots P^{R_J}_{i_1, \ldots, i_I}(\Delta j_J) \tag{3.22}$$

Let us consider calculation of the state probability P(2,2,1,1) in our example. The transition probabilities to this state are multiplied with respective state probabilities of the first salvo to determine individual contributions. For example, contribution of (1,3,2,3) to (2,2,1,1) is P(1,3,2,3) P((1,3,2,3)→ (2,2,1,1)) = 0.256145(0.002585) = 0.000662. When all such contributions are added, the state probability P(2,2,1,1) = 0.012824 is found as shown in Figure 3.6. Expected number and respective variances of number of survivors in blue units and red unit for the first ten salvos are given in Table 3.6.

Blue 1: i_1=3 Red

		j=3	j=2	j=1	j=0
	i_2=2	0.256145	0.240593	0.051948	0.002433
Blue 2	i_2=1	0.020222	0.018994	0.004101	0.000192
	i_2=0	0.000173	0.000162	0.000035	0.000002

Blue 1: i_1=2 Red

		j=3	j=2	j=1	j=0
	i_2=2	0.161436	0.151634	0.032740	0.001534
Blue 2	i_2=1	0.012745	0.011971	0.002585	0.000121
	i_2=0	0.000109	0.000102	0.000022	0.000001

Blue 1: i_1=1 Red

		j=3	j=2	j=1	j=0
	i_2=2	0.012915	0.012131	0.002619	0.000123
Blue 2	i_2=1	0.001020	0.000958	0.000207	0.000010
	i_2=0	0.000009	0.000008	0.000002	0.000000

Blue 1: i_1=0 Red

		j=3	j=2	j=1	j=0
	i_2=2	0.000000	0.000000	0.000000	0.000000
Blue 2	i_2=1	0.000000	0.000000	0.000000	0.000000
	i_2=0	0.000000	0.000000	0.000000	0.000000

Figure 3.5 State probabilities at the end of the first salvo

Blue 1: i_1=3 — Red

Blue 2		j=3	j=2	j=1	j=0
	i_2=2	0.065610	0.138426	0.106918	0.035692
Blue 2	i_2=1	0.010616	0.019933	0.013622	0.003864
	i_2=0	0.000530	0.000852	0.000492	0.000109

Blue 1: i_1=2 — Red

Blue 2		j=3	j=2	j=1	j=0
	i_2=2	0.093040	0.165145	0.101389	0.024036
Blue 2	i_2=1	0.015055	0.023675	0.012824	0.002586
	i_2=0	0.000751	0.001005	0.000459	0.000073

Blue 1: i_1=1 — Red

Blue 2		j=3	j=2	j=1	j=0
	i_2=2	0.041705	0.058073	0.026066	0.003951
Blue 2	i_2=1	0.006748	0.008312	0.003282	0.000421
	i_2=0	0.000337	0.000352	0.000117	0.000012

Blue 1: i_1=0 — Red

Blue 2		j=3	j=2	j=1	j=0
	i_2=2	0.005473	0.005091	0.001369	0.000116
Blue 2	i_2=1	0.000886	0.000727	0.000167	0.000011
	i_2=0	0.000044	0.000031	0.000006	0.000000

Figure 3.6 State probabilities at the end of the second salvo

Table 3.6 Results for the first ten salvos

	Blue 1		Blue 2		Red	
Salvo	Expected	Variance	Expected	Variance	Expected	Variance
0	3.000000	0.000000	2.000000	0.000000	3.000000	0.000000
1	2.565000	0.305775	1.925625	0.070093	2.361684	0.445876
2	2.219444	0.553571	1.866931	0.125699	1.832337	0.762890
3	1.955257	0.763204	1.821557	0.168868	1.416341	0.924106
4	1.765855	0.922107	1.786745	0.202334	1.114085	0.954461
5	1.636314	1.037463	1.759650	0.228842	0.904862	0.923533
6	1.550316	1.120050	1.737932	0.250571	0.763215	0.876079
7	1.494209	1.178646	1.719898	0.269064	0.667601	0.830500
8	1.457937	1.219800	1.704404	0.285351	0.602324	0.791651
9	1.434574	1.248394	1.690701	0.300097	0.556753	0.759398
10	1.419532	1.268066	1.678305	0.313723	0.523950	0.732382

3.4 Extensions of DSM

Two major and two minor extensions are presented in this section. Major extensions are concerned with noncombat loss and the engagement process for handling units' different rates of fire. They involve additional multiple single-dimensional discrete-time processes linked to DSM. Minor extensions are small changes in SSKPs to treat synergy effects, and shifts in states of military units that are subject to reinforcements.

3.4.1 Noncombat Loss

There are two sources of attrition in combat, combat loss and noncombat loss. The former is due to the opposing force's fire; it is the result of the interaction between two sides. The latter does not involve such an interaction. Noncombat loss occurs due to reasons such as illness, accidents and desertions. Such factors cause an additional decrease in the force level of each side. This decrease, however, depends only on own force levels.

Noncombat loss can also be handled in DSM by means of the binomial process. We assume that noncombat loss probabilities (or rates), $q_{B1},\ldots, q_{RJ}$ are specified for all military units and kept constant for all salvos. If a military unit is not subject to noncombat loss, its noncombat loss probability is zero. Single-dimensional noncombat loss transition probabilities for blue unit B_1 are calculated as

$$Q^{B_1}\left(\left(t, i_1 + \Delta i_1\right) \rightarrow \left(t, i_1\right)\right) = \binom{i_1 + \Delta i_1}{\Delta i_1} q_{B_1}^{\Delta i_1} \left(1 - q_{B_1}\right)^{i_1} \qquad (3.23)$$

The marginal probability distribution of B_1 after combat loss, $P_{B1}(t,i_1)$, $i_1=1,\ldots, m_1$, can be determined from the joint state probabilities $P(t, i_1,\ldots, i_I, j_1,\ldots, j_J)$. If there is noncombat loss, the single-dimensional state probabilities for B_1 can be updated as

$$Q_{B_1}\left(t, i_1\right) = \sum_{\Delta i_1 = 0}^{m_1 - i_1} P_{B_1}\left(t, i_1 + \Delta i_1\right) \binom{i_1 + \Delta i_1}{\Delta i_1} \; q_{B_1}^{\Delta i_1} \left(1 - q_{B_1}\right)^{i_1} \qquad (3.24)$$

Let $Q(t, i_1,\ldots, i_I, j_1,\ldots, j_J)$ be the joint state probabilities after noncombat loss at the end of salvo t. Then,

$$Q\left(t, i_1, \ldots, i_I, j_1, \ldots, j_J\right) = Q_{B_1}\left(t, i_1\right) \ldots Q_{B_I}\left(t, i_I\right) Q_{R_1}\left(t, j_1\right) \ldots Q_{R_J}\left(t, j_J\right) \qquad (3.25)$$

In order to incorporate noncombat loss in DSM, in the combat loss state probability calculation given by Equation (3.22), the state probabilities of the previous salvo, $P(t, i_1 + \Delta i_1,\ldots, j_J + \Delta j_J)$, should be replaced with the probabilities after noncombat loss, $Q(t, i_1 + \Delta i_1,\ldots, j_J + \Delta j_J)$, except for the first salvo.

3.4.2 Engagement Process

An assumption of DSM so far is that all combatants of a military unit fire in every salvo. Target detection time, weapon preparation time and rate of fire vary for different weapon systems, making perfect synchronization impossible. If we determine the salvo duration in terms of the most frequently firing weapon system, we can define an engagement probability for slower systems. For example, engagement probability would be 0.25 for a system that can fire once in every four salvos. Furthermore, command control problems and fatigue in weapon systems may give rise to stochastic engagement even if technical capabilities permit to fire in each salvo.

Engagement process can also be used to model combat in alternating volleys as suggested by Helmbold (1968) rather than simultaneous salvos. This can be achieved by setting one side's engagement probability to zero, while setting the other's to one. Other variations are also possible since engagement probabilities can be different for different units.

Let $e_{B1},\ldots, e_{RJ}$ be engagement probabilities of military units. If all combatants of a military unit fire in every salvo, its engagement probability is one. Engagement probabilities induce independent single-dimensional binomial processes as in noncombat loss. However, combatants that do not fire stay in combat and are subject to attrition. Hence, the engagement process affects all the targets although the number of firers may decrease.

In plain DSM, R_1 with j_1 combatants is subject to attrition due to blue units allocated to R_1 with the fractions $x_{B1,R},\ldots, x_{BI,R}$. Let us consider the interaction of B_1 and R_1. In plain DSM, only $\check{i}_1 = \lfloor x_{B1,R} \cdot i_1 \rfloor$ and $\hat{i}_1 = \lceil x_{B1,R} \cdot i_1 \rceil$ blue combatants are considered in calculating the combat loss of R_1. With the engagement process, one has to consider $\hat{i}_1, \check{i}_1, \check{i}_1 - 1, \ldots, 1, 0$ with respective probabilities calculated from the binomial engagement distribution.

The effect of the engagement process in our salvo calculations can be reflected in Equation (3.20), which is modified as

$$P^{R_1}_{i_1,\ldots,i_I}\left(\Delta j_1 \mid \left[x_{B1,R1} \cdot i_1\right],\ldots,\left[x_{BI,R1} \cdot i_I\right]\right) = \sum_{\nabla i_1 = 0}^{i_1} \cdots \sum_{\nabla i_I = 0}^{i_I} P^{R_1}_{i_1,\ldots,i_I}\Big(\Delta j_1 \mid \left[x_{B1,R1}\left(i_1 - \nabla i_1\right)\right],\ldots, \left[x_{BI,R1}\left(i_I - \nabla i_I\right)\right] \Big) \binom{i_1}{\nabla i_1} e_{B_1}^{\nabla i_1}(1-e_{B_1})^{i_1 - \nabla i_1} \cdots \binom{i_I}{\nabla i_I} e_{B_I}^{\nabla i_I}(1-e_{B_I})^{i_I - \nabla i_I} \tag{3.26}$$

The state probabilities for the first salvo are calculated using the combat loss probabilities after engagement. As the battle moves on to the second salvo, noncombat loss can also be applied. The reader may refer to the technical report by Kandiller et al. (2002) for an example where both engagement process and noncombat loss are considered.

3.4.3 Reinforcements

Jaiswal et al. (1997) modeled combat between two units under various rates of continuous reinforcement. We consider discrete reinforcements that simply extend dimension of the state space and incur a shift in the state probabilities. In our example, suppose R, initially having j=3 combatants, is reinforced by two new combatants at the end of salvo t. Then,

$n \leftarrow 3 + 2 = 5$, $P(t, i_1, i_2, j+2) \leftarrow P(t, i_1, i_2, j)$, and $P(t, i_1, i_2, 1) = P(t, i_1, i_2, 0) = 0$

3.4.4 Division and Combination Effects

Force division may yield a reduction in the attrition potentials whereas force combination may result in an increase in the potential due to synergy. Let λ_R denote the fractional loss in attrition power of R when it divides its force between two blue units as in our example. The SSKPs of R, $p_{k:R,B1}$ and $p_{k:R,B2}$, can be decreased by λ_R if the force division effect is to be observed. Let $p^{FD}_{k:R,B_1}$ and $p^{FD}_{k:R,B_2}$ be the SSKPs after force division. Then,

$$p^{FD}_{k:R,B_1} = (1-\lambda_R)p_{k:R,B_1} \text{ and } p^{FD}_{k:R,B_2} = (1-\lambda_R)p_{k:R,B_2} \qquad (3.27)$$

Let ϕ_{B1} denote the fractional gain in attrition power of B_1 combatants when B_1 and B_2 are combined against R. The SSKPs after force combination are

$$p^{FC}_{k:B_1,R} = 1-(1-\phi_{B_1})(1-p_{k:B_1,R}) \text{ and } p^{FC}_{k:B_2,R} = 1-(1-\phi_{B_2})(1-p_{k:B_2,R}). \qquad (3.28)$$

CHAPTER 4

DSM IMPLEMENTATION AND EXPERIMENTATION

Even though numeric solutions can be obtained by continuous time stochastic models available in the literature, they require excessive computation time and can handle very small force sizes most of the time. While we focus on the implementation of the model developed by Kandiller et al. (2002), we intend to handle larger force levels and a variety of weapon systems in combat simulation. Significant features of the DSM, namely heterogeneity of forces, application of directed, area and mixed fire, force allocation, noncombat loss and engagement process are all included in our implementation.

The DSM code is written in C++, using MS Visual C++ 6.0 integrated development environment. Data structures of the model are based on the dynamic arrays, created by utilizing the pointer abilities provided by the C++ programming language. The executable is a Windows application, however, with appropriate modifications to the source code, it can be made operating system-independent. Standard input, namely a simple DOS console entry through keyboard is employed in the executable of the model. The salvo statistics, including mean and variance of force levels, survival probabilities at the end of each salvo, and the computation time statistics for the combat model are stored in output files. These statistics are output to two separate files, one for force level statistics for each salvo, and the other for final force levels and CPU time results. The results are easily interpreted by means of Excel sheets.

Recall that we also intend to estimate ARCs for Lanchester model, such that the two models will yield similar figures for force levels. Therefore we also implemented a code for SLM, which is more straightforward. The input, output methods and executable format are the same as those used for DSM.

In the following sections, DSM and SLM implementation stages are described, followed by the results of experimentation with these models.

4.1 DSM Implementation

DSM implementation is based on the following proposed design. Initially, SSKPs are updated considering the force division and combination effects. Then, engagement probabilities for synchronization of different weapon systems are determined according to their firing rates. Salvo length and the desired combat duration are needed to determine the total number of salvos. In every salvo, the engagement process, the combat loss process, and the noncombat loss process are applied in this respective order. If a unit is reinforced at the end of a salvo, a state shift is made accordingly at the beginning of the next salvo. The salvo sequence is terminated when either the total number of salvos is reached or the expected number of survivors in any unit falls below a specified threshold value.

Our implementation is consistent with the above design, except for the reinforcement state shift and threshold specification steps. These are excluded for the sake of simplicity in input entry and output analysis, though they could easily be incorporated in the code. Also, the total number of salvos is a direct input. Following is a brief statement of the DSM algorithm.

S-0. Data input for the combat and the forces. These are:

- The number of different military units (I, J),

- Total number of salvos,
- The initial number of combatants in each military unit (m_i, i=1,…,I, n_j, j=1,…,J),
- The fire type of each unit (directed or area),
- The SSKPs of units ($p_{k:Bi,Rj}$,…, $p_{k:Rj,Bi}$),
- Fatal area radius / total area radius (r/R) for units employing area fire,
- Engagement probabilities of units (e_{B1},…, e_{RJ}),
- Noncombat loss probabilities of units (q_{B1},…, q_{RJ}), and
- Allocation fractions of units ($x_{Bi,Rj}$, $x_{Rj,Bi}$).

A screenshot of input entry is given in Figure 4.1.

S-1. Preparation (Prep.) for the salvo sequence. This involves:

- Computation of the casualty probabilities for every possible state,
- Application of the engagement process, attained by adjusting the casualty probabilities according to the engagement probabilities of units,
- Creation of the matrix of transition probabilities from the initial state,
- Creation of the matrix of noncombat loss probabilities.

S-2. Salvo sequence, which continues until the total number of salvos is reached. Each salvo includes:

- Application of the combat loss processes using the matrix of transition probabilities formed in S-1,
- Application of the noncombat loss processes using the matrix of noncombat loss probabilities formed in S-1,
- File output of the salvo statistics, namely the survival probabilities, expected number and variance of number of survivors for each military unit.

S-3. File output for final force levels and computation time statistics.

End {DSM}.

Stage S-0 is straightforward, only appropriate data entry by keyboard input is required. In stage S-1, the crucial calculations for combat salvo simulation are completed. The casualty probability computations are made using Equations (3.7), (3.12), (3.17) and (3.18), based on the fire types of military units in combat. Force allocation effects on these probabilities are reflected through Equation (3.20). Engagement process is applied utilizing Equation (3.26). The combat loss transition probability and noncombat loss probability matrices are created via Equations (3.21) and (3.23), respectively. S-2 is the salvo simulation stage. Combat loss and noncombat loss processes are applied through Equations (3.22) and (3.25). Marginal probabilities of surviving combatants for each military unit are listed, as well as the expected value and variance of the number of surviving combatants in the output file "salvos.txt". Stage S-3 ends the simulation by writing the final force level, namely the last salvo statistics and CPU time statistics in file "stats.txt". CPU time statistics include the time used in stage S-1 and individual salvo play times elapsed in stage S-2. Sample file output for the input in Figure 4.1 are presented in Appendix A.

```
D:\belgeler\ugur\tez\dsm.exe
Number of units in blue force:2
Number of units in red force:1
Initial number of elements in unit blue 1:5
Noncombat loss probability of unit blue 1:0.01
Engagement probability of unit blue 1:1
Fire type of unit blue 1 (1: area(exact places unknown), 2: directed :2
Initial number of elements in unit blue 2:4
Noncombat loss probability of unit blue 2:0
Engagement probability of unit blue 2:0.9
Fire type of unit blue 2 (1: area(exact places unknown), 2: directed :1
Initial number of elements in unit red 1:7
Noncombat loss probability of unit red 1:0
Engagement probability of unit red 1:1
Fire type of unit red 1 (1: area(exact places unknown), 2: directed :2
From blue 1 to red 1:.1
From blue 2 to red 1:.15
From blue 2 to red 1 fatal area radius (in percentage):40
From red 1 to blue 1:.15
From red 1 to blue 2:.1
From red 1 to blue 1 fire power allocation:.5
From red 1 to blue 2 fire power allocation:.5
Sum of fire power allocations (must be 1) = 1
Number of salvos to be calculated:10
```

Figure 4.1 An example of data input for DSM executable

4.2 SLM Implementation

In addition to experimentation with DSM, we try to estimate ARCs used in SLM, such that DSM for homogeneous combat and SLM will yield similar figures for force levels throughout the combat. This is attempted through comparison of expected force levels at the end of each DSM salvo with the respective force levels obtained by SLM using estimated ARCs.

The comparison of these models is based on the following equivalence, where we describe the case when the red unit employs directed fire against the blue unit. Let X be a Bernoulli random variable indicating whether or not a single blue target is killed by a single firer in a DSM salvo, i.e.

$$X = \begin{cases} 1, \text{if killed in one salvo,} \\ 0, \text{otherwise;} \end{cases} \qquad X : Ber\left(p_{DSM}\right), E\left(X\right) = p_{DSM};$$

where p_{DSM} is the SSKP of the red firer, namely $p_{k:R,B}$. Suppose we divide a DSM salvo into $1/\Delta t$ subintervals, each of length Δt. Let Y_i be a Bernoulli random variable indicating whether or not a blue target is killed by a red firer in subinterval i, that is $Y_i : Ber(p_{SLM})$. Here, p_{SLM} corresponds to the ARC of the red unit in subinterval i, which is taken constant throughout the combat. Then, $Y = \sum_{i=1}^{1/\Delta t} Y_i$ has binomial distribution with parameters $1/\Delta t$ and p_{SLM}. For small p_{SLM} and Δt, this distribution is approximated by Poisson distribution with parameter $\lambda = \frac{p_{SLM}}{\Delta t}$. Hence $E(Y) = \frac{p_{SLM}}{\Delta t}$. To make SLM and DSM comparable, we should have $E(Y)=E(X)$, therefore $p_{SLM}=\Delta t\ p_{DSM}$. In implementing SLM, Δt is taken as 1/100 of the unit DSM salvo length. We have also tried the ratios 1/10, 1/1000 and 1/10000 for several $p_{k:R,B}$ and $p_{k:B,R}$ combinations and concluded that SLM results are fairly robust to the choice of Δt, as seen in an example presented in Table 4.1. Therefore, the SLM force levels at the end of the 100^{th}, 200^{th}, … subintervals are compared to the DSM salvo force levels utilizing the ratio 1/100.

In the case where red unit employs area fire, the probabilities p_{DSM} and p_{SLM} are multiplied by ξ, the probability of the blue target being positioned in the fatal area. Then, having $E(X)=\xi\, p_{DSM}$ and $E(Y) = \frac{\xi\, p_{SLM}}{\Delta t}$, the equality $p_{SLM}=\Delta t\; p_{DSM}$ is valid again.

Table 4.1 SLM results for the first ten salvos with different Δt / Salvo length ratios

	Expected force levels for blue unit, $m = 5$, $p_{k:B,R} = 0.1$				Expected force levels for red unit, $n = 4$, $p_{k:R,B} = 0.3$			
	Δt / Salvo length							
Salvo	0.1	0.01	0.001	0.0001	0.1	0.01	0.001	0.0001
1	3.863682	3.870002	3.870659	3.870724	3.552329	3.557223	3.557710	3.557758
2	2.873896	2.892712	2.894602	2.894791	3.213783	3.222476	3.223332	3.223417
3	2.089412	2.120963	2.124037	2.124344	2.971899	2.982299	2.983319	2.983420
4	1.537657	1.572401	1.575782	1.576120	2.807584	2.817591	2.818575	2.818673
5	1.180506	1.210738	1.213711	1.214007	2.699380	2.707863	2.708703	2.708787
6	0.958918	0.982060	0.984365	0.984595	2.629150	2.635836	2.636503	2.636570
7	0.823323	0.839890	0.841559	0.841726	2.583754	2.588809	2.589316	2.589367
8	0.740099	0.751590	0.752758	0.752875	2.554372	2.558103	2.558479	2.558517
9	0.688432	0.696310	0.697117	0.697197	2.535281	2.537997	2.538272	2.538300
10	0.655882	0.661280	0.661835	0.661891	2.522819	2.524778	2.524977	2.524997

The implementation of SLM is straightforward compared to that of DSM. There is not any preparation stage as in DSM, which simplifies the code and decreases the computation time. The dimension of the state transition probability matrix is dramatically smaller due to force homogeneity. Moreover, only the combat loss process is implemented according to the model, which does not involve any extensions such as noncombat loss or engagement processes of DSM.

For each subinterval of length Δt, the state transition probabilities $P(t + \Delta t, i, j)$ are calculated using Equation (3.4), and the probabilities of possible states at the end of each subinterval are summed up to get the marginal probabilities of surviving combatants for both military units. The expected value and variance of the number of surviving combatants are evaluated at the end of every 100 subintervals, which correspond to a DSM salvo.

A screenshot of input entry for the example in Table 4.1 is given in Figure 4.2. The format of the salvo statistics file, "salvos.txt" is the same as of DSM output, the final statistics output file, "stats.txt" differs slightly from the one for DSM, which is presented in Appendix B.

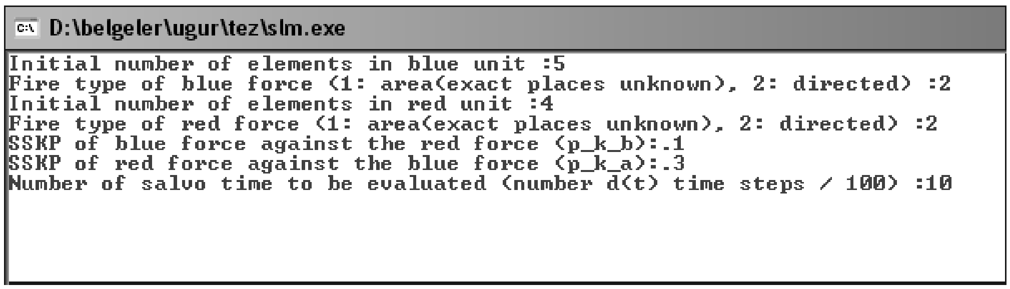


Figure 4.2 An example of data input for SLM executable

4.3 Experimentation

The DSM simulation runs for the purpose of comparison with SLM are based on homogeneous combats of length ten salvos. Both blue and red units employ directed fire or area fire against each other in the experimental runs. The experimental conditions are summarized in Table 4.2. In directed fire runs, for equal initial force levels $m = n = 5, 10, 20$, three SSKP combinations, namely $p_{k:B,R} = p_{k:R,B} = 0.1$; $p_{k:B,R} = p_{k:R,B} = 0.3$; and $p_{k:B,R} = 0.1$, $p_{k:R,B} = 0.3$, are used. In different initial force level cases, which are $m = 5$, $n = 10$, and $m = 10$, $n = 20$, SSKP values are $p_{k:B,R} = p_{k:R,B} = 0.1$; $p_{k:B,R} = p_{k:R,B} = 0.3$; $p_{k:B,R} = 0.1$, $p_{k:R,B} = 0.3$; and $p_{k:B,R} = 0.3$, $p_{k:R,B} = 0.1$. In addition, runs are made for the cases where there are two or three military units on both sides for computation time comparisons.

In area fire runs, the same force size and SSKP combinations are tried as in directed fire. These runs are made with three ξ values: 0.09, 0.25, 0.49, respectively corresponding to fatal area radius percentages (r/R_A) of 30%, 50% and 70%. The same ξ values are used for both sides.

Table 4.2 Experimental Conditions for DSM Runs

Homogeneous				Heterogeneous					
Blue		Red		Blue			Red		
m	$p_{k:B,R}$	n	$p_{k:R,B}$	m_1	m_2	m_3	n_1	n_2	n_3
5	0.1	5	0.1	5	5		5	5	
5	0.1	5	0.3	10	10		10	10	
5	0.3	5	0.3	3	3	3	3	3	3
10	0.1	10	0.1	4	4	4	4	4	4
10	0.1	10	0.3	5	5	5	5	5	5
10	0.3	10	0.3						
20	0.1	20	0.1						
20	0.1	20	0.3						
20	0.3	20	0.3						
5	0.1	10	0.1						
5	0.1	10	0.3						
5	0.3	10	0.1						
5	0.3	10	0.3						
10	0.1	20	0.1						
10	0.1	20	0.3						
10	0.3	20	0.1						
10	0.3	20	0.3						

Implementations of both model designs are tested on a Pentium III 1GHz PC with 256MB RAM. Tables 4.3 and 4.4 list the CPU times in seconds, respectively for directed and area fire. The upper halves of the tables compare SLM and DSM for homogeneous combat. DSM is executed with and without major extensions (engagement + noncombat loss) for both heterogeneous and homogeneous combat cases. As it was hinted previously, the “Prep.” columns give the time spent for preparation stage S-1 of DSM implementation. The “Salvo” column gives the average time per salvo spent in stage S-2 of DSM execution.

In homogeneous combat, CPU times become significant for initial force levels *m*, *n* > 10, reaching about 30 minutes for *m*=*n*=20, for directed fire. This duration is essentially spent for preparation stage S-1, using the CPU resource heavily. The preparation time differences between directed and area fire cases are significant for larger force sizes in homogeneous combat. This is due to different computations for individual fire effects. In the heterogeneous combat, however, since the same fire combination method is used, the computation times are similar. In this occasion, the time consuming task is finding the state probabilities for each salvo, namely stage S-2. As the state space dimension increases (see for example the last rows of Tables 4.3 and 4.4), time per salvo reaches a maximum of about 80 minutes. Salvo cycle consumes mostly the RAM of the computer rather than the CPU, as a result of memory allocation for the state probability matrices. In the case represented by the last row of Table 4.3, the execution of the program consumes almost all of the 256 MB memory. Homogeneous combat times are in favor of SLM, as clearly seen in the last columns of Tables 4.3 and 4.4, which are significantly shorter than DSM times especially for large force level cases.

Table 4.3 CPU Time statistics for DSM and SLM for Directed Fire

Scenario						Size of the State Space	DSM CPU times (seconds)				SLM CPU Times (seconds)
Blue			Red				Combat Loss		Engagement+ Combat+ Noncombat Loss		
m_1	m_2	m_3	n_1	n_2	n_3		Prep.	Salvo	Prep.	Salvo	
5			5			36	0.0	0.0	0.0	0.0	0.0
5			10			66	0.0	0.0	0.0	0.0	0.0
10			10			121	0.6	0.0	0.6	0.0	0.1
10			20			231	40.3	0.0	41.6	0.0	0.2
20			20			441	2001.6	0.0	2077.7	0.0	0.6
5	5		5	5		1296	0.6	0.1	0.9	0.2	-
10	10		10	10		14641	15.0	12.6	30.5	793.3	-
3	3	3	3	3	3	4096	1.8	0.9	2.6	1.8	-
4	4	4	4	4	4	15625	17.4	13.6	22.2	32.7	-
5	5	5	5	5	5	46656	194.2	2378.3	295.0	5067.8	-

Table 4.4 CPU Time statistics for DSM and SLM for Area Fire

Scenario						Size of the State Space	DSM CPU times (seconds)				SLM CPU Times (seconds)
Blue			Red				Combat Loss		Engagement+ Combat+ Noncombat Loss		
m_1	m_2	m_3	n_1	n_2	n_3		Prep.	Salvo	Prep.	Salvo	
5			5			36	0.0	0.0	0.0	0.0	0.0
5			10			66	0.0	0.0	0.0	0.0	0.0
10			10			121	0.6	0.0	0.6	0.0	0.1
10			20			231	1.1	0.0	1.1	0.0	0.1
20			20			441	4.4	0.0	4.5	0.0	0.4
5	5		5	5		1296	0.7	0.1	0.7	0.2	-
10	10		10	10		14641	14.9	12.8	29.8	634.3	-
3	3	3	3	3	3	4096	1.8	0.8	2.3	1.7	-
4	4	4	4	4	4	15625	16.4	13.8	20.4	30.8	-
5	5	5	5	5	5	46656	198.6	2394.7	319.1	5044.6	-

Expected force levels found by DSM and SLM, where both sides use directed fire, are plotted in Figure 4.3 for four representative scenarios. When both initial force levels and SSKPs are the same for two sides and SSKPs are small as in Figure 4.3 (a), then DSM and SLM produce very close results. This is expected since Poisson is the limiting distribution of binomial distribution when probability of success (SSKP) in a single trial (salvo) is small. However, when SSKPs are larger as in Figure 4.3 (b), DSM reaches steady state later than SLM, resulting in lower expected force levels. When one side (red) is stronger than the other (blue), either because the red unit's SSKP or its initial force level is higher than that of the blue unit as in Figure 4.3 (c) or 4.3 (d), then red kills blue targets faster with SLM than with DSM. Overall, DSM results in "closer combat", that is to say the difference between expected force levels of the two sides is smaller in DSM compared to that in SLM.

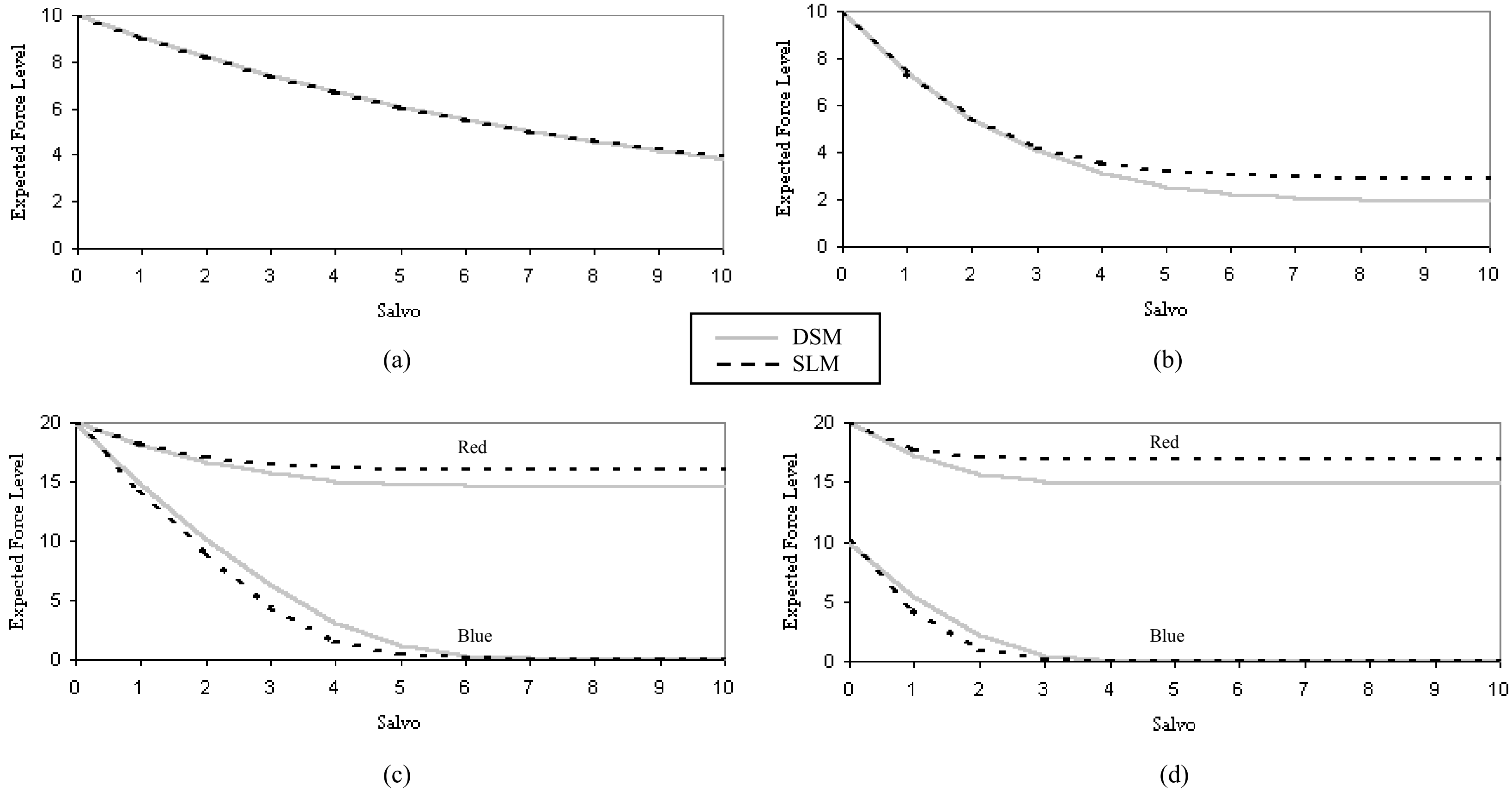


Figure 4.3 Expected force levels with DSM and SLM when directed fire is employed

(a) $m=n=10$, $p_{k:B,R}=p_{k:R,B}=0.1$, (b) $m=n=10$, $p_{k:B,R}=p_{k:R,B}=0.3$, (c) $m=n=20$, $p_{k:B,R}=0.1$, $p_{k:R,B}=0.3$, (d) $m=10$, $n=20$, $p_{k:B,R}=p_{k:R,B}=0.3$.

When both sides employ area fire, we observed the force levels for three different ξ values (0.09, 0.25, and 0.49) as stated in experimental conditions. As a common observation for all these area fire cases, DSM underestimates the force levels of both sides, compared to SLM. Expected force levels found by DSM and SLM, for two different scenarios, are plotted in Figure 4.4. The differences in force levels of the two models increase, as the ξ probabilities get larger. This situation is shown in force level plots of Figure 4.4. When one side is stronger than the other, this becomes very significant in the force level differences of the strong side (Figure 4.4 (d) and 4.4 (f)). The nature of DSM area fire allows a large attrition rate in the first salvo for both sides. Besides, the assumption of SLM, which allows just one casualty in a subinterval, apparently underestimates the casualty rate of the strong side. Hence, the difference maintained in the first salvo by forcing the weaker side to absorbing state does not diminish, preserving itself until the end of the combat.

In brief, DSM underestimates the force level of the stronger side and overestimates that of the weaker side compared to SLM, in directed fire. It usually underestimates each side's force levels in area fire, even when all three parameters, initial force level, SSKP, and ξ values of one side is larger than the other's. The execution time of SLM is significantly shorter due to the simplicity of the model, therefore the use of SLM in homogeneous combats with relatively larger initial force sizes, can be useful for risk analyses in operational applications. DSM on the other hand, implemented with its extensions to simulate the combat more precisely, handles satisfactorily larger levels compared to other stochastic models in the literature. Execution times grow prominently as the force sizes and weapon system types increase, but with the use of more optimized code and high configuration hardware, the model can be embedded in tactical level military planning applications.

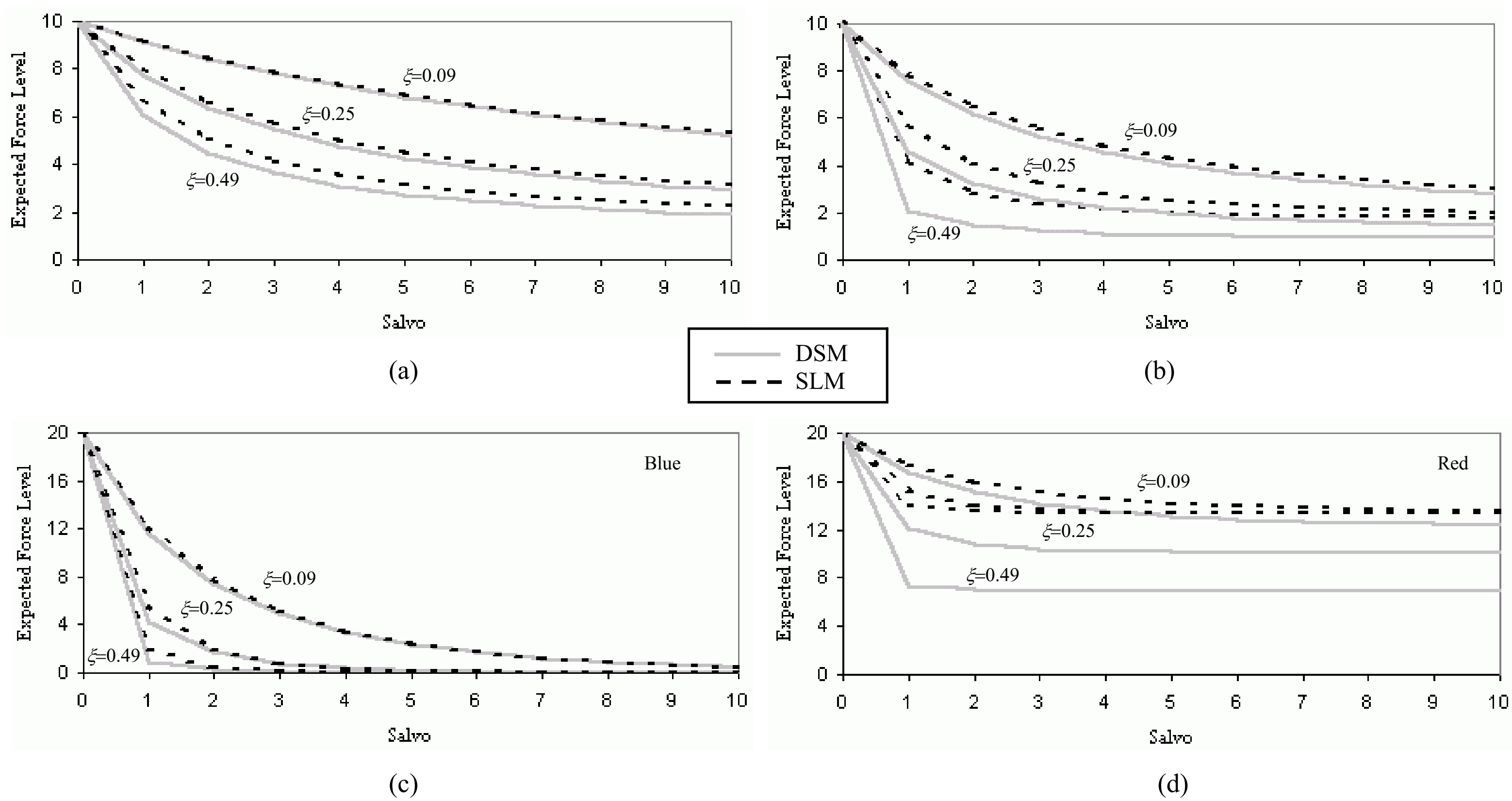
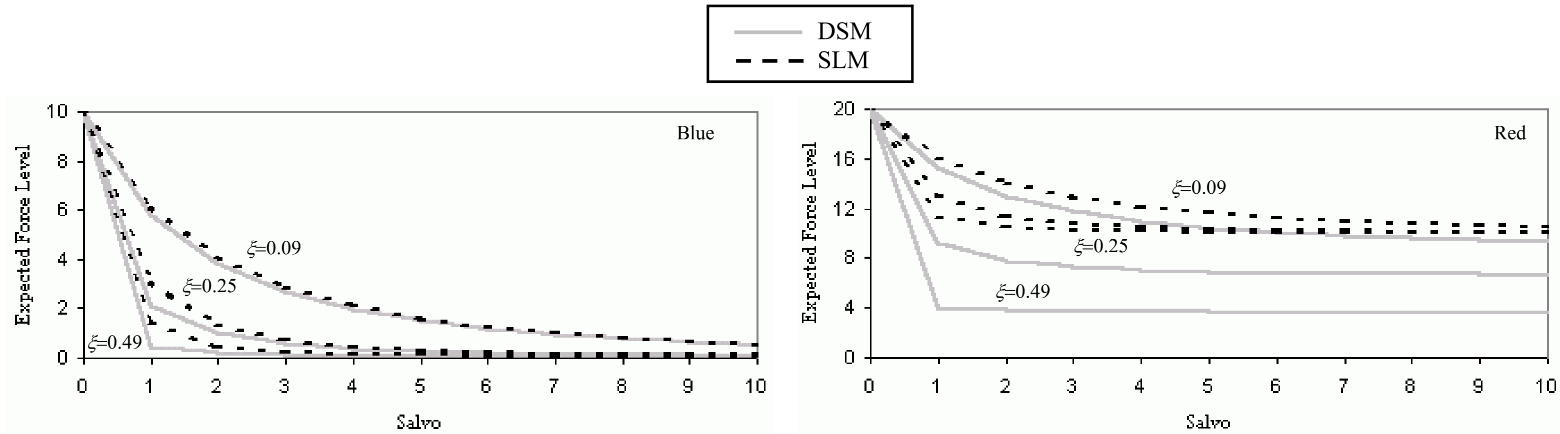


Figure 4.4 Expected force levels with DSM and SLM when area fire is employed

(a) m=n=10, $p_{k:B,R}$= $p_{k:R,B}$=0.1, (b) m=n=10, $p_{k:B,R}$= $p_{k:R,B}$=0.3, (c) Blue force levels for m=n=20, $p_{k:B,R}$=0.1, $p_{k:R,B}$=0.3, (d) Red force levels for the same conditions in (c).

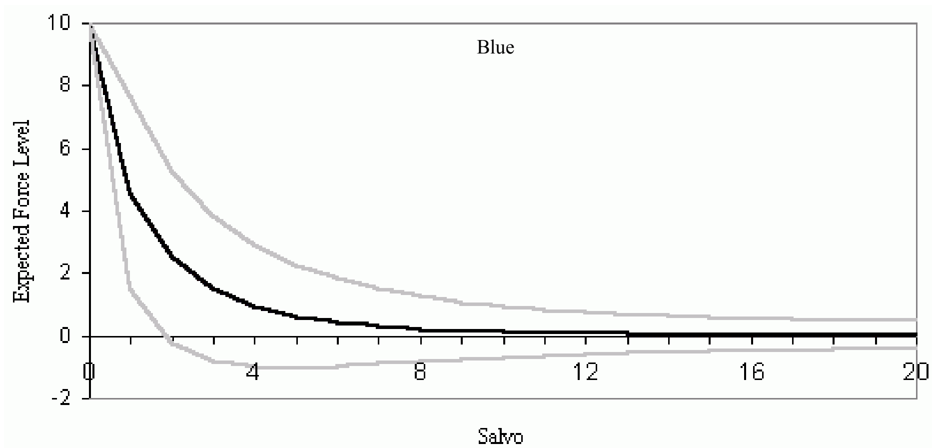


(e) (f)

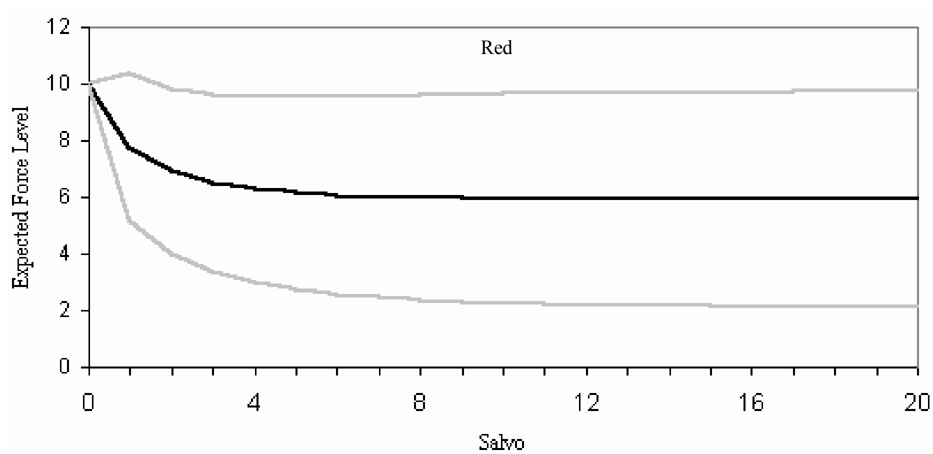
Figure 4.4 (continued) Expected force levels with DSM and SLM when area fire is employed

(e) Blue force levels for m=10, n=20, $p_{k:B,R}$= $p_{k:R,B}$=0.3, (f) Red force levels for the same conditions in (e).

The plots for 95% confidence intervals around the DSM expected values at the end of first 20 salvos, for initial conditions $m = n = 10$, $p_{k:B,R} = 0.1$, $p_{k:R,B} = 0.3$ are given in Figure 4.5. In Figures 4.5 (a) and (b) both sides employ directed fire, and in (c) and (d) both sides employ area fire with $\xi = 0.25$. The negative values for the lower bound of blue force levels are included for demonstrating the decrease in the variance. These kinds of plots allow conducting risk analyses for various combat scenarios. As the salvo number increases, the rate of change in the expected number of combatants, and the variance decrease indicating convergence.



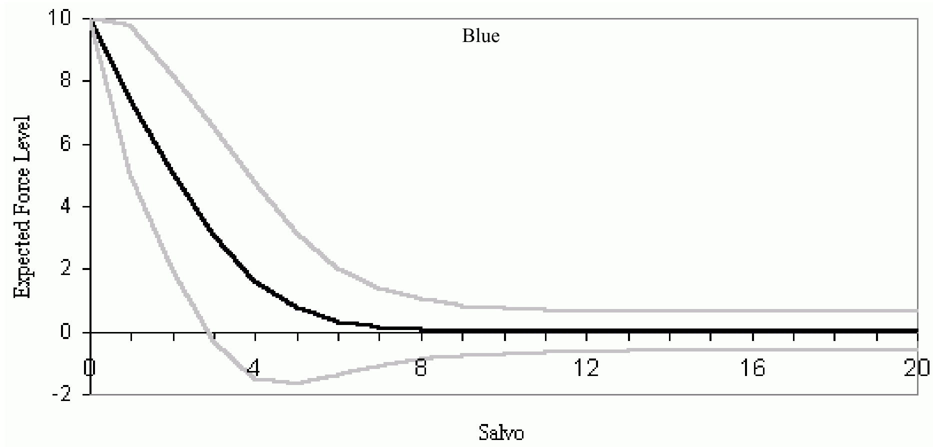
(a)



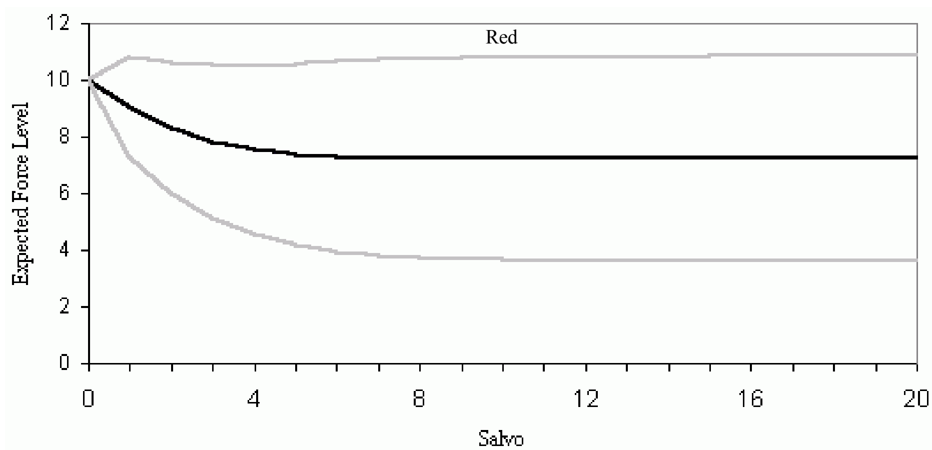
(b)

Figure 4.5 Expected force levels and 95% confidence interval limits with DSM,

(a) and (b) Blue and red when directed fire is employed, m=n=10, $p_{k:B,R}$=0.1, $p_{k:R,B}$=0.3



(c)



(d)

Figure 4.5 (continued) Expected force levels and 95% confidence interval limits with DSM,

(c) and (d) Blue and red when area fire is employed with $\xi = 0.25$, m=n=10, $p_{k:B,R}$=0.1, $p_{k:R,B}$=0.3

CHAPTER 5

A METHOD FOR ESTIMATING ATTRITION RATE COEFFICIENTS

In this chapter, we describe our methodology for estimating ARCs used in LM for red and blue units, namely *a* and *b*, respectively. Our motivation is that, if we succeed in finding good estimators by running DSM off-line, they can be used to form a parameter look-up library to quickly simulate combat by LM's difference equations. Our method is based on least squares estimation using the data for DSM force levels at the end of each salvo. Namely, we try to find estimators for the ARCs such that the mean squared errors (*MSE*s) in estimators of both sides are minimized, which is equivalent to minimizing sum of squared errors (*SSE*s). The method is characterized by the following equation.

$$SSE(a,b) = \sum_{t=1}^{T}\left(E[B_t] - \hat{B}_t\right)^2 + \sum_{t=1}^{T}\left(E[R_t] - \hat{R}_t\right)^2,$$

where $E[B_t]$ and $E[R_t]$ denote the expected force levels for blue and red forces at the end of t^{th} DSM salvo, and $\hat{B}_t, \hat{R}_t$ are the estimated force levels as a function of *a* and *b*. We want to choose *a* and *b* such that force levels predicted by LM using these coefficients will be as close to DSM expected values as possible.

Recall the systems of difference equations in LM for directed and area fire:

Directed fire:

blue → red $\hat{R}_{t+1} = \hat{R}_t - b\hat{B}_t$

red → blue $\hat{B}_{t+1} = \hat{B}_t - a\hat{R}_t$

Area fire:

blue → red $\hat{R}_{t+1} = \hat{R}_t - b\hat{B}_t\hat{R}_t$

red → blue $\hat{B}_{t+1} = \hat{B}_t - a\hat{R}_t\hat{B}_t$

In the *SSE* formula, we use expected force levels from DSM in finding $\hat{B}_t$ and $\hat{R}_t$ values since we want LM predictions to be close to DSM expectations. That is, when both forces apply directed fire, the difference equation for estimated force level, $\hat{B}_t = \hat{B}_{t+1} + a\hat{R}_t$, becomes $\hat{B}_t = E\left[B_{t+1}\right] + aE\left[R_t\right]$. Our estimate for $\hat{R}_t$ is similarly found. Substituting these in the *SSE* formula yields:

$$SSE(a,b) = \sum_{t=0}^{T-1}\left[\left(E\left[B_t\right] - aE\left[R_t\right] - E\left[B_{t+1}\right]\right)^2 + \left(E\left[R_t\right] - bE\left[B_t\right] - E\left[R_{t+1}\right]\right)^2\right].$$

To minimize *SSE*(*a*, *b*), differentiation with respect to *a* and *b* yields:

$$\frac{\partial SSE}{\partial a} = -2\sum_{t=0}^{T-1} E\left[R_t\right]\left(E\left[B_t\right] - aE\left[R_t\right] - E\left[B_{t+1}\right]\right), \text{ and}$$

$$\frac{\partial SSE}{\partial b} = -2\sum_{t=0}^{T-1} E\left[B_t\right]\left(E\left[R_t\right] - bE\left[B_t\right] - E\left[R_{t+1}\right]\right).$$

The Hessian becomes:

$$\nabla^2 SSE(a,b) = \begin{bmatrix} 2\sum_{t=0}^{T-1} E\left[R_t\right]^2 & 0 \\ 0 & 2\sum_{t=0}^{T-1} E\left[B_t\right]^2 \end{bmatrix},$$

which is positive semi-definite indicating convexity. Therefore, the least squares estimators for *a* and *b* become:

$$\frac{\partial SSE}{\partial a}=0 \qquad \Rightarrow -2\sum_{t=0}^{T-1}E[R_t]\left(E[B_t]-aE[R_t]-E[B_{t+1}]\right)=0$$

$$\Rightarrow \hat{a}=\frac{\sum_{t=0}^{T-1}E[R_t]\left(E[B_t]-E[B_{t+1}]\right)}{\sum_{t=0}^{T-1}E[R_t]^2}, \text{ and}$$

$$\frac{\partial SSE}{\partial b}=0 \qquad \Rightarrow -2\sum_{t=0}^{T-1}E[B_t]\left(E[R_t]-bE[B_t]-E[R_{t+1}]\right)=0$$

$$\Rightarrow \hat{b}=\frac{\sum_{t=0}^{T-1}E[B_t]\left(E[R_t]-E[R_{t+1}]\right)}{\sum_{t=0}^{T-1}E[B_t]^2}.$$

When blue force uses directed fire and red force applies area fire, the $SSE(a, b)$ becomes

$$SSE(a,b)=\sum_{t=0}^{T-1}\left[\left(E[B_t]-aE[R_t]E[B_t]-E[B_{t+1}]\right)^2+\left(E[R_t]-bE[B_t]-E[R_{t+1}]\right)^2\right].$$

Differentiation with respect to a and b yields:

$$\frac{\partial SSE}{\partial a}=-2\sum_{t=0}^{T-1}E[R_t]E[B_t]\left(E[B_t]-aE[R_t]E[B_t]-E[B_{t+1}]\right), \text{ and}$$

$$\frac{\partial SSE}{\partial b}=-2\sum_{t=0}^{T-1}E[B_t]\left(E[R_t]-bE[B_t]-E[R_{t+1}]\right).$$

The Hessian is:

$$\nabla^2 SSE(a,b)=\begin{bmatrix} 2\sum_{t=0}^{T-1}\left(E[R_t]E[B_t]\right)^2 & 0 \\ 0 & 2\sum_{t=0}^{T-1}E[B_t]^2 \end{bmatrix},$$

which is again positive semi-definite. Therefore, the least squares estimators for a and b are:

$$\frac{\partial SSE}{\partial a}=0 \qquad \Rightarrow -2\sum_{t=0}^{T-1}E\left[R_t\right]E\left[B_t\right]\left(E\left[B_t\right]-aE\left[R_t\right]E\left[B_t\right]-E\left[B_{t+1}\right]\right)=0$$

$$\Rightarrow \hat{a}=\frac{\sum_{t=0}^{T-1}E\left[R_t\right]E\left[B_t\right]\left(E\left[B_t\right]-E\left[B_{t+1}\right]\right)}{\sum_{t=0}^{T-1}\left(E\left[R_t\right]E\left[B_t\right]\right)^2}, \text{ and}$$

$$\frac{\partial SSE}{\partial b}=0 \qquad \Rightarrow -2\sum_{t=0}^{T-1}E\left[B_t\right]\left(E\left[R_t\right]-bE\left[B_t\right]-E\left[R_{t+1}\right]\right)=0$$

$$\Rightarrow \hat{b}=\frac{\sum_{t=0}^{T-1}E\left[B_t\right]\left(E\left[R_t\right]-E\left[R_{t+1}\right]\right)}{\sum_{t=0}^{T-1}E\left[B_t\right]^2}.$$

Finally, when both forces use area fire, the Hessian is

$$\nabla^2 SSE(a,b)=\begin{bmatrix} 2\sum_{t=0}^{T-1}\left(E\left[R_t\right]E\left[B_t\right]\right)^2 & 0 \\ 0 & 2\sum_{t=0}^{T-1}\left(E\left[R_t\right]E\left[B_t\right]\right)^2 \end{bmatrix},$$

which is clearly positive semi-definite. Least squares estimators for a and b become:

$$\hat{a}=\frac{\sum_{t=0}^{T-1}E\left[R_t\right]E\left[B_t\right]\left(E\left[B_t\right]-E\left[B_{t+1}\right]\right)}{\sum_{t=0}^{T-1}\left(E\left[R_t\right]E\left[B_t\right]\right)^2} \text{ and } \hat{b}=\frac{\sum_{t=0}^{T-1}E\left[B_t\right]E\left[R_t\right]\left(E\left[R_t\right]-E\left[R_{t+1}\right]\right)}{\sum_{t=0}^{T-1}\left(E\left[B_t\right]E\left[R_t\right]\right)^2}.$$

The results are summarized in Table 5.1. Note that, the estimator for the ARC of a unit depends only on the fire type of that unit. Namely, the estimator for a does not change with the fire type of the blue unit, but only with the fire type of red unit.

Table 5.1 Least squares estimators for ARCs

	Fire Type of Unit	
	Directed	**Area**
ARC of the Red Unit	$\hat{a} = \dfrac{\sum_{t=0}^{T-1} E[R_t]\left(E[B_t] - E[B_{t+1}]\right)}{\sum_{t=0}^{T-1} E[R_t]^2}$	$\hat{a} = \dfrac{\sum_{t=0}^{T-1} E[R_t]E[B_t]\left(E[B_t] - E[B_{t+1}]\right)}{\sum_{t=0}^{T-1} \left(E[R_t]E[B_t]\right)^2}$
ARC of the Blue Unit	$\hat{b} = \dfrac{\sum_{t=0}^{T-1} E[B_t]\left(E[R_t] - E[R_{t+1}]\right)}{\sum_{t=0}^{T-1} E[B_t]^2}$	$\hat{b} = \dfrac{\sum_{t=0}^{T-1} E[B_t]E[R_t]\left(E[R_t] - E[R_{t+1}]\right)}{\sum_{t=0}^{T-1} \left(E[B_t]E[R_t]\right)^2}$

For estimation of the LM attrition rate coefficients, both fire types are considered. Homogeneous DSM combats of length ten salvos, where blue and red units employ directed-directed, directed-area and area-area fire against each other, are run with initial parameters that are summarized in Table 5.2.Directed-area corresponds to the case where blue unit employs directed fire, and red unit area fire against the other side. The probability ξ in all area fire cases is taken to be 0.25, corresponding to a fatal area radius percentage 50%. Using the Cartesian product of the set of initial force levels {5, 10, 20} and the set of SSKPs {0.1, 0.3}, sufficient scenario data are produced for estimating the ARCs *a*, *b* in LM.

Table 5.2 Initial parameter sets for DSM runs used in LM ARC estimation

Blue		Red	
m (B_0)	$p_{k:B,R}$	n (R_0)	$p_{k:R,B}$
{5, 10, 20}	{0.1, 0.3}	{5, 10, 20}	{0.1, 0.3}

We run LM using the estimated ARCs and compare DSM and LM force levels. Since our method is based on minimizing the mean squared errors in estimating the ARCs a and b, the performance measure for a successful estimation is obtaining small MSE values. Looking at the results given in Figures 5.1 - 5.3, we observe that the MSE values vary depending on the fire type. Area-area type combats give the closest results, while directed-directed produce the worst results on the average.

In area-area fire ARC estimations, the average MSE between DSM and LM force level values is 0.0130, and the largest MSE is 0.1095 in 20-on-20 combat, where both units have SSKPs equal to 0.3. In directed-area case, average MSE is 1.4276, while 20-on-5 combat with blue and red units having 0.3 SSKPs produce a maximum MSE of 28.9991. Directed-directed case gives an average of 4.1489 for MSE, producing the largest difference, namely 28.4425, in the 10-on-20 or 20-on-10 combats, where both units' SSKP is 0.3.

A second observation concerning the MSE differences is the SSKP effect. When both units employ the same type of fire, namely in directed-directed and area-area cases, an increase in SSKP values boosts MSEs as well. The same applies for directed-area combat, but due to the asymmetry in fire type, the SSKP matchings 0.1-0.3 and 0.3-0.1 differ. Explicitly, the stronger in SSKP value is the side employing area fire, the closer the estimation results, and vice versa. As an example for these remarks, recall that the worst MSE value of all experimental runs is obtained in the case 20-on-5, directed-area combat with both SSKPs equal to 0.3.

For each fire type combination, estimated LM force levels versus expected DSM force levels for the blue unit are plotted in Figures 5.4 and 5.5. The almost perfect fit in area-area case is mainly due to replacement of the terms B_t and R_t with the product B_tR_t in ARC formulas in the case of area fire. More knowledge of both units' DSM salvo force levels give better estimators for LM ARCs to be used in the systems of difference equations.

directed - directed

0.1 x 0.1

	a			*b*			*MSE - Blue*			*MSE - Red*		
	R_0			R_0			R_0			R_0		
B_0	5	10	20	5	10	20	5	10	20	5	10	20
5	0.0895	0.0572	0.0268	0.0895	0.0970	0.0984	0.0027	0.5136	2.7958	0.0027	0.1401	0.8866
10	0.0970	0.0911	0.0584	0.0572	0.0911	0.0966	0.1401	0.0012	1.9366	0.5136	0.0012	0.5003
20	0.0984	0.0966	0.0912	0.0268	0.0584	0.0912	0.8866	0.5003	**0.0005**	2.7958	1.9366	0.0005

0.1 x 0.3

	a			*b*			*MSE - Blue*			*MSE - Red*		
	R_0			R_0			R_0			R_0		
B_0	5	10	20	5	10	20	5	10	20	5	10	20
5	0.1352	0.0567	0.0264	0.0944	0.0970	0.0983	1.2219	3.8584	5.2621	0.3475	1.2399	1.8199
10	0.2653	0.1371	0.0568	0.0730	0.0934	0.0964	0.1762	4.6050	14.8139	0.0749	1.2059	4.6780
20	0.2860	0.2661	0.1375	0.0311	0.0757	0.0928	6.0916	0.3734	17.7278	2.3057	0.1834	4.4937

0.3 x 0.1

	a			*b*			*MSE - Blue*			*MSE - Red*		
	R_0			R_0			R_0			R_0		
B_0	5	10	20	5	10	20	5	10	20	5	10	20
5	0.0944	0.0730	0.0311	0.1352	0.2653	0.2860	0.3475	0.0749	2.3057	1.2219	0.1762	6.0916
10	0.0970	0.0934	0.0757	0.0567	0.1371	0.2661	1.2399	1.2059	0.1834	3.8584	4.6050	0.3734
20	0.0983	0.0964	0.0928	0.0264	0.0568	0.1375	1.8199	4.6780	4.4937	5.2621	14.8139	17.7278

0.3 x 0.3

	a			*b*			*MSE - Blue*			*MSE - Red*		
	R_0			R_0			R_0			R_0		
B_0	5	10	20	5	10	20	5	10	20	5	10	20
5	0.2146	0.0739	0.0294	0.2146	0.2734	0.2851	0.1449	3.1649	5.1332	0.1449	7.7644	14.7802
10	0.2734	0.2226	0.0740	0.0739	0.2226	0.2694	7.7644	0.2469	12.0487	3.1649	0.2469	28.4425
20	0.2851	0.2694	0.2264	0.0294	0.0740	0.2264	14.7802	**28.4425**	0.3766	5.1332	12.0487	0.3766

Figure 5.1 Estimated ARC and Corresponding MSE Values for Directed-Directed Combat

directed - area

0.1 x 0.1

	a			b			MSE - Blue			MSE - Red		
	R_0			R_0			R_0			R_0		
B_0	5	10	20	5	10	20	5	10	20	5	10	20
5	0.0235	0.0224	0.0199	0.0900	0.0972	0.0985	0.0011	0.0001	0.0000	0.0022	0.0000	0.0000
10	0.0237	0.0224	0.0199	0.0668	0.0935	0.0966	0.1498	0.0011	0.0002	0.1954	0.0001	0.0001
20	0.0239	0.0225	0.0200	0.0351	0.0814	0.0931	6.2787	0.0507	0.0031	1.8427	0.0390	0.0013

0.1 x 0.3

	a			b			MSE - Blue			MSE - Red		
	R_0			R_0			R_0			R_0		
B_0	5	10	20	5	10	20	5	10	20	5	10	20
5	0.0645	0.0543	0.0395	0.0944	0.0969	0.0982	0.0023	0.0000	0.0000	0.0000	0.0000	**0.0000**
10	0.0644	0.0544	0.0395	0.0849	0.0933	0.0960	0.0354	0.0007	0.0002	0.0040	0.0002	0.0000
20	0.0650	0.0546	0.0396	0.0590	0.0867	0.0919	1.8260	0.0126	0.0032	0.2286	0.0024	0.0001

0.3 x 0.1

	a			b			MSE - Blue			MSE - Red		
	R_0			R_0			R_0			R_0		
B_0	5	10	20	5	10	20	5	10	20	5	10	20
5	0.0238	0.0223	0.0200	0.1403	0.2669	0.2864	0.2274	0.0032	0.0001	1.0830	0.0038	0.0005
10	0.0239	0.0225	0.0201	0.0678	0.1878	0.2708	3.0723	0.3555	0.0016	3.2909	1.0200	0.0049
20	0.0239	0.0226	0.0201	0.0326	0.0912	0.2391	**19.6840**	13.8354	0.0225	4.8400	8.2575	0.0305

0.3 x 0.3

	a			b			MSE - Blue			MSE - Red		
	R_0			R_0			R_0			R_0		
B_0	5	10	20	5	10	20	5	10	20	5	10	20
5	0.0642	0.0545	0.0395	0.2129	0.2732	0.2841	0.0701	0.0005	0.0001	0.0834	0.0009	0.0000
10	0.0651	0.0547	0.0396	0.1174	0.2447	0.2665	2.4726	0.0111	0.0017	1.1296	0.0037	0.0005
20	0.0650	0.0547	0.0396	0.0544	0.1862	0.2370	**28.9991**	0.0931	0.0229	3.0370	0.4396	0.0074

Figure 5.2 Estimated ARC and Corresponding MSE Values for Directed-Area Combat

area - area

0.1 x 0.1

B_0	*a* R_0 = 5	10	20	*b* R_0 = 5	10	20	*MSE - Blue* R_0 = 5	10	20	*MSE - Red* R_0 = 5	10	20
5	0.0236	0.0224	0.0200	0.0236	0.0237	0.0239	0.0003	0.0005	0.0002	0.0003	0.0008	0.0010
10	0.0237	0.0225	0.0201	0.0224	0.0225	0.0226	0.0008	0.0017	0.0029	0.0005	0.0017	0.0038
20	0.0239	0.0226	0.0202	0.0200	0.0201	0.0202	0.0010	0.0038	0.0316	0.0002	0.0029	0.0316

0.1 x 0.3

B_0	*a* R_0 = 5	10	20	*b* R_0 = 5	10	20	*MSE - Blue* R_0 = 5	10	20	*MSE - Red* R_0 = 5	10	20
5	0.0645	0.0545	0.0396	0.0237	0.0239	0.0238	0.0017	0.0002	0.0003	0.0003	0.0002	0.0002
10	0.0646	0.0547	0.0396	0.0224	0.0225	0.0224	0.0132	0.0040	0.0047	0.0016	0.0010	0.0019
20	0.0649	0.0548	0.0397	0.0201	0.0201	0.0199	0.0506	0.0490	0.0641	0.0027	0.0065	0.0112

0.3 x 0.1

B_0	*a* R_0 = 5	10	20	*b* R_0 = 5	10	20	*MSE - Blue* R_0 = 5	10	20	*MSE - Red* R_0 = 5	10	20
5	0.0237	0.0224	0.0201	0.0645	0.0646	0.0649	0.0003	0.0016	0.0027	0.0017	0.0132	0.0506
10	0.0239	0.0225	0.0201	0.0545	0.0547	0.0548	0.0002	0.0010	0.0065	0.0002	0.0040	0.0490
20	0.0238	0.0224	0.0199	0.0396	0.0396	0.0397	0.0002	0.0019	0.0112	0.0003	0.0047	0.0641

0.3 x 0.3

B_0	*a* R_0 = 5	10	20	*b* R_0 = 5	10	20	*MSE - Blue* R_0 = 5	10	20	*MSE - Red* R_0 = 5	10	20
5	0.0643	0.0546	0.0397	0.0643	0.0649	0.0647	0.0102	0.0033	0.0023	0.0102	0.0113	0.0114
10	0.0649	0.0546	0.0397	0.0546	0.0546	0.0543	0.0113	0.0196	0.0262	0.0033	0.0196	0.0325
20	0.0647	0.0543	0.0395	0.0397	0.0397	0.0395	0.0114	0.0325	0.1095	0.0023	0.0262	0.1095

Figure 5.3 Estimated ARC and Corresponding MSE Values for Area-Area Combat

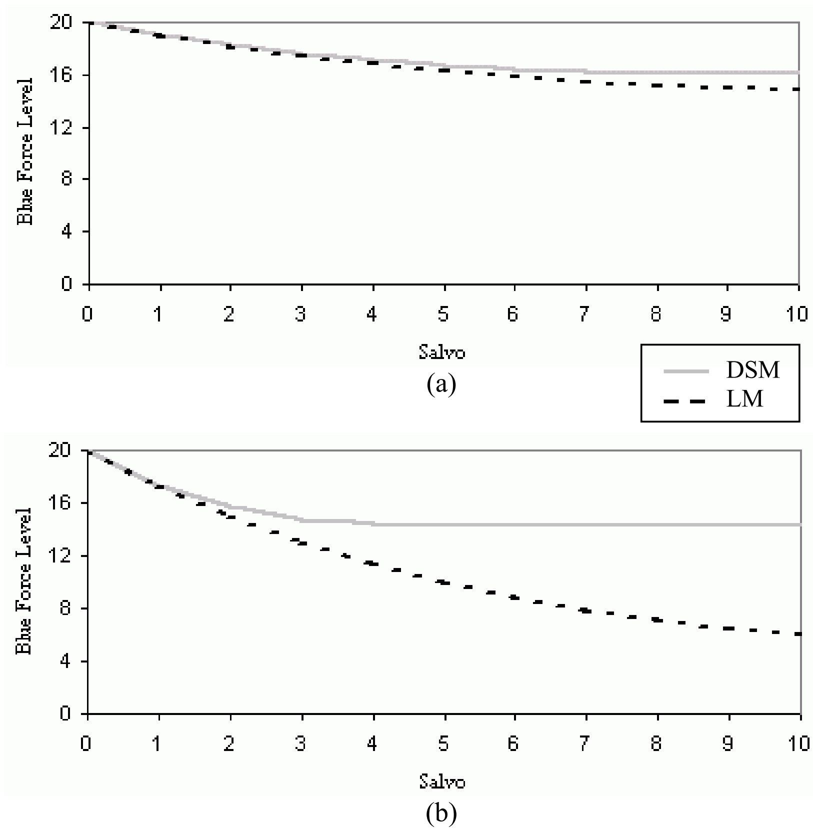


Figure 5.4 Blue unit force levels with DSM and LM for directed-directed fire

(a) m=20, n=10, $p_{k:B,R}$= $p_{k:R,B}$=0.1; LM ARCs, a=0.0966, b=0.0584,

(b) m=20, n=10, $p_{k:B,R}$= $p_{k:R,B}$=0.3; LM ARCs, a=0.2694, b=0.0740.

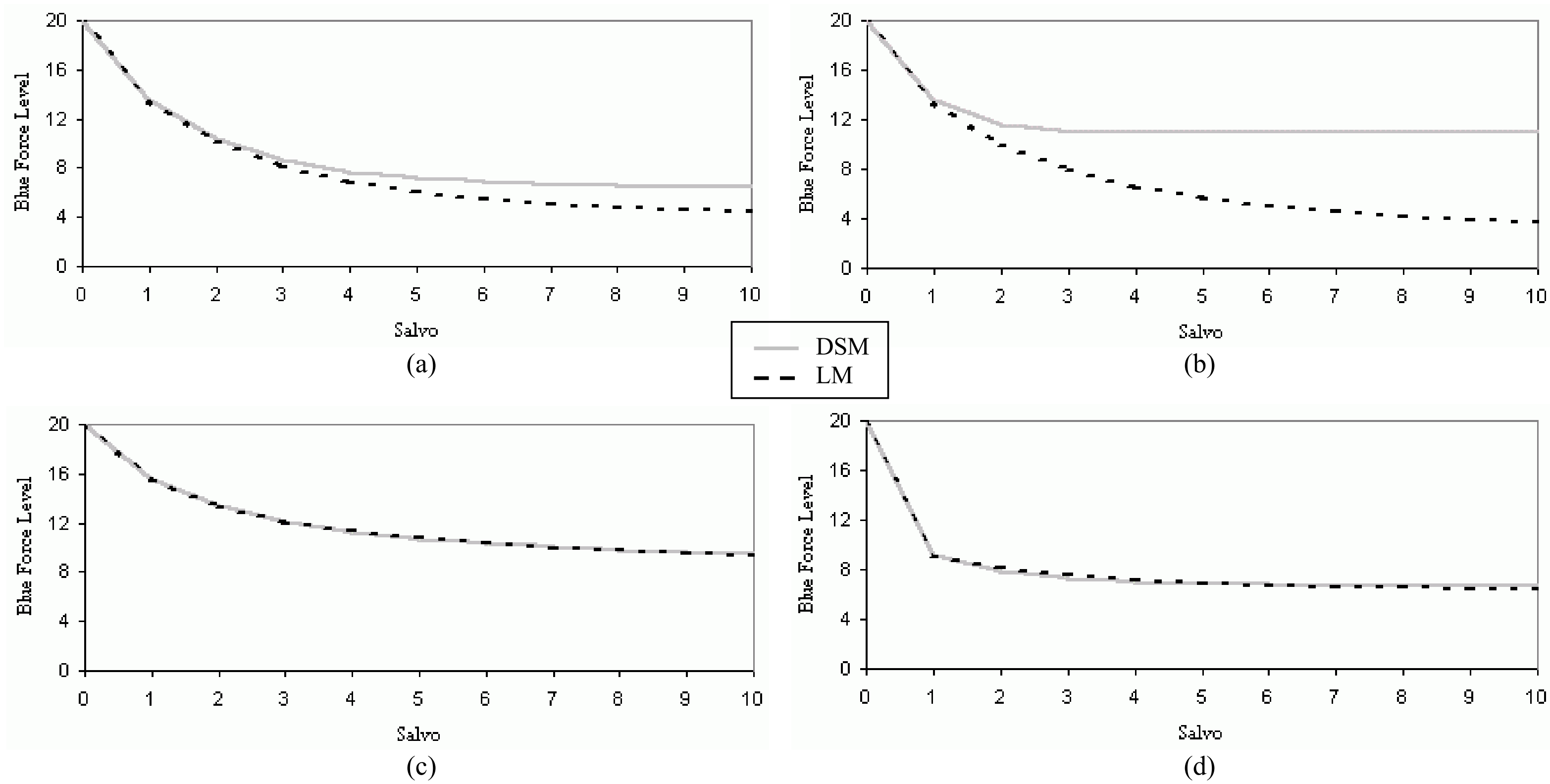


Figure 5.5 Blue unit force levels with DSM and LM for directed-area and area-area fires

(a) Directed-area fire with m=20, n=5, $p_{k:B,R}$=0.1, $p_{k:R,B}$=0.3; LM ARCs, a=0.0650, b=0.0590, (b) Directed-area fire with m=20, n=5, $p_{k:B,R}$= $p_{k:R,B}$=0.3; LM ARCs, a=0. 0650, b=0.0544, (c) Area-area fire with m=20, n=10, $p_{k:B,R}$= $p_{k:R,B}$=0.1, ξ = 0.25; LM ARCs, a=0.0226, b=0.0201, (d) Area-area fire with m=20, n=10, $p_{k:B,R}$= $p_{k:R,B}$=0.3, ξ = 0.25; LM ARCs, a=0. 0543, b=0.0397.

CHAPTER 6

CONCLUSION

In this thesis, we present the implementation and experimental analysis of a methodology (DSM) developed for modeling and analyzing tactical level land combat. DSM models heterogeneous land combat as a discrete-time stochastic process based on SSKPs to generate information for weapon and ammunition planning. Both directed fire and area fire are included in the model, and division and combination of military units are allowed. DSM integrates stochastic engagement, combat loss, and noncombat loss processes for calculating casualties in each salvo. Discrete reinforcements and adjustment of SSKPs to reflect division and combination effects are also possible.

DSM is implemented in C++, using MS Visual C++ 6.0 integrated development environment, utilizing the pointer abilities provided by the C++ programming language. The salvo statistics, including mean and variance of force levels, survival probabilities at the end of each salvo, and the computation time statistics for the combat model are stored in output files. In an attempt to estimate ARCs for Lanchester model, a code for SLM, which is based on the Poisson process, is also implemented. The input, output methods and executable format are the same as those used for DSM.

The simulation runs for the purpose of comparing DSM with SLM are based on homogeneous combats of ten salvos. Compared to SLM, DSM underestimates the force level of the stronger side, and overestimates the force level of the weaker side, resulting in closer combat. The execution time of SLM is significantly shorter due to the simplicity of the model, therefore the use of SLM in homogeneous combats with relatively larger initial force sizes, can be useful for risk analyses in operational applications. DSM on the other hand, implemented for heterogeneous combat with its extensions to simulate the combat more precisely, handles satisfactorily larger levels compared to other similar stochastic models in the literature. Execution times grow prominently as the force sizes and weapon system types increase, but with the use of more optimized code and state of the art configuration hardware, the model can be embedded in tactical level military planning applications.

In addition to experimentation with DSM, we estimate ARCs used in LM and SLM, such that DSM and LM will yield similar figures for force levels throughout the combat. This is attempted through comparison of force levels at the end of each DSM salvo with the respective force levels obtained by LM using estimated ARCs. The results for area fire are rather satisfactory in the sense that both models produce similar curves. The differences in force levels grow larger as SSKP values increase. This is expected since Poisson is the limiting distribution of binomial distribution when probability of success, here SSKP, in a single salvo is small. This is more apparent in area fire, where the attrition does not only depend on the firer’s force level, but also on the target’s force level distributed uniformly over the battlefield. The inclusion of fatal area probability values makes a slight modification of the SSKPs. The large differences are mainly due to the nature of DSM area fire and SLM area fire. The latter allows just one casualty in every small subinterval as in SLM directed fire case.

We identify the following possible further research directions. DSM should be applied for a real combat scenario to investigate the representation power of the model. Data gathered from military exercises or higher-resolution combat simulations can be used to compare DSM, SLM and other models, and for validation purposes.

As opposed to the assumptions of the model, SSKPs do not remain constant throughout the combat in real-life conditions. To reflect combatant skills, morale, or environmental conditions such as weather or terrain effects, time-varying and force-level dependent SSKPs may be considered.

It may be possible to formulate DSM as a Markov chain, and calculate results for any salvo directly by applying matrix geometric analysis to the special structure of the transition matrix.

Force aggregation methodologies in order to speed up DSM by reducing the dimensions could be developed. For instance, one side is kept heterogeneous and the other side is aggregated into a single homogeneous unit. Then, the salvo results are combined into heterogeneous combat state probability matrix. Another possible schema for approximated DSM is to discard the states having negligible marginal probabilities and consider mainly the states around the expected values by redistributing marginal probabilities of the discarded states.

Another topic is to develop a methodology to integrate LM with DSM, to model combat situations involving both military units with small number of combatants like artillery or tank, and military units with large number of combatants like infantry. Attrition rate coefficients required for modeling large scale combat problems with LM can be estimated with the developed procedures from DSM results, and an approximate LM combat can be run.

Finally, DSM can be used for risk analysis and estimation of munition requirements if different weapon systems are to be synchronized by means of the engagement process, and munition-dependent SSKPs are utilized.

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APPENDIX A1

FILE OUTPUT EXAMPLE FOR DSM SALVO STATISTICS: "SALVOS.TXT"

```
SALVO 1

BLUE 1
P (0 elements)    = 0.0000009261
P (1 elements)    = 0.0001425110
P (2 elements)    = 0.0057831838
P (3 elements)    = 0.0782617719
P (4 elements)    = 0.3755868501
P (5 elements)    = 0.5402247571
Sum               = 1.0000000000

Expected          = 4.4499653797
Variance          = 0.4404478397

BLUE 2
P (0 elements)    = 0.0000046876
P (1 elements)    = 0.0008906250
P (2 elements)    = 0.0296601562
P (3 elements)    = 0.2768945312
P (4 elements)    = 0.6925500000
Sum               = 1.0000000000

Expected          = 3.6610945312
Variance          = 0.2887688645

RED 1
P (0 elements)    = 0.0000009041
P (1 elements)    = 0.0000427863
P (2 elements)    = 0.0008485032
P (3 elements)    = 0.0091150767
P (4 elements)    = 0.0570846703
P (5 elements)    = 0.2075260773
P (6 elements)    = 0.4034875638
P (7 elements)    = 0.3218944183
Sum               = 1.0000000000

Expected          = 5.9692404014
Variance          = 0.8535269717
SALVO 2

BLUE 1
P (0 elements)    = 0.0003820268
P (1 elements)    = 0.0069025682
P (2 elements)    = 0.0535259927
P (3 elements)    = 0.2103248393
P (4 elements)    = 0.4111043546
P (5 elements)    = 0.3177602184
Sum               = 1.0000000000

Expected          = 3.9781475822
Variance          = 0.8099470418

BLUE 2
P (0 elements)    = 0.0007597823
P (1 elements)    = 0.0138186550
P (2 elements)    = 0.1054097232
P (3 elements)    = 0.3733430162
P (4 elements)    = 0.5066688233
Sum               = 1.0000000000

Expected          = 3.3713424432
Variance          = 0.5362959972

RED 1
P (0 elements)    = 0.0001945426
P (1 elements)    = 0.0026787721
P (2 elements)    = 0.0181805100
P (3 elements)    = 0.0742235299
P (4 elements)    = 0.1919412076
P (5 elements)    = 0.3099163566
P (6 elements)    = 0.2865861748
P (7 elements)    = 0.1162789064
Sum               = 1.0000000000

Expected          = 5.1125263887
Variance          = 1.4392234491
SALVO 3

BLUE 1
P (0 elements)    = 0.0046176907
P (1 elements)    = 0.0307856742
P (2 elements)    = 0.1200774874
P (3 elements)    = 0.2793388001
P (4 elements)    = 0.3625381571
P (5 elements)    = 0.2026421905
Sum               = 1.0000000000

Expected          = 3.5723206299
```

Variance = 1.1303354166

BLUE 2
P (0 elements) = 0.0054386953
P (1 elements) = 0.0400551327
P (2 elements) = 0.1702957496
P (3 elements) = 0.3944668949
P (4 elements) = 0.3897435275
Sum = 1.0000000000

Expected = 3.1230214263
Variance = 0.7540737949

RED 1
P (0 elements) = 0.0029058941
P (1 elements) = 0.0180634804
P (2 elements) = 0.0668517617
P (3 elements) = 0.1614083601
P (4 elements) = 0.2603753135
P (5 elements) = 0.2730797744
P (6 elements) = 0.1696511822
P (7 elements) = 0.0476642336
Sum = 1.0000000000

Expected = 4.3944489388
Variance = 1.8629536762

SALVO 4

BLUE 1
P (0 elements) = 0.0178078485
P (1 elements) = 0.0656484727
P (2 elements) = 0.1742318414
P (3 elements) = 0.2990634625
P (4 elements) = 0.3038506888
P (5 elements) = 0.1393976861
Sum = 1.0000000000

Expected = 3.2236937285
Variance = 1.4084989181

BLUE 2
P (0 elements) = 0.0159985929
P (1 elements) = 0.0703076391
P (2 elements) = 0.2144799000
P (3 elements) = 0.3855716703
P (4 elements) = 0.3136421977
Sum = 1.0000000000

Expected = 2.9105512407
Variance = 0.9453389100

RED 1
P (0 elements) = 0.0143289579
P (1 elements) = 0.0510936872
P (2 elements) = 0.1274506019
P (3 elements) = 0.2186337010
P (4 elements) = 0.2597279424
P (5 elements) = 0.2065017747
P (6 elements) = 0.0998526260
P (7 elements) = 0.0224107089
Sum = 1.0000000000

Expected = 3.7893073551
Variance = 2.1807598896

SALVO 5

BLUE 1
P (0 elements) = 0.0412033895
P (1 elements) = 0.1007302266
P (2 elements) = 0.2082516573
P (3 elements) = 0.2929609077
P (4 elements) = 0.2539661753
P (5 elements) = 0.1028876436
Sum = 1.0000000000

Expected = 2.9264191836
Variance = 1.6421056817

BLUE 2
P (0 elements) = 0.0318822136
P (1 elements) = 0.0982654906
P (2 elements) = 0.2408810337
P (3 elements) = 0.3662050329
P (4 elements) = 0.2627662292
Sum = 1.0000000000

Expected = 2.7297075737
Variance = 1.1105911515

RED 1
P (0 elements) = 0.0400926410
P (1 elements) = 0.0935912461
P (2 elements) = 0.1763782898
P (3 elements) = 0.2375167807
P (4 elements) = 0.2279960664
P (5 elements) = 0.1503409041
P (6 elements) = 0.0619083044
P (7 elements) = 0.0121757675
Sum = 1.0000000000

Expected = 3.2792671528
Variance = 2.4149336024

SALVO 6

BLUE 1
P (0 elements) = 0.0725228629
P (1 elements) = 0.1296080883
P (2 elements) = 0.2250805750
P (3 elements) = 0.2761918538
P (4 elements) = 0.2156639591
P (5 elements) = 0.0809326609

Sum = 1.0000000000

Expected = 2.6756639407
Variance = 1.8304194175

BLUE 2
P (0 elements) = 0.0514480778
P (1 elements) = 0.1212112546
P (2 elements) = 0.2545532186
P (3 elements) = 0.3447283428
P (4 elements) = 0.2280591062
Sum = 1.0000000000

Expected = 2.5767391450
Variance = 1.2513402918

RED 1
P (0 elements) = 0.0808633640
P (1 elements) = 0.1337995037
P (2 elements) = 0.2045841264
P (3 elements) = 0.2306464143
P (4 elements) = 0.1903266499
P (5 elements) = 0.1107402725
P (6 elements) = 0.0414056077
P (7 elements) = 0.0076340615
Sum = 1.0000000000

Expected = 2.8517870384
Variance = 2.5736685287

SALVO 7

BLUE 1
P (0 elements) = 0.1082959169
P (1 elements) = 0.1502694395
P (2 elements) = 0.2300412449
P (3 elements) = 0.2567487368
P (4 elements) = 0.1874157907
P (5 elements) = 0.0672288712
Sum = 1.0000000000

Expected = 2.4664056586
Variance = 1.9773906091

BLUE 2
P (0 elements) = 0.0729716854
P (1 elements) = 0.1386105828
P (2 elements) = 0.2598512765
P (3 elements) = 0.3245581490
P (4 elements) = 0.2040083063
Sum = 1.0000000000

Expected = 2.4480208078
Variance = 1.3703660546

RED 1
P (0 elements) = 0.1328385924
P (1 elements) = 0.1642627224
P (2 elements) = 0.2141513523
P (3 elements) = 0.2116696671
P (4 elements) = 0.1571140914
P (5 elements) = 0.0844506686
P (6 elements) = 0.0300623790
P (7 elements) = 0.0054505268
Sum = 1.0000000000

Expected = 2.4968120984
Variance = 2.6662381149

SALVO 8

BLUE 1
P (0 elements) = 0.1453622825
P (1 elements) = 0.1631990024
P (2 elements) = 0.2278233483
P (3 elements) = 0.2383743576
P (4 elements) = 0.1669185142
P (5 elements) = 0.0583224950
Sum = 1.0000000000

Expected = 2.2932553039
Variance = 2.0896003282

BLUE 2
P (0 elements) = 0.0950604263
P (1 elements) = 0.1509665513
P (2 elements) = 0.2599586684
P (3 elements) = 0.3068966046
P (4 elements) = 0.1871177494
Sum = 1.0000000000

Expected = 2.3400446995
Variance = 1.4709454612

RED 1
P (0 elements) = 0.1904918482
P (1 elements) = 0.1829491418
P (2 elements) = 0.2110674300
P (3 elements) = 0.1894261302
P (4 elements) = 0.1308988274
P (5 elements) = 0.0672608131
P (6 elements) = 0.0235759484
P (7 elements) = 0.0043298609
Sum = 1.0000000000

Expected = 2.2050264841
Variance = 2.7067111286

SALVO 9

BLUE 1
P (0 elements) = 0.1814081783
P (1 elements) = 0.1698799690
P (2 elements) = 0.2218329518

P (3 elements) = 0.2225534568
P (4 elements) = 0.1520803218
P (5 elements) = 0.0522451223
Sum = 1.0000000000

Expected = 2.1507531415
Variance = 2.1738670172

BLUE 2
P (0 elements) = 0.1167393457
P (1 elements) = 0.1591553137
P (2 elements) = 0.2570417527
P (3 elements) = 0.2919605997
P (4 elements) = 0.1751029882
Sum = 1.0000000000

Expected = 2.2495325713
Variance = 1.5562187444

RED 1
P (0 elements) = 0.2487815351
P (1 elements) = 0.1912589599
P (2 elements) = 0.2010332325
P (3 elements) = 0.1683348168
P (4 elements) = 0.1111532380
P (5 elements) = 0.0559683093
P (6 elements) = 0.0197438994
P (7 elements) = 0.0037260090
Sum = 1.0000000000

Expected = 1.9673298332
Variance = 2.7110329285

SALVO 10

BLUE 1
P (0 elements) = 0.2149837182
P (1 elements) = 0.1719759688
P (2 elements) = 0.2143159144
P (3 elements) = 0.2096346686
P (4 elements) = 0.1412382636
P (5 elements) = 0.0478514664
Sum = 1.0000000000

Expected = 2.0337221898
Variance = 2.2360245758

BLUE 2
P (0 elements) = 0.1373997864
P (1 elements) = 0.1640943683
P (2 elements) = 0.2525109664
P (3 elements) = 0.2795520246
P (4 elements) = 0.1664428543
Sum = 1.0000000000

Expected = 2.1735437921
Variance = 1.6288995080

RED 1
P (0 elements) = 0.3041151740
P (1 elements) = 0.1919051163
P (2 elements) = 0.1881041025
P (3 elements) = 0.1500743447
P (4 elements) = 0.0965384228
P (5 elements) = 0.0484558265
P (6 elements) = 0.0174173775
P (7 elements) = 0.0033896357
Sum = 1.0000000000

Expected = 1.7750008943
Variance = 2.6934906223

APPENDIX A2

FILE OUTPUT EXAMPLE FOR DSM FINAL STATISTICS: "STATS.TXT"

Preparation part took 15.000 cpu clocks
Salvo 1 took 0.000 cpu clocks
Salvo 2 took 0.000 cpu clocks
Salvo 3 took 0.000 cpu clocks
Salvo 4 took 16.000 cpu clocks
Salvo 5 took 0.000 cpu clocks
Salvo 6 took 0.000 cpu clocks
Salvo 7 took 16.000 cpu clocks
Salvo 8 took 0.000 cpu clocks
Salvo 9 took 0.000 cpu clocks
Salvo 10 took 0.000 cpu clocks
All salvos took 32.000 cpu clocks
Average salvo time is 3.20000 cpu clocks
1/CLOCKS_PER_SEC constant is 0.0010000000 seconds

FINAL RESULTS FOR THE FORCES

BLUE 1
P (0 elements) = 0.2149837182
P (1 elements) = 0.1719759688
P (2 elements) = 0.2143159144
P (3 elements) = 0.2096346686
P (4 elements) = 0.1412382636
P (5 elements) = 0.0478514664
Sum = 1.0000000000
Expected = 2.0337221898
Variance = 2.2360245758

BLUE 2
P (0 elements) = 0.1373997864
P (1 elements) = 0.1640943683
P (2 elements) = 0.2525109664
P (3 elements) = 0.2795520246
P (4 elements) = 0.1664428543
Sum = 1.0000000000
Expected = 2.1735437921
Variance = 1.6288995080

RED 1
P (0 elements) = 0.3041151740
P (1 elements) = 0.1919051163
P (2 elements) = 0.1881041025
P (3 elements) = 0.1500743447
P (4 elements) = 0.0965384228
P (5 elements) = 0.0484558265
P (6 elements) = 0.0174173775
P (7 elements) = 0.0033896357
Sum = 1.0000000000
Expected = 1.7750008943
Variance = 2.6934906223

APPENDIX B

FILE OUTPUT EXAMPLE FOR SLM FINAL STATISTICS: "STATS.TXT"

BLUE Force Information
5 combatants
p_d = 0.10
p_s (b) = 0.0010
Fire Type = Directed

Red Force Information
4 combatants
p_d = 0.30
p_s (a) = 0.0030
Fire Type = Directed

d(t) = 0.0100 salvo length
All combat took 31.000 cpu clocks
Average d(t) simulation time is 0.03100 cpu clocks
1/CLOCKS_PER_SEC constant is 0.0010000000 seconds

FINAL RESULTS FOR THE FORCES

BLUE FORCE
P (0 elements) = 0.7673525686
P (1 elements) = 0.0478343725
P (2 elements) = 0.0507643713
P (3 elements) = 0.0542154726
P (4 elements) = 0.0498954649
P (5 elements) = 0.0299377501
Sum = 1.0000000000
Expected = 0.6612801430
Variance = 1.8483108743

RED FORCE
P (0 elements) = 0.1676889111
P (1 elements) = 0.0786087435
P (2 elements) = 0.1500758574
P (3 elements) = 0.2684885303
P (4 elements) = 0.3351379577
Sum = 1.0000000000
Expected = 2.5247778798
Variance = 2.0830129261