PERIODIC SOLUTIONS AND STABILITY OF DIFFERENTIAL EQUATIONS WITH PIECEWISE CONSTANT ARGUMENT OF GENERALIZED TYPE

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

CEMİL BÜYÜKADALI

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY
IN
MATHEMATICS

JULY 2009

Approval of the thesis:

PERIODIC SOLUTIONS AND STABILITY OF DIFFERENTIAL EQUATIONS WITH PIECEWISE CONSTANT ARGUMENT OF GENERALIZED TYPE

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ABSTRACT

PERIODIC SOLUTIONS AND STABILITY OF DIFFERENTIAL EQUATIONS WITH PIECEWISE CONSTANT ARGUMENT OF GENERALIZED TYPE

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July 2009, 87 pages

In this thesis, we study periodic solutions and stability of differential equations with piecewise constant argument of generalized type. These equations can be divided into three main classes: differential equations with retarded, alternately advanced-retarded, and state-dependent piecewise constant argument of generalized type.

First, using the method of small parameter due to Poincaré, the existence and stability of periodic solutions of quasilinear differential equations with retarded piecewise constant argument of generalized type in noncritical case, that is, the unperturbed linear ordinary differential equation has not any nontrivial periodic solution, are investigated. The continuous and differential dependence of the solutions on an initial value and a parameter is considered. A new Gronwall-Bellmann type lemma is proved.

Next, quasilinear differential equations with alternately advanced-retarded piecewise constant argument of generalized type is addressed. The critical case, when associated linear homogeneous system admits nontrivial periodic solutions, is considered. Using the technique of Poincaré-Malkin, criteria of existence of periodic solutions of such equations are obtained. One of the main auxiliary results is an analogue of Gronwall-Bellmann Lemma for functions with alternately advanced-retarded piecewise constant argument. Dependence of solutions on an initial value and a parameter is investigated.

Finally, a new class of differential equations with state-dependent piecewise constant argument is introduced. It is an extension of systems with piecewise constant argument. Fundamental theoretical results for the equations: existence and uniqueness of solutions, the existence of the periodic solutions, the stability of the zero solution are obtained. Appropriate examples are constructed.

Keywords: Differential equations with piecewise constant argument of generalized type, Differential equations with state-dependent piecewise constant argument, The method of small parameter, Periodic solutions, Asymptotic stability.

ÖZ

GENEL TİPTEKİ PARÇALI SABİT ARGUMANLI DİFERENSİYEL DENKLEMLERİN PERİYODİK ÇÖZÜMLERİ VE KARARLILIĞI

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Temmuz 2009, 87 sayfa

Bu tezde, parçalı sabit argümanlı genel tipteki diferensiyel denklemlerin periyodik çözümlerinin varlığı ve kararlılığı incelenmiştir. Bu denklemler üç ana gruba ayrılabilir: genel tipteki gecikmeli, dönüşümlü ilerlemeli-gecikmeli, duruma bağımlı parçalı sabit argümanlı diferensiyel denklemler.

İlk olarak, Poincaré'nin küçük parametre metodu kullanılarak kritik olmayan durumda, diğer bir deyişle, pertürbesiz doğrusal adi diferensiyel denklemin sıfır çözümü haricinde periyodik çözümü olmadığında, hemen hemen doğrusal gecikmeli parçalı sabit argümanlı diferensiyel denklemlerin periyodik çözümlerinin varlığı ve kararlılığı incelenmiştir. Çözümlerin başlangıç koşuluna ve parametreye sürekli ve diferensiyel bağımlılığı araştırılmıştır. Yeni bir Gronwall-Bellmann tipi lemma ispatlanmıştır.

Daha sonra, hemen hemen doğrusal genel tipteki dönüşümlü ilerlemeli-gecikmeli parçalı sabit argümanlı diferensiyel denklemler göz önüne alınmıştır. Kritik durumdaki, ne zaman ilişkili doğrusal denklem sıfır çözümü haricinde periyodik çözümleri kabul ettiğindeki durum incelenmiştir. Poincaré-Malkin'in tekniği kullanılarak bu tipteki denklemlerin periyodik çözümlerinin varlılığı için koşullar elde edilmiştir. Dönüşümlü ilerlemeli-gecikmeli parçalı sabit argümanlı fonksiyonlar için Gronwall-Bellmann benzeri lemma önemli sonuçlardan biridir. Çözümlerin başlangıç koşuluna ve parametreye bağımlılığı araştırılmıştır.

Son olarak, duruma bağımlı parçalı sabit argümanlı diferensiyel denklemlerin yeni bir sınıfı tanımlanmıştır. Bunlar parçalı sabit argümanlı sistemlerin genişletilmiş halidir. Bu denklemler için temel sonuçlar: çözümlerin varlığı ve tekliği, periyodik çözümlerin varlığı, sıfır çözümünün kararlılığı, elde edilmiştir. Uygun örnekler kurulmuştur.

Anahtar Kelimeler: Genel tipteki parçalı sabit argümanlı diferensiyel denklemler, Duruma bağımlı parçalı sabit argümanlı diferensiyel denklemler, Küçük parametre metodu, Periyodik çözümler, Asimptotik kararlılık.

To my family

ACKNOWLEDGMENTS

I would like to express sincere thanks to my supervisor, Prof. Dr. Marat Akhmet for his motivation, helpful discussion, encouragement, patience, and constant guidance during this work.

I would like to thank all members of Applied Dynamics Group, Prof. Dr. Tanıl Ergenç, Duygu Aruğaslan, M. Onur Fen, Mehmet Turan and Enes Yılmaz for their friendship, valuable comments and suggestions during this work and my graduate studies.

To my family... To my father and my mother, Behçet and Emine Büyükadalı and my brother Bülent, and my sisters, Pınar, Nalan Nuriye, I own special thanks for their precious love, and encouragement during the period of my graduate studies.

I deeply thank my instructors and friends in the Department of Mathematics and Institute of Applied Mathematics in Middle East Technical University for their suggestions and continuous help during this period.

The financial support by Grant 106T418 from The Scientific and Technological Research Council of Turkey (TUBITAK) is also acknowledged.

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CHAPTER 1

INTRODUCTION AND PRELIMINARIES

1.1 Introduction

Differential equations with delay (DDE) provide a mathematical model for a physical, mechanical or biological system in which the rate of change of a system depends upon its past history. Although the general theory and fundamental results for DDE have by now been thoroughly explored and are available in the books [34, 35, 45, 50] and subsequent articles by many authors, the literature devoted to the theory and applications of DDE continues to grow very rapidly. Naturally, new important problems and directions arise continually in this intensively developing field. In 1977, the article by Myshkis [64], who observed that a substantial theory did not exist for differential equations with lagging arguments that are piecewise constant or continuous, pointed out differential equations with piecewise constant arguments (EPCA). Since that time many authors have investigated equations of this type.

A typical EPCA is of the form

$$x'(t) = f(t,\ x(t),\ x(h(t))), \tag{1.1}$$

where the argument $h(t)$ has *interval of constancy*. For example, equations with $h(t) = [t]$, $[t-n]$, $t-n[t]$ were investigated in [25], where n is a positive integer and [.] denotes the greatest integer function. Note that $h(t)$ is discontinuous, and although the equation fits within the general structure of DDE, the delays are discontinuous functions. Also note that the equation is nonautonomous, since the delays vary with t. Moreover, a solution is defined as a continuous, sectionally smooth function that satisfies the equation in the interval of constancy. Hence, the solutions are determined by a finite set of initial data, rather than by an initial function as in the case of DDE. In fact, EPCA have the structure of continuous dynamical systems within intervals of nonzero lengths. Continuity of a solution at a point joining any two consecutive intervals then implies recurrence relations for the solution of such points. Therefore, EPCA represent a hybrid continuous and discrete dynamical systems and combine the properties of both differential and difference equations.

An equation in which $x'(t)$ is given by a function x evaluated at t and at arguments $[t], \ldots, [t-n]$, where n is a non-negative integer, is called of retarded or delay type. If the arguments are t and $[t+1], \ldots, [t+n]$, then the equation is of advanced type. If both these types of arguments appear in the equation, it is called of mixed type. If the argument $h(t)$ is between $[t]$ and $[t+1]$ for each t, then the equation is of alternately advanced-retarded type. The equations may be linear or nonlinear. All types of EPCA share similar characteristics. First of all, it is natural to present the initial value problem for such equations not on an interval as in DDE but at a number of individual points. Secondly, for ordinary differential equations with a continuous vector field the solution exists to the right and left of the initial value. For EPCA, in general, two-sided solutions may not exist.

It is important to note that EPCA provide the simplest examples of differential equations capable of displaying chaotic behavior. Let us see the following example.

Example 1.1.1 *Consider the initial value problem*

$$\begin{aligned} x'(t) &= (\mu-1)x([t]) - \mu x^2([t]), \\ x(0) &= x_0. \end{aligned} \tag{1.2}$$

One can see that for $t \in [n, n+1)$, *the corresponding ordinary differential equation is of the form*

$$x'(t) = (\mu-1)x(n) - \mu x^2(n).$$

Then, by integrating the last equation from n to $n+1$, *we obtain the relation*

$$x(n+1) = \mu x(n)(1-x(n)), \qquad n = 0, 1, \ldots.$$

which is the famous logistic map. Therefore, we conclude that if we choose $\mu \geq 4$, *independent of choice of* x_0, *the unique solution of Eq.* (1.2) *exhibits chaos [33].*

Systematic study of theoretical and practical problems involving piecewise constant arguments was initiated in the early 1980's. Since then, differential equations with piecewise constant arguments have obtained great attention from the researchers in mathematics, biology, engineering and other fields. A mathematical model including piecewise constant argument was first considered by Busenberg and Cooke [23] in 1982. They constructed a first-order linear EPCA to investigate vertically transmitted diseases. Following this work, using the method of reduction to discrete equations, many authors have analyzed various types of EPCA.

The systems with first-order linear EPCA with constant coefficients of retarded type, of advanced type, and of alternately advanced-retarded type were first studied by Cooke and Wiener [25–27], Aftabizadeh and Wiener [1–4] and Shah and Wiener [77]. Existence and uniqueness of the solutions, their backward continuation on $(-\infty, 0]$ were proved. Moreover, stability and asymptotic stability of the trivial solution and oscillatory behavior of the corresponding solutions were analyzed. Based on the studies given by Cooke and Wiener, Zhang and Parni [86] considered the first-order linear EPCA with variable coefficients and studied the oscillatory and nonoscillatory properties of the solutions. Furthermore, Jayasree and Deo [48] established existence and uniqueness theorems, a variation of parameters formula, integral inequalities, the oscillation property, and some applications. The brief summary on theory can be found in the book by Wiener [81].

From the current literature, one can see that the interest on investigation of EPCA is continuously growing. Examples of research articles that have been done recently are on the existence of almost periodic solutions of retarded EPCA by Yuan [85], quasi-periodic solutions of EPCA by Küpper and Yuan [51], existence of periodic solutions of retarded EPCA by Wang [79], Green's function and comparison principles for first-order periodic EPCA by Cabada, Ferreiro and Nieto [24], Green's function for second-order periodic boundary value problems with piecewise constant arguments by Nieto and Rodriguez-Lopez [66] and by Yang, Liu and Ge [84], periodic solutions of a neutral EPCA by Wang [80], existence, uniqueness and asymptotic behavior of EPCA by Papaschinopoulos [71], stability of EPCA and the associated discrete equations using dichotomic map by Marconato [59].

Lakshmikhantham and Wiener [82] proved existence and uniqueness theorems for the initial value problem

$$x'(t) = f(x(t),\ x(g(t))), \qquad x(0) = x_0, \tag{1.3}$$

where f is a continuous function, and $g : [0, \infty) \to [0, \infty)$, $g(t) \leq t$, is a step function, that is, it is constant and equal to $g(t_n)$ on each interval $[t_n,\ t_{n+1})$, where $\{t_n\}$ is a strictly increasing sequence of real numbers with $\lim_{n\to\infty} t_n = \infty$.

The numerical approximation of differential equations is also one of the benefits of EPCA. For example, the simple Euler scheme for a differential equation $x'(t) = f(x(t))$ has the form $x_{n+1} - x_n = hf(x_n)$, where $x_n = x(nh)$ and h is the step size. This is equivalent to EPCA of the form $x'(t) = f(x([t/h]\,h))$. Györi [42] realized that equations with piecewise constant arguments can be used to approximate delay differential equations that contain discrete delays, and proved some limit relations between the solutions of delay differential equations with continuous arguments and the solutions of some retarded EPCA. The results were used to compute numerical solutions of ordinary and delay differential equations. Later, Györi, Hartung and Turi [43] generalized the results to approximate DDE with state dependent delays.

It is not surprising to expect that EPCA are used to construct mathematical models for the problems of biology, economics, or engineerings, as this was done using DDE [50]. In the papers of Dai and Singh [29–31], a direct analytical and numerical method independent of the existing classical methods for solving linear and nonlinear vibration problems was given with the introduction of a piecewise constant argument $[Nt]/N$. A new numerical method which produces sufficiently accurate results with good convergence was introduced. Development of the formula for numerical calculations was based on the original governing differential equations. For the details we refer to the book by Dai [32].

Murat and Celeste [62] investigated the damped loading system subjected to a piecewise constant voltage described by the equation of charge:

$$Lq''(t) + Rq'(t) + C^{-1}q(t) = Aq\left(\frac{[Nt]}{N}\right), \tag{1.4}$$

which was compared with a similar linear loading system governed by the following equation of charge

$$Lq''(t) + Rq'(t) + C^{-1}q(t) = Aq(t). \tag{1.5}$$

They considered, through numerical simulation, the phenomena of sensitivity on the initial data, stability and existence of oscillatory solutions.

The equations with piecewise constant arguments plays an important role in mathematical modeling of biological problems. Busenberg and Cooke [23] constructed a first-order linear EPCA to investigate vertically transmitted diseases. The authors like Gopalsamy, Ladas, Muroya, Seifert in several papers [38, 39, 63, 75, 76] investigated different types of population models based on logistic equations with piecewise constant arguments and obtained mathematical results. In [17], Akhmet et al studied an anticipatory extension of Malthusian model using first-order linear EPCA with constant coefficients of advanced type. They have found conditions for the solutions to be periodic, stable, or chaotic.

Lakshmikhantham and Wiener [83] studied the asymptotic behavior of a second-order EPCA of the form

$$x'' + \omega^2 x(t) = -bx'([t-1]), \tag{1.6}$$

where b and ω are positive constants. They found that last equation may generate periodic or even unbounded solutions whereas all solutions of the corresponding ordinary differential equation $x'' + bx'(t) + \omega^2 x(t) = 0$ tend to zero as $t \to \infty$.

Impulsive differential equations and loaded equations of control theory fit within the general paradigm of EPCA. Another application of EPCA is the stabilization of hybrid control systems with feedback delay. Some of these systems have been described in [28]. Moreover, Magni and Scattolini [56, 57] considered recently a new model predictive control (MPC) algorithm for nonlinear systems based on EPCA. The plant under control, the state and control constraints, and the performance index to be minimized are described in continuous time, while the manipulated variables are allowed to change at fixed and uniformly distributed sampling times. In so doing, the optimization is performed with respect to sequences, as in discrete-time nonlinear MPC, but the continuous-time evolution of the system is considered as in continuous-time nonlinear MPC.

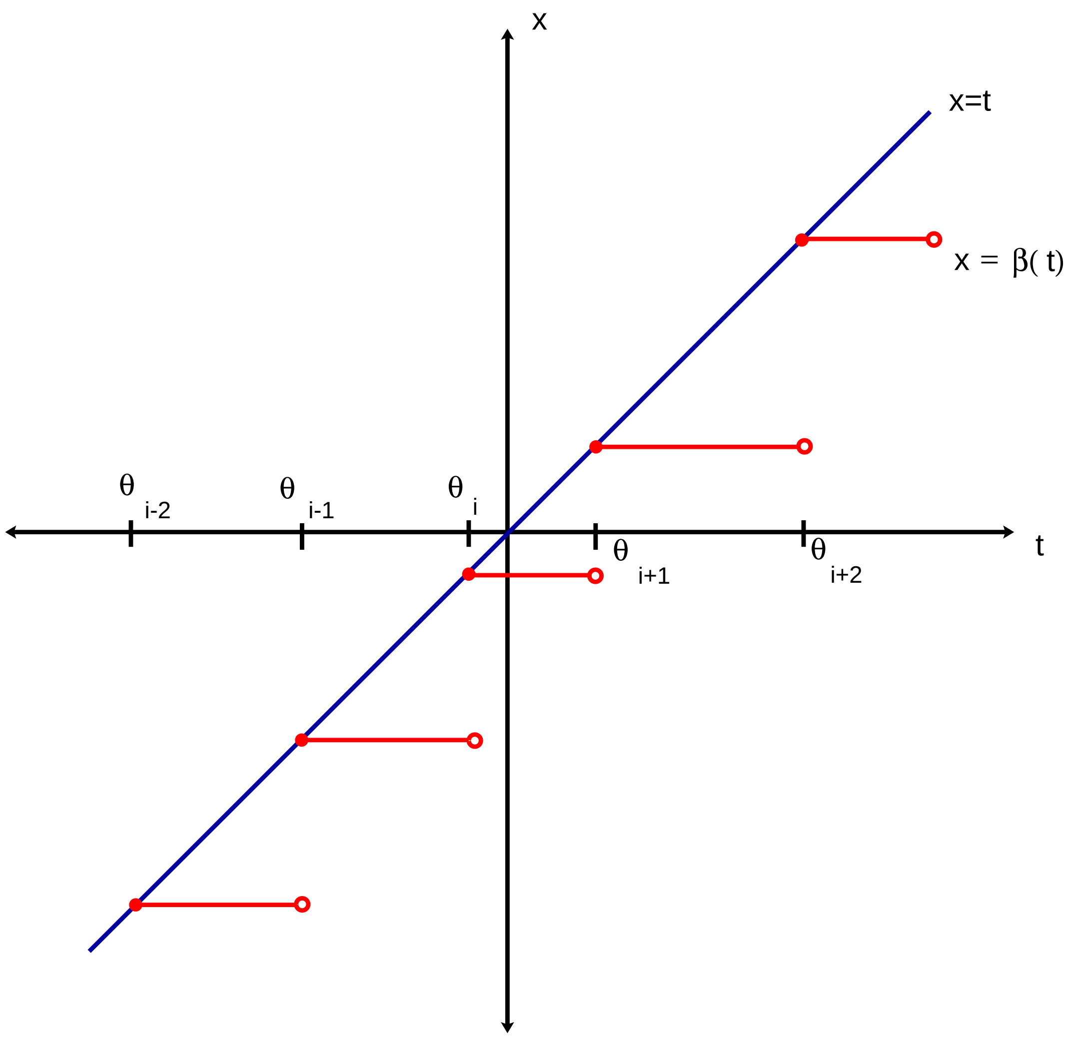

Figure 1.1: The graph of the argument $\beta(t)$.

Consequently, from the above mentioned theoretical and practical results we see that EPCA was generally investigated using the method of reduction to discrete equation by its founders and developers. This kind of investigation can be continued based on the theory of difference equation. However, there are some lacks of this method. For example, continuous and differential dependence, bifurcation theory, stability theory mainly need a different kind of investigation.

In [7–9], Akhmet proposed to investigate differential equations with piecewise constant argument of generalized type (EPCAG) of type

$$\frac{dx(t)}{dt} = f(t,\ x(t),\ x(\beta(t))), \tag{1.7}$$

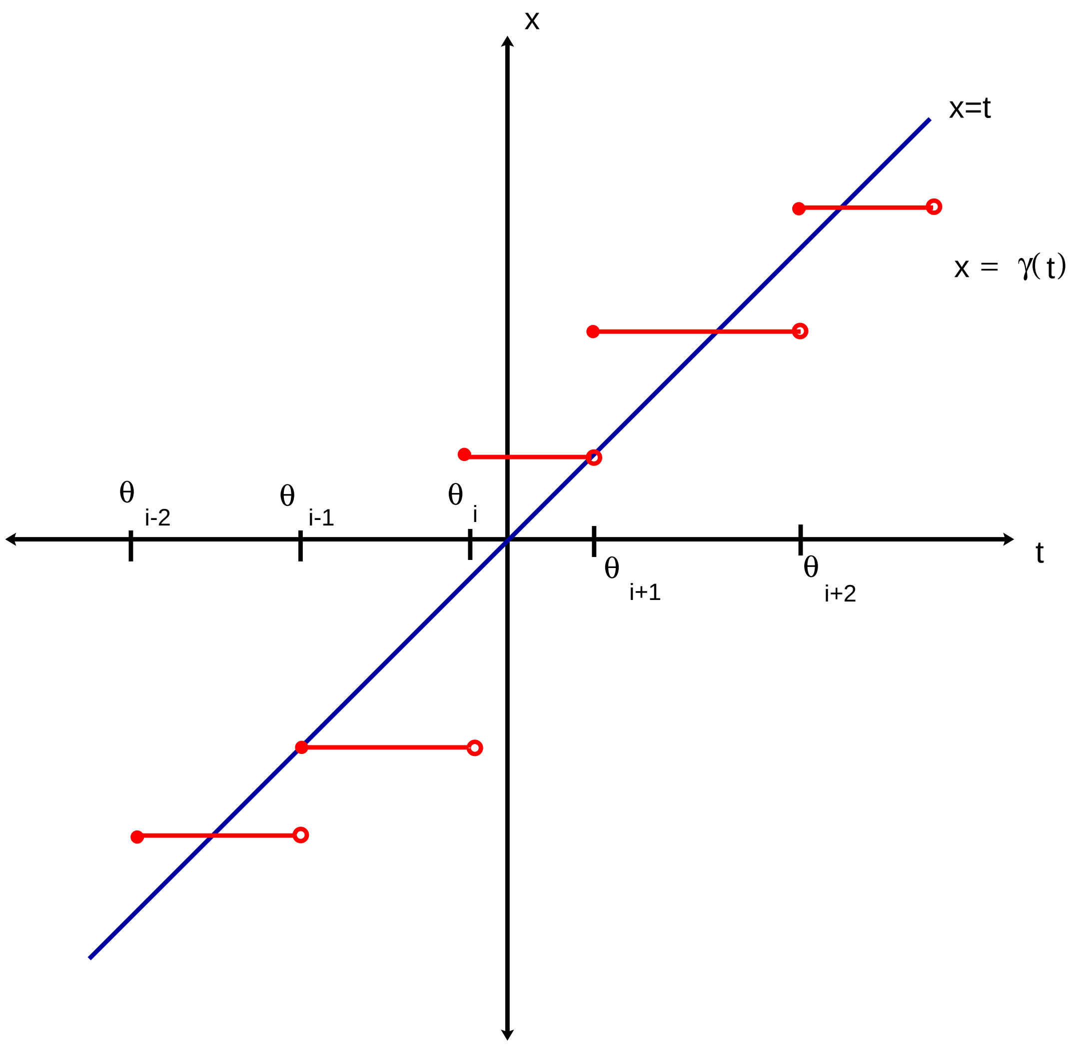

Figure 1.2: The graph of the argument $\gamma(t)$.

or

$$\frac{dx(t)}{dt} = f(t,\ x(t),\ x(\gamma(t))), \tag{1.8}$$

where $\beta(t) = \theta_i$ (see Fig. 1.1) and $\gamma(t) = \zeta_i$ (see Fig. 1.2) if $\theta_i \leq t < \theta_{i+1}$, i are integers, are piecewise constant functions, $\{\theta_i\}$ and $\{\zeta_i\}$ are strictly increasing sequence of real numbers, unbounded on the left and on the right such that $\theta_i \leq \zeta_i \leq \theta_{i+1}$ for all i. In papers [7–9], methods of investigation by constructing equivalent integral equations rather than using the method of reduction to discrete equations were introduced, and they have been applied for analysis of stability, existence of periodic and almost periodic solutions, integral manifolds in papers [10–12, 14–16]. These equations provide many opportunities for research of both theoretical and practical problems.

The problem of the existence of periodic solutions is one of the most interesting topics for applications. Poincaré [72] introduced the method of small parameter to investigate the problem and it has been developed by many authors (see, for example, [58, 73], and the references cited there) and this method remains as one of the most effective methods for this problem. For differential equations with discontinuities it was developed by Akhmet in [20, 21]. We apply his approach to the discontinuous processes in the investigation. It is important that the results obtained in this field can be extended to the bifurcation theory [22, 61].

In this thesis, we study periodic solutions and stability of differential equations with piecewise constant argument of generalized type by constructing an equivalent integral equations. These equations can be divided into three main classes: differential equations with retarded, alternately advanced-retarded, and state-dependent piecewise constant argument of generalized type.

This thesis is organized as follows.

In Chapter 2, using the method of small parameter, the existence and stability of the periodic solutions of quasilinear differential equations with retarded piecewise constant argument of generalized type in noncritical case, that is, when the unperturbed linear ordinary differential equations has not any nontrivial periodic solution, are investigated. The continuous and differential dependence of the solutions on an initial value and a parameter is considered. A new Gronwall-Bellmann type lemma is proved.

In Chapter 3, quasilinear differential equations with alternately advanced-retarded piecewise constant argument of generalized type and small parameter is addressed. The critical case, when the associated linear homogeneous system admits nontrivial periodic solutions, is considered. Using the technique of Poincaré-Malkin, criteria of existence of periodic solutions of such equations are obtained. One of the main auxiliary results is an analogue of Gronwall-Bellmann Lemma for functions with alternately advanced-retarded piecewise constant argument. Dependence of solutions on an initial value and a parameter is investigated.

In Chapter 4, a new class of differential equations with state-dependent piecewise constant argument is introduced. It is an extension of systems with piecewise constant argument. Fundamental theoretical results for the equations: existence and uniqueness of the solutions, the existence of the periodic solutions, the stability of the zero solution are obtained. Appropriate examples are constructed.

In Chapter 5, conclusion and future work topics are given.

The main parts of this thesis comes from the following papers:

M. U. Akhmet, C. Büyükadalı, Tanıl Ergenç, *Periodic solutions of the hybrid systems with small parameter*, Nonlinear Anal.: Hybrid Systems **2** (2008), 532-543.

M. Akhmet, C. Büyükadalı, *On periodic solutions of differential equations with piecewise constant argument*, Comput. Math. Appl. **56** (2008), 2034-2042.

M. U. Akhmet, C. Büyükadalı, *Differential equations with state-dependent piecewise constant argument*, Nonlinear Anal. TMA (Submitted).

1.2 Differential Equations with Piecewise Constant Arguments

Let $\mathbb{R}$, $\mathbb{N}$ and $\mathbb{Z}$ be the sets of all real numbers, natural numbers and integers, respectively. We will denote by $\|\,.\,\|$ the Euclidean norm for vectors in $\mathbb{R}^n$, $n \in \mathbb{N}$, and the uniform norm $\|C\| = \sup\{\|C\,x\| \mid \|x\| = 1\}$ for $n \times n$ matrices. Let I be an $n \times n$ identity matrix.

We shall now see some of the significant results previously established for EPCA.

1.2.1 Linear retarded EPCA with constant coefficients

The following results due to Cooke and Wiener [25] obtained by using the method of reduction to discrete equations.

Consider the scalar initial value problem

$$\begin{aligned} &x'(t) = ax(t) + a_0 x([t]) + a_1 x([t-1]), \\ &x(-1) = c_{-1}, \qquad x(0) = c_0 \end{aligned} \tag{1.9}$$

with constant coefficients. This equation is very closely related to impulsive and loaded equations. Indeed, Eq. (1.11) can be written as

$$x'(t) = ax(t) + \sum_{i=-\infty}^{+\infty} a_0 x(i) + a_1 x(i-1))(H(t-i) - H(t-i-1)),$$

where $H(t) = 1$ for $t > 0$ and $H(t) = 0$ for $t < 0$. If distributional derivatives are admitted, then by differentiating the last equation we have

$$x''(t) = ax' + \sum_{i=-\infty}^{+\infty} a_0 x(i) + a_1 x(i-1))(\delta(t-i) - \delta(t-i-1)),$$

where δ is the delta function. This impulsive equation contains the values of the unknown solution for the integral values of t. Let us introduce the following definition.

Definition 1.2.1 *A solution of Eq.* (1.9) *on* $[0, \infty)$ *is a function* $x(t)$ *that satisfies the conditions:*

(i) $x(t)$ *is continuous on* $[0, \infty)$.

(ii) *The derivative* $x'(t)$ *exists at each point* $t \in [0, \infty)$, *with the possible exception of the points* $[t] \in [0, \infty)$ *where the one sided derivative exists.*

(iii) *Eq.* (1.9) *is satisfied on each interval* $[n, n+1) \subset [0, \infty)$ *with integral endpoints.*

Let us consider Eq. (1.9).

Denote

$$b_0 = e^a + a^{-1} a_0 (e^a - 1), \qquad b_1 = a^{-1} a_1 (e^a - 1), \tag{1.10}$$

and let λ_1 and λ_2 be the roots of the equation

$$\lambda^2 - b_0 \lambda - b_1 = 0. \tag{1.11}$$

Theorem 1.2.1 *Problem* (1.9) *has on* $[0, \infty)$ *a unique solution*

$$x(t) = c_{[t]} e^{a(t-[t])} + a^{-1}(a_0 c_{[t]} + a_1 c_{[t]-1})(e^{a(t-[t])} - 1), \tag{1.12}$$

where

$$c_{[t]} = (\lambda_1^{[t]+1}(c_0 - \lambda_2 c_{-1})) + (\lambda_1 c_{-1} - c_0)\lambda_2^{[t]+1} / (\lambda_1 - \lambda_2). \tag{1.13}$$

Corollary 1.2.1 *The solution of* (1.9) *can not grow to infinity faster than exponentially as* $t \to +\infty$.

Since the solution of (1.9) on $[0, \infty)$ involves only the group e^{at}, it can be extended backwards on $(-\infty, 0]$.

Theorem 1.2.2 *If* $a_1 \neq 0$, *the solution of* (1.9) *has a unique backward continuation on* $(-\infty, 0]$ *given by the formulas* (1.12) *and* (1.13).

If $a_1 = 0$, it is formulated that

Theorem 1.2.3 *The problem*

$$x'(t) = ax(t) + a_0 x([t]), \qquad x(0) = c_0, \tag{1.14}$$

has on $[0, \infty)$ *a unique solution*

$$x(t) = u(t - [t])u^{[t]}(1)c_0, \tag{1.15}$$

where

$$u(t) = 1 + a^{-1}(e^{at} - 1)(a + a_0). \tag{1.16}$$

Theorem 1.2.4 *If* $u(1) \neq 0$, *the solution of Eq.* (1.14) *has a unique backward continuation on* $(-\infty, 0]$ *given by the formula* (1.15).

Theorem 1.2.5 *If* $u(1) \neq 0$ *and* $u(t_0 - [t_0]) \neq 0$, *then Eq.* (1.14) *with the initial condition* $x(t_0) = x_0$ *has on* $(-\infty, \infty)$ *a unique solution*

$$x(t) = u(\{t\})u^{[t]-[t_0]}(1)u^{-1}(\{t_0\})x_0, \tag{1.17}$$

where $\{t\}$ *is the fractional part of* t.

The last theorem establishes the fact that the initial value problem for Eq. (1.14) may be presented at any point, not necessarily integral. A similar proposition is true also for Eq. (1.9).

Theorem 1.2.6 *If* $a_1 \neq 0$ *and*

$$\lambda_i e^{a[t_0]} + a^{-1}(e^{a[t_0]} - 1)(\lambda_i a_0 + a_1) \neq 0, \qquad i = 1, 2 \tag{1.18}$$

where λ_i *are the roots of* (1.11), *then the problem* $x(t_0) = x_0$, $x(t_0 - 1) = x_{-1}$ *for Eq.* (1.9) *has a unique solution on* $(-\infty, \infty)$.

Theorem 1.2.7 *The solution* $x = 0$ *of Eq.* (1.9) *is asymptotically stable as* $t \to +\infty$ *if and only if the moduli of the roots of Eq.* (1.11) *satisfy the inequalities*

$$|\lambda_1| < 1, \qquad |\lambda_2| < 1. \tag{1.19}$$

Theorem 1.2.8 *If the solution* $x = 0$ *of Eq.* (1.9) *is asymptotically stable as* $t \to +\infty$, *then*

$$\begin{aligned} -a(2 + e^a)/(e^a - 1) \;&<\; a_0 \;<\; a(2 - e^a)/(e^a - 1), \\ |a_1| \;&<\; a/(e^a - 1). \end{aligned} \tag{1.20}$$

Theorem 1.2.9 *The solution* $x = 0$ *of Eq.* (1.9) *is asymptotically stable as* $t \to +\infty$, *if and only if any one of the hypothesis is satisfied:*

(i)

$$\begin{gathered} -a(2 + e^a)/(e^a - 1) < a_0 < a(2 - e^a)/(e^a - 1), \\ -\frac{a(e^a + a^{-1}(e^a - 1)a_0)^2}{4(e^a - 1)} \leq a_1 < \frac{a(e^a + 1)}{e^a - 1} + a_0; \end{gathered}$$

(ii)

$$\begin{gathered} -\frac{ae^a}{e^a - 1} < a_0 < \frac{a(2 - e^a)}{e^a - 1}, \\ -\frac{a(e^a + a^{-1}(e^a - 1)a_0)^2}{4(e^a - 1)} \leq a_1 < -a - a_0; \end{gathered}$$

(iii)

$$\begin{gathered} -\frac{a(2 + e^a)}{e^a - 1} < a_0 < \frac{a(2 - e^a)}{e^a - 1}, \\ -\frac{a}{e^a - 1} < a_1 < -\frac{a(e^a + a^{-1}(e^a - 1)a_0)^2}{4(e^a - 1)}. \end{gathered}$$

1.2.2 Approximation of equations with discrete delay

Equations with piecewise constant arguments can be used to approximate delay differential equations that contain discrete delays. In [42] some limit relations between the solutions of delay differential equations with continuous arguments and the solutions of some retarded EPCA have been proved. The results were used to compute numerical solutions of ordinary and delay differential equations. Let us see some of these results.

Consider the delay differential equation

$$x'(t) + p_0(t)x(t) + \sum_{i=1}^{m} p_i(t)x(t-\tau_i) = 0, \qquad t \geq 0, \tag{1.21}$$

where

(H1′) for $i = 1, \ldots, m$, τ_i are positive real numbers and $\tau = \max_{1 \leq i \leq m} \tau_i$;

(H2′) for $i = 0, 1, \ldots, m$, $p_i : [0, \infty) \to \mathbb{R}$ are continuous functions;

(H3′) for a fixed $k \in \mathbb{N}$, $k \geq 1$, set $h = \tau/k$.

Define delay differential equations with piecewise constant arguments as follows:

$$u'(t) + p_0(t)u(t) + \sum_{i=1}^{m} p_i(t)u\left(\left[\frac{t}{h} - \left[\frac{\tau_i}{h}\right]\right]h\right) = 0, \qquad t \geq 0, \tag{1.22}$$

and

$$v'(t) + p_0(t)v([\frac{t}{h}]h) + \sum_{i=1}^{m} p_i(t)v\left(\left[\frac{t}{h} - \left[\frac{\tau_i}{h}\right]h\right]\right) = 0, \qquad t \geq 0, \tag{1.23}$$

and

$$w'(t) + p_0\left(\left[\frac{t}{h}\right]h\right)w\left(\left[\frac{t}{h}\right]h\right) + \sum_{i=1}^{m} p_i\left(\left[\frac{t}{h}\right]h\right)w\left(\left[\frac{t}{h} - \left[\frac{\tau_i}{h}\right]\right]h\right) = 0, \qquad t \geq 0. \tag{1.24}$$

It is known [45] that Eq. (1.21) with the initial condition

$$x(s) = \phi(s),\ -\tau \leq s \leq 0,\ \phi \in C \equiv C([-\tau, 0], \mathbb{R}), \tag{1.25}$$

has a unique solution on $[-\tau, \infty)$, which is continuous on $[-\tau, \infty)$ and continuously differentiable on $[0, \infty)$.

With Eqs. (1.21)−(1.21), let us associate the following initial conditions, respectively:

$$u(jh) = \phi(jh) \qquad \text{for} \qquad j = -k, \ldots, 0 \tag{1.26}$$

and

$$v(jh) = \phi(jh) \qquad \text{for} \qquad j = -k, \ldots, 0 \tag{1.27}$$

and

$$w(jh) = \phi(jh) \qquad \text{for} \qquad j = -k, \ldots, 0. \tag{1.28}$$

Definition 1.2.2 *We say that a function* $u(t)$ *is a solution of* (1.22) *and* (1.26) *defined on the set* $\{-k, \ldots, 0\} \cup (0, \infty)$ *if*

(i) $u(t)$ *is continuous on* $[0, \infty)$*;*

(ii) *the derivative* $u'(t)$ *exists at each point* $t \in [0, \infty)$ *with the possible exception of the points* $t = nh$, $n \in \mathbb{N}$, *where finite one sided derivative exists;*

(iii) *the function* $u(t)$ *satisfies Eq.* (1.22) *on each interval* $[nh, (n+1)h]$ *for* $n \in \mathbb{N}$.

The definitions of the solutions $v(t)$ and $w(t)$ of the initial value problems (1.23) − (1.27) and (1.24) − (1.28), respectively, are analogues. The following lemma shows the existence and uniqueness result.

Lemma 1.2.1 *Assume that* (H1′) − (H3′) *hold. Then each one of the initial value problem* (1.22) − (1.26), (1.23) − (1.27) *and* (1.24) − (1.28) *has a unique solution.*

Let C_0^1 be defined by $C_0^1 = \left\{\psi \in C^1 : \psi'(0^-) + p_0(0)\psi(0) + \sum_{i=1}^{m} p_i(0)\psi(-\tau_i) = 0\right\}$, where C^1 denotes the set of continuously differentiable maps of $[-\tau, 0]$ into $\mathbb{R}$.

The following theorem shows the convergence result.

Theorem 1.2.10 *Assume that* (H1′) − (H3′) *hold. Then the following statements are valid:*

(a) *the solutions* $x(t)$, $u(t)$, $v(t)$ *and* $w(t)$ *of the initial value problems* (1.21) – (1.25), (1.22) – (1.26), (1.23) – (1.27) *and* (1.24) – (1.28), *respectively, satisfy the following relations for all* $T > 0$

$$\begin{aligned}\lim_{h\to 0}\max_{0\le t\le T}|x(t)-u(t)| &= \lim_{h\to 0}\max_{0\le t\le T}|x(t)-v(t)| \\ &= \lim_{h\to 0}\max_{0\le t\le T}|x(t)-w(t)| = 0. \end{aligned} \tag{1.29}$$

(b) *If* $\phi \in C_0^1$ *then for all* $T > 0$ *there exist constants* $L_0 = L_0(T_0, \phi)$ *and* $M_0 = M_0(T_0, \phi)$ *such that*

$$\|x(t) - u(t)\| \le L_0 h, \qquad 0 \le t \le T_0,\ h > 0, \tag{1.30}$$

and

$$\|x(t) - v(t)\| \le M_0 h, \qquad 0 \le t \le T_0,\ h > 0. \tag{1.31}$$

(c) *If for all* $i = 0, 1, \ldots, n$ *the functions* $p_i(t)$ *are Lipschitz-continuous on any compact subinterval of* $[0, \infty)$ *and* $\phi \in C_0^1$, *then there exists a constant* $N_0 = N_0(T_0, \phi)$ *such that*

$$\|x(t) - w(t)\| \le N_0 h, \qquad 0 \le t \le T_0,\ h > 0. \tag{1.32}$$

Remark 1.2.1 *It is known from* [41, Lemma 2.1] *that* C_0^1 *is a nonempty and dense set in* C.

In the next theorem a condition which guarantees that the first two approximations are uniform on the half line $[0, \infty)$.

Theorem 1.2.11 *Assume that* (H1′) – (H3′) *are satisfied and for all* $i = 0, 1, \ldots, m$

$$\int_0^\infty |p_i(t)|dt < \infty. \tag{1.33}$$

Let $\phi \in C$ *be a given function. Then the solutions* $x(t)$, $u(t)$ *and* $v(t)$ *of Eqs.* (1.21) – (1.25), (1.22) – (1.26), *and* (1.23) – (1.27), *respectively, satisfy the following relations*

$$\sup_{t\ge 0}|x(t) - u(t)| \to 0, \qquad \text{as} \qquad h \to 0, \tag{1.34}$$

and

$$\sup_{t\ge 0}|x(t) - v(t)| \to 0, \qquad \text{as} \qquad h \to 0. \tag{1.35}$$

The following results shows that Eqs. (1.22), (1.23), and (1.24) are strongly related to some discrete difference equations. Let us denote the first forward difference of a function $f(n)$ by $\Delta f(n) = f(n+1) - f(n)$.

Theorem 1.2.12 *Assume that* (H1′) − (H3′) *are satisfied and $k \geq 1$ is an integer. Set*

$$h = \tau/k \qquad \text{and} \qquad k_i = [\tau_i/h] \qquad \text{for all} \qquad i = 1, \ldots, m. \tag{1.36}$$

Then

(a) *the solution $u(t)$ of* (1.22) − (1.26) *is given by*

$$u(t) = a(n)e^{-\int_{nh}^{t} p_0(s)ds} - e^{-\int_0^t p_0(s)ds} \sum_{i=1}^{m} \int_{nh}^{t} p_i(s)e^{\int_0^s p_0(r)dr} ds\, a(n-k_i) \tag{1.37}$$

for all $nh \leq t < (n+1)h$ and $n \geq 0$, where $\{a(n)\}$ is a sequence which satisfies the difference equation

$$\begin{aligned} & a(n+1) - a(n)e^{-\int_{nh}^{(n+1)h} p_0(s)ds} \\ & + e^{-\int_0^{nh} p_0(s)ds} \sum_{i=1}^{m} \int_{nh}^{(n+1)h} p_i(s)e^{\int_0^s p_0(r)dr} ds\, a(n-k_i) = 0, \ n \geq 0 \\ & a(n) = \phi(nh), \qquad n = -k, \ldots, 0 \end{aligned} \tag{1.38}$$

(b) *the solution $v(t)$ of* (1.23) − (1.27) *is given by*

$$v(t) = (1 - \int_{nh}^{t} p_0(s)ds)\, b(n) + \sum_{i=1}^{m} \int_{nh}^{t} p_i(s)ds\, b(n-k_i) \tag{1.39}$$

for all $nh \leq t < (n+1)h$ and $n \geq 0$, where $\{b(n)\}$ is a sequence which satisfies the difference equation

$$\begin{aligned} & \Delta b(n) + \int_{nh}^{(n+1)h} p_0(s)ds\, b(n) + \sum_{i=1}^{m} \int_{nh}^{(n+1)h} p_i(s)ds\, b(n-k_i) = 0, \ n \geq 0 \\ & b(n) = \phi(nh), \qquad n = -k, \ldots, 0 \end{aligned} \tag{1.40}$$

(c) *the solution $w(t)$ of* (1.24) − (1.28) *is given by*

$$w(t) = c(n) - (p_0(nh)c(n) + \sum_{i=1}^{m} p_i(nh)c(n-k_i))(t-nh) \tag{1.41}$$

for all $nh \leq t < (n+1)h$ and $n \geq 0$, where $c(n)$ is a sequence which satisfies the difference equation

$$\begin{aligned} & \Delta c(n) + hp_0(nh)c(n) + h\sum_{i=1}^{m} p_i(nh)c(n-k_i) = 0, \qquad n \geq 0 \\ & c(n) = \phi(nh), \qquad n = -k, \ldots, 0. \end{aligned} \tag{1.42}$$

Let us illustrate the result with the following scalar equation with one constant delay.

Example 1.2.1 *Consider the equation*

$$x'(t) = -p(t)x(t-\tau), \qquad t \geq 0 \tag{1.43}$$

with the following initial condition

$$x(t) = \phi(t), \qquad -\tau \leq t \leq 0 \tag{1.44}$$

where ϕ is a given function in C. Let k be any positive integer and let $h = \tau/k$. This problem may be approximated by the EPCA

$$y'(t) = -p(t)y\left(\left[\frac{t}{h} - \left[\frac{t}{h}\right]\right]h\right) \tag{1.45}$$

with the initial condition

$$y(nh) = \phi(nh), \qquad n = -k, \ldots, 0. \tag{1.46}$$

Moreover, it can be found that $y(nh) = a_n$ satisfies the difference equation

$$a_{n+1} - a_n = -\int_{nh}^{(n+1)h} p(s)ds a_{n-k} \tag{1.47}$$

$$a_n = \phi(nh), \qquad n = -k, \ldots, 0. \tag{1.48}$$

Hence, using Theorem 1.2.10 *it is seen that the solution of* (1.45), (1.46) *provides uniform approximation to the solution of the problem* (1.43), (1.44) *on any compact interval* $[0, T_0]$, $T_0 > 0$.

1.2.3 Alternately advanced retarded EPCA

Differential equations of the form

$$x'(t) = f\left(x(t),\ x\left(\left[t + \frac{1}{2}\right]\right)\right) \tag{1.49}$$

have stimulated considerable interest and have studied by Aftabizadeh and Wiener [2, 4], Cooke and Wiener [27], Huang [47], Jayasree and Deo [48], Ladas, Partheniadas, and Schinas [52]. In this equations, the argument deviation $\tau(t) = t - \left[t + \frac{1}{2}\right]$ changes its sign in each interval $n - \frac{1}{2} < t < n + \frac{1}{2}$, $n \in \mathbb{Z}$. Indeed, $\tau(t) < 0$ for $n - \frac{1}{2} < t < n$, and

$\tau(t) > 0$ for $n < t < n + \frac{1}{2}$, which means that the equation is of alternately advanced-retarded type. It is of advanced type on $[n - \frac{1}{2}, n)$ and of retarded type on $(n, n + \frac{1}{2})$. Cooke and Wiener have studied in [27] the equation

$$x'(t) = ax(t) + a_0 x\left(2\left[\frac{t+1}{2}\right]\right), \qquad x(0) = c_0. \tag{1.50}$$

The argument deviation

$$\tau(t) = t - 2\left[\frac{t+1}{2}\right] \tag{1.51}$$

is negative for $2n - 1 \leq t < 2n$, and positive for $2n < t < 2n + 1$. Therefore Eq. (1.49) is of advanced type on $[2n - 1, 2n)$, and of retarded type on $(2n, 2n + 1)$.

Definition 1.2.3 [81] *A solution of Eq.* (1.49) *on* $[0, \infty)$ *is a function* $x(t)$ *that satisfies the conditions:*

(i) $x(t)$ *is continuous on* $[0, \infty)$;

(ii) *the derivative* $x'(t)$ *exists at each point* $t \in [0, \infty)$, *with the possible exception of the points* $t = 2n - 1$, $n \in \mathbb{Z}$, $n > 0$, *where one sided derivatives exist;*

(iii) *Eq.* (1.49) *is satisfied on each interval* $2n - 1 \leq t < 2n + 1$.

The following results are from [27]. In that paper, it was shown that Eq. (1.50) has a unique solution on $[0, \infty)$ and a unique backward solution on $(-\infty, 0]$. Also, it was determined that the set of (a, a_0) for which the zero solution is asymptotically stable as $t \to +\infty$, and the set of (a, a_0) such that all nontrivial solutions have no zeros in $(-\infty, \infty)$. The set of bounded solutions is characterized. Furthermore, the same equation with variable coefficients $a(t)$, $a_0(t)$ is examined, the condition for existence of a unique solution on $[0, \infty)$ is determined, and conditions are found under which all solutions are oscillatory.

Let

$$\lambda(t) = e^{at} + (e^{at} - 1)a^{-1}a_0, \qquad \lambda_{-1} = \lambda(-1),\ \lambda_1 = \lambda(1). \tag{1.52}$$

Theorem 1.2.13 *Problem* (1.50) *has on* $[0, \infty)$ *a unique solution*

$$x(t) = \lambda(\tau(t))\left(\frac{\lambda_1}{\lambda_{-1}}\right)^{[(t+1)/2]} c_0 \tag{1.53}$$

if $\lambda_{-1} \neq 0$, *where* $\tau(t)$ *is given by* (1.51).

Theorem 1.2.14 *The solution of Eq.* (1.50) *has a unique backward continuation on* $(-\infty, 0]$ *given by formula* (1.53) *if* $\lambda_1 \neq 0$.

Theorem 1.2.15 *The solution* $x = 0$ *of Eq.* (1.50) *is asymptotically stable as* $t \to +\infty$ *if and only if* $|\lambda_1/\lambda_{-1}| < 1$.

Theorem 1.2.16 *The solution* $x = 0$ *of Eq.* (1.50) *is asymptotically stable as* $t \to +\infty$ *if and only if any one of the following hypothesis is satisfied:*

(i) $a < 0, \quad a_0 > -\frac{a(e^{2a}+1)}{(e^a-1)^2}$ *or* $a_0 < -a$;

(ii) $a > 0, \quad -\frac{a(e^{2a}+1)}{(e^a-1)^2} < a_0 < -a$;

(iii) $a = 0, \quad a_0 < 0$.

Theorem 1.2.17 *All nontrivial solutions of Eq.* (1.50) *have no zeros in* $(-\infty, \infty)$ *if and only if*

$$-\frac{ae^a}{e^a - 1} < a_0 < \frac{a}{e^a - 1}. \tag{1.54}$$

Theorem 1.2.18 *The problem*

$$x'(t) = a(t)x(t) + a_0(t)x\left(2\left[\frac{t+1}{2}\right]\right), \qquad x(0) = c_0 \tag{1.55}$$

has a unique solution on $[0, \infty)$ *if* $a(t)$ *and* $a_0(t)$ *are continuous for* $t \geq 0$, *and*

$$\int_{2n-1}^{2n} U^{-1}(s)a_0(s)ds \neq U^{-1}(2n), \; n \in \mathbb{N}, \; n \geq 1, \tag{1.56}$$

where U^{-1} *is the reciprocal of* U *and*

$$U(t) = \exp\left(\int_0^t a(s)ds\right).$$

Theorem 1.2.19 *The differential inequality*

$$x'(t) + p(t)x(t) + q(t)x\left(2\left[\frac{t+1}{2}\right]\right) \leq 0, \tag{1.57}$$

with $p(t)$ *and* $q(t)$ *continuous on* $[0, \infty)$, *has no eventually positive solution if*

$$\limsup_{n\to\infty} \int_{2n}^{2n+1} q(t)\exp\left(\int_{2n}^{t} p(s)ds\right)dt > 1. \tag{1.58}$$

Theorem 1.2.20 *If condition* (1.58) *is satisfied, the differential inequality*

$$x'(t) + p(t)x(t) + q(t)x(2[(t+1)/2]) \geq 0 \tag{1.59}$$

has no eventually negative solution.

From Theorem (1.2.19) and (1.2.20) it follows that subject to hypothesis (1.58), the equation

$$x'(t) + p(t)x(t) + q(t)x(2[(t+1)/2]) = 0 \tag{1.60}$$

has no eventually negative solutions and therefore the following conclusion is valid.

Theorem 1.2.21 *Subject to condition* (1.58), *Eq.* (1.60) *has oscillatory solutions only.*

Corollary 1.2.2 *Eq.* (1.55) *has only oscillatory solutions on* $[0, \infty)$ *if*

$$\liminf_{n\to\infty} \int_{2n}^{2n+1} a_0(t) \exp\left(-\int_{2n}^{t} a(s)ds\right) dt < -1. \tag{1.61}$$

Remark 1.2.2 *Condition* (1.61) *is sharp. For Eq.* (1.50) *with constant coefficients,* (1.61) *becomes* $a_0 < -ae^a/(e^a - 1)$ *which is according to* (1.50), *one of the two "best possible" conditions for oscillation.*

Theorem 1.2.22 *Inequality* (1.57) *has no eventually negative solution if*

$$\liminf_{n\to\infty} \int_{2n}^{2n+1} q(t) \exp\left(\int_{2n}^{t} p(s)ds\right) dt < -1. \tag{1.62}$$

Theorem 1.2.23 *If condition* (1.62) *is satisfied,* (1.59) *has no eventually positive solution.*

Theorem 1.2.24 *Subject to condition* (1.62), *Eq.* (1.60) *has oscillatory solutions only.*

Corollary 1.2.3 *Eq.* (1.55) *has only oscillatory solutions on* $[0, \infty)$ *if*

$$\limsup_{n\to\infty} \int_{2n-1}^{2n} a_0(t) \exp\left(-\int_{2n}^{t} a(s)ds\right) dt > 1. \tag{1.63}$$

Theorem 1.2.25 *If* $a_0 > a/(e^a - 1)$, *then solution* (1.53) *with the condition* $x(0) = c_0$ *has precisely one zero in each interval* $2n - 1 < t < 2n$ *with integral endpoints. If* $a_0 < -ae^a/(e^a - 1)$, *then* (1.53) *has precisely one zero in each interval* $2n < t < 2n+1$.

Theorem 1.2.26 *All solutions of Eq.* (1.50) *that are bounded on* $-\infty < t < \infty$ *and that do not tend to zero as* $t \to \pm\infty$ *are periodic. They exist only for* $a_0 = -a$ *or* $a_0 = -a(e^{2a} + 1)/(e^a - 1)^2$. *In the first case, the solutions are constant; and in the second case, they are of period* 4.

1.3 Differential Equations with Piecewise Constant Arguments of Generalized Type

In this section we shall see some of the definitions and fundamental theorems established previously for differential equations with piecewise constant arguments of generalized type.

1.3.1 Retarded EPCAG

The following results due to Akhmet [7] obtained by constructing an equivalent integral equations.

Consider the quasilinear system

$$y' = A(t)y + f(t, y(t), y(\beta(t))), \tag{1.64}$$

where $y \in \mathbb{R}^n$, $t \in \mathbb{R}$, $\beta(t) = \theta_i$ if $\theta_i \le t < \theta_{i+1}$, $i \in \mathbb{Z}$, is an identification function, θ_i, $i \in \mathbb{Z}$, is a strictly ordered sequence of real numbers, $|\theta_i| \to \infty$ as $|i| \to \infty$, and there exist real numbers θ and $\overline{\theta} > 0$ such that $\theta \le \theta_{i+1} - \theta_i \le \overline{\theta}$, $i \in \mathbb{Z}$.

Let us introduce the following assumptions.

(H1′′) $A(t)$ is a continuous $n \times n$ matrix and $\sup_{t\in\mathbb{R}} \|A(t)\| = \kappa < \infty$;

(H2′′) $f(t, x, z)$ is continuous in the first argument, $f(t, 0, 0) = 0$, $t \in \mathbb{R}$, and f is Lipschitzian such that $\|f(t, y_1, w_1) - f(t, y_2, w_2)\| \le l_0(\|y_1 - y_2\| + \|w_1 - w_2\|)$;

(H3″) there exists a projection P_0 and positive constants K_0 and σ such that

$$\|X(t)P_0X^{-1}(s)\| \leq K_0 \exp(-\sigma(t-s)), \qquad t \geq s,$$

$$\|X(t)(I-P_0)X^{-1}(s)\| \leq K_0 \exp(\sigma(s-t)), \qquad t \leq s,$$

where $X(t)$ is a fundamental matrix of the associated linear homogeneous system.

Definition 1.3.1 *A solution $y(t) = y(t, \theta_i, y_0)$, $y(\theta_i) = y_0$, $i \in \mathbb{Z}$, of* (1.64) *on $[\theta_i, \infty)$ is a continuous function such that*

(i) *the derivative $y'(t)$ exists at each point $t \in [\theta_i, \infty)$, with the possible exception of the points $\theta_j,\ j \geq i$, where one-sided derivatives exist;*

(ii) *equation* (1) *is satisfied by $y(t)$ at each interval $[\theta_j, \theta_{j+1})$, $j \geq i$.*

Theorem 1.3.1 *Suppose conditions* (H1″) – (H3″) *are fulfilled. Then for every $y_0 \in \mathbb{R}^n$ and $i \in \mathbb{Z}$, there exists a unique solution $y(t)$ of* (1.64) *in the sense of Definition* 1.3.1.

One can be easily shown that there exist positive constants m, M, such that $m \leq \|X(t,\ s)\| \leq M$, if $|t-s| \leq \bar{\theta}$.

We need the assumptions:

(H4″) $2Ml_0\bar{\theta} < 1$;

(H5″) $Ml_0\bar{\theta}[1 + M(1 + l_0\bar{\theta})\exp(Ml_0\bar{\theta})] < m$.

Theorem 1.3.2 *Assume that conditions* (H1″) – (H5″) *are fulfilled. Then, for every $y_0 \in \mathbb{R}^n$, $t_0 \in \mathbb{R}$, $\theta_i < t_0 \leq \theta_{i+1}$, $i \in \mathbb{Z}$, there exists a unique solution $\bar{y}(t) = y(t, \theta_i, \bar{y}_0)$ of* (1.64) *in sense of Definition* 1.3.1 *such that $\bar{y}(t_0) = y_0$.*

Example 1.3.1 *Consider*

$$x'(t) = 3x(t) - x(t)x(\beta(t)), \tag{1.65}$$

where $\beta(t) = \theta_i$ *if* $\theta_i \le t < \theta_{i+1}$, $i \in \mathbb{Z}$, $\theta_{2j-1} = j - \frac{1}{5}$, $\theta_{2j} = j + \frac{1}{5}$, $j \in \mathbb{Z}$. *The distance* $\theta_{i+1} - \theta_i$, $i \in \mathbb{Z}$, *is equal either to* $\theta = \frac{3}{5}$ *or to* $\overline{\theta} = \frac{2}{5}$.

Let us find conditions when a solution $x(t)$ *of* (1.65) *can be continued to the left from* $t = \theta_{i+1}$. *If* $t \in [\theta_i, \theta_{i+1}]$ *for a fixed* $i \in \mathbb{Z}$, *then* $x(t)$ *satisfies the following equation*

$$x'(t) = 3x(t) - x(t)x(\theta_i).$$

Hence,

$$x(t) = x(\theta_i)\exp((3 - x(\theta_i))(t - \theta_i)). \tag{1.66}$$

From the last equality it implies that every nontrivial solution of (1.65) *are either positive or negative. That is why, without loss of generality, consider only positive solutions. For a fixed* $H > 0$ *denote* $G_H = \{x : 0 < x < H\}$.

If $x_1, x_2, y_1, y_2 \in G_H$, *then* $|x_1y_1 - x_2y_2| \le H(|x_1 - x_2| + |y_1 - y_2|)$. *Moreover, we have that*

$$m = \min_{|t-s|\le\theta} \exp(2(t-s)) = \exp(-9/5),\ M = \max_{|t-s|\le\theta} \exp(2(t-s)) = \exp(9/5).$$

Hence, condition $(H4'')$ *for continuation of solutions of* (1.65) *to the left in* G_H *has the form*

$$H < 5\exp(-9/5)/6. \tag{1.67}$$

Let us consider another way to define values $x(\theta_i)$ *such that the solution* $x(t)$ *can be continued to the left from* $t = \theta_{i+1}$.

Using (1.66) *we find that*

$$x(\theta_{i+1}) = x(\theta_i)\exp((3 - x(\theta_i))(\theta_{i+1} - \theta_i)). \tag{1.68}$$

Consider (1.68) *as an equation with respect to* $x = x(\theta_i)$. *Introduce the following functions* $F_1(x) = x\exp((3-x)\overline{\theta})$ *and* $F_2(x) = x\exp((3-x)\theta))$. *The critical values of* x *for the functions are* $x^{(1)}_{max} = \overline{\theta}^{-1} = \frac{5}{2} < 3$ *and* $x^{(2)}_{max} = \theta^{-1} = \frac{5}{3} < 3$ *respectively, and maximal values of these functions are*

$$F^{(1)}_{max} = F_1(x^{(1)}_{max}) = 5\exp(1/5)/2, \qquad F^{(2)}_{max} = F_2(x^{(2)}_{max}) = 5\exp(4/5)/3. \tag{1.69}$$

Denote $F_{max} = \min(F^{(1)}_{max}, F^{(2)}_{max})$.

If $x(\theta_{i+1}) \leq F_{max}$, *then the solution can be continued to* $t = \theta_i$.

Comparing (1.67) *and* (1.69) *we see that* $H < F_{max}$. *That is, the evaluation of* H *by* ($H4''$) *is reliable for equation* (1.65).

Let us introduce the following definition, which is a version of a definition from [69], adapted for the general case.

Definition 1.3.2 *A function* $y(t)$ *is a solution of* (1.64) *on* $\mathbb{R}$ *if:*

(i) $y(t)$ *is continuous on* $\mathbb{R}$;

(ii) *the derivative* $y'(t)$ *exists at each point* $t \in \mathbb{R}$ *with the possible exception of the points* θ_i, $i \in \mathbb{Z}$, *where one-sided derivatives exist;*

(iii) *equation* (1.64) *is satisfied on each interval* $[\theta_i, \theta_{i+1})$, $i \in \mathbb{Z}$.

Theorem 1.3.3 *Suppose that conditions* (H1$''$) − (H5$''$) *are fulfilled. Then, for every* $(t_0, y_0) \in \mathbb{R} \times \mathbb{R}^n$, *there exists a unique solution* $y(t) = y(t, t_0, y_0)$ *of* (1.64) *in sense of Definition* 1.3.2 *such that* $y(t_0) = y_0$.

The last theorem is of major importance in [7]. It arranges the correspondence between points $(t_0, y_0) \in \mathbb{R} \times \mathbb{R}^n$ and all solutions of (1.64), and there is not a solution of the equation out of the correspondence. Using the assertion we can say that definition of the IVP for the EPCAG is similar to the problem for an ordinary differential equation, although the EPCAG is an equation with delay argument.

1.3.2 Alternately advanced-retarded EPCAG

The following definitions and theorems due to Akhmet [10]. The results are obtained by constructing an equivalent integral equations.

Fix two real-valued sequences θ_i, ζ_i, $i \in \mathbb{Z}$, such that $\theta_i < \theta_{i+1}$, $\theta_i \leq \zeta_i \leq \theta_{i+1}$ for all $i \in \mathbb{Z}$, $|\theta_i| \to \infty$ as $|i| \to \infty$.

Consider the following two equations

$$z'(t) = A_0(t)z(t) + A_1(t)z(\gamma(t)), \tag{1.70}$$

and

$$z'(t) = A_0(t)z(t) + A_1(t)z(\gamma(t)) + f(t, z(t), z(\gamma(t))), \tag{1.71}$$

where $z \in \mathbb{R}^n$, $t \in \mathbb{R}$, $\gamma(t) = \zeta_i$, if $t \in [\theta_i, \theta_{i+1})$, $i \in \mathbb{Z}$.

The following assumptions are needed:

(C1′) $A_0, A_1 \in C(\mathbb{R})$ are $n \times n$ real valued matrices;

(C2′) $f(t, x, y) \in C(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n)$ is an $n \times 1$ real valued function;

(C3′) $f(t, x, y)$ satisfies the condition

$$\|f(t, x_1, y_1) - f(t, x_2, y_2)\| \leq L_0(\|x_1 - x_2\| + \|y_1 - y_2\|), \tag{1.72}$$

where $L_0 > 0$ is a constant, and the condition

$$f(t, 0, 0) = 0, \ t \in \mathbb{R}. \tag{1.73}$$

(C4′) matrices A_0, A_1 are uniformly bounded on $\mathbb{R}$;

(C5′) $\inf_{\mathbb{R}} \|A_1(t)\| > 0$;

(C6′) there exists a number $\overline{\theta} > 0$ such that $\theta_{i+1} - \theta_i \leq \overline{\theta}$, $i \in \mathbb{Z}$;

(C7′) there exists a number $\theta > 0$ such that $\theta_{i+1} - \theta_i \geq \theta$, $i \in \mathbb{Z}$.

One can easily see that equations (1.70) and (1.71) have the form of functional differential equations

$$z'(t) = A_0(t)z(t) + A_1(t)z(\zeta_i), \tag{1.74}$$

$$z'(t) = A_0(t)z(t) + A_1(t)z(\zeta_i) + f(t, z(t), z(\zeta_i)), \tag{1.75}$$

respectively, if $t \in [\theta_i, \theta_{i+1})$, $i \in \mathbb{Z}$.

Let us introduce the following definition, which is a version of a definition from [69], modified for the general case.

Definition 1.3.3 *A continuous function* $z(t)$ *is a solution of* (1.70) ((1.71)) *on* $\mathbb{R}$ *if:*

(i) *the derivative* $z'(t)$ *exists at each point* $t \in \mathbb{R}$ *with the possible exception of the points* $\theta_i,\ i \in \mathbb{Z}$, *where the one-sided derivatives exist;*

(ii) *the equation is satisfied for* $z(t)$ *on each interval* $(\theta_i, \theta_{i+1}),\ i \in \mathbb{Z}$, *and it holds for the right derivative of* $z(t)$ *at the points* $\theta_i,\ i \in \mathbb{Z}$.

Let I be an $n \times n$ identity matrix. Denote by $X(t, s)$, $X(s, s) = I$, $t, s \in \mathbb{R}$, the fundamental matrix of solutions of the system

$$x'(t) = A_0(t)x(t). \tag{1.76}$$

which is associated with systems (1.70) and (1.71). Let us introduce a matrix-function $M_i(t),\ i \in \mathbb{Z}$,

$$M_i(t) = X(t, \zeta_i) + \int_{\zeta_i}^{t} X(t, s)A_1(s)\,ds,$$

useful in what follows. From now on we make the assumption:

(C8′) For each fixed $i \in \mathbb{Z}$, $\det[M_i(t)] \neq 0$, for all $t \in [\theta_i, \theta_{i+1}]$.

Theorem 1.3.4 *Assume that condition* (C1′) *is fulfilled. For every* $(t_0, z_0) \in \mathbb{R} \times \mathbb{R}^n$ *there exists a unique solution* $z(t) = z(t, t_0, z_0)$ *of (1.70) in the sense of Definition 1.3.3 such that* $z(t_0) = z_0$ *if and only if condition* (C9′) *is valid.*

The last theorem is of major importance for [10]. It arranges the correspondence between points $(t_0, z_0) \in \mathbb{R} \times \mathbb{R}^n$ and the solutions of (1.70) in the sense of Definition 1.3.3, and there exists no solution of the equation out of the correspondence.

Theorem 1.3.5 *Assume that condition* (C1′) *is fulfilled, and a number* $t_0 \in \mathbb{R}$, $\theta_i \leq t_0 < \theta_{i+1}$, *is fixed. For every* $z_0 \in \mathbb{R}^n$ *there exists a unique solution* $z(t) = z(t, t_0, z_0)$ *of (1.70) in the sense of Definition 1.3.3 such that* $z(t_0) = z_0$ *if and only if* $\det[M_i(t_0)] \neq 0$ *and* $\det[M_j(t)] \neq 0$ *for* $t = \theta_j, \theta_{j+1},\ j \in \mathbb{Z}$.

Assume that (t_0, z_0) is fixed, and $\theta_i \le t_0 < \theta_{i+1}$ for a fixed $i \in \mathbb{Z}$. We suppose that $t_0 \neq \zeta_i$. The solution satisfies, on the interval $[\theta_i, \theta_{i+1}]$, the following functional differential equation

$$z'(t) = A_0(t)z + A_1(t)z(\zeta_i). \tag{1.77}$$

Formally we need the pair of initial points (t_0, z_0) and $(\zeta_i, z(\zeta_i))$ to proceed with the solution, but since $z_0 = M_i(t_0)z(\zeta_i)$, where matrix $M_i(t_0)$ is nonsingular, we can say that the initial condition $z(t_0) = z_0$ is sufficient to define the solution.

Theorem 1.3.4 implies that the set of the solutions of (1.70) is an $n-$dimensional linear space. Hence, for a fixed $t_0 \in \mathbb{R}$ there exists a fundamental matrix of solutions of (1.70), $Z(t) = Z(t, t_0)$, $Z(t_0, t_0) = I$ such that

$$\frac{dZ}{dt} = A_0(t)Z(t) + A_1(t)Z(\gamma(t)).$$

Without loss of generality assume that $\theta_i < t_0 < \zeta_i$ for a fixed $i \in \mathbb{Z}$, and define the matrix only for increasing t, as the construction is similar for decreasing t.

We have $Z(\zeta_i) = M_i^{-1}(t_0)I = M_i^{-1}(t_0)$. Hence, on the interval $[\theta_i, \theta_{i+1}]$, $Z(t, t_0) = M_i(t)M_i^{-1}(t_0)$. Then $Z(\zeta_{i+1}) = M_{i+1}^{-1}(\theta_{i+1})Z(\theta_{i+1}) = M_{i+1}^{-1}(\theta_{i+1})M_i(\theta_{i+1})M_i^{-1}(t_0)$, and then $Z(t, t_0) = M_{i+1}(t)Z(\zeta_{i+1}) = M_{i+1}(t)M_{i+1}^{-1}(\theta_{i+1})M_i(\theta_{i+1})M_i^{-1}(t_0)$ if $t \in [\theta_{i+1}, \theta_{i+2}]$. One can continue by induction to obtain

$$Z(t) = M_l(t)\Big[\prod_{k=l}^{i+1} M_k^{-1}(\theta_k)M_{k-1}(\theta_k)\Big]M_i^{-1}(t_0), \tag{1.78}$$

if $t \in [\theta_l, \theta_{l+1}]$, for arbitrary $l > i$.

Similarly, if $\theta_j \le t \le \theta_{j+1} < \ldots < \theta_i \le t_0 \le \theta_{i+1}$, then

$$Z(t) = M_j(t)\Big[\prod_{k=j}^{i-1} M_k^{-1}(\theta_{k+1})M_{k+1}(\theta_{k+1})\Big]M_i^{-1}(t_0). \tag{1.79}$$

One can easily see that

$$Z(t, s) = Z(t)Z^{-1}(s),\ t,\ s \in \mathbb{R}, \tag{1.80}$$

and a solution $z(t)$, $z(t_0) = z_0$, $(t_0, z_0) \in \mathbb{R} \times \mathbb{R}^n$, of (1.70) is equal to

$$z(t) = Z(t, t_0)z_0, \qquad t \in \mathbb{R}. \tag{1.81}$$

Let us consider system (1.71). One can easily see that (C4′) – (C7′) imply the existence of positive numbers $m_0,\ M$ and $\bar{M}$ such that $m_0 \leq \|Z(t,\ s)\| \leq \bar{M},\ \|X(t,\ s)\| \leq M$ if $t,\ s \in [\theta_i,\ \theta_{i+1}],\ i \in \mathbb{Z}$.

We need the assumption

(C9′) $2ML_0(1+\bar{M})\bar{\theta} < 1$.

Then, we can see that $M(1+\bar{M})L_0\bar{\theta}e^{ML_0(1+\bar{M})\bar{\theta}} < 1$, and the expression

$$\kappa(L_0) = \frac{\bar{M}e^{ML_0(1+\bar{M})\bar{\theta}}}{1 - M(1+\bar{M})L_0\bar{\theta}e^{ML_0(1+\bar{M})\bar{\theta}}}$$

can be introduced. The following assumption is also needed

(C10′) $2ML_0\bar{\theta}\kappa(L_0)(1+\bar{M}) < m_0$.

The following lemma is the most important auxiliary result of that paper.

Lemma 1.3.1 *Assume that conditions* (C1′) – (C10′) *are fulfilled, and fix $i \in \mathbb{Z}$. Then, for every $(\xi,\ z_0) \in [\theta_i,\ \theta_{i+1}] \times \mathbb{R}^n$, there exists a unique solution $z(t) = z(t,\ \xi,\ z_0)$ of (1.75) on $[\theta_i,\ \theta_{i+1}]$.*

Theorem 1.3.6 *Assume that conditions* (C1′) – (C10′) *are fulfilled. Then, for every $(t_0,\ z_0) \in \mathbb{R} \times \mathbb{R}^n$ there exists a unique solution $z(t) = z(t,\ t_0,\ z_0)$ of (1.71) in the sense of Definition 1.3.3 such that $z(t_0) = z_0$.*

Lemma 1.3.2 *Assume that conditions* (C1′) – (C10′) *are fulfilled. Then, the solution $z(t) = z(t,\ t_0,\ z_0),\ (t_0,\ z_0) \in \mathbb{R} \times \mathbb{R}^n$, of (1.71) is a solution on $\mathbb{R}$ of the following integral equation*

$$\begin{aligned} z(t) = {} & Z(t,\ t_0)[z_0 + \int_{t_0}^{\zeta_i} X(t_0,\ s)f(s,\ z(s),\ z(\gamma(s)))\,ds] + \\ & \sum_{k=i}^{k=j-1} Z(t,\ \theta_{k+1}) \int_{\zeta_k}^{\zeta_{k+1}} X(\theta_{k+1},\ s)f(s,\ z(s),\ z(\gamma(s)))\,ds + \\ & \int_{\zeta_j}^{t} X(t,\ s)f(s,\ z(s),\ z(\gamma(s)))\,ds, \end{aligned} \tag{1.82}$$

where $\theta_i \leq t_0 \leq \theta_{i+1}$ and $\theta_j \leq t \leq \theta_{j+1},\ i < j$.

CHAPTER 2

PERIODIC SOLUTIONS IN NONCRITICAL CASE

In this chapter, we investigate the existence and stability of periodic solutions of quasilinear system with a retarded piecewise constant argument of generalized type and a small parameter in noncritical case, when the corresponding linear ordinary differential equations have not any nontrivial periodic solution. We also prove theorems on continuous dependence of solutions with respect to an initial condition and a parameters. An example illustrating the obtained results is constructed as well.

2.1 Introduction and Preliminaries

The main purpose of this chapter is to apply the method of small parameter to the following quasilinear system

$$x'(t) = A(t)x(t) + f(t) + \mu g(t,\ x(t),\ x(\beta(t)),\ \mu), \tag{2.1}$$

where $t \in \mathbb{R},\ x \in \mathbb{R}^n$, and μ is a small parameter belonging to an interval $J \subset \mathbb{R}$ with $0 \in J$; the functions $f(t),\ g(t,\ x,\ y,\ \mu)$ are $n-$dimensional vectors, $A(t)$ is an $n \times n$ matrix for $n \in \mathbb{N}$; the argument $\beta(t) = \theta_j$ if $\theta_j \le t < \theta_{j+1},\ j \in \mathbb{Z}$, is the identification function. Here, $\theta_j,\ j \in \mathbb{Z}$, is a strictly ordered sequence of real numbers, $|\theta_i| \to \infty$ as $|i| \to \infty$, and there exist two positive real numbers $\theta,\ \overline{\theta}$ such that $\theta \le \theta_{j+1} - \theta_j \le \overline{\theta},\ j \in \mathbb{Z}$.

In [8], it was proposed to investigate differential equations of type (2.1), that is EPCAG, using a new method based on the construction of an equivalent integral equation.

We combine that method with the method of small parameter [58,61,72] to investigate the problem of the existence of a periodic solution of (2.1) in so called non-critical case, when the corresponding linear homogeneous system has not any nontrivial periodic solution.

The following assumptions for Eq. (2.1) will be needed throughout this chapter:

(H1) Functions $A(t),\ f(t)$ and $g(t,\ x,\ y,\ \mu)$ are continuous in all of their arguments.

(H2) The function $g(t,\ x,\ y,\ \mu)$ satisfies Lipschitz condition with a positive real constant L such that

$$\|g(t,\ \tilde{x},\ \tilde{y},\ \mu) - g(t,\ x,\ y,\ \mu)\| \leq L[\|\tilde{x} - x\| + \|\tilde{y} - y\|],$$

for all $t \in \mathbb{R},\ \tilde{x},\ x,\ \tilde{y},\ y \in \mathbb{R}^n,\ \mu \in J.$

This chapter is organized in the following way: In the next section, we consider the existence and uniqueness of a global solution of the equations defined on the real axis. In section three, continuous and differentiable dependence of the solutions on an initial values and a parameter is considered. The main result of this chapter: the existence of a unique periodic solution of the equations in noncritical case and its stability are investigated in the last section. Furthermore, an appropriate example is provided.

2.2 Existence and uniqueness of solutions

The following definitions are from [8]. They are similar to those in [68–70], adapted to EPCAG. Let us first consider solutions defined on a half line beginning at some member θ_i of the sequence $\{\theta_j\},\ j \in \mathbb{Z}$.

Definition 2.2.1 *We say that a function* $x(t) = x(t, \theta_i, x_0, \mu),\ x(\theta_i) = x_0,\ i \in \mathbb{Z}$ *for* $t \geq \theta_i,\ \mu \in J,\ i \in \mathbb{Z}$, *is a solution of the initial value problem* (2.1) *on* $[\theta_i, \infty)$ *if it is a continuous function satisfying the conditions:*

(i) *the derivative* $x'(t)$ *exists for all* $t \in [\theta_i, \infty)$ *with the possible exception of points* $\theta_j,\ j \geq i,\ j \in \mathbb{Z}$, *where one sided derivative exists;*

(ii) $x(t)$ *satisfies Eq.* (2.1) *for each interval* $[\theta_j, \theta_{j+1}),\ j \geq i$.

The following theorem is valid.

Theorem 2.2.1 *Suppose that conditions* (H1) *and* (H2) *hold. Then, for all* $x_0 \in \mathbb{R}^n$, $\mu \in J$ *and* $i \in \mathbb{Z}$, *there exists a unique solution* $x(t)$ *of an initial value problem* (2.1) *with* $x(\theta_i) = x_0$ *in the sense of Definition* 2.2.1.

Proof: Let us fix $x_0 \in \mathbb{R}^n$, $i \in \mathbb{Z}$, $\mu \in J$. To use mathematical induction, let us start with $t \in [\theta_i, \theta_{i+1}]$. The solution $x(t)$ of Eq. (2.1) satisfies the ordinary differential equation

$$\psi'(t) = A(t)\psi(t) + f(t) + \mu g(t, \psi(t), x_0, \mu), \tag{2.2}$$

$$\psi(\theta_i) = x_0, \tag{2.3}$$

where the functions $A(t)$, $f(t)$ and $g(t, \psi(t), x_0, \mu)$ satisfy the conditions of the classical existence and uniqueness theorems of Peano and Picard-Lindelöf. Consequently, $x(t)$ exists uniquely on $[\theta_i, \theta_{i+1}]$.

Suppose that $x(t)$ is a unique solution of (2.1) on the interval $[\theta_i, \theta_k]$ for some $k \in \mathbb{Z}$, $k \geq i+1$. If $t \in [\theta_k, \theta_{k+1}]$, then $x(t)$ is a solution of the following *IVP*:

$$\psi'(t) = A(t)\psi(t) + f(t) + \mu g(t, \psi(t), x(\theta_k), \mu), \tag{2.4}$$

$$\psi(\theta_k) = x(\theta_k). \tag{2.5}$$

For the same reason as that behind the existence and uniqueness of the solution of (2.2) and (2.3), we conclude that $x(t)$ is uniquely defined on this interval, too. Therefore, there exists a unique solution $x(t)$ of (2.1) for $t \geq \theta_i$, satisfying $x(\theta_i) = x_0$ for $x_0 \in \mathbb{R}^n$. The theorem is proved. □

Let $X(t)$ be a fundamental matrix solution of the homogeneous system, corresponding to Eq. (2.1),

$$x'(t) = A(t)x(t), \tag{2.6}$$

such that $X(0) = I$, where I is an $n \times n$ identity matrix. Denote by $X(t, s) = X(t)X^{-1}(s)$, $t, s \in \mathbb{R}$, the transition matrix of (2.6). Let $\kappa = \sup_{t\in\mathbb{R}} \|A(t)\| < \infty$.

Lemma 2.2.1 [44] Assume (H1) is satisfied. Then, the inequality

$$\|X(t,\ s)\| \le \exp(\kappa|t-s|),\ t,\ s \in \mathbb{R}, \tag{2.7}$$

holds.

Lemma 2.2.2 *Assume* (H1) *is satisfied. Then, the inequality*

$$m \le \|X(t,\ s)\| \le M,$$

where $m = \exp(-\kappa\overline{\theta}),\ M = \exp(\kappa\overline{\theta})$, *holds for* $|t-s| \le \overline{\theta}$.

Proof. Using (2.7) and the equality $X(t,\ s)X(s,\ t) = I$, it can be found immediately that the inequality

$$\|X(t,\ s)\| \ge \exp(-\kappa|t-s|),\ t,\ s \in \mathbb{R}, \tag{2.8}$$

is satisfied. By combining (2.7) with (2.8), the lemma is proved. □

The following definition is similar to those in [8, 68–70] adapted to EPCAG.

Definition 2.2.2 *We say that* $x(t)$ *is a solution of* (2.1) *on* $\mathbb{R}$ *if it satisfies the conditions:*

(i) $x(t)$ *is continuous on* $\mathbb{R}$;

(ii) *the derivative* $x'(t)$ *exists for all* $t \in \mathbb{R}$ *with the possible exception of the points* $\theta_j,\ j \in \mathbb{Z}$, *where one sided derivative exists;*

(iii) $x(t)$ *satisfies equation* (2.1) *for each interval* $[\theta_j,\ \theta_{j+1}),\ j \in \mathbb{Z}$.

Let us introduce the following two lemmas. We prove only the second of them, the proof of the first one is very similar.

Lemma 2.2.3 *Suppose* (H1) *is satisfied. A function* $x(t) = x(t,\ t_0,\ x_0,\ \mu)$, *where* t_0 *is a real fixed number, is a solution of* (2.1) *on* $\mathbb{R}$ *if and only if it is a solution on* $\mathbb{R}$ *of the following integral equation*

$$x(t) = X(t,\ t_0)x_0 + \int_{t_0}^{t} X(t,\ s)[f(s) + \mu g(s,\ x(s),\ x(\beta(s)),\ \mu)]\,ds. \tag{2.9}$$

Lemma 2.2.4 *Suppose* (H1) *is satisfied. A function* $x(t) = x(t, t_0, x_0, \mu)$, *where* t_0 *is a real fixed number, is a solution of* (2.1) *on* $\mathbb{R}$ *if and only if it is a solution on* $\mathbb{R}$ *of the following integral equation*

$$x(t) = x_0 + \int_{t_0}^{t} [A(s)x(s) + f(s) + \mu g(s, x(s), x(\beta(s)), \mu)]\, ds. \tag{2.10}$$

Proof: *Necessity.* Assume that $x(t)$ is a solution of (2.1) on $\mathbb{R}$. Denote

$$\phi(t) = x_0 + \int_{t_0}^{t} [A(s)x(s) + f(s) + \mu g(s, x(s), x(\beta(s)), \mu)]\, ds.$$

By straightforward evaluation, we can see that the integral exists.

Suppose $t \neq \theta_i$, $i \in \mathbb{Z}$. Then

$$\phi'(t) = A(t)x(t) + f(t) + \mu g(t, x(t), x(\beta(t)), \mu)$$

and

$$x'(t) = A(t)x(t) + f(t) + \mu g(t, x(t), x(\beta(t)), \mu).$$

Hence,

$$[\phi(t) - x(t)]' = 0.$$

Calculating the limit values at θ_i, $i \in \mathbb{Z}$, we find that

$$\phi'(\theta_i \pm 0) = A(\theta_i \pm 0)x(\theta_i \pm 0) + f(\theta_i \pm 0) + \mu g(\theta_i \pm 0, x(\theta_i \pm 0), x(\beta(\theta_i \pm 0)), \mu),$$

$$x'(\theta_i \pm 0) = A(\theta_i \pm 0)x(\theta_i \pm 0) + f(\theta_i \pm 0) + \mu g(\theta_i \pm 0, x(\theta_i \pm 0), x(\beta(\theta_i \pm 0)), \mu).$$

Consequently,

$$[\phi(t) - x(t)]'|_{t=\theta_i+0} = [\phi(t) - x(t)]'|_{t=\theta_i-0}.$$

Thus, $\phi(t) - x(t)$ is a continuously differentiable function on $\mathbb{R}$ satisfying the equation

$$[\phi(t) - x(t)]' = 0 \tag{2.11}$$

with the initial condition $\phi(t_0) - x(t_0) = 0$. This proves that $\phi(t) - x(t) = 0$ on $\mathbb{R}$.

Sufficiency. Suppose that (2.10) is valid. Fix $i \in \mathbb{Z}$ and consider the interval $[\theta_i, \theta_{i+1})$. If $t \in (\theta_i, \theta_{i+1})$, then by differentiating (2.10) one can see that $x(t)$ satisfies (2.1).

Moreover, by considering $t \to \theta_i + 0$, and taking into account that $x(\beta(t))$ is a right continuous function, we find that $x(t)$ satisfies (2.1) on $[\theta_i, \theta_{i+1})$. The lemma is proved. □

The following simple example shows that while a solution of EPCAG with small parameter exists in the sense of Definition 2.2.1, it may not exist in the sense of Definition 2.2.2, that is, a solution may exist on a half-axis and not exist on the whole real axis, unless we put some conditions.

Example 2.2.1 *Consider the following differential equation:*

$$x'(t) = \alpha x(t) - \mu x^2(\beta(t)), \tag{2.12}$$

where $x \in \mathbb{R}$, $t \in \mathbb{R}$, α is a real positive constant, and $\beta(t) = \theta_j$ if $\theta_j \leq t < \theta_{j+1}$, $j \in \mathbb{Z}$, $\theta_{2i-1} = 4i - 1$, $\theta_{2i} = 4i$, $i \in \mathbb{Z}$. The distance $\theta_{j+1} - \theta_j$, $j \in \mathbb{Z}$, is either equal to $\theta = 1$ or to $\overline{\theta} = 3$. Let us fix $x_0 \in \mathbb{R}$. We shall look for conditions on α and μ such that a solution $x(t) = x(t, \theta_0, x_0, \mu)$, $x(\theta_0) = x_0$, $x_0 > 0$, of (2.12) *exists in the sense of Definitions* 2.2.1 *and* 2.2.2.

If $\mu = 0$, it is easy to see that the solution $x(t)$ of (2.12) *exists uniquely, and it is positive and not bounded on $\mathbb{R}$.*

Suppose $\mu > 0$. Let us consider a transformation $x(t) = y(t)/\mu$. Using this transformation, we obtain the following equation from (2.12)

$$y'(t) = \alpha y(t) - y^2(\beta(t)). \tag{2.13}$$

Let $y(t) = y(t, \theta_0, y_0)$ be a solution of (2.13) *with $y(\theta_0) = y_0$, $y_0 > 0$. Denote $y_k = y(\theta_k)$, $k \in \mathbb{Z}$. We first consider the existence and uniqueness of the solution $y(t)$. Let us start with $t \in [\theta_0, \infty)$, that is, if time is increasing.*

If $t \in [\theta_0, \theta_1]$, then $y(t)$ is a solution of the equation

$$y'(t) = \alpha y(t) - y_0^2,$$

which is a linear nonhomogeneous differential equation with a constant coefficient, that is why the solution $y(t)$ is uniquely defined on $[\theta_0, \theta_1]$. The rest can be deduced from the arguments of mathematical induction. That is, the solution $y(t)$ and the corresponding solution $x(t)$ exist uniquely on $[\theta_0, \infty)$ in the sense of Definition 2.2.1.

Next, let us consider the solution for decreasing time. We will show that if

$$\begin{aligned} y_0 &\le \frac{\alpha\exp(2\alpha)}{4[\exp(\alpha)-1]} && (2.14)\\ \frac{\alpha\exp(\alpha)}{\exp(\alpha)-1} &\le \frac{\alpha\exp(6\alpha)}{4[\exp(3\alpha)-1]} && (2.15)\\ \frac{\alpha\exp(3\alpha)}{\exp(3\alpha)-1} &\le \frac{\alpha\exp(2\alpha)}{4[\exp(\alpha)-1]} && (2.16)\end{aligned}$$

are satisfied, then the solution $y(t) = y(t, \theta_0, y_0)$ *exists on* $(-\infty, \theta_0]$.

If $t \in [\theta_{-1}, \theta_0]$, *then* $y(t)$ *coincides with the solution of the following ordinary differential equation*

$$y'(t) = \alpha y(t) - y_{-1}^2. \qquad (2.17)$$

Using the equivalent integral equation of (2.17), *it can be written that*

$$y(t) = \exp(\alpha(t-\theta_{-1}))y_{-1} + \frac{1}{\alpha}[1-\exp(\alpha(t-\theta_{-1}))]y_{-1}^2. \qquad (2.18)$$

Denote $z = y_{-1}$. *It is easy to see that the solution* $y(t)$ *exists on* $[\theta_{-1}, \theta_0]$, *if the quadratic equation for* z, *obtained from* (2.18) *with* $t = \theta_0$,

$$z^2 - \frac{\alpha\exp(\alpha)}{\exp(\alpha)-1}z + \frac{\alpha}{\exp(\alpha)-1}y_0 = 0 \qquad (2.19)$$

has a real root. The last equation has a real root, if inequality (2.14) *is valid. Hence, if inequality* (2.14) *is valid, then the solution* $y(t)$ *exists on* $[\theta_{-1}, \theta_0]$, *but is not necessarily unique.*

Suppose inequality (2.14) *is valid. It is easy to check that the roots* $z_{1,2}$ *of equation* (2.19) *satisfy the inequality*

$$0 \le z_{1,2} \le \frac{\alpha\exp(\alpha)}{\exp(\alpha)-1}. \qquad (2.20)$$

Denote $z = y_{-2}$. *If* $t \in [\theta_{-2}, \theta_{-1}]$, *one can similarly obtain that the solution* $y(t)$ *exists on* $[\theta_{-2}, \theta_{-1}]$, *if the following quadratic equation*

$$z^2 - \frac{\alpha\exp(3\alpha)}{\exp(3\alpha)-1}z + \frac{\alpha}{\exp(3\alpha)-1}y_{-1} = 0 \qquad (2.21)$$

has a real root. The last equation has a real root, if

$$y_{-1} \le \frac{\alpha\exp(6\alpha)}{4[\exp(3\alpha)-1]} \qquad (2.22)$$

holds. Using inequalities (2.20) *and* (2.22), *it is clear that if inequality* (2.15) *is valid, then the solution* $y(t)$ *exists on* $[\theta_{-2}, \theta_{-1}]$.

Suppose inequality (2.15) *is valid. It is easy to see that the roots* $z_{3,4}$ *of equation* (2.21) *satisfy the inequality*

$$0 \leq z_{3,4} \leq \frac{\alpha \exp(3\alpha)}{\exp(3\alpha) - 1}. \tag{2.23}$$

If $t \in [\theta_{-3}, \theta_{-2}]$, *we then have a quadratic equation similar to* (2.19), *and*

$$y_{-2} \leq \frac{\alpha \exp(2\alpha)}{4[\exp(\alpha) - 1]} \tag{2.24}$$

holds. Therefore, the solution $y(t)$ *exists on* $[\theta_{-3}, \theta_{-2}]$. *Finally, using inequalities* (2.16), (2.23) *and* (2.24) *one can see that the solution* $y(t)$ *exists on* $[\theta_{-4}, \theta_{-3}]$.

By using the arguments of mathematical induction, we can conclude that if inequalities (2.14) – (2.16) *are satisfied, then the solution* $y(t, \theta_0, y_0)$ *exists on* $(-\infty, \theta_0]$, *but is not necessarily unique.*

Consequently, if inequalities (2.15), (2.16) *and the inequality*

$$0 < \mu \leq \frac{\alpha \exp(2\alpha)}{4x_0[\exp(\alpha) - 1]}, \tag{2.25}$$

obtained from (2.14), *are satisfied for* $x_0 > 0$, *then the solution* $x(t) = x(t, \theta_0, x_0, \mu)$ *exists in the sense of Definition* 2.2.2. *Moreover, if one of inequalities* (2.15), (2.16) *or* (2.25) *is violated, then the solution* $x(t)$ *exists in the sense of Definition* 2.2.1, *but it does not exist in the sense of Definition* 2.2.2.

From now on, we need the following assumptions:

(H3) $|\mu| < 1/(2ML\bar{\theta})$;

(H4) $|\mu|ML\bar{\theta}[1 + M(1 + L|\mu|\bar{\theta})\exp(ML|\mu|\bar{\theta})] < m$.

The following theorem provides the existence of a unique solution to the left when the initial moment ξ is an arbitrary real number.

Theorem 2.2.2 *Suppose that* (H1) – (H4) *hold. Then, for all* $x_0 \in \mathbb{R}^n$, $\xi \in \mathbb{R}$, $\theta_i < \xi \leq \theta_{i+1}$, $i \in \mathbb{Z}$, *there exists a unique solution* $\bar{x}(t) = x(t, \theta_i, \bar{x}_0, \mu)$ *of* (2.1) *in the sense of Definition* 2.2.1 *with* $\bar{x}(\xi) = x_0$.

Proof: *Existence.* Consider a solution $\psi(t) = x(t, \xi, x_0, \mu)$ with $\psi(\xi) = x_0$ of the equation

$$x'(t) = A(t)x(t) + f(t) + \mu g(t, x(t), \eta, \mu)$$

on $[\theta_i, \xi]$.

We need to prove that there is a vector $\eta \in \mathbb{R}^n$ such that the equation

$$\psi(t) = X(t, \xi)x_0 + \int_{\xi}^{t} X(t, s)[f(s) + \mu g(s, \psi(s), \eta, \mu)]\, ds \tag{2.26}$$

has a solution $\psi(t)$, defined on $[\theta_i, \xi]$, and satisfying $\psi(\theta_i) = \eta$.

Construct a sequence $\{\psi_k(t)\} \subset \mathbb{R}^n$, $k \in \mathbb{N}$ with $\psi_0(t) = X(t, \xi)x_0$ such that

$$\psi_{k+1}(t) = X(t, \xi)x_0 + \int_{\xi}^{t} X(t, s)[f(s) + \mu g(s, \psi_k(s), \psi_k(\theta_i), \mu)]ds,\ k \in \mathbb{N}.$$

By simple calculation, it can be found that

$$\max_{[\theta_i, \xi]} \|\psi_{k+1}(t) - \psi_k(t)\| \le (2ML\bar{\theta}|\mu|)^k \zeta,$$

where $\zeta = M\bar{\theta}\max_{[\theta_i, \xi]} \|f(s) + \mu g(s, \psi_0(s), \psi_0(\theta_i), \mu)\|$. That is, the sequence $\psi_k(t)$ is convergent, and its limit $\psi(t)$ satisfies (2.26) on $[\theta_i, \xi]$ with $\eta = \psi(\theta_i)$ whenever $|\mu| < 1/(2ML\bar{\theta})$. The existence is proved.

Uniqueness. It is sufficient to check that for each $t \in (\theta_i, \theta_{i+1}]$, and $x_1, x_2 \in \mathbb{R}^n$, $x_1 \neq x_2$, condition $x(t, \theta_i, x_1, \mu) \neq x(t, \theta_i, x_2, \mu)$ is valid.

Let $x_1(t) = x(t, \theta_i, x_1, \mu)$ and $x_2(t) = x(t, \theta_i, x_2, \mu)$ be two solutions of (2.1) such that $x_1 \neq x_2$. Suppose to the contrary that there exists $\tilde{t} \in (\theta_i, \theta_{i+1}]$ such that $x_1(\tilde{t}) = x_2(\tilde{t})$. This implies the equation

$$X(\tilde{t}, \theta_i)(x_1 - x_2) = -\mu \int_{\theta_i}^{\tilde{t}} X(\tilde{t}, s)[g(s, x_1(s), x_1(\theta_i), \mu) - g(s, x_2(s), x_2(\theta_i), \mu)]ds. \tag{2.27}$$

We have then inequalities

$$m\|x_1 - x_2\| \le \|X(\tilde{t}, \theta_i)(x_1 - x_2)\| \tag{2.28}$$

and

$$\|x_1(t) - x_2(t)\| \le M\|x_1 - x_2\| + \int_{\theta_i}^{t} ML\,|\mu|[\|x_1(s) - x_2(s)\| + \|x_1 - x_2\|]\, ds \tag{2.29}$$

for $t \in (\theta_i, \theta_{i+1}]$.

Hence, by applying Gronwall-Bellman Lemma to (2.29), we obtain

$$\|x_1(t) - x_2(t)\| \leq M(1 + L|\mu|\overline{\theta}) \exp(ML|\mu|\overline{\theta})\|x_1 - x_2\|,$$

which leads to the inequality

$$\left\| -\mu \int_{\theta_i}^{\tilde{t}} X(\tilde{t}, s)[g(s, x_1(s), x_1(\theta_i), \mu) - g(s, x_2(s), x_2(\theta_i), \mu)]ds \right\| \leq$$
$$|\mu|ML\overline{\theta}[1 + M(1 + L|\mu|\overline{\theta}) \exp(ML|\mu|\overline{\theta})]\|x_1 - x_2\|. \qquad (2.30)$$

Therefore, condition (H4) and inequalities (2.28), (2.30) contradict (2.27). The theorem is proved. □

Remark 2.2.1 *The last theorem provides us conditions* (H3) *and* (H4), *of smallness for the parameter* μ *such that the initial value problem has a unique solution defined on* $[t_0, \infty)$.

The following theorem is valid.

Theorem 2.2.3 *Suppose that* (H1) − (H4) *hold. Then, for all* $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$, *there exists a unique solution* $x(t)$ *of* (2.1) *in the sense of Definition* 2.2.2 *with* $x(t_0) = x_0$.

Proof: Fix a moment $t_0 \in \mathbb{R}$. Then, there is $i \in \mathbb{Z}$ such that $\theta_i < t_0 \leq \theta_{i+1}$. By Theorem 2.2.2, there is a unique solution $x(t) = x(t, \theta_i, {x_0}^i, \mu)$, $x(\theta_i) = x_0^i$ of (2.1) with $x(t_0) = x_0$. Similarly, by Theorem 2.2.2, there is a unique solution $\tilde{x}(t) = x(t, \theta_{i-1}, x_0^{i-1}, \mu)$, $\tilde{x}(\theta_{i-1}) = x_0^{i-1}$ with $\tilde{x}(\theta_i) = x_0^i$. Hence, $\tilde{x}(t_0) = x_0$. We can complete the proof by using mathematical induction. □

The last theorem is of major importance, since it supplies a one-to-one correspondence between points $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$ and solutions of (2.1), and there is no solution of (2.1) out of the correspondence. Although (2.1) is a delay differential equation, it has the properties of ordinary differential equations. We will make use of this correspondence in the rest of this chapter.

2.3 Dependence of solutions on initial value and parameter

Let us fix $t_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}^n$, and $\mu_0 \in J$. There exists $j \in \mathbb{Z}$ such that $\theta_j \le t_0 < \theta_{j+1}$. We denote by $\|\,.\,\|_t$ a sup-norm, $\|v(\xi)\|_t = \sup_{[\theta_j,\, t]} \|v(\xi)\|$. The theorem sets continuous dependence of solutions for (2.1) on an initial data. To prove the theorem, we consider the following assertion.

Lemma 2.3.1 *Let $v(t)$ be a continuous function for $t \ge \theta_j$, satisfying the inequality*

$$\|v(t)\| \le \eta + \int_{\theta_j}^{t} [a_1(s)\,\|v(s)\| + a_2(s)\,\|v(\beta(s))\|]\,ds, \tag{2.31}$$

where $\eta \ge 0$ is a real scalar, and $a_1(t)$, $a_2(t)$ are nonnegative piecewise continuous functions. Then,

$$\|v(\xi)\|_t \le \eta \exp\left(\int_{\theta_j}^{t} [a_1(s) + a_2(s)]ds\right),\ t \ge \theta_j. \tag{2.32}$$

Proof: Let us first show that

$$\|v(\xi)\|_t \le \eta + \int_{\theta_j}^{t} [a_1(s) + a_2(s)]\ \|v(\xi)\|_s ds,\ t \ge \theta_j. \tag{2.33}$$

Since $\theta_j \le \beta(s)) \le s$ for $s \ge \theta_j$, we have that

$$\|v(\beta(\xi))\|_t = \sup_{[\theta_j,\, t]} \|v(\beta(\xi))\| = \sup_{[\theta_j, \beta(t)]} \|v(\xi)\| \le \sup_{[\theta_j,\, t]} \|v(\xi)\| = \|v(\xi)\|_t.$$

Hence,

$$\|v(t)\| \le \eta + \int_{\theta_j}^{t} [a_1(s) + a_2(s)]\,\|v(\xi)\|_s\,ds$$

is satisfied.

If $\|v(t)\| = \|v(\xi)\|_t$ for a given $t \ge \theta_j$, then inequality (2.33) is valid. Suppose that $\|v(t)\| < \|v(\xi)\|_t$. By definition of sup-norm, there is a moment $\tilde{t} \in [\theta_j,\, t]$ such that $\|v(\tilde{t})\| = \|v(\xi)\|_t$. Hence, we have

$$\begin{aligned} \|v(\xi)\|_t &= \|v(\tilde{t})\| \\ &\le \eta + \int_{\theta_j}^{\tilde{t}} [a_1(s) + a_2(s)]\|v(\xi)\|_s\,ds \\ &\le \eta + \int_{\theta_j}^{t} [a_1(s) + a_2(s)]\|v(\xi)\|_s\,ds, \end{aligned}$$

as $\tilde{t} \leq t$. So, (2.33) is valid. Now, setting $\psi(t) = \|v(\xi)\|_t$ and applying Gronwall-Bellman Lemma to

$$\psi(t) \leq \eta + \int_{\theta_j}^{t} [a_1(s) + a_2(s)]\psi(s)\, ds,\ t \geq \theta_j,$$

we complete the proof. □

Let us fix a positive real number T.

Theorem 2.3.1 *Suppose* (H1) – (H4) *are valid. If* $x(t) = x(t, t_0, x_0, \mu_0)$ *and* $\tilde{x}(t) = x(t, t_0, x_0 + \Delta x, \mu_0)$ *are solutions of* (2.1), *where* Δx *is an n-dimensional real vector, then*

$$\|\tilde{x}(\xi) - x(\xi)\|_t \leq M\|\Delta x\| \exp(2|\mu_0|ML(t_0 + T - \theta_j)) \tag{2.34}$$

is satisfied for all $t \in [t_0, t_0 + T]$.

Proof: If $t \in [t_0, t_0 + T]$, then

$$\begin{aligned} \|\tilde{x}(t) - x(t)\| \ \leq\ & X(t, t_0)\|\Delta x\| + |\mu_0| \int_{t_0}^{t} X(t, s)\|g(s, \tilde{x}(s), \tilde{x}(\beta(s)), \mu_0) \\ & -g(s, x(s), x(\beta(s)), \mu_0)\|ds, \end{aligned}$$

Hence,

$$\|\tilde{x}(t) - x(t)\| \leq M\|\Delta x\| + |\mu_0|ML \int_{\theta_j}^{t} [\|\tilde{x}(s) - x(s)\| + \|\tilde{x}(\beta(s)) - x(\beta(s))\|]ds.$$

Applying Lemma 2.3.1 to the last inequality, we proved that (2.34) is valid. □

The differential dependence of a solution of (2.1) on an initial value is established by our next theorem. We need the following assumption:

(H5) The function $g(t, x, y, \mu)$ has continuous first partial derivatives in all its arguments $t \in \mathbb{R},\ x, y \in \mathbb{R}^n,\ \mu \in J$.

Let us introduce the following equations:

$$U'(t) = A(t)U(t) + \mu_0[A_1(t)U(t) + A_2(t)U(\beta(t))], \tag{2.35}$$

$$U(t_0) = I, \tag{2.36}$$

where $U \in \mathbb{R}^{n\times n}$, and the functions

$$A_1(t) = \frac{\partial g}{\partial x}(t,\ x(t),\ x(\beta(t)),\ \mu_0),\ A_2(t) = \frac{\partial g}{\partial y}(t,\ x(t),\ x(\beta(t)),\ \mu_0)$$

are $n \times n$ matrices.

Theorem 2.3.2 *Suppose* (H1) – (H5) *are valid. Let* $e_i = (0, \dots, 0, 1, 0, \dots, 0)^T$ *be the n-tuple whose i-th component is* 1 *and all others are* 0 *for* $i = 1, \dots, n$, *and* δ *a real positive constant. If* $U(t)$ *is the solution of* (2.35) *and* (2.36) *in the sense of Definition* 2.2.2, *and* $x(t) = x(t,\ t_0,\ x_0,\ \mu_0)$ *and* $\tilde{x}_i(t) = x(t,\ t_0,\ x_0 + \Delta x_i,\ \mu_0)$ *are the solutions of* (2.1), *where* $\Delta x_i = \delta e_i$ *is an n-dimensional vector, then*

$$\tilde{x}_i(t) - x(t) - U(t)\Delta x_i = o(\Delta x_i) \tag{2.37}$$

is satisfied for all $t \in [t_0,\ t_0 + T]$.

Proof: By the equivalence Lemma 2.2.4, $\tilde{x}_i(t)$, $x(t)$ and $U(t)$ satisfy the following integral equations:

$$\begin{aligned}
\tilde{x}_i(t) &= X(t,\ t_0)(x_0 + \Delta x_i) + \int_{t_0}^{t} X(t,\ s)[f(s) + \mu_0 g(s,\ \tilde{x}_i(s),\ \tilde{x}_i(\beta(s)),\ \mu_0)]ds,\\
x(t) &= X(t,\ t_0)x_0 + \int_{t_0}^{t} X(t,\ s)[f(s) + \mu_0 g(s,\ x(s),\ x(\beta(s)),\ \mu_0)]ds,\\
U(t) &= X(t,\ t_0) + \mu_0 \int_{t_0}^{t} X(t,\ s)[A_1(s)U(s) + A_2(s)U(\beta(s))]\,ds,
\end{aligned}$$

respectively. An easy computation shows that, if $t \in [t_0,\ t_0 + T]$,

$$\begin{aligned}
\tilde{x}_i(t) - x(t) - U(t)\Delta x_i = \mu_0 \int_{t_0}^{t} X(t,\ s)[g(s,\ \tilde{x}_i(s),\ \tilde{x}_i(\beta(s)),\ \mu_0)\\
-g(s,\ x(s),\ x(\beta(s)),\ \mu_0) - A_1(s)U(s)\Delta x_i - A_2(s)U(\beta(s))\Delta x_i]ds.
\end{aligned}$$

By expanding $g(s,\ \tilde{x}_i(s),\ \tilde{x}_i(\beta(s)),\ \mu_0)$ about $(s,\ x(s),\ x(\beta(s)),\ \mu_0)$, we write

$$\begin{aligned}
g(s,\ \tilde{x}_i(s),\ \tilde{x}_i(\beta(s)),\ \mu_0) = g(s,\ x(s),\ x(\beta(s)),\ \mu_0) + A_1(s)[\tilde{x}_i(s) - x(s)]\\
+A_2(s)[\tilde{x}_i(\beta(s)) - x(\beta(s))] + \xi(s),
\end{aligned}$$

where $\xi(t) = o(\Delta x_i)$. Hence,

$$\begin{aligned}
\|\tilde{x}_i(t) - x(t) - U(t)\Delta x_i\| \le \zeta + |\mu_0|M \int_{\theta_j}^{t} [\|A_1(s)\|\,\|\tilde{x}_i(s) - x(s) - U(s)\Delta x_i\|\\
+\|A_2(s)\|\,\|\tilde{x}_i(\beta(s)) - x(\beta(s)) - U(\beta(s))\Delta x_i\|]\,ds,
\end{aligned}$$

where $\zeta = |\mu_0| M \int_{t_0}^{t_0+T} \|\xi(s)\| ds$. Consequently, by applying Lemma 2.3.1 to the last inequality, we prove that (2.37) is valid. □

As a result of the last theorem, we have shown that the initial value problem (2.35) and (2.36) is a variation of (2.1). Moreover, we note that continuous dependence of solutions of (2.1) on a parameter follows from Theorem (2.3.1) and (H5) by adding the parameter μ to Eq. (2.1) as a new dependent variable and requiring that $\mu'(t) = 0$ and $\mu(t_0) = 0$.

2.4 Existence and stability of the periodic solutions

In this section, we prove the main theorem of this chapter. We need the following assumptions:

(H6) The functions $A(t),\ f(t)$ and $g(t,\ x,\ y,\ \mu)$ are ω-periodic in t, for some a positive real number ω.

(H7) The sequence $\{\theta_i\}$ satisfies an $(\omega,\ p)$-property, that is, $\theta_{i+p} = \theta_i + \omega,\ i \in \mathbb{Z}$, for some positive integer p.

Let us consider the following version of the Poincaré criterion.

Lemma 2.4.1 *Suppose that* (H1) – (H4) *and* (H6), (H7) *hold. Then, solution* $x(t) = x(t,\ t_0,\ x_0,\ \mu)$ *of* (2.1), *with* $x(t_0) = x_0$, *is* ω*-periodic if and only if*

$$x(\omega) = x(0). \tag{2.38}$$

Proof. If $x(t)$ is ω-periodic, then Eq. (2.38) is obviously satisfied. Suppose that Eq. (2.38) holds. Let $y(t) = x(t + \omega)$ on $\mathbb{R}$. Then, equation (2.38) can be written as $y(0) = x(0)$. One can show that $\beta(t + \omega) = \beta(t) + \omega$. Hence,

$$\begin{aligned} y'(t) &= x'(t+\omega) \\ &= A(t+\omega)x(t+\omega) + f(t+\omega) + \mu g(t+\omega,\ x(t+\omega),\ x(\beta(t+\omega)),\ \mu) \\ &= A(t)y(t) + f(t) + \mu g(t,\ y(t),\ y(\beta(t)),\ \mu). \end{aligned}$$

That is, $y(t)$ is a solution of (2.1). By uniqueness of the solution, $x(t) = y(t)$ on $\mathbb{R}$. The lemma is proved. □

The following theorem is a generalization of a classical theorem originally due to Poincaré [72] for EPCAG.

Theorem 2.4.1 *Assume that* $(H1) - (H7)$ *hold, and*

$$x'(t) = A(t)x(t) \tag{2.39}$$

has no nontrivial periodic solution with period ω. *Then, for sufficiently small* $|\mu|$, *equation* (2.1) *has a unique* ω*-periodic solution, which tends to the unique periodic solution with period* ω *of*

$$x'(t) = A(t)x(t) + f(t), \tag{2.40}$$

as $\mu \to 0$.

Proof: Let $x(t, \zeta, \mu)$ be a solution of equation (2.1), satisfying the initial condition $x(0, \zeta, \mu) = \zeta$, and let $x_0(t) = x(t, \zeta_0, 0)$ be a unique periodic solution of period ω of equation (2.40). To show, using Lemma 2.4.1, that for a sufficiently small μ the ω-periodic solution $x(t, \zeta, \mu)$ exists, it is necessary and sufficient that the equation

$$x(\omega, \zeta, \mu) - \zeta = 0 \tag{2.41}$$

be solvable with respect to ζ.

Let $P(\zeta, \mu) = x(\omega, \zeta, \mu) - \zeta$. In order to apply the implicit function theorem, we will show that the determinant of $P_\zeta'(\zeta_0, 0)$ exists and is different from zero.

Let $Z(t, \zeta, \mu) = (\partial x_i/\partial \zeta_k)$, $i, k = 1, \ldots, n$. Differentiating equation (2.1) with respect to ζ, we can see that $Z(t, \zeta_0, 0)$ is the fundamental matrix of equation (2.39). On the other hand, $P_\zeta'(\zeta_0, 0) = \det(Z(\omega, \zeta_0, 0) - I)$ and, since the eigenvalues of the matrix $Z(\omega, \zeta_0, 0)$ are different from unity, it follows that $P_\zeta'(\zeta_0, 0) \neq 0$. Therefore, in a sufficiently small neighborhood of the point $(0, \zeta_0)$, equation (2.41) is solvable with respect to ζ. The existence and uniqueness of an ω-periodic solution are proved. The fact that the solution $x(t, \zeta, \mu)$ tends to $x_0(t)$, when $\mu \to 0$, follows from Theorem 2.3.1. The theorem is proved. □

Let us demonstrate the last theorem by applying it to the following example.

Example 2.4.1 *Consider the following system of EPCAG*

$$x'(t) = \begin{pmatrix} \alpha & \gamma \\ -\gamma & \alpha \end{pmatrix} x(t) + \begin{pmatrix} sin(\pi t) \\ cos(\pi t) \end{pmatrix} + \mu g(t,\ x(t),\ x(\beta(t)),\ \mu), \tag{2.42}$$

where $x \in \mathbb{R}^2$, $\alpha \neq 0$, $\gamma, \mu \geq 0$, $\beta(t) = \theta_i$ *if* $\theta_i \leq t < \theta_{i+1}$, $i \in \mathbb{Z}$, *with* $\theta_i = i + (-1)^i/3$, $i \in \mathbb{Z}$; $g(t,\ x,\ y,\ \mu)$ *is a* 2−*periodic in* t, *continuous function, having continuous first partials in all of its arguments, and satisfying Lipschitz continuity with a constant* L, *that is,*

$$||g(t,\ x_1,\ y_1,\ \mu) - g(t,\ x_2,\ y_2,\ \mu)|| \leq L\left[||x_1 - x_2|| + ||y_1 - y_2||\right],$$

where $x_1,\ y_1,\ x_2,\ y_2 \in \mathbb{R}^2$. *One can see that the sequence* $\{\theta_i\}$ *fulfills* $\theta_{i+2} = \theta_i + 2$ *for all* $i \in \mathbb{Z}$. *By fixing a sufficiently small* $|\mu|$ *satisfying the inequalities*

$$|\mu| < 1/(2ML\overline{\theta}),$$
$$|\mu|ML\overline{\theta}[1 + M(1 + L|\mu|\overline{\theta})\exp(ML|\mu|\overline{\theta})] < m,$$

where $\overline{\theta} = 5/3$, $\kappa = \sqrt{\alpha^2 + \gamma^2}$, $m = e^{-5\kappa/3}$, *and* $M = e^{5\kappa/3}$, *we conclude that assumptions* (H1) − (H7) *are fulfilled. Therefore, through every point* $(t_0,\ \zeta)$ *of* $\mathbb{R}^3$, *there passes exactly one solution* $x(t,\ \mu) = x(t,\ t_0,\ \zeta,\ \mu)$, $x(t_0,\ \mu) = \zeta$ *of* (2.42) *in the sense of Definition* 2.2.2.

The monodromy matrix of (2.42) *is*

$$X(2) = \begin{bmatrix} e^{2\alpha}\cos(2\gamma) & e^{2\alpha}\sin(2\gamma) \\ -e^{2\alpha}\sin(2\gamma) & e^{2\alpha}\cos(2\gamma) \end{bmatrix},$$

and it has no unit multiplier for $\alpha \neq 0$. *Hence, there is a unique* 2*-periodic solution* $x_0(t)$ *of the system*

$$x'(t) = \begin{pmatrix} \alpha & \gamma \\ -\gamma & \alpha \end{pmatrix} x(t) + \begin{pmatrix} sin(\pi t) \\ cos(\pi t) \end{pmatrix},$$

with the initial value

$$x_0(\theta_0) = (I - X(2))^{-1} \int_{\theta_0}^{\theta_2} X(\theta_2 - s) \begin{pmatrix} sin(\pi s) \\ cos(\pi s) \end{pmatrix} ds.$$

Therefore, by Theorem 2.4.1, *there is a unique* 2*-periodic solution* $x(t,\ \mu)$ *of* (2.42), *satisfying* $x(t,\ \mu) \to x_0(t)$ *as* $\mu \to 0$.

Definition 2.4.1 *The solution* $x(t, x_0) = x(t, t_0, x_0, \mu)$, $x(t_0, x_0) = x_0$ *of* (2.1) *is said to be uniformly stable if for every* $\epsilon > 0$, *there exists a number* $\delta \in \mathbb{R}$, $\delta = \delta(\epsilon) > 0$ *such that* $\|\overline{x_0} - x_0\| < \delta$ *implies* $\|x(t, \overline{x_0}) - x(t, x_0)\| < \epsilon$ *for every* $t \geq t_0$.

Definition 2.4.2 *The solution* $x(t, x_0) = x(t, t_0, x_0, \mu)$, $x(t_0, x_0) = x_0$ *of* (2.1) *is said to be uniformly asymptotically stable if it is uniformly stable and there is a real number* $b > 0$ *such that for every* $\zeta > 0$ *there exists* $T(\zeta) > 0$ *such that* $\|\overline{x_0} - x_0\| < b$ *implies that* $\|x(t, \overline{x_0}) - x(t, x_0)\| < \zeta$ *if* $t > t_0 + T(\zeta)$.

Theorem 2.4.2 *Suppose that* (H1) − (H7) *hold. Let* $x(t) = x(t, t_0, x_0, \mu)$ *be a solution of* (2.1). *If all the characteristic multipliers of the equation*

$$x'(t) = A(t)x(t) \tag{2.43}$$

are less than unity in modulus, then for sufficiently small $|\mu|$, *the solution* $x(t)$ *is uniformly asymptotically stable.*

Proof. Let $u(t)$ be a solution of (2.1) with the initial condition $u(t_0) = x_0 + \eta$. Let us define $z(t) = u(t) - x(t)$. Since all the multipliers are less than unity in modulus,

$$\|X(t, s)\| \leq K \exp(-\alpha(t - s)), \ s \leq t,$$

where K and α are positive constants. By using the equivalence Lemma 2.2.3, one can find that

$$\|z(t)\| \leq \|X(t, t_0)\|\|\eta\| + \int_{t_0}^{t} \|X(t, s)\||\mu|[\|g(s, x(s) + z(s), x(\beta(s)) + z(\beta(s)), \mu) \\ -g(s, x(s), x(\beta(s)), \mu)\|]ds$$

and

$$\|z(t)\| \leq K \exp(-\alpha(t - t_0))\|\eta\| + \int_{t_0}^{t} \exp(-\alpha(t - s))|\mu|KL[\|z(s)\| + \|z(\beta(s))\|]ds.$$

Then,

$$\exp(\alpha t)\|z(t)\| \leq K \exp(\alpha t_0)\|\eta\| + \int_{\theta_j}^{t} \exp(\alpha s)\,|\mu|KL[\|z(s)\| + \|z(\beta(s))\|]ds.$$

Applying Lemma 2.3.1 to the last inequality, we have

$$\|z(t)\| \leq K \exp([-\alpha + 2|\mu|KL](t - \theta_j))\|\eta\|$$

Therefore, for $|\mu| < \alpha/(2KL)$, the solution $x(t)$ is uniformly asymptotically stable. The theorem is proved. □

CHAPTER 3

PERIODIC SOLUTIONS IN CRITICAL CASE

In this chapter, conditions are found for the existence of periodic solutions for forced weakly nonlinear ordinary differential equations with alternately advanced-retarded piecewise constant argument of generalized type. The resonant case is studied, that is, when the unperturbed linear ordinary differential equation has a nontrivial periodic solution. The dependence of solutions on initial values and parameters is also studied.

3.1 Introduction

The problem of the existence of periodic solutions is one of the most interesting topics for applications. Poincaré [72] introduced the method of small parameter to investigate the problem and it has been developed by many authors (see, for example, [58, 73], and the references cited therein) and this method remains as one of the most effective methods for this problem. It is important that the results obtained in this field can be extended to the bifurcation theory [22, 61].

Fix two real-valued sequences $\theta_i, \zeta_i, i \in \mathbb{Z}$, such that $\theta_i < \theta_{i+1}$, $\theta_i \le \zeta_i \le \theta_{i+1}$ for all $i \in \mathbb{Z}$, $|\theta_i| \to \infty$ as $|i| \to \infty$.

In this chapter we shall consider the following equation

$$z'(t) = A(t)z(t) + f(t) + \mu g(t, z(t), z(\gamma(t)), \mu), \tag{3.1}$$

where $z \in \mathbb{R}^n$, $t \in \mathbb{R}$, $\mu \in J \subset \mathbb{R}$, where J is an open interval containing 0, and $\gamma(t) = \zeta_i$, if $t \in [\theta_i, \theta_{i+1})$, $i \in \mathbb{Z}$.

The following assumptions will be needed throughout the chapter:

(C1) $A : \mathbb{R} \to \mathbb{R}^{n\times n}$, $f : \mathbb{R} \to \mathbb{R}^n$ and $g : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times J \to \mathbb{R}^n$ are continuous functions.

(C2) The function $g(t, x, y, \mu)$ satisfies Lipschitz continuity in the second and third arguments with a positive Lipschitz constant L such that

$$\|g(t, x_1, y_1, \mu) - g(t, x_2, y_2, \mu)\| \leq L(\|x_1 - x_2\| + \|y_1 - y_2\|)$$

for all $t \in \mathbb{R}$, $\mu \in J$ and $x_1, x_2, y_1, y_2 \in \mathbb{R}^n$.

(C3) The matrix A is uniformly bounded on $\mathbb{R}$.

(C4) There exists a number $\overline{\theta} > 0$ such that $\theta_{i+1} - \theta_i \leq \overline{\theta}$, $i \in \mathbb{Z}$.

(C5) There exists a number $\theta > 0$ such that $\theta_{i+1} - \theta_i \geq \theta$, $i \in \mathbb{Z}$.

In [8, 10], it was proposed to investigate differential equations of type (3.1), that is, the differential equations with piecewise constant argument of generalized type (EPCAG). Moreover, a new method based on the construction of an equivalent integral equation was used.

We combine that method with the method of small parameter [58,61,72] to investigate the problem of the existence of periodic solutions of Eq. (3.1) in the so called critical case, when the corresponding linear homogeneous system admits nontrivial periodic solutions.

This chapter is organized in the following way. In the next section, we give known definitions and results that will be needed further. Section three considers continuous and differentiable dependence of solutions on the initial value and the parameter. The main result of the chapter: the existence of periodic solutions of Eq. (3.1) is discussed in section four. Appropriate examples are given to illustrate the theory in the last section.

3.2 Preliminaries

In this section, we shall introduce some definitions and lemmas.

Definition 3.2.1 [10] *A continuous function $z(t)$ is a solution of Eq.* (3.1) *on $\mathbb{R}$ if:*

(i) *The derivative $z'(t)$ exists at each point $t \in \mathbb{R}$ with the possible exception of the points $\theta_i,\ i \in \mathbb{Z}$, where the one-sided derivatives exist.*

(ii) *The equation is satisfied for $z(t)$ on each interval $(\theta_i,\ \theta_{i+1}),\ i \in \mathbb{Z}$, and it holds for the right derivative of $z(t)$ at the points $\theta_i,\ i \in \mathbb{Z}$.*

The following lemmas of this section are similar to the assertions from [10]. That is why, we provide them without proof.

Let $X(t)$ be the fundamental matrix solution of the homogeneous system, corresponding to Eq. (3.1),

$$x'(t) = A(t)x(t),\ t \in \mathbb{R}, \tag{3.2}$$

such that $X(0) = I$, where I is an $n \times n$ identity matrix. Denote by $X(t,\ s) = X(t)X^{-1}(s)$, $t,\ s \in \mathbb{R}$ the transition matrix.

Let us now define the solutions of quasilinear system (3.1).

Lemma 3.2.1 *Suppose that* (C1) *is satisfied. A function $z(t) = z(t,\ t_0,\ z_0,\ \mu)$, where t_0 is a fixed real number, is a solution of* (3.1) *in the sense of Definition* 3.2.1 *if and only if it is a solution, on $\mathbb{R}$, of the following integral equation*

$$z(t) = X(t,\ t_0)z_0 + \int_{t_0}^{t} X(t,\ s)[f(s) + \mu g(s,\ z(s),\ z(\gamma(s)),\ \mu)]ds. \tag{3.3}$$

Denote $\kappa = \sup_{t \in \mathbb{R}} \|A(t)\| < \infty$. For the transition matrix, $X(t,\ s)$, one can obtain the following inequality [8, 44]:

$$m \le \|X(t,\ s)\| \le M, \tag{3.4}$$

where $m = \exp(-\kappa\overline{\theta})$ and $M = \exp(\kappa\overline{\theta})$, if $t,\ s \in [\theta_i,\ \theta_{i+1}]$ for all $i \in \mathbb{Z}$.

From now on we make the following assumption:

(C6) $2|\mu|ML\overline{\theta} < 1, \qquad |\mu|M^2L\overline{\theta}\left\{\dfrac{1 + |\mu|ML\overline{\theta}\exp\left(|\mu|ML\overline{\theta}\right)}{1 - |\mu|ML\overline{\theta}\exp\left(|\mu|ML\overline{\theta}\right)} + \exp\left(|\mu|ML\overline{\theta}\right)\right\} < m.$

Lemma 3.2.2 [10] *Assume that conditions* (C1) – (C6) *are fulfilled. Then for fixed* $i \in \mathbb{Z}$ *and every* $(\xi, z_0) \in [\theta_i, \theta_{i+1}] \times \mathbb{R}^n$ *there exists unique solution* $z(t) = z(t, \xi, z_0, \mu)$ *of Eq.* (3.1) *on* $[\theta_i, \theta_{i+1}]$.

From Lemma 3.2.2, one can obtain the following assertion.

Lemma 3.2.3 [10] *Assume that conditions* (C1) – (C6) *are fulfilled. Then for every* $(t_0, z_0) \in \mathbb{R} \times \mathbb{R}^n$ *there exists a unique solution* $z(t) = z(t, t_0, z_0, \mu)$ *of Eq.* (3.1) *in the sense of Definition* 3.2.1 *such that* $z(t_0) = z_0$.

3.3 Dependence of the solutions on initial value and parameter

Let us fix $t_0 \in \mathbb{R}$, $z_0 \in \mathbb{R}^n$ and $\mu_0 \in J$. There exists $j \in \mathbb{Z}$ such that $\theta_j \leq t_0 < \theta_{j+1}$. Let us denote by $\|\,.\,\|_t$ a max-norm, $\|v\|_t = \max_{\xi \in [\theta_j, t]} \|v(\xi)\|$. Define a function $\chi(t) = \max\{t, \gamma(t)\}$. The next theorem proves continuous dependence of solutions of (3.1) on an initial value z_0. To prove the theorems, we use the following assertion, which is analogue of Gronwall-Bellman Lemma.

Lemma 3.3.1 *Let* $u(t)$ *be continuous,* $\eta_1(t)$ *and* $\eta_2(t)$ *nonnegative piecewise continuous scalar functions defined for* $t \geq \theta_j$*. Suppose that* α *is a nonnegative real constant and that* $u(t)$ *satisfies the inequality*

$$\|u(t)\| \leq \alpha + \int_{\theta_j}^{t} [\eta_1(s)\,\|u(s)\| + \eta_2(s)\,\|u(\gamma(s))\|]\,ds, \tag{3.5}$$

for $t \geq \theta_j$*. Then the inequality*

$$\|u\|_{\chi(t)} \leq \alpha \exp\left(\int_{\theta_j}^{\chi(t)} [\eta_1(s) + \eta_2(s)]\,ds\right) \tag{3.6}$$

is satisfied for $t \geq \theta_j$.

Proof: Let us first show that

$$\|u\|_{\chi(t)} \leq \alpha + \int_{\theta_j}^{\chi(t)} [\eta_1(s) + \eta_2(s)]\|u\|_{\chi(s)}\,ds, \;\; t \geq \theta_j. \tag{3.7}$$

As $\chi(t) \geq \theta_j$, using (3.5), we have

$$\|u(\chi(t))\| \leq \alpha + \int_{\theta_j}^{\chi(t)} [\eta_1(s)\,\|u(s)\| + \eta_2(s)\,\|u(\gamma(s))\|]\,ds. \tag{3.8}$$

Since $\theta_j \leq \gamma(s) \leq \chi(s)$ for all $s \geq \theta_j$, we have that

$$\|u(\gamma)\|_{\chi(s)} = \max_{[\theta_j, \chi(s)]} \|u(\gamma(\xi))\| = \max_{[\zeta_j, \gamma(s)]} \|u(\xi)\| \leq \max_{[\theta_j, \chi(s)]} \|u(\xi)\| = \|u\|_{\chi(s)}.$$

Hence, using (3.8), the inequality

$$\|u(\chi(t))\| \leq \alpha + \int_{\theta_j}^{\chi(t)} [\eta_1(s) + \eta_2(s)]\|u\|_{\chi(s)}\, ds$$

is satisfied.

If $\|u(\chi(t))\| = \|u\|_{\chi(t)}$ is satisfied for a given $t \geq \theta_j$, then inequality (3.7) follows. Suppose that $\|u(\chi(t))\| < \|u\|_{\chi(t)}$ holds. One can see that by the definition of max-norm, there is a moment $\tilde{t} \in [\theta_j, \chi(t)]$ such that $\|u\|_{\chi(t)} = \|u(\tilde{t})\|$.

Then, using (3.5), we have

$$\begin{aligned}
\|u\|_{\chi(t)} &= \|u(\tilde{t})\| \\
&\leq \alpha + \int_{\theta_j}^{\tilde{t}} [\eta_1(s)\, \|u(s)\| + \eta_2(s)\, \|u(\gamma(s))\|]\, ds \\
&\leq \alpha + \int_{\theta_j}^{\chi(\tilde{t})} [\eta_1(s) + \eta_2(s)]\|u\|_{\chi(s)}\, ds \\
&\leq \alpha + \int_{\theta_j}^{\chi(t)} [\eta_1(s) + \eta_2(s)]\|u\|_{\chi(s)}\, ds,
\end{aligned}$$

as $\chi(\tilde{t}) \leq \chi(t)$. Hence, inequality (3.7) is valid. Now, set the function $\|u\|_{\chi(s)} = \psi(s)$, and note that $\psi(s) = \psi(\chi(s))$.

Thus we have the inequality

$$\psi(\chi(t)) \leq \alpha + \int_{\theta_j}^{\chi(t)} [\eta_1(s) + \eta_2(s)]\psi(\chi(s))\, ds.$$

Applying Gronwall-Bellman Lemma to the last inequality, we complete the proof. □

Let us fix a number $T > 0$. Now, we set continuous dependence of solutions of (3.1) on an initial value z_0 by the following theorem.

Theorem 3.3.1 *Suppose that* (C1) − (C6) *are valid. If* $z(t) = z(t, t_0, y_0, \mu_0)$ *and* $\tilde{z}(t) = z(t, t_0, z_0 + \Delta z, \mu_0)$ *are the solutions of Eq.* (3.1), *where* Δz *is an n-dimensional vector, then the inequality*

$$\|\tilde{z}(\xi) - z(\xi)\|_{\chi(t)} \leq M\|\Delta z\| \exp\Big(2|\mu_0| ML(\chi(t_0 + T) - \theta_j)\Big) \qquad (3.9)$$

is satisfied for $t \in [t_0, t_0 + T]$.

The last theorem can be proved by applying Lemma 3.3.1. The differential dependence of a solution of Eq. (3.1) on an initial value is established by our next theorem, which requires the following assumption:

(C7) $g(t, x, y, \mu)$ has continuous first partial derivatives in all of its arguments $t \in \mathbb{R}$, $x, y \in \mathbb{R}^n$, $\mu \in J$.

Let us introduce the following equations

$$U'(t) = A(t)U(t) + \mu_0[A_1(t)U(t) + A_2(t)U(\gamma(t))], \tag{3.10}$$

$$U(t_0) = I, \tag{3.11}$$

where $U \in \mathbb{R}^{n\times n}$ and the functions

$$A_1(t) = \frac{\partial g}{\partial x}(t, z(t), z(\gamma(t)), \mu_0), \qquad A_2(t) = \frac{\partial g}{\partial y}(t, z(t), z(\gamma(t)), \mu_0)$$

are $n \times n$ matrices.

Theorem 3.3.2 *Suppose that* (C1) – (C7) *are valid. Let* $e_i = (0, \dots, 0, 1, 0, \dots, 0)^T$ *be the n-tuple whose i-th component is* 1 *and all others are* 0 *for* $i = 1, \dots, n$, *and* δ *a real positive constant. If* $U(t)$ *is a solution of* (3.10) *and* (3.11) *on* $\mathbb{R}$, *and* $z(t) = z(t, t_0, z_0, \mu_0)$ *and* $\tilde{z}_i(t) = z(t, t_0, z_0 + \Delta z_i, \mu_0)$ *are solutions of Eq.* (3.1), *where* $\Delta z_i = \delta e_i$ *is an n-dimensional vector in the sense of Definition* 3.2.1, *then*

$$\tilde{z}_i(t) - z(t) - U(t)\Delta z_i = o(\Delta z_i) \tag{3.12}$$

is satisfied on a section $t \in [t_0, t_0 + T]$, $T > 0$.

Proof: By Lemma 3.2.1, the functions $\tilde{z}_i(t)$, $z(t)$ and $U(t)$ satisfy the following integral equations:

$$\begin{aligned}
\tilde{z}_i(t) &= X(t, t_0)(z_0 + \Delta z_i) + \int_{t_0}^{t} X(t, s)[f(s) + \mu_0 g(s, \tilde{z}_i(s), \tilde{z}_i(\gamma(s)), \mu_0)]\, ds, \\
z(t) &= X(t, t_0)z_0 + \int_{t_0}^{t} X(t, s)[f(s) + \mu_0 g(s, z(s), z(\gamma(s)), \mu_0)]\, ds, \\
U(t) &= X(t, t_0) + \mu_0 \int_{t_0}^{t} X(t, s)[A_1(s)U(s) + A_2(s)U(\gamma(s))]\, ds,
\end{aligned}$$

respectively. An easy computation shows that, if $t \in [t_0,\ t_0 + T]$, we have

$$\tilde{z}_i(t) - z(t) - U(t)\Delta z_i = \mu_0 \int_{t_0}^{t} X(t,\ s)[g(s,\ \tilde{z}_i(s),\ \tilde{z}_i(\gamma(s)),\ \mu_0)$$
$$-g(s,\ z(s),\ z(\gamma(s)),\ \mu_0) - A_1(s)U(s)\Delta z_i - A_2(s)U(\gamma(s))\Delta z_i]\, ds.$$

By expanding $g(s,\ \tilde{z}_j(s),\ \tilde{z}_j(\gamma(s)),\ \mu_0)$ about the point $(s,\ z(s),\ z(\gamma(s)),\ \mu_0)$, we write

$$g(s,\ \tilde{z}_i(s),\ \tilde{z}_i(\gamma(s)),\ \mu_0) = g(s,\ z(s),\ z(\gamma(s)),\ \mu_0) + A_1(s)[\tilde{z}_i(s) - z(s)]$$
$$+A_2(s)[\tilde{z}_i(\gamma(s)) - z(\gamma(s))] + \xi(s),$$

where $\xi(s) = o(\Delta z_i)$. Hence, the inequality

$$\|\tilde{z}_i(t) - z(t) - U(t)\Delta z_i\| \leq \zeta + |\mu_0| M \int_{t_0}^{t} [\|A_1(s)\|\, \|\tilde{z}_i(s) - z(s) - U(s)\Delta z_i\|$$
$$+\|A_2(s)\|\, \|\tilde{z}_i(\gamma(s)) - z(\gamma(s)) - U(\gamma(s))\Delta z_i\|]\, ds,$$

where $\zeta = |\mu_0| M \int_{t_0}^{t_0+T} \|\xi(s)\|\, ds$, is valid. Consequently, by applying Lemma 3.3.1 to the last inequality, we prove that (3.12) is true. □

As a result of the last theorem, we have shown that the initial value problem (3.10) and (3.11) is a variation of equation (3.1). Moreover, we note that continuous dependence of solutions of (3.1) on a parameter follows from Theorem (3.3.1) and (C7) by adding the parameter μ to Eq. (3.1) as a new dependent variable and requiring that $\mu'(t) = 0$ and $\mu(t_0) = 0$.

3.4 Existence of the periodic solutions

In this section, we prove the main result of this chapter. Let us introduce the following assumptions:

(C8) The functions $A(t)$, $f(t)$ and $g(t,\ x,\ y,\ \mu)$ are periodic in t with a fixed positive real period ω.

(C9) The sequences θ_i and ζ_i, $i \in \mathbb{Z}$, satisfy an $(\omega,\ p)$-property, that is there is a positive integer p such that the equations $\theta_{i+p} = \theta_i + \omega$ and $\zeta_{i+p} = \zeta_i + \omega$ hold for all $i \in \mathbb{Z}$.

We consider the following version of Poincaré criterion.

Lemma 3.4.1 *Suppose that* (C1) – (C6), (C8) *and* (C9) *hold. Then, the solution* $z(t) = z(t, t_0, x_0, \mu)$ *of Eq.* (3.1), *is* ω*-periodic if and only if*

$$z(\omega) = z(0). \tag{3.13}$$

Proof. If $z(t)$ is ω-periodic, then Eq. (3.13) is obviously satisfied. Suppose Eq. (3.13) holds. Let $y(t) = z(t + \omega)$ on $\mathbb{R}$. Then, Eq. (3.13) can be written as $y(0) = z(0)$.

One can show that $\gamma(t + \omega) = \gamma(t) + \omega$ for all $t \in \mathbb{R}$. Hence,

$$\begin{aligned} y'(t) &= z'(t + \omega) \\ &= A(t + \omega)z(t + \omega) + f(t + \omega) + \mu g(t + \omega, z(t + \omega), z(\gamma(t + \omega)), \mu) \\ &= A(t)y(t) + f(t) + \mu g(t, y(t), y(\gamma(t)), \mu). \end{aligned}$$

That is, $y(t)$ is a solution of Eq. (3.1). By the uniqueness of the solution, we have $z(t) = y(t)$ on $\mathbb{R}$. The lemma is proved. □

In the previous chapter, we considered the noncritical case. Now, we suppose that the homogeneous equation, corresponding to Eq. (3.1), has a nontrivial ω-periodic solution.

Let $\phi_j,\ j = 1, \ldots, k,\ k \le n$, be the solutions of Eq. (3.2), which form a maximal set of linearly independent ω-periodic solutions. Then, the corresponding adjoint system of (3.2),

$$x'(t) = -A^T(t)x(t), \tag{3.14}$$

has a maximal set of linearly independent ω-periodic solutions, $\psi_j,\ j = 1, \ldots, k$.

We compose an $n \times k$ matrix $K_1(t)$, whose columns are solutions $\psi_j,\ j = 1, \ldots, k$.

Let us introduce the following condition:

(C10) $\displaystyle\int_0^{\omega} K_1^T(s) f(s) ds = 0.$

Theorem 3.4.1 [58, 73] *Suppose that* (C1) – (C3), (C8) *and* (C10) *hold. Then, if Eq.* (3.2) *admits* $k \le n$ *linearly independent* ω*-periodic solutions, then there exists a family of* k *linearly independent* ω*-periodic solutions of the equation*

$$z'(t) = A(t)z(t) + f(t), \tag{3.15}$$

of the form $z(t, \alpha) = \alpha_1\phi_1(t) + \ldots + \alpha_k\phi_k(t) + \tilde{z}(t)$, *where* $\alpha = (\alpha_1, \ldots, \alpha_k)$ *is a real constant vector and* $\tilde{z}(t)$ *is a particular* ω*-periodic solution of Eq.* (3.15).

Now let us investigate the question of existence of periodic solutions of (3.1). The next theorem is a generalization of a classical theorem due to Malkin [58] for EPCAG.

Theorem 3.4.2 *Suppose that* (C1) – (C10) *hold and Eq.* (3.15) *admits a family of* ω*-periodic solutions* $z(t, \alpha)$. *Let* α^0 *be a solution of the equation* $h(\alpha) = 0$, *where the function* h *is given by*

$$h(\alpha) = \int_0^\omega K_1^T(s)g(s, z(s, \alpha), z(\gamma(s), \alpha), 0)ds, \tag{3.16}$$

such that

$$\det\left(\frac{\partial h}{\partial \alpha}\bigg|_{\alpha=\alpha^0}\right) \neq 0.$$

Then for sufficiently small $|\mu|$ *Eq.* (3.1) *has an* ω*-periodic solution that converges to* $z(t, \alpha^0)$ *when* $\mu \to 0$.

Proof. Let $z(t)$ be a solution of (3.1) and let us complete the matrix $K_1(t)$ by columns ψ_j, $j = k+1, \ldots, n$, which are solutions of (3.14) to obtain a fundamental matrix of solutions $K(t)$. Performing the substitution $y(t) = K^T(0)z(t)$ in (3.1), we obtain the equation

$$y'(t) = P(t)y(t) + r(t) + \mu F(t, y(t), y(\gamma(t)), \mu), \tag{3.17}$$

where

$$P(t) = K^T(0)A(t)K^T(0)^{-1}, \quad r(t) = K^T(0)f(t),$$
$$F(t, y(t), y(\gamma(t)), \mu) = K^T(0)g(t, K^T(0)^{-1}z(t), K^T(0)^{-1}z(\gamma(t)), \mu).$$

Denote $y(t, \alpha) = K^T(0)z(t, \alpha)$, $\beta = (\beta_{k+1}, \ldots, \beta_n)$ and let $v(t) = y(t, \alpha, \beta)$ be a solution of (3.17) with the initial condition $v(0) = y(0, \alpha) + (0, \beta)^T$. Further, let $L(t) = K^{-1}(0)K(t)$, $L_1(t) = K^{-1}(0)K_1(t)$, $L_2(t)$ be the matrix composed of the entries of the last $n-k$ columns and $n-k$ rows of the matrix $L(t)$, and $L_3(t)$ be the matrix composed of the last $n-k$ rows of $L^T(t)$. Denote

$$U(\alpha, \beta, \mu) = \int_0^\omega L_1^T(s)F(s, v(s), v(\gamma(s)), \mu)ds,$$
$$V(\alpha, \beta, \mu) = (L_2^T(\omega) - I)\beta - \mu\int_0^\omega L_3(s)F(s, v(s), v(\gamma(s)), \mu)ds.$$

Then the ω-periodicity condition for the solution $v(t)$ takes on the form of the equations

$$U(\alpha, \beta, \mu) = 0, \tag{3.18}$$

$$V(\alpha, \beta, \mu) = 0. \tag{3.19}$$

If, in (3.19), taking $\mu = 0$, we obtain $\beta = 0$, and then Eq. (3.18) has the form

$$U(\alpha, 0, 0) = \int_0^\omega L_1^T(s)F(s, y(s, \alpha), y(\gamma(s), \alpha), 0)ds = 0. \tag{3.20}$$

Let $\alpha^0 = (\alpha_1^0, \ldots, \alpha_k^0)$ be a solution of (3.20). Since the function U has continuous partial derivatives with respect to α_j, $j = 1, \ldots, k$, in a sufficiently small neighborhood of the point $(\alpha_0, 0, 0)$, it follows that under the assumption

$$\det\left(\frac{\partial U}{\partial \alpha}\bigg|_{\alpha=\alpha^0}\right) \neq 0$$

the system of equations (3.18) and (3.19) is solvable with respect to α and β so that the functions $\alpha_j(\mu)$ and $\beta_s(\mu)$, $j = 1, \ldots, k$, $s = k+1, \ldots, n$ are continuous and $\alpha_j(\mu) \to \alpha_j^0$, $\beta_s(\mu) \to 0$ as $\mu \to 0$.

Thus, we establish that for sufficiently small $|\mu|$, system (3.1) admits an ω-periodic solution, which converges to the solution $z(t, \alpha^0)$ of (3.15) as $\mu \to 0$. The theorem is proved. □

3.5 Illustrative examples

We will introduce appropriate examples in this section. These examples will show the feasibility of our theory. The equations of Duffing type are widely investigated in the field of nonlinear dynamics, and used to model many processes in mechanics and electronics [40, 65]. We construct the examples with Duffing equations below.

Example 3.5.1 *Let us consider the following EPCAG*

$$q''(t) = -q(t) + 3\sin^2(t) + \mu\left(q(t) + q'\left(2\pi\left[\frac{t+\pi}{2\pi}\right]\right)\cos t\right). \tag{3.21}$$

The form of the perturbation of the last equation is chosen to be linear since the simulation of the solutions for the equation with advanced and retarded argument is difficult in nonlinear case.

We write the last equation in the system form

$$z'(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} z(t) + \begin{pmatrix} 0 \\ 3\sin^2 t \end{pmatrix} + \mu \begin{pmatrix} 0 \\ z_1(t) + z_2(2\pi[\frac{t+\pi}{2\pi}])\cos t \end{pmatrix}. \tag{3.22}$$

Let us slightly generalize it as the following system

$$z'(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} z(t) + \begin{pmatrix} 0 \\ 3\sin^2 t \end{pmatrix} + \mu \begin{pmatrix} a\,z_1(2\pi[\frac{t+\pi}{2\pi}])\sin t + b\,z_2(t) \\ c\,z_1(t) + d\,z_2(2\pi[\frac{t+\pi}{2\pi}])\cos t \end{pmatrix}, \tag{3.23}$$

where a, b, c *and* d *are real constants.*

One can see that Eq. (3.22) *is a particular case of* (3.23) *when* $a = 0$, $b = 0$, $c = 1$, *and* $d = 1$.

If $\mu = 0$, *Eq.* (3.23) *takes the form*

$$z'(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} z(t) + \begin{pmatrix} 0 \\ 3\sin^2 t \end{pmatrix}. \tag{3.24}$$

It is easy to find 2π*-periodic solutions* ψ_j, $j = 1, 2$, *as*

$$\psi_1 = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}, \qquad \psi_2 = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

of the adjoint system of the last equation. Then, condition (C10) *can be verified*

$$\begin{aligned} \int_0^{2\pi} K_1^T(s) f(s) ds &= \int_0^{2\pi} \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix} \begin{pmatrix} 0 \\ 3\sin^2 s \end{pmatrix} ds \\ &= 0. \end{aligned}$$

Hence, the family of 2π*-periodic solutions of* (3.24) *is given by*

$$z(t, \alpha) = \begin{pmatrix} \alpha_1 \cos t + \alpha_2 \sin t + \frac{3}{2} + \frac{\cos 2t}{2} \\ -\alpha_1 \sin t + \alpha_2 \cos t - \sin 2t \end{pmatrix}, \tag{3.25}$$

where $\alpha_1, \alpha_2 \in \mathbb{R}$ *are parameters.*

Next, let us show that Eq. (3.23) *has a* 2π*-periodic solution.*

The function $h(\alpha)$ in Theorem 3.4.2 can be evaluated as

$$\begin{aligned}
h(\alpha) &= \int_0^{2\pi} K_1^T(s) g(s,\, z(s,\,\alpha),\, z(\gamma(s),\,\alpha),\, 0) ds, \\
&= \int_0^{2\pi} \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix} \begin{pmatrix} a\, z_1(2\pi[\frac{s+\pi}{2\pi}],\,\alpha) \sin s + b\, z_2(s,\,\alpha) \\ c\, z_1(s,\,\alpha) + d\, z_2(2\pi[\frac{s+\pi}{2\pi}],\,\alpha) \cos s \end{pmatrix} ds \\
&= \int_0^{2\pi} \begin{pmatrix} -c\,\alpha_2 \sin^2 s + b\,\alpha_2 \cos^2 s \\ (a\,(\alpha_1+2) - b\,\alpha_1) \sin^2 s + (c\,\alpha_1 + d\alpha_2) \cos^2 s \end{pmatrix} ds \\
&= \begin{pmatrix} \pi(b-c)\alpha_2 \\ \pi((a-b+c)\alpha_1 + d\alpha_2 + 2a) \end{pmatrix}.
\end{aligned}$$

Suppose that $b \neq c$ and $a \neq b - c$. By straight forward calculation one can see that the zero of the equation $h(\alpha) = 0$ is $\alpha^0 = (\frac{-2a}{a-b+c},\, 0)$, and the determinant is

$$\begin{aligned}
\det\left(\frac{\partial h}{\partial \alpha}\bigg|_{\alpha=\alpha^0} \right) &= \det \begin{pmatrix} 0 & \pi(b-c) \\ \pi(a-b+c) & d \end{pmatrix} \\
&= -\pi^2(b-c)(a-b+c) \\
&\neq 0.
\end{aligned}$$

Hence, using Theorem 3.4.2, we can conclude that for sufficiently small $|\mu|$ equation (3.23) has a 2π-periodic solution and this solution tends to $z(t,\,\alpha^0)$ as $\mu \to 0$. Since we know that the initial value of the solution is close to the initial value of the periodic solution of equation (3.24), and there is continuous dependence on parameter μ, one can make the following simulations with identical initial data, $z(0) = (2,\,0)^T$. They can be seen from Fig. 3.1, where the solid lines are graphs of the periodic solution of equation (3.24), and graphs of two coordinates of the periodic solution of equation (3.23) are near the dashed lines.

Example 3.5.2 *Let us consider another example when the perturbation is nonlinear. In this case, we can not provide a numerical simulation, but we can show the existence of periodic solutions following the result of this chapter.*

Consider the equation

$$z'(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} z(t) + \begin{pmatrix} 0 \\ 3\sin^2 t \end{pmatrix} + \mu \begin{pmatrix} z_1(2\pi[\frac{t+\pi}{2\pi}])^2 \sin t + z_2(t) \\ 2\, z_1(t) + \, z_2(2\pi[\frac{t+\pi}{2\pi}])^2 \cos t \end{pmatrix}. \qquad (3.26)$$

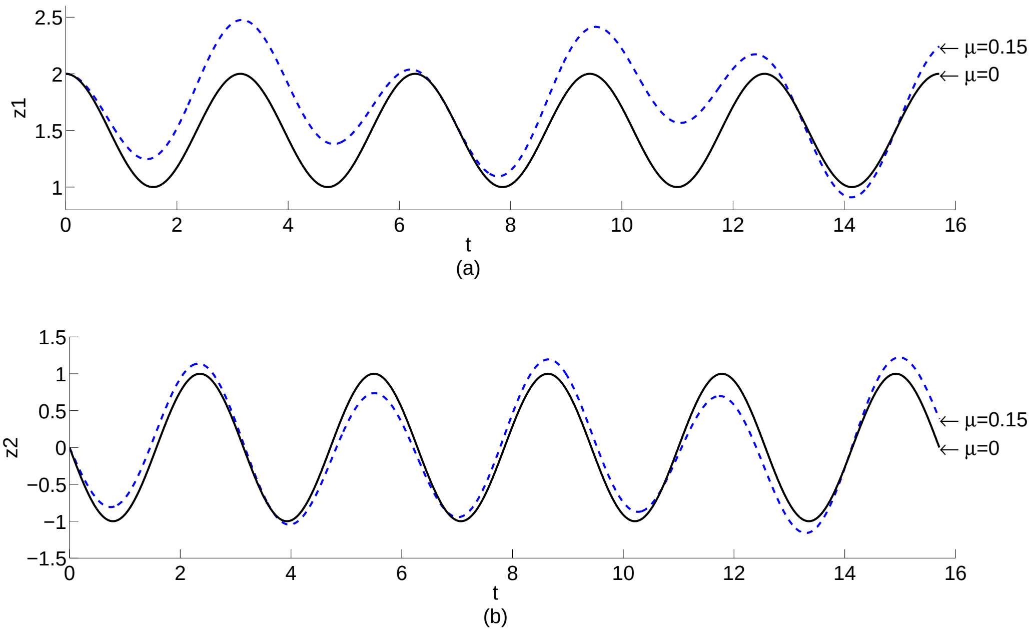

Figure 3.1: Simulation of the periodic solution of (3.24) (solid) and the solution of (3.23) (dashed) which is near the periodic solution of the perturbed system if $a = 0$, $b = 0$, $c = 1$, $d = 1$, with identical initial data, $z(0) = (2, 0)^T$. In (a) the first coordinates are shown, and second coordinates of the solutions are given in (b).

Similar to the previous example, one can see that conditions (C1) – (C10) hold. The function $h(\alpha)$ can be evaluated as

$$\begin{aligned} h(\alpha) &= \int_0^{2\pi} K_1^T(s) g(s, z(s,\alpha), z(\gamma(s),\alpha), 0)ds, \\ &= \int_0^{2\pi} \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix} \begin{pmatrix} z_1(2\pi[\frac{s+\pi}{2\pi}], \alpha)^2 \sin s + z_2(s,\alpha) \\ 2\, z_1(s,\alpha) + z_2(2\pi[\frac{s+\pi}{2\pi}], \alpha)^2 \cos s \end{pmatrix} ds \\ &= \int_0^{2\pi} \begin{pmatrix} -2\,\alpha_2 \sin^2 s + \alpha_2 \cos^2 s \\ ((\alpha_1+2)^2 - \alpha_1)\sin^2 s + (2\,\alpha_1 + \alpha_2^2)\cos^2 s \end{pmatrix} ds \\ &= \begin{pmatrix} -\pi\,\alpha_2 \\ \pi((\alpha_1+2)^2 + \alpha_1 + \alpha_2^2) \end{pmatrix}. \end{aligned}$$

Then, the zeros of the equation $h(\alpha) = 0$ are $\alpha^1 = (-1, 0)$ and $\alpha^2 = (-4, 0)$. By straightforward calculation one can see that the determinant

$$\det\left(\frac{\partial h}{\partial \alpha}\Big|_{\alpha=\alpha^i} \right) \neq 0, \; i = 1, 2.$$

Hence, using Theorem 3.4.2, we conclude that for sufficiently small $|\mu|$ Eq. (3.26) has two 2π-periodic solutions and these solutions tend to $z(t, \alpha^1)$ and $z(t, \alpha^2)$, respectively, as $\mu \to 0$.

CHAPTER 4

DIFFERENTIAL EQUATIONS WITH STATE-DEPENDENT PIECEWISE CONSTANT ARGUMENT

4.1 Introduction

In this chapter we generalize the mentioned equations in previous chapters to a new type of systems. They are differential equations with state-dependent piecewise constant argument (ESPA), where intervals of constancy of the independent argument are not prescribed and they depend on the present state of a motion. The method of analysis for equations was initiated in [7–11]. We are confident that introduction of these equations will provide new opportunities for the development of theory of differential equations and for applications [29, 32, 42, 54–57]. One must say that present results use the rich experience accumulated for dynamical systems with discontinuities [36, 37, 53, 60, 67, 74, 78] and strongly influenced by theoretical concepts developed for different type of equations with discontinuities [5, 6, 18, 19]. Since the systems are to be introduced have a complicated structure: they involve variable, state dependent delays, and discontinuities of the argument realized on certain surfaces, we call them *constancy switching surfaces*, this chapter consists of two main parts. In Section 4.2 we introduce the most general, for the present time, form of the equations. Basic properties of ordinary differential equations, constancy switching surfaces are defined, which give a start of investigation. One of them is called *extension property*. The definition of solutions is given. In the rest part of the manuscript we realize the general concepts for a particular type of equations, namely, quasilinear systems. Existence and uniqueness theorem, periodicity, and stability of the zero solution are discussed.

4.2 Generalities

Let $\mathcal{I} = (a, b) \subseteq \mathbb{R}$ and $\mathcal{A} = \{p, p+1, \ldots, r\} \subseteq \mathbb{Z}$ be nonempty intervals of real numbers, and integers, correspondingly. Let $\mathcal{G} \subseteq \mathbb{R}^n$ be an open connected region. Denote by $C(\mathcal{G}, \mathcal{I})$ and $C^1(\mathcal{G}, \mathcal{I})$ the set of all continuous and continuously differentiable functions from $\mathcal{G}$ to $\mathcal{I}$, respectively. Fix a sequence of real valued functions $\{\tau_i(x)\} \subset C(\mathcal{G}, \mathcal{I})$, where $i \in \mathcal{A}$.

We introduce the following assumption.

(A1) There exist two positive real numbers θ and $\bar{\theta}$ such that $\theta \leq \tau_{i+1}(x) - \tau_i(y) \leq \bar{\theta}$ for all $x, y \in \mathcal{G}$ and $i \in \mathcal{A}$.

Set the surfaces $S_i = \{(t, x) \in \mathcal{I} \times \mathcal{G} : t = \tau_i(x)\}$, $i \in \mathcal{A}$, in $\mathcal{I} \times \mathcal{G}$, and define the regions $D_i = \{(t, x) \in \mathcal{I} \times \mathcal{G} : \tau_i(x) \leq t < \tau_{i+1}(x)\}$, $i \in \mathcal{A}$, and $D_r = \{(t, x) \in \mathcal{I} \times \mathcal{G} : \tau_r(x) \leq t\}$ if $\max \mathcal{A} = r < \infty$. Because of (A1), one can see that D_i's, $i \in \mathcal{A}$ are nonempty disjoint sets.

We consider the equation

$$\frac{dx(t)}{dt} = f(t, x(t), x(\beta(t, x))), \tag{4.1}$$

where $t \in \mathcal{I}$, $x \in \mathcal{G}$, and $\beta(t, x)$ is a functional such that if $x(t) : \mathcal{I} \to \mathcal{G}$ is a continuous function, and $(t, x(t)) \in D_i$ for some $i \in \mathcal{A}$, then $\beta(t, x) = \eta_i$, where η_i satisfies the equation $\eta = \tau_i(x(\eta))$. From the description made for role of functions τ, it implies that one can call surfaces $t = \tau_i(x)$ as *constancy switching surfaces,* since the solution's piecewise constant argument changes its value at the moment of meeting one of the surfaces.

We call system (4.1) as *a system of differential equations with state-dependent piece-wise constant argument, ESPA.*

Let us define the following conditions, which are necessary to define a solution of Eq. (4.1) on $\mathcal{I}$.

(A2) For a given $(t_0, x_0) \in \mathcal{I} \times \mathcal{G}$, there is an integer $j \in \mathcal{A}$ such that $t_0 \geq \tau_j(x_0)$, and $j \geq k$ if $t_0 \geq \tau_k(x_0)$, $k \in \mathcal{A}$.

One can see that the functional $\beta(t, x) \leq t$ for all $t \in \mathcal{I}$, $x \in \mathcal{G}$. Indeed, to define system (4.1), the point (t, x) must be in D_j for some $j \in \mathcal{A}$.

Consider the ordinary differential equation

$$\frac{dy(t)}{dt} = f(t, y(t), z), \tag{4.2}$$

where z is a constant vector in $\mathcal{G}$.

We impose the following assumption.

(B0) For a given $(t_0, x_0) \in \mathcal{I} \times \mathcal{G}$, solution $y(t) = y(t, t_0, x_0)$ of Eq. (4.2) exists and is unique in any interval of existence, and it has an open maximal interval of existence such that any limit point of the set $(t, y(t))$, as t tends to the endpoints of the maximal interval of existence, is a boundary point of $\mathcal{I} \times \mathcal{G}$.

Let us remind that condition (B0) is valid, if, for example, the function f is continuous in t, and satisfies the local Lipschitz condition in y.

We shall need the following conditions:

(A3) for a given $(t_0, x_0) \in \mathcal{I} \times \mathcal{G}$ satisfying (A2), there exists a solution $y(t) = y(t, t_0, x_0)$ of Eq. (4.2) such that $\eta_j = \tau_j(y(\eta_j))$ for some $\eta_j \leq t_0$;

(A4) for each $z \in \mathcal{G}$ and $j \in \mathcal{A}$ solution $y(t, \tau_j(z), z)$ of Eq. (4.2) does not meet the surface S_j if $t > \tau_j(z)$.

(A5) for a given $(t_0, x_0) \in \mathcal{I} \times \mathcal{G}$ belonging to S_j, $j \in \mathcal{A}$, there exist a surface $S_{j-1} \subset \mathcal{I} \times \mathcal{G}$, a solution $y(t) = y(t, t_0, x_0)$ of Eq. (4.2) such that $\eta_{j-1} = \tau_{j-1}(y(\eta_{j-1}))$ for some $\eta_{j-1} < t_0$.

If a point $(t_0, x_0) \in \mathcal{I} \times \mathcal{G}$ satisfies (A2) and (A3), then we say that this point has *extension property.*

Fix $(t_0, x_0) \in \mathcal{I} \times \mathcal{G}$. Assume that it has extension property. We consider the problem of global existence of solution $x(t) = x(t, t_0, x_0)$ of (4.1).

Let us investigate the problem for increasing t. The point (t_0, x_0) is either in S_j, or there is a ball $B((t_0, x_0); \epsilon) \subset D_j$ for some real number $\epsilon > 0$, and $j \in \mathcal{A}$. The solution

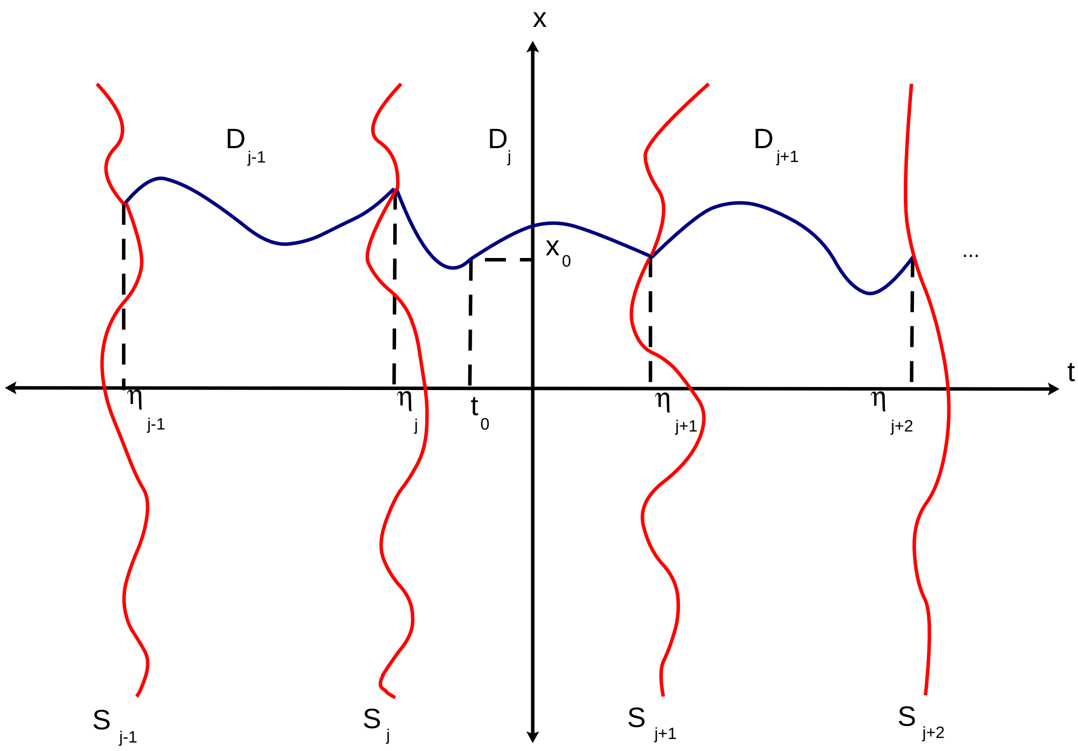

Figure 4.1: A solution of differential equations with state-dependent argument.

$x(t)$ is defined on an interval $[\eta_j, t_0]$, $\eta_j \le t_0$ by extension property, and satisfies the initial value problem (IVP)

$$y'(t) = f(t, y(t), y(\eta_i)),$$
$$y(\eta_i) = x(\eta_i), \tag{4.3}$$

such that $\eta_i = \tau_i(x(\eta_i))$ for $i = j$ (see Fig. 4.1). By using (A4) and (B0), there exists a solution $\psi(t) = \psi(t, \eta_j, x(\eta_j))$ of (4.3) defined on the right maximal interval of existence, $[t_0, \beta)$. If $\psi(t)$ does not intersect S_{j+1}, or the constancy switching surface S_{j+1} does not exist, then the right maximal interval of $x(t)$ is $[t_0, \beta)$, $\beta > t_0$. Otherwise, there is some $\xi \in \mathcal{I}$ such that $t_0 < \xi < \beta$, and $\xi = \tau_{j+1}(\psi(\xi))$. Then by denoting $\eta_{j+1} = \xi$, we define the solution $x(t)$ as $\psi(t)$ on $[t_0, \eta_{j+1}]$. Now, one can apply the above discussion for (t_0, x_0) to the point $(\eta_{j+1}, x(\eta_{j+1}))$.

Proceeding in this way, we shall come either to the case when for some $k \in \mathcal{A}$, $k > j$, solution $\psi(t) = \psi(t, \eta_k, x(\eta_k))$ has a right maximal interval $[\eta_k, \gamma)$ and this solution does not meet S_{k+1}, and then $[t_0, \gamma)$, $\gamma > \eta_k$, is the right maximal interval of existence of $x(t)$. If there is no such k, then either $x(t)$ is continuable to $+\infty$ if the set $\mathcal{A}$ is unbounded from above, or the solution achieves the point $(\eta_r, x(\eta_r))$, $\eta_r = \tau_r(x(\eta_r))$ and then $x(t)$ has the right maximal interval $[t_0, \kappa)$, $\kappa > \eta_r$ where $[\eta_r, \kappa)$ is the right maximal interval of solution $\psi(t)$ of Eq. (4.3) for $i = r$.

On the basis of the above discussion we can conclude that if extension property for (t_0, x_0) and conditions (A4) and (B0) are valid, then solution $x(t, t_0, x_0)$ of Eq. (4.1) has a right maximal interval of existence, and it is open from the right.

Now consider decreasing t. Assume, again, that (t_0, x_0) satisfies extension property. Let us consider first for $(t_0, x_0) \in S_j$. If condition (A5) is not valid, then the solution $x(t, t_0, x_0)$ does not exist for $t \leq t_0$. Otherwise, it is continuable to η_{j-1} such that $\eta_{j-1} = \tau_{j-1}(x(\eta_{j-1}))$, and satisfies Eq. (4.3) for $i = j - 1$. Then, again, as for $(\eta_j, x(\eta_j))$, we may make the same discussion for the point $(\eta_{j-1}, x(\eta_{j-1}))$. Finally, we may conclude that either there exists η_k, $k \leq j$ such that the left maximal interval of $x(t)$ is $[\eta_k, t_0]$ (It is true also if there exists $k = \min \mathcal{A}$), or the solution is continuable to $-\infty$. Let us now consider the case when (t_0, x_0) is an interior point of D_j, and satisfies extension property. Then, it is continuable to the left till S_j, and then, we can repeat the above made discussion. So, we can make a conclusion that the left maximal interval of existence of $x(t)$ is either a closed interval $[\eta_k, t_0]$, $k \in \mathcal{A}$, or an infinite interval $(-\infty, t_0]$. By combining the left and right maximal intervals, we define the solution $x(t)$ on the maximal interval of existence.

Now, we can introduce the definition of a solution of (4.1).

Definition 4.2.1 *A function $x(t)$ is said to be a solution of Eq.* (4.1) *on an interval $\mathcal{J} \subseteq \mathcal{I}$ if:*

(i) *it is continuous on $\mathcal{J}$,*

(ii) *the derivative $x'(t)$ exists at each point $t \in \mathcal{J}$ with the possible exception of the points η_i, $i \in \mathcal{A}$, for which the equation $\eta = \tau_i(x(\eta))$ is satisfied, where the one sided derivatives exist.*

(iii) *the function $x(t)$ satisfies Eq.* (4.1) *on each interval (η_i, η_{i+1}), $i \in \mathcal{A}$, and it holds for the right derivative of $x(t)$ at the points η_i.*

4.3 Quasilinear systems

In this section, we investigate the existence and uniqueness of solutions of quasilinear ESPA.

Let $\mathcal{I} = \mathbb{R}$, $\mathcal{G} = \mathbb{R}^n$ and $\mathcal{A} = \mathbb{Z}$. Fix a sequence of real numbers $\{\theta_i\} \subset \mathbb{R}$ such that $\theta_i < \theta_{i+1}$ for all $i \in \mathbb{Z}$. Take a sequence of functions $\xi_i(x) \in C(\mathbb{R}^n, \mathbb{R})$. Set $\tau_i(x) = \theta_i + \xi_i(x)$. Define the constancy switching surfaces $S_i = \{(t, x) \in \mathbb{R} \times \mathbb{R}^n : t = \theta_i + \xi_i(x)\}$, $i \in \mathbb{Z}$, and the regions $D_i = \{(t, x) \in \mathbb{R} \times \mathbb{R}^n : \theta_i + \xi_i(x) \le t < \theta_{i+1} + \xi_{i+1}(x)\}$, $i \in \mathbb{Z}$.

Let us now consider the following quasilinear differential equation

$$x'(t) = A(t)x(t) + F(t, x(t), x(\beta(t, x))), \tag{4.4}$$

where $t \in \mathbb{R}$, $x \in \mathbb{R}^n$, and $\beta(t, x)$ is a functional such that if $x(t) : \mathbb{R} \to \mathbb{R}^n$ is a continuous function, and $(t, x(t)) \in D_i$ for some $i \in \mathbb{Z}$, then $\beta(t, x) = \eta_i$, where η_i satisfies the equation $\eta = \theta_i + \xi_i(x(\eta))$.

Fix $H \in \mathbb{R}$, $H > 0$, and denote $K_H = \{x \in \mathbb{R}^n : \|x\| < H\}$. We introduce the following assumptions:

(Q1) there exist positive real numbers c, d such that $c \le \theta_{i+1} - \theta_i \le d$, $i \in \mathbb{Z}$;

(Q2) there exists $l \in \mathbb{R}$, $0 \le 2l < c$, such that $|\xi_i(x)| \le l$, $i \in \mathbb{Z}$, for all $x \in K_H$.

(Q3) the functions $A : \mathbb{R} \to \mathbb{R}^{n\times n}$ and $F : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous;

(Q4) there exists a Lipschitz constant $L_1 > 0$ such that

$$\|F(t, x_1, y_1) - F(t, x_2, y_2)\| \le L_1 [\|x_1 - x_2\| + \|y_1 - y_2\|]$$

for $t \in \mathbb{R}$ and $x_1, y_1, x_2, y_2 \in K_H$;

(Q5) $\sup_{t\in\mathbb{R}} \|A(t)\| = \kappa < \infty$;

(Q6) $\sup_{t\in\mathbb{R}} \|F(t, 0, 0)\| = N < \infty$;

(Q7) there exists a Lipschitz constant $L_2 > 0$ such that

$$|\xi_i(x) - \xi_i(y)| \le L_2\|x - y\|$$

for all $x, y \in K_H$ and $i \in \mathbb{Z}$.

One can see that conditions (Q1) and (Q2) imply (A1) with $\theta = c - 2l$ and $\overline{\theta} = d + 2l$. Also, Eq. (4.2) for system (4.4) has the form

$$y'(t) = A(t)y(t) + F(t, y(t), z), \tag{4.5}$$

where $z \in \mathbb{R}^n$ is a constant vector. Hence, under conditions (Q1)-(Q4), it is not difficult to see that (A2) and (B0) are valid for the last equation.

Let $X(t)$ be a fundamental matrix solution of the homogeneous system, corresponding to Eq. (4.5),

$$x'(t) = A(t)x(t), \tag{4.6}$$

such that $X(0) = I$, where I is an $n \times n$ identity matrix. Denote by $X(t, s) = X(t)X^{-1}(s)$, $t, s \in \mathbb{R}$, the transition matrix of (4.6). For the transition matrix $X(t, s)$, one can obtain the following inequalities:

$$m \leq X(t, s) \leq M, \tag{4.7}$$

$$\|X(t, s) - X(\overline{t}, s)\| \leq \kappa M|t - \overline{t}|, \tag{4.8}$$

where $m = \exp(-\kappa\overline{\theta})$ and $M = \exp(\kappa\overline{\theta})$ if $t, \overline{t}, s \in [\theta_j - l, \theta_{j+1} + l]$ for some $j \in \mathbb{Z}$.

Let us fix $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$. The following lemma is an auxiliary result of this chapter.

Lemma 4.3.1 *Suppose that* (Q1) – (Q3) *are fulfilled. Then,* $x(t)$ *is a solution of Eq.* (4.4) *with* $x(t_0) = x_0$ *for* $t \geq t_0$*, if and only if it satisfies the equation*

$$x(t) = X(t, t_0)x_0 + \int_{t_0}^{t} X(t, s)F(s, x(s), x(\beta(s, x)))ds. \tag{4.9}$$

Proof. *Necessity.* Assume that $x(t)$ is a solution of Eq. (4.4) such that $x(t_0) = x_0$, $(t_0, x_0) \in D_j$ for some $j \in \mathbb{Z}$. Denote

$$\phi(t) = X(t, t_0)x_0 + \int_{t_0}^{t} X(t, s)F(s, x(s), x(\beta(s, x)))ds. \tag{4.10}$$

Assume that $(t, x(t)) \in D_j \backslash S_j$. Then, there exists a moment $\eta_j \in \mathbb{R}$ such that $\beta(s, x) = \eta_j$ for all $(s, x(s)) \in D_j$. Also, we have

$$\phi'(t) = A(t)\phi(t) + F(t, x(t), x(\eta_j)),$$

and

$$x'(t) = A(t)x(t) + F(t,\ x(t),\ x(\eta_j)).$$

Hence,

$$[\phi(t) - x(t)]' = A(t)[\phi(t) - x(t)].$$

Calculating the limit values at $\eta_j,\ j \in \mathbb{Z}$, we can find that

$$\phi'(\eta_j \pm 0) = A(\eta_j \pm 0)\phi(\eta_j \pm 0) + F(\eta_j \pm 0,\ x(\eta_j \pm 0),\ x(\beta(\eta_j \pm 0,\ x(\eta_j \pm 0)))),$$
$$x'(\eta_j \pm 0) = A(\eta_j \pm 0)x(\eta_j \pm 0) + F(\eta_j \pm 0,\ x(\eta_j \pm 0),\ x(\beta(\eta_j \pm 0,\ x(\eta_j \pm 0)))).$$

Consequently,

$$[\phi(t) - x(t)]'|_{t=\eta_j+0} = [\phi(t) - x(t)]'|_{t=\eta_j-0}\,.$$

Thus, $[\phi(t)-x(t)]$ is a continuously differentiable function defined for $t \geq t_0$ satisfying (4.6) with the initial condition $\phi(t_0) - x(t_0) = 0$. Using uniqueness of solutions of Eq. (4.6) we conclude that $\phi(t) - x(t) \equiv 0$ for $t \geq t_0$.

Sufficiency. Suppose that $x(t)$ is a solution of (4.9) for $t \geq t_0$. Fix $j \in \mathbb{Z}$ and consider the region D_j. If $(t,\ x(t)) \in D_j \backslash S_j$, then by differentiating (4.9) one can see that $x(t)$ satisfies Eq. (4.4). Moreover, considering $(t,\ x(t)) \to S_j$, and taking into account that $x(\beta(t,\ x))$ is a right-continuous function, we find that $x(t)$ satisfies Eq. (4.4) in D_j. The lemma is proved. □

The following example shows that for even simple linear ESPA we have difficulties with uniqueness of solutions.

Example 4.3.1 *Consider the equation*

$$x'(t) = -2x(\beta(t,\ x)), \tag{4.11}$$

where $\beta(t,\ x)$ *is defined by using the sequences* $\theta_j = 2j$ *and* $\xi_j(x) = \cos x/4,\ j \in \mathbb{Z}$. *Fix* $(t_0,\ x_0) \in \mathbb{R} \times \mathbb{R}^n$, *that satisfies the equation* $t = (\cos x)/4$. *Then solution* $x(t)$ *of* (4.11) *with* $x(t_0) = x_0$, *is of the form* $x(t) = (1 - 2(t - \cos x_0/4))x_0$ *for* $t \in [t_0,\ 5/4)$. *Particularly, for* $(t_0,\ x_0) = (1/4,\ 0)$ *and* $(1/4,\ 2\pi)$, *the correspond solutions are* $x_1(t) = 0$ *and* $x_2(t) = \pi(3 - 4t)$, *each of which passes through the point* $(3/4,\ 0)$. *Hence, the uniqueness is not the case.*

Denote $\tilde{M} = 2L_1H + N$. From now on we need the following assumption:

(Q8) $2M\bar{\theta}L_1 < \min\{1 - 2(\kappa H + M\tilde{M})L_2,\ 1 - N\bar{\theta}M/H\}$.

Let $h \in \mathbb{R}$, $0 < h < \left(\frac{1-2ML_1\bar{\theta}}{M}H - N\bar{\theta}\right)$. The following lemma impose sufficient conditions for Eq. (4.4) to satisfy extension property.

Lemma 4.3.2 *Suppose that conditions* (Q1) – (Q8) *are fulfilled, and* $(t_0, x_0) \in D_j$ *for some* $j \in \mathbb{Z}$ *such that* $\|x_0\| < h$. *Then there exists a solution* $y(t) = y(t, t_0, x_0)$ *of Eq. (4.4) such that* $\eta_j = \theta_j + \xi_j(y(\eta_j))$ *for some* $\eta_j \le t_0$, *and* $y(t) \in K_H$ *for all* $t \in [\theta_j - l, \theta_{j+1} + l]$.

Proof. If $(t_0, x_0) \in S_j$, then by taking $\eta_j = t_0$ we can conclude the result directly. Suppose that $(t_0, x_0) \in D_j \backslash S_j$. Let us construct the following sequences. Take $\eta^0 = \theta_j$, $y_0(t) = X(t, t_0)x_0$, and define

$$\eta^{k+1} = \theta_j + \xi_j(y_k(\eta^k)), \tag{4.12}$$

$$y_{k+1}(t) = X(t, t_0)x_0 + \int_{t_0}^{t} X(t, s)F(s, y_k(s), y_k(\eta^k))ds \tag{4.13}$$

for all $k \in \mathbb{Z}$, $k \ge 0$.

Let $\|\,.\,\|_0 = \max_{t\in[\theta_j - l, \theta_{j+1}+l]} \|\,.\,\|$. It is straightforward to see that

$$\begin{aligned}\|y_{k+1}\|_0 &\le M\|x_0\| + \left\|\int_{t_0}^{t} \|X(t, s)\|\|F(s, y_k(s), y_k(\eta^k))\|ds\right\|_0 \\ &\le Mh + NM\bar{\theta} + 2ML_1\bar{\theta}\|y_k\|_0 \\ &\le \frac{1 - (2ML_1\bar{\theta})^{k+2}}{1 - 2ML_1\bar{\theta}}(Mh + NM\bar{\theta}).\end{aligned}$$

Using (Q8), we see $y_k(t) \in K_H$ for all $t \in [\theta_j - l, \theta_{j+1} + l]$, $k \in \mathbb{Z}$, $k \ge 0$.

Now, we will show that the sequence $\{y_k(t)\}$ is uniformly convergent. Eqs. (4.12) and (4.13) imply that

$$\begin{aligned}|\eta^{k+1} - \eta^k| &= |\xi_j(y_k(\eta^k)) - \xi_j(y_{k-1}(\eta^{k-1}))| \\ &\le L_2\|y_k(\eta^k) - y_{k-1}(\eta^{k-1})\|,\end{aligned}$$

$$\begin{aligned}\|y_{k+1} - y_k\|_0 &\le \max_{t\in[\theta_j - l, \theta_{j+1}+l]} \left|\int_{t_0}^{t} M\|F(s, y_k(s), y_k(\eta^k)) \right.\\ &\quad \left. -F(s, y_{k-1}(s), y_{k-1}(\eta^{k-1}))\|ds\right| \\ &\le ML_1\bar{\theta}\left[\|y_k - y_{k-1}\|_0 + \|y_k(\eta^k) - y_{k-1}(\eta^{k-1})\|\right],\end{aligned}$$

$$
\begin{aligned}
\|y_{k+1}(\eta^{k+1}) - y_k(\eta^k)\| &\leq \left\|X(\eta^{k+1},\, t_0) - X(\eta^k,\, t_0)\right\| \|x_0\| \\
&+ \left| \int_{\eta_k}^{\eta_{k+1}} \|X(\eta^{k+1},\, s)F(s,\, y_k(s),\, y_k(\eta^k))\| ds \right| \\
&+ \left| \int_{t_0}^{\eta_k} \|X(\eta^{k+1},\, s)F(s,\, y_k(s),\, y_k(\eta^k)) \right. \\
&\left. - X(\eta^k,\, s)F(s,\, y_{k-1}(s),\, y_{k-1}(\eta^{k-1}))\right\| ds \Big| \\
&\leq (\kappa h + \tilde{M}(1 + \kappa\bar{\theta}))M|\eta_{k+1} - \eta_k| \\
&+ ML_1\bar{\theta}[\|y_k - y_{k-1}\|_0 + \|y_k(\eta^k) - y_{k-1}(\eta^{k-1})\|] \\
&\leq M(L_2(\kappa h + \tilde{M}(1 + \kappa\bar{\theta})) \\
&+ L_1\bar{\theta})[\|y_k - y_{k-1}\|_0 + \|y_k(\eta^k) - y_{k-1}(\eta^{k-1})\|] \\
&\leq (L_2(\kappa H + M\tilde{M}) + ML_1\bar{\theta})[\|y_k - y_{k-1}\|_0 \\
&+ \|y_k(\eta^k) - y_{k-1}(\eta^{k-1})\|].
\end{aligned}
$$

Then,

$$
|\eta^{k+1} - \eta^k| \leq \left[2(L_2(\kappa H + M\tilde{M}) + ML_1\bar{\theta})\right]^{k-1} \bar{\theta}M\tilde{M}, \tag{4.14}
$$

$$
\|y_{k+1}(\eta^{k+1}) - y_k(\eta^k)\| \leq \left[2(L_2(\kappa H + M\tilde{M}) + ML_1\bar{\theta})\right]^{k} \bar{\theta}M\tilde{M}, \tag{4.15}
$$

$$
\|y_{k+1} - y_k\|_0 \leq \left[2(L_2(\kappa H + M\tilde{M}) + ML_1\bar{\theta})\right]^{k} \bar{\theta}M\tilde{M}. \tag{4.16}
$$

Thus, there exist a unique moment η_j, and a solution $y(t)$ of Eq. (4.4) with $y(t_0) = x_0$ such that $\eta_j = \theta_j + \xi_j(y(\eta_j))$, and η^k and y_k converge as $k \to \infty$, respectively. The lemma is proved. □

In what follows, we will consider the differential equations of type (4.4) such that the solutions intersect each constancy switching surface not more than once. In the previous section this assumption coincides with (A4). The following lemma defines the sufficient condition for this property.

From now on we shall need the following condition.

(Q9) $L_2\left[\kappa MH + M\tilde{M}\right] < 1.$

Lemma 4.3.3 *Suppose that* (Q1) – (Q7), (Q9) *hold. Then every solution* $x(t) \in K_H$ *of Eq.* (4.4) *meets any constancy switching surface not more than once.*

Proof. Suppose the contrary. Then, there exist a solution $x(t) \in K_H$ of (4.4) and a surface S_j, $j \in \mathbb{Z}$ such that $x(t)$ meets this surface more than once. Let the first intersection be at $t = t_0$ and another intersection at $t = t^*$ so that we have $t_0 = \theta_j + \xi_j(x(t_0))$ and $t^* = \theta_j + \xi_j(x(t^*))$ for $t_0 < t^*$. Then, we have

$$\begin{aligned} |t^* - t_0| &\leq L_2 \|X(t^*, t_0)x(t_0) + \int_{t_0}^{t^*} X(t, s)F(s, x(s), x(\beta(s, x)))ds - x(t_0)\| \\ &\leq L_2 \left[\kappa MH + M\tilde{M}\right] |t^* - t_0|, \end{aligned}$$

which contradicts (Q9). The lemma is proved. □

From the above lemmas we conclude the following theorem.

Theorem 4.3.1 *Assume that conditions* (Q1) – (Q9) *are fulfilled, and* $(t_0, x_0) \in D_j$ *for some* $j \in \mathbb{Z}$ *such that* $\|x_0\| < h$. *Then there exists a unique solution* $x(t) = x(t, t_0, x_0)$ *of Eq.* (4.4) *on* $[\eta_j, \eta_{j+1}]$ *such that* $\eta_j = \theta_j + \xi_j(x(\eta_j))$, $\eta_{j+1} = \theta_{j+1} + \xi_{j+1}(x(\eta_{j+1}))$, *and* $x(t) \in K_H$.

4.4 Periodic solutions

In this section, we investigate periodic solutions of quasilinear ESPA of type (4.4).

Let ω and p be fixed positive real number and integer, respectively. We shall introduce the following assumptions:

(Q10) the functions $A(t)$ and $F(t, x, y)$ are ω-periodic in t;

(Q11) the sequence $\theta_i + \xi_i(x)$ satisfies (ω, p)-periodicity, i.e. $\theta_{i+p} = \theta_i + \omega$ and $\xi_{i+p}(x) = \xi_i(x)$ for all $i \in \mathbb{Z}$ and $x \in \mathbb{R}^n$;

(Q12) $\det(I - X(\omega)) \neq 0$; that is, system (4.6) does not have any w-periodic solution.

We define, if (Q12) is fulfilled, the function

$$G(t, s) = \begin{cases} X(t)(I - X(\omega))^{-1}X^{-1}(s), & 0 \leq s \leq t \leq \omega, \\ X(t+\omega)(I - X(\omega))^{-1}X^{-1}(s), & 0 \leq t < s \leq \omega, \end{cases} \tag{4.17}$$

which is known as *Green's function* [46]. Let $\max_{t, s \in [0, \omega]} \|G(t, s)\| = K$.

We need the following lemma to prove the main theorem. This lemma can be proved using Lemma 4.3.1.

Lemma 4.4.1 *Suppose that* (Q1) – (Q12) *are fulfilled. Then the solution* $x(t)$ *of Eq.* (4.4) *is w-periodic if and only if it satisfies the integral equation*

$$x(t) = \int_0^{\omega} G(t,\ s)F(s,\ x(s),\ x(\beta(s,\ x)))ds. \tag{4.18}$$

Let $\|\,.\,\|_{\omega} = \max_{t\in[0,\,\omega]} \|\,.\,\|$. Denote by Φ the set of all continuous and piecewise continuously differentiable ω–periodic functions on $\mathbb{R}$ such that if $\phi \in \Phi$, then $\|\phi(t)\|_{\omega} < H$, and $\|\frac{d\phi(t)}{dt}\|_{\omega} < N + (2L_1 + \kappa)H$.

We introduce the following assumption to prove the next theorem.

(Q13) $(2KL_1\omega - 1)H + NK\omega < 0$;
$L_2(N + (2L_1 + \kappa)H) < 1$;
$KL_1(2 - L_2(N + (2L_1 + \kappa)H))\omega + 2KHL_1L_2p + L_2(N + (2L_1 + \kappa)H) < 1$.

Theorem 4.4.1 *Suppose that* (Q1) – (Q13) *hold. Then Eq.* (4.4) *has a unique* ω*-periodic solution* $\phi(t)$ *such that* $\phi(t) \in K_H$.

Proof. Suppose that for all $x \in K_H$ and $k = j, \ldots, j + p - 1$, for some $j \in \mathbb{Z}$ and $p > 1$, we have $0 \le \theta_k + \xi_k(x) \le \omega$. The other cases are similar. Define an operator T on Φ as

$$T[\phi] = \int_0^{\omega} G(t,\ s)F(s,\ \phi(s),\ \phi(\beta(s,\ \phi)))ds. \tag{4.19}$$

Using (Q13), it is easy to see that $\|T[\phi]\|_{\omega} < H$ and $\|\frac{dT[\phi]}{dt}\|_{\omega} < N + (2L_1 + \kappa)H$. That is, $T[\phi] \in \Phi$.

Now, we will show that the operator T is contractive on Φ. Let $\phi_1,\ \phi_2 \in \Phi$. One can see that using (Q13), the function $\phi_i(t)$ intersects any constancy switching surface S_k exactly once at $t = \eta_k^i$ for all $i = 1, 2$ and $k = j, \ldots, j + p - 1$. Without loss of generality suppose that $\eta_k^1 \le \eta_k^2$.

Also, one can show that using Mean Value Theorem and (Q13), the inequality

$$\|\phi_1(\eta_k^1) - \phi_2(\eta_k^2)\| \le \frac{1}{1 - L_2(N + (2L_1 + \kappa)H)}\|\phi_1 - \phi_2\|_{\omega} \tag{4.20}$$

is satisfied.

Using (4.19) and (Q11), we write

$$
\begin{aligned}
T[\phi_i(t)] &= \int_0^{\eta_j^i} G(t,\, s)F(s,\, \phi_i(s),\, \phi_i(\eta_{j+p-1}^i))ds \\
&\quad + \sum_{k=j}^{j+p-2} \int_{\eta_k^i}^{\eta_{k+1}^i} G(t,\, s)F(s,\, \phi_i(s),\, \phi_i(\eta_k^i))ds \\
&\quad + \int_{\eta_{j+p-1}^i}^{\omega} G(t,\, s)F(s,\, \phi_i(s),\, \phi_i(\eta_{j+p-1}^i))ds
\end{aligned}
$$

for $i = 1,\, 2.$

Then, using (4.20), we obtain

$$
\begin{aligned}
\|T[\phi_1] - T[\phi_2]\|_\omega \leq\ & K\Big[\int_0^{\eta_j^1} \|F(s,\, \phi_1(s),\, \phi_1(\eta_{j+p-1}^1)) - F(s,\, \phi_2(s),\, \phi_2(\eta_{j+p-1}^2))\|ds \\
&+ \sum_{k=j}^{j+p-2} \int_{\eta_k^2}^{\eta_{k+1}^1} \|F(s,\, \phi_1(s),\, \phi_1(\eta_k^1)) - F(s,\, \phi_2(s),\, \phi_2(\eta_k^2))\|ds \\
&+ \int_{\eta_{j+p-1}^2}^{\omega} \|F(s,\, \phi_1(s),\, \phi_1(\eta_{j+p-1}^1)) - F(s,\, \phi_2(s),\, \phi_2(\eta_{j+p-1}^2))\|ds \\
&+ \sum_{k=j}^{j+p-1} \int_{\eta_k^1}^{\eta_k^2} \|F(s,\, \phi_1(s),\, \phi_1(\beta(s,\, \phi_1))) \\
&-F(s,\, \phi_2(s),\, \phi_2(\beta(s,\, \phi_2)))\|ds\Big] \\
\leq\ & \left[\frac{KL_1\omega(2 - L_2(N + (2L_1 + \kappa)H)) + 2KHL_1L_2p}{1 - L_2(N + (2L_1 + \kappa)H)}\right]\|\phi_1 - \phi_2\|_\omega.
\end{aligned}
$$

Hence, T is contractive. Because of Lemma 4.4.1, we see that the fixed point is $\omega-$periodic solution of Eq. (4.4). The theorem is proved. □

Let us illustrate the last theorem by the following example.

Example 4.4.1 *Consider the equation*

$$
\begin{aligned}
x'(t) &= -x(t) - a\sin(2\pi t + y(\beta(t,\, x,\, y))) \\
y'(t) &= -2y(t) + a\sin(2\pi t + x(\beta(t,\, x,\, y))),
\end{aligned}
\tag{4.21}
$$

where $t,\, x,\, y \in \mathbb{R}$, and a is a positive real number. Here, $\beta(t,\, x,\, y)$ is defined by $\theta_j = j,\, \xi_j(x,\, y) = -a\cos(x + y)$. The corresponding parameters in conditions of Theorem 4.4.1 are $L_1 = a\sqrt{2},\, L_2 = a,\, \bar{\theta} = 1 + 2a,\, \kappa = 2,\, N = a\sqrt{2},\, M = e^{2+4a},\, \tilde{M} = (2H + 1)a\sqrt{2},\, \omega = 1,\, p = 1,\, K = e^2(1 - e^{-1})^{-1}$. One can show that conditions (Q1) − (Q13) *are satisfied for* $a = e^{-4}$, $H = 1$. *Hence, by Theorem* 4.4.1, *we ensure that there is an* $1-$*periodic asymptotically stable solution of* (4.21). *Figure* 4.2 *shows a solution* $(x(t), y(t))$ *of* (4.21) *with an initial condition* $(x(-e^{-4}), y(-e^{-4})) = (0.02, -0.02)$ *that approaches this periodic solution.*

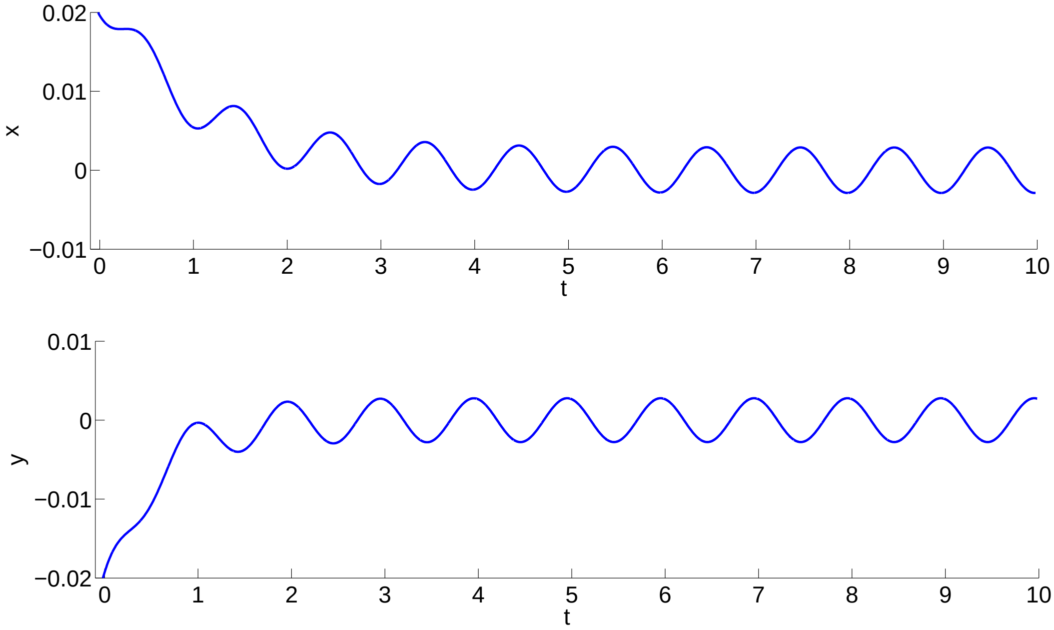

Figure 4.2: A solution $(x(t), y(t))$ of ESPA that approaches the $1-$periodic solution as time increases.

4.5 Stability of the zero solution

In this section we give sufficient conditions for stability of the zero solution.

Let us introduce the following conditions:

(Q14) $F(t, 0, 0) = 0$ for all $t \in \mathbb{R}$;

(Q15) $M\left[(1+\overline{\theta}L_1)(e^{ML_1\overline{\theta}}-1)+L_1\overline{\theta}\right] < 1.$

Define

$$K(L_1, \overline{\theta}) = \frac{M}{1 - M\left[(1+\overline{\theta}L_1)(e^{ML_1\overline{\theta}}-1)+L_1\overline{\theta}\right]}.$$

The following lemma plays a significant role in this chapter. Using the technique in [8] and similar to [13, Lemma 1.2], the following lemma can be proved.

Lemma 4.5.1 *Suppose that* (Q1) – (Q9), (Q14), (Q15) *are fulfilled. Then, every solution $x(t)$ of Eq.* (4.4) *satisfies the inequality*

$$\|x(\beta(t, x))\| \leq K(L_1, \bar{\theta})\|x(t)\| \tag{4.22}$$

for all $t \in \mathbb{R}$.

Proof. Fix $t \in \mathbb{R}$. Let $x(t)$ be a solution of (4.4). Then, there are $k \in \mathbb{Z}$, and $\eta_k \in \mathbb{R}$ such that $(t, x(t)) \in D_k$, and $\beta(t, x) = \eta_k$. Using Lemma 4.3.1, we have

$$x(t) = X(t, \eta_k)x(\eta_k) + \int_{\eta_k}^{t} X(t, s)F(s, x(s), x(\eta_k))ds.$$

Then,

$$\begin{aligned} \|x(t)\| &\leq M\|x(\eta_k)\| + ML_1 \int_{\eta_k}^{t} (\|x(s)\| + \|x(\eta_k)\|)\, ds \\ &\leq M(1 + \bar{\theta}L_1)\|x(\eta_k)\| + ML_1 \int_{\eta_k}^{t} \|x(s)\|ds. \end{aligned}$$

Hence, using Gronwall-Bellman Lemma, we obtain

$$\|x(t)\| \leq M(1 + \bar{\theta}L_1)e^{ML_1(t-\eta_k)}\|x(\eta_k)\|.$$

Moreover,

$$x(\eta_k) = X(\eta_k, t)x(t) - \int_{\eta_k}^{t} X(\eta_k, s)F(s, x(s), x(\eta_k))ds.$$

Then,

$$\begin{aligned} \|x(\eta_k)\| &\leq M\|x(t)\| + ML_1 \int_{\eta_k}^{t} (\|x(s)\| + \|x(\eta_k)\|)ds \\ &\leq M\|x(t)\| + M\left[(1 + \bar{\theta}L_1)(e^{ML_1\bar{\theta}} - 1) + L_1\bar{\theta}\right]\|x(\eta_k)\|. \end{aligned}$$

Thus, for $(t, x(t)) \in D_k$, we have $\|x(\eta_k)\| \leq K(L_1, \bar{\theta})\|x(t)\|$. The lemma is proved. □

Definition 4.5.1 *The zero solution of* (4.4) *is said to be uniformly stable if for any $\epsilon > 0$ and $t_0 \in \mathbb{R}$, there exists a $\delta = \delta(\epsilon) > 0$ such that $\|x(t, t_0, x_0)\| < \epsilon$ whenever $\|x_0\| < \delta$ for $t \geq t_0$.*

Definition 4.5.2 *The zero solution of* (4.4) *is said to be uniformly asymptotically stable if it is uniformly stable, and there is a* $b > 0$ *such that for every* $\zeta > 0$ *there exists* $T(\zeta) > 0$ *such that* $\|x_0\| < b$ *implies that* $\|x(t, t_0, x_0)\| < \zeta$ *if* $t > t_0 + T(\zeta)$.

Theorem 4.5.1 *Suppose that* (Q1) – (Q9), (Q14), (Q15) *hold. If the zero solution of Eq.* (4.6) *is uniformly asymptotically stable, then for sufficiently small Lipschitz constant* L_1*, the zero solution of Eq.* (4.4) *is uniformly asymptotically stable.*

Proof. Suppose that the zero solution of Eq. (4.6) is uniformly asymptotically stable. Then, there exist positive real numbers α and σ such that for $t > s$,

$$\|X(t, s)\| \leq \alpha e^{-\sigma(t-s)}. \tag{4.23}$$

Let $x(t)$ be a solution of (4.4) with the initial condition $x(t_0) = x_0$ such that $\|x_0\| \leq h$. We have for $t \geq t_0$,

$$\begin{aligned} \|x(t)\| &= \left\| X(t, t_0)x(t_0) + \int_{t_0}^{t} X(t, s)F(s, x(s), x(\beta(s, x)))ds \right\| \\ &\leq \alpha e^{-\sigma(t-t_0)}\|x_0\| + L_1 \int_{t_0}^{t} \alpha e^{-\sigma(t-s)}(1 + K(L_1, \overline{\theta}))\|x(s)\|ds. \end{aligned}$$

Then,

$$e^{-\sigma t}\|x(t)\| \leq \alpha e^{-\sigma t_0}\|x_0\| + \alpha L_1(1 + K(L_1, \overline{\theta})) \int_{t_0}^{t} e^{-\sigma s}\|x(s)\|ds.$$

Hence, using Gronwall-Bellman Lemma, we have

$$\|x(t)\| \leq \alpha e^{(\alpha L_1(1+K(L_1, \overline{\theta}))-\sigma)(t-t_0)}\|x_0\|.$$

Since for sufficiently small L_1, we have $\alpha L_1(1 + K(L_1, \overline{\theta})) - \sigma < 0$, the theorem is proved. □

The following example validates the last result.

Example 4.5.1 *Consider the equation*

$$\begin{aligned} x'(t) &= -x(t) - a\sin^2(y(\beta(t, x, y))) \\ y'(t) &= -2y(t) + a\sin^2(x(\beta(t, x, y))). \end{aligned} \tag{4.24}$$

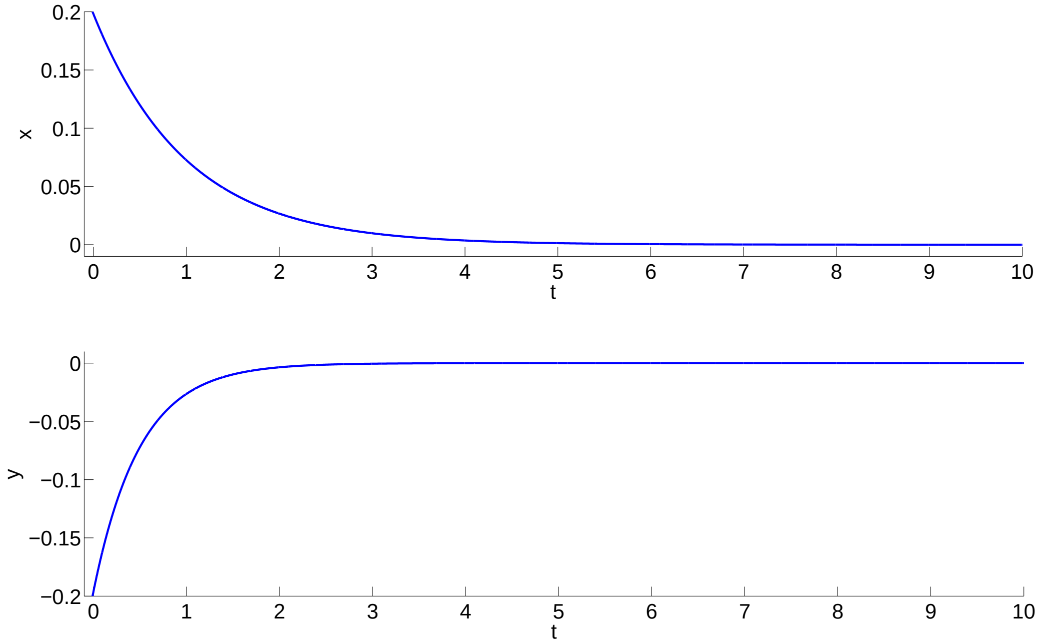

Figure 4.3: A solution $(x(t), y(t))$ of ESPA that approaches the zero solution as time increases.

where $t, x, y \in \mathbb{R}$, *and* a *is a positive real number. Here,* $\beta(t, x, y)$ *is defined by* $\theta_j = j$ *and* $\xi_j(x, y) = -a\cos(x + y)$. *The corresponding parameters in conditions of Theorem* 4.5.1 *are* $L_1 = 2\sqrt{2}a$, $L_2 = a$, $\overline{\theta} = 1 + 2a$, $\kappa = 2$, $N = 0$, $M = e^{2+4a}$, $\tilde{M} = 4\sqrt{2}aH$. *One can show that conditions* (Q1) – (Q9), (Q14), (Q15) *are satisfied for* $a = e^{-4}$, $H = 1$. *Hence, by Theorem* 4.5.1, *the zero solution is uniformly asymptotically stable. Figure* 4.3 *shows a solution* $(x(t), y(t))$ *of* (4.24) *with initial condition* $(x(-e^{-4}), y(-e^{-4})) = (0.02, -0.02)$ *that approaches the zero solution.*

CHAPTER 5

CONCLUSION AND FUTURE WORKS

This thesis studies periodic solutions and stability of differential equations with piecewise constant argument of generalized type. We divide these equations into three main parts: differential equations with piecewise constant argument of retarded, alternately retarded-advanced, and state-dependent piecewise constant argument of generalized type. The investigation is carried out by constructing equivalent integral equations rather than using the method of reduction to discrete equations, which was employed by the founders and developers of systems with piecewise constant arguments. The results are new and better than the existing ones.

Chapter 2 analyzes, using the method of small parameter, the periodic solutions and stability of quasilinear differential equations with retarded piecewise constant argument of generalized type in noncritical case, when the corresponding linear ordinary differential equations have no nontrivial periodic solution. The smoothness of the solutions with respect to initial value and parameter was presented as well.

In Chapter 3, conditions are found for the existence of periodic solutions for forced weakly nonlinear ordinary differential equations with alternately retarded-advanced piecewise constant argument of generalized type. The resonant case is studied, that is, when the unperturbed linear ordinary differential equation has a nontrivial periodic solution. The dependence of solutions on initial values and parameters is also studied. Examples with Duffing equations show the feasibility of our theory.

Chapter 4 generalizes the mentioned equations in the previous chapters to a new type of systems, differential equations with state-dependent piecewise constant argument, where intervals of constancy of the independent argument are not prescribed and they depend on the present state of a motion. The general concepts for a particular type of equations, namely, quasilinear systems: Existence and uniqueness theorem, periodicity, and stability of the zero solution, are discussed.

Some future works can be summarized as follows:

- The results obtained in Chapter 2 and 3 can be extended to the bifurcation theory [22, 61], in particular, when Eq. (2.41) does not satisfy the conditions of implicit function theorem. Moreover, averaging in resonant case [65] will be the next step of our results in Chapter 3.
- The stability of periodic solutions of systems with state-dependent piecewise constant argument is deserved to be analyzed as the neighborhood solutions have different meeting moments with the constancy switching surfaces defined in Chapter 4.
- It is interesting to study impulsive differential equations with piecewise constant arguments of generalized type. The method of construction of integral equations needs the knowledge of theory of both impulsive differential equations and differential equations with piecewise constant argument. One of the results that shows the existence of oscillatory and periodic solutions of a class of first order scalar impulsive delay differential equations with piecewise constant argument was reported in [49]. The investigation of such systems leads to the modeling some engineering problems, such as discharging capacitor, driven Froude pendulum, Work-piece cutter system, as it was done using systems with piecewise constant argument [29–32, 62]. Moreover, the problems of optimal control for the process with piecewise constant argument can be of great interest [55, 87, 88].

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