HOMOLOGY OF REAL ALGEBRAIC VARIETIES AND MORPHISMS TO SPHERES

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ABSTRACT

HOMOLOGY OF REAL ALGEBRAIC VARIETIES AND MORPHISMS TO SPHERES

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Let X and Y be affine nonsingular real algebraic varieties. One of the classical problems in real algebraic geometry is whether a given C^∞ mapping $f : X \to Y$ can be approximated by regular mappings in the space of C^∞ mappings. In this thesis, we obtain some sufficient conditions in the case when Y is the standard sphere S^n.

In the second part of the thesis, we study mainly the kernel of the induced map on homology $i_* : H_k(X, R) \to H_k(X_{\mathbb{C}}, R)$, where $i : X \to X_{\mathbb{C}}$ is a nonsingular projective complexification. First, using Lefshcetz Hyperplane Section Theorem we study $KH_k(X \cap H, R)$, where H is a hyperplane. In the remaining part, we relate $KH_k(X, R)$ to the realization of cohomology classes of $X_{\mathbb{C}}$ by harmonic forms.

Keywords: Real Algebraic Variety, Algebraic Homology, Complexification.

ÖZ

REEL CEBİRSEL VARYETELERİN HOMOLOJİLERİ VE KÜRELERE GÖNDERİMLER

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X ve Y birer affin tekil olmayan reel cebirsel varyete olsun. Reel cebirsel geometrinin klasik konularından biri, C^∞ gönderimlerin regüler gönderimlerle yaklaşılması problemidir. Bu tezde, verilen C^∞ bir $f : X \to Y$ gönderiminin regüler bir gönderime homotopik olması için Y'nin standart küre S^n olması durumunda, bazı yeter şartlar elde edilmiştir.

Tezin ikinci kısmında, verilen projektif tekil olmayan bir $i : X \to X_{\mathbb{C}}$ kompleksifikasyonu için homolojide elde edilen $i_* : H_k(X, R) \to H_k(X_{\mathbb{C}}, R)$ homomorfizmasının çekirdeği çalışılmıştır. İlk önce Lefshcetz Hiper Düzlem Kesiti Teoremi kullanılarak H bir hiper düzlem olmak üzere $KH_k(X \cap H, R)$ incelenmiştir. Tezin son bölümünde, $X_{\mathbb{C}}$' nin kohomoloji sınıflarının harmonik formlarla temsil edilebilinmesini $KH_k(X, R)$ grubuyla ilişkilendirdik.

Anahtar Kelimeler: Reel Cebirsel Varyete, Cebirsel Homoloji, Kompleksifikasyon.

To my family

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CHAPTER 1

INTRODUCTION

Throughout this thesis, X will denote an affine real algebraic variety, that is, isomorphic to an algebraic subset of $\mathbb{R}^n$.

If X is compact and nonsingular, by the classical Stone-Weierstrass approximation theorem, every C^∞ mapping $f : X \to \mathbb{R}^n$ can be approximated by polynomial maps. Indeed, every C^∞ map from X into Euclidean space can be approximated by regular maps. One of the main problems in real algebraic geometry is to extend this approximation theorem to different target spaces. We are interested in the following question: Given a compact nonsingular real algebraic variety X and C^∞ mapping $f : X \to S^n$, can f be approximated by regular maps?

This question has been studied for the cases $n = 1, 2$ and 4 in [4] and [6].

If X is compact and nonsingular, it is known that there exists a nonsingular projective complexification of X. We will denote such a complexification by $X_\mathbb{C}$.

The subgroup of the cohomology group $H^2(X, \mathbb{Z})$ that consists of the cohomology classes which are pullbacks (via the homomorphism induced by the inclusion mapping $X \hookrightarrow X_\mathbb{C}$) of the cohomology classes in $H^2(X_\mathbb{C}, \mathbb{Z})$ whose Poincaré duals are represented by complex algebraic hypersurfaces of $X_\mathbb{C}$ will be denoted by $H^2_{\mathbb{C}-alg}(X, \mathbb{Z})$. This subgroup is the main tool to answer the above question in some cases. We have the following sufficient condition for the smooth map $f : X^{2n} \to S^{2n}$: If there is a cohomology class $u \in H^2_{\mathbb{C}-alg}(X, \mathbb{Z})$ such that

$u^n = f^*(\alpha)$, where $\alpha \in H^{2n}(S^{2n}, \mathbb{Z})$ is the generator, then f is homotopic to a regular map. For the reverse direction we need more restrictions. This result and its related consequences will be given in Chapter 3.

We also work on the group $KH_k(X, R)$ which is the kernel of the homomorphism $i_* : H_k(X, R) \to H_k(X_{\mathbb{C}}, R)$, induced by the complexification. Dually, we denote the image of the homomorphism $i^* : H^*(X_{\mathbb{C}}, R) \to H^*(X, R)$ by $ImH^*(X, R)$. Both of these groups were studied before by Ozan in [15], [14] and [16]. In Chapter 4, we will give some applications of these groups. In the first section, we examine the group $KH_k(X \cap H, R)$ where H is a real hyperplane. In the second section, we give an obstruction in terms of $ImH^k(X, \mathbb{Z})$ and the Euler characteristic $\chi(X)$, to the harmonicity of products of harmonic forms representing cohomology classes on $X_{\mathbb{C}}$.

CHAPTER 2

PRELIMINARIES

In this chapter basic theory of real algebraic varieties will be reviewed. We will also introduce the cohomology rings H^*_{alg} and $H^{even}_{\mathbb{C}-alg}$. We refer to [1] and [3] for the standard terminology.

An affine real algebraic variety X is an irreducible Zariski closed subset of $\mathbb{R}^n$. Indeed, X is the common zero locus of finitely many irreducible polynomials taken from $\mathbb{R}[x_1, ..., x_n]$. Morphisms between two real algebraic varieties are regular maps. Similarly a projective real algebraic variety is a locally ringed space isomorphic to an algebraic subset of $\mathbb{RP}^n$ endowed with the Zariski topology and the sheaf of real valued regular functions.

Definition 2.1. A general real algebraic variety, not necessarily affine or projective, is a topological space X equipped with a sheaf of $\mathbb{R}$-valued regular functions such that there exist a finite open cover $(U_i)_{i\in I}$ of X with each $(U_i, \mathbb{R}_X \mid_{U_i})$ being an affine real algebraic variety and it is denoted by $\mathbb{R}_X$.

Definition 2.2. Let U be a Zariski open subset of V which is an algebraic subset of $\mathbb{R}^n$. A regular function on U is the quotient $f = g/h$, where g and h are in the ring of $\mathbb{R}$-valued regular functions on V, and $h^{-1}(0) \cap U = \emptyset$. If $(X, \mathbb{R}_X)$ and $(Y, \mathbb{R}_Y)$ are two real algebraic varieties, then the regular mapping from X to Y is a continuous map $\varphi : X \to Y$ such that $f \circ \varphi \mid_{\varphi^{-1}(U)}$ is in $\mathbb{R}_X(\varphi^{-1}(U))$ where U is opensubset of Y and $f \in \mathbb{R}_Y$.

Every Zariski locally closed subset of the real projective space $\mathbb{RP}^n$ can be viewed as an affine real algebraic variety (Proposition 2.4.1 of [1]). In particular, every projective real algebraic variety is an affine real algebraic variety. Moreover, compact affine real algebraic varieties are projective (Corollary 2.5.14 of [1]). We deal with compact and nonsingular real algebraic varieties. Therefore, we will not distinguish between real compact affine varieties and real projective varieties.

Definition 2.3. Let X be an affine real algebraic variety. A complexification of X is a pair (V, j), where V is a quasi-projective complex algebraic variety defined over $\mathbb{R}$ and $j : X \to V$ is an injective map such that $j(X) = V(\mathbb{R})$ is Zariski dense in V and j is viewed as a map from X into $V(\mathbb{R})$ which is an isomorphism of real algebraic varieties. Here, $V(\mathbb{R})$ denotes the set of real points of V.

Given an affine nonsingular real algebraic variety X, by the resolution of singularities theorem [10], we can always find a nonsingular projective complexification (V, j) of X. So, up to isomorphism, any compact affine nonsingular real algebraic variety can be viewed as a set of real points of a nonsingular projective complex algebraic variety defined over $\mathbb{R}$.

The theory of algebraic morphisms from real affine algebraic varieties into the standard sphere is closely related to the theory of algebraic vector bundles over such varieties. For this reason, we give basic definitions and some results about algebraic vector bundles.

Let F be $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ and X be an affine real algebraic variety.

Definition 2.4. An algebraic F-vector bundle ζ over X consists of a real algebraic variety $E(\zeta)$ and a regular map $\pi : E(\zeta) \to X$ such that for each point in X, the fiber has the structure an F-vector space. Moreover, it is required that for each x in X, there exists a Zariski open neighbourhood U of x, a nonnegative integer p and a commutative diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\phi} & U \times F^p \\ \pi \downarrow & & \downarrow pr \\ U & \xrightarrow{\sim} & U, \end{array}$$

where ϕ is a biregular map which is an F-linear isomorphism on every fiber.

An algebraic F-vector bundle ζ over an affine real algebraic variety is said to be strongly algebraic if ζ is isomorphic to an algebraic subbundle of a trivial bundle.

Example 2.1. For $p \leq n$ define

$$G_{n,p}(F) = \{A \in M(n,F) \mid A^2 = A = A^*,\ \mathrm{trace} A = p\}$$

as the Grassmannian of a p-dimensional F vector subspace of F^n. It is a nonsingular real algebraic variety. The natural F vector bundle over $G_{n,p}(F)$ is defined as

$$\gamma_{n,p}(F) = (E(\gamma_{n,p}(F), \pi, G_{n,p}(F)) \text{ over } G_{n,p}(F)$$

where

$$E(\gamma_{n,p}(F)) = \{(A,x) \in G_{n,p}(F) \times F^n \mid Ax = x\},\ \ \pi(A,x) = A$$

is strongly algebraic. Moreover, if $f : X \to G_{n,p}(F)$ is a regular map then the pullback vector bundle $f^*(\gamma_{n,p}(F))$ is strongly algebraic too.

The following result will be useful in the sequel.

Theorem 2.1. ([3]) *Let X be a compact affine nonsingular real algebraic variety. Given a continuous (respectively $\mathbb{C}^\infty$) mapping $f : X \to G_{n,p}(F)$, the following properties are equivalent:*

(i) The induced F- vector bundle $f^*(\gamma_{n,p}(F))$ is topologically isomorphic to an algebraic F- vector bundle.

(ii) The mapping f can be approximated in the $\mathcal{C}^0$ (respectively $\mathbb{C}^\infty$) topology by a regular mappings $X \to G_{n,p}(F)$.

(iii) The mapping f is homotopic to a regular mapping $X \to G_{n,p}(F)$.

The algebraic homology group $H_k^{alg}(X, R)$ ($R = \mathbb{Z}$ or $\mathbb{Z}_2$) is defined as the subgroup of $H_k(X, R)$ generated by a compact real algebraic subset of X. We set

$$H_*^{alg}(X, R) := \bigoplus_{k\geq 0} H_k^{alg}(X, R).$$

If X is a compact affine nonsingular real algebraic variety of dimension d, we define $H^k_{alg}(X, \mathbb{Z}_2)$ to be the Poincaré dual of the groups $H^{alg}_{d-k}(X, \mathbb{Z}_2)$. We define

$$H^k_{alg}(X, \mathbb{Z}_2) = D^{-1}(H^{alg}_{d-k}(X, \mathbb{Z}_2) \text{ and } H^*_{alg}(X, \mathbb{Z}_2) = \bigoplus_{k\geq 0} H^k_{alg}(X, \mathbb{Z}_2).$$

Real algebraic variety has a totally algebraic homology if $H_*^{alg}(X, \mathbb{Z}_2) = H_*(X, \mathbb{Z}_2)$. For example, homology of Grassmannian is totally algebraic. For the functorality we give the following theorem.

Theorem 2.2. ([3]) *For every regular map $f : X \to Y$ between compact affine real algebraic varieties,*

(i) $f^*(H^*_{alg}(Y, \mathbb{Z}_2)) \subseteq H^*_{alg}(Y, \mathbb{Z}_2)$, provided that X and Y are nonsingular;

(ii) $f_*(H_*^{alg}(Y, \mathbb{Z}_2)) \subseteq H_*^{alg}(Y, \mathbb{Z}_2)$.

For a compact nonsingular affine real algebraic variety X, $H^k_{\mathbb{C}-alg}(X, \mathbb{Z})$ is defined to be the subgroup of $H^k(X, \mathbb{Z})$ generated by the restriction of the classes $H^k_{alg}(X_{\mathbb{C}}, \mathbb{Z})$ by the nonsingular complexification $(X_{\mathbb{C}}, i)$ of X, that is,

$$H^*_{\mathbb{C}-alg}(X,\mathbb{Z}) = i^*(H^*_{alg}(X_{\mathbb{C}},\mathbb{Z})).$$

For more information please see [5]. By construction, the image of the complex class map is contained in the ring H^{even}, so $H^*_{\mathbb{C}-alg}(X,\mathbb{Z})$ will indeed be a subring of $H^{even}(X,\mathbb{Z})$. $H^{even}_{\mathbb{C}-alg}(X,\mathbb{Z})$ does not depend on the choice of nonsingular complexification of X. If $f : X \to Y$ is a regular mapping between nonsingular compact affine real algebraic varieties, then the induced homomorphism of the cohomology rings maps $H^*_{\mathbb{C}-alg}(Y,\mathbb{Z})$ into $H^*_{\mathbb{C}-alg}(X,\mathbb{Z})$.

In the next chapter we will need the following result characterizing algebraic line bundles in terms of their characteristic classes.

Theorem 2.3. *Let X be a nonsingular compact real algebraic variety.*

i) A smooth real line bundle $E \to X$ admits a strongly algebraic structure if and only if $w_1(E) \in H^1_{alg}(X,\mathbb{Z}_2)$.

ii) A smooth complex line bundle $E \to X$ admits a strongly algebraic structure if only if $c_1(E) \in H^2_{\mathbb{C}-alg}(X,\mathbb{Z})$.

We refer to Theorem 12.4.5 of [3] for the proof of (i) and to Remark 5.4. of [5] for the proof of (ii).

Remark 2.1. *In general, if $E \to X$ is a strongly algebraic real (complex) vector bundle then the total Stiefel-Whitney class (the total Chern class) is in $H^*_{alg}(X,\mathbb{Z}_2)$ (resp. in $H^*_{\mathbb{C}-alg}(X,\mathbb{Z})$).*

CHAPTER 3

REGULAR MAPPINGS FROM REAL ALGEBRAIC VARIETIES TO SPHERES

Given two nonsingular affine real algebraic varieties X and Y, we regard the set $R(X,Y)$ of all regular maps from X into Y as a subset of the space $C^\infty(X,Y)$ of all C^∞ maps from X into Y endowed with the C^∞-topology.

We will study the following question: When C^∞ maps between nonsingular affine real algebraic varieties can be approximated by regular maps? If X is compact and nonsingular, as indicated by the classical Stone-Weierstrass approximation theorem, every C^∞ mapping $f : X \to \mathbb{R}^n$ can be approximated by the polynomial maps in $C^\infty(X, \mathbb{R}^n)$. In particular, every C^∞ mapping from X into Euclidean space can be approximated by regular maps in the C^∞-topology. The general idea is to try to extend this result to different target spaces. The next natural case is to take the standard n-dimensional unit sphere

$$S^n = \{x_0, ..., x_n \in \mathbb{R}^{n+1} \mid x_0^2 + ... + x_n^2 = 1\},$$

as a target space. In this case, the approximation problem becomes very difficult. There are some positive results in this direction. First, Ivanov prove that smooth map $f : X \to S^1$ can be approximated by regular maps from X to S^1 if and only if $f^*(u)$ belongs to $H^1_{alg}(X, \mathbb{Z}_2)$ where u is a generator of $H^1_{alg}(S^1, \mathbb{Z}_2)$ [11].

After that Bochnak and Kucharz improve this result to S^2 and get some partial results for S^4 [6], [4]. There are also some negative results. Loday shows that any polynomial map from T^n to S^n is null homotopic [13]. Bochnak and Kucharz prove that any regular map from $X \times S^{2n-k}$ to S^{2n} is null homotopic, where k is dimension of X and $k < 2n$ [4]. Proving these results they use the algebraic K-theory. Ozan avoid this technique by using the group $KH(X, R)$ [14].

We examine this approximation problem for regular maps to spheres that factors through the real and the complex projective spaces.

First we have a purely topological result.

Lemma 3.1. *Let X^{2n} be a closed oriented manifold and $f : X \to S^{2n}$ be any smooth map. Then there is a smooth map $\tilde{f} : X \to \mathbb{CP}^n$ such that the diagram*

$$\begin{array}{ccc} & & \mathbb{CP}^n \\ & {\scriptstyle \tilde{f}}\nearrow & \downarrow \pi \\ X & \xrightarrow{f} & S^{2n} \end{array}$$

commutes up to homotopy if and only if there is a cohomology class $u \in H^2(X, \mathbb{Z})$ such that $u^n = f^*(\alpha)$, where $\alpha \in H^{2n}(S^{2n}, \mathbb{Z})$ is a generator.

Before the proof, we construct the map $\pi = \mathbb{CP}^n \to S^{2n}$ of degree one which is given by

$$\pi([z_0 : ... : z_n]) = \begin{cases} \|z\|^{-2}(2z_0 z_n, ..., 2z_{n-1}z_n, (\sum_{i=0}^{n-1} z_i^2) - z_n^2), & \text{for } z_n \neq 0 \\ (0, .., 0, 1), & \text{for } z_n = 0 \end{cases}$$

We obtain the map π by composing the inverse of the stereographic projection from the point $(0, .., 0, 1) \in S^{2n}$ with local chart on $\mathbb{CP}^n$.

Proof. (**lemma 3.1**) Assume that such an $\tilde{f}$ exists. Then,

$$\begin{aligned} f^*(\alpha) &= (\pi \circ \tilde{f})^*(\alpha) \\ &= (\tilde{f}^* \circ \pi^*)(\alpha) \\ &= \tilde{f}^*(a^n) \qquad (\pi \text{ is a degree one map}) \\ &= (\tilde{f}^*(a))^n \\ &= u^n, \end{aligned}$$

here $a \in H^2(\mathbb{CP}^n, \mathbb{Z})$ is a generator and $u = \tilde{f}^*(a)$. So, one side is proved.

Conversely, assume that there is a cohomology class $u \in H^2(X, \mathbb{Z})$ in the form of $u^n = f^*(\alpha)$. Let $\tilde{f} : X \to \mathbb{CP}^\infty = K(\mathbb{Z}, 2)$ be a map such that $\tilde{f}^*(a) = u$, where $a \in H^2(\mathbb{CP}^\infty, \mathbb{Z})$ is the generator. Since X is $2n$-dimensional, we can change $\tilde{f}$ by a homotopy so that $\tilde{f}(X) \subseteq \mathbb{CP}^n \subseteq \mathbb{CP}^\infty$, where $\mathbb{CP}^n$ is the $2n$-th skeleton of $\mathbb{CP}^\infty$. Now we assume that $\tilde{f} : X \to \mathbb{CP}^n$ is a map such that $\tilde{f}^*(a) = u$. Then, $(\tilde{f}^*(a))^n = u^n$, where $a^n \in H^{2n}(\mathbb{CP}^n, \mathbb{Z})$. Since $a^n = \pi^*(\alpha)$, we get

$$\begin{aligned} (\pi \circ \tilde{f})^*(\alpha) &= \tilde{f}^*(\pi^*(\alpha)) \\ &= u^n \\ &= f^*(\alpha). \end{aligned}$$

Thus, $\pi \circ \tilde{f}$ and f have the same degree so that $\pi \circ \tilde{f}$ and f are homotopic. □

Theorem 3.1. *Let X^{2n} be a nonsingular compact orientable real algebraic variety and $f : X^{2n} \to S^{2n}$ be a continuous map. If there is a cohomology class $u \in H^2_{\mathbb{C}-alg}(X, \mathbb{Z})$ such that $u^n = f^*(\alpha)$, where $\alpha \in H^{2n}(S^{2n}, \mathbb{Z})$ is a generator, then f is homotopic to a regular map.*

Proof. By Lemma 3.1, there is an $\tilde{f} : X \to \mathbb{CP}^n$ such that $\pi \circ \tilde{f}$ is homotopic to f. The pull back complex line bundle $\tilde{f}^*(\gamma_{n,1})$, where $(\gamma_{n,1}) \to \mathbb{CP}^n$ is the canonical line bundle over $\mathbb{CP}^n$, is strongly algebraic because its Chern class $c_1(\tilde{f}^*(\gamma_{n,1})) = u$ is in $H^2_{\mathbb{C}-alg}(X, \mathbb{Z})$ (by Theorem 2.3 (ii)). Now by Theorem 2.1 the map $\tilde{f}$ classifying the pull back bundle can be homotoped to a regular map.

□

Theorem 3.2. *Let $f : X^{2n} \to S^{2n}$ be a continuous map where X^{2n} is a non-singular compact orientable real algebraic variety. If there is a regular map $\tilde{f} : X \to \mathbb{CP}^n$ such that $\pi \circ \tilde{f}$ is homotopic to f, then there is a cohomology class $u \in H^2_{\mathbb{C}-alg}(X, \mathbb{Z})$ such that $u^n = f^*(\alpha)$, where α is generator of $H^{2n}(S^{2n}, \mathbb{Z})$.*

Proof. Since $\pi : \mathbb{CP}^n \to S^{2n}$ has degree one $\pi^*(\alpha) = a^n$, where $a \in H^2(\mathbb{CP}^n, \mathbb{Z})$ is a generator. However $H^2(\mathbb{CP}^n, \mathbb{Z}) = H^2_{\mathbb{C}-alg}(\mathbb{CP}^n, \mathbb{Z})$. Now, let $u = \tilde{f}^*(a)$. Then $u \in H^2_{\mathbb{C}-alg}(X, \mathbb{Z})$ because $\tilde{f}$ is a regular map. By assumption $\pi \circ \tilde{f}$ is homotopic to f we get

$$f^*(\alpha) = (\tilde{f})^* \pi^*(\alpha) = \tilde{f}^*(a^n) = (\tilde{f}^*(a))^* = u^n.$$

□

Example 3.1. Let M^{2n} be a closed orientable manifold and $u \in H^2(M^{2n} \sharp \mathbb{CP}^n, \mathbb{Z})$ be such that $u^n \in H^{2n}(M^{2n} \sharp \mathbb{CP}^n, \mathbb{Z})$ is a generator. Then by Theorem 1.2 of [7], $M^{2n} \sharp \mathbb{CP}^n$ has an algebraic model X such that $u \in H^2_{\mathbb{C}-alg}(X, \mathbb{Z})$.

In the remaining part of this chapter we give a similar results using the real projective space instead of the complex projective space. Let $\pi : \mathbb{RP}^n \to S^n$ be a regular map defined by

$$\pi([x_0 : ... : x_n]) = \|x\|^{-2}(2x_0 x_n, ..., 2x_{n-1} x_n, (\textstyle\sum_{i=0}^{n-1} x_i^2) - x_n^2).$$

Then the following diagram commutes:

$$\begin{array}{ccc} \mathbb{RP}^n & \xrightarrow{\pi} & S^n \\ \varphi \uparrow & & \uparrow i \\ \mathbb{R}^n & \xrightarrow{\phi^{-1}} & S^n - (N), \end{array}$$

where $N = (0, 0, ..., 1)$ is the north pole of S^n, ϕ is stereographic projection, φ is the embedding defined by $\varphi(x_1, ..., x_n) = (x_1 : ... : x_n : 1)$, and i is the inclusion map. We may consider π as an extension of ϕ^{-1} so that $\deg(\pi) = 1$ for n odd and modulo two reduction for n even.

Lemma 3.2. *Let M^n be a nonorientable manifold and $f : M^n \to S^n$ be a continuous map. Then there is a continuous map $\tilde{f} : M^n \to \mathbb{RP}^n$ such that the diagram*

$$\begin{array}{ccc} & & \mathbb{RP}^n \\ & {}^{\tilde{f}}\nearrow & \downarrow \pi \\ M^n & \xrightarrow{f} & S^n \end{array}$$

commutes up to homotopy if and only if there is a cohomology class $v \in H^1(M, \mathbb{Z}_2)$ such that $v^n = f^*(\alpha)$, where $\alpha \in H^n(S^n, \mathbb{Z}_2)$ is a generator.

Proof. Assume that such $\tilde{f}$ exists. Then,

$$\begin{aligned} f^*(\alpha) &= (\pi \circ \tilde{f})^*(\alpha) \\ &= (\tilde{f}^* \circ \pi^*)(\alpha) \\ &= \tilde{f}^*(a^n) \qquad (\pi \text{ is a degree one map}) \\ &= (\tilde{f}^*(a))^n \\ &= v^n, \end{aligned}$$

here $a \in H^1(\mathbb{RP}^n, \mathbb{Z}_2)$ is a generator and $v = \tilde{f}^*(a)$.

Conversely, assume that $v \in H^1(M^n, \mathbb{Z}_2)$ such that $v^n = f^*(\alpha)$. Let $\tilde{f} : M^n \to \mathbb{RP}^\infty = K(\mathbb{Z}_2, 1)$ be a map such that $\tilde{f}^*(a) = v$, where $a \in H^1(\mathbb{RP}^\infty, \mathbb{Z}_2)$ is the generator. Since M^n is n-dimensional, we can change $\tilde{f}$ by a homotopy so that $\tilde{f}(M^n) \subseteq \mathbb{RP}^n \subseteq \mathbb{RP}^\infty$, where $\mathbb{RP}^n$ is the n-th skeleton of $\mathbb{RP}^\infty$. Now we assume that $\tilde{f} : M^n \to \mathbb{RP}^n$ is a map such that $\tilde{f}^*(a) = v$, where $a \in H^1(\mathbb{RP}^n, \mathbb{Z}_2)$. Then, $(\tilde{f}^*(a))^n = v^n$. Since $a^n = \pi^*(\alpha)$, we get

$$\begin{aligned}(\pi \circ \tilde{f})^*(\alpha) &= \tilde{f}^*(\pi^*(\alpha)) \\ &= v^n \\ &= f^*(\alpha).\end{aligned}$$

Thus, we get that $\pi \circ \tilde{f}$ and f have the same $\mathbb{Z}_2$ degree and thus they are homotopic. □

Theorem 3.3. *Let X^n be a nonorientable, closed, nonsingular variety and $f : X \to S^n$ a continuous map. If there is some $v \in H^1_{alg}(X, \mathbb{Z}_2)$ such that $v^n = f^*(\alpha)$ and $\alpha \in H^n(S^n, \mathbb{Z}_2)$ is a generator, then f is homotopic to a regular map.*

Proof. By Lemma 3.2, there is an $\tilde{f} : X \to \mathbb{RP}^n$ such that $\pi \circ \tilde{f}$ is homotopic to f. The pull back real line bundle $\tilde{f}^*(\gamma_{n,1})$, where $(\gamma_{n,1}) \to \mathbb{RP}^n$ is the canonical line bundle over $\mathbb{RP}^n$, is strongly algebraic because its Stiefel-Whitney class $w_1(\tilde{f}^*(\gamma_{n,1})) = v$ is in $H^1_{alg}(X, \mathbb{Z}_2)$ (by Theorem 2.3 (i)). Now by Theorem 2.1 the map $\tilde{f}$ classifying the pull back bundle can be homotoped to a regular map. □

CHAPTER 4

HOMOLOGY OF REAL ALGEBRAIC VARIETIES

4.1 Homology of real hyperplane sections

Let R be any commutative ring with unity. For an R orientable nonsingular compact real algebraic variety X, define $KH_*(X, R)$ to be the kernel of the induced map

$$i_* : H_*(X, R) \to H_*(X_{\mathbb{C}}, R)$$

on homology, where $i : X \to X_{\mathbb{C}}$ is the inclusion map into some nonsingular projective complexification. In [15], it is shown that $KH_*(X, R)$ is independent of the complexification $X \subseteq X_{\mathbb{C}}$ and thus is an isomorphism invariant of X. Dually, denote the image of the homomorphism

$$i^* : H^*(X_{\mathbb{C}}, R) \to H^*(X, R)$$

by $ImH^*(X, R)$, which is also isomorphism invariant. For the properties of these groups, we recall the following theorem.

Theorem 4.1. ([15]) *Let X and Y be compact, R oriented, nonsingular real algebraic varieties with $dim(X) = n$, and k and l be nonnegative integers.*

1. If $f : X \to Y$ *is a regular map, then* $f_*(KH_k(X,R)) \subseteq KH_k(Y,R)$ *and* $f^*(ImH^k(Y,R)) \subseteq ImH^k(X,R)$.

2. If $V \subseteq \mathbb{CP}^N$ *is any compact nonsingular complex algebraic variety then* $KH_k(V_{\mathbb{R}}, R) = 0$.

3. $H^{2k}_{\mathbb{C}-alg}(X,R) \subseteq Im^{2k}(X,R)$.

4. Assume that R *is a field or* $R = \mathbb{Z}$*, and* X *has a complexification* $X_{\mathbb{C}}$ *so that* $H_*(X_{\mathbb{C}}, \mathbb{Z})$ *is torsion free. Then, for any* $\alpha \in H_k(X,R)$ *and* $\beta \in H_l(Y,R)$, $\alpha \times \beta \in KH_{k+l}(X \times Y, R)$ *if and only if* $\alpha \in KH_k(X,R)$ *or* $\beta \in KH_l(Y,R)$.

5. Assume that X *is connected and the Euler characteristic* $\chi(X)$ *of* X *in* R *coefficients is not zero. Then* $KH_n(X,R) = 0$.

6. Suppose X *has dimension* $n \geq 3$ *with a complete intersection complexification* $X_{\mathbb{C}}$*. Then,* $KH_k(X,\mathbb{Z}) = \bar{H}_k(X,\mathbb{Z})$ *for* $0 \leq k \leq n-2$.

As usual $\bar{H}(W,\mathbb{R})$ denotes the reduced homology for any space.

Example 4.1. (Example 2.4 of [15])

i) Let X be an algebraic model for the real projective space $\mathbb{RP}^{2n}$. The first Stiefel-Whitney class $w_1 \in H^1_{alg}(X,\mathbb{Z}_2)$ is nontrivial and hence by taking powers of w_1 we get $\mathbb{Z}_2 = H^k_{alg}(X,\mathbb{Z}_2) = H^k(X,\mathbb{Z}_2)$ for $k \leq 2n$. In [2] Akbulut and King showed that $H^k_{alg}(X,\mathbb{Z}_2)^2 \subseteq H^{2k}_{\mathbb{C}-alg}(X,\mathbb{Z}_2)$ for all k. By this result, $H^{2k}(X,\mathbb{Z}_2) = H^{2k}_{alg}(X,\mathbb{Z}_2) = H^k_{alg}(X,\mathbb{Z}_2)^2 \subseteq H^{2k}_{\mathbb{C}-alg}(X,\mathbb{Z}_2)$ and therefore $H^{2k}(X,\mathbb{Z}_2) = H^{2k}_{\mathbb{C}-alg}(X,\mathbb{Z}_2)$. By Theorem 4.1 (3), we conclude that $KH_{2k}(X,\mathbb{Z}_2) = 0$. Moreover, if $X = \mathbb{RP}^{2n}$, the standard real projective space, then $X_{\mathbb{C}} = \mathbb{CP}^{2n}$ and thus $KH_{2k+1}(X,\mathbb{Z}_2) = H_{2k+1}(X,\mathbb{Z}_2)$ for all k.

ii) Let X be an algebraic model for the smooth manifold $\mathbb{CP}^n$. We know that all Pontrjagin classes of X are nonzero and by the Corollary 2 in [2] they belong to $H^{4k}_{\mathbb{C}-alg}(X,\mathbb{Z})$. Therefore, by Theorem 4.1 (3) we see that $KH_{4k}(X,\mathbb{Z}) = 0$.

This result is the best possible one: Indeed, there exist an algebraic model of the complex projective plane $\mathbb{CP}^2$ such that $KH_2(X,\mathbb{Z}) \neq 0$ (Remark 1.6 in [16]).

iii) Let $X = T^n = S^1 \times \cdots \times S^1$, where S^1 is the standard unit circle in $\mathbb{R}^2$. Since S^1 bounds in its complexification $S^1_{\mathbb{C}} = S^2$, for all nonzero k we have $KH_k(X,\mathbb{Z}) = H_k(X,\mathbb{Z})$. Let Y be another algebraic model for T^n, where S^1 is replaced by $A = \{(x,y) \in \mathbb{R}^2 \mid x^4 + y^4 = 1\}$ which does not bound in its complexification. (Note that $A_{\mathbb{C}}$ is a nonsingular curve of degree 4 in $\mathbb{CP}^2$ and thus genus 3. If A is a dividing curve, then $A_{\mathbb{C}} - A$ has two connected components that are permuted under complex conjugation, which implies that the surface $A_{\mathbb{C}}$ is of even genus, a contradiction.) In this case, by Theorem 4.1 (4), we have $KH_k(Y,\mathbb{Z}) = 0$ for all k.

Theorem 4.2. ([9]) *Let M be an n-dimensional compact, complex manifold and $V \subset M$ a smooth hypersurface with $L = [V]$ positive line bundle. Then the map*

$$H^q(M,\mathbb{Q}) \to H^q(V,\mathbb{Q})$$

induced by the inclusion $i : V \hookrightarrow M$ is an isomorphism for $q \leq n-2$ and injective for $q = n-1$.

The classical Lefschetz Hyperplane Theorem 4.2 yields the following result for real hyperplanes.

Theorem 4.3. *Let $X \subseteq \mathbb{RP}^n$ be a nonsingular real algebraic variety of dimension n with a nonsingular complexification $X_{\mathbb{C}} \subseteq \mathbb{CP}^n$. Let $H \subseteq \mathbb{RP}^n$ be a hyperplane meeting X transversely at a nonempty set. Assume further that both X and $X \cap H$ are oriented and that $i \leq n-2$. Then,*

1. If $i_* : H_i(X,\mathbb{Q}) \to H_i(X_{\mathbb{C}},\mathbb{Q})$ or $j_* : H_i(X \cap H,\mathbb{Q}) \to H_i(X,\mathbb{Q})$ is trivial then $KH_i(X \cap H,\mathbb{Q}) = \bar{H}_i(X \cap H,\mathbb{Q})$;

2. If $H_i(X,\mathbb{Q})=0$, *then* $KH_i(X\cap H,\mathbb{Q})=\bar{H}_i(X\cap H,\mathbb{Q})$*;*

3. If $i_*: H_i(X,\mathbb{Q})\to H_i(X_{\mathbb{C}},\mathbb{Q})$ *is injective, then* $KH_i(X\cap H,\mathbb{Q})=ker(j_*): H_i(X\cap H,\mathbb{Q})\to H_i(X,\mathbb{Q}))$*;*

4. $KH_i(X\cap H,\mathbb{Q})=0$ *if and only if the composition* $i_*\circ j_*: H_i(X\cap H,\mathbb{Q})\to H_i(X_{\mathbb{C}},\mathbb{Q})$ *is injective.*

Proof. By the assumption, $H\subseteq\mathbb{RP}^n$ is a hyperplane meeting X transversely at a nonempty set. Hence, $\dim(X\cap H)=\dim(X)-1=n-1$. By the Lefschetz Hyperplane Section Theorem for generic H, the pair $X_{\mathbb{C}}\cap H_{\mathbb{C}}\subseteq X_{\mathbb{C}}$ is $(n-1)$-connected. In particular, for $i\leq n-2$, we have the following commutative diagram

$$\begin{array}{ccc} H_i(X\cap H,\mathbb{Q}) & \xrightarrow{j_*} & H_i(X,\mathbb{Q}) \\ i_{H*}\downarrow & & \downarrow i_* \\ H_i(X_{\mathbb{C}}\cap H_{\mathbb{C}},\mathbb{Q}) & \xrightarrow{j_{\mathbb{C}*}} & H_i(X_{\mathbb{C}},\mathbb{Q}), \end{array}$$

where $j_{\mathbb{C}*}$ is an isomorphism. This diagram gives the result. □

It is clear that the real hyperplane sections of the standard sphere S^n or of the projective space $\mathbb{RP}^n$ are again standard S^{n-1} and $\mathbb{RP}^{n-1}$. It is of interest whether the same holds for non standard spheres and projective spaces. Note that a smooth closed manifold has uncountably many nonisomorphic algebraic models [8]. The following examples give a partial answer to this question.

Example 4.2. i) Let X be an algebraic model of S^n. Then $X\cap H$ is an $(n-1)$-dimensional closed submanifold of S^n and hence of $\mathbb{R}^n$. In particular, it is orientable. Now by Theorem 4.3 (2) , for $i\leq n-2$, we have $KH_i(X\cap H,\mathbb{Q})=\bar{H}_i(X\cap H,\mathbb{Q})$, just as in the case of the standard sphere.

ii) If $X=X_1\times...\times X_n$, where each X_i is a dividing (respectively non-dividing) nonsingular real algebraic curve, then by part (iii) of Example 4.1 and by part (2) (respectively (3)) of the Theorem 4.3, for $i \leq n-2$, we have $KH_i(X \cap H, \mathbb{Q}) = \bar{H}_i(X \cap H, \mathbb{Q})$ (respectively $KH_i(X \cap H, \mathbb{Q}) = \ker(j_*)$, $i \leq n-2$).

iii) Let X be an algebraic model for the complex projective space $\mathbb{CP}^n$. Example 4.1 (ii) and part (3) of the Theorem 4.3 yield that, for $4i \leq 2n-2$, $KH_{4i}(X \cap H, \mathbb{Q}) = \ker(j_*)$,and for odd values of $i \leq 2n-2$ we have $KH_i(X \cap H, \mathbb{Q}) = \bar{H}_i(X \cap H, \mathbb{Q})$.

Moreover, if $X = \mathbb{CP}^n$ and is regarded as a real algebraic variety, then for $2i \leq 2n-2$ we have $KH_{2i}(X \cap H, \mathbb{Q}) = \ker(j_*)$.

4.2 Algebraic models of smooth manifolds and geometric formality

Let X be an n-dimensional nonsingular compact oriented real algebraic variety and $X_{\mathbb{C}}$ be a nonsingular projective complexification of X. Theorem 4.6 below gives an obstruction in terms of $ImH^k(X, \mathbb{Z})$ and the Euler characteristic $\chi(X)$, to the harmonicity of products of harmonic forms representing cohomology classes on $X_{\mathbb{C}}$.

Let M be an n-dimensional closed, oriented, Riemannian manifold. We let $E^p(M)$ be the vector space of all smooth p-forms on M. The metric and the orientation give a volume element $d\mu \in E^n(M)$. The Hodge star operator $*$ is defined by comparing the natural metric on the forms with the wedge product

$$\alpha \wedge *\beta = (\alpha, \beta) d\mu \quad \text{for all } \alpha, \beta \in E^i(M).$$

The $*$ operator interchanges the forms on complementary degrees. So, we have the linear operator $* : E^p(M) \to E^{n-p}(M)$ satisfying $** = (-1)^{p(n-p)}$. The formal adjoint δ of d is defined by

$$\int_M (\delta\alpha, \beta) d\mu = \int_M (\alpha, d\beta) d\mu$$

for all $\alpha \in E^p(M)$ and $\beta \in E^{p-1}(M)$. It is easy to see that δ is given by the formula $\delta = (-1)^{n(p+1)+1} * d*$ on $E^p(M)$.

The Laplace-Beltrami operator $\triangle$ on $E^p(M)$ is defined as $\triangle = \delta d + d\delta$. One can check easily that $\triangle$ is self adjoint and commutes with the star operator. Moreover $\triangle\alpha = 0$ if and only if $d\alpha = 0$ and $\delta\alpha = 0$.

Definition 4.1. Let $H^p := \{w \in E^p(M) : \triangle w = 0\}$. The elements of H^p are called harmonic p-forms.

There is a relation between harmonic forms and the de Rham cohomology class. For this, we give the following classical theorem.

Theorem 4.4. ([17]) *Each de Rham cohomolgy class on a compact oriented Rimennian manifold M contains a unique harmonic representative.*

Remark 4.1. *If $\mu \neq 0$ is a harmonic form then $[\mu] \neq 0$.*

A Riemannian manifold is called (metrically) formal if all wedge products of harmonic forms are harmonic. A closed manifold is geometrically formal if it admits a formal Riemannian metric. In 2001 D. Kotschick proved the following theorem.

Theorem 4.5. ([12]) *A closed, oriented manifold admits a nonformal Riemannian metric if and only if it is not a rational homology sphere.*

Next we state the result of this subsection.

Theorem 4.6. *Let X^n be a nonsingular compact oriented real algebraic variety. Let $i : X \to X_{\mathbb{C}}$ be a nonsingular projective complexification. Assume that $\chi(X) \neq 0$ and $i^*(a)$ is a nonzero class in $ImH^k(X, \mathbb{Z})$, where $a \in H^k(X_{\mathbb{C}}, \mathbb{Z})$. Let $x = PD([X]) \in H^n(X_{\mathbb{C}}, \mathbb{Z})$. Assume further that the product $u \wedge v$ is a nonzero harmonic form, where u and v are harmonic representative for $x \in H^n(X_{\mathbb{C}}, \mathbb{Z})$, and $a \in H^k(X_{\mathbb{C}}, \mathbb{Z})$,respectively. Then $i^*([\mu]) \neq 0$ in $H^{n-k}(X, \mathbb{Z})$ where $\mu = *(u \wedge v)$.*

Proof. The Poincaré dual of $i^*(a)$ is represented by $X^n \pitchfork L^{2n-k}$ where $L = PD(a) \in H_{2n-k}(X_{\mathbb{C}}, \mathbb{Z})$. But $X^n \pitchfork L^{2n-k}$ is also the $PD(x \cup a)$. By assumption, $u \wedge v$ is harmonic and therefore so is $\mu = *(u \wedge v)$, and

$$\int_{X_{\mathbb{C}}} \mu \wedge u \wedge v = \int_{X_{\mathbb{C}}} *(u \wedge v) \wedge (u \wedge v) = \|u \wedge v\| \neq 0.$$

So $[\mu \wedge u \wedge v]([X_{\mathbb{C}}]) = \mu(PD([u \wedge v]))$ is not zero. Finally, since $PD([u \wedge v])$ is represented by $X^n \pitchfork L^{2n-k}$, a class in $H_{n-k}(X, \mathbb{Z})$, we get $i^*([\mu]) \neq 0$. □

Example 4.3. Let Y be an algebraic model for $\mathbb{CP}^3$ with $ImH^2(Y, \mathbb{Z}) = 0$. We can construct such an algebraic model in the following way: Let $T^2 \subset \mathbb{CP}^3$ be a smoothly embedded submanifold realizing the homology class of $[\mathbb{CP}^1] \in H_2(\mathbb{CP}^3, \mathbb{Z})$. Such a T^2 can be obtained by attaching a one-handle to $\mathbb{CP}^1$ in a disc neighbourhood of a point $p \in \mathbb{CP}^1$ in $\mathbb{CP}^3$. Embed $\mathbb{CP}^3$ smoothly into some Euclidian space $\mathbb{R}^n$, so that the submanifold T^2 maps diffeomorphically onto $S^1 \times S^1 \subseteq \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^{n-4}$, where S^1 is the standard unit circle. Now recall the following theorem: Let $L \subseteq M \subseteq \mathbb{R}^k$, where L is a nonsingular real algebraic variety and M an embedded closed smooth manifold. Then there is a smooth embedding $g : M \to \mathbb{R}^k \times \mathbb{R}^l$ such that $X = g(M)$ is a nonsingular real algebraic variety with $g(x) = x$ for all $x \in L$ if and only if the normal bundle

$N_M(L)$ of L in M has a strongly algebraic structure. For the proof of this fact we refer to Theorem 2.8.4 of [1]. By Corollary 12.5.4 and Remark 12.6.8 of [3], if X is a nonsingular real algebraic variety of dimension less than or equal to 3 such that X has totally algebraic homology then any smooth vector bundle over X is strongly algebraic. In our case $L = T^2$ has totally algebraic homology. Hence its normal bundle in $\mathbb{CP}^3$ has strongly algebraic structure. Moreover, $ImH^2(Y, \mathbb{Z}) = 0$ because S^1 bounds in its complexification $S^1_{\mathbb{C}} = \mathbb{CP}^1 = S^2$ and we have the following commutative diagram:

$$\begin{array}{ccc} S^1 \times S^1 & \xrightarrow{e} & Y \\ i \downarrow & & \downarrow i \\ S^1_{\mathbb{C}} \times S^1_{\mathbb{C}} & \xrightarrow{e_{\mathbb{C}}} & Y_{\mathbb{C}}. \end{array}$$

It then follows that $i^* : H^2(Y_{\mathbb{C}}, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$ is the zero homomorphism. Thus we have found an algebraic model Y of $\mathbb{CP}^3$ such that $ImH^2(Y, \mathbb{Z}) = 0$. Since $\chi(Y) = 4 \neq 0$ and $p_1(Y) = 2c_2 - c_1^2 = -4 \neq 0$, we can find nonzero harmonic forms representing $x = PD([X])$ and that $a = p_1(X) \in ImH^4(Y, \mathbb{Z})$. Let u and v be such representatives, respectively. If the product of these harmonic forms were harmonic then $\mu = *(u \wedge v) \in H^2(Y_{\mathbb{C}}, \mathbb{Z})$ would satisfy $i^*([\mu]) \neq 0$ in $H^2(Y, \mathbb{Z})$, which is a contradiction.

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