

A STUDY OF THE DAY-AHEAD ENERGY MARKET AUCTIONS FROM A
MULTI-OBJECTIVE PERSPECTIVE

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MULTI-OBJECTIVE PERSPECTIVE**

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ABSTRACT

A STUDY OF THE DAY-AHEAD ENERGY MARKET AUCTIONS FROM A MULTI-OBJECTIVE PERSPECTIVE

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In this study, we develop approaches for the market clearing problem in European day-ahead electricity markets. We first present the surplus maximization problem and extend it with pricing constraints that limit market loss or missed surplus associated with paradoxically accepted and rejected bids. We develop a Benders decomposition algorithm with price-based Benders infeasibility cuts to solve the problem. Our algorithm outperforms the state-of-the-art Benders decomposition algorithms and the primal-dual approach on practical-sized market instances. Then, we develop a multi-objective formulation of the problem with market surplus, market loss and market missed surplus objectives where the first one is to be maximized, and the last two are to be minimized. We develop a cone-based search algorithm to solve three-objective mixed-integer linear programming problems where at least one objective takes discrete values, and apply the algorithm on the three-objective day-ahead electricity market clearing problem. We examine the characteristics of the nondominated set of the problem and derive insights for market operators related to the design of the market rules.

Keywords: Combinatorial auctions, day-ahead electricity market, electricity pricing, multi-objective mixed-integer linear programming, Benders decomposition

ÖZ

GÜN ÖNCESİ ENERJİ PİYASASI İHALELERİNİN ÇOK AMAÇLI BAKIŞ AÇISI İLE İNCELENMESİ

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Bu çalışmada, Avrupa Gün Öncesi Elektrik Piyasalarındaki piyasa takas problemi için yaklaşımlar geliştirdik. İlk olarak piyasa fazlası ençoklama problemini sunuyor ve problemi fiyatlandırma kısıtları altında inceliyoruz. Fiyatlandırma kısıtları, reddilmesi gerekirken kabul edilen veya kabul edilmesi gerekirken reddedilen teklifler sonucu ortaya çıkan piyasa zararı veya kaçırılan piyasa fazlasını sınırlandırmak amacı ile kullanılmaktadır. Bu problem için fiyat tabanlı olursuzluk kesileri ile bir Benders ayrıştırma algoritması geliştirdik. Geliştirdiğimiz algoritma, gerçek boyutlu problem örneklerinde mevcut Benders ayrıştırma algoritmalarından ve asal-eşiz problem yaklaşımından daha iyi performans göstermektedir. Sonrasında, piyasa fazlası ençoklama, piyasa zararı ve kaçırılan piyasa fazlası enazlama amaçları ile problemin çok amaçlı formülasyonunu sunuyoruz. En az bir amaç fonksiyonunun kesikli değerler aldığı üç amaçlı karışık tamsayı doğrusal programlama problemlerini için koni tabanlı bir arama algoritması geliştirdik, ve bu algoritmayı gün öncesi elektrik piyasası takas problemi üzerinde uyguladık. Baskın noktalar kümesini inceledik ve piyasa

iřletmecisi iin piyasa kurallarına ynelik ıkarımlar yaptık.

Anahtar Kelimeler: Kombinatoriyal ihaleler, gn ncesi elektrik piyasası, elektrik fiyatlandırması, ok amalı karıřık tamsayı dođrusal programlama, Benders ayrıřtırması

To my niece Ece and my nephew Emir...

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I started this study in 2017 when I was working for EPIAS as one of the developers of the Turkish day-ahead electricity market clearing software. Both EPIAS and my current employer, Unscrambl, Inc, supported my research in this area. I thank them for their support.

I hope that our research and findings help researchers or practitioners in the field to make things better.

TABLE OF CONTENTS

ABSTRACT	v
ÖZ	vii
ACKNOWLEDGMENTS	x
TABLE OF CONTENTS	xi
LIST OF TABLES	xv
LIST OF FIGURES	xvii
LIST OF ABBREVIATIONS	xx
CHAPTERS	
1 INTRODUCTION	1
2 BACKGROUND AND LITERATURE REVIEW	7
2.1 Day-ahead electricity markets	9
2.1.1 European day-ahead electricity markets	11
2.1.2 US day-ahead electricity markets	11
2.1.3 Turkish day-ahead electricity market	12
2.2 Day-ahead market basics	13
2.2.1 Players	13
2.2.2 Auction	14
2.2.3 Bidding	14

2.2.4	Clearing	16
2.2.5	Pricing	17
2.2.6	Market surplus	19
2.2.7	Equilibrium	19
2.2.8	Uplift	20
2.2.9	Settlement	21
2.2.10	Market efficiency	21
2.3	Literature review	22
2.3.1	Day-ahead electricity market clearing problem	22
2.3.2	Multi-objective mixed-integer linear programming	29
3	SOLVING THE EUROPEAN DAY-AHEAD ELECTRICITY MARKET CLEARING PROBLEM EFFICIENTLY WITH BENDERS DECOMPOSITION	33
3.1	Surplus maximization problem	35
3.1.1	Hourly bids	35
3.1.2	Block bids	37
3.1.3	Problem formulation	38
3.1.4	Linear relaxation	39
3.1.5	Integer case	41
3.2	Surplus maximization problem under pricing constraints	43
3.3	Benders decomposition	47
3.3.1	Solving (SMILP-NoPAB) problem	50
3.4	Computational Results	54
3.5	Surplus maximization problem with no PRB	60

3.6	More sophisticated bid types and network constrained markets	61
3.7	Conclusions	65
4	FINDING THE NONDOMINATED SET AND EFFICIENT INTEGER VEC- TORS FOR A CLASS OF THREE OBJECTIVE MIXED-INTEGER LIN- EAR PROGRAMS	67
4.1	Problem definition	69
4.2	Set dominance	76
4.2.1	Case 1. Dominance between two points	77
4.2.2	Case 2. Dominance of a continuous frontier compared to a point	78
4.2.3	Case 3. Dominance of a point compared to a continuous frontier	80
4.2.4	Case 4. Dominance between two continuous frontiers	81
4.3	A cone-based search algorithm	84
4.3.1	Search region	84
4.3.2	Search problem	87
4.3.3	Search region selection and update	88
4.3.4	Update of the nondominated set	93
4.3.5	The algorithm	94
4.3.6	Example 1.	97
4.3.7	Example 2.	99
4.4	Implementation	100
4.4.1	Solving super search problems	102
4.4.2	Search region coupling	104
4.4.3	Further search region elimination: bi-objective problem	105
4.5	Computational study	106

4.6	Approximating the nondominated frontier	114
4.7	Conclusions	116
5	MULTI-OBJECTIVE DAY-AHEAD ENERGY MARKET CLEARING PROBLEM	119
5.1	Literature Review	120
5.2	Surplus maximization problem and the market disequilibrium	120
5.3	European DAM clearing in practice	125
5.4	Multi-objective market clearing problem	130
5.4.1	Alternative surplus-maximizing market clearing prices	133
5.4.2	Characterizing the nondominated edges	137
5.4.2.1	Case 1. A single period with alternative surplus-maximizing market clearing prices	144
5.4.2.2	Case 2. Multiple periods with alternative surplus-maximizing market clearing prices	146
5.4.3	Generating the nondominated set	148
5.5	Computational study	149
5.6	Application of the findings in practice	158
5.7	Conclusions	160
6	CONCLUSIONS	163
	CURRICULUM VITAE	175

LIST OF TABLES

TABLES

Table 2.1	The bid types offered by different European power exchanges	17
Table 3.1	Performance results of the primal-dual and the Benders decomposition approaches for (SMILP-NoPAB)	57
Table 3.2	Performance results of the primal-dual and the Benders decomposition approaches for (SMILP-NoPAB) on large instance set ^a	60
Table 4.1	Results for CBSA on TOMILPs for 3O-C _m -I1	108
Table 4.2	Results for CBSA on TOMILPs for 3O-C _m -I1-Int	109
Table 4.3	Results for CBSA on TOMILPs for 3O-C _m -I2	110
Table 4.4	Results for CBSA on BOMILP instances in comparison to the state-of-the-art algorithms	111
Table 4.5	Results for CBSA on TOKP instances in comparison to KS	113
Table 4.6	Results for CBSA on modified TOKP instances with 10 items	114
Table 4.7	Results for ϵ CBSA on 30-C80-I1 instances	116
Table 5.1	Market surplus, loss, missed surplus and price gap values for the feasible block bid decisions in Example 3	125
Table 5.2	Results for the market disequilibrium with optimal (SMILP) solutions	126

Table 5.3 Results for market disequilibrium with (SMILP-NoPAB) and (SMILP-NoPRB) solutions in comparison to (SMILP) solution	129
Table 5.4 Results for the nondominated set of (MOMCP) with the original instances	151
Table 5.5 Characteristics of the nondominated points generated for the original instances	155
Table 5.6 Results for the nondominated set of (MOMCP) with block bids only	156

LIST OF FIGURES

FIGURES

Figure 2.1	Long term vs short-term electricity trade (Europex, 2017)	9
Figure 3.1	A step function for hourly supply bids	36
Figure 3.2	A step function for hourly demand bids	36
Figure 3.3	A piece-wise linear function with supply hourly bids	37
Figure 3.4	The structure of a block bid	37
Figure 4.1	The solutions for the slice problems in Case 1, $ I^{\neq} = 1$	74
Figure 4.2	The solutions for the slice problems in Case 2, $ I^{\neq} = 2$	74
Figure 4.3	The solutions for the slice problems in Case 3, $ I^{\neq} = 3$	75
Figure 4.4	Different cases displaying the dominance of an edge relative to a point	79
Figure 4.5	Different dominance relations between an edge and a continuous frontier	82
Figure 4.6	Illustration of a search region and its components	86
Figure 4.7	Illustration of the lower bound update procedure	91
Figure 4.8	Partitioning of \mathcal{R}^1 for different cases of $\mathcal{Z}_{ND}(\mathbf{y}^1, \mathcal{R}^1)$	93
Figure 4.9	Illustration of the search regions for Example 1	98
Figure 4.10	The feasible criterion space of the problem in Example 2	99

Figure 4.11	The evolution of the search space for the problem in Example 2	101
Figure 4.12	Illustration of super search region definition	103
Figure 4.13	Illustration of search region coupling	105
Figure 4.14	Comparison of CBSA and ODS under different parameter settings of the algorithms (Points dominate their north-west region.)	112
Figure 5.1	The aggregate market supply and demand functions constructed from the hourly bids given in Example 3	123
Figure 5.2	The aggregate market supply and demand functions for solution s_1 in Table 5.1	124
Figure 5.3	$s_h(\pi_t)$ for a demand hourly bid, $q_{h,t} > 0$	135
Figure 5.4	$s_h(\pi_t)$ for a supply hourly bid, $q_{h,t} < 0$	135
Figure 5.5	Illustration of $S_{H_t}(\bar{\mathbf{y}}, \pi_t)$ with unique optimal market clearing prices, $\pi_t^* = p_{h_3}$	136
Figure 5.6	Illustration of $S_{H_t}(\bar{\mathbf{y}}, \pi_t)$ when there are multiple optimal market clearing prices, $\pi_t^* = [p_{h_2}, p_{h_3}]$	137
Figure 5.7	Illustration of $z_2^s(\boldsymbol{\pi}^*, \bar{\mathbf{y}})$ with respect to π_t^*	138
Figure 5.8	Illustration of $z_2^d(\boldsymbol{\pi}^*, \bar{\mathbf{y}})$ with respect to π_t^*	139
Figure 5.9	Illustration of $z_3^s(\boldsymbol{\pi}^*, \bar{\mathbf{y}})$ with respect to π_t^*	140
Figure 5.10	Illustration of $z_3^d(\boldsymbol{\pi}^*, \bar{\mathbf{y}})$ with respect to π_t^*	140
Figure 5.11	Trade-offs between $z_2(\boldsymbol{\pi}^*, \bar{\mathbf{y}})$ and $z_3(\boldsymbol{\pi}^*, \bar{\mathbf{y}})$ as π_t^* changes	142
Figure 5.12	The nondominated set and the nondominated sets of slice problems for Example 3	143
Figure 5.13	Examples of nondominated sets generated by CBSA	154
Figure 5.14	Visualization of the nondominated set for Instance 7	157

Figure 5.15 Visualization of the nondominated set for Instance 8 158

LIST OF ABBREVIATIONS

BDSP	Bounded dual sub problem
BOMBLP	Bi-objective mixed-binary linear program
BOMCP	Bi-objective market clearing problem
CBSA	Cone-based search algorithm
CPU	Central processing unit
DAM	Day-ahead electricity market
DM	Decision maker
DSP	Dual sub problem
EMRA	Energy market regulatory authority
EXIST	Energy Exchange Istanbul
ILP	Integer linear program
ISO	Independent system operator
LMP	Locational marginal prices
LP	Linear program
MAR	Minimum acceptance ratio
MILP	Mixed-integer linear program
MIQCP	Mixed-integer quadratically constrained program
MIQP	Mixed-integer quadratic program
MO	Market operator
MOILP	Multi-objective integer linear program
MOIP	Multi-objective integer program
MOLP	Multi-objective linear program
MOMBLP	Multi-objective mixed-binary linear program
MOMCP	Multi-objective market clearing problem

MOMILP	Multi-objective mixed-integer linear program
MP	Master problem
NEMO	Nominated electricity market operator
ODS	One directional search algorithm
PAB	Paradoxically accepted bid
PD	Primal-dual
PRB	Paradoxically rejected bid
SDAC	Single day-ahead coupling
SMILP	Surplus maximizing integer linear program
SMLP	Surplus maximizing linear program
SP	Sub problem
SR	Search-and-remove algorithm
TEIAS	Turkish Electricity Transmission Company
TOKP	Three-objective knapsack problem
TOMILP	Three-objective mixed-integer linear program
TSA	Triangle-splitting algorithm

CHAPTER 1

INTRODUCTION

In this thesis, we study a practical problem in day-ahead electricity market (DAM) auctions. DAM auctions are held daily one day prior to the actual generation or consumption of electricity to determine the electricity trade between the buyers and the sellers of the market. Market players submit their bids for different hours of the auctioned day and the market operator (MO) determines the winning bids and the market clearing prices. The objective of MO is to maximize the total surplus resulting from the trade between the market players while balancing the supply and demand of electricity for each time period.

Different MOs may employ different pricing mechanisms to solve the market clearing problem. In European DAMs, the common approach is to find the surplus maximizing trade such that every bidder is exposed to the same market prices. The problem is a mixed-integer linear program (MILP) or mixed-integer quadratic problem (MIQP) depending on the type of bids allowed in the auction and contains partial market equilibrium constraints as well as supply-demand balance constraints. Solving the problem requires specially developed approaches to be able to generate optimal or at least high quality solutions under strict time limits that MOs have to follow. The occurrence of suboptimal solutions is not rare in practice. Although there is an ongoing progress in the literature towards more efficient solution methods, it seems that more has to be done to be able to solve optimally the real-life problems of growing size.

As we discuss the current studies on combinatorial auctions, and day-ahead electricity markets in particular in the next chapter, it seems that there is no single best pricing rule for combinatorial auctions. The nonconvexity of the winner determination prob-

lem prevents the auctioneer from having the best values at all desired outcomes. This has caused auctioneers in different parts of the world to trade-off between the desired auction outcomes in different ways.

Under the uniform pricing scheme employed by the European DAMs, the market clearing prices at the surplus maximizing solution may not guarantee the market equilibrium. Bidders may incur financial losses as a result of their accepted bids or they may be exposed to missed surpluses due to their rejected bids when those bids are favorable with respect to the market clearing prices. These side effects of surplus maximization needs to be included in the market clearing algorithm. The current market designs associate constraints that prevent the occurrence of exactly one of the side effects to guarantee feasibility, but also may lead to very poor solutions in terms of other criteria. Therefore, multi-objective modeling of this problem can reveal important insights about the characterization of the solutions that might be more appealing for the market designers and operators.

In this thesis, we contribute to the literature in the following areas:

- *We develop an efficient Benders decomposition algorithm to solve the surplus maximization problem under pricing constraints.* The existing methods to solve the surplus maximization problem in European markets construct pricing constraints that ensure none of out-of-the-money bids is accepted. That is, the model guarantees to yield market clearing prices under which each bidder does not have any financial loss. Differently, the model developed for the market clearing problem in Turkish market generates a solution such that there does not exist any rejected in-the-money bids, hence total missed surplus is guaranteed to be zero. In this thesis, we define more general pricing constraints such as setting upper bounds on the total loss or the total missed surplus associated with the auction outcome. We develop *price-based* Benders infeasibility cuts for the special cases. We show that the price-based cuts are stronger than the *no-good* cuts used in the literature and improve the performance of the Benders decomposition algorithm on the real-size problem instances substantially.
- *We develop a criterion-space search algorithm to generate nondominated sets of a class of three-objective MILPs.* The three-objective MILP formulation of

the day-ahead market clearing problem that we present in Chapter 5 has the property that the surplus maximization objective function takes discrete values only. Hence, nondominated sets of the problem can only include points or edges in parallel planes of feasible surplus values. Motivated by this problem and also the consideration of similarly structured problems in other domains, we develop a cone-based search algorithm (CBSA) to solve three-objective MILPs of this class by exploiting the special structure. We generate problem instances by following a common instance generation schema in the literature and show that CBSA is able to generate the nondominated set and the set of efficient integer variable vectors in finite number of steps. CBSA is one of the few studies that aim to generate nondominated sets of three-objective MILPs and is the first one that considers this special structure. We compare our algorithm to the state-of-the-art bi-objective MILP (BOMILP) algorithms as well as multi-objective integer programming (MOIP) algorithms and show that CBSA is competitive to those algorithms although it is designed to solve a special case of three-objective MILPs.

- *We formulate a multi-objective day-ahead electricity market clearing problem.* There exist several criteria in day-ahead energy market clearing problem in addition to market surplus such as market loss, market missed surplus and maximum difference in the prices that two different players are settled for the same unit energy. The smaller these amounts are the better for the market with respect to the competition in the market, the fairness among the players and the transparency of prices. The existing methods maximize total surplus by eliminating exactly one of the market loss or missed surplus and leaving another unrestricted. However, the generated solutions may perform poorly in unrestricted criteria. In this thesis, we formulate a three-objective mixed-integer linear program for the market clearing problem and analyze the nondominated set with market surplus, market loss and missed surplus criteria. We apply CBSA to generate the nondominated sets. We examine the degree of compromise from the market surplus to improve the market loss and missed surplus at different regions of the criterion space.

In Chapter 2, we provide the basics of the day-ahead electricity markets and examine

the characteristics of the common market designs. Then, we present a review of the literature in day-ahead electricity market pricing and multi-objective optimization techniques.

In Chapter 3, we develop a Benders decomposition algorithm to solve the market clearing problem in European day-ahead electricity markets (DAMs). The problem is a large scale mixed-integer linear or quadratic program (depending on the types of bids available in the market) and the problem needs to be solved in about 10 minutes in order to implement the results within the tight time frame the market is operating in.

Benders decomposition is the most studied solution approach for this problem in the literature (Martin et al., 2014; Madani and Van Vyve, 2014, 2015; Madani and Van Vyve, 2018; Euphemia, 2016). This is mainly because of the complexity of solving a compact formulation when there are hundreds or thousands of binary variables, complex bid types and the equilibrium constraints that have to be satisfied. Benders decomposition algorithm reduces the complexity by solving simpler models and introducing constraints (cuts) as necessary to enforce the feasibility of the original model. However, the performances of the Benders decomposition algorithms are not up to the task of solving the problem within the required time frame. This is mainly due to the use of “no-good” cuts (Martin et al., 2014) that cause weak relaxation bounds or locally-valid cuts that can only be used in the sub-trees (Madani and Van Vyve, 2015; Madani and Van Vyve, 2018).

We develop a Benders decomposition algorithm based on *price-based* cuts that we generate utilizing the market clearing prices associated with an integer solution. We prove that the price-based cuts are valid and stronger than the “no-good” cuts. We test the performance of our algorithm on practical-sized instances and show that our algorithm is superior to the existing Benders decomposition algorithms and the primal-dual approach. The improved performance implies substantial surplus increases in European DAMs with millions of Euros of daily trade and provides an efficient algorithm for MOs that operate under strict timelines. We also evaluate the performance of our algorithm using two leading commercial mixed-integer programming solvers, IBM ILOG Cplex and Gurobi. We show that our algorithm outperforms the compared

algorithms in both cases, and performs best when Gurobi is employed as the solver.

We develop CBSA in Chapter 4. CBSA picks one of the discrete valued objectives and generates nondominated points or edges in non-increasing order of their values in the selected objective. The search space is projected to the two-dimensional feasible criterion space and partitioned into cone-based convex search regions. CBSA searches these regions to find the next nondominated point and solves a *slice* problem to generate the nondominated edges if any. We show that CBSA terminated in finite number of iterations generating the nondominated set and all the efficient integer vectors.

We test the performance of CBSA on BOMILPs and three-objective binary knapsack problem instances as well as generated instances for TOMILPs. We show that CBSA is competitive both to the state-of-the algorithms for BOMILPs (Boland et al., 2015; Fattahi and Turkay, 2018; Soylu, 2018) and to the one of the best performing algorithms to solve MOIPs (Kirlik and Sayin, 2014). We also extend the algorithm as an approximation algorithm to generate a representative set with less computational effort.

In Chapter 5, we present a multi-objective formulation of the European DAM clearing problem and apply CBSA to solve the problem. We consider three objectives: surplus maximization, loss minimization and missed surplus minimization. We show that a feasible integer variable vector determines the market surplus, but alternative market clearing may exist and lead to trade-offs between market loss and market missed surplus. As a result, the nondominated set may include both points and edges in parallel planes. We generate the nondominated sets of 20 instances from Turkish DAM and examine the characteristics of the nondominated sets.

CHAPTER 2

BACKGROUND AND LITERATURE REVIEW

Day-ahead electricity markets have been at the core of the liberalized energy systems as the deregulation of the sector has spread all over the world in last 30 years. The aim of deregulation process has been claimed to build secure, reliable and low-cost energy systems. To achieve those targets, privatization, market-based mechanisms and competition was put forward as the key elements. The deregulation process has led to vertical unbundling of state-owned large utility companies into single or multiple companies at four main layers: generation, transmission, distribution and retail:

- *Generation* layer was opened to competition for private investment and state-owned assets were privatized. Investment decisions and the resulting technology mix was left to the hands of market-based incentives. A typical technology mix combined capital-intensive nuclear and coal power plants having low marginal costs, less capital-intensive natural gas power plants having higher marginal costs and renewable energy generation assets like hydro, solar and wind power plants.
- *Transmission* system operations were left to a single state-owned or regulated private company due to its natural-monopoly characteristic. The system operator has to provide non-discriminatory access of all parties to the grid. In the long term, it is supposed to make necessary transmission capacity expansion investments. In the short term, it has to ensure that real-time electricity demand can be satisfied in a reliable and efficient manner.
- *Distribution* systems exhibit local monopoly situation and operated by single regulated private or state-owned company. Distribution companies have to pro-

vide non-discriminatory access to end consumers and to small producers connected to the distribution grids. They have to make necessary investments in the region they operate. Distribution companies' main operations include measuring and recording of end-user electricity consumption, and transfer of the data to retailer companies. The regulatory authority determines a tariff for each distribution region to cover the costs incurred by the distribution companies plus a reasonable profit margin.

- *Retail* companies are the suppliers of end customers. If the retail layer is not opened to competition, then the price of the electricity consumed by the end-users is regulated and the regulatory authority determines the tariffs. Else, many retail companies can compete in the market and consumers are free to choose their suppliers.

The cost of consuming a unit energy in a particular geographical area includes electricity generation costs, transmission costs, distribution costs and retail operations cost. A regulatory authority issues all the necessary regulatory framework to be able to operate the whole system in an effective and efficient manner. It designs the market structure and organizes the market places. A market structure defines the type of agents, their functions and interactions with each other. Market places are where the sellers and buyers of electricity come together and trade with each other.

Day-ahead electricity markets are *organized* market places for the *wholesale* of electricity for the next day. They are *spot* markets in the sense that physical delivery and financial settlement is realized in a short period of time. Figure 2.1 exhibits the main features of long and short term energy trade. Since spot market prices are generally volatile, traders try to hedge their price risks by means of forward and future contracts and determine their positions in the long term. As the contract time approaches, they try to close their positions by participating to the short-term electricity markets like day-ahead, intra-day and real-time electricity markets.

Day-ahead markets have prevailed in spot wholesale markets. Since thermal generators have constituted a significant part of the generation mix in many markets and their operational constraints required advance planning, it has been necessary to give the commitment decisions for those plants at least one day before the delivery day.

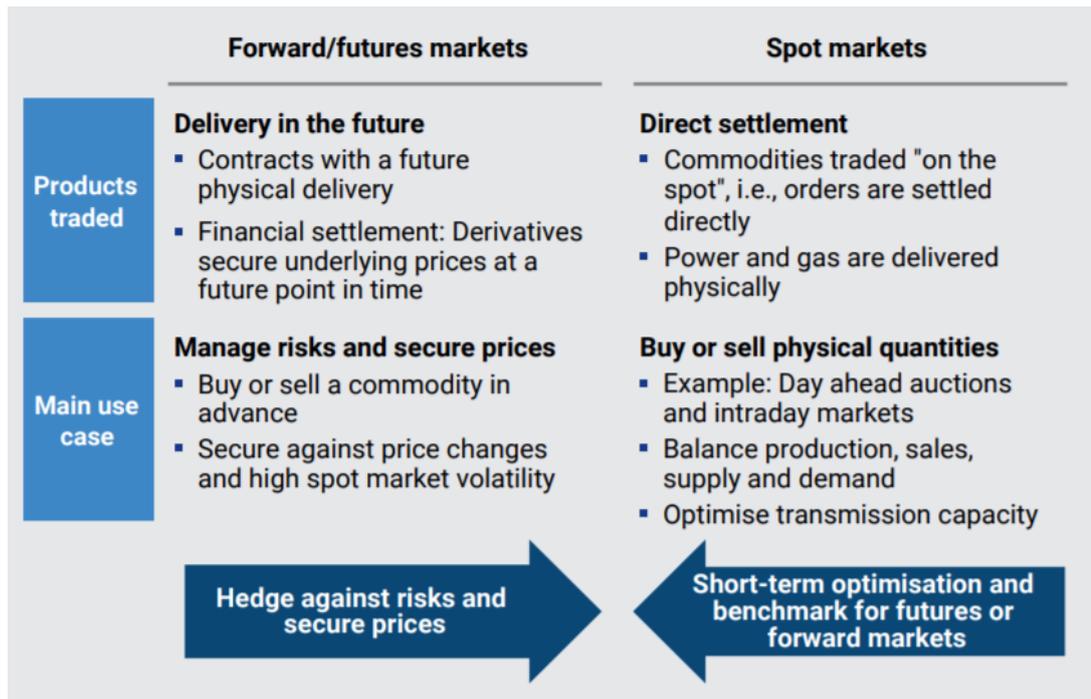


Figure 2.1: Long term vs short-term electricity trade (Europex, 2017)

Those plants could generate power in a steady and continuous manner. Hence, the main generation-consumption schedule is determined in day-ahead stage and minor deviations from the plan are handled in intra-day and real-time markets. Those deviations are regarded as minor contingencies and the wholesale price of electricity is based on the marginal cost resulted in the day-ahead markets. Although the growing share of renewable energy generation, the uncertainty and the intermittency associated with them can disrupt day-ahead markets in the long-term, their current role is still prominent.

2.1 Day-ahead electricity markets

Day-ahead electricity markets provide short-term physical trading possibilities for energy traders. Traders have the obligation to realize their traded volume in the real time. Otherwise, they incur penalty costs based on the real-time energy prices and their imbalances. It is the main function of day-ahead markets to balance the electricity supply and demand in the most efficient way at a short-time before the real-time

balancing. This requires an effective market design which mostly covers the auction design, bidding mechanisms and pricing rules.

There are two main day-ahead market designs across the world and we can categorize them as *pool-type* and *exchange-type*:

- *Pool-type*: Participation of the generation units into the market is compulsory. All generating units state their cost functions and operational constraints in a detailed way like minimum up/down times, minimum and maximum generation limits, start-up costs, no-load costs and variable costs. They can freely bid on these cost items and are expected to be truthful bidders under an effective auction design. On the other hand, demand function can be either elastic or inelastic depending on the market. Given the demand function, *independent system operator (ISO)* schedules the units in a way to minimize total electricity generation cost. In this type of markets, physical constraints on the electricity flow and voltage/frequency stability requirements of the transmission system are included in the problem in detail. This cost minimization problem is called *security-constrained unit commitment and economic dispatch problem* and US regional markets is a good example for this type of day-ahead markets.
- *Exchange-type*: Electricity is more regarded as a commodity than an entity with its specific physical properties. Market participation is not compulsory and the participants can pool their assets and form portfolios. They bid in the auction to buy or sell electricity not necessarily associated with a particular asset, but a portfolio. Consumption units can also participate into the market and bid their valuations of electricity consumption. Market participants can use multiple of different types of energy bids which are assumed to include fixed costs and operational constraints of the underlying assets. The *market operator* solves *market clearing problem* and generates surplus maximizing energy trade. It is then market participants' problem to schedule their assets to realize their day-ahead trade in the real-time. We see many examples of exchange-type markets in the European countries.

2.1.1 European day-ahead electricity markets

Norway and UK are the leading countries in Europe in deregulation of energy markets. Norway established its day-ahead electricity market in 1993, NordPool. From then, the design of the Norway day-ahead market has dominated the other European day-ahead markets.

The distinguishing feature of this market design is that the market participants self-schedule their power generation assets according to their trading volumes in the energy and capacity markets. Their bids in the day-ahead market are not necessarily associated with a particular unit among their power generation assets. By using the set of available bid types in the market, the bidders form a portfolio of bids. This is called *portfolio-based bidding*. In addition, not all the transmission network capacity constraints are included into the market clearing problem in the day-ahead market, except some critical transmission lines. The transmission system operator defines *bidding zones* which are separated by those critical transmission lines. The configuration of the bidding zones are to be determined such a way that the intra-zonal transmission capacity constraints are non-binding for any possible production-consumption schedule in the day-ahead stage. In case the inter-zonal transmission lines are congested, the day-ahead market clearing prices are differentiated between the bidding zones in addition to the market time unit.

2.1.2 US day-ahead electricity markets

A large part of the transmission system in the US is administered by the *regional transmission organizations (RTOs)*, as Pennsylvania-New Jersey-Maryland (PJM), New York ISO (NYISO), New England ISO (ISO-NE), Midcontinent ISO (MISO), Southwest Power Pool (SPP), Electricity Reliability Council of Texas (ERCOT), California ISO (CAISO). Independent system operators of RTOs provide grid services to the users in a non-discriminatory way and operate the energy pools to generate a least cost unit commitment and dispatch to satisfy the demand. The generating units bid their cost functions and operational constraints to the ISO and the ISO solves the security-constrained unit commitment and economic dispatch problem. The con-

straints imposed by the transmission system on the electricity flow and transmission losses are modelled in a granular way so that the cost of energy can be differentiated between the the small nodes of the grid like buses. The approach is called *nodal pricing* and the prices are referred as *locational marginal prices (LMPs)*.

2.1.3 Turkish day-ahead electricity market

The Turkish day-ahead electricity market is an exchange-type market being operated since 2011-2012. In 2001, the first electricity market law was issued and Turkish electricity sector has undergone the deregulation process onward. The law enacted the energy market regulatory authority (EMRA) to provide all the necessary regulatory frameworks throughout the deregulation process. The Turkish Electricity Transmission Company (TEIAS) was founded as the grid operator company in the role of *system operator* and was also put in charge of planning the day-ahead production/-consumption schedule. In 2015, Energy Exchange Istanbul (EXIST), a public sector undertaking, was established and given the *market operator* role. Since then, EXIST operates *day-ahead* and *intra-day* markets whereas TEIAS operates the *real-time* market.

EXIST has to complete a list of tasks everyday and follows the timeline below in doing so:

- **(D-5, 12:30)** Market participants submit their bids for the next day's auction. They can bid as early as 5 days before the delivery day, D. Bid submission window closes at 12:30 on D-1.
- **(12:30, 13:00)** Bid verification and collateral check. EXIST calculates the amount of financial guarantees to be given by each bidder to be allowed to trade in the market.
- **(13:00, 13:30)** Market clearing window. The market clearing algorithm finds the allocation and the market clearing prices. The results are announced at 13:30 as the preliminary market results.
- **(13:30, 13:50)** Collects objections to the preliminary market results.

- **(13:50, 14:00)** Finalized market results are announced.

Day-ahead auction closes at 14:00 on D-1 if every step can be successfully completed. Otherwise, EXIST has the right to postpone gate closure times. After the results are finalized, the market participants are supposed to schedule their assets and nominate their production/consumption schedules. EXIST aggregates those schedules and form the *finalized day-ahead program* which shows how much each unit connected to the grid will generate or consume at each hour of the following day. This program is continuously updated as trade occurs in the intra-day market. The system operator, TEIAS, uses this program to conduct the electricity flow analysis and may counter-trade in the real-time market to resolve real-time imbalances and grid capacity violations.

2.2 Day-ahead market basics

Day-ahead electricity markets provide short-term physical trading possibilities for energy traders. Traders have the obligation to realize their traded volume in the real time. Otherwise, they incur penalty costs based on the real-time energy prices and their imbalances. It is the main function of day-ahead markets to balance the electricity supply and demand in the most efficient way a short-time before the real-time balancing. This requires an effective market design which mostly covers the auction design, bidding mechanisms and pricing rules. Our study focuses on the pricing rules.

2.2.1 Players

The typical day-ahead market participants are producers, wholesalers, retailers and large-industrial consumers. They bid in the market to sell their excess energy to the market or buy energy from the market to meet the load of their customers. They are assumed to be rational players acting individually and seeking to maximize their profits resulting from their trading activities in the market.

2.2.2 Auction

The day-ahead market operator conducts a daily auction to determine the trading volumes to be delivered next day. The auction is done in a *single-round* and the bids are anonymous. The day-ahead auctions are called *double-sided* when the demand side is also allowed to participate in the market. Otherwise, the auction is called *one-sided* where only supply bids are allowed and the demand function is constructed by the system operator. The underlying product of the auction is one megawatt-hour of energy to be delivered or consumed for a specific time period, and in some markets, for a specific location. A unit energy in different time periods and at different locations are considered different items so that the auction is a *multi-item* auction. The time period is one hour in many markets and the definition of location can be as large as a country or as small as a network bus. There are also some day-ahead markets where reserve capacity products are auctioned simultaneously with energy products. In our study, we only consider the energy products.

2.2.3 Bidding

The players reveal their valuations of energy via their bids. A simple bid consists of unit energy price, the quantity and the time period the bid is offered. Each bid can be associated with a portfolio or generation unit that changes according to the underlying market design. In *portfolio-based* markets, players can submit multiple bids of different types which are not required to be related with a specific generation or consumption unit.

The technical complexities of power generation require the introduction of complex bid types in energy markets. The generators with different technologies and fuel types have different variable costs and operating principles. Thermal generators cannot start generation instantaneously or be stopped immediately. They require some time (hours) to be able to generate energy in a stable manner which may change based on their initial state (e.g. warm-start, cold-start). They have ramping constraints that allow only a limited amount of output change between sequential time periods. Those generators are associated with high start-up costs that makes short period of

commitment not economically feasible.

In U.S. markets like PJM (Pennsylvania-New Jersey-Maryland) day-ahead market, each unit present a detailed information about variable costs at different output levels, no-load costs (the generation cost at minimum output level), start-up costs, minimum run time, maximum number of daily starts and maximum/minimum economical output levels PJM (2017). The system operator takes into account all the cost parameters and operational constraints into account when solving the unit commitment and economical dispatch problem.

The European energy markets introduced *block bids* to handle the fixed-costs and operational constraints of thermal power plants in addition to the *hourly bids* mostly used by more flexible generation plants. An hourly bid can only be offered for a single period and consists of a set of price-quantity pairs which are used to form a stepwise function or a piecewise linear function in different markets. On the other hand, a block bid includes a single price-quantity pair but can include multiple periods. It can be either accepted at full quantity in every period offered or completely rejected. The bidders are supposed to internalize the fixed-costs and operational constraints associated with their plants to the price and quantity vectors. Binary nature of the block bids prevent bidders from volume and price risk that may occur with partial acceptance of the offered quantities.

In the past few years, new features are added to the block bids in order to better represent the bidders' valuations and give some flexibility to the market operator in the commitment decisions. In the European power exchanges like Nord Pool (operates the markets in Norway, Sweden, Finland, Estonia, Latvia, Lithuania, Denmark and UK) and EPEX SPOT (operates the markets in Germany, Austria, France, UK, The Netherlands, Belgium and Switzerland), block bids are allowed to include different quantities for each period, *profile block bids*. They can be associated with a minimum acceptance ratio (MAR) which indicates that if the auctioneer accepts that block bid, then it must accept at least the quantity corresponding to the offered quantity multiplied by this ratio. The bidders can include a set of block bids in their portfolio and present the condition that at most one of them can be accepted (*exclusive block bids*). In addition, there can also be a tree-structured block bid set such that a child block

bid can only be accepted if its parent block bid is accepted (*linked block bids*).

In Turkish day-ahead market and NordPool, there are also *flexible bids* for use of market participants. Flexible bids are similar to block bids with the additional information of time window that must be larger than the bid duration. The market operator can accept the flexible bid at any consecutive subset of periods in the time-window specified by the bidder.

In Iberian market (Spanish-Portuguese joint market) operated by OMIE (OMI-Polo Español S.A.), the players submit what is called *complex bids*. A complex bid is a series of several consecutive hourly bids, but includes many restrictions on the feasible dispatch quantities for the associated generating unit. One of the restrictions is the minimum income condition. The bidder specifies a monetary amount so that any dispatch to be determined by the market operator must bring the bidder a revenue greater than or equal to that. The Italian market operated by GME (Gestore dei Mercati Energetici) also has its own specific bid type called *PUN order*. A PUN order is a demand order and its distinguishing feature is that it must be evaluated based on the national single price (“Prezzo Unico Nazionale” in Italian) instead of the locational price of the zone it is offered for.

In Table 2.1, we give the list of bid types offered by different power exchanges in Europe, NordPool (2017), EPEXSpot (2017), OMIE (2017), GME (2017), EXIST (2017). In addition to the different bid types, power exchanges usually differ on the restrictions of the bid parameters and the size of the portfolios. For example, the maximum and minimum bid price limits, maximum allowable bid quantity and the maximum number of bids per portfolio may change across the European power exchanges.

2.2.4 Clearing

The day-ahead market trade must balance supply and demand bids so that if the forecasts are perfect and no contingencies occur, the real time system security can be attained. The following definition is taken from Martin et al. (2014).

Definition 1. “The clearing condition of a commodity c is an equation that ensures

Table 2.1: The bid types offered by different European power exchanges

Bid Types	EXIST	NordPool	EPEXSpot	OMIE	GME
Hourly bid	✓	✓	✓	✓	✓
Block bid	✓	✓	✓		
Profile block bid	✓	✓	✓		
Block bid with MAR		✓			
Linked block bids	✓	✓	✓		
Exclusive block bids		✓	✓		
Flexible bid	✓	✓			
Complex bid				✓	
PUN order					✓

that the number of bought units of commodity c is equal to the number of sold units of commodity c ".

If the time unit of the market is one hour, than the market clearing problem contains 24 clearing conditions. In some European exchanges (NordPool and GME), supply-demand balance is required for smaller geographical zones inside a country due to the transmission capacity bottlenecks. In this case, clearing condition also includes the energy inflow to the zone and outflow from the zone. The number of clearing conditions becomes as many as the number of zones (generally a few) multiplied by the number of periods. In U.S. markets, the transmission network is fully integrated to the day-ahead market clearing problem so that the problem includes the clearing conditions for each network bus, which can be in thousands.

2.2.5 Pricing

The auctioneer has to announce a price vector per unit of energy in order to remunerate accepted supply bids and charge the accepted demand bids. The European target model for the union of energy markets favors the use of uniform prices. The Capacity Allocation and Congestion Management Network Code (EU Commission Regula-

tion, 2015) states in Article 38 that “...uses the marginal pricing principle according to which all accepted bids will have the same price per bidding zone per market time unit.” In uniform pricing, the bidders pay or get paid an amount equal to the unit item price multiplied by the traded quantity of the item. This means that the trade volume of a bidder is a linear function of unit energy prices. In Martin et al. (2014), linear and strict linear pricing is differentiated as follows:

Definition 2. Linear (uniform or non-discriminatory) pricing “Consider m distinct commodities. A pricing schedule $I : \mathbb{R}^m \rightarrow \mathbb{R}$ is a map that returns the total amount of money to be paid by a consumer depending on his consumption vector $\mathbf{q} \in \mathbb{R}^m$. The schedule is called a linear pricing schedule if the map is linear, that is, $I(\mathbf{q}) = \boldsymbol{\pi}^T \mathbf{q}$. In this case, $\boldsymbol{\pi}$ is called a linear price vector and π_c is the price per unit for commodity c . The definition is also applicable to producers if we use $-q_c$ consumption units to model q_c production units of commodity c ”.

Definition 3. Strict linear pricing “A pricing schedule is a strict linear pricing schedule if it is linear and the number of commodities m is equal to the number of clearing conditions in the auction model. In the electricity market, the clearing conditions are the flow conservation equations for each network node and time slot.”

Definition 4. Non-linear (non-uniform or discriminatory) pricing A pricing scheme is called non-linear if more than one linear pricing schedule is used for a single commodity.

The U.S. day-ahead markets deviate from linear pricing as the auctioneer pays additional money to the bidders for which the linear prices and the dispatch commands are not optimal. This leads to non-uniform prices across the bidders since the average price per unit of energy traded can be different for each bidder.

There are different applications for the determination of the price vector. The most common approach is to set the price to the marginal cost of energy. In a market where there are clearing conditions only for the time periods, the marginal cost of energy at a period is the marginal power generation cost of the power plant producing the last unit of energy cleared in that period. In case the prices are differentiated on space in addition to time, the marginal price at period t and location l is the summation of the marginal cost of power generation at period t , shadow price of transmission capacity

to location l at period t and the price of marginal transmission loss of energy delivered to location l and period t PJM State & Member Training Department (2013).

2.2.6 Market surplus

The objective of day-ahead market clearing problems is to maximize the total surplus resulting from the trade between the bidders under the constraints associated with the clearing conditions. The market surplus is equivalent to the difference between the total amount of money offered by the bidders to buy as much as the cleared energy and the total amount of money asked by the bidders to sell as much as the cleared energy.

Let \mathbf{q}_p be the quantity vector traded by player $p \in P$ in the day-ahead market and let $\boldsymbol{\pi}$ be the market clearing price vector. Given a value function $V_p : \mathbb{R}^m \rightarrow \mathbb{R}$ for player p , the surplus of player p is equal to $V_p(\mathbf{q}_p) - I(\mathbf{q}_p)$ where $V_p(\mathbf{q}_p) \geq 0$ for consumers and $V_p(\mathbf{q}_p) \leq 0$ for producers and the players are assumed to bid their true value functions. Then, the market surplus is the summation of individual surplus values of the players:

$$S(\mathbf{q}) = \sum_{p \in P} \{V_p(\mathbf{q}_p) - I(\mathbf{q}_p)\} \quad (2.1)$$

Since the clearing conditions ensure that $\sum_{p \in P} \mathbf{q}_p = 0$ and $\sum_{p \in P} I(\mathbf{q}_p) = \boldsymbol{\pi}^T (\sum_{p \in P} \mathbf{q}_p) = 0$, Equation (2.1) can be reduced and the market surplus can be written as:

$$S(\mathbf{q}) = \sum_{p \in P} V_p(\mathbf{q}_p) \quad (2.2)$$

2.2.7 Equilibrium

In order to achieve an efficient and fair allocation, the auctioneer needs to determine the prices and the commitment/dispatch decisions such a way that each player must be perfectly satisfied. That is, there should not be any better trade opportunity for any

player under the current prices so that they do not want to deviate from the allocation determined by the auctioneer.

Suppose that $S_p(\boldsymbol{\pi}, \mathbf{q}_p)$ be the surplus of player p under the price vector $\boldsymbol{\pi}$ and quantity vector \mathbf{q}_p , $S_p(\boldsymbol{\pi}, \mathbf{q}_p) = V_p(\mathbf{q}_p) - I(\mathbf{q}_p)$. Let $S_p^*(\boldsymbol{\pi})$ be the maximum surplus that the player p can achieve under the price vector $\boldsymbol{\pi}$, $S_p^*(\boldsymbol{\pi}) = \max_{\mathbf{q}_p \in X_p} \{V_p(\mathbf{q}_p) - I(\mathbf{q}_p)\}$ where X_p represents the set of feasible quantity vectors that can be allocated to player p . The market is said to be in equilibrium if the following conditions hold:

$$S_p(\boldsymbol{\pi}, \mathbf{q}_p) = S_p^*(\boldsymbol{\pi}) \quad \forall p \in P \quad (2.3)$$

Since the bidders reveal their value functions by their bids in the auction, the equilibrium of the allocation can also be defined in terms of the accept/reject decisions on the bids. We first need to make the following definitions:

Definition 5. (In-the-money bid) Given a market clearing price vector $\boldsymbol{\pi}$, a bid with price vector \mathbf{v} and quantity vector \mathbf{q} is in-the-money if $(\mathbf{v} - \boldsymbol{\pi})^T \mathbf{q} > 0$.

Definition 6. (At-the-money bid) Given a market clearing price vector $\boldsymbol{\pi}$, a bid with price vector \mathbf{v} and quantity vector \mathbf{q} is at-the-money if $(\mathbf{v} - \boldsymbol{\pi})^T \mathbf{q} = 0$.

Definition 7. (Out-of-the-money bid) Given a market clearing price vector $\boldsymbol{\pi}$, a bid with price vector \mathbf{v} and quantity vector \mathbf{q} is out-of-the-money if $(\mathbf{v} - \boldsymbol{\pi})^T \mathbf{q} < 0$.

Based on the definitions given above, the auction results achieve equilibrium if and only if all the in-the-money bids are accepted and all the out-of-the-money bids are rejected.

2.2.8 Uplift

When the market clearing problem is non-convex, there may not exist a price vector $\boldsymbol{\pi}$ such that $S_p(\boldsymbol{\pi}, \mathbf{q}_p) = S_p^*(\boldsymbol{\pi}) \quad \forall p \in P$. The auctioneer cannot even guarantee that $V_p(\mathbf{q}_p) \geq I(\mathbf{q}_p)$ for each player $p \in P$. The auctioneer accepts to pay as much as $S_p^*(\boldsymbol{\pi}) - S_p(\boldsymbol{\pi}, \mathbf{q}_p)$ to player p which is called the *uplift* payment to player p ,

$U_p(\boldsymbol{\pi}, \mathbf{q}_p)$.

$$U_p(\boldsymbol{\pi}, \mathbf{q}_p) = S_p^*(\boldsymbol{\pi}) - S_p(\boldsymbol{\pi}, \mathbf{q}_p) \quad \forall p \in P \quad (2.4)$$

In U.S. electricity markets, the system operator accepts to pay the full uplift $U_p(\boldsymbol{\pi}, \mathbf{q}_p)$ whereas in European markets the auctioneer does not pay any uplift Van Vyve (2011). However, the auctioneer includes constraints into the surplus maximization problem to guarantee that $S_p(\boldsymbol{\pi}, \mathbf{q}_p) \geq 0$ (Euphemia, 2016). We call these constraints as “no-loss constraints”. No-loss constraints prevent any player p from being allocated a quantity vector \mathbf{q}_p whose market value $I(\mathbf{q}_p)$ exceeds the value assigned by player p , $V_p(\mathbf{q}_p)$. However, player p can still miss potential surplus that it could achieve, which is equal to $S_p^*(\boldsymbol{\pi}) - S_p(\boldsymbol{\pi}, \mathbf{q}_p) : S_p(\boldsymbol{\pi}, \mathbf{q}_p) \geq 0$. We call this amount as the *missed surplus* by player p . On the contrary, the Turkish market operator includes constraints that ensure zero missed surplus for each market player. In this case, any market player can have negative surplus, *loss*, that is fully compensated by the market operator. That is, market operator pays as much as $S_p(\boldsymbol{\pi}, \mathbf{q}_p)$ to player p if $S_p(\boldsymbol{\pi}, \mathbf{q}_p) < 0$ (Energy Exchange Istanbul, 2016).

2.2.9 Settlement

Settlement is the process of calculating the total payments to be made to each market participant in return of their energy supply into the market and total receivables to be taken from each market participant in return of their energy demand from the market. In a uniform pricing scheme, the cash inflow and outflow cancel out each other so that the market operator’s financial status does not get affected from the auction results. However, if uplift payments are to be made, then they generate a cash outflow for the market operator and create the *missing money* problem.

2.2.10 Market efficiency

Market efficiency is a measure of the total value created in the market by trading the associated commodity. Trade should occur between the seller agents with minimum

valuations of the commodity and the buyer agents with maximum valuations. Competition is one of the key drivers of market efficiency. On top of that, in organized market places the auctioneer should ensure *allocative efficiency*. This can happen only if the auction design is effective enough to incentivize the players to bid their true valuations, *incentive compatible*, and the auctioneer can find the value maximizing allocations. These incentives must be high enough that the expected utility of each bidder from truthfull bidding is always greater than or equal to the utility when the valuation is misrepresented. Assuming that the former one holds, then an efficient allocation is the one that maximizes the total value given the players' bids.

Let $Q^* = [\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_{|P|}]$ be the allocation determined by the auctioneer. Then, Q^* is efficient if the following holds:

$$Q^* = \operatorname{argmax}_{\mathbf{q}} \sum_{p \in P} V_p(\mathbf{q}_p) \quad (2.5)$$

2.3 Literature review

In this section, we first review the studies on European DAM clearing problem, and then the algorithms developed so far to solve multi-objective mixed-integer programs.

2.3.1 Day-ahead electricity market clearing problem

Day-ahead electricity market auctions are combinatorial in their nature. Therefore, the literature on the combinatorial auctions is of interest for the study of day-ahead energy markets. Generally speaking, the day-ahead energy market auctions are single-round, single-attribute, multi-item and multi-unit sealed-bid combinatorial auctions where each single item is priced at a single uniform price and a bundle of items is priced as the sum of the prices of individual items. In the context of day-ahead electricity market, an item is a unit energy to be generated or consumed at a particular period of the day and, in some markets, at a particular location of the grid. The multi-period bids like block bids bundle items from different time periods to handle the nonconvex cost structure of thermal generating units.

A review of pricing approaches in multi-item combinatorial auctions are examined in Xia et al. (2004) under two classes: bundle pricing and individual item pricing. In bundle pricing, a price for each bundle in the auction is calculated instead of price per item. In case of individual pricing, a price for each item is calculated and price of a bundle is determined as the sum of the prices of items the bundle comprises. The authors argue the advantages and disadvantages of both approaches. Since market clearing individual prices cannot be guaranteed in combinatorial auctions, Briskorn et al. (2016) propose a non-linear anonymous pricing approach in which a set of price vectors are calculated and announced to the market instead of a single price vector. They define market clearing in terms of this set of price vectors and show that it exists.

Pekei and Rothkopf (2003) review different applications of combinatorial auctions and emphasizes the complexity of winner determination problem and the cooperative aspect of iterative combinatorial auctions. The authors state the desirable properties of auctions as allocative efficiency, revenue maximization, low transaction costs, fairness, failure freeness, and scalability. However, they argue that the complexity of these auctions may require to trade-off between these desirable properties.

de Vries and Vohra (2003) present the state of the knowledge about the design of combinatorial auctions and emphasizes the relation of auctions to the duality theory of optimization problems. In the first part, the authors give an integer programming formulation to achieve efficient allocation in a general single-unit, single-attribute and multi-item combinatorial auctions. They note that the problem is an instance of the Set Packing Problem (SPP) and further analyze the complexity of SPP.

In the second part, the authors distinguish between quantity-setting and price-setting type auctions and interpret the price-setting auctions as primal-dual algorithms for solving the winner determination problem. They examine the classical results of duality to derive the properties that the prices must satisfy to produce an allocation that solves the combinatorial auction problem. The authors interpret the exposure problem in combinatorial auctions as the violation of complementary slackness and note that any auction scheme that relies on prices for individual items alone will face this problem. Additionally, they implement column generation idea in an auction setting and assert that column generation subproblem can be viewed as the generation

of bundles by the bidders based on the value functions of the bidders and announced item prices by the auctioneer.

In the last section of de Vries and Vohra (2003), the authors focus on the auction mechanisms that give bidders incentive to reveal their valuations truthfully. They point out that to achieve efficient outcome, the auctioneer needs true valuations of the bidders but those are private information to bidders. However, they present the mechanism of Vickrey-Clarke-Groves (VCG) scheme which is known to result in efficient allocation since bidding true valuations are weakly dominant strategy for the bidders. In addition to that, VCG mechanism also gives the maximum revenue to the seller among the all the auctions that implement the efficient allocation. On the other hand, Parkes et al. (2001) focus on the budget-deficiency problem of the auctioneer in Vickrey pricing schemes due to the Vickrey payments. They propose a pricing problem in which a distance function between the real payments and Vickrey payments are minimized under the budget balance of the auctioneer and the individual rationality of the bidders, that is all bidders have positive expected utility to participate into the market.

In the context of day-ahead electricity markets, non-convex cost functions are defined in order to represent complementarity relations between energy volumes in consecutive time periods. For thermal power plants, there are significant start-up costs and it requires a few hours to start-up or shut down a thermal power plant. So, as the number of periods a thermal power plant is operating increases, the average energy price per unit of energy is expected to decrease. In addition, there are additional constraints on the periodic output levels of a thermal power plant when it is turned on.

For market participants to be able to model their operational constraints efficiently, many day-ahead electricity markets have introduced complex bid types that led to a non-convex problem to determine the optimal allocation of resources. Non-convexity of the problem has brought both computational challenges to determine the winning set and pricing challenges. Where the problem is convex, it is guaranteed that there exist uniform prices that is compatible with the optimal allocation. On the other hand, the existence of uniform market clearing prices that supports the optimal allocation are not guaranteed once the problem is non-convex. The non-convexity is attributed

to the “lumpiness” of the day-ahead electricity markets and that lumpiness creates a debate on the definition of the “right” price (Elmaghraby et al., 2004). One may have to sacrifice from some desirable properties of an auction outcome in order to achieve further in some other conflicting goals.

In the U.S. electricity markets, the independent system operator (ISO) first solves the unit commitment and economic dispatch problem in order to minimize the cost of satisfying a certain level of load. Afterwards, ISO needs to announce market clearing prices that must give no incentive to the bidders to deviate from their determined schedules. However, there may exist no such uniform prices. If any market participant is able to get more revenue under the announced market clearing prices by changing its allocation, then the ISO compensates the difference. These payments are called uplift payments.

Gribik et al. (2007) examine the impact of alternative pricing models on the volatility of prices and the total uplift payments. They study three pricing approaches: In their “restricted” model, they fix the integer variables to their optimal values and solves the problem again. This problem is convex and the optimal values of the dual variables associated with the periodic supply demand balance are defined as the market clearing prices. In the second model, “dispatchable model”, they just relax the integrality constraints and solve the relaxed problem. They note that these pricing rules are the common industry practices at the U.S. electricity markets (Current practice in the European electricity markets including Turkey is to solve the restricted model). Lastly, Gribik et al. (2007) propose the “convex-hull” pricing model which yields uplift minimizing prices. In this model, lagrangean dual problem is solved to obtain prices and the corresponding duality gap gives the minimum uplift. Herrero et al. (2015) focuses on the long-term incentives of these pricing rules and evaluates the restricted and dispatchable model by comparing the long-term market technology mix under the prices generated by these approaches.

Liberopoulos and Andrianesis (2016) classify the existing pricing schemes under *uniform pricing with external uplifts*, *zero-sum uplift pricing* and *revenue-adequate pricing*. In the first one, a uniform market price vector is calculated. The price may be the marginal price or not and the bidders are compensated with uplifts if they incur loss at

the allocation assigned to them (O'Neill et al., 2005; Gribik et al., 2007; Bjørndal and Jörnsten, 2008). In zero-sum uplift pricing (Motto and Galiana, 2002; Galiana et al., 2003; Liberopoulos and Andrianesis, 2016; Van Vyve, 2011), uplifts are determined during the pricing by equating the total uplift paid by and the total uplift paid to the bidders so that the market equilibrium is achieved and there does not occur an external uplift. On the other hand, in revenue-adequate pricing, prices are determined in such a way that they are enough to guarantee that all bidders get non-negative profits and uplifts are not needed (Araoz and Jörnsten, 2011; Ruiz et al., 2012).

There are major differences in the pricing models of the U.S. and European day-ahead markets as well as in their market structures, regulatory framework and the modelling of transmission systems. Van Vyve (2011) focus on the pricing differences and compares the properties of resulting auction outcomes. Although the uniform pricing is prevalent in both markets, there are different restrictions applied on the prices. In the U.S. model, the allocation problem and the pricing problem are solved sequentially. Once the optimal allocation is found, the associated uniform prices and uplift payments are calculated with the preferred pricing model discussed in Gribik et al. (2007). The advantages of this model is that the allocation is welfare maximizing and equilibrium is satisfied via the uplift payments. However, the uplift payments lead to a missing money problem for the auctioneer since it has to pay to bidders more than it receives. The author also points out that the uplift payments for the opportunity costs of bidders may cause bidders not to bid truthfully and bid strategically.

Day-ahead market operators in European markets, on the other hand, refuse to pay uplift to bidders and requires each market participant to be subject to exactly the same prices. Equilibrium requirement is relaxed in these markets by allowing the rejection of bids that need to be accepted at the announced prices. However, the reverse is not acceptable since it may lead to financial losses for some bidders. Under these requirements, they have to include pricing variables and constraints into the allocation problem in order to prevent the occurrence of losses. The resulting problem becomes more difficult to solve and the optimal solution may not be welfare maximizing. Furthermore, the real instances show that the inclusion of constraints to reject any “out-of-the money” bids may cause the rejection of low volume bids that are “too in-the-money”. The advantage of this model is that the auctioneer always has a bud-

get balance and the bidders are given incentives to bid truthfully since no other side payment is in place.

An alternative pricing model proposed in Van Vyve (2011) merges good properties of both models. The main idea is the integration of uplift mechanism into the pricing problem and let the bidders with positive surplus compensate the loss of any bidder if there exist any. The objective of the pricing problem is to minimize the maximum uplift contribution of a bidder. Although the proposed model is desirable in terms of the aspects listed above, the unit uplift payments and contributions may differ from bidder to bidder that makes the resulting pricing scheme non-linear.

The Turkish day-ahead electricity market is very similar to the European counterparts as the Scandinavian energy market design, NordPool, was the target design during the liberalization of the Turkish electricity market. In Turkish day-ahead market pricing, rejection of in-the-money bids are prevented in contrast to common European practice and the bidders incurring losses due to acceptance of out-of-the-money bids are compensated with uplifts. The underlying motivation behind this design choice is to eliminate the objections of market participants having in-the-money bids rejected. At the same time, none of the bidders can be better off by changing its allocation so that the equilibrium solution is achieved at the expense of sub-optimal allocation in terms of total market surplus. Ceyhan et al. (2017) compares the Turkish pricing model with the common European pricing model on some performance criteria that are of interest to the market operator. They implement the models on the real Turkish market data and report that the two pricing models may end up with very different market clearing prices although they do not differ too much on the average. The results of the tests reveal an important observation: The distribution of total market surplus among the market participants can be significantly affected by the pricing design choice of the market operator even if the total surplus of the market does not change considerably.

To solve the market clearing problem in European DAMs, one needs to develop dedicated solution methods as the off-the-shelf commercial solvers are not very effective at solving the practical size problems in a limited amount of time. The complexity of the problem for the European markets has been increasing with the number of markets being coupled. During the last few years, seven power exchanges operating the

day-ahead markets of 23 countries have merged their bid sets and solving a single but a much bigger size problem Euphemia (2016). That way, the capacity of the cross-border transmission lines is utilized more efficiently. That also requires, however, to devise more efficient solution methods.

The “no-loss” requirement in the auction outcomes necessitates the inclusion of price variables which are actually the dual variables corresponding to the supply-demand balance constraints in the European day-ahead electricity market coupling problem. This leads to a primal-dual formulation and integration of non-linear complementarity constraints into the problem. The prevailing method to solve this problem is the Benders decomposition method in which dual problem variables and complementarity constraints are handled using the sub-problems. When an integer feasible solution is found to the master problem (primal problem), a subproblem is solved to find the corresponding market clearing prices and check if there is any bidder incurring any loss. If losses occur, an infeasibility cut is added to the master problem to exclude the generated solution. Martin et al. (2014), Madani and Van Vyve (2015), Euphemia (2016), and Madani and Van Vyve (2018) present similar Benders decomposition schemes for the problem. Martin et al. (2014) develops exact infeasibility cuts which are further improved by Madani and Van Vyve (2015) to be only valid for the sub-tree associated with the current node in the branch-and-bound tree. Martin et al. (2014) also proposes heuristic cuts for which they report optimal or near-optimal solutions in a small amount of time at the expense of no-guarantee for the optimal solution.

There are also attempts to solve directly the compact MIP formulation of the problem by linearizing the non-linear complementarity constraints. Derinkuyu (2015), Ceyhan et al. (2017) and Derinkuyu et al. (2019) use big-M approach to linearize those constraints whereas Madani and Van Vyve (2014) replaces complementary-slackness constraints with strong-duality constraint. For the case where hourly bids are step-wise functions, it results in an MILP. However, the existence of piecewise hourly bids make the strong duality constraint quadratic and the resulting problem MIQCP. The Turkish market operator, EXIST, solves the formulation (M3) in Ceyhan et al. (2017) with additional preprocessing algorithms and meta-heuristics to supply an initial solution to the mathematical programming solver Energy Exchange Istanbul (2016).

We see that total surplus maximization, or cost minimization when the demand is assumed to be not price-sensitive, is the main objective function to be considered for this problem except a few studies. This is not surprising since any auction design is supposed to achieve economic efficiency which can be measured by the surplus in the market assuming that the bidders bid their true valuations. However, Derinkuyu (2015) sets the objective function to average market clearing price minimization claiming that high market clearing prices are politically undesirable. This kind of objective function is highly debatable since the goal of markets must be to find the most efficient outcome and prices must be just an information that markets signal to buyers and sellers. However, it can be investigated that how much the surplus maximizing and price minimizing solutions differ by testing both models on some real instances. Madani and Van Vyve (2014) conducts this kind of analysis for the opportunity cost minimization problem. They define the opportunity cost as the foregone surplus associated with the rejected in-the-money bids based on the calculated market clearing prices. Based on the ten real market instances of Central Western European markets in 2011, they report that the trade-off between two criteria is often small and surplus maximizing solution turned up to be the opportunity cost minimizing solution as well for two instances out of ten.

2.3.2 Multi-objective mixed-integer linear programming

It was not too long ago that we had to resort to heuristics or approximation algorithms to solve moderate-size MILPs from practice due to their computational complexities. With the developments in optimization solvers and advances in processors, much larger MILPs can be solved to optimality today. In recent years, large scale multi-objective MILPs (MOMILPs) have also been addressed in practice to search for preferred solutions of a decision maker (DM) considering their trade-offs between objectives. In the existence of multiple objectives, it is rare to find a solution that is best in all objectives. Rather, there are a set of meaningful *nondominated* points, each outperforming any other solution in at least one objective.

Many approaches have been developed to solve multi-objective linear programming problems (MOLPs), where all the objective functions and the constraints are linear.

They differ from each other in aspects such as the characteristics of the solutions searched for and the involvement of a DM in the search process. Many approaches aim to generate a representative set of nondominated points or converge to points preferred by a DM. A simple and common approach to generate nondominated points is the *weighted-sum* that tries to capture the DM's preferences by linearly aggregating the objectives with positive weights. Alternatively, multiple nondominated points can be generated by systematically changing the weights (Marler and Arora, 2010; Kim and De Weck, 2006) without requiring an input from the DM.

Another commonly used method to solve MOLPs is *goal programming* (Charnes and Cooper, 1977), where a goal is defined to be attained in each criterion. Then, the a single objective optimization minimizes the deviations from the goals (penalties). Ehrgott (2005) reviews different types of scalarization methods as well such as *ϵ -constraint method* (Haimes et al., 1971), *Benson's method* (Benson, 1978), *reference point methods* (Zeleny, 2012), and *direction-based methods* (Korhonen and Walleenius, 1988).

Generating the nondominated set of large scale MOILPs is challenging due to computational difficulties of solving integer programs repetitively. Earlier approaches addressed bi-objective problems (Ulungu et al., 1995; Ehrgott and Gandibleux, 2007). More recently, researchers have been working on more than two objectives and many efficient algorithms are available today (Przybylski et al., 2010a,b; Lokman and Köksalan, 2013; Ozlen et al., 2014; Kirlik and Sayin, 2014; Boland et al., 2016, 2017). The algorithms developed in Lokman and Köksalan (2013), Ozlen et al. (2014), Kirlik and Sayin (2014), Klamroth et al. (2015), and Dächert et al. (2017) are applicable to MOILPs with any number of objective functions. There are also implementations of parallelized algorithms (see for example, Turgut et al., 2019). Due to the extensive computational effort in generating the whole nondominated set, some recent approaches focus on generating representative subsets of the nondominated set (Sylva and Crema, 2007; Masin and Bukchin, 2008; Ceyhan et al., 2019).

In the case of MOMILPs, the nondominated set includes facets as well as points or edges. It is not straightforward to separate the regions dominated by such nondominated sets. The initial research in this area focused on generating the set of

extreme supported nondominated points (Aneja and Nair, 1979; Przybylski et al., 2010a; Özpeynirci and Köksalan, 2010; Alves and Costa, 2016). Assuming that all objectives are of maximization type, each such point have the property of uniquely maximizing some positively weighted sum of the objectives. Other studies that aim to find the whole nondominated set for MOMILPs are restricted to the bi-objective case (Mavrotas and Diakoulaki, 2005; Vincent et al., 2013; Belotti et al., 2013; Stidsen et al., 2014; Boland et al., 2015; Soylu and Yıldız, 2016; Soylu, 2018; Fattahi and Turkay, 2018).

Our multi-objective day-ahead market clearing problem formulation in Chapter 5 is a class of MOMILP with three objectives (TOMILP). Hence, we mostly focus on the algorithms that are developed to generate nondominated sets of such problems. Rasmi and Türkay (2019) presents the only existing approach aimed at finding the nondominated facets for MOMILPs with more than two objectives. They develop a two-stage algorithm that starts with finding the efficient integer vectors, and then generates the nondominated edges and facets associated with each such vector. However, due to their construction, the generated facets are not guaranteed to be nondominated as a whole; they may contain dominated regions as well.

CHAPTER 3

SOLVING THE EUROPEAN DAY-AHEAD ELECTRICITY MARKET CLEARING PROBLEM EFFICIENTLY WITH BENDERS DECOMPOSITION

In this chapter, we develop a Benders decomposition algorithm to solve the market clearing problem in European day-ahead electricity markets (DAMs). European DAMs are spot markets that are organized to trade electricity between sellers and buyers one day prior to the actual generation and consumption. Market participants can submit combinations of different types of bids with different prices and quantities for different periods of the delivery day. Market operators (MOs) solve the market clearing problem, and find the surplus maximizing electricity trade and the market clearing prices.

European MOs have undertaken a major market coupling process in the last decade to create a single pan European DAM, called Single Day-Ahead Coupling (SDAC). It was initiated by eight power exchanges and now accounts for 95% of the EU consumption. The value of the daily traded electricity is around 200 million Euros on average (NEMO Committee, 2020). The resulting problem is a large scale mixed-integer linear or quadratic program (depending on the types of bids available in the market) and the problem needs to be solved in about 10 minutes in order to implement the results within the tight time frame the market is operating in.

Benders decomposition is the most studied solution approach for this problem in the literature (Martin et al., 2014; Madani and Van Vyve, 2014, 2015; Madani and Van Vyve, 2018; Euphemia, 2016). This is mainly because of the complexity of solving a compact formulation when there are hundreds or thousands of binary variables, complex bid types and the equilibrium constraints that have to be satisfied. Ben-

ders decomposition algorithm reduces the complexity by solving simpler models and introducing constraints (cuts) as necessary to enforce the feasibility of the original model. The performances of the Benders decomposition algorithms are not up to the task of solving the problem within the required time frame. This is mainly due to the use of “no-good” cuts (Martin et al., 2014) that cause weak relaxation bounds or locally-valid cuts that can only be used in the sub-trees (Madani and Van Vyve, 2015; Madani and Van Vyve, 2018). SDAC uses the Euphemia algorithm that associates heuristic cuts with the aim of generating a high-quality solution within the time limit (Euphemia, 2016).

We develop a Benders decomposition algorithm based on *price-based* cuts that we generate utilizing the market clearing prices associated with an integer solution. We prove that the price-based cuts are valid and stronger than the “no-good” cuts. We test the performance of our algorithm on practical-sized instances and show that our algorithm is superior to the existing Benders decomposition algorithms and the primal-dual approach. The improved performance implies substantial surplus increases in European DAMs with millions of Euros of daily trade and provides an efficient algorithm for MOs that operate under strict timelines. We also evaluate the performance of our algorithm using two leading commercial mixed-integer programming solvers, IBM ILOG Cplex and Gurobi. We show that our algorithm outperforms the compared algorithms in both cases, and performs best when Gurobi is employed as the solver.

In the next section, we examine the well-known surplus maximization problem and elaborate on the properties of its linear relaxation. In Section 3.2, we present a primal-dual formulation of the surplus maximization problem under pricing constraints. In Section 3.3, we present a Benders decomposition algorithm to solve the surplus maximization problem under pricing constraints and develop strengthened Benders infeasibility cuts that are globally valid. In Section 3.4, we test the new cuts on practical-sized problem instances. We discuss the extensions of our findings in Sections 3.5 and 3.6.

3.1 Surplus maximization problem

In this section, we formulate the surplus maximization problem as a mixed-integer program. We examine the linear relaxation of the model and investigate the economical properties of the optimal solution. We show that the optimal solution is also the equilibrium solution for the linear case. We then elaborate on the mixed-integer model and discuss the cases when the duality-gap is non-zero.

In the rest of the paper, we concentrate on hourly and (profile) block bids. These are the most commonly-used bid types in the European day-ahead markets. Limiting the analysis with these two bid types will simplify the expositions of formulations and findings while capturing the essence of the auctions. Nevertheless, we show in Section 3.6 that our findings are more general and are applicable in the presence of the more sophisticated bid types and network-constrained markets.

3.1.1 Hourly bids

An hourly bid h is a price-quantity pair, (p_h, q_h) . $q_h < 0$ implies that the bidder is willing to supply an amount $|q_h|$ to the market at a minimum price of p_h . Similarly, $q_h > 0$ implies that the bidder is willing to buy an amount $|q_h|$ from the market at a maximum price of p_h . A bidder can specify a sequence of hourly bids in the increasing order of the price for supply bids and decreasing order of the price for demand bids. Such bids form step functions as shown in Figures 3.1 (x-axis shows absolute supply quantities) and 3.2. The supply (demand) function indicates the amount of energy the bidder is willing to sell (buy) at different market clearing prices.

In some markets (EPEXSpot, EXIST), the market operator accepts a piece-wise linear function instead of a step function. In this case, each hourly bid is defined by two price values and a single quantity. In Figure 3.3, we show a piece-wise linear supply function obtained from hourly bids in increasing order of prices. This function indicates the exact amounts of energy the bidder is willing to supply at different market clearing prices. We initially restrict our discussions to hourly bids that are represented by step functions. We show in Section 3.6 that our findings are applicable to piece-wise hourly bid functions as well.

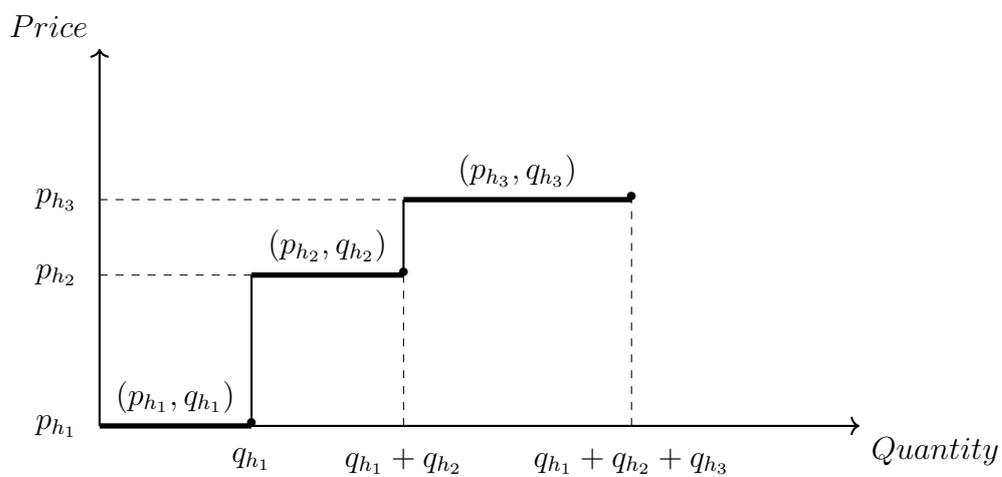


Figure 3.1: A step function for hourly supply bids

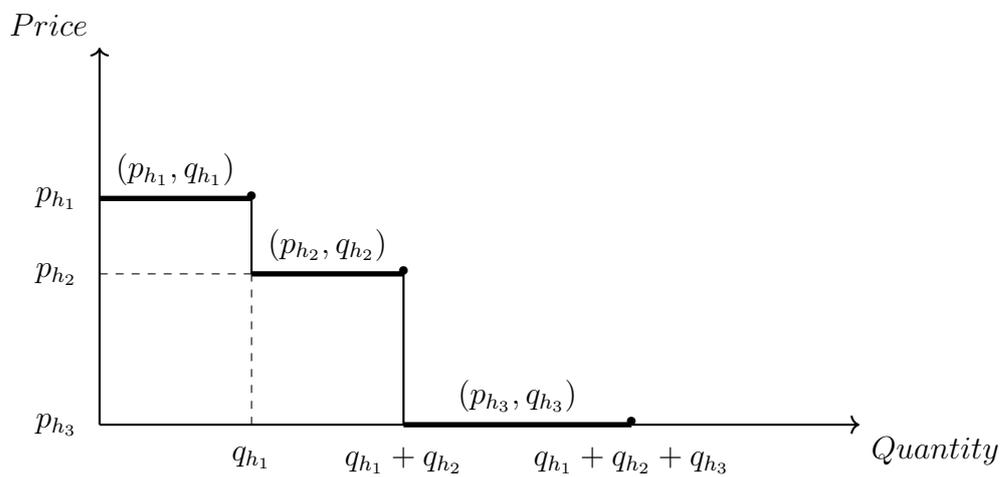


Figure 3.2: A step function for hourly demand bids

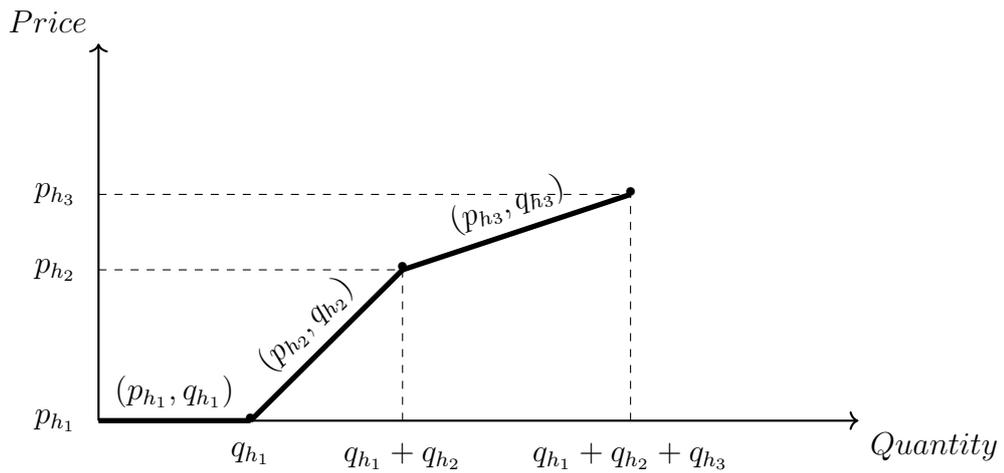


Figure 3.3: A piece-wise linear function with supply hourly bids

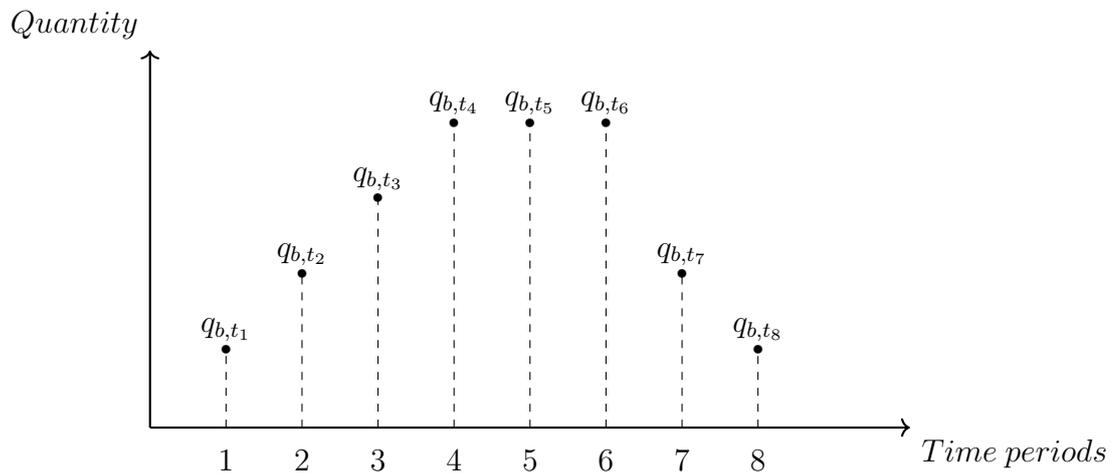


Figure 3.4: The structure of a block bid

3.1.2 Block bids

Block bids are collections of single hourly bids offered for consecutive time periods. For a block bid, a single price applies to all periods it is offered for. However, the quantities for different periods need not be the same. A block bid needs to be either accepted or rejected as a whole (at full quantity for each period). There is no partial acceptance of a block bid in terms of quantity or the set of time periods. We show an example of block bid in Figure 3.4.

3.1.3 Problem formulation

We use the following sets, parameters, and decision variables in our problem formulation:

- T : set of time periods
- H : set of hourly bids
- B : set of block bids
- $p_h, q_{h,t}$: price and quantity for time period $t \in T$, for an hourly bid $h \in H$ ($q_{h,t} = 0, \forall t \in T, t \neq t'$ for a particular period t')
- $p_b, q_{b,t}$: price and quantity for time period $t \in T$, for a block bid $b \in B$
- T_b : the set of time periods spanned by block bid $b \in B, T_b \subseteq T$ ($q_{b,t} = 0, \forall t \notin T_b$)
- x_h : decision variable representing the accepted fraction of hourly bid $h \in H, x_h \in [0, 1]$
- y_b : decision variable for block bid $b \in B, 1$ if accepted, and 0 if rejected.

The market surplus can be calculated as follows:

$$S_L(\mathbf{x}, \mathbf{y}) = \sum_{t \in T} \left\{ \sum_{h \in H} p_h q_{h,t} x_h + \sum_{b \in B} p_b q_{b,t} y_b \right\} \quad (3.1)$$

In Equation (3.1), quantities are negative (positive) for all supply (demand) bids. Therefore, the right-hand side of the equation shows the difference between the total value assigned by the buyers to the accepted demand bids and the total value assigned by the sellers to the same quantity of accepted supply bids. The market surplus function is linear in \mathbf{x} and \mathbf{y} . We formulate the surplus-maximizing mixed-integer linear program as:

(SMILP):

$$\begin{aligned} \text{Max} \quad & S_L(\mathbf{x}, \mathbf{y}) \\ \text{s.to.} \quad & \end{aligned}$$

$$\begin{aligned}
\sum_{h \in H} q_{h,t} x_h + \sum_{b \in B} q_{b,t} y_b &= 0 & \forall t \in T \\
x_h &\leq 1 & \forall h \in H \\
x_h &\geq 0 & \forall h \in H \\
y_b &\in \{0, 1\} & \forall b \in B
\end{aligned}$$

The first constraint balances the supply and demand in each period. The next two inequalities force \mathbf{x} variables to fractional values. The problem is a mixed-integer program due to the binary variables, \mathbf{y} . We next relax the integrality requirements of \mathbf{y} variables and investigate the properties of the optimal solution of the relaxed linear problem.

3.1.4 Linear relaxation

In this section, we present the primal, **(SMLP)**, and the dual, **(D-SMLP)**, formulations of the linear relaxation of **(SMILP)**, and provide the complementary slackness conditions.

(SMLP):

$$\begin{aligned}
\text{Max } & S_L(\mathbf{x}, \mathbf{y}) \\
\text{s.to. } & \\
& \sum_{h \in H} q_{h,t} x_h + \sum_{b \in B} q_{b,t} y_b = 0 & \forall t \in T \quad [\pi_t] \\
& x_h \leq 1 & \forall h \in H \quad [s_h] \\
& y_b \leq 1 & \forall b \in B \quad [s_b] \\
& x_h \geq 0 & \forall h \in H \\
& y_b \geq 0 & \forall b \in B
\end{aligned}$$

In **(SMLP)**, we show the associated dual variable for each constraint in brackets. The dual problem is:

(D-SMLP):

$$\begin{aligned}
\text{Min} \quad & \sum_{h \in H} s_h + \sum_{b \in B} s_b \\
\text{s.to.} \quad & \\
& s_h \geq \sum_{t \in T} (p_h - \pi_t) q_{h,t} \quad \forall h \in H \\
& s_b \geq \sum_{t \in T} (p_b - \pi_t) q_{b,t} \quad \forall b \in B \\
& s_h \geq 0 \quad \forall h \in H \\
& s_b \geq 0 \quad \forall b \in B
\end{aligned}$$

Based on the primal and dual problems, the following complementary slackness conditions must hold at the optimal solution of **(SMLP)**:

(CS-SMLP):

$$\begin{aligned}
s_h(1 - x_h) &= 0 \quad \forall h \in H \\
s_b(1 - y_b) &= 0 \quad \forall b \in B \\
x_h(s_h - \sum_{t \in T} (p_h - \pi_t) q_{h,t}) &= 0 \quad \forall h \in H \\
y_b(s_b - \sum_{t \in T} (p_b - \pi_t) q_{b,t}) &= 0 \quad \forall b \in B
\end{aligned}$$

The optimality conditions of **(SMLP)** imply the market equilibrium since no player can be better off by deviating from the quantities allocated to them under the market clearing prices. We further elaborate on this below:

1. If an hourly bid is fully rejected, then it is not in-the-money.

$$x_h^* = 0 \implies \sum_{t \in T} (p_h - \pi_t^*) q_{h,t} \leq 0 \quad (3.2)$$

2. If an hourly bid is partially accepted, then it is at-the-money.

$$0 < x_h^* < 1 \implies \sum_{t \in T} (p_h - \pi_t^*) q_{h,t} = 0 \quad (3.3)$$

3. If an hourly bid is fully accepted, then it is not out-of-the-money.

$$x_h^* = 1 \implies \sum_{t \in T} (p_h - \pi_t^*) q_{h,t} \geq 0 \quad (3.4)$$

4. If an hourly bid is in-the-money, then it must be fully accepted.

$$\sum_{t \in T} (p_h - \pi_t^*) q_{h,t} > 0 \implies x_h^* = 1 \quad (3.5)$$

5. If an hourly bid is out-of-the-money, then it must be fully rejected.

$$\sum_{t \in T} (p_h - \pi_t^*) q_{h,t} < 0 \implies x_h^* = 0 \quad (3.6)$$

6. If a block bid is rejected, then it is not in-the-money.

$$y_b^* = 0 \implies \sum_{t \in T} (p_b - \pi_t^*) q_{b,t} \leq 0 \quad (3.7)$$

7. If a block bid is accepted, then it is not out-of-the-money.

$$y_b^* = 1 \implies \sum_{t \in T} (p_b - \pi_t^*) q_{b,t} \geq 0 \quad (3.8)$$

8. If a block bid is in-the-money, then it must be accepted.

$$\sum_{t \in T} (p_b - \pi_t^*) q_{b,t} > 0 \implies y_b^* = 1 \quad (3.9)$$

9. If a block bid is out-of-the-money, then it must be rejected.

$$\sum_{t \in T} (p_b - \pi_t^*) q_{b,t} < 0 \implies y_b^* = 0 \quad (3.10)$$

These properties ensure that each bidder has a non-negative surplus as a result of its participation in the day-ahead market, guaranteeing that none of the accepted demand bids is overvalued and none of the accepted supply bids is undervalued by the market operator. Furthermore, none of the in-the-money bids are rejected, implying that there are no missed potentials for additional surplus for any bidders.

3.1.5 Integer case

With the binary restrictions on \mathbf{y} , the problem becomes mixed-integer program and properties (3.7) - (3.10) may not hold at the optimal solution of **(SMILP)**. Under a marginal pricing scheme, we may first solve **(SMILP)**. Afterwards, we may fix the values of the binary variables obtained in **(SMILP)** and solve **(SMLP)**. Then, the

optimal dual variable vector π represents the market clearing price vector. However, some in-the-money block bids may end up being rejected or some out-of-the-money block bids may end up being accepted in this case. These bids are called *paradoxically* accepted or rejected bids (Martin et al., 2014).

Definition 8. Paradoxically accepted bid (PAB) Let \mathbf{q} be the bid quantity vector, π^* be the vector of market clearing prices, and \mathbf{p} be the bid price vector. The bid is paradoxically accepted if it is accepted and $(\mathbf{p} - \pi^*)^T \mathbf{q} < 0$. That is, the accepted quantities of the bid generates negative surplus at the given market clearing prices.

Definition 9. Paradoxically rejected bid (PRB) Let \mathbf{q} be the bid quantity vector, π^* be the vector of market clearing prices, and \mathbf{p} be the bid price vector. The bid is paradoxically rejected if it is rejected and $(\mathbf{p} - \pi^*)^T \mathbf{q} > 0$. That is, the bid is rejected even though its acceptance would generate positive surplus at the given market clearing prices.

At the optimal solution of (SMILP), both PABs and PRBs may occur among the block bids. The market clearing prices and the surplus maximizing quantities for hourly bids are at equilibrium since the problem is convex once the block bid decisions are fixed. The market operator may choose to compensate PABs by paying as much as the associated loss and PRBs by paying as much as the missed surplus. The total payments made by the market operator is called the *uplift payments*. Uplift payments create the missing money problem for the market operator since it pays more to sellers and receives less from the buyers than accounted for by the solution.

Due to the uplift payments, the market operator deviates from uniform pricing since different bidders may be settled with different energy prices for the same unit of energy. Although this is not an issue in markets with non-uniform pricing schemes, it creates an unfair energy pricing between market participants in markets that are designed with a uniform pricing scheme.

Let B_{pab} and B_{prb} be the sets of PABs and PRBs, respectively. The total uplift payment for PABs becomes:

$$TU_{pab} = \sum_{b \in B_{pab}} (\pi^* - \mathbf{p}_b)^T \mathbf{q}_b \quad (3.11)$$

Similarly, the total uplift payment for PRBs becomes:

$$TU_{prb} = \sum_{b \in B_{prb}} (\mathbf{p}_b - \boldsymbol{\pi}^*)^T \mathbf{q}_b \quad (3.12)$$

We next define and formulate the surplus maximization problem under constraints that prevent or limit TU_{pab} or TU_{prb} .

3.2 Surplus maximization problem under pricing constraints

In this section, we give a primal-dual formulation of the surplus maximization problem with pricing variables, and define a generalized problem, **(SMILP-GU)**, that sets upper bounds on both TU_{pab} and TU_{prb} . We call these upper bounds as the *pricing constraints*. The formulation can be used to enforce market design rules such as rejecting all out-of-the-money bids (as in the EU markets) or accepting all in-the-money bids (as in the Turkish market). Similarly, it can be used to limit the total market loss associated with the accepted out-of-the-money bids, TU_{pab} , or to limit total opportunity cost associated with the rejected in-the-money bids, TU_{prb} .

We use the primal-dual formulation given by Madani and Van Vyve (2014). Let $\bar{\mathbf{y}}$ be a given commitment vector for the set of block bids and **(SMLP($\bar{\mathbf{y}}$))** be the linear program obtained by setting $\mathbf{y} = \bar{\mathbf{y}}$. Let B_0 and B_1 be a partition of B such that $B_0 = \{b \in B : \bar{y}_b = 0\}$ and $B_1 = \{b \in B : \bar{y}_b = 1\}$. Consider the following linear program:

(SMLP($\bar{\mathbf{y}}$)):

$$\text{Max } S_L(\mathbf{x}, \mathbf{y}) = \sum_{t \in T} \left\{ \sum_{h \in H} p_h q_{h,t} x_h + \sum_{b \in B} p_b q_{b,t} y_b \right\}$$

s.to.

$$\sum_{h \in H} q_{h,t} x_h + \sum_{b \in B} q_{b,t} y_b = 0 \quad \forall t \in T \quad [\pi_t]$$

$$x_h \leq 1 \quad \forall h \in H \quad [s_h]$$

$$y_b \leq 1 \quad \forall b \in B \quad [s_b]$$

$$\begin{aligned}
-y_b &\leq -1 && \forall b \in B_1 \quad [l_b] \\
y_b &\leq 0 && \forall b \in B_0 \quad [m_b] \\
x_h &\geq 0 && \forall h \in H \\
y_b &\geq 0 && \forall b \in B
\end{aligned}$$

The dual of **(SMLP(\bar{y}))** is:

(D-SMLP(\bar{y})):

$$\begin{aligned}
\text{Min} \quad & \sum_{h \in H} s_h + \sum_{b \in B} s_b - \sum_{b \in B_1} l_b \\
\text{s.to.} \quad & \\
s_h &\geq \sum_{t \in T} (p_h - \pi_t) q_{h,t} && \forall h \in H \\
s_b - l_b &\geq \sum_{t \in T} (p_b - \pi_t) q_{b,t} && \forall b \in B_1 \\
s_b + m_b &\geq \sum_{t \in T} (p_b - \pi_t) q_{b,t} && \forall b \in B_0 \\
s_h &\geq 0 && \forall h \in H \\
s_b, l_b, m_b &\geq 0 && \forall b \in B
\end{aligned}$$

If $(\mathbf{x}^*, \bar{\mathbf{y}})$ is a feasible solution to **(SMLP(\bar{y}))**, then **(D-SMLP(\bar{y}))** is also feasible. Then, $(\mathbf{x}^*, \bar{\mathbf{y}})$ is an optimal solution to **(SMLP(\bar{y}))** if it satisfies the following complementary slackness constraints.

(CS-SMLP(\bar{y})):

$$\begin{aligned}
s_h(1 - x_h^*) &= 0 && \forall h \in H \\
s_b(1 - \bar{y}_b) &= 0 && \forall b \in B \\
l_b(1 - \bar{y}_b) &= 0 && \forall b \in B_1 \\
m_b \bar{y}_b &= 0 && \forall b \in B_0 \\
x_h^* (s_h - \sum_{t \in T} (p_h - \pi_t) q_{h,t}) &= 0 && \forall h \in H \\
\bar{y}_b (s_b - l_b - \sum_{t \in T} (p_b - \pi_t) q_{b,t}) &= 0 && \forall b \in B_1 \\
\bar{y}_b (s_b + m_b - \sum_{t \in T} (p_b - \pi_t) q_{b,t}) &= 0 && \forall b \in B_0
\end{aligned}$$

The complementary slackness constraints ensure that $l_b m_b = 0, \forall b \in B$. Madani and Van Vyve (2014) show that l_b and m_b are upper bounds on the loss and missed surplus of bid b , respectively. In addition, they replace the complementary slackness constraints with the strong duality constraint and state the following mathematical program with equilibrium constraints:

(E-SMILP):

$$\begin{aligned}
\text{Max} \quad & \sum_{t \in T} \left\{ \sum_{h \in H} p_h q_{h,t} x_h + \sum_{b \in B} p_b q_{b,t} y_b \right\} \\
\text{s.to.} \quad & \\
& \sum_{h \in H} q_{h,t} x_h + \sum_{b \in B} q_{b,t} y_b = 0 && \forall t \in T \\
& x_h \leq 1 && \forall h \in H \\
& y_b \leq 1 && \forall b \in B \\
& s_h \geq \sum_{t \in T} (p_h - \pi_t) q_{h,t} && \forall h \in H \\
& s_b - l_b + m_b \geq \sum_{t \in T} (p_b - \pi_t) q_{b,t} && \forall b \in B \\
& m_b \leq M_b (1 - y_b) && \forall b \in B \\
& l_b \leq M_b y_b && \forall b \in B \\
& \sum_{t \in T} \left\{ \sum_{h \in H} p_h q_{h,t} x_h + \sum_{b \in B} p_b q_{b,t} y_b \right\} \geq \sum_{h \in H} s_h + \sum_{b \in B} s_b - \sum_{b \in B} l_b \\
& x_h, s_h \geq 0 && \forall h \in H \\
& y_b \in \{0, 1\} && \forall b \in B \\
& s_b, l_b, m_b \geq 0 && \forall b \in B
\end{aligned}$$

In this model, the dual problem constraints associated with B_1 and B_0 are combined and the property that $l_b m_b = 0, \forall b \in B$ is modeled by the big-M constraints where M_b is an appropriate upper bound on the loss or missed surplus of a block bid b . Denoting the maximum and the minimum allowable bid prices as p^{max} and p^{min} , respectively, $M_b \geq \sum_{t \in T} |q_{b,t}| (p^{max} - p^{min})$ is an upper bound on the loss or missed surplus of a block bid, and is a sufficiently large big-M value. We represent the feasible set of **(E-SMILP)** by Ψ , and define the surplus maximization problem under generalized

uplift limits by imposing bounds on both the total market loss and the total missed profit:

(SMILP-GU):

$$\begin{aligned} \text{Max} \quad & \sum_{t \in T} \left\{ \sum_{h \in H} p_h q_{h,t} x_h + \sum_{b \in B} p_b q_{b,t} y_b \right\} \\ \text{s.to.} \quad & (\mathbf{x}, \mathbf{y}, \boldsymbol{\pi}, \mathbf{s}, \mathbf{l}, \mathbf{m}) \in \Psi \\ & \sum_{b \in B} l_b \leq \overline{TU}_{pab} \quad \forall b \in B \\ & \sum_{b \in B} m_b \leq \overline{TU}_{prb} \quad \forall b \in B \end{aligned}$$

(SMILP-GU) with $\overline{TU}_{pab} = 0$ and $\overline{TU}_{prb} > M = \sum_{b \in B} M_b$ corresponds to a special case, **(SMILP-NoPAB)**, which we define next. In this case, the market loss is prevented by eliminating solutions with PABs and there is no binding constraint on the market missed surplus. In order to prevent PABs, it is sufficient to add constraints $l_b \leq 0, \forall b \in B$ to **(E-SMILP)**. We define this restricted model below:

(SMILP-NoPAB):

$$\begin{aligned} \text{Max} \quad & \sum_{t \in T} \left\{ \sum_{h \in H} p_h q_{h,t} x_h + \sum_{b \in B} p_b q_{b,t} y_b \right\} \\ \text{s.to.} \quad & (\mathbf{x}, \mathbf{y}, \boldsymbol{\pi}, \mathbf{s}, \mathbf{l}, \mathbf{m}) \in \Psi \\ & l_b \leq 0 \quad \forall b \in B \end{aligned}$$

(SMILP-NoPAB) is always feasible since the feasible set of **(SMLP($\bar{\mathbf{y}}$))** for $\bar{\mathbf{y}} = \mathbf{0}$ and $\bar{l}_b \leq 0, \forall b \in B$ is non-empty. Let $(\mathbf{x}^*, \mathbf{y}^*)$ be the optimal values of the primal variables.

$TU_{prb} = \min \left\{ \sum_{b \in B} m_b : (\mathbf{x}, \mathbf{y}, \boldsymbol{\pi}, \mathbf{s}, \mathbf{l}, \mathbf{m}) \in \Psi, l_b \leq 0, \forall b \in B, (\mathbf{x}, \mathbf{y}) = (\mathbf{x}^*, \mathbf{y}^*) \right\}$ is the minimum total missed surplus, and $TU_{pab} = 0$ is the total loss. In European markets, the market operator does not pay uplift to PRBs and TU_{prb} is regarded as a foregone opportunity. In a similar manner, **(SMILP-NoPRB)** can be stated as **(SMILP-GU)** with $\overline{TU}_{pab} > M$ and $\overline{TU}_{prb} = 0$.

3.3 Benders decomposition

Since the surplus maximization problem is easy to solve in the absence of the pricing variables and constraints, Benders decomposition (Benders, 1962) has the potential to perform well in solving **(SMILP-GU)**. We define **(SMILP)** as the *master problem* and the problem of finding a market clearing price vector satisfying the pricing constraints as the *subproblem*. A feasible solution to the master problem will also be feasible to **(SMILP-GU)** if there exists a set of prices in the subproblem that satisfy the hourly bid equilibrium and the uplift constraints. Otherwise, the solution cannot be feasible for **(SMILP-GU)** and must be eliminated by adding an appropriate cut. Every feasible solution of **(SMILP-GU)** should satisfy a valid cut, and the cut should eliminate the solution identified as infeasible, at the minimum. We next present the master problem **(MP)** and subproblems **(SP)**:

(MP):

$$\begin{aligned}
 & \text{Max} \quad S_L(\mathbf{x}, \mathbf{y}) \\
 & \text{s.to.} \\
 & \quad \sum_{h \in H} p_h q_{h,t} x_h + \sum_{b \in B} p_b q_{b,t} y_b = 0 \quad \forall t \in T \\
 & \quad x_h \leq 1 \quad \forall h \in H \\
 & \quad x_h \geq 0 \quad \forall h \in H \\
 & \quad y_b \in \{0, 1\} \quad \forall b \in B
 \end{aligned}$$

Let Z be the set of feasible points of **(SMILP)**, Ω be the set of feasible points of **(SMILP-GU)**, $\rho \in P$ be the projection of $\omega \in \Omega$ onto Z and $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \in Z$ be such that $\bar{\mathbf{x}}$ is the optimal solution of **(SMLP)($\bar{\mathbf{y}}$)**. For a given $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$, the solution of the linear program **(SP)($\bar{\mathbf{x}}, \bar{\mathbf{y}}$)** reveals if feasible market clearing prices that satisfy the uplift constraints associated with block bids and hourly bid equilibrium constraints exist.

(SP)($\bar{\mathbf{x}}, \bar{\mathbf{y}}$):

$$\begin{aligned}
 & \text{Min} \quad 0 \\
 & \text{s.to.}
 \end{aligned}$$

$$\begin{aligned}
s_h + \sum_{t \in T} \pi_t q_{h,t} &\geq \sum_{t \in T} p_h q_{h,t} && \forall h \in H \quad [x_h^d] \\
s_b - l_b + \sum_{t \in T} \pi_t q_{b,t} &\geq \sum_{t \in T} p_b q_{b,t} && \forall b \in B_1 \quad [y_b^{d,1}] \\
s_b + m_b + \sum_{t \in T} \pi_t q_{b,t} &\geq \sum_{t \in T} p_b q_{b,t} && \forall b \in B_0 \quad [y_b^{d,0}] \\
-\sum_{h \in H} s_h - \sum_{b \in B} s_b + \sum_{b \in B_1} l_b &\geq -S_L(\bar{\mathbf{x}}, \bar{\mathbf{y}}) && [\phi] \\
-\sum_{b \in B_1} l_b &\geq -\overline{TU}_{pab} && [\alpha] \\
-\sum_{b \in B_0} m_b &\geq -\overline{TU}_{prb} && [\beta] \\
s_h &\geq 0 && \forall h \in H \\
s_b, l_b, m_b &\geq 0 && \forall b \in B
\end{aligned}$$

The variables in brackets show the dual variables associated with each constraint of the subproblem. The dual problem is:

(DSP($\bar{\mathbf{x}}, \bar{\mathbf{y}}$)):

$$\text{Max} \quad S_L(\mathbf{x}^d, \mathbf{y}^d) - \phi S_L(\bar{\mathbf{x}}, \bar{\mathbf{y}}) - \alpha \overline{TU}_{pab} - \beta \overline{TU}_{prb}$$

s.to.

$$\begin{aligned}
x_h^d - \phi &\leq 0 && \forall h \in H \\
y_b^{d,1} - \phi &\leq 0 && \forall b \in B_1 \\
y_b^{d,0} - \phi &\leq 0 && \forall b \in B_0 \\
-y_b^{d,1} + \phi - \alpha &\leq 0 && \forall b \in B_1 \\
y_b^{d,0} - \beta &\leq 0 && \forall b \in B_0 \\
\sum_{h \in H} q_{h,t} x_h^d + \sum_{b \in B_1} q_{b,t} y_b^{d,1} + \sum_{b \in B_0} q_{b,t} y_b^{d,0} &= 0 && \forall t \in T \\
x_h^d, y_b^{d,1}, y_b^{d,0}, \phi, \alpha, \beta &\geq 0
\end{aligned}$$

where $S_L(\mathbf{x}^d, \mathbf{y}^d) = \sum_{t \in T} \left\{ \sum_{h \in H} p_h q_{h,t} x_h^d + \sum_{b \in B_1} p_b q_{b,t} y_b^{d,1} + \sum_{b \in B_0} p_b q_{b,t} y_b^{d,0} \right\}$, and $\mathbf{y}^d = (y_b^{d,1}, y_b^{d,0})$. **(DSP($\bar{\mathbf{x}}, \bar{\mathbf{y}}$))** is feasible since the trivial solution, $\mathbf{x}^d = y_b^{d,1} = y_b^{d,0} = 0$, $\phi = \alpha = \beta = 0$. The optimal objective function value of this trivial solution is non-negative. Hence, **(SP($\bar{\mathbf{x}}, \bar{\mathbf{y}}$))** is infeasible if and only if **(DSP($\bar{\mathbf{x}}, \bar{\mathbf{y}}$))** is unbounded.

If $(\mathbf{DSP}(\bar{\mathbf{x}}, \bar{\mathbf{y}}))$ is unbounded, there exists feasible solutions, $(\bar{\mathbf{x}}^d, \bar{\mathbf{y}}^{d,1}, \bar{\mathbf{y}}^{d,0}, \bar{\phi}, \bar{\alpha}, \bar{\beta})$, such that $S_L(\bar{\mathbf{x}}^d, \bar{\mathbf{y}}^d) - \bar{\phi}S_L(\bar{\mathbf{x}}, \bar{\mathbf{y}}) - \bar{\alpha}\overline{TU}_{pab} - \bar{\beta}\overline{TU}_{prb} > 0$. For $\bar{\phi} > 0$, $\mathbf{d} = (\bar{\mathbf{x}}^d, \bar{\mathbf{y}}^{d,1}, \bar{\mathbf{y}}^{d,0}, \bar{\phi}, \bar{\alpha}, \bar{\beta})$ is a direction of unboundedness. Then, $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \in Z$ has to satisfy

$$S_L(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \geq \frac{1}{\bar{\phi}}S_L(\bar{\mathbf{x}}^d, \bar{\mathbf{y}}^d) - \frac{\bar{\alpha}}{\bar{\phi}}\overline{TU}_{pab} - \frac{\bar{\beta}}{\bar{\phi}}\overline{TU}_{prb} \quad (3.13)$$

if $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \in P$. By setting $\phi = 1$ and dropping the fixed term $S_L(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ in the objective function, we can rewrite the dual subproblem as a bounded dual subproblem, $(\mathbf{BDSP}(\bar{\mathbf{x}}, \bar{\mathbf{y}}))$, as follows:

$(\mathbf{BDSP}(\bar{\mathbf{x}}, \bar{\mathbf{y}}))$:

$$\begin{aligned} \text{Max} \quad & S_L(\mathbf{x}^d, \mathbf{y}^d) - \alpha\overline{TU}_{pab} - \beta\overline{TU}_{prb} \\ \text{s.to.} \quad & \\ & x_h^d \leq 1 \quad \forall h \in H \\ & y_b^{d,1} \leq 1 \quad \forall b \in B_1 \\ & y_b^{d,0} \leq 1 \quad \forall b \in B_0 \\ & \alpha \geq 1 - y_b^{d,1} \quad \forall b \in B_1 \\ & \beta \geq y_b^{d,0} \quad \forall b \in B_0 \\ & \sum_{h \in H} q_{h,t}x_h^d + \sum_{b \in B_1} q_{b,t}y_b^{d,1} + \sum_{b \in B_0} q_{b,t}y_b^{d,0} = 0 \quad \forall t \in T \\ & x_h^d, y_b^{d,1}, y_b^{d,0}, \alpha, \beta \geq 0 \end{aligned}$$

In this case, the objective function of $(\mathbf{BDSP}(\bar{\mathbf{x}}, \bar{\mathbf{y}}))$ is bounded from above and there is no direction of unboundedness. If there exist $(\bar{\mathbf{x}}^d, \bar{\mathbf{y}}^{d,1}, \bar{\mathbf{y}}^{d,0}, \bar{\alpha}, \bar{\beta})$ such that $S_L(\bar{\mathbf{x}}^d, \bar{\mathbf{y}}^d) - \bar{\alpha}\overline{TU}_{pab} - \bar{\beta}\overline{TU}_{prb} > S_L(\bar{\mathbf{x}}, \bar{\mathbf{y}})$, then $(\mathbf{DSP}(\bar{\mathbf{x}}, \bar{\mathbf{y}}))$ becomes unbounded in the direction $\mathbf{d} = (\bar{\mathbf{x}}^d, \bar{\mathbf{y}}^{d,1}, \bar{\mathbf{y}}^{d,0}, 1, \bar{\alpha}, \bar{\beta})$. Otherwise, the problem cannot be unbounded for any other $\phi \neq 1$ as well and we can conclude that $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \in P$. Let $(\mathbf{x}^{d*}(\bar{\mathbf{x}}, \bar{\mathbf{y}}), \mathbf{y}^{d*}(\bar{\mathbf{x}}, \bar{\mathbf{y}}), \alpha^*(\bar{\mathbf{x}}, \bar{\mathbf{y}}), \beta^*(\bar{\mathbf{x}}, \bar{\mathbf{y}}))$ be an optimal solution to $(\mathbf{BDSP}(\bar{\mathbf{x}}, \bar{\mathbf{y}}))$. We can rewrite equation (3.13) as:

$$S_L(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \geq S_L(\mathbf{x}^{d*}(\bar{\mathbf{x}}, \bar{\mathbf{y}}), \mathbf{y}^{d*}(\bar{\mathbf{x}}, \bar{\mathbf{y}})) - \alpha^*(\bar{\mathbf{x}}, \bar{\mathbf{y}})\overline{TU}_{pab} - \beta^*(\bar{\mathbf{x}}, \bar{\mathbf{y}})\overline{TU}_{prb} \quad (3.14)$$

Proposition 1. Let $\overline{TU}_{pab} = \overline{TU}_{prb} = 0$ and $\omega \in \Omega$. Then, $\rho \in P$ is optimal to (\mathbf{SMLP}) .

Proof. Let $(\bar{x}, \bar{y}) = \rho \in P$ and consider $(\mathbf{BDSP}(\bar{x}, \bar{y}))$. Since $\overline{TU}_{pab} = \overline{TU}_{prb} = 0$, $\alpha \geq 1 - y_b^{d,1}, \forall b \in B_1$ and $\beta \geq y_b^{d,0}, \forall b \in B_0$ constraints become redundant and can be removed from the problem together with α and β variables. Then, $(\mathbf{BDSP}(\bar{x}, \bar{y}))$ is equivalent to (\mathbf{SMLP}) . This implies that $S_L(\bar{x}, \bar{y}) \geq S_L(\mathbf{x}^*, \mathbf{y}^*), \forall \rho = (\bar{x}, \bar{y}) \in P$, where $(\mathbf{x}^*, \mathbf{y}^*)$ is the optimal solution of (\mathbf{SMLP}) . \square

Proposition 1 implies that the market operator can achieve equilibrium only when the duality gap of (\mathbf{SMILP}) is zero. Otherwise, there exist a positive market loss or market missed surplus for the optimal allocation, and the optimal allocation is not surplus maximizing.

For any $(\hat{x}, \hat{y}) \in Z$ such that $\hat{y} = \bar{y}$, $S_L(\hat{x}, \hat{y}) = S_L(\bar{x}, \bar{y})$ and the optimal objective function value of $(\mathbf{BDSP}(\bar{x}, \bar{y}))$ is equal to that of $(\mathbf{BDSP}(\hat{x}, \hat{y}))$. Hence, if $(\bar{x}, \bar{y}) \notin P$ the following inequality is valid for $(\mathbf{SMILP-GU})$.

$$\sum_{b \in B_1} (1 - y_b) + \sum_{b \in B_0} y_b \geq 1 \quad (3.15)$$

Inequality (3.15) ensures that for a feasible solution of $(\mathbf{SMILP-GU})$, at least one of the rejected block bids at \bar{y} must be accepted or at least one of the accepted block bids at \bar{y} must be rejected. These so-called ‘‘no-good’’ cuts have been frequently used in the literature for solving MILP problems, and Martin et al. (2014) and Madani and Van Vyve (2015) also use these cuts in their proposed Benders decomposition algorithms to solve $(\mathbf{SMILP-NoPAB})$.

3.3.1 Solving $(\mathbf{SMILP-NoPAB})$ problem

In this section, we examine a special case of $(\mathbf{SMILP-GU})$ in which $\overline{TU}_{pab} = 0$ and $\overline{TU}_{prb} > M$. At a feasible solution of this problem, there are no PABs so that every bidder will be settled with the same unit energy price, the market clearing price.

Reconsidering $(\mathbf{BDSP}(\bar{x}, \bar{y}))$, let $(\mathbf{x}^{d*}(\bar{x}, \bar{y}), \mathbf{y}^{d*}(\bar{x}, \bar{y}), \alpha^*(\bar{x}, \bar{y}), \beta^*(\bar{x}, \bar{y}))$ be an optimal solution to $(\mathbf{BDSP}(\bar{x}, \bar{y}))$. $\beta^*(\bar{x}, \bar{y}) = 0$ when $\overline{TU}_{prb} > M$. As a result, $\mathbf{y}^{d,0*}(\bar{x}, \bar{y}) = \mathbf{0}$. In addition, $\mathbf{y}^{d,1*}(\bar{x}, \bar{y})$ is not restricted by $\alpha^*(\bar{x}, \bar{y})$ since $\overline{TU}_{pab} = 0$. Using these properties, we can simplify $(\mathbf{BDSP}(\bar{x}, \bar{y}))$ as follows:

(BDSP(\bar{x}, \bar{y})-NoPAB):

$$\text{Max } S_L(\mathbf{x}^d, \mathbf{y}^d)$$

s.to.

$$\begin{aligned} x_h^d &\leq 1 && \forall h \in H \\ y_b^{d,1} &\leq 1 && \forall b \in B_1 \\ y_b^{d,0} &= 0 && \forall b \in B_0 \\ \sum_{h \in H} q_{h,t} x_h^d + \sum_{b \in B_1} q_{b,t} y_b^{d,1} &= 0 && \forall t \in T \\ x_h^d, y_b^{d,1} &\geq 0 \end{aligned}$$

For **(SMILP-NoPAB)**, $S_L(\bar{x}, \bar{y}) \leq S_L(\mathbf{x}^{d*}(\bar{x}, \bar{y}), \mathbf{y}^{d*}(\bar{x}, \bar{y}))$ since $(\bar{x}, \bar{y}) \in Z$ is a feasible solution to **(BDSP(\bar{x}, \bar{y})-NoPAB)**. If $(\bar{x}, \bar{y}) \in P$, then it must satisfy the following: $S_L(\bar{x}, \bar{y}) \geq S_L(\mathbf{x}^{d*}(\bar{x}, \bar{y}), \mathbf{y}^{d*}(\bar{x}, \bar{y}))$. This implies that $S_L(\bar{x}, \bar{y}) = S_L(\mathbf{x}^{d*}(\bar{x}, \bar{y}), \mathbf{y}^{d*}(\bar{x}, \bar{y}))$, $\forall (\bar{x}, \bar{y}) \in P$.

Inequality (3.15) is valid for **(SMILP-NoPAB)** as well. In Madani and Van Vyve (2015), a strengthened version of inequality (3.15) is shown to be valid but only in the sub-tree associated with the node solution (\bar{x}, \bar{y}) of **(MP)**. So, inequality (3.16) can be added to each node of the sub-tree if solution (\bar{x}, \bar{y}) is infeasible for **(SMILP-NoPAB)**.

$$\sum_{b \in B_1} (1 - y_b) \geq 1 \tag{3.16}$$

It is possible to extend (3.16) into a globally valid inequality when the optimal objective function value of **(BDSP(\bar{x}, \bar{y})-NoPAB)** is larger than that of **(SMILP)**. We show this formally in the following proposition.

Proposition 2. *Let $(\mathbf{x}^*, \mathbf{y}^*)$ be an optimal solution to **(SMILP)**. Given $(\bar{x}, \bar{y}) \in Z$, if $S_L(\mathbf{x}^{d*}(\bar{x}, \bar{y}), \mathbf{y}^{d*}(\bar{x}, \bar{y})) > S_L(\mathbf{x}^*, \mathbf{y}^*)$, then $\sum_{b \in B_1} (1 - y_b) \geq 1$ is a valid inequality for **(SMILP-NoPAB)**.*

Proof. Suppose $(\mathbf{x}', \mathbf{y}') \in Z$ such that $B_1 \subseteq B'_1 = \{b \in B : y'_b = 1\}$. Then, the optimal objective function value of **(BDSP(\mathbf{x}', \mathbf{y}')-NoPAB)** is at least as large as that of **(BDSP(\bar{x}, \bar{y})-NoPAB)**, $S_L(\mathbf{x}^{d*}(\mathbf{x}', \mathbf{y}'), \mathbf{y}^{d*}(\mathbf{x}', \mathbf{y}')) \geq S_L(\mathbf{x}^{d*}(\bar{x}, \bar{y}), \mathbf{y}^{d*}(\bar{x}, \bar{y}))$, and

hence $S_L(\mathbf{x}^{d*}(\mathbf{x}', \mathbf{y}'), \mathbf{y}^{d*}(\mathbf{x}', \mathbf{y}')) > S_L(\mathbf{x}^*, \mathbf{y}^*)$. Due to the optimality of $(\mathbf{x}^*, \mathbf{y}^*)$ for **(SMILP)**, $S_L(\mathbf{x}^*, \mathbf{y}^*) \geq S_L(\mathbf{x}, \mathbf{y})$ for any $(\mathbf{x}, \mathbf{y}) \in Z$. Then, $S_L(\mathbf{x}', \mathbf{y}') < S_L(\mathbf{x}^{d*}(\mathbf{x}', \mathbf{y}'), \mathbf{y}^{d*}(\mathbf{x}', \mathbf{y}'))$ and $(\mathbf{x}', \mathbf{y}') \notin P$. Therefore, at least one accepted block bid must be rejected in a feasible solution to P , $\sum_{b \in B_1} (1 - y_b) \geq 1$.

Since $S_L(\mathbf{x}^*, \mathbf{y}^*) \geq S_L(\mathbf{x}, \mathbf{y})$ for any $(\mathbf{x}, \mathbf{y}) \in Z$ due to the optimality of $(\mathbf{x}^*, \mathbf{y}^*)$ for **(SMILP)**, $S_L(\mathbf{x}^{d*}(\mathbf{x}', \mathbf{y}'), \mathbf{y}^{d*}(\mathbf{x}', \mathbf{y}')) > S_L(\mathbf{x}', \mathbf{y}')$ and $(\mathbf{x}', \mathbf{y}') \notin P$. Therefore, at least one accepted block bid must be rejected in a feasible solution to P , $\sum_{b \in B_1} (1 - y_b) \geq 1$. \square

Solving **(SMILP)** before solving **(SMILP-NoPAB)** and incorporating (3.16) as a globally-valid inequality (when the condition in Proposition 2 holds) could be beneficial if **(SMILP)** is an easy problem to solve. We next develop a *strengthened* version of cut (3.15) and show that it is *globally valid*. We call these cuts as “price-based” cuts.

Let B^s and B^d be the set of supply and demand block bids, respectively. Let indicator variable $\delta_{b, \hat{b}} = 1$ denote that the block bids b and \hat{b} have at least one common period, and 0 otherwise. For each \hat{b} , we add inequality (3.17), if it is a supply PAB, and inequality (3.18), if it is a demand PAB, to **(MP)**.

$$(1 - y_{\hat{b}}) + \sum_{b \in B^s: \bar{y}_b=1, \delta_{b, \hat{b}}=1} (1 - y_b) + \sum_{b \in B^d: \bar{y}_b=0, \delta_{b, \hat{b}}=1} y_b \geq 1 \quad (3.17)$$

$$(1 - y_{\hat{b}}) + \sum_{b \in B^s: \bar{y}_b=0, \delta_{b, \hat{b}}=1} y_b + \sum_{b \in B^d: \bar{y}_b=1, \delta_{b, \hat{b}}=1} (1 - y_b) \geq 1 \quad (3.18)$$

The idea here is that once we have a supply (demand) PAB at the new incumbent solution obtained at a node of the master problem branch-and-bound tree, we must either reject this bid or increase (decrease) the average market clearing price of the periods for which this bid is offered. In order to increase (decrease) the average market clearing price for those periods, we must either reject at least one accepted supply (demand) block bid in one of those periods or accept at least one rejected demand (supply) block bid in one of those periods. We show in Proposition 3 that inequalities (3.17) are both valid and stronger than the “no-good” cuts. A similar

proof can also be made for the demand PABs to prove the same for inequalities (3.18). In order to assess if a block bid is a PAB or PRB, we need to find market clearing prices corresponding to $\bar{\mathbf{y}}$, by solving **SMLP**($\bar{\mathbf{y}}$) (the corresponding market clearing price vector is the optimal value of the dual variable $\boldsymbol{\pi}$).

Proposition 3. *Inequalities (3.17) are valid for (SMILP-NoPAB) and stronger than the “no-good” cuts.*

Proof. Part 1. We first show that inequality (3.17) is a valid inequality for **(SMILP-NoPAB)**. Given $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \in Z$ and $\boldsymbol{\pi}^*$ be the corresponding market clearing price vector, let $\hat{b} \in B_1$ be a supply PAB, $\mathbf{q}_{\hat{b}} < 0$, $l_{\hat{b}}^* = -\sum_{t \in T} (p_{\hat{b}} - \pi_t^*) q_{\hat{b},t} > 0$. For $\boldsymbol{\omega}' \in \Omega$, \mathbf{y}' and l' have to satisfy the complementary slackness constraints, $l'_b(1 - y'_b), \forall \hat{b} \in B_1$. There are two cases:

Case 1. If $y'_{\hat{b}} = 0$, $(1 - y'_{\hat{b}}) = 1 \geq 1$ satisfies inequality (3.17).

Case 2. If $y'_{\hat{b}} = 1$, $l'_{\hat{b}} = 0$ holds only if $\sum_{t \in T} (p_{\hat{b}} - \pi'_t) q_{\hat{b},t} \geq 0 > \sum_{t \in T} (p_{\hat{b}} - \pi_t^*) q_{\hat{b},t}$. That is, $\sum_{t \in T} \pi'_t q_{\hat{b},t} < \sum_{t \in T} \pi_t^* q_{\hat{b},t}$. This implies that there exists $\hat{t} \in T_{\hat{b}}$ such that $\pi'_{\hat{t}} > \pi_{\hat{t}}^*$ since $q_{\hat{b},t} \leq 0, \forall t \in T$. Let $Q_{\mathbf{y}', \hat{t}} = \sum_{b \in B} q_{b, \hat{t}} y'_b$ and $Q_{\bar{\mathbf{y}}, \hat{t}} = \sum_{b \in B} q_{b, \hat{t}} \bar{y}_b$ be the total accepted block bid quantities in period $\hat{t} \in T$. Similarly, let $Q_{\mathbf{x}', \hat{t}} = \sum_{h \in H} q_{h, \hat{t}} x'_h$ and $Q_{\bar{\mathbf{x}}, \hat{t}} = \sum_{h \in H} q_{h, \hat{t}} \bar{x}_h$ be the total accepted hourly bid quantities in period $\hat{t} \in T$. Since $Q_{\mathbf{y}', \hat{t}} + Q_{\mathbf{x}', \hat{t}} = 0$ and $Q_{\bar{\mathbf{y}}, \hat{t}} + Q_{\bar{\mathbf{x}}, \hat{t}} = 0$, $\pi'_{\hat{t}} > \pi_{\hat{t}}^*$ implies that $Q_{\mathbf{x}', \hat{t}} < Q_{\bar{\mathbf{x}}, \hat{t}}$ due to the hourly bid equilibrium constraints. Therefore, $Q_{\mathbf{y}', \hat{t}} > Q_{\bar{\mathbf{y}}, \hat{t}}$. This can only be achieved if either at least one accepted supply block bid covering period \hat{t} is rejected or at least one rejected demand block bid covering period \hat{t} is accepted. That is,

$$\sum_{b \in B^s: \bar{y}_b=1, \delta_{b, \hat{b}}=1} (1 - y_b) + \sum_{b \in B^d: \bar{y}_b=0, \delta_{b, \hat{b}}=1} y_b \geq 1.$$

Either *Case 1* or *Case 2* must hold for every feasible solution of **(SMILP-NoPAB)**. This proves that inequality (3.17) is a valid inequality for **(SMILP-NoPAB)**.

Part 2. We next show that inequality (3.17) is stronger than inequality (3.15). For any \mathbf{y} , the left-hand side of inequality (3.15) is at least as large as the left-hand side of inequality (3.17). If \mathbf{y} violates inequality (3.17), it also violates inequality (3.15). However, the reverse is not true when none of the accepted supply bids at $\bar{\mathbf{y}}$ covering a period in $T_{\hat{b}}$ is rejected at \mathbf{y} , or none of the rejected demand bids at $\bar{\mathbf{y}}$ covering a

period in T_b is accepted at \mathbf{y} . □

Proposition 4. *Inequalities (3.18) are valid for (SMILP-NoPAB) and stronger than the “no-good” cuts.*

Proof. Proof is similar to that of Proposition 3. □

3.4 Computational Results

In this section, we present the computational results for the approaches from the literature as well as our approach.. In particular, we evaluate the surplus maximization problem with no PAB, (SMILP-NoPAB), employing:

1. Primal-dual formulation: **PD**
2. Benders decomposition with
 - (a) *No-Good* cut, cut (3.15), used in Martin et al. (2014): **BD-NG**
 - (b) *Locally Valid* cut, cut (3.16), used in Madani and Van Vyve (2015): **BD-LV**
 - (c) *Price-Based* cuts we developed, cuts (3.17) and (3.18): **BD-PB**

We run our models using two leading commercial solvers available, IBM ILOG Cplex and Gurobi to assess the impact of the mixed-integer programming solver on the performance of the solution methods, if any. Our aim is to reveal the best-performing solution approach to solve the DAM clearing problem with no-PAB constraints and to investigate whether the performances of the approaches depend on the solver used. We also compare the Benders decomposition using the price-based cuts we developed with the available Benders decomposition algorithms in the literature for the problem we address.

We run each solution approach on a test set of 20 instances generated based on the real market data published by EXIST on the transparency platform (EXIST, 2016). EXIST publishes the full set of hourly bids submitted to the auction daily. In order

to create representative bids, we take 20 separate instances from different months of 2017 and 2018. In terms of block bids, EXIST only publishes total supply and total demand volumes of block bids accepted and rejected on an hourly basis. To generate the block bids for the selected days, we use market report EXIST (2018) and the aggregate data to approximate the real block bid data as discussed below.

The yearly market statistics report, (EXIST, 2018), contains the daily average number of supply and demand block bids in the auction as well as the share of the block bid volume in the total volume. Additionally, there are market rules limiting the price, the quantities, and the number of periods of the bids that can be submitted into the auction (EXIST, 2017). We generate block bids for our experiments maintaining all these properties. For each instance of our experiments, we generate 15,000 hourly and 150 block bids that resemble the actual bids.

We programmed our solution approaches in Python 3.5 and used Python APIs of the solvers to create the models and solve the instances. In the experiments, we used IBM ILOG Cplex 12.8.0 and Gurobi 8.0.1. We conducted the tests on a MacBook Pro with 2.6 GHz Intel Core i5 processor and 8 GB 1600 MHz DDR3 memory with a time limit of 600 seconds and 10^{-6} relative mip gap tolerance. In the Turkish DAM instances, the average market surplus is around 5×10^8 Turkish liras. A solution within 10^{-6} relative gap will have market surplus within 500 Turkish liras (less than 100 USD as of 2020) of the maximum surplus on average, which is an immaterial amount within the context of this problem. In order to examine the impact of parallel tree search on the performances of the approaches, we conducted tests with both single and multi-thread (4 threads that were available in the computation environment) settings of the solvers. We could have conducted experiments on multiple threads only but some solver-method combinations do not work in this case and the comparisons would not have been comprehensive. For all the other parameters, we used the default configurations of the corresponding solver.

We present the solution statuses of the problem instances in the tables using the following abbreviations:

- the number of cases an optimal solution (a feasible solution satisfying the relative gap tolerance) was found (*Opt*),

- the number of cases an optimal solution could not be found but a feasible solution (that does not satisfy the relative gap tolerance) was found (*Feas*), and
- the number of cases a feasible solution could not be found within the time limit (*Inf*).

In our results to follow, we display average and maximum run times as well as the relative gaps. Run time statistics are calculated only over the instances where the solution status is *Opt*, and relative gap statistics are calculated only over the instances where the solution status is *Feas* under both solvers.

We present the results of different approaches over 20 different instances of the surplus maximization problem under no-PAB constraints (**SMILP-NoPAB**) using two different solvers in Table 3.1. The first part of the table presents the results with single thread (Th=1) execution of the solvers. When using the **PD** approach, we found an optimal solution in 17 and 14 of the 20 instances with Cplex and Gurobi, respectively. The 14 instances Gurobi solved to optimality turned out to be a subset of the 17 instances Cplex solved to optimality. Over those 14 instances, Cplex solved faster than Gurobi both on average and in the worst case (Cplex solved another instance to optimality in 446.56 seconds) and found higher quality solutions for the instances neither solver could solve to optimality. Gurobi could not find a feasible solution within the time limit for two of the instances, whereas Cplex found high quality feasible solutions (with relative gaps below 15×10^{-6}) for all three instances it could not solve to optimality. In multi-thread (Th=4) execution of **PD**, the results are about the same.

For the Benders decomposition-based approaches (**BD-NG**, **BD-LV**, and **BD-PB**) we present in Table 3.1, there are differences in the implementations of the solvers due to the differences in their interfaces. In Cplex, we solve the master problem and use a *LazyConstraintCallback* to call the subproblem whenever an integer feasible solution is found at a node of the search tree. Inside the callback function, we first solve the subproblem and then add the corresponding cut(s) if the subproblem indicates that the solution is not feasible for (**SMILP-NoPAB**). Otherwise, the solution becomes the best candidate solution. Cplex automatically disables some problem reduction routines when a lazy constraint callback is included. This is necessary since the

Table 3.1: Performance results of the primal-dual and the Benders decomposition approaches for (SMILP-NoPAB)

Th	Appr	Solver	Solution			Run Time (secs)			Relative Gap ($\times 10^{-6}$)			
			Opt	Feas	Inf	BO ^a	Avg	Max	BF ^b	Avg	Max	
1	PD	Cplex	17	3	0	14	20.30	108.66	2	10	11	
		Gurobi	14	4	2		76.67	341.86		18	24	
	BD-NG	Cplex	5	15	0	5	90.33	449.08	3	86	245	
		Gurobi	7	3	10		2.70	12.68		83	142	
	BD-LV ^c	Cplex	11	9	0	11	60.42	254.61	9	11,320	27,893	
	BD-PB	Cplex	19	1	0	18	56.48	320.29	1	33	33	
		Gurobi	18	2	0		30.73	151.29		792	792	
	4	PD	Cplex	16	4	0	15	21.53	79.78	2	3	3
			Gurobi	15	2	3		64.02	462.29		36	67
		BD-NG ^d	Gurobi	7	6	7	7	29.22	143.78	6	38	88
BD-PB ^d		Gurobi	20	0	0	20	18.49	166.29	0	-	-	

^[a] BO (Both Optimal): Number of instances both solvers solved to optimality (*Opt* status)

^[b] BF (Both Feasible): Number of instances both solvers found a feasible solution but neither reached optimality (*Feas* status)

^[c] Gurobi results are not available as Gurobi MIPSOL callback API does not support adding local cuts.

^[d] Cplex results are not available as Cplex disables parallel MIP search when there exist lazy constraint callbacks.

model is not complete without the lazy constraints and the problem reductions on the incomplete model can cut off the true optimal solution. In addition, it defaults to single thread in the existence of lazy constraint callbacks since Cplex does not guarantee thread-safety of these types of callbacks. Hence, there is no straightforward multi-thread implementation of Benders decomposition approaches with Cplex.

In Gurobi, we implement a MIPSOL callback to initiate the subproblem and add the cuts when necessary. Since Gurobi Python API does not provide the capability to add locally valid inequalities in a node of the branch-and-bound tree, we cannot present test results of inequality (3.16) on the Gurobi solver. We set *LazyConstraints=1* to disable problem reductions.

With **BD-NG**, the number of instances for which an optimal solution was found is roughly the same for the two solvers. Gurobi was substantially faster than Cplex in the 5 instances for which both solvers found an optimal solution. On another 3 instances where both solvers could only find feasible solutions, Gurobi was able to find slightly higher quality solutions compared to Cplex. However, Gurobi could not find a feasible solution in half of the instances. The quality of the solutions is rather poor when we consider all 15 instances Cplex found only a feasible solution for. The average and the maximum relative gaps are 1.74×10^{-3} and 1.13×10^{-2} , respectively, over these 15 instances.

We report only Cplex tests for **BD-LV** as Gurobi MIPSOL callback API does not allow to add local cuts. Cplex found an optimal solution for 11 of the instances and found a feasible solution for the remaining 9 instances. The solution quality is rather poor for the 9 instances for which only a feasible solution could be generated.

With the **BD-PB** approach we developed, Cplex and Gurobi found an optimal solution in 19 and 18 of the 20 instances, respectively. The quality of the feasible solution Cplex found in the remaining instance was good with a relative gap of 33×10^{-6} . On the other hand, when we compare the solvers on 18 instances for which both solvers found an optimal solution, Gurobi outperforms Cplex substantially in the run time. Furthermore, when Gurobi is run with four threads, it solves all 20 instances to optimality under 20 seconds on average and about 3 minutes at maximum.

Among the Benders decomposition approaches, we observe that **BD-NG** and **BD-LV** do not perform as well as **PD** in finding an optimal solution within the imposed time limit. However, **BD-PB** we developed outperforms all other approaches in terms of all performance measures. We observe that Gurobi is much faster than Cplex when Benders decomposition is implemented, whereas the performance is flipped when the primal-dual algorithm is implemented, considering the problems solved to optimality in all cases.

To put the relative gaps into perspective, recall that 10^{-6} corresponds to a little under \$100, on average, and the problem is solved daily. Based on this, the worst case performance obtained with **BD-LV** represents a magnitude in the order of roughly $100 \times 27,893 = \$2,789,300$. The average absolute gap of **BD-LV** is about \$1,132,000

over the 9 problems for which it could find a feasible solution. The corresponding values for **BD-NG** with Cplex are around \$8,600 on average over the 3 problems and \$24,500 in the worst case. Similarly, for **BD-NG** with Gurobi and single-thread execution, the average and maximum relative gap over the 3 problems correspond to about \$8,300 and \$14,200.

In the computational tests presented above, the problem size represents the market size of the Turkish DAM. In order to test the performance of the algorithms for larger problem sizes, we create larger instances extrapolating the instances from the Turkish DAM. In order to create instances that are similar to central-western European market instances that have around 50,000 hourly bids and 600 block bids (used by Madani and Van Vyve (2015)), we merged four instances of the Turkish DAM. We generated 10 large instances in such a way that any pair of large instances have half their bids the same and the other half different.

In Table 3.2, we present the results of the created large instances for **PD** and **BD-PB**. We did not test the performances of **BD-NG** and **BD-LV** on large-sized problem instances as those performed poorly on the instances from the Turkish DAM. We run the solvers with four threads, and hence, only Gurobi results are available for **BD-PB**. Out of 10 instances, **BD-PB** was able to solve 8 instances to optimality compared to 6 instances by **PD** approach, with both solvers. Over the 5 instances solved to optimality under all settings, **BD-PB** solved the instances in a fraction of the time of **PD**. In addition to the average and maximum computational times, **BD-PB** outperforms **PD** in every single instance that was solved to optimality by both approaches. The remaining 3 instances solved to optimality by **BD-PB** took 152.40 seconds on average, and 441.11 seconds at maximum. The average and maximum relative gaps of the four feasible solutions generated by **PD** are 2×10^{-6} and 4×10^{-6} with Cplex and 6×10^{-6} and 9×10^{-6} with Gurobi. Whereas the relative gaps for the two feasible solutions generated by **BD-PB** are 2×10^{-6} and 3×10^{-6} , respectively.

We observe from all results that **BD-PB** outperforms all other approaches both in terms of solution quality and computational performance. Its computational performance is a small fraction of its competitors, especially when solved using Gurobi.

Table 3.2: Performance results of the primal-dual and the Benders decomposition approaches for (SMILP-NoPAB) on large instance set^a

Approach	Solver	Solution			Run Time (secs)			Relative Gap ($\times 10^{-6}$)		
		Opt	Feas	Inf	AO ^b	Avg	Max	AF ^c	Avg	Max
PD	Cplex	6	4	0		96.82	239.03		6	6
	Gurobi	6	4	0	5	85.10	160.54	1	2	2
BD-PB ^d	Gurobi	8	2	0		11.51	49.61		3	3

^[a] Executed with parallel MIP search utilizing 4 threads

^[b] AO (All Optimal): Number of instances both solvers solved to optimality (*Opt* status)

^[c] AF (All Feasible): Number of instances both solvers found a feasible solution but neither reached optimality (*Feas* status)

^[d] Cplex results are not available as Cplex disables parallel MIP search when there exist lazy constraint callbacks.

3.5 Surplus maximization problem with no PRB

Surplus maximization with no PRB is the current market rule in the Turkish market (Energy Exchange Istanbul, 2016). The PABs are settled from the bid prices instead of the marketing clearing prices through the uplift payments so that their “loss” is fully compensated. This creates a missing money problem for the market operator, but supports equilibrium in the market. There is no foregone opportunity for any bidder and the allocation determined for each bidder is surplus maximizing for them under the market clearing prices and the uplift payments.

We can generalize the Benders decomposition algorithm in order to solve (**SMILP-NoPRB**). The basic combinatorial cuts of the form (3.15) are also valid for this problem as they only eliminate the current integer variable vector from the feasible space of block bid decisions. We can rewrite the dual subproblem, (**BDSP**(\bar{x}, \bar{y})), for this case as follows:

(**BDSP**(\bar{x}, \bar{y})-**NoPRB**):

$$\text{Max } S_L(\mathbf{x}^d, \mathbf{y}^d)$$

s.to.

$$x_h^d \leq 1 \quad \forall h \in H$$

$$\begin{aligned}
y_b^{d,1} &= 1 & \forall b \in B_1 \\
y_b^{d,0} &\leq 1 & \forall b \in B_0 \\
\sum_{h \in H} q_{h,t} x_h^d + \sum_{b \in B_1} q_{b,t} y_b^{d,1} &= 0 & \forall t \in T \\
x_h^d, y_b^{d,0} &\geq 0
\end{aligned}$$

Suppose that $(\bar{x}, \bar{y}) \notin P$. In this case, one cannot get a feasible solution to **(SMILP-NoPRB)** in a sub-tree of a (\bar{x}, \bar{y}) unless at least one of the rejected block bids is accepted in the new solution. Hence, cut (3.19) is a valid inequality in the subtree of the node associated with (\bar{x}, \bar{y}) .

$$\sum_{b \in B_0} y_b \geq 1 \quad (3.19)$$

In addition, the following price-based cuts are valid for **(SMILP-NoPRB)**. Once we have a supply (demand) PRB, \hat{b} , we must either accept this bid or decrease (increase) the average market clearing price of the periods for which this bid is offered. In order to decrease (increase) the average market clearing price for those periods, we must either accept at least one rejected supply (demand) block bid or reject at least one accepted demand (supply) block bid in one of those periods.

$$y_{\hat{b}} + \sum_{b \in B^s: \bar{y}_b=0, \delta_{b,\hat{b}}=1} y_b + \sum_{b \in B^d: \bar{y}_b=1, \delta_{b,\hat{b}}=1} (1 - y_b) \geq 1 \quad (3.20)$$

$$y_{\hat{b}} + \sum_{b \in B^s: \bar{y}_b=1, \delta_{b,\hat{b}}=1} (1 - y_b) + \sum_{b \in B^d: \bar{y}_b=0, \delta_{b,\hat{b}}=1} y_b \geq 1 \quad (3.21)$$

3.6 More sophisticated bid types and network constrained markets

In the previous sections, we concentrated on hourly and block bids in order to keep the presentation simple. We now show that the Benders decomposition algorithm with all the cuts reviewed in Section 3.3 is a valid approach to solve the European DAM clearing problem under the existence of piece-wise linear hourly bids, linked block bids, exclusive block bids and flexible bids. We also show that the algorithm can solve the problem in the network constrained markets as well.

If there are piece-wise linear hourly bids in the market, the market surplus function becomes a quadratic function. Let p_h^0 and p_h^1 be the starting and ending prices, respectively, for an hourly bid such that $p_h^1 > p_h^0$ for supply bids and $p_h^0 > p_h^1$ for demand bids. Then, equation (3.22) gives the market surplus function:

$$S_Q(\mathbf{x}, \mathbf{y}) = \sum_{t \in T} \left\{ \sum_{h \in H} \left\{ p_h^0 q_{h,t} x_h + (p_h^1 - p_h^0) q_{h,t} \frac{x_h^2}{2} \right\} + \sum_{b \in B} p_b q_{b,t} y_b \right\} \quad (3.22)$$

In this case, the surplus maximization problem is a mixed-integer quadratic program (MIQP) instead of a mixed-integer linear program. Once the integrality requirements are relaxed, strong duality still applies as the problem is still convex. Similarly, **BDSP**(\mathbf{x}, \mathbf{y}) becomes an MIQP as well for any $(\mathbf{x}, \mathbf{y}) \in Z$. Essentially, nothing changes apart from the type of the master and subproblem solved in the Benders decomposition algorithm, and all the cuts are still valid. On the other hand, the primal-dual formulation becomes a mixed integer quadratically-constrained program (MIQCP) and can be solved by the available solvers only if the objective function and constraints satisfy certain conditions. (IBM, 2020).

If block bid \hat{b} is linked to block bid b , then \hat{b} can only be accepted when b is accepted. This can be modelled by adding $y_{\hat{b}} - y_b \leq 0$ to the constraint set for each such linked block bid pair. Similarly, if E represents a set of exclusive block bids in the same group, then we need to add $\sum_{b \in E} y_b \leq 1$ to the model for each such exclusive block bid group. A flexible bid can also be regarded as a set of exclusive block bids, e.g., creating as many single period block bids as the number of periods and adding a constraint to ensure that at most one of them can be accepted. We can generalize these constraints by assuming a constraint of the form:

$$\sum_{b \in B} \mathbf{a}_b y_b \leq \mathbf{e} \quad (3.23)$$

Assuming there are m such constraints, $\mathbf{a}_b \in \mathbb{Z}^m$ with entries $a_{b,i} \in \{-1, 0, 1\}$, $i = 1, \dots, m$, and $\mathbf{e} \in \mathbb{Z}_{\geq 0}^m$ where each entry is either 0 or 1, $e_i \in \{0, 1\}$, $\forall i = 1, \dots, m$. Following similar steps to those in Section 3.3 for the surplus maximization problem with the additional constraint (3.23), we end up with the following bounded dual subproblem for the no-PAB case:

(BDSP(\bar{x}, \bar{y})-NoPAB):

$$\begin{aligned}
& \text{Max} && S_L(\mathbf{x}^d, \mathbf{y}^d) \\
& \text{s.to.} && \\
& && x_h^d \leq 1 && \forall h \in H \\
& && y_b^{d,1} \leq 1 && \forall b \in B_1 \\
& && y_b^{d,0} = 0 && \forall b \in B_0 \\
& && \sum_{b \in B_1} \mathbf{a}_b y_b^{d,1} \leq \mathbf{e} \\
& && \sum_{h \in H} q_{h,t} x_h^d + \sum_{b \in B_1} q_{b,t} y_b^{d,1} = 0 && \forall t \in T \\
& && x_h^d, y_b^{d,1} \geq 0
\end{aligned}$$

In this case, **(BDSP(\bar{x}, \bar{y})-NoPAB)** has additional constraint $\sum_{b \in B_1} \mathbf{a}_b y_b^{d,1} \leq \mathbf{e}$. All the valid inequalities ((3.15), (3.16), (3.17), (3.18)) presented above are still valid.

European DAMs have undergone a coupling process in the last decade, with the intention of creating a single “pan-european” power market (Euphemia, 2016). To account for the constraints imposed by the capacity of the transmission network elements on the flow of electricity, European power exchanges have adopted the *zonal-pricing* methodology. In this methodology, the capacity constraints for only a set of critical transmission lines are included in the DAM clearing problem. The transmission system operator defines *bidding zones* that are separated by those critical transmission lines. The configuration of the bidding zones are to be determined such a way that the intra-zonal transmission capacity constraints are non-binding for any possible production-consumption schedule in the day-ahead stage. In case the inter-zonal transmission lines are congested, the DAM clearing prices may differ between the bidding zones.

The algorithm that is used to clear the European single DAM, Euphemia, embeds a network model in which each bidding zone is a node and the nodes are connected to each other via *arcs*. In the early days of the market coupling, the flow capacity on each arc was determined, making sure that the energy imbalance in any node is equal to the sum of the flows on the arcs connected to that node. This model is known as the Available Transmission Capacity (ATC) model. This model totally ignored the

physical laws of electricity flow, and was replaced by the *Flow-Based* model (FB) that approximates the physical flow of electricity between the nodes better. In this model, power transmission distribution factors (PTDFs) that disclose the marginal energy change on an arc due to a unit energy exchange between two nodes are used. These factors can take different values for different arcs associated with a node, in contrast with the all-equal assumption of the ATC model. Nevertheless, both network designs can be modelled as a set of linear constraints and can be embedded in the surplus maximization problem as follows:

(SMILP-Multi-node):

$$\begin{aligned}
& \text{Max} && S_L(\mathbf{x}, \mathbf{y}) \\
& \text{s.to.} && \\
& && \sum_{h \in H} q_{h,n,t} x_h + \sum_{b \in B} q_{b,n,t} y_b + \delta_{n,t} = 0 && \forall n \in N, \forall t \in T \\
& && \sum_{n \in N} \sigma_{n,t}^a \delta_{n,t} \leq C_t^a && \forall a \in A, \forall t \in T \\
& && x_h \leq 1 && \forall h \in H \\
& && x_h \geq 0 && \forall h \in H \\
& && y_b \in \{0, 1\} && \forall b \in B
\end{aligned}$$

In this model, $\delta_{n,t}$ represents the net energy export of node n at time period t and $\sigma_{n,t}^a$ is the PTDF associated with node n , time period t , and arc a . The total energy flow induced on arc a is calculated as $\sum_{n \in N} \sigma_{n,t}^a \delta_{n,t}$, and is restricted by arc capacity C_t^a .

The network capacity constraints presented above define a convex set and preserve the convexity of the surplus maximization problem. Complementary slackness constraints can be satisfied by the net energy export variables for any given $\bar{\mathbf{y}}$, similar to the case of hourly bids, and therefore the net energy export values and the zonal prices $\pi_{n,t}$ will be at equilibrium for any set of accepted block bids. For example, under the ATC model, the price in the exporting node is at most as big as the price in the importing node, and the prices of two nodes connected by an arc must be equal if the arc capacity constraint is non-binding.

The bounded dual subproblem will be very similar to **(SMILP-Multi Node)** and will additionally include variable bounds imposed by \bar{y} found by the master problem. Since the net energy export variables are continuous, there is nothing that invalidates the Benders infeasibility cuts developed in Section 3.3. The primal-dual formulation can also be easily adapted with the network constraints specified here. The problem can be still formulated as an MILP.

3.7 Conclusions

In this chapter, we studied the market clearing problem in the exchange-type electricity market design, the preferred market mechanism by many European countries. Although the surplus maximizing mixed-integer linear programming problem can be solved efficiently by today's most powerful solvers like Cplex and Gurobi, the optimal solution may include some accepted block bids that bring negative profits, paradoxically-accepted bids (PABs), to their bidders at the market prices. This implies a non-equilibrium market outcome. To prevent solutions with PABs, the market operator typically imposes additional constraints, settling for a lower total market surplus. With the addition of such constraints, the computational burden increases substantially.

We developed Benders infeasibility cuts that use the market clearing prices over the periods of a PAB to find the set of block bid variables of which at least one needs to be changed to eliminate the PAB. We call these price-based cuts and show that they are stronger than the cuts proposed by Martin et al. (2014). The computational results on practical-size instances from the Turkish DAM show that using price-based cuts as the infeasibility cuts in the Benders decomposition algorithm outperform using the no-good cuts of Martin et al. (2014) and the locally-valid cuts of Madani and Van Vyve (2015). The improved Benders decomposition algorithm solved all instances to optimality within about one minute when Gurobi solver was used. The tests on larger instances also showed that the improved Benders decomposition algorithm not only found feasible solutions for all instances but also solved more instances to optimality in a fraction of time of the primal-dual approach.

In practice, market operators operate under a time restriction to solve these problems and to announce market clearing prices (typically around 10 minutes). The Benders decomposition algorithm using price-based cuts appears to be the reasonable algorithm to implement. As a matter of fact, the Benders decomposition algorithm with heuristic cuts is used as the approach with the aim of finding high quality feasible solutions for the European market coupling problem (Euphemia, 2016). Furthermore, the Benders decomposition algorithm is the only practical approach in the existence of piece-wise linear hourly bids that lead to a quadratic objective function. Such bids are allowed in some of the markets such as Nord-Pool.

We also developed an improved Benders decomposition algorithm for the market designs where no paradoxically-rejected bid (PRB) constraints replace no PAB constraints or when the market includes more sophisticated bid types and transmission network constraints. We show that the price-based cuts are valid under all these market designs. We believe that, implementing our improved Benders decomposition approaches will improve the solution quality for the market clearing problems in the exchange-type electricity market designs. We also believe that, these developments will trigger new research in this area to further improve the results. We intend to conduct further research and tests on the extensions we developed.

CHAPTER 4

FINDING THE NONDOMINATED SET AND EFFICIENT INTEGER VECTORS FOR A CLASS OF THREE OBJECTIVE MIXED-INTEGER LINEAR PROGRAMS

In the existence of multiple objectives, it is rare to find a solution that is best in all objectives. Rather, there are a set of meaningful *nondominated* points, each outperforming any other solution in at least one objective. Generating nondominated sets of different classes of problems with multiple objectives has been an important research area, both from theoretical and practical perspectives. Many authors address multi-objective linear programs (MOLPs), where all objective functions and constraints are linear. Some attempt to characterize nondominated sets of MOLPs (see Wiecek et al., 2016, for a review).

Generating the nondominated sets of large scale multi-objective integer linear programs (MOILPs) is challenging due to computational difficulties of solving integer programs repetitively. Several efficient algorithms are available (Lokman and Köksalan, 2013; Kirlik and Sayın, 2014; Klamroth et al., 2015; Dächert et al., 2017). Due to the extensive computational effort in generating the whole nondominated set, some recent approaches focus on generating representative subsets of the nondominated set (Masin and Bukchin, 2008; Ceyhan et al., 2019).

In the case of multi-objective mixed-integer linear programs (MOMILPs), the nondominated set includes facets as well as points or edges. Previous research in this area focused on generating the set of extreme supported nondominated points (see for example, Przybylski et al., 2010a; Özpeynirci and Köksalan, 2010; Alves and Costa, 2016). Assuming that all objectives are of maximization type, each such point has the property of uniquely maximizing some positive-weighted-sum of the objectives.

Other studies that aim to find the whole nondominated set for MOMILPs are restricted to the bi-objective case (Stidsen et al., 2014; Boland et al., 2015; Soylu, 2018; Fattahi and Turkay, 2018).

To the best of our knowledge, Rasmi and Türkay (2019) is the only approach that aims to find the nondominated facets for MOMILPs for more than two objectives. They develop a two-stage algorithm that starts with finding the efficient integer vectors, and then generates the nondominated edges and facets associated with each such vector. However, by construction, the generated facets are not guaranteed to be nondominated; they may contain some dominated regions.

In this study, we consider a subclass of three-objective mixed-integer linear programs (TOMILPs) that have at least one of the three objectives discrete-valued. This is a more general version of the bi-objective mixed-binary linear programs (BOMBLPs) studied by Stidsen et al. (2014) that have one real and one discrete-valued objective. We were inspired by the multiple objective version of the day-ahead electricity market clearing problem. In this problem, the surplus maximization objective can have both discrete and continuous decision variables but has a unique value corresponding to each fixed integer decision variable vector (O'Neill et al., 2005). For each surplus value, there are, typically, alternative optimal market clearing prices that lead to different allocations of costs among market participants.

Other problems where there is an objective to minimize a countable resource, such as the number of facilities, vehicles, idle machines, uncovered demand points, tasks, and there are relevant conflicting continuous-valued objectives based on the context of the application have structures similar to that of the day-ahead electricity market clearing problem. (see, e.g., Görmez et al. 2011, Daskin and Maass 2015, Kalita and Datta 2017). Our problem is more general than Stidsen et al. (2014)'s in two aspects: (i) it can handle up to three objectives, and (ii) none of the objectives is restricted to integer variables only. All three objectives can include continuous variables so long as at least one of the objectives has a discrete feasible set.

We can highlight our contributions in this chapter as follows:

1. We develop a criterion-space search algorithm that generates the exact nondom-

inated set of the BOMILPs and a subset of TOMILPs.

2. The algorithm generates all efficient integer vectors if it is configured to do so.
3. We develop an open-source software that is available in a public repository (Ceyhan, 2020).
4. We create a publicly available set of benchmark problems and demonstrate the performance of our algorithm on these problems (also available in Ceyhan, 2020).

In the next section, we define our problem and study the characteristics of its non-dominated set. In Section 4.2, we define the dominance relations between two sets of points in the criterion space, and present theory and establish conditions on dominance relations between sets. We develop a cone-based search algorithm using the established conditions in Section 4.3. In Section 4.4, we present the implementation details and provide illustrative examples. In Section 4.5, we develop benchmark problems and report our computational results. We give an extension of the algorithm to approximate nondominated sets in Section 4.6 and make concluding remarks in Section 4.7.

4.1 Problem definition

We present the following general MOMILP, and then define our problem as a subclass of MOMILP. We assume, without loss of generality, that all objectives are of maximization type throughout the paper.

$$\begin{aligned}
 \mathcal{P} : \quad & \text{Max} \quad \mathbf{z}(\mathbf{x}) = (z_1(\mathbf{x}), z_2(\mathbf{x}), \dots, z_m(\mathbf{x})) \\
 & \text{s.to.} \quad \mathbf{x} \in \mathcal{P} \\
 & \quad \quad x_u \in \mathbb{R}, \forall u \in \mathcal{U} \\
 & \quad \quad x_v \in \mathbb{Z}, \forall v \in \mathcal{V}
 \end{aligned}$$

where $z_i(\mathbf{x})$ is a linear function of \mathbf{x} denoting the i^{th} objective, $i = 1, 2, \dots, m$, $\mathcal{P} \subseteq \mathbb{R}^{|\mathcal{V}|+|\mathcal{U}|}$ is a polyhedron, \mathcal{V} is the index set of integer decision variables and \mathcal{U} is the index set of real-valued decision variables.

We denote the feasible set with \mathcal{X} , $\mathcal{X} := \{\mathbf{x} = (\mathbf{x}^c, \mathbf{x}^d) \in \mathcal{P} : \mathbf{x}^c \in \mathbb{R}^{|\mathcal{U}|}, \mathbf{x}^d \in \mathbb{Z}^{|\mathcal{V}|}\}$, the image of \mathcal{X} in the objective space with $\mathcal{Z} \subseteq \mathbb{R}^m$, and the image of $\mathbf{x} \in \mathcal{X}$ in the objective space with $\mathbf{z} \in \mathcal{Z}$. We assume \mathcal{X} is non-empty and \mathcal{Z} is bounded.

- If $\mathcal{V} = \emptyset$, \mathcal{P} is an MOLP.
- If $\mathcal{U} = \emptyset$, \mathcal{P} is an MOILP.
- If $x_v \in \{0, 1\} \forall v \in \mathcal{V}$, \mathcal{P} is a multi-objective mixed-binary linear program (MOMBLP).

We define the following three subsets of \mathbb{R}^m :

$$\mathbb{R}_{>}^m := \{\mathbf{u} \in \mathbb{R}^m : u_i > 0, \forall i = 1, 2, \dots, m\}$$

$$\mathbb{R}_{\geq}^m := \{\mathbf{u} \in \mathbb{R}^m : u_i \geq 0, \forall i = 1, 2, \dots, m, \mathbf{u} \neq \mathbf{0}\}$$

$$\mathbb{R}_{\geq}^m := \{\mathbf{u} \in \mathbb{R}^m : u_i \geq 0, \forall i = 1, 2, \dots, m\}$$

Additionally, we define the following sets that represent the dominating or dominated cones of point $\mathbf{z} \in \mathcal{Z}$:

- ◇ $\mathbf{z}^> := \{\mathbf{u} \in \mathbb{R}^m : \mathbf{u} - \mathbf{z} \in \mathbb{R}_{>}^m\}$, the set of real-valued vectors that are larger than \mathbf{z} in each criterion
- ◇ $\mathbf{z}^{\geq} := \{\mathbf{u} \in \mathbb{R}^m : \mathbf{u} - \mathbf{z} \in \mathbb{R}_{\geq}^m\}$, the set of real-valued vectors that are at least as large as \mathbf{z} in each criterion, and larger than \mathbf{z} in at least one criterion
- ◇ $\mathbf{z}^{\geq} := \{\mathbf{u} \in \mathbb{R}^m : \mathbf{u} - \mathbf{z} \in \mathbb{R}_{\geq}^m\}$, the set of real-valued vectors that are at least as large as \mathbf{z} in each criterion
- ◇ $\mathbf{z}^{<} := \{\mathbf{u} \in \mathbb{R}^m : \mathbf{z} - \mathbf{u} \in \mathbb{R}_{>}^m\}$, the set of real-valued vectors that are smaller than \mathbf{z} in each criterion
- ◇ $\mathbf{z}^{\leq} := \{\mathbf{u} \in \mathbb{R}^m : \mathbf{z} - \mathbf{u} \in \mathbb{R}_{\geq}^m\}$, the set of real-valued vectors that are at most as large as \mathbf{z} in each criterion, and smaller than \mathbf{z} in at least one criterion
- ◇ $\mathbf{z}^{\leq} := \{\mathbf{u} \in \mathbb{R}^m : \mathbf{z} - \mathbf{u} \in \mathbb{R}_{\geq}^m\}$, the set of real-valued vectors that are at most as large as \mathbf{z} in each criterion

Definition 10. z is a nondominated point and x is an efficient solution if $z^{\geq} \cap \mathcal{Z} = \emptyset$.

Definition 11. z is a weakly nondominated point and x is a weakly efficient solution if $z^{>} \cap \mathcal{Z} = \emptyset$.

In a multi-objective problem, the complete set or a subset of nondominated points is of primary interest. The set of *nondominated points*, \mathcal{Z}_{ND} , can be defined as follows:

$$\mathcal{Z}_{ND} := \{z \in \mathcal{Z} : z^{\geq} \cap \mathcal{Z} = \emptyset\}$$

Similarly, the set of *weakly nondominated points*, \mathcal{Z}_{WND} , can be defined as

$$\mathcal{Z}_{WND} := \{z \in \mathcal{Z} : z^{>} \cap \mathcal{Z} = \emptyset\}$$

where $\mathcal{Z}_{ND} \subseteq \mathcal{Z}_{WND}$.

Definition 12. If $\exists \lambda \in \mathbb{R}_{>0}^m$ such that z^* is an optimal solution of $\max_{z \in \mathcal{Z}} \lambda z$, then z^* is a supported nondominated point. If z^* is unique, then z^* is an extreme supported nondominated point.

Definition 13. $z^I \in \mathbb{R}^m$ is the ideal point of \mathcal{Z} such that $z_i^I = \max_{z \in \mathcal{Z}} z_i, \forall i = 1, 2, \dots, m$.

Given $\mathcal{V} \neq \emptyset$, \mathcal{V} denotes the set of feasible integer vectors, $\mathcal{V} := \{x^d : (x^c, x^d) \in \mathcal{X}\}$, and the *slice problem* (Belotti et al., 2013; Soylu, 2018; Rasmi and Türkay, 2019) of \mathcal{P} corresponding to integer vector $\bar{y} \in \mathcal{V}$ as $\mathcal{P}(\bar{y})$:

$$\begin{aligned} \mathcal{P}(\bar{y}) : \quad & \text{Max} \quad z(x) = \{z_1(x), z_2(x), \dots, z_m(x)\} \\ & \text{s.to.} \quad x \in \mathcal{X} \\ & \quad \quad x^d = \bar{y} \end{aligned}$$

$\mathcal{P}(\bar{y})$ is an MOLP. The image set of feasible solutions of $\mathcal{P}(\bar{y})$ in the objective space is $\mathcal{Z}(\bar{y}) := \{z(x) : x \in \mathcal{X}, x^d = \bar{y}\}$, and the nondominated set of $\mathcal{P}(\bar{y})$ is $\mathcal{Z}_{ND}(\bar{y}) := \{z \in \mathcal{Z}(\bar{y}) : z^{\geq} \cap \mathcal{Z}(\bar{y}) = \emptyset\}$. Note that $\mathcal{Z}(\bar{y})$ is a polyhedron, and $\mathcal{Z}_{ND}(\bar{y})$ is connected (Naccache, 1978) and supported. To generate $\mathcal{Z}_{ND}(\bar{y})$, one can use one of the algorithms developed to find the nondominated set of an MOLP, such as *multi-objective simplex algorithm* (Evans and Steuer, 1973; Rudloff et al., 2017),

primal-dual simplex method (Ehrgott et al., 2007) or *dichotomic search algorithm* (Aneja and Nair, 1979; Cohon, 2004).

Since $\mathcal{X} \neq \emptyset$, there must exist some $\mathbf{y} \in \mathcal{Y} : \mathcal{Z}_{ND} \cap \mathcal{Z}_{ND}(\mathbf{y}) \neq \emptyset$. Let $\mathcal{Z}_{ND}^{\mathcal{P}}(\mathbf{y}) = \mathcal{Z}_{ND} \cap \mathcal{Z}_{ND}(\mathbf{y})$, and $\mathcal{Z}_{WND}^{\mathcal{P}}(\mathbf{y}) = \mathcal{Z}_{WND} \cap \mathcal{Z}_{ND}(\mathbf{y})$. We are particularly interested in the integer vectors \mathbf{y} for which $\mathcal{Z}_{ND}^{\mathcal{P}}(\mathbf{y})$ is not empty.

Definition 14. For $\mathbf{y} \in \mathcal{Y}$, \mathbf{y} is an *efficient integer vector* if $\mathcal{Z}_{ND}^{\mathcal{P}}(\mathbf{y}) \neq \emptyset$, and *weakly efficient integer vector* if $\mathcal{Z}_{WND}^{\mathcal{P}}(\mathbf{y}) \neq \emptyset$.

Let \mathcal{Y}_E be the set of efficient vectors, $\mathcal{Y}_E := \{\mathbf{y} \in \mathcal{Y} : \mathcal{Z}_{ND}^{\mathcal{P}}(\mathbf{y}) \neq \emptyset\}$. Note that both \mathcal{Y} and $\mathcal{Y}_E \subseteq \mathcal{Y}$ are finite sets. In the following proposition, we show that the nondominated set of a MOMILP problem consists of the nondominated sets of the slice problems of its efficient integer vectors.

Proposition 5. $\mathcal{Z}_{ND} = \bigcup_{\mathbf{y} \in \mathcal{Y}_E} \mathcal{Z}_{ND}^{\mathcal{P}}(\mathbf{y})$.

Proof. $\mathcal{Z}_{ND} \subseteq \bigcup_{\mathbf{y} \in \mathcal{Y}_E} \mathcal{Z}_{ND}^{\mathcal{P}}(\mathbf{y})$ since $\exists \bar{\mathbf{y}} = \bar{\mathbf{x}}_v \in \mathcal{Y}_E : \bar{\mathbf{z}} = \mathbf{z}(\bar{\mathbf{x}}) \in \mathcal{Z}_{ND}^{\mathcal{P}}(\bar{\mathbf{y}})$ for any $\bar{\mathbf{z}} \in \mathcal{Z}_{ND}$. $\bigcup_{\mathbf{y} \in \mathcal{Y}_E} \mathcal{Z}_{ND}^{\mathcal{P}}(\mathbf{y}) \subseteq \mathcal{Z}_{ND}$ since $\bigcup_{\mathbf{y} \in \mathcal{Y}_E} \mathcal{Z}_{ND}^{\mathcal{P}}(\mathbf{y}) = \bigcup_{\mathbf{y} \in \mathcal{Y}_E} (\mathcal{Z}_{ND}(\mathbf{y}) \cap \mathcal{Z}_{ND}) = \left(\bigcup_{\mathbf{y} \in \mathcal{Y}_E} \mathcal{Z}_{ND}(\mathbf{y}) \right) \cap \mathcal{Z}_{ND} \subseteq \mathcal{Z}_{ND}$. \square

In the rest of the paper, we consider the special case of three-objective mixed-integer programming problems (TOMILPs). Furthermore, we restrict our attention to the case where the slice problems are bi-objective linear programs (BOLPs) that are obtained by setting the third objective function to a specific feasible value. Let I be the index set of objective functions and $\bar{\mathbf{y}} \in \mathcal{Y}$. For two distinct feasible points \mathbf{z}' and \mathbf{z}'' in $\mathcal{Z}(\bar{\mathbf{y}})$, we define the index set of objectives for which both points have the same objective value as $I^=(\bar{\mathbf{y}})$, that is, $I^=(\bar{\mathbf{y}}) := \{i \in I : \forall \mathbf{z}, \mathbf{z}' \in \mathcal{Z}(\bar{\mathbf{y}}), z_i = z'_i, \mathbf{z} \neq \mathbf{z}'\}$. Let $q \in I^=(\bar{\mathbf{y}})$ and $z_q(\bar{\mathbf{y}}) = z_q, \forall \mathbf{z} \in \mathcal{Z}(\bar{\mathbf{y}})$. Then, $\mathcal{Z}_{ND}(\bar{\mathbf{y}})$ can only include a single point or a set of connected edges in the $z_q = z_q(\bar{\mathbf{y}})$ plane as formalized with the next corollary.

Proposition 6. If $I^=(\bar{\mathbf{y}}) \neq \emptyset$ for $\bar{\mathbf{y}} \in \mathcal{Y}$, then $\dim(\mathcal{Z}(\bar{\mathbf{y}})) \leq 2$.

Proof. Since $\mathcal{Z}(\bar{\mathbf{y}}) \subseteq \mathbb{R}^3$, $\dim(\mathcal{Z}(\bar{\mathbf{y}})) + \text{rank}(M^=) = 3$, where $M^=$ is the set of inequalities in the description of polyhedron $\mathcal{Z}(\bar{\mathbf{y}})$ that are satisfied at equality. Since $I^=(\bar{\mathbf{y}}) \neq \emptyset$, $\exists q \in I^=(\bar{\mathbf{y}})$, and $\{z_q = z_q(\bar{\mathbf{y}})\} \subseteq M^=$. Hence, $\text{rank}(M^=) \geq 1$, and $\dim(\mathcal{Z}(\bar{\mathbf{y}})) \leq 2$. \square

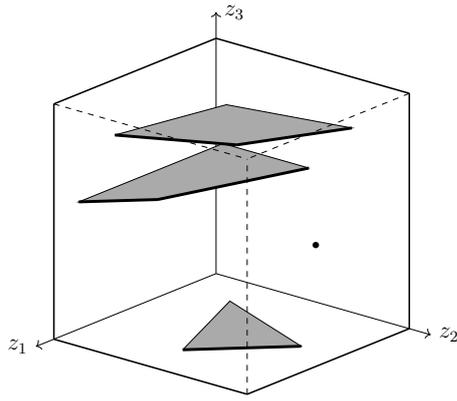
Let $\mathcal{F}^{\bar{\mathbf{y}},d}$ be the d dimensional proper faces of $\mathcal{Z}(\bar{\mathbf{y}})$, $d = 0, 1$. Also, let $\mathcal{F}_{ND}^{\bar{\mathbf{y}},d}$ be the d dimensional nondominated faces of $\mathcal{Z}(\bar{\mathbf{y}})$, $d = 0, 1$, $\mathcal{F}_{ND}^{\bar{\mathbf{y}},d} \subseteq \mathcal{F}^{\bar{\mathbf{y}},d}$. As $\mathcal{Z}_{ND}(\bar{\mathbf{y}}) = \mathcal{F}_{ND}^{\bar{\mathbf{y}},0} \cup \mathcal{F}_{ND}^{\bar{\mathbf{y}},1}$, there can be two different cases for the nondominated set of the slice problem $\mathcal{P}(\bar{\mathbf{y}})$:

- *Case 1:* $\mathcal{F}_{ND}^{\bar{\mathbf{y}},1} = \emptyset$. There is no edge in the nondominated set. Then, the nondominated set must be a singleton as $\bar{\mathbf{y}}$ is a feasible integer vector by assumption and $\mathcal{Z}_{ND}(\bar{\mathbf{y}})$ is connected, $\mathcal{Z}_{ND}(\bar{\mathbf{y}}) = \mathcal{F}_{ND}^{\bar{\mathbf{y}},0} = \{z^{\bar{\mathbf{y}}}\}$.
- *Case 2:* $\mathcal{F}_{ND}^{\bar{\mathbf{y}},1} \neq \emptyset$. The nondominated set consists of connected edges. In this case, $\forall F \in \mathcal{F}_{ND}^{\bar{\mathbf{y}},0}, \exists \hat{F} \in \mathcal{F}_{ND}^{\bar{\mathbf{y}},1} : F \subseteq \hat{F}$. Hence, we can define $\mathcal{Z}_{ND}(\bar{\mathbf{y}})$ in terms of the one-dimensional faces (edges) in $\mathcal{F}_{ND}^{\bar{\mathbf{y}},1}$.

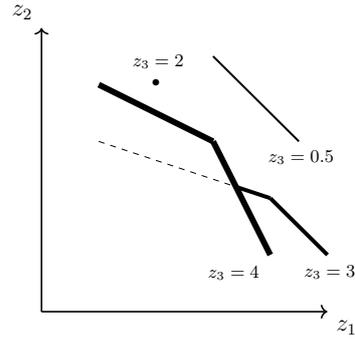
We next define $I^= := \bigcap_{\bar{\mathbf{y}} \in \mathcal{Y}} I^=(\bar{\mathbf{y}})$. Given that $I^= \neq \emptyset$, three cases are possible in terms of the characteristics of the nondominated set:

Case 1: $|I^=| = 1$. Suppose $q = 3$. Let $\mathcal{Z}_{ND,q}$ be the projection of \mathcal{Z}_{ND} on the q axis of the criterion space, $\mathcal{Z}_{ND,q} := \{z_q : z \in \mathcal{Z}_{ND}\}$. Then, $\mathcal{Z}_{ND,q} \subseteq \{z_q(\mathbf{y}), \mathbf{y} \in \mathcal{Y}\}$, and it is a discrete set of points. In Figure 4.1, we present an example for this case. The polygons (since $\dim(\mathcal{Z}(\bar{\mathbf{y}})) \leq 2$) in the planes parallel to $z_1 - z_2$ axes represent the feasible regions of the slice problems, $\mathcal{Z}(\bar{\mathbf{y}})$ for some $\bar{\mathbf{y}} \in \mathcal{Y}$, and the dark edges of those polygons show the corresponding nondominated sets of the slice problems, $\mathcal{Z}_{ND}(\bar{\mathbf{y}})$ (if $\mathcal{Z}(\bar{\mathbf{y}})$ is a singleton, then $\mathcal{Z}_{ND}(\bar{\mathbf{y}}) = \mathcal{Z}(\bar{\mathbf{y}})$). In Figure 4.1b, we show the projection of \mathcal{Z}_{ND} onto $z_1 - z_2$ plane. Solid lines and point represent the nondominated set, with thickness proportional to z_3 value. The dashed line segment of the frontier with the second highest z_3 value is dominated by the frontier with the highest z_3 value, and hence is not part of the nondominated set.

Case 2: $|I^=| = 2$. Although the problem is a TOMILP, the nondominated set consists of only disconnected points. Suppose $I^= = \{2, 3\}$. In Figure 4.2a, we show that the



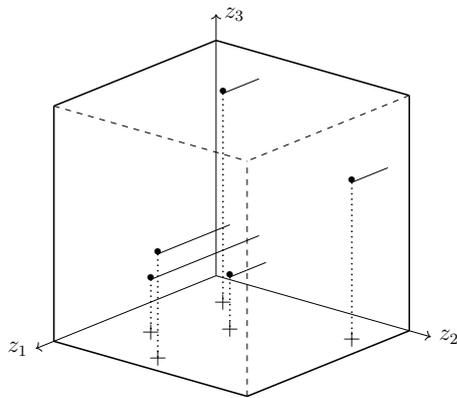
(a) The slices in the criterion space



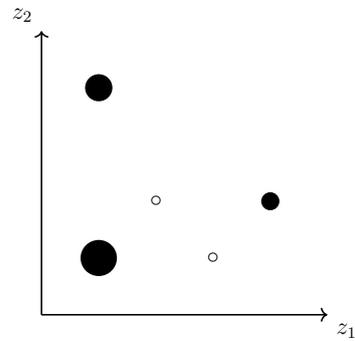
(b) The projection of the nondominated set onto $z_1 - z_2$ plane

Figure 4.1: The solutions for the slice problems in Case 1, $|I^=| = 1$

feasible set of a slice problem is a line segment (since $\dim(\mathcal{Z}(\bar{\mathbf{y}})) \leq 1$) perpendicular to $z_2 - z_3$ plane (or, it can be either empty or a single point), and its nondominated set is the point with maximum z_1 value. In this case, the nondominated set is a discrete set of points. Figure 4.2b illustrates the projection of the nondominated set onto the $z_1 - z_2$ plane. Solid points represent nondominated points, with their radius proportional to z_3 value.



(a) The slices in the criterion space



(b) The projection of the nondominated set onto $z_1 - z_2$ plane

Figure 4.2: The solutions for the slice problems in Case 2, $|I^=| = 2$

Case 3: $|I^=| = 3$. In this case, the problem is a three-objective integer linear program (TOILP). As we show in Figure 4.3, both the feasible set and the nondominated set are discrete (since $\dim(\mathcal{Z}(\bar{\mathbf{y}})) \leq 0$).

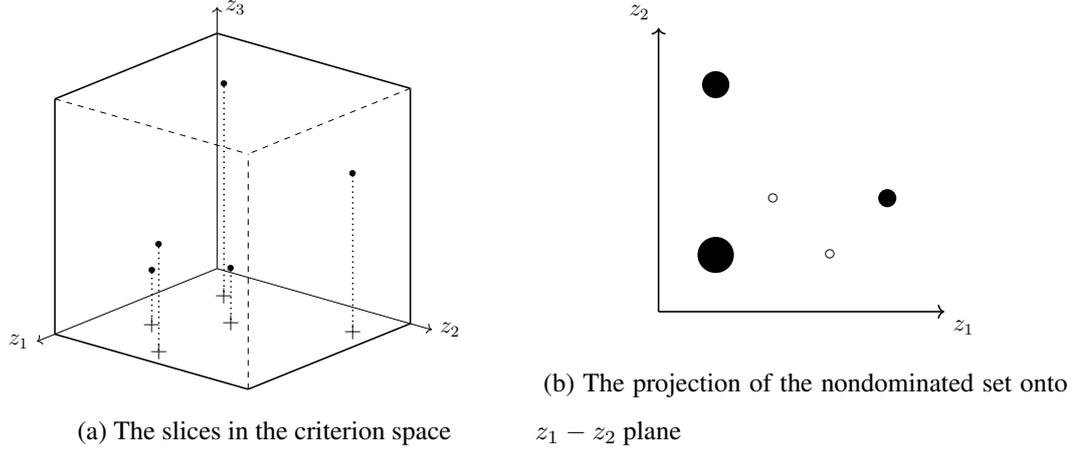


Figure 4.3: The solutions for the slice problems in Case 3, $|I^=| = 3$

Among the above, the first case provides continuous trade-offs between two of the objectives at different parts of the criterion space as shown in Figure 4.1. The algorithms that generate extreme supported points (Özpeynirci and Köksalan, 2010; Przybylski et al., 2010a) do not provide this information. An algorithm that can also generate the nondominated edges can empower the DM with the full set of efficient values of the continuous decision variables. Although there are efficient algorithms developed to solve the problems with a discrete nondominated set as in Case 2 and 3, it may not be straightforward to identify a priori that the nondominated set of the problem is discrete.

In the remainder of this chapter, we assume $q = 3$, without loss of generality. Let $E^{\bar{y},k}$ be the k^{th} nondominated edge in $\mathcal{Z}_{ND}(\bar{y})$, and $z^{\bar{y},k,nw}$, $z^{\bar{y},k,se}$ be its north-west and south-east extreme points such that $z_1^{\bar{y},k,nw} < z_1^{\bar{y},k,se}$ and $z_2^{\bar{y},k,nw} > z_2^{\bar{y},k,se}$. Then, $E^{\bar{y},k} = \{\lambda z^{\bar{y},k,nw} + (1 - \lambda)z^{\bar{y},k,se}, 0 \leq \lambda \leq 1\}$. Also, let $g^{\bar{y},k}u = h^{\bar{y},k}$ be the supporting hyperplane of $E^{\bar{y},k}$ perpendicular to $z_1 - z_2$ plane, $g_3 = 0$.

If there are n nondominated edges in $\mathcal{Z}_{ND}(\bar{y})$, we index the edges such that $z^{\bar{y},k,se} = z^{\bar{y},k+1,nw}$ for $1 \leq k < n$. Let $K(\bar{y}) := \{1, 2, \dots, n\}$ be the set of such indices. Then, $\mathcal{Z}_{ND}(\bar{y}) = \{E^{\bar{y},k}, k \in K(\bar{y})\}$ is the continuous nondominated frontier of the slice problem of \bar{y} . In the following sections, we also make use of the north-west extreme point, $z^{\bar{y},nw} = z^{\bar{y},1,nw}$, south-east extreme point, $z^{\bar{y},se} = z^{\bar{y},|K(\bar{y})|,se}$, the

ideal point, $z^{\bar{y},ne}$ where $z_i^{\bar{y},ne} = \max_{z \in \mathcal{Z}(\bar{y})} z_i, i = 1, 2, 3$, and the nadir point, $z^{\bar{y},sw}$ where $z_i^{\bar{y},sw} = \min_{z \in \mathcal{Z}_{ND}(\bar{y})} z_i, i = 1, 2, 3$, of this frontier.

In the next section, we examine the dominance relations between the nondominated sets of the slice problems through the set dominance rules and the models we develop.

4.2 Set dominance

In order to define dominance relations between different sets of points in the criterion space, we first define sets: $\mathcal{Z}^> := \bigcup_{z \in \mathcal{Z}} z^>$, $\mathcal{Z}^{\geq} := \bigcup_{z \in \mathcal{Z}} z^{\geq}$, $\mathcal{Z}^{\cong} := \bigcup_{z \in \mathcal{Z}} z^{\cong}$, $\mathcal{Z}^< := \bigcup_{z \in \mathcal{Z}} z^<$, $\mathcal{Z}^{\leq} := \bigcup_{z \in \mathcal{Z}} z^{\leq}$ and $\mathcal{Z}^{\leq} := \bigcup_{z \in \mathcal{Z}} z^{\leq}$. We define the following dominance relations of two sets relative to each other:

Definition 15. \mathcal{Z}^1 is nondominated relative to \mathcal{Z}^2 if $\mathcal{Z}^{1\geq} \cap \mathcal{Z}^2 = \emptyset$.

Definition 16. \mathcal{Z}^1 is weakly nondominated relative to \mathcal{Z}^2 if $\mathcal{Z}^{1>} \cap \mathcal{Z}^2 = \emptyset$.

Definition 17. \mathcal{Z}^1 is partially nondominated relative to \mathcal{Z}^2 if $\mathcal{Z}^1 \cap \mathcal{Z}^{2\leq} \neq \mathcal{Z}^1$.

Definition 18. \mathcal{Z}^1 is dominated by \mathcal{Z}^2 if $\mathcal{Z}^1 \cap \mathcal{Z}^{2\leq} = \mathcal{Z}^1$.

Definition 19. \mathcal{Z}^1 is strictly dominated by \mathcal{Z}^2 if $\mathcal{Z}^1 \cap \mathcal{Z}^{2<} = \mathcal{Z}^1$.

Since we assume that $|I^=| \geq 1$, we compare points or edges against each other. If we compare the nondominated sets of two different slice problems, we define the efficiency of the related integer vectors as follows:

Definition 20. $\mathbf{y}^1 \in \mathcal{Y}$ is an efficient integer vector relative to $\mathbf{y}^2 \in \mathcal{Y}$ if $\mathcal{Z}_{ND}(\mathbf{y}^1)$ is partially nondominated relative to $\mathcal{Z}_{ND}(\mathbf{y}^2)$.

Given $\mathbf{y}^1, \mathbf{y}^2 \in \mathcal{Y}$, we first examine the case where $z_3(\mathbf{y}^1) \neq z_3(\mathbf{y}^2)$. Without loss of generality, we assume that $z_3(\mathbf{y}^1) > z_3(\mathbf{y}^2)$ and consider the four cases below. In each of these cases, $\mathcal{Z}_{ND}(\mathbf{y}^1)$ is nondominated relative to $\mathcal{Z}_{ND}(\mathbf{y}^2)$ and \mathbf{y}^1 is an efficient integer vector relative to \mathbf{y}^2 since $z_3(\mathbf{y}^1) > z_3(\mathbf{y}^2)$. In order to identify the dominance status of $\mathcal{Z}_{ND}(\mathbf{y}^2)$ relative to $\mathcal{Z}_{ND}(\mathbf{y}^1)$, we consider the following linear program once for each $i = 1, 2$:

$$\begin{aligned} \mathcal{P}(\mathbf{y}^1, \mathbf{y}^2, S, i) : \quad & \text{Max} \quad z_i \\ & \text{s.to.} \quad \mathbf{z} \in \Phi(\mathbf{y}^1, \mathbf{y}^2, S) \end{aligned}$$

where $\Phi(\mathbf{y}^1, \mathbf{y}^2, S) = S \cap \mathcal{Z}_{ND}(\mathbf{y}^1) \stackrel{\leq}{\subseteq} S$, S is a point if $\mathcal{Z}_{ND}(\mathbf{y}^2)$ is a singleton, otherwise an edge of $\mathcal{Z}_{ND}(\mathbf{y}^2)$. By the following proposition, we show that $\mathcal{Z}_{ND}(\mathbf{y}^1) \stackrel{\leq}{\subseteq}$ is a polyhedron, hence $\mathcal{P}(\mathbf{y}^1, \mathbf{y}^2, S, i)$ is a linear program.

Proposition 7. *Let n be the number of edges in $\mathcal{Z}_{ND}(\bar{\mathbf{y}})$. Then, $\mathcal{Z}_{ND}(\bar{\mathbf{y}}) \stackrel{\leq}{\subseteq}$ is a polyhedron defined by $n + 3$ half-spaces.*

Proof. If $\dim(\mathcal{Z}_{ND}(\bar{\mathbf{y}})) = 1$, let $\mathcal{Z}_{ND}(\bar{\mathbf{y}}) = \{\mathbf{z}^{\bar{\mathbf{y}}}\}$. Then, $\mathcal{Z}_{ND}(\bar{\mathbf{y}}) \stackrel{\leq}{\subseteq}$ is defined by the intersection of the three half-spaces in \mathbb{R}^3 , $\{\mathbf{u} \in \mathbb{R}^3 : u_1 \leq z_1^{\bar{\mathbf{y}}}\}$, $\{\mathbf{u} \in \mathbb{R}^3 : u_2 \leq z_2^{\bar{\mathbf{y}}}\}$ and $\{\mathbf{u} \in \mathbb{R}^3 : u_3 \leq z_3^{\bar{\mathbf{y}}}\}$. Else, let $\mathcal{Z}_{ND}(\bar{\mathbf{y}}) = \{E^{\bar{\mathbf{y}},k}, k \in K(\bar{\mathbf{y}})\}$ be the set of edges. Then,

$$\mathcal{Z}_{ND}(\bar{\mathbf{y}}) \stackrel{\leq}{\subseteq} = \{\mathbf{u} \in \mathbb{R}^3 : u_3 \leq z_3(\bar{\mathbf{y}}), u_1 \leq z_1^{\bar{\mathbf{y}},se}, u_2 \leq z_2^{\bar{\mathbf{y}},nw}, \mathbf{g}^{\bar{\mathbf{y}},k} \mathbf{u} \leq h^{\bar{\mathbf{y}},k}, \forall k \in K(\bar{\mathbf{y}})\}.$$

If we let $n = |K(\bar{\mathbf{y}})|$, $\mathcal{Z}_{ND}(\bar{\mathbf{y}}) \stackrel{\leq}{\subseteq}$ is defined by the intersection of $n + 3$ half-spaces and it is a polyhedron. \square

Although this simple linear program can be solved as necessary, to determine the status of $\mathcal{Z}_{ND}(\mathbf{y}^2)$ relative to $\mathcal{Z}_{ND}(\mathbf{y}^1)$ as discussed later, we will develop rules that help extract the same information more efficiently for some of the cases. We will implement the rules first (*rule-based dominance test*) and resort to models only for those cases the rules are inconclusive (*model-based dominance test*).

For the below analyses, recall that \mathbf{y}^2 is an efficient integer vector relative to \mathbf{y}^1 if $\mathcal{Z}_{ND}(\mathbf{y}^2)$ is at least partially nondominated relative to $\mathcal{Z}_{ND}(\mathbf{y}^1)$.

4.2.1 Case 1. Dominance between two points

Let $\mathcal{Z}_{ND}(\mathbf{y}^1) = \{\mathbf{z}^{\mathbf{y}^1}\}$ and $\mathcal{Z}_{ND}(\mathbf{y}^2) = \{\mathbf{z}^{\mathbf{y}^2}\}$. $S = \mathbf{z}^{\mathbf{y}^2}$ and either $\Phi(\mathbf{y}^1, \mathbf{y}^2, S) = \emptyset$, in which case $\mathcal{Z}_{ND}(\mathbf{y}^2)$ is nondominated relative to $\mathcal{Z}_{ND}(\mathbf{y}^1)$, or $\Phi(\mathbf{y}^1, \mathbf{y}^2, S) = \mathbf{z}^{\mathbf{y}^2}$, in which case $\mathcal{Z}_{ND}(\mathbf{y}^2)$ is dominated by $\mathcal{Z}_{ND}(\mathbf{y}^1)$. The following rules specify the necessary and sufficient conditions to determine the dominance of $\mathcal{Z}_{ND}(\mathbf{y}^2)$.

Rule 1.1. If $z_1(\mathbf{y}^2) \geq z_1(\mathbf{y}^1)$ or $z_2(\mathbf{y}^2) \geq z_2(\mathbf{y}^1)$, then $\mathcal{Z}_{ND}(\mathbf{y}^2)$ is weakly nondominated relative to $\mathcal{Z}_{ND}(\mathbf{y}^1)$. $\mathcal{Z}_{ND}(\mathbf{y}^2)$ is nondominated relative to $\mathcal{Z}_{ND}(\mathbf{y}^1)$ if at least one of the inequalities is strict.

Rule 1.2. If $z_1(\mathbf{y}^2) \leq z_1(\mathbf{y}^1)$ and $z_2(\mathbf{y}^2) \leq z_2(\mathbf{y}^1)$, then $\mathcal{Z}_{ND}(\mathbf{y}^2)$ is dominated by $\mathcal{Z}_{ND}(\mathbf{y}^1)$.

4.2.2 Case 2. Dominance of a continuous frontier compared to a point

Let $\mathcal{Z}_{ND}(\mathbf{y}^1) = \{z^{\mathbf{y}^1}\}$ and $\mathcal{Z}_{ND}(\mathbf{y}^2) = \{E^{\mathbf{y}^2, k}, k \in K(\mathbf{y}^2)\}$. If $E^{\mathbf{y}^2, k}$ is nondominated relative to $\mathcal{Z}_{ND}(\mathbf{y}^1) \forall k \in K(\mathbf{y}^2)$, then $\mathcal{Z}_{ND}(\mathbf{y}^2)$ is nondominated relative to $\mathcal{Z}_{ND}(\mathbf{y}^1)$. Else, if $\exists \hat{k}$ such that $E^{\mathbf{y}^2, \hat{k}}$ is partially nondominated relative to $\mathcal{Z}_{ND}(\mathbf{y}^1)$, then $\mathcal{Z}_{ND}(\mathbf{y}^2)$ is partially nondominated relative to $\mathcal{Z}_{ND}(\mathbf{y}^1)$. If $\mathcal{Z}_{ND}(\mathbf{y}^2)$ is neither nondominated, nor partially nondominated relative to $\mathcal{Z}_{ND}(\mathbf{y}^1)$, then it is dominated by $\mathcal{Z}_{ND}(\mathbf{y}^1)$. If any of the following rules identify the dominance of an edge relative to a point, we can do away with solving the corresponding model.

As $z_3(\mathbf{y}^1) > z_3(\mathbf{y}^2)$ by assumption, we test $E^{\mathbf{y}^2, k}$ with respect to z_1 and z_2 values only. We define the following sets in the $z_1 - z_2$ plane that we use in the exposition of the rules:

- $NW := \{\mathbf{u} \in \mathbb{R}^2 : u_1 \leq z_1(\mathbf{y}^1), u_2 \geq z_2(\mathbf{y}^1)\}$,
- $\widetilde{NW} := \{\mathbf{u} \in \mathbb{R}^2 : u_1 < z_1(\mathbf{y}^1), u_2 > z_2(\mathbf{y}^1)\}$,
- $NE = z_1(\mathbf{y}^1) \stackrel{\cong}{=} \widetilde{NE} = z_1(\mathbf{y}^1) \stackrel{>}{}$,
- $SW = z_1(\mathbf{y}^1) \stackrel{\leq}{=} \widetilde{SW} = z_1(\mathbf{y}^1) \stackrel{<}{}$,
- $SE := \{\mathbf{u} \in \mathbb{R}^2 : u_1 \geq z_1(\mathbf{y}^1), u_2 \leq z_2(\mathbf{y}^1)\}$,
- $\widetilde{SE} := \{\mathbf{u} \in \mathbb{R}^2 : u_1 > z_1(\mathbf{y}^1), u_2 < z_2(\mathbf{y}^1)\}$.

Rule 2.1. If $z^{\mathbf{y}^2, k, nw}, z^{\mathbf{y}^2, k, se} \in SW$, then $E^{\mathbf{y}^2, k}$ is dominated by $z^{\mathbf{y}^1}$. (See Figure 4.4a)

Rule 2.2. If $z^{\mathbf{y}^2, k, nw} \in \widetilde{NW}$ and $z^{\mathbf{y}^2, k, se} \in SW$, then $[z^{\mathbf{y}^2, k, nw}, z^*]$ is a half-open

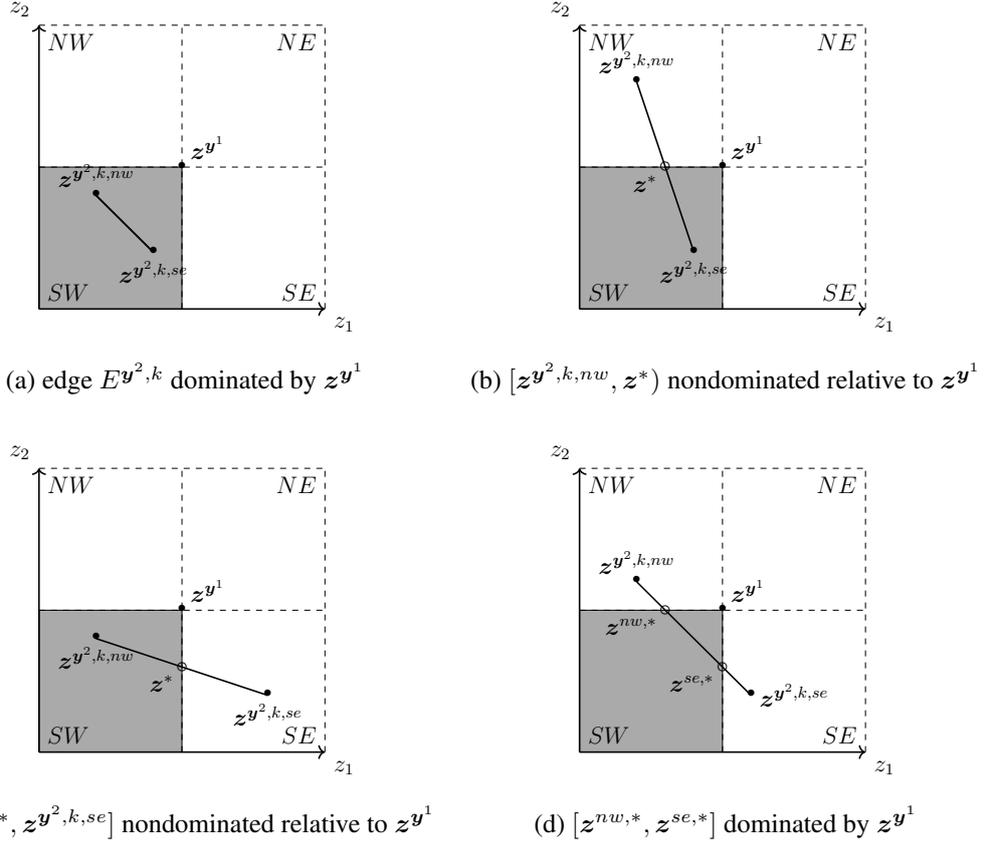


Figure 4.4: Different cases displaying the dominance of an edge relative to a point

edge that is nondominated relative to z^{y^1} , where z^* is the intersection point of $E^{y^2, k}$ and $z_2 = z_2^{y^1}$ line. (See Figure 4.4b)

Rule 2.3. If $z^{y^2, k, nw} \in SW$ and $z^{y^2, k, se} \in \widetilde{SE}$, then $(z^*, z^{y^2, k, se}]$ is a half-open edge that is nondominated relative to z^{y^1} , where z^* is the intersection point of $E^{y^2, k}$ and $z_1 = z_1^{y^1}$ line. (See Figure 4.4c)

Rule 2.4. If $z^{y^2, k, nw} \in \widetilde{NW}$, $z^{y^2, k, se} \in \widetilde{SE}$ and $g^{y^2, k}(z - z^{y^1}) \leq 0$ for $z \in E^{y^2, k}$, then $[z^{y^2, k, nw}, z^{nw, *})$ and $(z^{se, *}, z^{y^2, k, se}]$ are the half-open edges that are nondominated relative to z^{y^1} , where $z^{nw, *}$ and $z^{se, *}$ are the intersection points of $z_2 = z_2^{y^1}$ and $z_1 = z_1^{y^1}$ lines with $E^{y^2, k}$, respectively. (See Figure 4.4d)

Rule 2.5. If none of the above rules hold, then $E^{y^2, k}$ is nondominated relative to z^{y^1} .

Before applying the above rules for each edge of the frontier, we can check if the frontier is dominated by the point. We present the following proposition for this purpose:

Proposition 8. $\mathcal{Z}_{ND}(\mathbf{y}^2) = \{E^{\mathbf{y}^2,k}, k \in K(\mathbf{y}^2)\}$ is dominated by $\mathcal{Z}_{ND}(\mathbf{y}^1) = \{z^{\mathbf{y}^1}\}$ if and only if $z_2^{\mathbf{y}^2,nw} \leq z_2^{\mathbf{y}^1}$ and $z_1^{\mathbf{y}^2,se} \leq z_1^{\mathbf{y}^1}$.

Proof. (\Rightarrow) If $\mathcal{Z}_{ND}(\mathbf{y}^2)$ is dominated by $\mathcal{Z}_{ND}(\mathbf{y}^1)$, then $E^{\mathbf{y}^2,k}$ is dominated by $z^{\mathbf{y}^1}$, $\forall k \in K(\mathbf{y}^2)$. If $E^{\mathbf{y}^2,k}$ is dominated by $z^{\mathbf{y}^1}$, then $E^{\mathbf{y}^2,k} \in z^{\mathbf{y}^1 \leq}$ and, hence, $z_2^{\mathbf{y}^2,nw} \leq z_2^{\mathbf{y}^1}$, $z_1^{\mathbf{y}^2,se} \leq z_1^{\mathbf{y}^1}$.

(\Leftarrow) Given that $z_1^{\mathbf{y}^2,k,se} \leq z_1^{\mathbf{y}^2,se} \leq z_1^{\mathbf{y}^1}$ and $z_2^{\mathbf{y}^2,k,nw} \leq z_2^{\mathbf{y}^2,nw} \leq z_2^{\mathbf{y}^1}$ for $k \in K(\mathbf{y}^2)$, we will show that $E^{\mathbf{y}^2,k}$ is dominated by $z^{\mathbf{y}^1}$, $\forall k \in K(\mathbf{y}^2)$. Let e be a point on the edge $E^{\mathbf{y}^2,k}$, $e = \lambda z^{\mathbf{y}^2,k,nw} + (1 - \lambda) z^{\mathbf{y}^2,k,se}$, $0 \leq \lambda \leq 1$. Then, $e_1 \leq z_1^{\mathbf{y}^1}$ and $e_2 \leq z_2^{\mathbf{y}^1}$, because $z_1^{\mathbf{y}^2,k,nw} \leq z_1^{\mathbf{y}^2,k,se} \leq z_1^{\mathbf{y}^1}$ and $z_2^{\mathbf{y}^2,k,se} \leq z_2^{\mathbf{y}^2,k,nw} \leq z_2^{\mathbf{y}^1}$. Since $e_3 < z_3(\mathbf{y}^1)$, e is dominated by $z^{\mathbf{y}^1}$, $\forall e \in E^{\mathbf{y}^2,k}$, $\forall k \in K(\mathbf{y}^2)$ and, hence, $\mathcal{Z}_{ND}(\mathbf{y}^2) = \{E^{\mathbf{y}^2,k}, k \in K(\mathbf{y}^2)\}$ is dominated by $\mathcal{Z}_{ND}(\mathbf{y}^1) = \{z^{\mathbf{y}^1}\}$. \square

4.2.3 Case 3. Dominance of a point compared to a continuous frontier

Let $\mathcal{Z}_{ND}(\mathbf{y}^1) = \{E^{\mathbf{y}^1,k}, k \in K(\mathbf{y}^1)\}$ and $\mathcal{Z}_{ND}(\mathbf{y}^2) = \{z^{\mathbf{y}^2}\}$. If the condition in the following proposition holds, then $\mathcal{Z}_{ND}(\mathbf{y}^2)$ is dominated by $\mathcal{Z}_{ND}(\mathbf{y}^1)$. Otherwise, it is nondominated relative to $\mathcal{Z}_{ND}(\mathbf{y}^1)$.

Proposition 9. $\mathcal{Z}_{ND}(\mathbf{y}^2) = \{z^{\mathbf{y}^2}\}$ is dominated by $\mathcal{Z}_{ND}(\mathbf{y}^1) = \{E^{\mathbf{y}^1,k}, k \in K(\mathbf{y}^1)\}$ if and only if

1. $z_1^{\mathbf{y}^2} \leq z_1^{\mathbf{y}^1,se}$,
2. $z_2^{\mathbf{y}^2} \leq z_2^{\mathbf{y}^1,nw}$, and
3. $g^{\mathbf{y}^1,k} z^{\mathbf{y}^2} \leq h^{\mathbf{y}^1,k}, \forall k \in K(\mathbf{y}^1)$.

Proof. Let $R = \{\mathbf{u} \in \mathbb{R}^3 : u_1 \leq z_1^{\mathbf{y}^1,se}, u_2 \leq z_2^{\mathbf{y}^1,nw}, g^{\mathbf{y}^1,k} \mathbf{u} \leq h^{\mathbf{y}^1,k}, \forall k \in K(\mathbf{y}^1)\}$. Since $z_3(\mathbf{y}^2) < z_3(\mathbf{y}^1)$, $\mathcal{Z}_{ND}(\mathbf{y}^1) \subseteq R$.

(\Rightarrow) If z^{y^2} is dominated by $\mathcal{Z}_{ND}(y^1)$, then $z^{y^2} \cap \mathcal{Z}_{ND}(y^1)^\leq = z^{y^2} \cap R = z^{y^2}$ and, hence, $z^{y^2} \in R$.

(\Leftarrow) Since $z_3^{y^2} < z_3(y^1)$, if $z^{y^2} \in R$, then $z^{y^2} \in \mathcal{Z}_{ND}(y^1)^\leq$. \square

4.2.4 Case 4. Dominance between two continuous frontiers

Let $\mathcal{Z}_{ND}(y^1) = \{E^{y^1,k}, k \in K(y^1)\}$ and $\mathcal{Z}_{ND}(y^2) = \{E^{y^2,k}, k \in K(y^2)\}$. We first analyze the case when an edge of $\mathcal{Z}_{ND}(y^2)$, $E^{y^2,k}$ for some $k \in K(y^2)$, is dominated by $\mathcal{Z}_{ND}(y^1)$. In Figure 4.5, we illustrate different dominance relations between an edge and a continuous frontier.

Proposition 10. *$E^{y^2,k} \in \mathcal{Z}_{ND}(y^2)$ is dominated by $\mathcal{Z}_{ND}(y^1) = \{E^{y^1,k}, k \in K(y^1)\}$ if and only if both $z^{y^2,k,nw}$ and $z^{y^2,k,se}$ are dominated by $\mathcal{Z}_{ND}(y^1)$.*

Proof. (\Rightarrow) If $E^{y^2,k}$ is dominated by $\mathcal{Z}_{ND}(y^1)$ for $k \in K(y^2)$, then $z^{y^2,k,nw}, z^{y^2,k,se} \in E^{y^2,k} \subseteq \mathcal{Z}_{ND}(y^1)^\leq$ and, hence, both $z^{y^2,k,nw}$ and $z^{y^2,k,se}$ are dominated by $\mathcal{Z}_{ND}(y^1)$.

(\Leftarrow) If $z^{y^2,k,nw}$ and $z^{y^2,k,se}$ are dominated by $\mathcal{Z}_{ND}(y^1)$, then $z^{y^2,k,nw}, z^{y^2,k,se} \in \mathcal{Z}_{ND}(y^1)^\leq$. Since $\mathcal{Z}_{ND}(y^1)^\leq$ is convex, $E^{y^2,k} \subseteq \mathcal{Z}_{ND}(y^1)^\leq$, and $E^{y^2,k}$ is dominated by $\mathcal{Z}_{ND}(y^1)$. \square

We can conclude whether $\mathcal{Z}_{ND}(y^2)$ is dominated by $\mathcal{Z}_{ND}(y^1)$ or not by applying Proposition 10 to every edge in $\mathcal{Z}_{ND}(y^2)$. $\mathcal{Z}_{ND}(y^2)$ is dominated by $\mathcal{Z}_{ND}(y^1)$ if every edge of it is dominated by $\mathcal{Z}_{ND}(y^1)$. We can also conclude that $\mathcal{Z}_{ND}(y^2)$ is nondominated relative to $\mathcal{Z}_{ND}(y^1)$ if the following proposition applies:

Proposition 11. *Let $z^{y^1,ne}$ be the ideal point of $\mathcal{Z}_{ND}(y^1)$ and $z^{y^2,sw}$ be the nadir point of $\mathcal{Z}_{ND}(y^2)$. If $z^{y^2,sw}$ is nondominated relative to $z^{y^1,ne}$, then $\mathcal{Z}_{ND}(y^2)$ is nondominated relative to $\mathcal{Z}_{ND}(y^1)$.*

Proof. By definition, $\mathcal{Z}_{ND}(y^1) \subseteq (z^{y^1,ne})^\leq$ and $\mathcal{Z}_{ND}(y^2) \subseteq (z^{y^2,sw})^\geq$. If $z^{y^2,sw}$ is nondominated relative to $z^{y^1,ne}$, then $(z^{y^2,sw})^\geq \cap z^{y^1,ne} = \emptyset$ by Definition 15. Since $\mathcal{Z}_{ND}(y^1) \subseteq (z^{y^1,ne})^\leq$, $(z^{y^2,sw})^\geq \cap \mathcal{Z}_{ND}(y^1) = \emptyset$. Similarly, $(\mathcal{Z}_{ND}(y^2))^\geq \cap \mathcal{Z}_{ND}(y^1) = \emptyset$ since $\mathcal{Z}_{ND}(y^2) \subseteq (z^{y^2,sw})^\geq$, and, in turn, $\mathcal{Z}_{ND}(y^2)^\geq \subseteq (z^{y^2,sw})^\geq$.

Then, $\mathcal{Z}_{ND}(\mathbf{y}^2)$ is nondominated relative to $\mathcal{Z}_{ND}(\mathbf{y}^1)$ since $(\mathcal{Z}_{ND}(\mathbf{y}^2))^\geq \cap \mathcal{Z}_{ND}(\mathbf{y}^1) = \emptyset$ by Definition 15. \square

In case $\mathcal{Z}_{ND}(\mathbf{y}^2)$ cannot be labeled by means of the above propositions, we apply the model-based dominance test stated in Algorithm 1.

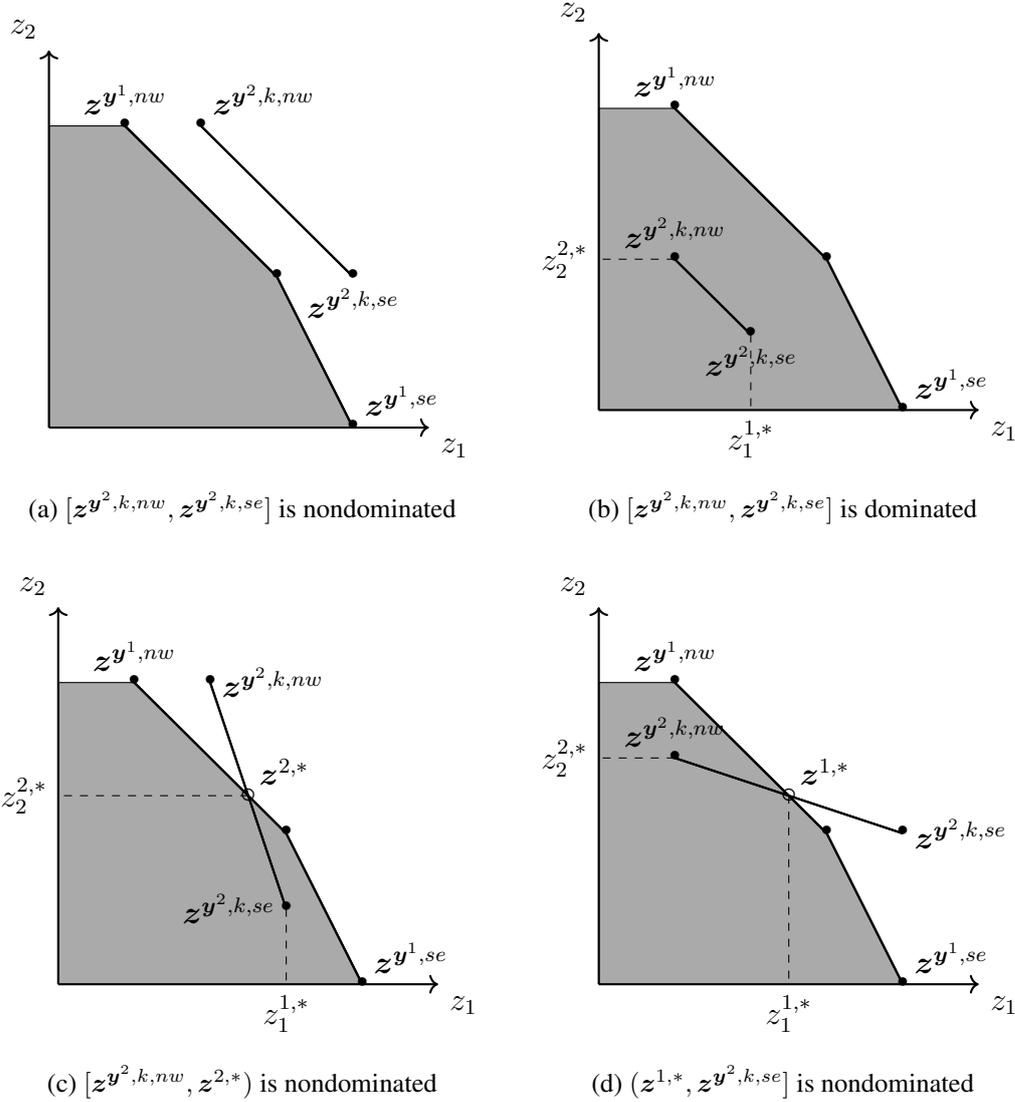


Figure 4.5: Different dominance relations between an edge and a continuous frontier

Until this point in this section, we assumed that $z_3(\mathbf{y}^2) < z_3(\mathbf{y}^1)$ while presenting the dominance relations between the nondominated sets of two slice problems. In case of $z_3(\mathbf{y}^1) = z_3(\mathbf{y}^2)$ for $\mathbf{y}^1, \mathbf{y}^2 \in \mathcal{Y}$, \mathbf{y}^1 is not guaranteed to be an efficient integer vector

Algorithm 1 Model-based dominance test between two continuous frontiers

Let $\mathcal{Z}_{ND}(\mathbf{y}^1) = \{E^{\mathbf{y}^1,k}, k \in K(\mathbf{y}^1)\}$ and $\mathcal{Z}_{ND}(\mathbf{y}^2) = \{E^{\mathbf{y}^2,k}, k \in K(\mathbf{y}^2)\}$. Also, let k_{nd} and k_d be the number of edges of $\mathcal{Z}_{ND}(\mathbf{y}^2)$ nondominated relative to $\mathcal{Z}_{ND}(\mathbf{y}^1)$ and dominated by $\mathcal{Z}_{ND}(\mathbf{y}^1)$, respectively. Initially, $k_{nd} = k_d = 0$.

for $k \in K(\mathbf{y}^2)$ **do**

Let $S = E^{\mathbf{y}^2,k}$ and solve $\mathcal{P}(\mathbf{y}^2, S, \mathbf{y}^1, 1)$.

if $\mathcal{P}(\mathbf{y}^2, S, \mathbf{y}^1, 1)$ is infeasible **then**

$E^{\mathbf{y}^2,k}$ is nondominated relative to $\mathcal{Z}_{ND}(\mathbf{y}^1)$ (see Figure 4.5a), $k_{nd} = k_{nd} + 1$.

else

Solve $\mathcal{P}(\mathbf{y}^2, S, \mathbf{y}^1, 2)$. Let $z^{i,*}$ be the optimal solution of $\mathcal{P}(\mathbf{y}^2, S, \mathbf{y}^1, i)$, $i \in \{1, 2\}$.

if $z^{1,*} = z^{\mathbf{y}^2,k,se}$ and $z^{2,*} = z^{\mathbf{y}^2,k,nw}$ **then**

$E^{\mathbf{y}^2,k}$ is dominated by $\mathcal{Z}_{ND}(\mathbf{y}^1)$ (see Figure 4.5b), $k_d = k_d + 1$.

else if $z^{1,*} = z^{\mathbf{y}^2,k,se}$ **then**

$[z^{\mathbf{y}^2,k,nw}, z^{2,*})$ is nondominated relative to $\mathcal{Z}_{ND}(\mathbf{y}^1)$ (see Figure 4.5c).

else if $z^{2,*} = z^{\mathbf{y}^2,k,nw}$ **then**

$(z^{1,*}, z^{\mathbf{y}^2,k,se}]$ is nondominated relative to $\mathcal{Z}_{ND}(\mathbf{y}^1)$ (see Figure 4.5d).

else

$[z^{\mathbf{y}^2,k,nw}, z^{2,*})$ and $(z^{1,*}, z^{\mathbf{y}^2,k,se}]$ are nondominated relative to $\mathcal{Z}_{ND}(\mathbf{y}^1)$.

end if

end if

end for

if $k_{nd} = |K(\mathbf{y}^2)|$ **then**

$\mathcal{Z}_{ND}(\mathbf{y}^2)$ is nondominated relative to $\mathcal{Z}_{ND}(\mathbf{y}^1)$.

else if $k_d = |K(\mathbf{y}^2)|$ **then**

$\mathcal{Z}_{ND}(\mathbf{y}^2)$ is dominated by $\mathcal{Z}_{ND}(\mathbf{y}^1)$.

else

$\mathcal{Z}_{ND}(\mathbf{y}^2)$ is partially nondominated relative to $\mathcal{Z}_{ND}(\mathbf{y}^1)$.

end if

relative to \mathbf{y}^2 . Hence, we also need to check if $\mathcal{Z}_{ND}(\mathbf{y}^1)$ is dominated by $\mathcal{Z}_{ND}(\mathbf{y}^2)$, and, if it is not, to find the nondominated subsets of it.

In the next section, we present our algorithm to find \mathcal{Z}_{ND} . The algorithm generates efficient integer vectors $\mathbf{y}^1, \mathbf{y}^2, \dots, \mathbf{y}^n$ in non-increasing order of z_3 values, $z_3(\mathbf{y}^j) \geq z_3(\mathbf{y}^{j+1})$. It uses the dominance tests presented in this section to conclude whether $\mathcal{Z}_{ND}(\mathbf{y}^j) \subseteq \mathcal{Z}_{ND}$, or to eliminate the subsets of $\mathcal{Z}_{ND}(\mathbf{y}^j)$ dominated by $\mathcal{Z}_{ND}(\mathbf{y}^k)$, $k > j$, where $z_3(\mathbf{y}^j) = z_3(\mathbf{y}^k)$.

4.3 A cone-based search algorithm

As \mathcal{Z}_{ND} may include disconnected edges (closed, half-open, open) as well as points, it is a non-convex set, and hence it is difficult to separate it from the dominated region in the criterion space. The dimension of the feasible criterion space of any slice problem of \mathcal{P} is at most two and this feasible criterion space is a subset of a plane parallel to the z_1 and z_2 axes. Let $z_{1,2} \in \mathbb{R}^2$ be the projection of $\mathbf{z} \in \mathbb{R}^3$ on the $z_1 - z_2$ plane. Similarly, let $\mathcal{Z}_{1,2}$ be the projection of \mathcal{Z} on the $z_1 - z_2$ plane, $\mathcal{Z}_{1,2} := \{(z_1, z_2) : \mathbf{z} \in \mathcal{Z}\}$. We develop an iterative algorithm that sequentially finds weakly nondominated points or edges in the non-increasing order of z_3 values and partitions $\mathcal{Z}_{1,2}$ into smaller search regions that include the rest of the nondominated frontier. At each iteration, we select a search region, solve the slice problem in that region and update the search regions and the nondominated set. The algorithm stops when none of the search regions includes any nondominated points or edges.

4.3.1 Search region

Without loss of generality, we assume that $\mathcal{Z}_{1,2}$ is in the non-negative quadrant, $\mathcal{Z}_{1,2} \in \mathbb{R}_{\geq}^2$. We partition the non-negative quadrant into convex cones and search within each cone for nondominated points or edges. Each cone containing a weakly nondominated point or edge is further partitioned into child cones and the dominated regions in the parent cone are eliminated from the child cones.

We illustrate the search region definition in Figure 4.6. The dashed lines represent the

boundary of the dominated space. In Figure 4.6a, search region \mathcal{R}^j is separated from the dominated space by means of cone \mathcal{C}^j and edge \mathcal{E}^j . A lower bound vector, l^j , is also active in defining the search region in Figure 4.6b. We formally define search regions and their components next.

Let \mathcal{R} be the set of search regions and J denote the index set of regions in \mathcal{R} . We denote $\mathcal{R}^j \in \mathcal{R}$ as the j^{th} search region, $\mathcal{R}^j \subseteq \mathbb{R}_{\geq}^2, \forall j \in J$. We represent \mathcal{R}^j by the tuple $(\mathcal{C}^j, \mathcal{E}^j, l^j)$ as demonstrated in Figure 4.6, where:

- $\mathcal{C}^j \subseteq \mathbb{R}_{\geq}^2$ is the enclosing convex cone of \mathcal{R}^j , $\mathcal{R}^j \subseteq \mathcal{C}^j$, such that $\mathcal{C}^j := \{z : z = \mu^0 r^{j,0} + \mu^1 r^{j,1}, \mu^0, \mu^1 \geq 0\}$ where $r^{j,0}$ and $r^{j,1}$ are its extreme rays. Each search region is enclosed by exactly one convex cone and $\text{int}(\mathcal{C}^j) \cap \text{int}(\mathcal{C}^k) = \emptyset$, for $k \neq j$. Let $\alpha(r)$ be the acute angle of ray r with the unit vector $(1, 0)$. Then, $\alpha(r^{j,0}) > \alpha(r^{j,1})$.
- \mathcal{E}^j is the edge of the search region \mathcal{R}^j with a non-empty interior. Each search region has at most one such edge in its boundary (\mathcal{E}^j does not exist in the definition of \mathcal{R}^j if \mathcal{R}^j can be characterized by a lower bound vector). If \mathcal{E}^j exist, and $e^{j,0}$ and $e^{j,1}$ are the two extreme points of \mathcal{E}^j such that $e_1^{j,0} < e_1^{j,1}$, then $r^{j,0}$ and $r^{j,1}$ pass through $e^{j,0}$ and $e^{j,1}$, respectively.
- $l^j = (l_1^j, l_2^j)$ is the lower bound vector of \mathcal{R}^j , where l_1^j and l_2^j are the lower bounds on z_1 and z_2 , respectively. A lower bound vector does not exist for \mathcal{R}^j if \mathcal{C}^j and \mathcal{E}^j are sufficient to separate \mathcal{R}^j from the dominated space.

We define the j^{th} search region as $\mathcal{R}^j := \{u \in \mathbb{R}^2 : u \in \mathcal{C}^j \cap (\mathcal{E}^j)^{\geq} \cap (l^j)^{\geq}\}$. We next show that \mathcal{R}^j is a polyhedron.

Proposition 12. \mathcal{R}^j is a polyhedron in $\mathbb{R}_{\geq 0}^2$ and is the intersection of at most 5 half-spaces.

Proof. We will show that \mathcal{R}^j is the intersection of $n \leq 5$ half-spaces, $\mathcal{S}^1, \mathcal{S}^2, \dots, \mathcal{S}^5$,

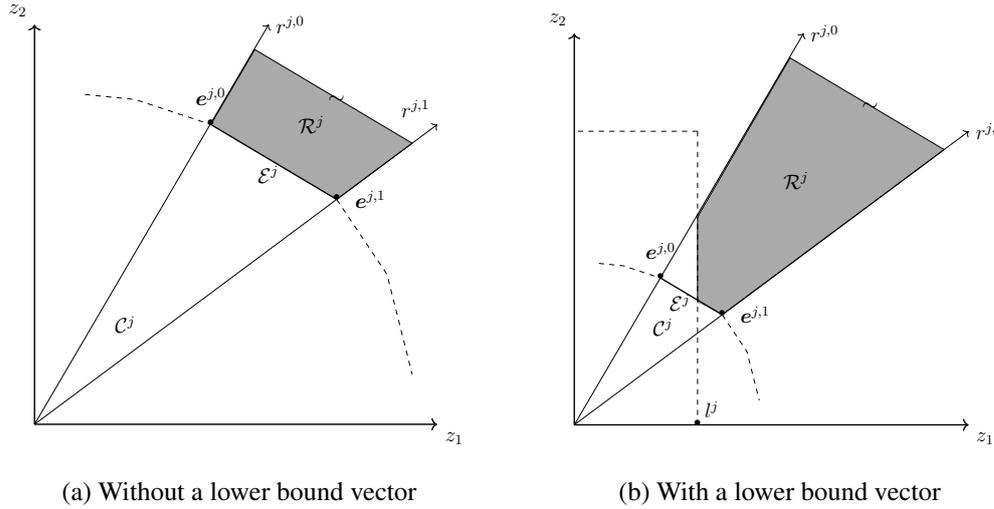


Figure 4.6: Illustration of a search region and its components

and it will suffice to prove that \mathcal{R}^j is a polyhedron. These half-spaces are

$$\begin{aligned} \mathcal{S}^1 &= \{ \mathbf{u} \in \mathbb{R}_{\geq 0}^2 : \tan(\alpha(r^{j,0})) \times u_1 \geq u_2 \}, \\ \mathcal{S}^2 &= \{ \mathbf{u} \in \mathbb{R}_{\geq 0}^2 : \tan(\alpha(r^{j,1})) \times u_1 \leq u_2 \}, \\ \mathcal{S}^3 &= \{ \mathbf{u} \in \mathbb{R}_{\geq 0}^2 : \mathbf{u}^T \mathbf{g}^j \geq 0 \}, \\ \mathcal{S}^4 &= \{ \mathbf{u} \in \mathbb{R}_{\geq 0}^2 : u_1 \geq l_1^j \}, \\ \mathcal{S}^5 &= \{ \mathbf{u} \in \mathbb{R}_{\geq 0}^2 : u_2 \geq l_2^j \}, \end{aligned}$$

where \mathbf{g}^j is the gradient vector of \mathcal{E}^j . The enclosing convex cone of the region is $\mathcal{C}^j = \mathcal{S}^1 \cap \mathcal{S}^2$. If \mathcal{E}^j and \mathbf{l}^j does not exist, then $n = 2$. Otherwise, $\mathcal{C}^j \cap (\mathcal{E}^j)^\cong = \mathcal{C}^j \cap \mathcal{S}^3$ since $e^{j,0}$ and $e^{j,1}$ are on the extreme rays $r^{j,0}$ and $r^{j,1}$ of the cone, respectively. If \mathbf{l}^j does not exist, then $n = 3$. Otherwise, by definition, $(\mathbf{l}^j)^\cong = \mathcal{S}^4 \cap \mathcal{S}^5$ and $n = 5$. Then, $\mathcal{R}^j = \{ \mathbf{u} \in \mathbb{R}^2 : \mathbf{u} \in \mathcal{C}^j \cap (\mathcal{E}^j)^\cong \cap (\mathbf{l}^j)^\cong \} = \bigcap_{k=1, \dots, 5} \mathcal{S}^k$. \mathcal{R}^j is a polyhedron in $\mathbb{R}_{\geq 0}^2$ defined by the intersection of $n \leq 5$ half spaces in $\mathbb{R}_{\geq 0}^2$. \square

Throughout the search, we maintain a set of search regions and guarantee that none of them include points that are strictly dominated by the previously generated non-dominated points or edges. This is achieved by searching the regions that are not dominated by \mathbf{l}^j or \mathcal{E}^j . In addition, nondominated points or edges that have not been identified yet must exist in some of the current search regions. Hence, we are able to separate the dominated regions from the feasible criterion space throughout the iter-

ations of the algorithm. We represent the boundary between \mathcal{R}^j and the dominated space, \mathcal{Z}_D , as $\mathcal{B}^j = \mathcal{R}^j \cap \mathcal{Z}_D$. We will use \mathcal{B}^j to identify which of the weakly nondominated points in \mathcal{R}^j may be nondominated.

4.3.2 Search problem

To search for a nondominated point that has a projection on \mathcal{R}^j , we solve the following problem:

$$\begin{aligned} \mathcal{P}^j : \quad & \text{lexmax} \quad (z_3(\mathbf{x}), z_1(\mathbf{x}), z_2(\mathbf{x})) \\ & \text{s.to.} \quad \mathbf{x} \in \mathcal{X} \\ & \quad \quad H(\mathbf{x}^d, \mathbf{y}^t) \geq 1, \forall \mathbf{y}^t \in \mathcal{Y}_j^t \\ & \quad \quad (z_1(\mathbf{x}), z_2(\mathbf{x})) \in \mathcal{R}^j \end{aligned}$$

where $H(\mathbf{x}^d, \mathbf{y}^t)$ is the Hamming distance (Hamming, 1950) between \mathbf{x}^d and $\mathbf{y}^t \in \mathcal{Y}_j^t$, and \mathcal{Y}_j^t is the set of previously generated integer vectors with the nondominated set of their slice problems having intersection with \mathcal{B}^j . Restricting the Hamming distance between \mathbf{x}^d and \mathbf{y}^t to be at least one, we eliminate the integer vectors in \mathcal{Y}_j^t from the feasible set of \mathcal{P}^j . The constraints of type $H(\mathbf{x}^d, \mathbf{y}^t) \geq 1$ are called *tabu constraints* (Fischetti and Lodi, 2003) or *no-good constraints* (Hooker, 2011). Stidsen et al. (2014) uses tabu constraints to make integer fathoming in their branch and bound algorithm to solve a class of BOMILP.

The tabu constraints support the exact nature of the algorithm. \mathcal{P}^j either produces a new integer vector in $\mathcal{R}^j \subseteq \mathcal{C}^j$ (if any) or is infeasible. When the algorithm completes searching all regions, it reveals all efficient integer vectors as well as all nondominated points and edges.

The hamming distance of an integer decision variable vector, \mathbf{x}^d to an integer vector \mathbf{y} , $H(\mathbf{x}^d, \mathbf{y})$, can be computed as $H(\mathbf{x}_v, \mathbf{y}) = \sum_{v \in \mathcal{V}} |x_v^d - y_v|$. Fischetti et al. (2005) linearizes it for binary vectors as follows:

$$H(\mathbf{x}^d, \mathbf{y}) = \sum_{v \in \mathcal{V}: y_v=0} x_v^d + \sum_{v \in \mathcal{V}: y_v=1} (1 - x_v^d)$$

For a general integer vector \mathbf{x}^d where $l_v \leq x_v^d \leq u_v$, $v \in \mathcal{V}$, Soylu (2018) presents a linear inequality system that requires an additional binary variable and two linear

constraints for each integer variable x_v^d , $v \in \mathcal{V}$ such that $l_v < x_v^d < u_v$.

$$\begin{aligned}
H(\mathbf{x}^d, \mathbf{y}) &= \sum_{v \in \mathcal{V}: y_v = l_v} (x_v^d - l_v) + \sum_{v \in \mathcal{V}: y_v = u_v} (u_v - x_v^d) + \sum_{v \in \mathcal{V}: l_v < y_v < u_v} (x_v^{d,+} + x_v^{d,-}) \\
x_v^d - x_v^{d,+} + x_v^{d,-} &= y_v \\
x_v^{d,+} &\leq (u_v - l_v)\beta_v \\
x_v^{d,-} &\leq (u_v - l_v)(1 - \beta_v) \\
x_v^{d,+}, x_v^{d,-} &\geq 0, \beta_v \in \{0, 1\}
\end{aligned}$$

The objective function of \mathcal{P}^j lexicographically uses the objectives in the order of z_3 , z_1 , and z_2 . That is, the model is used to maximize z_3 , and first z_1 , then z_2 are used to break ties, without sacrificing from the achievements of preceding objective(s). \mathcal{P}^j is an MILP since \mathcal{R}^j is a polyhedron, and the tabu constraints are modeled as linear constraints with binary variables.

We denote the set of points that have not yet proven to be nondominated as $\tilde{\mathcal{Z}}_{ND}$ and the integer vectors corresponding to these points as $\tilde{\mathcal{Y}}_E$. Let \mathbf{y}^n be the integer vector for which we solve the slice problem at iteration n . We next show that, at each iteration feasible search regions exist, we guarantee to find a new weakly efficient integer vector.

4.3.3 Search region selection and update

At iteration n , we maintain a set of feasible search regions, \mathcal{R}^j , and candidate points \mathbf{z}^j , $\mathbf{z}_{1,2}^j \in \mathcal{R}^j$. For region j , \mathbf{z}^j is found by solving \mathcal{P}^j , and it is nondominated relative to the other feasible points in the region. Let \mathbf{y}^j be the integer vector of \mathbf{z}^j . We select search region j^* that contains the point that has the maximum z_3 value, $j^* = \operatorname{argmax}_{j \in J} z_3^j$. We break ties in favor of the point and the region with the higher z_1 value first and higher z_2 value next. Denoting $\mathbf{z}^n = \mathbf{z}^{j^*}$ and $\mathbf{y}^n = \mathbf{y}^{j^*}$, we show that \mathbf{z}^n is a weakly nondominated point of \mathcal{P} and establish the condition required for \mathbf{z}^n to be nondominated.

Proposition 13. *$\mathbf{z}^n = \mathbf{z}^{j^*}$ is a weakly nondominated point and $\mathbf{y}^n = \mathbf{y}^{j^*}$ is a weakly efficient integer vector. If \mathbf{z}^{j^*} is not a boundary point, $\mathbf{z}^{j^*} \notin \mathcal{B}^{j^*}$, then \mathbf{z}^n is a nondominated point and \mathbf{y}^n is an efficient integer vector.*

Proof. For the first iteration, $n = j = 1$, \mathbf{z}^1 is a nondominated point since $\mathcal{Z}_{1,2} \subseteq \mathcal{R}^1 = \mathbb{R}_{\geq}^2$, and $\mathcal{Y}_1^t = \emptyset$. For $n > 1$, let $\mathcal{Z}_D^{n-1} \subseteq \mathcal{Z}$ be the feasible objective space dominated by the points and edges generated by the algorithm in previous iterations. The search region definition and the tabu constraints added to exclude the previously generated integer vectors in $\mathcal{Y}_{j^*}^t$ guarantees that \mathbf{z}^{j^*} is a weakly nondominated point relative to any point generated in a previous iteration m , $m < n$. At iteration n , \mathbf{z}^n is selected in such a way that $\mathbf{z}_3^n \geq \mathbf{z}_3^m$ for $m \geq n$, and $\nexists \mathbf{z}^m, \mathbf{z}^m \neq \mathbf{z}^n : \mathbf{z}_3^m = \mathbf{z}_3^n, \mathbf{z}_1^m \geq \mathbf{z}_1^n, \mathbf{z}_2^m \geq \mathbf{z}_2^n$. That is, \mathbf{z}^n will be nondominated relative to any point to be generated after iteration n . Therefore, \mathbf{z}^n is a weakly nondominated point and \mathbf{y}^n is a weakly efficient integer vector. Furthermore, when $\mathbf{z}^{j^*} \notin \mathcal{B}^{j^*}$, it is guaranteed that \mathbf{z}^{j^*} is a nondominated point relative to any point generated in a previous iteration m , $m < n$. Since, \mathbf{z}^{j^*} is already nondominated relative to \mathbf{z}^m for $m > n$, \mathbf{z}^n is a nondominated point and \mathbf{y}^n is an efficient integer vector. \square

After generating the integer vector \mathbf{y}^{j^*} at iteration n , we next generate the nondominated set of the slice problem of \mathbf{y}^{j^*} , $\mathcal{P}(\mathbf{y}^{j^*})$. While solving slice problem $\mathcal{P}(\mathbf{y}^{j^*})$, we also include the constraint $z_{1,2}(\mathbf{x}) \in \mathcal{R}^{j^*}$ to only generate the nondominated subset of the slice problem that resides in the current search region. This simplifies the search region update procedure as we do not have to update the adjacent cones that can include a part of the nondominated set of the slice problem, and also prevents generating points in the dominated region. We define the slice problem of \mathbf{y}^j in region \mathcal{R}^j , $\mathcal{P}(\mathbf{y}^j, \mathcal{R}^j)$, as follows:

$$\begin{aligned} \mathcal{P}(\mathbf{y}^j, \mathcal{R}^j) : \quad & \text{Max} \quad \mathbf{z}(\mathbf{x}) = \{z_3(\mathbf{x}), z_1(\mathbf{x}), z_2(\mathbf{x})\} \\ & \text{s.to.} \quad \mathbf{x} \in \mathcal{X} \\ & \quad \mathbf{x}^d = \mathbf{y}^j \\ & \quad z_{1,2}(\mathbf{x}) \in \mathcal{R}^j \end{aligned}$$

Let $\mathcal{Z}_{ND}(\mathbf{y}^j, \mathcal{R}^j)$ be the nondominated set of $\mathcal{P}(\mathbf{y}^j, \mathcal{R}^j)$. We know that $\forall \mathbf{z} \in \mathcal{Z}(\mathbf{y}^j)$, $z_3 = z_3(\mathbf{y}^j)$, and $\dim(\mathcal{Z}(\mathbf{y}^j)) \leq 2$. Therefore, we solve $\mathcal{P}(\mathbf{y}^j, \mathcal{R}^j)$ as a BOLP, and use the dichotomic search algorithm (Aneja and Nair, 1979) to generate $\mathcal{Z}_{ND}(\mathbf{y}^j, \mathcal{R}^j)$. Since the selected point in the region, \mathbf{z}^j , is already a point of $\mathcal{Z}_{ND}(\mathbf{y}^j)$, $\mathcal{Z}_{ND}(\mathbf{y}^j, \mathcal{R}^j) \cap \mathcal{Z}_{ND}(\mathbf{y}^j) \neq \emptyset$. In the following proposition, we show that $\mathcal{Z}_{ND}(\mathbf{y}^j, \mathcal{R}^j)$ is a subset of $\mathcal{Z}_{ND}(\mathbf{y}^j)$.

Proposition 14. *If there exists $z \in \mathcal{Z}_{ND}(\mathbf{y}^j)$ such that $z_{1,2} \in \mathcal{R}^j$, then $\mathcal{Z}_{ND}(\mathbf{y}^j, \mathcal{R}^j) \subseteq \mathcal{Z}_{ND}(\mathbf{y}^j)$.*

Proof. If $\mathcal{Z}_{ND}(\mathbf{y}^j)$ is a singleton, then $\mathcal{Z}_{ND}(\mathbf{y}^j, \mathcal{R}^j) \subseteq \mathcal{Z}_{ND}(\mathbf{y}^j)$ as $\mathcal{Z}_{ND}(\mathbf{y}^j, \mathcal{R}^j) \cap \mathcal{Z}_{ND}(\mathbf{y}^j) = z$. Else, let $\tilde{\mathcal{Z}}(\mathbf{y}^j, \mathcal{R}^j) := \{z \in \mathcal{Z}(\mathbf{y}^j) : z_{1,2} \in \mathcal{R}^j\}$ and $\tilde{\mathcal{Z}}_{ND}(\mathbf{y}^j, \mathcal{R}^j) := \{z \in \mathcal{Z}_{ND}(\mathbf{y}^j) : z_{1,2} \in \mathcal{R}^j\}$. We need to show that:

1. $\tilde{\mathcal{Z}}_{ND}(\mathbf{y}^j, \mathcal{R}^j) \subseteq \mathcal{Z}_{ND}(\mathbf{y}^j, \mathcal{R}^j)$
2. $\mathcal{Z}_{ND}(\mathbf{y}^j, \mathcal{R}^j) \subseteq \tilde{\mathcal{Z}}_{ND}(\mathbf{y}^j, \mathcal{R}^j)$

Proof of (1): For any $\bar{z} \in \tilde{\mathcal{Z}}_{ND}(\mathbf{y}^j, \mathcal{R}^j)$, there does not exist $\hat{z} \in \mathcal{Z}(\mathbf{y}^j)$ such that $\bar{z} \in (\hat{z})^{\leq}$. Then, there does not exist $\hat{z} \in \mathcal{Z}(\mathbf{y}^j, \mathcal{R}^j)$ such that $\bar{z} \in (\hat{z})^{\leq}$. Hence, $\bar{z} \in \mathcal{Z}_{ND}(\mathbf{y}^j, \mathcal{R}^j)$, and $\tilde{\mathcal{Z}}_{ND}(\mathbf{y}^j, \mathcal{R}^j) \subseteq \mathcal{Z}_{ND}(\mathbf{y}^j, \mathcal{R}^j)$.

Proof of (2): We use the fact that the nondominated set of a multi-objective linear program is connected and supported. By assumption, there exists $z \in \mathcal{Z}_{ND}(\mathbf{y}^j)$ such that $z_{1,2} \in \mathcal{R}^j$. Let $\bar{z} \in \tilde{\mathcal{Z}}_{ND}(\mathbf{y}^j, \mathcal{R}^j)$. Suppose that \bar{z}^l is the extreme supported nondominated point of $\mathcal{Z}_{ND}(\mathbf{y}^j)$ being left adjacent to \bar{z} . Then, $[\bar{z}^l, \bar{z}]$ is a nondominated edge of $\mathcal{Z}_{ND}(\mathbf{y}^j)$. If $\bar{z}_{1,2}^l \in \mathcal{R}^j$, then $[\bar{z}^l, \bar{z}] \subseteq \mathcal{Z}_{ND}(\mathbf{y}^j, \mathcal{R}^j)$ and $[\bar{z}^l, \bar{z}] \subseteq \tilde{\mathcal{Z}}_{ND}(\mathbf{y}^j, \mathcal{R}^j)$. We can let $\bar{z} = \bar{z}^l$ and continue in the same manner as long as the left adjacent extreme supported nondominated point is in the considered search region. All the visited edges are in both $\mathcal{Z}_{ND}(\mathbf{y}^j, \mathcal{R}^j)$ and $\tilde{\mathcal{Z}}_{ND}(\mathbf{y}^j, \mathcal{R}^j)$.

If $\bar{z}_{1,2}^l \notin \mathcal{R}^j$, then $\exists z^* \in [\bar{z}^l, \bar{z}]$ such that $z_{1,2}^*$ is a boundary point of \mathcal{R}^j . Then, z^* is an extreme supported nondominated point in $\mathcal{Z}_{ND}(\mathbf{y}^j, \mathcal{R}^j)$, and there does not exist any nondominated point in $\mathcal{Z}_{ND}(\mathbf{y}^j, \mathcal{R}^j)$ being left adjacent to it. Therefore, $[z^*, \bar{z}] \subseteq \mathcal{Z}_{ND}(\mathbf{y}^j, \mathcal{R}^j)$, and also $[z^*, \bar{z}] \subseteq \tilde{\mathcal{Z}}_{ND}(\mathbf{y}^j, \mathcal{R}^j)$ as $[z^*, \bar{z}] \subseteq \mathcal{Z}_{ND}(\mathbf{y}^j)$. A similar procedure can be followed for any extreme supported nondominated point in $\mathcal{Z}_{ND}(\mathbf{y}^j, \mathcal{R}^j)$ being right adjacent to \bar{z} .

We showed that every edge of $\mathcal{Z}_{ND}(\mathbf{y}^j, \mathcal{R}^j)$ is also an edge of $\tilde{\mathcal{Z}}_{ND}(\mathbf{y}^j, \mathcal{R}^j)$. This implies that $\mathcal{Z}_{ND}(\mathbf{y}^j, \mathcal{R}^j) \subseteq \tilde{\mathcal{Z}}_{ND}(\mathbf{y}^j, \mathcal{R}^j)$. \square

Having generated $\mathcal{Z}_{ND}(\mathbf{y}^j, \mathcal{R}^j)$ for $j = j^*$, the next step is to exclude the dominated space by $\mathcal{Z}_{ND}(\mathbf{y}^j, \mathcal{R}^j)$ from the search regions in \mathcal{R} . We do this by updating the lower

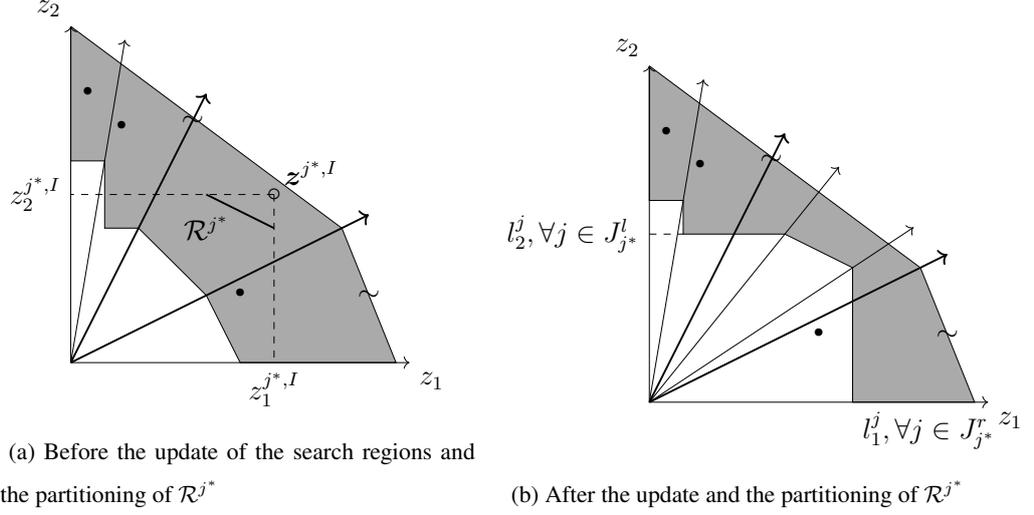


Figure 4.7: Illustration of the lower bound update procedure

bound vectors of the search regions. Let $z^{j^*,I}$ be the ideal point of $\mathcal{Z}_{ND}(\mathbf{y}^{j^*}, \mathcal{R}^{j^*})$, and $J_{j^*}^l := \{j \in J : \alpha(r^{j,1}) \geq \alpha(r^{j^*,0})\}$ and $J_{j^*}^r := \{j \in J : \alpha(r^{j,0}) \leq \alpha(r^{j^*,1})\}$ be the index set of cones on the left and right hand-side of \mathcal{C}^{j^*} , respectively. Note that $z^{j^*,I} \in \mathcal{R}^{j^*}$, and $z_3^{j^*,I} \geq z_3, z \in \mathcal{R}^j, j \in J$.

For each $j \in J_{j^*}^l$, if $l_2^j < z_2^{j^*,I}$, then we update $l_2^j = z_2^{j^*,I}$. Similarly, for each $j \in J_{j^*}^r$, if $l_1^j < z_1^{j^*,I}$, then we update $l_1^j = z_1^{j^*,I}$. We illustrate the update procedure in Figure 4.7. After updating the lower bound vectors of the search regions, if there exists k such that $z_{1,2}^k \notin \mathcal{R}^k$ (that is z^k is revealed to be dominated after the update), then we solve \mathcal{P}^k again with the updated lower bound vector. This situation is depicted in Figure 4.7b, where the search problem must be solved again for the right-most cone as it turns out that the candidate point in that cone is dominated by $\mathcal{Z}_{ND}(\mathbf{y}^{j^*}, \mathcal{R}^{j^*})$ as shown by the following two propositions:

Proposition 15. For $z^k \in \mathcal{R}^k : k \in J_{j^*}^l$, if $z_2^k < z_2^{j^*,I}$, then z^k is dominated by $\mathcal{Z}_{ND}(\mathbf{y}^{j^*}, \mathcal{R}^{j^*})$.

Proof. Let r^{j^*} and r^k be the rays originating from $(0,0)$ and passing through $z^{j^*,I}$ and z^k , respectively. Since $k \in J_{j^*}^l$, $\alpha(r^k) \geq \alpha(r^{j^*})$. If $z_2^k < z_2^{j^*,I}$, then $z_1^k < z_1^{j^*,I}$ as $\tan(\alpha(r^k)) \geq \tan(\alpha(r^{j^*}))$, $z_2^k = z_1^k \tan(\alpha(r^k))$, and $z_2^{j^*,I} = z_1^{j^*,I} \tan(\alpha(r^{j^*}))$. We also know that $z_3^{j^*,I} \geq z_3^k$ due to the selection of $z^{j^*,I}$. Then, z^k is dominated by $z^{j^*,I}$,

and hence, dominated by $\mathcal{Z}_{ND}(\mathbf{y}^{j^*}, \mathcal{R}^{j^*})$. \square

Proposition 16. For $\mathbf{z}^k \in \mathcal{R}^k : k \in J_{j^*}^r$, if $z_1^k < z_1^{j^*,I}$, then \mathbf{z}^k is dominated by $\mathcal{Z}_{ND}(\mathbf{y}^{j^*}, \mathcal{R}^{j^*})$.

Proof. Let r^{j^*} and r^k be the rays originating from $(0, 0)$ and passing through \mathbf{z}^{j^*} and \mathbf{z}^k , respectively. Since $k \in J_{j^*}^r$, $\alpha(r^k) \leq \alpha(r^{j^*})$. If $z_1^k < z_1^{j^*}$, then $z_2^k < z_2^{j^*}$ as $\tan(\alpha(r^k)) \leq \tan(\alpha(r^{j^*}))$, $z_2^k = z_2^k \tan(\alpha(r^k))$ and $z_2^{j^*,I} = z_1^{j^*,I} \tan(\alpha(r^{j^*}))$. We also know that $z_3^{j^*} \geq z_3^k$ due to the selection of \mathbf{z}^{j^*} . Then, \mathbf{z}^k is dominated by $\mathbf{z}^{j^*,I}$, and hence, dominated by $\mathcal{Z}_{ND}(\mathbf{y}^{j^*}, \mathcal{R}^{j^*})$. \square

The last step is to partition \mathcal{R}^{j^*} into smaller child regions to eliminate the dominated region in \mathcal{R}^{j^*} . Let $\mathcal{R}_k^{j^*}, k \in J_{j^*}$ be the k^{th} child search region of \mathcal{R}^{j^*} . We update $\mathcal{R} = \mathcal{R} \setminus \mathcal{R}^{j^*} \cup \{\mathcal{R}_k^{j^*}, k \in J_{j^*}\}$ and add $H(\mathbf{x}^d, \mathbf{y}^{j^*}) \geq 1$ constraint to each search problem $\mathcal{P}^k, k \in J_{j^*}$. We also update $J = J \setminus j^* \cup J_{j^*}$. In the next iteration, we only solve \mathcal{P}^k , for $k \in J_{j^*}$ as the other search regions have not been modified and already contain their candidate points.

We illustrate how we partition $\mathcal{R}^1 = \mathbb{R}_{\geq}^2$ after the generation of $\mathcal{Z}_{ND}(\mathbf{y}^1, \mathcal{R}^1)$ in Figure 4.8. We partition \mathcal{R}^1 into $e + 2$ search regions and update $\mathcal{R} = \mathcal{R} \setminus \mathcal{R}^1 \cup \{\mathcal{R}^2, \dots, \mathcal{R}^{e+3}\}$, where e is the number of edges in $\mathcal{Z}_{ND}(\mathbf{y}^1, \mathcal{R}^1)$. Figure 4.8a shows the case where $\mathcal{Z}_{ND}(\mathbf{y}^1, \mathcal{R}^1) = \{\mathbf{z}^1\}$. The ray passing through \mathbf{z}^1 partitions the parent cone \mathcal{C}^1 to child cones \mathcal{C}^2 and \mathcal{C}^3 . The region dominated by $\mathcal{Z}_{ND}(\mathbf{y}^1, \mathcal{R}^1)$ is eliminated and two new search regions are defined, $\mathcal{R}^2 = (\mathcal{C}^2, _, (_, l_2^2))$ and $\mathcal{R}^3 = (\mathcal{C}^3, _, (l_1^3, _))$.

If $\mathcal{Z}_{ND}(\mathbf{y}^1, \mathcal{R}^1)$ is a single edge, then \mathcal{R}^1 is partitioned as shown in Figure 4.8b. Rays $r^{2,1} = r^{3,0}$ and $r^{3,1} = r^{4,0}$ pass through $e^{3,0}$ and $e^{3,1}$, and partition \mathcal{C}^1 into three child cones $\mathcal{C}^2, \mathcal{C}^3$ and \mathcal{C}^4 . The new search regions are defined as $\mathcal{R}^2 = (\mathcal{C}^2, _, (_, l_2^2))$, $\mathcal{R}^3 = (\mathcal{C}^3, \mathcal{E}^3, _)$ and $\mathcal{R}^4 = (\mathcal{C}^4, _, (l_1^4, _))$. If $\mathcal{Z}_{ND}(\mathbf{y}^1, \mathcal{R}^1)$ includes multiple edges, then there would be more child search regions as depicted in Figure 4.8c.

We should also note that if \mathcal{P}^j finds a point on the boundary of \mathcal{R}^j and if the non-dominated set of the associated slice problem is only that point, then we do not need to partition \mathcal{R}^j . It is sufficient to add the corresponding tabu-constraint to \mathcal{P}^j . In this

case, the set of child search regions of \mathcal{R}^j is itself.

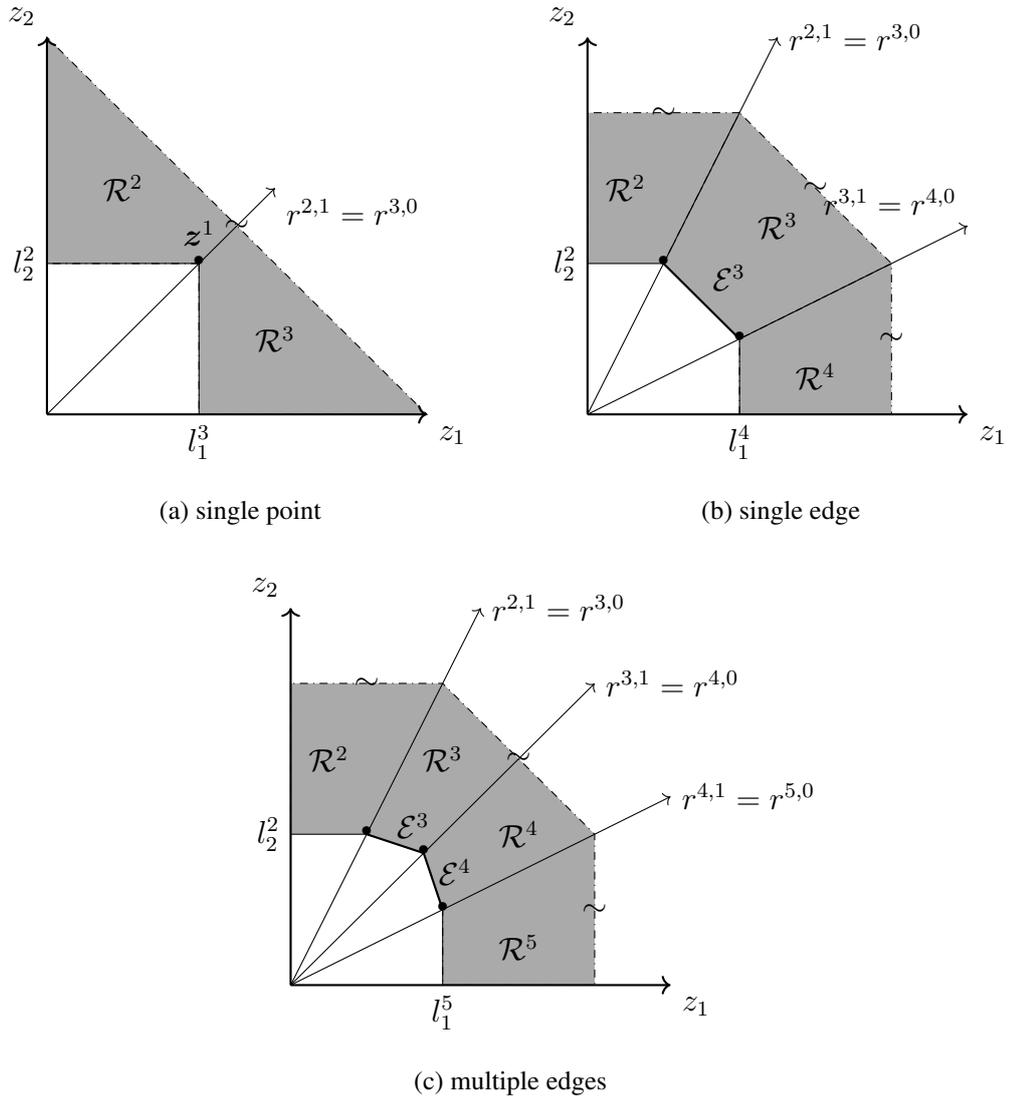


Figure 4.8: Partitioning of \mathcal{R}^1 for different cases of $\mathcal{Z}_{ND}(\mathbf{y}^1, \mathcal{R}^1)$

4.3.4 Update of the nondominated set

In this section, we present our nondominated set update procedure. Proposition 13 indicates that we find at least one weakly nondominated point, z^n , (if any) at iteration $n > 1$ (z^1 is nondominated), and we find a subset of the nondominated set of the corresponding slice problem, $\mathcal{Z}_{ND}(\mathbf{y}^n, \mathcal{R}^{j^*})$, in region \mathcal{R}^{j^*} where z^n was found. We know that $\mathcal{Z}_{ND}(\mathbf{y}^n, \mathcal{R}^{j^*})$ is a weakly nondominated set since its points have the high-

est z_3 value in the feasible criterion space not dominated by the previously generated points. We first check the current weakly nondominated set, if it is non-empty, to eliminate the ones that are dominated by $\mathcal{Z}_{ND}(\mathbf{y}^n, \mathcal{R}^{j^*})$, or to move the ones that are proved to be nondominated to the set of nondominated points. There are two cases to consider:

Case 1. If $z_3^n < z_3^{n-1}$, then $\mathcal{Z}_{ND} = \mathcal{Z}_{ND} \cup \mathcal{Z}_{WND}$, $\mathcal{Y}_E = \mathcal{Y}_E \cup \mathcal{Y}_{WE}$ and $\mathcal{Z}_{WND} = \mathcal{Y}_{WE} = \emptyset$.

Case 2. If $z_3^n = z_3^{n-1}$ and $\mathcal{Z}_{WND} \neq \emptyset$, then we need to conduct the dominance tests of Section 4.2 between $\mathcal{Z}_{ND}(\mathbf{y}^n, \mathcal{R}^{j^*})$ and each element of \mathcal{Z}_{WND} . Based on the dominance test results, we eliminate the dominated ones, move the nondominated ones to \mathcal{Z}_{ND} and their corresponding integer vectors to \mathcal{Y}_E , and update \mathcal{Z}_{WND} and \mathcal{Y}_{WE} . A subset S of \mathcal{Z}_{WND} is proved to be nondominated only if $\min_{z \in S} z_1 > \max_{z \in \mathcal{Z}_{ND}(\mathbf{y}^n, \mathcal{R}^{j^*})} z_1$ as $z_1(\mathbf{x})$ is assumed to be the objective function with the second highest priority in the lexicographic maximization of the search problems after $z_3(\mathbf{x})$.

As the last step, we add $\mathcal{Z}_{ND}(\mathbf{y}^n, \mathcal{R}^{j^*})$ and \mathbf{y}^n to the weakly nondominated set, \mathcal{Z}_{WND} , and to the weakly efficient integer vectors set, \mathcal{Y}_{WE} , respectively, if $\mathcal{Z}_{ND}(\mathbf{y}^n, \mathcal{R}^{j^*}) \cap \mathcal{B}^{j^*} = \emptyset$. Otherwise, we first eliminate every point $\hat{\mathbf{z}}$ in $\mathcal{Z}_{ND}(\mathbf{y}^n, \mathcal{R}^{j^*})$ such that $\hat{z}_{1,2} \in \mathcal{B}^{j^*}$, and then update \mathcal{Z}_{WND} and \mathcal{Y}_{WE} if the remaining set is non-empty.

4.3.5 The algorithm

We present the pseudo-code of the algorithm as Algorithm 2. The algorithm consists of four main steps:

1. Generate a candidate point for each region.
2. Determine the next integer vector and the region for the slice problem. Then, solve the slice problem and update the lower bounds of the search regions.
3. Update the nondominated set.
4. Partition the selected search region.

The algorithm terminates and produces the nondominated set when there are no search regions left. Throughout the iterations, the algorithm maintains the nondominated and weakly nondominated sets, and moves the weakly nondominated points and edges to the nondominated set as soon as the nondominance is proved.

With the next two propositions, we prove that the algorithm is exact and it terminates in a finite number of iterations.

Proposition 17. *If $\mathcal{R} = \emptyset$, \mathcal{Z}_{ND} is the nondominated set and \mathcal{Y}_E is the set of efficient integer vectors of \mathcal{P} .*

Proof. Let $\mathcal{Z}_{ND}^n \subseteq \mathcal{Z}_{ND}$ and $\mathcal{Z}_{WND}^n \subseteq \mathcal{Z}_{WND}$ be the nondominated set and weakly nondominated set at iteration n . Also, let $\mathcal{S} = \bigcup_{j \in J} \mathcal{R}^j$ and $\mathcal{S}_d \subseteq \mathcal{Z}_{1,2}$ be dominated by the points in $(\mathcal{Z}_{ND}^n \cup \mathcal{Z}_{WND}^n)_{1,2}$. Initially, $\mathcal{Z}_{1,2} \subseteq \mathcal{S}$ and $\mathcal{S}_d = \emptyset$. At each iteration, $((\mathcal{Z}_{ND}(\mathbf{y}^n, \mathcal{R}^{j^*}))_{1,2})^{\leq}$, which does not include any nondominated point, is separated from \mathcal{S} and added to the dominated search region, $\mathcal{S}_d \cup ((\mathcal{Z}_{ND}(\mathbf{y}^n, \mathcal{R}^{j^*}))_{1,2})^{\leq}$. Hence, $\nexists z \in \mathcal{Z}_{ND} \setminus \mathcal{Z}_{ND}^n$ such that $z_{1,2} \in \mathcal{S}_d$, and if $\exists z \in \mathcal{Z}_{ND} \setminus \mathcal{Z}_{ND}^n$, then $z_{1,2} \in \mathcal{S}$. If $\mathcal{R} = \emptyset$ and hence $\mathcal{S} = \emptyset$, then $\mathcal{Z}_{ND} \setminus \mathcal{Z}_{ND}^n = \emptyset$, and $\mathcal{Z}_{ND} = \mathcal{Z}_{ND}^n$. Since $((\mathcal{Z}_{ND}(\mathbf{y}^n, \mathcal{R}^{j^*}))_{1,2})^{\leq}$ is separated from the search region instead of $((\mathcal{Z}_{ND}(\mathbf{y}^n, \mathcal{R}^{j^*}))_{1,2})^{\leq}$, the algorithm allows generation of all efficient integer vectors corresponding to the same nondominated point. Therefore, \mathcal{Y}_E is the set of efficient integer vectors of \mathcal{P} . \square

Proposition 18. *The algorithm terminates in a finite number of iterations.*

Proof. Let $\mathcal{Y}_E^{\mathcal{R}} \subseteq \mathcal{Y}_E$ be the set of efficient integer vectors that can be found in the set of finitely many search regions \mathcal{R} . Initially, $\mathcal{Y}_E^{\mathcal{R}} = \mathcal{Y}_E$. By Proposition 17, given $\mathbf{y}^e \in \mathcal{Y}_E^{\mathcal{R}}$, $\exists \mathcal{R}_e \subseteq \mathcal{R}$ such that \mathbf{y}^e is a feasible integer vector for some \mathcal{P}^k , $\mathcal{R}^k \in \mathcal{R}_e$. The number of times \mathbf{y}^e is generated is bounded by $|\mathcal{R}_e|$ since each search region containing \mathbf{y}^e is partitioned into child search regions and \mathbf{y}^e is eliminated from the feasible set of each child search region by the addition of the corresponding tabu-constraint. That is, each efficient integer vector is eliminated from the feasible search space in a finite number of iterations. Since \mathcal{Y}_E is finite, Algorithm 2 terminates in a finite number of iterations with $\mathcal{R} = \mathcal{Y}_E^{\mathcal{R}} = \emptyset$. \square

Algorithm 2 Cone-based search algorithm (CBSA)

$n = 1$, $\mathcal{Z}_{ND} = \mathcal{Z}_{WND} = \mathcal{Y}_E = \mathcal{Y}_{WE} = \emptyset$, $J = \{1\}$, $\mathcal{R}^1 = (\mathbb{R}_{\geq}^2, -, -)$, $\mathcal{R} = \{\mathcal{R}^1\}$

while $\mathcal{R} \neq \emptyset$ **do**

*/****** Search the regions *****/*

for $j \in J$ **do**

If \mathcal{R}^j does not have a candidate point, solve \mathcal{P}^j . If \mathcal{P}^j is infeasible, $J = J \setminus \{j\}$ and $\mathcal{R} = \mathcal{R} \setminus \{\mathcal{R}^j\}$. Otherwise, z^j is the candidate point in \mathcal{R}^j , and \mathbf{y}^j is the corresponding integer vector.

end for

if $\mathcal{R} = \emptyset$ **then**

$\mathcal{Z}_{ND} = \mathcal{Z}_{ND} \cup \mathcal{Z}_{WND}$, $\mathcal{Y}_E = \mathcal{Y}_E \cup \mathcal{Y}_{WE}$

break

end if

*/****** Select the next integer vector, solve the slice problem and update the search space *****/*

$j^* = \underset{j \in J}{\operatorname{arglex\,max}} z^j$, $\mathbf{y}^n = \mathbf{y}^{j^*}$. Solve $\mathcal{P}(\mathbf{y}^n, \mathcal{R}^{j^*})$ and find $\mathcal{Z}_{ND}(\mathbf{y}^n, \mathcal{R}^{j^*})$.

for $j \in J \setminus \{j^*\}$ **do**

Update l^j .

if $z^j \notin \mathcal{R}^j$ after the update **then**

Solve \mathcal{P}^j and update \mathbf{y}^j . If \mathcal{P}^j is infeasible, $J = J \setminus \{j\}$, $\mathcal{R} = \mathcal{R} \setminus \{\mathcal{R}^j\}$.

end if

end for

*/****** Update the nondominated set *****/*

if $n > 1$ and $z_3(\mathbf{y}^n) < z_3(\mathbf{y}^{n-1})$ **then**

$\mathcal{Z}_{ND} = \mathcal{Z}_{ND} \cup \mathcal{Z}_{WND}$, $\mathcal{Y}_E = \mathcal{Y}_E \cup \mathcal{Y}_{WE}$, $\mathcal{Z}_{WND} = \mathcal{Y}_{WE} = \emptyset$

else

Eliminate the subset of \mathcal{Z}_{WND} dominated by $\mathcal{Z}_{ND}(\mathbf{y}^n, \mathcal{R}^{j^*})$, and update \mathcal{Y}_{WE} .

end if

$\mathcal{Z}_{WND} = \mathcal{Z}_{WND} \cup \mathcal{Z}_{ND}(\mathbf{y}^n, \mathcal{R}^{j^*})$, $\mathcal{Y}_{WE} = \mathcal{Y}_{WE} \cup \{\mathbf{y}^n\}$.

*/****** Partition the selected cone *****/*

Partition \mathcal{R}^{j^*} and update $J = J \cup J_{j^*} \setminus \{j^*\}$, $\mathcal{R} = \mathcal{R} \cup \{\mathcal{R}^j : j \in J_{j^*}\} \setminus \{\mathcal{R}^{j^*}\}$.

Add $H(\mathbf{x}^d, \mathbf{y}^n)$ to each \mathcal{P}^j , $j \in J_{j^*}$.

end while

4.3.6 Example 1.

Consider the following simple problem.

$$\begin{aligned}
 \mathcal{P} : \quad & \text{Max} \quad \mathbf{z}(\mathbf{x}) = \{z_1, z_2, z_3\} \\
 & \text{s.to.} \\
 & x_1 + x_2 + x_3 \leq 2 \\
 & z_1 = x_1 \\
 & z_2 = x_2 \\
 & z_3 = x_3 \\
 & x_1, x_2, x_3 \in \mathbb{R}_{\geq} \\
 & x_3 \in \mathbb{Z}
 \end{aligned}$$

Since one of the objectives, $z_3 = x_3$ is restricted to integer values, our algorithm can generate the nondominated set as well as all efficient integer vectors. We demonstrate our algorithm's progress below:

Iteration 1: $\mathcal{R} = \{\mathcal{R}^1\}$, where $\mathcal{R}^1 = (\mathbb{R}_{\geq}, -, -)$, $j^* = 1$, $\mathbf{y}^1 = (x_3) = 2$, $\mathbf{z}^1 = (0, 0, 2)$. The slice problem $\mathcal{P}(\mathbf{y}^1, \mathcal{R}^1)$ yields the nondominated set, $\mathcal{Z}_{ND}(\mathbf{y}^1, \mathcal{R}^1)$, as a singleton, the point previously found in \mathcal{R}^1 . Hence $\mathcal{Z}_{WND} = \{(0, 0, 2)\}$, $\mathcal{Y}_{WE} = \{(2)\}$. Since the nondominated frontier is a singleton, and the point is the boundary point of \mathcal{R}^1 , only a single child problem \mathcal{P}^2 is created with region $\mathcal{R}^2 = \mathcal{R}^1$ (see Figure 4.9a). We add the tabu-constraint $|x_3 - 2| \geq 1$ to \mathcal{P}^2 .

Iteration 2: $\mathcal{R} = \{\mathcal{R}^2\}$, where $\mathcal{R}^2 = (\mathbb{R}_{\geq}, -, -)$, $j^* = 2$, $\mathbf{y}^2 = (x_3) = 1$, $\mathbf{z}^2 = (1, 0, 1)$. The slice problem yields the nondominated edge $[(0, 1, 1), (1, 0, 1)]$. As $z_3^2 < z_3^1$, $\mathcal{Z}_{ND} = \{(0, 0, 2)\}$, $\mathcal{Y}_E = \{(2)\}$, $\mathcal{Z}_{WND} = \{[(0, 1, 1), (1, 0, 1)]\}$, $\mathcal{Y}_{WE} = \{(1)\}$. As the extreme points of the generated edge is on the boundary of the search region, only a single child search problem \mathcal{P}^3 is created with region \mathcal{R}^3 (see Figure 4.9b). We add the tabu-constraint $|x_3 - 1| \geq 1$ to \mathcal{P}^3 .

Iteration 3: $\mathcal{R} = \{\mathcal{R}^3\}$, $\mathcal{R}^3 = (\mathbb{R}_{\geq}, \mathcal{E}^3, -)$, $j^* = 3$, $\mathbf{y}^3 = (x_3) = 0$, $\mathbf{z}^3 = (2, 0, 0)$. The slice problem has the nondominated edge $[(0, 2, 0), (2, 0, 0)]$. As $z_3^3 < z_3^2$, $\mathcal{Z}_{ND} = \{(0, 0, 2), [(0, 1, 1), (1, 0, 1)]\}$, $\mathcal{Y}_E = \{(2), (1)\}$, $\mathcal{Z}_{WND} = \{[(0, 2, 0), (2, 0, 0)]\}$, and

$\mathcal{Y}_{WE} = \{(0)\}$. As the extreme points of the generated edge is on the boundary of the search region, only a single child search problem \mathcal{P}^3 is created with region \mathcal{R}^4 (see Figure 4.9c). We add the tabu-constraint $|x_3 - 0| \geq 1$ to \mathcal{P}^4 .

Iteration 4: $\mathcal{R} = \{\mathcal{R}^4\}$, where $\mathcal{R}^4 = (\mathbb{R}_{\geq}, \mathcal{E}^4, _)$. Since \mathcal{P}^4 is infeasible, $\mathcal{R} = \emptyset$, and the algorithm terminates with a single disconnected nondominated point, two nondominated edges, and three efficient integer vectors, that is,

$$\mathcal{Z}_{ND} = \{(0, 0, 2), [(0, 1, 1), (1, 0, 1)], [(0, 2, 0), (2, 0, 0)]\}$$

and $\mathcal{Y}_E = \{(2), (1), (0)\}$. We illustrate the nondominated set in Figure 4.9d.

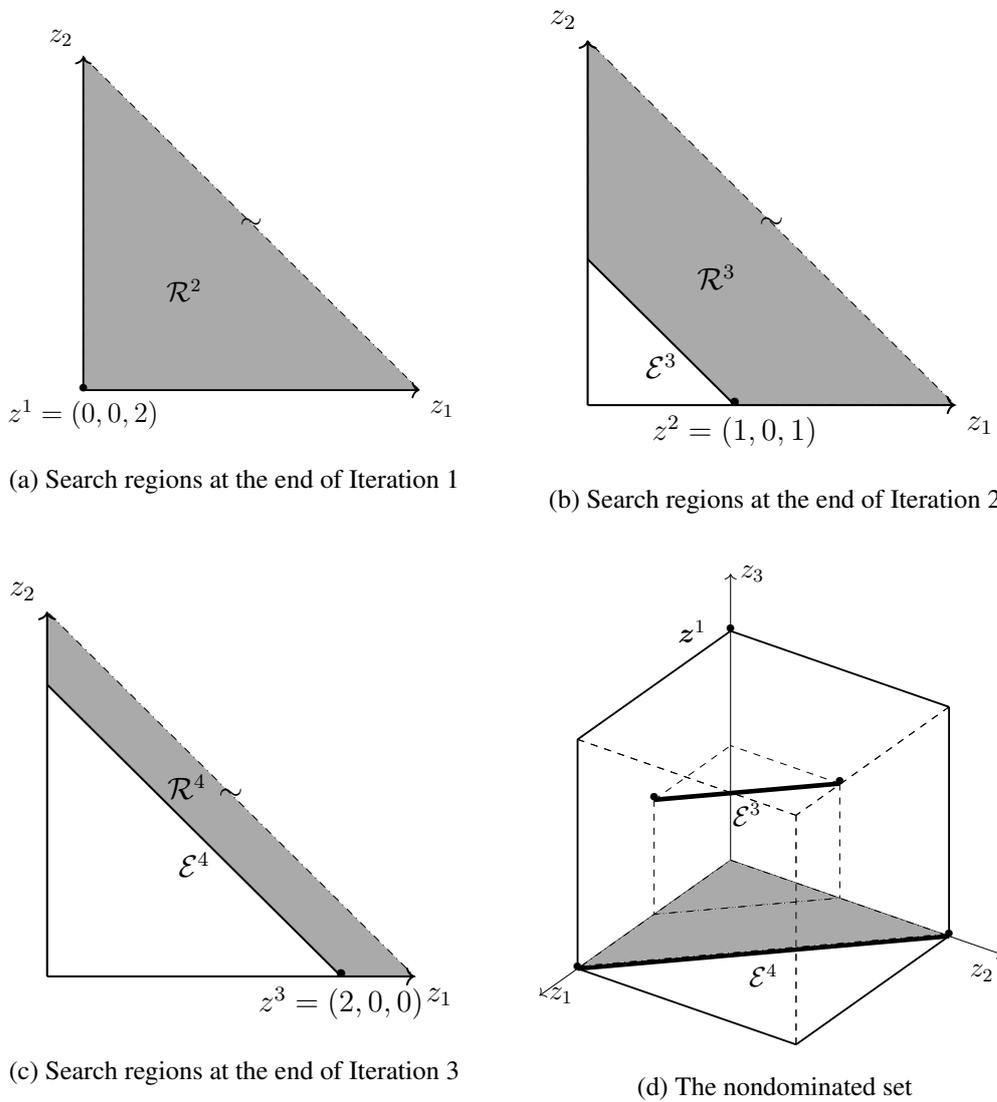


Figure 4.9: Illustration of the search regions for Example 1

4.3.7 Example 2.

Lets consider a three-objective MILP with a feasible criterion space as depicted in Figure 4.10. In this problem, $\mathcal{Y} = \{\mathbf{y}_1, \mathbf{y}_3, \mathbf{y}_3, \mathbf{y}_4\}$,

- $\mathcal{Z}(\mathbf{y}_1) = \text{conv}(\mathcal{Z}_1)$, $\mathcal{Z}_1 = \{z_{1,k}, k = 1, \dots, 5\}$, $z_{1,1} = (1, 1, 4)$, $z_{1,2} = (1, 5, 4)$, $z_{1,3} = (2, 4, 4)$, $z_{1,4} = (3, 2, 4)$, $z_{1,5} = (3, 1, 4)$,
- $\mathcal{Z}(\mathbf{y}_2) = \mathcal{Z}_2 = \{z_{2,1}\}$, $z_{2,1} = (1.5, 3, 3)$,
- $\mathcal{Z}(\mathbf{y}_3) = \text{conv}(\mathcal{Z}_3)$, $\mathcal{Z}_3 = \{z_{3,k}, k = 1, \dots, 4\}$, $z_{3,1} = (2, 0, 2)$, $z_{3,2} = (2, 3, 2)$, $z_{3,3} = (5, 2, 2)$, $z_{3,4} = (5, 0, 2)$,
- $\mathcal{Z}(\mathbf{y}_4) = \text{conv}(\mathcal{Z}_4)$, $\mathcal{Z}_4 = \{z_{4,k}, k = 1, \dots, 3\}$, $z_{4,1} = (0, 3, 0)$, $z_{4,2} = (1.5, 1.5, 0)$, $z_{4,3} = (0, 0, 0)$.

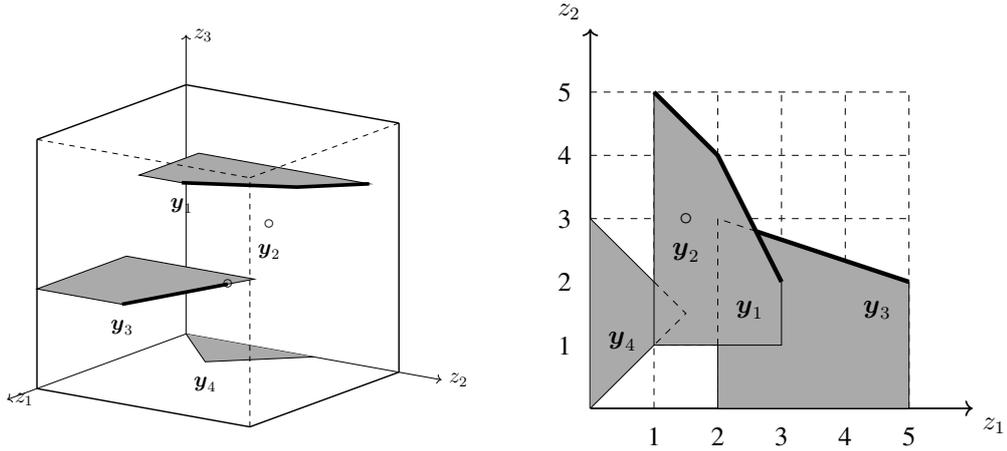


Figure 4.10: The feasible criterion space of the problem in Example 2

Iteration 1: The algorithm finds $\mathbf{z}^1 = z_{1,1} = (3, 2, 4)$ and $\mathbf{y}^1 = \mathbf{y}_1$. We solve $\mathcal{P}(\mathbf{y}^1)$ and obtain the set $\mathcal{Z}_{ND}(\mathbf{y}^1) = \{[z_{1,2}, z_{1,3}], [z_{1,3}, z_{1,4}]\}$. We update \mathcal{Y}_{WE} and \mathcal{Z}_{WND} as $\mathcal{Y}_{WE} = \{\mathbf{y}^1\}$, $\mathcal{Z}_{WND} = \mathcal{Z}_{ND}(\mathbf{y}^1)$. The search region is updated as in Figure 4.11a. Since $\mathcal{Z}_{ND}(\mathbf{y}^1)$ dominates $\mathcal{Z}(\mathbf{y}_2)$ and $\mathcal{Z}(\mathbf{y}_4)$, they are eliminated from the updated search region. In addition, \mathbf{y}^1 is eliminated from \mathcal{R}^1 and \mathcal{R}^2 with the addition of the corresponding tabu-constraint in the problems \mathcal{P}^1 and \mathcal{P}^2 .

Iteration 2: We solve \mathcal{P}^1 , \mathcal{P}^2 , and \mathcal{P}^3 . \mathcal{P}^1 is infeasible, and therefore we can

eliminate \mathcal{R}^1 from the search region, $\mathcal{R} = \mathcal{R} \setminus \mathcal{R}^1$. Both \mathcal{P}^2 and \mathcal{P}^3 yields the same integer vector, \mathbf{y}_3 , but find different points of $\mathcal{Z}_{ND}(\mathbf{y}_3)$. Since there is a tie in terms of their z_3 values, the algorithm picks the point with a higher value in z_1 , which is $z_{3,3} = (5, 2, 2)$ in \mathcal{R}^3 . Then, $\mathbf{z}^2 = z_{3,3}$, $\mathbf{y}^2 = \mathbf{y}_3$, and we solve $\mathcal{P}(\mathbf{y}^2, \mathcal{R}^3)$. We find $\mathcal{Z}_{ND}(\mathbf{y}^2, \mathcal{R}^3)$ and update the search region \mathcal{R}^2 since a part of it is dominated by $\mathcal{Z}_{ND}(\mathbf{y}^2, \mathcal{R}^3)$. Since $z_3(\mathbf{y}^2) < z_3(\mathbf{y}^1)$, $\mathcal{Z}_{ND} = \mathcal{Z}_{ND}(\mathbf{y}^1)$, $\mathcal{Y}_E = \{\mathbf{y}^1\}$, $\mathcal{Z}_{WND} = \mathcal{Z}_{ND}(\mathbf{y}^2, \mathcal{R}^3)$, and $\mathcal{Y}_{WE} = \{\mathbf{y}^2\}$. \mathcal{R}^3 is partitioned to its child search regions \mathcal{R}^4 and \mathcal{R}^5 . \mathbf{y}^2 is eliminated from \mathcal{R}^4 and \mathcal{R}^5 with the addition of the corresponding tabu-constraint in the problems \mathcal{P}^4 and \mathcal{P}^5 . The new search region is illustrated in Figure 4.11b.

Iteration 3: We solve \mathcal{P}^4 and \mathcal{P}^5 . Both problems are infeasible and therefore we can eliminate \mathcal{R}^4 and \mathcal{R}^5 from the search region, $\mathcal{R} = \mathcal{R} \setminus \{\mathcal{R}^4, \mathcal{R}^5\}$. \mathcal{R}^2 is the only feasible search region, and \mathbf{y}_3 is a feasible integer vector in \mathcal{R}^2 . Then, $\mathbf{z}^3 = (3.65, 2.45, 2)$, $\mathbf{y}^3 = \mathbf{y}_3$, and we solve $\mathcal{P}(\mathbf{y}^3, \mathcal{R}^2)$ to find $\mathcal{Z}_{ND}(\mathbf{y}^3, \mathcal{R}^2)$ so that we are able to obtain $\mathcal{Z}_{ND}(\mathbf{y}_3) = \mathcal{Z}_{ND}(\mathbf{y}_3, \mathcal{R}^2) \cup \mathcal{Z}_{ND}(\mathbf{y}_3, \mathcal{R}^3)$. Since point $(2.6, 2.8, 2)$ is in \mathcal{B}^2 and hence dominated by edge $[(2, 4, 4), (3, 2, 4)]$, we do not add it to the weakly nondominated set, $\mathcal{Z}_{WND} = \mathcal{Z}_{WND} \cup \mathcal{Z}_{ND}(\mathbf{y}^3, \mathcal{R}^2) \setminus \{(2.6, 2.8, 2)\}$. \mathcal{R}^2 is partitioned to its child search regions \mathcal{R}^6 and \mathcal{R}^7 . \mathbf{y}^3 is eliminated from \mathcal{R}^6 and \mathcal{R}^7 with the addition of the corresponding tabu-constraint in the problems \mathcal{P}^6 and \mathcal{P}^7 . The new search region is illustrated in Figure 4.11c.

Iteration 4: In the final iteration, both \mathcal{P}^6 and \mathcal{P}^7 is infeasible, and $\mathcal{R} = \emptyset$. We update $\mathcal{Z}_{ND} = \mathcal{Z}_{ND} \cup \mathcal{Z}_{WND}$ and $\mathcal{Y}_E = \mathcal{Y}_E \cup \mathcal{Y}_{WE}$. Since $\mathcal{R} = \emptyset$, the algorithm terminates with $\mathcal{Y}_E = \{\mathbf{y}_1, \mathbf{y}_3\}$, and

$$\mathcal{Z}_{ND} = \{[(1, 5, 4), (2, 4, 4)], [(2, 4, 4), (3, 2, 4)], ((2.6, 2.8, 2), (5, 2, 2))\}.$$

4.4 Implementation

In this section, we provide some details regarding the implementation of the algorithm and develop it further to improve its computational performance.

We implement the algorithm in Python 3.5.2, and use Gurobi 8.0.1 to solve the search

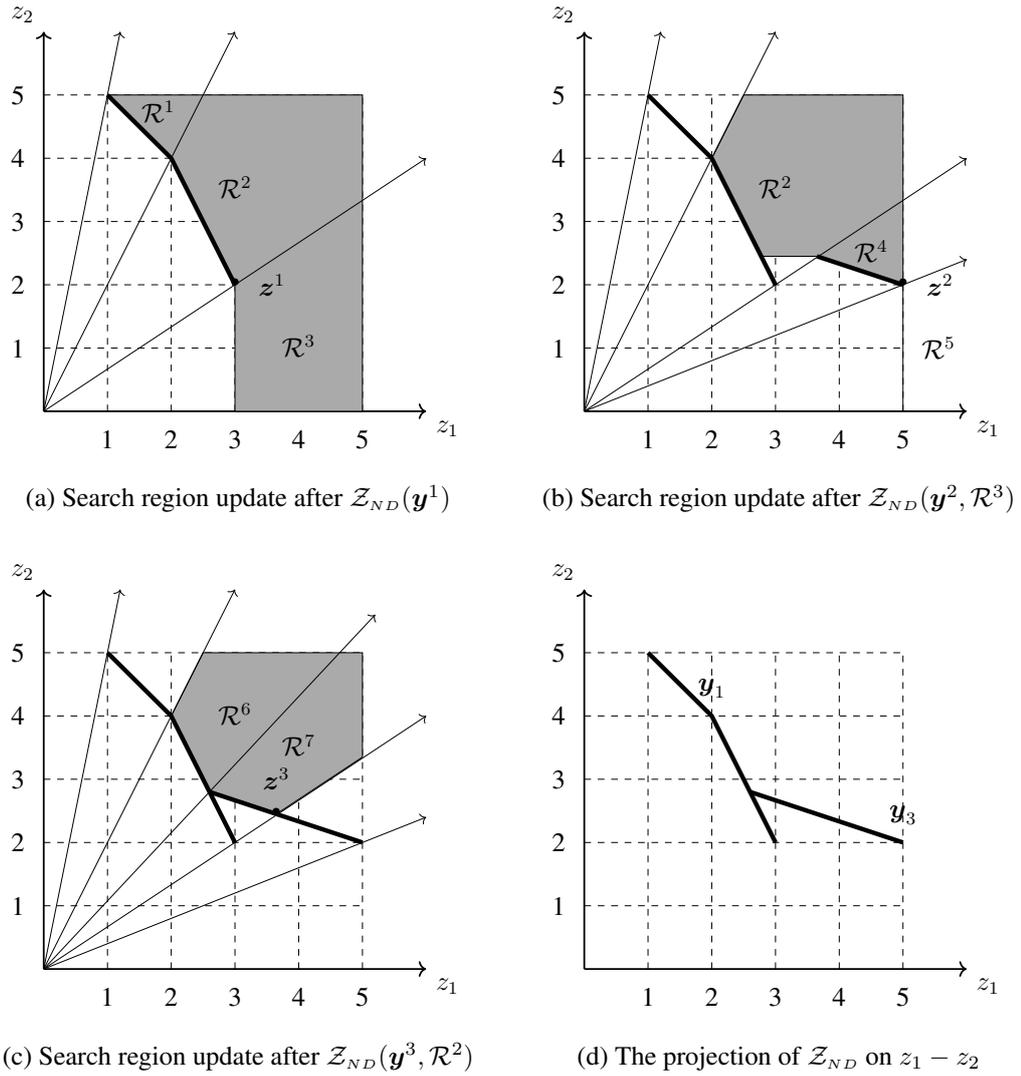


Figure 4.11: The evolution of the search space for the problem in Example 2

problems and conduct model-based dominance tests. To reduce the memory requirements of the algorithm, we maintain a single search and slice problem and keep updating it with the associated constraints of the search region before solving the problem, rather than creating a Gurobi model object for each search or slice problem. Gurobi 8.0.1 allows specifying multiple objectives, and setting priorities, weights and tolerances on each objective. In our search problems, we implement lexicographic maximization by assigning appropriate priorities to the three objectives. On the other hand, while solving the two-objective slice problem by the dichotomic search, we assign the same priority to both objectives and modify their weights to accommodate

for the objective to be maximized to generate the next extreme supported point.

While testing for the equality of two values, we use a relative error term of $\gamma = 10^{-6}$. That is, we assume two values x and y are equal if $|x - y| \leq \gamma \max\{|x|, |y|\}$. This is in compliance with our choice of feasibility tolerance of 10^{-6} in Gurobi models. We also use the same error value as the MIP relative gap parameter, and the stopping criterion for the dichotomic search while solving the slice problem.

We conduct additional iterations of dichotomic search in the two-dimensional feasible criterion space characterized by two nondominated points \mathbf{z}^1 and \mathbf{z}^2 ($z_1^1 > z_1^2$) provided that $\mathbf{w}^T \mathbf{z}^* > (1+\beta)\mathbf{w}^T \mathbf{z}^1$, \mathbf{z}^* is the currently generated point, $\mathbf{w}^T \mathbf{z}^1 = \mathbf{w}^T \mathbf{z}^2$, $w_1 = z_2^2 - z_2^1$, $w_2 = z_1^1 - z_1^2$, and $\beta > 0$. The magnitude of β directly impacts the quality of the representation of the true nondominated frontier. The smaller values of β improves the approximations of the nondominated set at the expense of increased computational effort, due to additional steps in the dichotomic search and the increased number of search regions leading to more MILP problems to be solved.

Our algorithm can also handle bi-objective problems with a simple adjustment. In this case, we create an additional artificial objective function consisting of a single artificial variable to transform the problem into a three-objective problem. We set the lower and upper bounds of the artificial variable to zero. Assigning the highest priority to the artificial objective in the lexicographic maximization fixes $z_3(\mathbf{x}) = 0$ for any feasible solution \mathbf{x} of the problem and the algorithm works with the remaining two objectives. We next develop procedures to improve the computational performance of the algorithm.

4.4.1 Solving super search problems

After solving the slice problem in the selected search region, we partition the search region into child search regions to exclude the regions dominated by the nondominated set of the slice problem. In case the search region does not include further candidate points, all the child search problems will be infeasible. But, to prove this, we need to solve as many search problems as the number of child search regions. To improve on this requirement, we consider a *super search problem* with the hope to

detect infeasibility without having to search each child region. We solve the super search problem whenever the frontier consists of multiple edges.

Let $\mathcal{Z}_{ND}(\mathbf{y}^{j^*}, \mathcal{R}^{j^*}) = \{\mathcal{E}^1, \mathcal{E}^2, \dots, \mathcal{E}^n\}$ be the set of edges in the nondominated set of the slice problem, $\mathcal{S} = \bigcup_{k=1,2,\dots,n} \mathcal{R}^k$ be the union of the respective child search regions defined by these edges, $\mathcal{R}^k \subseteq \mathcal{R}^{j^*}, \forall k = 1, 2, \dots, n$, and $\mathbf{z}^{nw}, \mathbf{z}^{se}$ be the north-west and south-east extreme points of the frontier, respectively. We create edge $\tilde{\mathcal{E}} = [\mathbf{z}^{nw}, \mathbf{z}^{se}]$ and cone $\tilde{\mathcal{C}}$ with its extreme rays passing through \mathbf{z}^{nw} and \mathbf{z}^{se} . Then, we define a search region $\tilde{\mathcal{R}} = (\tilde{\mathcal{C}}, \tilde{\mathcal{E}}, _)$ and the associated search problem, $\tilde{\mathcal{P}}$, adding to it the tabu-constraint that eliminates \mathbf{y}^{j^*} . We illustrate the super search region associated with the super search problem in Figure 4.12. We show that \mathcal{P}^k is infeasible for $k = 1, 2, \dots, n$ and we can remove $\mathcal{R}^k, k = 1, 2, \dots, n$ from the search space if $\tilde{\mathcal{P}}$ is infeasible by the following proposition:

Proposition 19. *If $\tilde{\mathcal{P}}$ is infeasible, then \mathcal{P}^k is also infeasible for $k = 1, 2, \dots, n$.*

Proof. Since $\mathcal{Z}_{ND}(\mathbf{y}^{j^*}, \mathcal{R}^{j^*})$ is an MOLP, the set of edges, $\{\mathcal{E}^1, \mathcal{E}^2, \dots, \mathcal{E}^n\}$, is connected and supported. Therefore, $\mathcal{S} \subseteq \tilde{\mathcal{R}} \subseteq (\tilde{\mathcal{E}})^\geq$ and if $\tilde{\mathcal{P}}$ is infeasible, then \mathcal{P}^k is also infeasible for $k = 1, 2, \dots, n$. \square

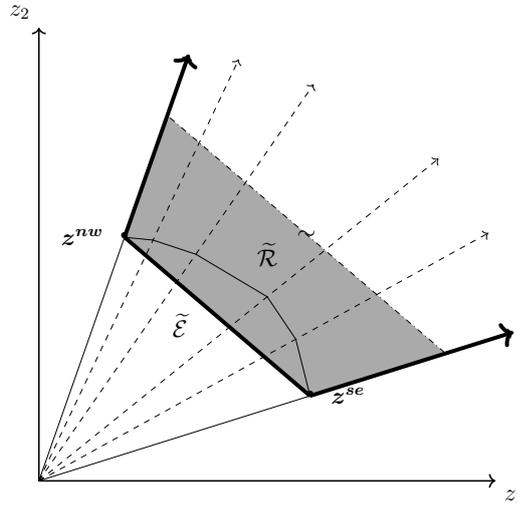


Figure 4.12: Illustration of super search region definition

If \mathcal{S} does not include any feasible solution, we are able to prove infeasibility by only solving the super search problem instead of five child search problems. We should

also note that solving the super search problem does not necessarily require solving an additional MILP problem. In case $\widetilde{\mathcal{P}}$ is feasible, the solution may still be useful if the generated point is in one of the child search regions. Then, we can use that solution as if we had solved that child search problem.

Alternatively, one can define a smaller super search region by considering only a subset of the edges in the frontier that are adjacent to each other, or a larger super search region by also including the left-most or right-most child search regions in case the extreme points of the frontier are not at the boundaries of the search region \mathcal{R}^{j^*} . The trade-off is that the larger the super search region is, the lower the chances are of finding the super search problem infeasible (even though some or all of the child problems could be infeasible), but the higher the number of child search regions that can be dropped from the search space when the super search problem is infeasible.

Another alternative is to define a search problem such that $\widetilde{\mathcal{R}} = \mathcal{S}$. Although the super search region is identical to the union of the child search regions, the search region becomes non-convex, in this case. To handle this, we can use disjunction constraints by integrating a set of additional binary variables. We did not follow this approach in order not to introduce additional binary variables into the search problem.

4.4.2 Search region coupling

During the execution of the algorithm, the child search regions are created, and the lower bound vectors are updated. The lower bound update procedure poses an opportunity to couple multiple adjacent search regions into a single, and larger search region, and consolidate multiple searches into a single one. We couple all adjacent search regions that form a convex search region when coupled.

In Figure 4.13a, there are several search regions on both sides of the selected region \mathcal{R}^{j^*} . All but one of those search regions share lower bound produced by the lower bound update procedure in their respective cones. The three shaded search regions to the left of \mathcal{R}^{j^*} can be coupled into a single search region, $\widehat{\mathcal{R}}_1$. Similarly, the five shaded search regions to the right of \mathcal{R}^{j^*} can be coupled into the larger search region, $\widehat{\mathcal{R}}_2$.

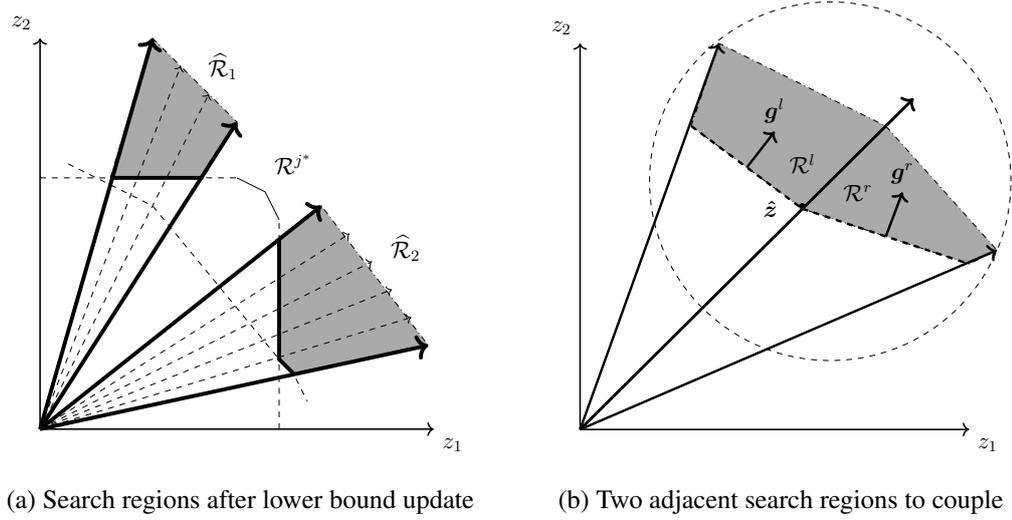


Figure 4.13: Illustration of search region coupling

In Figure 4.13b, we illustrate the necessary and sufficient conditions for region coupling. Two adjacent search regions, \mathcal{R}^l and \mathcal{R}^r , can be coupled only if they contain a common point \hat{z} on their boundaries with the dominated region, \mathcal{Z}_D . That is, there exist \hat{z} such that $\hat{z} = \mathcal{R}^l \cap \mathcal{R}^r \cap \mathcal{Z}_D$. Another necessary condition is that the union of the search regions must form a convex set. Let \mathbf{g}^l and \mathbf{g}^r be the normal vectors to the boundary lines of \mathcal{R}^l and \mathcal{R}^r respectively, and $\tilde{\mathcal{R}} = \mathcal{R}^l \cup \mathcal{R}^r$. $\tilde{\mathcal{R}}$ is convex if $m(\mathbf{g}^l) \leq m(\mathbf{g}^r)$, where $m(\cdot)$ is the slope of the normal vector. In other words, $\tilde{\mathcal{R}}$ is convex if and only if the boundaries of \mathcal{R}^l or \mathcal{R}^r form an obtuse angle at \hat{z} .

While coupling two search regions, we also consider the candidate solutions to the corresponding search problems if any. If both search problems have candidate solutions, the better one is set as the candidate solution of the coupled search problem. Otherwise, the coupled search problem needs to be resolved to determine the next candidate point.

4.4.3 Further search region elimination: bi-objective problem

When solving a bi-objective problem, we know that $z_3^n = z_3^{n-1}$ and $z_1^n \leq z_1^{n-1}$ for $n > 1$. Hence, if \mathbf{z}^{j^*} is the candidate point of search region \mathcal{R}^{j^*} , $\nexists \mathbf{z} : z_1 > z_1^{j^*}, z_{1,2} \in \mathcal{R}^k, k \in J_{j^*}^r$. By Proposition 16, this implies that there cannot be any nondominated

point in the search regions to the right (where $z_1 > z_1^{j*}$) of the selected search region \mathcal{R}^{j*} , and those search regions can be eliminated from the search space.

4.5 Computational study

We perform the computational experiments to compare our cone-based search algorithm (CBSA) with the available algorithms on three different problem sets:

- **Bi-objective mixed-binary programming problems (BOMBLPs):** We use the instances created by Boland et al. (2015) based on the instance generation scheme of Mavrotas and Diakoulaki (1998). Some other recent studies on bi-objective mixed-integer programs such as Soylu (2018) and Fattahi and Turkey (2018) employ this instance set as well. We denote the subclasses as Cm indicating that there are m constraints in the subclass. We use $m = 20, 40, 80, 160,$ and 320 and there are five instances in each subclass. In each instance, there are $m/2$ binary and $m/2$ continuous variables. The constraint and objective function coefficients are generated from uniform distributions with different ranges for the variable type in the constraints and the objective ranges.
- **Three-objective mixed-integer linear programming problems (TOMILPs):** Extending the instance generation scheme of Mavrotas and Diakoulaki (1998), we create an additional objective function that has only binary variables. We denote these problem subclasses as $O3-Cm-Ik$. $O3$ denotes that there are 3 objectives and Ik denotes that only k of the objectives are made up of binary variables only. We experiment with $k = 1$ and 2 . We also experiment with a case where we replace the binary variables with general integer variables. We denote these subclasses as $O3-Cm-I1-Int$. For each subclass we generate five instances and use $m = 20, 40,$ and 80 .
- **Three-objective knapsack problems (TOKPs):** We use the 0/1 single dimensional knapsack problem instances from Kirlik and Sayin (2014). We test with 10 randomly generated instances from problems with 10, 20, 30, 40, 50, and 100 items.

We implemented CBSA in Python 3.5.2, and used Gurobi 8.0.1 as the mathematical programming solver. We run CBSA on MacBook Pro 2.6 GHz Dual-Core Intel Core i5 processor and 8 GB 1600 MHz DDR3 memory. We use the tolerances and improvements of CBSA as discussed in Section 4.4 throughout our experiments. We report the run time statistics in seconds. In the model statistics, a model refers to the lexicographic optimization problem where an MILP is solved as many as the number of objective functions.

We present results on the performance of CBSA on TOMILPs, where one of the objectives has a discrete feasible set in Table 4.1. The first three columns of the table present the instance, the number of models solved, and the number of iterations conducted. Then, the next three columns show the number of nondominated edges, disconnected nondominated points and efficient integer vectors, respectively, generated for the problem instance. Next, we report the total run time as well as its distribution among the search problems, slice problems, and the model-based dominance tests.

Table 4.1 shows that the nondominated sets of the instances mostly consist of edges. The number of models to solve is affected by the number of edges as the search region including a subset of the nondominated edges is partitioned into child search regions as many as the number of edges. There are 4.59, 8.42 and 15.78 edges on average per efficient integer vector, and CBSA solves 6.44, 10.75 and 23.89 models per efficient integer vector for $m = 20, 40$ and 80 , respectively. The number of models per nondominated edge is similar in all problems and average to 1.51. Most of the computational effort is spent in solving the MILPs corresponding to the search problems. The linear programs solved in the slice problems and the model-based dominance tests consume a limited amount of the overall computational effort.

For our next test, we remove the binary constraints on the integer variables and repeat the experiments in order to see the effect of binary restrictions on the nondominated set and the computational performance of the algorithm. We report the results in Table 4.2 for $m = 20$ and 40 . When the instances include general integer variables, the number of nondominated edges and the efficient integer vectors increase on average, and CBSA needs to solve more models to generate those compared to the binary case.

To investigate the performance of CBSA on discrete nondominated sets, we also con-

Table 4.1: Results for CBSA on TOMILPs for 3O-C m -II

m	Instance	Iterations	Models	Nondominated Set			Run Time (secs)			
				Edges	Points	$ \mathcal{V}_E $	Total	Search	Slice	Dominance
20	1	11	52	44	1	10	1.34	0.90	0.23	0.00
	2	69	207	148	1	22	6.48	5.05	0.88	0.07
	3	34	143	115	0	32	4.81	4.04	0.47	0.00
	4	48	172	123	0	23	5.70	4.74	0.59	0.02
	5	36	154	88	1	26	6.01	5.31	0.39	0.01
	Average	39.6	145.6	103.6	0.6	22.6	4.87	4.01	0.51	0.02
40	1	274	1206	860	4	131	180.76	173.33	4.39	0.34
	2	121	550	408	0	56	53.87	50.52	2.00	0.18
	3	203	767	617	3	80	71.71	66.77	3.01	0.30
	4	262	1424	1251	2	99	210.14	199.35	5.50	0.89
	5	235	923	678	7	87	115.83	109.28	3.77	0.40
	Average	219	974	762.8	3.2	90.6	126.46	119.85	3.73	0.42
80	1	6561	21962	14129	74	1221	14246.79	13198.66	127.95	587.54
	2	10220	35141	23859	51	1271	35686.72	31615.48	220.92	3061.26
	3	20138	70210	46586	204	2493	67197.79	54246.37	435.92	10227.82
	4	13378	42379	27542	68	1410	32105.22	27111.17	267.5	3830.67
	5	6330	25093	16561	126	1760	13922.61	13042.72	139.53	341.07
	Average	11325.4	38957	25735.4	104.6	1631	32631.83	27842.88	238.36	3609.67

ducted tests for the case of two objective functions having discrete feasible value sets. As discussed in Section 4.1, the nondominated sets of the problem instances generated under this setting are discrete. We present the results in Table 4.3. There are no nondominated edges as expected, and hence CBSA does not have to solve slice problems. In addition, there are no LPs solved in dominance tests as a weakly nondominated but dominated point can be detected by simple comparisons if there exist any such point. The number of models solved per efficient integer vector is 1.90, 2.01 and 2.03 on average for $m = 20, 40,$ and 80 respectively. For $m = 80$, there are more efficient integer vectors compared to the continuous case although it is similar when $m = 20$ and 40 . This might be because of the increased density of the nondominated set as the problem size increases, and the smaller hypervolumes dominated by the nondominated points compared to nondominated edges.

Even though we designed CBSA is order to solve a special class of TOMILPs, we

Table 4.2: Results for CBSA on TOMILPs for 3O- Cm -I1-Int

m	Instance	Iterations	Models	Nondominated Set			Run Time (secs)			
				Edges	Points	$ \mathcal{Y}_E ^b$	Total	Search	Slice	Dominance
20	1	32	158	111	0	25	16.99	16.17	0.38	0.01
	2	76	249	164	1	29	47.43	45.89	0.74	0.06
	3	48	212	146	0	47	32.96	31.51	0.51	0.00
	4	87	306	215	0	41	51.26	49.89	0.84	0.02
	5	101	338	186	3	49	99.48	97.6	0.86	0.06
	Average	68.8	252.6	164.4	0.8	38.2	49.62	48.21	0.67	0.03
40	1	1428	7371	4213	19	533	1500.64	1449.59	17.83	0.01
	2	78	405	296	0	61	17.53	15.75	1.26	0.00
	3	20	94	87	1	20	4.08	3.42	0.41	0.00
	4	312	2684	1979	6	223	285.08	273.61	7.20	0.01
	5	41	294	243	2	39	16.81	15.53	0.94	0.00
	Average	375.8	2169.6	1363.6	5.6	175.2	364.83	351.58	5.53	0.01

^[b] We disabled the search for alternative efficient integer vectors since the extra variables and constraints that have to be added to model the tabu-constraints increase the complexity of the models solved substantially.

showed that it can be used to solve bi-objective problems. To test its performance on bi-objective problems, we conduct experiments to compare CBSA to the performances of the solvers that have been developed for these problems. We use the BOMBLP instances from Boland et al. (2015), and compare our results to those of their triangle splitting algorithm (TSA), as well as the results of search-and-remove (SR) algorithm in Soylu (2018) and one-directional search (ODS) algorithm in Fatahi and Turkay (2018) on these instances. We report the average number of non-dominated edges generated by each algorithm (except for SR as the authors did not report this information) as well as the average number of models solved to generate the nondominated set. We do not include the run time statistics in the comparison as the computational environments and the mathematical programming solvers used are different. The number of models solved is representative of the computational effort since the complexity of the models solved by each algorithm is similar and almost all the computational effort is spent in solving the models.

In Soylu (2018), the performance of SR algorithm is compared with the TSA's, hence we assume that SR algorithm generates nondominated edges as many as TSA. TSA

Table 4.3: Results for CBSA on TOMILPs for 3O-C m -I2

m	Instance	Iterations	Models	Nondominated Set		Run Time (secs)	
				Points	$ \mathcal{Y}_E $	Total	Search
20	1	10	18	10	10	0.42	0.25
	2	23	43	23	23	1.11	0.90
	3	32	61	32	32	1.37	1.21
	4	24	47	24	24	1.28	1.06
	5	26	50	26	26	1.54	1.36
	Average	23	43.8	23	23	1.14	0.96
40	1	148	291	144	144	36.25	35.72
	2	62	124	62	62	9.02	8.76
	3	91	179	90	90	12.67	12.26
	4	110	212	104	104	21.77	21.2
	5	90	178	89	89	15.67	15.36
	Average	100.2	196.8	97.8	97.8	19.08	18.66
80 ^a	1	1000	2020	995	995	3552.6	3543.28
	2	1000	2008	990	990	2839.04	2830.15
	3	1000	2012	988	988	2962.35	2953.83
	4	1000	2013	997	997	2285.35	2276.48
	5	1000	2016	986	986	2195.25	2186.24
	Average	1000	2013.8	991.2	991.2	2766.92	2758.00

^[a] We put an upper limit of 1000 iterations while running CBSA for $m = 80$.

outperforms SR in the largest problems although SR performs better in the small to mid-size problems. In Fattahi and Turkay (2018), the authors test sensitivity of ODS with respect to ξ and μ parameters by which they control the approximation quality of the generated set with respect to the true nondominated set. We report the ODS results corresponding to $\xi = 10^{-5}$ and $\mu = 10^{-4}$ as they are the suggested values by the authors to achieve a good balance between the approximation quality and the computational effort. ODS solves less models per edge compared to TSA at different levels of approximation quality.

In CBSA, we control the approximation quality with β parameter in the dichotomic search that is used to terminate the search in the regions characterized by the adjacent extreme supported points. We experimented with different values of this parameter and selected $\beta = 3 \times 10^{-6}$ that leads to nondominated sets having slightly better approximation quality than those generated by ODS.

In Table 4.4, we report the number of edges generated and the number of models solved by each algorithm as well as the number of models solved per nondominated edge (M/E) generated on average. CBSA is able to generate more nondominated edges by solving less number of models than ODS as well as the other algorithms except problem $m = 320$. For $m = 320$, CBSA generates a better quality nondominated set but at the expense of solving more models than ODS. CBSA solves 1.72 models per edge on average compared to 1.60 models solved by ODS for $m = 320$.

Table 4.4: Results for CBSA on BOMILP instances in comparison to the state-of-the-art algorithms

m	CBSA ^a			ODS ^b			TSA ^c			SR ^d	
	Edges	Models	M/E	Edges	Models	M/E	Edges	Models	M/E	Models	M/E
20	37.8	31.6	0.84	34.8	58.2	1.67	44.2	133.2	3.01	104.2	2.36
40	164.8	116.2	0.71	149.4	195	1.31	183.2	428.6	2.34	298.4	1.63
80	679.2	525.4	0.77	650.2	804.6	1.24	744.2	1609.6	2.16	968.6	1.30
160	1402.4	1746.8	1.25	1301.6	1782.4	1.37	1397.4	2839.6	2.03	3071.2	2.20
320	2670.8	4589.2	1.72	2374	3803.2	1.60	3055.8	5388.2	1.76	10152.2	3.32

^[a] We set $\beta = 3 \times 10^{-6}$ for CBSA runs. It took 0.77, 4.79, 47.99, 380.09 and 5850.23 seconds on average for CBSA to solve instances with $m = 20, 40, 80, 160$ and 320 , respectively. In addition, CBSA found 10, 19, 49.6, 122.6 and 412 efficient integer vectors on average for $m = 20, 40, 80, 160$ and 320 , and 0.6 and 3.6 isolated nondominated points on average for $m = 160$ and 320 , respectively.

^[b] $\xi = 10^{-5}$ and $\mu = 10^{-4}$ for ODS runs.

^[c] The number of edges after post-processing step are 33.2, 119.6, 285.4, 351.6, 449.8 for $m = 20, 40, 80, 160$ and 320 , respectively. TSA does not generate a minimal representation of the nondominated frontier, and the authors develop a post-processing algorithm to convert the representation generated by the algorithm into a minimal representation.

^[d] We assume that SR algorithm generates the same number of edges as generated by TSA since the authors do not generate the corresponding edge statistics and they compare their algorithm with TSA.

We compare CBSA and ODS in terms of their computational efficiency to approximate the nondominated sets of $m = 320$ instances at different levels of representation quality. Fattahi and Turkay (2018) present computational results with $\xi \in$

$\{10^{-3}, 3 \times 10^{-4}, 10^{-4}, 3 \times 10^{-5}, 10^{-5}, 3 \times 10^{-6}, 10^{-6}\}$, where lower values of ξ implies better representation quality. Similarly, we run CBSA for $m = 320$ with $\beta \in \{10^{-3}, 3 \times 10^{-5}, 10^{-5}, 10^{-6}, 10^{-7}\}$. In Figure 4.14, we plot the average number of nondominated edges generated and the average number of models solved by both algorithms for the instances with $m = 320$ at different ξ and β values of ODS and CBSA, respectively. CBSA is nondominated relative to ODS at all β values experimented with. However, ODS points with $\xi = 10^{-3}$ and 3×10^{-4} are dominated by CBSA with $\beta = 10^{-3}$.

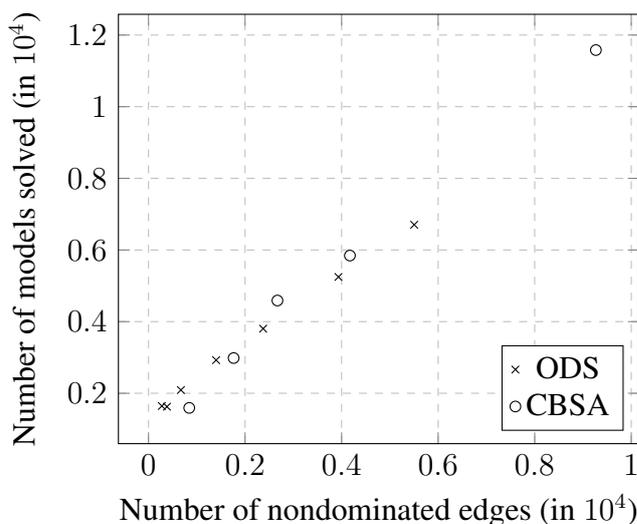


Figure 4.14: Comparison of CBSA and ODS under different parameter settings of the algorithms (Points dominate their north-west region.)

In Table 4.5, we report the performance of CBSA on TOKP instances in comparison to KS algorithm (Kirlik and Sayın, 2014). KS algorithm is one of the best performing algorithms to solve multi-objective integer programs (MOIPs) (Dächert and Klamroth, 2015). It is the best algorithm in terms of the number of models solved among the studies that report empirical test results. Since most of the computational effort is spent in solving the single objective problems in combinatorial optimization problems, we compare KS and CBSA in terms of the number of single objective models solved. We report the average number of nondominated points in the instances and the average number of models solved by each algorithm. Furthermore, we present the average run time in seconds spent by CBSA and the ratio of time spent in solving the models with respect to the total algorithm execution time.

In terms of the number of models, both algorithms perform similarly (CBSA solved slightly more models on average). Dächert and Klamroth (2015) shows that $2|Z_N| - 1$ is an upper bound on the number of models to solve in three-objective integer programs by ε -constraint scalarization, where $|Z_N|$ is the number of nondominated points. The number of models solved by CBSA is within this upper bound for all problem sizes except the 30 item instances where there is a slight difference. CBSA solves 1.97 models on average to generate the nondominated set. Except the small-sized problem instances with 10 items, most of the computation time of CBSA is spent in solving these models. The run time per model is independent of the algorithm as we observed that the complexity of the model does not change over the iterations in CBSA. On the other hand, Dächert and Klamroth (2015) reports that although KS solves the least number of models among the state-of-the-art algorithms, it performs poorly in terms of CPU time as it scans every search region twice in each iteration. They show that their algorithm, Algorithm-2 (EC), and OBS (Ozlen et al., 2014) outperforms KS in terms of CPU times. In this respect, CBSA is very competitive since its management of the search regions is very efficient and scales very well with the size of the nondominated set.

Table 4.5: Results for CBSA on TOKP instances in comparison to KS

Items	Nd points	KS	CBSA ^a		
		Models	Models	Run Time (seconds)	Model Run Time (%)
10	9.80	18.60	18.40	0.53	61%
20	38.00	74.80	75.30	3.56	93%
30	115.80	230.00	232.00	23.26	98%
40	311.20	617.20	621.10	84.43	98%
50	444.20	878.90	886.90	131.37	98%
100	5849.10	11536.20	11568.10	4728.17	97%

^[a] We disabled the search for alternative efficient integer vectors in CBSA since KS does not aim to find them.

CBSA differs from the current state-of-the-art algorithms developed to solve MOIPs in the way that it partitions the feasible criterion space. CBSA uses cones to define search regions instead of rectangles or regional lower bounds used in the previous algorithms. Furthermore, CBSA is able to identify all efficient solutions as it uses tabu-constraints to eliminate the generated solutions instead of shifting the lower

bounds of the search regions in the criterion space by an error term to eliminate the dominated space. KS and the other competitive algorithms for MOIPs do not have this capability. In Table 4.6, we report the number of efficient solutions for instances with 10 items that are slightly modified to create alternative efficient solutions. In those instances, there are two efficient solutions on average per nondominated point. The number of models solved per nondominated point, 3.58 on average, increases as generating alternative efficient solutions requires additional models to solve.

Table 4.6: Results for CBSA on modified TOKP instances with 10 items

Instance	Nd points	Eff Solutions	Models	Run Time (secs)
1	11	27	48	0.87
2	16	26	47	1.21
3	7	17	36	0.81
4	7	7	11	0.28
5	14	32	60	2.15
6	8	20	36	0.85
7	10	20	35	0.85
8	5	7	13	0.32
9	4	4	6	0.22
10	8	16	30	0.82
Average	9	17.6	32.2	0.84

The source code of the algorithm and all the instances used in our computational study are available in Ceyhan (2020).

4.6 Approximating the nondominated frontier

CBSA is an exact algorithm in the sense that it generates the nondominated point set and all efficient integer vectors. However, generating the nondominated set can be computationally expensive when the nondominated set is large and the single objective problem is a difficult MILP. It is possible to use CBSA to generate an approximate nondominated set with less computational effort. We use a variant of *binary ε -indicator* measure defined in Zitzler et al. (2003) to guarantee a certain level of representation of the nondominated set with the generated approximation. For a maximization type problem, we restate the *binary ε -indicator* measure as follows:

Definition 21. A criterion vector $\mathbf{z}^1 \in \mathcal{Z}$ is said to ε -dominate another criterion vector $\mathbf{z}^2 \in \mathcal{Z}$, written as $\mathbf{z}^1 \succeq_\varepsilon \mathbf{z}^2$, if $(1 + \varepsilon)z_i^1 \geq z_i^2, \forall i = 1, \dots, m$ for a given $\varepsilon \geq 0$.

Based on the binary ε -dominance definition given above, we define ε -dominance between an approximation set and the nondominated set as follows:

Definition 22. An approximation set \mathcal{Z}_A ε -dominates the nondominated set \mathcal{Z}_{ND} , $\mathcal{Z}_A \succeq_\varepsilon \mathcal{Z}_{ND}$, if $\exists \hat{\mathbf{z}} \in \mathcal{Z}_A$ such that $\hat{\mathbf{z}} \succeq_\varepsilon \mathbf{z}$ for all $\mathbf{z} \in \mathcal{Z}_{ND}$.

The ε -indicator value for approximation set \mathcal{Z}_A , $I_\varepsilon(A)$, is the minimum $\varepsilon \in \mathbb{R}$ value such that \mathcal{Z}_A ε -dominates the nondominated set \mathcal{Z}_{ND} . For any approximation set \mathcal{Z}_A , $I_\varepsilon(A) \geq 0$.

Let $f_\varepsilon: \mathcal{Z} \rightarrow \mathbb{R}^m$, $f_\varepsilon(\mathbf{z}) = \varepsilon\mathbf{z}$. We define an ε -neighborhood, $\varepsilon > 1$, around $S \subseteq \mathcal{Z}$, $\mathcal{N}_\varepsilon(S)$, such that $\mathcal{N}_\varepsilon(S) = \{f_\varepsilon(\mathbf{z}) : \mathbf{z} \in S\} \setminus S$. By construction, $S \succeq_\varepsilon \mathcal{N}_\varepsilon(S)$. We generate an approximation set $\mathcal{Z}_{ND}^\varepsilon$ by restricting CBSA not to search $\mathcal{N}_\varepsilon(\mathcal{Z}_{ND}(\mathbf{y}^n, \mathcal{R}^{j^*}))$ at each iteration n , where $\mathcal{Z}_{ND}(\mathbf{y}^n, \mathcal{R}^{j^*})$ is the subset of the nondominated set of the slice problem associated with integer vector \mathbf{y}^n at iteration n . We construct these neighborhoods by applying f_ε on $\mathcal{Z}_{ND}(\mathbf{y}^n, \mathcal{R}^{j^*})$ while separating the dominated space by $\mathcal{Z}_{ND}(\mathbf{y}^n, \mathcal{R}^{j^*})$ from the search space. We next show that $\mathcal{Z}_{ND}^\varepsilon$ ε -dominates \mathcal{Z}_{ND} .

Proposition 20. Given $\varepsilon \geq 0$, $\mathcal{Z}_{ND}^\varepsilon \succeq_\varepsilon \mathcal{Z}_{ND}$.

Proof. Let S^n be the set of points generated by CBSA as candidate nondominated points at iteration n , $S^n \subseteq \mathcal{Z}$ (For $n = 1$, $S^n \subseteq \mathcal{Z}_{W_{ND}}$), and $\mathcal{N}_\varepsilon(S^n)$ be the neighborhood of S^n that is eliminated from the search space. Also, let $\mathcal{Z}_{ND}(\mathcal{N}_\varepsilon(S^n))$ be the subset of \mathcal{Z}_{ND} in this neighborhood that will not be searched by CBSA. $\mathcal{Z}_{ND}(\mathcal{N}_\varepsilon(S^n))$ is ε -dominated by S^n . Since $\mathcal{Z}_{ND}^\varepsilon \succeq_0 S^n$, $\mathcal{Z}_{ND}^\varepsilon \succeq_\varepsilon \mathcal{Z}_{ND}(\mathcal{N}_\varepsilon(S^n))$ for any iteration n . If we let $\tilde{\mathcal{Z}}_{ND}$ be the union of the nondominated points in all the restricted neighborhoods, then $\mathcal{Z}_{ND}^\varepsilon \succeq_\varepsilon \tilde{\mathcal{Z}}_{ND}$. The nondominated points not in the restricted neighborhoods will be generated by CBSA. That is, if $\mathcal{Z}_{ND} \setminus \tilde{\mathcal{Z}}_{ND}$ is non-empty, then $(\mathcal{Z}_{ND} \setminus \tilde{\mathcal{Z}}_{ND}) \subseteq \mathcal{Z}_{ND}^\varepsilon$. Therefore, $\mathcal{Z}_{ND}^\varepsilon \succeq_\varepsilon \mathcal{Z}_{ND}$. \square

We denote this version of CBSA as CBSA- ε . We test CBSA- ε with $\varepsilon = 10^{-5}, 10^{-4}, 10^{-3}, 10^{-2}$, and 10^{-1} on 30-C80-II instances, and report the results in Table 4.7. The last three columns represent the corresponding figures relative to the case when $\varepsilon = 10^{-5}$. For $\varepsilon = 10^{-3}$, CBSA- ε generates around 23×10^3 edges by solving 32×10^3 models in 5118.55 seconds on average. By assuming $\mathcal{Z}_{ND}^\varepsilon \approx \mathcal{Z}_{ND}$ for $\varepsilon = 10^{-5}$, any nondominated edge will be ε -dominated, $\varepsilon = 10^{-3}$, by a set of edges being less than 60% of the nondominated set in size and generated with up to 60% less computational effort.

Table 4.7: Results for ε CBSA on 30-C80-II instances

ε	Edges	Models	Run Time (secs)	Percentage (%)		
				Edges	Models	Run Time (secs)
10^{-5}	40025.0	64523.8	12230.88	-	-	-
10^{-4}	30065.2	42686.8	7211.41	75.12	66.16	58.96
10^{-3}	23077.4	32007.6	5118.55	57.66	49.61	41.85
10^{-2}	8159.2	10001.4	1329.98	20.39	15.50	10.87
10^{-1}	636.8	612.8	81.36	1.59	0.95	0.67

4.7 Conclusions

In this chapter, we present a new criterion space search algorithm to solve a class of three-objective mixed-integer linear programs, TOMILPs, where at least one of the objectives take discrete values. We develop a novel search space partitioning scheme that utilizes convex cones and the already generated edges to create polyhedral search regions in the two-dimensional projection of the feasible criterion space. Our cone-based search algorithm, CBSA, finds a weakly nondominated point at each iteration in the worst case, and generates the nondominated set and the set of all efficient integer vectors.

We test CBSA on randomly generated instances of TOMILPs as well as on three-objective 0/1 knapsack problems, TOKPs, and bi-objective mixed-binary linear programs, BOMBLPs. We present results for different special cases of TOMILPs such

as (i) when all integer variables are binary, (ii) when all integer variables are general integer variables, and (iii) when there are two objectives that take discrete values.

We compare the results of CBSA on BOMBLP instances to that of ODS (Fattahi and Turkay, 2018), TSA (Boland et al., 2015) and SR (Soylu, 2018). CBSA outperforms TSA and SR in terms of the number of MILPs solved to generate the nondominated set. CBSA also outperforms ODS at all experimented problem sizes except the largest one, where CBSA results for different levels of representation quality are nondominated relative to ODS and dominates ODS for high-level approximations.

On the TOKP instances we experimented, CBSA solves nearly the same number of models with KS (Kirlik and Sayin, 2014). The majority of the computation time of CBSA is spent for solving the search problems (above 97% for medium to large problem sizes). That is, CBSA does not suffer from increasing computational overhead due to search region management, whereas KS is reported to perform poor in terms of CPU time due to significant amounts of time spent in the management of the search regions for large problems (Dächert and Klamroth, 2015). CBSA can also find all efficient solutions to three-objective integer programs if it is configured to do so.

We also present an extension of CBSA, CBSA- ε , to generate ε -dominating approximations of the nondominated set with smaller computation effort. The generalization of our search space partitioning scheme to more than three objectives awaits further research.

CHAPTER 5

MULTI-OBJECTIVE DAY-AHEAD ENERGY MARKET CLEARING PROBLEM

In this chapter, we study the DAM clearing problem under multiple objectives. We first examine the surplus-maximizing solutions of the problem in terms of some auxiliary measures that we use to assess the degree of market disequilibrium, such as market loss, missed surplus, number of PABs, number of PRBs, and the differences in the implicit prices perceived by the market participants. We then elaborate on the common market designs in European DAMs that eliminate some of the elements of market disequilibrium, and show their inadequacy regarding to resolving market disequilibrium.

We develop a multi-objective formulation of the market clearing problem. We use market surplus, market loss, and missed surplus as three main objectives of MOs, where the market surplus is to be maximized and the market loss and missed surplus are to be minimized. We employ the cone-based search algorithm, CBSA, which we present in Chapter 4 to find the nondominated set and the set of efficient binary decision variable vectors. We present the conditions for the existence of continuous trade-off regions in the market loss and missed surplus criteria and develop methods to find those.

In our multi-objective formulation, we use the surplus maximization problem under the pricing constraints that we develop in Chapter 3. CBSA updates the problem with the associated constraints of the search regions that are created in a systematical manner during the execution of the algorithm. We test the algorithm with the instances we generate preserving the characteristics of the Turkish DAM and report the results.

5.1 Literature Review

Many existing studies and the applications developed for the European DAM clearing problem use surplus maximization as the single objective. Some of them also associate additional constraints to eliminate either PABs or PRBs (Martin et al., 2014; Madani and Van Vyve, 2015; Yörükoğlu et al., 2018; Derinkuyu et al., 2019; Energy Exchange Istanbul, 2016; Euphemia, 2016).

Multi-objective approaches for the DAM clearing problem is scarce. Yörükoğlu (2015) studies the European DAM clearing problem as a bi-objective problem, where the number of PRBs are reduced in an iterative manner while maximizing the total surplus at each step. Madani et al. (2016) considers minimizing the opportunity cost due to the rejection of PRBs and compares the surplus-maximizing and opportunity-cost-minimizing solutions. Derinkuyu (2015) develops a model to minimize the average market clearing prices under the constraints to prevent both PABs and PRBs although the problem may be infeasible in many cases.

To the best of our knowledge, our study is the first study that addresses more than two objectives for the European DAM clearing problem. We develop a multi-objective formulation and characterize the trade-offs between market loss, missed surplus, and market surplus. These tradeoffs provide valuable insights to develop alternative market designs.

In the next section, we revisit the surplus maximization problem and examine the properties of the surplus-maximizing solutions.

5.2 Surplus maximization problem and the market disequilibrium

In this section, we restate the surplus-maximizing mixed-integer linear program given in Chapter 3 and discuss the market disequilibrium that occurs when there are PABs or PRBs. We first provide the associated sets, parameters, and decision variables:

- T : set of time periods
- H : set of hourly bids

- B : set of block bids
- $p_h, q_{h,t}$: price and quantity for time period $t \in T$, for an hourly bid $h \in H$ ($q_{h,t} = 0, \forall t \in T, t \neq t'$ for a particular period t')
- $p_b, q_{b,t}$: price and quantity for time period $t \in T$, for a block bid $b \in B$
- T_b : the set of time periods spanned by block bid $b \in B, T_b \subseteq T$ ($q_{b,t} = 0, \forall t \notin T_b$)
- x_h : decision variable representing the accepted fraction of hourly bid $h \in H, x_h \in [0, 1]$
- y_b : decision variable for block bid $b \in B, 1$ if accepted, and 0 if rejected.

The quantities are negative for supply bids ($q_{h,t}, q_{b,t} < 0$) and positive for demand bids ($q_{h,t}, q_{b,t} > 0$). Then, the surplus-maximizing mixed-integer linear program, **(SMILP)**, is stated as follows:

(SMILP):

$$\begin{aligned} \text{Max} \quad & \sum_{t \in T} \left\{ \sum_{h \in H} p_h q_{h,t} x_h + \sum_{b \in B} p_b q_{b,t} y_b \right\} \\ \text{s.to.} \quad & \sum_{h \in H} q_{h,t} x_h + \sum_{b \in B} q_{b,t} y_b = 0 & \forall t \in T \\ & x_h \leq 1 & \forall h \in H \\ & x_h \geq 0 & \forall h \in H \\ & y_b \in \{0, 1\} & \forall b \in B \end{aligned}$$

The first constraint balances the supply and the demand in each period. The next two inequalities force x variables to fractional values. Binary variable constraints for the block bids prevent their partial acceptance.

Let $(\mathbf{x}^*, \mathbf{y}^*)$ be an optimal solution of **(SMILP)** and $s(\mathbf{x}^*, \mathbf{y}^*)$ be the corresponding market surplus. Under marginal pricing scheme, we solve the restricted surplus maximization linear program, **(SMLP)(\mathbf{y}^*)**, by fixing the binary variable vector to the surplus-maximizing block bid decisions, $\mathbf{y} = \mathbf{y}^*$. Then, the optimal value of the

dual variable associated with the supply-demand balance constraint for period t , π_t^* , represents the market clearing price for period $t \in T$.

The optimality conditions of **(SMLP(\mathbf{y}^*))** imply equilibrium for the hourly bids. However, some block bids might be over or undervalued at the market clearing prices, π^* . A block bid b is PAB if it generates a negative surplus at the market clearing prices, $s_b(\pi^*) = \sum_{t \in T_b} (p_b - \pi_t^*)q_{b,t} < 0$, but accepted, and PRB if it implies a positive surplus at the market clearing prices, $s_b(\pi^*) = \sum_{t \in T_b} (p_b - \pi_t^*)q_{b,t} > 0$, but rejected.

Let B_{pab} and B_{prb} be the sets of PABs and PRBs, respectively. In order to compensate for negative surpluses that occur for the bidders of PABs, the MO values PABs at their bid prices instead of the market clearing prices. In turn, the MO faces a missing money problem since the amount paid to sellers exceed the amount received from the buyers. We denote the total of this amount as *market loss*, $l(\pi^*)$, and calculate it as:

$$l(\pi^*) = \sum_{b \in B_{pab}} l_b(\pi^*) = \sum_{b \in B_{pab}} \sum_{t \in T_b} (\pi_t^* - p_b)q_{b,t} \quad (5.1)$$

In the European DAMs, PRBs are not compensated for. Bidders of PRBs miss the potential surplus they could have gained at the market clearing prices. Missed surplus for PRB b , m_b , is equal to the implied surplus at the market clearing prices, $m_b(\pi^*) = s_b(\pi^*) = \sum_{t \in T_b} (p_b - \pi_t^*)q_{b,t}$. Then, the market missed surplus, $m(\pi^*)$, becomes:

$$m(\pi^*) = \sum_{b \in B_{prb}} m_b(\pi^*) = \sum_{b \in B_{prb}} \sum_{t \in T_b} (p_b - \pi_t^*)q_{b,t} \quad (5.2)$$

PABs and PRBs are accounted for by different prices than the actual market clearing prices. PAB b is valued at its own price p_b , deviating from the quantity-weighted average market clearing price over the periods in T_b by $\delta_b(\pi^*) = |p_b - \sum_{t \in T_b} \pi_t^* \frac{q_{b,t}}{Q_b}|$, where $Q_b = \sum_{t \in T_b} q_{b,t}$. Similarly, the difference in the price accounted for the bidder of PRB b and the average market clearing price over the periods in T_b is $\delta_b(\pi^*)$. Then, we define the *market price gap*, $\delta(\pi^*)$, as the maximum of those values:

$$\delta(\pi^*) = \max_{b \in B_{pab} \cup B_{prb}} \delta_b(\pi^*) \quad (5.3)$$

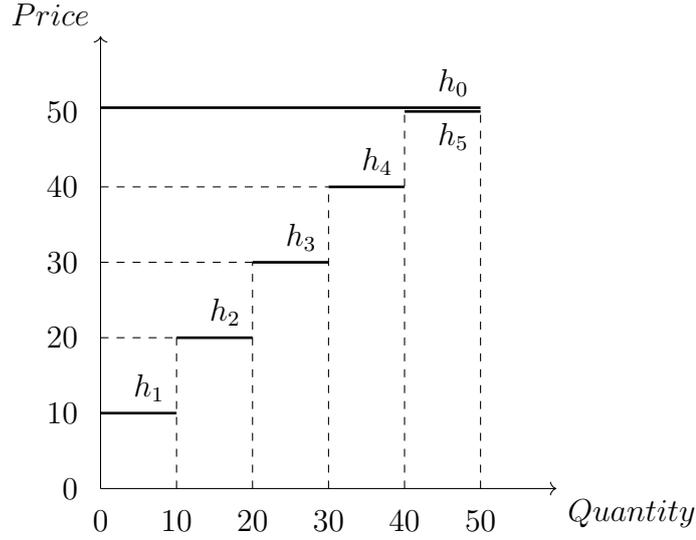


Figure 5.1: The aggregate market supply and demand functions constructed from the hourly bids given in Example 3

In case PABs or PRBs occur at the market clearing prices associated with the surplus-maximizing solution, the surplus-maximizing solution cannot establish market equilibrium. There may not exist even a feasible solution that satisfy the market equilibrium conditions. The following example illustrates such a case:

Example 3. Let the set of hourly bids consist of a single demand bid h_0 with $(p_{h_0}, q_{h_0}) = (50, 50)$ and five supply bids, $(p_{h_1}, q_{h_1}) = (10, -10)$, $(p_{h_2}, q_{h_2}) = (20, -10)$, $(p_{h_3}, q_{h_3}) = (30, -10)$, $(p_{h_4}, q_{h_4}) = (40, -10)$ and $(p_{h_5}, q_{h_5}) = (50, -10)$. Let there also be three block bids, b_1 , b_2 and b_3 , where $(p_{b_1}, q_{b_1}) = (10, -10)$, $(p_{b_2}, q_{b_2}) = (37, -10)$ and $(p_{b_3}, q_{b_3}) = (29, -50)$.

We show the aggregate market supply and the demand functions constructed from the hourly bids in Figure 5.1. In Table 5.1, we enumerate all feasible block bid decision vectors and report the associated market clearing price(s), number of PABs and PRBs, market surplus, market loss, missed surplus, and price gap. Ignoring the block bids, the intersection point(s) of the aggregate supply and demand functions determine the market clearing price(s). In case of rejecting all block bids (solution s_5 in Table 5.1), the market clearing price is 50. Accepting a supply block bid shifts the aggregate supply function to the right on the quantity axis by the quantity of the accepted block bid. For example, in solution s_1 , b_1 and b_2 are accepted and bring a total supply of 20

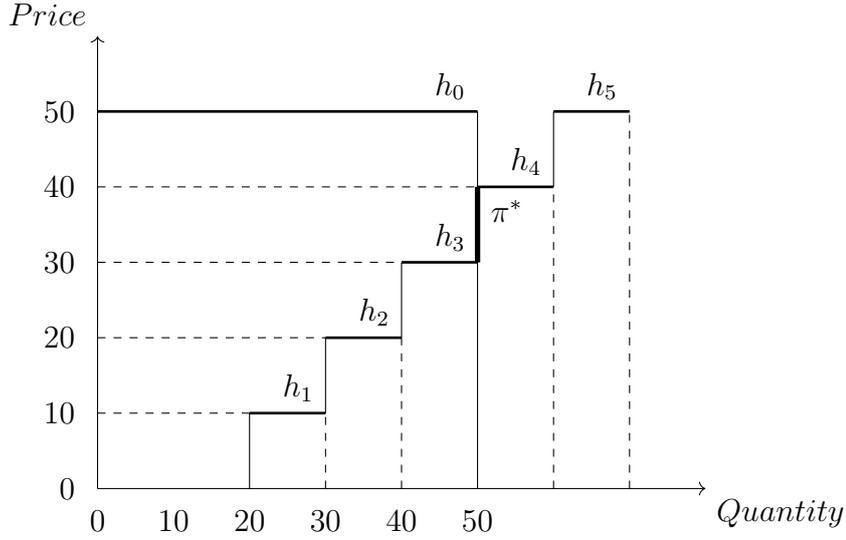


Figure 5.2: The aggregate market supply and demand functions for solution s_1 in Table 5.1

units. As shown in Figure 5.2, the aggregate supply and demand functions intersect in the interval between 30 and 40, implying alternative optimal market clearing prices. In fact, there exist alternative market clearing prices for all solutions except s_5 .

In Table 5.1, the solutions are sorted in the non-increasing order of their surplus values. The surplus-maximizing solution is s_1 . Assuming that the MO checks for alternative market clearing prices, if the minimum market clearing price is selected, then there is a PAB and a PRB. However, the MO could also choose the maximum market clearing price and prevent PABs. This, in turn, would increase $m(\pi^*)$ to 550 and $\delta(\pi^*)$ to 11.

We next report similar measures of market disequilibrium for the 20 instances we generated by representing the characteristics of the bids in the Turkish DAM. In each instance, there are approximately 15,000 hourly and 150 block bids. Hourly bids are actual bids from selected days in years 2017 and 2018 (EXIST, 2016), and the block bids are generated randomly by reproducing the characteristics of the bid sets in the Turkish DAM. In Table 5.2, we present the number of PABs and PRBs, and the corresponding market loss and missed surplus, respectively, for each instance. We also report the average and the maximum price gaps for PABs and PRBs.

Table 5.1: Market surplus, loss, missed surplus and price gap values for the feasible block bid decisions in Example 3

Sol	y_1	y_2	y_3	π	π^*	PAB	PRB	$s(\pi^*)$	$l(\pi^*)$	$m(\pi^*)$	$\delta(\pi^*)$
s_1	1	1	0	[30, 40]	30	1	1	1430	70	50	7
					40	0	1	1430	0	550	11
s_2	1	0	0	[40, 50]	40	0	2	1400	0	580	11
					50	0	2	1400	0	1180	21
s_3	0	1	0	[40, 50]	40	0	2	1130	0	850	30
					50	0	2	1130	0	1450	40
s_4	0	0	1	[0, 10]	0	1	0	1050	1450	0	29
					10	1	0	1050	950	0	19
s_5	0	0	0	50	50	0	3	1000	0	1580	40

Among the 20 instances, there is only one instance (Instance 16) where the market is in equilibrium at the surplus maximizing solution. There are no PABs or PRBs for that instance. There are 3 more instances where there are no PABs, but there is at least one PRB in 19 of the 20 instances. The maximum market loss occurs in Instance 10 and Instance 6 has the highest number of PABs. Instance 12 has 8 PRBs and the highest missed surplus. The maximum price gap occurs in Instance 1 with 15.25 Turkish liras where the average market clearing price for the experimented instances is 151.86 Turkish liras.

5.3 European DAM clearing in practice

European energy legislation favors uniform pricing in DAMs (EU Commission Regulation, 2015), and does not allow financial settlement of the traded amounts at prices different from the market clearing prices. European MOs have to eliminate the solutions with PABs in order to prevent bidders' losses that may occur due to the use of a single price vector. To eliminate PABs, the dual variables of the surplus maximization problem, such as market clearing prices and the surpluses, have to be integrated into

Table 5.2: Results for the market disequilibrium with optimal (SMILP) solutions

Ins	PAB				PRB			
	$ B_{pab} $	$l(\pi^*)^a$	Avg $\delta_b(\pi^*)$	Max $\delta_b(\pi^*)$	$ B_{prb} $	$m(\pi^*)^a$	Avg $\delta_b(\pi^*)$	Max $\delta_b(\pi^*)$
1	0	0.00	0.00	0.00	7	135.34	5.73	15.25
2	4	15.46	1.17	1.94	5	13.15	1.32	4.01
3	2	0.76	1.56	2.75	5	17.52	1.54	2.38
4	2	23.16	4.46	5.09	5	15.92	2.44	5.54
5	3	5.32	1.61	3.08	8	64.98	3.50	4.29
6	8	45.43	1.23	2.29	1	3.32	1.89	1.89
7	2	86.50	8.08	13.87	2	4.45	1.12	1.80
8	2	17.87	1.20	1.72	5	38.98	3.62	5.12
9	0	0.00	0.00	0.00	2	10.92	1.59	1.80
10	4	129.84	5.78	10.51	2	36.94	4.53	7.45
11	4	23.40	1.63	2.74	1	30.49	3.19	3.19
12	1	8.99	3.32	3.32	8	263.25	8.61	14.92
13	2	1.65	1.33	1.33	4	2.46	0.60	0.60
14	1	8.48	2.12	2.12	3	25.79	3.82	5.23
15	5	24.19	1.67	2.88	2	2.94	0.18	0.18
16	0	0.00	0.00	0.00	0	0.00	0.00	0.00
17	4	9.52	2.82	9.65	6	92.21	4.92	11.30
18	0	0.00	0.00	0.00	1	20.39	5.37	5.37
19	1	0.17	0.13	0.13	5	5.65	0.56	1.33
20	2	1.53	0.76	1.40	5	38.56	2.07	2.49

^[a] Displayed in thousand Turkish liras. The average market surplus of 20 instances is 5.37×10^8 Turkish liras.

the primal problem. The resulting model becomes a primal-dual model with partial equilibrium constraints. We present below the primal-dual model that we develop in Section 3.2.

(E-SMILP):

$$\text{Max} \quad \sum_{t \in T} \left\{ \sum_{h \in H} p_h q_{h,t} x_h + \sum_{b \in B} p_b q_{b,t} y_b \right\}$$

s.to.

$$\sum_{h \in H} q_{h,t} x_h + \sum_{b \in B} q_{b,t} y_b = 0 \quad \forall t \in T$$

$$\begin{aligned}
x_h &\leq 1 && \forall h \in H \\
y_b &\leq 1 && \forall b \in B \\
s_h &\geq \sum_{t \in T} (p_h - \pi_t) q_{h,t} && \forall h \in H \\
s_b - l_b + m_b &\geq \sum_{t \in T} (p_b - \pi_t) q_{b,t} && \forall b \in B \\
m_b &\leq M_b(1 - y_b) && \forall b \in B \\
l_b &\leq M_b y_b && \forall b \in B \\
\sum_{t \in T} \left\{ \sum_{h \in H} p_h q_{h,t} x_h + \sum_{b \in B} p_b q_{b,t} y_b \right\} &\geq \sum_{h \in H} s_h + \sum_{b \in B} s_b - \sum_{b \in B} l_b \\
x_h, s_h &\geq 0 && \forall h \in H \\
y_b &\in \{0, 1\} && \forall b \in B \\
s_b, l_b, m_b &\geq 0 && \forall b \in B
\end{aligned}$$

In this model, s_h and s_b denote the surplus corresponding to hourly bid $h \in H$ and block bid $b \in B$, respectively. l_b and m_b represent upper bounds on the market loss and the missed surplus associated with block bid b . Loss and missed surplus variables are not defined for the hourly bids as the equilibrium conditions are enforced for them. However, l_b and m_b variables relax the equilibrium conditions for block bids making the partial equilibrium a feasible solution. M_b is an appropriate upper bound on the loss or missed surplus of a block bid b . Denoting the maximum and the minimum allowable bid prices as p^{max} and p^{min} , respectively, $M_b \geq \sum_{t \in T} |q_{b,t}| (p^{max} - p^{min})$ is an upper bound on the loss or missed surplus of a block bid, and is a sufficiently large big-M value. The last constraint is the strong-duality constraint that ensures the feasible hourly bid decisions are surplus-maximizing when the block bid decisions are fixed, and it is a binding constraint for all feasible solutions. We represent the feasible set of **(E-SMILP)** by Ψ , and define the surplus maximization problem with no PAB next:

(SMILP-NoPAB):

$$\begin{aligned}
\text{Max} \quad & \sum_{h \in H} s_h + \sum_{b \in B} s_b - \sum_{b \in B} l_b \\
\text{s.to.} \quad &
\end{aligned}$$

$$\begin{aligned}
& (\mathbf{x}, \mathbf{y}, \boldsymbol{\pi}, \mathbf{s}, \mathbf{l}, \mathbf{m}) \in \Psi \\
& \mathbf{l} \leq \mathbf{0}
\end{aligned}$$

We use the dual objective function for **(SMILP-NoPAB)** instead of the primal objective function used in **(E-SMILP)**. They are equivalent due to the strong-duality constraint in the constraint set. Solution s_1 in Table 5.1 is feasible for **(SMILP-NoPAB)** with market clearing prices in the range $[37, 40]$. It is a coincidence that the surplus-maximizing solution has feasible market clearing prices for **(SMILP-NoPAB)**. However, elimination of PAB causes an increase in the missed surplus value and the price gap of b_3 .

Turkish MO adopts a different pricing rule than the rest of its European counterparts. In Turkish DAM, PRBs are eliminated, whereas PABs are allowed (Energy Exchange Istanbul, 2016). The MO of the Turkish DAM, EXIST, compensates the losses of those bidders having a PAB. This is achieved by solving a similar model, **(SMLP-NoPRB)**, to prevent PABs while maximizing the surplus:

(SMILP-NoPRB):

$$\begin{aligned}
\text{Max} \quad & \sum_{h \in H} s_h + \sum_{b \in B} s_b - \sum_{b \in B} l_b \\
\text{s.to.} \quad & \\
& (\mathbf{x}, \mathbf{y}, \boldsymbol{\pi}, \mathbf{s}, \mathbf{l}, \mathbf{m}) \in \Psi \\
& \mathbf{m} \leq \mathbf{0}
\end{aligned}$$

The optimal solution of **(SMLP-NoPRB)** for the problem given in Example 3 is s_4 . The MO has to sacrifice from the market surplus as much as 380 units to eliminate PRBs. In addition, the market loss and price gap increase dramatically in this case. There are alternative optimal market clearing prices, and the results are the worst if the generated market clearing price is at minimum.

In Table 5.3, we present the results for the optimal solutions of **(SMILP-NoPAB)** and **(SMILP-NoPRB)** by subtracting the corresponding values of optimal **(SMILP)** solutions. For **(SMILP-NoPAB)**, we report the changes in the market surplus, number of PRBs, missed surplus, and market price gap. Similarly for **(SMILP-NoPRB)**, we

report the changes in the market surplus, number of PABs, loss, and market price gap.

Table 5.3: Results for market disequilibrium with **(SMILP-NoPAB)** and **(SMILP-NoPRB)** solutions in comparison to **(SMILP)** solution

Ins	(SMILP-NoPAB)				(SMILP-NoPRB)			
	$s(\pi^*)^a$	$ B_{prb} $	$m(\pi^*)^a$	Max $\delta_b(\pi^*)$	$s(\pi^*)^a$	$ B_{pab} $	$l(\pi^*)^a$	Max $\delta_b(\pi^*)$
1	0.00	0	0.00	0.00	-3.25	6	169.00	31.84
2	-4.10	2	9.95	4.10	-10.02	4	14.96	4.12
3	-1.14	1	13.75	0.22	-6.84	6	34.34	7.43
4	-20.27	2	45.31	9.70	-27.43	6	29.28	6.81
5	-12.72	4	53.75	20.13	-4.99	7	78.01	11.29
6	-13.54	3	34.22	5.41	-4.41	-1	-6.64	89.39
7	-28.87	3	75.95	13.73	-7.42	1	8.26	20.53
8	-2.52	2	4.31	0.38	-5.20	3	37.45	14.97
9	0.00	0	0.00	0.00	-3.71	2	11.54	3.64
10	-29.95	5	101.71	14.99	-48.23	1	-36.64	11.84
11	-5.61	2	-8.47	-0.78	-12.13	2	14.61	8.47
12	-17.87	3	34.31	18.02	-10.30	6	98.94	12.95
13	-1.16	1	1.19	20.95	-1.26	1	1.37	25.60
14	-15.55	7	92.73	5.59	-36.70	5	50.02	30.05
15	-22.50	1	39.92	8.26	-8.21	1	6.66	46.12
16	0.00	0	0.00	0.00	0.00	0	0.00	0.00
17	-9.59	4	37.90	11.29	-7.42	4	57.89	69.28
18	-0.38	3	17.30	1.86	-6.44	5	127.57	12.71
19	-11.04	3	98.36	7.60	-2.98	3	7.70	19.16
20	-3.94	5	72.01	7.94	-7.81	4	33.05	59.31
avg	-10.04	2.55	36.21	7.47	-10.74	3.3	36.87	24.27
max	-29.95 ^b	7	101.71	20.95	-48.23 ^b	7	169.00	89.39
std	9.80	1.85	35.80	7.10	12.37	2.34	49.23	24.10

^[a] Displayed in thousand Turkish liras. The average market surplus of 20 instances is 5.37×10^8 Turkish liras.

^[b] Maximum denotes the negative of maximum decrease in market surplus.

Based on the results in Table 5.3, the relatively small changes in the market surplus for both **(SMILP-NoPAB)** and **(SMILP-NoPRB)** imply that there exist solutions with

near optimal market surplus that have either only PABs or only PRBs. However, the optimal solutions of **(SMILP-NoPAB)** and **(SMILP-NoPRB)** show a strong trade-off between market loss and missed surplus. While eliminating PABs (PRBs), the market missed surplus (loss) increase by more than 36×10^3 Turkish liras on average, and more than 10^5 (1.69×10^5) Turkish liras in the worst case. In addition, no PRB constraints lead to extreme price gaps in the PABs.

The two models adopted in the European DAMS, **(SMILP-NoPAB)** and **(SMILP-NoPRB)**, correspond to only two extreme practices to resolve the market disequilibrium. Our computational results show that these extreme solutions perform poorly in terms of the relevant criteria they have not accounted for. There may exist other solutions providing more desirable compromises between the market surplus, market loss, and missed surplus. In the following sections, we investigate such solutions.

5.4 Multi-objective market clearing problem

In the previous section, we discussed that MOs need to handle multiple conflicting objectives. It is not possible to achieve market equilibrium in many cases, and the MOs have to relax some of the equilibrium conditions. The pricing rules in practice do not aim for compromise solutions, instead, they choose one of the two extreme solutions. However, it turns out that such solutions perform poorly considering other criteria relevant for the MOs. In order to investigate the extent of possible trade-offs between the feasible solutions of the problem, we formulate a multi-objective version of the DAM clearing problem.

We consider the following multi-objective market clearing problem, **(MOMCP)**:

(MOMCP):

$$\begin{aligned} \text{Max } z(\psi) &= (z_1, -z_2, -z_3) \\ \text{s.to.} \\ z_1 &= \sum_{h \in H} s_h + \sum_{b \in B} s_b - \sum_{b \in B} l_b \\ z_2 &= \sum_{b \in B} l_b \end{aligned}$$

$$z_3 = \sum_{b \in B} m_b$$

$$\boldsymbol{\psi} = (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\pi}, \boldsymbol{s}, \boldsymbol{l}, \boldsymbol{m}) \in \Psi$$

where $\boldsymbol{\psi} \in \Psi$ is a feasible solution for **(E-SMILP)**, $\boldsymbol{z} = z(\boldsymbol{\psi}) = (z_1, -z_2, -z_3) \in \mathcal{Z}$ represents the image of $\boldsymbol{\psi}$ in the three-dimensional feasible objective space of market surplus, $z_1 = z_1(\boldsymbol{\psi})$, market loss, $z_2 = z_2(\boldsymbol{\psi})$, and market missed surplus, $z_3 = z_3(\boldsymbol{\psi})$. We next define some special solutions for our problem:

Definition 23. $\boldsymbol{\psi}^k \in \Psi$ is an efficient solution if $\nexists \boldsymbol{\psi}^j \in \Psi$ such that $z_1(\boldsymbol{\psi}^j) \geq z_1(\boldsymbol{\psi}^k)$, $z_i(\boldsymbol{\psi}^j) \leq z_i(\boldsymbol{\psi}^k)$ for $i = 2, 3$, and $z_1(\boldsymbol{\psi}^j) > z_1(\boldsymbol{\psi}^k)$ or $z_i(\boldsymbol{\psi}^j) < z_i(\boldsymbol{\psi}^k)$ for at least one $i = 2, 3$. If $\boldsymbol{\psi}^k$ is efficient, then $\boldsymbol{z}^k = z(\boldsymbol{\psi}^k)$ is said to be nondominated. On the other hand, if there exists such $\boldsymbol{\psi}^j$, then $\boldsymbol{\psi}^k$ is said to be inefficient and \boldsymbol{z}^k is said to be dominated.

Definition 24. $\boldsymbol{\psi}^k \in \Psi$ is a weakly efficient solution if $\nexists \boldsymbol{\psi}^j \in \Psi$ such that $z_1(\boldsymbol{\psi}^j) > z_1(\boldsymbol{\psi}^k)$, $z_i(\boldsymbol{\psi}^j) < z_i(\boldsymbol{\psi}^k)$ for $i = 2, 3$. If $\boldsymbol{\psi}^k$ is weakly efficient, then $\boldsymbol{z}^k = z(\boldsymbol{\psi}^k)$ is said to be weakly nondominated. On the other hand, if there exists such $\boldsymbol{\psi}^j$, then $\boldsymbol{\psi}^k$ is said to be strictly inefficient and \boldsymbol{z}^k is said to be strictly dominated.

Definition 25. Nondominated point $\boldsymbol{z}^k = z(\boldsymbol{\psi}^k)$ is an extreme nondominated point if $\exists \boldsymbol{w} \in \mathbb{R}_{>0}^3$ such that $\boldsymbol{\psi}^k = \operatorname{argmax}_{\boldsymbol{\psi} \in \Psi} \boldsymbol{w}z(\boldsymbol{\psi})$ is the unique optimal solution.

Let $\mathcal{Z}_{ND} \subseteq \mathcal{Z}$ be the set of nondominated points for **(MOMCP)**, and \mathcal{Y} be the set of feasible block bid decisions. We first investigate the properties of the slice problems (Belotti et al., 2013), that are three-objective linear programs with fixed integer (binary) variables. For a given block bid decision vector $\bar{\boldsymbol{y}} \in \mathcal{Y}$, the corresponding slice problem, **(MOMCP)($\bar{\boldsymbol{y}}$)**, is defined as:

(MOMCP)($\bar{\boldsymbol{y}}$):

$$\text{Max } z(\boldsymbol{\psi}) = (z_1, -z_2, -z_3)$$

s.to.

$$z_1 = \sum_{h \in H} s_h + \sum_{b \in B} s_b - \sum_{b \in B} l_b$$

$$\begin{aligned}
z_2 &= \sum_{b \in B} l_b \\
z_3 &= \sum_{b \in B} m_b \\
\boldsymbol{\psi} &= (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\pi}, \boldsymbol{s}, \boldsymbol{l}, \boldsymbol{m}) \in \Psi \\
\boldsymbol{y} &= \bar{\boldsymbol{y}}
\end{aligned}$$

Let $\mathcal{Z}(\bar{\boldsymbol{y}}) \subseteq \mathcal{Z}$ and $\mathcal{Z}_{ND}(\bar{\boldsymbol{y}}) \subseteq \mathcal{Z}(\bar{\boldsymbol{y}})$ be the feasible objective space and the nondominated set of $(\mathbf{MOMCP}(\bar{\boldsymbol{y}}))$, respectively. $\bar{\boldsymbol{y}} \in \mathcal{Y}$ is an efficient block bid decision vector if $\mathcal{Z}_{ND}(\bar{\boldsymbol{y}}) \cap \mathcal{Z}_{ND} \neq \emptyset$. We denote the set of efficient block bid decision vectors with $\mathcal{Y}_E \subseteq \mathcal{Y}$.

We next analyze the properties of the feasible solutions for $(\mathbf{MOMCP}(\bar{\boldsymbol{y}}))$, $\Psi(\bar{\boldsymbol{y}})$. We first present the complementary slackness conditions for $\mathbf{SMLP}(\bar{\boldsymbol{y}})$, $(\mathbf{CS-SMLP}(\bar{\boldsymbol{y}}))$, that are satisfied by each $\boldsymbol{\psi} \in \Psi(\bar{\boldsymbol{y}})$:

(CS-SMLP($\bar{\boldsymbol{y}}$)):

$$\begin{aligned}
s_h^*(1 - x_h^*) &= 0 & \forall h \in H \\
s_b^*(1 - \bar{y}_b) &= 0 & \forall b \in B \\
m_b^* \bar{y}_b &= 0 & \forall b \in B_0 \\
l_b^*(1 - \bar{y}_b) &= 0 & \forall b \in B_1 \\
x_h^*(s_h^* - \sum_{t \in T} (p_h - \pi_t^*) q_{h,t}) &= 0 & \forall h \in H \\
\bar{y}_b(s_b^* + m_b^* - \sum_{t \in T} (p_b - \pi_t^*) q_{b,t}) &= 0 & \forall b \in B_0 \\
\bar{y}_b(s_b^* - l_b^* - \sum_{t \in T} (p_b - \pi_t^*) q_{b,t}) &= 0 & \forall b \in B_1
\end{aligned}$$

B_0 and B_1 denote the set of rejected and accepted block bids, respectively. x_h^* is the optimal value of x_h for $h \in H$, and π_t^* is the optimal market clearing price for period $t \in T$ of $\mathbf{SMLP}(\bar{\boldsymbol{y}})$. The following proposition shows that the market surplus is fixed for a block bid decision vector, and hence it takes discrete values in \mathcal{Z} and \mathcal{Z}_{ND} .

Proposition 21. *For any $\bar{\boldsymbol{y}} \in \mathcal{Y}$ and $\boldsymbol{z}^1, \boldsymbol{z}^2 \in \mathcal{Z}(\bar{\boldsymbol{y}})$, $z_1^1 = z_1^2$.*

Proof. The constraint set of $\mathbf{MOMCP}(\bar{\mathbf{y}})$ includes the strong-duality constraint associated with the surplus maximization linear program for a given $\bar{\mathbf{y}}$, ($\mathbf{SMILP}(\bar{\mathbf{y}})$). Let s^* be the optimal objective value of ($\mathbf{SMILP}(\bar{\mathbf{y}})$). Then, $z_1 = s^*$ for any $\mathbf{z} \in \mathcal{Z}(\bar{\mathbf{y}})$. \square

There might exist multiple feasible market clearing price vectors for ($\mathbf{MOMCP}(\bar{\mathbf{y}})$) for some $\bar{\mathbf{y}} \in \mathcal{Y}$. Let $\Pi^*(\bar{\mathbf{y}})$ be the set of alternative market clearing price vectors in the slice problem corresponding to $\bar{\mathbf{y}}$. We first analyze the case when $\Pi^*(\bar{\mathbf{y}})$ is a singleton. We show in Proposition 22 that \mathcal{Z}_{ND} is a finite set consisting of isolated nondominated points when there is a unique market clearing price for each block bid decision vector.

Proposition 22. *If $\Pi^*(\bar{\mathbf{y}})$ is a singleton, then $|\mathcal{Z}_{ND}(\bar{\mathbf{y}})| = 1$ and \mathcal{Z}_{ND} is a finite discrete set.*

Proof. By Proposition 21, the market surplus is constant for $\bar{\mathbf{y}}$. Let π^* be the market clearing price. Then, $s_b^* - l_b^* = \sum_{t \in T} (p_b - \pi_t^*) q_{b,t}$, $\forall b \in B_1$, and $s_b^* = l_b^* = 0$, $\forall b \in B_0$. In an efficient solution, if $\sum_{t \in T} (p_b - \pi_t^*) q_{b,t} \geq 0$, then $l_b^* = 0$. Else, $l_b^* = \sum_{t \in T} (\pi_t^* - p_b) q_{b,t}$. Similarly, $m_b^* = \max \left\{ 0, \sum_{t \in T} (p_b - \pi_t^*) q_{b,t} \right\}$, $\forall b \in B_0$ for an efficient solution $\psi \in \Psi$. This proves that π^* uniquely determines the market loss and missed surplus. Hence, $\mathcal{Z}_{ND}(\bar{\mathbf{y}})$ is a singleton. Since \mathcal{Y} is a finite set and $\mathcal{Z}_{ND}(\bar{\mathbf{y}})$ is a singleton for each $\bar{\mathbf{y}} \in \mathcal{Y}$, \mathcal{Z}_{ND} is a finite discrete set. \square

We next investigate the conditions that lead to alternative market clearing prices. Then, we characterize the nondominated sets of the slice problems in terms of the market clearing prices.

5.4.1 Alternative surplus-maximizing market clearing prices

Given $\bar{\mathbf{y}} \in \mathcal{Y}$, let $Q_t(\bar{\mathbf{y}}) = \sum_{b \in B} \bar{y}_b q_{b,t}$. We reconsider the restricted surplus maximization linear program in Section 3.2 by excluding the constant value $\sum_{b \in B} \sum_{t \in T} p_b q_{b,t} \bar{y}_b$ from the objective function:

(SMLP(\bar{y})):

$$\begin{aligned}
\text{Max} \quad & \sum_{h \in H} \sum_{t \in T} p_h q_{h,t} x_h \\
\text{s.to.} \quad & \sum_{h \in H} q_{h,t} x_h = -Q_t(\bar{y}) & \forall t \in T \quad [\pi_t] \\
& x_h \leq 1 & \forall h \in H \quad [s_h] \\
& x_h \geq 0 & \forall h \in H
\end{aligned}$$

Since an hourly bid is offered for a single period (it is zero in all other periods), we can decompose **(SMLP(\bar{y}))** into $|T|$ single period problems. Let H_t be the set of hourly bids offered for period $t \in T$. The single period restricted surplus maximization linear program can be written as:

(SMLP(\bar{y}, t)):

$$\begin{aligned}
\text{Max} \quad & \sum_{h \in H_t} p_h q_{h,t} x_h \\
\text{s.to.} \quad & \sum_{h \in H_t} q_{h,t} x_h = -Q_t(\bar{y}) & [\pi_t] \\
& x_h \leq 1 & \forall h \in H_t \quad [s_h] \\
& x_h \geq 0 & \forall h \in H_t
\end{aligned}$$

The dual restricted surplus maximization linear program for period $t \in T$ is:

(D-SMLP(\bar{y}, t)):

$$\begin{aligned}
\text{Min} \quad & \sum_{h \in H_t} s_h - Q_t(\bar{y}) \pi_t \\
\text{s.to.} \quad & s_h \geq (p_h - \pi_t) q_{h,t} & \forall h \in H_t \\
& s_h \geq 0 & \forall h \in H_t
\end{aligned}$$

Since at least one of the constraints of **(D-SMLP(\bar{y}, t))** is binding for each hourly bid at an optimal solution, $s_h^* = \max \{0, q_{h,t}(p_h - \pi_t^*)\}$ holds for each $h \in H$. Let $s_h(\pi_t) = \max \{0, q_{h,t}(p_h - \pi_t)\}$. Then, $s_h(\pi_t)$ is a non-increasing piece-wise linear function of π_t for a demand bid ($q_{h,t} > 0$) and a non-decreasing piece-wise linear function of π_t for a supply bid ($q_{h,t} < 0$). Letting p_t^{min} and p_t^{max} represent the lower

and upper limits for π_t , respectively, $s_h(\pi_t)$ is depicted in Figures 5.3 and 5.4 for demand and supply bids, respectively.

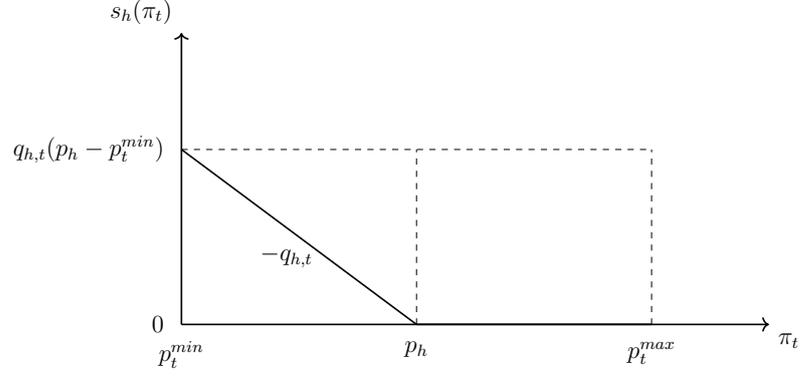


Figure 5.3: $s_h(\pi_t)$ for a demand hourly bid, $q_{h,t} > 0$

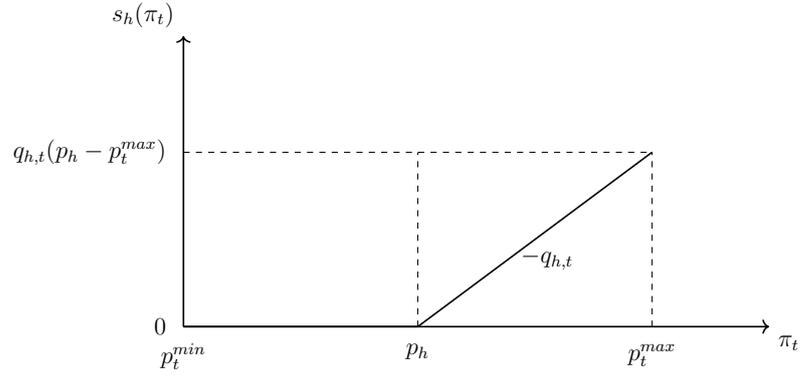


Figure 5.4: $s_h(\pi_t)$ for a supply hourly bid, $q_{h,t} < 0$

Let $S_{H_t}(\bar{\mathbf{y}}, \pi_t) = \sum_{h \in H_t} s_h(\pi_t) - Q_t(\bar{\mathbf{y}})\pi_t$. Then, the optimal objective function value of **(D-SMLP)($\bar{\mathbf{y}}, t$)** can be found by solving the following problem:

$$\min_{\pi_t \in [p_t^{min}, p_t^{max}]} S_{H_t}(\bar{\mathbf{y}}, \pi_t) = \min_{\pi_t \in [p_t^{min}, p_t^{max}]} \left\{ \sum_{h \in H_t} \max \{0, q_{h,t}(p_h - \pi_t)\} - Q_t(\bar{\mathbf{y}})\pi_t \right\}$$

We can rewrite $S_{H_t}(\bar{\mathbf{y}}, \pi_t)$ as:

$$\begin{aligned}
S_{H_t}(\bar{\mathbf{y}}, \pi_t) &= \sum_{h \in H_t, q_{h,t} > 0, p_h > \pi_t} [q_{h,t}(p_h - p_t^{min}) - (\pi_t - p_t^{min})q_{h,t}] + \\
&\quad \sum_{h \in H_t, q_{h,t} < 0, p_h < \pi_t} q_{h,t}(p_h - \pi_t) - Q_t(\bar{\mathbf{y}})\pi_t \\
&= \sum_{h \in H_t, q_{h,t} > 0, p_h > \pi_t} q_{h,t}(p_h - \pi_t) + \\
&\quad \sum_{h \in H_t, q_{h,t} < 0, p_h < \pi_t} q_{h,t}(p_h - \pi_t) - Q_t(\bar{\mathbf{y}})\pi_t
\end{aligned}$$

Let $S'_{H_t}(\bar{\mathbf{y}}, \hat{\pi}_t)$ be the subderivative of $S_{H_t}(\bar{\mathbf{y}}, \pi_t)$ with respect to π_t at point $\hat{\pi}_t$. Note that $S_{H_t}(\bar{\mathbf{y}}, \pi_t)$ is differentiable at any point $\pi_t \in (p_t^{min}, p_t^{max}) : \pi_t \neq p_h, \forall h \in H_t$. Then,

$$S'_{H_t}(\bar{\mathbf{y}}, \hat{\pi}_t) = \sum_{h \in H_t, q_{h,t} > 0, p_h > \hat{\pi}_t} -q_{h,t} + \sum_{h \in H_t, q_{h,t} < 0, p_h < \hat{\pi}_t} -q_{h,t} - Q_t(\bar{\mathbf{y}})$$

for any $\hat{\pi}_t$ where it is differentiable. As $S'_{H_t}(\bar{\mathbf{y}}, \pi_t)$ is a non-decreasing function of π_t (since the number of demand hourly bids with $p_h > \hat{\pi}_t$ is non-increasing and the number of supply hourly bids with $p_h < \hat{\pi}_t$ is non-decreasing as π_t increases), $S_{H_t}(\bar{\mathbf{y}}, \pi_t)$ is a convex piece-wise linear function of π_t . Let $\partial S_{H_t}(\bar{\mathbf{y}}, \hat{\pi}_t)$ be the set of subgradients of $S_{H_t}(\bar{\mathbf{y}}, \pi_t)$ at point $\hat{\pi}_t$. Then, $\pi_t^* = \{\pi_t \in [p_t^{min}, p_t^{max}] : 0 \in \partial S_{H_t}(\bar{\mathbf{y}}, \pi_t)\}$ gives the set of optimal market clearing prices. In Figure 5.5, we illustrate the characteristics of $S_{H_t}(\bar{\mathbf{y}}, \pi_t)$.

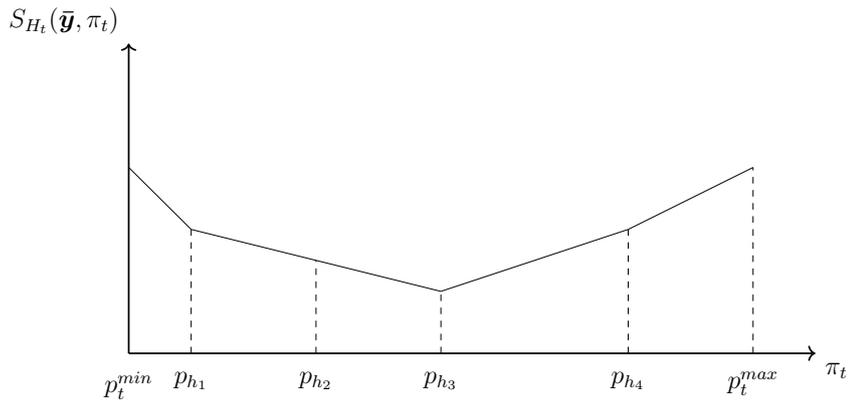


Figure 5.5: Illustration of $S_{H_t}(\bar{\mathbf{y}}, \pi_t)$ with unique optimal market clearing prices, $\pi_t^* = p_{h_3}$

In Figure 5.6, the set of optimal market clearing prices is in the interval $[p_{h_2}, p_{h_3}]$. This case occurs when the aggregate supply and demand functions intersect at a vertical line as shown in Figure 5.2.

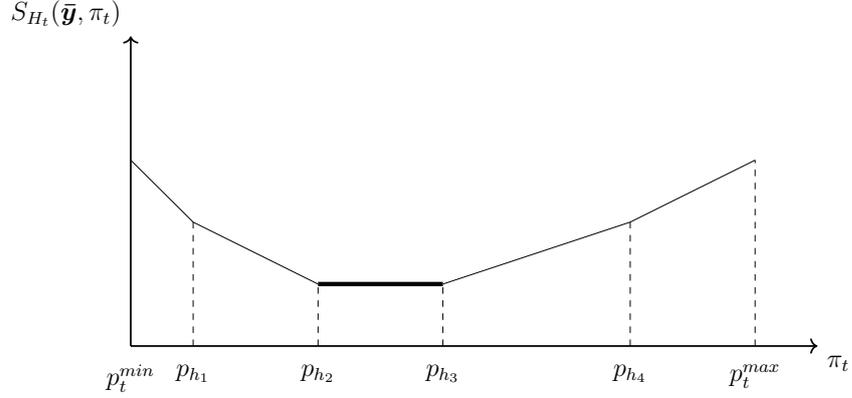


Figure 5.6: Illustration of $S_{H_t}(\bar{y}, \pi_t)$ when there are multiple optimal market clearing prices, $\pi_t^* = [p_{h_2}, p_{h_3}]$

Let $\Pi_t^*(\bar{y})$ denote the set of alternative optimal market clearing prices for period $t \in T$ and $\bar{y} \in \mathcal{Y}$ such that $\Pi_t^*(\bar{y}) = [\pi_t^l(\bar{y}), \pi_t^u(\bar{y})]$. To find $\pi_t^l(\bar{y})$ and $\pi_t^u(\bar{y})$, we minimize and maximize the market clearing price problem, $(\mathbf{P}(\bar{y}, t))$, respectively:

$(\mathbf{P}(\bar{y}, t))$:

$$\text{Min (Max)} \quad \pi_t$$

s.to.

$$s_h \geq (p_h - \pi_t)q_{h,t} \quad \forall h \in H_t$$

$$\sum_{h \in H_t} s_h - Q_t(\bar{y})\pi_t = S_{H_t}^*(\bar{y})$$

$$s_h \geq 0 \quad \forall h \in H_t$$

where $S_{H_t}^*(\bar{y})$ is the optimal objective value of $(\mathbf{D-SMLP}(\bar{y}, t))$.

5.4.2 Characterizing the nondominated edges

For a fixed block bid decision vector, alternative surplus-maximizing market clearing prices may generate feasible trade-offs between market loss and missed surpluses.

Such trade-offs may be nondominated.

Let $l_b(\boldsymbol{\pi}^*, \bar{\mathbf{y}})$ and $m_b(\boldsymbol{\pi}^*, \bar{\mathbf{y}})$ be the market loss and missed surplus, respectively, associated with block bid $b \in B$, $\boldsymbol{\pi}^* \in \Pi^*(\bar{\mathbf{y}})$, and $\bar{\mathbf{y}} \in \mathcal{Y}$. Also let π_b^* be the quantity-weighted average market clearing price for block bid b , $\pi_b^* = \sum_{t \in T_b} \pi_t^* \frac{q_{b,t}}{Q_b}$. Note that $l_b(\boldsymbol{\pi}^*, \bar{\mathbf{y}})$ is zero for each rejected block bid $b \in B_0$. Similarly, $m_b(\boldsymbol{\pi}^*, \bar{\mathbf{y}})$ is zero for each accepted block bid $b \in B_1$. We next consider the form of the market loss function for demand block bids, B^d , and supply block bids, B^s :

Case 1. Suppose $\bar{y}_b = 1$ and $b \in B^d$. Then,

$$l_b(\boldsymbol{\pi}^*, \bar{\mathbf{y}}) = \begin{cases} 0 & \text{if } \pi_b^* \leq p_b \\ \sum_{t \in T} (\pi_t^* - p_b) q_{b,t} & \text{if } \pi_b^* > p_b \end{cases} \quad \text{(PAB)} \quad (5.4)$$

Case 2. Suppose $\bar{y}_b = 1$ and $b \in B^s$. Then,

$$l_b(\boldsymbol{\pi}^*, \bar{\mathbf{y}}) = \begin{cases} 0 & \text{if } \pi_b^* \geq p_b \\ \sum_{t \in T} (\pi_t^* - p_b) q_{b,t} & \text{if } \pi_b^* < p_b \end{cases} \quad \text{(PAB)} \quad (5.5)$$

We express the total market loss, z_2 , as a function of $\boldsymbol{\pi}^*$, and investigate its change with respect to π_t^* for any $t \in T$. We partition z_2 into the loss of supply block bids, z_2^s , and the loss of demand block bids, z_2^d . Let $z_2^s(\boldsymbol{\pi}^*, \bar{\mathbf{y}}) = \sum_{b \in B^s} l_b(\boldsymbol{\pi}^*, \bar{\mathbf{y}})$ and $z_2^d(\boldsymbol{\pi}^*, \bar{\mathbf{y}}) = \sum_{b \in B^d} l_b(\boldsymbol{\pi}^*, \bar{\mathbf{y}})$. We illustrate $z_2^s(\boldsymbol{\pi}^*, \bar{\mathbf{y}})$ and $z_2^d(\boldsymbol{\pi}^*, \bar{\mathbf{y}})$ as a function of π_t^* for $t \in T$ in Figures 5.7 and 5.8, respectively.

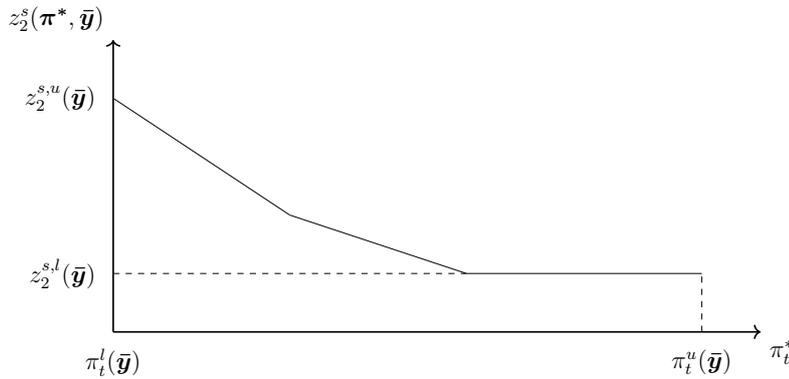


Figure 5.7: Illustration of $z_2^s(\boldsymbol{\pi}^*, \bar{\mathbf{y}})$ with respect to π_t^*

In Figure 5.7, $z_2^{s,l}(\bar{\mathbf{y}})$ and $z_2^{s,u}(\bar{\mathbf{y}})$ represent lower and upper bounds on $z_2^s(\boldsymbol{\pi}^*, \bar{\mathbf{y}})$ with respect to $\pi_t^* \in \Pi_t^*(\bar{\mathbf{y}})$, respectively. $z_2^s(\boldsymbol{\pi}^*, \bar{\mathbf{y}})$ is a nonincreasing piecewise linear convex function of π_t^* since the total number of supply PABs and the associated total absolute quantity are nonincreasing in π_t^* .

Similarly, we show that $z_2^d(\boldsymbol{\pi}^*, \bar{\mathbf{y}})$ is a nondecreasing piecewise linear convex function of π_t^* . In Figure 5.8, $z_2^{d,l}(\bar{\mathbf{y}})$ and $z_2^{d,u}(\bar{\mathbf{y}})$ represent lower and upper bounds on $z_2^d(\boldsymbol{\pi}^*, \bar{\mathbf{y}})$ for $\pi_t^* \in \Pi_t^*(\bar{\mathbf{y}})$, respectively. $z_2^d(\boldsymbol{\pi}^*, \bar{\mathbf{y}})$ is a nondecreasing piecewise linear convex function of π_t^* since the total number of demand PABs and the associated total absolute quantity are nondecreasing in π_t^* .

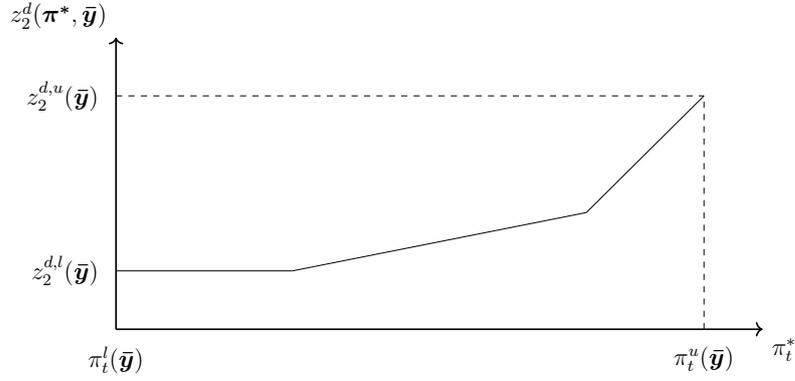


Figure 5.8: Illustration of $z_2^d(\boldsymbol{\pi}^*, \bar{\mathbf{y}})$ with respect to π_t^*

Since both $z_2^s(\boldsymbol{\pi}^*, \bar{\mathbf{y}})$ and $z_2^d(\boldsymbol{\pi}^*, \bar{\mathbf{y}})$ are convex functions of π_t^* , $z_2(\boldsymbol{\pi}^*, \bar{\mathbf{y}}) = z_2^s(\boldsymbol{\pi}^*, \bar{\mathbf{y}}) + z_2^d(\boldsymbol{\pi}^*, \bar{\mathbf{y}})$ is also a convex function of π_t^* .

We next analyze the market missed surplus, $z_3(\boldsymbol{\pi}^*, \bar{\mathbf{y}})$, as a function of the market clearing price, $\boldsymbol{\pi}^* \in \Pi^*(\bar{\mathbf{y}})$, and investigate its change with respect to $\pi_t^* \in \Pi_t^*(\bar{\mathbf{y}})$ for $t \in T$.

Case 1. Suppose $\bar{y}_b = 0$ and $b \in B^d$. Then,

$$m_b(\boldsymbol{\pi}^*, \bar{\mathbf{y}}) = \begin{cases} 0 & \text{if } \pi_b^* \geq p_b \\ \sum_{t \in T} (p_b - \pi_t^*) q_{b,t} & \text{if } \pi_b^* < p_b \quad \text{(PRB)} \end{cases} \quad (5.6)$$

Case 2. Suppose $\bar{y}_b = 0$ and $b \in B^s$. Then,

$$m_b(\boldsymbol{\pi}^*, \bar{\mathbf{y}}) = \begin{cases} 0 & \text{if } \pi_b^* \leq p_b \\ \sum_{t \in T} (p_b - \pi_t^*) q_{b,t} & \text{if } \pi_b^* > p_b \quad \text{(PRB)} \end{cases} \quad (5.7)$$

We partition z_3 into missed surplus by supply block bids, z_3^s , and missed surplus by demand block bids, z_3^d . Then, $z_3^s(\boldsymbol{\pi}^*, \bar{\mathbf{y}}) = \sum_{b \in B^s} m_b(\boldsymbol{\pi}^*, \bar{\mathbf{y}})$ and $z_3^d(\boldsymbol{\pi}^*, \bar{\mathbf{y}}) = \sum_{b \in B^d} m_b(\boldsymbol{\pi}^*, \bar{\mathbf{y}})$. We illustrate $z_3^s(\boldsymbol{\pi}^*, \bar{\mathbf{y}})$ and $z_3^d(\boldsymbol{\pi}^*, \bar{\mathbf{y}})$ as a function of π_t^* for $t \in T$ in Figures 5.9 and 5.10, respectively.

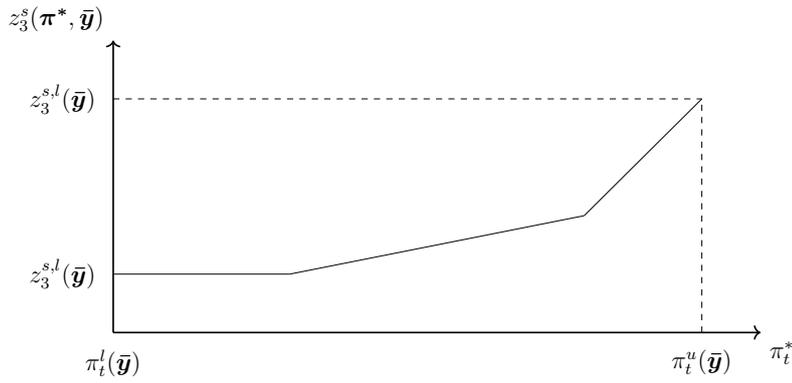


Figure 5.9: Illustration of $z_3^s(\boldsymbol{\pi}^*, \bar{\mathbf{y}})$ with respect to π_t^*

In Figure 5.9, $z_3^{s,l}(\bar{\mathbf{y}})$ and $z_3^{s,u}(\bar{\mathbf{y}})$ represent lower and upper bounds on $z_3^s(\boldsymbol{\pi}^*, \bar{\mathbf{y}})$ with respect to $\pi_t^* \in \Pi_t^*(\bar{\mathbf{y}})$, respectively. $z_3^s(\boldsymbol{\pi}^*, \bar{\mathbf{y}})$ is a nondecreasing piecewise linear convex function of π_t^* since the total number of supply PRBs and the associated total absolute quantity are nondecreasing in π_t^* .

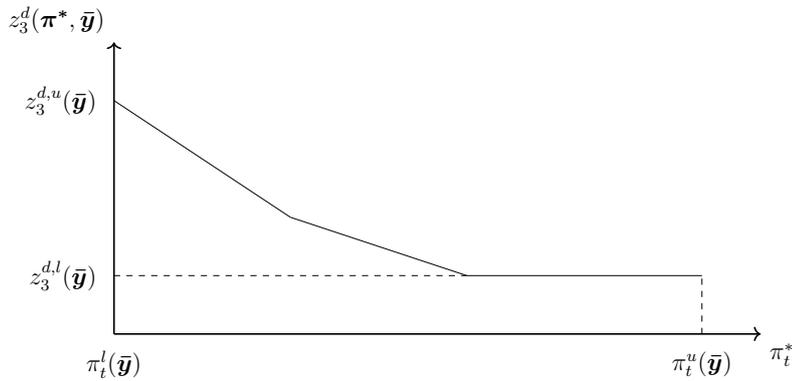


Figure 5.10: Illustration of $z_3^d(\boldsymbol{\pi}^*, \bar{\mathbf{y}})$ with respect to π_t^*

Similarly, we show that $z_3^d(\boldsymbol{\pi}^*, \bar{\boldsymbol{y}})$ is a nonincreasing piecewise linear convex function of π_t^* . In Figure 5.10, $z_3^{d,l}(\bar{\boldsymbol{y}})$ and $z_3^{d,u}(\bar{\boldsymbol{y}})$ represent lower and upper bounds on $z_3^d(\boldsymbol{\pi}^*, \bar{\boldsymbol{y}})$ with respect to $\pi_t^* \in \Pi_t^*(\bar{\boldsymbol{y}})$, respectively. $z_3^d(\boldsymbol{\pi}^*, \bar{\boldsymbol{y}})$ is a nonincreasing piecewise linear convex function of π_t^* since the total number of demand PRBs and the associated total absolute quantity are nonincreasing in π_t^* .

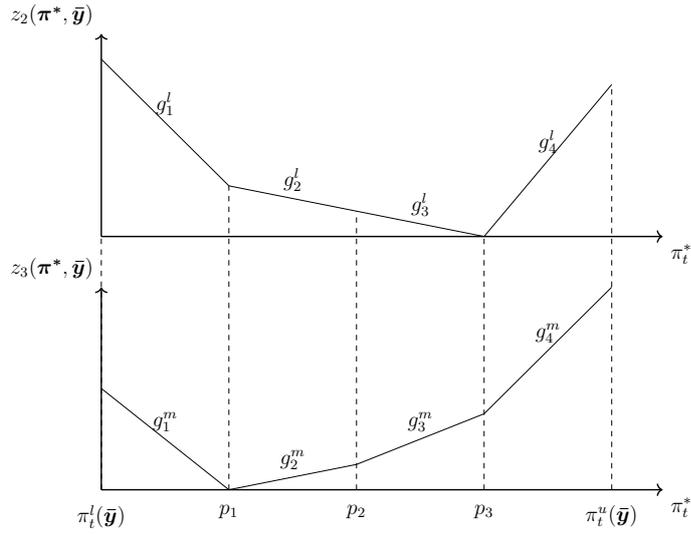
Since both $z_3^s(\boldsymbol{\pi}^*, \bar{\boldsymbol{y}})$ and $z_3^d(\boldsymbol{\pi}^*, \bar{\boldsymbol{y}})$ are convex functions of π_t^* , $z_3(\boldsymbol{\pi}^*, \bar{\boldsymbol{y}}) = z_3^s(\boldsymbol{\pi}^*, \bar{\boldsymbol{y}}) + z_3^d(\boldsymbol{\pi}^*, \bar{\boldsymbol{y}})$ is also a convex function of π_t^* .

Based on the above characterizations of $z_2(\boldsymbol{\pi}^*, \bar{\boldsymbol{y}})$ and $z_3(\boldsymbol{\pi}^*, \bar{\boldsymbol{y}})$ with respect to π_t^* for $t \in T$ where alternative market clearing prices exist, the trade-offs between them could provide valuable insights. Assume $z_2(\boldsymbol{\pi}^*, \bar{\boldsymbol{y}})$ and $z_3(\boldsymbol{\pi}^*, \bar{\boldsymbol{y}})$ are as shown in Figure 5.11a. The derivatives of both functions, g_r^l, g_r^m in intervals $r = 1, \dots, 4$ respectively, are specified on the corresponding line segments. For this example, there exist market clearing prices where both functions have the derivatives with the same sign (for $r = 1$ and 4) as well as with opposite signs (for $r = 2$ and 3). The corresponding values of $z_2(\boldsymbol{\pi}^*, \bar{\boldsymbol{y}})$ and $z_3(\boldsymbol{\pi}^*, \bar{\boldsymbol{y}})$ in $z_2 - z_3$ plane are shown in Figure 5.11b.

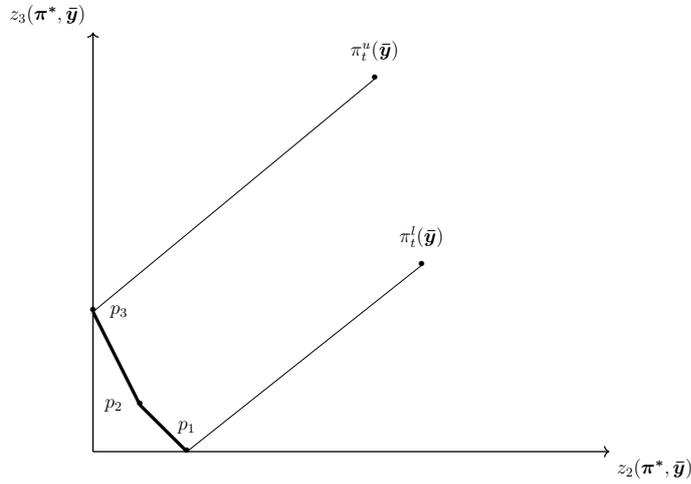
Given that alternative surplus-maximizing market clearing prices exist for a single period in slice problem **(MOMCP($\bar{\boldsymbol{y}}$))**, the dark edges in Figure 5.11b show $\mathcal{Z}_{ND}(\bar{\boldsymbol{y}})$. The market clearing prices in intervals $r = 2$ and 3 are efficient for slice problem **(MOMCP($\bar{\boldsymbol{y}}$))**.

The nondominated frontier of problem **(MOMCP($\bar{\boldsymbol{y}}$))** exhibits increasing marginal cost as we need to sacrifice more from one objective to further decrease another. The rate of decrease in one of the functions decreases as the rate of increase in the other function increases due to the convexity of market loss and missed surplus as a function of market clearing prices.

In Figure 5.12, we display the nondominated set of **(MOMCP)** for the bid set given in Example 3. In this problem, we assume there is a single period and there are alternative surplus-maximizing market clearing prices for all feasible block bid decision vectors except for s_5 . However, efficient trade-offs between the market loss, $l(\boldsymbol{\pi}^*)$, and the market missed surplus, $m(\boldsymbol{\pi}^*)$, exist only for $\bar{\boldsymbol{y}} = (1, 1, 0)$ associated with



(a) $z_2(\pi^*, \bar{y})$ and $z_3(\pi^*, \bar{y})$



(b) Trade-offs

Figure 5.11: Trade-offs between $z_2(\pi^*, \bar{y})$ and $z_3(\pi^*, \bar{y})$ as π_t^* changes

solution s_1 . Hence, \mathcal{Z}_{ND} is discrete except for the region corresponding to s_1 . We illustrate $\mathcal{Z}_{ND}(\bar{y})$ in the $z_2 - z_3$ plane for each $\bar{y} \in \mathcal{Y}$ (solid points and dark edges are the nondominated points and edges of the slice problems, respectively). The nondominated sets of the slice problems corresponding to s_2 , s_3 , and s_5 are dominated by point $(1430, 0, 400)$ in solution s_1 . The nondominated set consists of an edge from $\mathcal{Z}_{ND}((1, 1, 0))$ and the single nondominated point of $\mathcal{Z}_{ND}((0, 0, 1))$.

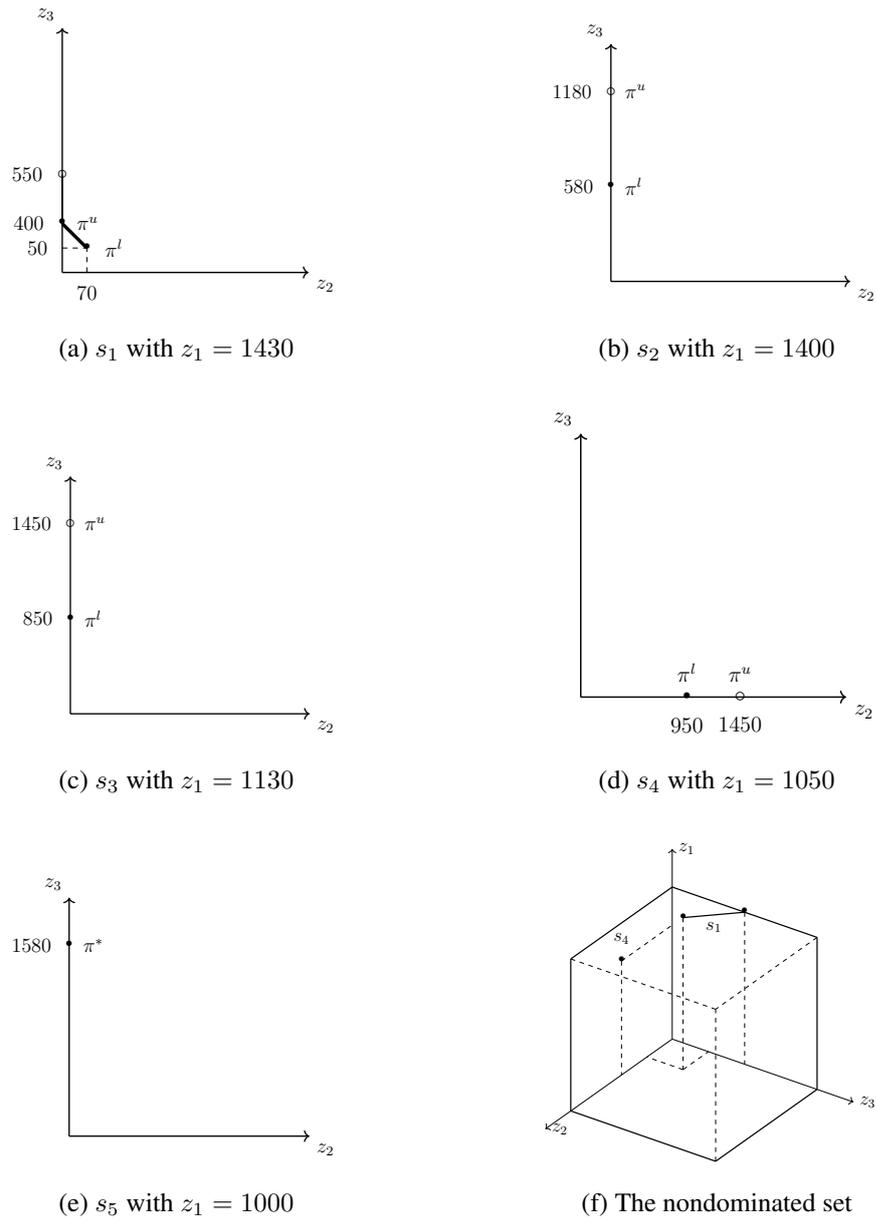


Figure 5.12: The nondominated set and the nondominated sets of slice problems for Example 3

We next characterize the efficient market clearing prices and the nondominated edges for a slice problem. We drop \bar{y} from the notation in the remaining of this section since we present the following results in the context of a slice problem.

5.4.2.1 Case 1. A single period with alternative surplus-maximizing market clearing prices

Let $t^a \in T$ be the single period where there exist alternative surplus-maximizing market clearing prices. Let $B_{pab}(\Pi^*, t^a)$ be the set of block bids that span period t^a and become a PAB at one of the market clearing price vectors, $B_{pab}(\Pi^*, t^a) = \{b \in B : t^a \in T_b, l_b(\boldsymbol{\pi}^*) > 0, \boldsymbol{\pi}^* \in \Pi^*\}$. Then, the set of possible change points (the points defining the intervals in a piecewise linear function) for the loss function is $P^l(\Pi^*, t^a) = \{\pi_{t^a}^* \in \Pi_{t^a}^* : l_b(\boldsymbol{\pi}^*) = 0, b \in B_{pab}(\Pi^*, t^a)\}$. In a similar manner, we define $B_{prb}(\Pi^*, t^a) = \{b \in B : t^a \in T_b, m_b(\boldsymbol{\pi}^*) > 0, \boldsymbol{\pi}^* \in \Pi^*\}$, and $P^m(\Pi^*, t^a) = \{\pi_{t^a}^* \in \Pi_{t^a}^* : m_b(\boldsymbol{\pi}^*) = 0, b \in B_{prb}(\Pi^*, t^a)\}$.

We next consider the following set of prices: $P(\Pi^*, t^a) = P^l(\Pi^*, t^a) \cup P^m(\Pi^*, t^a) \cup \{\pi_{t^a}^l, \pi_{t^a}^u\}$. Assuming n such prices, we order the prices from the lowest to the highest such that $p_{(i)} \in P(\Pi^*, t^a)$ denotes the i^{th} lowest price, $i = 1, 2, \dots, n$, $p_{(1)} = \pi_{t^a}^l$ and $p_{(n)} = \pi_{t^a}^u$. Note that $n \geq 2$ by the assumption of alternative market clearing prices.

Let g_i^{l,t^a} be the derivative of market loss with respect to the market clearing price in period t^a and interval $i = 1, 2, \dots, n-1$, $g_i^{l,t^a} = z'_2(p_{(i)} + \varepsilon)$, where $\varepsilon > 0$ is a small constant. Similarly, let g_i^{m,t^a} be the derivative of market missed surplus with respect to the market clearing price in interval $i = 1, 2, \dots, n-1$, $g_i^{m,t^a} = z'_3(p_{(i)} + \varepsilon)$. We can calculate g_i^{l,t^a} and g_i^{m,t^a} as follows:

$$g_i^{l,t^a} = \sum_{b \in B_{pab}(\Pi^*, t^a, i)} q_{b,t^a},$$

$$g_i^{m,t^a} = \sum_{b \in B_{prb}(\Pi^*, t^a, i)} q_{b,t^a},$$

where $B_{pab}(\Pi^*, t^a, i) = \{b \in B_{pab}(\Pi^*, t^a) : t^a \in T_b, l_b(\boldsymbol{\pi}^*) > 0, \pi_{t^a}^* \in [p_{(i)}, p_{(i+1)}]\}$ and $B_{prb}(\Pi^*, t^a, i) = \{b \in B_{prb}(\Pi^*, t^a) : t^a \in T_b, m_b(\boldsymbol{\pi}^*) > 0, \pi_{t^a}^* \in [p_{(i)}, p_{(i+1)}]\}$.

We next define $\pi_{t^a}^{e,l}$ and $\pi_{t^a}^{e,u}$ that denote the minimum and the maximum market clearing price for period t^a , respectively, in an efficient solution of the slice problem:

$$\pi_{t^a}^{e,l} = \min_{i=1,2,\dots,n-1} \left\{ p_{(i)} \in P(\Pi^*, t^a) : g_i^{l,t^a} g_i^{m,t^a} < 0 \vee g_i^{l,t^a} = g_i^{m,t^a} = 0 \right\}$$

$$\pi_{t^a}^{e,u} = \max_{i=2,\dots,n} \left\{ p_{(i)} \in P(\Pi^*, t^a) : g_{i-1}^{l,t^a} g_{i-1}^{m,t^a} < 0 \vee g_{i-1}^{l,t^a} = g_{i-1}^{m,t^a} = 0 \right\}$$

Proposition 23. *Given that alternative market clearing prices exist only for a single period $t^a \in T$, an efficient solution for the slice problem must have a market clearing price $\pi^e \in \Pi^*$ such that $\pi_{t^a}^e \in [\pi_{t^a}^{e,l}, \pi_{t^a}^{e,u}]$.*

Proof. Let $\psi^k = (\mathbf{x}^k, \mathbf{y}^k, \boldsymbol{\pi}^k, \mathbf{s}^k, \mathbf{l}^k, \mathbf{m}^k)$ be an efficient solution of **(MOMCP)(\mathbf{y}^k)** and $\Pi_{t^a}^*$ be the set of surplus-maximizing market clearing prices for the slice problem in period t^a , $\boldsymbol{\pi}^k \in \Pi^*$, with a non-empty interior. Suppose that $\pi_{t^a}^k \notin [\pi_{t^a}^{e,l}, \pi_{t^a}^{e,u}]$. Let $\pi_{t^a}^k \in (p_{(\bar{i})}, p_{(\bar{i}+1)})$ for $p_{(\bar{i})}, p_{(\bar{i}+1)} \in P(\Pi^*, t^a)$. Then, $g_{\bar{i}}^{l,t^a} g_{\bar{i}}^{m,t^a} \geq 0$ with at least one of the terms being non-zero by definition of $[\pi_{t^a}^{e,l}, \pi_{t^a}^{e,u}]$.

Case 1: $g_{\bar{i}}^{l,t^a}, g_{\bar{i}}^{m,t^a} \geq 0$. For a sufficiently small $\varepsilon > 0$, let $\boldsymbol{\pi}^j \in [p_{(\bar{i})}, p_{(\bar{i}+1)}]$ such that $\pi_t^j = \pi_t^k - \varepsilon$ for $t = t^a$ and $\pi_t^j = \pi_t^k$ for $t \neq t^a$. Then, $z_2(\boldsymbol{\pi}^j, \mathbf{y}^k) \leq z_2(\boldsymbol{\pi}^k, \mathbf{y}^k)$ and $z_3(\boldsymbol{\pi}^j, \mathbf{y}^k) \leq z_3(\boldsymbol{\pi}^k, \mathbf{y}^k)$, where at least one of the inequalities is strict. Since, $z_1(\boldsymbol{\pi}^j, \mathbf{y}^k) = z_1(\boldsymbol{\pi}^k, \mathbf{y}^k)$, ψ^j dominates ψ^k and ψ^k cannot be an efficient solution.

Case 2: $g_{\bar{i}}^{l,t^a}, g_{\bar{i}}^{m,t^a} \leq 0$. For a sufficiently small $\varepsilon > 0$, let $\boldsymbol{\pi}^j \in [p_{(\bar{i})}, p_{(\bar{i}+1)}]$ such that $\pi_t^j = \pi_t^k + \varepsilon$ for $t = t^a$ and $\pi_t^j = \pi_t^k$ for $t \neq t^a$. Then, $z_2(\boldsymbol{\pi}^j, \mathbf{y}^k) \leq z_2(\boldsymbol{\pi}^k, \mathbf{y}^k)$ and $z_3(\boldsymbol{\pi}^j, \mathbf{y}^k) \leq z_3(\boldsymbol{\pi}^k, \mathbf{y}^k)$, where at least one of the inequalities is strict. Since, $z_1(\boldsymbol{\pi}^j, \mathbf{y}^k) = z_1(\boldsymbol{\pi}^k, \mathbf{y}^k)$, ψ^j dominates ψ^k and ψ^k cannot be an efficient solution.

Since Case 1 and Case 2 cannot hold for an efficient solution ψ^k , $\pi_{t^a}^k \in [\pi_{t^a}^{e,l}, \pi_{t^a}^{e,u}]$.

□

As we show above, the market clearing price for period t^a lies in the price range $[\pi_{t^a}^{e,l}, \pi_{t^a}^{e,u}]$ in all efficient solutions. We call $\boldsymbol{\pi}^e \in \Pi^*$ as an *efficient price vector* for the considered slice problem, if $\pi_{t^a}^e \in [\pi_{t^a}^{e,l}, \pi_{t^a}^{e,u}]$, and $[\pi_{t^a}^{e,l}, \pi_{t^a}^{e,u}]$ as the *efficient price range* for period $t^a \in T$.

Let $\mathbf{z} = (z_2(\boldsymbol{\pi}^e), z_3(\boldsymbol{\pi}^e))$ be the corresponding nondominated point for efficient price $\boldsymbol{\pi}^e$, $\pi_{t^a}^e \in [\pi_{t^a}^{e,l}, \pi_{t^a}^{e,u}]$, where $z_2(\boldsymbol{\pi}^e)$ and $z_3(\boldsymbol{\pi}^e)$ are the associated market loss and missed surplus, respectively. Let $\boldsymbol{\pi}^{e,l} = \boldsymbol{\pi}^e \in \Pi^* : \pi_{t^a}^e = \pi_{t^a}^{e,l}$ and $\boldsymbol{\pi}^{e,u} = \boldsymbol{\pi}^e \in \Pi^* : \pi_{t^a}^e = \pi_{t^a}^{e,u}$. Representing market loss and missed surplus on abscissa and ordinate, respectively, $\mathbf{z}^l = (z_2(\boldsymbol{\pi}^{e,l}), z_3(\boldsymbol{\pi}^{e,l}))$ and $\mathbf{z}^u = (z_2(\boldsymbol{\pi}^{e,u}), z_3(\boldsymbol{\pi}^{e,u}))$ are the north-west (south-east) and south-east (north-west) extreme nondominated points, respectively, if $z_2(\boldsymbol{\pi}^{e,l}) < z_2(\boldsymbol{\pi}^{e,u})$ ($z_2(\boldsymbol{\pi}^{e,l}) > z_2(\boldsymbol{\pi}^{e,u})$). Additionally, we define $P_{t^a}^e =$

$P(\Pi^*, t^a) \cap (\pi_{t^a}^{e,l}, \pi_{t^a}^{e,u})$ as the set of prices in the efficient price range at which derivatives of market loss or missed surplus functions change. Assuming m such prices, let $P_{t^a}^e = \{p_{(j_1)}, p_{(j_2)}, \dots, p_{(j_m)}\}$. We next show that $\mathbf{z}^k = (z_2(\boldsymbol{\pi}), z_3(\boldsymbol{\pi}))$, $\pi_{t^a}^k = p_{(j_k)}$, $k = 1, 2, \dots, m$ are extreme nondominated points.

Proposition 24. $\mathcal{Z}_{END} = \{\mathbf{z}^k, \forall p_{(j_k)} \in P_{t^a}^e\} \cup \{\mathbf{z}^l, \mathbf{z}^u\}$ is the set of extreme nondominated points for the slice problem.

Proof. Note that \mathbf{z}^l and \mathbf{z}^m are defined by the lower and upper limits of the efficient price range, respectively, and they are extreme nondominated points. Let $\mathbf{g}^k = (|g_k^{m,t^a}|, |g_k^{l,t^a}|)$ be the absolute gradient vector for the k^{th} interval in the efficient price range, $k = 1, 2, \dots, m + 1$. Consider weight vector $\mathbf{w}^k = (\mathbf{g}^k + \mathbf{g}^{k+1})$ for $k = 1, 2, \dots, m$. Then, $p_{j_k} = \operatorname{argmin}_{\boldsymbol{\pi} \in \Pi^*: \pi_{t^a} \in [\pi_{t^a}^{e,l}, \pi_{t^a}^{e,u}]} \{\mathbf{w}^k \mathbf{z}, \mathbf{z} = z(\boldsymbol{\pi})\}$ is the unique optimal solution and, hence, \mathbf{z}^k is an extreme nondominated point for $k = 1, 2, \dots, m$.

Let $\Pi_{t^a}^1 = (\pi_{t^a}^{l,e}, p_{(j_1)})$, $\Pi_{t^a}^{m+1} = (p_{(j_m)}, \pi_{t^a}^{u,e})$, and $\Pi_{t^a}^k = (p_{(j_k)}, p_{(j_{k+1})})$ for $k = 1, 2, \dots, m - 1$. Consider weight vector $\mathbf{w}^k = \mathbf{g}^k$ for $k = 1, 2, \dots, m + 1$. Then, $\Pi_{t^a}^k = \operatorname{argmin}_{\boldsymbol{\pi} \in \Pi^*: \pi_{t^a} \in [\pi_{t^a}^{e,l}, \pi_{t^a}^{e,u}]} \{\mathbf{w}^k \mathbf{z}, \mathbf{z} = z(\boldsymbol{\pi})\}$ is the set of alternative optimal market clearing prices and, hence, the corresponding nondominated points are not extreme nondominated points. \square

5.4.2.2 Case 2. Multiple periods with alternative surplus-maximizing market clearing prices

In the previous section, we examined the case where alternative market clearing prices exist in a single period. We showed that efficient market clearing prices exist in a closed interval where either the market loss and missed surplus functions change in opposite directions or they both stay the same. In this section, we consider the case where alternative market clearing prices exist in multiple periods.

Let $[\pi_t^l, \pi_t^u] \in \mathbb{R}$ be the set of surplus-maximizing market clearing prices in period $t \in T$ in a slice problem. The set of alternative market clearing price vectors, Π^* , is a hyper-rectangle in $\mathbb{R}^{|T|}$ such that $\pi_t \in [\pi_t^l, \pi_t^u]$ for $t \in T$. We consider the following bi-objective market clearing price problem, **BOMCP**:

(BOMCP):

$$\text{Min } z(\boldsymbol{\pi}) = (z_2(\boldsymbol{\pi}), z_3(\boldsymbol{\pi}))$$

s.to.

$$m_b \geq \sum_{t \in T} (p_b - \pi_t) q_{b,t} \quad \forall b \in B_0$$

$$l_b \geq \sum_{t \in T} (\pi_t - p_b) q_{b,t} \quad \forall b \in B_1$$

$$\pi_t - \pi_t^l \geq 0 \quad \forall t \in T$$

$$\pi_t - \pi_t^u \leq 0 \quad \forall t \in T$$

$$m_b \geq 0 \quad \forall b \in B_0$$

$$l_b \geq 0 \quad \forall b \in B_1$$

(BOMCP) is a bi-objective linear program. We employ dichotomic search (DS) algorithm (Aneja and Nair, 1979) to find the set of extreme supported nondominated points of **(BOMCP)**. Let abscissa and ordinate represent $z_2(\boldsymbol{\pi})$ and $z_3(\boldsymbol{\pi})$, respectively. We first generate the north-west (\boldsymbol{z}^{NW}) and south-east (\boldsymbol{z}^{SE}) extreme nondominated points. To generate the north-west extreme nondominated point, we first minimize $z_2(\boldsymbol{\pi})$, then minimize $z_3(\boldsymbol{\pi})$ by restricting $z_2(\boldsymbol{\pi}) \leq z_2^*(\boldsymbol{\pi})$. For the south-east extreme nondominated point, we first minimize $z_3(\boldsymbol{\pi})$, then minimize $z_2(\boldsymbol{\pi})$ by restricting $z_3(\boldsymbol{\pi}) \leq z_3^*(\boldsymbol{\pi})$.

If $\boldsymbol{z}^{NW} = \boldsymbol{z}^{SE}$, then there is a single nondominated point. Otherwise, let $\tilde{\mathcal{Z}}_{END}$ be the set of extreme nondominated points generated so far and $\boldsymbol{z}^{(i)} \in \tilde{\mathcal{Z}}_{END}$ be the extreme nondominated point with the i^{th} -best value in z_2 among the generated points. Initially, $\boldsymbol{z}^{(1)} = \boldsymbol{z}^{NW}$ and $\boldsymbol{z}^{(2)} = \boldsymbol{z}^{SE}$. To check if there exist $\boldsymbol{z}^j \in \mathcal{Z}_{END}$ such that $z_2^{(1)} < z_2^j < z_2^{(2)}$, we minimize a weighted-sum objective function $z(\boldsymbol{\pi}) = w_2 z_2(\boldsymbol{\pi}) + w_3 z_3(\boldsymbol{\pi})$, where $w_2 = z_3^{(1)} - z_3^{(2)}$ and $w_3 = z_2^{(2)} - z_2^{(1)}$. We denote the corresponding problem as **(W-BOMCP)**. If such \boldsymbol{z}^j is the nondominated point associated with the optimal solution of **(W-BOMCP)**, then $\boldsymbol{z}^{(2)} = \boldsymbol{z}^j$ and $\boldsymbol{z}^{(3)} = \boldsymbol{z}^{(2)}$.

Let \mathcal{L} be the set of nondominated point pairs to consider for **(W-BOMCP)**. The following DS algorithm generates \mathcal{Z}_{END} . Then, the set of nondominated edges, \mathcal{Z}_{NE} , are defined by each pair of adjacent extreme nondominated points. If there are n extreme nondominated points, $n = |\mathcal{Z}_{END}|$, then $\mathcal{Z}_{NE} = \{[\boldsymbol{z}^{(i)}, \boldsymbol{z}^{(i+1)}], i = 1, 2, \dots, n - 1\}$.

Algorithm 3 Dichotomic search (DS) algorithm (Aneja and Nair, 1979)

$$\tilde{\mathcal{Z}}_{END} = \{z^{(1)}, z^{(2)}\}, \mathcal{L} = \{(z^{(1)}, z^{(2)})\}$$

while $\mathcal{L} \neq \emptyset$ **do**

 Select $(z^{(i)}, z^{(k)}) \in \mathcal{L}, \mathcal{L} = \mathcal{L} \setminus \{(z^{(i)}, z^{(k)})\}$

 Solve **(W-BOMCP)** with $w_2 = z_3^{(i)} - z_3^{(k)}, w_3 = z_2^{(k)} - z_2^{(i)}$. Let z^* be the generated point.

if $z_2^{(i)} < z_2^* < z_2^{(k)}$ **then**

$k = k + 1$ and $k' = k' + 1$ for $z^{k'} \in \tilde{\mathcal{Z}}_{NDE}, k' > k$

$z^j = z^*, j = i + 1$

$\mathcal{L} = \mathcal{L} \cup \{(z^{(i)}, z^{(j)}), (z^{(j)}, z^{(k)})\}$

$\tilde{\mathcal{Z}}_{END} = \tilde{\mathcal{Z}}_{END} \cup \{z^{(j)}\}$

end if

end while

5.4.3 Generating the nondominated set

In the previous section, we present methods to generate the nondominated set of **(MOMCP)(\bar{y})**, $\mathcal{Z}_{ND}(\bar{y})$, for some $\bar{y} \in \mathcal{Y}$. To generate \mathcal{Z}_{ND} , we employ the *cone-based search algorithm (CBSA)* that we develop in Chapter 4. CBSA selects the market surplus objective as the primary objective since the market surplus is constant whereas market loss and missed surplus objectives may exhibit continuous tradeoffs when there are alternative surplus-maximizing market clearing prices for a block bid decision vector. CBSA iteratively finds the nondominated points or edges in non-increasing order of their market surplus. CBSA maintains a set of polyhedral search regions that are mutually exclusive and collectively exhaustive of the feasible objective space that may contain further nondominated points or edges.

Over all feasible search regions, CBSA selects the point with the maximum surplus value and solves the slice problem for the corresponding block bid decision vector of the point. If there are ties, then the market loss and missed surplus objectives are used as the first and the second tie-breakers, respectively, where the smaller is the better. The slice problem is solved for that block bid decision vector in the corresponding search region. If the nondominated set of the slice problem is a single point, then it is added to the nondominated set provided that it is not on the boundary between the

search region and the dominated region.

If the nondominated set of the slice problem includes edges, then the edges are weakly nondominated and stored in a separate list. If the maximum market surplus value in the next iteration is less than the current market surplus, then the edges are nondominated. Else, the edges are checked for dominance by the rules and the model presented in Section 4.2. The dominated edges or the dominated segments of the edges are eliminated and the rest is added to the nondominated set. Lastly, the regions dominated by the nondominated set of the slice problem are separated from the search space by means of convex cones defined by the extreme points of the nondominated set of the slice problem solved.

5.5 Computational study

In this section, we present the results of our computational experiments regarding the generation of the nondominated sets for instances representative of the Turkish DAM. We use the same 20 instances in Chapter 3 and conduct the following experiments:

1. *Investigate the alternative surplus maximizing market clearing prices:* For each instance, we generate a set of feasible block bid decision vectors and find the minimum and maximum surplus maximizing market clearing price (to reveal if it is possible to have multiple efficient solutions) for each period of the problem and for each block bid decision vector.
2. *Generate the nondominated set of (MOMCP):* Employing CBSA, we generate the nondominated set for each instance and report statistics summarizing the characteristics of the nondominated sets.
3. *Analyze the impact of the bid structure on the nondominated sets:* We modify the bid sets of the instances to analyze the associated changes in the characteristics of the nondominated sets. Specifically, we eliminate hourly bids and employ only block bids in order to capture cases with rich occurrences of nondominated edges.

We implement the tests with Python 3.7, employing Gurobi 9 as the single-objective

mathematical programming solver for the CBSA algorithm. We conduct the tests on a workstation with Windows Server 2012 R2 operating system, Intel(R) Xeon(R) E5 1.90 GHz CPU and 32 GB RAM.

In our first set of experiments, we find the block bid decision vectors associated with the best n solutions of **(SMILP)**, $\bar{\mathbf{y}}_{(1)}, \dots, \bar{\mathbf{y}}_{(n)}$, and solve **(P($\bar{\mathbf{y}}, t$))** twice to find the maximum and minimum market clearing prices for each $t \in T$ and $\bar{\mathbf{y}}_{(j)}, j = 1, \dots, n$. We calculate the number of solutions where the maximum market clearing price is larger than the minimum for at least one period. We also calculate the maximum differences for such periods.

For $n = 100$, there exist alternative market clearing prices in 6 of the 20 instances (30%). Among those 6 instances, there are 26 block bid decision vectors, on average, that lead to alternative market clearing price vectors. We observe alternative market clearing price ranges of up to 7.00 Turkish liras and 3.20 Turkish liras, on average.

We generate the nondominated set of **(MOMCP)** for each instance employing CBSA on the 20 instances. In CBSA, we disable the search for alternative efficient integer vectors. We set $\alpha = 10^{-5}$ and $\beta = 10^{-6}$ that control the error in the representation of the true nondominated points and edges by the generated ones by CBSA, respectively. Additionally, we use a MIP relative gap of 10^{-6} and 600 seconds time limit for the single objective optimization runs conducted by CBSA.

In Table 5.4, we present the results for the generated nondominated sets and the associated computational efforts spent by CBSA on each instance. We display the results in the following order: number of iterations conducted by CBSA, where each iteration results with a nondominated point or weakly nondominated edges, the number of nondominated points and edges (if any), the number of efficient integer vectors (block bid decision vectors), the total number of models solved, and the total run time in seconds spent by CBSA.

Based on the results in Table 5.4, the number of nondominated points for **(MOMCP)** is scarce. In addition, there are no nondominated edges within the tolerances set for CBSA. On average, CBSA solves about 4 models in 30 mins to generate 2.65 nondominated points. When we associate the number of nondominated points with

the properties of the surplus maximizing solutions reported in Table 5.2, we observe that the instances with fewer number of points correspond to those where the surplus maximizing solution has small market loss and missed surplus. In Instance 16, there is a single nondominated point that is the ideal point.

Table 5.4: Results for the nondominated set of (**MOMCP**) with the original instances

Instance	# of Iterations	Nondominated Set ^a		CBSA ^b	
		# of Points	$ \mathcal{Y}_E $	# of MILPs ^b	Run Time (secs)
1	7	7	7	11	506.73
2	2	2	2	3	801.69
3	2	2	2	3	1814.85
4	3	3	3	4	3293.12
5	5	5	5	8	3950.64
6	3	3	3	4	2063.93
7	2	2	2	3	2409.22
8	2	2	2	3	583.83
9	2	2	2	3	28.33
10	3	3	3	4	2744.27
11	4	4	4	7	8431.01
12	4	4	4	6	1830.09
13	1	1	1	1	10.16
14	1	1	1	1	992.41
15	2	2	2	3	288.18
16	1	1	1	1	3.24
17	3	3	3	4	1436.75
18	3	3	3	5	3611.4
19	1	1	1	1	6.72
20	2	2	2	3	811.05
avg	2.65	2.65	2.65	3.9	1780.88
max	7	7	7	11	8431.01

^[a] The generated nondominated sets do not include any edges.

^[b] Represents the computational effort spent by CBSA while generating the nondominated set.

^[c] An MILP corresponds to solving lexicographically with three objectives.

In Figure 5.13, we present the nondominated sets of the four instances for which the

highest number of nondominated points are generated. On the abscissa, we display the *surplus gaps* of the points corresponding to the difference between the market surplus of the considered nondominated point and the maximum market surplus. On ordinate, we plot the corresponding missed surplus and the market loss values (in negative amounts) in the upper and lower figures, respectively. The label below each point indicates the order of the corresponding point with respect to its surplus gap.

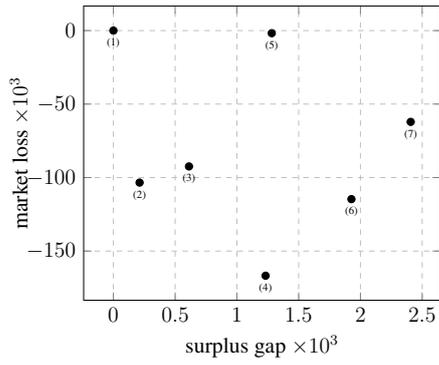
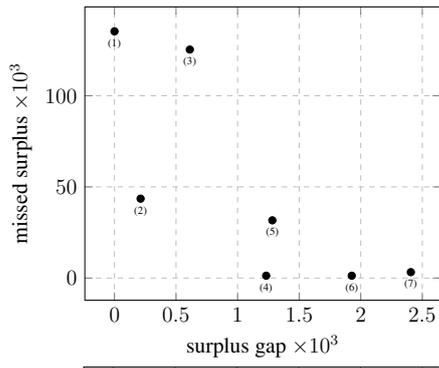
Based on the instances displayed in Figure 5.13, we analyze the tradeoffs with respect to the surplus maximizing point, Point (1). In Instance 1, the surplus maximizing point has a large missed surplus. It can be improved substantially with a small increase in surplus gap and without worsening market loss if Point (5) is selected. The surplus maximizing point has a high market loss in Instance 5. It can be improved substantially with a small increase in surplus gap if Point (3) is selected. In this case, the missed surplus improves substantially. In Instance 11, it is also possible to improve missed surplus by a large amount with a small increase in surplus gap, without worsening market loss (Point (4)). Differently from the previous instances, improving market loss costs a substantial increase in missed surplus in Instance 12.

In Table 5.5, we investigate the tradeoffs between the surplus maximizing solution and the other nondominated points generated by CBSA. We first report the characteristics of the surplus maximizing solutions in terms of market loss and missed surplus. We consider the values smaller than 10^4 Turkish liras as *small*, the values between 10^4 and 10^5 as *moderate*, and the values larger than 10^5 as *large*. Then, we describe the typical tradeoffs observed in the three criteria with respect to the surplus maximizing solution. Single, double, and triple up (down) arrows represent small, moderate, and large increases (decreases) for the associated criterion, respectively. \leftrightarrow represents no change. We also check if any of the models (**SMILP-NoPAB**) and (**SMILP-NoPRB**) is able to capture such tradeoffs as well. Lastly, we categorize instances with respect to the characteristics of the tradeoffs observed.

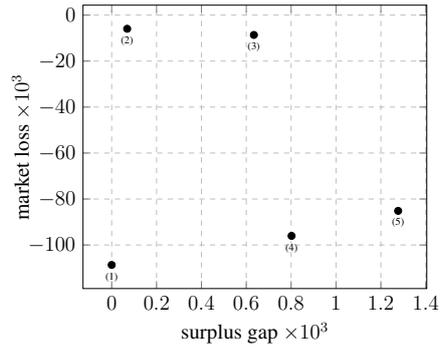
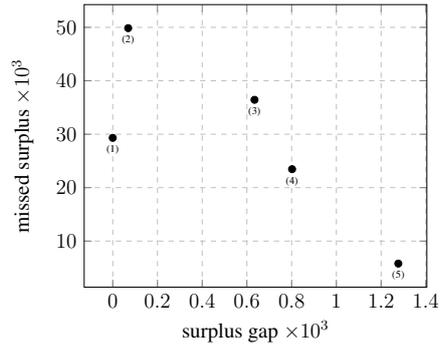
Category A represents instances where the surplus maximizing solution performs well in all criteria. Hence, there is a single best solution for the market clearing problem. In **Category B** instances, market loss or missed surplus can be improved substantially in exchange for a small decrease in surplus, without compromising from the

other criterion. Tradeoffs show increasing marginal return in market loss with respect to missed surplus in **Category C** instances. Similarly, **Category D** instances characterize the tradeoffs exhibiting increasing marginal return in missed surplus with respect to market loss, and **Category E** instances are associated with constant marginal return. Lastly, **Category F** instances characterize tradeoffs with decreasing marginal return in market loss with respect to missed surplus.

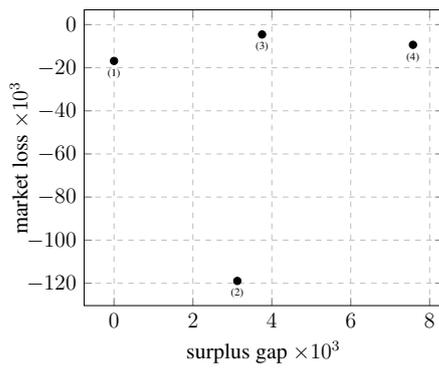
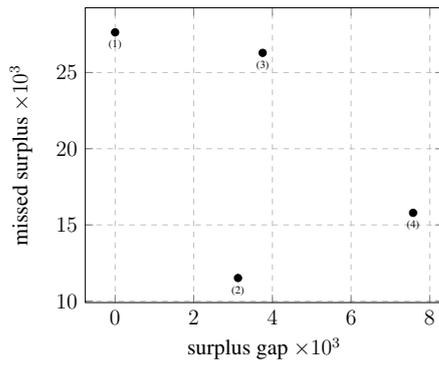
Although the ideal instance for the market clearing problem would be a Category A instance, occurrences of such cases are not common. In other situations, nondominated points with characteristics as in categories B, C, and D instances may be desirable by the MOs, especially Category B. Neither **(SMILP-NoPAB)** nor **(SMILP-NoPRB)** have been able to capture such tradeoffs in our experiments.



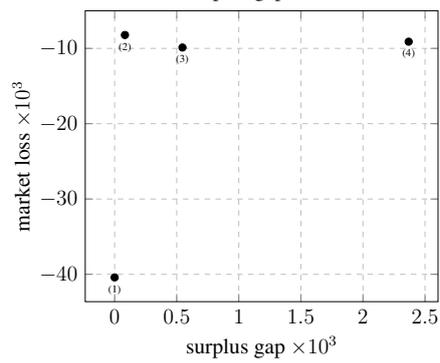
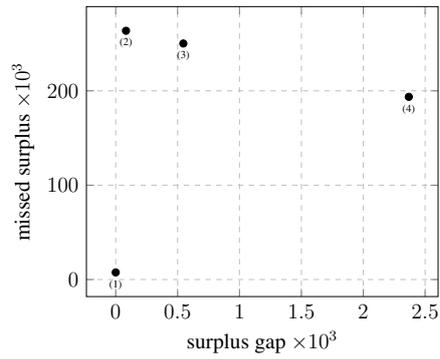
(a) Instance 1



(b) Instance 5



(c) Instance 11



(d) Instance 12

Figure 5.13: Examples of nondominated sets generated by CBSA

Table 5.5: Characteristics of the nondominated points generated for the original instances

Instance	Surplus maximizing solution ^a		Tradeoffs ^b			Current designs ^c		Category ^d
	Loss	Missed surplus	Surplus	Loss	Missed surplus	SMILP-NoPAB	SMILP-NoPRB	
1	-	Large	↓	↔	↓↓↓	x	x	B
2	Moderate	Moderate	↓	↓↓	↑	✓	x	C
3	Small	Moderate	↓	↑	↓↓	x	✓	D
4	Moderate	Moderate	↓	↓↓	↑↑	x	x	E
5	Large	Moderate	↓	↓↓↓	↓↓	x	x	B
6	Moderate	Small	↓	↓↓	↔	x	x	B
7	Moderate	Small	↓↓	↓↓	↑↑	✓	x	E
8	Moderate	Moderate	↓	↑↑	↓↓	x	✓	E
9	Small	Moderate	↓	↑	↓	x	x	E
10	Large	Moderate	↓	↓↓↓	↓↓	x	x	B
11	Moderate	Moderate	↓	↔	↓↓	x	x	B
12	Moderate	Small	↓	↓↓	↑↑↑	✓	x	F
13	Small	Small	-	-	-	✓	✓	A
14	Small	Moderate	-	-	-	x	x	A
15	Moderate	Small	↓	↓↓	↑↑	✓	x	E
16	-	-	-	-	-	✓	✓	A
17	Small	Moderate	↓	↑↑	↓↓	x	✓	E
18	-	Moderate	↓↓	↑↑	↓↓	x	x	E
19	Small	Small	-	-	-	x	✓	A
20	Small	Moderate	↓	↑↑	↓↓	x	✓	E

^[a] Null entries (-) represent zero values.

^[b] Single, double, and triple up (down) arrows represent small ($\leq 10^4$), moderate ($10^4 < . \leq 10^5$), and large ($> 10^5$) increases (decreases), respectively. ↔ represents no change. Null entries (-) imply that there is a single nondominated point.

^[c] An instance is marked with (✓) if the corresponding model is able to generate a nondominated point with the described tradeoffs, else with (x).

^[d] A: Surplus maximizing solution performs well in all criteria. B: Market loss or missed surplus can be improved substantially in exchange for a small decrease in surplus, without compromising from the other criterion. C: Tradeoffs exhibit increasing marginal return in market loss with respect to missed surplus. D: Tradeoffs exhibit increasing marginal return in missed surplus with respect to market loss. E: Tradeoffs exhibit constant marginal return in both market loss and missed surplus. F: Tradeoffs exhibit decreasing marginal return in market loss with respect to missed surplus.

In Table 5.6, we present the corresponding results when we restrict the problem to only the block bids in the instances. When there are no hourly bids, the problem becomes a pure binary program. Since the equilibrium constraints of hourly bids that restrict the market clearing prices into narrow ranges are not part of the problem in this case, the chances of having alternative market clearing prices and the range of those prices increase. This, in turn, leads to the existence of nondominated edges in such instances. In our experiments, CBSA finds 22.5 and 1.6 nondominated edges and points on average, respectively. There are 2.56 nondominated edges generated per efficient block bid decision vector, on average.

Table 5.6: Results for the nondominated set of (MOMCP) with block bids only

Instance	# of Iterations	Nondominated Set			CBSA ^a	
		# of Edges	# of Points	$ \mathcal{Y}_E $	# of MILPs ^b	Run Time (secs)
1	21	38	2	12	70	794.56
2	37	37	6	16	77	1313.87
3	1	7	0	1	2	583.69
4	24	37	2	13	65	565.44
5	9	16	2	3	27	357.56
6	3	2	1	3	6	147.03
7	49	59	3	24	123	1268.74
8	38	50	1	29	101	1833.09
9	3	10	0	3	15	1211.24
10	18	25	2	12	53	740.48
11	18	30	1	8	53	1108.72
12	17	26	3	8	50	297.58
13	11	15	1	6	28	551.28
14	7	13	1	5	22	382.35
15	8	17	0	5	28	580.44
16	7	12	1	6	24	533.77
17	7	17	2	5	24	806.62
18	3	8	0	2	11	347.42
19	6	12	2	5	20	103.55
20	15	19	2	10	37	528.73
avg	15.1	22.5	1.6	8.8	41.8	702.81
max	49	59	6	29	123	1833.09

^[a] Represents the computational effort spent by CBSA while generating the nondominated set.

^[b] An MILP corresponds to solving lexicographically with three objectives.

In Figures 5.14 and 5.15, we display the nondominated set for two of the instances, Instances 7 and 8, when we only include the block bids in the problem. We plot the projection of all nondominated edges in Instance 7 with respect to market loss and missed surplus in Figure 5.14a. Existence of long and connected edges imply existence of alternative market clearing prices in wide ranges. In Figures 5.14b and 5.14c, we plot the missed surplus and market loss with respect to the surplus gap, respectively. A vertical line in Figure 5.14b (Figure 5.14c) represents the projected missed surplus (market loss) of a nondominated edge.

In Instance 7, we observe that decreasing market loss below 0.05×10^6 Turkish liras causes a substantial increase in the missed surplus above 30×10^6 Turkish liras. A similar behavior is also observed for missed surplus as well. The marginal cost of

improving market loss (missed surplus) at its lower spectrum is very large in terms of missed surplus (market loss). On the other hand, there exist many compromise solutions around the “knee” point of the frontier (where a small sacrifice in one objective brings a big gain in the other) that can be more preferable for the MOs.

In Instance 8, the surplus maximizing solution has a relatively large market loss as we show in Figure 5.15. It can be decreased to its minimum either allowing for a substantial decrease in market surplus (as much as 0.5×10^6 Turkish liras) or settling for a substantial amount of missed surplus (as much as 30×10^6 Turkish liras). However, “good” compromise solutions also exist for Instance 8. It is possible to achieve low market loss and missed surplus values allowing for only a small decrease in market surplus.

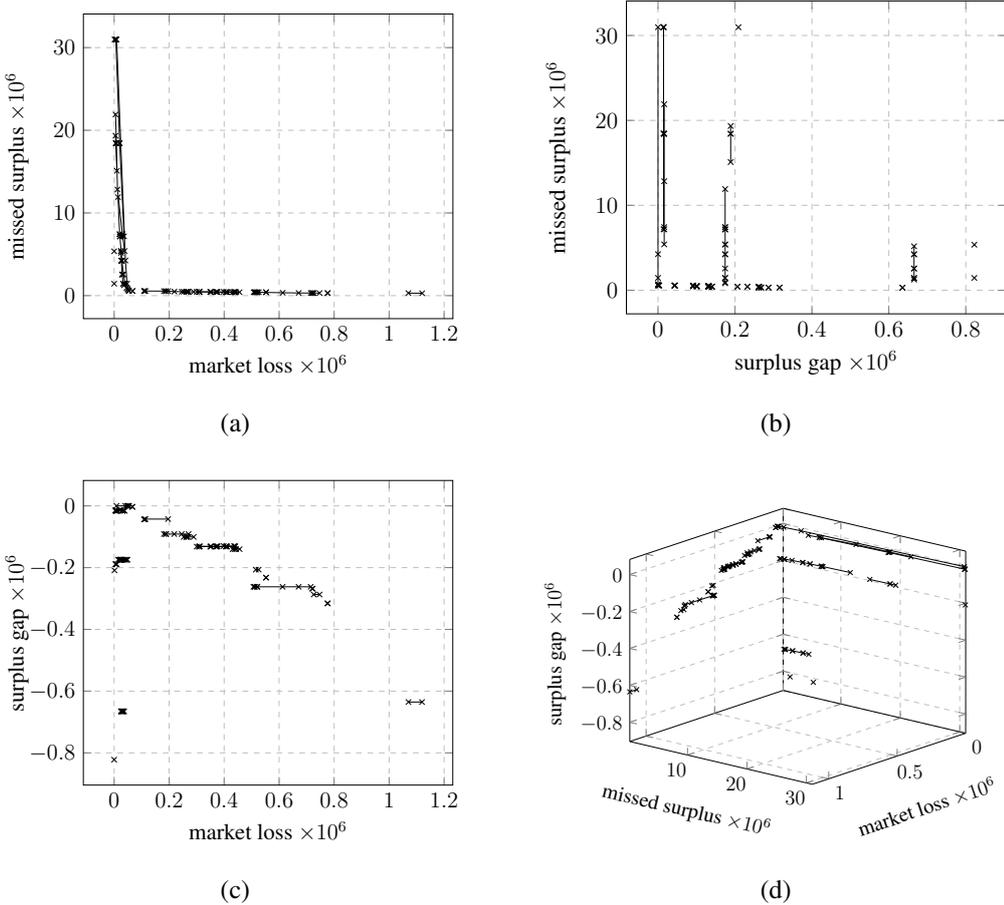


Figure 5.14: Visualization of the nondominated set for Instance 7

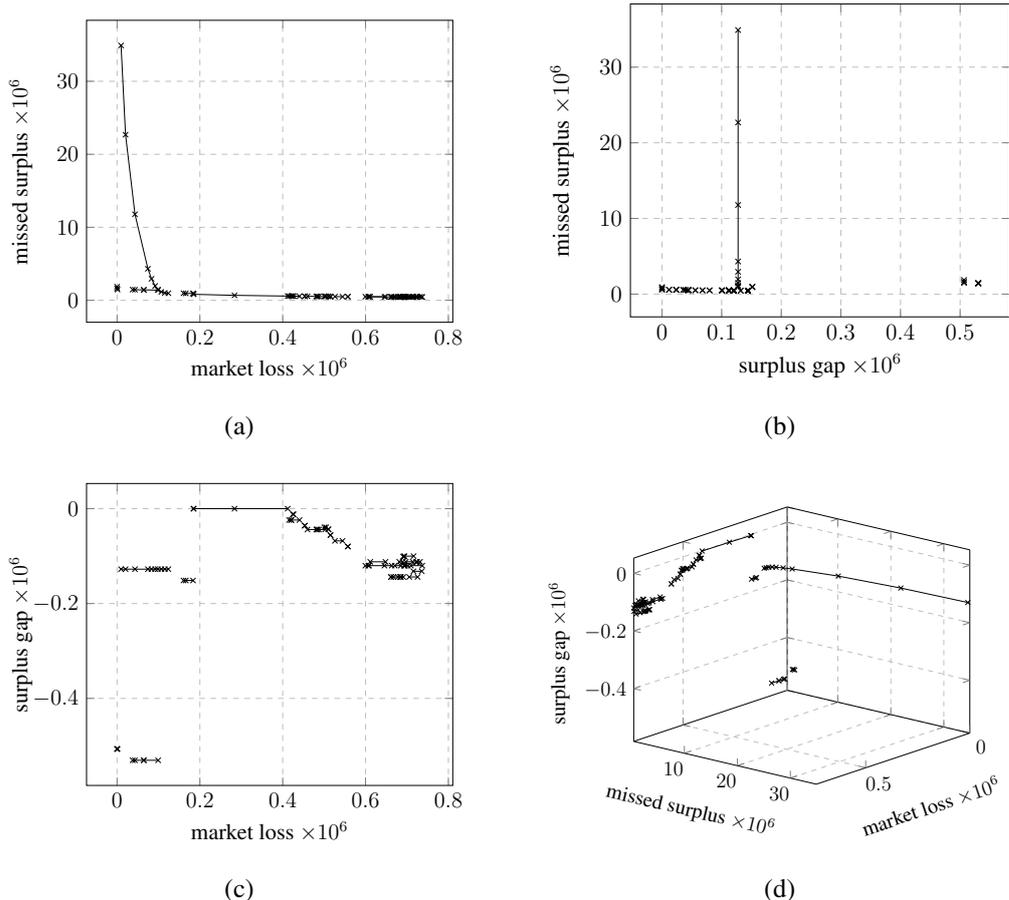


Figure 5.15: Visualization of the nondominated set for Instance 8

5.6 Application of the findings in practice

In the previous section, we generated the nondominated sets of the multi-objective DAM clearing problems and analyzed the properties of those sets. In a practical setting, the procedure to obtain a solution for the market clearing problem and the properties of that solution have to be determined apriori and made public for market participants. In a multi-objective problem setting, the MO needs to define a set of conditions that needs to be satisfied by the selected nondominated point and the efficient solution to be announced as the auction result. In addition, the conditions should be strict enough to adequately represent the properties of the announced solutions to the market.

The problem is a difficult one to solve even for a single objective when there are bounds on market loss or missed surplus. Due to the restricted time available for MOs to announce a solution for the problem (around 10 mins), it may be impractical to generate all nondominated points and then select one of them. Based on the insights we obtain in the previous section, we propose a set of conditions and a procedure below that can be applied by MOs to find a solution for the problem that satisfy these conditions provided that sufficient time exists.

Properties of the selected solution:

- The selected solution should be nondominated, that is there does not exist any other solution that performs at least as good as the selected solution in market surplus, loss and missed surplus, and outperforms the selected solution in at least one of these criteria.
- Let z_s^* , z_l^* and z_m^* represent the market surplus, loss and missed surplus of the selected solution. Then, there does not exist any other feasible solution that can improve the market loss and missed surplus of the selected solution by at least $\max \{ \alpha_l^{rel} z_l^*, \alpha_l^{abs} \}$ and $\max \{ \alpha_m^{rel} z_m^*, \alpha_m^{abs} \}$, respectively, and having a market surplus at most $\min \{ \beta^{rel} z_s^*, \beta^{abs} \}$ worse than that of the selected solution, where α_l^{rel} , α_l^{abs} , α_m^{rel} , α_m^{abs} , β^{rel} and β^{abs} to be determined by the MO.

A procedure to obtain a solution satisfying the above properties:

1. Find the nondominated point with the maximum surplus value. Set this solution as the selected solution.
2. Set lower bounds on the objectives:

$$z_s \geq z_s^* - \min \{ \beta^{rel} z_s^*, \beta^{abs} \}$$

$$z_l \geq z_l^* + \max \{ \alpha_l^{rel} z_l^*, \alpha_l^{abs} \}$$

$$z_m \geq z_m^* + \max \{ \alpha_m^{rel} z_m^*, \alpha_m^{abs} \}$$

3. Repeat Steps 1 and 2 as long as a new solution is found or the time limit is reached.

5.7 Conclusions

In this chapter, we study the surplus maximization problem, **(SMILP)**, with a multi-objective perspective. We analyze the surplus maximizing solutions in terms of their associated market loss and the missed surplus that occur due to PABs and PRBs, if any. We then examine the surplus maximization problem under no-PAB, **(SMILP-NoPAB)**, and no-PRB constraints, **(SMILP-NoPRB)**, which are the models used by the European MOs in their market clearing activities. Based on our experiments with the 20 instances representative of the Turkish DAM, the surplus maximizing solutions for **(SMILP-NoPAB)** lead to substantial amounts of missed surplus in many cases. Similarly, the surplus maximizing solutions for **(SMILP-NoPRB)** may lead to substantial amounts of market loss. We show that this results from increasing marginal cost of improving one of the objectives further.

The above two cases, that is, zero market loss but large missed surplus in case of **(SMILP-NoPAB)** and zero missed surplus but large market loss in case of **(SMILP-NoPRB)** may not be desirable by MOs. To explore for more desirable solutions, we develop a multi-objective market clearing problem, **(MOMCP)**, considering three objectives: maximizing market surplus and minimizing market loss and missed surplus. We investigate the existence of “good” compromise solutions with a small decrease in market surplus and also with small amounts of market loss and missed surplus.

We show that the nondominated set of **(MOMCP)** is discrete if alternative market clearing prices do not exist for the surplus maximization problem in the slice problems. We identify the conditions for the existence of alternative market clearing prices and develop a model to find the maximum and minimum market clearing prices. If alternative market clearing prices exist only for a single period, then we characterize the efficient range of market clearing prices in that period and derive the corresponding nondominated frontier of the slice problem. If alternative market clearing prices occur in multiple periods, then we consider a bi-objective linear program with market loss and missed surplus as a function of the market clearing price vector and apply a dichotomic search algorithm to generate the extreme nondominated points of the slice problem.

We use our cone-based search algorithm (CBSA) to generate the nondominated set of (**MOMCP**) for each of the 20 instances. We observe different tradeoff characteristics especially between the market loss and missed surplus. We categorize these tradeoffs based on their similarity. In the four Category A instances, the surplus maximizing solution performs well in the other objectives as well leading to a single nondominated point. However, there exist moderate (above 10^4 Turkish liras) to large (above 10^5 Turkish liras) amounts of market loss or missed surplus in surplus maximizing nondominated point in the five Category B instances and there exist other nondominated points that can improve at least one of the market loss or missed surplus substantially in exchange for a small decrease in market surplus, without compromising from the other objective. In the two Category C and D instances, it is possible to improve market loss or missed surplus at a higher rate than the degradation in the compromised objective, leading to increasing marginal returns. In the eight instances of Category E and one instance of Category F, tradeoffs show constant and decreasing marginal returns, respectively.

We also experimented with instances including only block bids in order to analyze the nondominated sets that include edges. Since there exist many hourly bids (around 15,000) in the original instances, the hourly bid equilibrium constraints restrict the market clearing prices into narrow intervals and decrease the chances of observing alternative market clearing prices. By restricting the bid sets to only block bids and studying such a combinatorial auction, we observe 22.5 nondominated edges on average. In the market loss and missed surplus objectives, we similarly observe “good” compromise solutions with small market loss, missed surplus, and surplus gap. However, further decreasing market loss (missed surplus) towards its minimum sharply increases missed surplus (market loss). Therefore, (**SMILP-NoPAB**) and (**SMILP-NoPRB**) type solutions perform much worse in such combinatorial auction settings.

As a future research, we would like to examine the characteristics of the efficient solutions in addition to the nondominated points. We also plan to extend our experiments with instances including other types of bids used in the European DAMs. It awaits further research to derive new pricing methodologies for the MOs based on the insights we have from the characteristics of the nondominated sets.

CHAPTER 6

CONCLUSIONS

We study the market clearing problem in the exchange-type day-ahead electricity market design, the market mechanism implemented by many European countries. Although the surplus maximizing mixed-integer linear programming problem can be solved efficiently by today's powerful solvers like Cplex and Gurobi, the optimal solution may include some accepted block bids that bring negative profits, paradoxically-accepted bids (PABs), to their bidders at the market prices. This implies a non-equilibrium market outcome. To prevent solutions with PABs, the market operator typically imposes additional constraints, settling for a lower total market surplus. This increases the computational burden substantially.

In Chapter 3, we develop Benders infeasibility cuts that use the market clearing prices over the periods of a PAB to find the set of block bid variables, of which at least one needs to be changed to eliminate the PAB. We call these price-based cuts and show that they are stronger than the cuts proposed by Martin et al. (2014). The computational results on practical-size instances representative of the Turkish DAM show that using price-based cuts as the infeasibility cuts in the Benders decomposition algorithm outperforms both using the no-good cuts of Martin et al. (2014) and the locally-valid cuts of Madani and Van Vyve (2015). The improved Benders decomposition algorithm solved all instances to optimality within about one minute when Gurobi solver was used. The tests on larger instances also showed that the improved Benders decomposition algorithm not only found feasible solutions for all instances but also solved more instances to optimality in a fraction of time of the primal-dual approach.

We next study a multi-objective formulation of the European DAM clearing problem.

We develop a criterion-space search algorithm in order to generate its nondominated set that includes edges as well as isolated points. To the best of our knowledge, there is no available algorithm that can solve this class of problems.

In Chapter 4, we present a new criterion-space search algorithm to solve a class of three-objective mixed-integer linear programs, TOMILPs, where at least one of the objectives take discrete values. We develop a novel search space partitioning scheme that utilizes convex cones and the already generated edges to create polyhedral search regions in the two-dimensional projection of the feasible criterion space. Our cone-based search algorithm, CBSA, finds a weakly nondominated point at each iteration in the worst case, and generates the nondominated set and the set of all efficient integer vectors. We test CBSA on randomly generated instances of TOMILPs as well as on three-objective 0/1 knapsack problems, TOKPs, and bi-objective mixed-binary linear programs, BOMBLPs. We present results for different special cases of TOMILPs such as (i) when all integer variables are binary, (ii) when all integer variables are general integer variables, and (iii) when there are two objectives that take discrete values. Our experiments reveal that CBSA can efficiently solve the special class of TOMILPs. Furthermore, CBSA performance is very competitive with the state-of-the-art algorithms designed specifically for bi-objective problems or pure integer problems.

In Chapter 5, we present a multi-objective version of the market clearing problem, (**MOMCP**), and apply CBSA to generate its nondominated set for the instances representative of the Turkish DAM bids. We study the characteristics of the nondominated sets and derive insights for the MOs. In particular, we show that good compromise solutions may exist in many cases that may be more desired for the MOs compared to the extreme solutions generated with the current practices. As a future study, we would like to transform these insights to practical approaches to be used in the European DAM auctions.

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PUBLICATIONS

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