THE EVOLUTION OF MATHEMATICAL PRACTICES IN A SEVENTH-GRADE CLASSROOM: ANALYZING STUDENTS' DEVELOPMENT OF PROPORTIONAL REASONING

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ABSTRACT

THE EVOLUTION OF MATHEMATICAL PRACTICES IN A SEVENTH-GRADE CLASSROOM: ANALYZING STUDENTS' DEVELOPMENT OF PROPORTIONAL REASONING

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The first purpose of this study is to develop, test, and revise a Classroom Hypothetical Learning Trajectory and related instructional sequence for teaching proportional reasoning in seventh-grade. The second purpose is to explain students' communal ways of reasoning with informal tools and how this reasoning evolves over time to reasoning with formal tools in line with a Realistic Mathematics Education perspective. The third purpose is to document students' collective development of mathematical ideas and concepts related to proportional reasoning (i.e., documenting mathematical practices). To those ends, an instructional sequence developed in the United States by Stephan and colleagues was tested and revised in two subsequent design experiments in two years with a design-based research perspective. Data were collected from an experienced middle school mathematics teacher's two seventh-grade classrooms in two years in a public school in Altındağ District of Ankara. The videotapes of the classroom implementation in the second experiment, which lasted six weeks, were analyzed by an adaptation of Toulmin's argumentation model (Classroom Mathematical Practices Analysis). Findings showed that the instructional sequence has extensive potential in supporting a classroom community's proportional reasoning in increasingly sophisticated ways. In particular, it revealed that the classroom community started to make sense of proportional situations by reasoning with pictures and ratio tables (informal tools), and this reasoning evolved to reasoning with the symbolic representations of ratio and proportion (formal tools). Lastly, it presented that five mathematical practices were established in the classroom.

Keywords: Classroom Mathematical Practices, Proportional Reasoning, Realistic Mathematics Education, Design Research, Hypothetical Learning Trajectories

ÖZ

BİR YEDİNCİ SINIFTA MATEMATİKSEL UYGULAMALARIN GELİŞİMİ: ÖĞRENCİLERİN ORANTISAL AKIL YÜRÜTMELERİNİN GELİŞİMİNİN İNCELENMESİ

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Bu çalışmanın birinci amacı, yedinci sınıfta orantısal akıl yürütmenin öğretilmesi için bir varsayıma dayalı öğrenme rotası ve ilgili etkinlik dizisinin geliştirilmesi, test edilmesi ve düzenlenmesidir. Çalışmanın ikinci amacı öğrencilerin informel ve formel araçlarla ortaklaşa akıl yürütmelerinin ve bu akıl yürütmenin Gerçekçi Matematik Eğitimi perspektifi doğrultusunda formel araçlarla akıl yürütmeye doğru gelişiminin açıklanmasıdır. Üçüncü amaç ise, öğrencilerin orantısal akıl yürütmeye yönelik fikir ve kavramların ortaklaşa gelişiminin ortaya konulmasıdır (sınıf içi matematiksel uygulamalar analizi). Bu amaçlar doğrultusunda, Amerika Birleşik Devletleri'nde Stephan ve arkadaşları tarafından geliştirilen etkinlik dizisi Tasarı Tabanlı Araştırma perspektifiyle iki yıl içerisinde ardışık iki tasarı deneyinde test edilmiş ve düzenlenmiştir. Çalışmanın verileri, Ankara'nın Altındağ ilçesinde bir devlet okulunda görev yapan deneyimli bir ortaokul matematik öğretmeninin iki yedinci sınıfında iki yıl içerisinde toplanmıştır. İkinci tasarı deneyinde altı haftalık sürede gerçekleşen sınıf içi uygulamanın video kayıtları Toulmin'in argümantasyon modelinin Stephan ve Rasmussen tarafından uyarlanan analiz yöntemi ile analiz edilmiştir (sınıf içi matematiksel uygulamalar analizi). Sınıf içi matematiksel uygulamalar analizine yönelik bulgular, en genelde, geliştirilen etkinlik dizisinin bir sınıfın orantısal akıl yürütme becerilerinin basitten karmaşığa doğru ilerleyen bir biçimde geliştirilmesinde önemli derecede potansiyele sahip olduğunu göstermiştir. Özelde ise, öğrencilerin resimler ve oran tablolarıyla (informel araçlar) orantısal durumları anlamlandırmaya başladıkları ve bu durumun süreç içerisinde oran ve orantının sembolik gösterimleriyle (formel araçlar) akıl yürütmeye doğru geliştiğini göstermiştir. Son olarak, çalışmanın bulgularına göre sınıf içerisinde etkinlik dizisinin uygulanması süresince beş matematiksel uygulamanın ortaya konduğunu göstermiştir.

Anahtar Kelimeler: Sınıf içi Matematiksel Uygulamalar, Orantısal Akıl Yürütme, Gerçekçi Matematik Eğitimi, Tasarı Araştırması, Varsayıma Dayalı Öğrenme Rotaları

To My Family

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LIST OF ABBREVIATIONS

CCSSI	Common Core State Standards Initiative
MoNE	Ministry of National Education
NCTM	National Council of Teachers of Mathematics
HLT	Hypothetical Learning Trajectory
RME	Realistic Mathematics Education
CMP	Classroom Mathematical Practice
CLT	Classroom Learning Trajectories
NAEP	National Assessment of Educational Progress
TL	Turkish Lira
TAS	Taken-as-Shared
HRT	Horizontal Ratio Table
VRT	Vertical Ratio Table
MCF	Multiplicative Conceptual Field
METU	Middle East Technical University

CHAPTER 1

INTRODUCTION

Ross (1998) stresses the importance of teaching students mathematical reasoning with the following: "The foundation of mathematics is reasoning…If reasoning ability is not developed in the student, then mathematics simply becomes a matter of following a set of procedures and mimicking examples without thought as to why they make sense" (p. 254). The term reasoning itself is defined as "the line of thought, the way of thinking, adopted to produce assertions and reach conclusions" (Lithner, 2000, p. 166). Therefore, reasoning, in particular mathematical reasoning, is an essential skill that needs to be developed in students in order to ensure meaningful learning and communicating that learning.

Proportional reasoning is a type of mathematical reasoning (Cramer & Post, 1993a, 1993b; English, 2004) that entails an understanding of covariation and multiplicative comparisons (Lesh, Post, & Behr, 1988). In general terms, proportional reasoning is referred to as reasoning and making inferences about the essential structural relationships in situations that can be represented by a proportion (i.e., $\frac{a}{b} = \frac{c}{d}$). In particular, it involves "detecting, expressing, analyzing, explaining, and providing evidence in support of assertions about proportional relationships" through a thoughtful and sense-making approach (Lamon, 2007, p. 647). It is also a term that refers to "reasoning in a system of two variables between which there exists a linear functional relationship" (Karplus, Pulos, & Stage, 1983, p. 219).

Proportional reasoning lies at the heart of many mathematical structures, especially those included in the primary and middle school mathematics curricula (Lesh et al., 1988). These include rational numbers, fractions, scaling, basic algebra, geometry, probability and statistics, and measurement (Beswick, 2011; Empson, 1999; Greenes & Fendell, 2000; Lamon, 1995, 1999; Lesh et al., 1988). It is also essential in the development of higher-level geometry and calculus concepts such as functions, vectors, and trigonometry (Karplus et al., 1983; Lamon, 1995, 2007; Lesh et al., 1988; Vergnaud, 1988). In addition, proportional reasoning is inherent in many of the foundational concepts in science, geography, and daily life (Cramer & Post, 1993a; Hart, 1988; Lesh et al., 1988). Nevertheless, the utmost importance of proportional reasoning is rooted in its potential for promoting problem-solving skills (Lesh et al., 1988).

Thus, proportional reasoning is a comprehensive, unifying, and integrative concept and a key skill in the development of other mathematical and scientific concepts. This emphasis on proportional reasoning and a focus on its conceptual development is also highlighted in many of the national and international standards and curricular documents (Common Core State Standards Initiative [CCSSI], 2010; Finnish National Board of Education, 2003; Ministry of National Education [MoNE], 2013, 2018; Ministry of Education Singapore, 2012; National Council of Teachers of Mathematics [NCTM], 2000). For instance, NCTM (2000) considers proportionality among foundational ideas "that should have a prominent place in the mathematics curriculum because they enable students to understand other mathematical ideas and connect ideas across different areas of mathematics" (p. 15). Similarly, CCSSI (2010) refers to proportional reasoning as one of the four critical areas that instructional time should be devoted to in grades six and seven. Similar to the US Standards, the Turkish Middle School Mathematics Curriculum highlights the importance of proportional reasoning and devotes a substantial portion of course time to it, especially in the seventh grade (MoNE, 2013, 2018).

Despite the importance attached to proportional reasoning and its pervasiveness, there is a pile of studies showing that students experience a great deal of challenges in responding to proportional reasoning tasks and understanding the concepts of ratio and proportion (Ayan & Isiksal-Bostan, 2018; Ben-Chaim, Fey, Fitzgerald, Benedetto, & Miller, 1998; Brousseau, 2002; Hart, 1981, 1988; Kaput & West, 1994; Karplus et al., 1983; Lobato & Thanheiser, 2002; Misailidou & Williams, 2003; Resnick & Singer, 1993, Steinthorsdottir & Sriraman, 2009; Thompson & Preston, 1994; Tourniaire & Pulos, 1985; Tourniaire, 1986; van Dooren, De Bock, & Verschaffel, 2010). For instance, Thompson and Preston (1994) indicate that students experience challenges in covarying quantities while keeping the relationship the same while dealing with proportional reasoning tasks. Moreover, Lobato and Thanheiser (2002) state that using incorrect or irrelevant data in computations while solving proportional problems is one of the common errors.

Other difficulties include the inability of discerning proportional and non-proportional situations and an overreliance of proportional strategies for nonproportional situations (Ayan & Isiksal-Bostan, 2018; De Bock, Verschaffel, & Janssens, 1998; De Bock, van Dooren, Janssens, & Verschaffel, 2002; Freudenthal, 1983; Hadjidemetriou &Williams, 2010; Modestou & Gagatsis, 2007, 2009, 2010; van Dooren, De Bock, Verschaffel, & Janssens, 2003; van Dooren, De Bock, Hessels, Janssens, & Verschaffel, 2004; van Dooren, De Bock, Janssens, & Verschaffel, 2007). In another study, Ben-Chaim et al. (1998) found that students responded to the numbers and performed meaningless operations instead of attending to the context of a given problem. Besides, centration on one of the variables and ignoring the other part, and providing subjective and irrelevant responses were among the reported challenges of students (Ben-Chaim et al., 1998). Moreover, Misailidou and Williams (2003) revealed students' major errors in proportional reasoning as incorrect build-up method, magical halving/doubling, and incomplete reasoning. To be more specific, they reported that students correctly used build-up strategies to some point and then added a constant difference to both variables in order to handle the remainder (e.g., 3 yellow - 6 red, 6 yellow – 12 red, 6+1 = 7 yellow and 12+1 = 13 red), which was referred to as an incorrect build-up method. Magical halving/doubling was used to refer to students' tendency to double/halve to arrive at the answer when doubling/halving was inappropriate (Misailidou & Williams, 2003).

Apart from those, erroneous additive reasoning was reported as the most common difficulty while dealing with proportional reasoning tasks in a great number of studies starting from the early studies by Piaget and colleagues (Atabaş & Öner, 2017; Ben-Chaim et al., 1998; Duatepe, Akkuş-Çıkla, & Kayhan, 2005; Fernández, Llinares, van Dooren, De Bock, & Verschaffel, 2012; Harel, Behr, Lesh, & Post, 1994; Hart, 1981, 1984, 1988; Inhelder & Piaget, 1958; Kahraman, Kul, & İskenderoglu, 2019; Kaplan, İşleyen, & Öztürk, 2011; Kaput & West, 1994; Karplus et al., 1983; Kayhan, Duatepe & Akkuş-Çıkla, 2004; Mersin, 2018; Misailidou & Williams, 2003; Noelting, 1980a, 1980b; Özgün-Koca & Altay, 2009; Piaget & Beth, 1966; Piaget & Inhelder, 1975; Resnick & Singer, 1993; Steinthorsdottir & Sriraman, 2009; Tourniaire & Pulos, 1985, Tourniaire, 1986; van Dooren et al., 2010). Indeed, "the early preference for additive solutions to proportion problems is a robust finding, replicated in several studies" (Resnick & Singer, 1993, p. 123). For instance, Tourniaire (1986) found that several students used inappropriate additive reasoning for the following problem:

> There are two mixtures of orange juice and water. One is made with two glasses of orange juice and four glasses of water. The other is made with six glasses of orange juice. How much water should be used to get the same taste? (p. 404)

Those students who adopted an incorrect additive approach answered the problem as "8, because there should be 2 more water than orange juice" (p. 404). Students' tendency to reason additively was also inherent in other contexts, including scaling (Kaput & West, 1994). Indeed, Misailidou and Williams (2003) pointed out that incorrect additive reasoning was the most common error for all types of proportional reasoning problems. Lastly, van Dooren et al. (2010) found that students showed a greater tendency to use additive strategies for multiplicative problems, especially when the numbers given formed non-integer ratios.

Therefore, it is understood that students might fail to think about what is going on in proportional situations (Smith III, 2002). Similar to students, teachers have been found inadequate in both understanding the essential elements of proportional reasoning and providing students with a rich environment for effective teaching and learning (Canada, Gilbert, & Adolphson, 2008; Harel & Behr, 1995; Hilton, & Hilton, 2019; Hines & McMahon, 2005; Kastberg, D'Ambrosio, & Lynch-Davis, 2012; Nagar, Weiland, Orrill, & Burke, 2015; Simon & Blume, 1994a, 1994b; Sowder, Armstrong, Lamon, Simon, Sowder, & Thompson, 1998; Sowder & Philipp, 1995; Thompson, & Thompson, 1994, 1996; Weiland, Orrill, Nagar, Brown, & Burke, 2020). Besides, it is reported that teachers rely heavily on procedural algorithms for solving proportional reasoning tasks (Fisher, 1988; Lobato, Orrill, Druken, & Jacobson, 2011; Orrill & Burke, 2013), and their knowledge is isolated (Hilton, & Hilton, 2019; Nagar et al., 2015). Moreover, it was revealed that some inservice teachers had inadequate understanding of the difference between additive and multiplicative reasoning and how to support students' understanding of those concepts (Sowder & Philipp, 1995). Furthermore, even having a strong mathematics background was not enough to teach proportional reasoning conceptually (Thompson & Thompson, 1994, 1996).

Given teachers' lack of conceptual understanding for the teaching of proportional reasoning, the difficulties experienced by students might be attributed to how instruction takes place in classrooms (Hilton, Hilton, Dole, & Goos, 2016). It is well known that the teaching of proportional reasoning is mostly based on procedural algorithms without making connections to other topics. Concerning this, Karplus et al. (1983) note that in classrooms, "students are shown how to represent the information in proportion word-problems as an equivalent fraction equation, and to solve it by cross multiplying and then dividing" (p. 79). Similarly, for the instruction in textbooks, Lamon (1995) points out that proportional reasoning is taught "in a single chapter of the mathematical textbook, in which symbols are introduced before sufficient groundwork has been laid for students to understand them" (p. 167). In other words, students are often imposed cross-multiplication algorithm that includes operating with the equality of multiplication of cross values (i.e., if $\frac{a}{b} = \frac{c}{d}$; then, a×d = b×c) (Lamon, 1995; Lesh et al., 1988).

However, it is consistently reported that students do not make sense of the cross-multiplication algorithm (Lamon, 1995; Post, Behr, & Lesh, 1988), it is not a student-generated algorithm (Hart, 1984), and does not enhance proportional reasoning; rather, precludes it (Lesh et al., 1988). Therefore, “teaching children how to solve proportion problems by correctly placing three of the four quantities into the equation $\frac{a}{b} = \frac{c}{d}$, then cross multiplying and dividing, does nothing to promote proportional reasoning” (Lamon, 1995, p. 167). Therefore, it is clear that the current instruction is inadequate in addressing the essential elements of proportional reasoning, and there is a need for improved instruction through which students’ conceptual understanding of proportional reasoning is nurtured.

NCTM (2000) suggests that instruction in proportional reasoning should entail methods that have a powerful and intuitive basis. There is an amassed number of studies that reveal young children’s intuitive knowledge in proportional reasoning. This intuitive knowledge refers to “knowledge that does not depend on formal instruction, knowledge that children construct on the basis of their everyday experience in the world” (Resnick & Singer, 1993, p. 107). Piaget and colleagues (Inhelder & Piaget, 1958; Piaget, 1968; Piaget & Beth, 1966; Piaget & Inhelder, 1975) proposed that children have only basic qualitative reasoning mostly based on an understanding of similarity until 11-12 years of age and cannot construct a common identity element to make proportional judgments to connect two ratios that fall into the same equivalence class. Besides, they claimed that proportional reasoning is a late achievement in the development of pupils since it includes higher-order reasoning with an understanding of "relationships of relationships" (Piaget & Inhelder, 1975, p. 160). Katz and Beilin (1976) found similar findings that young children (aged 3-5) do not have a solid understanding of proportionality and invariance.

However, Bryant (1974) suggested a contrary view that young children (aged 3-5) can deal with invariance both qualitatively and quantitatively. Bryant's (1974) assertion was based on the presumption that "young children can on the whole register and remember relative values with great ease but have problems in situations in which they must remember absolute values along any continuum" (p. 14). Therefore, he suggested that the development of proportional schemes progresses from the initial ideas related to making relative judgments to quantifying these in absolute manners. This showed evidence that young children can make inferences regarding proportional relations and operate with two linked quantities by using one to one correspondence (Bryant, 1974). Bryant noted that although this correspondence is an instance of relative coding, proportional reasoning requires making relative judgments that are quantified multiplicatively (e.g., there are twice as many black counters as white counters).

Similarly, Muller (1977, 1978) found that young children aged 5-11 can create a common identity element (e.g., size, color, or proportion) and connect two quantities logically by using this element (i.e., given two ratios 2: 6 and 3: 9, the identity element of 1:3 is constructed). Thus, he showed evidence contrary to Piaget and colleagues' assertions regarding the inability of young children to reason proportionally. In addition, he proposed that children can quantify continuous and discrete proportions through a three-level-trajectory: (1) very young children cannot make successful relative choices, (2) young children can maintain more/less relationships and quantify those judgments multiplicatively, (3) children can make inferences regarding proportional situations.

Therefore, these early studies showed that young children have intuitive and informal proportional reasoning as opposed to the claims made by Piaget and colleagues. Since then, there has been a boost in studies that examine young children's intuitive conceptions related to proportional reasoning in the past three decades (e.g., Boyer, Levine, & Huttenlocher, 2008; Boyer, & Levine, 2012; Fujimura, 2001; Gouet, Carvajal, Halberda, & Peña, 2020; Ham, & Gunderson, 2019; Lamon, 1994, 1995; Möhring, Newcombe, & Frick, 2015; Möhring, Newcombe, Levine, & Frick, 2016; Ng, Heyman, & Barner, 2011; Resnick & Singer, 1993; Singer, Kohn, & Resnick, 1997; Singer & Resnick, 1992; Spinillo, & Bryant, 1991, 1999).

To begin with, Resnick and Singer (1993) suggest that "quite young children have protoquantitative relational schemas that, in principle, could serve as the basis for quantified ratio schemas" (p. 126). That is, young children have an intuitive and informal knowledge of proportional reasoning, on which more formal knowledge can be built. However, early strategies used by children to deal with proportional situations are based on "protoratio" strategies that do not require constructing ratios (Resnick & Singer, 1993). These strategies stem from children's informal knowledge about the factorial structures of numbers or number relations that they are familiar with, which are also referred to as children's existing strengths (Lamon, 1995). Lamon (1995) considers counting, matching, and partitioning skills among these strengths. Kaput and West (1994) consider that these early experiences are associated with "natural build-up reasoning patterns rooted in counting, skip counting, and grouping… and unit factor approach" (p. 283). Similarly, Tourniaire (1986) refers to those as elementary methods that do not involve multiplication and division.

Spinillo and Bryant (1991) also agree with those researchers by claiming that young children can make judgments based on proportional reasoning as young as six years old, provided that they can make sense of the first-order relations that they have difficulties with when dealing with second-order relations (i.e., proportions). They also suggest that young children's proportional reasoning is mostly based on part-part relationships and their images of “half,” rather than part-whole relationships.

Therefore, there is a striking discrepancy among the findings of the research studies that focus on the developmental course of proportional reasoning. These striking discrepancies cannot be attributed to the differences in the design of studies or participants of those studies (Spinillo & Bryant, 1999). A more tenable explanation is that “there are radically different kinds of proportional reasoning, some of which are readily available to young children, whereas others continue to be difficult for them in adolescence and even into adulthood” (Spinillo & Bryant, 1999, p. 192). For instance, Noelting (1980a, 1980b) conducted a study in which he asked children (aged 5-16) to decide which of the two orange punches would have a more orangey taste. According to the findings of the study, even children as young as seven years old could successfully compare the two mixtures: Mixture A (one glass of orange juice-one glass of water) and Mixture B (one glass of orange juice-two glasses of water). Since the amount of orange juice is equal in both mixtures, very young children could compare them by focusing on the amounts of water in both without any calculation. However, even some of the older children aged 10-12 were not able to compare the two mixtures: Mixture A (3 glasses of orange juice-two glasses of water) and Mixture B (4 glasses of orange juice and 3 glasses of water) since this comparison required a multiplicative and relative reasoning (Lamon, 1994). Therefore, the numbers used in a task is also a determinant of kinds of proportional reasoning that is required (Noelting 1980a, 1980b).

The strategies that are inherent in young children’s reasoning are mostly based on an understanding of covariation between two quantities (Spinillo & Bryant, 1999). Children can construct a simple form of covariation for many proportional problems: “when one changes, the other one also changes in a precise way with the first quantity” (Lamon, 2007, p. 648). For instance, let us say, in order to make an orange punch, for every glass of orange juice, one should add two glasses of water. A young child can keep track of the covariation between the amount of water and orange juice as follows: 1-2, 2-4, 3-6, 4-8… by build-up reasoning and find the other amount when the other amount is known (Kaput & West, 1994; Thompson, 1994). In this process, students experience ratio and proportion "through a concrete activity such as counting and pairing two sets of objects" that is "an important prerequisite to abstracting the concept of ratio, an important step in a learning cycle" (Lamon, 1994, p. 115). In the following stages of the development, students should understand that while two quantities vary together (i.e., covary), the relationship between these quantities does not change (i.e., is invariant) (Lamon, 1995).

At the heart of covariation and building up is forming composite units and being able to work with composite units (Battista & van Auken Borrow, 1995; Lamon, 1994; Steffe, 1988). This process includes constructing a unit of units, taking it as one thing (i.e., composite unit), and operating with the composite unit by keeping track of how many times this composite unit is iterated. Therefore, iterating composite units is defined as the ability to take one group as a unit and iterate this unit without changing the nature of its elements (Steffe, 1994). Lamon (1994) refers to this process of forming a composite unit as a reference unit as "unitizing" and considers it one of the critical abilities for the development of proportional reasoning.

The ultimate goal for creating composite units is reinterpreting a situation in relation to that unit, which is referred to as norming (Freudenthal, 1983; Lamon, 1994). One of the most common uses of norming is working with the scalar relationship within the same family of quantities. For instance, back to the Noelting's (1980a, 1980b) example-comparing Mixture A (3 glasses of orange juice-two glasses of water) and Mixture B (4 glasses of orange juice and 3 glasses of water)-, one can take 3 glasses of orange juice as a composite unit and reinterpret 4 glasses of orange juice in the second punch as $\frac{4}{3}$ of 3 glasses (i.e., $4 = \frac{4}{3}(3)$). Then, seeing that the same scale factor does not apply to the amounts of water in both mixtures (i.e., $3 \neq \frac{4}{3}(2)$), she or he can decide that the two mixtures do not taste the same. Moreover, by finding the scale factor that applies to the amounts of water in both mixtures as $\frac{3}{2}$ (i.e., 3 = $\frac{3}{2}$ (2)), she or he can understand that Mixture A has a stronger taste of orange since the scale factor used for the amount of orange juice is greater than that of the amount of water. These relationships that involve working with the scale factors within the same measure space are referred to as scalar ratios (Lamon, 1994; Lesh et al., 1988; Vergnaud, 1994), internal ratios (Freudenthal, 1973), within measures ratios (Lamon, 1994), or between ratio (Karplus et al., 1983; Noelting, 1980a, 1980b).

On the other hand, one can also focus on the functional relationship between the amount of orange juice and water within each mixture. Then, she or he could reason like, “the amount of orange juice in Mixture A is $\frac{3}{2}$ of the amount of water in Mixture A; yet, the amount of orange juice in Mixture B is $\frac{4}{3}$ of the amount of water in Mixture B. Therefore, there is more orange juice than water in Mixture A in relative terms.” This type of interpretation that entails reasoning with functional relationships between quantities that belong to different measure spaces are called functional relationships/rates (Lamon, 1994; Lesh et al., 1988; Vergnaud, 1994), external ratios (Freudenthal, 1973), between measures ratios (Lamon, 1994), or within ratio (Karplus et al., 1983; Noelting, 1980a, 1980b). This understanding focuses on the part-part relationships within each mixture. It is also possible to work with part-whole relationships for this problem. In this case, one can reason such as, "In Mixture A, 2 of the 5 glasses of liquid is orange juice while 4 of the 7 glasses of liquid is orange juice. Hence, less than half of the liquid in Mixture A is orange, whereas more than half of the liquid is orange juice in Mixture B. Therefore, Mixture A has a stronger taste." Therefore, working with scalar or functional relationships is critical for the development of proportional reasoning since students approach proportional problems either of these two ways, whatever strategy they use (Lamon, 1994).

Therefore, it is seen that there is a mass of research that documents students' difficulties in proportional reasoning and their strengths crucial for the development of proportional reasoning. In addition, it is clear to see that there is convergence in the developmental course of proportional reasoning. Taken all together, these issues and concerns suggest a need to find ways to "promote the development of new mathematical concepts (e.g., ratio, derivative, variation), particularly whose development is often unsure" (Simon & Tzur, 2004, p. 92). Concerning this, Simon (1995) proposed the development and use of Hypothetical Learning Trajectories (HLT) that are referred to as "predictions as to the path by which learning might proceed" (Simon, 1995, p. 135). He also suggested that they include "the learning goal, the learning activities, and the thinking and learning in which students might engage" (p. 133). In doing so, he offered the essential aspects of how to plan mathematics lessons. In a later study, Simon and Tzur (2004) explicated the underlying assumptions of HLTs as follows:

> 1. Generation of an HLT is based on understanding of the current knowledge of the students involved.
> 2. An HLT is a vehicle for planning learning of particular mathematical concepts.
> 3. Mathematical tasks provide tools for promoting learning of particular mathematical concepts and are, therefore, a key part of the instructional process.
> 4. Because of the hypothetical and inherently uncertain nature of this process, the teacher is regularly involved in modifying every aspect of the HLT. (p. 93)

Therefore, it is critical to consider students' current understanding of a mathematical topic and their likely progressions in their development of that topic. In this process, research on student thinking and learning and their development progress in a certain mathematical topic can be used as essential resources (Clements & Sarama, 2004; Confrey, Maloney, & Corley, 2014; Daro, Mosher, & Corcoran, 2011; Simon, 1995). Besides, instructional tasks should be selected carefully and purposefully since they play a critical role in students' development of mathematical concepts (Clements & Sarama, 2004; Lamon, 1995; Simon & Tzur, 2004). In addition, since learning trajectories (i.e., students' increasingly sophisticated ways of reasoning) are hypothetical and uncertain, teachers should engage in a cyclic process of assessing student thinking and revising the

HLT as they interact with students (Clements & Sarama, 2004; Simon, 1995; Simon & Tzur, 2004).

Since Simon's (1995) introduction, HLTs have constituted a significant part of research area that focus on curriculum development, measurement and assessment, professional development, and improving the quality of instruction and learning (Corcoran, Mosher, & Rogat, 2009; Lobato & Walters, 2017; Sarama, Clements, Barrett, Van Dine, & McDonel, 2011). Based on Simon's notion of HLTs, Clements and Sarama (2004) described HLTs as:

> descriptions of children's thinking and learning in a specific mathematical domain and a related, conjectured route through a set of instructional tasks designed to engender those mental processes or actions hypothesized to move children through a developmental progression of levels of thinking, created with the intent of supporting children's achievement of specific goals in that mathematical domain. (p. 83)

Moreover, they stressed the importance of using an ordered set of instructional tasks to engender the "mental processes or actions hypothesized to move children through a developmental progression of levels of thinking," which is referred to as an instructional sequence (Clements & Sarama, 2004, p. 83). Based on this, they suggest that "a complete hypothetical learning trajectory includes all three aspects: the learning goal, developmental progressions of thinking and learning, and sequence of instructional tasks" (Clements & Sarama, 2004, p. 84).

There is a considerable amount of HLTs developed in mathematical ideas/subjects including length measurement (Battista, 2006), area and volume measurement (Battista, 2004), linear measurement and flexible arithmetic (Gravemeijer, Bowers, & Stephan, 2003a), rational numbers (Wright, 2014), proportional reasoning (Carpenter, Gomez, Rousseau, Steinthorsdottir, Valentine, & Wagner et al., 1999; Steinthorsdottir & Sriraman, 2009), equipartitioning (Confrey, Maloney, Nguyen, & Rupp, 2014), geometric and spatial thinking (Battista, 2007), place value (Bowers, Cobb, & McClain, 1999), statistics (Cobb, McClain, & Gravemeijer, 2003), whole number calculation (van den Heuvel-Panhuizen, 2008), multiplication (Sherin & Fuson, 2005), spatial thinking (Cross, Woods, & Schweingruber, 2009), exponential growth (Ellis, Ozgur, Kulow, Dogan, & Amidon, 2016), integer addition and subtraction (Stephan & Akyuz, 2012), relational thinking about addition, subtraction, and division (Carpenter, & Moser, 1984; Stephens & Armanto, 2010), three-dimensional shapes and their properties (Sahin Dogruer & Akyuz, 2020). Besides, Clements and Sarama (2009) address several HLTs for a variety of early mathematical ideas (i.e., recognition of number and subitizing, counting, composing number and multidigit addition and subtraction, shapes, composition and decomposition of shapes, geometric measurement, spatial thinking, comparing, ordering, and estimating number, addition and subtraction) in their book, "*Learning and teaching early math: The learning trajectories approach.*"

In each of these studies, HLTs are approached differently. In their systematic review of literature, Lobato and Walters (2017) outlined seven approaches to learning trajectories including (1) Cognitive levels, (2) Levels of discourse, (3) Schemes and operations, (4) Hypothetical learning trajectory, (5) Collective mathematical practices, (6) Disciplinary logic and curricular coherence, and (7) Observable strategies and learning performances. Although most of these approaches focus on the gradual development of mathematical learning at the individual level, HLTs in Approach 5 (i.e., Collective mathematical practices) describe the collective progress of a community of learners (Lobato & Walters, 2017). An HLT in this perspective includes a "sequence of classroom mathematical practices together with conjectures about the means of supporting their evolution from prior practices" (Cobb, 1999, p. 9). Besides, classroom mathematical practices in this approach are described as "taken-as-shared ways of reasoning, arguing, and symbolizing established while discussing particular mathematical ideas" (Cobb, Stephan, McClain, & Gravemeijer, 2001, p. 12). The term, taken-as-shared used here, is to emphasize that the focus is on a community's specific ways of reasoning that are established and no longer need justification (Rasmussen, & Stephan, 2008; Stephan & Rasmussen, 2002).

Participating in classroom mathematical practices takes place through engaging in social interactions in which students and teachers make mathematical claims and justify their claims by providing evidence. This process is called a Collective Argumentation (Krummheuer, 1995). A collective argumentation does not develop in a linear way; instead, disagreements might take place that would eventually result in the processes of correcting, modifying, retracting, and replacing (Krummheuer, 1995). Hence, the result of such a process is called an argument (Krummheuer, 1995), while argumentation is defined as "the substantiation, the part of the reasoning that aims at convincing oneself, or someone else, that the reasoning is appropriate" (Lithner, 2000, p. 166).

Recent studies in mathematics education, based on Toulmin's (1958) model of argumentation and following Krummheuer (1995), have drawn special attention to collective argumentation as an essential element of teaching and learning process (Brown, 2017; Conner, Singletary, Smith, Wagner, & Francisco, 2014a, 2014b; Dede, 2018; Krummheuer, 1995, 2007; Rasmussen & Stephan, 2008; Stephan & Rasmussen, 2002; Wagner, Smith, Conner, Singletary, & Francisco, 2014; Weber, Maher, Powell, & Lee, 2008; Whitenack & Knipping, 2002; Yackel, 2001). This emphasis is also seen in *Principles and standards for school mathematics* (NCTM, 2000) that sees making conjectures and developing sense-making arguments as essential components of mathematical thinking and reasoning. Concerning this, NCTM (2000) calls for attention to creating environments that promote making conjectures and constructing arguments and responding to others' arguments by asking questions such as "Why do you think it is true?" and "Does anyone think the answer is different, and why do you think so?" (p. 56). Hence, by engaging in the process of argumentation in such an environment, students can "see that statements need to be supported or refuted by evidence and learn and agree on what is acceptable as an adequate argument in the mathematics classroom" (NCTM, 2000, p. 56). Similarly, the (Turkish) Middle School Mathematics Curriculum (2018) highlights the importance of raising students who question, express their thoughts verbally, and produce claims based on evidence (MoNE, 2018). Also, it includes the statement, students… "should express their reasoning easily and detect deficiencies and gaps in others' reasoning" (MoNE, 2018, p. 5) as one the specific goals of the curriculum. Therefore, it is critical to provide students with learning environments that would support their argumentation skills (Krummheuer, 2007).

In a collective argumentation environment, it is essential to support a classroom community's collective ways of reasoning and learning by providing them with the opportunities to explore informal material and move to more formal mathematics by engaging in the processes of negotiation, collaboration, and discussion (Gravemeijer, Cobb, Bowers, & Whitenack, 2000; Streefland, 1991). In this process, it is critical to develop a sequence of instructional tasks that would support students' development in increasingly sophisticated ways (Clements & Sarama, 2004; Gravemeijer, & Stephan, 2002). Recent research suggests that the domain-specific instructional theory of Realistic Mathematics Education can guide the design and development of instructional sequences (Cobb, 2003; Gravemeijer et al., 2000, Gravemeijer, & Stephan, 2002; Gravemeijer et al., 2003a, 2003b; Stephan & Akyuz, 2012).

Realistic Mathematics Education (RME) is a theory that is founded by Freudenthal (1973, 1991), who is opposed to seeing mathematics as a set of rules, symbols, and algorithms. According to Freudenthal, no "teaching matter should be imposed upon student as a ready-made product" (Freudenthal, 1973, p. 118). Instead, mathematics should be seen "as an activity, in particular as the activity of a learner" (Freudenthal, 1973, p. 114). Freudenthal (1973) termed that kind of teaching method "that is built on interpreting and analyzing mathematics as an activity" as the method of reinvention (p. 120).

In a reinvention process, students start with exploring realistic situations that are unmathematical or less mathematical matters and organize them into a more mathematical structure that would allow for refinements, which is referred to as mathematizing (Freudenthal, 1973, 1991). Mathematizing should take place in rich contexts that are referred to as the domains of reality and presented to the learner to be mathematized (Freudenthal, 1991). There are two forms of mathematizing: horizontal and vertical mathematizing (Freudenthal, 1991; Treffers, 1978). According to this distinction, horizontal mathematizing refers to moving from the real-world situations to the symbolic world, whereas vertical mathematizing refers to manipulating symbols within the mathematical matter to arrive at more efficient procedures (Freudenthal, 1991; Streefland, 1991; Treffers, 1978).

Models are important in the process of mathematizing in such a way that a shift from a *model of* to *model for* should support the process of mathematizing (Freudenthal, 1973, 1991; Gravemeijer, 1999; Streefland, 1991). In this process, models represent the reality explored in unmathematical or less mathematical situations at the beginning. Later on, as the model is found helpful in applying to other situations, it becomes a model for more mathematical reasoning (Freudenthal, 1991; Streefland, 1991). Hence, models do not directly apply to mathematics; instead, they should be used as intermediaries in order to arrive at mathematical formulas (Freudenthal, 1991).

Reinvention can be an individual or collective activity. In a collective reinvention activity that takes place in a classroom setting, students and the teacher engage in discussions where they make conjectures and justify/refute these conjectures in order to arrive at taken-as-shared meanings (Gravemeijer et al., 2000). In order to foster a collective reinvention process with an RME perspective, it is critical to design instructional sequences that start from realistic situations and move to the intended formal mathematics. In this process, a specific set of RME heuristics should be followed (Gravemeijer et al., 2000; Gravemeijer, & Stephan, 2002). These principles are (1) guided reinvention through progressive mathematizing, (2) didactical phenomenology, and (3) emergent models (Freudenthal, 1983; Gravemeijer et al., 2000; Gravemeijer, & Stephan, 2002; Streefland, 1991). Regarding the first principle, it is essential to consider the historical and informal development of a mathematical topic in order to see potential strengths and obstacles that would be helpful in designing instructional sequences (Gravemeijer et al., 2000; Streefland, 1991). The second principle guides the anticipation of the relationships between students' informal thinking and critical elements of a mathematical phenomenon and design of experientially real contexts through which students can arrive at mathematically advanced strategies in increasingly sophisticated ways (Freudenthal, 1983; Gravemeijer et al., 2000; Lamon, 1995). Lastly, the third principle is helpful in anticipating students' informal use of models/tools and reasoning with these models/tools and in supporting the transition from these informal models to more formal models/tools (Gravemeijer et al., 2000; Gravemeijer & Stephan, 2002).

1.1. Purposes and Research Questions of the Study

The purposes of this study are manifold: (1) to develop, test, and revise a classroom HLT and related instructional sequence for teaching proportional reasoning in seventh grade (2) to explain students' communal ways of reasoning with informal tools and how this reasoning evolves over time to reasoning with formal tools in line with an RME perspective, (3) to document students' collective development of mathematical concepts related to proportional reasoning (i.e., documenting mathematical practices). It should be noted that, although proportional reasoning may also include reasoning about inverse proportional relationships, the scope of this study is limited to direct proportional relationships.

In particular, the first and second purposes of the study seek answers to propose an instructional sequence and HLT and associated local instructional theory that explains how the instructional sequence and the HLT can be effective in teaching and learning of proportional reasoning. In this process, they attempt at blending the critical components and perspectives of HLT and RME and building the instruction on the accumulated knowledge of students' informal ways of reasoning in proportional reasoning (including the tool use) and its developmental course in young children. More specifically, the initial points of departure and informal tools for the teaching and learning of proportional reasoning, and how students rely on their informal knowledge and informal use of tools as they try to mathematize (horizontally and vertically) these initial situations are the aspects under investigation within the context of the first two purposes.

The third purpose is associated with understanding how the hypothesized HLT takes place in the classroom setting by documenting the classroom mathematical practices. Particularly, the potentials and barriers provided by the hypothesized instructional sequence in supporting the collective mathematization of students are the foci of the analysis as part of the third purpose. Therefore, the study attempts to arrive at shreds of evidence from the classroom experiments conducted by using the HLT and the instructional sequence to support the viability of the proposed local instructional theory for teaching proportional reasoning and to suggest refinements to the HLT and the instructional sequence. To these purposes, the research questions of this study are phrased as follows:

1. What would an optimal HLT and instructional sequence for proportional reasoning look like?
    - What would be the initial points of departure for teaching proportional reasoning based on RME?
    - How do students rely on their informal knowledge in order to mathematize that knowledge?
    - How does the instructional sequence foster this process of mathematization?
    - What opportunities and barriers does the instructional sequence provide for realization of the hypothesized learning trajectory?
    - How do student-generated solutions provide opportunities for horizontal and vertical mathematization?
    - What evidences emerge from the classroom experiments conducted by using the HLT and instructional sequence?

2. What are the mathematical practices as students engage in the instructional sequence?

Given these purposes and research questions, the rationale for conducting this study is elucidated in the next part.

1.2. Significance of the Study

As aforementioned, proportional reasoning is an essential domain for students' academic success, especially in mathematics and science (Cramer & Post, 1993a, 1993b; Kilpatrick, Swafford, & Findell, 2001; Lesh et al., 1988). In addition, it is critical in order to deal with daily life situations (Cramer & Post, 1993a, 1993b; Karplus et al., 1983; Spinillo & Bryant, 1999). Indeed, NCTM (1989) argues that the development of proportional reasoning is so important that "it merits whatever time and effort that must be expended to assure its careful development" (p. 82).

The "ability to reason proportionally develops in students throughout grades 5-8" (NCTM, 1989, p. 82). That is, middle school years are the critical years for the development of proportional reasoning. On the other hand, the development of proportional reasoning is also a challenging process. Lamon (2007) points out that:

> of all the topics in the school curriculum, fractions, ratios and proportions arguably hold the distinction of being the most protracted in terms of development, the most difficult to teach, the most mathematically complex, the most cognitively challenging, the most essential to success in higher mathematics and science, and one of the most compelling research sites (p. 629).

Moreover, Resnick and Singer (1993) state that the concepts of ratio and proportion "constitutes one of the stumbling blocks of the middle school curriculum, and there is a good possibility that many people never come to understand them" (Resnick & Singer, 1993, p. 107). Concerning this, Lamon (2006) states that more than half of the adult population is not able to reason proportionally.

These difficulties and challenges faced in the teaching and learning of proportional reasoning are attributed to the traditional instruction that mostly approaches ratio and proportion as isolated topics and focuses on procedural algorithms such as the cross multiplication. Nevertheless, the literature is consistent in showing that these algorithms do not address the essential understandings of proportional reasoning (Hart, 1984; Lamon, 1995; Post et al.,1988) On the other hand, there is also evidence that a carefully developed instruction can improve students' proportional reasoning in a great extent (Adjiage, & Pluvinage, 2007; Bentley, & Yates, 2017; Fujimura, 2001; Jitendra, Star, Starosta, Leh, Sood, Caskie, ... & Mack, 2009; Jitendra, Star, Rodriguez, Lindell, & Someki, 2011; Lamon, 1995; Ng et al., 2011; Smith, Silver, Stein, Boston, Henningsen, & Hillen, 2005).

This study aims at proposing an HLT and related instructional sequence for teaching proportional reasoning together with a local instructional theory outlining the rationale for those. This instructional sequence was designed based on the theory of RME that objects to the traditional type of instruction that starts with procedural skills and algorithms and puts the most emphasis on them. With an RME perspective, this study aims at designing and implementing instruction that values conceptual understanding through the processes of reinvention and mathematization. In this process, students start with exploring realistic situations and use informal tools (i.e., ratio tables) to represent and organize the phenomenon in those situations. Later on, the tools they use evolve into more formal and mathematical tools (i.e., symbolic ratio and proportion) together with how they reason with the tools. Therefore, this study has the potential for improving the classroom instruction in proportional reasoning and, in return, accelerating student understanding and achievement since it highlights meaningful and gradual learning in line with a reinvention and mathematization perspective.

Many studies showed that teachers have similar misconceptions and difficulties as their students, and lack conceptual understanding required to understand and teach proportional reasoning (Canada et al., 2008; Harel, & Behr, 1995; Hilton, & Hilton, 2019; Hines & McMahon, 2005; Kastberg et al., 2012; Nagar et al., 2015; Sowder et al., 1998; Simon & Blume, 1994a, 1994b; Sowder & Philipp, 1995; Thompson & Thompson, 1994, 1996; Weiland et al., 2020). Moreover, a number of studies revealed that their strategies for dealing with proportional situations rely excessively on procedural algorithms (Fisher, 1988; Lobato et al., 2011; Orrill & Burke, 2013). These constitute a serious problem since it is critical for teachers to have a conceptual understanding of proportional reasoning "to employ explicit teaching strategies to promote students' proportional reasoning and to enhance the underlying foundational concepts" (Hilton et al., 2016, p. 194).

More salient rationale arises when considered that "successful teaching requires, at minimum, that teachers possess the schemes we hope children will build" (Thompson & Thompson, 1996, p. 21). Therefore, in addition to being an essential skill, proportional reasoning is also difficult to learn and teach, and it is most likely that teachers need to improve their understandings of proportional reasoning and how to support student understanding in instruction. This study provides teachers with knowledge of the essential understandings of proportional reasoning together with their developmental course and how to support those understandings using the developed HLT and the instructional sequence. Besides, it also provides informal and formal tools that would support the development of proportional reasoning in increasingly sophisticated ways. Hence, it is expected that this study could be helpful in improving teachers’ subject matter knowledge and pedagogical content knowledge in proportional reasoning as it outlines the essential understandings of proportional reasoning in an increasingly sophisticated way and how these understandings can be supported in a classroom context.

Carpenter, Franke, Jacobs, and Fennema (1996) suggest that teachers' knowledge of student thinking and learning enhances teachers' instruction, and, hence, advances students' achievement in mathematics. This study provides teachers with the opportunity to learn about students' developmental track in proportional reasoning and to make use of the existing literature on student thinking and meaningful learning and how to support students along this track. Therefore, this study has the potential to foster teachers' understanding of the meaning of the track in some detail and "knowing what is likely to help keep a student moving forward on it, or to get him or her back on it if they are having problems" (Daro et al., 2011, p. 55). That is to say, it is expected that this study would help teachers learn about and recognize research studies in proportional reasoning, students' conceptual progressions, and the resources they need in order to support learning along that progression (Confrey et al., 2014). This is very critical since teachers are not provided with the tools they need to teach (Daro et al., 2011). Hence, it is believed that the study would help teachers integrate the knowledge of their students' learning and the tools they need to support that learning into their instructional practices and to enhance student achievement in proportional reasoning.

Even though the elements of student learning (i.e., how students learn) are not unfamiliar issues in educational research for particular domains, the implementation and testing of this accumulated knowledge on learning in real classroom contexts do not happen much frequently (Wilson, 2009). Given that proportional reasoning is essential in mathematical understanding of students from elementary school to advanced years, a vast amount of knowledge related to students' success, failures, misconceptions, and learning of proportional reasoning is available in the literature. In other words, there is a considerable amount of studies amassed over many decades in which students' conceptions and misconceptions in proportional reasoning and its developmental course have been revealed.

Despite the richness of this line of research, there has been a gap in educational research and practice in such a way that "teaching practice has been only marginally influenced by this research" (Misialidou & Williams, 2003, p. 336); that is, "implications for teaching ratio and proportion have been slow to emerge" (Lamon, 1993, p. 152). Thus, the questions of how to support students’ development of proportional reasoning and what characterizes a full conceptual understanding of proportional reasoning, and how this understanding develops, and what kinds of instructional activities are helpful for promoting this understanding have not been fully addressed in the research literature (Resnick & Singer, 1993). This study utilizes a design research in a real classroom environment in order to develop a research-based learning trajectory that characterizes the development of seventh-grade students' conceptual understanding of essential aspects of proportional reasoning and to propose a set of instructional activities promoting that developmental process together with an instructional theory that outlines their rationale. Therefore, it is believed that this study could be helpful in enhancing the theoretical and practical issues of learning of proportional reasoning by acting as a bridge between theory and practice so that productive and practical outcomes will be reached.

Particularly, this study attempts to address how best to support students’ development of proportional reasoning and reduce the gap between theory and practice by making use of the previous research results on student thinking. In doing so, it pursues "balancing the theoretical analysis by examining children's knowledge in clinical interviews either before formal instruction, after limited instruction, or in teaching experiments to determine which informal knowledge forms a useful foundation upon which instruction might be built" (Lamon, 1993, p. 132). That is to say, this study provides an instructional theory for the teaching and learning of proportional reasoning that is informed by the accumulated knowledge of children's informal ways of thinking and by students’ intuitive knowledge that develops before formal instruction.

In relation to this, this study also has the potential to address how students' informal and intuitive knowledge can inform instructional design and how students can rely on that knowledge as they engage in this instructional design. This is a significant contribution since it might be helpful in addressing Resnick and Singer's (1993) following concern: "Although mathematics educators in recent years have been receptive to the idea that considerable mathematics knowledge is rooted in everyday practice, a good theory has not been developed to suggest how that experience could give rise to formal mathematical thinking" (p. 107). Given that this study provides teachers with an instructional sequence and related instructional resources together with a local instructional theory that suggests the rationale for the instructional sequence developed (Cobb & Jackson, 2015), it has the potential to inform teachers and researchers about how students can rely on their informal experiences and develop more mathematical knowledge in gradually increasing sophistication.

In regard to this, Stephan, Bowers, Cobb, and Gravemeijer (2003) suggest that one way to assess the viability of an instructional sequence is to document the Classroom Mathematical Practices (CMPs) since CMPs manifest how the hypothesized learning trajectory are realized or actualized in classroom settings. Thus, documenting mathematical practices has the potential to provide evidence related to the revisions/refinements that are necessary for improving the emerging instructional theory on proportional reasoning. The third purpose of the study deals with documenting the classroom mathematical practices that emerge as the teacher and the students interact around the instructional sequence. In doing so, it promises to provide evidence that would be helpful in revising the instructional sequence so that a better version of it would be available to be used in classrooms.

Lamon (1993) stresses that one of the major goals of research is "to identify important mechanisms by which thinking becomes progressively more sophisticated from early childhood through adulthood" (p. 132). Similarly, Steffe (2004) notes that constructing learning trajectories of children's mathematical reasoning constitutes "one of the most daunting but urgent problems facing mathematics education today" (p. 130). Besides, it is noted that there is limited knowledge about the key steps in the development of mathematical knowledge and how to support students who experience problems along that developmental path (Daro et al., 2011). However, one can make use of this existing body of knowledge to design instruction and see how it works out so that this gap can be filled over time (Daro et al., 2011). One of the major gaps in students' developmental progress and the ways to support that progress occurs in the topics of ratio, rate, and proportion (Daro et al., 2011). Although substantial work has been carried out in developing HLTs to address students' conceptual development of other mathematical topics, few studies have focused on the development of HLTs in proportional reasoning and related concepts (Carpenter, Gomez, Rousseau, Steinthorsdottir, Valentine, Wagner, et al., 1999; Steinthorsdottir & Sriraman, 2009; Wright, 2014).

Nevertheless, these studies have focused on the development of individuals' understanding of these concepts. That is to say, none of the studies above have focused on the gradual development of proportional reasoning in collective ways as it takes place in a classroom environment. Given students' difficulties in proportional reasoning and related concepts, this study has the potential to contribute to the theory and practice in providing a picture of how a community of learners' gradual development in proportional reasoning takes place in a classroom environment and how this reasoning is supported in increasingly sophisticated ways with an RME perspective.

Several studies showed that students' informal experiences of natural build-up strategies and "situationally grounded presentations, accompanied by tabular or systematic forms of record-keeping, may support the eventual discovery of new number relationships" that are essential in the development of proportional reasoning (Resnick & Singer, 1993, p. 127). However, research studies that propose an instructional theory highlighting the initial points of instruction with these kinds of experiences are scarce (Resnick & Singer, 1993). Therefore, there is a "strong need for the kinds of concrete representations that support and extend students' natural build-up reasoning patterns rooted in counting, skip counting, and grouping" (Kaput & West, 1994, p. 283). Students should engage in these kinds of experiences in order to have an informal and conceptual background before formal instruction on ratio and proportion takes place (Kaput & West, 1994).

Although there is a number of available learning trajectories in proportional reasoning and related concepts (Carpenter et al., 1999; Steinthorsdottir & Sriraman, 2009; Wright, 2014), they are not readily available to teachers to be used as instructional tools (Daro et al., 2011) since they outline individual students' developmental progress. Nevertheless, there is a need to develop learning trajectories "that stress the learning supports, the key mathematics ideas, and the key questions for students so that they can support classroom teachers and students through the learning paths" (Daro et al., 2011, p. 57). To this end, this study has the potential to fill this gap since it provides the teachers with a complete picture of the key ideas and essential understandings of proportional reasoning together with the learning supports including a sequence of instructional tasks and the tools that would support the development of those key ideas in increasingly sophisticated ways.

It is stressed that this study aims at developing a local instructional theory for the teaching of proportional reasoning by using the theory of RME as a guideline and inspiration. In this sense, RME is "worked out" in a local instructional theory for proportional reasoning within the context of this study (Gravemeijer & Stephan, 2002). Since local instruction theories "comprise newly created instances of how RME can be worked out, these local theories can, in turn, form the raw material for the construction of a more refined version of the general theory" (Gravemeijer & Stephan, 2002, p. 148). In relation to this, the findings obtained from this study can inform the general theory of RME since this study provides a "reconstruction of a theory in action" and an instance of how RME can inform the design of an HLT for proportional reasoning. In doing so, it helps develop the guiding theory (i.e., RME) itself, although the central phenomenon of seeing mathematics as an activity of learners remains the same (Gravemeijer & Stephan, 2002).

Moreover, it is well known that prior experience, knowledge, and cultural background have an influence on learning, and there is a need to understand the mathematical development of students with different knowledge and cultural backgrounds (Daro et al., 2011). Although this study does not attempt to examine how students' mathematical development is affected by their academic and cultural backgrounds, it provides a picture of a Turkish classroom community's mathematical development of proportional reasoning. Therefore, it has the potential to enhance the understanding of mathematical development in a different context.

Above all, the Middle School Mathematics Curriculum was revised in 2013 and 2018 in Turkey, and many changes were made regarding the concepts of ratio, rate, and proportion, which are the foci of proportional reasoning. Since then, not many studies have been conducted that would help teachers how to integrate those changes into their instruction. Furthermore, although the objectives related to ratio and proportion in the curriculum are presented in order, there is no clear empirical evidence about whether that order is helpful in supporting students' development of proportional reasoning. Moreover, the curriculum does not include any information related to how to best support students in a classroom environment in achieving those objectives. These, together with the fact that the curriculum (MoNE, 2018) does not include all of the critical components of proportional reasoning (e.g., additive and multiplicative reasoning, qualitative reasoning), it is expected that this study has potential in informing the objectives in the curriculum and their order.

In particular, in this study, the essential understandings of proportional reasoning and the instructional tasks that would support students' development of those understandings were determined based on empirical knowledge available in the literature. Also, these instructional tasks were ordered in increasingly sophisticated ways by taking students' informal ways of reasoning and developmental course of proportional reasoning into consideration. Therefore, it is expected that the HLT and the instructional sequence developed in this study would be helpful in guiding teachers to teach proportional reasoning in conceptual and comprehensive ways in an increasingly sophisticated manner. In doing so, it could provide a “road map that helps teachers guide students to increasing levels of sophistication and depths of knowledge… and understand the mathematics that has been studied by students at the previous level and what is to be the focus at successive levels" (NCTM, 2000, p. 16). In this respect, the study has the potential to contribute to both the theory and practice regarding the teaching and learning of proportional reasoning. Therefore, the hypothetical learning trajectory and related instructional sequence to be developed might have a role as a source not only for access to related literature but also for making sense of the curriculum objectives related to proportional reasoning.

1.3. Definitions of Important Terms of the Study

For the sake of ease of understanding and avoiding vagueness, the terms and concepts that are pertinent to the purposes of this study are defined both constitutively and operationally in this section.

1.3.1. Proportional Reasoning

In general, proportional reasoning is referred to as making inferences about the essential structural relationships in situations that can be represented by a proportion (i.e., $\frac{a}{b} = \frac{c}{d}$). It is also a term that refers to "reasoning in a system of two variables between which there exists a linear functional relationship" (Karplus et al., 1983, p. 219). In this study, proportional reasoning is referred to as a type of mathematical reasoning that requires "a sense of co-variation and of multiple comparisons, and the ability to mentally store and process several pieces of information" (Lesh et al., 1988, p. 93). Besides, an understanding of proportional reasoning in this study involves making inferences and predictions about the holistic relationships between two rational expressions that are ratios, rates, quotients, and fractions through both qualitative and quantitative ways of reasoning (Lesh et al., 1988).

1.3.2. Ratio, rate, and proportion

A ratio, in general sense, is "a comparative index that conveys the abstract notion of relative magnitude" (Lamon, 1995, p. 169). Concerning this, a proportion is usually referred to as "the statement of equality between one ratio and another in the sense that both convey the same relationship" (Lamon, 1995, p. 171). With a different interpretation, it is also possible to define proportion as a function that satisfies the isomorphic properties "$f(x+y) = f(x) + f(y)$" and "$f(ax) = af(x)$" (Vergnaud, 1988, 1994). Also, a proportion can also be defined based on interpretations with external and internal ratios: a proportion, also referred to as linear mapping, is either obtained by mapping one magnitude upon another by preserving the internal ratios or by postulating the constancy of the external ratio (Freudenthal, 1978). Hence, ratio refers to a relative comparison between quantities, and proportion to the equality of these comparisons; in other words, equality of ratios or ratio preservation (Freudenthal, 1978).

In mathematics education literature, rate is usually defined based on its distinctions from ratio. Several classifications define ratios and rates and how they differ from each other. The early attempts consider ratio as a comparison of quantities that belong to the same measure space and rate as a comparison of quantities that belong to different measure spaces (Vergnaud, 1988). Another perspective imposes using the terms intensive and extensive quantities instead of ratios and rates. According to this perspective, extensive quantities express the extent of a quantity (i.e., how much) of an object, whereas intensive quantities tell relationships between a quantity relative to a unit of the other quantity. Therefore, in this perspective, rate refers to a single intensive quantity while ratio refers to a relationship between two quantities (Freudenthal, 1973; Kaput, Luke, Poholsky, & Sayer, 1986; Kaput & West, 1994; Schwartz, 1988). Lesh et al. (1988) suggest a different perspective by referring to rates as intensive quantities that could be "recognized by the "per" in their unit labels," and ratios as "binary relations which involve ordered pairs of quantities (of either the extensive, intensive or scalar types)" (p. 112). Thus, rates include only the unit rates (i.e., so many A's per 1 B) in Lesh et al.'s (1988) perspective.

Although all of these perspectives and associated definitions are valuable, Thompson's (1994) perspective that stresses the mental operations for describing rates and ratios is adopted in this study. Based on this perspective, a ratio is referred to as "the result of comparing two quantities multiplicatively" (Thompson, 1994, p. 191) while rate as "a reflectively abstracted constant ratio" regardless of the measure spaces of the quantities (p. 192). Therefore, in this way of interpretation, a comparison of two specific and fixed (i.e., non-varying) quantities is considered as a ratio. When this comparison between the two quantities is abstracted in order to interpret the ratio in relation to the invariant (i.e., constant) result of the multiplicative comparison, then it is referred to as rate (Thompson, 1994).

1.3.3. Within and between ratios

There are two main ways to define within and between ratios, and they are almost the opposites of each other. Mathematics Education Literature refers to within ratios as comparisons within the same measure space, whereas between ratios are interpreted as comparisons between different measure spaces (Freudenthal, 1973; Lamon, 1994; Vergnaud, 1994). However, research that follows a science tradition (e.g., Karplus et al., 1983; Noelting, 1980a, 1980b) uses a different terminology basing the distinction on whether or not the two quantities belong to the same system that they define as a series of interacting elements. According to this perspective, within ratios include comparisons within a system while between ratios involve comparisons between two systems that interact with each other. Therefore, two interpretations are the opposites of each other. However, this confusion can be eradicated by using a specific terminology as "within or between measure spaces" (Lamon, 2007, p. 634). In this study, this specific terminology will be used in order to avoid this conflict and for the sake of ease of understanding.

1.3.4. Multiplicative and additive reasoning

Multiplicative reasoning, in general terms, is defined as "making multiplicative comparisons between quantities" (Wright, 2005, p. 363). In multiplicative reasoning "the terms within a ratio are related multiplicatively and then this relation is extended to the second ratio" (Tourniaire & Pulos, 1985, p. 184), while in additive reasoning "the relationship within the ratios is computed by subtracting one term from another, and then the difference is applied to the second ratio" (Tourniaire & Pulos, 1985, p. 186). Therefore, in general terms, multiplicative reasoning is a type of reasoning that underlies proportional reasoning, while additive reasoning does not apply to proportional situations.

Another perspective that is helpful in understanding multiplicative and additive reasoning lies in making distinctions between understanding change in relative and absolute manners (Lamon, 1995). In line with this perspective, thinking about change in absolute manners involves additive reasoning while thinking about change in relative terms entails multiplicative reasoning. To illustrate, let us say one observes that a tree was 30 cm last year, and now, a year later, it is 40 cm. That person can think that "the tree has grown 10 cm in a year" (actual growth-absolute thinking) or "the tree has grown $\frac{1}{3}$ of its initial length" (relative growth-relative thinking). Therefore, multiplicative reasoning is associated with relative thinking, while additive reasoning is connected to absolute thinking. Since proportional reasoning requires interpreting change in relative terms, additive reasoning is usually considered as an incorrect type of reasoning for proportional situations. Similarly, additive reasoning is interpreted as an erroneous strategy when applied to proportional reasoning tasks in this study.

In general, multiplicative reasoning is associated with being able to work with within and between ratios (Lamon, 1994). More specifically, multiplicative reasoning includes being able to find and operate with a scalar operator- the number that transposes a measure within a single measure space- and with a function operator- the number that represents the coefficient of a mapping (i.e., a linear function) from a measure space to another (Lamon, 1994). That is to say, multiplicative reasoning is usually referred to as understanding the multiplicative relationships within and between measure spaces between two equal ratios/rates. To illustrate, multiplicative reasoning includes understanding and operating with the horizontal “times two relationships” (i.e., ×2) and the vertical “times five relationship” (i.e., ×5) represented in the following figure:

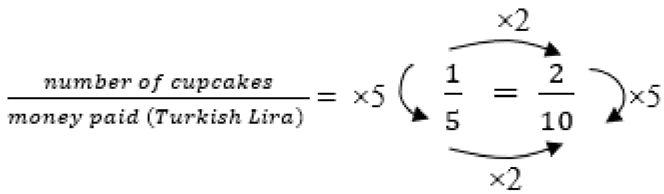


Figure 1.1. The components of multiplicative reasoning

The horizontal times-two-relationship requires reasoning with the questions of “how many one-cake in 2-cakes? How many one-cake goes into 2-cakes? What is the scale factor that transposes the number of cupcakes from 1 to 2?” while the vertical times-five-relationship involves reasoning with the question “What is the relationship between the number of cupcakes and money paid?”

Therefore, in common, multiplicative reasoning is defined as understanding this two-way multiplicative nature of the rational numbers (Lamon, 2007). As different from this understanding, in this study, understanding and being able to operate with the multiplicative relationship between variables in different measure spaces (i.e., functional relationships) is considered as multiplicative reasoning, whereas understanding and being able to operate with the multiplicative relationship between variables within the same measure space is interpreted as pre-multiplicative reasoning. The rationale behind this is on account of the fact that working with the scalar operator is associated with a short way for building up strategies, which will be explained in more detail in Chapter 4 (i.e., Findings) and Chapter 5 (i.e., Discussion) of this dissertation.

1.3.5. Hypothetical Learning Trajectories

The term Hypothetical Learning Trajectories (HLTs) was coined by Simon (1995) who defined them as “predictions as to the path by which learning might proceed" (p. 135), which includes “the learning goal, the learning activities, and the thinking and learning in which students might engage” (p. 133). Another definition by National Research Council (2007) refers to learning trajectories as “descriptions of the successively more sophisticated ways of thinking about a topic that can follow one another as children learn about and investigate a topic over a broad span of time" (p. 214). In another study, Confrey, Maloney, and Corley (2014) refer to HLTs as “research-based frameworks developed to document in detail the likely progressions, over long periods of time, students’ reasoning about big ideas in mathematics” (p. 720). Clements and Sarama (2004) define HLTs as:

> descriptions of children’s thinking and learning in a specific mathematical domain and a related, conjectured route through a set of instructional tasks designed to engender those mental processes or actions hypothesized to move children through a developmental progression of levels of thinking, created with the intent of supporting children’s achievement of specific goals in that mathematical domain (p. 83).

Moreover, they suggest that “a complete hypothetical learning trajectory includes all three aspects: the learning goal, developmental progressions of thinking and learning, and sequence of instructional tasks” (Clements & Sarama, 2004, p. 84).

While all of these definitions emphasize the learning routes of individuals, the definition has also been adapted to refer to a learning path of a social community (Cobb, 2001). This perspective reconceptualizes the term learning trajectory as a "sequence (or set) of (taken-as-shared) classroom mathematical practices that emerge through interaction (especially through classroom discourse-with the proactive involvement of the teacher)" (Clements & Sarama, 2004, p. 85). In relation to this, Stephan (2015) uses the term C*lassroom Learning Trajectories* (CLT) defined as “conjectures about the mathematical ideas that become taken-as-shared and individuals' ways of participating in and contributing to them" (Stephan, 2015, p. 908). She goes on to suggest that these conjectures include "the mathematical goals, and tool use as students engage with the instructional tasks" (Stephan, 2015, p. 908). Furthermore, CLTs include an outline of instructional supports in order to foster student learning along a learning path.

In this study, the perspectives by Clements and Sarama (2004) and Stephan (2015) are followed in order to define and interpret HLTs. In other words, within the context of this study, HLT is used to refer to CLT since the use of the term HLT is more common.

1.3.6. (Collective) Argumentation

I follow Krummheuer (1995), who sees argumentation as a social phenomenon in which individuals present justifications for their actions and make adjustments in their intentions. Argumentation in such an environment is referred to as techniques and methods to establish a claim and seen as a specific aspect of social interaction (Krummheuer, 1995). Therefore, a successful process of argumentation involves challenging claims and arriving at a consensual and acceptable claim for all individuals (Krummheuer, 1995). This kind of argumentation in which several individuals contribute to the development of mathematical arguments through social interaction is called a collective argumentation (Krummheuer, 1995). A collective argumentation does not develop in a linear way; instead, disagreements might take place that would eventually result in the processes of correcting, modifying, retracting, and replacing (Krummheuer, 1995). Hence, the result of such a process is called an argument (Krummheuer, 1995).

1.3.7. (Collective) Reinvention and Mathematization

Freudenthal (1973) coined the terms "reinvention" and "mathematization" and referred to the former as a process that includes understanding and analyzing mathematics as a human activity and the latter as the activity of organizing nonmathematical or inadequately mathematical matters into a structure that would allow for mathematical refinements. Later on, Gravemeijer et al. (2000) drew attention to the activities of reinvention and mathematizing as being both individual and collective activities.

I follow Gravemeijer et al. (2000), who focus on reinvention and mathematization as they take in the social context of a classroom context. In such collective activities, students participate in whole-class discussions as they engage in the processes of conjecturing, explaining, and justifying (Gravemeijer et al., 2000). In this process, the teacher capitalizes on students' reasoning in order to guide their progress to intended mathematics so as to foster the process of classroom negotiation in order to support the emerging of taken-as-shared meanings when certain social and sociomathematical norms are established (Cobb & Yackel, 1996; Gravemeijer et al., 2000; Yackel & Cobb, 1996).

1.3.8. Horizontal and Vertical Mathematization

Horizontal mathematization is referred to as moving from the real world to the symbolic world, whereas vertical mathematization as the acts of mechanical and reflective shaping, reshaping, and manipulation of symbols that take place in the world of symbols (Freudenthal, 1973, 1991; Treffers, 1978). Streefland (1991) stresses that starting from reality and engaging in the processes of structuring, arranging, symbolizing, visualizing, schematizing, and hence moving to mathematics is horizontal mathematization. Working within this mathematical matter in order to arrive at more efficient procedures, abbreviations, and symbolic language of mathematics by abstracting, generalizing, unifying, and specifying is associated with vertical mathematization (Freudenthal, 1973, 1991; Streefland, 1991).

CHAPTER 2

LITERATURE REVIEW

The main purpose of this study is to develop, test, and revise a classroom HLT and related instructional sequence in order to propose a potentially viable local instructional theory for supporting the development of proportional reasoning in seventh grade. In relation to this purpose, another purpose of the study is to document seventh grade students' collective growth in proportional reasoning by an analysis of Classroom Mathematical Practices. In addition, the third purpose includes expounding seventh-grade students' communal ways of reasoning with informal and formal tools and how this reasoning is supported in increasingly sophisticated ways with an RME perspective.

This chapter presents the literature that is germane to these purposes. The research agenda includes three main research fields: Proportional Reasoning, Realistic Mathematics Education, and Hypothetical Learning Trajectories. First, it puts forward the essential concepts and skills in proportional reasoning and elaborates on them with a focus on students' informal ways of reasoning in proportional reasoning and its developmental course. Second, it expounds on the theory of RME and its principles. Third, it explores the definitions of Hypothetical Learning Trajectories and introduces the literature that discusses different understandings and uses of HLTs. Lastly, previously developed HLTs in proportional reasoning are discussed.

2.1. Proportional Reasoning

Proportional reasoning is a type of mathematical reasoning that requires "a sense of co-variation and of multiple comparisons, and the ability to mentally store and process several pieces of information" (Lesh et al., 1988, p. 93). It involves making inferences and predictions through both qualitative and quantitative ways of reasoning (Lesh et al., 1988). Proportional reasoning is referred to as a watershed concept, a cornerstone of higher mathematics, a capstone of elementary concepts (Lesh et al., 1988) and a gateway to higher levels of mathematics success (Kilpatrick et al., 2001; Lesh et al., 1988). In particular, proportional reasoning is “a capstone of elementary arithmetic, number, and measurement concepts and cornerstone of algebra and other higher levels areas of mathematics” and "a watershed concept that separates elementary from more advanced concepts: it is both one of the most elementary higher-order understandings and one of the highest-level elementary understandings" (Lesh et al., 1988, p. 97). Moreover, proportional reasoning is viewed as a herald of a conceptual transition from concrete operational level to a formal operation level in Piaget’s theory of learning (Hart, 1988; Piaget & Beth, 1966) since it is associated with a relationship between two relationships (i.e., a second-order relationship) (Piaget & Inhelder, 1975). Therefore, proportional reasoning is “a complex phenomenon both in terms of mathematical relationships and in terms of the experiences that give rise to the mathematics” (Lamon, 1995, p. 167).

The importance of proportional reasoning is on account of the fact that it is related many of the foundational concepts in mathematics, science, geography, and art, as well as situations in everyday life (Cramer & Post, 1993; Hart, 1988; Lesh et al., 1988). Basic scientific concepts related to proportional reasoning include but are not limited to temperature, density, concentrations, velocities, and chemical compositions (Karplus et al., 1983; Spinillo & Bryant, 1999). Everyday life situations include deciding on the best buy, grocery purchases, personal finances (Spinillo & Bryant, 1999), medicine dosages, and economic and sociological predictions (Valverde & Castro, 2012). In addition, the relationships between the sales tax paid and the item cost, the amount of paycheck and the number of hours worked, and the distance in real life and representation of that distance on a map are typical situations that include proportional relationships in their nature (Lamon, 1995). Peculiar to mathematics, proportional reasoning is an essential integrative concept that connects many mathematics topics in grades 6-8 (Lesh et al., 1988; NCTM, 2000). Besides, it is a key and unifying concept in a wide variety of essential topics beyond middle school (Lesh et al., 1988; van de Walle, Karp, Bay-Williams, & Wray, 2013).

To begin with, Lamon (1999) stated that “proportional reasoning is one of the best indicators that a student has attained understanding of rational numbers” (p. 3). Other mathematical topics related to proportional reasoning include ratios, fractions, percent, similarity, scaling, trigonometry (Beswick, 2011); basic algebra, geometry, problem-solving (Empson, 1999; Fuson & Abrahamson, 2005; Hasemann, 1981; Lamon, 1995; Lesh et al., 1988; Saxe, Gearhart, & Seltzer, 1999); functions, graphing, variables, algebraic equations, measurement, and vectors (Karplus et al., 1983; Lamon, 1995, 2007; Lesh et al., 1988; Vergnaud, 1988); probability and statistics, scale drawing, similar figures, measurement conversions (Greenes & Fendell, 2000); and steepness (Cheng, Star, & Chapin, 2013). Moreover, proportional reasoning is assumed in order to achieve higher-level mathematics and science, including "geometry, calculus, statistics, chemistry, and physics" (Lamon, 1995, p. 172). On the other hand, it is associated with some of the most conceptual stumbling blocks in the curriculum including, “(equivalent) fractions, long division, place value and percents, measurement conversion, ratios, and rates" (Lesh et al., 1988, p. 95). Thus, proportional reasoning is a comprehensive, unifying, and integrative concept and a key skill in the development of other mathematical and scientific concepts.

Despite being such an inclusive, comprehensive, and essential concept, the definition of proportional reasoning is too superficial and restrictive in such a way that it is conventionally referred to as solving missing-value problems (Lesh et al., 1988). In other words, students who are able to solve problems asking for the fourth value when three values related to a situation are given are conventionally considered as reasoning proportionally. This limited understanding of proportional reasoning is also present in most of the national and international textbooks. Lamon (1995) notes that proportional reasoning has typically been taught in "a single chapter of the mathematical textbook, in which symbols are introduced before sufficient groundwork has been laid for students to understand them" (p. 167). Besides, "students are shown how to represent the information in proportion word-problems as an equivalent fraction equation, and to solve it by cross multiplying and then dividing" (Karplus et al., 1983, p. 79).

In particular, many mathematics textbooks in Turkey begin with the definitions of ratio and proportion, make students find ratios of quantities by division, and include missing value proportional problems. These explorations require only computational skills. Parallel to the instruction in textbooks, students are often imposed cross-multiplication algorithm that includes operating with the equality of multiplication of cross values (i.e., if $\frac{a}{b} = \frac{c}{d}$; then, $a \times d = b \times c$) in classrooms (Lamon, 1995; Lesh et al., 1988). However, it is consistently reported that: students do not make sense of cross-multiplication algorithm (Lamon, 1995; Post et al., 1988), it is not a student-generated algorithm (Hart, 1984), and does not enhance proportional reasoning; rather, precludes it (Lesh et al., 1988). Therefore, “teaching children how to solve proportion problems by correctly placing three of the four quantities into the equation $\frac{a}{b} = \frac{c}{d}$, then cross multiplying and dividing, does nothing to promote proportional reasoning” (Lamon, 1995, p. 167).

Therefore, solving proportional problems by memorized algorithms cannot be regarded as an indicator of proportional reasoning (Cramer & Post, 1993a, 1993b; Lesh et al., 1988). It is essential for proportional reasoning to reason about the holistic relationships between two rational expressions that are ratios, rates, quotients, and fractions (Lesh et al., 1988). In other words, it is necessary for a child to be considered as reasoning proportionally that he or she "presents valid reasons in support of claims made about the structural relationships that exist when two ratios are equivalent" (Lamon, 1995, p. 173) either on a quantitative or qualitative level. These structural relationships are depicted in Figure 2.1 below.

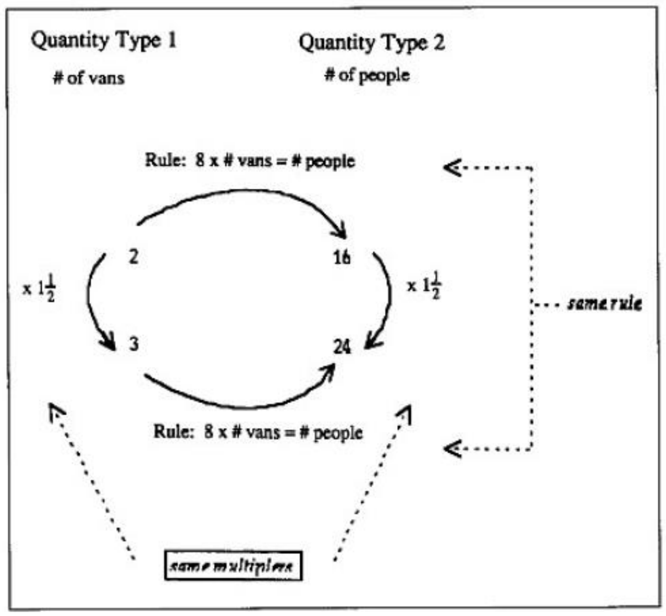


Figure 2.1. Structural relationships in a proportional situation (Lamon, 1995, p. 172)

In Figure 2.1 given above, the relationships between the number of vans and the number of people, and between the number of vans in the first situation and the second situation, and between the number of people in the first situation and second situation should be interpreted. These interpretations are based on structural relationships among those four vales that are referred to as within comparisons and between comparisons (Lamon, 1994). Regardless of the strategy used, students usually solve a proportion question by viewing the initial ratio as in either of the two ways: between and within comparisons (Lamon, 1994). For instance, for the situation represented in Figure 2.1 above, a within comparison would be comparing the two scalar multipliers within each measure space (i.e., $1\frac{1}{2}$) to see if they are the same. Also, it would require reasoning in such a way that having the same scalar multiples would mean having the same number of groups within each quantity (i.e., $1\frac{1}{2}$ groups of 2 vans in 3 vans and $1\frac{1}{2}$ groups of 16 people in 24 people). Moreover, a between comparison would be making sense of the functional relationship connecting the number of vans and the number of people and forming the rule 1:8 (i.e., the reduced form of 2:16 or 3:24) and using it to find an unknown value.

For the same relationships within each type of measure space and between measure spaces, Vergnaud (1994) uses the terms scalar ratios (within quantity types) and functional ratios (between quantity types), as shown in Figure 2.2 below.

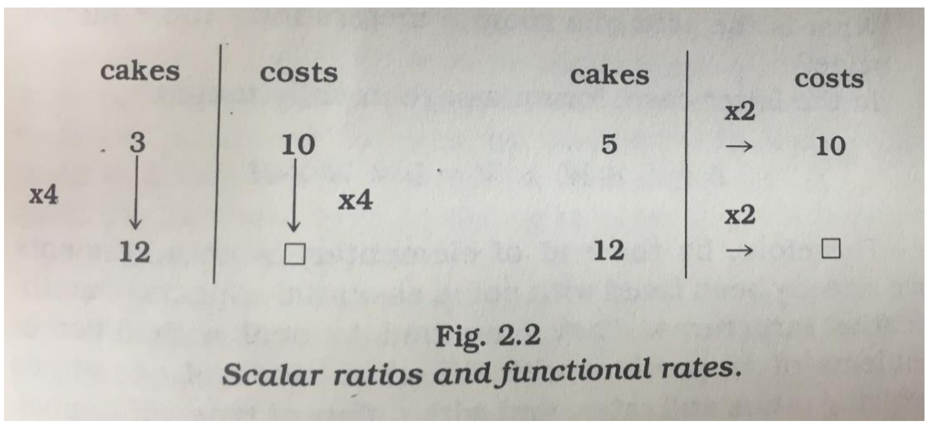


Figure 2.2. Scalar ratios and functional rates (Vergnaud, 1994, p. 51)

For this situation, including the number of cakes and the corresponding cost, reasoning with a scalar ratio would mean "How many 3-cakes are there in 12-cakes?" or "How many 3-cakes goes into 12-cakes?" Then, seeing that 4 units of 3-cakes would go into 12-cakes, the corresponding cost should be multiplied by the same scale factor. Furthermore, reasoning with a functional rate would require thinking, "What is the relationship between the number of cakes and cost?" Then, upon formulating the relationship between the number of cakes and the corresponding cost as 1:2, the corresponding cost in the second situation should be multiplied by 2.

Similar to the examples above, Freudenthal (1973) distinguishes internal (within measures) ratios and external (between measures) ratios based on whether the quantities belong to the same measure space. More specifically, if two quantities belong to the same measure space, then they form an internal (within measures) ratio; if two quantities belong to different measures spaces, then they form an external (between measures) ratio. He also stresses that “internal ratios are ‘abstract’ numbers, whereas external ratios are in general ‘concrete’ numbers.” (Freudenthal, 1978, p. 293), meaning that external ratios form a single entity on their own (e.g., velocity).

However, there is confusion and convergence in the literature about this distinction between within comparison/ratio/strategy and between comparison/ratio/strategy. Research that follows a science tradition (e.g., Karplus et al., 1983; Noelting, 1980a, 1980b) uses different terminology. Particularly, in this tradition, the concept of a system is interpreted differently as a series of elements that interact (Lamon, 2007). Hence, according to this interpretation, the first of the following examples corresponds to a within ratio, while the second example corresponds to a between ratio.

(1) 2 glasses of orange juice:3 glasses of water = 4 glasses of orange juice:6 glasses of water
(2) 2 glasses of orange juice:4 glasses of orange juice = 3 glasses of water:6 glasses of water

According to Karplus et al.’s and Noelting’s approach, the first example (1) is interpreted as a within ratio since it includes the relationships between the two systems (i.e., the amount of water and the amount of orange juice) that interact with each other (Noelting, 1980a, 1980b). Besides, the second example (2) is interpreted as a between ratio since it deals with comparisons between the systems. Therefore, as can be seen in the examples above, the two interpretations are the opposites of each other. However, this confusion can be eradicated by using a specific terminology as "within or between systems" or "within or between measure spaces" (Lamon, 2007, p. 634). In this study, this specific terminology will be used in order to avoid this conflict and for the sake of ease of understanding.

In this section, the structural relationships in a proportion and corresponding interpretations of these relationships were explained as essential understandings for proportional reasoning and clarified for the purposes of this study. Concerning this, the concepts of ratio and rate and the distinctions between them are described in the next section.

2.1.1. Ratio, rate, and proportion

A ratio, in general sense, is “a comparative index that conveys the abstract notion of relative magnitude” (Lamon, 1995, p. 169). In relation to this, a proportion is “the statement of equality between one ratio and another in the sense that both convey the same relationship” (Lamon, 1995, p. 171). With a different interpretation, it is also possible to define proportion as a function that satisfies the isomorphic properties “$f(x+y) = f(x) + f(y)$” and “$f(ax) = af(x)$” (Vergnaud, 1988, 1994). Besides, Freudenthal (1978) defines proportion based on the distinction he makes between external and internal ratios: a proportion, also referred to as linear mapping, is either obtained by mapping one magnitude upon another by preserving the internal ratios or by postulating the constancy of the external ratio (Freudenthal, 1978). In other words, the two ways to define proportions are as follows "by the equality of corresponding internal ratios" and "by the constancy of the external ratio of corresponding values" (Freudenthal, 1978, p. 294). Hence, ratio refers to a relative comparison between quantities, and proportion to the equality of these comparisons; in other words, equality of ratios or ratio preservation (Freudenthal, 1978).

Defining rate is rather difficult since there is a tendency of defining rate based on how it differs from ratio in the literature. It is specifically more difficult since there is a disagreement about how rates differ from ratios (Lesh et al., 1988; Thompson, 1994). In earlier studies based on the tradition of Greeks, ratios and rates were defined in relation to comparisons within and between measure spaces (Lamon, 2007). According to this type of definitions, ratio is thought as a comparison of quantities that belong to the same measure space, while rate is thought as a comparison of quantities that belong to different measure spaces. In this tradition, some researchers argue that rates include quantities in two different measure spaces (e.g., 20 km/4 hours) while ratios involve quantities within a single measure space (e.g., 10 books/6 books) (Vergnaud, 1983, 1988). For instance, Vergnaud (1988) refers to fractions as part-whole relationships between two quantities that have the same nature where "one is the part of the other" (p. 158) and stresses that the value of fractions is always less than one. Furthermore, he refers to ratios as part-part relationships between two quantities that have the same nature where one is not included in the other (e.g., number of girls/number of boys) and argues that its value can be anything (i.e., less than/equal to/greater than one). Lastly, Vergnaud (1988) defines rates as relationships between two quantities that have different natures and refers to them as functions.

On the other hand, some researchers follow Gauss in making distinctions between the two types of quantities: extensive quantities and intensive quantities rather than stressing whether the quantities belong to the same measure space or different measure spaces (Freudenthal, 1973; Kaput et al., 1986; Kaput & West, 1994; Schwartz, 1988). In this distinction, while extensive quantities express the extent of a quantity (i.e., how much) of an object (e.g., 4 apples or 10 kilometers, etc.), intensive quantities tell relationships between a quantity relative to a unit of the other quantity (e.g., 20 kilometers per hour, 30 students per a teacher, etc.). In this case, two quantities that belong to the same measure space can be intensive if they are compared in relation to another (e.g., getting an extra 5 points per ten-points earned). Thus, according to this interpretation, rate refers to a single intensive quantity, while ratio refers to a relationship between two quantities (Schwartz, 1988).

Lesh et al. (1988) argue that there are problems in both types of distinctions and attempt to clarify the concepts of ratios, rates, fractions, and quotients. According to Lesh et al. (1988), *rates* are intensive quantities that could be "recognized by the per in their unit labels," and ratios are "binary relations which involve ordered pairs of quantities (of either the extensive, intensive or scalar types)" (p. 112). Furthermore, fractions are "special kinds of extensive quantities; they tell the size of a single object" (p. 112). Lastly, quotients are "binary operations which combine two quantities (extensive, intensive, or scalar) by mapping them to a quantity in a third measure space" (p. 113). Thus, rates include only the unit rates (i.e., so many A's per 1 B) in Lesh et al.'s (1988) perspective.

Thompson (1994) stresses that while all of the researchers above focus on situations in order to make sense of the distinction between ratios and rates, it is also important to focus on the mental operations by which people make sense of them. Based on this notion, Thompson (1994) refers to a ratio as "the result of comparing two quantities multiplicatively" (p. 191) and rate as "a reflectively abstracted constant ratio" regardless of the measure spaces of the quantities (p. 192). The following quote highlights this interpretation and the change in the way of thinking about the relationship between the two quantities:

> When one conceives of two quantities in multiplicative comparison and conceives of the compared quantities as being compared in their *independent, static* states, one has made a ratio. As soon as one reconceives the situation as being that the ratio generally applies outside of the phenomenal bounds in which it was originally conceived, then one has generalized ratio to a rate. (Thompson, 1994, p. 192, emphasis in original)

Thus, in this way of interpretation, a comparison of two specific and fixed (i.e., non-varying) quantities is considered as a ratio. The comparison might be made between two quantities as wholes or in terms of comparison of one quantity relative to the units of the other quantity (Thompson, 1994). On the other hand, when this comparison between the two quantities is abstracted in order to interpret the ratio in relation to the invariant (i.e., constant) result of the multiplicative comparison, then, it is referred to as rate.

Thompson (1994) sees build-up strategies as a rich environment to facilitate students' process of reconceptualizing rate in terms of the constant ratio. In this process, while a student builds up in a succession of equivalent ratios (i.e., two oranges to three apples, four oranges to six apples, ….twenty oranges to thirty apples), he or she can abstract this relationship between the number of oranges and apples as "two oranges for three apples" (i.e., an iterable ratio between the number of oranges and apples that may vary) and ultimately they can arrive at the conclusion, "there should be 2/3 of an apple for every orange" or "the number of oranges will be 2/3 of the number of apples." This is a shift in students' ways of thinking from co-varying the number of oranges and apples to understanding that they co-vary in a constant (i.e., invariant) ratio to the other. While the conception at the beginning is referred to as an internalized ratio, the emerging conception that emphasizes the invariant relationship between the quantities is referred to as an *interiorized ratio,* or a rate (Thompson, 1994). In this study, the concepts of ratio and rate are interpreted in line with Thompson’s (1994) point of view since this interpretation fits the classroom data of this study.

In the previous sections, the within and between measure spaces types of comparisons and related approaches to the concepts of rate and ratio are expounded for the aim of clearing up those constructs before embarking on other essential understandings in proportional reasoning. In the following section, other critical components of proportional reasoning are presented from a Didactical Phenomenology perspective.

2.1.2. Didactical Phenomenology of proportional reasoning

According to Freudenthal (1983), a didactical phenomenological perspective should be followed for the teaching and learning of a subject/idea. In this process, critical elements of a subject area should be organized around learning sites that would help children gain essential understandings of that subject area. Therefore, Didactical Phenomenology describes the possible experiences and learning sites through which a student enters into the process of learning by organizing the phenomena and reconstructs the intended mathematical idea (Freudenthal, 1983).

Based on this notion, Lamon (1995) examines the types of experiences and learning sites that facilitate proportional reasoning and outlines a didactical phenomenology of ratio and proportion by stressing the critical components for the development of those concepts. These include relative and absolute change, covariance and invariance, and ratio sense. Besides, she refers to partitioning, relations, and unitizing as three interrelated areas of didactical activities that have the potential to foster children's understanding of those critical components.

The mathematical contents and early didactical activities related to the development of proportional reasoning proposed by Lamon (1995) are presented in Figure 2.3 below. According to Lamon (1995), it is crucial to facilitate the essential mathematical components of proportional reasoning in the first level (i.e., absolute and relative thinking, covariance and invariance, ratio appropriateness) by building the proportional reasoning instruction on the didactical activities on the second level (i.e., relationships, unitizing, partitioning) for the development of proportional reasoning (Lamon, 1995).

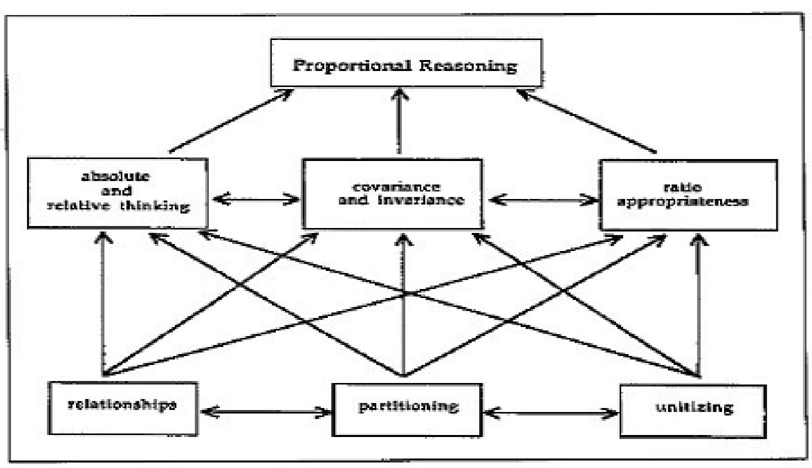


Figure 2.3. Some mathematical and didactical dimensions of proportional reasoning (Lamon, 1995, p. 177).

These mathematical components and didactical activities are explained in the following sections. Besides, another critical component (i.e., qualitative and quantitative reasoning) and didactical dimension of proportional reasoning (i.e., linking and iterating composite units) are discussed in order to extend this groundwork by Lamon (1995).

2.1.2.1. Critical mathematical components of proportional reasoning

Three mathematical components of proportional reasoning were documented by Lamon (1995) as absolute and relative thinking, covariance and invariance, and ratio appropriateness. One mathematical component (i.e., qualitative and quantitative reasoning) was added in this study in order to extend this framework. These four components are explained in the following sections.

2.1.2.1.1. Absolute and relative thinking

NCTM (2000) calls for attention to analyzing patterns of change in a variety of contexts. It recommends that students understand the change and develop a "deeper understanding of the ways in which changes in quantities can be represented mathematically" (NCTM, 2000, p. 305). In particular, analyzing and interpreting change in terms of both absolute and relative manners is one of the most critical types of thinking for the development of proportional reasoning (Lamon, 1995). The example given below might help understand the difference in these two types of thinking:

> Jo has two snakes, String Bean and Slim. Right now, String Bean is 4 feet long and Slim is 5 feet long. Jo knows that two years from now both snakes will be fully grown. At her full length, String Bean will be 7 feet long, while Slim's length when he is fully grown will be 8 feet. Over the next two years, will both snakes grow the same amount? (Lamon, 1995, p. 174).

It is possible to approach the problem from two perspectives: both snakes will grow 3 feet of length, which is the same amount (absolute change). This change represents the actual growth, regardless of anything else. However, another perspective is associated with expected growth relative to their present length (relative change). More specifically, String Bean is expected to grow $\frac{3}{4}$ of her present length (3 feet = $\frac{3}{4}$ of 4 feet), and Slim is expected to grow $\frac{3}{5}$ of his present length (3 feet = $\frac{3}{5}$ of 5 feet). Thus, relatively, they will grow in different amounts (Lamon, 1995). Therefore, it is essential to make sense of change in these two different perspectives for proportional reasoning.

Most importantly, it is vital for students to shift to an understanding in relative manners since ratio involves a relative and hence multiplicative comparison (e.g., “How many times its present length did each snake grow?”) (Lamon, 1995, p. 175). Although, Freudenthal (1978) asserts that the concepts “relatively” or “comparatively” are “rooted independently of ratio and proportion” (p. 297), students’ transition from thinking in absolute ways to relative ways might be difficult. Indeed, this transition might proceed through the following levels suggested by Freudenthal (1978):

> Understanding that what matters in certain orders is comparative order;
> understanding 'relatively' in the sense of 'in relation to... .', with the criterion of comparison filled in in the blank space;
> completing ‘relatively’ and ‘in relation to’ in a context;
> knowing what ‘relatively’ and ‘in relation to’ mean in general;
> explaining what ‘relatively’ and ‘in relation to . . .’ mean in general (p. 297).

In addition to interpreting change within a single quantity, it is also essential to construct an image of a quantity and coordinate images of two quantities and form an image of change in both quantities (Thompson, 1994). This is related to the concept of variation, more specifically, covariation, which will be explained in the following part.

2.1.2.1.2. Covariance and invariance

The quantities that compose a ratio vary together (i.e., covary); yet, the relation between them does not change (i.e., is invariant) (Lamon, 1995). For instance, buying three tickets for four dollars represents the same relationship as buying six tickets for eight dollars. Although the quantities that compose the ratio (i.e., the number of tickets and the amount of money) change, the relationship stays the same (i.e., one ticket for $\frac{4}{3}$ dollars). Therefore “all at the same time, something changes, but something else doesn’t change” (Lamon, 1995, p. 176). This implies that students should accommodate both covariation and invariation within a single situation (Lamon, 1995). Moreover, they need to understand that the second ratio (i.e., six tickets for eight dollars) is two sets of the first ratio (i.e., three tickets for four dollars).

Proportional relationships entail a simple form of covariation: two linked quantities change together (i.e., covary) in such a way that "when one changes, the other one also changes in a precise way with the first quantity" (Lamon, 2007, p. 648). In other words, covariation in proportional relationships includes interpretations such as “y varies as x” or “y is directly proportional to x” (Lamon, 2007, p. 648). For instance, let’s say, in a cake recipe, for every glass of sugar, one adds $2\frac{1}{2}$ glasses of milk. If he or she doubles the recipe and uses 2 glasses of sugar, then he or she has to double the amount of milk and use 5 glasses of milk so that the amount of sugar compared to the amount of milk always remains the same regardless of the size of the cake. Indeed, in any size of the cake made by using this recipe, there is always 2 times as much milk as there is sugar. That is to say, the amount of sugar and milk can vary together (i.e., covariation), conditioned that the relationship between them is preserved (invariation). It is important for proportional reasoning to "look beyond the given quantities to construct a new quantity… that derives from the relationship of the two changing and connected quantities; this new quantity remains constant" (Lamon, 2007, p. 649). Therefore, being able to interpret and work with covariance and invariance at the same time in a proportional situation is an essential component of proportional reasoning.

Concerning understanding covariation and invariation, there is a mass of literature on covariational reasoning, which is defined as "the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other” (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002, p. 354). Although it is mostly seen as a way to think about functions (Confrey & Smith, 1994), its development starts in middle school or even earlier within the context of early covariational reasoning (Ellis et al., 2016).

To discuss a few, Confrey and Smith (1994) describe covariation as a process that includes coordination of movements from y_m to y_{m+1} and from x_m to x_{m+1}. That is to say, a covariational approach "entails moving between successive values of one variable and coordinating this with moving between corresponding successive values of another variable" (Saldanha & Thompson, 1998, p. 298). According to Confrey and Smith (1994), a covariation approach is easy and intuitive for the development of the concept of function. Moreover, coordinating the invariant relationship between the two quantities in tables by building up or down can "also lead nicely to an algebraic coding of the correspondence rule for a function" (Confrey & Smith, 1994, p. 33). Therefore, exploring covariance and invariance in tables have the potential to foster the algebraic notation of a function that represents a proportional relationship. For instance, building-up on a table by covarying the quantities such as 1-3, 2-6, 3-9, and so on, students can obtain the relationship between these two variables as $y = 3x$.

Coulombe and Berenson (as cited in Saldanha & Thompson, 1998) describe that covariational reasoning entails

> (a) the identification of two data sets, (b) the coordination of two data patterns to form associations between increasing, decreasing, and constant patterns, (c) the linking of two data patterns to establish specific connections between data values, and (d) the generalization of the link to predict unknown data values (p. 88).

Therefore, at the heart of covariation is coordinating sequences in successive manners. Tables can be effective tools to present these successive states of quantities that vary together. In doing so, seeing covariation can be described as "holding in mind a sustained image of two quantities' values (magnitudes) simultaneously. It entails coupling the two quantities, so that, in one's understanding, a multiplicative object is formed of the two" (Saldanha & Thompson, 1998, p. 298). Saldanha and Thompson (1998) argue that these images of covariation are developmental, and "in early development one coordinates two quantities' values- think of one, then the other, then the first, then the second, and so on" (p. 298).

2.1.2.1.3. Ratio appropriateness (ratio sense)

Ratio appropriateness or ratio sense deals with the ability to distinguish between situations that "are appropriately organized by ratios and those that are not" (Lamon, 1995, p. 176). The first stage includes being aware of the relationship between two quantities and the invariance of this relationship. Then, it is critical to distinguish examples and nonexamples of proportional relationships and to determine the conditions under which a situation can be represented by proportional relationships. In addition, ratio sense includes the use of correct mathematical language and vocabulary for discussing proportional relationships. This should include informal ways of expressing ratios (Lamon, 1995). These informal ways include "per language" (e.g., 3 food bars per 5 fish) and other ways to compare quantities intensively (e.g., for every 10 students there are 2 teachers, out of every 12 people 2 of them are teachers, etc.). Thus, ratio sense refers to an “intuitive sense about the contexts and the mathematical relationships associated with proportions” (Lamon, 1995, p. 176) by analyzing various relationships that are proportional or nonproportional.

One way to distinguish proportional and nonproportional situations is to focus on additive and multiplicative relationships and being able to reason additively and multiplicatively for appropriate situations. In addition, thinking about change in absolute manners involves additive reasoning while thinking about change in relative terms entails multiplicative reasoning. These terms will be used instead of relative and absolute change throughout the dissertation. Besides, being able to discern additive and multiplicative relationships and to apply the correct strategy will be considered in terms of having a ratio sense.

2.1.2.1.4. Quantitative and qualitative reasoning

Quantitative reasoning is "the analysis of a situation into a quantitative structure - a network of quantities and quantitative relationships" that focuses on relationships among quantities rather than numbers (Thompson, 1993, p. 165). Hence, quantitative reasoning entails an analysis of quantities and their relationships, the creation of new quantities, and making inferences based on those (Thompson, 1994). First, it is important to understand what a quantity is in order to grasp quantitative reasoning. A quantity "is not the same as a number. A person constitutes a quantity by conceiving of a quality of an object in such a way that he or she understands the possibility of measuring it" (Thompson, 1993, p. 165). In other words, although quantities might have numerical values when measured, one does not have to know those values in order to think about them (Thompson, 1993). For instance, it is possible to compare two people's heights without actually knowing them. Therefore, quantitative reasoning should not be confused with numerical reasoning since quantitative reasoning can be numeric or nonnumeric in nature (Lobato, & Siebert, 2002; Thompson, 1994).

Though it is not the opposite of quantitative reasoning, maybe a specific case of it, qualitative reasoning involves deciding about the order relations between ratios/rates without depending on the numbers (Behr, Harel, Post, & Lesh, 1992). These situations are represented by the equation $a/b = c$, and the task requires children to reason about the direction of change (or no change at all) rather than the amount when one or two of the three values change (i.e., decrease or increase). The required reasoning should be as follows: when a remains the same and b increases, a/b decreases; when b remains the same and a increases, a/b increases; when a and b both increase, the situation is indeterminant, and quantitative type of reasoning is required (Behr et al., 1992). Then, determinability and determination are essential aspects of qualitative reasoning. Determinability is related to the question "Can the order relation requested in the problem be determined through qualitative reasoning?" while determination is related to the question: "What is the order relation requested in the problem, if it can be determined?"

(Behr et al., 1992, p. 317). Once determinability is assured, then the relation can be ascertained as less than, equal to, or greater than (Behr et al., 1992). The nine ways in which the effects of increasing/decreasing numerator and/or denominator on the value of a rate are presented in Figure 2.4 below.

Changes in the Value of a Rate When the Numerator and Denominator of the Rate Changes

	Denominator		
Numerator	Increases	Stays same	Decreases
Increases	Can't tell	Increases	Increases
Stays same	Decreases	Stays same	Increases
Decreases	Decreases	Decreases	Can't tell

Figure 2.4. Changes in the value of a rate when the numerator and/or denominator of the rate changes (Heller, Post, Behr, & Lesh, 1990)

As seen in Figure 2.4 above, two of these nine cases result in a situation in which the direction of change in the value of a rate is indeterminant. These are the cases when both components of the rates increase or decrease at the same time.

Streefland (1985) argues that qualitative comparisons are at the onset of learning of ratio and proportion. Young children (aged 6-7) can informally and qualitatively deal with situations involving ratio and proportion by a natural form of mathematical intuition that develops independently of instruction (Lamon, 1994; Streefland, 1985; Van den Brink & Streefland, 1979). This intuitive and informal knowledge is mostly based on a visual understanding related to similarity and congruence (Harel et al., 1992; Lamon, 1994; Streefland, 1985). On the other hand, Kaput and West (1994) report that, of the many missing value problems, similarity problems are the most challenging problems for sixth-grade students.

As different from this intuitive and informal knowledge of qualitative reasoning that is rooted in a visual understanding of congruence, qualitative reasoning of children has been the subject of studies through a different interpretation that includes making inferences about the direction of change and determinability of change. To begin with, Heller, Ahlgren, Post, Behr, and Lesh (1989) examined seventh-grade students' reasoning in both numerical and qualitative tasks that were either missing value or comparison tasks. Two examples, one of which deals with a ratio change and the other with qualitative comparison, are presented in Figure 2.5 below.

Qualitative Ratio Change:

If Cathy ran less[2] laps in more time than she did yesterday, her running speed would be
(a) faster
(b) slower
(c) exactly the same
(d) there is not enough information to tell

Qualitative Comparison:

Bill ran the same number of laps as Greg. Bill ran for more time than Greg. Who was the faster runner?
(a) Bill
(b) Greg
(c) they ran at exactly the same speed
(d) there is not enough information to tell

Figure 2.5. Qualitative directional reasoning problems (Heller et al., 1989, p. 211)

Heller et al. (1989) found that some of the participants performed well on numerical reasoning tasks, whereas the same participants performed worse in qualitative reasoning tasks. They concluded that while problems including numerical values can be approached by rotely memorized algorithms without having good skills in qualitative reasoning (Heller et al., 1989), qualitative reasoning tasks require approaching problems in unusual ways (Billings, 2002). Therefore, one can solve numerical problems without being able to reason qualitatively. In other words, numerical reasoning does not warrant qualitative reasoning. Another conclusion was that qualitative reasoning might improve performance on numerical reasoning tasks; yet, it is not sufficient for proportional reasoning. Therefore, a person should be able to reason both qualitatively and quantitatively in order to become a proficient proportional reasoner.

In another study by Larson, Behr, Harel, Post, and Lesh (1989), seventh-grade students' qualitative reasoning was explored through tasks asking students to decide if the value of a ratio would change and, if changes, to determine the direction of the change (i.e., decrease, stay the same, increase). According to the results of the study, some of the students indicated that the value of a ratio would change in the same direction as its components when the two components change in the same direction. In addition, some students stated that the direction of change in the value of a ratio depends on the amount of change in numerator and denominator: if the denominator increases more than the numerator, then, the value of the ratio would decrease; if the numerator increases more than the denominator, then the value of the ratio would decrease. Therefore, it was concluded that seventh-grade students had a sense of the direction of change, though it is incomplete.

In a third study, Heller, Post, Behr, and Lesh (1990) investigated 467 seventh grade and 522 eighth grade students' performance on quantitative and qualitative reasoning tasks. Two sample problems for qualitative reasoning tasks that were used in the study are given in Figure 2.6 below.

QR[b]: If Nick ran fewer laps in more time than he did yesterday, his running speed would be—
- **a) faster.**
- **b) slower.**
- **c) exactly the same.**
- **d) There is not enough information to tell.**

QC: Nancy's movie line has more people in it than Kathy's line. Nancy's line is longer than Kathy's line. In which line are the people closer together?
- **a) Nancy's line**
- **b) Kathy's line**
- **c) The people are spaced exactly the same.**
- **d) There is not enough information to tell.**

Figure 2.6. Qualitative directional reasoning problems (Heller et al., 1990, p. 391)

The results of the study revealed that approximately 60% of the seventh and eighth-grade students correctly answered the qualitative reasoning tasks with a significantly higher performance of eighth-graders. Besides, seventh grade and eighth-grade students were found to use similar response patterns in these tasks. Moreover, only one-fifth of the students in grade seven and one-fourth of the students in grade eight were able to answer the questions related with the two indeterminate situations correctly (i.e., increasing/increasing and decreasing/decreasing). Lastly, the results of this study confirmed the results of a previous study by Heller et al. (1989) stressing that numerical reasoning tasks can be solved by students who performed low in qualitative reasoning tasks, although a high performance in qualitative reasoning tasks assures greater success in numerical reasoning tasks.

Based on the results of these studies, the development of qualitative reasoning and whether it precedes or proceeds numerical reasoning is ambiguous. As can be interpreted from the discussions above, there is an aspect of qualitative reasoning that develops long before students can deal with proportional situations numerically (Lamon, 1994; Streefland, 1985). However, it is also reported that many students could deal with numerical reasoning tasks but failed in qualitative reasoning tasks (Heller et al., 1989, 1990). Therefore, there is a need to make a distinction between expert qualitative reasoning and intuitive and informal qualitative reasoning as follows: while qualitative reasoning applied by experts is mostly rooted in scientific principles that are used to form relationships and based on principles of the content domain, qualitative reasoning applied by novices is based on intuitive knowledge and superficial relationships among the problem components (Behr et al., 1992).

2.1.2.1.5. Reasoning within and between measure spaces (multiplicative reasoning)

It is stressed in many studies that at the heart of multiplicative reasoning is creating composite units and being able to work with composite units (Battista & van Auken

Borrow, 1995; Lamon, 1994; Steffe, 1988). The goal for creating a composite unit is usually to reinterpret a situation in relation to that unit, which is referred to as norming (Freudenthal, 1983; Lamon, 1994). One of the most prevalent uses of norming is to determine a scale factor within a measure space wherein one value is a scalar multiple of the other. In this process, one value is reinterpreted in relation to the other value by "using the process of scalar decomposition" (Lamon, 1994, p. 95), which is depicted in Figure 2.7 below.

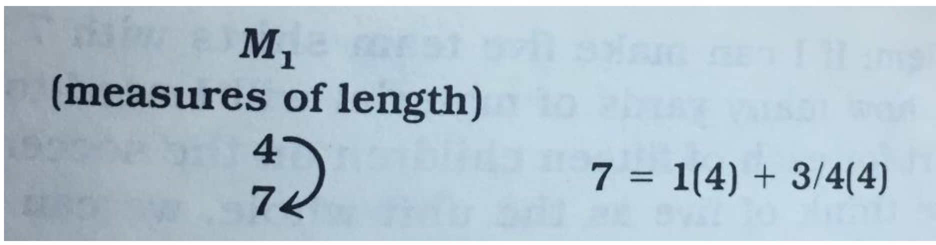


Figure 2.7. A scalar operator transposes a measure within a single measure space (Lamon, 1994, p. 95)

As seen in the figure above, the unit whole is 4, and 7 is reinterpreted in relation to four units, which includes a whole-four-unit and a certain fractional part (i.e., $\frac{3}{4}$) of the four-unit. This ability of norming and working with a scalar operator is critical for understanding proportional situations since the values within the same measure spaces are linked by the same scalar operator in a proportional situation, as illustrated in Figure 2.8 below.

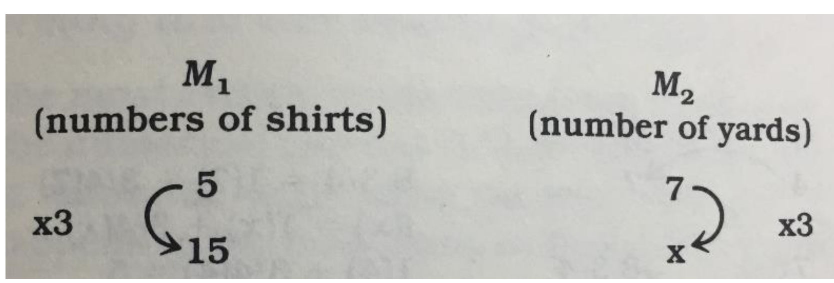


Figure 2.8. The scalar method for finding a missing value in a proportion: an instance of norming (Lamon, 1994, p. 95)

This relationship can be used to find a missing value in a proportional situation (i.e., "If I can make five team shirts with 7 yards of material, how many yards of material will I need to make a team shirt for each of fifteen children on the soccer team?" (Lamon, 1994, p. 96), which is referred to as a within strategy or a scalar method (Lamon, 1994). In this method shown in Figure 2.8, five is thought as the unit whole, and fifteen is considered as three of the five-unit whole. Therefore, the scalar operator in this situation is three.

On the other hand, a between strategy or a functional method for the same situation would include equating two ratios in two different measure spaces (i.e., between measure spaces). This relationship can also be useful for finding a missing value in a proportional situation (i.e., "The pharmacist gave you 7 ounces of medicine for $8.75. What would you expect to pay for a bottle containing 4 ounces?" (Lamon, 1994, p. 96), as illustrated in Figure 2.9 below.

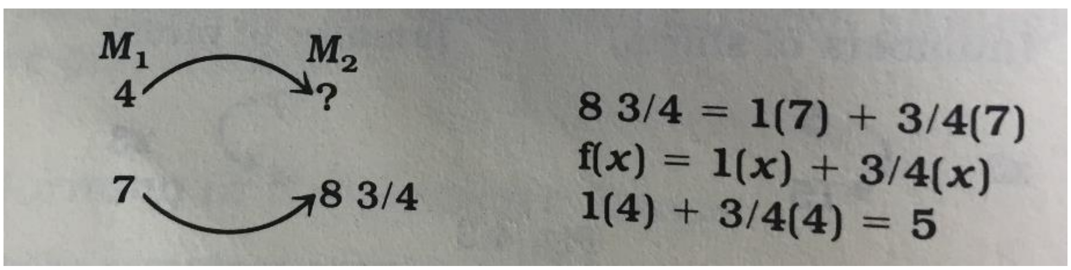


Figure 2.9. Finding the function operator through a norming process (Lamon, 1994, p. 96)

As seen in the figure above, the process of norming also applies to this method since 7 ounces is considered as a whole unit and $8.75 (i.e., 8 dollars + $\frac{3}{4}$ dollars) is reinterpreted in terms of 7. However, this interpretation is a little bit different than the previous one. Here, the function operator does not represent a scaling; rather, it is the coefficient of the mapping (i.e., a linear function) from M_1 to M_2 (Lamon, 1994).

Several studies in the literature suggest that a within-measures-strategy is more natural than a between-measures-strategy (Karplus et al., 1983; Vergnaud, 1980). A piece of further evidence that a within-measures-strategy is more natural comes from the fact that it was the only accepted form of proportional reasoning until the 14th century (Freudenthal, 1978; Karplus et al., 1983). However, it should be noted that a between-measures-strategy is “more faithful to the problem’ structure” (Karplus et al., 1983, p. 221) since it deals with a functional relationship.

In the previous sections, the essential mathematical components of proportional reasoning were described. Another level in Lamon's (1995) work related to the didactical phenomenology of proportional reasoning includes didactical dimensions of proportional reasoning through which those components are to be facilitated. In the next section, these dimensions are explained in detail.

2.1.2.2. Didactical dimensions of proportional reasoning

Didactical activities that are supposed to facilitate students’ understanding of proportional situations include students’ already existing strengths prior to instruction. The earliest and intuitive experiences of children in relation to ratio and proportion include understanding relationships. This can involve making judgments regarding scaled drawings based on visual interpretation and noticing distortion, and interpreting covariation in basic terms. Other experiences include partitioning (e.g., fair-sharing, especially halving and doubling) and unitizing (creating units, both 1-unit and other units- and deciding on efficient units) (Lamon, 1995). These didactical activities have the potential to facilitate children’s understanding of the mathematical components discussed above. In the following sections, how those mathematical components can be fostered through the early didactical activities will be discussed in detail.

2.1.2.2.1. Relationships

Among the earliest experiences young children have in relation to ratio and proportion is a visual and intuitive level that includes evaluating the relative size relationships between the parts of an object or between various objects (Lamon, 1995). For instance, it is possible that most preschool children can intuitively reason about whether a drawing or a picture appear right or wrong in terms of scaling or enlargement (Lamon, 1995). That is, they can make judgments regarding the distortion of pictures. Besides, they can make intuitive judgments about basic proportional relationships (e.g., the more money you have, the more food you can buy, the closer the object is, the bigger it looks). Although students have these kinds of intuitive understandings of different relationships, they might have difficulties in verbalizing those relationships (Lamon, 1995). Hence, the instruction should be built upon children's existing strengths related to these understandings of relationships in order to move students to a more formal and mathematical understanding of those relationships. In this process, early experiences related to counting, matching, and partitioning can play an important role (Lamon, 1995).

2.1.2.2.2. Partitioning

Another early experience of children that is essential in the development of proportional reasoning is partitioning that entails subdividing a whole into equal parts (Lamon, 1995; Pothier & Sawada, 1983). Children deal with sharing among their siblings, starting from early childhood (Lamon, 1995). These experiences are mostly based on halving and splitting in to equal parts whose denominators are powers of two (Pothier & Sawada, 1983). In later years, they can deal with other fractions that have odd denominators (Pothier & Sawada, 1983). During the instruction on proportional reasoning, students should discuss various ways to partition a whole and decide which partition is the most effective one (Lamon, 1995).

Recently, the term "equipartitioning" has been used more often to refer to the cognitive behaviors in order to produce "equal-sized groups (from collections) or equal-sized parts (from continuous wholes), or equal-sized combinations of wholes and parts, such as is typically encountered by children initially in constructing fair shares for each of a set of individuals" (Confrey, Maloney, & Corley, 2014a, p. 724). Therefore, even though it is not the same as partitioning, it refers to a more general term used for splitting or producing equal shares since partitioning can involve breaking into unequal parts that have the same size (Confrey et al., 2014a). These researchers suggest that equipartitioning is foundational for rational number reasoning; hence, proportional reasoning (Confrey, Maloney, Nguyen, & Rupp, 2014b).

Confrey et al. (2014a) outlines the proficiency levels of equipartitioning and proposes a sixteen-level learning trajectory. According to this trajectory, the lower levels (i.e., Levels 1-5) entail equipartitioning of collections and wholes, being able to justify strategies and results, name each fair share in terms of fractional language (i.e., $\frac{1}{n}$ th), and identify the size of the whole based on a single fair share (i.e., n times as many/much). Besides, the middle levels (i.e., Levels 6-11) consist of a variety of relationships and properties regarding the equipartitioning of single wholes (e.g., geometrical shapes). These also include the composition of splits, judging the inverse relationship between the number of shares and size of shares, and understanding that a whole can be shared into any number of pieces. Lastly, the upper levels (i.e., Levels 12-16) include being able to work with multiple wholes based on the experiences gained in the lower levels.

Therefore, partitioning is one of the essential didactical activities that is inherent in young children's early experiences. Another didactical activity is unitizing, which is the opposite process of partitioning (Lamon, 1995). The activity of unitizing is explained in the following section.

2.1.2.2.3. Unitizing and norming

Lamon (1994) points to the ability of "unitizing" as one of the critical abilities for the development of proportional reasoning. Unitizing is defined as "the ability to construct a reference unit or a unit whole, and then reinterpret a situation in terms of that unit" (Lamon, 1994, p. 92). In this process, children, starting from early childhood, create composite units in progressively sophisticated ways to form complex structures of quantities (Lamon, 1994). The beginning of this process of creating composite units possibly traces back to early childhood, where visual quantifying (e.g., subitizing) takes place and then is extended to counting (Lamon, 1994). It is required to conceptually coordinate multiple compositions in order to develop addition and subtraction schemes into multiplicative structures (Lamon, 1994). Below is an example of a simple multiplicative structure, as illustrated in Figure 2.10 below:

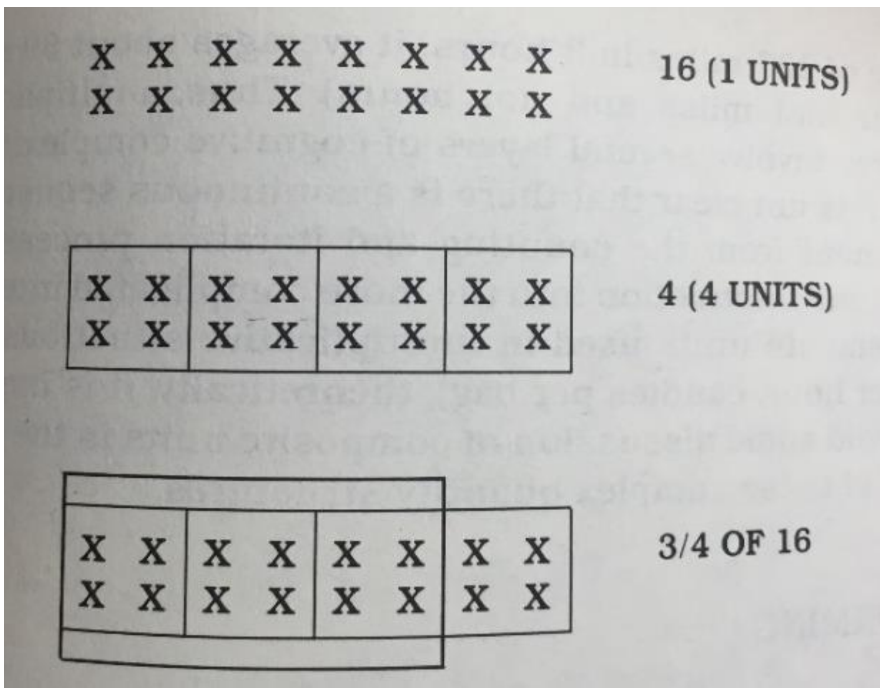


Figure 2.10. A simple multiplicative structure (Lamon, 1994, p. 93)

Students begin with seeing 16 objects as 16 singleton units (i.e., 16 one units), form units of units (i.e., four composite units each including 4-one singleton units), and then form units of units of units (i.e., one three-unit including three of the four four-units). In addition, multiplicative structures may consist of two entities in different measure spaces to form an entity different from either of the entities. For instance, if an airplane travels

1800 kilometers in two hours, its average speed is 90 kilometers per hour. While the two entities are the distance (in kilometers) and time (in hours), they form a totally different entity (i.e., speed) having a different measure (i.e., kilometers per hour) when they come together in a multiplicative situation.

While seeing a relationship as a single entity and operating with it seems to be the basic ability, making sense of the relative nature of entities in a ratio seems to be a higher level for understanding ratio and proportion (Lamon, 1994). Based on this, a number of processes in which students' reasoning evolves into more sophisticated levels as they interpret ratio and proportion is summarized in Figure 2.11 below.

TABLE 4.3. THE INCREASING COMPLEXITY OF THE UNITIZING PROCESS AS REVEALED BY CHILDREN'S THINKING IN THE ADDITIVE AND MULTIPLICATIVE CONCEPTUAL FIELDS

	+ or −	× or ÷	*Ratio*
Modeling and Counting	Counting objects 1, 2, 3, 4, 5 Subitizing: The visual formation of composite units △△△ △△ 5	Segmented counting: forming experential composites How many 3s in 12: 1, 2, 3; 4, 5, 6; 7, 8, 9; 10, 11, 12	Counting triggered by pictorial presentation; forming experential ratios: matching three aliens and five food pellets
Composing	Constructing a numerical composite: (hand) A A A 5 6 7 8	Early multiplication scheme: repetitively forming and counting three units: 1 − 2 − 3 1 1 − 2 − 3 2 1 − 2 − 3 3 4 × 3 1 − 2 − 3 4	Repetitively matching and counting three units and five units 1 − 2 − 3 1 − 2 − 3 − 4 − 5 1 1 − 2 − 3 1 − 2 − 3 − 4 − 5 2 1 − 2 − 3 1 − 2 − 3 − 4 − 5 3
Abstracting	Constructing an abstract numerical composite: thinking of one five-unit without considering its constituent one units	Forming iterable units: 3 is stripped of its composite quality: 3 is 1 6 is 2 9 is 3 double counting 12 is 4	Iterable ratios: 6 : 10 and 9 : 15 represent the same relationship as 3 : 5
Relating	PPW: the whole is a numerical composite whose parts are also composites	PPW, the distributive property: eight four units can be achieved by adding five four units and three four units; parts are multiples of the same composite units	PPW, a multiple of the whole is composed of multiples of different composite units: 5(8-units) = 5(3-units) + 5(4-units) + 5(1-units)

Figure 2.11. The increasing complexity of the unitizing process as revealed by children's thinking in the additive and multiplicative conceptual fields (Lamon, 1994, p. 114)

In order to interpret the table above in terms of increasing levels of sophistication in student thinking, the route from left to right between the columns and from top to bottom between the rows should be followed (Lamon, 1994). Early counting activities that include grouping of objects (i.e., subitizing) are the early foundations of forming composite units. This process involves grouping single items to construct a single quantity on itself. A more sophisticated level includes matching quantities and using them simultaneously. Using pictorial representations enhances the ability to coordinate these entities, two at a time (Lamon, 1994). The next level is abstracting wherein operating with the composite unit takes place without attending to its elements in such a way that the composite unit is an entity on itself. The more sophisticated level includes relating the abstract numerical composites to others. For instance, understanding the parts as composite units as well as the whole in part-whole relationships is a sophisticated level of thinking that is included in this stage.

2.1.2.2.4. Linking composite units and iterating linked composites

Another didactical activity essential for proportional reasoning is linking composite units and iterating linked composites, which students' proportional reasoning is said to be built on (Battista & van Auken Borrow, 1995; Park & Nunes, 2001; Steffe, 1988). In order to understand these, it is essential to look at the teaching experiment studies of Steffe (1988, 1994) related to the construction of number sequences and multiplying schemes wherein the early foundations for proportional reasoning is outlined.

In his study, Steffe (1988, 1994) focuses on young children's transition from initial number sequences (i.e., counting by ones 1-2-3-4-5 and so on) to creating composite and iterable units. In this process, children first construct a unit of units, take it as one thing, and operate with that abstract composite unit by keeping track of how many times this composite unit is iterated. According to Steffe (1994), coordinating at least two composite units and distributing one of them over the elements of the other is necessary in order to establish a situation as multiplicative. Therefore, the ability to iterate composite units is crucial for the development of multiplicative reasoning, and hence proportional reasoning.

Iterating composite units is defined as the ability to take one group as a unit and iterate this unit without changing the nature of its elements (Steffe, 1994). An example of a process of iterating composite units by Steffe (1994) might support understanding of this skill better. A big red piece of paper and several blue pieces of paper of the same size and in rectangular forms, six of which would fit on the red paper, are placed in front of a student. The student is asked to find the number of blue pieces that would fit on the red paper. After she places six blue pieces on the red paper and answers the question as six, the implementer removed three of these blue pieces and put two orange square pieces on a blue piece. The student, then, is asked how many orange squares would fit on the red paper by considering the blue pieces and without concretely fitting the pieces. To answer this question, the student reasons like 1-2, 3-4, 5-6, 7-8, 9-10, 11-12, and gives the answer 12, in which she coordinates two-for-ones and distributes the units of two over the units of six.

According to Steffe (1994), this type of reasoning is not repeated addition since it includes linking units by making one-to-many correspondences and iterating those linked composites. When a student is asked, "*What is the number of total cubes when there are 9 rows and 3 cubes in each row?*" and he is able to iterate this unit such as 3, 6, 9, 12, 15, … nine times; then, it is evident that he takes 3 cubes as one unit (iterating unit) and does not need to distribute a unit of three over the units of nine. In other words, the child is able to create a unit of three and iterate this unit by keeping track of these iterations in order to establish nine units of three. Steffe (1994) calls this process "an iterating concept of multiplication and take it as a root of repeated addition" (p. 23). Steffe (1994) concludes that multiplication should not be taught depending solely on either repeated addition or distributing a (composite) unit over the elements of another (composite) unit.

Continuing the works of Steffe; Battista and van Auken Borrow (1995) claimed that once students gain the ability to iterate composite units, this thinking can be extended to the ratio concept and proportional reasoning. In order to describe this transfer to proportional situations, they conducted a study with a second-grade student in which the interviewer created a bundle of 5 white and 3 red sticks and asked the student the number of the same kind of bundles behind his back if he had 10 white sticks. The student immediately answered as "5, 10, … then 2." Then, she was asked the number of white sticks if there were 12 red sticks. The student answered as "you need for bundles to get 12 reds. Then 5, 10, 15, 20" and hence is considered as having the ability to iterate composite units of 3 and 5. Battista and van Auken Borrow (1995) claimed that the student extended his counting scheme to construct a "linked composite" counting sequence since she was able to iterate a composite consisting of a composite of 3 linked together with a composite of 5. Therefore, she managed to iterate linked composites in order to solve a proportional situation.

Lamon (2007) refers to this process as "reasoning up and down" and gives an example of this type of thinking with two students' works for the problems as follows:

If a box of detergent contains 80 cups of powder and your washing machine recommends $1\frac{1}{4}$ cups per load, how many loads can you do with one box?

Think: $1\frac{1}{4}$ cups do 1 load
5 cups do 4 loads
40 cups do 32 loads
80 cups do 64 loads

These are $\frac{2}{3}$ of Joan's pennies. How many pennies are $\frac{1}{2}$ of all she has?

The problem gives information about $\frac{2}{3}$ of the unit; from there, one can reason down to $\frac{1}{3}$, then to 1 (because three $\frac{1}{3}$s make the whole set of pennies), then to $\frac{1}{2}$.

Think: 8 pennies are $\frac{2}{3}$
4 pennies are $\frac{1}{3}$
12 pennies are 1 whole set
6 pennies are $\frac{1}{2}$ of Joan's pennies

Figure 2.12. Reasoning up and down (Lamon, 2007)

As seen in Figure 2.12 above, students link the number of cups and loads and create a list of corresponding amounts by iterating the linked composites of the number of cups and loads in order to arrive at the amount they need to find. In this process, they do not reason with the symbolic representation of proportion (i.e., $\frac{a}{b} = \frac{c}{d}$). These kinds of strategies are also called building-up or down strategies in the literature (Kaput & West, 1994; Thompson, 1994).

To sum up, the essential mathematical components of proportional reasoning include absolute and relative thinking (Lamon, 1995), understanding covariance and invariance (Carlson et al., 2002; Confrey & Smith, 1994; Ellis et al., 2016; Lamon 1995, 2007; Saldanha & Thompson, 1998), ratio sense (Lamon, 1995), qualitative and quantitative reasoning (Behr et al., 1992; Heller et al., 1990, Thompson, 1993), and reasoning within and between measure spaces (Freudenthal, 1978; Karplus et al., 1983; Lamon, 1994; Vergnaud, 1981). Besides, the informal and intuitive experiences that young children have prior to instruction, which are referred to as didactical activities in this study, are mostly based on relationships (Lamon, 1995), partitioning / equipartitioning (Confrey et al. 1994a, 1994b; Lamon, 1995), unitizing and norming (Lamon, 1994, 1995), and iterating linked composites (Battista & van Auken Borrow, 1995; Steffe, 1994). These mathematical components and didactical activities were explained in general terms in the previous parts.

It is also essential to understand informal knowledge that students have before instruction and their developmental progress in proportional reasoning for designing and implementing instructional sequences. In particular, it is vital to determine the informal knowledge of students so that effective instruction could be built on that intuitive knowledge and strategies (Kaput & West, 1994; Lamon, 1994). Therefore, in the next section, students' informal strategies for solving specific proportion problems prior to ratio and proportion instruction will be portrayed in order to understand the roots of proportional reasoning and its developmental progress.

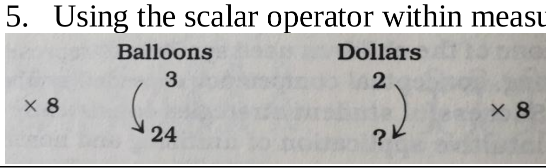
2.1.3. Students' informal strategies for solving proportion problems

Students' strategies for solving proportional problems are mostly based on the mathematical components and didactical activities that were discussed in the previous parts. However, it is helpful to delve into how they make use of those kinds of reasoning and activities in order to solve proportional problems. In this part, students' strategies for solving specific problems will be expounded, starting from students' informal strategies before they were instructed on ratio and proportion.

Lamon (1994) reports the results of her study in which she conducted clinical interviews with 24 sixth grade children prior to formal instruction in ratio and proportion. The purpose of the study was to outline increasingly sophisticated ways of reasoning as students solve ratio and proportion problems. In particular, through these problems, the goal was to investigate students' tendency to work with singleton units or composite units and to use within or between strategies and to examine the ability to deal with multiple compositions. The problems used in the study and related informal strategies of students are presented in the following pages.

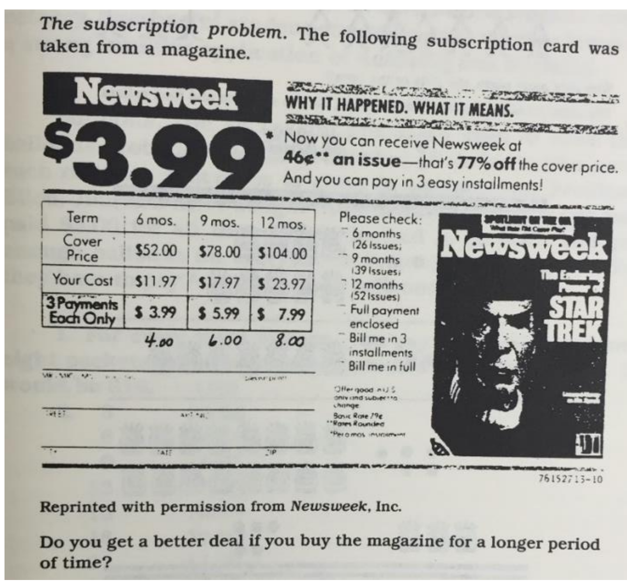
The first problem that is "The Balloon Problem" included a situation about finding the cost for 24 balloons, when the cost of 3 balloons was known. This problem and the strategies used by the students are shown in Table 2.1 below.

Table 2.1. The balloon problem and students' informal strategies (Lamon, 1994)

Problem	Strategies
Problem 1: "The balloon problem: Ellen, Jim, and Steve bought 3 helium-filled balloons and paid $2.00 for all three. They decided to go back to the store and get enough balloons for everyone in their class. How much did they have to pay for 24 balloons?" (Lamon, 1994, p. 101).	1. Reasoning with a composite unit of 3 "For every third balloon, you pay $2.00. So you have eight packets (sets, groups, or bunches of eight) and 8 × 2 would be $16." (p. 103) 2. Constructing a table to keep track of doubling counting (p. 103) 3 — $2.00 6 — 4 9 — 6 12 — 8 15 — 10 18 — 12 21 — 14 24 — 16 3. Two-step unit rate strategy with an inaccurate answer (p. 103) $2.00/3 = .6666 … or .66 r2 .66 × 24 = $15.84 4. Reasoning with the unit rate (p. 103) "Three balloons were $2.00 and 2 divided by 3 is 2/3, so I asked myself how many 24ths is 2/3? The answer is 16." 5. Using the scalar operator within measures space (p. 104) 

Therefore, as seen in the table above, the first informal strategy for the balloon problem was related to working with a composite unit of 3 and reasoning with the number of composite units of 3 that would go into 24 and multiplying this number with 2 that is the amount for every third balloon. It was reported that 10 of the 24 students employed this strategy for this problem. The second strategy for the balloon problem, which was used by the two of the participants, included keeping track of a double-counting scheme by constructing a table and extending the table upon pattern recognition. Lamon (1994) reported that the two of the students that used this strategy were not able to recognize the functional relationship between the number of balloons and the related cost; hence, they could not find the cost for a given number of balloons. The third strategy employed by three students was named as two-step unit rate strategy in which the students reasoned with single units but ended up with an inaccurate answer. Lamon (1994) reported that this kind of reasoning with single units was less powerful than reasoning with composite units, though it was more sophisticated. However, another student was able to reason with the unit rate without converting it to decimal and thinking as, *"how many 24ths would go into 2/3*?" The work that involved recognizing the structural relationships within the measure spaces and operating with the scalar operator by a student was reported as the last strategy for the balloon problem.

The second problem in Lamon's (1994) study, The Subscription Problem, involved information regarding different costs for monthly (i.e., 6 months, 9 months, or 12 months) subscriptions to a magazine and asked about comparing these deals. This problem is presented in Figure 2.13 below, and students' informal strategies for solving this problem is depicted in the table that follows.


The subscription problem. **The following subscription card was taken from a magazine.**

Newsweek

$3.99*

WHY IT HAPPENED. WHAT IT MEANS.

Now you can receive Newsweek at **46¢** an issue**—that's **77% off** the cover price. And you can pay in 3 easy installments!

Term	6 mos.	9 mos.	12 mos.
Cover Price	$52.00	$78.00	$104.00
Your Cost	$11.97	$17.97	$ 23.97
3 Payments Each Only	$ 3.99	$ 5.99	$ 7.99
	4.00	6.00	8.00

Please check:
- 6 months (26 Issues)
- 9 months (39 Issues)
- 12 months (52 Issues)
- Full payment enclosed
- Bill me in 3 installments
- Bill me in full

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Reprinted with permission from *Newsweek*, Inc.

Do you get a better deal if you buy the magazine for a longer period of time?

Figure 2.13. The Subscription Problem (Lamon, 1994, p. 101)

Students' informal strategies for the subscription problem (Lamon, 1994, p.104);

1. Reasoning with equal scale factors within measure spaces
"The price for two 6-month subscription is the cost of 12 months and 9 is just the middle of that."
2. Reasoning with the equivalent fractional parts of the wholes
"4 is 2/3 of $6.00 and 6 is 2/3 of 9."
3. Scaling up and down the cost for different monthly subscriptions
"Three payments of $ 4.00 is $12.00; three payments of $6.00 is $18.00; three payments of $8.00 is $24.00. Divide each in half and you get 6, 9, and 12."
4. Recognizing the number patterns of increase in both entities
"The months go up by 3 and the dollars go up by 6, so it's the same any way you do it"
5. Finding the unit rate
"In each case, if you figure out how much it costs a month, you get $2.00 a month."
6. Equating one of the quantities and comparing the corresponding costs
"For the 6-month subscription, every 3 months would be $2.00; for the 9-month subscription, every 3 months is $2.00; the same for the 12-month subscription."

Lamon (1994) reported that 13 of the 24 students were successful in solving the subscription problem. Various solution strategies of those students were also reported. To begin with the first strategy, five of the students reasoned with the equal scale factor within each of the measure spaces (i.e., the number of months and the corresponding cost for that number of months). The second strategy for the subscription problem, which was employed only by a student, included reasoning with the equivalent fractional parts of the wholes and comparing the scalar operators in both measure spaces. The third strategy used by a student for answering the subscription problem included finding the total cost for three payments and seeing that the number of months and corresponding cost in dollars have the same values (i.e., the cost is $1 per month in each situation). Another student used a strategy in which he focused on the number patterns related to the increase in both measures. The fifth strategy used by a student was related to finding the cost for the subscription per month (i.e., unit rate) in each case. The most common strategy employed by 13 students who correctly answered the subscription problem included equating one of the quantities (i.e., the number of months) in each situation and comparing the corresponding cost in each situation. Lamon (1994) stressed that in most of these strategies, students showed a tendency to find and compare rates, only one of which is focused on the unit rate. In particular, while the fifth strategy includes a one-month unit, the remaining involves working with three-month or six-month units. Lamon (1994) concluded that the preference of the most suitable unit depends on the problem situation since most of the strategies made use of 3-month unit since 3 was the greatest common factor of the three number of months given in the problem (i.e., 6, 9, and 12).

The third problem on which students work while they were engaged in clinical interviews was the apartment problem. This problem asked students to find the number of one-, two-, or three-bedroom apartments satisfying a rule for the communities' needs. This problem is depicted in Figure 2.14 below.

> Problem 3. "The apartment problem. In a certain town, the demand for rental units was analyzed and it was determined that, to meet the community's needs, builders would be required to build apartments in the following way: Every time they build 3 one-bedroom apartments, they should build 4 two-bedroom apartments and 1 three-bedroom apartment. Suppose a builder is planning to build a large apartment complex containing 30-40 apartments. How many apartments should be built to meet this regulation?
>
> Suppose one built 32/40 apartments (choose one). How many one-bedroom, two-bedroom, and three-bedroom apartments would the apartment building contain?
>
> Suppose one built 14 one-bedroom apartments, 18 two-bedroom apartments, and 4 three-bedroom apartments. Would the requirement be satisfied?"

Figure 2.14. The apartment problem (Lamon, 1994, p. 101)

This problem could be solved by creating units (i.e., 1- bedroom apartments, 3-bedroom apartments, 4- bedroom apartments) and units of units (i.e., three units of 1-bedroom apartments, four units of 3-bedroom apartments, one unit of, 4-bedroom apartments), and units of units of units (i.e., one eight-unit of apartments), and units of units of units of units (i.e., 5 of that eight-unit apartments). According to the findings of Lamon's (1994) study, eleven students were able to use related composition and decomposition processes.

A sample student answer wherein she constructed one eight-unit of 1-, 3- and 4-bedroom apartments is provided below in Figure 2.15.

Laurie: He should build fifteen one-bedroom apartments, twenty two-bedroom apartments, and five three-bedroom apartments.

Researcher: Tell me how you thought about that.

Laurie: Well, 3 and 4 and 1 make like one building and you can make five buildings. So there would be 5 of each kind of apartment.

Researcher: What do you mean 5 of each kind?

Laurie: There would be three one-bedrooms in each building, and five buildings, so that's fifteen one-bedrooms and like that for the other sizes.

Figure 2.15. Creating units of units of units for the apartment problem (Lamon, 1994, p. 107)

Besides, four students were able to add columns of units of units without creating one eight-unit of units of units in order to obtain a total of 40 apartments. A sample student answer is provided in Figure 2.16 below.

Kari: (Worked for a few minutes on paper.) 15 one-bedroom apartments, 20 two-bedroom apartments and 5 three-bedroom apartments.

Researcher: Would you explain that answer? What were you doing here?

Kari: (Showed the following work)

3 one-bedroom	6	9	15
4 two-bedroom	8	12	20
1 three-bedroom	2	3	5
	16	24	40

First I doubled each number and added them up. That was 16, so I could get more. . . . So I did three more of each and I just added them up. Then I decided to try each one times 5 and that gave me 40.

Figure 2.16. Adding columns of units of units to obtain a total of 40 apartments for the apartment problem (Lamon, 1994, p. 107)

The fourth problem used by Lamon (1994) to understand students' informal knowledge in ratio and proportion concepts was the pizza problem that asked the comparison of the amount of pizza received in two situations. This problem is presented in Figure 2.17 below.

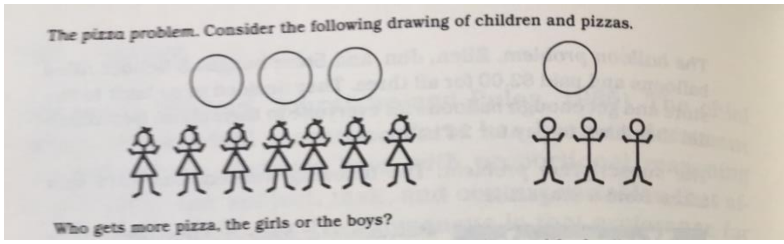


Figure 2.17. The pizza problem (Lamon, 1994, p. 102)

Lamon (1994) reported that eighteen of the students that are the ones who correctly solved the problem took a pizza and boy as a linked composite unit (i.e., unitizing) and reinterpreted the case for girls in relation to the number of boys and a pizza relationship (i.e., norming) as illustrated in Figure 2.18 below.

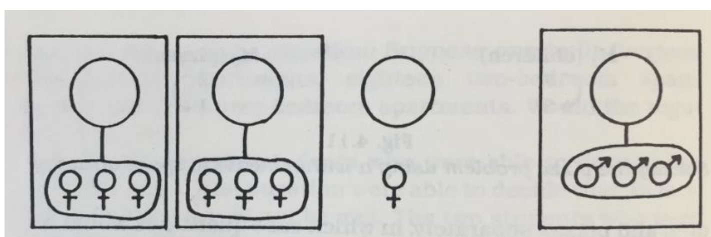


Figure 2.18. Unitizing and norming for the pizza problem (Lamon, 1994, p. 110)

The last problem used in the study of Lamon (1994) was the Alien Problem that is presented in Figure 2.19 below.

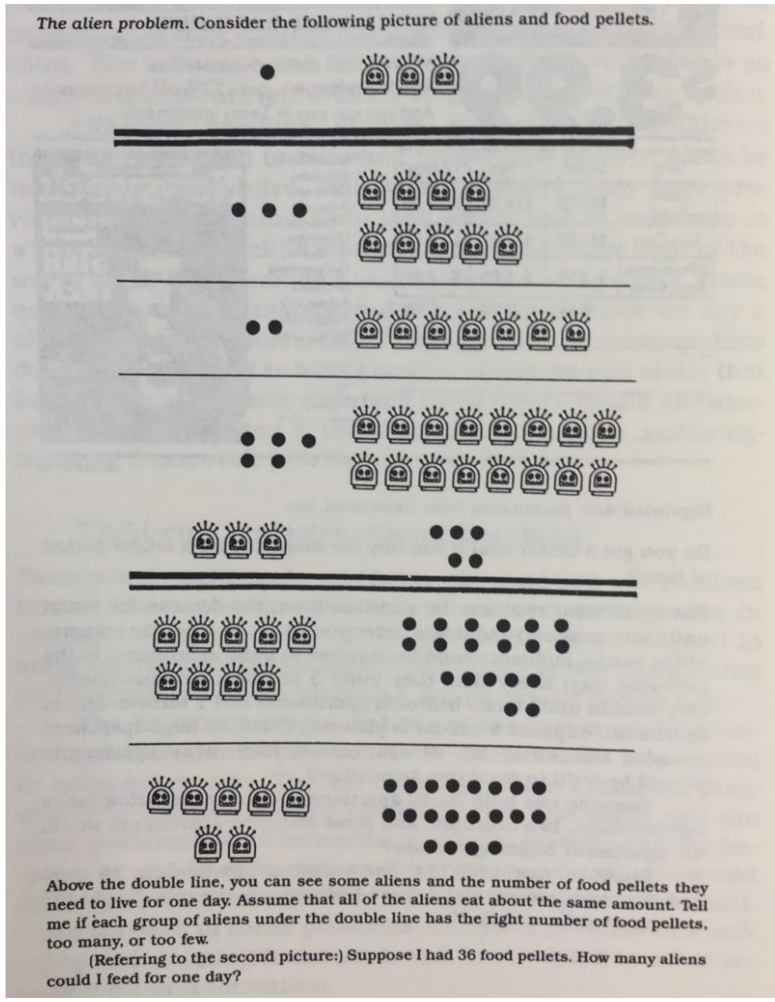
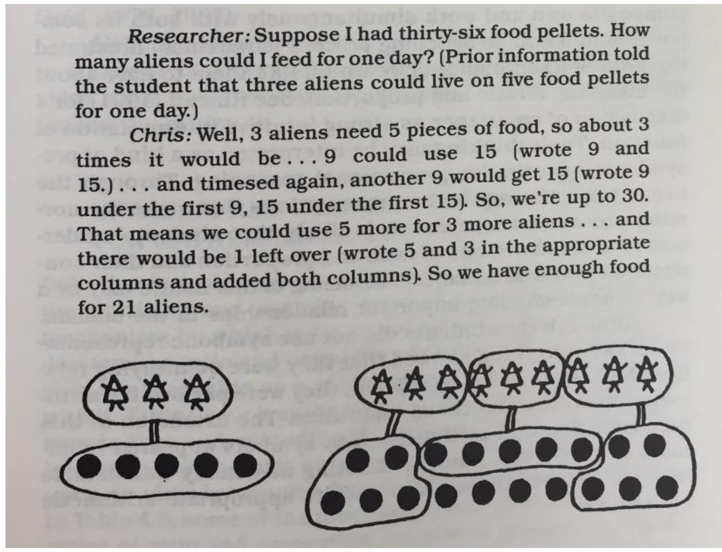


Figure 2.19. The alien problem (Lamon, 1994, p. 102)

The Alien Problem was solved correctly and in the same way by all of the 24 students (Lamon, 1994). The strategies used for solving this question included taking 3 aliens and 5 food pellets as a linked composite unit and re-interpreted other situations in terms of this unit. A sample interview transcript and related drawing of the student is presented in Figure 2.20 below.



Researcher: Suppose I had thirty-six food pellets. How many aliens could I feed for one day? (Prior information told the student that three aliens could live on five food pellets for one day.)

Chris: Well, 3 aliens need 5 pieces of food, so about 3 times it would be . . . 9 could use 15 (wrote 9 and 15.) . . . and timesed again, another 9 would get 15 (wrote 9 under the first 9, 15 under the first 15). So, we're up to 30. That means we could use 5 more for 3 more aliens . . . and there would be 1 left over (wrote 5 and 3 in the appropriate columns and added both columns). So we have enough food for 21 aliens.

Figure 2.20. The ratio 3:5 used as a norming unit for the alien problem (Lamon, 1994, p. 111)

In order to interpret the results of Lamon's (1994) study in relation to ratio and proportion, the students were able to consider ratio as "an invariant composite unit" and work simultaneously with both its components in a double matching process (covariance), which is the kind of understanding that is necessary for the development of ratio and proportion concepts. In addition, they were able to choose the most appropriate unit and operate with it. In this process, unitizing and norming were the processes by which more sophisticated levels of reasoning evolved. It is important to note that the divisibility relationship between 3 aliens in the first situation and 9 aliens in the second situation paves the way for using 3:5 as the norming unit (Lamon, 1994). Lamon (1994) concluded that children have informal and intuitive knowledge about ratio and proportion prior to instruction: They were able to use invented strategies that were based on the abilities of unitizing and norming without employing any symbol use to communicate their reasoning, which might be interpreted as "presymbolic quantitative proportional reasoning."

Thus, Lamon's (1994) study provides a comprehensive and detailed outline for the specific types of proportional problems and students' informal strategies for those problems. Although Lamon (1994) does not classify these problems, proportional reasoning problems can be classified under three main categories (Behr et al., 1992; Cramer, Post, & Currier, 1993; Cramer, & Post, 1993b; Heller et al., 1990; Kaput & West, 1994; Noelting, 1980a; Post et al., 1988). Below is the classification of these three types of problems:

1. "Missing value problems: Problem includes an implicit or explicit instance of a ratio and one of the two values in another ratio that is equivalent to the first one. It asks to find the corresponding fourth value in the second ratio (Cramer et al., 1993; Kaput & West, 1994)
2. Numerical comparison problems: All the values of two ratios/rates are given, and the task is to compare these ratios/rates rather than finding a numerical value (Cramer et al., 1993; Karplus et al., 1983; Noelting, 1980a)
3. Qualitative prediction/comparison problems: The problem does not require computing with numerical values; instead, the task is to reason about the direction of change by making comparisons" (Behr et al., 1992; Cramer et al., 1993)

Freudenthal (1978) stresses that missing value and numerical comparison problems can be approached in three ways: (1) reasoning with an internal (within measures) ratio (e.g., $\frac{distance1}{distance2} = \frac{time1}{time2}$) and applying the result for solving the problem, (2) reasoning with an external (between measures) ratio (e.g., $\frac{distance1}{time1} = \frac{distance2}{time2}$) and applying the result for solving the problem, and (3) avoiding computation until the result has been found or setting up a relationship that includes all the data and computing.

By referring to these three types of problems, Ben-Chaim et al. (1998) suggest that these types of problems can appear in three broad categories:

1. Comparison of two (or more) parts of a single whole (e.g., ratio of the number of girls to boys in a classroom)
2. Comparison of different quantities that are connected in some ways (e.g., unit price, miles per gallon, density)
3. Comparison of two conceptually related quantities that are not parts of a whole (e.g., scaling up/down)

Additionally, in their study, Ben-Chaim et al. (1998) investigated seventh-grade students' strategies (including incorrect ones) for solving proportional reasoning problem types and categories mentioned above. In this study, Ben-Chaim et al. (1998) worked with approximately 2000 students who were taught with different types of curricula (i.e., mainly two types as traditional and reform-based curricula). They examined these students' written responses to a paper-pencil test and in individual interviews. According to the results of the study, they identified nine different strategies for numerical comparison problems that are listed in the following pages.

Strategy 1. "*Comparing the ratio of two different variables using external ratios or a functional method*" (Ben-Chaim et al., 1998, p. 258): In this strategy, students work with price per unit or unit per price in making comparisons. Another name for this strategy is unit rate since students deal with external or within measures ratios, including units. Ben-Chaim et al. (1998) report that this is one of the most frequently used strategies for comparison problems that develop independently of instruction.

Strategy 2. "*Comparing ratios of the same variable using internal ratios or a scalar method*" (Ben-Chaim et al., 1998, p. 259): In this strategy, students deal with ratios including values that belong to the same measure space. Ben-Chaim et al. (1998) reported that fewer students used this strategy and concluded that it was more complicated than the first strategy.

Strategy 3. “*Comparing the cost of the same quantity by finding common factor or common multiple quantities such as price per unit”* (Ben-Chaim et al., 1998, p. 260): In this strategy, students equate the amounts (e.g., kg, ounce) in both situations and compare the corresponding costs. The type of reasoning here is: “If I buy the same amount in both, in which option do I get a cheaper deal?”

Strategy 4. “*Comparing amounts for the same cost by finding common factor or common multiple costs such as unit per price*" (Ben-Chaim et al., 1998, p. 260): In this strategy, rather than equating the amounts, students equate the costs in both situations and compare the corresponding amounts. The type of reasoning here is: "If I pay the same money, in which option do I get more amount of what I would like to buy?"

Strategy 5. “*Building up strategy*” (Ben-Chaim et al., 1998, p. 260): In this strategy, students create a list of the linked quantities of cost and money by building up (e.g., 2-16, 4-32, 6-48, etc.) in both situations. When they reach a common multiple, they compare the other value in order to make a decision. Ben-Chaim et al. (1998) reported that a small number of students used this strategy. They concluded that this might be due to the presence of non-integer ratios in the problems, which might have discouraged students from using a build-up strategy. In addition, they suggested that another possible explanation might be that building up strategy and constructing a ratio table do not develop naturally in students.

Strategy 6. “*Looking at ratios of differences between the same variables*" (Ben-Chaim et al., 1998, p. 261): This strategy involves students' erroneous focus on comparing the differences between values that belong to the same measure spaces, which is known as incorrect additive strategy. Ben-Chaim et al. (1998) reported that students used this strategy almost for all of the problems, including the problems that ask students to compare change in relative terms.

Strategy 7. "*Responding to the numbers but not the context of a given problem*" (Ben-Chaim et al., 1998, p. 261): In this strategy, students only apply algorithms including multiplication, that is close to cross multiplication and do not perform operations related to the meaning of the problem (e.g., multiplying time 1 and distance 2, which does not make sense for the problem context). Apparently, students who used this strategy lack essential understandings of proportional reasoning.

Strategy 8. "*Relating to only one variable by ignoring part of the data in the problem*" (Ben-Chaim et al., 1998, p. 262): In this strategy, students centrate on one of the variables and ignore the other. Thus, this erroneous strategy lacks thinking ratio as a single entity on its own that includes a relative comparison.

Strategy 9. "*Affective responses to numerical data and questions*" (Ben-Chaim et al., 1998, p. 262): This strategy includes students' subjective responses including irrelevant and nonmathematical information that did not make relation to comparing ratios (e.g., I like it better).

Thus, the study by Ben-Chaim et al. (1998) has provided a comprehensive and detailed analysis of seventh-grade students' correct and incorrect and informal and formal strategies. In summary, students' correct strategies are mostly based on working with within or between measures ratios (also called a multiplicative strategy), unit factor (also called unit rate) approach, and building up strategies. Indeed, these are the three basic types of strategies that students use for most proportional problems (Tourniaire & Pulos, 1985). Besides, students' incorrect strategies include ignoring part of the data, providing an irrelevant response, and erroneous additive reasoning (Ben-Chaim et al., 1998; Tourniaire & Pulos, 1985).

In another study, Tourniaire (1986) proposed that there are basically two kinds of erroneous reasoning for proportional reasoning, which consist of misuse of an incorrect strategy (e.g., using an inappropriate multiplier) and incorrect use of additive reasoning.

While the former is reported in a few studies, there is an amassed number of studies that report additive reasoning as an incorrect type of reasoning for proportional reasoning tasks. Indeed, when the literature on ratio and proportion is reviewed, it is seen that the most frequently reported incorrect type of reasoning for proportional problems is based on an erroneous additive reasoning (Brousseau, 2002; Hart, 1981, 1988; Kaput & West, 1994; Karplus et al., 1983; Misailidou & Williams, 2003; Resnick & Singer, 1993; Steinthorsdottir & Sriraman, 2009; Tourniaire & Pulos, 1985; Tourniaire, 1986; van Dooren et al., 2010). In later years, this type of reasoning where multiplicative reasoning is required, is called as an obstacle in the development of proportional reasoning (Ayan, & Isiksal-Bostan, 2018).

2.1.4. Erroneous additive reasoning: The obstacle in the development of proportional reasoning

Multiplicative reasoning is defined as “making multiplicative comparisons between quantities” (Wright, 2005, p. 363). In another study, it is defined as “the functioning of a person’s multiplicative operations, multiplying schemes, and multiplicative concepts in ongoing interaction in her experiential world” (Hackenberg, 2010, p. 391). In multiplicative reasoning "the terms within a ratio are related multiplicatively and then this relation is extended to the second ratio" (Tourniaire & Pulos, 1985, p. 184) while in additive reasoning "the relationship within the ratios is computed by subtracting one term from another, and then the difference is applied to the second ratio" (Tourniaire & Pulos, 1985, p. 186). Therefore, in general terms, multiplicative reasoning is a type of reasoning that underlies proportional reasoning, while additive reasoning does not apply to proportional situations.

Students learn addition and subtraction before multiplication and division in primary school. The questions that they deal with in these early years of the primary school include the following: "How many more (less) is A than B?" and "How many are A and B all together?" Based on these explorations, they learn multiplication as a short way for repeated addition. For instance, for the problem, "There are 3 boxes each of which contains 4 eggs. How many eggs are there altogether?" students learn about the equation $4 + 4 + 4 = 3 \times 4$. Thus, some researchers suggest that multiplicative reasoning is based on repeated addition or additive reasoning (Fishbein, Deri, Nello, & Marino, 1985). However, recent approaches propose that this understanding is sketchy and superficial, although it is helpful intuitively for students (Clark & Kamii, 1996; Park & Nunes, 2001; Van Dooren et al., 2010).

According to Park and Nunes (2001), repeated addition is only a procedural skill to solve multiplication problems. Thus, in primary school and the early years of middle school, students' reasoning is expected to change from additive to multiplicative (NCTM, 2000; Harel & Confrey, 1994; Fernandez & Llinares, 2009; Park & Nunes, 2001). Singh (2000) stresses that two changes should be ensured while moving from additive to multiplicative reasoning: they are the "changes in what numbers are and changes in what the numbers are about" (p. 273).

Even though repeated addition is not accepted as lying in the roots of multiplicative reasoning, additive reasoning is seen as a prior stage for multiplicative reasoning (Fernandez, Llineares, van Dooren, et al., 2010). Many researchers point to a pre-proportional reasoning stage in which additive reasoning is applied by building-up strategies to respond to multiplicative situations (Lesh et al., 1988; Piaget & Inhelder, 1975; Steffe, 1994). Therefore, it is important to build multiplicative reasoning on students' additive reasoning skills (Fernandez et al., 2010) in order to move students from additive reasoning to multiplicative reasoning (Harel & Confrey, 1994).

On the other hand, Misailidou and Williams (2003) pointed out that additive reasoning is the strategy that was most commonly reported as an inappropriate strategy in solving proportional reasoning problems. In another study, it is reported that one-third of 12-15 years old students applied incorrect additive reasoning (i.e., focusing on the difference) for several tasks, including the highly known Mr. Tall and Mr. Short problem (Hart, 1988). Therefore, although additive reasoning is essential for the development of proportional reasoning, it is also an obstacle in the development of proportional reasoning (Ayan & Isiksal Bostan, 2018).

Fernandez and Llinares (2009) state that discerning additive and multiplicative relationships from each other is a sign of mathematical maturity. However, many students are incapable of interpreting and distinguishing additive and multiplicative relationships and tend to use them in inappropriate situations. This misuse might occur in two ways: either using additive strategies for multiplicative problems or using multiplicative strategies for additive problems. For instance, for the problem, "Grandma adds 2 spoonfuls of sugar to the juice of 10 lemons to make lemonade. How many lemons are needed if 6 spoonfuls of sugar are used?" (van Dooren et al., 2010, p. 362) students might erroneously think that the second mixture should include 6-2=4 more spoonfuls of sugar and, hence, it should include 10+4 = 14 lemons.

On the other hand, for the problem, "Sue and Julie were running equally fast around a track. Sue started first. When she had run 9 laps, Julie had run 3 laps. When Julie completed 15 laps, how many laps had Sue run?" (Cramer & Post, 1993, p. 344) students might think that the correct answer is 45 by considering a multiplicative relationship as $\frac{a}{b} = \frac{c}{d}$. Nevertheless, it can be understood that the context of the problem requires an additive reasoning instead of a multiplicative one, and the result is 21 laps.

Kaput and West (1994) asserted that the area of geometry and measurement is one of the most vulnerable areas to erroneous additive reasoning. This means that students might use additive strategies for geometry and measurement problems, which are multiplicative in nature. A problem used in the study of Kaput and West (1994) is presented in Figure 2.21 below.

The two sides of Figure A are 9 cm high and 15 cm long. Figure B is the same shape but bigger. If one side of Figure B is 24 cm high, how long is the other side?

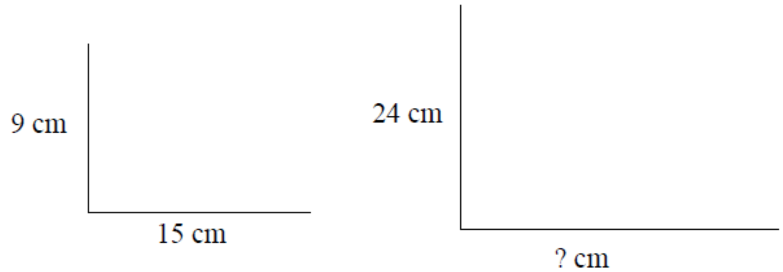


Figure 2.21. Missing value geometry problem (Kaput & West, 1994, p. 268-269)

A student who applies additive reasoning to this problem might think that since the height is increased by the amount of 15 (24 – 9 = 15), the length is also increased by 15 and, the result is 30 (15 + 15 = 30 cm). However, it is not the case for this problem since the shapes are similar, and the lengths should increase by the same ratio, not the same amount. Therefore, the correct solution for the problem should be $\frac{9}{24} = \frac{15}{?}$ or $\frac{9}{15} = \frac{24}{?}$ that is 40 cm. Kaput and West (1994) conclude that the incorrect use of additive reasoning, in this case, is rooted in an inadequate comprehension of similarity and its quantitative implications. On the other hand, it is also possible that a student uses addition by default when he or she is confused about them (Kaput & West, 1994). Besides, he or she might tend to apply addition when the sizes of two units are large, but there is a small difference between them (Kaput & West, 1994).

In addition, van Dooren et al. (2010) conducted a study in order to investigate 325 third, fourth, fifth, and sixth-grade students' additive strategies to solve proportional problems. The researchers also aimed at investigating the proportional strategies of the students in order to solve additive problems and also students' progress from additive to multiplicative ways of thinking. The researchers administered a test in which half of the problems required additive strategies, and half of them required proportional strategies.

The two sample questions, one of which is additive, and the other is multiplicative, are provided in Figure 2.22 below.

Tom and his sister Ana have the same birthday. Tom is 15 years old when Ana is 5 years old. They are wondering how old Ana will be when Tom is 75 … (= additive situation).

Rick is at the fish store to buy tuna. The customer before him bought 250 grams of tuna and had to pay 10 euro. Rick needs 750 grams of tuna, and he wonders what he will have to pay … (= multiplicative situation).

Figure 2.22. Additive and multiplicative problems (Van Dooren et al., 2010, p. 361)

Van Dooren and colleagues described the first situation by a function f (x) = x+a and explained that the situation is additive since the numbers are related by addition and subtraction. Therefore, the correct solution strategy for the first problem is looking at the differences between the ages of the two persons and applying the difference to the second value. Moreover, they described the second situation by a function f (x) = bx and justified that the situation was multiplicative (proportional or linear) since the variables are related by multiplication and division. Thus, the correct solution strategy for the second problem is writing a proportion between the given variables or applying the ratio of the first two variables to the second variable. The researchers pointed out that the required reasoning in these situations is very distinct since the first one deals with a difference, and the second one deals with a ratio between the two values. The findings of the study revealed that students showed a tendency to use additive strategies for multiplicative problems. Specifically, 46.6% of the students in the third grade and 6.4% of the students in the sixth grade were additive reasoners. Another finding of the study was that the tendency to use additive strategies for multiplicative problems decreased with age, whereas the tendency to use multiplicative strategies for additive problems increased with age. Karplus et al. (1983) urge that proportional reasoning requires being able to differentiate these two types of systems of relationships.

In another study conducted by Misailidou and Williams (2003), constructing an instrument to determine the misconceptions of the students in the domain of proportional reasoning was aimed. The researchers hypothesized that the misconception of the use of additive strategy for proportional problems would occur in students' answers frequently. Three hundred three students between 10 and 14 years of age were given a test prepared to determine their misconceptions related to proportional reasoning. Students' answers were coded as correct or erroneous for each item, and the results were analyzed using the Rasch model in order to scale the most common errors. The findings of the study revealed that the "tendency to additive strategy" was the strongest and the most frequent misconception.

Thus, it is understood that additive reasoning constitutes one of the stumbling blocks in the development of proportional reasoning. The literature is rich in providing evidence that additive reasoning becomes more apparent when specific types of task variables are used in the problems. Thus, in the following section, these task variables that affect student success in proportional reasoning and cause students to fall back to additive reasoning will be explained in order to guide the development of the tasks used in this study.

2.1.5. Task variables affecting student success in proportional reasoning

Noelting (1980a, 1980b) called for attention to the numbers used in a task. He ordered the orange juice task in terms of difficulty based on the values that it included. The task included pictures and required children to decide which of the two orange juice concentrations that included different amounts of orange juice and water had a more orangey taste. Karplus et al. (1983) used a similar task, including lemon juice concentrations, wherein students were asked to find a missing value as well as make comparisons. According to the results of their study, 60% of the answers to the comparison questions included a within recipe approach, while less than 20 % included a between recipe approach. Moreover, it was reported that almost 60% of the participants could correctly answer the questions with integer ratios, whereas only 10% of them could correctly answered the questions that required creating a common denominator (or numerator). Twenty-five percent of the participants could solve integer ratios correctly by multiplicative reasoning, but they applied incorrect additive reasoning when the question included a non-integer ratio. Thus, they concluded that, the numbers used in a task is a significant factor that determines the task difficulty. Particularly, if the numbers in a task are divisible, form an integer ratio or a unit rate, the task is considered to be less difficult, and it is possible to enhance symbolic representations of proportions by using these number relationships (Karplus et al., 1983; Noelting, 1980).

Kaput and West (1994) outline a hierarchical analysis of proportional problems in terms of problem difficulty. In particular, they lay out the task variables that facilitate and hinder the solution of problems in terms of numerical and semantic features. To begin with, the first task variable that is considered to enhance problem solution is the reduced form of ratio that has to do with the numerical feature of a task. Having a whole number quotient between the quantities (i.e., the reduced form of ratio) enables multiplicative comparison and the use of build-up strategies. The second task variable is including a familiar multiple of one quantity of the other quantity either within the measure space or between the measure spaces in terms of numerical features. This makes it easy to notice that one of the quantities is a multiple of the other quantity directly and without calculation. In addition, this could also pave the way for the application of (abbreviated) build-up strategies. Concerning semantic features, containment, for every/each statement, and familiar rates are discussed to facilitate problem-solving. In particular, it is stressed that if a problem context involves a containment wherein it is possible to associate two entities, then build-up strategies would be more applicable, which would, in return, facilitate problem solution. Secondly, if the explicit use of "for every/each statement" appears in the problem, then it makes it easier to conceptualize the situation as rate or ratio. Lastly, a problem situation that involves a familiar rate (e.g., speed, price) enables students to understand the situation as rate-ratio and to use the unit-factor approach (Kaput & West, 1994).

On the other hand, some task features hinder the process of problem solution that are related to numerical or semantic features. In relation to the numerical features, if a task includes two quantities, either of which does not divide the other evenly, then, that task would be more difficult for the students. In addition, the tasks that involve quantities with a small difference between them yield students' tendency to use additive strategies, which would increase the task difficulty. When semantic features that hinder the problem-solving process is in question, ambiguous grouping might be responsible for increasing the task difficulty. If a problem involves ambiguous groups that lose their own identities (e.g., mixture problems), it might be hard for children to keep their original measures separately. Besides, including continuous units in the tasks that are hard to be visualized and identified as separate makes the task more difficult for students (Kaput & West, 1994; Tourniaire, 1986).

Up to this point, the essential elements of proportional reasoning and related didactical activities were discussed with a didactical phenomenology perspective. Then, different types of proportional reasoning tasks and students' informal and formal strategies for these tasks were elaborated. Other topics discussed include additive reasoning as an obstacle for proportional reasoning and task variables that hinder/facilitate students' success in proportional tasks. In the following section, the developmental course of proportional reasoning will be explained in order to shape the learning trajectory and instructional sequence in this study.

2.1.6. Development of Proportional Reasoning

The development of a concept occurs “not in isolation but in relationship with other concepts, through several kinds of problems and with the help of several wordings and symbolisms” (Vergnaud, 1988, p. 142). In other words, it is not possible to describe a single and linear route for the development of understanding complex mathematical domains independent of other concepts; rather, “development looks more like a tree with an intricate branching system” (Lamon, 1994, p. 90). Kieren (1976) suggests essential subconstructs of rational number understanding as fractions, equivalence classes of fractions, ratio, operators or mappings, quotients, measures, and decimals. He argues that all of these subconstructs are related to and interact with each other (Kieren, 1976). Similarly, Vergnaud (1983, 1988) puts forward the term “Multiplicative Conceptual Field” [MCF] that includes situations that “can be analyzed as simple and multiple proportion problems and for which one needs to multiply or divide” (Vergnaud, 1988, p. 141). This field consists of the concepts “linear and n-linear functions, vector spaces, dimensional analysis, fraction, ratio, rate, rational number, and multiplication and division” (Vergnaud, 1988, p. 141). Therefore, it is essential to look at the development of proportional reasoning in relation to other concepts.

Additionally, in order to understand children’s proportional reasoning, it is also significant to make sense of the developmental course of it. However, there is no clear developmental time course of proportional reasoning in the literature (Boyer, Levine, & Huttenlocher, 2008). The original works by Piaget and colleagues (Inhelder & Piaget, 1958; Piaget & Beth, 1966; Piaget & Inhelder, 1975) suggest that proportional reasoning is a late achievement, and children are not capable of reasoning proportionally until 11-12 years of age. On the other hand, later studies have consistently shown that young students have an informal and intuitive knowledge in proportional reasoning and can deal with proportional situations as early as grades 3-5 (Boyer et al., 2008; Kaput & West, 1994; Lamon, 1994). Nevertheless, these strategies might be “primitive, context bound, relatively symbol free, and based upon counting, adding, and halving” (Lamon, 1994, p. 99).

Students have a bulk of prior knowledge and experience in proportional situations by the time they engage in instruction on proportional reasoning (Lamon, 1994, 1995). By the time they are in middle school, they already have powerful skills to count, match, and partition. Therefore, it is essential to build new knowledge of proportional reasoning on powerful use of those skills (Lamon, 1995). However, "the development of proportional reasoning depends on more than a child's existing collection and organization of past informal experiences" (Lamon, 1995, p. 178). It is also evident that proportional reasoning is developed and facilitated through a teaching and learning process (Lamon, 1994, 1995). Thus, though students have rich and extensive knowledge before instruction, it is the teacher who should utilize and guide this knowledge to mathematically significant directions progressively over time in accordance with didactical phenomena (Freudenthal, 1983, 1978; Lamon, 1994, 1995). Hence, essential questions for the teaching of learning of proportional reasoning should include: “How can we view ratio, and eventually proportional reasoning, as an extension of some basic mathematical idea(s)?” and “What intuitive, informal, or existing knowledge aids the learning of rational number concepts?” (Lamon, 1994, pp. 90-91). I already tried to answer these questions in the previous sections.

The growth in proportional reasoning takes place in a progressive increase in local competence (Karplus et al., 1983; Tourniaire & Pulos, 1985). The mastery of proportionality is gained in a small and limited set of problem situations and then progressively extended to other sets of problems (Lesh et al., 1988). This progressive increase in local competence helps guide research and instruction on proportional reasoning (Lesh et al., 1988). Therefore, a variety of frameworks that lay out the developmental stages of students in proportional reasoning that have potential in guiding the design of the instructional sequence and the teaching/learning environment are described in the following parts.

To begin with, Lesh et al. (1988) outline essential stages in children’s development of proportional reasoning as follows:

(1) At the most primitive stage, students neglect part of information. For instance, they might ignore denominators and only compare numerators in the proportion A/B = C/D.
(2) In the second stage, students might recognize relations among the four values in the proportion A/B = C/D; yet, relating those values might be solely in qualitative ways.
(3) In the third stage, quantification starts with recognizing constant differences in additive manners.
(4) The fourth stage is referred to as the earliest stage of multiplicative reasoning that includes pattern recognition and replication or a build-up strategy (i.e., 2 pieces for 8 cents, 4 pieces for 16 cents, 6 pieces for 24 cents, etc.).
(5) At the fifth stage, the child notices a multiplicative relationship between two values, and then, this relationship is applied to the other pair of values.

Although these stages are essential in understanding students' development of proportional reasoning, "the level of reasoning that a child uses is often not consistent across tasks or even within a given task" …which is referred to as "horizontal decalage" (Lesh et al., 1988, p. 105). In other words, these stages are not linear, and the type of reasoning students employ depends highly on the tasks used.

These five stages can be summarized in a two-stage distinction in students' development of proportional reasoning as pre-proportional (additive) reasoning and proportional reasoning (Lesh et al., 1988; Piaget, & Inhelder, 1975; Steffe, 1994). Pre-proportional reasoning involves coordination of functions, whereas proportional reasoning includes reversible operations (Piaget & Inhelder, 1975; Steffe, 1994). At the pre-proportional stage, students might recognize a pattern and apply this pattern to find the missing value. In other words, they have a sense that the values change according to their sizes, and the nature of this change is multiplicative. However, they may not notice that the difference between these values constantly increases (Lesh et al., 1988). In this stage, children can deal with multiplicative relationships that can be represented by the equations A/B = C/D or A*B = C*D "without recognizing the structural similarity of the two sides of the equation" (Lesh et al., 1988, p. 102-103). This type of reasoning is a weak indicator of proportional reasoning (Lesh et al., 1988). On the other hand, the foremost characteristic of proportional reasoning entails recognition of "the invariance of a simple mathematical system" (Lesh et al., 1988, p. 101). Hence, the critical difference between pre-proportional reasoning and proportional reasoning depends on whether or not the child can change one of the remaining three values in order to preserve the equality (i.e., compensate) when one of the four values in a proportion change (Lesh et al., 1988).

In another study, Kaput and West (1994) delineate competent but informal reasoning of students in proportional tasks, which is defined as reasoning patterns of students that were not based on the formal symbolism (i.e., cross multiplication or formal division). In particular, they highlight three basic types of reasoning: "(1) Coordinated build-up/build-down processes, (2) Abbreviated build-up/build-down processes using multiplication and division, (3) Unit factor approaches" (Kaput & West, 1994, p. 244). Kaput and West (1994) consider the first of these forms of reasoning (i.e., coordinated build-up/down) as the most fundamental one in the development of proportional reasoning. Also, they stress that the second form of reasoning is cognitively based on the first one concerning the repeated addition interpretation of multiplication, especially while dealing with discrete quantities and integer ratios. The development of the first two build-up strategies might appear independent of formal instruction; yet, the development of the unit factor approach can be facilitated through engagement in carefully designed tasks (Kaput & West, 1994). The three ways of reasoning can be facilitated through the use of pictures and tables and in situations that include discrete or continuous variables (Kaput & West, 1994). These three forms of reasoning processes are illustrated by the help of the strategies used for solving the following problem:

> Placemat Problem (1): A restaurant sets tables by putting seven pieces of silverware and four pieces of china on each placemat. If it used thirty-five pieces of silverware in its table settings last night, how many pieces of china did it use? (Kaput & West, 1994, p. 245)

According to Kaput and West (1994), the basic build-up process to solve this problem includes increasing both quantities in coordination in terms of double skip counting as "For seven silver there is four china, for fourteen silver there is eight china, for twenty-one silver there is twelve china, for twenty-eight silver there is sixteen china, for thirty-five silver there is twenty china" (p. 246), which can be organized in such tables provided in Figure 2.23 below.

Silver	7	14	21	28	35
China	4	8	12	16	20

Silver	China
7	4
14	8
21	12
28	16

Figure 2.23. Double skip counting by sevens and fours for the Placemat Problem (Kaput & West, 1994, p. 246)

This type of reasoning involves two processes as an initial conceptualization of the correspondence relationships between the quantities and computation of increment (or decrement) in both quantities (Kaput & West, 1994). In order to handle the process of incrementing (or decrementing) in more efficient ways, this process can be abbreviated or consolidated in terms of multiplication (as an efficient way of repeated addition). This kind of process is referred to as the abbreviated build-up process (Kaput & West, 1994).

An abbreviated build-up strategy for a different version of the Placemat Problem above when the number of pieces of china for 392 pieces of silverware is asked would be thinking as:

> "We are given 392 pieces of silverware, so 392 silverware divided by 7 silverware per placement gives 56 placemats. There are 4 pieces of china per placemat, so there were 4 china per placemat times 56 placemats, which gives 224 pieces of china" (Kaput & West, 1994, p. 248).

This would be written symbolically as follows:

$$\frac{392 \text{ silverware}}{7 \text{ silverware/placemat}} = 56 \text{ placemats}$$

$$56 \text{ placemats} \cdot 4 \text{ china/placemat} = 224 \text{ china}$$

Figure 2.24. Abbreviated build-up strategy for the Placemat Problem (Kaput & West, 1994, p. 248)

It should be noted that the unit size of one quantity is divisible to the other quantity, that is, the quotient is a whole number in this example. Students might deal with a divisibility failure, when the quotient is not a whole number, by making an adjustment in unit size at the beginning and operate with the adjusted unit or by making an adjustment later. This process of making adjustments can lead to the unit factor approach wherein the unit size of a quantity is used as a divisor (i.e., unit factor) to find the unknown quantity. A related strategy for the Placemat problem is illustrated in Figure 2.25 below:

$$\frac{4\ pieces\ of\ china}{7\ pieces\ of silverware} \approx 0.\ 57 \text{ china/silverware}$$

$$0.\ 57 \text{ china/silverware} \times 35 \text{ silverware} \approx 20 \text{ china}$$

Figure 2.25. The unit factor approach for the Placemat Problem (Kaput & West, 1994)

While the informal strategies for the Placemat Problem would be related to one or more of the three strategies above, the formal approach to the problem would involve setting up a formal equation, either within a measure of between measures. A within-measure formal equation for the Placemat Problem is presented in Figure 2.26 below.

$$\frac{7\ silverware}{392\ silverware} = \frac{4\ china}{x\ china}$$

Figure 2.26. A formal equation for the Placemat Problem (Kaput & West, 1994, p. 253)

Last but not least, particularly, the teaching experiment study conducted by Lo and Watanabe (1997) can give essential insight into the informal understandings of children prior to ratio and proportion instruction. This study focuses on the development of one-fifth grader's ratio and proportion concepts as he gains knowledge in the multiplicative conceptual field and how he schematizes his informal knowledge over the course of six months. The researchers posed several proportion tasks, including a variety of task variables (e.g., the numbers, the context, availability of physical materials) to the fifth-grade student, Bruce, in face-to-face interviews. The findings of the study showed that, at the beginning of the study, he was able to use physical materials in order to group and link them and to find the answers using multiplication and missing multiplicand approaches, ratio-unit/build-up method, or an intelligent guess. Nevertheless, he avoided working with division and fractions while using physical manipulatives. After the removal of the manipulatives, he started to draw pictures for the candy-buying task, "Yesterday, I bought 28 candies with 12 quarters. Today, if I go to the same store with 15 quarters, how many candies can I buy?" (Lo & Watanabe, 1997, p. 218). Below is an example of his work that included drawing circles to represent candies and quarters and coordinating equivalent relationships between the number of candies and the number of quarters:

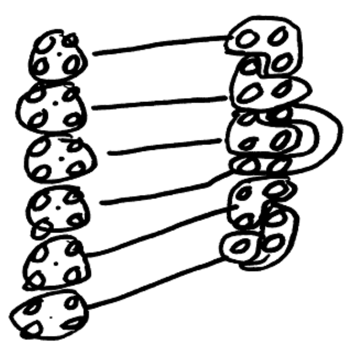


Figure 2.27 Bruce’s drawing for the candy-buying task (Lo & Watanabe, 1997, p. 223)

In a follow-up interview, Bruce was asked to solve a different version of the candy-buying task, "Yesterday, I bought 8 candies with 12 quarters. Today, if I go to the same store with 9 quarters, how many candies can I buy?" (Lo & Watanabe, 1997, p. 223) without drawing circles. However, he was not able to correctly find the answer; instead, he applied an erroneous additive reasoning and concluded that the answer was 5 by applying the difference between the number of candies and quarters in the first situation (i.e., 12-8 = 4) to the second situation (i.e., 5 is 4 less than 9). Nevertheless, for the following question, "How many candies can six quarters buy?" (Lo & Watanabe, 1997, p. 224), he was able to realize his mistake. Instead of drawing circles, he wrote down numbers 1 to 12 in a column-like manner and 1 to 8 in another column next to the first one. Then, he grouped the numbers in both columns to find equivalent relations and coordinate the two sets of numbers. However, he was still unaware of the common factor between the number of candies and quarters. Below is an illustration of this process:

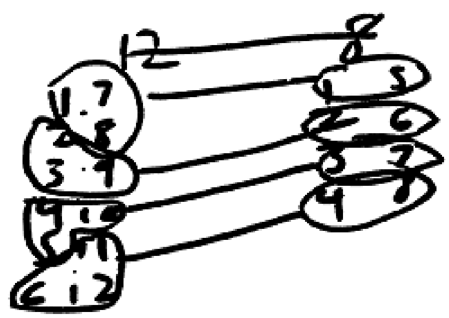


Figure 2.28 Coordinating the two sets of numbers to find equivalent relations (Lo & Watanabe, 1997, p. 224)

After four more sessions of the teaching experiment, Bruce wrote down a table-like representation and employed coordinated build-up reasoning while working on the following task:

> A house was 24 feet tall and had a window that was 12 feet above the ground. This house became 18 feet tall after a certain amount of magic liquid was applied. How tall would the window be above the ground after the magic liquid was applied? (Lo & Watanabe, 1997, p. 225)

Below is Bruce's table-like representation and coordinated build-up reasoning:

4	8	12	16	20	24
3	6	9	12	15	18

Figure 2.29 Bruce's table like representation and coordinated build-up reasoning (Lo & Watanabe, 1997, p. 225)

In particular, as he was reasoning with the table-like representation above, Bruce focused on the multiplicative relationship between the old and new heights of the house (i.e., 24 feet and 18 feet). Eventually, he concluded that "the house went down one fourth, so the window went down one fourth." (Lo & Watanabe, 1997, p. 225). The researchers concluded that he identified 6 as a common factor between the old and new heights of the house, which was referred to as norming (Lo & Watanabe, 1997). They also stressed that even though he was able to find a common factor through a norming process, he was still unable to curtail the build-up process through multiplication and division, which is referred to as abbreviated build-up reasoning, especially when the numbers got bigger. However, during the next interviews, while he was working on the question, "A helicopter flies 16 miles from the airport to a downtown hotel in 10 minutes. At this rate, how far could the helicopter fly in 2 hours?" (Lo & Watanabe, 1997, p. 229), he was able to curtail this process to scale both values within their measure spaces upon the interviewer's probing in the following table-like representation:

10	20	30	...	110	120
16	32	48	...	176	192

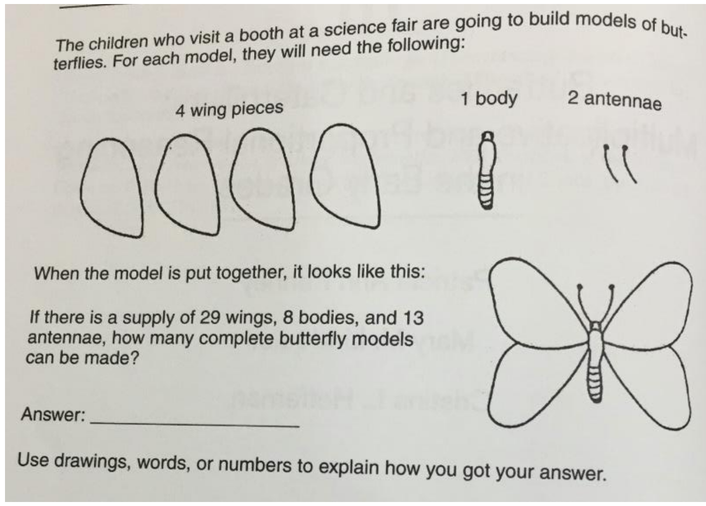
Figure 2.30 Curtailment of the build-up reasoning by multiplication within measures spaces (Lo & Watanabe, 1997, p. 229),

More precisely, Bruce came to the point that he had to multiply 16 by 12 in order to find the distance the helicopter could fly in 2 hours since 120 was 10 multiplied by 10. The researchers concluded that Bruce developed several methods to tackle a variety of proportional tasks, including scalar and functional methods. Moreover, his methods were influenced by several task variables, including the size of the numbers involved, the type of ratio, and the context. Lastly, they concluded that Bruce's difficulties were rooted in his limited understanding of several concepts in the multiplicative conceptual field: multiplication, division, fractions, and decimals.

Therefore, the findings of the study by Lo and Watanabe (1997) suggest that the development of ratio and proportion concepts occurs in relation to other concepts in the multiplicative conceptual field. Moreover, the findings of this study propose that the development of these concepts happen in relation to the use of informal tools such as pictures and table-like representations in order to link composite units and coordinate different kinds of linked composites. Moreover, several other studies cited above pointed out that students can naturally use tables and table-like representations in order to make sense of the proportional relationships by drawing their informal experiences of building up and iterating linked composites. Similarly, in this study, ratio tables are the overarching model that deserves particular attention on its own. Hence, the role of ratio tables in the development of proportional reasoning and related concepts of ratio and proportion is elucidated in more detail in the next section.

2.1.7. The Role of Ratio Tables in the Development of Proportional Reasoning

Kenney, Lindquist, and Heffernan (2002) reported fourth-grade students' informal strategies for two proportional reasoning tasks from the 1996 National Assessment of Educational Progress (NAEP) in mathematics that can be helpful to understand how elementary school students approach proportional reasoning tasks. The first one of these tasks named “the Butterfly Task” asks grade four students to find the number of complete butterfly models that can be made from the given supply of 29 wings, 8 bodies, and 13 antennae when a complete model of butterfly requires 4 wing pieces, 1 body, and 2 antennae. Below is the “Butterfly Task”:


The children who visit a booth at a science fair are going to build models of butterflies. For each model, they will need the following:

4 wing pieces

1 body

2 antennae

When the model is put together, it looks like this:

If there is a supply of 29 wings, 8 bodies, and 13 antennae, how many complete butterfly models can be made?

Answer: ____________________

Use drawings, words, or numbers to explain how you got your answer.

Figure 2.31. The Butterfly Task from NAEP 1996 for Grade 4 (Kenney et al., 2002, p. 88)

Although many strategies that include pictorial models with or without words were reported, the two of these strategies were helpful in order to understand fourth-grade students' use of informal tools. These two strategies are presented in Figure 2.32 below.

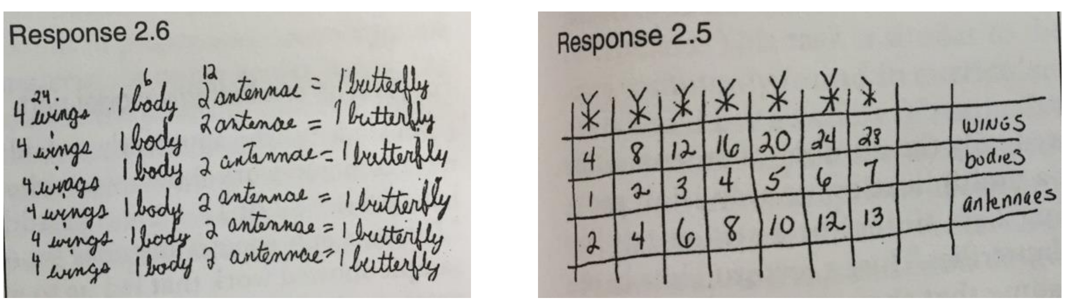


Figure 2.32. Two examples of responses to the Butterfly Task that include the organization of the information with informal tools (Kenney et al., 2002, p. 91)

As seen in the strategy on the left in Figure 2.32, the fourth-grade student linked 4 wings, 1 body, and 2 antennae that composed a complete butterfly model and wrote these linked quantities until there is not enough of one of the supplies (i.e., wings, body, and antennae). In this process, he or she represented these linked quantities by writing them next to each other. Besides, as a similar but more organized strategy, another student wrote these in a table in order to keep track of the iterations he or she made with the linked supplies.

The second task, named "the Caterpillar Task," asks students how many leaves would be necessary to feed 12 caterpillars if 5 leaves are needed to feed 2 caterpillars. Below is this task:

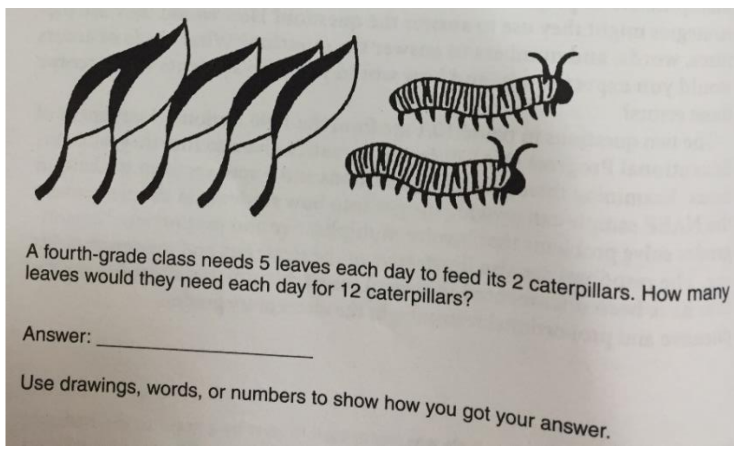


Figure 2.33. The Caterpillar Task from NAEP 1996 for Grade 4 (Kenney et al., 2002, p. 88)

The fourth-grade students were able to come up with several strategies for solving this task, including pictorial answers and verbal explanations (Kenney et al., 2002). However, their responses that include the use of horizontal and ratio tables gave essential ideas about informal tool use for proportional reasoning tasks. Below is an example of students' use of horizontal and vertical ratio tables in order to respond to the Caterpillar Task:

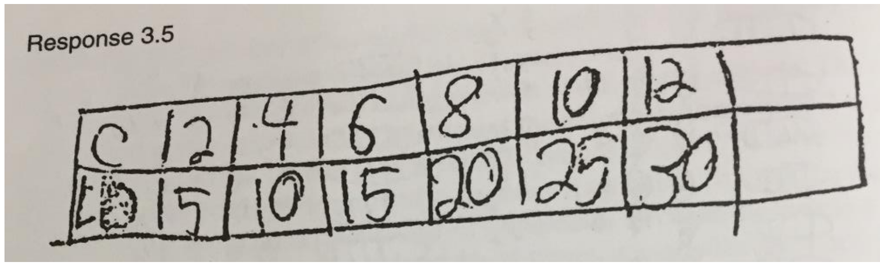


Figure 2.34. Fourth-grade student's use of the horizontal ratio table (Kenney et al., 2002, p. 96)

As seen in Figure 2.34, a fourth-grade student was able to link the number of caterpillars and required leaves (i.e., 2-5) and iterate this link through the columns of the horizontal ratio table. Besides, some of the students structured these tables vertically, as presented in Figure 2.35 below.

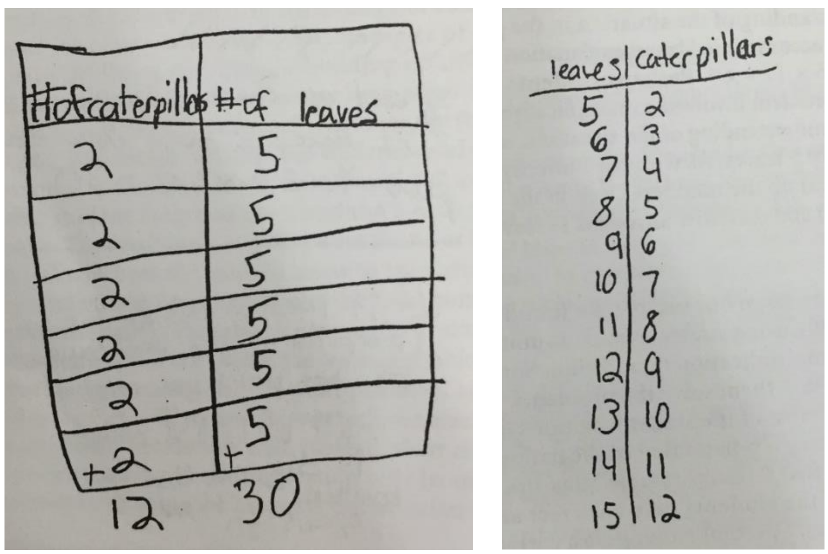


Figure 2.35. Fourth-grade student's use of the vertical ratio table (Kenney et al., 2002, p. 96-97)

Therefore, the data from a big scale assessment program revealed that fourth-grade students were able to deal with proportional reasoning tasks by iterating linked composites and building up strategies. In addition, it was also seen that these students were able to organize their build-up and iteration processes in horizontal and vertical ratio tables before receiving any formal instruction on ratio and proportion.

In another study, Streefland (1984) elucidates a teaching experiment with third graders (aged 8-9), which he suggests that can serve as building blocks for a teaching and learning theory for ratio. In this study, he particularly exhibits tools for outlining and shaping the long-term process of learning ratios in which he focuses on fostering schematization of ratio. In this experiment, the classroom discussion revolves around the theme "with the giant's regards ratio as a phenomenon" (Streefland, 1984, p. 327). In particular, one of the situations in this story is as follows:

> "The giant has a son. Sometimes they take a walk with the baker. This is not easy. For three steps of the giant the baker needs fifteen… Take the distance from the giant's dwelling place to the baker's house, The giant can do it in 18 steps. What about the baker?" (Streefland, 1984, p. 329-330).

Below is the schematization of this situation as it emerged in the classroom discussion initially:

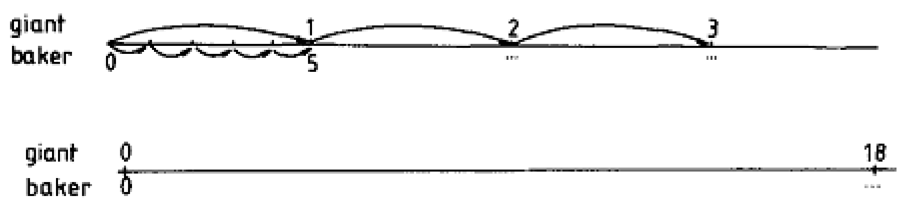


Figure 2.36. The initial schematization of the steps of the giant and the baker on the number line (Streefland, 1984, p. 330).

In the following instances of the discussion, the teacher asks where the giant has done ten steps, five steps, and one step tell while moving her finger along the number line from 18 towards 0 as in the following figure:

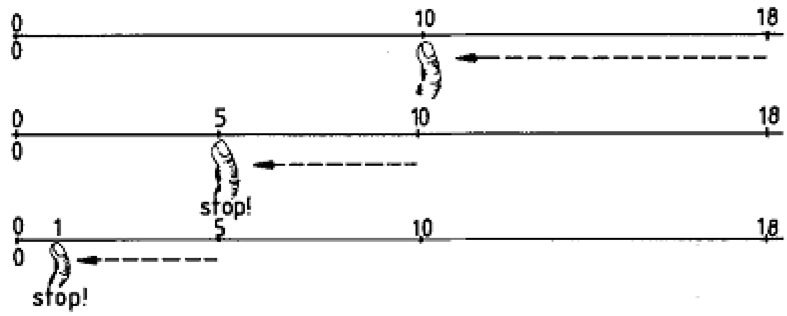


Figure 2.37. Moving fingers along the number line in order to represent the walk (hand gesture) (Streefland, 1984, p. 330).

Streefland (1984) suggests that once the Giant's and Baker's walk together is represented on the number line, the same process can be coordinated on the number line as follows:

Figure 2.38. Building up the walk of the Giant and Baker on the number line (Streefland, 1984, p. 330).

Moreover, he suggests that the corresponding ratio table can also be helpful to represent the walk of the two as in the following figure:

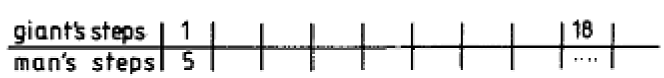


Figure 2.39. Building up the walk of the Giant and Baker in the table (Streefland, 1984, p. 330).

Streefland (1984) refers to this approach that includes gradually building up with number lines and tables as an "open approach" and suggests that this approach has the potential to invite children to fill in the values for other situations by obeying the given number relationships while "ratios remain invariant" (p. 333).

In a follow-up study, Streefland (1985) suggests that making coffee with a fixed recipe is a mapping (i.e., ϕ) of numbers of scoops on numbers of cups; that is, a ratio-anticipating activity. For this task, various situations of coffee by using the same recipe (i.e., $\frac{3}{4}$ of a scoop per cup) can be rendered in a table as follows:

scoops	3	6	9	12	15			
cups	4	8	12	16	20			

Figure 2.40. Rendering various situations of coffee with the same strength (Streefland, 1985, p. 80)

As seen in Figure 2.40, by using the given recipe (i.e., $\frac{3}{4}$ of a scoop per cup), a variety of situations can be represented in the table while the relationship between the number of scoops and cups remains the same. Furthermore, several properties of proportional relationships can be observed by using the relationships in the table (Streefland, 1985). For instance, the property "$\phi(na) = n\phi(a)$- if you want doubling or otherwise multiplying the number of cups, double or, correspondingly, multiply the numbers of scoops" (Streefland, 1985, p. 81) can be represented in the table as follows:

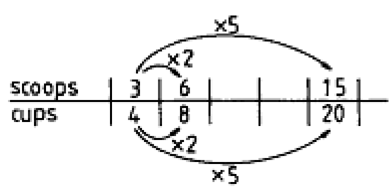


Figure 2.41 Scaling up the numbers in the ratio table (representing the property of $\phi(na) = n\phi(a)$ (Streefland, 1985, p. 81)

As seen in Figure 2.41, the process of scaling up the number of scoops and cups with the same scale factor can easily be observed by using ratio tables. Moreover, a process that requires scaling down can also be easily represented in the table as follows, where the scale factor is a fraction. This can be represented algebraically by "$\phi(\frac{1}{m}a) = \frac{1}{m}\phi(a)$- half of some other (integral) part of cups requires half or the corresponding part of scoops respectively" or "of ϕ(ra) = rϕ(a) where r is a fraction" (Streefland, 1985, p. 81).

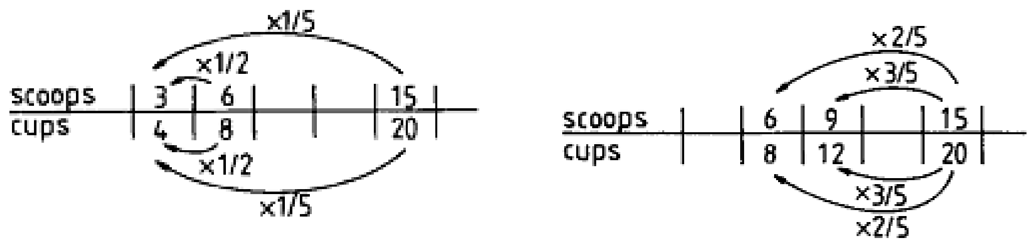


Figure 2.42. Scaling down the numbers in the ratio table (representing the property of "$\phi(\frac{1}{m}a) = \frac{1}{m}\phi(a)$ and of ϕ(ra) = rϕ(a), respectively (Streefland, 1985, p. 81)

Hence, in a similar vein to Kenney and colleagues' (2002) study, Streefland's (1984, 1985) studies provided useful insights regarding students' informal ways of reasoning in proportional tasks and how they make use of ratio tables in order to organize, represent, and communicate their reasoning. Thus, in this study, the instruction is built on these ways of reasoning and tool use in such a way that in the hypothesized trajectory ratio tables are anticipated as informal tools that would support students' understanding of proportional situations and covariation and invariation relationships that occur among these situations.

In summary, the review of the literature on proportional reasoning revealed essential mathematical components of proportional reasoning (i.e., absolute and relative thinking, covariance and invariance, ratio appropriateness-ratio sense, quantitative and qualitative reasoning, and reasoning within and between measure spaces-multiplicative reasoning) and related didactical activities (i.e., relationships, partitioning, unitizing and norming, linking composite units and iterating linked composites). Besides, it declared the formal and informal strategies of students for solving proportion tasks as mostly based on working with within or between measures ratios (also called the multiplicative strategy), unit factor (also called the unit rate) approach, and building up strategies. Besides, it presented the incorrect types of student reasoning when dealing with proportional tasks as ignoring part of the data, providing an irrelevant response, and erroneous additive reasoning with a focus on the last one. As another concern, it displayed the task variables that affect student performance, and that should get attention while designing instructional tasks. Lastly, it portrayed the developmental course of proportional reasoning, especially in young students. Thus, all this information provided a comprehensive background in order to guide the development of an instructional sequence and related hypothetical learning trajectory. Another tenet that guided the design of this study is the theory of Realistic Mathematics Education. Therefore, this theory will be described with a focus on its principles in the following section.

2.2. Realistic Mathematics Education

The domain-specific instructional theory of Realistic Mathematics Education was used to develop, revise, and implement the instructional sequence and HLT developed in this study. This theory is elucidated in the following sections by a focus on its developmental course.

2.2.1. Early works by Treffers, Freudenthal, and Streefland

In a traditional mathematics classroom, [formal] mathematics (i. e., definitions and notation) is usually the starting point, and concrete problems appear at the end as applications (Freudenthal, 1973, 1991). In this process, students passively listen to what the teacher prescribes to the students, and students imitate as they learn "ready-made mathematics" (Freudenthal, 1973, 1991). However, Freudenthal is opposed to this type of teaching by claiming that no "teaching matter should be imposed upon student as a ready-made product" (Freudenthal, 1973, p. 118) since this type of teaching is not compatible with a didactical perspective. As the opposite of ready-made mathematics, he uses the term "the method of re-invention" for a teaching method that focuses on understanding and analyzing mathematics as a human activity, particularly as the activity of a learner.

In a re-invention process, concrete problems should be the starting point, and they should be explored in terms of mathematical matters in a process that he termed as *re-invention*. Similarly, Treffers (1987) and Streefland (1991) emphasize a learning process of students, which would start from problems that are open and generative in their nature and move to construction and production of mathematics on their own. In this process, the role of reality should be to act as a source for producing mathematics and hence to serve as a domain of application (Streefland, 1991). He goes on to stress that this is also a path that is compatible with the historical learning process since it includes the process of reconstructing the historical progress of mathematical knowledge (Streefland, 1991). Following this path makes sure that students can gradually progress in their process of mathematization as they engage in a learning process (Streefland, 1991).

Freudenthal (1991) exemplifies a process of re-inventing geometry as follows. Let us say a student starts with analyzing a number of parallelograms and discovers many common properties of those shapes (i.e., opposite sides are parallel, measures of opposite angles are equal, adjacent angles are supplementary, diagonals bisect mutually, certain triangles obtained by dividing of bisectors are congruent, it is possible to pave the plane by using congruent parallelograms, etc.). Later on, he or she discovers connections among those properties and logically organizes these properties in such a way that he or she finally discovers one property among all that can be used to derive all other properties from, which would be the formal definition of a parallelogram. Therefore, this student involves in the *activity* of defining rather than being imposed a definition that would be an instance of antididactical inversion (Freudenthal, 1991).

As can be observed in Freudenthal's example of a re-invention process given above, the learning has to proceed through structured levels of *directed invention* organized around perspectives of students in which the student himself/herself re-invents mathematics (Freudenthal, 1973). Bottom level activities include real situations that include unmathematical matter. On this level, students construct models to the real situation that is to be mathematized. These activities include performing actions that are not directly related to mathematical ideas. However, bottom level activities are not unmathematical; rather, they are pre-mathematical since they are precursors that prepare students for higher-level mathematics (Freudenthal, 1973). Students may not even realize they are doing mathematics as they operate with the concepts on this level; however, the teacher would know how children's actions relate to higher degree mathematical ideas. Freudenthal (1973) argues that passing over this pre-mathematical level is a mistake, which is very common in traditional mathematics instruction.

On the next level, the organization of the activities done on the bottom level and reflections on these activities should take place in a more mathematical manner so that children become aware of what they have done previously. Otherwise, they would believe what they have been doing is irrelevant and would not be able to move to a higher level; that is, the higher level would not be accessible (Freudenthal, 1973). Eventually, higher-level operations can be introduced as algorithms and routines to provide automatism and autonomy as long as the didactic principle of re-invention is followed. Freudenthal typifies this process by giving an example of a child who starts to work intuitively on finding 8 + 5 with suitable material and splits the expression as (8+2) + 3, possibly as unconsciously. As soon as the child becomes conscious of what he is doing, feels the need to split, and develops the algorithm for such additions, he moves to the next level. Eventually, when the child is able to formulate the splitting algorithm, he even moves to a higher level. Thus, the child discovers an algorithm, which is the best way to understand it (Freudenthal, 1973).

Therefore, starting from the bottom level activities and moving to higher-level activities, formalizing takes place progressively. Students work with objects, construct models to these objects, and operate with these objects and models on the bottom level. As objects and operations become routine, they might provoke short-cut methods and streamlining, which in return might eventually lead to a common language and symbolism. This process is what Freudenthal (1991) terms as progressive formalizing. Engaging in this process of re-invention and progressive formalizing as one's own activity is essential in order to make sense of mathematizing (or mathematization), which is explained in the next section.

2.2.1.1. Mathematization

Mathematizing has always been one of the main interests of mathematicians. Freudenthal (1973) suggests that students should also learn to mathematize wherein they begin with real situations and eventually mathematize those situations at the end. He even claims that "there is no mathematics without mathematising" (Freudenthal, 1973, p. 134). Mathematizing can include mathematizing unmathematical things and/or not adequately mathematical things, which need to be mathematized better and more perspicuously (Freudenthal, 1991). Hence, mathematizing is referred to as the activity of organizing nonmathematical or inadequately mathematical matters into a structure that would allow for mathematical refinements (Freudenthal, 1991).

Therefore, mathematizing could occur at different levels: Understanding spatial gestalts as figures is a mathematization of space, organizing the properties of parallelograms to obtain a definition is a mathematization of the conceptual field of parallelogram, organizing the geometrical theorems to arrive at all of those from a number of those is a mathematization of geometry, arranging this system into a language is a mathematization of a subject, which is called formalizing (Freudenthal, 1991). Hence, how far students should mathematize is a question to be considered; however, they should undoubtedly engage in mathematizing on the lowest level that includes working with unmathematical matter and on the next level where an organization of mathematical matter into a structure takes place (Freudenthal, 1973).

Freudenthal (1991) mentions modeling as an essential aspect of mathematizing, which includes creating schemes to fit reality and where the emphasis is on the content in lieu of the form. Within the context of mathematization, a model should act as an "intermediary by which a complex reality or theory is idealized or simplified in order to become accessible to more formal mathematical treatment" (Freudenthal, 1991, p. 34). Hence, models do not directly apply to mathematics; instead, they should be used as intermediaries in order to arrive at mathematical formulas (Freudenthal, 1991). According to him, urns (i.e., small bags that are used to draw lots from) are models for mathematizing things that are conditioned by chance (Freudenthal, 1991).

Similarly, Streefland (1991) highlights the importance of models to support mathematization and proposes a process of a shift from *a model of* to *model for*. According to him, reality is first simplified to a model as an *after-image*, and as the model is found as applicable to other situations, it becomes a model for higher-level mathematical reasoning. This shift from a *model of* to *model for* can be observed in students' work in (fair-)sharing pizzas (Streefland, 1991). In sharing pizzas, students firstly draw circles to represent pizzas and divide those circles into equal parts, in which circles act as *models of* pizzas. As students reason with similar drawings about the relations between fractions, they become *models for* higher-level mathematical reasoning. In addition to the importance of using models to reason with in order to arrive at more mathematical knowledge, production and creation of models, schemas, and symbols is also crucial to communicate the obtained knowledge (Streefland, 1991).

2.2.1.1.1. Horizontal and Vertical Mathematization

Treffers' (1978) study is known to be the first attempt to distinguish horizontal and vertical mathematizing where horizontal mathematizing is referred to as making a problem situation accessible to mathematical treatment, and vertical mathematizing is referred to as processing more or less sophisticated mathematics. Freudenthal (1991) clarifies this distinction by referring to horizontal mathematization as moving from the real world to the symbolic world and vertical mathematization as the acts of mechanical and reflective shaping, reshaping, and manipulation of symbols that take place in the world of symbols.

Likewise, according to Streefland (1991), students should begin with reality and then engage in the processes of structuring, arranging, symbolizing, visualizing, schematizing, and hence move to mathematics, through which they engage in horizontal mathematization. Besides, they should also work within the mathematical matter in order to arrive at more efficient procedures, apply abbreviations, and use the symbolic language of mathematics by engaging in the processes of abstraction, generalization, unification, and specification, which Freudenthal (1973, 1991) mentions as vertical mathematization. These worlds can get wider or narrower, and there is no clear line between these worlds (Freudenthal, 1991). While something can belong to the real world on one occasion, in another, it can belong to the symbolic world. For instance, natural numbers can belong to the real world; yet, symbolic addition with natural numbers belongs to the world of symbols. On the other hand, mathematical objects and operations can be a natural part of a mathematician's life, while this may not be the case for students. Therefore, the distinguishing of horizontal and vertical mathematization is highly dependent on the situation, the individual that is engaged in the process, and the environment (Freudenthal, 1991).

We can look at the examples provided by Freudenthal (1991) in order to better understand the distinction between these two types of mathematizations. For instance, if a student works on the operations 2+9 and 9+2 visually and/or mentally and discovers that these two can be replaced by each other, this is horizontal mathematization. If the student applies the law of commutativity generally; then, that would be vertical mathematization. He also gives several examples that would apply to the concepts of ratio and linearity. For example, schematizing and graphing the linear functions as straight lines would be vertical mathematization of the concept of ratio, while the concept of ratio can be mathematized horizontally through daily life situations. In this situation, understanding the relationship between constant ratio and straightness is related to vertical mathematization. Besides, grasping the relationship between the ratio’s value and the graph’s steepness is an example of vertical mathematization as well. Furthermore, in a context involving similar shapes, going back and forth through visual geometrical, and arithmetical similarity might lead to alternating of horizontal and vertical mathematization with each other as students start with interpretations such as "what is double in size here, must also be double in size here" (Freudenthal, 1991, p. 43).

Freudenthal (1983) suggests that the process of mathematizing should be based on *Didactical Phenomenology*. He alludes to the use of the term "didactical" instead of educational. The aim of this is to stress the relationship between teaching and learning since the didactics of a subject area is related to the organization of the process of teaching and learning in that area (Freudenthal, 1991). According to Freudenthal (1983), the process of teaching and learning should include learning sites that help children to gain critical insights into essential mathematics. These learning sites constitute the didactical phenomenology of a mathematical domain, which includes a series of realistic situations in which critical mathematical ideas are rooted in real phenomena. In other words, these learning sites serve as contexts wherein meaningful learning of mathematics occurs through an organization of the real phenomena being investigated. Therefore, didactical phenomenology describes the possible experiences and learning sites through which a student enters into the process of learning by organizing the phenomena and reconstructs the intended mathematical idea (Freudenthal, 1983). In a later study, Freudenthal (1991) outlines a series of didactical principles of mathematics, among which guided re-invention and bonds with reality is highlighted. These principles will be explained in further detail in the following sections.

2.2.1.2. Guided re-invention

Freudenthal (1991) stresses that the term discovery might be preferred in the context of teaching (i.e., discovery learning); yet, the term "invention" puts an emphasis on both the interactions between content and form and discovering and organizing. That means, while the term discovery as in discovery learning is related to uncovering things covered up by someone, the term invention involves the process of discovering the content and then organizing the content in mathematical forms. Furthermore, while the term "invention" in re-invention stands for the steps in the process of learning, the prefix "re-" puts a restriction on what the learner would invent since he or she is supposed to invent something new for him but known and intended by the guide (Freudenthal, 1991).

Re-invention is also related to the “historical learning process” through which “insightful construction of the system” of mathematics should take place as how the construction of mathematics took place in history (Streefland, 1991, p. 19). However, it should be noted that it does not mean that students must go through exactly the same experience as in history (Freudenthal, 1991; Streefland, 1991). Instead, they can retrace what took place in the history of the development of mathematics according to the spirit of historical learning process (Streefland, 1991); that is, “not as it factually took place but rather as it would have done if people in the past had known a bit more of what we know now” (Freudenthal, 1991, p. 48). However, re-invention does not always have to refer to the historical learning process; children's informal ways of thinking can also be sources for the intended formal knowledge (Streefland, 1991).

Some children can re-invent some of the mathematical knowledge (e.g., arithmetic) on their own; however, it is not reasonable to expect every child to reinvent all the mathematics on their own. So, they need the guidance of others- adults and their peers (Freudenthal, 1991). Hence, the term "guided" implies the instructional environment organized by the guide to allow the process of re-invention. Guided re-invention includes acquiring knowledge and ability as a result of the learner's own activity. Therefore, the learner retains and uses that knowledge and skills better. Besides, it might provide with higher motivation. Third, by engaging in the process of re-invention, learners can better experience and see mathematics as a human activity (Freudenthal, 1991).

2.2.1.3. Bonds with reality

Freudenthal stresses that if a child learns mathematics in an unrelated way in his/her lived-through reality, that learned knowledge would disappear since "what is unrelatedly learned does not last long" (Freudenthal, 1973, p. 77). As opposed to the teaching of those unrelated sets of mathematical subjects, he proposes the teaching of "mathematics fraught with relations." Similarly, Streefland (1991) urges that instruction should be intertwined or interwoven in such a way that the material should be related to other mathematical concepts. Engaging in intertwined teaching of mathematics fraught with relations, students could learn faster and deeper, and retain that learned knowledge (Freudenthal, 1973; Streefland, 1991). Therefore, the teaching of coherent and connected material rather than isolated teaching should be valued. These connections can be within mathematics in order to teach mathematics in a unified manner. Another type of connection can be made to physics or daily life, which is considered to be more natural and essential (Freudenthal, 1973, 1991). However, these connections should involve a lived-through reality instead of a "dead mock reality" (Freudenthal, 1973, p. 78) that is made-up to serve as an example. Therefore, it is crucial to create, strengthen, and maintain bonds with reality (Freudenthal, 1991).

In order to teach *mathematics fraught with relations*, mathematizing should take place in *rich contexts* that are referred to as the domains of reality that are presented to the learner to be mathematized (Freudenthal, 1991). Streefland (1991) also highlights the role of realistic contexts for concept formation and highlights the importance of embedding concepts in daily life contexts. These contexts can include cases such as location (e.g., Disneyland-like-island), story (e.g., Gulliver in Lilliput), project (e.g., building a bungalow), themes (e.g., flying), and clippings (e.g., information from newspapers, books and/or other media) that are demarcated by the teacher in order for the learner to reinvent specific processes and formal knowledge (Freudenthal, 1991). Freudenthal (1991) also lists teaching and learning processes among the didactic principles of mathematics, in which he stressed individual and group learning processes.

2.2.2. Recent Work on RME

Freudenthal (1973) sowed the seeds of the theory of Realistic Mathematics Education (RME), and Treffers (1987) and Streefland (1991) have contributed to the theory substantially, as aforementioned in the section above. However, RME is not a rigid theory, but it has been continuously subject to essential elaborations and refinements "in an on-going process of designing, experimenting, analyzing, and reflecting" (Gravemeijer, 1991, p. 157). Based on the works of Freudenthal related to guided re-invention, progressive formalization, and mathematization, Gravemeijer (1991) refers to the RME theory as a theory of knowledge construction, in which students start from experientially real contexts and arrive at mathematical knowledge through progressive formalization.

Freudenthal (1991) stressed the importance of models in mathematizing and described their role as intermediaries that idealize and simplify realities in order to make reality accessible for mathematical refinements. Since then, a growing interest has been expressed in the role of models among RME researchers. To begin with, Gravemeijer (1991) made a distinction between the role of models in RME practices and other approaches. According to Gravemeijer (1991), while models are used as the concrete examples of formal mathematics to be taught in other approaches, it is intended that models grounded in the contexts are created by students and used to re-invent more formal mathematics in RME approach.

It is also essential that a model should act as a catalyst for a shift from *a model of* to *model for,* together with a shift in the ways students reason with the models (Gravemeijer, 1991). While models encourage thinking about the context at the beginning; later, they should support a focus on mathematical relations. An example would be the emergence of the ruler as a model where students start to work with the empty number line model in the activity of measuring. Therefore, the term emergent in emergent models is overarching: “it refers both to the process by which models emerge within RME and to the process by which these models support the emergence of formal mathematical knowledge" (Gravemeijer, 1991, p. 175). Therefore, the emergent model heuristic of RME is attributed to three intertwined processes: (1) global transition, wherein the model emerges as a model of students' informal activity at the beginning and develops into a model for higher-level mathematical thinking progressively, (2) emerging of new mathematical reality as a shift from model of to model for takes place (3) emerging of a chain of signification, which implies a series of signs occurring in a recursive manner.

As Freudenthal (1991) highlights the importance of individual and group learning processes, Gravemeijer, Cobb, Bowers, and Whitenack (2000) draws attention to the activity of reinvention as being both an individual and collective activity. In a collective re-invention activity, students participate in whole-class discussions as they engage in the processes of conjecturing, explaining, and justifying, which is referred to as collective mathematizing (Gravemeijer et al., 2000). In such an activity, the teacher should capitalize on students' reasoning in order to guide their progress to intended mathematics. In this process, the teacher and students negotiate in order to support the emerging of taken-as-shared meanings when certain social and sociomathematical norms are established (Cobb & Yackel, 1996; Gravemeijer et al., 2000; Yackel & Cobb, 1996). In such normative ways, the classroom community engages in horizontal mathematization as they come up with informal taken-as-shared ways of reasoning and communicating; that is, as they mathematize the reality in the served context. When these ways of reasoning and communication are subject to further mathematization, students engage in the activity of vertical mathematization in collective ways (Gravemeijer et al., 2000).

In order to support a classroom community's collective ways of reasoning and learning through an RME perspective, it is essential to design instructional sequences, including situations that would enhance the progress from these situations to intended mathematics.

In order to design such sequences that are compatible with the spirit of RME, a specific set of heuristics RME heuristics should be followed (Gravemeijer et al., 2000, Gravemeijer, & Stephan, 2002). The first principle is "guided reinvention through progressive mathematizing," which guides the exploration of the history of mathematics in order to understand its development over time. In this way, it is possible to see and make sense of the potential obstacles and breakthroughs in the case of designing instructional sequences. The designer, then, should check if students may go through the same path to develop an understanding of the subject. In addition to history, this principle also includes thinking about possible informal understandings of students that may anticipate more mathematical practices. Therefore, the historical development and/or students' informal reasoning may be the starting points and sources of insight while designing an instructional sequence (Gravemeijer et al., 2000; Streefland, 1991).

When mathematics is interpreted as a product of the activity of a community of learners trying to find solutions to problems in a progressive manner, it is essential to develop real-life problems with rich contexts (Gravemeijer et al., 2000). However, finding problems with rich contexts that would support progressive mathematizing in designing instructional sequences is challenging. The heuristics of didactical phenomenology (Freudenthal, 1983) can be used as a guide in the process of developing rich contexts. In this process, the goal is to analyze the relations between thinking processes and mathematical phenomenon from a didactical standpoint. That means, the developers try to create experientially real contexts wherein students can negotiate gradually advanced solutions to problems in collective and normative ways (Gravemeijer et al., 2000).

The third principle that helps guide the design of instructional sequences for collective mathematization is the heuristic of *emergent models*. In the context of collective mathematization, models emerge as situated in students' informal ways of reasoning and develop over time to more mathematical models including symbols as negotiation takes place in order to arrive at "taken-as-shared ways of symbolizing" (Gravemeijer et al., 2000, p. 240). The goal in this process is to come up with ways of symbolizing that can fit and foster students' informal reasoning (Gravemeijer & Stephan, 2002). Another goal is to capitalize on the informal use of students' own models and support the shift to using conventional symbols by introducing symbolizations that are apt to the ways they reason (Gravemeijer et al., 2000). In this way, a transition from a model of to model for would be supported. Therefore, this taken together with the previous principles, it is essential to create situations "in which symbolizations and meaning co-evolve" (Gravemeijer & Stephan, 2002, p. 146) and wherein students could invent their own models that eventually evolve into more mathematical models and/or symbols as hypothesized in the proposed learning trajectory. Such an approach is called a "bottom-up approach" (Gravemeijer & Stephan, 2002, p. 146) as opposed to the classical “top-down approach” wherein fixed models are emanated from formal mathematical knowledge.

Therefore, in this section, the domain-specific theory of RME was elaborated with a focus on its principles and its developmental course. The literature review on RME showed that embedding mathematical concepts in realistic contexts is necessary in order to support students ‘processes of mathematization and reinvention. Moreover, it also revealed that the use of models and tools should support these processes in such a way that a transition from a model of to model for is supported. Especially, the recent work on RME depicted the principles that should be followed while designing and implementing instructional sequences within the context of classroom-based research in order to support students’ reasoning in collective ways. Since the purposes of this study include developing, testing, and revising an instructional sequence based on an RME perspective, these guidelines were followed in the design and implementation of the tasks in this instructional sequence. More specifically, in order to support students' processes of mathematization and reinvention, the essential mathematical ideas that were described in the previous section were embedded in realistic tasks as bottom level activities. In this process, students' informal strategies (i.e., build-up, abbreviated build-up, unit factor approach, within- and between-measures comparisons, etc.) were taken as the starting points since the review on its historical progress did not yield specific guidelines to build instruction on. Besides, ratio tables were hypothesized to be used as models that would emerge from students' build-up strategies and foster their processes of mathematization.

In the next section, Hypothetical Learning Trajectories will be explained by focusing on the different definitions made to define them and the areas in which they are used and approached.

2.3. Hypothetical Learning Trajectories

Since the constructivist views started to dominate the educational area, more and deeper knowledge about learning and learners have been available (Simon, 1995). These changing views about learning have resulted in a significant reform in mathematics teaching and learning worldwide (National Council of Teachers of Mathematics [NCTM], 1989). Even though reform curricula highlight student reasoning and problem-solving as essential aspects of teaching and learning, they do not adequately address how student reasoning develops over time in each topic (Lobato & Walters, 2017). Consequently, even though reform efforts achieved great success in changing practices in selected cases, they have failed to cause a broader effect (Ball, Lubienski, & Mewborn, 2001). Moreover, many international and national studies continue to report low student achievement (National Assessment of Education Progress [NAEP], 2007; Yıldırım, Yıldırım, Ceylan, &Yetişir, 2013). Therefore, there might be some gap between the constructivism theory and practice.

Simon (1995) stresses that constructivism "does not tell us how to teach mathematics" (p. 114); that is, it does not impose a particular way of any instructional method. Therefore, it might be hard to integrate a constructivist approach into instructional decisions. Based on this concern, Simon (1995) proposes that teachers design mathematics lessons in line with related research findings on student thinking and learning and anticipated ways of reasoning. Simon (1995) suggests a way to integrate research and educational practice by the development and use *of Hypothetical Learning Trajectories (HLT)*. In this process of developing an HLT, a teacher engages in developing conjectures about the ways of his/her students might reason and their possible challenges about a specific subject. Upon the anticipation of student's potential learning route, the teacher creates learning activities that would support their learning of the targeted subject along the hypothesized route that includes sophisticated levels of understanding. Simon (1995) stresses that HLTs are hypothetical since it is not possible to know the real learning trajectory. He defines learning trajectories as "predictions as to the path by which learning might proceed" (p. 135). He goes on to suggest that they include "the learning goal, the learning activities, and the thinking and learning in which students might engage" (p. 133). Learning goal specifies the direction, learning activities are the tools that will be used to reach the goal, and the learning process includes the ways students' thinking and understanding will be developed by the help of these activities (Simon, 1995).

It should be noted that "the development of a learning process and the development of learning activities have a symbiotic relationship" (Simon, 1995, p. 136). That means, the development of learning activities depends on the thinking and learning processes of students, and the hypotheses of students' learning routes are dependent on the nature of the learning activities. Moreover, "all forms of teacher support, appropriate tasks and tools, peer-to-peer discourse, and the language necessary to specify and build ideas" have a central role in students' progress (Confrey, Maloney, & Corley, 2014, p. 721). Therefore, the nature of instructional activities and the teaching and learning process is critical for students' development of mathematical ideas.

Since Simon (1995)'s introduction, HLTs have been considered as an essential research area in curriculum development, measurement and assessment, professional development, improving the quality of instruction and learning (Corcoran, Mosher, & Rogat, 2009; Lobato & Walters, 2017; Sarama, Clements, Barrett, Van Dine, &

McDonel, 2011). This gaining interest in HLTs urges a transition from partitioned and isolated teaching of mathematical facts and skills to organized and sequenced teaching where the focus is on developing longer and coherent sequences of instruction that connects knowledge vertically across grades and horizontally within a grade (Duschl, Maeng, & Sezen, 2011).

The definition of HLTs has been subject to change, and researchers have focused on different aspects over time. For instance, Clements and Sarama (2004) define HLTs as "descriptions of children's thinking and learning in a specific mathematical domain and a related, conjectured route through a set of instructional tasks designed to engender those mental processes or actions hypothesized to move children through a developmental progression of levels of thinking, created with the intent of supporting children's achievement of specific goals in that mathematical domain" (p. 83) and maintain that HLTs should include a learning goal, related learning activities, and possible thinking of students. Also, they stress that the learning activities should comprise key tasks that would support student learning along sophisticated and succeeding levels of learning, which they refer to as an instructional sequence (Clements & Sarama, 2004). Based on this, they suggest that "a complete hypothetical learning trajectory includes all three aspects: the learning goal, developmental progressions of thinking and learning, and sequence of instructional tasks" (Clements & Sarama, 2004, p. 84). Additionally, it should be noted that learning trajectories are hypothetical at the beginning; thus, teachers "must construct new models of children's mathematics as they interact with children around the instructional tasks and thus alter their own knowledge of children and future instructional strategies and paths" (Clements & Sarama, 2004, p. 85). Hence, the actual learning trajectory and the taken-as-shared ways of reasoning are emergent.

Another definition by National Research Council (2007) refers to learning trajectories as "descriptions of the successively more sophisticated ways of thinking about a topic that can follow one another as children learn about and investigate a topic over a broad span of time" (p. 214). In another study, Confrey et al. (2014) refer to LTs as "research-based frameworks developed to document in detail the likely progressions, over long periods of time, students' reasoning about big ideas in mathematics" (p. 720). While all of these definitions emphasize the individualistic routes of learning, the definition has also been adapted to refer to a learning path of a social community (Cobb, 2001). This perspective reconceptualizes the term learning trajectory as a "sequence (or set) of (taken-as-shared) classroom mathematical practices that emerge through interaction (especially through classroom discourse-with the proactive involvement of the teacher)" (Clements & Sarama, 2004, p. 85).

Having different definitions of learning trajectories in the literature is closely related to how they are approached. Concerning this, Lobato and Walters (2017) classified seven approaches to learning trajectories based on a systematic review of literature as follows: (1) Cognitive levels, (2) Levels of discourse, (3) Schemes and operations, (4) Hypothetical learning trajectory, (5) Collective mathematical practices, (6) Disciplinary logic and curricular coherence, and (7) Observable strategies and learning performances.

According to Lobato and Walters (2017), the Cognitive Levels Approach includes qualitative identification of distinct hierarchic levels of cognition, although the hierarchies can be weak. In this approach, researchers integrate the results of a series of empirical research in order to come up with levels of sophisticated reasoning. In the second approach, namely Levels of Discourse, hierarchic levels of communication, including discursive practices, are under investigation. In the third approach, the focus is on initial schemes and (mental) operations of students and how they are modified over time. The fourth approach, namely Hypothetical Learning Trajectories, is rooted in Simon’s (1995) work on instructional decisions and related to making decisions that support student learning. In this approach, the teacher anticipates a learning route of his/her students by taking into account their present understandings regarding the learning goal. Based on this anticipation, the teacher creates learning activities that would support students' transition in sophisticated ways of thinking and learning. In the fifth approach, Collective Mathematical Practices, a classroom's taken-as-shared ways of reasoning, operating, arguing, and tool use are being explored. The sixth approach is referred to as Disciplinary Logic and Curricular Coherence Approach, wherein expert knowledge informed by long-time research is used in generating levels of how students think in sophisticated levels in a particular subject and/or domain. In the seventh and last approach, namely Observable Strategies and Learning Performances, describing proficiency levels of observable behaviors and/or learning performances is the primary goal.

Although the approaches to and definitions of HLT have changed over time, some commonalities exist in all approaches/definitions. When these commonalities are in question, it can be seen that all of them include the following features: (1) HLTs are research-based, (2) HLTs include learning goals, activities, and possible reasoning of students as they engage with those activities, (3) HLTs include long-term learnings of students. On the other hand, learning trajectories can focus solely on learning (i.e., approaches 1, 2, 3, 6, 7) or the interplay between teaching and learning (i.e., approaches 4-5). Furthermore, they can also focus on an individual’s learning (i.e., Steffe, 2004) or emerging mathematical practices of communities of learners in a classroom (i.e., Gravemeijer et al., 2003b).

Concerning this, different terms are used in order to emphasize the interaction of teaching and learning along a trajectory. For instance, Cross, Woods, and Schweingruber (2009) adapt the term learning trajectory as "teaching-learning paths" in their report. Similarly, Van den Heuvel-Panhuizen (2008) uses the term "learning-teaching trajectories" in order to put an emphasis on teaching and learning. Lastly, when descriptions of normative ways of reasoning and learning of a classroom are in question, Stephan (2015) uses the term *classroom learning trajectories* (CLT) defined as “conjectures about the mathematical ideas that become taken-as-shared and individuals' ways of participating in and contributing to them” (Stephan, 2015, p. 908). These conjectures include "the mathematical goals, and tool use as students engage with the instructional tasks" (Stephan, 2015, p. 908).

Furthermore, CLTs include an outline of instructional supports in order to foster student learning along a learning path. After the implementation in a classroom takes place, the developer documents the actual learning trajectory (i.e., classroom mathematical practices) and make refinements to the CLT and the instructional sequence based on his/her analysis of the classroom implementation. With this refined and hypothetical learning trajectory (HLT), another classroom experiment is conducted to inform future research (Stephan, 2015).

An example of a CLT for teaching addition and subtraction with integers, which was developed, revised, and refined after a series of implementations, was reported in Stephan and Akyuz (2012). A portion of the table that belongs to this CLT is presented in the following figure.

A Hypothetical Learning Trajectory for the Integer Sequence

Phase	Tool	Imagery	Activity/ taken-as-shared interests	Possible topics of mathematical discourse	Possible gesturing and metaphors
1	Net worth statements	Assets and debts are quantities that have opposite effect on net worth.	Learning finance terms	• Conceptualizing an asset as something owned and a debt as something owed • Conceptualizing a net worth as an abstract quantity (not tangible)	
2	Net worth statements	Differences in collections of assets and collections of debts	• Determining a person's net worth • Who is worth more?	Different strategies for finding net worths	Pay off
3	Symbols (+ and –)	+ means asset and – means debt.	Determining and comparing net worths	• Different strategies for finding net worths • Creating additive inverses as objects	Pay off
4		Good decisions increase net worth. Bad decisions decrease net worth.	Which transactions have good and bad effects on net worth?	When taking away an asset; is this good or bad? When taking away a debt; is this good or bad? • Judging the results of transactions and, therefore, direction to move on a number line	Arms moving up and down to indicate good or bad movements

Figure 2.43. A portion of the CLT table for the integer sequence (Stephan & Akyuz, 2012, p. 434)

As can be seen in the figure above, a CLT includes tools, imagery, activity/taken-as-shared interests, possible topics of mathematical discourse, and possible gesturing and metaphors for the related mathematics subject. Besides, when CLTs are in question, it is necessary to develop the related instructional sequence that would support the hypothesized development through interaction with the instructional sequence. The initial instructional sequence includes a set of ordered instructional tasks together with guidelines regarding the order of the tasks and anticipated thinking and learning processes of students as they interact with these tasks. In the second phase, these tasks are tested in a classroom setting and subject to revisions and refinements throughout classroom testing on a daily basis based on the experience obtained from the implementation of previous tasks (Clements & Sarama, 2004). These revisions can also be related to the teacher's role and /or the classroom culture (Clements & Sarama, 2004). In the third phase, the "best-case instructional sequence" is developed in order to come up with a more general HLT that describes the local instructional theory that can be used by the teachers in their classrooms (Clements & Sarama, 2004).

In a similar way, the learning trajectory that is developed in this study is a CLT that includes the same elements (i.e., tools, imagery, activity/taken-as-shared interests, possible topics of mathematical discourse, and possible gesturing and metaphors) in it. In addition, the related instructional sequence that students would interact with is developed in order to support students' development along the hypothesized trajectory. However, there are some differences, as well. First of all, the CLT developed in this study includes big ideas as learning goals in line with Simon's (1995) allusion to that learning trajectories should include learning goals. Second, tools and imageries are presented together in the same column since they are closely related to each other. For instance, in this experiment, students started the instructional sequence by circling pictures of fish and food bars in order to group them and linking the two groups with arrows. Therefore, the pictures of the fish and food bars were the tools students used to link and iterate composite units. However, as the numbers got bigger, the students quit drawing pictures and imagined the grouping and linking processes in their minds to divide and multiply the number of food bars and/or fish. Therefore, the physical tools turned in to mental images that could then be called imageries.

Up to this point, different approaches to HLTs were described. Concerning this, the different definitions and uses of HLTs were elaborated. In the next section, a literature review of HLTs developed for proportional reasoning or related fields will be portrayed in order to describe the convergences of the HLT developed in this study and in the previous studies.

2.3.1. HLT in ratio and proportion

Carpenter, Gomez, Rousseau, Steinthorsdottir, Valentine, Wagner, et al. (1999) proposed a four-level learning trajectory in proportional reasoning developed through studying a combined classroom of 4th and 5th graders in a 2-week period. According to this trajectory, Level 1 includes a demonstration of limited knowledge of ratio. This limited knowledge might include performing random calculations or reasoning additively (i.e., focusing on the difference). At the next level, Level 2, students perceive ratio as a single unit and find equivalent ratios by repeated addition or by multiplication with an integer. However, students at this level cannot perform operations related to partitioning the given ratio (i.e., noninteger ratios). Therefore, this level is characterized mostly by build-up strategies in which repeated addition, multiplication, or a combination of both is performed when integer ratios exist. Furthermore, at Level 3, students can perceive ratio as a single unit and perform operations with both types of ratios, integer, and non-integer ratios. Therefore, Level 3 is an extension of Level 2, wherein ratios are reduced. For instance, while working on a problem that can be represented by the proportion $\frac{8}{12} = \frac{42}{x}$, a student at Level 2 cannot proceed to the solution while a student at Level 3 can reduce the ratio $\frac{8}{12}$ to $\frac{2}{3}$ and perform within or between strategies on the proportion $\frac{2}{3} = \frac{42}{x}$. As an alternative, multiplying the ratio $\frac{8}{12}$ by a non-integer value (i.e., $5\frac{1}{4}$) to find the missing value is a characteristic of a student at Level 3. Besides, a student at Level 3 can build up from $\frac{8}{12}$ to $\frac{40}{60}$, reduces $\frac{8}{12}$ to $\frac{2}{3}$, and then adds this reduced ratio to $\frac{40}{60}$ in order to find the missing value. The representation of this kind of thinking on a ratio table is presented in Figure 2.44 below.

Hours	8	16	32	*(+ 8)*	40	2	*(+40)*	42
Km	12	24	48	*(+ 12)*	60	3	*(+60)*	63

Figure 2.44. A typical strategy at Level 3 in Carpenter et al.'s (1999) trajectory

Lastly, Level 4 involves thinking ratios as more than just units and recognizing within and between relationships in the proportion $\frac{a}{b} = \frac{c}{d}$. Therefore, students do not only use some kind of build-up strategies, but they can also make sense of preserving the relationships within a ratio, regardless of the fact that these relationships form an integer value or not.

In a later study, Steinthorsdottir and Sriraman (2009) investigated the developmental trajectory of two classes of 5th-grade girls through the implementation of an instructional unit in 12 weeks in order to validate, explicate, and extend the learning trajectory developed by Carpenter et al. (1999). According to the results of the study, the four-level learning developmental trajectory for proportional reasoning was validated; yet, extending the trajectory by adding an emerging Level 3 between Level 2 and Level 3 was found necessary. The results showed that emerging Level 3 students could solve a problem that included a scaling-down process (i.e., $\frac{8}{24} = \frac{2}{x}$) while Level 2 students could not. However, these students at emerging Level 3 could not perform a scaling-down process with noninteger ratios, while Level 3 students could deal with noninteger relationships. Therefore, an emerging level between Level 2 and Level 3 was suggested to represent the competence that students displayed.

In another study, Wright (2014) proposes a hypothetical learning trajectory for rational numbers based on a literature review wherein he takes rates and ratios as a sub-construct of rational numbers along with measures, quotients, and operators. The first phase of this trajectory is associated with improper centration on one of the measures or no application of ratio/rate. Besides, focusing on the difference of measures (i.e., inappropriate additive reasoning) is also inherent in this first phase (Wright, 2014). In the second phase, a composite unit is formed and treated as a ratio, and this ratio is built up by repeated addition. In the third phase, the composite unit treated as a ratio is built up multiplicatively in such a way that the process of building up is abbreviated. In the last phase (i.e., Phase 4), the ratios/rates are treated as iterable units, and the within and between measures relationships are used flexibly irrespective of whether there exists an integral or non-integral operator.

An example might be helpful to make sense of and distinguish between these phases. For instance, while comparing the lightness of the colors obtained by mixing 2 units of blue and 3 units of yellow and by mixing 3 units of blue and 5 units of yellow, a student might judge that the color obtained by mixing the colors with a ratio of 2:3 would yield in a lighter color since that includes less amount of blue in Phase 1. This is rooted in improper centration on the absolute amount of the colors rather than the relative amount. However, it is possible to correctly judge the lightness of the color when one measure is equal (i.e., 2:3 and 2:4). In order to answer the same question, a student at the second phase would be able to replicate 2:3 in order to form 4:6, and 6:9 and compare 6:9 and 6:10 (which is equivalent to 3:5) and make a correct judgment.

Moreover, a student at the third phase would correctly obtain 16:24 as equivalent to 2:3 by scaling by 8 and 15:25 as equivalent to 3:5 by scaling by 5. Then, he or she would be able to compare these 16:24 and 15:25 by focusing on the parts of a whole since both have a whole composed of 40 parts. Lastly, a student at the fourth phase would be able to compare 2:3 and 3:5 by reconceptualizing these ratios as part-whole relationships ($\frac{2}{5}$ and $\frac{3}{8}$) and make use of fraction comparison.

Therefore, although it is limited to a small number of studies, the literature review on the previously developed HLTs in proportional reasoning revealed that the development of proportional reasoning proceeds along a path starting from focusing on the differences (i.e., absolute change) or no relevant reasoning at all. In the middle levels, students start to build up by iterations or repeated addition while they might be unaware of the multiplicative relationships between the quantities. In the upper levels, students start to work with within- or between-measures comparisons. In this phase, the presence of an integer or non-integer ratio is a critical determinator of students' performance to result in their falling back to the erroneous additive reasoning. In the top levels, students can make comparisons between different ratios/rates.

Additionally, it was seen that the already developed HLTs mostly focused on individuals' gradual development. In other words, none of the studies above describes collective ways of development in progressively sophisticated manners, although they might be useful in informing those. Therefore, this study has the potential to contribute to the literature in providing a developmental path of a group of students in communal ways.

CHAPTER 3

METHODOLOGY

The purposes of this study are multifaceted: (1) to develop, test, and revise a classroom HLT and related instructional sequence for proportional reasoning in seventh grade (2) to explain students' communal ways of reasoning with informal tools and how this reasoning evolves over time to reasoning with formal tools in line with an RME perspective, and (3) to document students' collective development of mathematical concepts related to proportional reasoning (i.e., documenting mathematical practices). To these ends, the research questions of this study are phrased as follows:

1. What would an optimal HLT and instructional sequence for proportional reasoning look like?
    - What would be the initial points of departure for teaching proportional reasoning based on RME?
    - How do students rely on their informal knowledge in order to mathematize that knowledge?
    - How does the instructional sequence foster this process of mathematization?
    - What opportunities and barriers does the instructional sequence provide for realization of the hypothesized learning trajectory?
    - How do student-generated solutions provide opportunities for horizontal and vertical mathematization?

- What evidences emerge from the classroom experiments conducted by using the HLT and instructional sequence?

2. What are the mathematical practices as students engage in the instructional sequence?

In order to achieve these purposes and to answer these research questions, a design research was conducted. In the following sections, this methodological approach, the context and participants of the study, data collection procedures, data collection tools, and data analysis methods are explained in detail. In addition, the issues related to the trustworthiness of the study, researcher role, and ethical considerations are elaborated in the sections that follow.

3.1. Design of the Study

Design research has attracted a great deal of attention recently, especially in the educational field (van den Akker, Gravemeijer, McKenney, & Nieveen, 2006). This was mostly due to the "disappointment with the impact of conventional approaches to research in education" and "the availability of promising new theories of learning and technologies through which these theories can be applied" (Walker, 2006, p. 8). The motivation for the development of design research has been reducing the gap between educational research and educational policy and practice (Bakker, & van Eerde, 2015; Lagemann & Shulman, 1999; van den Akker et al., 2006) and the wish to handle theoretical issues related to the nature of learning in the real contexts with more meaningful and broader measures of learning (Collins, Joseph, & Bielaczyc, 2004). With a design research perspective, researchers and practitioners can create promisingly effective interventions by a careful study of learning in the target settings and inform the results together with principles that form the basis of their effectiveness (Collins et al., 2004; van den Akker et al., 2006). In doing so, a greater chance of improved policy and practice is ensured (van den Akker et al., 2006).

Another motivation for design research is related to the desire to develop empirically grounded theories (i.e., local instruction theories) by a study of the forms of learning and means of supporting and organizing the process of learning (Cobb, 2003; Gravemeijer, & Cobb, 2006). Hence, the aim in design research is both "developing theories about domain-specific learning and the means that are designed to support that learning" (Bakker & van Eerde, 2015, p. 430). In doing so, it aims at producing both useful educational products and theoretical understandings of how these products can be useful in education (Bakker & van Eerde, 2015). In this process, the design of educational products is an essential part of the research itself (Bakker & van Eerde, 2015). That is to say, “the design of learning environments is interwoven with the testing or developing of theory” (Bakker & van Eerde, 2015, p. 430).

Design research is defined as “a series of approaches, with the intent of producing new theories, artifacts, and practices that account for and potentially impact learning and teaching in naturalistic settings” (Barab & Squire, 2004, p. 2). In another study, it is referred to as “the study of learning in context through the systematic design and study of instructional strategies and tools” (Design-Based Research Collective, 2003, p. 5). Another definition emphasizes the interplay between design research and developing instructional sequences while describing design research as “an iterative process of integrating socially situated analyses of students’ learning within the design of classroom environments, of which instructional sequences are one part” (Stephan & Cobb, 2003, p. 36). In this study, this approach is followed in order to interpret and carry out a design research study.

Design research has been conducted for various goals in the educational research area, including designing and examining innovations like activities, institutions, interventions, or curricula (Design-Based Research Collective, 2003). More specifically, it is a promising research method for searching for innovative environments of teaching and learning in a complex system, generating context-based theories of learning and teaching, building up knowledge of design, and improving the capacity for novelty in education (Design-Based Research Collective, 2003). However, design research is more than designing and examining specific interventions; it also serves for an understanding of how theory, interventions, and practice are related to each other in order to contribute to the existing theories of teaching and learning (Design-Based Research Collective, 2003). Therefore, the main concern in design research is applying enactments that have the potentials to result in knowledge that applies to educational practices.

Design research is used as a common name in this study for a series of related research approaches including design studies/experiments (Brown, 1992), development/developmental research (Freudenthal, 1991; Gravemeijer, 1994), formative research (van den Akker et al., 2006), engineering research (van den Akker et al., 2006), and transformational research (Gravemeijer, 1994; National Council of Teachers of Mathematics Research Advisory Committee, 1988). Although different names are used, and the related terminology has not been fully established, van den Akker et al. (2006) lists the characteristics that are applicable to most design research and related studies as follows:

> • Interventionist: the research aims at designing an intervention in the real world;
> • Iterative: the research incorporates a cyclic approach of design, evaluation, and revision;
> • Process oriented: a black box model of input–output measurement is avoided, the focus is on understanding and improving interventions;
> • Utility oriented: the merit of a design is measured, in part, by its practicality for users in real contexts; and
> • Theory oriented: the design is (at least partly) based upon theoretical propositions, and field testing of the design contributes to theory building (p. 5).

As slightly different from those, Design-Based Research Collective (2003) also puts forward some characteristics that a satisfactory design research should have. First, there must be an interplay between the main goals of designing learning contexts and establishing theories related to learning. Second, there should be iterative cycles of "design, enactment, analysis, and redesign" (p. 5). Third, the design study should end up with theories that would be helpful for practitioners and other educational stakeholders. Fourth, the research study should also stress the interactions between issues related to learning in addition to the accomplishments or deficiencies for the aim of leading to a better design. In other words, design research studies should have the goal to address the complex nature of educational settings by designing key elements and evaluating these elements operating as a system to improve learning by means of multiple iterations (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003).

It is believed that design research is an appropriate methodology for this study by considering many issues from the purposes of the study to the nature of the study. First, design research is defined as the study of context-based learning in a systematically designed environment (Design-Based Research Collective, 2003), and it makes use of theories and the findings of previous studies. This study aims to examine the teaching and learning process designed and implemented based on related literature in proportional reasoning. Thus, design research would help evaluate what the previous research suggests to integrate into a systematically designed teaching and learning process and understand the teaching and learning process within a specific domain and context. On the other hand, one of the main goals of design research is to develop theories and improve educational practice. Therefore, the study has a promising aspect that would also help to "increase the impact, transfer, and translation of education research into improved practice" (Anderson & Shattuck, 2012, p. 16). Hence, the design research approach would have the potential not only to help us to use the theory and literature but also to use what is evidenced in the learning environment to develop theories and improve practice.

Teachers do not have time or required training to conduct accurate research studies, and researchers do not have the knowledge of complexities of learning environments to design an effective intervention (Anderson & Shattuck, 2012). Given that design research studies deal with the critical design elements of a learning environment from the perspectives of both researchers and a teacher, it enables us to blend and complement the experiences of the researchers and teacher in order to have a holistic understanding of these elements from theoretical and practical bases. Moreover, since design research is very much context-dependent, it would allow us to collect rich data related to the teaching and learning process of proportional reasoning in complex and natural settings.

As aforementioned, design research is iterative in nature (Design-Based Research Collective, 2003). Similarly, the purposes of this study require an iterative nature in its process in order to design, implement, refine, and evaluate a hypothetical learning trajectory and related instructional sequence. In addition, it is critical to have multiple iterations in order to identify critical elements, examine the opportunities and barriers of the designed HLT and instructional sequence, and test and revise them. Moreover, it is essential to take all of these considerations as they place in the complex nature of a classroom environment. Therefore, design research would help us understand the gradual development of proportional reasoning of a classroom community as it is situated in a classroom context.

In summary, there is a match between the nature of this study and design research in two major ways. First, the study aims to explore the processes of design and development, implementation, and evaluation. Second, the study has the purpose of filling the gap in our understanding of the design of a learning environment, HLT, and related instructional sequence developed based on empirical knowledge of student thinking and how students' learning in communal ways can be supported through those.

So far, the purposes and research questions of the study were stated. Besides, the design research methodology was explained with a focus on its general characteristics and purposes. Lastly, the considerations regarding why design research is the most appropriate methodology were expounded. In the following section, phases of a design research study, and the principles and procedures that are followed in each phase will be elaborated. In the pages that follow, the principles of this study and the procedures that were followed in this study are explicated in each of these phases.

3.1.1. Phases of Design Research

There are different classifications for the phases of design research studies suggested by several researchers. For instance, McKenney and Reeves (2012) outline the three phases of a design research study as baseline, intervention, and retrospective analysis. In another study, Plomp and Nieveen (2013) list three phases named preliminary research, development or prototyping phase, and assessment phase. In line with the purposes of this study, the phases for conducting a design research study specified by Gravemeijer and Cobb (2006) were followed. These phases are elaborated in the following sections, together with explanations of the corresponding aspects of this study.

Gravemeijer and Cobb (2006) list the three phases of a design research study as: "1) preparing for the experiment, (2) experimenting in the classroom, and (3) conducting retrospective analyses" (p. 19). These phases are explained in further detail in the following sections. The procedures followed in each of these phases within the context of this study are explained in further sections.

3.1.1.1. Phase 1- Preparing for the experiment

The aim of the first phase in a design research study is to clarify the theoretical intent of the study (research perspective) and to create a local instruction theory that is open to elaboration and refinement (from a design perspective) (Gravemeijer & Cobb, 2006). To this purpose, the starting point should be the determination and clarification of the learning goals in a mathematical domain. Taking the learning goals from curricula as they are and searching for the ways to arrive at these goals is not the goal of design researchers; instead, the research team has to come up with the most relevant, essential, and useful goals. Thus, it is essential for the researcher to ask the following question:

“What are the core ideas in this domain?” (Gravemeijer & Cobb, 2006, p. 19). It is also essential to assess the consequences of earlier instruction in this phase (Gravemeijer & Cobb, 2006).

After a determination of learning goals (i.e., potential endpoints) and analysis of the consequences of earlier instruction (i.e., instructional starting points), the research team needs to conjecture a local instruction theory (Gravemeijer & Cobb, 2006). This local instruction theory includes "conjectures about a possible learning process, together with conjectures about possible means of supporting that learning process" (Gravemeijer & Cobb, 2006, p. 21). Therefore, a local instructional theory entails both students' learning processes and the instructional tasks and tools that are developed as a means to support these. The team anticipates the evolution process of students' reasoning and understanding prior to the implementation of the planned instructional tasks in the classroom. Additionally, the research team makes conjectures about the means of supporting student reasoning and understanding. These entail productive and effective instructional tasks, tools including technological ones, classroom culture and norms, and the components of the proactive role of the teacher (Gravemeijer & Cobb, 2006).

Thus, a local instruction theory and related means of supporting students' learning are the components of instructional design. In addition to conjecturing an instructional design, the research team should also devise the theoretical intent of the design research study since the goal in a design study is not only describing what happened in the classroom but also "to define cases of more general phenomena that can inform design or teaching in other situations" (Gravemeijer & Cobb, 2006, p. 22). Therefore, in order to prepare for the experiments of this study, these steps and suggestions were followed. The procedures conducted regarding this phase are explained in the section “Data Collection Procedures” in the following pages.

3.1.1.2. Phase 2- The Design Experiment

The second phase of a design research entails carrying out the design experiment in the classroom, in which the research team takes the responsibility of the learning of students for a specific time period (Gravemeijer & Cobb, 2006). The aim of carrying out the design experiment lies in the desire to test and refine the conjectured local instructional theory and to understand how it works rather than seeing whether it works (Gravemeijer & Cobb, 2006). This process of testing, refining, and understanding happens through an "iterative sequence of tightly integrated cycles of design and analysis" (Gravemeijer & Cobb, 2006, p. 24).

Therefore, a cyclical process of designing, testing, and refining lies at the heart of design research studies. The research team engages in anticipatory thought experiments in each cycle wherein the actual realization of the prepared instructional activities and tools, and possible student learning through these activities are envisioned. Moreover, during the classroom implementations and after each class session, actual learning of students and their participation behaviors are analyzed. Based on these on-going and retrospective analyses, the research team assesses if the proposed conjectures regarding the instructional task and the classroom norms are valid and tries to refine these aspects of the design. Hence, the design research study involves cyclical processes of anticipatory thought experiments and instruction experiments, which are referred to as microcycles of design and analysis (Gravemeijer & Cobb, 2006).

These microcycles are associated with Simon's "mathematical teaching cycle" that refers to a similar process of a mathematics teacher's anticipation, enactment, and refinement of hypothetical learning trajectories (Simon, 1995). In such a teaching cycle, a mathematics teacher hypothesizes students' mental activities when they are engaged in proposed instructional tasks, then assesses the extent the actual student reasoning is consistent with the anticipated reasoning during the implementation of the tasks, and finally makes decisions about follow-up tasks (Simon, 1995). However, there are two differences between a local instructional theory and Simon's interpretation of HLT: (1) an HLT, in Simon's notion, includes small number of instructional tasks while the local instructional theory comprises a whole instructional sequence; and (2) an HLT is formulated for the teacher's own classroom at a specific instance while the local instruction theory is a more general framework that informs other situations (Gravemeijer, 1999). Therefore, although Simon's notion undergirds the HLT that is developed in this study, the instructional theory developed in this study provides a more general picture that have the potential to inform other contexts since that kind of theory includes an instructional sequence (and HLT) together with the rationale regarding how the instructional sequence might facilitate student learning.

The microcycles in a design experiment that includes anticipatory thought experiments and instruction experiments lay the basis for creating a local instruction theory (Gravemeijer & Cobb, 2006). Besides, the conjectured local instruction theory serves as a guide for these thought and instruction experiments. Thus, there exists a "reflexive relation between the thought and instruction experiments and the local instruction theory that is being developed" (Gravemeijer & Cobb, 2006, p. 28). Below is a figure that summarizes this reflexive relationship:

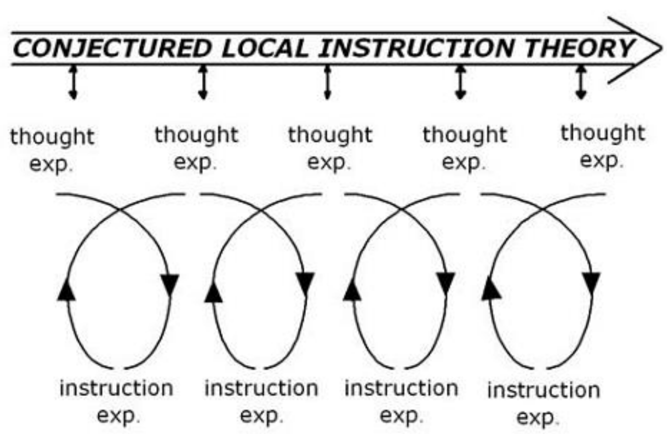


Figure 3.1. Reflexive relation between theory and experiment (Gravemeijer, & Cobb, 2006, p. 28).

These microcycles of thought and instruction experiments consist of on-going analyses of students' individual and collective activities and social aspects of a classroom in order to inform ensuing thought experiments, the creation and refinement of instructional tasks, and reshaping the learning goals (if necessary) (Gravemeijer & Cobb, 2006).

In addition to the assessment and refinement of a local instruction theory during a single design experiment, it is also possible to conduct a subsequent design experiment that is informed by the retrospective analysis of the previous experiment in order to obtain a more robust yet still revisable local instructional theory (Gravemeijer & Cobb, 2006). In such cases, macrocycles that span all levels of the experiments can be conducted. This process of conducting microcycles and macrocycles and the reflexive relationship between those and the emerging local instructional theory is summarized in Figure 3.2 below.

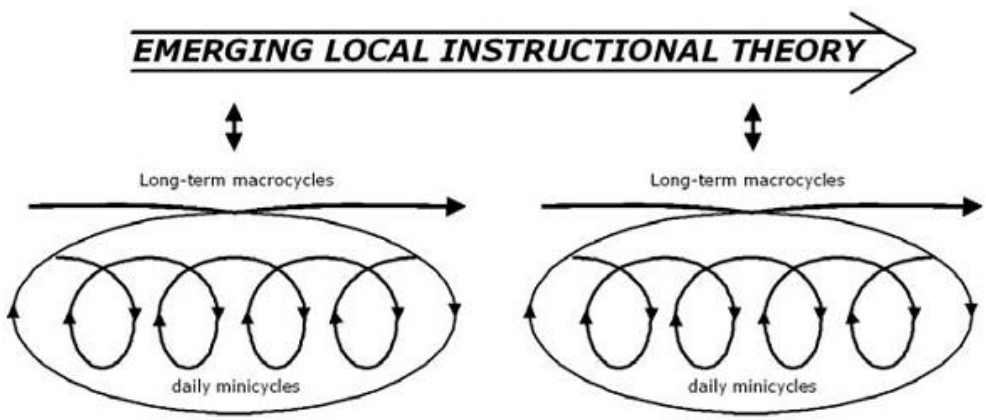


Figure 3.2. Micro- and macro-design cycles (Gravemeijer & Cobb, 2006, p. 29)

Therefore, adopting this perspective of design research that entails micro- and macro-cycles, this study was conducted in two macrocycles that are composed of two design experiments. The details of these two experiments are explained in the section “Data Collection Procedures” in the following pages.

Conducting micro- and macro-cycles in a design research results in a big set of data in a complex environment of classrooms. In order to make sense of the nature of this complex and messy classroom events, it is essential to have an interpretative framework through which the learning and engagement of students and the means of supporting and organizing that process is interpreted (Gravemeijer & Cobb, 2006). This interpretative framework can be useful in interpreting these events during the experiment, as well as in the process of retrospective analysis. In doing so, it is crucial to describe explicitly how the classroom observations are translated into scientific interpretations. Such a framework should encompass components that enable the researchers to interpret the emerging learning environment in addition to mathematical thinking and learning of students (Gravemeijer & Cobb, 2006). Two frameworks guided the design and development and interpretation of the findings of this study. In particular, the domain-specific instructional theory of Realistic Mathematics Education was used to design and develop the tasks in the instructional sequence, whereas Emergent Perspective was used in order to interpret the learning of the classroom community as it took in the classroom contexts. These frameworks are described in more detail in the section “Data Collection Procedures."

3.1.1.3. Phase 3- The Retrospective Analysis

The last phase of a design research study consists of conducting a retrospective analysis of the complete set of data collected throughout the experiment in order to inform the local instruction theory. This type of analysis should entail analyzing the complete set of data in an iterative process. This set of data mostly consists of video recordings of classroom implementation sessions and individual student interviews prior to and following the experiment, duplicates of entire written works of students, field notes, and audio recordings of debriefing sessions with the teacher and research team meetings (Gravemeijer & Cobb, 2006). Based on the systematic and detailed analysis of this set of data, the conjectures are refined and/or refuted, and the local instructional theory is reconstructed (Gravemeijer & Cobb, 2006) since the ultimate goal of retrospective analysis is to obtain a revised local instruction theory and "potentially optimal instructional sequence" (Gravemeijer & Cobb, 2006, p. 42). This process includes eliminating the instructional activities that did not function as expected and revising the instructional tasks that have proven effective. In this process, on-going analyses that were conducted during the experiment and retrospective analyses that were conducted at the end of the design should act as complementary to each other so that the results of the design experiment are empirically grounded. The inferences made from these results can sometimes require the need for a subsequent experiment (i.e., macrocycle).

In addition to the development of local instruction theories, design research also provides a contribution to the formulation of a domain-specific instruction theory. This theory development can occur at three different levels: "the instructional activities (micro theories) level, the instructional sequence (local instruction theories) level, the domain-specific theory level" (Gravemeijer & Cobb, 2006, p. 46). In this study, after the revised instructional sequence was implemented in the second experiment, students' collective ways of reasoning along this sequence were analyzed by a documentation of the mathematical practices. These practices, then, guided the process of making final revisions for the best case HLT and the instructional sequence (local instruction theory). The final revisions are explained in the results section.

3.2. Context and Participants of the Study

This study was conducted in two seventh grade classrooms in a public middle school in Altındağ District of Ankara, Turkey, that includes students aged 11-14 and in grades 5-8. Almost 1000 students from the low socioeconomic background were attending to this school. The (revised) instructional sequence was implemented in two macrocycles in two consecutive years: the first one during January-February 2016 (including a semester break) in five weeks (25 class hours) and the second one during January-February 2017 in six weeks (30 class hours) according to the school's yearly plan of instruction. The Turkish Middle School Curriculum (2013) devotes 8 class hours for the development of the ratio concepts (part-whole, part-part, structuring ratios) in sixth grade and 24 class hours to proportional relationships (including inverse proportion) and 10 hours to linear equations and their graphs in seventh grade. Therefore, 5-6 weeks (25-30 class hours) was a realistic and justifiable period of time for implementing the instructional sequence.

The teacher was selected based on her willingness to take part in the study and collaborate with the researchers. Besides, she was selected because she had ambitious, innovative, and inquiry-based teaching practices, and was open for collaboration and improvements. The teacher had eight-year teaching experience in teaching seventh grades and 10-year experience in teaching middle school in general when the first cycle was conducted. This teacher got a master's degree in 2008 and conducted a classroom teaching experiment in her own class as a fulfillment of this degree. In addition, she took part in the revision of the Turkish Middle School Mathematics Curriculum (MoNE, 2013) and developing middle school mathematics textbooks in line with the reform movement of mathematics education in Turkey.

The first experiment of the study was conducted in one of this teacher's seventh-grade classrooms that included 12 girls and 15 boys starting from January 2016. In the second experiment, a revised version of the instructional sequence was implemented in a seventh-grade classroom of the same teacher that included 14 boys and 11 girls. There were two boys and a girl with learning disabilities, and a few students were known to perform above the seventh-grade level. Almost half of the remaining students were known to perform at their grade level and the other half considerably below their grade level. Both of these two groups of seventh graders were taught by the same teacher when they were in fifth and sixth grades. Therefore, the targeted social classroom norms and sociomathematical classroom norms were established prior to the experiments conducted for this study. Moreover, since both experiments in two macrocycles started in January, that is three and a half months after the start of the school year, these norms were stable during the experiments.

The teacher took a proactive role in the two phases of this design research study. Particularly, in the preparation phase, she collaborated with the design research team in order to anticipate likely student responses and communal learning paths of her classroom, including tool use in line with an RME perspective and design, sequence, and revise instructional tasks to support that path of learning. In the design experiment phase of this study, she took a proactive role in the teaching and learning process wherein she introduced the task, gave time for small-group work explorations, monitored the group work and orchestrated the whole class discussion by selecting and sequencing particular students to explain their reasoning and justifications, and helping students make mathematical connections among these ideas. Looking from another perspective, the teacher's role was similar to the five practices for facilitating mathematical discussions suggested by Stein and colleagues (Stein, Engle, Smith, & Hughes, 2008).

In addition to those practices, the teacher had a specific role in facilitating the argumentation process in whole-class discussions. These roles included managing students' turn-taking in order to provide everyone with the access to the discussion, repeating or clarifying students' contribution, asking for agreement/disagreement, selecting students in order to support mathematically elegant discussions, and introducing the formal symbolism in order to enable students (re)organize their thinking (Stephan & Akyuz, 2012). While these were the hypothesized roles of the teacher, the analysis of the classroom data collected in this study indicated that she also took another role in order to facilitate the mathematical argumentation process during the whole-class discussions. This role included providing a claim and asking students to provide other elements (i.e., data, warrant, backing, rebuttal) for the claim when necessary.

Prior to and following each experiment, pre- and post-tests were implemented to all students in the classes where the experiments are conducted in. The same test used as a pretest and posttest. This test included a set of questions that were related to the big ideas in the instructional sequence. Based on students' answers to the pre-test and the teacher's suggestion, eight students were selected to be involved in pre-and-post- interviews. These students were selected according to the following criteria: ensuring variability in terms of academic performance and having different reasoning patterns so as to contribute to the whole-class discussions significantly.

3.3. Data Collection Procedures

As aforementioned, a design research study entails three phases: "1) preparing for the experiment, (2) experimenting in the classroom, and (3) conducting retrospective analyses" (Gravemeijer & Cobb, 2006, p. 19). In the following sections, the procedures followed, and the steps taken in each of these phases within the context of this study are described in detail.

3.3.1. Phase 1- Preparing for the experiment

As suggested by Gravemeijer and Cobb (2006), the first attempt in order to prepare for the experiment was made in relation to determining and clarifying learning goals for proportional reasoning in seventh grade. To this end, a research team was created, including the researchers (the doctoral student and two advisors) and the teacher, in order to review the curricula and literature to find out the most relevant, essential, and useful goals (Gravemeijer & Cobb, 2006).

The researchers started to work on an instructional sequence that was developed for teaching ratio and proportion in the USA by Stephan et al. (2015) and engaged in design team meetings related to the adaptation of this instructional sequence. The teacher also participated in some of these meetings since it is crucial in design research that the selection and design of the intervention is a mutual and joint work of researchers and the teacher (Anderson & Shattuck, 2012). In this study, this joint work of the teacher and the researchers was ensured “from initial problem identification, through literature review, to intervention design and construction, implementation, assessment, and to the creation and publication of theoretical and design principles” (Anderson & Shattuck, 2012, p. 17).

In the meantime, a literature review was conducted on proportional reasoning and related concepts such as ratio, rate, proportion, and rational numbers in order to come up with other big ideas and key learning goals of proportional reasoning. Also, the related objectives in the Turkish Middle School Mathematics Curriculum (MoNE, 2013) and several other curricula were examined. It was seen that the Turkish Middle School Mathematics Curriculum has 11 objectives directly related to proportional reasoning. The three of these objectives are in sixth grade, seven in seventh grade, and one in eighth grade. According to the Turkish Middle School Mathematics Curriculum, sixth-grade students deal with the concepts of rate and ratio only and not the proportion or proportional relationships. They use ratios for comparing quantities, find the part-part and part-whole ratios, and solve related problems. In the seventh grade, a huge emphasis is given to proportions and solving problems related to proportional situations. More specifically, seventh-grade students find the value of one quantity when the value of the other quantity and their ratio are provided and also find the unit rate and proportionality constant between quantities. Besides, seventh-grade students determine whether two quantities form a proportional situation by analyzing ratio tables and line graphs and express the proportional relationship between two quantities in tables and equations. Next, seventh-grade students determine whether two quantities form an inverse relationship by analyzing ratio tables and line graphs and solve related problems. As a last consideration, it was agreed that the objective in the eighth grade related to similarity and finding the lengths of similar shapes is associated with proportional reasoning even though this objective is placed under the domain of geometry.

A cross-analysis of curriculum objectives and related literature on proportional reasoning and related concepts was conducted in order to determine big ideas of proportional reasoning. This analysis yielded several big ideas that were organized as:

> Linking composite units and iterating linked composites while the link is preserved (also referred to as unitizing and norming in the literature) (Battista & van Auken Borrow, 1995; Lamon, 1994; Steffe, 1988, 1994; Stephan et al., 2015)

Covariation and invariation (Bryant, 1974; Carlson et al., 2002; Confrey & Smith, 1994; Ellis et al., 2016; Lamon, 1995, 2007; Muller, 1977, 1978; Saldanha & Thompson, 1998)
Absolute and relative thinking (Freudenthal, 1978; Lamon, 1995; NCTM, 2000)
Additive reasoning and multiplicative reasoning (Clark & Kamii, 1996; Harel & Confrey, 1994; Hart, 1988; Park & Nunes, 2001; Stephan et al., 2015; Tourniaire & Pulos, 1985; van Dooren et al., 2010)
Partitioning (Lamon, 1995; Pothier & Sawada, 1983)
Ratio appropriateness (ratio sense) (Lamon, 1995)
Linear relationships and their representations (Cramer & Post, 1993; Cramer, Post, Currier, 1993; Karplus et al., 1983; MoNE, 2013; Stephan et al., 2015)
Part-part and part-whole relationships (Spinillo & Bryant, 1991; Stephan et al., 2015)
Within and between measures comparisons (Freudenthal, 1973, 1978; Karplus et al., 1983; Lamon, 1994, 2007; Noelting, 1980a, 1980b; Stephan et al., 2015; Vergnaud, 1981, 1988)
Distinguishing rates and ratios (Lamon, 2007; Lesh et al., 1988; Stephan et al., 2015; Thompson, 1994),
Similarity and distortion (Kaput & West, 1994; MoNE, 2013; Stephan et al., 2015; Streefland, 1985; van den Brink & Streefland, 1979)
Quantitative and qualitative reasoning (Behr et al., 1992; Heller et al., 1989, 1990; Lamon, 1994; Larson et al., 1989; Lobato, & Siebert, 2002; Thompson, 1993, 1994)

In addition to determining big ideas, it is important to comprehend the consequences of already existing instruction in order to conjecture a local instruction theory (Gravemeijer & Cobb, 2006). In this process, existing research results might be useful (Gravemeijer & Cobb, 2006). In addition, researchers themselves should conduct assessments (e.g., written tests, interviews, whole class performance assessments) before initiating a design research study (Gravemeijer & Cobb, 2006). In this study, the literature review revealed that the instruction of proportional reasoning is procedural, isolated, and superficial, and there is a need to build on students' informal understandings of linking composite units, unitizing and norming, build-up strategies, and within and between measures comparisons.

Moreover, the literature review provided an insight into the informal tools that can be used as models of organizing students' iteration and build-up processes. According to the reviewed studies, children intuitively work with linked quantities either pictorially or numerically and use table-like representations in order to keep track of the linked quantities and iterations with them (Kenney et al., 2002; Middleton, & Van den Heuvel-Panhuizen, 1995; Misailidou & Williams, 2003; Stephan et al., 2015; Streefland, 1984, 1985). In a similar vein, the pre-test and the pre-interviews conducted with the students prior to the study gave essential information about students' intuitive ideas of build-up strategies and the use of tables as informal tools to keep track of the building-up or -down processes. This information that reveals young children's intuitive and informal ways of reasoning and tool use were used in shaping the instruction in such a way that it was hypothesized that the students would start making iterations with linked quantities pictorially and in table-like representations. Moreover, it was anticipated that ratio tables would also help support the transition from building-up or -down by ones to building-up or --down by many (Middleton & Van den Heuvel-Panhuizen, 1995; Stephan et al., 2015; Streefland, 1985).

As suggested by Gravemeijer and Cobb (2006), the research team worked on conjecturing a local instructional theory subsequent to determining learning goals and analyzing the consequences of earlier instruction. It was stressed that this local instruction theory should include "conjectures about a possible learning process, together with conjectures about possible means of supporting that learning process" (Gravemeijer & Cobb, 2006, p. 21). In particular, it is critical that a local instructional theory includes the evolution of students' learning and the instructional tasks and tools that have the potential to foster this process. Moreover, the means of supporting this learning process should also entail classroom culture and norms, and the components of the proactive role of the teacher (Gravemeijer & Cobb, 2006).

Concerning these issues, researchers should also clarify the theoretical perspectives of the study in order to make sense of what is going on in classrooms (Gravemeijer & Cobb, 2006). Two frameworks were used in this study: (1) to design the instructional tasks and (2) to interpret students' mathematical development as socially constructed in the classroom. While the domain-specific theory of Realistic Mathematics Education was used to guide the former, Emergent Perspective was used to guide and interpret the latter in this study. These two frameworks, together with their principles, are explained in the following sections.

3.3.1.1. Realistic Mathematics Education

The theory that undergirds the design and development of the instructional sequence developed in this study is RME. The core of RME is based on understanding mathematics as a human activity. Freudenthal (1968) noted that mathematics should be seen not "as a closed system, but rather as an activity, the process of mathematizing reality and if possible even that of mathematizing mathematics" (p. 7). Within the context of RME studies, students need to be guided to reinvent mathematical ideas and concepts by organizing didactically rich contexts that are realistic (Gravemeijer, 1994). Besides, they need to be encouraged to reason with models and imagery related with the physical tools, inscriptions and activities they deal with.

In such kinds of processes, models are important to support mathematization (Streefland, 1991). In particular, models should act as simplified versions of reality at the beginning, and as models are applied to other situations, they should act as models for higher-level mathematical thinking. This kind of process is named as a transition from *a model of* to *model for* in the RME literature (Gravemeijer, 1991, 1994; Streefland, 1991). In such a process of transition, the normative reasoning in the classroom is also expected to change in such a way that while models encourage thinking about the context at the beginning; later, they should support a focus on mathematical relations (Gravemeijer, 1991).

In this study, the processes of reinvention and mathematization are perceived as both individual and collective activities in line with Gravemeijer and colleagues' (2003) perspective. That is, a social environment that includes conjecturing, explaining, challenging, and justifying where a teacher and students negotiate in order to arrive at taken-as-shared meanings and model use is valued. In such communal ways, a community of learners engages in horizontal and vertical mathematization as they reason in increasingly sophisticated ways while they interact around an instructional sequence. This instructional sequence was designed in order to support students' mathematization by a transition from a model of to model for. In particular, it was anticipated that ratio tables would be introduced as effective ways of representing and organizing the reality (i.e., linking the number of fish and food bars in line with the rule and iterating this link) at the beginning. In the following instances, these ratio tables are curtailed in order to arrive at more efficient procedures (i.e., abbreviated build-up). Lastly, as short ratio tables are found as applicable to finding missing values in proportional situations, they are used in order to support the evolution of reasoning with symbolic proportion representation. Therefore, ratio tables are used as *models of* organizing linked composites at the onset. As they are found applicable to finding missing values by scalar and functional operators (i.e., within and between ratios), they become *models for* structuring symbolic representation of ratios and proportions.

3.3.1.2. Emergent Perspective

The second interpretative framework employed in this study is *Emergent Perspective* that is useful in examining the mathematical growth of students "as it occurs in the social context of the classroom" (Cobb & Yackel, 1996, p. 176). In this framework, the two distinct theoretical perspectives, namely constructivist and sociocultural perspectives, are coordinated in such a way that it is based on a social constructivist perspective (Cobb & Yackel, 1996). That is to say, in this perspective, both individual and social aspects of learning are essential "with neither taking primacy over the other" (Stephan, 2003, p. 28). This interpretative framework is summarized in Figure 3.3 below.

SOCIAL PERSPECTIVE	PSYCHOLOGICAL PERSPECTIVE
Classroom social norms	Beliefs about own role, others' roles, and the general nature of mathematical activity in school
Sociomathematical norms	Mathematical beliefs and values
Classroom mathematical practices	Mathematical conceptions and activity

Figure 3.3. An interpretative framework for analyzing individual and collective activity at the classroom level (Cobb & Yackel, 1996, p. 177)

The social perspective (see the left column in Figure 3.3) emphasizes an interactionist perspective of seeing classroom processes in collective ways whereas the psychological perspective (see the right column in Figure 3.3) accentuates a constructivist perspective of students' individual activities as they engage in and contribute to the evolving collective activities (Cobb & Yackel, 1996). As seen in the figure above, the social perspective includes three subconstructs as classroom social norms, sociomathematical norms, and classroom mathematical practices, which indicate the three aspects of a classroom's culture. In relation to these social subconstructs, the corresponding individual aspects are listed below the psychological perspective on the right column. Thus, each row in the figure above denotes a dual relationship between an aspect of classroom culture and the individual activities of those who engage in and contribute to it (Cobb & Yackel, 1996).

According to Cobb and Yackel (1996), classroom social norms, as a subconstruct of social perspective, are the norms that characterize patterns in collective activities that are established by all members of the classroom (i.e., the teacher and students) through negotiation. The teacher and students can renegotiate the classroom social norms as the instruction progresses. In participating in this renegotiation process, students engage in the process of reorganizing their beliefs about their roles as well as others', and the general nature of mathematical activity in school. Some of the classroom norms in this study included articulating and justifying solution ways, listening to others' ideas and trying to understand them, showing agreement/disagreement, and searching for alternative ways to solve problems (Cobb & Yackel, 1996). These practices had been established in the classroom community prior to the study since the collaborating teacher had attempted to establish an inquiry-based practice and valued meaningful student learning and argumentation.

Classroom social norms that are specific to mathematics and mathematical activities are referred to as sociomathematical norms (Cobb & Yackel, 1996; Yackel & Cobb, 1996). Sociomathematical norms in this study involved establishing claims based on a mathematical argumentation process rather than referring to the rules and/or any authority such as a textbook. In addition, negotiation of "what counts as a different mathematical solution, a sophisticated mathematical solution, an efficient mathematical solution, and an acceptable mathematical solution" (Cobb & Yackel, 1996, p. 178) were among the sociomathematical norms established in this study. Even though related sociomathematical norms had been established in the classroom, these types of sociomathematical norms were renegotiated throughout the experiment. In this renegotiation process, students adopt beliefs and values that are specific to mathematics, which in return facilitates student autonomy (Cobb & Yackel, 1996; Yackel & Cobb, 1996). Therefore, mathematical beliefs and values are considered as corresponding psychological constructs for sociomathematical norms, as seen in Figure 3.3.

The third subconstruct of social perspective in the Emergent Perspective is classroom mathematical practices that are related to the communal mathematical growth of a classroom. Analyzing classroom mathematical practices are compatible with the aims of design research since this type of analysis focuses on the documentation of the hypothesized instructional sequence as it is realized in the classroom (Cobb & Yackel, 1996). It also enables to connect instructional development and theory by informing the current efforts based on a description of mathematical learning as situated in the social context (Cobb & Yackel, 1996). Mathematical conceptions and activities are considered as corresponding psychological constructs for classroom mathematical practices as seen in Figure 3.3, since in such a process student “actively contribute to the evolution of classroom mathematical practices as they reorganize their individual mathematical activities and, conversely, that these reorganizations are enabled and constrained by the students’ participations in the mathematical practices” (Cobb & Yackel, 1996, p. 180). Therefore, there exists a reflexive relation between classroom mathematical practices and individuals' mathematical conceptions and activities as individual students engage in and contribute to the evolution of classroom mathematical practices. Thus, a documentation of classroom mathematical practices gives a picture of how the instructional sequence is actualized in the classroom and how individuals contribute to the evolving of those practices (Stephan et al., 2003).

Therefore, using RME as design theory and Emergent Perspective as an interpretative framework to interpret learning in a classroom setting, a local instructional theory including students’ reasoning in sophisticated ways for proportional reasoning and possible means of supporting this learning was conjectured in this study. This theory also provides a rationale for the HLT, and instructional sequence developed and how they can be helpful in supporting students' gradual development in the classroom context by taking into consideration classroom culture and norms and the proactive role of the teacher. In the development process of this local instructional theory, various research studies in the literature, and particularly, the Hypothetical Learning Trajectory and the related instructional sequence developed by Stephan, McManus, Smith, and Dickey (2015) were used as guides to shape instruction on proportional reasoning. In this study, this HLT and the instructional sequence was extended and enriched by the help of the related research on proportional reasoning and related concepts. In the following sections, the original instructional sequence developed by Stephan et al. (2015) and the revisions and extensions made to the original instructional sequence, together with their rationale, are explained in detail.

3.3.1.3. The original instructional sequence

The original instructional sequence and the related HLT were developed in the United States by Stephan et al. (2015) in order to support the teaching of rate and ratio for understanding and in ways that are consistent with the Common Core State Standards (CCSSI, 2010). That is to say, this sequence was developed for practical concerns and published on the website of the university as an instructional resource for teachers.

The instructional design theory that undergirds this sequence is RME. In this sequence, instruction begins with a story about a bad dream in which aliens were chasing the teacher, and a bar of food was enough to satisfy three aliens. Then, the teacher introduces the anchor activity, which is referred to as alien-food bar activity throughout the dissertation. This activity is considered as an anchor activity since the informal, more formal, and formal tools (i.e., long and short ratio tables, symbolic proportion) would be introduced through this activity. Below is a small portion of this anchor activity:

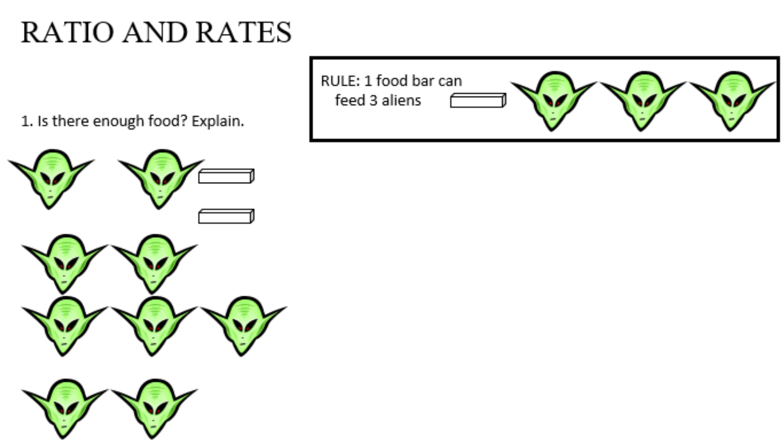


Figure 3.4. A small portion of the alien-food bar activity (Stephan et al., 2015)

This activity aims to support students' linking of one food bar with three aliens and understanding that this rule cannot be broken. In the following stages, the rule changes so as to include different unit rates, reducible ratios/rates, and non-integer ratios/rates (i.e., 1-5, 2-4, 2-3, and 2-5). As students solve problems, they are encouraged to organize their pictures or numbers, and eventually, a (long) ratio table is introduced as an efficient way to keep track of how the two quantities may scale up or down. In the following instances, this long ratio table is shortened to include four values only. In doing so, it encourages more efficient and faster calculations, as suggested by Streefland (1995). Below is an illustration of a transition from a long ratio table to shortened ratio table by putting an emphasis on an abbreviated build-up process in Stephan and colleagues' (2015) HLT:

aliens	3	6	9	12	15	18	21	24	27	30	33	36	39
FB	2	4	6	8	10	12	14	16	18	20	22	24	26

⇨

x13

aliens	3	39
FB	2	26

x13

Figure 3.5. The transition from long ratio tables to shortened ratio tables (Stephan et al., 2015)

The second task in Stephan and colleagues' (2015) sequence includes new contexts wherein different wordings of ratio (i.e., informal ratio language related to part-part relationships) and the formal terminology are used. The learning goals associated with this task include applying proportional reasoning in these new contexts and using the informal and formal ratio language correctly to make sense of the part-part relationships. The first context includes the relationships between the number of infants and teachers and toddlers and teachers in a fictional daycare called Tiny Tots. In the second context, similar use of informal ratio language exists to represent part-part relationships between the number of people who are in favor of and against the war. A similar context that includes relationships between amounts of sugar and flour in a recipe is included as the third context. The last context includes part-part relationships between the number of seventh and eighth graders who preferred action movies. Therefore, the second task includes different contexts where part-part relationships are explored in both informal and formal use of ratio language. Most of these contexts also include a question that required students to decide whether the given ratios were equal to the ratios given in the problems. The use of the shortened ratio table is also encouraged in these contexts (Stephan et al., 2015).

In the following parts of the instructional sequence, proportionality problems, including both integer and non-integer ratios and having different contexts, are given place. During explorations on these problems, students are introduced the symbolism and definition of proportion by erasing the lines in between the ratios in the table. Below is the summary of this transition from shortened ratio tables to the symbolic representation of proportion:

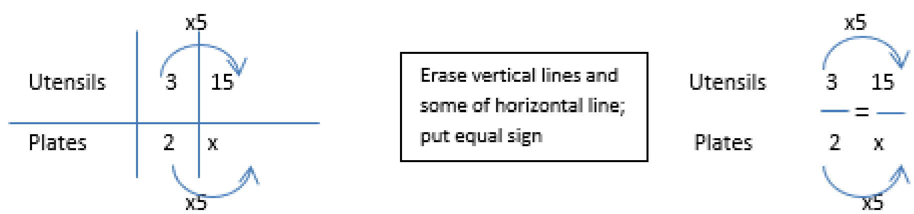


Figure 3.6. The transition from shortened ratio tables to the symbolic representation of proportion (Stephan et al., 2015)

Also, throughout these discussions, distinguishing ratios (part-whole) from rates (part-part) is targeted. Scaling (i.e., stretching) problems and comparing ratios in the contexts of comparing the tastes and density of different mixtures are also included in the following parts. Moreover, concerning comparing rates/ratios, several contexts that include comparing different speeds and deciding on the best buy are presented. Lastly, the final tasks in the sequence target moving from ratio tables to graphs in order to represent proportional relationships graphically.

Therefore, the original instructional sequence is comprehensive in the sense that it encompasses most of the essential big ideas for proportional reasoning. These ideas include but are not limited to linking composite units, iterating linked composites by building up by ones or abbreviatedly (i.e., scaling), additive and multiplicative reasoning, analyzing equivalent ratios/rates, and comparing ratio/rates. The corresponding HLT that includes components including these big ideas, related tool/imagery use, and possible topics of discourse is presented in Figure 3.7 below.

Big idea	***Tools/imagery***	***Possible Topics of Discourse***	***Activity Pages***
Linking composite units	Connecting Pictures of aliens to food bars	If the rule is 1 food bar feeds 3 aliens, the rule cannot be broken if we add more food bars	Page 1
Iterating linked composites **Build up Strategy** **Abbreviated Build up Strategy**	Informal symbolizing (e.g., tables, two columns of numbers, pictures of aliens and food bars)	How students keep track of two quantities while making them bigger	Pages 2-4
Build up Strategy **Abbreviated Build up Strategy**	Ratio table	Keeping track of two linked quantities while they grow additively	Pages 3-4
Additive versus multiplicative reasoning	Fold back to pictures; Shortened ratio table	Adding or multiplying to build-up	Page 5
Structuring ratios multiplicatively	Shortened ratio tables through multiplication and division with scale factors	Efficient ways of curtailing long ratio tables What does the horizontal scale factor represent? What does the vertical scale factor represent?	Pages 3-7
Creating equivalent ratios	Ratio tables with missing values; Traditional proportion representation (two ratios separated with an equal sign)	Adding versus multiplying; Meanings of scale factors; What do decimal scale factors mean?	Pages 8-10
	Ratio tables	Adding versus multiplying; horizontal and vertical scale factors	Pages 11-12 and Pages 15-17
Analyzing equivalent ratios	"fraction" imagery	Reducing ratios; Vertical and horizontal scale factors	Page 11 (bottom) and Page 14 (top)
Comparing ratios	Ratio table No ratio tables, but can use arrow notations	Finding common numerators or denominators; the size of the scale factors; unit ratios	Pages 18-23
Comparing rates	Ratio table; arrow notation; standard proportion notation	Difference between a ratio and rate; unit rates; common denominator and numerators	Pages 24-27

Figure 3.7. Ratio and rates hypothetical learning trajectory (Stephan et al., 2015)

As seen in Figure 3.7, this HLT includes big ideas, tools/imagery, and possible topics of (mathematical) discourse for the related instructional sequence. In this HLT table, the big idea column describes the learning goal for that portion of the instruction, and the tools/imagery column outlines the specific inscriptions and/or notations that are intended to support the corresponding learning goal. The third column, possible topics of discourse, is meant to guide teachers in the types of questions or mathematical conversations that are important for that section of the sequence. More information about this HLT, instructional sequence and the materials is available on https://cstem.uncc.edu/sites/cstem.uncc.edu/files/media/Ratio%20T%20Manual.pdf

3.3.1.4. Revisions to the original instructional sequence (prior to the experiments)

The original sequence is of good quality in several ways. First, it is one of the first attempts to propose a classroom HLT and instructional sequence for teaching ratio and proportion that was developed by a collaboration of teachers and a researcher in the field of mathematics education. Second, it was developed to explicitly move students from informal reasoning to formal knowledge; hence, it sits well with RME. In particular, although ratio tables have been stated as effective tools for fostering students' proportional reasoning in other studies (e.g., Kenney et al., 2002; Middleton, & Van den Heuvel-Panhuizen, 1995; Misailidou & Williams, 2003; Streefland, 1984, 1985) and the use of ratio tables for supporting building-up/down process is emphasized in some of these studies (e.g., Middleton & Van den Heuvel-Panhuizen, 1995; Streefland, 1985), none of these studies specifically focus on a transition from a model of to a model for perspective in line with an RME perspective.

Therefore, one of the most significant strengths of the HLT and the instructional sequence developed by Stephan et al. (2015) lies in its potential to support a transition from using informal ratios (i.e., ratio tables) to reasoning with formal tools (i.e., symbolic representation of proportion, $\frac{a}{b} = \frac{c}{d}$). Another strength of the HLT is emanated from the fact that it involves engaging contexts for a variety of big ideas in ratio and proportion. Fourth, the fact that it is a ready-made tool for teachers since it includes guidelines for teachers in order to implement the sequence in classrooms makes it an essential tool to work on. Therefore, the HLT and the instructional sequence were very helpful in guiding the instructional sequence and the local instruction theory developed in this study.

However, in spite of all its qualities, the research team had some concerns about a few drawbacks that the original sequence had and that it would not directly apply to the Turkish context. To begin with, it was questionable whether the alien-food bar context would be engaging for Turkish students. Another drawback was concerned with the scope of the sequence. Even though the original sequence is comprehensive, it was anticipated that it lacks some of the essential big ideas related to proportional reasoning. For instance, although the sequence includes a question emphasizing the relationships among verbal, tabular, and graphical representations of proportional situations, it lacks explorations related to writing the symbolic equation representation of proportional relationships (i.e., $y = mx$). However, representing proportional situations, algebraically, is an objective that seventh-grade students should attain according to the Turkish Middle School Mathematics Curriculum (MoNE, 2013).

Another big idea that is not included in Stephan and colleagues' (2015) instructional sequence is related to distinguishing between proportional relationships with the form $y = mx$ and linear non-proportional relationships with the form $y = mx + n$. Besides, qualitative reasoning tasks are not included in the scope of Stephan and colleagues' (2015) instructional sequence. Even though these key ideas are not highlighted in the national curriculum (MoNE, 2013), the literature on proportional reasoning considers them among essential understandings of proportional reasoning and main types of tasks that foster proportional reasoning.

Therefore, several revisions were made to the instructional sequence developed by Stephan et al. (2015) prior to the first experiment of this study in order to address those drawbacks. First of all, the story of the anchor context was changed from the alien-food bar context to the fish-food bar context in order for it to be more engaging for Turkish students. By doing so, it was hypothesized that this story would lay the ground for students to make sense of why the rule for feeding the fish could not be broken -if the fish are underfed or overfed, they might get sick or even die from underfeeding or overfeeding. Besides, the contexts of some of the remaining tasks were also changed for similar reasons. Second, several tasks were added to the sequence in order to extend its scope. These tasks include explorations on the tabular, graphical, and algebraic representations of proportional relationships of the form $y = mx$ and linear non-proportional relationships of the form $y = mx + n$. These tasks are provided in the instructional sequence that is given in the Appendix section. Moreover, a task that requires students to rely on their qualitative reasoning skills was added to the instructional sequence. This task is the last in the sequence (See Appendix A).

Additionally, another task that was added to the instructional sequence before the first experiment included measuring different lengths with long and short sticks whose actual lengths were not known to the students. Within the context of this activity, it was aimed that students would measure various lengths and record how many long sticks and short sticks were those lengths in ratio tables. It was anticipated that students would make sense of proportional relationships between the results of their measurements with short and long sticks in ratio tables by drawing on their knowledge of measurement and experiences in measuring.

On the other hand, although several tasks were added to the original instructional sequence before the first experiment, a couple of tasks were also removed from the sequence. In particular, the tasks related to the concept of percent were not included in this study in order to have a more focused sequence on proportional reasoning and the concepts of rate, ratio, and proportion. Thus, even though these changes were made to the instructional sequence, the initial version of the HLT for this study was structured as the same as the one in Stephan and colleagues' (2015) study (see Figure 3.7).

3.3.2. Phase 2. Design experiment

Gravemeijer and Cobb (2006) refer to the second phase of a design research study as the "Design Experiment" phase and stress that the aim in this phase is to test and refine the conjectured local instruction theory and see that how it works out in the classroom. As aforementioned, this phase includes a cyclical process of designing, testing, and revising, where a series of anticipatory thought experiments and instruction experiments guide this process (Gravemeijer & Cobb, 2006). The microcycles of a design experiment consist of these anticipatory thought experiments, and instruction experiments lay the basis for creating a local instruction theory, which entails on-going analyses of students' individual and collective activities and social aspects of the classroom (Gravemeijer & Cobb, 2006).

A design research study can have several macrocycles, each of which is comprised of many microcycles in order to arrive at a more robust local instructional theory (see Figure 3.2). Following this perspective, this study was conducted in two macrocycles that are composed of two design experiments (i.e., macrocycles). First of these design experiments took place in five weeks during January-February 2016 and the second in six weeks during January-February 2017. These design experiments provided insights to understand how the hypothesized instructional sequence works out in the complex nature of classrooms.

3.3.2.1. Design Experiment 1

In accordance with the principles of a design research study, the researchers and the teacher worked on the instructional sequence and engaged in thought experiments in order to anticipate likely growth of the classroom community in mathematical reasoning and tool use as they interact with the instructional sequence long before the experiment starts. After the completion of these procedures in the initial preparation phase, the instructional sequence was implemented in one of the teacher's seventh-grade classes for five weeks (i.e., 25 class hours) from January 2016 to February 2016. Before each class session, the teacher and the researcher engaged in thought experiments in order to review the learning goals of the classroom session and possible topics of mathematical discourse, including aspects regarding the possible use of tools and imagery.

The instructional sequence that was implemented in the first experiment included nine activities, each of which targeted several big ideas that were determined as the essential aspects of proportional reasoning. The order of these activities, together with the learning goals they target, are summarized in the table below.

Table 3.1. The content of the instructional sequence implemented in the first experiment

Instructional tasks	Learning goals
1. Let's feed the fish	Linking composite units, iterating linked composites, unitizing and norming, covariation and invariation, within and between measures comparisons, distinguishing rates and ratios, ratio appropriateness (ratio sense), additive and multiplicative reasoning
2. What do the survey results tell?	Linking composite units, iterating linked composites, unitizing and norming, covariation and invariation, within and between measures comparisons, distinguishing rates and ratios, ratio appropriateness (ratio sense), additive and multiplicative reasoning, informal ratio language
3. Measuring lengths with sticks	Linking composite units, iterating linked composites, unitizing and norming, covariation and invariation, within and between measures comparisons, distinguishing rates and ratios, ratio appropriateness (ratio sense)
4. Learning ratio and proportion	Linking composite units, iterating linked composites, unitizing and norming, covariation and invariation, within and between measures comparisons, distinguishing rates and ratios, ratio appropriateness (ratio sense), additive and multiplicative reasoning, formal ratio language, symbolic use of ratio and proportion

Table 3.1. (Continued)

5.	Representing proportional situations with graphs and equations	Iterating linked composites, unitizing and norming, covariation and invariation, linear relationships and their representations
6.	Do the pictures look alike?	Iterating linked composites, unitizing and norming, covariation, and invariation, within and between measures comparisons, ratio appropriateness (ratio sense), additive and multiplicative reasoning, similarity, and distortion.
7.	Comparing oranginess	Iterating linked composites, unitizing and norming, covariation and invariation, within and between measures comparisons, ratio appropriateness (ratio sense), additive and multiplicative reasoning, distinguishing rates and ratios
8.	Comparing speeds and deciding on best buy	Iterating linked composites, unitizing and norming, covariation and invariation, within and between measures comparisons, ratio appropriateness (ratio sense), additive and multiplicative reasoning, distinguishing rates and ratios
9.	Comparing coffee strengths	Unitizing and norming ratio appropriateness (ratio sense), absolute and relative thinking, additive and multiplicative reasoning, qualitative reasoning

As can be seen in the table above, most of the activities include many of the big ideas stated as essential understandings of proportional reasoning. The first task is an adaptation of the first activity (i.e., alien-food bar activity) in Stephan and colleagues' (2015) sequence. The second task aims at the exploration of part-part and part-whole relationships similar to the second task in the original instructional sequence, although the contexts of the questions are quite different. In addition, it makes use of informal ratio language (i.e., per language, for every…, out of every…) and does not include any formal terminology. The third task requires students to measure different lengths with short and long sticks whose actual lengths are not known, keep a record of the measurement results in tables, and make sense of the proportional relationships between the results of measurements within long and short sticks. This task was developed by the researcher as inspired by the well-known Mr. Short and Mr. Tall problem (Karplus, Karplus, & Wollman, 1974).

After making explorations regarding iterations, covariation and invariation, within and between measures comparisons in the first three activities, the fourth task was designed to foster the formal language and symbolic use of ratio and proportion. In the last part of this task, students are required to decide which of the given ratios belong to the same equivalence class as the given ratio. Therefore, they are required to analyze whether or not given two ratios form a proportion (i.e., are proportional). The fifth task entails making explorations with proportional relationships and their tabular, graphical, and algebraic representations and understanding the relationships among those. The first part of the sixth task includes finding the missing lengths of two similar shapes, while the second part is related to deciding whether or not the given two shapes are similar. These tasks were developed by the researchers based on the activities in Stephan and colleagues' (2015) instructional sequence.

As seen in Table 3.1 above, the last three tasks involve comparing ratios/rates in addition to previous key ideas. Particularly, the seventh task involves comparing pairs of orange juice mixtures containing different amounts of water and orange juice. This task was adapted from Noelting's (1980a, 1980b) task. The eighth task includes comparing speeds of people and cars based on given information related to how far people/cars travel in a specific time interval. Besides, deciding on the best buy by analyzing the information regarding the price of products sold in different amounts in different supermarkets was another context in the eighth task. This task was developed based on the activities in the original instructional sequence (Stephan et al., 2015). Lastly, the ninth task was added to the instructional sequence in order to support students' reasoning in qualitative ways; that is, independent of numbers. In the ninth task, students are required to investigate whether adding coffee and/or milk affects the strength of coffees and determine the change of direction if it does. This task is adapted from Billings' (2002) study.

These instructional tasks were implemented in the classroom in five weeks that include 25 class hours (i.e., 2+2+1 hours each week). The first experiment of the study took place in one of the seventh-grade classes of the collaborating teacher for five weeks from January 2016 to February 2016. Posttests and post interviews were conducted from March 2016 to April 2016. Also, the teacher was interviewed for assessing the whole process until the end of the first experiment. From April 2016 to January 2017, the researchers conjectured the necessary revisions for the subsequent experiment by working on the classroom data, teacher interviews, pre- and post-tests and interviews, and design research team meetings. The classroom events that took place as students and the teacher interact around these activities were analyzed in line with the theoretical lenses of the Emergent Perspective and Realistic Mathematics Education during and after the first experiment. These analyses revealed the need for revising the instructional sequence to be implemented in the second design experiment in the next macro-cycle. These revisions are explained in the following section.

3.3.2.1.1. Revisions to the instructional sequence during and after the first experiment

In line with the spirit of design research, the teacher and the researcher(s) conducted debriefing sessions in order to assess the learning of students as it took place in normative ways. During each mini-cycles of the first experiment and after the first experiment was completed, the classroom videos were watched, and the audio recordings of the teacher and student interviews, and design research team meetings were listened. This helped the researchers to improve the HLT and the instructional sequence and have a better case for the subsequent experiment. In between the mini-cycles, referred to as thought experiments (see Figure 3.1), the design team made conjectures about the sequence of the activities, the social and sociomathematical classroom norms, how students moved from informal ways of reasoning to more formal ways of reasoning together with the tool use, and whether and how the instructional sequence supported this shift. For instance, after the implementation of the first four parts of the first task (i.e., fish-food bar), the following dialogue took place between the collaborating teacher (T) and the researcher (R) in a debriefing session:

> *R:* We finished the first four parts of the first activity, and we will implement the last part in the next class. First, can I have your general opinion? How did the implementation go? What went well? What could be improved?
> *T*: I think the activates are sequenced well (so far). First, students dealt with the pictures to make sense of the big ideas (linking and iterating), then they moved to reasoning with numbers. They already had prior knowledge… They are familiar with the subjects. So they could draw on their previous knowledge of problem-solving.
> *R:* What should we improve for the next implementation?
> *T:* I think we should give the pictures in the third part (i.e., 2 food bars-4 fish) in order to make students feel the need for the unit ratio like we did in the first part. They can group the pictures of fish and food bars and link them and see that 2-4 is the same as 1-2 on the pictures.
> *R*: Do you think we should also give the pictures for the last part (2 food bars-3 fish, for the next class)?
> *T:* Yes, I think that would be helpful too. And I think we should include more questions that are not multiples of 4 (for the number of fish)? Like 6 or 22…
> *R:* OK. Any other suggestions?
> *T:* We already introduced a long ratio table. So, we can include ratio tables in the following parts. Well. I asked students myself, but that would be helpful to include directions related to the relationships in the table (in the activity sheet). We draw the tables on the board, but they could fill in the tables in their activity sheets and explore the relationships on the table. We can keep track of their work more easily on the tables, and also, they can see the relationships in a more organized way.

In the excerpt above, the teacher and the researcher debriefed on the instruction experiment on Day 2 of the first design experiment. In this debriefing session, the teacher assessed the sequence of the activities and students' reasoning with the informal tools by drawing on their informal and prior knowledge related to multiplicative reasoning. She also made suggestions for the next macro-cycle, and together with the researcher, they decided to include pictures and long ratio tables in the activity sheet for the next instruction experiment in the same cycle. The teacher and the researcher engaged in a great number of similar debriefing sessions during the design experiments.

Therefore, one of the revisions to the instructional sequence after the first implementation was related to the inclusion of pictures and tables in students' activity sheets. Another revision that can be given as an example of the revisions made was related to the sequence of the instructional activities. In the instructional sequence implemented in the first design experiment, the third activity was related to measuring with sticks whose actual lengths were unknown. The researchers had anticipated that the activity would support students' understanding of the multiplicative relationships between the lengths of the sticks and the results of measurements with those sticks. After the implementation of the task, the design team conjectured that this activity did not successfully support students' understanding of the direct proportional relationships since it includes complex sets of relations about relations (i.e., the length of the sticks, the measurement results, the relationships between the lengths of the sticks and the measurement results). On the other hand, the teacher and the researcher conjectured that this task would have great potential in supporting students' understandings of inverse proportional relationships and its relationship with direct proportion (i.e., A×B = C×D and $\frac{A}{C} = \frac{D}{B}$). Below is a dialogue between the researcher and the teacher in which they discuss the potentials and drawbacks of the measurement activity:

> *T:* I think we should not put this activity (i.e., measurement activity) in this place. The students got confused about long and short sticks and measurement results with long sticks and short sticks. Very confusing terms. Nevertheless, we can use it for inverse proportion so that they can arrive at the idea that is measuring with long sticks results in a smaller measure while measuring with short sticks results in a greater measurement result.
> *R:* I agree that the students got confused about the terms. They even had a hard time stating their explorations (verbally). Do you think they can make sense of the invariance of the products of these (the lengths of the sticks and measurement results with these sticks) in a further implementation?
> *T:* I think they can. Multiplying the length of the stick by the (measurement) result gives the length of the (measured) object, and they can figure it out.
> *R:* Then, we should give the lengths of the sticks.
> *T:* I think so.

In the debriefing sessions, the researchers and the teacher agreed on many revisions for the following days within the first experiment (i.e., mini-cycles) and the subsequent experiment (i.e., second macrocycle). The two excerpts given above can be given as evidence to this, although many revisions were made in many of those debriefing sessions.

In many other debriefing sessions and design research team meetings, the members of the design research team decided on the necessary changes in order to obtain a more viable instructional sequence. Although the dialogues in which these changes were conjectured will not be provided from now on, these changes will be summarized together with their rationale in a few sentences that follow. As aforementioned, the activity that included measuring lengths with short and long sticks was removed from this sequence that aims at fostering understanding of direct proportional relationships since it had greater potential to facilitate making sense of inverse proportional relationships.

Another revision was made to the activity named "Representing proportional situations with graphs and equations" (i.e., the fifth activity in the first version). Particularly, the scope of this activity was extended by the inclusion of a context that included finding weights on the Moon given the weights on the Earth with a fractional scale factor (i.e., the weight on the Moon is $\frac{1}{6}$ of the weight on the Earth). It was conjectured that this context would foster students' understandings of representing proportional situations using graphs and equations. Furthermore, an activity that includes non-proportional linear situations (i.e., a situation that can be represented by the equation $y = mx + n$, $n \neq 0$) was added to the sequence following this activity. It was anticipated that this activity would enhance students' understanding of proportionality and linearity and how their tabular, graphical, and algebraic representations differ from each other.

Lastly, the order of the activities "Comparing oranginess" and "Comparing speeds and deciding on best buy" were switched places since the activities related to comparing speeds and deciding on best buy were found as more intuitive than comparing the tastes of orange punches. Therefore, it was moved forward so that it appears before the activity named “Comparing oranginess.” The content of the instructional sequence in order to be implemented in the second experiment is provided in the following section named “Design Experiment 2.”

Another modification made after the first design experiment was related to the structure and content of the HLT table in the original sequence. As seen in Figure 3.7 above, the original HLT table includes big ideas, tools/imagery, and possible topics of (mathematical) discourse. In the first design experiment, it was seen that the classroom discussions revolved around some specific activities changing almost every day. Moreover, it was observed that some gestures and metaphors emerged in the classroom discussion, which were efficient in supporting the collective reasoning of the students. One of these was related to making hand gestures (i.e., moving fingers horizontally along the horizontal ratio tables) in order to point to making iterations with and scaling of values that belong to the same measure space in the ratio tables and the symbolic proportion representations. Moreover, moving fingers vertically in the horizontal ratio tables took place in the classroom discussion frequently in order to point to the invariant (i.e., functional) relationships between values in different measure spaces. Other gestures and metaphors were also found useful in order to support students' reasoning in normative ways.

Therefore, the design research team decided to add the components of “activities/taken-as-shared interest” and “possible gestures and metaphors” to the HLT table. Therefore, the HLT developed in this study is organized in a table that includes five components: big ideas, tools/imagery, activity/taken-as-shared interest, possible topics of mathematical discourse, and possible gestures and metaphors similar to some of the studies in related domain (e.g., Gravemeijer et al., 2003; Rasmussen, Stephan, & Allen, 2004; Stephan & Akyuz, 2012). Moreover, the HLT was also broken into six phases in order to specify the shifts in students’ mathematical reasoning along this HLT. The revised HLT that guided the implementation in the second experiment is provided in the following section named "Design Experiment 2."

3.3.2.2. Design Experiment 2

Upon the completion of making the necessary revisions to the first version of the instructional sequence, the second experiment of this design research study was conducted in six weeks (i.e., 30 class hours) from the first days of January 2017 to the last days of February 2017 (including a 2-week semester break). One of the collaborating teacher's seventh-grade classrooms was selected since the teacher indicated that that specific classroom had effective argumentation practices than any other seventh-grade class of the teacher. Below is the revised version of the instructional sequence that the classroom discussion took place around:

Table 3.2. The content of the instructional sequence implemented in the second experiment

Instructional tasks	Learning goals
1. Let's feed the fish	Linking composite units, iterating linked composites, unitizing and norming, covariation and invariation, within and between measures comparisons, distinguishing rates and ratios, ratio appropriateness (ratio sense), additive and multiplicative reasoning
2. What do the survey results tell?	Linking composite units, iterating linked composites, unitizing and norming, covariation and invariation, within and between measures comparisons, distinguishing rates and ratios, ratio appropriateness (ratio sense), additive and multiplicative reasoning, informal ratio language
3. Learning ratio and proportion	Linking composite units, iterating linked composites, unitizing and norming, covariation and invariation, within and between measures comparisons, distinguishing rates and ratios, ratio appropriateness (ratio sense), additive and multiplicative reasoning, formal ratio language, symbolic use of ratio and proportion

Table 3.2. (Continued)

4.	Let's solve problems	Linking composite units, iterating linked composites, unitizing and norming, covariation and invariation, within and between measures comparisons, ratio appropriateness (ratio sense), additive and multiplicative reasoning, formal ratio language, symbolic use of ratio and proportion
5.	Representing proportional situations with graphs and equations	Linking composite units, iterating linked composites, unitizing and norming, covariation and invariation, linear relationships and their representations
6.	Proportionality and linearity	Linking composite units, iterating linked composites, unitizing and norming, covariation and invariation, linear relationships and their representations, additive and multiplicative reasoning
7.	Do the pictures look alike?	Iterating linked composites, unitizing and norming, covariation and invariation, within and between measures comparisons, ratio appropriateness (ratio sense), additive and multiplicative reasoning, similarity and distortion.
8.	Comparing speeds and deciding on best buy	Iterating linked composites, unitizing and norming, covariation and invariation, within and between measures comparisons, ratio appropriateness (ratio sense), additive and multiplicative reasoning, distinguishing rates and ratios
9.	Comparing oranginess	Iterating linked composites, unitizing and norming, covariation and invariation, within and between measures comparisons, ratio appropriateness (ratio sense), additive and multiplicative reasoning, distinguishing rates and ratios
10.	Comparing coffee strengths	Unitizing and norming ratio appropriateness (ratio sense), absolute and relative thinking, additive and multiplicative reasoning, qualitative reasoning

As can be seen from the table above, the instructional sequence implemented in the second design experiment included ten tasks. In particular, two of these tasks (Task 4 and Task 6) were implemented for the first time in the second experiment. Moreover, some of the tasks switched places in order to provide a more productive learning path for students. Although it cannot be seen in the table above, the contents of some of these activities were also revised. The final versions of these activities are provided as an appendix to this study (see Appendix A).

As afore-stated, the HLT presented in Stephan and colleagues' (2015) study was used as the backbone for the first experiment. However, this HLT was also subject to revisions in between two design experiments. The revisions made to the HLT were explained in the previous part, together with their rationales. The revised version of the HLT that used as a guide for the second experiment, and that includes five components: big ideas, tools/imagery, activity/taken-as-shared interest, possible topics of mathematical discourse, and possible gestures and metaphors are presented in Figure 3.8 through Figure 3.13 below.

Big Idea	Tools/Imagery	Activity-Taken-as-shared interest	Possible Topics of Mathematical Discourse	Possible Gesturing and Metaphors
Linking Composite Units	-Grouping pictures by circling and connecting two composite units with arrows -Drawing shapes	If 1 food bar feeds 3 fish, the rule is kept the same regardless of the number of food bars and fish	-If the rule is 1 food bar feeds 3 fish, the rule cannot be broken if we add more food bars	-Matching concrete objects for illustration (i.e., 3 pencils for 1 student) -Making hand gestures for linking
Iterating Linked Composite Units **Covarying two linked quantities**	-Informal symbolizing (matched numbers, i.e. 1-3,2-6 etc.; tabular-like-representations) -(Mental)images of objects -Build up (Additive) Strategy -(Horizontal and vertical) Long ratio tables (numerical patterns, showing increments in tables) -Abbreviated Build Up Strategy -Shortened ratio tables (scaling values within the same measure spaces on the tables) -Within comparisons -Informal proportionality constant	-Keeping track of two linked quantities while making them bigger (additively or by multiplications as a short way for repeated addition) -Increasing both quantities by the same scale factor	-How to keep track of two quantities while making them bigger? -What are the efficient ways to build up? -What are the efficient ways for curtailing long ratio tables?	-Matching concrete objects and iterating for illustration -Counting up/down with fingers horizontally along the horizontal tables for building up/down -Making hand gestures to show the horizontal relationships in tables (i.e., scaling up/down values within same measure spaces)
Multiplicative Reasoning	-(Horizontal and vertical) Long and shortened ratio tables (numerical patterns) -Informal unit rate (finding the unit number of values) -Between comparisons	-Multiplying/dividing to build up/down by keeping track of quantities -Recognizing the ease of working with unit rate	-While two quantities covary, there exists a third quantity that remains invariant (i.e., it doesn't change).	Making hand gestures to show the vertical relationships in tables (i.e., functional relationships)

Figure 3.8. Phase 1 of the revised HLT for the proportional reasoning instructional sequence

Big Idea	Tools/Imagery	Activity-Taken-as-shared interest	Possible Topics of Mathematical Discourse	Possible Gesturing and Metaphors
Reasoning in part-whole contexts	-Short and long ratio tables -Informal ratio language -Within and between comparisons -part-whole relationships -Fraction imagery	-Extending covariation and invariation to the relationship between parts and the whole	-Parts among themselves and parts with their whole covary in a precise way -There is an invariant relationship between parts and their whole and among the parts themselves. -Part-whole relationships represent fractions	Horizontal and vertical hand gestures

Figure 3.9. Phase 2 of the revised HLT for the proportional reasoning instructional sequence

Big Idea	Tools/Imagery	Activity-Taken-as-shared interest	Possible Topics of Mathematical Discourse	Possible Gesturing and Metaphors
Structuring ratios and proportions	-Short ratio tables -Fractional representation -Symbolic proportion representation	-Moving from short ratio tables to proportions -Structuring ratios and proportions multiplicatively	-Ratio refers to how the two values compare to each other -Proportion is the equality of two ratios -There is an invariant relationship between proportional situations as the values change together in precise ways	Moving fingers to draw a straight line for the fractional division line
Creating equivalent ratios	-Ratio tables with missing values -Symbolic proportion representation with missing values -Vertical and horizontal relationships in proportion representation	-Extending covariance and invariance to proportion -Finding missing values by within and between measures comparisons	-What do the horizontal and vertical relationships mean?	Horizontal and vertical hand gestures
Analyzing equivalent ratios	-Within and between measures comparisons -Equal sign with a question mark on top of it	-Reasoning with within and between measures comparisons -Determining proportionality by covariational and multiplicative reasoning	Do the horizontal/vertical relationships hold the same for the two ratios?	-Horizontal and vertical hand gestures -Drawing question marks with fingers

Figure 3.10. Phase 3 of the revised HLT for the proportional reasoning instructional sequence

Big Idea	Tools/Imagery	Activity-Taken-as-shared interest	Possible Topics of Mathematical Discourse	Possible Gesturing and Metaphors
Representing proportional relationships on linear graphs	-Long ratio tables -Linear graphs on the coordinate plane	-Moving from tables to linear graphs to represent proportional relationships -Representing proportional relationships on linear graphs of the form $y = mx$ and recognizing that they pass through the origin	-How to represent proportional situations on graphs? -What are the features of the graphs of proportional situations?	- Horizontal and vertical hand gestures -Drawing a straight line with fingers
Representing proportional relationships algebraically.	-Long ratio tables -Algebraic equation of the form $y = mx$	-Formalizing the invariant multiplicative relationship in a proportional situation into an algebraic equation	-How to represent proportional situations with algebraic equations? -What are the features of the equations of proportional situations?	
Analyzing linearity and proportionality	-Long ratio tables -Linear graphs on the coordinate plane -Algebraic equations of the forms $y = mx$ and $y = mx + n$	-Distinguishing between linearity and proportionality -Distinguishing the features of proportional and linear relationships	What destroys proportionality?	

Figure 3.11. Phase 4 of the revised HLT for the proportional reasoning instructional sequence

Big Idea	Tools/Imagery	Activity-Taken-as-shared interest	Possible Topics of Mathematical Discourse	Possible Gesturing and Metaphors
Comparing ratios/rates	-Within shapes and between shapes comparisons -Fraction imagery (part-whole ratios) -Anchoring to whole and/or half -Greater than, less than symbols (<, >) -Images of density and strength -Informal ratio language - Orange juice and water neutralize each other (Canceling out each other)	-Conceptualizing similarity and distortion - Comparing rates/ratios and deciding which one is bigger /smaller /equal -Comparing the strength of mixtures with different amounts of ingredients -Equalizing the numerator or denominator of ratios/rates	-Are the shapes similar or one of them is distorted? -Which of the mixtures tastes stronger? -What do part-part and part-whole ratios tell? -What do the tastes of the mixtures have to do with proportionality?	

Figure 3.12. Phase 5 of the revised HLT for the proportional reasoning instructional sequence

Big Idea	Tools/Imagery	Activity-Taken-as-shared interest	Possible Topics of Mathematical Discourse	Possible Gesturing and Metaphors
Qualitative reasoning	-Dispersion -Images of density and strength		-How does adding coffee/milk affect the strength of liquids? -How to reason about the direction of change and determinability? -Is the change of direction determinable? -In what direction does the value of ratio change?	

Figure 3.13. Phase 6 of the revised HLT for the proportional reasoning instructional sequence

According to the first phase (i.e., Phase 1) of the HLT table created before the second experiment, students start with grouping and linking the pictures of groups of fish and food bars as composite units and make iterations with linked composites. In this phase, it is hypothesized that students would start using numerical iterations (i.e., 1-3, 2-6, 3-9, etc.) when the numbers get bigger. Then, the first tool, the horizontal long ratio table, will be introduced in order to organize these iterations and keep track of them. As students deal with different rules for the number of food bars and fish, it is envisioned that they will reason with building up by ones and eventually abbreviatedly (i.e., using scale factor within measure spaces). Upon the emerging of abbreviated build-up strategies, shortened ratio tables will be introduced as the second tool in order to do more efficient calculations. In the first phase, it is anticipated that students will make sense of the covariation (i.e., building up/scaling up within measure spaces) and invariation (i.e., unit ratio/rate) in the long and short ratio tables.

After students make sense of the covariation and invariation in the short ratio tables in the contexts that include discrete variables, in Phase 2, it is envisioned that these understandings regarding covariation and invariation would be extended to part-whole contexts. In the next phase, Phase 3, formal representation of ratio and proportion will be introduced as formal tools by removing some of the borders of the short ratio table. It is envisioned that students would draw on their experiences related to covariance and invariance in the ratio tables to make sense of the functional and scalar relationships in the proportions in order to create and analyze equivalent ratios. Phase 4 of the HLT anticipates that students will move from tabular and formal representations of proportional relationships to representing proportional relationships graphically and algebraically. Also, it envisions that students will make sense of the relationships between multiple representations (i.e., tabular, numerical, graphical, algebraic) of proportional relationships. The next phase, Phase 5, envisions that students will move from creating equivalent proportions by scalar and functional relationships to compare ratios/rates and decide which one is bigger/smaller/equal in different contexts, including similar/similar shapes and taste comparisons. Lastly, it is hypothesized that in Phase 6, students would reason independent of numerical values in order to reason about the determinability of change in the values of ratios and find the direction of change if it is determinable in a context that includes comparing tastes of liquids. Thus, this HLT gives a picture of the hypothesized growth of learning in normative ways. Documentation of Classroom Mathematical Practices is used in order to see how this hypothesized learning path will be actualized in the second experiment of this study and suggest revisions to the HLT and the instructional sequence. This method of analysis is described in the following sections under the subsection named Data Analysis.

In the previous sections, the procedures and the events that took place during the first two phases of the study were explained. Below is a table that summarizes this process in a time table:

Table 3.3. Time schedule of the study

Time period	Events and activities
January-December 2015	Preparation for the experiments (Literature review, development of the HLT and the instructional sequence for the first experiment)
December 2015	Pre-test and pre-interviews
January-February 2016 (5 weeks)	Design experiment 1
March-April 2016	Post-test and post-interviews
March-December 2016	Revisions to the HLT and the instructional sequence
December 2016	Pre-test and pre-interviews
January-February 2017(6 weeks)	Design experiment 2
March 2017	Post-test and post-interviews
April-September 2017	Transcription of classroom and interview data
December 2017-May 2018	Retrospective analysis (Mathematical Practices Analysis)
June 2018-June 2019	Final revisions to the HLT and the instructional sequence

3.3.3. Data Collection Tools

Design research makes use of a variety of data collection tools by utilization of various research techniques in order to evaluate the outcomes of design and refine the design process (Anderson & Shattuck, 2012; Design-Based Research Collective, 2003). More specifically, the key elements of a learning environment, which constitutes the units of analysis of design research, are "the tasks or problems that students are asked to solve, the kinds of discourses that are encouraged, the norms of participation that are established, the tools and related means provided, and the practical means by which classroom teachers can orchestrate relations among these elements" (Cobb et al., 2003, p. 9). Additionally, it might be helpful to incorporate suitable anchor assessment items that are utilized by other researchers (Gravemeijer & Cobb, 2006).

In this study, the data collection tools consist of video recordings of classroom implementation sessions, pre-and post-tests of all students, individual student interviews prior to and following the experiments, duplicates of entire written works of students, field notes, and audio recordings of debriefing sessions with the teacher and research team meetings (Gravemeijer & Cobb, 2006). While all of this data helped to understand and conjecture sophisticated ways of communal reasoning and how to support this gradual process, some of the data collection tools had other major aims. The details of these data collection tools, together with the aim of collecting such kinds of data are stated in the following subsections.

3.3.3.1. Video recordings of classroom implementations

The classroom implementations during the two design experiments of this design research study were video recorded. After the implementations, the researcher herself transcribed these video recordings into words by a systematic and persistent observation in which the teacher's actions, students' mathematical thinking and learning, the learning environment, gesture use, tool use, and the connections among these were noted next to the transcripts.

3.3.3.2. Pre- and post-tests

Pre-and post-tests are usually implemented in order to assess success rates before and after any intervention and to see if there is a difference between the results. In this study, apart from this aim, the major goal in collecting data through pre-test was to understand the informal and intuitive ways of reasoning of students and their tool use in proportional reasoning in order to build the instruction on those (Kaput & West, 1994; Lamon, 1994). To these ends, a test that included questions regarding most of the big ideas determined as critical understandings of proportional reasoning was constructed to be used as a pre-test and post-test. Particularly, this test consisted of questions that measured the big ideas of iterating linked composites, unitizing and norming, multiplicative reasoning, unit rate, within and between measure comparisons, and qualitative reasoning.

In the first few questions of the test, it was given that three balloons could be bought with one Turkish Lira (the national currency of Turkey). Students were required to find the missing values, either the money needed to buy a specific number of balloons or the number of balloons that could be bought with a specific amount of money by using this rule. Besides, a set of questions that included informal ratio language (i.e., per language or part-whole relationships) related to part-whole and part-part relationships were included in this test. Moreover, the test involved questions related to finding missing lengths of similar shapes and deciding on the best buy. Lastly, the test included qualitative reasoning tasks wherein no numbers were present, and the students were required to find the direction of change.

The analysis of students' work in the pretest revealed that students had some informal knowledge related to linking composite units and iterating linked composites with pictures and on tables. All of the students were able to group the pictures of balloons and linked them with 1 Turkish Lira (TL) when it was given that three balloons could be bought with one TL. When the pictures were not provided, and bigger numbers were included in the questions, some of the students continued to draw pictures of balloons and coins for the questions. Besides, some of the students were able to link 1 and 3 numerically and build up by ones and threes numerically (e.g., 1-3, 2-6, 3-9, etc.), some of them were able to divide and multiply as a procedural way for grouping and iterating. Nevertheless, these informal ways of reasoning were quite naïve in that they were mostly based on building up and did not include the invariant functional relationship between the amounts of balloons and money. Furthermore, for a considerable number of students, these informal ways of reasoning yielded in an incorrect additive reasoning, especially when the story did not include a unit ratio, as shown in Figure 3.14a and Figure 3.14b below.

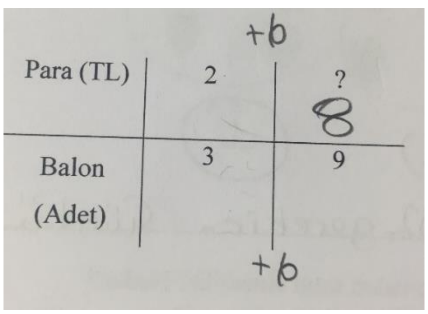


Figure 3.14a. Additive reasoning in the pre-test

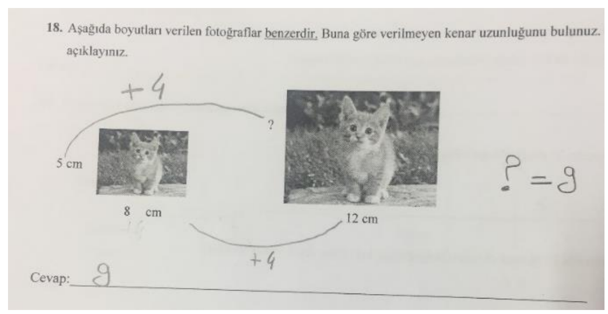


Figure 3.14b. Additive reasoning in the pre-test

Therefore, it was anticipated that one of the most challenging obstacles would be related to incorrect additive reasoning during the experiments. Thus, it was conjectured that the numbers had to be selected strategically to include unit ratios and integer ratios at the beginning and also non-integer ratios in the following instances.

Also, the pretest and pre-interview data showed that almost all of the students were able to reason qualitatively for basic comparisons, such as deciding on the faster person when it was known that one of them took the same distance in a shorter time. Furthermore, some of the students were also successful in comparing the ratios/rates by finding the unit rates. However, it was also the case that some of the students compared two situations by focusing only on the amount of one quantity in both situations instead of focusing on their relative amounts. Therefore, the data collected before the experiments consistently showed that students had different amounts of knowledge in the big ideas of proportional reasoning, but they were all able to make iterations with the linked composites either with pictures or numbers by building up. Therefore, these data supported that the starting point for the instructional sequence had to be linking composite units and iterating linked composites based on build-up strategies.

The purpose of the post-test was to gain insight into students' learning after the experiment in order to see the shifts in their reasoning processes and the use of tools and symbols. However, the posttests were not analyzed in detail within the context of this study.

3.3.3.3. Pre- and post-interviews

After each student in the classroom in which the experiments would be conducted answered the written pre-tests, eight students were selected to be involved in pre-and post-interviews in both experiments based on their answers in the pre-tests. Besides, the teacher was consulted for the selection of these students. The selection criteria for student interviews were related to ensuring variability in terms of academic performance and having different reasoning patterns so as to contribute to the whole-class discussions significantly. In the pre- and post- interviews, the students were asked to solve the problems in the written pretest and think aloud as they solved those problems. Moreover, a few extra questions were asked where necessary. In this way, there was a chance to delve into students' informal ways of reasoning, together with their informal tool use.

The goal of conducting pre and post-test interviews was the same as the pre- and post-test. In particular, pre-interviews were conducted in order to make sense of students' informal and intuitive knowledge and tool use for proportional reasoning tasks. This information, then, was used to shape the instruction (Kaput & West, 1994; Lamon, 1994). Similar to the pretests, the pre-interviews also revealed that students made use of table-like representations in order to link and iterate composite units. Moreover, they had challenges when the problems included non-integer and/or nonreducible ratios and misapplied additive reasoning for proportional situations.

3.3.3.4. Student's written work

In both experiments, the students worked on the tasks in the instructional sequence, both individually and in small groups. After each class session, the sheets that included students' work were copied. These duplicates of students' work as they interact with the instructional tasks provided extra insight into their mathematical growth and reasoning, including the use of models and tools. In addition, they were also helpful in understanding individual students' contributions to the classroom mathematical practices in detail.

3.3.3.5. Audio recordings of debriefing and research team meetings

The teacher and the researcher met before and after each classroom session. They engaged in anticipatory thought experiments before each classroom implementation in order to go over the learning goals of the lesson and envision students' likely progressions in that class. In addition, the teacher and the researcher met after each class session in order to assess the benefits and drawbacks of the prior class and to conjecture the revisions necessary for HLT and the instructional sequence. Besides, the design research team met occasionally in order to evaluate instruction and come up with conjectures about students' collective learning and the ways to facilitate this learning for the subsequent experiment. All of these meetings were audiotaped. The audiotapes were listened carefully and frequently in order to make revisions to the HLT and the instructional sequence.

3.3.3.6. Fieldnotes

Field notes are “the written account of what the researcher hears, sees, experiences, and thinks in the course of collecting and reflecting on the data in a qualitative study” (Bogdan & Biklen, 2007, p. 118-119). In this study, I, as the researcher, was present in the classroom during all the implementation sessions in both experiments. As a participant-observer in the classroom, in addition to talking with the teacher in order to resolve a situation in the classroom, I always took field notes related to the classroom events and revised those notes after each classroom session.

3.4. Data Analysis

Having a vast amount of data (i.e., videotapes of almost 30 hours of classroom implementation, audiotapes of almost 20 hours of research team meetings and debriefing sessions, etc.) requires a careful organization and analysis of the data. While these data were analyzed continuously as a part of the on-going analysis process, the retrospective analysis focused on understanding the taken-as-shared ways of learning of a classroom community and documenting the mathematical practices. To this end, transcripts of the classroom video data were created as the starting point. The classroom data were analyzed by an adaptation of Toulmin's (1958, 2003) argumentation model by Stephan and Rasmussen (2002) and Rasmussen and Stephan (2008). This process of analysis method includes three phases. In the first phase, the videotapes of every class session were watched, and the instances in which a claim is made were noted, and the whole-class discussions were coded according to Toulmin’s argumentation model. Therefore, it is important to explain this model at first.

A basic argumentation process involves three components as a claim (C), data (D), and warrant (W) in Toulmin's (1958) model of argumentation. According to this model, a claim is an assertion or a conclusion that is made based on the data (Toulmin, 1958). Therefore, data point out the ground on which the argument is constructed. Warrants are different from data in that they describe and justify how one gets from the data to the claim (Toulmin, 1958, 2003). Therefore, warrants act as bridges between data and claims in that they show that "taking these data as a starting point, the step to the original claim or conclusions is an appropriate and legitimate one." (Toulmin, 2003, p. 91). These warrants are usually rules or principles that authorize the steps taken to move from the data to the claim (Toulmin, 2003). Below is a skeleton proposed by Toulmin (1958, 2003) for analyzing:

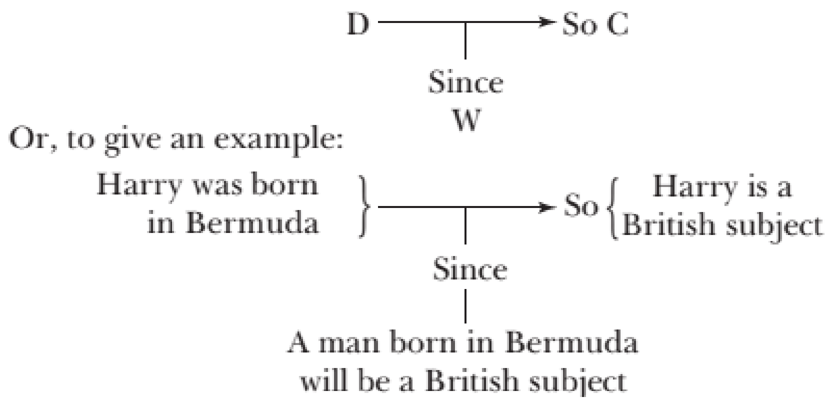


Figure 3.15. Basic skeleton for analyzing arguments (Toulmin, 2003, p. 92)

As seen in the basic argumentation model shown in Figure 3.15 datum and claim are connected by an arrow to emphasize their relation, and a warrant is written right below the arrow. In the example provided in Figure 3.6, the warrant functions as explanatory in that its task is to legitimize the step taken from the data to the claim and to refer to the other groups of steps that are supposedly legitimate (Toulmin, 2003). However, warrants can be of different types, and "may confer different degrees of force on the conclusions they justify." (Toulmin, 2003, p. 92). In other words, warrants can function differently and justify the claim in different degrees- some of them authorize the claim in a way that leaves no doubt, whereas some of them authorize the claim tentatively or under certain conditions or qualifications (Toulmin, 2003). Therefore, an argument might be more complex, and it may not be enough to include only data, claim, and warrants in the corresponding skeleton. Therefore, a more complex argumentation scheme includes the components of qualifier (Q), rebuttal (R), and backing (B). A qualifier describes the degree of and the extent to which the claim is justified by the warrant, whereas a rebuttal indicates the circumstances wherein the warrant fails to confer the claim. Besides, a backing supports the warrant's acceptability and legitimacy. Therefore, a more complex argumentation, including all these elements can be structured as follows:

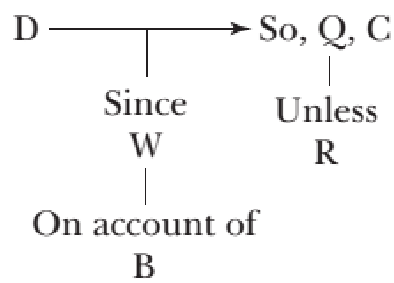


Figure 3.16. A comprehensive skeleton for analyzing arguments (Toulmin, 2003, p. 97)

As seen in the figure above, the qualifier is written beside the claim it qualifies, and the rebuttal is written right below the qualifier to specify the exceptional conditions that can defeat or rebut the warranted conclusion (Toulmin, 2003). Therefore, below is an extension of the example depicted in Figure 3.16, which includes all the additional elements:

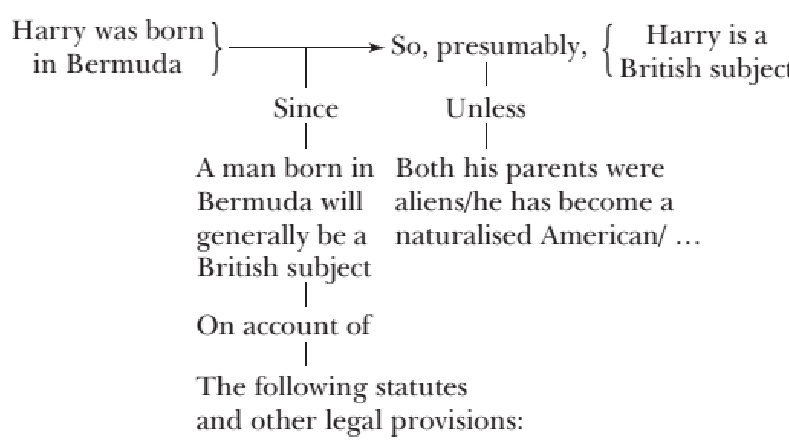


Figure 3.17. An exemplary argument including elements of an argument (Toulmin, 2003, p. 97)

Although Toulmin (1958) proposes this model of argumentation for the field of law and for individual argumentation processes, he stresses that some features of this model are field-invariant. That means this model can be applied to other fields. Taking this notion, Krummheuer (1995) adapted Toulmin's model for analyzing mathematical arguments. Besides, he also focused on the social aspects of an argumentation process where he sees argumentation as a social phenomenon in which individuals present justifications for their actions and make adjustments in their intentions (Krummheuer, 1995). Argumentation in such an environment is referred to as techniques and methods to establish a claim and seen as a specific aspect of social interaction (Krummheuer, 1995). Therefore, a successful process of argumentation involves challenging claims and arriving at a consensual and acceptable claim for all individuals (Krummheuer, 1995). This kind of argumentation in which several individuals contribute to the development of mathematical arguments through social interaction is called a collective argumentation (Krummheuer, 1995). A collective argumentation does not develop in a linear way; instead, disagreements might take place that would eventually result in the processes of correcting, modifying, retracting, and replacing (Krummheuer, 1995). Hence, the result of such a process is called an argument (Krummheuer, 1995). In line with this perspective, Yackel (2001) states that "what constitutes data, warrants, and backing is not predetermined but is negotiated by the participants as they interact" (p. 7), emphasizing that the elements of the arguments are "situation-specific, emergent, and co-constituted." (Cole, Becker, Towns, Sweeney, Wawro, & Rasmussen, 2012, p. 197).

Krummheuer (1995) refers to the skeleton of an argument that contains only data, conclusion (claim), and warrant as the core of an argument. He gives an example of a symbolic representation of the core of an argument as follows:

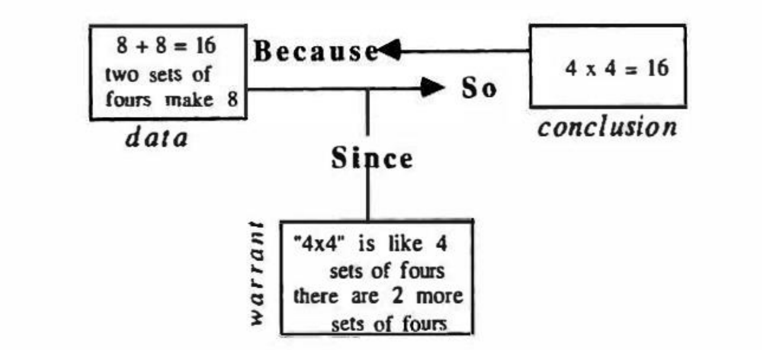


Figure 3.18. Symbolic representation of the core of an argument (Krummheuer, 1995, p. 243)

In addition to the core of an argument, Krummheuer (1995) points to the importance of backing to ratify and authorize the claim and considers an argumentation including data, claim, warrant, and backing as sufficiently elaborated. An example of backing for the core of the argument above and how it is attached to it is presented in Figure 3.19 below.

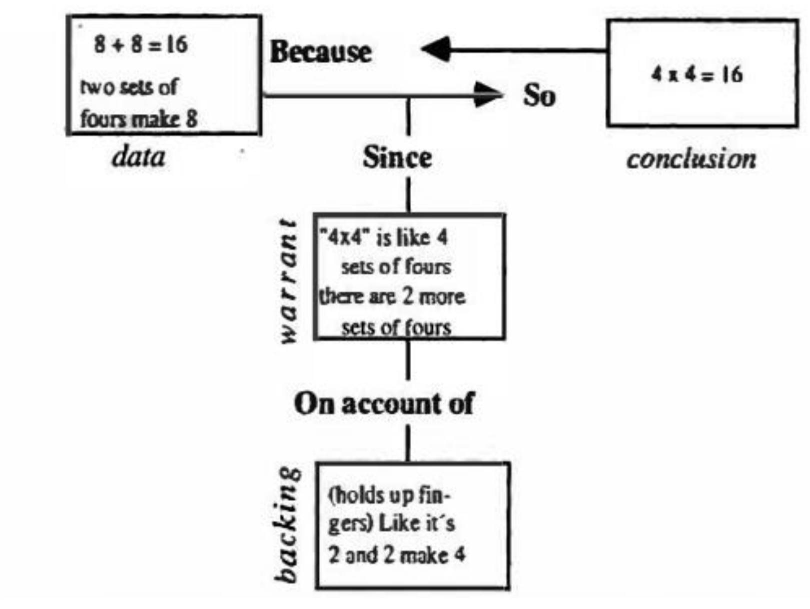


Figure 3.19. Complete schematic representation of a collective argument (Krummheuer, 1995, p. 246)

Similar to Krummheuer's (1995) approach, I coded the whole-class discussions as claim, data, warrant, and backing (if necessary) and created the corresponding argumentation schemes. In this process, I consulted my co-advisor, Dr. Stephan, in order to verify or refute the argumentation schemes. We went over the argumentation schemes, and came to an agreement on the argumentation schemes or drew new argumentation schemes for the purposes of reliability (Rasmussen & Stephan, 2008). In this process of identification of the elements of the argumentation schemes, we paid attention to the function of the contribution that the teacher or the students made (Rasmussen & Stephan, 2008). For instance, if a contribution functioned as a bridge between the data and claim, it was identified as a warrant. In addition, the aim of the contribution was also the focus. For instance, if a student answered the problem, it was identified as a claim, whereas it was identified as a warrant if he or she answered a question that the teacher asked for elaboration. Hence, at the end of the first phase, an argumentation log ordering all the argumentation schemes in succession across all whole-class discussions was obtained.

In line with the second phase of the three-phase method developed by Stephan and Rasmussen (2002) and Rasmussen and Stephan (2008), the argumentation log obtained in the first phase was analyzed across the class sessions in order to see which of these mathematical ideas became taken-as-shared (i.e., become a part of the class' communal reasoning). In this phase, the following two criteria developed by Stephan and Rasmussen (2002) Rasmussen and Stephan (2008) was used:

> 1. When the backings and/or warrants for an argumentation no longer appear in students' explanations (i.e., they become implied rather than stated or called for explicitly, no member of the community challenges the argumentation, and/or if the argumentation is contested and the student's challenge is rejected), we consider that the mathematical idea expressed in the core of the argument stands as self-evident.
> 2. When any of the four parts of an argument (the data, warrant, claim, or backing) shifts position (i.e., function) within subsequent arguments and is unchallenged (or, if contested, challenges are rejected), the mathematical idea functions as if it were shared (p. 200).

For instance, on Day 6, the idea of formalizing the invariant multiplicative relationship into an equation was noted as a new mathematical idea when a student claimed that the relationship between the amounts of apples in kg and its price as y= 2x and provided a warrant related to the invariant "times two" relationship. Then, on the very same day, several students wrote the corresponding equations representing the relationships between the amounts of vegetables/fruits and their price, and no warrant/backing came out, or nobody required any warrants/backings. Moreover, the same idea never got challenged on the following days. Hence, based on the first criteria related to dropping off of the warrants, we were able to conclude that the idea of formalizing the invariant multiplicative relationship into an equation was taken-as-shared among the classroom community. In addition, when students used a previously justified claim in their data/warrant/backing for subsequent arguments (i.e., the idea shifted place from claim to become data/warrant/backing), it was evident that the idea was taken-as-shared.

Although using these two criteria was very helpful for identifying the mathematical ideas that became taken-as-shared in the classroom community, an additional criterion improved the analysis method. This criterion was put forward by Cole et al. (2012) as it emerged when they utilized the three-phase methodological approach by Stephan and Rasmussen (2008) into their Chemistry classrooms. This criterion is as follows: "Criterion 3 When a particular idea is repeatedly used as either data or warrant for different claims across multiple days" (Cole et al., 2012, p. 200). In particular, this criterion was helpful in order to conclude that the initial mathematical ideas in the instructional sequence were taken-as-shared. For example, the initial task (fish-food bar task) in the instructional sequence required students to find the numerical answers to the questions that asked the number of food bars needed to feed the specific number of fish or the number of fish that could be fed with the given amount of food bars. In order to make claims about the numerical answers, students used linking and iterating with pictures, numbers, and operations, and on tables in their data and warrants. Since this is an informal and intuitive idea that develops in grades 3-5 (Kaput & West, 1994; Lamon, 1994), none of the students in the seventh-grade classroom used a warrant or asked for a warrant on how they arrived at the answers based on these intuitive ideas. In other words, the ideas related to iterating and linking appeared only in data or warrant in the first place and never got challenged, which made it impossible to use the two criteria by Stephan and Rasmussen (2002) and Rasmussen and Stephan (2002). However, by using the third criterion, we were able to conclude that iterating composite units and linking composite units was taken-as-shared since this idea was repeatedly used in students' data or warrants for a variety of claims across multiple days.

The identification of the taken-as-shared ideas took place in a cyclical manner in which we went back and forth across all class sessions. In this process, a three-column-chart was created including the elements (1) the mathematical ideas that were identified as taken-as-shared, (2) the mathematical ideas emerged in the discussion and to be kept an eye on to see if there would be further evidence for them to be taken-as-shared on the following days, (3) additional comments (Rasmussen & Stephan, 2008; Stephan & Rasmussen, 2002). Below is a portion of this table that belonged to Day 6:

Table 3.4. A portion of the mathematical ideas chart for Day 6 of the instruction

Taken-as-shared ideas	Keep an eye on	Additional comments
Structuring ratios and proportions multiplicatively and extending invariance and covariance to proportion	Checking for proportionality	Students heavily drew on their knowledge of comparing fractions while they checked whether the given two ratios were proportional. For instance, while checking whether or not $\frac{2}{3}$ and $\frac{10}{20}$ were proportional, students drew on their knowledge in fractions by referring to the $\frac{10}{20}$ as half and $\frac{2}{3}$ as larger than a half.

Table 3.4 is only a portion of a series of charts that belonged to each day of the experiment. The complete set of tables was used in order to see if the ideas in the second

column (i.e., keep an eye on) moved to the first column (i.e., taken-as-shared) across subsequent days. In doing so, the conclusions related to the mathematical ideas' being taken-as-shared were made based on the comparison of the elements of the mathematical ideas chart, which is also consistent with the Constant Comparison Method of analysis by Glaser and Strauss (1967).

Upon the completion of the identification of all the taken-as-shared ideas, these ideas were organized around common activities, which were later referred to as classroom mathematical practices in the third phase of the analysis method. When considered together, these classroom mathematical practices comprise the collective development of the classroom community in sophisticatedly increasing ways. For instance, it was observed that the students grouped the objects (i.e., fish and food bars) to form composite units, linked those composite units, and made sense of the covariation between those by making iterations with linked composites by using numerical, pictorial and tabular representations. Later on, these iterations took shifted from building up/down by ones to scaling up/down abbreviatedly. Then, the invariant multiplicative relationship between the linked quantities was interpreted and conceptualized as unit rate and used as a benchmark for finding missing values. Lastly, these understandings of covariance and invariance were reconceptualized in part-whole contexts. These mathematical ideas that were taken-as-shared on the first four days of instruction felt like being related to the general activity of reasoning with pictures and tables to find missing values. Therefore, these five mathematical ideas were put together and organized around the common activity of reasoning with pictures and tables to find missing values as they emerged and became taken-as-shared. Hence, the first mathematical practice in this study was called "reasoning with pictures and tables to find missing values (preserving link and it's one situation, not comparing two)" and consisted of the following five taken-as-shared ideas:

- Linking composite units and iterating linked composites while the link is preserved

- Pre-multiplicative reasoning - covarying the linked composites by a scale factor (i.e., building up abbreviatedly)
- Multiplicative reasoning- invariance of the multiplicative relationship between the two units as they covary-
- Conceptualizing the invariant relationship between the two linked composites as *unit rate* and *constant of proportionality* and using it as a tool/benchmark/anchor for finding missing values
- Reasoning with ratio tables and symbolic proportion representation to extend covariation and invariation to the relationship between parts and their whole.

Five classroom mathematical practices, including the above one, were identified in this study. It was found that these classroom mathematical practices emerged in a network-like manner, as found by Stephan and Rasmussen (2002), in which the separation of the practices was not always possible. For instance, there were a couple of times where more than one mathematical practice was established by the same taken-as-shared ideas. Moreover, it was possible that a specific mathematical idea became taken-as-shared, whereas a different idea emerged in the discussion during the same day. In other words, these mathematical practices "can have structural overlap, rather than a timing overlap of when the practices are initiated and constituted" (Rasmussen & Stephan, 2008, p. 201).

3.5. Trustworthiness

The trustworthiness of a research study is related to the procedures followed to ensure validity and reliability (Patton, 2002). In particular, trustworthiness in a design research study is related to the extent that the inferences and conjectures are justified and reasonable (Gravemeijer & Cobb, 2006). Briefly, validity is concerned with the question of whether the researchers measure what they intend to measure, whereas reliability concerns the independence of the researcher (Bakker, & van Eerde, 2015). Lincoln and Guba (1985) elaborates on the issues of validity and reliability by referring to the terms, credibility, transferability, dependability, and confirmability. In the following parts, these issues and how they are handled in this study will be explained in detail.

First, credibility is related to internal validity (i.e., how truthful are the findings?) and deals with the “quality of data and the soundness of the reasoning that has led to the conclusions" (Bakker & van Eerde, 2015, p. 444). The credibility of a study can be established by a description of prolonged engagement of the researcher with students and teachers in the context (Gravemeijer & Cobb, 2006; Lincoln & Guba, 1985; McKenney & Reeves, 2012), which is one of the strengths of design research studies (Gravemeijer & Cobb, 2006). In particular, according to Bakker and van Eerde (2015), internal validity of a design research study can be improved by testing the conjectures to multiple episodes and checking if different data collection tools yield to the same conjectures, which is referred to as triangulation (Lincoln & Guba, 1985; McKenney & Reeves, 2012, Patton, 2002). In this study, the researcher took part actively in the designing, classroom experimenting, and revising processes for almost three years. In this process, the researcher conducted interviews with the students and the teacher periodically before, during, and after the design experiments in two cycles, which in return helped the researcher understand the context with its potentials and drawbacks and come up with the means to support learning in that context. Besides, this study was conducted in two macro-cycles in two consecutive years, which made it possible to test the early developed conjectures in successive experiments, which contributed to the improvement of the internal validity of the study (Bakker & van Eerde, 2015).

Additionally, the data were collected through a variety of tools (i.e., student and teacher interviews, field notes, classroom observations, design research team meetings, debriefing sessions, etc.), all of which were used in developing and testing conjectures of the study. In this process, the research design team, including the researcher, the two advisors, and the collaborating teacher, participated in developing and testing the conjectures related to student learning and the means of supporting this learning. In addition, debriefing sessions with the teacher after each class session were conducted. Particularly, in order to ensure the credibility of data analysis, a plausible method of data analysis (i.e., three-phase-method for documenting mathematical practices) was used, and all the phases of the analysis process were documented in detail (Gravemeijer &

Cobb, 2006). Moreover, in this process of data analysis, the researcher worked with Dr. Stephan, who is one of the designers of the three-phase method.

Second, transferability is related to external validity (Lincoln & Guba, 1985), generalizability (Bakker & van Eerde, 2015), or ecological validity (Gravemeijer & Cobb, 2006). It deals with the question “how do the findings inform other contexts?” and "to what extent do the lessons learned in a specific experiment can be useful for subsequent experiments" (Bakker & van Eerde, 2015). In a design research study, classroom activities and classroom events are the issues that are associated with generalizability; yet, it should be noted that each classroom has its own characteristics (Gravemeijer & Cobb, 2006). Thus, in a design research study, “what is generalized is a way of interpreting and understanding specific cases that preserves their individual characteristics” (Gravemeijer & Cobb, 2006, p. 47). That is to say, the insights obtained from this study can be used to interpret and inform other cases that are relevant in terms of students’ mathematical learning. In this way, the developed local instruction theory and instructional activities can be used to foster the learning of students in other classrooms (Gravemeijer & Cobb, 2006). Also, the design of this study can inform further research and future instructional design. To put it differently, it is the desire in a design research study that the developed local instruction theory "can function as a frame of reference for teachers who want to adapt the corresponding instructional sequence to their own classrooms, and their personal objectives" (Gravemeijer & Cobb, 2006, p. 45).

One of the major ways to address transferability or ecological validity is “thick description” (Bakker & van Eerde, 2015; Firestone, 1987; Gravemeijer & Cobb, 2006; Lincoln & Guba, 1985; McKenney & Reeves, 2012). Particularly in design research studies, these descriptions should include the details of participation behaviors of students, teaching-learning process (Gravemeijer & Cobb, 2006) and “failures and successes, procedures followed, the conceptual framework used, and the reasons for certain choices” (Bakker & van Eerde, 2015, p. 445). In addition, an analysis of how the teaching-learning process might be affected by these aspects should be provided in detail (Gravemeijer & Cobb, 2006). Also, the design principles followed, and both the processes of evaluation and the context should be articulated theoretically in order to help the audience to estimate the extent that transferring from the reported situation to theirs is possible (Miles & Huberman, 1994). In line with these suggestions, in this study, the interpretative frameworks (i.e., RME, emergent perspective), procedures followed, and revisions to the sequence in each phase with their rationale were explained in detail. Besides, students' communal ways of reasoning as they interact with the instructional sequence and how this reasoning is interpreted within the interpretative frameworks were explained in detail. Moreover, collective ways of participating in the classroom mathematical practices and how these participation behaviors were categorized in terms of Toulmin's argumentation scheme were described thoroughly. In addition, transcripts from the classroom experiments were provided where necessary in order to help readers understand the context and the theoretical claims made (Bakker & van Eerde, 2015).

Third, dependability relates to reliability and deals with the consistency and replicability of the findings (Lincoln & Guba, 1985). Gravemeijer and Cobb (2006) point out that design research studies do not aim replication of the instructional strategies in other classrooms since replicability is not possible and even desirable (Simon, 1995). Instead, they refer to ecological validity as the aim in design research, wherein adaptations are made to other situations based on the explanations of the results of a specific study (Gravemeijer & Cobb, 2006). One way to establish the dependability of a study might be possible through the use of the same design principles and multiple iterations in design research (McKenney & Reeves, 2012). Concerning this, this study is conducted in two macrocycles in two consecutive years. In these two cycles, the same design principles (e.g., the principles of RME) were followed.

Forth, confirmability is associated with objectivity and deals with the question "how much are the findings shaped by the participants and conditions and not due to researcher bias?" (Bakker & van Eerde, 2015; Lincoln & Guba, 1985; McKenney & Reeves, 2012). The same issue is referred to as internal reliability, which is associated with the "degree of how independently of the researcher the data are collected and analyzed" (Bakker & van Eerde, 2015, p. 445). According to Bakker and van Eerde (2015), it can be improved by audio- or videotaping during data collection and peer examination (i.e., inter-rater reliability). Besides, the triangulation of data collection methods can be helpful in ensuring confirmability (Lincoln & Guba, 1985; McKenney & Reeves, 2012). In this study, data were collected in a variety of ways. Besides, the classroom sessions, teacher and student interviews, and design team meetings were recorded via audio- or video recording, and the transcripts were provided where necessary. Lastly, two coders worked collaboratively in analyzing the data in order to establish a peer examination.

Above all, one of the critical considerations related to the trustworthiness of qualitative studies is that the researcher's skills, competence, personal beliefs, and experiences affect his interpretations (Patton, 2002). This issue will be minimized by a detailed description of the researcher's role throughout all stages of the study.

3.6. Researcher Role

Researcher's ontological, epistemological, and methodological beliefs, or what is called paradigm, shape and guide the research (Denzin & Lincoln, 1994). In design experiments, researchers should be present in the classroom during the experiment and engage in debriefing sessions with the teacher subsequent to each classroom session in order to come up with common interpretations of the classroom events (Gravemeijer, & Cobb, 2006). Therefore, in line with the principles of a design research study, I was present in each class session during the two experiments and conducted meetings before and after each class session.

Particularly, throughout this study, I took two roles: a participant-observer in collecting and analyzing data and a data source as a supporter of the teacher in designing the course.

In other words, I was an active participant in all of the phases of this design research study. Even though it might be regarded as biased through quantitative paradigms, this active participant in the process provided me with the opportunity to interpret the study through my own experience in the context (Marshall & Rossman, 1999). In particular, even though being in the setting seems like a threat to the trustworthiness of the study, it might also be considered as an advantage to support the design process and solve the design problems in the setting (Barab & Squire, 2004).

3.7. Ethical Considerations

In advance of the design experiment, the necessary permissions were taken from Middle East Technical University Human Subjects Ethics Committee (see Appendix B) and the Head of Elementary Mathematics Education program of the university. Then, official permissions needed for conducting the study were taken from the Ministry of National Education (see Appendix C). The collaborating teacher was informed about the purposes and procedures of the study and asked for her consent. In relation to the procedures, she was explicitly informed that her classes would be video recorded, and debriefing sessions will be audio recorded. Also, she was assured about the confidentiality of her identity and that the results will be used only for scientific purposes. See Appendix D for the informed consent form through which her consent was taken.

Additionally, the students in both macrocycles were informed about the same issues (i.e., purposes, data collection procedures, confidentiality, voluntarism). In order to ensure confidentiality, pseudonyms were used for the students. Moreover, permissions from the parents of the students that were selected to be interviewed at the beginning and end of the study were taken via parent permission form (See Appendix E). In this form, it was explicitly written that the interviews conducted with these students would be videotaped without capturing their faces. They were also assured that the identities of their children would be kept confidential. After the necessary permissions were obtained from the parents, these children that were selected to be interviewed were informed about the same issues and were asked for their consent.

CHAPTER 4

FINDINGS

The overall purposes of this dissertation are to develop, test, and revise a classroom HLT and related instructional sequence for supporting seventh-graders' proportional reasoning, to explain collective development of proportional reasoning of a seventh-grade classroom community by a documentation of classroom mathematical practices, and to describe the emergence of communal ways of reasoning with informal and formal tools (i.e., models, imageries, gestures, and metaphors). This chapter is devoted to the elaboration of the findings obtained by the Classroom Mathematical Practices analysis. Particularly, this chapter explains the taken-as-shared ideas and the classroom mathematical practices that emerged as the students interacted around the instructional sequence and the HLT. A particular focus is given to the tool use and transition from reasoning with informal tools to reasoning with formal tools. It also delineates how the classroom community's reasoning is supported in increasingly sophisticated ways with an RME perspective and what opportunities and barriers the instructional sequence provides for realization of the hypothesized learning trajectory and the process of mathematization. In doing so, it also embodies how the students rely on their informal knowledge in order to mathematize their informal and intuitive knowledge and self-generated solutions.

Five mathematical practices were documented over the course of 15 class periods that were focused on ratio, proportion, and proportional reasoning. The process of mathematical ideas being taken-as-shared and the emergence of the five mathematical practices occurred in a network-like manner, as found by Stephan and Rasmussen (2002). Thus, the separation of the practices was not always possible. This was because the instructional sequence was designed in order for students to develop a web of ideas in pictorial, tabular, graphical, numerical, and symbolic ways of representations. For example, there were a couple of times where more than one mathematical practice was established by the same taken-as-shared ideas. Moreover, it was possible that a specific mathematical idea became taken-as-shared, whereas a different idea emerged in the discussion during the same day as Stephan and Rasmussen (2002) stressed. The mathematical ideas that were taken-as-shared (TAS) throughout the establishment of these mathematical practices are described in the following parts.

Table 4.1. Five Classroom Mathematical Practices

Classroom Mathematical Practices
Practice 1. Reasoning with pictures and tables to find missing values • Linking composite units and iterating linked composites while the link is preserved • Pre-multiplicative reasoning- covarying the linked composites by a scale factor (i.e., abbreviated build-up) • Multiplicative reasoning- invariance of the multiplicative relationship between the two units as they covary • Conceptualizing the invariant relationship between two linked composites as *unit rate* and using it as a tool/benchmark/anchor for finding missing values • Reasoning with ratio tables and symbolic proportion representation to extend covariation and invariation to the relationship between parts and their whole
Practice 2. Reasoning with tables and symbols to determine proportional situations • Structuring ratios and proportions multiplicatively and extending invariance and covariance to symbolic proportion representation. • Determining proportionality by covariational and multiplicative reasoning

Table 4.1 (Continued)

Practice 3. Coordinating the relationships among the representations • Linking composite units and iterating linked composite units • Multiplicative reasoning- the relationship between the two composite units is invariant • Formalizing the invariant multiplicative relationship into an equation • Representing proportional relationships with linear equations of the type y=mx and graphs passing through the origin • Representing non-proportional linear relationships with linear equations of the type y=mx+b and graphs not passing through the origin
Practice 4. Extending covariation and invariation to continuous contexts • Reasoning with within-shape and between-shapes ratios to find missing side lengths of similar shapes • Conceptualizing distortion of shapes
Practice 5. Comparing rates/ratios and deciding which one is bigger/smaller/equal • Creating and reasoning with equivalent ratios to compare quantities.

4.1. CMP 1. Reasoning with pictures and tables to find missing values

Five ideas became taken-as-shared as the students mathematized the fish-food bar situations and part-whole relationships on the first five days of the instruction.

- Linking composite units and iterating linked composites while the link is preserved (i.e., building up)
- Pre-multiplicative reasoning- covarying the linked composites by a scale factor (i.e., abbreviated build-up)
- Multiplicative reasoning- invariance of the multiplicative relationship between the two units as they covary
- Conceptualizing the invariant relationship between the two linked composites as *unit rate* and *constant of proportionality* and using it as a tool/benchmark/anchor for finding missing values
- Reasoning with ratio tables and symbolic proportion representation to extend covariation and invariation to the relationship between parts and their whole.

The instructional sequence started with an experientially real context that required students to link composite units and make iterations with linked composites. In order to launch the lesson, the teacher asked the students if they had pets. After getting answers from the students, she told a story as follows:

> I have pet fish, and I had a dream last night. In my dream, there was a noise coming from the living room where the aquarium is. I went to see what the noise was about and saw that the fish jumped out of the aquarium and were wandering around the living room screaming, “we want food.” I thought that I had given them enough food, but I felt sorry that it was not enough. The fish were so mad at me that they attacked me. That woke me up, and I was so frightened that I went to the living room and checked the fish, but they seemed fine. I was so upset that I promised myself I would be cautious about the amount of food I give them. Besides, I thought that could be a delicate topic for today's math class.

The teacher then asked the students how they feed their pets. The students suggested that pets had to be fed with a certain amount of food each day. If they were underfed, they would be hungry; if they were overfed, they would get sick. In the worst case, they could even die from under- or overfeeding. Below is a classroom dialogue that illustrates this process:

> *Teacher*: Is there anyone who pets fish (A couple of students raise hands). How do you feed them? What happens when you overfeed?
> *Berk*: They die when overfed. We need to give a specific amount of food each day.
> *Teacher*: That’s right. Our pet fish died due to overfeeding. What happens if we underfeed?
> *Ozan*: Then, they get hungry.
> *Berk*: They can even die from being underfed.

As could be deduced from the dialogue above, this story laid the ground for the students to make sense of why the rule for feeding the fish could not be broken. Then, the teacher introduced an activity that explored situations involving feeding fish with food bars each time using the given specific rule.

TAS Idea 1: Linking composite units and iterating linked composites while the link is preserved (i.e., building up). The first rule was one food bar for three fish, which included a whole number ratio between the food bars and the fish as the starting point. The pictures of fish and food bars were also included at the beginning in order to help students group the pictures and link the composites concretely, as shown in the following figure.

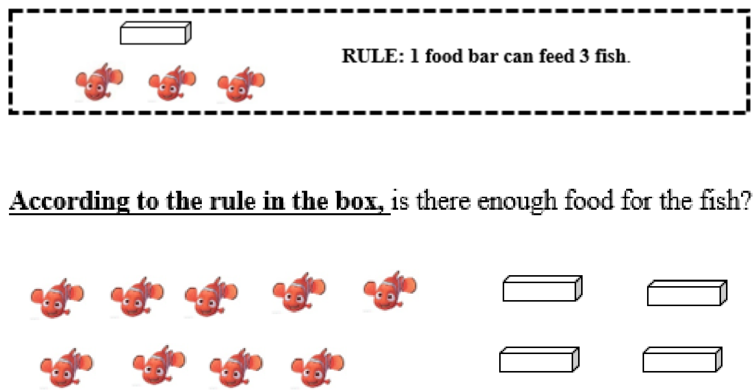


Figure 4.1. A sample question with pictures of food bars and fish for grouping and linking

Even though this might be a simple problem for seventh graders, it was posed so that there was a chance for the teacher to capitalize on student thinking related to taking three fish as a unit and linking with one food bar. Including the pictures was also helpful for the students to make sense of the rule and the linking process in the first place by concretely grouping the three fish and matching each group with one food bar with arrows. Below is a dialogue that is helpful to understand how including the pictures helped the students:

> *Teacher*: How did you solve the problem?
> *Seval*: (Writes her solution on the board) 3 fish eat one food bar. I circled three fish like this and wrote 1,2,3 on the bars. The last bar was not used (draws circles to represent the fish and rectangles to represent the bars. Then, she draws closed curves to group each three fish and writes 1, 2,3 on each group, and writes 1,2,3 on the three of the four food bars. She leaves the fourth rectangle blank and writes one extra food bar next to it, See Figure 4.2 below)
> *Elif*: Yes, there is enough. We have even one extra (food bar).
> *Berk*: Yes, we can feed three more fish. That means there is enough and even more.

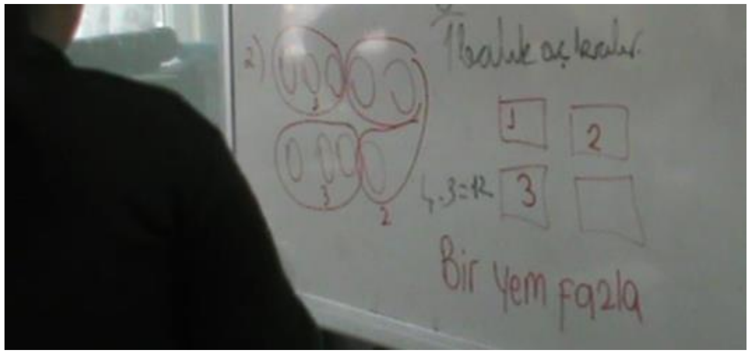


Figure 4.2. Drawing closed curves to group three fish and linking with food by using numerals

As it was deduced from the dialogue and the figure given above, Seval grouped the pictures of fish in 3s and wrote 1,2,3 on each of the groups and also on the food bars in order to show the link between each group and a food bar. When a Toulmin analysis was conducted on this dialogue, Seval's solution, including the pictorial and verbal explanations, was considered as a data to the claim “there is enough food bar.” Elif and Berk explained why having an unmatched food bar led to the claim that there is enough food bar, which in return was regarded as a warrant to Seval's data and claim. This interpretation is illustrated with Toulmin's analysis scheme presented in Figure 4.3 below:

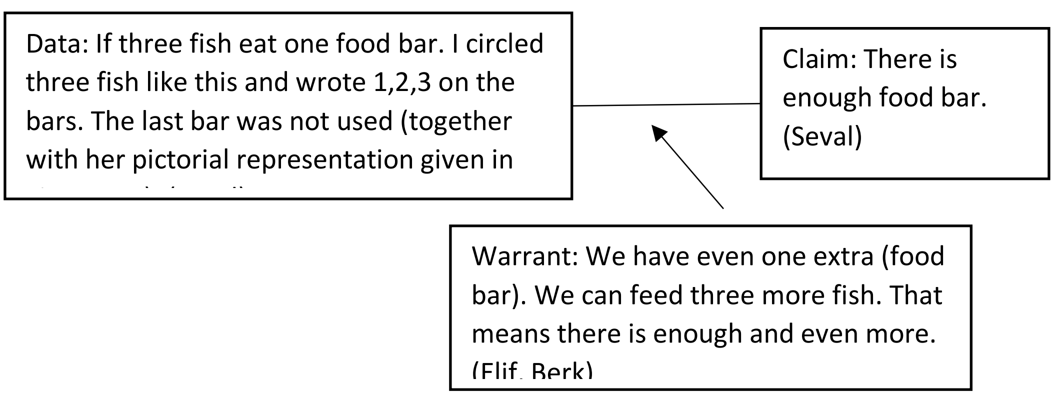


Figure 4.3. Toulmin Analysis scheme regarding linking composite units

Even though Seval used numerals in order to link a group of 3 fish and a food bar during the whole class discussion, many students also used arrows for showing the same link during the small group discussions. This can be seen in the following figure that includes Ceyda's work on the activity sheet.

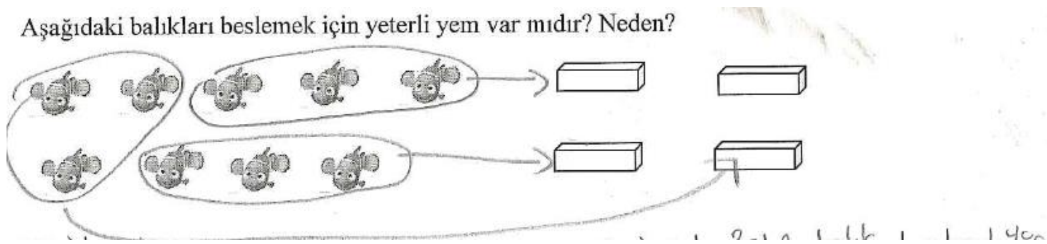


Figure 4.4. Using arrows for linking a group of fish and a food bar

As the students engaged with similar questions that included the pictures of food bars and fish, they continued to make claims in which they answered if there were enough food bars for the given number of fish. They also referred to the rule and the fact that there were not enough food bars if any fish was left unfed or the number of food bars was enough if all the fish were fed in their data and/or warrants.

In the following problems on Day 1, pictures of either food bars or fish were provided in order to support students' move to numerical linking and iterating gradually, as shown in the following figures:

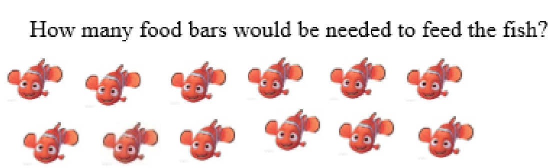


Figure 4.5a. Only the pictures of fish were given in the problem

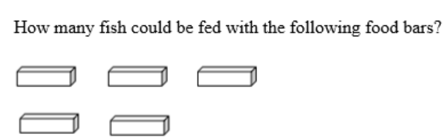


Figure 4.5b. Only the pictures of food bars were given in the problem

In order to answer the problem in Figure 4.5a, Sinem came to the board and drew 12 circles in order to represent the fish and wrote 1-2-3-4 on each group, as shown in Figure 4.6 below.

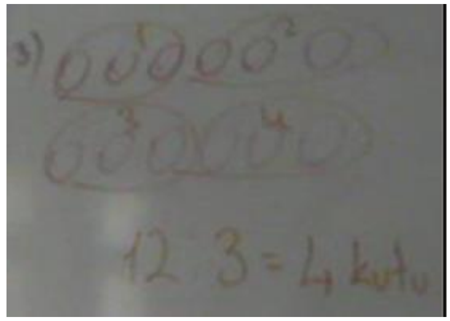


Figure 4.6. Grouping the fish in 3s and linking with a food bar

Then, Merve suggested that she had an alternative way of solving the problem, and the following dialogue happened in the classroom discourse, which led to making connections between grouping and division operation:

> *Sinem*: There are 12 fish, and four food bars are needed to feed them. I grouped the fish in 3s, and a food bar is needed for each (group).
> *Merve*: I divided 12 by three and found 4. Four food bars are needed.
> *Teacher*: Why did you divide 12 by 3?
> *Merve*: I used the given rule. It is the same as grouping. I grouped the fish in threes and obtained four groups (shows Sinem’s solution on the board).
> *Giray*: Yes, we don’t need to draw fish all the time.

When the analysis of this dialogue in terms of Toulmin’s model in question, Sinem claimed that four food bars were needed (in order to feed 12 fish). Her data for this claim included grouping the pictures of fish in 3s and linking each group with a food bar by using numerals. Noone challenged Sinem, and Merve provided alternative data for the same claim by suggesting to divide 12 by 3. The teacher asked Merve to justify her strategy, which led us to note how her data related to the claim for two reasons. The first one was that dividing as a procedural way for grouping emerged for the first time in the classroom, and the second one was due to the fact that it was a procedural strategy. Upon this request, Merve provided a warrant in which she made a connection between the division operation and grouping by 3s. Based on this kind of interpretation, the following Toulmin scheme was constructed for the dialogue above:

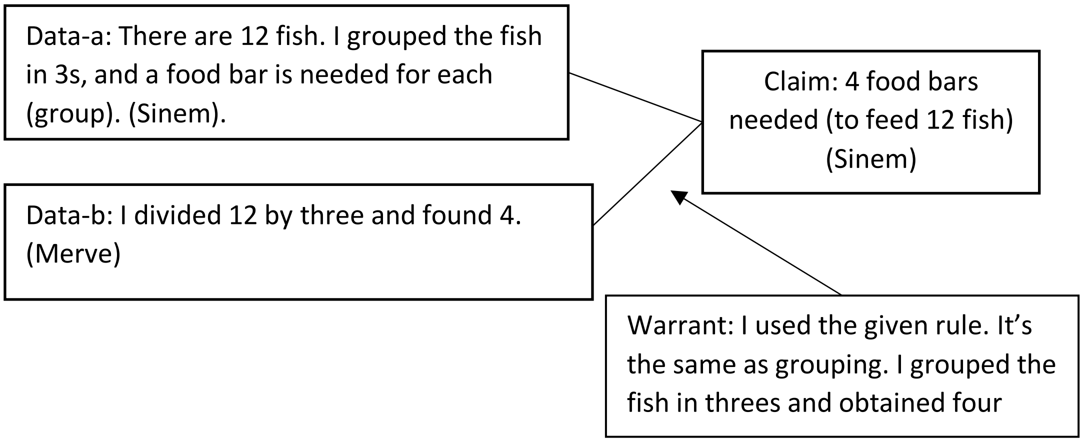


Figure 4.7. Toulmin's Analysis scheme on discussion related to grouping and linking and corresponding division operation to maintain the link and iterating

Similarly, when only the pictures of food bars were given as seen in Figure 4.5b, Gizem claimed that 15 fish could be fed (with five food bars) by providing data that included multiplying three by five and referring to the rule. Since this was procedural data, the teacher asked the class why Gizem multiplied three by five. As an answer to the teacher's question, Faruk drew five rectangles to represent five food bars and wrote three on top of each rectangle to establish a link between a food bar and three fish. He further explained that adding five threes would mean three multiplied by five, which acted as a warrant for the data-claim pair that Gizem established. The operational answer by Gizem and the drawing by Faruk on the board is illustrated in Figure 4.8 below.

Figure 4.8. Linking and pictorial/numerical iterations to maintain the link

Therefore, the symbolic division as a procedural way for grouping and multiplication as a procedural strategy for iterating emerged in the classroom at the very beginning of the instruction on Day 1. Moreover, the students felt the need to provide warrants for the claims related to these relationships. In addition to the classroom discourse, students' work on the activity sheets as the classroom discussion took place also gave evidence related to this. Some of the students provided pictorial representations as data to their claims regarding finding the corresponding number of fish/food bars in relation to the given number of food bars/fish, as seen in Figure 4.9a and 4.9b below. As seen in Figure 4.9a, the link between three fish and one food bar is illustrated with a grouping of three fish and drawing a rectangular prism to represent the food bars beside the fish. Similarly, the same link between a food bar and three fish is represented by drawing circles, as shown in Figure 4.9b.

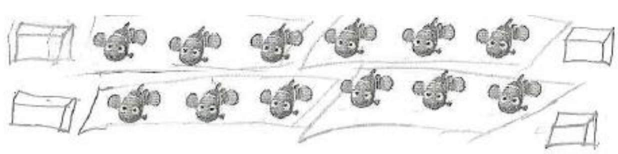


Figure 4.9a. Drawing rectangular prisms to represent the food bars and linking these with a group of three fish

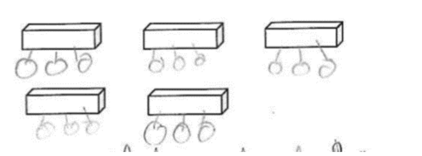


Figure 4.9b. Drawing circles to represent the fish and linking these with a food bar

Rather than drawing pictures, some of the students used numbers and operations with the number of fish/food bars as data in order to make claims about the answers to the questions, as illustrated in Figure 4.9c and 4.9d.

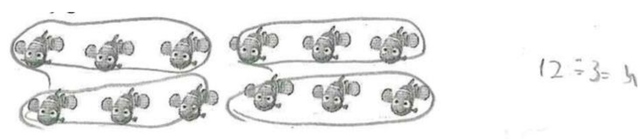


Figure 4.9c. Initial numerical iterations by making connections to grouping

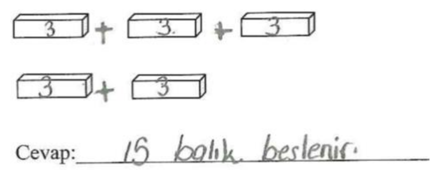


Figure 4.9d. Initial numerical iterations by making connections to iterating

As understood from both dialogues and the figures, the strategies related to division (see Figure 4.6) and multiplication (see Figure 4.8) were short ways for linking a food bar and three fish and making iterations with this link. It was clear that the students were maintaining the link between 1 food bar and three fish by making iterations with this link. This became more apparent, especially when Merve pointed to Sinem's solution with grouping and made a connection between grouping and division operation. Moreover, it was also obvious when Faruk wrote 3s on each of the rectangles that represent the food bars and pointed out that he had to add three five times. In other words, even though these strategies included multiplication and division, they were not regarded as multiplicative reasoning since they were related to making numerical iterations with the help of the given pictures. Instead, these types of strategies were considered as short numerical ways for iterating linked composites as the students did in the previous parts with the given pictures.

In the following problems, questions with bigger numbers (e.g., How many fish can be fed with nine food bars? How many fish can be fed with ten food bars?) were posed. No picture was included in these problems in order to promote students' process of creating mental images of linking composite units and iterating this link. As seen in small group work, Sezin and Faruk continued to draw pictures of food bars and/or fish. They worked with the corresponding numerical operations, in which they continued to link a food bar with three fish and iterate this link in pictorial and pictorial/numerical ways in their providing of data for their claims, as demonstrated in Figure 4.10a&10b below.

Q. How many fish can be fed with nine food bars?

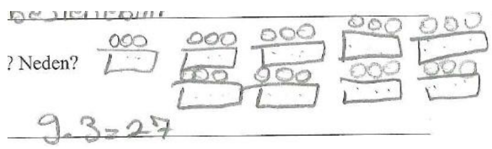


Figure 4.10a. Sezin continued to draw pictures and link a food bar with three fish and iterate

Q. How many fish can be fed with ten food bars?

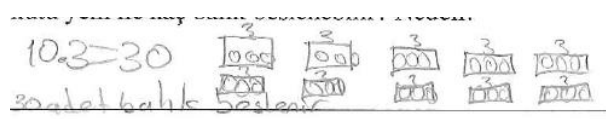


Figure 4.10b. Faruk continued to draw pictures and link a food bar with three fish and iterate

The teacher had Sezin and Faruk share their pictorial strategies on the board, and Seval came to the board and said that she multiplied three by 10. Nobody challenged Sezin or Faruk about linking and iterating with pictures or Seval about the multiplication as a short way for making iterations. Then, the teacher gave the floor to Giray and let him share his strategy, which the teacher had observed in the small group discussion for the question "How many fish can be fed with ten food bars?" Below is a classroom dialogue took place while Giray was sharing his strategy to this problem:

> *Teacher*: Giray solved the question in a different way. Giray, how did you come up with this?
> *Giray*: We had found that nine food bars could feed 27 fish (in the previous question). There are 10 bars here, and 10 bars can feed 30 fish. I added 3 to 27.
> *Teacher*: Is there anyone who didn't get what Giray did?
> *Selim*: Why did you add 3 to 27?
> *Giray*: Since 1 (food) bar can feed three fish, it goes up by 3s. When a food bar is added, 3 (more) fish can be fed.
> *Selim*: OK.

In order to interpret this dialogue in terms of Toulmin Analysis, it could be said that the claim for the previous question [27 fish can be fed with nine food bars] became the data for a later claim [30 fish can be fed with 10 bars]. Together with a previous claim, Giray stated that he added 3 to 27, which also acted as data for that claim. When he was challenged to justify this strategy, he provided a warrant implying the link between a food bar and three fish and the fact that the link had to be maintained even when the number of food bar/fish changed with a build-up strategy. Therefore, this was the first time a building up strategy emerged in the classroom discussion. This interpretation is summarized in the Toulmin scheme in Figure 4.11 below:

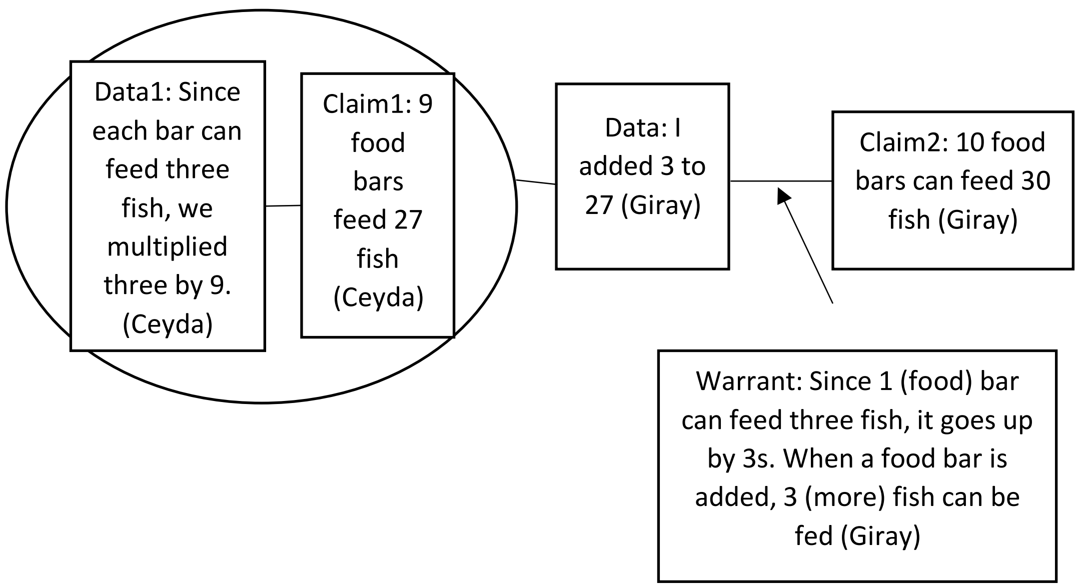


Figure 4.11. A claim becomes data for a subsequent claim while a student is using a build-up strategy

On the second day of the instruction, as a different application of build-up strategy, it was observed that Berk started to make pictorial/numerical linking in small group work. Particularly, he started to build-up the number of fish as 3, 6, 9, 12, 15, etc. with a numerical/pictorial representation that was presented in Figure 4.12 below:

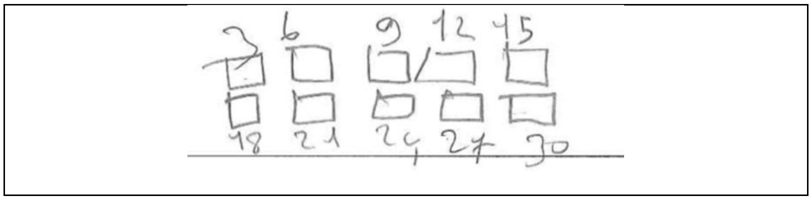


Figure 4.12. Berk’s building up with pictures and numbers (pictorial/numerical build-up)

The teacher had Berk share this strategy on the board and moved to the next question. When the students were working on the question “how many food bars would be needed in order to feed 18 fish?” in small groups, the teacher observed that Aylin linked the number of food bars and the number of fish and iterated this link numerically as shown in Figure 4.13 below. In contrast, other students made iterations by grouping pictures and division as a short way for grouping.

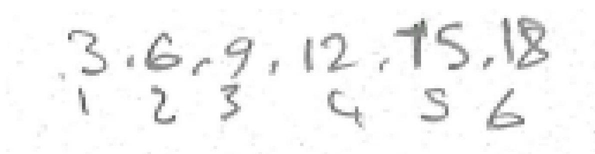


Figure 4.13. Aylin’s iteration of the 1-3 link numerically

After the students shared their strategies related to iterating by grouping and/or division on the board, the teacher showed Aylin’s iteration on the board. The teacher took this opportunity to introduce a long ratio table. Below is a classroom dialogue in which the long ratio tables were introduced on the second day of the instruction:

> *Teacher*: Aylin, can you explain what you did on your paper?
> *Aylin*: I started from one and three and went on like two-six, three-nine, four-twelve, and so on and found that 18 fish could be fed with six bars.
> *Teacher*: Did you understand what Aylin did with these numbers?
> Students: Yes.
> *Teacher*: She aligned them vertically, fish and food bars. We can see how many fish could be fed with each of the numbers (number of food bars). I would like to put these numbers into a table to organize them better (draws the table presented in Figure 4.14 below). Nevertheless, I would like to write the number of food bars

in the first row and the number of fish on the second. Can you help me fill in these blanks?
Students: (Saying aloud all together) 1 bar 3 fish, 2 bars 6 fish, 3-9, 4-12, 5-15, 6-18.
Teacher: How did you fill in those blanks?
Elif: I went up according to the rule.
Musa: I skip-counted.
Merve: I multiplied one by three, two by three, three by three.

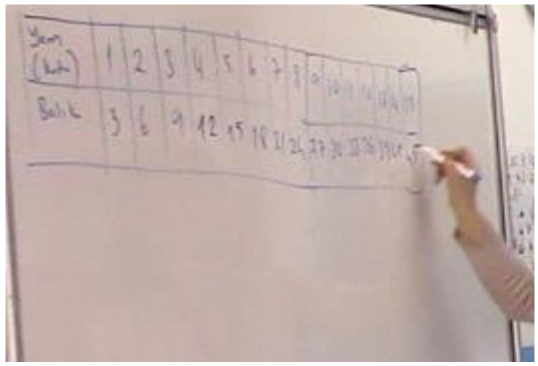


Figure 4.14. Introduction of long ratio table to organize the linked numbers (Day 2)

As understood from the dialogue above, the teacher introduced a long horizontal ratio table (HRT) on the board in order to organize the linked quantities and keep track of the iterations with this link by using Aylin's informal table-like representation. Therefore, with an RME perspective, long ratio tables functioned as *models of* an organized way of iterations with linked composites in the first place. Then, the teacher had all the students in the class say aloud the numbers that would go in the table. After constructing the table on the board, a bunch of students said aloud all together "1 food bar 3 fish, two food bars 6 fish, three food bars 9 fish, etc.", in which they used build-up reasoning in their data. Even though Merve touched upon the multiplicative relationship between the number of food bars and fish, neither the teacher nor the students paid attention to that idea that emerged for the first time in the instruction.

In the second period of the second day of the instruction, after filling in the table by building up strategies, the teacher encouraged students to explore the relationships in the table by posing the following prompt: "Is there a relationship between the change in the number of food bars and the number of fish?" This prompt was presented to help students use correct mathematical language related to the covariation between the number of food bars and fish. Merve made a claim saying that "For every one food bar added, three more fish are added" and provided data by referring to the rule as "since one food bar feeds three fish." She went on expressing her reasoning in different words using a specific hand gesture: "when the number of food bars goes up by one [points at number 1 in the first row in the table and moves her hand until it points to number 2 in the first row, repeating consecutively], the number of fish goes up by three" [points at number 3 in the second row in the table and moves her hand until it points to number 6 in the second row in the table, repeating consecutively], as demonstrated in Figure 4.15a and 4.15b below.

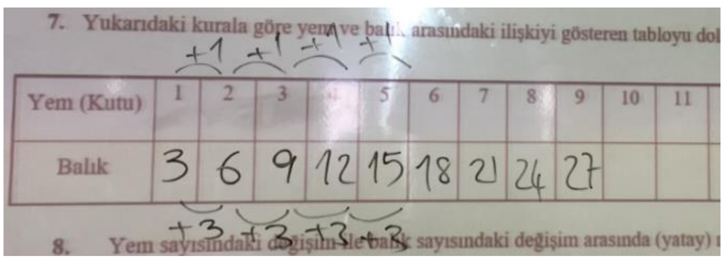


Figure 4.15a. Merve building up by ones and threes in the long ratio table

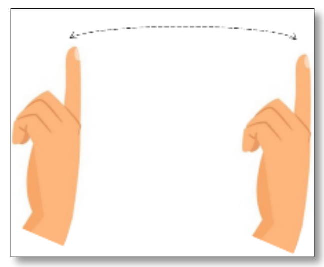


Figure 4.15b. The hand gesture for building up in the long ratio table

No student challenged Merve's claim related to iterating the number of food bars by one and the number of fish by three. Therefore, the idea of "linking composite units and iterating linked composites while the link is preserved" never got challenged in the instruction. In other words, as the students answered the questions by linking and iterating with pictures, numbers and operations, and on tables, they just referred to the given rule as data. Moreover, none of the students ever asked for a warrant or used a warrant for why they linked one food bar with three fish and preserved the link throughout the first two days of the instruction. Therefore, the two criteria by Stephan and Rasmussen (2002) and Rasmussen and Stephan (2008) were not met in order to conclude that this idea was taken-as-shared. On the other hand, this idea was used in tables and with pictures and/or numbers as data for different claims across different days. Hence, it was evident that this idea was taken-as-shared since the third criterion by Cole et al. (2012) was met. Besides, the episodes from the subsequent days also gave further evidence that this idea was repeatedly used as data for different claims. Therefore, linking composite units and iterating linked composites while the link is preserved was taken-as-shared among the classroom community by the end of the second day of the instruction.

TAS Idea 2. Pre-multiplicative reasoning- covarying the linked composites by a scale factor (i.e., abbreviated build-up, reasoning with within measures ratio). As the students continued to explore the relationships in the table, Sinem stated a different relationship by claiming that "While the number of food bars is increasing by 5 [points at number 1 in the first row in the table and moves her hand until it points to number 6 in the first row in the table], the number of fish increased by 15 [points at number 3 in the second row in the table and moves her hand until it points to number 6 in the second row]." That means the number of fish grows three times more when compared to the number of food bars.” These procedures are illustrated in Figure 4.16a and 4.16b below:

+5

Number of food bars	1	2	3	4	5	6	7	8
Number of fish	3	6	9	12	15	18	21	24

+15

Figure 4.16a. Sinem's building up by multiples of one and three on the long ratio table

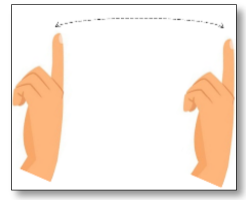


Figure 4.16b. The hand gesture for building up on the long ratio table

Since this was a procedural claim, the teacher asked the other students whether or not they understood what Sinem was trying to tell. Emre challenged Sinem by asking the reasoning behind this relationship with the following question: "I can see that it increased by five above and 15 below, but I don't understand why it is times three below" (referring to Sinem's claim regarding growing three times more). Sinem provided a warrant for her claim by referring to the rule and stressing that "since one food bar feeds three fish, the number of fish always increases three times more in relation to the number of (food) bars." Therefore, even though multiplying the number of food bars by three as a short and symbolic way for iterating with pictures emerged on Day 1, the "times three relationship" between the increment in the number of food bars and the number of fish emerged on Day 2 and was challenged by a classmate. Later on, the teacher asked the students to explore other relationships in the table, and the following dialogue took place in the classroom:

Teacher: Did you observe any other relationship in the table?
Ceyda: I shaded two first and then 10 (in the table). For the food bars, for instance, ten is two multiplied by five. When we look at the (number of) fish, five times six is 30 (Ceyda explains her group's solution in her seat and moves her hands as if she was shading, and the teacher shades 2 and 6 in the second column in the table and then 10 and 30 in the sixth column in the table according to Ceyda's explanations) (See Figure 4.17 below).

Teacher: Then, what kind of a relationship do you think is there?
Ceyda: Both numbers are multiplied by the same number.
İlter: We took 12 and 36. (İlter explains his group's solution in his seat and moves his hands as if he was shading, and the teacher shades 12 and 36 in the twelfth column in the table. Later, İlter moves his hand backwards so as to point to 6 in the sixth column). We divided 12 by two and got 6. Then, we divided 36 by two and got 18 (the teacher draws a curve from 12 to 6 and then from 36 to 18 backwards according to İlter's explanations) (See Figure 4.17 below).
Teacher: Then, you went from here (points to the twelfth column of the table) to here (points to the sixth column of the table).
İlter: Yes, we went backwards. We went from 12 to 6 (in the first row) and 36 to 18 (in the second row). Both are divided by 2.
Teacher: Then, this group found that both numbers are multiplied by the same number, and this group found that both numbers are divided by the same number. Why does it happen that way?
Ceyda: Because, for instance, when we go from 2 to 10, there are five groups, each including two food bars. When five groups of fish are added (linked) (moves her hand in circles to iterate five groups of fish) to those, the number of fish is quintupled.
Elif: That means that they (fish and food bars) are added in groups instead of one by one.

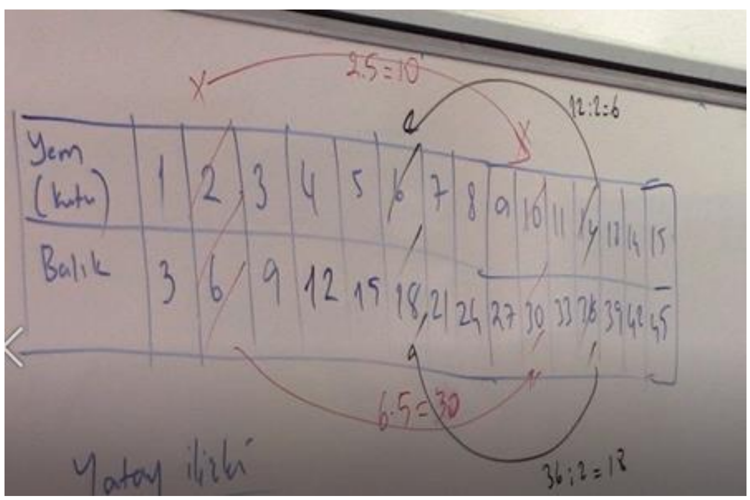


Figure 4.17. Representations of abbreviated build-up strategies on the long ratio table

In order to interpret the dialogue above in terms of Toulmin's model of argumentation, it is understood that Ceyda grouped a column of food bar and fish and compared it to another grouped column of food bar and fish in the table. She observed that the numbers in the second column were five times larger than the ones in the first column. Based on this data, she claimed that both numbers are multiplied by the same number (in order to obtain the numbers in the second column). Then, İlter grouped two columns of food bar and fish (6-18 and 12-36) and moved backwards from 12 to 6 (in the first row) and 36 to 18 (in the second row) and claimed that both numbers were divided by 2. However, since these were procedural answers, the teacher asked those students to provide warrants for those claims. Ceyda provided a warrant by referring to the times five relationship as adding groups of fish to 5 groups of food bars. After that, Elif provided backing by making a connection between the times five relationship and adding groups instead of ones. In doing so, she implicitly referred to multiplication as an abbreviated way for iterating linked composites by addition. In other words, by reasoning that way, as it is clear from Ceyda's warrant, the students were scaling the number of food bars and fish at the same time. However, there still needs some higher-order reasoning that deals with the invariant relationship between the number of food bars and fish. This type of thinking could not be considered as complete multiplicative reasoning since it does not include the invariant relationship between the number of food bars and fish. These strategies that included scaling both quantities as a short way for doing iterations were called *horizontal relationships in the HRT*. These horizontal relationships deal with the increase/decrease within the same unit. They do not involve the relationship between the number of different units, which is usually referred to as within measures ratio reasoning in the literature.

As soon as the students discovered the scaling relationships within each measure spaces, the teacher took this opportunity to provoke them to shorten the table so that they were representing two different situations involving four variables in order to make calculations easier. The teacher's goal was to encourage the students to begin to create equivalent ratios in shorter and more efficient ways with multiplication or division (Battista & Van Auken Borrow, 1995). The dialogue that took place in the classroom as the teacher encouraged students to shorten the long ratio tables is provided below:

Teacher: We don't need to write all the columns, do we? Let's say we want to find how many fish would 20 food bars feed, the number of food bars is multiplied by…how many (draws an arrow from 1 to 20 in the first row of the table as shown in Figure 4.18a below)?
Students: 20
Teacher: Then, the number of fish should be multiplied by… how many?
Students: 20
Teacher: Why 20?
Giray: Since (the number of) food bars is multiplied by 20, so should (the number of) fish.
Teacher: Then, how many fish can be fed with 20 food bars?
Giray: 60.
Teacher: We don't even have to put all the columns in between. We can just write 1 and 20 and 3 and ? in the table as we only need them (see Figure 4.18b below).

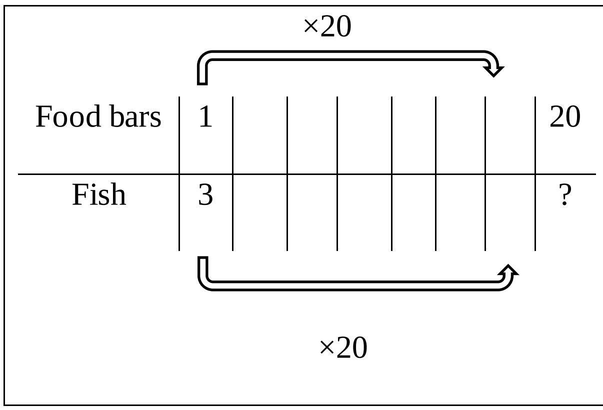


Figure 4.18a. The teacher's representation to provoke a shortened ratio table

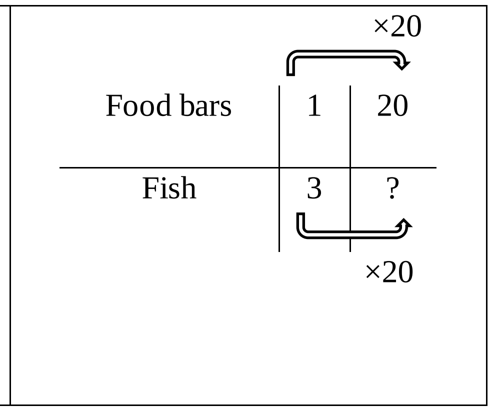


Figure 4.18b. Shortened ratio tables

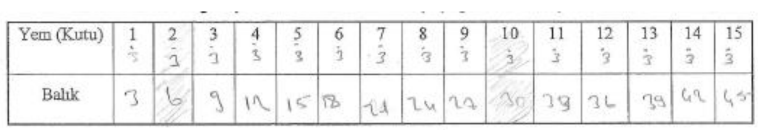
As deduced from the dialogue and the figures above, the teacher wrote down a number that was not in the previous table (i.e., 20 food bars) and asked the students to find the number of fish that could be fed with 20 food bars. She stressed that there was no need to write down the numbers in each column and asked the students to find the scale factor (even though she did not use the term) for the number of food bars and fish. Many students said all together that the number of food bars was multiplied by 20 and so the number of fish had to be multiplied by 20 as well. In the end, Giray referred to this relationship as a "times 20" relationship. This acted as data to find 60 as the number of fish that could be fed with 20 food bars, and no student challenged Giray. Moreover, the fact that the idea of scaling both quantities with the same factor shifted place from claim to data provided with initial evidence about that idea (i.e., *covarying the linked composites by a scale factor)* being taken-as-shared.

On the other hand, looking from an RME perspective, long and short ratio tables functioned as *models of* iterating the number of food bars and fish in more effective ways (i.e., scaling) as well as organizing information and making iterations one by one. This also helped the teacher to keep track of students' single and abbreviated iterations.

TAS Idea 3. Multiplicative reasoning- invariance of the multiplicative relationship between the two units as they co-vary (reasoning with between measures ratio). Throughout the first two days of the instruction, the students moved from grouping and linking the pictures of fish and food bars to making grouping and iterating with numerical values. Then, the long ratio table was introduced as a tool for organizing iterations with numbers and (abbreviated) build-up strategies. After the students discussed these relationships in long ratio tables for some time, long ratio tables were shortened to represent abbreviated build-up strategies and to make calculations in more efficient ways. On Day 3, the students worked with the same rule (i.e., 1-3) and continued to make claims about the number of food bars/fish for the given situations by scaling both values by the same factor in long and short ratio tables. These ideas appeared in data, and no warrants were provided or required. Later on, the teacher encouraged students to start thinking about the invariant multiplicative relationship between the numbers of food bars and fish, as illustrated in the following dialogue:

> *Teacher*: Is there a relationship between the number of food bars and the number of fish?
> *Seval*: Since one food bar feeds three fish, when we multiply the number of food bars with 3, we get the number of fish.
> *Teacher*: Do you think you can always get the number of fish when you multiply the number of food bars by three regardless of the numbers (number of food bars and/or fish)?
> *Seval*: Yes, the number of fish is always three times that of food bars, since one food bar feeds three fish. We can see on the table. When we multiply the number of food bars by three, it always gives the number of fish (shows her previous work

on the activity sheet by making a hand gesture to draw vertical arrows from the first row to the second row as illustrated in Figure 4.19a & Figure 4.19b below).
Teacher: Can you come to the board and show your table to everyone?
Seval: See (shows her work in Figure 4.19aa by making hand gestures as shown in Figure 4.19b)? It's always correct.
Gizem: Or we get the number of food bars when we divide the number of fish by 3.
Teacher: Do you think it always gives the (correct) result?
Gizem: Yes, when we think it from the other way around, we get the number of food bars when we divide the number of fish by 3.



Yem (Kutu)	1 ·3	2 ·3	3 ·3	4 ·3	5 ·3	6 ·3	7 ·3	8 ·3	9 ·3	10 ·3	11 ·3	12 ·3	13 ·3	14 ·3	15 ·3
Balık	3	6	9	12	15	18	21	24	27	30	33	36	39	42	45

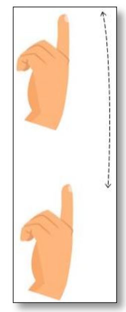


Figure 4.19a. The invariant times three relationship on the long ratio table

Figure 4.19b. Using hand gestures to show the vertical times three relationship on the long ratio table

As deduced from the dialogue and the associated figures above, when asked to explore the relationship between the number of food bars and fish, Seval claimed that when the number of food bars is multiplied by three, the result gives the number of fish. This implied an operational relationship. When she was asked to make a conjecture about whether or not this relationship always holds, she claimed that the number of fish is always three times that of food bars. This implied a more conceptual relationship than the previous one. Then, she provided a warrant by showing her calculations in the long ratio table and the vertical times three relationship by hand gestures. Then, Gizem made a claim about the reciprocal "divided by three" relationship between the number of fish and food bars.

In the next parts of the first task, the students worked with shortened ratio tables, including different relationships between the number of food bars and fish (i.e., 2-4 and 2-3). In this process, they found the missing values in short ratio tables by exploring both the horizontal and vertical relationships without providing warrants/backings. Hence, they made claims about the missing values in the tables by using the invariant relationship between the number of food bars and fish, which was a previous claim, in their warrants. This indicated that the invariant relationship between the number of food bars and fish was taken-as-shared at this point. Still, there is further evidence from the following classes that would support this conclusion.

TAS Idea 4. Conceptualizing the invariant relationship between two linked composites as unit rate and using it as a tool/benchmark/anchor for finding missing values. Up to this point, the number of food bars in the rule was one (e.g., 1-3). This automatically made the students reason with the number of fish that can be fed with one food bar (unit rate). In the second period of Day 3, the instructional sequence continued with different rules that required constructing different types of links between the number of food bars and fish. The first rule that did not include a unit rate was "two food bars for four fish," which was intentionally chosen for the students to make sense of unit rate. While some students used the given rule in order to find missing values for different situations in similar ways as they did previously, some of the students claimed that they could change the rule to "one food bar for two fish." Below is a conversation in which multiple students negotiated about changing the rule while the question was asking how many food bars were needed to feed 12 fish (when the rule was two food bars for four fish):

> *Teacher*: How many food bars do we need to feed 12 fish?
> *Sezin:* 6 food bars are needed. Because I divided 12 by 2.
> *Teacher*: Why did you divide 12 by 2?
> *Sezin*: Because four fish can be fed with two food bars.
> *Berk*: Then, you have to divide it by 4, right?
> *Teacher*: Why do you think you have to divide it by 4?
> *Berk*: (Writes and explains his solution on the board by drawing 12 circles to represent 12 fish and groups each four fish by a bigger circle) I grouped the fish by

4s and obtained three groups [of 4 fish]. I know that I need two food bars for each group. So, I multiplied three by two and obtained 6.
Aylin: We changed the rule. If two food bars feed four fish, then, one food bar feeds two fish. So, we divided each value by 2.
Teacher: What does everyone think about this?
Gizem: It is easily seen with the pictures (links 1 food bar with two fish with pictures). It is easily seen that one food bar feeds two fish. It is easier to use this rule. We can group fish by 2s [for the following questions] after we change the rule. (See Figure 4.20 below)
Berk: Oh, OK. I see it now.

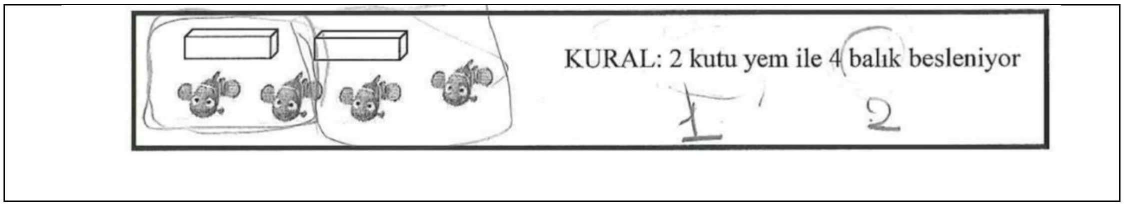


Figure 4.20. Changing the rule while preserving the link between the number of food bars and fish

In order to interpret this dialogue in terms of the Toulmin analysis scheme, it is considered that Sezin claimed that six food bars were needed for 12 fish by drawing on her data that involved directly dividing the number of fish by two. This was the first instance that included operating with values different than the ones in the given rule. Therefore, Berk challenged her with his own warrant (12 divided by 4) to solve the question using the given rule as they did previously. Even though Berk provided a warrant for his reasoning, no student focused on his warrant since it was a previously taken-as-shared idea and that Berk's solution functioned as a warrant for his own strategy than a challenge to Sezin's strategy. Thus, Aylin joined in the conversation with backing for Sezin's claim by referring to the equivalence of the two rules: "2 food bars-4 fish" and "1 food bar-2 fish." Then, Gizem came in by using the pictures in order to show that both rules are grounded on the same link between the number of food bars and fish, as shown in Figure 4.20. This acted as a further backing for Sezin's data and claim. Therefore, these students provided warrants related to changing the rule while preserving the link and the invariant relationship between the number of food bars and fish. This gave evidence about the strength of the first and second taken-as-shared ideas. On the other hand, even though she did not use the term, Gizem also referred to the concept of unit rate and how using unit rate makes calculations easier. Hence, this was the first instance in which the concept of unit rate was explicitly used as a tool/benchmark/anchor for finding missing values. For finding the missing values in all of the remaining questions related to the rule "2 food bars for four fish," multiple students used the altered data in the data and/or warrants that they provided.

Following the rule "2 food bar for four fish", the students were given a new rule "2 food bars for six fish" in a short ratio table in order to support their use of short ratio tables and interpretations of horizontal and vertical relationships in them. When the number of food bars to feed 15 fish was asked, interpretation of the "vertical times three relationship" as unit rate emerged in the classroom as illustrated in the following dialogue:

Teacher: How did you find the missing value in this short table?
Elif: Since there is three times the number of fish as the number of food bars (draws a vertical arrow from 2 to 6 and writes ×3 next to it, see Figure 4.21), and there is "times three." So, I thought like "3 multiplied by what makes 15?" Then, I divided 15 by three and found 5.
Murat: I found a different way. I didn't think two food bars for six fish but thought one food bar for three fish (draws a new column and writes 1 and 3 in it, see Figure 4.21 below). It was easier to find like this. I solved all the (remaining) questions like this.
Teacher: Do you see a connection between this 3 (points to the ×3 next to the vertical arrow) and that (1 food bar for three fish)?
Murat: Yes, when I divide six by 3, I find the number of fish that can be fed with a food bar.

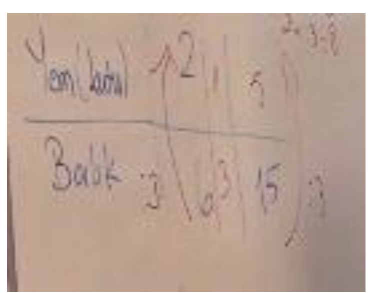


Figure 4.21. Vertical relationships on the shortened ratio table

As deduced from the dialogue above, Elif claimed that five food bars could feed 15 fish. For this claim, she provided data, including the application of the "vertical times three relationship" between the number of food bars and fish with a hand gesture to the second value. Then, Murat proposed different data for the same claim by referring to the unit rate and how it made calculations easier. When asked to interpret the relationship between this unit rate and “vertical times three relationship” (or divided by 3), he connected these two by implying that dividing by three gives the unit rate. Thus, further evidence was obtained regarding the interpretation of the invariant multiplicative relationship since it appeared in the data for another claim.

As another variation of the rule, the instructional sequence continued with the values that did not form an integer ratio (i.e., two food bars for three fish) in order to strengthen the students’ making sense of the concepts of iteration, covariation, and multiplicative reasoning. Throughout the activity, multiple students made claims about missing values of the number of fish/food bars by using (i) build-up strategies (i.e., adding up by 2s and 3s on the long ratio table); (ii) abbreviated build-up strategies (i.e., scaling the number of food bars and fish with the same factor); (iii) numerical grouping and iterations (i.e., dividing the number of food bars (or fish) by 2 (or 3) and multiplying the result by 3 (or 2)); and (iv) using the invariant multiplicative relationship between the two units (i.e., the number of fish is always 1.5 times of the number of food bars, which is also the unit rate) in their data/warrants. A sample classroom dialogue that included these TAS ideas in the students’ data/warrants is given below for illustration:

> *Emre*: (Fills in the long ratio table by iterating the link- 2 food bars for three fish) Since 2 bars feed three fish, this (first row) increases by 2s (draws horizontal curves between each value and makes a corresponding hand gesture). This (the second row) increases by 3s (draws horizontal curves between each value and makes a corresponding hand gesture, see Figure 21a below). I can continue by skip counting.
> *Teacher*: Anybody thinking differently?
> *Ceyda*: I did like this. Let’s say we would like to find how many fish could be fed with 4 bars. Since the number of food bars is multiplied by two (makes a hand gesture for drawing horizontal arrows from 2 in the first row and moving it until 4

in the same row), so should the number of fish (makes hand gesture for drawing horizontal curves starting from 3 in the second row and moving it until 6 in the same row). If the number of food bars is multiplied by 3, so should the number of fish.

Teacher: OK. How did you find the answer to this question? (How many fish can be fed with 16 food bars?)

Tolga: (Looks at the table that Emre filled in) We can look at the table. Sixteen food bars feed 24 fish.

Sinem: We found (the answer) in a different way. When we divide 16 by 2, the result is 8. When we add 16 and 8, we find 24.

Teacher: Can you explain your strategy on the board?

Sinem: 16 divided by 2 is 8. Then, I add the result (on to 16). It is valid for all the other numbers.

Teacher: Anybody understood her strategy? I didn't understand it.

Researcher: (Intervenes by talking to the teacher in private and says: She finds three over 2 of the number of food bars by adding its half on to itself, and the teacher agrees.)

Teacher: What happens when you divide by two and then add the result (to the number itself)?

Sinem: That means, I find the half and add on to itself.

Berk: What is it that you found, then?

Sinem: Then, I found three over 2 of it (the number of food bars).

Ceyda: That means you found 1.5 times of it.

Sinem: Yes, that means I can find the number of fish when I divide (the number of food bars) by two and multiply the result by 3.

Teacher: How can you relate it with the rule?

Ceyda: 3 fish can be fed with two food bars in the rule. That means the number of fish is always three over 2 of the number of food bars. This is always true.

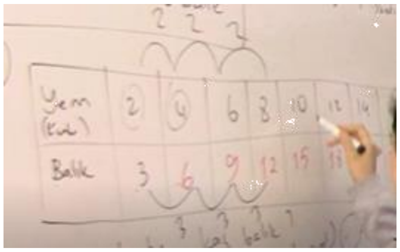
Teacher: Anybody who found a different way?

Ozan: We divided 16 by 2. By doing that, I grouped 16 by 2s and found that there are eight groups of 2. We know that three fish can be fed with each group. Hence, I multiplied 8 with three and found 24.

Seval: (Shows her work on the activity sheet in front of the class, see Figure 4.22b below) We found the answer in a different way. If two food bars can feed three fish, then, one food bar can feed 1.5 fish. Therefore, we multiplied the number of food bars by 1.5 to find the number of fish in each situation (points her hand to the number 2 in the first row and moves vertically until the number 3 in the second row, through which she traced the vertical arrows drawn on her activity sheet, see Figure 4.22b below).

Teacher: Can you find a relationship between the two strategies?

Seval: Ozan divided [the number of food bars] by two and then multiplied (the result) by 3. This is the same thing as multiplying by 1.5.



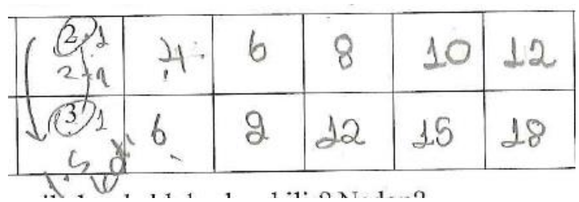


Figure 4.22a. Building up by 2s and 3s in the long ratio table

Figure 4.22b. Using the unit rate to find missing values in the long ratio table

In order to get a sense of the dialogue in Toulmin's model of argumentation, it is inferred that Emre provided data to make claims about the values that would go in the table, which included building up by 2s and 3s (i.e., TAS Idea 2). Then, Ceyda worked with equal scale factors within the same measure spaces to find missing values. Then, Sinem made a claim about the relationship between 16 and 24, which was related to adding the half of 16 on to itself and obtaining the number of the fish that would be fed with 16 bars. She further claimed that that relationship holds for all the other numbers as well. Since this was a procedural claim, which was not anticipated prior to the implementation, the teacher was not sure about what Sinem was trying to tell. As the researcher in the classroom understood that the teacher was struggling to understand her claim, she talked to the teacher in private and stated that Sinem was calculating $\frac{3}{2}$ of 16 (the number of food bars) by adding its half on to itself. Then, the teacher asked Sinem what those operations meant. She provided a warrant by expressing that she added half of 16 on to itself, which was eventually explained as finding $\frac{3}{2}$ of 16. Ceyda joined in the discussion by stressing that that would mean finding 1.5 times of 16, and this functioned as a backing to Sinem's claims. Then, Sinem agreed to Ceyda's warrant and made a conjecture about how to find the number of fish when the number of food bars was known (i.e., divide by two and multiply the result by 3). Upon this conjecture, the teacher asked the students to make a connection between Sinem's conjecture and the given rule, which resulted in Ceyda's folding back to the invariant relationship between the number of food bars and fish (i.e., the number of fish is always three over 2 of the number of food bars- TAS Idea 3).

After it was evident that the confusion was resolved, the teacher went back to the problem and asked if anyone could suggest an alternative method. Ozan claimed that 24 fish could be fed with 16 food bars based on his data that included grouping 16 food bars in 2s and multiplying the result by 3 to link each group with groups of 3 fish and iterate. This was a previous claim (i.e., TAS idea 1) that appeared now in data for a subsequent claim. Seval provided different data regarding the "invariant 1.5 times relationship" (i.e., unit rate) between the number of food bars and fish. This was also related to the previous claims (i.e., TAS idea 3 & TAS idea 4). Furthermore, Seval expressed the relationship between grouping and iterating, and the unit rate, which showed the strength of the previous ideas being taken-as-shared.

Therefore, it was evident that the first four ideas were taken-as-shared among the classroom community at the end of the third day of the instruction. This allowed the teacher to alter the symbolization to introduce the symbolic representations of ratio and proportion. Hence, on Day 4, the teacher started with writing a shortened ratio table including the "2-3 relationship" on the board, and had the students summarize the horizontal and vertical relationships in the table. While the short ratio table was on the board, she suggested removing some of the borders of the table to obtain the traditional ratio and proportion representations. Below is a classroom dialogue that illustrates this process of introduction of the symbolic representations of ratio and proportion:

> *Teacher*: We have been using short ratio tables to find the number of food bars and fish in different situations. What we are going to do now is quite fancy. Musa said, "the ratio of the number of food bars to fish" the other day, remember?
> Students: Yes.
> *Teacher*: How can we write the ratio of (the number of) food bars to fish in this situation?
> *Musa*: It is two over three.
> *Teacher*: Yes, we need to write it in the fractional form. So, I wrote two over three here. That means, I erased some of the borders of the table. And I am going to

remove the other border here (erases all of the vertical lines and some of the horizontal in the short ratio table). What do I need to write?
Seda: 16 over question mark.
Teacher: OK. First, let's find the question mark. What number is it?
Seda: It's 24. We found it before.
Teacher: Good. Then, I am writing two over three here and 16 over 24 here (writes $\frac{2}{3}$. and $\frac{16}{24}$ next to each other). Which one is bigger?
Student: 16 over 24.
(Several students say "no" in the background)
Teacher: Selim, which one is bigger?
Selim: They are the same.
Teacher: We don't say they are the same, right? Can we say that they are equivalent?
Selim: Yes, they are equivalent.
Teacher: Why did you think they are equivalent?
Selim: They are reduced to the same thing when simplified.
Merve: They are just like equivalent fractions, then.
Ceyda: Actually, do we need to simplify? We found it by multiplying (both values) with the same number, didn't we?
Teacher: What do you mean, Ceyda? Can you explain it in more detail?
Ceyda: We looked at the relationships in the table. Since the number of food bars is multiplied by 8, we also multiplied the number of fish by 8. That means, we expanded the first fraction.
Teacher: So, can I put an equal sign between these two (ratios)? (Puts the equal sign "=" in between the two ratios $\frac{2}{3}$ and $\frac{16}{24}$ to obtain the symbolic representation of proportion $\frac{2}{3} = \frac{16}{24}$, See Figure 4.23 below).
Students: (All together) Yes.
Teacher: So, we will say that two ratios are proportional when they are equivalent and write the symbolic representation as equality of two ratios like this (points to the symbolic representation on the board). We can show all the relationships in the tables that we explored from the beginning by establishing proportions. We call each of the comparisons here (points to the ratios $\frac{2}{3}$ and $\frac{16}{24}$) as "ratios." You learned last year what ratio means.
Elif: The ratio between two quantities.
Teacher: Yes, what does "the ratio between two quantities" mean? (No student could define ratio). We have explored the relationships between (the number of) food bars and (the number of) fish since the beginning of this task. What kinds of relationships have we explored?
Ozan: We looked at how many times it is multipled.
Ali: We explored the "times" relationship between (the number of) food bars and (the number of) fish. Like "times 3", "times 1.5" relationships.
Teacher: Yes. We compared the quantities multiplicatively. We name these multiplicative comparisons like this as "ratio." Now, I have a question for you. Can we talk about the horizontal and vertical relationships here in the proportion? (draws vertical and horizontal arrows in the proportion representation, see Figure 4.23).

Aylin: Yes.
Teacher: What does this (horizontal) relationship mean (points to the horizontal arrow that she drew from 2 to 16, and then, 3 to 24, See Figure 4.23).
Aylin: It means that the number of food bars is scaled by 8.
Teacher: OK. We will call this "times 8" relationship scale factor since we scale both quantities with 8. What does the vertical relationship mean (points to the vertical arrow that she drew from 2 to 3 and then 16 to 24, See Figure 4.23)?
Merve: It means that the number of fish is times 1.5 of the number of food bars.
Teacher: This "vertical times 1.5 relationship" or "two over three relationship," you can call it the constant of proportionality.

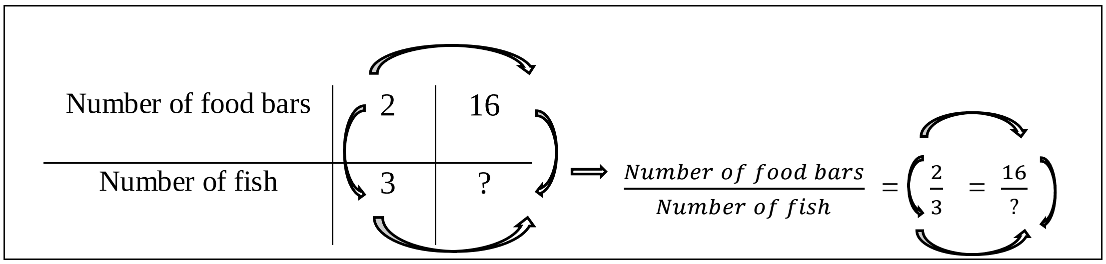


Figure 4.23. Using shortened ratio tables to introduce the symbolic representations of ratio and proportion (Day 4)

Therefore, as deduced from the dialogue above, the fractional representation was introduced as the formal representation of ratio based on a previous classroom discussion. In addition, the equality of two ratios was introduced as the formal representation of proportion by removing some of the borders of the short ratio table for the first time on Day 4. Moreover, the "horizontal relationship" in the short ratio table, which the students discovered as a short way for making iterations with linked composites, was introduced as the scale factor. On the other hand, the "vertical relationship," which was referred to as the invariant multiplicative relationship between the two linked composites, was introduced as the constant of proportionality for the first time in the instruction. Therefore, this implied an incipient shift from a *model of* to *model for* since the students were introduced the conventional proportion representation as a tool for solving a variety of proportional problems. This is due to the fact that ratio tables were used as a *model of* an organized way of linking composite units and iterating linked composites at the onset of the instruction; yet, they became a *model for* structuring symbolic representations of ratios and proportions.

TAS Idea 5: Reasoning with ratio tables and proportion representation to extend covariation and invariation to the relationship between parts and their whole. On the first four days of the instruction, the students engaged in the fish-food bar context in which they made iterations, reasoned about covariation and invariation by using pictures and long and short ratio tables. Eventually, the students formalized these explorations into formal representations of ratio and proportion. On Day 5, a new task, including situations with different ratio language usages in part-whole contexts, was introduced. Throughout the activity, the students employed their previous strategies for solving the problems given in the new part-whole contexts.

In the first question, it was given that “for every three students who commute to school by bus, there are seven students who walk to school," which included the use of "per language" as a type of informal ratio language. The students were required to find the numbers of students in each group and the total number of students for different situations. Sezin and Faruk solved the first problem that asked the number of students who walked to school on the board with different strategies when it was given that 45 students took the school bus. Particularly, Faruk drew a short ratio table, wrote three and seven in the cells in the first column, and said, "If 45 corresponds to three in the table, we need to find what corresponds to seven. Since [the number of] students who take the school bus is scaled by 15 [the number of], [the number of] students who walk to school should also be scaled by 15." Based on this data, he claimed that the answer was 105, as seen in Figure 4.24 below. No one challenged Faruk, and Sezin took the floor and said that she wanted to provide an alternative solution. This showed that she agreed upon Faruk’s solution and wanted to add on it. She divided 45 by three and multiplied the result by seven, which was related to grouping and iterating. No one challenged Faruk and Sezin and nor asked for warrants/backings for this question, which gave evidence that the ideas regarding linking and iterating, and covariance (i.e., TAS idea 1 & TAS idea 2) were taken-as-shared among the students.

Ebru implemented a survey in her school and saw that there were 3 students who commute to school by bus for every 7 students who walk to school. If there are 45 students who ride the bus, how many students walk to school?

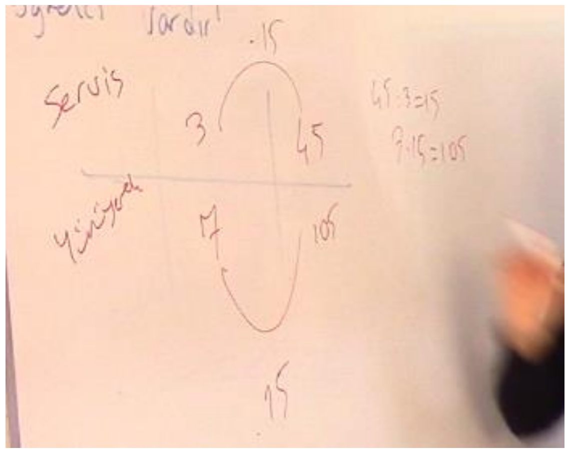


Figure 4.24. Iterating linked composites by scaling or abbreviated build-up (i.e., pre-multiplicative reasoning) in part-whole contexts

Even though the fish-food bar context did not involve a relationship between parts of a whole, the students were able to draw on their experiences of grouping, linking, and iterating in the fish-food bar context. Moreover, they did not require any warrants/backings for this question that included a part-part relationship. However, when the next question included only the number of the total students in the school as 120 and asked for the number of students who walked to school and took the bus, that was not the case. To answer this question, Giray claimed that the number of students who took the school bus was 36, and the number of students who walked to school was 84. When asked for the corresponding data and warrant, he said that he did trial and error in such a way that he multiplied three and seven by 11 (random multiplier) at first, found the corresponding numbers as 33 and 77, added these numbers, and found the total number of students in the school as 110. He continued to explain his reasoning that he had to try a bigger multiplier since the value should have been bigger. For the second trial, he multiplied three and seven by 12 and found the corresponding numbers as 36 and 84, and the total number of students as 120. Therefore, he only used the link between the parts and iterated these linked composites by a scale factor, but did not think about the link between the part(s) and the whole. In other words, there was no evidence of extending his reasoning about covariation and invariation to the relationships between a whole and its parts. Thus, the teacher asked Giray and the rest of the students in the class to find a different way other than trial and error. Upon this request, Berk proposed a different method by drawing a table that included an extra row with a value of 10, representing the total number of students. This solution can be seen in Figure 4.25 below, and the associated explanations regarding his reasoning can be understood from the following excerpt.

Ebru implemented a survey in her school and saw that there were 3 students who commute to school by bus for every 7 students who walk to school. If there are 120 students in this school, how many students ride the bus?

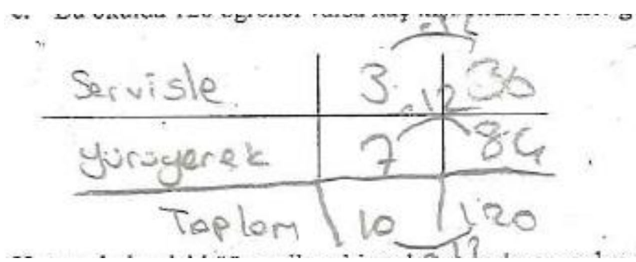


Figure 4.25 Representing the whole in terms of the link between its parts

Berk: We inserted an extra row for the total number of students in the school and wrote 3, 7, and 10 in the first cells, respectively. We wrote 120 next to 10 since the total number of students was given as 120. Since the total number of students is multiplied [scaled] by 12, so should the number of students who walk and take the bus (using hand gestures for the horizontal relationships). Hence, there are 36 students that commute to school by bus and 84 students who walk to school.
Emre: Where did ten come from?
Berk: There are as many students in the school as the sum of the number of students who walk to school and who take the school bus. Hence, we add three and seven. The corresponding number for the total number of students is 10.
Teacher: Why did you add three and seven Berk?

Berk: It is given in the smallest generalization in the question. For every seven students that walk to school, three students take the bus. That means, there are ten students in the general form.
Teacher: Did you understand where ten came from, Emre?
Emre: Yes, I did.
Teacher: Is there anyone who didn't understand where ten came from?
Students: No.
Teacher: Is there anyone who wants to explain it differently?
Seval: There are seven students who walk to school for every three students who take the school bus. That means that out of every ten students, seven students walk to school, and three students take the school bus. Therefore, the corresponding number for the total number of students is 10.
Teacher: Yes, this brings us to the next question.

As deduced from the dialogue above, when Berk added an extra value (i.e., 10) corresponding to the total number of students, Emre required him to clarify his data regarding where that ten came from. It was the first time that the students were required to explore the relationships between the whole and its parts and to infer a value for the whole by using the values that represented the link between the two parts of a whole. When Berk clarified his data by referring to the part-whole relationship in the given information, the students accepted it. Then, Seval expanded the discussion by opening up a different use of an informal ratio language that focused on the relationship between the whole and its parts. In the following questions, no one required a warrant or a backing for the inferred value of the total number of students. Moreover, multiple students used this idea in their data/warrants to find the missing values in the same part-whole context.

The second part of the task gave the result of a survey as “five out of every eight students have at least one sibling, and others don’t have any siblings.” Therefore, this information included the use of informal ratio language that is focused on part-whole relationships. The first question in this part asked the number of students who didn’t have any siblings if there were 65 students who had at least one sibling. To answer this question, Erdem drew a ratio table including three rows and named those rows as siblings, no siblings, and total. Moreover, he wrote five, three, and eight in the corresponding cells, respectively, as Berk did in the previous question. He also wrote 65 next to five in the third column to link the number of students who had at least one sibling. He argued that the number of students who had at least one sibling was scaled by 13, so the number of students who did not have any siblings also had to be scaled by 13 by using the horizontal relationships in the table. Then, he claimed that the number of students who did not have any siblings was 39. No student challenged Erdem, which provided initial evidence about the idea of extending covariation to part-whole contexts being taken-as-shared.

Then, the teacher selected Mehmet to express the answer he found since she observed that he made a mistake while he was working with his peers in small groups. Mehmet said that he found the answer as 104. The teacher asked him to explain his solution on the board. Mehmet drew a (short) table and named the cells as sibling and no siblings. He wrote five and eight next to these cells, as illustrated in Figure 4.26 below and explained his reasoning as in the excerpt that follows:

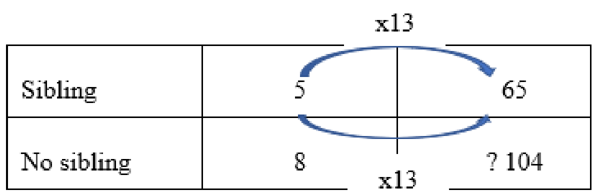


Figure 4.26. Establishing an incorrect link between the parts

Mehmet: I wrote five and eight in the table since five out of every eight students had siblings. Since the (number of) students with siblings is multiplied [scaled] by 13 (points to the horizontal arrow in the first row of the table), so should the students with no siblings (points to the horizontal arrow in the second row of the table). So, I found the answer as 104.
Students: (Many students show disagreement with Musa by shaking their heads side to side) No, that is not true.
Musa: It can't be 104 anyway. The number of students who don't have siblings should be less than the number of students who have siblings.
Teacher: Why do you think it should be less than that?
Musa: There are three students who don't have any siblings for every five students who have siblings. That means there are fewer students who don't have any siblings than students who have.
Seval: Yes, it should be less than 65, since it is given that there are 65 students who have siblings.
Mehmet: Ok. I got it. I read it wrong. Eight is for [the number of] all students.

The analysis of the dialogue above gave further evidence that the first two ideas were taken-as-shared for a couple of reasons. First, when Mehmet established an incorrect link between the parts and the whole, many students disagreed with him. Moreover, Musa and Seval rebutted him by providing an argument related to the link between the number of students who had siblings and who didn't (i.e., the parts of the whole). Moreover, even though Mehmet assigned incorrect values for representing the parts, he was able to scale the parts within the same measure spaces, and none of the students rebutted him about this scaling. Besides, considering the whole process, it was deduced that the multiple students agreed that the number of students who had no siblings could be represented by three. This was because none of the students were challenged or requested any warrant or backing regarding this link and the covariation between the whole and its parts in the following sub-questions. Thus, more salient evidence about these three ideas' (TAS Idea 1 & TAS Idea 2) being taken-as-shared was obtained at this point.

The following question asked the number of students who had at least one sibling if the total number of students in the school was 168 (the link is the same, that is, five out of every eight students have at least one sibling, and others don't have any siblings). Ceyda drew a short table, including two variables for the number of students with sibling(s) and the total number of students. She proposed to draw the table vertically as different from the previous days, as demonstrated in Figure 4.27 below.

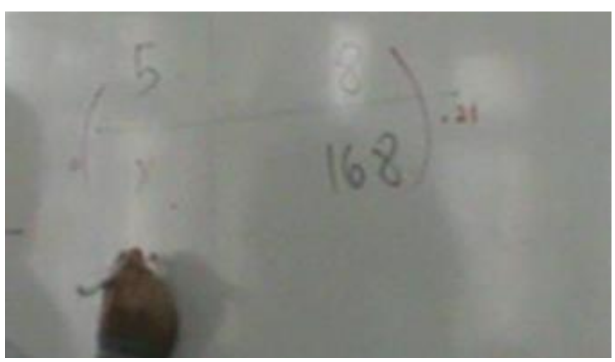


Figure 4.27. The emergence of a vertical ratio table (Day 5)

Ceyda told that since the total number of students is scaled by 21, so should the number of students who had siblings by using hand gestures to show the vertical relationship. She said that it did not matter if the table was drawn horizontally or vertically. However, this time the vertical relationship was related to scaling of the values within the same measure space, not the invariant relationship between the values in different measure spaces. Hence, she emphasized that the horizontal relationship in the horizontal table corresponded to the vertical relationship in the vertical table. So, the use of the vertical ratio table (VRT) emerged in the classroom naturally on Day 5, and a discussion about the horizontal and vertical relationships in vertical and horizontal ratio tables was initiated.

In the second period of the class on Day 5, the instructional sequence posed a question in which it was given that out of every nine students, four support BJK (soccer team 1-ST1), three support GS (soccer team 2-ST2), and two support FB (soccer team 3-ST3). Therefore, this question included part-whole relationships wherein there were three parts of a whole. The first sub-question asked the number of people who support GS and FB when it was given that there were 44 BJK supporters. As an answer to this question, Hande drew the corresponding horizontal table, including three parts and a whole, and used the horizontal relationships in this table. In some of the following sub-questions, either a value for one of the parts was given, and the other parts and the whole were asked, or a value for the whole was given, and the corresponding values for all of the parts were asked. The students drew horizontal ratio tables and used the horizontal relationships (i.e., scaling within measure spaces) in those tables in their data to come up with claims to answer the questions, as illustrated in Figure 4.28 below.

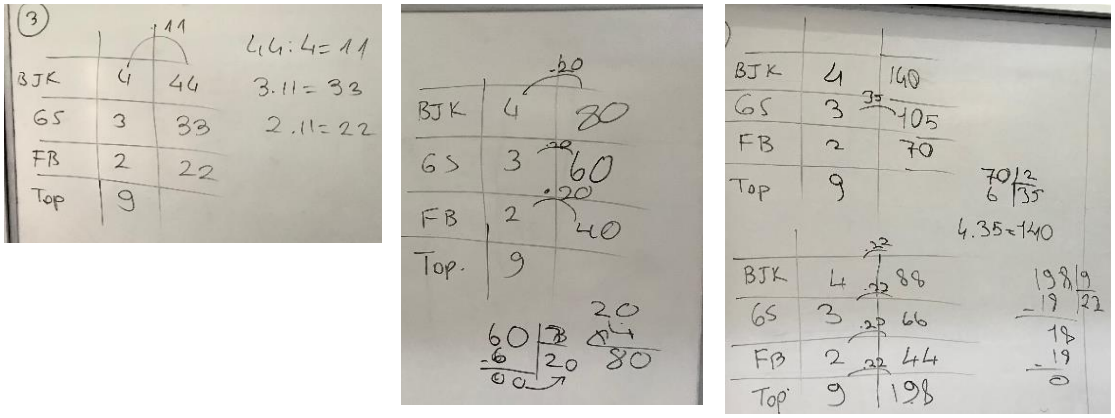


Figure 4.28. Horizontal relationships in the short ratio tables including three parts of a whole

As seen in Figure 4.28 above, the students made use of the horizontal relationships in the short ratio tables to find parts and/or whole, and none of these strategies were challenged. This provided further evidence that TAS Idea 1 and TAS Idea 2 were taken-as-shared. On the other hand, this also gave further evidence about the students' extending their understanding of covariation to part-whole contexts. This allowed the teacher to encourage the students to reason about the invariant multiplicative relationship in this part-whole context. Thus, she asked the classroom if there was a relationship between the numbers of BJK supporters and FB supporters. Hande expressed that the number of BJK supporters is twice as much as the number of FB supporters in each situation. The teacher then asked if they were able to interpret such relationships between the other quantities. Ceyda claimed that the number of GS supporters is always $\frac{1}{3}$ of the total number of students. Musa claimed that the number of FB supporters is $\frac{2}{3}$ of the number of GS supporters. Erdem claimed that the number of FB supporters is $\frac{1}{2}$ of the number of BJK supporters. Seval claimed that the number of BJK supporters is $\frac{4}{9}$ of the total number of the students. Therefore, all of these students were able to abstract the relationships between the pairs of parts or between a specific part and the whole. Thompson (1994) refers to those abstractions as interiorized ratios or rates while he refers to a multiplicative comparison of two particular values as a ratio. Hence, at this point, the students abstracted their understandings of ratios between fixed values to arrive at interiorized ratios or rates in this part-whole context.

Even though this was the first time that the students reasoned about the invariant multiplicative relationships between parts and the whole and its parts, there is no evidence that the students needed warrants or backings for these relationships. However, a later discussion that took place on the same day might give an idea about this issue. After Tunay answered the question related to finding the number of BJK supporters when it was given that "there were 60 GS supporters," based on the horizontal relationships (see Figure 4.29), the teacher challenged her to reason with the invariant multiplicative relationship between the two parts. Moreover, when she used this invariant relationship to find a missing value, another student challenged her as well. Below is a classroom discussion that took place regarding this process:

x20

$$\frac{GS}{BJK} = \frac{3}{4} = \frac{60}{}$$

x20

Figure 4.29. Using the symbolic representation of proportion to find missing values in part-whole contexts

Tunay: Since the number of the GS supporters is scaled by 20, so should the number of BJK supporters. The answer is 4×20 = 80.
Teacher: Can you find the answer by using the relationship between the number of GS supporters and BJK supporters, which you mentioned a short time ago?
Tunay: Well, we didn't find the relationship between GS and BJK. But I can find it. [The number of] GS supporters is $\frac{3}{4}$ of [the number of] BJK supporters.
Ahmet: Yes.

Tunay: Then, to find [the number of] BJK supporters, we need to divide 60 by three and multiply (the result) by four.
Ahmet: Or should we divide it by four and multiply (the result) by three?
Tunay: No. GS is $\frac{3}{4}$ of BJK, so GS is the smaller part. We need to find the whole when $\frac{3}{4}$ (of that whole) is 60.
Ahmet: OK, like we did with fractions.
Tunay: Yes. So, the number of BJK supporters is 60:3=20 and 20×4=80.

As inferred from the excerpt above, Tunay wrote the symbolic representation of proportion to find the missing value and used the horizontal relationships in the proportion. When the teacher asked her to use the (invariant) relationship between these parts to find that value, she was able to state the invariant multiplicative relationship between the two parts. However, when she used this relationship to find a missing value, Ahmet required her to provide a warrant regarding the multiplicative relationship between the two variables. This resulted in Tunay's making a connection to fractions. Therefore, even though no student required warrants for the invariant multiplicative relationship between the parts, Tunay was challenged to provide a warrant for using this relationship to find another part when one of the parts was known. Hence, it could be that the students were able to draw on the previously established idea regarding the invariant multiplicative relationship between the quantities from the fish food bar context to reason in the part-whole context.

Nevertheless, when this invariant multiplicative relationship (i.e., unit rate) was used as an anchor to find the missing value, it was challenged. Even so, it wasn’t for sure whether or not extending the invariance to the part-whole contexts was taken-as-shared in the classroom community at that time. However, it became more salient on the ensuing days (e.g., Day 6, Day 8, and Day 13) that students were able to extend their understanding of invariance to the part-whole contexts. For instance, on Day 13, multiple students included the invariant multiplicative relationships between the amounts of water and orange juice in their data/warrants for making claims about comparing the tastes of mixtures.

To sum up, during the first five days of the instruction, the students started to group objects to form composite units, linked those composite units, and made sense of the covariation between those by making iterations with linked composites by using numerical, pictorial and tabular representations. These iterations evolved from building up/down by ones to abbreviated build-up/down strategies (i.e., scaling up/down by the same factor). Then, the invariant multiplicative relationship between the linked quantities was interpreted and conceptualized as unit rate. In addition, it was used as a tool/benchmark/anchor for finding missing values. Lastly, these understandings of covariance and invariance were reconceptualized in several part-whole contexts. These mathematical ideas felt like being related to the general activity of reasoning with pictures and tables to find missing values as the students worked collectively. Therefore, these five mathematical ideas were put together and organized around the common activity of reasoning with pictures and tables to find missing values as they emerged and became taken-as-shared.

4.2. CMP 2. Reasoning with tables and symbols to write and solve proportions

On Day 6, the instruction focused on interpreting the relationships between the values in the proportion representation and making inferences regarding covariation and invariation among those values. As the students engaged in discussions regarding those, two ideas became taken-as-shared among the classroom community:

- Structuring ratios and proportions multiplicatively and extending covariance and invariance to symbolic proportion representation,
- Determining proportionality by covariational and multiplicative reasoning.

TAS Idea 1. Structuring ratios and proportions multiplicatively and extending covariance and invariance to symbolic proportion representation. The teacher started Day 6 by asking Berk to count the number of students in the class and to write the ratio of the number of girls to the number of boys. Berk stated that there were 12 girls and 15 boys in the class and wrote the corresponding ratio as $\frac{12}{15}$ on the board. Therefore, this was a claim regarding structuring ratios multiplicatively. Then, the teacher asked the class if they could express the same ratio in different ways. After she let the students work for a while in small groups, she took an answer from Emre, and the following dialogue took place in the discussion:

> *Emre*: We made a table and found the other values, 24-30, 36-45, and so on. However, the teacher asked us (in the small group discussion) if we could represent $\frac{12}{15}$ with smaller numbers. So, we extended the table backward. We couldn't divide both (numbers) by 2, so we tried 3. Then, we divided both (numbers) by three and got $\frac{4}{5}$.
> *Teacher*: Did you try to go backwards again?
> *Emre*: Yes, you asked us to go backwards again, but we couldn't divide $\frac{4}{5}$ by any other number.
> *Teacher*: What do you all think about Emre's strategy?
> *Ozan*: We found the same thing, but we did not draw a table. We wrote $\frac{12}{15}$ is equal to $\frac{24}{30}$. This is equal to $\frac{36}{45}$ and so on (writes these proportions as an equivalence class on the board, i.e., $\frac{12}{15} = \frac{24}{30} = \frac{36}{45} = \frac{48}{60}$). That means we expanded the given ratio as we do with fractions. Then, when the teacher asked us (in small group discussion) if there could be smaller numbers, we reduced it to $\frac{4}{5}$ (adds $\frac{4}{5}$ to the left to obtain the equivalence class $\frac{4}{5} = \frac{12}{15} = \frac{24}{30} = \frac{36}{45} = \frac{48}{60}$). We couldn't reduce it anymore, either.
> *Teacher*: Anyone who could find a smaller ratio?
> Students: No.
> *Teacher*: So, we call the most simplified ratio as the base ratio. Then, what is the base ratio for the number of girls to the number of boys?
> Students: $\frac{4}{5}$.

As understood from the dialogue above, Emre provided data related to drawing a ratio table and making iterations to make claims about representing the given ratio in different ways. Ozan made these iterations in the symbolic representation of proportion to obtain an equivalence class of ratios by making references to expanding and reducing fractions in his warrants. Eventually, the teacher introduced the term "base ratio" as the most simplified ratio. The argumentation analysis of the dialogue above is summarized in the following argumentation schema illustrated in Figure 4.30 below.

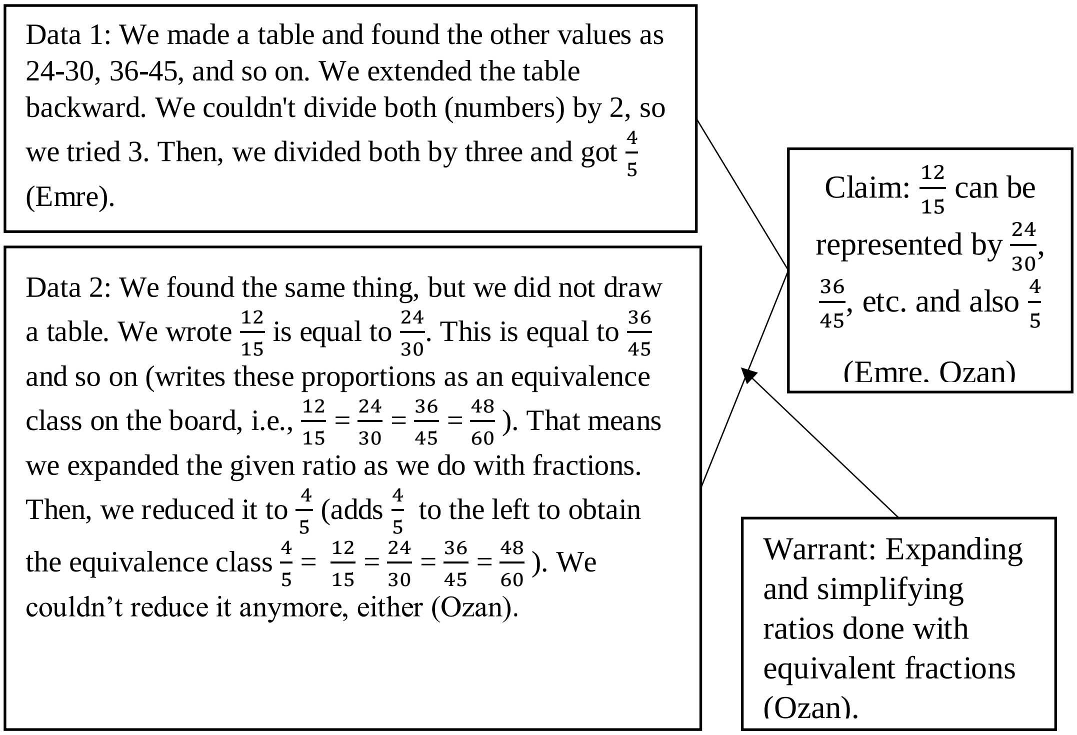


Figure 4.30. Argumentation schema for structuring ratios and proportions multiplicatively and making iterations with the symbolic representations

Structuring ratios and proportions multiplicatively and using covariance to obtain proportional situations appeared on Day 4 for the first time when the classroom discussion was shifted from reasoning with short ratio tables to using the proportion representation and talking about the horizontal and vertical relationships in it. Thus, on that day, the idea of "structuring ratios and proportions multiplicatively and using covariance to obtain proportional situations" appeared in the students' claims. However, as deduced from Figure 4.30, this idea appeared in data to make claims about equivalent ratios (i.e., proportion) on Day 6. This gave initial evidence that the idea of structuring ratios and proportions multiplicatively was taken-as-shared among the classroom community. Moreover, extending covariance to the proportion representation was taken-as-shared as well based on the same concerns. The classroom discussion on Day 6 continued with the teacher's question given below. The teacher wanted the class to make interpretations from the base ratio of the number of girls to the number of boys in their class by using informal ratio language. Below is a dialogue in which two students used informal part-part-whole relationships to make sense of the relationship between the number of girls, boys, and all students in the classroom:

> *Teacher*: So, we made interpretations yesterday like "there are seven students who walk to school for every three students who take the bus." Remember? Can you use the same language for the number of boys and girls in this class?
> *Sinem*: There are four girls for every five boys.
> *Teacher*: Yes. Another one?
> *Ozan*: Out of every nine students, four are girls, and five are boys.

As seen in the dialogue above, Sinem and Ozan made claims about the part-part-whole relationships by using informal ratio language in a context that included the comparisons among the number of boys, girls, and all students in the class without any need for warrants/backings. Therefore, this could be given as further evidence that the idea of extending covariance relationship to part-whole contexts was taken-as-shared. After finding equivalent ratios and making connections to part-part-whole relationships, the students were encouraged to work in small groups. They were asked to find the numbers of girls, boys, and all students in a given class, not their class anymore, by using the following information “the ratio of the number of girls to the number of boys is 2:3.” Particularly, the first question asked the students to find the number of boys when there were 16 girls in that class. Merve drew a short ratio table and placed two next to the number of girls (K) and three next to the number of boys (E). Then, she used the horizontal relationships to find the number of boys as 24, when there were 16 girls. She also wrote the corresponding symbolic proportion representation, as shown in Figure 4.31 below.

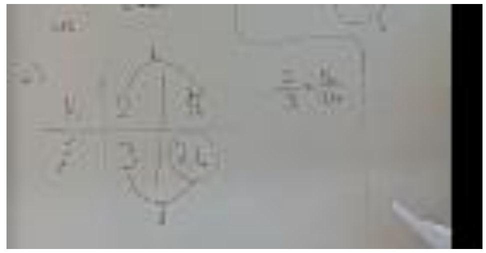


Figure 4.31. Moving from a short ratio table to symbolic proportion representation to find a missing value

The class discussion proceeded by making claims about finding the number of girls/boys/all students (part-part-whole) for different situations. For this, the students used the formal ratio language, structured the formal representation of proportion, and made sense of the horizontal/vertical relationships in the proportion in their data. At the end of the first session on Day 6, the classroom discussion was shifted to deciding whether the given ratios belong to the same ratio (i.e., $\frac{2}{3}$). This included checking for proportionality and deciding if the given two ratios were proportional. One of the questions requested the students to determine if $\frac{2}{3}$ and $\frac{10}{20}$ were proportional. Below is a dialogue that took place when the classroom discussion revolved around comparing those two ratios:

Teacher: So, what do you think about $\frac{10}{20}$?
Faruk: It is not proportional (to $\frac{2}{3}$) since two is multiplied by five to obtain ten, but three is not multiplied by five.
Teacher: Do you all agree with Faruk?
Ceyda: Yes. They cannot be proportional anyway because $\frac{10}{20}$ is half, and the other one is $\frac{2}{3}$.

In this dialogue, Faruk claimed that the two ratios were not proportional since $\frac{10}{20}$ could not be obtained by scaling $\frac{2}{3}$ by 5. Then, the teacher asked the class if they agreed, and Ceyda provided a warrant for the same claim by making connections to fraction comparison. Therefore, Ceyda used the invariant multiplicative relationship between the values in her warrant to determine the proportionality of the give two ratios. Therefore, this gave initial evidence that the invariant relationships were extended to the symbolic representation of proportion. However, there is further evidence from the ensuing days to suggest that the invariant relationships between the values in different measure spaces were extended to the symbolic representation of proportion.

As an answer to the following question, Selim claimed that $\frac{2}{3}$ is proportional to $\frac{8}{12}$ since eight is four times two, and 12 is four times two (i.e., the equivalence of the scale factors). As Selim was providing his data, the teacher drew horizontal arrows on the board and put a checkmark next to the proportion since Selim claimed that they were proportional. The classroom discussion progressed with deciding on whether each of the given ratios was proportional to the ratio of the number of girls to the number of boys given in the task. Throughout these discussions, the students made claims about checking if the two ratios were proportional by using the horizontal relationships in the two proportions that were related to the processes of scaling (i.e., abbreviated build-up). The students based these claims on the data that were focused on whether the given ratios could be obtained by scaling $\frac{2}{3}$ with any factor. Particularly, they claimed that the two ratios were proportional if the second one was the scaled version of $\frac{2}{3}$, and they were not proportional if the second one was not a scaled version of $\frac{2}{3}$. Therefore, on Day 6, the students made claims about checking for proportionality and determining if the given two ratios were proportional. These claims were initially challenged. This was an idea that needed to be kept an eye on whether or not it would be taken-as-shared in the following days.

TAS Idea 2. Determining proportionality by covariational and multiplicative reasoning. On Day 7, the students engaged in solving problems in a variety of contexts. The first problem asked the number of pens that could be bought with 12 Turkish Liras (TLs) if two pens could be purchased with four TLs. The teacher encouraged the students to use multiple strategies. Below is a conversation in which the students used ratio tables and symbolic proportion, and interpreted covariation and invariation by referring to the horizontal and vertical relationships on those:

> *Teacher*: You should be able to solve these (problems) in several ways. Let's solve all the problems in several ways.
> *Ceyda*: I set up a proportion including $\frac{4}{2}$. That means I wrote the ratio between TL and (the number of) pens. Since it asks how many pens could be bought with 12 TLs, I put an equal sign and wrote $\frac{12}{x}$. Since 12 is three times as much as 4, I should multiply the (number of) pens by 3, so that they could be equal (The teacher writes the proportion on the board while Ceyda was explaining her strategy from her seat, See Figure 4.32a below). So, I can buy six pens.
> *Merve*: We can also simplify $\frac{4}{2}$ to $\frac{2}{1}$. That means one pen can be bought with 2 TLs. So, six pens can be purchased with 12 TLs.
> *Hande*: Can I tell another strategy?
> *Teacher*: Say it.
> *Hande*: We can also think vertically. We can reduce $\frac{4}{2}$, and then one pen could be bought with 2 TLs. So, six pens could be bought with 12 TL, since it should be its half.
> *Giray*: We can also write the ratio between the amount of money (in the first situation) and the amount of money (in the second situation) and (the number of) pens (in the first situation) and (the number of) pens (in the second situation).
> *Teacher*: How can you write the ratios between the amounts of money and the number of pens? What does that mean?
> *Giray*: If we draw a table for money to money and pen to pen (tells his strategy from his seat, the teacher draws the tables on the board, See Figure 4.32b below); then, (the amount of) money is tripled (makes a hand gesture to show the vertical relationship from 4 to 12 in the VRT), so should the (number of) pens. It will be 6.
> *Teacher*: Yes, Ceyda had drawn a table vertically like this in a previous class. Can you tell me the ratio that could be obtained from this table (see Figure 4.33b)?
> *Giray*: $\frac{4}{12}$ is equal to $\frac{2}{x}$. So, it's the ratio of money to money and pen to pen.
> Teacher: Ceyda, can you tell me which proportion could be obtained from this table (points to the table in Figure 4.33a)?
> *Ceyda*: It's the one that I wrote. $\frac{4}{2}$ is equal to $\frac{12}{6}$ (points to the table in Figure 31a).
> *Teacher*: So, we can draw tables vertically and horizontally. Each table gives a different proportion, but they give the same result. Can you complete my sentences? Horizontal relationships in the horizontal table (points to the horizontal arrows in the horizontal table shown in Figure 4.33a) correspond to the…
> *Students*: (all together) vertical relationships in the vertical table (teacher points to the vertical arrows in the vertical table shown in Figure 4.33b).
> *Teacher*: Yes, and the vertical relationships in the horizontal table corresponds to the…
> *Students*: (all together) horizontal relationships in the vertical ratio table.

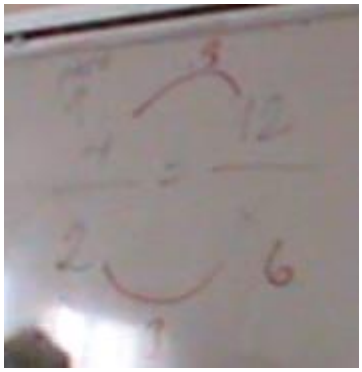


Figure 4.32a. Structuring of between measures ratios

Figure 4.32b. Structuring of within measures ratios

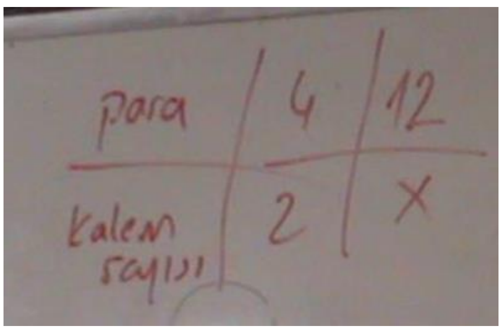


Figure 4.33a. HRT (model for between measures ratio)

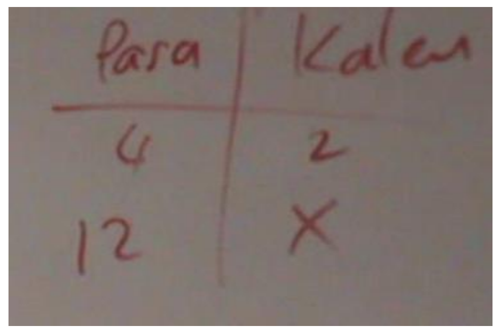


Figure 4.33b. VRT (model for within measures ratio)

As understood from the dialogue and the related figures above, Ceyda made a claim about the missing value in proportion by referring to the horizontal relationship (i.e., scaling) in the proportion (i.e., within measures ratio) in her data. Then, Merve provided data regarding simplifying the ratio that included numerical values to obtain the unit rate (i.e., the amount of money needed to buy one pen) in the same proportion. Hande provided backing to Merve's argument by referring to the same relationship as the vertical relationship and the invariant times $\frac{1}{2}$ relationship. Afterward, Giray suggested drawing a VRT that would vertically align the values within the same measure spaces and referred to the vertical relationship in the vertical table (i.e., scaling) in his data to obtain the same claim. Later, the teacher made a connection to a previous discussion regarding Ceyda's drawing of the table vertically. Eventually, the teacher had students make connections between HRT and between measures ratios, and VRT and within measures ratios. Also, the correspondences between the horizontal relationships in the HRT and the vertical relationships in the VRT, and the vertical relationships in the HRT and the horizontal relationships in the VRT were re-negotiated and established.

The students continued to engage in the process of solving proportional problems in different contexts in small groups by considering the teacher's request in the second period of the instruction on Day 7: solving the problems in as many ways as possible. After they tried to come up with a variety of solution ways, each group shared their solutions with their classmates on the board for each question. Then, after each group finished explaining different solutions, the teacher asked the whole class if they could add any other strategy. In sharing their solutions on the board, each group provided alternative data that included horizontal/vertical ratio tables and symbolic proportion representation, including within/between measures ratios, by referring to the horizontal/vertical relationships in those to make claims about the answers to the problems. None of the students was challenged or asked for further warrant/backing regarding their arguments, which showed further evidence that the ideas of "structuring ratios and proportions multiplicatively and extending invariance and covariance to proportion" and "determining proportionality by covariational and multiplicative reasoning" became taken-as-shared. To illustrate, a dialogue that took place in the whole class discussion for answering the question "A real distance of 9 kilometers is represented by 5 centimeters on a map. Then, how long is a distance in real life if it is represented by 20 cm on the map?" is presented below:

> *Remzi*: (Writes a VRT including a within measures ratios, draws vertical arrows to show the vertical relationships, and writes ×4 next to the vertical arrows, See Figure 4.34a). We made a vertical table for real (distance) and (distance on the)

map. It became 20 when it was 5, so nine should become 36 since it should also be multiplied by 4.
Teacher: What is the unit of 36?
Remzi: Kilometers.
Teacher: OK. What kind of ratios did you establish?
Remzi: Ratios of real distance to real distance and (distances on the) map to map.
Ozan: We wrote a proportion, not a table. Still, we looked at the times four relationships as well. Since this was five and became 20 (writes $\frac{9}{5} = \frac{x}{20}$ on the board and makes a hand gesture from 5 to 20 in the denominators of the two ratios, see Figure 4.34b); that means, it is multiplied by four. This (points to x in the numerator of the second ratio) should be four times 9; that is, 36.
Seval: As an alternative solution, I wrote the proportion differently (writes $\frac{9}{x} = \frac{5}{20}$ on the board, see Figure 4.34c). Then, since this side is multiplied by 4 (draws a vertical arrow from 5 to 20 and writes ×4 next to the arrow), nine multiplied by 4 becomes 36.
Berk: The fourth method is the horizontal of the first method (draws an HRT and horizontal arrows from 5 to 20, see Figure 4.34d). I thought what multiple of 5 is 4. It is times four. Therefore, times four above gives the result as 36.
Murat: We looked at the vertical relationship in the proportion (writes $\frac{9}{5} = \frac{x}{20}$ on the board, draws a vertical arrow from 5 to 9 and writes $\frac{9}{5}$ next to the arrow, see Figure 4.34e). We wrote the proportion $\frac{9}{5}$ is equal to $\frac{x}{20}$. Then, we thought what multiplied by five is equal to 9 and found it as $\frac{9}{5}$ (shows the vertical arrow and $\times\frac{9}{5}$ next to it). Then, we divided 20 by five and multiplied (the result) by 9. It is 36.
Teacher: Yes, we talked about it before. We can look at the non-integer relationships like this. But I have a question for you, how did you write the proportion?
Murat: We wrote the ratios of real distances to the distances on the map.
Teacher: Can you interpret these ratios?
Murat: 9 kilometers (points to the number 9 in the proportion $\frac{9}{5} = \frac{x}{20}$) in real is shown by 5 (points to the number 5 in the proportion) centimeters on the map, 36 kilometers (points to the number 36 in the proportion) in real is shown by 20 (points to the number 20 in the proportion) centimeters on the map.
Teacher: Is this relationship true for only 9 km and five cm and 36 km and 20 cms?
Murat: No. For all the distances. Every five cm on the map is actually 9 km in real life.
Teacher: Good, thank you. Have you ever seen a number in the bottom corner of a map?
Students: Yes.
Teacher: What does that number mean?
Student 1: For example, it says 250 kilometers is 1 centimeter. That means a distance of 250 kilometers in real life is represented by 1 centimeter on the map.
Ceyda: It gives a unit rate. This rate applies to all numbers (distances) on the map. For instance, a distance of 2 centimeters on the map is 500 kilometers in real life.

Teacher: Yes, we call this number the "scale of a map." So, what is the scale of our map (the one that they were working on)?
Ceyda: 5 centimeters over 9 kilometers is the scale of this map.
Teacher: OK. Let's bring a map in the following class and talk about that.

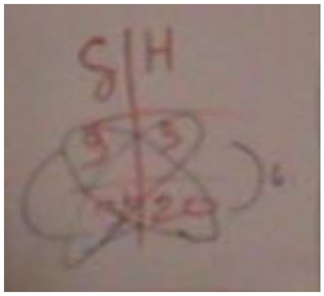


Figure 4.34a. Remzi drawing a VRT

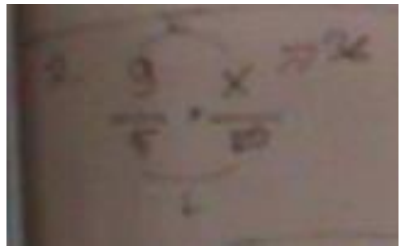


Figure 4.34b. Ozan writing a proportion including within ratios

Figure 4.34c. Seval writing a proportion including between ratios

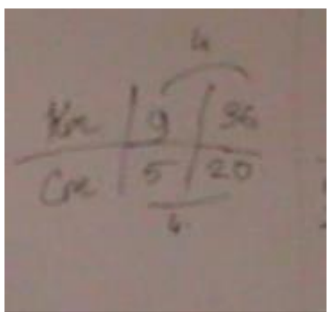


Figure 4.34d. Berk drawing an HRT

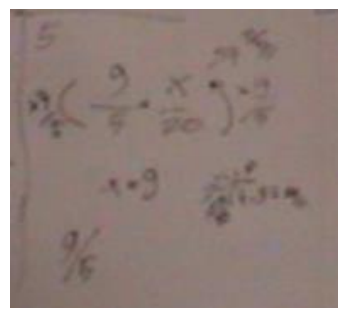


Figure 4.34e. Murat referring to the vertical relationships on the proportion

As inferred from the dialogue and the related figures above, multiple students made the same claim for the answer to the scaling problem by providing a variety of previous ideas in their data without a need for any further warrant/data. First, Remzi drew a VRT including the within measures ratios, and claimed that the distance that was represented by 20 cm on the map was 36 km in real life by referring to the vertical relationships (i.e., scaling). Then, Ozan wrote a proportion including the ratios between measure spaces and referred to the horizontal relationships (i.e., scaling) in his data. Seval also wrote a proportion including the ratios within measure spaces and made use of the vertical relationships (i.e., scaling). Later, Berk drew an HRT and referred to the horizontal relationships (i.e., scaling) in his data. Up to that point, either using horizontal or vertical relationships, the students referred to the relationships regarding working with the same scale factors in the tables and proportions. Then, Murat referred to the vertical relationships in the proportion that included between measures ratios, which corresponded to the invariant multiplicative relationship or the unit rate between the distances on the map and in real life. Then, the teacher took this opportunity to introduce the term “scale of a map” and asked the students to express the scale of the map they had been working on.

In sum, during Days 6 and 7, the classroom discussion and the communal ways of reasoning were moved from finding missing values in ratio tables to reason with the symbolic representation of proportion by drawing on their experiences of scaling (i.e., abbreviated build-up/iteration), invariant multiplicative reasoning (i.e., unit rate) in the contexts of fish-food bar and part-whole. In addition, the emergence of VRT and corresponding proportion representations contributed to the shift from reasoning with ratio tables to the structuring of within measures and between measures ratios and related representations of proportion. Therefore, even though the discussions that took place on Days 6 and 7 were built on the TAS Ideas in CMP 1, the ways students reasoned with the tools and the nature of the classroom discourse were altered to a more mathematical way of thinking. Therefore, the mathematical ideas of “Structuring ratios and proportions multiplicatively and extending invariance and covariance to proportion” and “Determining proportionality by covariational and multiplicative reasoning” felt different than the TAS Ideas in CMP 1. Moreover, it was inferred that those ideas belonged to the general activity of “Reasoning with tables and symbols to determine proportional situations.” Thus, these two mathematical ideas were put together and organized around the common activity of “Reasoning with tables and symbols to determine proportional situations” as they emerged and became taken-as-shared during the discussion of Days 6 and 7.

4.3. CMP 3: Coordinating the relationships among the representations

Throughout Days 8 to 11, the students were engaged in a series of activities that included representing proportional and non-proportional situations using tables, graphs, and equations. Therefore, three mathematical ideas were taken-as-shared as the students worked on these activities:

- Formalizing the invariant multiplicative relationship into an equation,
- Representing proportional relationships with linear equations of the type y=mx and graphs passing through the origin,
- Representing non-proportional linear relationships with linear equations of the type y=mx+b and graphs not passing through the origin.

TAS Idea 1. Formalizing the invariant multiplicative relationship into an equation. On the eighth day of the instruction, the students were formed into five groups, and each group was given a different situation related to the unit price of a vegetable/fruit. Their task was to fill a long ratio table for different amounts of the given vegetable/fruit and their price. The information given to each group included one of the following five unit prices: 1 kg of potatoes is 1 Turkish Lira (TL), 1 kg of apples is 2 TLs, 1 kg of tomatoes is 3 TLs, 1 kg of bananas is 5 TLs, and 1 kg of kiwi is 6 TLs. Even though the price for each vegetable/fruit was different, the idea was the same in each situation: finding the amount of money required to buy specific amount of vegetables/fruits in kilograms when the unit price was known; and representing the relationship between the amount of fruit/vegetable and the money using tables, graphs, and equations. After working in small groups, each shared their strategies and answers on the board. Ceyda came to the board and explained how they filled in the table and obtained the algebraic equation as illustrated in the following exchange:

Ceyda: One kg of apple is 2 TLs in our example. Since 1 kg of apple is 2 TLs, we thought about how much 2 kg would cost (shows the numbers in the long ratio table they filled in in her activity sheet). Since this (the amount of apples) is doubled, the money would be 4 TLs. And we continued to fill in the table like this. Here (points to the x on the last column of the table), we thought what the price would be for an unknown amount x. That would be 2x since the price is always (makes a hand gesture to point to the vertical relationships on the table) twice the amount.
Teacher: Yes. We have explored the relations so far, but now, we are writing the relationship algebraically. Then, we need to draw it on the coordinate plane. Which value should be on the y-axis on the coordinate plane?
Student: y
Teacher: Then, in order to represent this relationship on the coordinate plane, I should say that y = 2x. That becomes the equation of the relationship. Ceyda, can you explain what this equation means?
Ceyda: The twice the amount (of apples) is equal to the price.
Teacher: Yes. We are going to represent relationships with algebraic equations. Can you explain the graph, Ceyda?
Ceyda: 1 kg is 2 TLs, so these are (points to the values on the table) 1 to 2, 2 to 4, 3 to 6, and so on. These became the points on the coordinate plane.
Teacher: I see you drew a graph on your paper. How did you obtain that graph?
Ceyda: We connected the points 1-2, 2-4, 3,6, 4-8 (points to the points on the graph, see Figure 33), and so on and obtained a line.
Teacher: How did you know it was a line?
Ceyda: 1-2, 2-4, 3-6, 4-8, and so on. It could go forever since we can buy any amount.
Teacher: Well. I see the origin on the graph. How did you obtain that?
Ceyda: When we don't buy anything, we don't have to pay. Then, it becomes the point zero to zero.

In this exchange, Ceyda claimed that the general algebraic expression for the price of the apples of any amount would be 2x by looking at the values on the table that they filled in by building up (i.e., iteration). She also provided a warrant for this argument that included the "invariant times two relationship" between the amount and the price of the apples. Therefore, her argument could be summarized as in the following Toulmin's scheme:

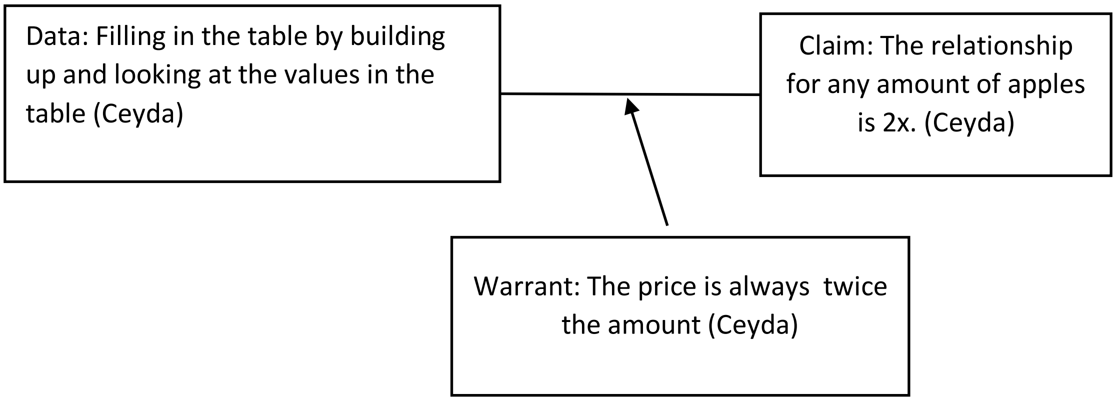


Figure 4.35. Toulmin's argumentation scheme for formalizing the invariant relationship into an algebraic expression

Therefore, Ceyda formalized the invariant multiplicative relationship into an algebraic expression for the first time in the instruction and provided a warrant in the first place. However, since it was not in an equation form, the teacher introduced the corresponding equation as y=2x by referring to the coordinate plane. Then, the teacher asked Ceyda to explain how they obtained the line graph that they had on their paper and especially how they figured out to include the origin in their graph (see Figure 4.36 below).

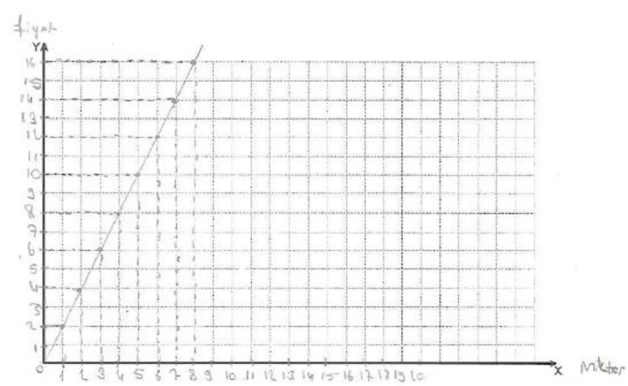


Figure 4.36. Ceyda's line graph for the equation y = 2x

Ceyda made a connection between the linked composites 1-2, 2-4, 3-6, 4-8… in the table and the points that included the corresponding ordered pairs on the coordinate plane.

Based on this, she claimed that they obtained a line (to represent the relationship between the amount and price of the apples). Besides, she stressed that they included the origin in their graph since they interpreted the origin as paying nothing when buying nothing. Therefore, in this instance, the claim was the graph drawn by the students, and the data were related to the procedures regarding how they obtained the graph (i.e., plotting the points on the coordinate plane, connecting the points and obtaining a line and including the origin). The teacher took this opportunity to revise the claim and introduce the term, "linear relationship," as understood from the following excerpt:

> *Teacher*: Then, what can you say about this graph? Where does it pass through?
> *Students*: The origin.
> *Teacher*: (To the class) Does your graph also pass through the origin?
> Students: Yes.
> *Teacher*: Well. Yes, these are all graphs that pass through the origin. Since the relationship between the amount of apples and its price can be represented by a line, we will call this a linear relationship.

Therefore, in the excerpt above, the teacher claimed that all the situations that the classroom worked on formed linear relationships since they could be represented by lines. Thus, this was the first time that the issue of representing proportional situations by linear graphs passing through the origin emerged in the discussion. Hence, this was an idea to keep an eye on whether it becomes taken-as-shared later.

In the following instances on Day 8, a student from each group continued to share their answers, strategies, and graphs. As the spokesperson of the second group that worked with the information that included the unit price for a kilogram of bananas as 5 TLs, Gizem claimed that the algebraic equation for their situation was $y=5x$. The corresponding graph was also linear by referring to the "invariant times five relationship" between the amount of bananas and the related price. Similarly, the students from all remaining groups wrote the corresponding algebraic equations correctly and drew the corresponding line graphs, and no warrant/backing was presented. Therefore, dropping off of warrants in students' answers showed that the idea of formalizing the invariant multiplicative relationship into an equation was taken-as-shared. There is also evidence that these ideas were taken-as-shared from the following class, where students used these claims to provide their data/warrants.

TAS Idea 2. Representing proportional relationships with linear equations of the type y=mx and graphs passing through the origin. As the students formalized the invariant multiplicative relationships into algebraic equations, they also drew the related line graphs that passed through the origin, as mentioned above. After each group presented their answers and explanations for the five different situations on the board, the answers (tables, graphs, and equations) for each situation were hung on the board as shown in the following figure:

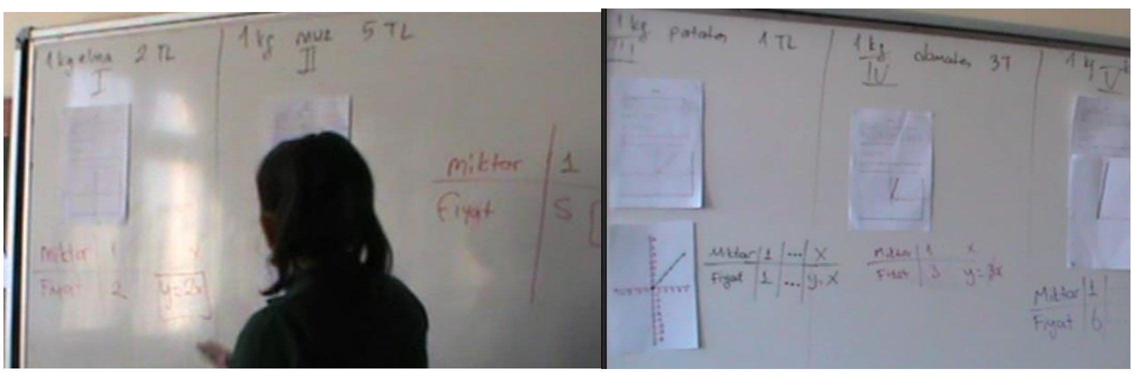


Figure 4.37. Formalizing the multiplicative relationships into linear equations of the type y = mx and drawing the corresponding line graphs that pass through the origin

Now, the students were engaged in a discussion that focused on the similarities/differences among all of the situations and their tabular, algebraic, and graphical representations. Multiple students made a variety of claims regarding the similarities and differences, as illustrated in the following excerpt:

> *Berk*: We used the same region (quadrant) in all of those.
> *Merve*: That means all of them are in the first quadrant.
> *Teacher*: Why do you think we didn't use the other quadrants?

A bunch of students: There is no minus (value).
Seval: We cannot buy minus tomatoes, nor can we pay minus amounts.
Teacher: Did you understand why we only used the first quadrant, Elif?
Elif: Yes. There is no such thing as buying minus kg (of things).
Teacher: OK. Does anyone want to add anything?
Berk: All of them are linear relationships.
İlter: All (the graphs) pass through the origin.
Selim: The (unit) amount is the same (i.e., 1 kg) in all, but the (unit) price is different.
Teacher: What did you mean by saying that the amount was the same?
Selim: The kg (amount) increases by 1s in all of the graphs and tables, but the price increases differently.
Merve: For instance, in y=2x and y=5x, the increment in the amounts and the prices are different. The increment is 2 there (in y=2x), but 5 here (in y= 5x).
Teacher: How does the x increase in each?
Merve: It goes up by 1s in both.
Teacher: Well. Can you tell the increment in x and y all together (for both)?
Merve: x goes up by 1s whereas y, which is the price, goes up by 2s in this one (points to the equation y=2x). X goes up by 1s as well in this one (points to y=5x), but y goes up by 5s.
Teacher: How can we observe this (difference) on the graphs?
Merve: Their alignment becomes different.
Teacher: How is their alignment different?
Merve: One of them is more inclined (makes hand gestures to illustrate the inclination in each) than the other.
Teacher: Yes. We will learn this next year. We will call this the slope of the lines. Now, let's continue with the similarities and differences.
Sinem: All have different slopes, then.
Teacher: What makes the slopes different (in each)?
Sinem: Price.
Teacher: Well. What kind of a relationship is there among their equations?
İlter: All of them are; y is equal to x, or something is equal to x (meaning y=x or y=mx).
Teacher: What are the values in these equations?
Student: The amount and price.
Teacher: Yes. The relationship between the amount that I buy and the price that I have to pay forms a linear equation.

In the exchange above, Berk and Merve claimed that all the graphs were only in the first quadrant. The teacher asked why all the graphs were in the first quadrant. Therefore, this was an instance where the teacher provided the claim and asked students to provide data and/or warrant for that claim. A bunch of students provided data saying that they did not have minus values in the problem. However, that was not enough for other students to understand what it had to do with the claim. Therefore, Seval provided a warrant saying that it was not possible to buy a minus amount of tomatoes and pay a minus amount of money, hence that they were not supposed to use quadrants other than the first one.

To continue to look for similarities and/or differences among the five different situations and their representations, Berk claimed that all of the relationships are linear relationships. Ilter added on to that by claiming that all the graphs pass through the origin. Besides, Selim told that the amount of vegetables/fruits increases by ones in each situation while the price increases differently, for which the teacher requested a warrant. Upon this request, Merve explained what Selim meant by taking the equations y=2x and y=5x as an example. She referred to the increment in the former equation as 2 and the latter as 5. In order to support students' correct use of mathematical language, the teacher asked students to take the increment in x into consideration as well, which resulted in Merve's claim related to the rate of change in both situations. Then, the teacher asked Merve how the difference in the increments could be observed in the graphs. As an answer to this question, Merve pointed to the differences in their alignments. Even though Berk and Merve touched upon the concept of slope, the teacher did not push for any warrants/backings from these students in this class or the following classes since the idea of the slope was beyond the trajectory and the seventh-grade curriculum. Instead, she introduced the term slope briefly as a topic that would be discussed in the following school year. Therefore, although these two students brought about a new idea regarding the slope of lines, there is no solid evidence whether they were taken-as-shared. Continuing to make claims about the similarities and differences among all the graphs, İlter claimed that all the equations have the form of, "y equals x multiplied by something" (i.e., y=mx). Thus, the fact that the students did not need to provide warrants/backings while comparing all five situations also gave initial evidence that the idea of representing proportional relationships by linear equations of the type y=mx and linear graphs passing through the origin was taken-as-shared in the classroom. There is also evidence from the subsequent classes that would support this conclusion.

At the end of the discussion about the similarities/differences among all five situations and their representations, the teacher posed the following question: “Well, you said that all of these graphs pass through the origin. Do you think all graphs (in the world) pass through the origin?” and the following discussion occurred:

> *Teacher*: Well, you said that all of these graphs passed through the origin. Do you think all graphs (in the world) pass through the origin?”
> Student1: Yes.
> *Musa*: No.
> *Teacher*: When do you think that it won’t pass?
> *Musa*: When it corresponds to zero.
> *Student2*: It may pass through the numbers.
> *Sinem*: I think it always passes through the origin.
> *Student1*: I agree.
> *Teacher*: I would like you to justify your answers. Musa, what do you mean by, “it won’t pass when it corresponds to zero?”
> *Musa*: The point A (the point on the origin), it can be 0 comma 2, can’t it?
> *Teacher*: Can you think of a situation where A would be (0,2).
> *Musa*: Well…I cannot think right now.
> *Teacher*: Let's think this way. In this situation, you don't pay anything when you don't buy anything. Can there be situations that you have to pay when you get nothing?
> *Sinem*: If we have a debt…
> *Ceyda*: If we call a taxi, we have to pay even if we don’t go anywhere.
> *Teacher*: Interesting. Then, the relationship is between which values?
> *Ceyda*: The distance traveled and the money paid. Like the taximeter.
> *Teacher*: What happens with the taximeter?
> *Ceyda*: We need to pay at the beginning, even if we don’t go anywhere. We pay extra money for the distance we travel.
> *Teacher*: Then, you say that it won’t pass through the origin.
> *Ceyda*: No. That would go from 0 and 2 since even if the distance is 0, we have to pay 2.
> *Teacher*: OK. We can talk about these relationships in the next class.

In the excerpt above, the teacher encouraged students to think whether there would be graphs that won’t pass through the origin to see their intuitive conceptions about non-proportional linear relationships and their readiness for the next class. Some of the students said that all graphs would pass through the origin while some of them said that not all graphs would go through the origin. Students engaged in a debate where they tried to think of situations that could be represented by a graph that would not go through the origin. Sinem told that all the graphs would go through the origin, and a couple of students agreed with her. Musa challenged her, saying that it could go through the point (0, 2), which required further justification. He stated that when something is zero, the other thing could be two, which also required further clarification. Ceyda took the floor and told that they would have to pay an amount when they got a taxi even if the taxi driver did not take them anywhere. It means that she was able to exemplify a situation to justify her claim that not every graph would pass through the origin. The teacher asked her to say more, and she went on by saying that there is a starting price for the taxi, and extra money would have to be paid for the distance taken. She concluded by saying that the graph of the money and distance taken on a taxi would not go through the origin, but it would go through the point (0,2). This was a new idea regarding non-proportional linear relationships to be kept an eye on in the following classes to see whether it would be taken-as-shared. On the other hand, this paved the way for the teacher and the researcher to shape the task for the next class to bring a task regarding the taximeter example in the class to engage students in discussions regarding the linear situations that were not proportional.

In the second block of the lesson on Day 8, the students worked on another similar task in which the information "a person weighs 72 kg on the Earth and 12 kg on the Moon" was given, and the students were asked to find the weight of a person on the moon if he weighs 48 kg on the Earth. Elif claimed that the person would weigh 8 kg on the Moon by providing procedural data and including the invariant multiplicative relationship between the weights on the Earth and the Moon, and several students rephrased this relationship. Below is an excerpt that includes those discussions:

> *Elif*: I used the given information (a person weighs 72 kg on the Earth and 12 kg on the Moon) and divided 72 by 6. Then, I divided 48 by 6 and found that he would weigh 8 kg on the Moon.
> *Teacher*: What does this "6" mean?
> *Gizem*: The relationship between the weight on the Earth and the Moon.
> *Musa*: It means that our weights on the Moon would be one-sixth of our weights on the Earth.

Merve: We find it by dividing the weight on the Earth by 6.
Sinem: That means that we find the weight on the Moon by dividing the weight on the Earth by 6. Then, the weight on the Moon is one-sixth of the weight on the Earth.
İlter: Our weights on the Moon would be six divided by our weights on the Earth, which is one-sixth.

The argumentation process above can be summarized as in the following Toulmin scheme:

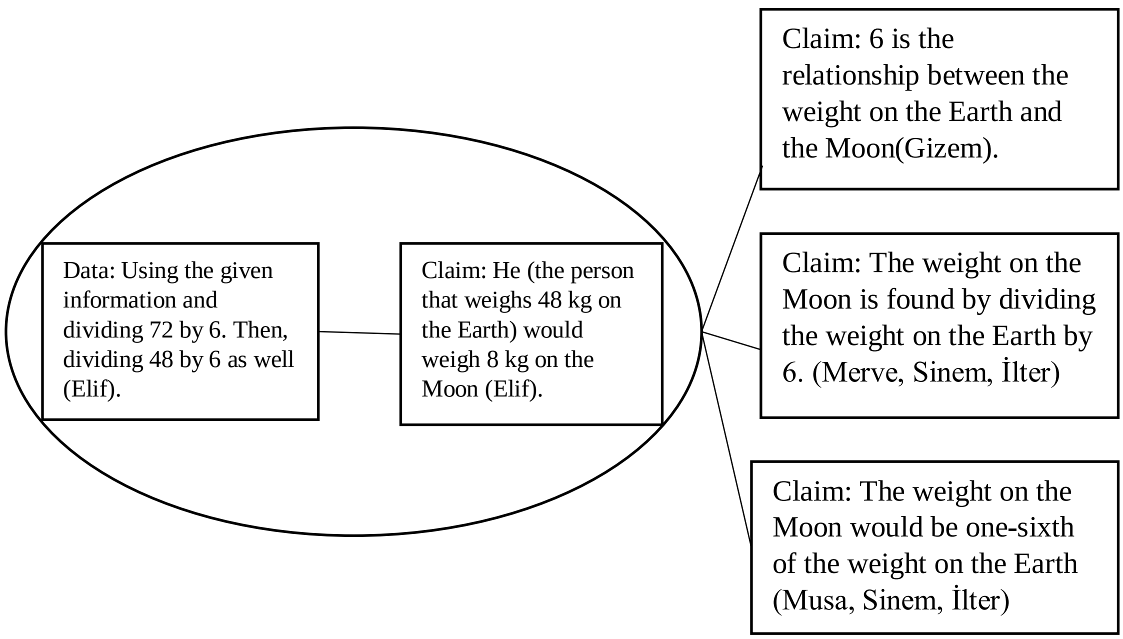


Figure 4.38. The Toulmin scheme for finding the weight on the Moon

As deduced from the excerpt above and the associated Toulmin argumentation scheme, Elif found the weight of a person on the Moon by dividing the weight on the Earth by six. Then, upon the teacher’s question regarding the meaning of 6, multiple students made three new claims regarding the "invariant times six relationship" between the weight on the Earth and the weight on the Moon without needing any data/warrant to back those claims. Therefore, this provided further evidence about the idea that reasoning with the invariant multiplicative relationship between the two units was taken-as-shared.

In the following instances, the students were asked to represent this relationship with equations and draw its graph. The classroom discussion above, then, was maintained as in the following exchange:

> *Teacher*: Then, how did you write the equation for this (relationship)?
> *İlter*: y is the weight on the Moon, and x is the weight on the Earth, and the equation is y equals x over 6.
> *Teacher*: OK. Let's draw the graph of this relationship then and see if it would be linear.
> *Ceyda*: (Fills in the table including two units as the weight on the Earth and the weight on the Moon and plots the points on the coordinate plane) The first point is 24 and 4, the second point is 30 and 5, the third point is 36 and 6, then 42 and 7, 48 and 8, 54 and 9.
> *Teacher*: What would be the next point?
> *Musa*: 66 and 11.
> *Teacher*: How did you draw the graph?
> *Merve*: We connected these points, but it didn't pass through the origin.
> *İlter*: No, think about the smaller values too.
> *Ceyda*: Yes, it should go through the origin.
> *Teacher*: How many of you think that it would pass through the origin? (Most of the students raised their hands). İlter, can you explain why you think it would pass through the origin?
> *İlter*: If it (the weight on the Earth) was 18 kg then it (the weight on the Moon) would be 3; if it was 12 kg, then it would be 2 kg on the Moon, 6 kg would be 1 kg, 0 kg would be 0 kg on the Moon too.
> *Merve*: Then, that passed through the origin.
> *İlter*: Yes, indeed.
> *Teacher*: Then, what kind of a relationship is there (between the weight on the Earth and the weight on the Moon) in this question?
> *Students*: Linear.
> *Teacher*: We will maybe talk about when it won't be linear later.

After the invariant multiplicative relationship between the weight on the Earth and the weight on the Moon was stressed in students' warrants and backings, the teacher asked the students to formalize this relationship into an equation. İlter claimed that the equation would be, "y over 6," without providing a warrant. Then the teacher asked the class to draw the graph of this relationship, and Ceyda considered the linked composites of the weights on the Earth and the Moon as ordered pairs and plotted the corresponding points on the coordinate plane. Merve claimed that the graph wouldn't pass through the origin. However, İlter and Ceyda immediately disagreed with her. Then, the teacher asked everyone to raise their hands if they thought that the graph would pass through the origin, and most of the students raised hands. İlter justified his claim by plotting the points with the smaller values on the coordinate plane, including (0,0), and Merve changed her claim by saying that the graph passed through the origin. Eventually, the teacher asked the students to name the type of the relationship, and many students named the relationship as, "linear relationship." In terms of the ideas of formalizing the invariant relationship into an equation and representing proportional relationships by linear equations of the type y=mx and linear graphs passing through the origin being taken-as-shared, a couple of things may be asserted. First, İlter did not provide a warrant to make a claim about the equation $y = \frac{x}{6}$, and no one challenged him to do it. This, together with the fact that students were able to extend their understandings of formalizing the multiplicative relationship into an equation to situations represented by y=mx with a non-integer m, gave us further evidence about the idea of formalizing the invariant multiplicative relationship into an equation being taken-as-shared. Therefore, this could also be given as evidence of the idea of representing proportional relationships by linear equations of the type y=mx was taken-as-shared.

On the other hand, even though Merve claimed that the graph of that relationship would not pass through the origin at the first place, several students disagreed with her by providing data including plotting the points for the corresponding smaller values for their claims that the graph would indeed pass through the origin. In other words, they were not challenged to provide warrants for their claims about the graph of $y = \frac{x}{6}$. Therefore, it was evident that the idea of representing proportional relationships by linear equations of the type y=mx and linear graphs passing through the origin was taken-as-shared in the classroom.

TAS Idea 3. Representing non-proportional linear relationships with linear equations of the type y=mx+b and graphs not passing through the origin. On Day 9, the instruction started with a context in which the students explored the relationship between the amount of money given and the distance taken on a taxi where there was an opening price of three Turkish Liras (TLs) and the taximeter charged two TLs for each kilometer of distance traveled. Therefore, the focus was on the non-proportional linear relationships and the differences between non-proportional and proportional relationships and their algebraic and graphical representations. This context was chosen since a student had brought up the context in the previous class. As the first task, the students were required to fill in the long ratio table for the given situation, and the following discussion occurred in the class:

> *Teacher*: So, let's fill in the (ratio) table together. How did you fill in the table?
> *Erdem*: We get on the taxi, and there is a constant 3 TLs that we need to pay no matter how far we travel. I need to pay 2 TLs for each km I travel. Everyone will pay this 3 TLs regardless of how far they go. Then, I have to pay 2 TLs per each km traveled. So, I move further by adding 2 (makes a hand gesture to show horizontal relationships for building up by 2s) at each step.
> *Teacher*: What did you write for the relationship on the last row in the table?
> *Erdem*: I wrote 3+2, then 5+2, and 7+2, and so on.
> *Teacher*: Could you make a relationship between the distance traveled then and the money paid? Like, to find the money for traveling 100 km?
> *Erdem*: … (No answer)
> *Teacher*: Can you find the money that would be paid for traveling 10 km?
> *Erdem*: Well…
> *Teacher*: Ceyda, can you tell how you found the money for 10 km?
> *Ceyda*: I multiplied 10 by 2 and then added 3.
> *Selim*: I didn't understand why you multiplied it by 2.
> *Ceyda*: We don't have to add one by one. 20 TLs is for paid the distance traveled, and then we add the constant 3.
> *Teacher*: İlter did it in another way so that he can find the money to be paid for any km traveled. İlter, can you tell us how you did it (filled in the table)?
> *İlter*: We paid some money for the distance traveled. 2 multiplied by 2 equals 4. If we travel 3km, we multiply 2 by 3 and obtain 6. When we finally add the 3 in the beginning, it makes 9 TLs.
> *Teacher*: How do you find it (the money to be paid for any distance) then?
> *İlter*: That is, I find the money that I have to pay for the distance I travel, and I add the constant money at the beginning. Doing this way is easier than adding them one by one each time.
> *Teacher*: Did you understand? Ali, did you understand?

Ali: Yes
Teacher: Then, can you tell how much money you have to pay for traveling 60 km?
Ali: I will multiply 60 by 2 and find the money for the distance I travel. I should also add the constant at the beginning. Then, I have to pay 123 TLs.
Teacher: Then, let's imagine that we don't know the distance we travel or find the money for any distance we travel. That is, let's say x for the distance we travel and find the equation.
Faruk: Since we always multiply by 2 and add 3 (to the result), the equation is 2x+3.
Teacher: OK. How can you relate it to the other value?
Faruk: Then, it becomes y = 2x+3.
Teacher: Is there a difference between this and the equation that we learned in the previous class?
Sinem: This (the equation) has something plus next to it since we pay 3 TLs even if we don't travel anywhere.
Teacher: OK. How did you solve the next questions then?
Giray: It asks how much we need to pay when 12 km is traveled. I multiplied 12 by 2 and added 3, and it is 27.
Teacher: Well. Let's do the following.
Gizem: It asks how much money needs to be paid to the taxi driver for a distance of 40 km. Again, I multiplied 40 by 2 and added three, which equals to 83.

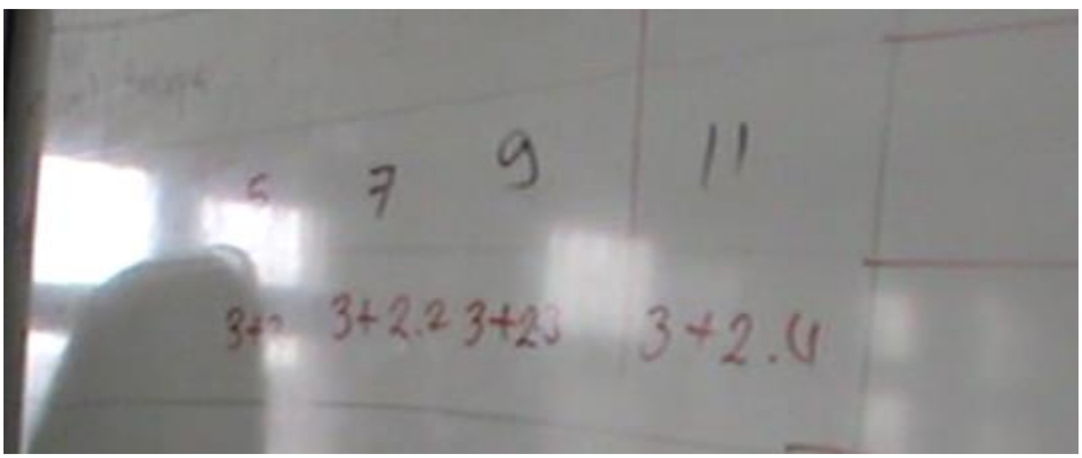


Figure 4.39. Filling in the table and representing the non-proportional linear relationship with an equation

In the exchange above, Erdem filled in the table that included the values regarding the distance traveled and the corresponding price by stressing the constant at the beginning and building up by 2s for each kilometer traveled and found the money that would be paid for the distances of 1 km, 2 km, 3 km, 4 km, and 5 km. He also symbolically showed this building up by 2s at each step in order to express the relationship between the distance and the money. However, the teacher asked him whether he was able to find a direct relationship between the distance and the money to find the money required for the larger values of distance traveled (i.e., 10 km). As Erdem was struggling to find the money for traveling 10 km directly, the teacher asked the same question to Ceyda. Ceyda claimed that she found the answer by multiplying 10 by 2 and adding 3 to the result. Selim did not immediately understand what these calculations meant and asked Ceyda to provide an explanation (warrant) for her procedural solution. Ceyda said that they did not have to add 2s one by one and that it would be enough to multiply 10 by 2 for the distance traveled and add the constant 3 to the result. Then, the teacher went back to the strategies for filling in the table and gave the floor to İlter to share his strategy. He used the same strategy with Ceyda to find the money that would be paid to travel 2 km, 3 km, 4 km, and 5 km. Then the teacher asked him to make a generalization about how he could find the money that would be paid for any distance traveled. Upon this request, he stated the general strategy to find the money to be paid to the taxi driver for traveling a distance as multiplying it by 2 and adding 3. For the following question that asked the money required to travel 60 kilometers, Ali claimed that he would have to pay 123 TLs based on the data that included multiplying 60 by 2 and then adding 3. Then, Faruk claimed that the equation for that relationship would be $y= 2x + 3$ upon the teacher's question and guidance. The students did not ask for a further warrant for the equation obtained for the given situation, and several students used the equation to find the money paid for the given distances without needing to provide data.

The discussion was moved to drawing the graph of the equation $y=2x+3$. Sinem told that the first point on the graph would be (0,3) since they had to pay three TLs even if the taxi did not take them anywhere. She went on by saying the points (1,5), (2,7), (3,9), (4,11), (5,13) on the coordinate plane, and claimed that they would form a line when connected. All students agreed with her and, students didn't ask for any explanation. The teacher asked the classroom whether the relationship between the distance traveled and the amount of money paid for that distance was linear. A bunch of students claimed that the relationship was linear by looking at the type of the graph, which was a line. Then, the teacher asked if the students could give evidence to the claim that the relationship was linear by looking at the table, and the following dialogue took place in the discussion:

> *Teacher*: Well, we saw that connecting the dots resulted in a line graph. Can we understand that it is linear by looking at the table?
> *Sinem*: (Shows the values on the table drawn on the board). When we look at the values on the table vertically, they go up at the same proportion.
> *Teacher*: What do you mean by, "they go up at the same proportion?"
> *Sinem*: It means that while one value goes up by 1-1-1, the other one goes up by 2-2-2 all the time.
> *Seval*: That is, the amount of change (in both values) is the same.
> *Teacher*: So, the amount of change is 1 here (on the first row on the table) and 2 here.
> Students: Yes
> *Teacher*: And that results in being linear, you say.
> *Sinem*: Yes.
> *Teacher*: Like we did in yesterday's graphs.

In the dialogue above, Sinem's data included the relationship between the change in the amounts of the two values (the distance traveled and the money paid), and she referred to it as "going up at the same proportion." Since it was not clear what she meant by this, the teacher asked her to provide a warrant for her argument. Upon this request, she expressed that while one of the values (i.e., distance) goes up by ones, the other value (i.e., money) goes up by twos in her warrant. Then, Seval provided backing by stressing that the amount of change was the same for both values. Therefore, these students' claims included the idea of representing the non-proportional linear relationships with linear graphs whose equations were in the form of $y = mx + n$. Furthermore, they referred to the idea of the constant rate of change (i.e., slope) in linear graphs; yet, they were not pushed for further warrants/backings since the ideas related to the rate of change was beyond the grade level and the trajectory.

In the following part of the lesson, the students were asked whether the relationship between the amount of money paid, and the distance traveled was proportional in this situation. Several students claimed that the relationship was proportional, whereas most students claimed that it was not proportional. The teacher engaged students in a debate by asking everyone to raise their hands for either of the claims. A majority of the students raised their hands for the second claim, and a few students raised their hands for the first claim. Students from the supporters of both claims were challenged to provide data and warrants for their claims, and the following argumentation process was observed in the classroom:

> *Merve*: It asks if the distance traveled, and the money that needs to be paid is proportional.
> *Ozan and a few students*: Yes, it is proportional.
> *Seval*: No, it is not proportional.
> *Teacher*: Raise your hand if you say it is proportional.
> (Ceyda, Aylin, Ilter, Ozan, Musa, Giray and a few more students raised their hands)
> *Teacher*: Raise your hand if you say it is not proportional.
> *Teacher*: Why do you think that it is proportional, Musa?
> *Musa*: A proportional relationship is related to the multiplication of multiples.
> *Merve*: It is not about the multiples here. (Writes the values in the table as ratios) 1 over 5, 2 over 7, 3 over 9. These are not proportional.
> *Gizem*: But it increases by 1 in the upper part while it increases by 2 in the lower part.
> *Merve*: Look (showing the ratios she wrote on the board), it is not proportional. When I expand 1 over 5, I cannot obtain 2 over 7. If it was proportional, the price would be doubled when the distance was doubled.
> *Ceyda*: So, the 3 at the beginning destroys the proportionality, right?

In the dialogue above, Musa provided data to his claim that the relationship between the distance traveled and the money to be paid to the taxi driver for the distance traveled was proportional by saying that proportional relationships include multiples. Merve immediately disagreed with him by saying that it had nothing to do with multiples and justified her reasoning by writing the values in the table as ratios $\frac{1}{5}$, $\frac{2}{7}$, and $\frac{3}{9}$ and saying that these ratios were not proportional. Gizem challenged her by saying that the upper part was increasing by ones while the lower part was increasing by twos, so it had to do something with being proportional. Merve rebutted her by showing the ratios she wrote on the board and stressing that expanding $\frac{1}{5}$ would not give the ratio $\frac{2}{7}$. Ceyda agreed with her and concluded that the 3 at the beginning ruined the proportionality. This was an instance where different rebuttals yielded a new claim regarding the impact of the constant at the beginning on ruining the proportionality, which was schemed as the following:

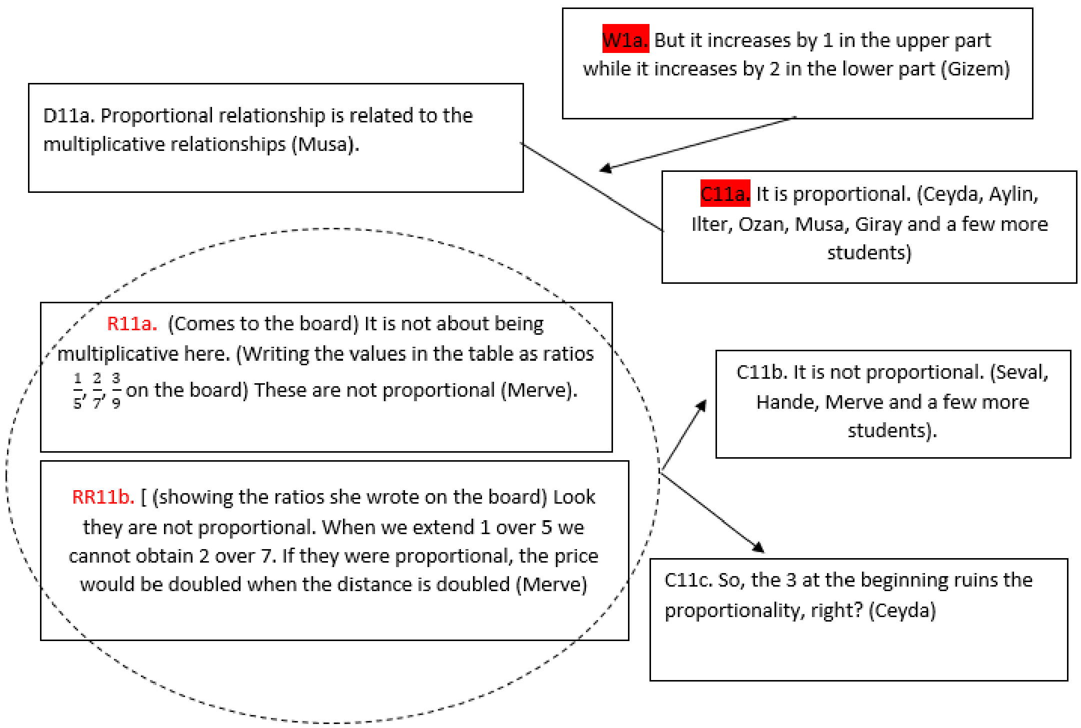


Figure 4.40. Toulmin's analysis model where different rebuttals yield a new claim

The discussion was moved to another direction when the teacher asked the classroom how they could decide whether the relationship was proportional by looking at the graph, and the following discussion emerged in the classroom:

> *Teacher*: How can you understand whether it is proportional by looking at the graph?
> *Sinem*: (For a relationship) To be proportional, it should pass through the origin.
> *Students*: Yes. To be proportional, it should pass through the origin.

In this dialogue, Sinem claimed that [to be proportional] the graph had to pass through the origin, and many students agreed with her. Continuing to work with non-proportional and proportional relationships and their representations, the students were engaged in a task wherein they were required to fill in tables and write corresponding equations and real-life situations for the given line graphs on Day 10. The first graph was related to a multiplicative relationship with an equation of y = 3x, and the students filled in the table by using the points on the graph and interpreting them as linked composites to be written in the table. As the first claim, Ozan claimed that the equation would not have a constant value since the graph passed through the origin. In the following instances, as they filled in the table as 1-3, 2-6, 3-9, and so on, most of them immediately found the equation as y = 3x. Emre made a claim that the equation would be y=3x in the whole class discussion based on the explorations with the values written in the table (see Figure 4.41 below), and none of the students challenged him to provide a warrant. Then, as real-life situations, several students made claims about situations that would represent the relationship given on the graph: a domino moving 3 meters in a minute (Giray), a person skating 3 meters in a minute (Musa), an old woman walking 3 meters in a minute (Ali), a tailor sewing three-meters-long curtain in a minute (Tunay). None of the students asked for any warrants, and no one asked them to do so.

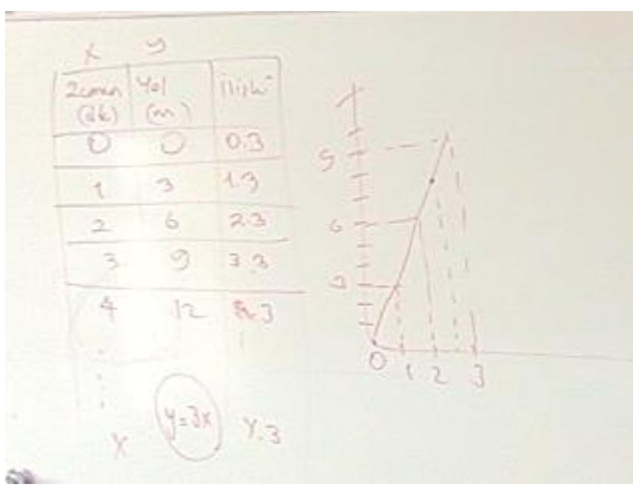


Figure 4.41. Filling in the table and writing the equation of the multiplicative relationship given on a line graph

As another type of graph, students were given the line graph of the equation y = 2x +8 (only the graph not the equation) and asked to fill in the table, write the equation, and propose a related real-life situation similar to the task above. Looking at the graph, Berk immediately claimed that the equation would have a constant since the graph did not pass through the origin, and several students went on by making claims about the equation of the relationship as follows:

> *Teacher*: Berk, what do you see when you look at the graph given?
> *Berk*: (Looking at the graph given) It doesn't pass through the origin, so there is a constant in its equation. Let's say that it is a relationship where a plant grows 2 cm in a month since it goes up by 2s on the y-axis.
> *Teacher*: What do you think, Elif?
> *Elif*: I determined the points that are above the numbers. 0 to 8, 1 to 10, 2 to 12, 3 to 14, and filled in the table. So, it goes up by 2s each time.
> *Teacher*: OK. Let's decide on the real-life situation first. You explain it, Ilter.
> *İlte*r: We buy fish from a pet store. Its height is 8 cm at the beginning, 10 cm after a month, and 12 cm in the second month.
> *Esra*: A newborn lizard is 8 cm, and it grows 2 cm each month.
> *Teacher*: How did you fill in the table, Esra?
> *Esra*: 0 to 8; the lizard is 8 cm at the beginning. One month passes, and it grows 2 cm, and its length is 10 cm, so it is 8 +2 = 10. Then, at the end of two months, it again grows by 2 cm and becomes 8 +2 +2 12 cm; then after three months, it is 8 + 2 +2 + 2 = 14 cm and so on, and then at the end of x months, it becomes 8 + 2x cm in terms of length since we add 2 in each step. It's easier to multiply by 2 than to write 2 plus 2 plus 2 for each month.
> *Teacher*: Then, I will write the steps as 8 + 2, 8+2.2, 8+2.3 in the table (See Figure 39 below).
> *Teacher*: Then, what is the equation, Faruk?
> *Faruk*: y = 8 + 2x
> *Ceyda*: Or, y = 2x + 8

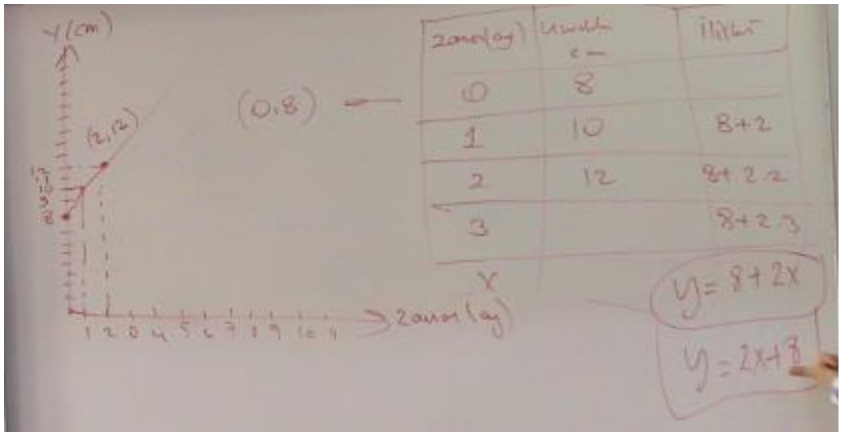


Figure 4.42. Filling in the table and writing the equation of the non-proportional relationship given on a line graph

As deduced from the classroom discussion above, Berk immediately claimed that the equation would have a constant in it since it did not pass through the origin, and referred to the rate of change as 2 cm in a month. Then, Elif took the points on the graph and put them in the tablet to interpret them as linked composites and agreed to Berk's idea about the rate of change. Then, upon the teacher's request, İlter came up with a real-life situation that included the length of a pet fish in months, and Esra came up with a similar context that included the length of a newborn lizard in months. Esra continued to interpret the points on the graph in accordance with the context she proposed by referring to the constant change in each month. She then claimed that the algebraic expression for the length of the lizard would be 8 +2x at the end of x months. Eventually, Faruk and Ceyda expressed the corresponding equations as y = 8 + 2x or y = 2x + 8. Therefore, when the two tasks and related dialogues on Day 10 were explored, it was seen that the students collectively constructed the equations for multiplicative and non-proportional relationships in the forms of y = mx, and y = mx + n respectively, and the warrants were removed from the conversation. In addition, there is also similar evidence from Day 11 wherein the students worked on several problems that included proportional and non-proportional relationships, including the famous problem "Sue and Julie were running equally fast around a track. Sue started first. When she had run nine laps, Julie had run three laps. When Julie had completed fifteen laps, how many laps had Sue run?" (Cramer & Post, 1993). Therefore, it was evident that the two ideas related to representing proportional relationships with linear equations of the form y=mx and graphs passing through the origin and non-proportional relationships with linear equations of the form y=mx+b and graphs not passing through the origin were taken-as-shared among the classroom.

Thus, Throughout Days 8 to 11, the focus of the classroom discussion and students' reasoning were shifted from finding the missing values by using covariation and invariation on the ratio tables and symbolic proportion representation to representing multiplicative and non-proportional relationships in algebraic, tabular and graphical ways and exploring the relationships among these different representations. Accordingly, students' reasoning was altered in such a way that the focus of the classroom discussion was on representing relationships with symbols, tables, and graphs. However, as students reasoned about and with these different representations, they drew on the two previous ideas: Linking composite units and iterating linked composite units and Invariant multiplicative reasoning- the relationship between the two composite units is invariant while filling in the tables and discovering the relationships on these tables. However, the three ideas, which are (1) Formalizing the invariant multiplicative relationship into an equation, (2) Representing proportional relationships with linear equations of the type y=mx and graphs passing through the origin, (3) Representing non-proportional linear relationships with linear equations of the type y=mx+b and graphs not passing through the origin, felt different than those and every other idea, and it was understood that these three ideas were related to the general activity of "Coordinating the relationships among the algebraic, tabular and graphical representations of multiplicative and non-proportional relationships." Therefore, the three ideas, together with the previous two ideas that belonged to CMP 1, were put together and organized around the common activity of "Coordinating the relationships among the representations" as they emerged in the discussion and became taken-as-shared in the classroom community.

4.4. CMP 4: Extending covariation and invariation to continuous contexts

Although the previous contexts (i.e., fish-food bar, missing value word problems, etc.) were helpful for the ideas of iterating composite units, they were fairly less helpful for the invariant multiplicative relationships since the quantities were mostly discrete (i.e., it does not make sense to have 3,2 fish or students) rather than continuous. Therefore, scaling problems with continuous values were introduced for the first time on Day 12 of the instruction in order to support students' understanding of the invariant multiplicative relationship between the quantities. Hence, Mathematical Practice 4 emerged as the classroom discussion revolved around finding the missing length when the two shapes were known to be similar and deciding on whether the given pictures were similar to the original picture. This mathematical practice included three taken-as-shared ideas:

- Reasoning with within-shape and between-shapes ratios to find missing side lengths of similar shapes,
- Conceptualizing the distortion of shapes.

TAS idea 1: Reasoning with within-shape and between-shapes ratios to find missing side lengths of similar shapes. The first question included two pictures, and it was stated that the two pictures were similar. The first shape had side lengths of 8 cm and 6 cm, and the second one had its short side length as 3 cm. The students were asked to find the missing long side length of the second shape, and the following collective argumentation process took place in the classroom:

> *Ozan*: It is 3 (cm) since the shape is shrunk to its half.
> *Emre*: What do you mean by "it is shrunk to its half?"
> *Ozan*: We can make a table. A table of the long side length and the short side length (Draws a short ratio table on the board, See Figure 40). 8 became 4, and it means that it was reduced to its half (Makes a hand gesture like drawing a horizontal curve from 8 to 4 and on the second row of the table). Then, 6 should also be reduced to its half and become 3.
> *Berk*: They should be proportional.
> *Teacher*: What does it mean, "they should be proportional?"

Berk: When we write the ratios, they should be proportional (Writes the proportion $\frac{Long}{Short} = \frac{8}{6} = \frac{x}{3}$ on the board). So, the long side length should be 4 (cm).
İlter: Why do they have to be proportional? It says the shapes are similar, not proportional.
Berk: The side lengths of the second shape is half of the side lengths of the first shape since the shape is reduced to its half from the sides and the top (makes a hand gesture to shrink the shape to its half horizontally and vertically).
Teacher: Yes, the first picture is halved in both directions horizontally and vertically, and the second shape is obtained. So, the shape is shrunk to its half. Halving the shape means multiplying the side lengths with $\frac{1}{2}$. Each of the lengths is multiplied by $\frac{1}{2}$ in order to obtain a similar shape. We will call this number that we multiply the lengths with, "scale factor." It is the number that we use to scale the lengths in order to obtain a similar shape.

In this dialogue, Ozan claimed that the missing length was 4 cm since the second shape was obtained by shrinking the first shape to its half. When he was challenged to further justify his reasoning by Emre, he came to the board and drew a short ratio table as demonstrated in Figure 4.42 below:

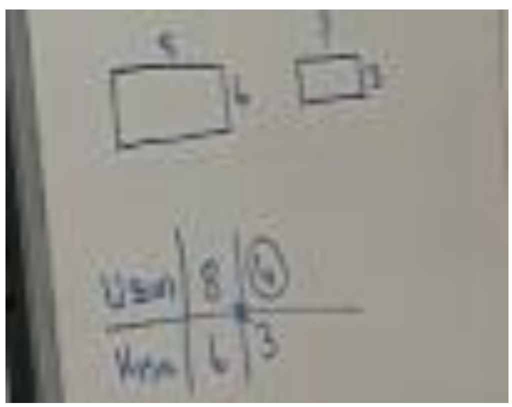


Figure 4.42. Partnering values between situations in the context of similar shapes

In this short ratio table, Ozan partnered the corresponding lengths (long-long and short-short) between the two shapes by making a hand gesture. Then, he justified his claim by reducing both side lengths into half. Then, Berk provided backing by stressing that the shapes had to be proportional. When he was asked what it meant for the shapes to be proportional, he set up a proportion by writing the ratios of the long side lengths to the short side lengths within each shape as $\frac{Long}{Short} = \frac{8}{6} = \frac{x}{3}$, which acted as alternative data for Ozan's claim. It was still not clear how the shapes' being similar required that their side lengths be proportional in the classroom community as deduced from İlter's challenge about what being proportional had to do with being similar for the two pictures. Then, Berk provided a further backing by saying that both the short side length and the long side length of the first picture were halved in order to obtain the second picture, which meant that the picture was halved in both directions horizontally and vertically. He also used a hand gesture for shrinking to halve the side lengths in both directions. Here, even though he wrote the ratios of lengths within each shape, he partnered the lengths between the two shapes and hence reasoned with the scale factor *between* the two shapes. Therefore, the teacher revisited the term *scale factor* in this context and referred to it as the number used to scale the lengths of a shape in order to obtain a similar shape.

In the following few questions with whole number scale factors, the students partnered the values between shapes and reasoned with between-shapes ratio without needing further warrants and/or backings. However, when a question included a non-integer scale factor (i.e., $\frac{3}{4}$ or $\frac{4}{3}$), the students started to investigate the ratios within the shapes. The third question included the side lengths of the first picture as 8 cm and 4 cm, and the long side length of the second shape was given as 12 cm. The short side length of the second figure was asked. Gizem started to reason with within-shapes-ratio, and below is a dialogue in which multiple students contributed to the discussion:

> *Gizem*: The missing length is 6 cm. Since the ratio between the lengths of the first shape is 2, the ratio between the lengths of the second shape should be 2 as well.
> *Murat*: You mean, like a vertical relationship?
> *Gizem*: Yes. A vertical relationship.
> *Teacher*: What does the vertical relationship mean here? Can you explain it on the board?
> *Gizem*: (Comes to the board and writes the proportion $\frac{Long}{Short} = \frac{8}{4} = \frac{12}{x}$) Since this (points to 4 on the denominator) is half of this (points to 8 on the numerator), this (points to x on the denominator) should be half of this (points to 12 on the

numerator). Therefore, (draws a vertical arrow from 8 to 4) since this is divided by 2, 12 should be divided by 2 as well (draws a vertical arrow from 12 to x).
Teacher: Yes, but what does it mean for the shapes?
Ali: The long side lengths are twice as much as the short side lengths in both shapes.
Teacher: Yes, that's right. This vertical relationship is again about the invariant relationship between the values.
Sinem: We can also say that there are 2 units of horizontal lengths for every 1 unit of vertical lengths.
Teacher: What do you mean, Sinem? Can you explain it on the board?
Sinem: (Comes to the board and draws two rectangles having the lengths of 8 cm and 4 cm, and 12 cm and 6 cm on the board) There is 1 unit of vertical length for every 2 units of horizontal lengths in this shape (makes a hand gesture to trace the side lengths of the first shape). There should be 1 unit of vertical length for every 2 units of horizontal lengths in this shape, too (makes a hand gesture to trace the side lengths of the second shape, see Figure 4.43 below).
Teacher: Does everybody understand this?
Students: Yes.

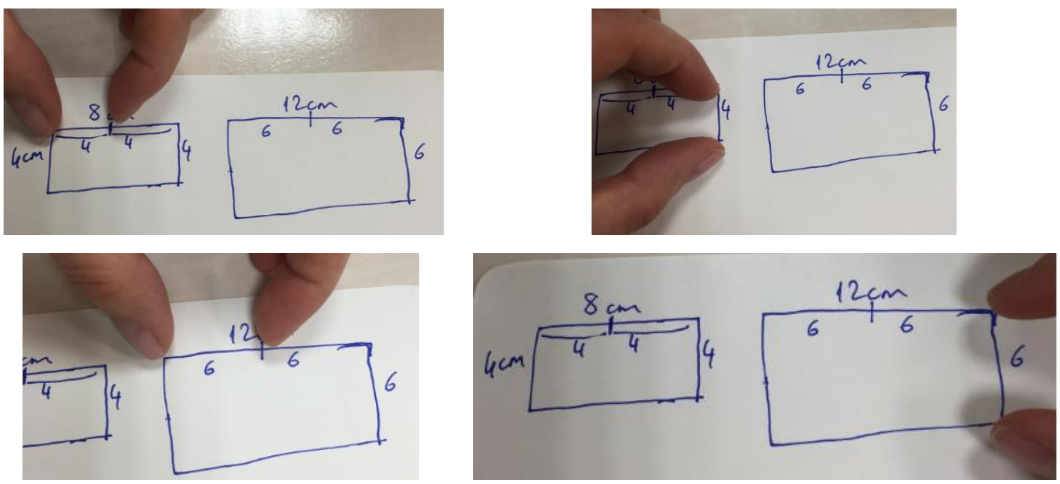


Figure 4.43. Tracing the unit lengths in similar shapes (the researcher's drawing)

In terms of Toulmin's analysis, the discussion above was schemed as follows in the Figure 4.44:

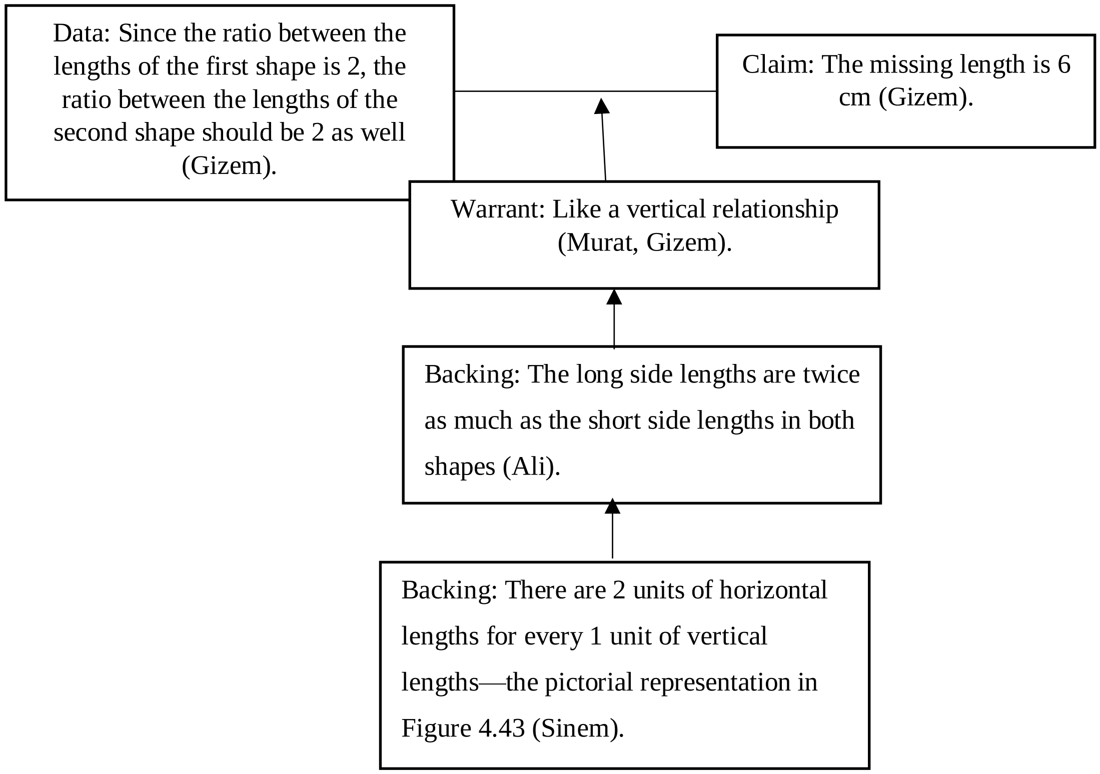


Figure 4.44. Toulmin's analysis scheme for the idea of reasoning with within-shapes-ratio

As understood from the discussion above and the related figures, Gizem claimed that the missing length was 6 cm by providing data and expressing the ratio of the lengths within the first shape and applying the same ratio in the second shape. Murat provided a warrant by referring to this ratio as the vertical relationship. When the teacher asked what the vertical relationship meant in that context, she referred to the "vertical times $\frac{1}{2}$ relationship" between the lengths of each shape. However, the teacher was not satisfied with this answer and asked for further backing regarding the meaning of the vertical relationships for the (similar) shapes. Upon this request, Ali referred to the invariant multiplicative relationship between the lengths of each shape, which acted as data to Gizem's claim. In addition, Sinem provided a backing in which she stressed that there was 1 unit of vertical lengths (short length side drawn vertically) for every 2 units of horizontal lengths (the long length side drawn horizontally) in both shapes and used hand gestures to trace each of the vertical 1 and horizontal 2 units in both shapes as seen in Figure 4.43. None of the students further challenged Ali or Sinem. Therefore, all students partnered the values within a shape and reasoned with within-shapes-ratio in their data, warrants, and backing(s).

In the following questions, multiple students continued to partner the lengths of the shapes and reason with within-shape and between-shapes ratios without needing and/or providing warrants. This showed us that partnering the values within a situation/setting or between situations/settings and reasoning with within-shape ratios and between-shapes ratios were taken-as-shared among the classroom. In addition, these classroom discussions also showed us that the previous ideas regarding the iteration of composite units and multiplicative reasoning (i.e., reasoning with the invariant multiplicative relationship between the values) were taken-as-shared since they shifted place from claims to data, warrants, and backing in this session.

TAS Idea 2. Conceptualizing the distortion of shapes. In the first session of Day 12, the students found the missing lengths of similar shapes by reasoning with within-shape and between-shapes ratios. In the second session, the students were presented with an original picture and its various copies. The students were asked to determine whether each copy was [mathematically] similar to the original picture. Students had already discussed checking for proportionality on Day 6 while they were checking whether the given ratios of the number of boys and girls belonged to their class. Now they were continuing to check whether the side lengths of two shapes were proportional in order to decide whether the two shapes were similar. The teacher launched the second session by saying that all the pictures were the same printed differently, as illustrated in the following dialogue:

> *Teacher*: My question is this: There are various copies of this (original) picture. Some of them are stretched, some of them are shrunk. In some of them, the (original) picture is stretched only from its length, in some of them, from its width,

and sometimes from both. Which of these resembles the kid (in the picture) the most?
Berk: The original picture.
Teacher: Yes. Nevertheless, there is one original picture, and it's hung on the board. You have its different copies.
Student: The one that is stretched from both directions resembles the most.
Teacher: Is it enough to stretch from both directions? I would like you to pay attention to this. Should I stretch more from one direction than the other direction? Should I stretch in the same amount from both directions? Or what should we do? Let's discuss with our group members. I would like you to explain why they are similar or not.

As understood from the dialogue above, a student claimed that the pictures that were stretched from both directions would resemble the kid himself the most as soon as the teacher launched the problems. Then, the teacher asked her whether it was enough to stretch from both directions and how this stretching had to be done in order to provoke the students about the rate of change in each direction. After the students discussed with their peers in small groups, the whole class discussion was started with the first picture. The original picture is provided in Figure 4.45 below.

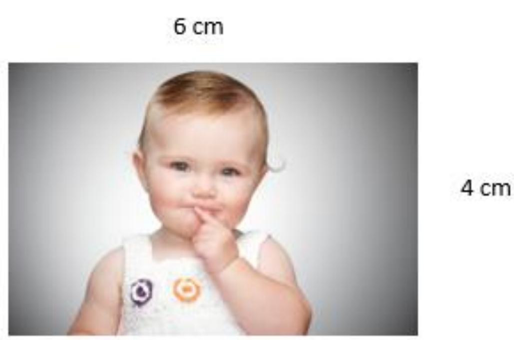


Figure 4.45. The original picture to be compared to its copies in order to check for similarity

As seen in the figure, the side lengths of the original picture were 6 cm and 4 cm. The first picture to be explored included side lengths of 9 cm and 4 cm. Hande claimed that the two pictures were not similar since the short side lengths were the same in both pictures, but the long side lengths were different. Since how this data led to that claim did not make sense for all students, the teacher asked Hande for a warrant. She justified her reasoning by saying that the [original] picture was stretched only in one direction to obtain the second picture and made the corresponding hand gesture for stretching horizontally. She also said that the kid in the picture looked fatter since the picture was stretched only horizontally. She was not challenged after this warrant, and the teacher took answers for the second picture.

The second picture had both side lengths of 6 cm. Giray claimed that the two pictures were not similar since the second picture was stretched only in one way, similar to the first picture. He further justified his answer by saying that the kid in the picture looked taller and thinner compared to the original picture since it was stretched only vertically, and he made a hand gesture for stretching vertically. Ozan brought up the idea that people do these kinds of tricks, especially on social media, in order to look taller and thinner, but they look different on those photoshopped photos than they actually do. Elif took the floor and provided alternative data for why the two pictures were not similar by saying that the second picture was a square, whereas the original picture was a rectangle. Therefore, in the first two questions, the students intuitively reasoned about the distortion of the shapes when they were stretched only in one direction.

The third picture, compared with the original picture, had side lengths of 8 cm and 12 cm, whereas the original picture had side lengths of 4 cm and 6 cm. Sezin claimed that the two pictures were similar since the kids in the two pictures looked exactly the same, neither fatter nor taller. Musa claimed that the shapes were similar since that copy was the doubled version of the original copy. He also stressed that the short and long side lengths of the two shapes formed a proportion such as $\frac{Short}{Long} = \frac{4}{6} = \frac{8}{12}$, and no one challenged him to provide a warrant. The fourth and the last picture had side lengths of 13 cm and 15 cm whereas the original picture had side lengths of 4 cm and 6 cm. Merve immediately checked whether the between-shapes ratios were equal such as : $\frac{4}{13} \stackrel{?}{=} \frac{6}{15}$

and then stressed that they were not equal $\frac{4}{13} \neq \frac{6}{15}$. This was the first time that the not equivalent symbol (i.e., ≠) emerged in the discussion throughout the implementation. Since these values for the side lengths were intentionally chosen to reveal incorrect additive reasoning but it did not come out at the first place, the teacher intentionally posed the following question: "A student from another seventh-grade class told that the shapes were similar since both lengths of the original picture were increased by 9 cm. What do you think about his thinking?" Below is a discussion in which multiple students contributed to the rebuttal of an incorrect additive reasoning strategy:

> *Merve*: That is wrong. The original shape was distorted when both lengths were increased by 9 cm. 9 cm is more than two times of 4 cm (the short side length of the original picture) but less than two times of 6 cm (the long side length of the original picture). So, the long side length should be increased by a greater amount than 9 cm.
> *Sinem*: It is like making a 20 TL discount from both 100 TL and 50 TL. It affects this (50 TL) more. We should shrink more from the longer side (than the shorter side)

Therefore, as understood from the discussion above, Merve and Sinem were able to provide rebuttals for an incorrect claim mooted by the teacher. Moreover, in these rebuttals, they reasoned multiplicatively and with between-shapes- and within-shapes-ratios, which showed that the ideas of multiplicative reasoning (i.e., reasoning with the invariant relationship between the values) and reasoning with within-shape and between-shapes-ratios were taken-as-shared.

On the other hand, in terms of evaluating the instruction on Day 12, it could be deduced that the idea that the shapes were distorted when the shapes were stretched additively was very intuitive at the beginning of the class since the students only mentioned distortion when the shapes were stretched (or shrunk) in one direction. They claimed that the shapes (persons or animals in the pictures) got fatter when the shape was stretched only horizontally, and they got taller when it was stretched only vertically. As the discussion moved on, they came to the point that the shapes had to be scaled with a particular factor in both directions in order to obtain similar shapes. In other words, they made claims about the fact that the shapes should be stretched (or shrunk) in both ways proportionally. Then, they used these claims as data and/or warrants for other claims for the following questions and in the following classes.

Therefore, the mathematical practice analysis of the classroom instruction on Day 12 showed that the classroom discussion shifted from coordinating the representations of the proportional and additive relationships to reasoning about the concepts of similar shapes and distortion on that day. Thus, the two ideas of (1) Reasoning with within-shape and between-shapes ratios to find missing side lengths of similar shapes and (2) Conceptualizing the distortion of shapes, were put together to form the fourth classroom mathematical practice of "Extending covariation and invariation to continuous contexts" as they emerged and became taken-as-shared among the classroom community. On the other hand, this dialogue shows that this task posed the term, "similar" without letting students explore what it means in informal contexts. This implied a revision in the order of this task in order to let students explore the ideas of stretching and shrinking before they are presented with the word "similar" and use tables and proportion to construct within-shape and between-shapes ratios.

4.5. CMP 5: Comparing rates/ratios and deciding which one is bigger /smaller /equal.

The students already compared different rates/ratios on Day 6 and Day 12 of the instruction; yet, the main focus was deciding whether the rates/ratios formed a proportion. Hence, they did not decide which rate/ratio was bigger/smaller/equal. They worked on three different contexts on Days 13, 14, and 15, all of which were focused on comparing rates/ratios and deciding which one was bigger/smaller/equal. The classroom discussion on Day 13 started with a task that would require using unit rate as a tool/anchor/benchmark for comparing the speeds of objects/people intuitively on the first day. In the following days, the students drew on their experiences with the concept of speed in different contexts, such as the situations that involved deciding on best buy and comparing the tastes of mixtures. Throughout the three days, students made connections with their fraction knowledge, especially while comparing the rates/ratios and deciding which one was bigger/smaller/equal. However, the speed and best buy contexts did not involve the fraction concept since there were no part-whole relationships in these contexts. During the last day (i.e., Day 15), the students made a distinction between fractions and ratios/rates as they were engaged in a mixture context that included part-part and part-whole ratios/rates. Therefore, Mathematical Practice 5 emerged as students worked on comparing different rates/ratios and determined which one was bigger/smaller/equal in different contexts on Days 13, 14, and 15. One mathematical idea became taken-as-shared while this practice became established in the classroom community on the last three days of instruction:

- Creating and reasoning with equivalent ratios to compare quantities.

TAS Idea 1: Creating and reasoning with equivalent ratios to compare ratios/rates On Day 13, the students worked on finding the interval(s) in which the car had the highest speed when it was given that the car completed a travel of 180-km-distance (Interval 1-Ankara/City 1-Bolu/City 2) in 2 hours, 70-km-distance (Interval 2-Bolu/City 2-Adapazarı/City 3) in 1 hour, and 180-km-distance (Interval 3-Adapazarı/City 3-İstanbul/City 4) in 3 hours. The values were selected to form integer ratios. Elif claimed that the car was the fastest in the first interval since dividing the distances by the corresponding times gave the greatest result for the first interval among all such that the speed in the first interval was 180:2 = 90, the speed in the second interval was 70:1 = 70 and the speed in the third interval was 180:3 = 60. When the teacher asked her what those numbers meant, she explained that these numbers were the unit ratios/rates which stood for the speeds of the car in three intervals. Ilter added to that answer by saying that the car had the slowest speed in the third interval since 60 was the smallest number among the three. These class argumentations showed that the idea of creating and using unit rate as a tool/anchor/benchmark for comparisons was taken-as-shared since this idea was used as a warrant for a new claim.

The teacher then asked the classroom to compare the speeds of the car in the first and second intervals as she wanted to encourage students to develop different strategies. Selim suggested a solution method in which he made use of the equal distances of these two intervals, which acted as data for his claim. Below is a dialogue in which this argumentation process took place:

> *Teacher*: Well. Can you compare the speeds in the first and third intervals by using a different strategy?
> *Selim*: Since the distances are the same, I can look at the time. 180 km is traveled in 2 hours in one, and in 3 hours in the other.
> *Merve*: It means that we would be faster when we travel (180 km) in 2 hours. It is like dividing a cake into two pieces and eating one slice and dividing the same cake into three pieces and eating one slice.
> *Sinem*: I can say that it is faster (in the first interval) since it travels the same distance in a shorter amount of time.

As understood from the dialogue above, Selim claimed that the car had a higher speed in the first interval since it traveled 180 km in 2 hours in the first interval and 3 hours in the third interval. Merve provided a warrant for this claim by making a connection with fraction imagery by saying that it was like dividing a cake into two pieces and eating one slice and dividing the same cake into three pieces and eating one slice, which acted as a strategy for deciding on the bigger rate. Even though the concept of speed was not conceptually connected with the concept of fractions, she drew on her previous knowledge regarding ordering the fractions while comparing and ordering rates/ratios. Sinem made it more explicit by providing a warrant for Selim's data and claim by saying that the car traveled the same distance in a shorter time in the first interval.

In the second question, the information about different distances walked by four people in different time periods was given, and the students were asked to order the four people from the slowest to the fastest. It was given that Ahmet walked 10 km in 2 hours, Beren walked 14 km in 4 hours, Ceylan walked 15 km in 5 hours, and Derya walked 24 km in 6 hours. As an answer to the question, Faruk claimed that the four people could be ordered as Ceylan, Beren, Derya, and Ahmet from the slowest to the fastest based on his work that included the division of the distances by the time periods and comparison of the results. He also included informal language regarding the concept of unit rate (i.e., the distance walked in an hour) in his answer as can be understood from the following classroom discussion:

> *Faruk*: (Divides the distances by the time periods for each people) We should divide 10 by 2 and obtain 5. This is the distance (in kilometers) that he walks in an hour.
> *Teacher*: Can you write it using the proportion representation on the board?
> *Faruk*: (Writes the proportion $\frac{2}{10} = \frac{1}{5}$ on the board).
> *Teacher*: Can you write the units next to them?
> *Faruk*: (Writes hour next to 1 and km next to 5). I am dividing the second one, too. I divided 14 by 4, and it is 3.5. It means that she walks 3.5 km in an hour. I divided 15 by 5, and it means that she takes 3 km in an hour. When 24 is divided by 6, the result is 4 (writes the corresponding unit rates, i.e., speeds, on the board, as shown in Figure 4.46). Then, the slowest is Ceylan, then Beren, Derya, and Ahmet (orders these people from the slowest to the fastest by using less-than symbols as Ceylan < Beren < Derya < Ahmet).

He also wrote the unit rates (i.e., speeds) and used less-than symbols, as shown in Figure 4.46 and Figure 4.47 below.

Ahmet	Beren	Ceylan	Derya
$\frac{10\ km}{2\ hours} = \frac{5\ km}{1\ hour}$	$\frac{14\ km}{4\ hours} = \frac{3.5\ km}{1\ hour}$	$\frac{15\ km}{5\ hours} = \frac{3\ km}{1\ hour}$	$\frac{24\ km}{6\ hours} = \frac{4\ km}{1\ hour}$

Figure 4.46. Writing the unit rates in order to find the speeds of the persons

Later on, the teacher made the same move by asking the students to compare the speeds of Ahmet and Ceylan to encourage the use of different strategies. Musa stated that "Ahmet walks 10 km in 2 hours and Ceylan walks 15 km in 5 hours; so I multiplied 10 by 3 and 15 by 2 and made the distances equal. Ahmet walks 30 km in 6 hours, whereas Ceylan walks 30 km in 10 hours." He claimed that "Ahmet is faster than Ceylan since he walked the same distance in a shorter time." When the teacher asked Musa to come to the board and write the symbolic calculations of his reasoning, he came to the board and wrote the corresponding rates as in the following figure:

Ahmet	Ceylan	
$\frac{2\ hours}{10\ km} = \frac{6\ hours}{30\ km}$	$\frac{5\ hours}{15\ kms} = \frac{10\ hours}{30\ km}$	Ahmet > Ceylan
(3)	(2)	

Figure 4.47. Equalizing the distances walked in order to compare the speeds

As can be seen from his solution in Figure 4.47 above, Musa compared the speeds of two persons by writing the rates of time periods to the distances taken and then changing each distance to a common numerator of 30 km. On the other hand, "less than" and "greater than" signs (i.e., <, >) emerged on Day 13, only a day after the not-equal-sign (i.e., ≠) emerged in the classroom discussion.

In the second session of Day 13, the students worked on a context that involved deciding on the best buy for different kitchen ingredients such as rice, yogurt, sugar, and vegetable oil. The prices for different amounts of these ingredients in three different supermarkets were given, and the students were asked to decide on the best buy for each of the ingredients. Ali wrote the corresponding amounts of rice per Turkish Lira (TL) in three supermarkets as $\frac{18\ TL}{3\ kg} = \frac{6\ TL}{1\ kg}$ (Supermarket Sevgi); $\frac{14\ TL}{2\ kg} = \frac{7TL}{1\ kg}$ (Supermarket Dost); $\frac{25TL}{5\ kg} = \frac{5\ TL}{1\ kg}$ (Supermarket Cicek) and claimed that the best buy could be made from Supermarket Cicek. The teacher asked for a warrant for those calculations from Tolga, and he said that these unit rates showed how much rice could be purchased for 1 Turkish

Lira in all of the three markets. None of the students challenged him for further warrants/backings.

When the next question was asked to the students to determine the supermarket for the best buy for yogurt, Faruk wrote the corresponding unit rates on the board as $\frac{4\,TL}{1\,kg}$ (Supermarket Sevgi); $\frac{6\,TL}{2\,kg} = \frac{3\,TL}{1\,kg}$ (Supermarket Dost); $\frac{8\,TL}{4\,kg} = \frac{2\,TL}{1\,kg}$ (Supermarket Cicek) and claimed that the best buy for yogurt could be made from Supermarket Cicek as well. Nobody asked him for a warrant, and Ozan provided different data for the same claim by suggesting to make all the denominators 4kg as follows: as $\frac{4\,TL}{1\,kg} = \frac{16\,TL}{4\,kg}$ (Supermarket Sevgi); $\frac{6\,TL}{2\,kg} = \frac{12\,TL}{4\,kg}$ (Supermarket Dost); $\frac{8\,TL}{4\,kg}$ (Supermarket Cicek). Once more, nobody asked for any warrants. Therefore, the discussion revolved around equating the amounts of kitchen ingredients and comparing the prices for equal amounts of products. The class worked on deciding on the best buys for sugar and vegetable oil using the same strategies: creating common denominators of the amount and choosing the smallest numerator of the price, creating unit rates, and choosing the smallest numerator without proving or asking for any warrants and/or backings. However, none of the following possible strategies emerged in the discussions that revolved around the best buy context: creating common numerators and choosing the biggest numerator; writing the rates as kg/TL, finding the common denominators and choosing the largest denominator; finding the amount of kitchen ingredients that could be purchased with 1 TL (i.e., unit rate); and focusing on the invariant relationship between the numerator and denominator (i.e., ×2, ×3, etc.). This might be a consequence of the "best buy" context, which was based on saving money.

On Day 14, students engaged in a task wherein the mixtures of varying numbers of glasses of orange juice and water in five pitchers were given (see Figure 4.48 below). The task required the students to compare the relative orange taste of each pair of mixtures and decide which one had a stronger taste of orange.

	Pitcher A	Pitcher B	Pitcher C	Pitcher D	Pitcher E
					
Amount of orange juice	3 glasses	3 glasses	4 glasses	5 glasses	2 glasses
Amount of water	2 glasses	3 glasses	3 glasses	5 glasses	3 glasses

Figure 4.48. Five pitchers in the Orange Juice Task

Musa started by comparing the mixtures in pitchers A and B and claimed that A was more orangey since both had the same amount of orange juice, but A had less amount of water. This comparison was named as "basic comparison" since it did not include a comparison of the relative amounts of the ingredients; instead, it included a comparison of the amounts of one ingredient only. Merve suggested alternative data by constructing the ratios between the amounts of orange juice and water in each pitcher such that $\frac{orange}{water} = \frac{3}{2} > \frac{3}{3}$. When she was asked to provide a warrant regarding what those ratios had to do with the taste of oranginess, she explained that "*the bigger the ratio was, the more orangey taste the mixture had since the numerator was the amount of orange juice.*"

In order to compare the tastes of the mixtures in pitchers B and D, a classroom discussion was held including basic comparison and use of unit rate as a benchmark for the comparison, as presented below:

> *Berk*: (Pitcher) B includes 3 glasses of orange juice and 3 glasses of water. The amounts of orange juice and water are equal in (Pitcher) D as well. Then B and D are equal (in terms of the tastes of the mixtures they contain).
> *Teacher*: But D has 5 glasses of orange juice; that is, there is more orange juice (in D than in B). How come do they have equal tastes?

Berk: There is more orange juice, but there is more water, too.
Merve: Both are equal to one whole when they are compared in ratios. I am comparing them in terms of unit rate, and there is one glass of water for a glass of orange juice in both.
Sinem: Yes, there is a 1-1 ratio in both. A glass of water corresponds to a glass of orange juice in both.
Musa: Yes, there is one glass to one glass correspondence. Hence, their tastes are equal

The corresponding Toulmin argumentation scheme is constructed for the dialogue above, as shown below in Figure 4.49:

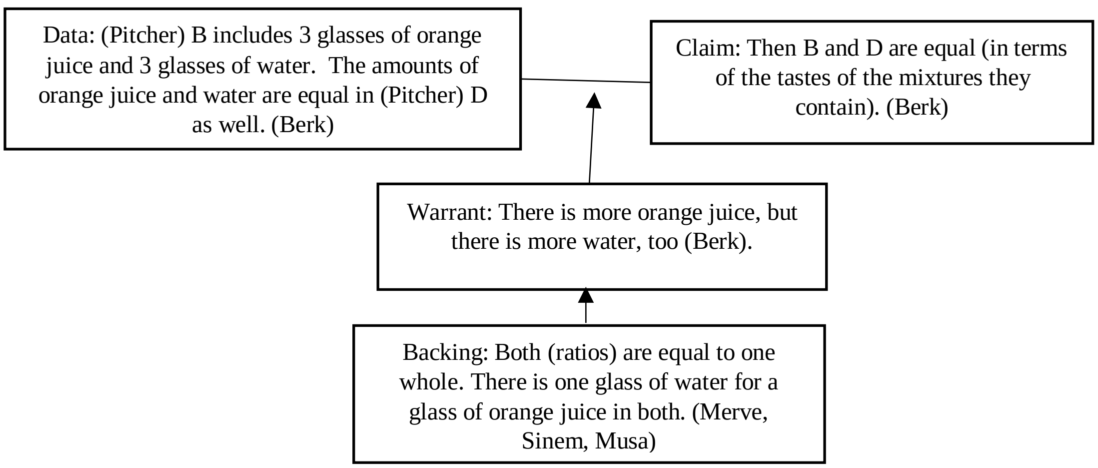


Figure 4.49. Toulmin's argumentation scheme for comparing ratios in the Orange Juice Task

In order to interpret the dialogue and the related argumentation scheme, Berk claimed that the two mixtures in Pitchers B and D had the same taste since the amount of water was equal to the amount of orange juice in both pitchers. Upon the teacher's request, Berk provided a warrant to his data and claimed that it included a comparison of the relative amount of orange juice and water in both pitchers. In order to justify this warrant, Merve and Sinem reasoned with the part-part ratios within the pitchers (i.e., the ratios of the amount of orange juice and water within each shape) and made a connection with the unit rate.

The values in pitchers A and C were selected in such a way that the difference between the amount of water and orange juice was the same in both, which would reveal incorrect additive reasoning. While comparing A and C, the teacher intentionally gave the floor to Gizem, whom she observed as using incorrect additive reasoning in the small group discussion. Below is a discussion in which Gizem was challenged and rebutted by multiple students:

Teacher: Gizem, can you tell us how you thought and solved this question?
Gizem: (Comparing the mixtures in Pitchers A and C) 3 glasses of orange juice and 2 glasses of water and 4 glasses of orange juice and 3 glasses of water. I think they are equal (in terms of orange taste).
A couple of students: I don't think they are equal.
Teacher: If you also think that their tastes of oranginess are the same, raise your hands.
(A couple of students raise their hands)
Teacher: Why do you think they are equal, Gizem?
Gizem: Because the difference is 1 in both.
Sinem: But we shouldn't look at the difference.
Ozan: If we write them as ratios $\frac{3}{2}$ and $\frac{4}{3}$, $\frac{3}{2}$ is $\frac{1}{2}$ more than one whole. $\frac{4}{3}$ is $\frac{1}{3}$ more than one whole. $\frac{3}{2}$ is larger than $\frac{4}{3}$. Hence, (the mixture in) Pitcher A is more orangey than (the mixture in) Pitcher C since the numerator is (the amount of) orange.
Merve: When I check if they form a proportion, $\frac{4}{3}$ and $\frac{3}{2}$ are not proportional.
Teacher: So, what do you mean by, "they are not proportional."
Merve: So, they don't taste the same.
Teacher: So, who thinks they taste the same?
(None of the students raise their hands)
Teacher: OK. So, we agree that they do not taste the same. Then, which one is more orangey?
Seval: A is less orangey than C. I equalized the (amount of) orange juice (in both pitchers) at 12 (glasses). I multiplied (the amounts of orange juice and water in Pitcher) A by 4 and got 12 glasses of orange juice and 8 glasses of water. When I made (the amount of orange juice in Pitcher C) 12, the (amount of) water became 9 (glasses). The more the water there is in it, the less orangey it is. So, since C includes more amount of water, it is less orangey (See Figure 4.50 below).

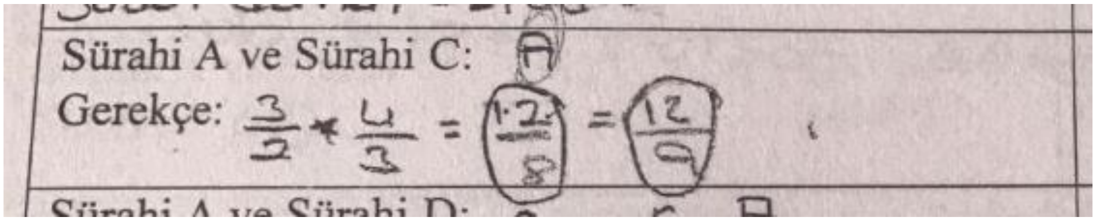


Figure 4.50. Equalizing the numerators in order to compare the ratios in the Orange Juice Task

In order to interpret this dialogue in terms of Toulmin's argumentation model, it is deduced that Gizem made an incorrect claim by using an incorrect additive reasoning in which she focused on the difference between the amounts of orange juice and the water in both pitchers. A couple of students immediately disagreed with her by claiming that their tastes were not equal. The teacher asked the students to raise their hands if they agreed with Gizem, and only a couple of students raised their hands. Sinem rebutted Gizem by stressing that the difference method should not be applied. Then, Ozan wrote the corresponding part-part ratios between the amounts of orange juice and the water in both pitchers and made a connection with the fraction imagery in order to compare the ratios. Merve took the floor and stressed that the two ratios did not form a proportion (i.e., $\frac{4}{3} \neq \frac{3}{2}$), so they did not taste the same. Upon this claim, the teacher asked how they could decide which of the mixtures was more orangey. Seval pointed an alternative strategy in order to decide which ratio was bigger, in which she changed the amount of orange juice and water so that both mixtures had equal amount of orange juice (i.e., 12 glasses of orange juice in both; 8 glasses of water in A, 9 glasses of water in C). Then, she claimed that A had a stronger taste of orange since the mixture in Pitcher C had more water in it than the mixture in Pitcher A.

For the comparison of another pair of mixtures (i.e., the mixtures in Pitchers C and D), Selim claimed that C had a stronger taste of orange than D because there was a one to one match in the Mixture D, but there was more than one glass of juice for a glass of water in the Mixture D. Nobody challenged Selim, and Ali immediately wrote and compared the corresponding ratios as follows: $\frac{orange}{water} = \frac{4}{3} > \frac{5}{5}$ in order to provide an alternative data to Selim's data and claim.

After a while, since the solutions of all students included the ratios between the amount of orange juice and the amount of water (i.e., the amount of orange is always the numerator) within a mixture, the teacher asked the class whether they could write the amount of water in the numerator and the amount of orange juice in the denominator. Merve quickly responded that when the ratio is constructed in such a way that the water is on the top and the orange juice is on the bottom; then, the smaller ratio would give a more orangey taste. Ceyda contributed to this discussion by stressing that the one with the smaller numerator would be more orangey. The teacher asked both students to justify their reasoning. Ceyda justified her reasoning by stressing that the ratio, then, would give the amount of water per one glass of orange juice. To this point on Day 14, the students discussed the relative amount of water and orange juice *within* each mixture. These comparisons included direct basic comparison (i.e., when either amount is the same in both mixtures), equalizing the amount of one ingredient in both mixtures and then making a basic comparison, and constructing common denominators/numerators in order to equalize either amount and then comparing it with the amount of the other ingredient. In other words, students reasoned with part-part ratios and the relative amount of one ingredient to the other within each mixture. On the other hand, the warrants/backings were removed from the class discussion by the end of the first session.

The second session of Day 14 started with the students' reasoning about the part-part ratios as similar to the discussions in the first session while comparing the mixtures A and E. In order to encourage the use of the part-whole ratios (i.e., fractions), the teacher asked the class to compare the orange juice concentrate in the mixtures. Erdem claimed that A had a more orangey taste than E since it had a larger concentration of orange juice. When he was asked to clarify the relationship between this claim and the data he provided, he stressed that both mixtures had 5 glasses of liquid. He went on by saying that, out of 5 glasses of liquid in both pitchers, there were 3 glasses of juice in A and 2 glasses of juice in E. Therefore, Erdem used the previous idea of reconceptualizing/coordinating the linked composites in a ratio in terms of the whole in a warrant for a later claim, which also gave us further evidence that the previous idea was taken-as-shared. Besides, it was the first time in the instructional sequence that the term concentration, which is the ratio of an ingredient to the total mixture and a part-whole ratio, emerged in the classroom discussion. At that instance in the instruction, the teacher highlighted the difference between a fraction and a ratio/rate by stating that the part-whole ratios could be considered as fractions while part-part ratios were not fractions since fractions are focused on the relationships between a whole and its parts.

Throughout the rest of the second session, the students worked on the second part of the activity, which was designed to reveal and remedy any incorrect additive reasoning. The students were asked to decide whether the tastes of the mixtures in five pitchers would change when a glass of water and a glass of orange juice were added to each of the mixtures and to determine the direction of the change if there is any change. Below is a portion of a class discussion in which a few students discussed whether adding a glass of water and a glass of orange juice would affect the taste of the mixture in Pitcher A:

> *Musa*: Dividing a cake into two pieces and eating one piece and dividing the cake into three pieces, and eating one piece would not be the same. So, it (the taste) would not be the same.
> *Teacher*: What do you mean, Musa? How did you relate this to the question?
> *Musa*: The difference is 1 in both, but it (the 1 in both) is not the same for both (Having a hard time explaining his strategy).
> *Teacher*: Did you compare in relation to the amount before adding?
> *Musa*: Yes. There were 3 glasses of orange juice, but 2 glasses of water.
> *Teacher*: And, we add a glass of orange juice and a glass of water.
> *Musa*: It doesn't have the same effect. Increasing the two by one and increasing the three by one don't have the same effect.
> *Teacher*: What is the difference?
> *Musa*: Increasing the three by one has (the relative amount of orange juice) less effect. So, I think that the taste of orange would decrease.
> *Teacher*: Let's think this way. When you add a glass of orange juice to the already existing amount of 3 glasses of orange juice, how much of the already existing amount does it make?

Musa: One over three. Yes, that's what I was trying to say.
Teacher: Then, can you interpret it the same way for the (amount of) water as well?
Musa: There are two glasses of water at the beginning. When we add a glass of water to it, it means that we add half of the already existing amount. This will have more effect.
Teacher: What do you think, Erdem?
Erdem: I wrote the new amounts (after a glass of orange juice and water are added) next to the old amounts on the table (See Figure 4.51a below). For A, it (the part-whole ratio) was $\frac{3}{5}$, and now it is $\frac{4}{7}$. I am looking at the orange juice in both pitchers and the total amount; that is the concentration of the orange juice. I equalized the denominators, and this became $\frac{21}{35}$, and this became $\frac{20}{35}$ (See Figure 4.51b below). So, they cannot taste the same.
Teacher: What do $\frac{21}{35}$ and $\frac{20}{35}$ mean?
Erdem: Of the 35 glasses of the mixture, there would be 21 glasses of orange juice, but after adding them, there would be 20 glasses of orange juice. So, the (orange) taste decreased.
Teacher: OK. I would like you to think this way now. How much orange juice are there per a glass of water in each situation?
Ceyda: There is 1.5 glasses (of orange juice) for a glass of water in the first situation. Here (after adding), I am dividing 4 by 3, and there is 1.33 glasses of orange juice for a glass of water.
Teacher: (Asks the whole class) So, what is the answer?
A bunch of students: The orange taste is less compared to the first situation.

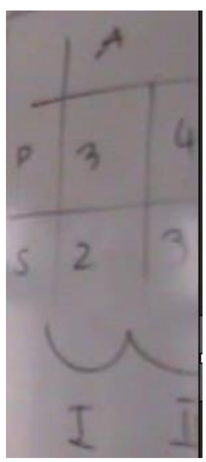


Figure 4.51a. Drawing a table in order to see the amounts before and after adding a glass of water and a glass of orange juice

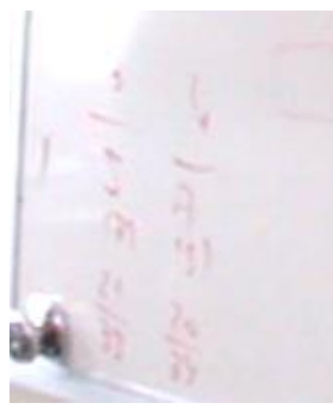


Figure 4.51b. Comparing part-whole relationships before and after adding a glass of water and a glass of orange juice

In order to interpret this dialogue in terms of Toulmin's analysis, Musa claimed that the taste of the mixture A would change when a glass of orange juice and a glass of water were added to it. For this claim, he provided data that included making connections to the comparison of fractions $\frac{1}{2}$ and $\frac{1}{3}$. When he was asked what those had to do with the question, he expressed that the difference between the amount of orange juice and water in both situations would be one; yet, it did not mean that they would taste the same. Since he had a hard time justifying his reasoning, the teacher helped him to think about the relative amount of added ingredients and the already-existing amounts of the ingredients. Upon this probe, Musa justified his reasoning by referring to the different impacts of adding a glass of orange juice and water to a mixture that included three glasses of orange juice and two glasses of water on the taste of the mixture. Specifically, he stressed that a glass of orange juice is one-third of the already-existing amount of orange juice, but a glass of water is half of the already-existing amount of water. He concluded that adding equal amounts of water and orange juice would reduce the oranginess of the mixture, and less amount of water had to be added to keep the taste the same. In other words, he made use of the multiplicative relationship between the amount of already-existing-ingredients and the amount of added ingredients, which included reasoning with part-part ratios. In order to elaborate more on the discussion, the teacher asked Erdem to talk about the problem. Erdem reasoned with part-whole ratios before and after adding a glass of water and orange juice. In order to compare the two ratios, he made use of his previous knowledge about comparing fractions by equalizing their denominators. Since none of the students included unit rate in their data and/or warrants, the teacher encouraged the class to reason with the unit rate in each situation. Ceyda and a bunch of other students claimed that the oranginess would be less compared to the first situation by drawing on the comparison between the amounts of orange juice per glass of water in both situations. Thus, while Musa and Ceyda focused on part-part ratios, Erdem reasoned with the part-whole ratios and referred to this relationship as "concentration." In order to compare these part-part and part-whole ratios, all of these students drew on their knowledge in comparing fractions.

In the following instances, Ali claimed that the taste of B would not change after a glass of water and orange juice were added since there would be one glass of orange juice per glass of water in both situations. Giray repeated the same claim by referring to the equality of the ratios of orange juice to the total mixture in both situations, which he also referred to as orange juice concentration. The students made claims about the change in the tastes of the mixtures for all other mixtures in the second sessions of Day 14, and warrants were dropped off of the class discussion as the students referred to the part-part ratios (unit rate, multiplicative comparisons) and part-whole ratios in their data. Therefore, they incorporated the previous taken-as-shared ideas (covariation, multiplicative idea, coordinating parts into a whole, creating and using unit rate as a tool/anchor/benchmark for comparisons) throughout the lesson, which were taken-as-shared by the end of Day 14.

On Day 15, the classroom discussion centered around comparing the tastes of two coffees that involved different amounts of coffee and milk and had varying strengths of taste. However, no numerical value was provided in any of the questions in order for students to reason qualitatively, that is, without depending on numbers. The first question included the mugs A and B that were in different amounts and had different tastes, which could also be understood from the pictures given on the activity sheet. It was known that the mug A contained less liquid than B and had a stronger taste of coffee. The students were asked a question to determine which of the mugs would contain the liquid with a stronger taste after a spoon of coffee was added to the Mug A, and a spoon of dried milk was added to the Mug B. Musa claimed that the liquid in A would have a stronger taste since it was already stronger in the first place, and adding more coffee would make it stronger. Merve added to that claim by saying that the liquid in Mug A would be way stronger than the liquid in Mug B since adding coffee to the stronger coffee would make it stronger, and adding milk to the weaker liquid would make it weaker. The aim for starting with this simple question was to capitalize on the following ideas: "adding coffee makes the mixture stronger" and "adding dried milk makes the mixture less strong," as stressed by Musa and Merve.

In the second question, both mugs contained an equal amount of liquid, but the liquid in Mug B had a stronger coffee taste. The students were asked to reason about the relative taste of the two mixtures after a spoon of coffee was added to each mug. Seda claimed that the situation would not change since their amounts were the same at the beginning. When she was asked to provide a warrant for this claim, she justified her reasoning by saying that Mug A contained less coffee than Mug B since their amounts were the same, but the one in A was weaker. After adding a spoon of coffee to each mug, both mugs would again contain the same amount of mixture, but Mug B would contain more coffee as it was the case in the first place. Hence, though it was implicit, she reasoned with the part-whole ratio. In relation to TAS Ideas, Seda referred to the previous taken-as-shared idea regarding the conceptualization of part/whole ratios in her warrant, and this gave further evidence for this idea being TAS.

Even though all the questions on Day 15 were mostly about distinguishing additive and multiplicative relationships (and overcoming incorrect additive reasoning) since adding ingredients might cause incorrect additive reasoning, the third question directly aimed to uncover these issues. In the third question, two mugs included two liquid with the same strength of taste but in different amounts. While Mug B contained less of the same liquid, the students were asked to compare the tastes of the mixtures in two mugs after a spoon of coffee was added to both mugs. This question is presented in Figure 4.52 below:

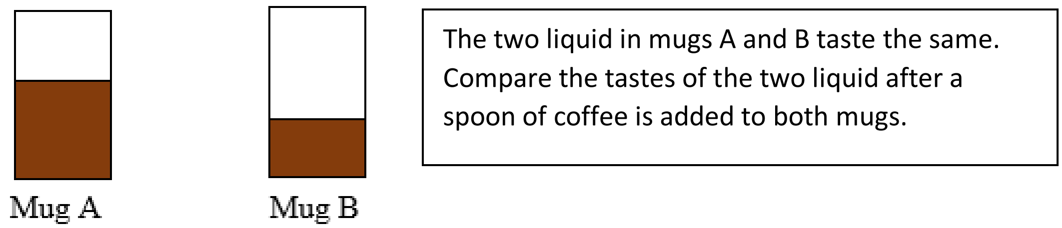


Figure 4.52. The third question in the Coffee Task

This question resulted in a debate in the class where some of the students claimed that both mixtures would still taste the same, and some of them claimed that the mixture in Mug B would have a stronger taste. The teacher gave the floor to Musa, who claimed that the two mixtures would again taste the same after a spoon of coffee was added to both. He explained that the two mixtures had the same taste in the first place, and the same amount of coffee was added to both, which acted as data for his claim. Below is a dialogue in which multiple students rebut Berk for this incorrect data and claim:

> *Ceyda*: Even though we are adding the same amount of coffee to both cups, B will have stronger coffee since the amount is less. When we add the coffee, it will affect B more, and it will be denser.
> *Teacher*: Your friend is saying that they taste the same, but they have different amounts. Therefore, the added-coffee will affect the mixture in Mug B more. What do you think?
> *A couple of students*: No, they taste the same.
> *A couple of students*: Yes, they would be equal (concerning the strength of the mixture).
> *Musa*: What does it have to do with being in less amount? They will taste the same. They had equal amounts of orange juice and water in the first place and after they were added.
> *Ceyda*: Let's assume that Mug A has 3 spoons of coffee and 3 spoons of dried milk, and Mug B has 1 spoon of coffee and 1 spoon of dried milk so that they would taste the same. When a spoon of coffee is added to both, Mug A would contain 4 spoons of coffee and 3 spoons of dried milk, and Mug B would contain 2 spoons of coffee and 1 spoon of coffee, hence, B would be denser and stronger.
> (A bunch of students showed agreement by nodding)
> *Musa*: Aha, I got it. Only coffee is added but not the dried milk.
> *Ozan*: Yea, it is just like we did with the orange juice yesterday.
> *Teacher*: Let's think this way: You make coffee in a big coffee pot and pour some to a small cup. You add a spoon of coffee to each. How are the two coffees affected? Which one has a stronger taste?

Musa: (The mixture in) The small cup would have a stronger taste. It will disperse less in the small cup.
Ceyda: That is what I meant by denser. The less it disperses, the denser it is.

As understood from the dialogue above, Ceyda tried to rebut Musa by referring to Mug B's having less amount and to the concept of density. The teacher asked the classroom if they understood what Ceyda said about being denser. No student elaborated on the idea of density, and then, Ceyda needed to draw on numbers. However, the teacher wanted to encourage their qualitative reasoning by raising the case of adding a spoon of coffee in a small cup and a big pot for the students to think about. Then, Musa claimed that the small cup would contain a stronger mixture by referring to the idea of "dispersion." Upon this interpretation, Ceyda made a connection between being dense and dispersion. Thus, this coffee pot-small cup example helped the students draw on their daily life experiences and relate to the idea of density in general and specifically to the question above. The students came to the point that even though the same amount was added to both mugs, including the mixtures with the same taste, it would affect the mixture with the smaller amount more.

In the following question, two mugs with the same amount of liquid were given. However, Mug B contained a stronger taste of milk coffee. The students were asked to reason about the tastes of the two mixtures after a spoon of coffee was added to Mug A, and a spoon of dried milk was added to Mug B with a stronger taste of coffee. Seval immediately claimed that they [the tastes of the two mixtures] could be equal. Seda added that they [the tastes of the two mixtures] might not be equal. Seval explained that it was not possible to make a specific conclusion since the amount [the amount of added ingredients compared to the ones already in the mixtures] was not known. She added that it also was not known how stronger the first mixture was than the second one. Ceyda provided a warrant by stressing that it was not possible to know if the added milk would suppress the strength of the first coffee and if the added coffee would strengthen the weaker coffee enough. She also emphasized that since neither the amounts of ingredients in the two mugs nor the amount of added ingredients were known, any of the following was possible: The relative strengths might stay the same (i.e., the coffee in Mug A could still be stronger), their strengths might be equalized (i.e., they could taste the same), the relative strength might be reversely changed (i.e., the coffee in Mug B could be stronger). A bunch of students nodded and claimed that there was missing information.

To sum up, the discussions that took place on Day 15 were hard to analyze in terms of the TAS ideas since the debate regarding the third question (i.e., same taste-different amount-one spoon of coffee is added to both mugs, see Figure 4.52) got resolved not by using qualitative reasoning but by reasoning with numbers. Even though Ceyda was able to reason qualitatively by referring to the density concept that is the amount of particles per volume, which is a unit rate, she needed to fold back to the orange juice concept in order to convince her classmates. Since the students had trouble reasoning qualitatively for this question, the teacher offered an example in another context, including the small cup-big pot case. This supported the students' conceptions of "density" and "dispersion.”

On the other hand, the potentials of the milk coffee context are still evident. First, this context enabled students to fold back to the orange juice context in which part-part and part-whole relationships were established by using numerical values. However, the students focused on the part-whole relationships in the orange juice context as an alternative way to obtain the result, and not because they had to. Unlike the orange juice context, the students had to focus on the whole as well as the parts in this context since the parts and the whole were unknown and inseparable. Secondly, this context really pushed additive reasoning, and what came out of it is the concept of density where it did not come out this strongly before even though the students established part-part and part-whole relationships on the previous days. Ceyda and some other students were able to reason intuitively that adding to a bigger mixture would not make an impact as much as it does to a smaller mixture.

This argument, related to the concept of density, came out in Ceyda’s warrant. Similarly, other ideas related to qualitative reasoning appeared in students’ data and/or warrants during Day 15 since the claims were related to deciding on the relative taste of the coffees in two mugs after a specific amount of coffee/milk was added. Therefore, qualitative reasoning did not appear as a claim; hence, it was not possible to use the two criteria by Stephan and Rasmussen (2002) for deciding on whether qualitative reasoning was taken-as-shared. This might be due to the fact that only one task was posed qualitatively. Therefore, adding another task that is focused on qualitative reasoning is suggested and discussed in the final chapter.

CHAPTER 5

CONCLUSIONS AND DISCUSSION

In this study, a classroom Hypothetical Learning Trajectory (HLT) and associated instructional sequence for proportional reasoning were developed, tested, and revised in order to arrive at a potentially viable local instructional theory for the teaching of proportional reasoning. In addition, the development of the seventh-grade classroom community's proportional reasoning in normative and increasingly sophisticated ways was analyzed by documentation of Classroom Mathematical Practices (CMPs). Particular emphasis was given to the development of seventh graders' communal ways of reasoning with informal and formal tools (i.e., models, imageries, gestures, and metaphors) and how this reasoning was supported in increasingly sophisticated ways with an RME perspective. The need for several revisions for the instructional sequence and the HLT emerged out of the CMP analysis.

This chapter presents a synopsis of the findings of this study concerning these purposes and compares those findings with the related literature in three major sections: (1) Development of proportional reasoning in the social context (Classroom Mathematical Practices), (2) The role of tools, models, imageries, gestures, and metaphors in supporting student understanding and learning, (3) Revisions to the instructional sequence and the HLT. Furthermore, it features implications for practice as wells as suggestions for further research in the sections that follow.

5.1. Development of Proportional Reasoning in the Social Context

This study includes two subsequent design experiments that focus on the development of seventh-grade classroom community's proportional reasoning and the means of supporting and organizing that development in line with a design-based research perspective (Bakker & van Eerde, 2015; Cobb, 2003; Gravemeijer & Cobb, 2006). In particular, this study presents what a classroom learning trajectory and related instructional sequence for proportional reasoning look like when developed, tested, and revised in a classroom setting by providing a picture of how the hypothetical learning trajectory was actualized in the classroom as the teacher and the classroom community interacted around the instructional sequence.

Prior research of students' learning of the concepts of ratio, rate, and proportion and the development of proportional reasoning was used to develop and revise the HLT and the instructional sequence. The Classroom HLT and the instructional sequence developed by Stephan and colleagues (2015) served as the backbone for the HLT and the instructional sequence developed in this study. The instructional sequence was designed and revised so that seventh-graders could work through realistic contexts to build an understanding of proportional reasoning and related concepts such as ratio, rate, and proportion as they interact with the instructional tasks.

The difficulties in understanding and learning mathematics are ascribed to the discrepancies between informal knowledge based on real-life experiences and formal mathematics that are taught through instruction (Gravemeijer, 1991, 1999). Nonetheless, within the context of this study, informal and formal knowledge of mathematics is not perceived as distinct from each other. Instead, the students reinvented formal mathematical knowledge regarding proportional reasoning, drawing on the informal and intuitive knowledge that were reported by other researchers who explored young children's understanding and development of proportional reasoning. Some researchers also refer to this informal and intuitive knowledge as qualitative knowledge (Lesh et al., 1992; Resnick, 1986; van den Brink & Streefland, 1979) or personal knowledge (Kieren, 1988).

In addition, pre-tests and pre-interviews conducted prior to this study shed light on students' informal ideas regarding proportional reasoning. Upon a review of the related literature and drawing on the pre-assessment results, these informal ideas were determined as forming composite units and iterating linked composites (Battista & van Auken Borrow, 1995; Park & Nunes, 2001; Steffe, 1988, 1994), unitizing and norming (Freudenthal, 1983; Lamon, 1994, 1995), and (abbreviated) build-up strategies (Ben-Chaim et al., 1998; Carpenter et al., 1999; Kaput & West, 1994; Lamon, 2007; Lesh et al., 1988; Thompson, 1994), which point to similar mental operations.

Although prior research showed that several factors play role in determining students' proportional reasoning, most of those "do not provide illumination into how and why these factors operate as they do" (Akatugba & Wallace, 1999, p. 305). Moreover, there has been little attention to the social factors that are associated with students' proportional reasoning (Akatugba & Wallace, 1999). However, there has been a call for studying learning in socially situated contexts as the social and cultural aspects have a significant influence on students' construction of understandings (Cobb & Bauersfeld, 1995; Hufferd-Ackles, Fuson, & Sherin, 2004; Krummheuer, 1995, 2007; Saxe, 1991; Stephan, 2003; Vygotsky, 1978). In this study, learning and mathematical development of students are viewed through a social constructivist perspective, named Emergent Perspective (Cobb & Yackel, 1996), which sees learning both as an individual and social activity. Although I accept that both individual and social aspects of learning are essential at the same level and learning cannot be described by focusing only on the individual or social aspects, in this study, I put the social perspective in the foreground by focusing on the collective development of the classroom community. To this end, I described the emergence of the classroom mathematical practices that were established by a seventh-grade classroom community. In doing so, I described the taken-as-shared ways of reasoning and arguing of the community instead of how an individual was reasoning. The documentation of the classroom mathematical practices, therefore, gave a picture of the patterns in the classroom community's processes of reasoning, structuring, arranging, arguing, symbolizing, visualizing, and schematizing as they interacted with the instructional sequence.

Five mathematical practices were documented over the course of six weeks that included 30 class hours by using an analysis method proposed by Stephan and Rasmussen (2002) and Rasmussen and Stephan (2008). The documentation of the classroom mathematical practices revealed that the instructional sequence has extensive potential in supporting a classroom community's proportional reasoning in increasingly sophisticated ways. Particularly, CMP1 showed that the students were able to reason with pictures and tables to find missing values of fish/food bars in normative ways by drawing on their informal knowledge of grouping, linking, iterating, unitizing, norming, and building up. Within the context of the first practice, it became taken-shared-that when the number of food bars changes, the number of fish that can be fed with that specific amount of food bars changes in a precise way, that is, in line with the given rule. This includes a naïve understanding of what the literature terms as "covariation" or "covariance" (Carlson et al., 2002; Ellis et al., 2016; Lamon, 1995, 2007). At the onset, this understanding of covariance was based on coordinated build-up strategies (i.e., when the number of food bars goes up by ones, the number of fish goes up by threes) (Kaput & West, 1994) or a skip counting process (1-3, 2-6, 3-9, etc.). As students moved through the sequence, these naïve interpretations evolved into building up in more efficient and abbreviated ways. These efficient ways included using multiplication and division or working with a scalar operator within the same measure spaces (i.e., norming).

For instance, on Day 2, when it was known that one food bar could feed three fish and the students were asked to find the number of food bars required to feed 12 fish (pictures of 12 fish were given), Merve suggested that she could divide 12 by three, instead of grouping and linking, in order to find the number of the food bars needed by referring to the grouping with pictures. Besides, when the pictures of five fish were given, and the question asked how many food bars were needed to feed those 15 fish, Gizem suggested to multiply five by three in order to do more effective operations, and multiple students were able to provide warrants making connections between multiplication and iterating with pictures. These kinds of reasoning correspond to what Kaput and West (1994) refer to as "abbreviated build-up/build-down processes using multiplication and division" (p. 244). On the following days, the classroom community clearly made a connection between these processes that include multiplication and division and reasoning with equal scale factors within measure spaces (i.e., doubling, tripling, or otherwise multiplying the values by the same factor within their own measure spaces). There were multiple instances that the classroom community used division within the same measure spaces in order to find the scale factor and multiplied the value that belonged to the other measure space with that scale factor in order to arrive at equivalent ratios. Moreover, both types of reasoning were also associated with the initial processes of grouping and linking with pictures.

Particularly, reasoning with a scale factor emerged naturally in the classroom discussion on Day 2 when Sinem instigated a new type of reasoning with the (long) ratio table by stating, "the number of fish always increases three times more in relation to the number of (food) bars." In the following instances, multiple students made claims about finding more efficient ways of building up by ones. For instance, Ceyda observed that moving from 2 to 10 in the first row of the table (i.e., the row that includes the number of fish in each situation) and from 6 to 30 in the second row of the table (i.e., the row that includes the number of food bars in each situation), both numbers are multiplied by the same number, that is five. Moreover, İlter stated that while moving between the cells within the same rows, both numbers were divided by the same number as he went backward through the table. Also, Ceyda and Elif associated these processes with making iterations with linked composites. The analysis of the classroom data showed that this kind of reasoning with a scalar operator appeared more frequently than the first one (i.e., abbreviated build-up/build-down processes using multiplication and division). Therefore, in this study, the term "abbreviated build-up strategies" is mostly used to refer to working with scalar operators within the same measure spaces as differently than Kaput and West (1994), although it should be noted that they correspond to similar mental operations.

Therefore, the CMP analysis showed that these types of abbreviated build up strategies that include multiplication and division or operating with a scale factor within the same measure space are short ways for building up strategies and making iterations with linked quantities. In particular, it was evident when the students referred to their previous work of groupings and iterations with pictures in their warrants to justify the rationale behind those operations. Therefore, reasoning with abbreviated build-up strategies is not interpreted as multiplicative reasoning in this study. It was rather considered as pre-multiplicative reasoning, as divergent from several studies (Lamon, 2007; Vermont Mathematics Partnership's Ongoing Assessment Project, 2011; Wright, 2014). This is due to the fact that it entails similar characteristics (e.g., coordination of quantities, pattern recognition without a recognition of the structural invariant relationships) to what several researchers refer to as pre-proportional reasoning (Lesh et al., 1988; Piaget & Inhelder, 1975; Steffe, 1994).

Although covariation of the number of fish and food bars in relation to each other by making coordinated and abbreviated iterations, that is building up by ones and abbreviated build-up strategies, emerged naturally on the first day of the instruction and became taken-as-shared on the same day, the idea regarding the invariant relationship between the number of food bars and fish did not emerge naturally on the first day of the instruction. Instead, the teacher stimulated the students to think about the relationship between the number of food bars and fish on Day 2. Upon the teacher's stimulation, Seval provided the following operational claim "when we multiply the number of food bars with three, we get the number of fish" in the first place, and then, a more conceptual claim "the number of fish is always three times that of food bars." The second type of reasoning focuses on the invariant relationship between the number of food bars and fish; hence, it shows a more robust understanding of the functional relationship between the number of food bars and fish.

In the following instances, multiple students used this functional relationship between the number of food bars and fish (i.e., the number of fish is always three times that of food bars) in order to find missing values. As the students moved through the instructional sequence, they were given a reducible ratio between the number of food bars and fish, that is, 2 food bars for 4 fish. The idea that this rule is the same as the rule 1 food bar for 2 fish emerged shortly after the students started to work with the rule 2 food bars for 4 fish. In addition, the equality of the functional relationship between the number of food bars and fish in both rules was emphasized. A shred of evidence comes from Gizem's illustration with the pictures of food bars and fish that provides more backing for this claim (i.e., showing that 1-2 is the same as 2-4 with pictures, see Figure 4.20). Then, this equivalent functional relationship between the number of food bars and fish (i.e., the number of fish is always two times of the number of food bars) was used in order to find missing values instead of the initial rule (i.e., 2 food bars for 4 fish). This was when it was revealed that the idea that the invariant relationship between the number of fish and food bars was conceptualized as unit rate and used as an anchor to find missing values was taken-as-shared among the classroom community. This taken-as-shared idea is associated with what the literature refers to reasoning with a functional rate or unit rate (Ben-Chaim et al., 1998; Lamon, 1994; Tourniaire & Pulos, 1985; Vergnaud, 1994).

Therefore, on the first four days of the instruction, the ideas related to linking composite units that included coordinated build-up strategies, and working with a scale factor that included abbreviated build-up strategies, and reasoning with the invariant functional relationship between two distinct values, and using this invariant relationship to find missing values were taken-as-shared among the classroom community. Then, drawing on all of these experiences, these ideas were extended to the part-whole contexts on Day 5. The classroom mathematical practice analysis suggested that these five ideas revolved around the same ideas of reasoning with covariation within the same measure spaces and with the invariant relationship between different measure spaces. Thus, these ideas were put together and organized around the common activity of reasoning with pictures and tables to find missing values as they emerged and became taken-as-shared.

Mathematical Practice 2 showed the strength of these ideas for providing the foundation for reasoning within measures ratios and between measures ratios in order to find a missing value and determine proportionality by using the symbolic proportion representation. Reasoning with a scale factor within the same measure spaces (i.e., abbreviated build-up reasoning) and reasoning with the invariant and functional relationship between two linked values in different measure spaces became powerful strategies for solving proportional problems in a variety of contexts on Day 7. Particularly, the classroom discussion that emerged while the classroom community was working on deciding whether the given ratio (i.e., $\frac{10}{20}$) belonged to the same equivalence class as the ratio between the number of girls to the boys (i.e., $\frac{2}{3}$) can provide evidence for this. Faruk claimed that these ratios could not be proportional since the scale factors within the same measures were not the same (i.e., 2 is multiplied by 5 to obtain 10, but 3 is not multiplied by 5). Moreover, Ceyda provided a warrant for this claim by focusing on the functional relationship between the number of girls and boys (i.e., $\frac{10}{20}$ is equal to a half, but the other one is not). Other mathematically significant discussions that took place on Day 7 included structuring within and between ratios based on those two types of relationships.

Throughout days 8 to 11, the classroom community's normative ways of reasoning shifted from finding missing values in tables or proportions to representing proportional relationships on tables and graphs and with algebraic equations and coordinating the relationships among them. In addition, understanding the nature of the nonproportional linear relationships of the form y = mx + n and how their algebraic and graphical representations differ from those of proportional relationships were among the topics of classroom discourse. In particular, formalizing the proportional and nonproportional linear relationships into algebraic equations was a new idea that was taken-as-shared on these days. Therefore, this shift observed in the classroom community's reasoning suggested the emergence of a new practice, coordinating the relationships among the representations (CMP 3). However, again, it should be noted that the students drew on the previous taken-as-shared ideas of working with equivalent scale factors (i.e., abbreviated build-up) and reasoning with the invariant relationship between quantities that belong to different measure spaces throughout the establishment of the third mathematical practice.

Similarly, during the remaining days in which CMP 4 and CMP 5 were established, the students heavily relied on their understanding of working with scale factors within the same measure spaces and the invariant functional relationship between quantities in different measure spaces. More specifically, while students decided if the given two shapes were similar or distorted and found the missing lengths in a pair of similar shapes, they reasoned with the between-shapes scale factors and/or the invariant multiplicative relationship between the lengths of a single shape (i.e., within shapes relationships) on the twelfth day of instruction. Moreover, on the following days, they also drew on these ideas while comparing different ratios/rates and deciding which one was bigger or smaller or if they were equal. For example, while the classroom was working on comparing the orange tastes in Mixture B (3 glasses of orange juice-3 glasses of water) and Mixture D (5 glasses of orange juice-5 glasses of water) on Day 14, Berk claimed that the two mixtures in Pitchers B and D had the same taste since the amount of water was equal to the amount of orange juice in both pitchers. Then, multiple students referred to the invariant one to one relationship in both mixtures (i.e., for every glass of orange juice, there is one glass of water in both).

The classroom mathematical practices analysis suggests that the idea of comparing ratios/rates demands higher cognitive reasoning than creating equivalent ratios/rates and finding the missing value in a pair of equivalent ratios/rates in two ways. First, as afore-stated, reasoning within and between measure spaces underlie creating equivalent ratios/rates. In order to make claims about comparing different ratios/rates, one needs to analyze those within and between measures comparisons and evaluate whether both comparisons are the same. Second, because a ratio is a comparison itself, comparing ratios becomes a comparison of comparison of quantities, that is, a second-order act of comparing. Remembering Piaget's utterance to ratio as "relationships of relationships" (Piaget & Inhelder, 1975, p. 160), or a second-order relationship, comparing ratios include relationships of relationships of relationships or a third-order relationship. However, it is noteworthy that a comparison of ratios can also include comparing the amount of one quantity when the other is equal. For example, when comparing the oranginess of Mixture A (3 glasses of orange juice and 2 glasses of water) and Mixture B (3 glasses of orange juice and 3 glasses of water) on Day 14, Namık only focused on comparing the amounts of water in two mixtures stating that the amount of orange juice was the same in both. In this study, this type of comparison was referred to as basic comparison since it did not include a comparison of the relative amounts of the ingredients; instead, a comparison of the amounts of one ingredient only. This kind of comparison is easier for students as Noelting (1980a) stressed that even children as young as seven years old could successfully compare those kinds of mixtures.

The mathematical practices analysis can also provide insight into the essential mathematical components of proportional reasoning determined by the review of the literature. A thorough understanding of reasoning about change involves understanding covariance and invariance. As several researchers (e.g., Carlson et al., 2002; Confrey & Smith, 1994; Ellis et al., 2016; Lamon 1995, 2007; Saldanha & Thompson, 1998) highlighted the importance of understanding covariation and invariation, the findings of this study suggest that a conceptual understanding of covariance and invariance lies at the heart of proportional reasoning. As shown by the classroom mathematical practices, the students in this study began making sense of the first task as they covaried the number of fish and food bars and eventually recognized and reasoned with the invariant relationship between the number of food bars and fish. In each of the following tasks, they relied on these notions of covariation and invariation.

In regards to covariance and invariance, the findings of this study also have the potential to confirm the covariational reasoning levels suggested by Coulombe and Berenson (as cited in Saldanha & Thompson, 1998). In terms of the development of covariational reasoning through the fish-food bar activity, although there was no explicit statement, the classroom community first must have identified two data sets as the number of fish and food bars before linking these two sets. Then, they coordinated these two sets of data and inferred that when one increased, the other one also increased. Next, the classroom community constructed links between these data patterns number patterns (e.g., 1-3, 2-6, 3-9, …etc.), which were referred to as coordinated build-up strategies. Lastly, this link is generalized in order to find missing values. Therefore, the mathematical practices analysis showed that the covariational reasoning levels suggested by Coulombe and Berenson were valid. However, it also revealed that these levels were too general that they do not include abbreviated build-up strategies.

As another aspect associated with covariation and invariation, the mathematical practices analysis revealed that reasoning about change in relative terms rather than absolute terms paved the way for a deep understanding of proportional reasoning, as Lamon (1995) pointed out. For instance, on Day 12, when the classroom community worked on determining whether or not the two rectangles (the first rectangle had lengths 4 cm and

6 cm, the second rectangle had lengths 13 cm and 15 cm) were similar, Merve reasoned with the between shapes ratios and claimed that the shapes were not proportional. Since the values for the side lengths were intentionally chosen to reveal incorrect additive reasoning but it did not emerge in the discussion, the teacher asked the class whether it was possible to reason about the change in the two shapes in absolute terms (i.e., the differences of short and long side lengths between the two shapes are the same, that is 9 cm). Multiple students rebutted this type of reasoning by referring to the relative relationship between the amount of change, that is 9 cm, and the short and long side lengths of the two shapes by stating that "9 cm is more than two times of 4 cm (the short side length of the original picture) but less than two times of 6 cm (the long side length of the original picture)."

Ratio sense was another essential component for the development of proportional reasoning, as suggested by Lamon (1995). The mathematical practices analysis revealed that the classroom community was able to apply ratio reasoning to several tasks in which the term ratio was not used. They were able to organize the proportional situations in tables and with numbers and reasoned about the ratio relationships within and between measure spaces. Moreover, they were able to use correct mathematical language regarding its informal and formal uses. The classroom community was also able to distinguish proportional relationships from nonproportional relationships as they worked with the linear non-proportional relationships on Days 8-11.

As the analysis of the emerging mathematical practices showed that the strategies that emerged in the classroom were based on either reasoning within measure spaces or between measure spaces (or within shape or between shapes in the context of similar shapes), the findings of the study showed how these understandings were essential for the development of proportional reasoning. Similarly, ideas associated with qualitative and quantitative reasoning emerged in the discussion often across multiple days.

In relation to these essential understandings, informal activities/experiences of relationships (Lamon, 1995), unitizing and norming (Lamon, 1994, 1995), and iterating linked composites (Battista & van Auken Borrow, 1995; Steffe, 1994) were mainly found helpful to promote proportional reasoning. More specifically, in terms of relationships, the classroom community drew on their intuitive experiences of relationships when they made sense of the relationships between the quantities and recognized that "when one increases, the other also increases." Moreover, the classroom community relied on their experiences of basic comparisons related to recognizing distortion (i.e., looking fatter/thinner or taller/shorter) that took place on Day 12. Secondly, an example for the use of unitizing and norming can be given from Day 12, where Sinem reasoned with the within-shapes-ratio in order to refer to the invariant times two relationship between the within-shapes-lengths. More precisely, Sinem provided a warrant to a previous claim by saying that "there are 2 units of horizontal lengths for every 1 unit of vertical lengths." In this process, she also represented her reasoning with pictures (See Figure 4.43) in which she took a 4-cm-vertical-length in the first shape as a unit (unitizing) and reinterpreted the 8-cm-horizontal-length in the same shape in terms of that 4-cm-length (norming). Lastly, throughout the establishment of all the mathematical practices, the students relied on their intuitive knowledge of linking composite units and iterating linked composites (Battista & van Auken Borrow, 1995; Steffe, 1994). Therefore, the findings of the study confirm the extension of the groundwork by Lamon (1995) in terms of mathematical dimensions and informal activities of proportional reasoning.

Thus, the classroom mathematical practices analysis provided parallel results with studies conducted at the individual level showing that formal proportional reasoning is based on the informal (didactical) activities of basic qualitative comparisons, creating composite units and iterating linked composites, building up strategies, reasoning with number patterns, unitizing and norming, working with within or between measures ratios (also called as multiplicative strategies), and unit factor (also called as unit rate) approach (Battista & van Auken Borrow, 1995; Ben-Chaim et al., 1998; Kaput & West, 1994;

Lamon, 1994, 1995; Lo & Watanabe, 1997; Park & Nunes, 2001; Steffe, 1988, 1994; Streefland, 1985; Tourniaire & Pulos, 1985). Notably, this study provided further evidence that the roots of proportional reasoning lie at the ideas of creating composite units (i.e., unitizing) and iterating linked composites (i.e., norming) in parallel with several researchers (Battista & Van Auken Borrow, 1995; Park & Nunes, 2001; Steffe 1988, 1994) since all the five practices documented in this study were built on these ideas. On the other hand, this conclusion is in striking contrast with the arguments made by researchers that suggest that the basic idea behind multiplicative reasoning is repeated addition (Fischbein et al., 1985).

On the other hand, prior research revealed students' incorrect strategies and difficulties as ignoring part of the data, providing irrelevant response, and erroneous additive reasoning (Ben-Chaim et al., 1998; Tourniaire & Pulos, 1985), with the last one as being the most prevalent (Brousseau, 2002; Hart, 1981, 1988; Kaput & West, 1994; Misailidou & Williams, 2003; Resnick & Singer, 1993; Steinthorsdottir & Sriraman, 2009; Tourniaire & Pulos, 1985; Tourniaire, 1986; van Dooren et al., 2010). Resnick and Singer (1993) attribute this high incidence of incorrect erroneous reasoning to two factors: (1) slow development of multiplicative relations compared to additive relations and (2) children' initial experiences of quantifying additive relations of numbers than their multiplicative relations. In other words, beneath students' tendency to reason additively for proportional tasks lie their early experiences and familiarity with additive relationships as well as the lower demand it requires than that of multiplicative reasoning requires.

It is noteworthy that the erroneous additive reasoning was only suggested by a few students in this study, and when it was suggested, multiple students rebutted this idea by using the ideas of scaling within measure spaces and invariant relationships between measure spaces. For instance, when the classroom community worked on comparing the oranginess of Mixture A (3 glasses of orange juice and 2 glasses of water) and Mixture

C (4 glasses of orange juice and 3 glasses of water) on Day 14, Gizem claimed that the two mixtures would taste the same based on her observation that the difference of the values was 1 in both mixtures. It was not for sure whether she referred to the one unit of difference within measures spaces or between measures spaces. Regardless of this issue, several students showed disagreement with her and rebutted her claim. More precisely, Ozan referred to the part-part ratios within a mixture (i.e., between measure spaces), and the previous experiences with fraction comparisons ($\frac{3}{2}$ is bigger than $\frac{4}{3}$ since $\frac{3}{2}$ is $\frac{1}{2}$ more than a whole, while $\frac{4}{3}$ is $\frac{1}{3}$ more than a whole) in his rebuttal. Merve stated that $\frac{3}{2}$ and $\frac{4}{3}$ do not form a proportion, so they would not taste the same. Then, Seval revised that claim to say that A is less orangey than C by equalizing the amounts of orange juice in both mixtures and comparing the amounts of water in both mixtures with the same amount of orange juice.

This finding significantly differs from the studies that report a high incidence of erroneous additive reasoning in proportional tasks (Atabaş & Öner, 2017; Duatepe, et al., 2005; Fernández et al., 2012; Hart, 1981, 1988; Kahraman et al., 2019; Kaplan et al., 2011; Kaput & West, 1994; Kayhan et al., 2004; Mersin, 2018; Misailidou & Williams, 2003; Özgün-Koca & Altay, 2009; Resnick & Singer, 1993; Steinthorsdottir & Sriraman, 2009; Tourniaire & Pulos, 1985; Tourniaire, 1986; van Dooren et al., 2010) although a low frequency of erroneous additive reasoning was also found in a few other studies (Karplus et al., 1983). This divergence from many studies might be attributed to several reasons ranging from the participants of this study to the nature of this study.

First, it is noteworthy that some of these studies suggest that additive strategies dominate student thinking in primary school (Fernández et al., 2012; van Dooren et al., 2010). Since this study was conducted in a seventh-grade classroom, it might be possible that these students have already changed their additive reasoning schemas to multiplicative ones. Second, it should be noted that most of these studies that report a high incidence of additive reasoning in students' thinking involve a one-shot collection of data from students through surveys or interviews. However, in this study, a classroom community's development of proportional reasoning as they engaged in argumentation processes was examined. In such an environment, students engage in certain participation behaviors to present their ideas, question and evaluate others' ideas, compare, contrast, justify, confirm, and rebut the emerging ideas (Brown, 2017). Thus, they may find opportunities to revise and reconstruct their ideas and make conjectures in collective ways (Whitenack & Knipping, 2002). Therefore, similar to the arguments made by these researchers, it is quite clear that when an incorrect additive reasoning appeared in the classroom discussion in this study, it was challenged, rebutted, and revised by the classroom community. Therefore, similar to the findings of the study conducted by Karplus, Karplus, Formisano, and Paulsen (1979), the findings of this study suggest that "additive reasoning does not lie on a invariant development sequence but is strongly influenced by instruction and represents an effort by students to deal with a task in an ad hoc rather than a systematic way" (p. 65). More precisely, the findings of this study conclude that, through a carefully designed instruction, erroneous additive reasoning might be prevented and eliminated.

It should also be noted that no mention has been made of the cross-multiplication algorithm for solving the proportional reasoning tasks in this study. Although a small number of students attempted to suggest such types of procedural solutions as they provided alternative data to their claims, the classroom community required more conceptual answers due to the pre-established sociomathematical norm associated with the features of an acceptable mathematical solution (Cobb & Yackel, 1996). Therefore, this finding is in line with the findings of a few studies that reported a low frequency of the use of cross multiplication strategy (Karplus et al., 1983; Vergnaud, 1983). On the other hand, this finding is also in stark contrast to the findings of many studies that report overreliance of students on cross multiplication and similar procedural algorithms (Arıcan, 2019; Atabaş & Öner, 2017; Ben-Chaim et al., 1998; Cramer & Post, 1993; Cramer, Post, & Currier, 1993; Duatepe et al., 2005; Kahraman et al., 2019; Kaplan et al., 2011; Kayhan et al., 2004; Özgün-Koca & Altay, 2009). Therefore, the findings of this study suggest that students can deal with a variety of proportional tasks without using the cross-multiplication algorithm or any other procedural strategy in a social setting. As another concern, it is critical to highlight that the cross-multiplication strategy did not naturally emerge in the classroom discussion as the other ideas did and became taken-as-shared. Therefore, the findings of this study also conclude that cross multiplication is not a natural and student-generated algorithm, as Hart (1984) concluded.

Multiple studies showed similar developmental paths starting from recognizing patterns in qualitative ways (i.e., when one quantity increases, the other one also increases), moving through quantifying this pattern in additive ways through coordinated build-up strategies (i.e., when one quantity goes up by ones, the other goes up by threes) and abbreviated build-up strategies (i.e., when one quantity is scaled by n, the other quantity should also be scaled by n), and finally understanding the invariant multiplicative relationship (including unit rate) between two values and applying this to obtain equivalent ratios (Kaput & West, 1994; Lesh et al., 1988; Lo & Watanabe, 1997; Piaget, & Inhelder, 1975; Steffe, 1994; Tourniaire & Pulos, 1985). Moreover, several researchers pointed out that the development of proportional reasoning occurs in relation to other concepts, including fractions, rational numbers, linear mappings, multiplication, and division (Kieren, 1976; Lamon, 1994; Lo & Watanabe, 1997; Steffe, 1988, 1994; Vergnaud, 1988). Although these studies were conducted at the individual level, the findings of this study suggest that the development of proportional reasoning in a social setting might follow a similar path. Thus, based on the CMP analysis, it is safe to conclude that studies that investigated the informal knowledge and strategies, conceptions, and misconceptions and developmental trajectories that were conducted at the individual level can be useful in informing the ways to support the development of a mathematical subject at the classroom level. Nonetheless, it should be noted that the development of proportional reasoning depends on a teaching and learning process (Lamon, 1994, 1995), in particular, the tasks used in the instruction (Lesh et al., 1988).

Since the instructional sequence was designed based on the findings of the studies conducted at the individual level, it is no surprise that the development in a social setting follows a similar path as individual development.

As afore-stated, this study was conducted based on an intensive literature review that resulted in the emergence of several issues for discussion and reflection. The findings obtained from the analysis of classroom mathematical practices can also be helpful to reflect on the afore-mentioned divergences in the definitions of several terms. To begin with, it was stated in the first two chapters that there is confusion in defining rates and ratios and how they differ from each other. To recapitulate, some researchers consider ratio as a comparison of quantities that belong to the same measure space and rate as a comparison of quantities that belong to different measure spaces (Vergnaud, 1988). Other researchers make distinctions between extensive and intensive quantities: while the former is referred to as the extent of a quantity, the latter is referred to as the relationships between a quantity relative to a unit of the other quantity (Freudenthal, 1973; Kaput et al., 1986; Kaput & West, 1994; Schwartz, 1988). Therefore, they accept that rate refers to a single intensive quantity, while ratio refers to a relationship between two quantities (Schwartz, 1988). As different from those perspectives, Thompson (1994) focuses on mental operations in order to make a distinction between ratios and rates. According to him, a ratio is the result of a multiplicative comparison of two specific and fixed quantities, whereas rate is a reflectively abstracted constant ratio, that is, a generalization of the invariant ratio. In this study, I follow Thompson (1994) in making distinctions between ratios and rates since it is the most useful interpretation in order to make sense of the classroom data of this study.

More precisely, as described in Chapter Four and briefly restated above in this chapter, the students in this study started to make sense of the fish-food bar task by making iterations (i.e., build-up strategies) in pictorial, numerical, and tabular forms. In this process, they reasoned with equivalent ratios successively (one food bar- three fish, two food bars-six fish, three food bars-nine fish…). Indeed, this type of thinking constituted the first taken-as-shared idea in this study. As the students reasoned in more sophisticated ways in collective ways, the classroom discussion evolved into abstracting the invariant relationship between the number of food bars and fish (i.e., the number of fish should always be three times of the number of food bars). This was a shift in the classroom community's ways of reasoning from covarying the number of food bars and oranges in successive ways to understanding the invariant nature of this covariation. Therefore, these successive moves from covariation to invariation conclude that Thompson's (1994) perspective can be adopted in order to interpret a classroom community's communal ways of reasoning with ratios and rates. More precisely, the students' mental operations were related to the concept of ratio when they made iterations within the same spaces and to the concept of rate when they abstracted this understanding of covariation to make sense of the invariant relationship between the number of food bars and fish.

Another discrepancy revealed by the literature review was related to defining within and between ratios. As mentioned earlier, some researchers use the term within ratio to refer to a comparison of quantities that belong to the same measure space and between ratio to a comparison of quantities that belong to different measure spaces (Freudenthal, 1973; Lamon, 1994; Vergnaud, 1994). However, other researchers use the term within ratio as a synonym for a comparison of quantities that belong to the same system (i.e., a series of interacting elements) and between ratio for a comparison of quantities that belong to different systems (Karplus et al., 1983; Noelting, 1980a, 1980b). In this study, a specific terminology suggested by Lamon (2007) was used in order to avoid this confusion. This uses the terms within measure (spaces) ratio/comparison and between measure (spaces) ratio/comparison. The mathematical practice analysis showed that the classroom community started to make iterations within the same measure spaces by abbreviated build-up strategies, and then, moved to reasoning between different measure spaces by making sense of the invariant relationship between quantities that belonged to different measure spaces. In this process, while within measures comparisons emerged naturally in the discussion, the teacher had to stimulate students in order to think about the between measures comparisons. Therefore, the mathematical practice analysis conducted in this study revealed that within measure spaces ratios are more natural than between measure spaces ratios as concluded by other researchers (Freudenthal, 1978; Karplus et al., 1983; Noelting, 1980a, 1980b; Vergnaud, 1980). Furthermore, the mathematical practice analysis also lent additional support to the conclusion that between measures ratios are more sophisticated in terms of mathematical thinking and more representative of the problem structure than within measures ratios the in the same direction with those researchers (Freudenthal, 1978; Karplus et al., 1983; Noelting, 1980a, 1980b; Vergnaud, 1980) based on the same concerns stated above.

Thus, the mathematical practices analysis shed light on several issues mentioned in the literature. It is noteworthy to restate that these practices evolved over time as the taken-as-shared ways of reasoning were established in the classroom community, each of which emerged as growing out of the previous practices established by the classroom community. In the following section, the evolution of use of tools, inscriptions, models, and imageries in the classroom community is described. Moreover, the communal use of gestures and how it is co-evolved with students' sophisticated ways of reasoning is presented.

5.2. The Role of Tools, Inscriptions, Models, Imageries, and Gestures in Supporting Student Understanding and Learning

Although the development of the classroom community's proportional reasoning was discussed without an emphasis on the use of tools, inscriptions, models, imageries, and gestures in the previous section, I do not attempt to claim that development occurs in isolation from those. On the contrary, the findings of this study provide further evidence that learning and development occurs in relation to reasoning with tools, models, imageries, and gestures as stressed in many other studies (e.g., Cobb, 2003; Gravemeijer, 1999; Gravemeijer et al., 2000, 2003; Lehrer, Schauble, Carpenter, & Penner, 2000; Lo

& Watanabe, 1997; Rasmussen et al., 2004; Stephan, 1998; Thompson, 1994, 1996). In other words, the findings of this study conclude that the creation of symbols and meaning co-evolve, as concluded in the studies cited above. Thus, I conclude that an increasingly sophisticated use of tools and models constitute a crucial element of an instructional sequence similar to Gravemeijer et al. (2003) and Stephan and Akyuz (2012).

As aforesaid, in this study, the development and revision processes of the instructional sequence were guided by the instructional principle of "Emergent Models" (Gravemeijer, 1991, 1994, 1999). Thus, it was anticipated that the classroom community would participate in and contribute to the development of emerging mathematical practices as they engage in increasingly sophisticated ways of tool use (Gravemeijer et al., 2003). Therefore, an increasingly sophisticated use of tools and models constitute a crucial element of the instructional sequence in this study. Although the ratio table was the overarching model, it took several forms of tools and inscriptions throughout the HLT (Gravemeijer, 1999), supporting the emergence of more formal tools and gestures, as explained in the following pages.

The CMP analysis revealed that the introduction and use of ratio tables could support the classroom community's ways of organizing the linked composites and keeping track of iterations with those linked composites. After the classroom community worked with the pictures of fish and food bars in order to make iterations with those on the first day of instruction, Berk and Aylin started to make table-like representations (see Figures 4.12 and Figure 4.13) that included iterations with the numerical values of the number of fish and food bars. The teacher showed these students' works and had these students explain their representations in front of the classroom community. Then, she took this opportunity to introduce a long ratio table, which was a similar representation to those of Aylin and Berkay, as a more organized way of their representations. The students immediately started to make iterations with the linked units in the tables (i.e., building up, 1-3, 2-6, 3-9…). As they made iterations in the tables, the first relationship recognized in the table was related to how the two quantities change in relation to each other (i.e., for every one food bar added, three more fish are added). This was the beginning of reasoning about the covariation of the two quantities (i.e., how one changes in relation to the other). Moreover, this was also the time that the use of hand gestures to point to the successive numbers in each row of the table emerged in the classroom discussion. This suggests that the introduction of the ratio table incented students' reasoning about the relative change of the two quantities as well as the use of hand gestures to make sense of and represent this change.

As the classroom community continued to investigate the relationships in the table, Sinem made a different claim regarding the relative relationship between the changes in the two quantities. Instead of looking at the relationship between the two consecutive numbers, she focused on the relationships between pairs of linked quantities in the table and claimed that "While the number of food bars increases by 5, the number of fish increases by 15. That means the number of fish always increases three times more in relation to the number of (food) bars." She also used hand gestures to move from one number to another within each row. Shortly after this, multiple students reasoned with the scale factor within each measure spaces by using abbreviated build-up strategies (i.e., when the number of fish is multiplied by n, so should the number of food bars) and expressing their reasoning with hand gestures that pointed to moving in between measure spaces. Moreover, this process was also connected to grouping, linking, and iterating made with the pictures previously upon the teacher's question. Thus, it was clear that how the iterations in the ratio tables were related to the previous activities of making iterations with food bars and fish.

As the students reasoned with scale factors within the measure spaces in the (horizontal) ratio tables and made corresponding hand gestures, these relationships were, then, referred to as horizontal relationships in the classroom community. After the teacher made sure that the long ratio tables and the horizontal relationships in these tables were discussed and used as a way for finding missing values, she provoked students to do more effective calculations to find a missing value by introducing a short(ened) ratio table stressing that "we don't even have to put all the columns in between." As the students worked on the short ratio tables, they drew on their experiences with the long ratio tables in order to reason with equal scale factors within each measure spaces and corresponding hand gestures and referred to those processes as horizontal relationships.

Therefore, the introduction of short ratio tables by a curtailment of the long ratio tables supported the classroom community's search for short and efficient ways of building up. More specifically, the imagery that underlay the short ratio tables was an abbreviated build-up process instead of building up by ones and singletons. Therefore, the advantages of this new tool were discussed, and the classroom community decided that it could be used as a tool for representing a shortcut way for abbreviated build-up reasoning. Just as they used multiplication as an easier way for repeated addition, they used the "times n" relationships in the short ratio tables as a shortened way for building up by ones and singles.

On the second day of the instruction, the teacher prompted students to reason about the invariant multiplicative relationship between the number of food bars and fish (i.e., regardless of the number of food bars and fish, the number of fish is always three times that of the number of food bars). Upon this request, Seval referred to the long ratio table they drew the previous day and claimed that "when we multiply the number of food bars with 3, we get the number of fish" by making hand gestures to draw vertical arrows from the first row to the second row (See Figure 4.19a & 4.19b). In the following instances, the classroom community reasoned in more conceptual ways by referring to the invariant functional relationship between the number of food bars and fish in normative ways. For example, when it was given that 2 food bars could feed 6 fish, multiple students referred to the invariant "times many relationship" (Confrey et al., 2014a; Dole, Downton, Cheeseman, & Sawatzki, 2018) by writing ×3 next to the table and made hand gestures to draw vertical arrows along the rows of the table. Moreover, they connected this “times three relationship” with the concept of unit rate. As the classroom community solved problems by using these invariant functional relationships between quantities and made corresponding hand gestures to refer to those, these relationships were then referred to as vertical relationships in the classroom community.

However, on Day 4, the ways the classroom community was reasoning diverted by the introduction of the symbolic representation of proportion. More precisely, after the teacher made sure that the students were ready to move to the formal and symbolic notation, she introduced the symbolic proportion representation by removing some of the borders in the short ratio table. Then, the teacher and the students negotiated the horizontal and vertical relationships in the symbolic proportion representation by drawing on the previous taken as shared ideas (see Figure 4.23). Therefore, the students mathematized the relationships within measure spaces and between measure spaces to arrive at and reason with the symbolic proportion representation. In this process, the tools and gestures played significant roles in supporting learning in increasingly sophisticated ways.

On the following days of the instruction, the classroom community continued to work with the symbolic representation and ratio tables. In particular, these ratio tables were drawn horizontally until the fifth day of the instruction. However, on Day 5, Ceyda suggested to draw the short table vertically (see Figure 4.27). This was the first time that the use of a vertical ratio table emerged in the classroom discussion. Two days later, on Day 7, a variety of problems were solved by using vertical and horizontal tables.

Additionally, the classroom community (including the teacher and the students) negotiated about the structuring of between measures ratio from horizontal ratio tables and within measures ratio from vertical ratio tables. Moreover, horizontal relationships in horizontal ratio tables were associated with vertical relationships in vertical ratio tables, and vertical relationships in horizontal ratio tables were associated with horizontal relationships in vertical ratio tables. Therefore, from that point, the terms horizontal and vertical relationships were not enough to refer to the previously taken as shared ideas. The classroom community had to re-negotiate what those relationships referred to in horizontal and vertical ratio tables. The associated gestures also needed to be renegotiated.

Therefore, from an RME perspective, long and short ratio tables functioned as *models of* organizing and keeping track of iterations with food bars and the number of fish in more efficient ways at the onset of the instruction. As the classroom community reasoned with these tables in order to solve a variety of problems, they became *models for* structuring symbolic representation of ratios and proportions. In particular, horizontal ratio tables became models for the structuring of between-measures ratios, and vertical ratio tables became models for the structuring of within measures ratios. Below is a summary of this transition process:

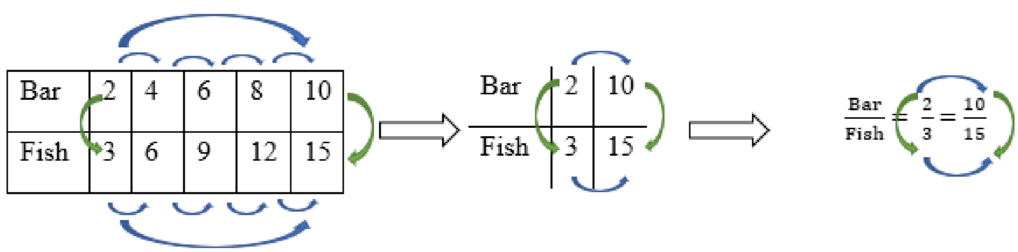


Figure 5.1. The transition from a *model of* to *model for* (Structuring of between measures ratios)

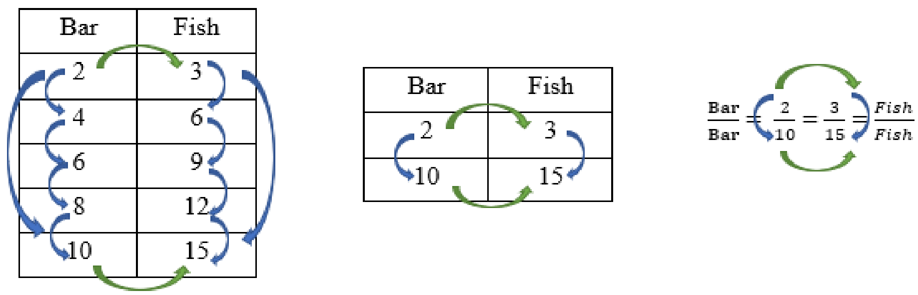


Figure 5.2. The transition from a *model of* to *model for* (Structuring of within measures ratios)

Therefore, the findings obtained from the CMP analysis suggest that the timings for the introduction of the semi-formal and formal notation were essential to foster the classroom community's reasoning as Gravemeijer et al. (2003) pointed out. The teacher had to make sure that the classroom community was ready to move to more formal ways of reasoning with the tools.

Kendon (2000) defines gestures as "the range of visible bodily actions that are, more or less generally regarded as part of a person's willing expression" (p. 49), which might involve facial expressions, body movements, and in particular hand movements. In this study, the use of gestures in this study mostly happened with the movements made by hand, similar to Rasmussen et al. (2004), although a few body movements also appeared in the classroom. The findings of this study suggest that the development and learning of a mathematical concept are associated with gestures in such a way that gesturing and learning, in particular mathematical practices, develop simultaneously as stressed by Rasmussen et al. (2004).

Although the most commonly used hand gestures were the ones to show the horizontal and vertical relationships in the tables and symbolic proportion, other uses of hand gestures also appeared in the classroom discussion. For example, while the symbolic representation of the proportion was introduced by the teacher, the classroom community used hand gestures to show the fractional division line by moving fingers to draw a straight line segment. Besides, as the classroom community interacted with the instructional sequence on Days 8 to 11, they used different hand gestures to show the graphical representation of proportional and linear relationships by drawing straight lines with fingers. Moreover, Merve made a hand gesture in order to show the different inclinations of the proportional relationships with the form $y = mx$. Nevertheless, since the idea of steepness was beyond the trajectory and the seventh-grade curriculum, the teacher did not push for further warrants and backings.

Several other uses of hand gestures occurred on Day 12 when the classroom community worked on reasoning within shapes ratios and between shapes ratios in order to determine whether the two shapes were similar or distorted and finding the missing lengths in pairs of similar shapes. For instance, Berk used hand gestures to shrink the rectangular shape to its half both vertically and horizontally. Giray used hand gestures to stretch the shape vertically. Moreover, Sinem made a hand gesture to show the composite units of 4 cm and 6 cm that she unitized and traced these units to make comparisons (i.e., norming) with the other lengths in two rectangles. Another use of hand gesture appeared while the classroom community reasoned about the direction of change in the last two tasks (i.e., orange juice task and qualitative reasoning task) while they made hand gestures by moving their hands upwards to show increase in the tastes of the liquids and by moving their hand downwards to show decrease in the tastes of the liquids. Moreover, apart from the use of hand gestures, another use of gestures included body movements, nodding their heads in order to show agreement and shaking their heads (side to side movement) in order to show disagreement in the classroom discussion.

Thompson (1996) refers to an image, rather than only being a mental picture, “as being constituted by experiential fragments from kinesthesis, proprioception, smell, touch, taste, vision, or hearing… and fragments of past cognitive experiences, such as judging, deciding, inferring, or imagining” (pp. 267-268). Moreover, he suggests that imagery has an essential influence on the development of mathematical reasoning in two aspects: “students’ immediate understandings of the situations” and “more global aspects of their development of mental operations” (p. 274). Concerning this, seeing covariation is described as “holding in mind a sustained image of two quantities’ values (magnitudes) simultaneously. It entails coupling the two quantities, so that, in one’s understanding, a multiplicative object is formed of the two” (Saldanha & Thompson, 1998, p. 298). The findings of this study provide further evidence for Saldanha and Thompson’s (1998) arguments regarding that the images of covariation are developmental, starting from holding an image of two linked quantities and coordinating those quantities in successive ways.

In particular, the findings of this study confirm the four successive images and operations that appear between ratio and rates suggested by Thompson and Thompson (1992) and Thompson (1994) with one exception. More specifically, these four successive images that appear between ratio and rates can be summarized as follows. At the first level, ratio, students compare two fixed quantities by using the criterion "as many times as." At the second level, internalized ratio, students construct accumulations of co-varying quantities where the accumulations occur additively, while no conceptual relationship between within measure spaces is made. At the third level, interiorized ratio, children construct covarying amounts of quantities additively, but they can anticipate that the ratio of the accumulations remains invariant. At the fourth level, rate, students can conceptualize the constant ratio variation between a pair of quantities as a single entity, that is, as a reflectively abstraction of ratio. In this study, the classroom community did not focus on the “as many times as” relationship between the number of food bars and fish in the first place. They started to explore the fish-food bar context by taking one food bar and three fish as a linked composite, and then, constructed accumulations of this linked composite by building up (i.e., additively) without referring to the invariant relationship between the number of food bars and fish (i.e., internalized ratio). Next, they constructed covarying amounts of quantities by building up by ones or operating with equal scale factors (i.e., interiorized ratio). Lastly, they were able to abstract the invariant relationship between the number of fish and food bars (i.e., rate). In other words, the findings of this study do not provide evidence concerning the first level in Thompson's successive images but confirm the remaining three images in their exact orders. The reason for this might be due to the discrete nature of the two quantities (i.e., fish and food bar) and how they are linked to each other. That is, since the context is about feeding a specific number of fish with a specific number of food bars, it might be possible that it has led students to think about the question “How many groups of that number of food bars and fish can be created?” rather than “What is the relationship between the number of food bars and fish?” in the first place.

5.3. The instructional sequence

The instructional sequence included several contexts in which essential understandings of proportional reasoning were embedded. In particular, most of the fundamental ideas of proportional reasoning were embedded in the fish-food bar context. The CMP analysis revealed that five mathematical ideas were taken-as-shared and transition from informal tools to more formal tools were made as students made explorations within this context on the first four days of the instruction. On the following days, while the classroom community engaged in discussions while they interacted with other contexts, they referred to the explorations regarding the vertical and horizontal relationships made in the fish-food bar context. Therefore, the findings of this study suggest that a context can serve as an aid for supporting student understanding and recalling mathematical ideas, in particular, taken-as-shared ideas as other researchers stressed (Meyer, Dekker, & Querelle, 2001; Reinke, 2019). In particular, Reinke (2019) refers to those contexts that foster learners’ making sense of new mathematical ideas and ensuing examples and tasks as *conceptual anchor contexts.* Thus, the findings of this study conclude that the fish-food bar context can serve as a conceptual anchor context in supporting a seventh-grade classroom community's development of proportional reasoning in increasingly sophisticated ways.

The instructional sequence was developed and revised based on an extensive literature review so as to include all types of problems stated in the literature. Based on the literature review, three major types of problems were identified as missing value problems, numerical comparison problems, and qualitative prediction/comparison problems (Behr et al., 1992; Cramer et al., 1993; Cramer & Post, 1993b; Heller et al., 1990; Kaput & West, 1994; Noelting, 1980a; Post et al., 1988). Therefore, it can be seen that the instructional sequence involves all types of proportional problems. More precisely, the Tasks 1-5 and 9 involve missing value problems, the Tasks 10-11 include numerical comparison problems, and the last task (i.e., Task 12) involves qualitative prediction/comparison problems. Moreover, the instructional sequence includes all the contexts suggested by Ben-Chaim et al. (1998): comparison of part-whole relationships (Tasks 3 and 4), comparison of two connected quantities (Tasks 10 and 11), and comparison of two conceptually related quantities that does not include part-whole relationships (Tasks 1, 5, and 9). Also, the tasks included the ratios of discrete quantities (Tasks 1, 3, 4, 5) and continuous quantities (Tasks 2, 5, 6, 8, 9, 10, 11).

Additionally, the literature on how task variables affect proportional reasoning was taken into consideration when the instructional sequence was designed, implemented, and revised. Nevertheless, since task variables were not the focus of this study, only a limited number of observations can be offered for consideration. Compatible with the conclusion of several researchers (e.g., Kaput & West, 1994; Tourniaire, 1986; Tourniaire & Pulos, 1985), the students were able to reason with discrete quantities (e.g., fish and food bars) more naturally and easily than they did with continuous quantities (e.g., lengths of shapes). As Tourniaire and Pulos (1985) concluded, this was because discrete quantities can be visualized more easily. Notably, throughout the first few days of the instruction, the classroom community linked the pictures of fish and food bars and iterate them coordinately. On the ensuing days, they were able to work with the mental pictures of food bars and fish and made these coordinated iterations in tables and with numbers. The Toulmin Analysis showed that while making iterations with (mental) pictures of fish and food bars, no warrants were provided or required in the classroom discussion. That is, the students were able to naturally and collectively work with the pictures of food bars and fish to make iterations. When the questions included continuous units such as lengths of similar shapes, the students provided warrants regarding their claims, and some of them were required to do so. This suggests that reasoning with continuous units required more demanding reasoning within the classroom community.

The findings of this study can also provide evidence to reflect on the hierarchical analysis of task variables by Kaput and West (1994). To rehash the previously stated arguments, the numerical task variables that enhance success in proportional tasks are (1) having a reduced form of ratio (i.e., integer ratio) between the quantities, (2) having a familiar multiple of one quantity of the other quantity either within the measure space or between the measure spaces, (3) involving a familiar rate. Similar to the arguments made by Kaput and West (1994), the findings of this study suggest that having a reduced form of ratio enhances students' proportional reasoning and increase the number of strategies they can use. For instance, on Day 2, while the students were working on the fish-food bar and the rule was 2 food bars-4 fish, the classroom community made use of the equivalence of the ratios $\frac{2}{4}$ and $\frac{1}{2}$ in solving the problems and stated that working with a unit rate made the calculations easier (see Figure 4.20).

Moreover, the findings of this study provide further evidence that the classroom community was able to work with quantities easily, one of which was a familiar multiple of the other either within the measure space or between the measure spaces, as suggested by Kaput and West (1994). The Toulmin analysis can provide evidence for this conclusion. When the rule was 1 food bar for three fish (3 is a familiar multiple of 1), the classroom community's reasoning with the whole number scale factors and functional relationships naturally emerged in the classroom across several days, and not many warrants/backings were provided or requested. However, the findings of this study also suggest that when a non-familiar multiple was present in the problem, the classroom community's reasoning increased in terms of sophistication. Hart (1981) makes a similar conclusion stated as: "when the question required operations more complicated than doubling or trebling the number of methods used by the children increased" (p. 91). For example, as afore-stated, the students came up with a variety of solution methods (including an incorrect one) with a variety of sophistication when comparing the tastes of Mixture A (3 glasses of orange juice and 2 glasses of water) and Mixture C (4 glasses of orange juice and 3 glasses of water). More specifically, these solution methods involved erroneous additive thinking, reasoning with part-part ratios within measure spaces, structuring ratios and comparing them based on fraction comparison, and equalizing the amounts of orange juice in both mixtures and comparing the amounts of water in both mixtures. It should be noted that neither within measure spaces ratio nor between measures spaces ratio yield in doubling or tripling. Instead, these comparisons required working with non-integer ratios.

Regarding the semantic task variables, no specific conclusion can be made in relation to whether containment and explicit use of for every/each statement in the problem fosters problem-solving performance in proportional tasks. However, the findings of this study conclude that including a familiar rate such as speed or unit price can foster proportional reasoning. For example, on Day 8, the classroom community was able to work with a variety of unit prices, represent the relationships between the amounts of vegetables/fruits and their price in tables and on graphs, and eventually formalize these relationships into algebraic equations. Moreover, on Day 13, they were able to naturally reason about the best buy for various kitchen ingredients in a variety of strategies including creating common denominators of the amount and choosing the smallest numerator of the price, creating unit rates and choosing the smallest numerator without proving or asking for any warrants and/or backings.

However, it should be noted that none of the following possible strategies emerged in the discussions that revolved around the best buy context: creating common numerators and choosing the biggest numerator; writing the rates as kg/TL, finding the common denominators, and choosing the largest denominator; finding the amount of kitchen ingredients that could be bought with 1 TL (i.e., unit rate); and focusing on the invariant relationship between the numerator and denominator (i.e., ×2, ×3, etc.). This might be a consequence of the "best buy" context, which was based on saving money. Therefore, the findings of this study conclude that the strategies used by students are highly influenced by the contexts of the problems. This can also be supported by the classroom discussions that took place in the speed contexts. Particularly, when the classroom community worked on deciding the interval in which a car had the largest speed, reasoning with between measure spaces ratios (i.e., the ratio between distance and time) dominated the classroom discussion.

Another task variable that was found to affect performance in proportional reasoning in this study, which was not mentioned by Kaput and West (1994), was including pictures in the problems. As hypothesized in the anticipatory thought experiments, the classroom community used representations with pictures in their warrants and backings in order to support their claims. This helped the students construct more salient claims and convince peers. For instance, when Sezin claimed that 6 food bars were needed to feed 12 fish, while the rule was 2 food bars for 4 fish, by dividing 12 by two. Berk did not understand why she divided by two, mainly because the rule was 2-4. Aylin provided a backing saying that the rules 1-2 and 2-4 are equivalent, and Gizem referred to the pictures in order to show that the two rules were equivalent, which acted as a further backing. Indeed, the Toulmin analysis showed that there were multiple instances where the pictures were used in providing data/warrant/backing in order to make/justify claims. Thus, the findings of the study conclude that including pictures in the problems facilitates students' problem-solving performance in proportional tasks.

5.3.1. Suggested Revisions to the Instructional Sequence and the HLT

The CMP analysis showed that the instructional sequence was actualized in the classroom as hypothesized to a great extent. In particular, it showed that the classroom community moved through the fish-food bar activity smoothly without any revision required. The documentation of the first classroom mathematical practice showed that the students could work with the scalar relationships within measure spaces and functional relationships between measure spaces in the fish-food bar context and various part-whole contexts. As the analysis of the first classroom mathematical practice revealed the ideas of linking composite units and iterating linked composites, covarying the linked composites by a scale factor (pre-multiplicative reasoning), invariance of the multiplicative relationship between the two units (multiplicative reasoning), conceptualizing the invariant relationship between the two linked composites as *unit rate* and *constant of proportionality* and using it as a tool/benchmark/anchor for finding missing values, and reasoning with ratio tables and symbolic proportion representation to extend covariation and invariation to the relationship between parts and the whole were taken-as-shared at the end of the first five days of the instruction.

However, the findings obtained by the CMP analysis suggest a few revisions to have a more viable instructional sequence for future uses. First of all, in the instructional sequence implemented in the second experiment, the students worked with discrete quantities until Day 12. This might have hindered the classroom community's reasoning with functional relationships in the form of decimals. Therefore, in order to foster students' reasoning with functional relationships in decimal forms, including a task with continuous variables in a recipe context as the second task of the instructional sequence is suggested. An example of such a task is provided in the final instructional sequence that is given in the appendix (see Appendix A).

Understanding and interpreting change in both relative and absolute terms was stated as an essential component of reasoning proportionally (Freudenthal, 1978; Lamon, 1995;

NCTM, 2000). Although there were several instances in which the classroom discussion revolved around interpreting change in relative terms, a task that specifically focuses on absolute and relative thinking and how they differ from each other is suggested to be included as the eighth task in the instructional sequence. In this way, further evidence could be obtained from the classroom mathematical analysis to conclude that these ideas were taken-as-shared.

The seventh task in the instructional sequence, the similar shapes task, was designed so that students could work on similar shapes in order to extend previous ideas regarding proportion to the concept of similarity. Particularly, the first part of this activity was designed for students to draw on their previous experiences of creating equivalent ratios in order to find the missing lengths in pairs of similar shapes. The second part was designed so that the students would compare the two shapes and determine whether those shapes would be similar (i.e., have proportional lengths within and between shapes). Therefore, it was hypothesized that the flow of this activity would be parallel to the flow of the instructional sequence that first poses problems about creating equivalent ratios and then comparing ratios. However, the analysis of the fourth classroom mathematical practice (CMP 4) revealed that the similarity task posed the term "similar" without students exploring what that meant beforehand. The classroom dialogue that took place on Day 12 pointed that although a few students made connections between the lengths of shapes' being proportional and the shapes' being similar, it did not naturally occur to every student in the classroom community. This was evident when İlter challenged Berk's solution that included writing a proportion between the short and long side lengths within the shapes to find the missing length with the following question: "Why do they have to be proportional? It says the shapes are similar, not proportional." Although Berk and Onur were able to provide warrants for their solutions, including that the shape was shrunk to its half, the teacher had to explain what shrinking a shape to its half had to with being similar.

On the other hand, in the second period of the instruction, the students were presented with an original picture and its various copies and asked to determine which of the copies resembled the original picture. Particularly, the teacher launched the task with the question "Which of these resembles the kid (in the picture) the most?" without using the mathematical term similar, and one of the students immediately answered that “the one that is stretched from both directions resembles the most," which showed that he or she made an intuitive connection between stretching and resemblance. Upon this claim, the teacher posed the big question for students to explore “Is it enough to stretch from both directions? I would like you to pay attention to this. Should I stretch more from one direction than the other direction? Should I stretch in the same amount from both directions? Or what should we do? Let’s discuss with our group members.”

After a small group discussion time, the students made claims about whether or not the pictures were similar without creating equivalent ratios but looking at whether or not the shapes were stretched/shrunk from both directions in the first place. Several students claimed that a shape could not be similar to the original picture if it was shrunk/stretched from only one direction. For instance, while exploring whether the shape that had lengths 6 cm and 6 cm was similar to the original picture that had side lengths 6 cm and 4 cm, Giray claimed that the two pictures were not similar since the second picture was stretched only in one direction. He provided a warrant to this pair of data and claim by stressing that the kid in the picture looked taller and thinner compared to the original picture since it was stretched only vertically. Ozan made a connection to his daily life experience of people photoshopping pictures in order to look taller and thinner than they actually are.

Furthermore, Elif drew attention to another point based on the fact that the picture was a square while the original picture was a rectangle. Therefore, multiple students intuitively reasoned about the distortion of the shapes when they were stretched only in one direction based on their informal knowledge in the first two questions. Then, for the third shape that had lengths 8 cm and 12 cm, Sezin claimed that they were similar since the kids in the two pictures looked exactly the same, neither fatter nor taller. Moreover, Mehmet provided a warrant for this argument by stating that that copy was the doubled version of the original picture and created equivalent ratios to provide a backing (i.e., $\frac{Short}{Long} = \frac{4}{6}$ $= \frac{8}{12}$). As the discussion moved on, the classroom community came to the point that the shapes had to be scaled with a certain factor in both directions in order to obtain similar shapes; that is, the shapes had to be stretched (or shrunk) in both ways proportionally. Besides, within-shapes ratios were discussed, and connections between within shapes ratios and being proportional were established.

Therefore, the CMP analysis showed that the ideas of similarity and distortion naturally emerged in the classroom discussion based on students' intuitive and informal knowledge of resemblance and stretching/shrinking as they tackled with the second part of the task. This is no surprise when Lamon's (1995) claim that even most preschool children can intuitively reason about whether a drawing or a picture appear right or wrong in terms of scaling or enlargement is considered. Therefore, it is suggested that the order of the two parts of this task is reversed so that students can draw on their informal ideas regarding stretching/shrinking and resemblance/distortion before they start using tables and proportions to reason with within and between shapes ratios.

In addition to the revisions for the instructional sequence, the CMP analysis suggested a few revisions for the HLT that served as the mainstay of this study. To begin with, although reasoning in part-whole contexts was stated as the second phase before conducting CMP analysis, the findings of the study showed that the ideas that took place in the part-whole contexts revolved around the ideas of iterating linked composites by (abbreviated) build-up strategies and establishing and reasoning with the invariance between quantities, as similar to the ideas occurred in the fish and food bar context. That is, similar mental operations and reasoning took place as students moved to the part-whole contexts, and the only difference was about the quantities having part-whole relationships. Therefore, reasoning in part-whole contexts did not suggest a new phase, so, it was merged to the first phase. On the other hand, the CMP analysis made sure that structuring symbolic representations of ratios and proportions suggested a new phase since the classroom community's reasoning as well as their tool use shifted direction on Day 6. Therefore, the second phase of the HLT table started when the idea of structuring ratios and proportions emerged in the classroom discussion and continued with as this idea was taken-as-shared as well as the ideas of creating equivalent ratios and analyzing equivalent ratios.

Another revision about the phases of the HLT was adding a new phase that started when the similarity tasks were posed to the students on Day 12. The classroom mathematical practice analysis showed that the students' thinking shifted from reasoning about proportional and linear relationships and their algebraic and graphical representations to reasoning about similarity and distortion by using within shapes and between shapes ratios on Day 12. More specifically, the occurrence of students' acts of challenging and justifications that included the relationships between proportionality and similarity/distortion provided evidence that their reasoning was shifted to another direction. Moreover, reasoning about continuous quantities in a geometrical context and using new gestures to shrink/stretch gave further evidence. Therefore, a new phase named analyzing proportionality, similarity, and distortion was added to the HLT table as the fourth phase. In addition, related hand gestures for stretching, shrinking, and tracing were added as the components of this phase.

As the classroom instruction approached the last day, the students worked on the last task named Comparing coffee strengths that was posed qualitatively so that the idea of qualitative reasoning would emerge and become taken-as-shared in the classroom community. The CMP analysis showed that although the task fostered the use of previous ideas in students' data, warrants, and backings in a new context and showed further evidence that those ideas were taken-as-shared. Moreover, it revealed that this task really got students really challenged and talk about erroneous additive reasoning and paved the way for the emergence of the imageries of density and dispersion. However, even though this task had great potential for the issues above, the CMP analysis suggested that it did not cause the starting of a new phase. More precisely, since there was only one task posed qualitatively that did not cause a shift in the normative ways of reasoning of the classroom community, it did not merit a new phase on its own. Therefore, it was merged as an idea to the previous phase of the HLT. In addition, the CMP analysis failed to provide salient evidence of whether it was taken-as-shared in the classroom community.

To conclude, the CMP analysis not only documented the collective growth of the classroom community but also provided a retrospective outline of the mathematical content that arose over the course of six weeks of implementation. Moreover, it unfolded the necessary revisions for the content and order of the instructional sequence and the Hypothetical Learning Trajectory for future uses. The revised version of the instructional sequence is given as an appendix to this study (see Appendix A). The phases of the revised HLT are provided below in Figure 5.3 through Figure 5.7.

Big Idea	Tools/Imagery	Activity-Taken-as-shared interest	Possible Topics of Mathematical Discourse	Possible Gesturing and Metaphors
Linking Composite Units	-Grouping pictures by circling and connecting two composite units with arrows -Drawing shapes	If 1 food bar feeds 3 fish, the rule is kept the same regardless of the number of food bars and fish	-If the rule is 1 food bar feeds 3 fish, the rule cannot be broken if we add more food bars	-Matching concrete objects for illustration (i.e., 3 pencils for 1 student) -Making hand gestures for linking
Iterating Linked Composite Units Covarying two linked quantities	-Informal symbolizing (matched numbers, i.e. 1-3,2-6 etc.; tabular-like-representations) -(Mental)images of objects -Build up (Additive) Strategy -(Horizontal and vertical) Long ratio tables (numerical patterns, showing increments in tables) -Abbreviated Build Up Strategy -Shortened ratio tables (scaling values within the same measure spaces on the tables) -Within comparisons -Informal proportionality constant	-Keeping track of two linked quantities while making them bigger (additively or by multiplications as a short way for repeated addition) -Increasing both quantities by the same scale factor	-How to keep track of two quantities while making them bigger? -What are the efficient ways to build up? -What are the efficient ways for curtailing long ratio tables?	-Matching concrete objects and iterating for illustration -Counting up/down with fingers horizontally along the horizontal tables for building up/down -Making hand gestures to show the horizontal relationships in tables (i.e., scaling up/down values within same measure spaces)
Multiplicative Reasoning	-(Horizontal and vertical) Long and shortened ratio tables (numerical patterns) -Informal unit rate (finding the unit number of values) -Between comparisons	-Multiplying/dividing to build up/down by keeping track of quantities -Recognizing the ease of working with unit rate	-While two quantities covary, there exists a third quantity that remains invariant (i.e., it doesn't change).	Making hand gestures to show the vertical relationships in tables (i.e., functional relationships)
Reasoning in part-whole contexts	-Short and long ratio tables -Informal ratio language -Within and between comparisons -part-whole relationships -Fraction imagery	-Extending covariation and invariation to the relationship between parts and the whole	-Parts among themselves and parts with their whole covary in a precise way -There is an invariant relationship between parts and their whole and among the parts themselves. -Part-whole relationships represent fractions	Horizontal and vertical hand gestures

Figure 5.3. Phase 1 of the revised HLT

Big Idea	Tools/Imagery	Activity-Taken-as-shared interest	Possible Topics of Mathematical Discourse	Possible Gesturing and Metaphors
Structuring ratios and proportions	-Short ratio tables -Fractional representation -Symbolic proportion representation	-Moving from short ratio tables to proportions -Structuring ratios and proportions multiplicatively	-Ratio refers to how the two values compare to each other -Proportion is the equality of two ratios -There is an invariant relationship between proportional situations as the values change together in precise ways	Moving fingers to draw a straight line for the fractional division line
Creating equivalent ratios	-Ratio tables with missing values -Symbolic proportion representation with missing values -Vertical and horizontal relationships in proportion representation	-Extending covariance and invariance to proportion -Finding missing values by within and between measures comparisons	-What do the horizontal and vertical relationships mean?	Horizontal and vertical hand gestures
Analyzing equivalent ratios	-Within and between measures comparisons -Equal sign with a question mark on top of it	-Reasoning with within and between measures comparisons -Determining proportionality by covariational and multiplicative reasoning	Do the horizontal/vertical relationships hold the same for the two ratios?	-Horizontal and vertical hand gestures -Drawing question marks with fingers

Figure 5.4. Phase 2 of the revised HLT

Big Idea	Tools/Imagery	Activity-Taken-as-shared interest	Possible Topics of Mathematical Discourse	Possible Gesturing and Metaphors
Representing proportional relationships on linear graphs	-Long ratio tables -Linear graphs on the coordinate plane	-Moving from tables to linear graphs to represent proportional relationships -Representing proportional relationships on linear graphs of the form $y = mx$ and recognizing that they pass through the origin	-How to represent proportional situations on graphs? -What are the features of the graphs of proportional situations?	- Horizontal and vertical hand gestures -Drawing a straight line with fingers
Representing proportional relationships algebraically.	-Long ratio tables -Algebraic equation of the form $y = mx$	-Formalizing the invariant multiplicative relationship in a proportional situation into an algebraic equation	-How to represent proportional situations with algebraic equations? -What are the features of the equations of proportional situations?	
Analyzing linearity and proportionality	-Long ratio tables -Linear graphs on the coordinate plane -Algebraic equations of the forms $y = mx$ and $y = mx + n$	-Distinguishing between linearity and proportionality -Distinguishing the features of proportional and linear relationships	What destroys proportionality?	

Figure 5.5. Phase 3 of the revised HLT

Big Idea	Tools/Imagery	Activity-Taken-as-shared interest	Possible Topics of Mathematical Discourse	Possible Gesturing and Metaphors
Analyzing proportionality, similarity, and distortion	-Within shapes and between shapes comparisons in ratio tables or proportion - use of not equivalent symbol (i.e., ≠) for ratios that are not proportional	-Conceptualizing similarity and distortion -Establishing relationships between proportionality and similarity	-Are the shapes similar or one of them is distorted? -What is the relationship between similarity and proportionality? -What does the within shapes ratio mean? -What does the between shapes ratio mean?	- Hand gestures for shrinking vertically or horizontally -Hand gestures for stretching vertically or horizontally -Hand gestures for tracing the unit lengths (i.e., unitizing and norming)

Figure 5.6. Phase 4 of the revised HLT

Big Idea	Tools/Imagery	Activity-Taken-as-shared interest	Possible Topics of Mathematical Discourse	Possible Gesturing and Metaphors
Comparing ratios/rates	-Fraction imagery (part-whole ratios) -Anchoring to whole and/or half -Greater than, less than symbols (<, >) -Images of density and strength -Informal ratio language - Orange juice and water neutralize each other (Canceling out each other)	- Comparing rates/ratios and deciding which one is bigger /smaller /equal -Comparing the strength of mixtures with different amounts of ingredients -Equalizing the numerator or denominator of ratios/rates	-Which of the mixtures tastes stronger? -What do part-part and part-whole ratios tell? -What do the tastes of the mixtures have to do with proportionality?	
Qualitative reasoning	-Dispersion -Images of density and strength -Coffeepot/cup imagery		-How does adding coffee/milk affect the strength of liquids? -How to reason about the direction of change and determinability? -Is the change of direction determinable? -In what direction does the value of ratio change?	-Hand gestures for showing the direction of change

Figure 5.7. Phase 5 of the revised HLT

5.4. Implications of the Study

The findings and conclusions of this study provide essential insights into increasing the quality of instruction of proportional reasoning, which would be interest to many people in the mathematics education field. These people include mathematics teachers, primary school teachers, preservice teachers, teacher educators, curriculum developers, educational stakeholders, and mathematics education researchers.

First and foremost, the findings of this study demonstrated that the instructional sequence and the related Hypothetical Learning Trajectory have the potential to support a seventh-grade classroom community's development of proportional reasoning in increasingly sophisticated ways in an argumentative classroom environment. The instructional sequence and the associated local instructional theory that conjectures about a likely learning path and possible means of supporting that path are provided as a readily usable instructional tool. Therefore, teachers can easily integrate these into their ratio and proportion instruction in seventh grade. Lower grades teachers can also let their students explore some of the tasks and reason about the ideas (e.g., covariation and invariation) without formally structuring ratios and proportions in order to facilitate students' early proportional reasoning. Notably, the findings may help teachers anticipate students' informal and formal strategies and their difficulties in the development of proportional reasoning. Teacher educators can also integrate those into their instruction in order to help preservice teachers develop their subject matter knowledge and pedagogical content knowledge so that they can use those in their future instructions. This is significant when teachers' lack of understanding of the essential elements of proportional reasoning is considered.

Although there is a mass of literature stressing the importance of proportional reasoning and documenting students' conceptions and misconceptions in this area, studies documenting essential understandings of proportional reasoning and how those understandings can be supported in a social context are rare. The literature review conducted in this study and the analysis of classroom mathematical practices outlined the essential understandings of proportional reasoning and didactical activities. The examination of the Turkish Middle School Mathematics Curriculum (MoNE, 2013, 2018) indicated that the curriculum lacks several of those understandings (e.g., iterating linked composites, absolute and relative thinking, qualitative reasoning, and several others). Therefore, based on the findings of this study, several objectives can be added to the topic “ratio and proportion” in sixth and seventh grades. Furthermore, several objectives can be added to other topics in a variety of grades since the findings of this study indicated that proportional reasoning develops in relation to several concepts and topics, including fractions, rational numbers, and multiplication and division. Concerning this, a promising teaching approach for enhancing proportional reasoning would be taking and grasping every opportunity to promote the development of proportional reasoning from preschool through middle school (Lamon, 1995). This requires seeing proportional reasoning as an umbrella skill that should be revisited in almost every topic in mathematics and many topics in science rather than approaching it as an isolated topic on its own. Moreover, it calls for linking applications of proportional reasoning with the general act of problem-solving in much wider contexts (Akatugba & Wallace, 1999).

Several researchers pointed out that problems used in classrooms and textbooks and how they are taught do not help the conceptual development of students. More precisely, I agree with Freudenthal (1973) who stressed that textbooks include irrealistic problems that dictate routine solution methods, and also with Nunes and Bryant (1996) who state that "the problems that are used at school in mathematics exercise books for teaching children about proportions are often more an excuse to use the arithmetic than a content for the youngsters to think about" (p. 182-183). Over and above, these problems are treated in classrooms by focusing on fluent execution of algorithmic procedures without consideration and elaboration of the relationships inherent in those problems (van Dooren et al., 2010). The findings of this study suggest that if a variety of problem types (i.e., missing value, comparison, and qualitative comparison) in a variety of realistic contexts are used in instruction in a successively sophisticated order, a classroom community can develop a meaningful and comprehensive understanding of the concepts of ratio, rate, and proportion in normative ways.

Particularly, the findings of this study demonstrated that the cross-multiplication algorithm did not emerge naturally in the classroom discourse. Indeed, the classroom community could establish a variety of conceptual relationships in order to make sense of the proportional tasks and develop methods to solve those tasks. Therefore, the findings of this study suggest that a seventh-grade classroom can tackle with proportional tasks without using procedural algorithms such as cross-multiplication. Nevertheless, as stressed in NCTM (2000), I agree that "the so-called cross-multiplication method can be developed meaningfully if it arises naturally in students' work, but it can also have unfortunate side effects when students do not adequately understand when the method is appropriate to use" (p. 221). That is, the method can only be an endpoint and used in instruction only after students have had a great deal of experiences and opportunities in a variety of contexts since it is a mechanical process being efficient but devoid of meaning (Cramer, Post, & Behr, 1989; Lamon, 1993; Post et al., 1988; Streefland, 1985).

The findings of the study indicated that building the instruction on students' informal and intuitive understandings (e.g., grouping, building up strategies, unitizing, and norming) have significant potential in order to support a classroom community's transition from informal to more formal knowledge regarding ratio and proportion. Therefore, I agree with several researchers who points out to the "need for the kinds of concrete representations that support and extend students' natural build-up reasoning patterns rooted in counting, skip counting, and grouping" (Kaput & West, 1994, p. 283) and "matching and partitioning" (Lamon, 1995, p. 178) in grades 3-5 before formal instruction on ratio and proportion takes place. That is, I suggest that the formal representation of ratios and proportions should be built on what students already know and understand. In this way, students' difficulties, especially erroneous additive reasoning, can be prevented, rather than remedied before it constitutes serious problems (Resnick & Singer, 1993; van Dooren et al., 2010). It should be noted that the complexity of students' knowledge and reasoning "starts out small and, with effective instruction, becomes much larger over time, and that the amount of growth clearly varies with experience and instruction" (Daro et al., 2011, p. 23). That is, as students move through the instruction over time, they will be more competent and proficient. Therefore, improving conceptual understanding of proportional reasoning would have the potential to facilitate understandings of several concepts in mathematics and science.

The findings of the study showed that the numbers used in tasks are essential to support the conceptual growth of the classroom community. More specifically, starting with integer ratios (e.g., 1-3) can support making sense of problems at the onset. In addition, including pictures as students start out their explorations with proportional tasks can foster their use of a variety of strategies. Moving through using reducible ratios (e.g., 2-4) and using non-integer values (e.g., 2-3, 3-5) can support the development of more sophisticated relationships and solution methods.

Last but not least, the findings of the study indicated that the use of tools, models, imageries, and gestures have great potential in supporting a classroom community's conceptual growth. Especially a transition from a model of to model for perspective in line with the theory of Realistic Mathematics Education has proved effective in supporting the development of the classroom community's proportional reasoning in increasing sophistication. Thus, it is suggested that this perspective should be adopted in teaching other concepts as well.

5.5. Suggestions for Further Research

This study provided a picture of how a seventh-grade classroom community developed essential ideas of proportional reasoning in increasingly sophisticated ways, how those ideas became taken-as-shared as the students interacted with the instructional sequence, and how this development was supported. Thus, the findings of the study raise several questions and lack several answers to other questions. Therefore, in this part, additional explorations to find answers to those questions raised or remained unanswered by this study are suggested as further research.

First of all, the data for this study were collected within a particular Turkish middle school context. It would be possible to see if the patterns in the development of proportional reasoning would be different in other contexts where the ratio and proportion topics are handled and sequenced differently. Thus, it would be interesting to examine whether the same developmental path would be followed or to what extent it would be followed as students interact with the same instructional sequence in another context. This would be a significant topic of study since cultural aspects might have a role in students' learning and development in general (Trueba, 1988; Werstsch & Toma, 1992), and, in particular, proportional reasoning (Aktugba & Wallace, 1999).

The documentation of the classroom mathematical practices gave essential insight into the collective development of a classroom community's proportional reasoning. It should be noted that this development took place in a sociocultural setting wherein social interactions contributed to learning (Lave, 1992). Observing and documenting the learning that took place in an intact class might fail to address some of the legitimate questions about individual students' understanding and development (Lesh & Doerr, 2000). Therefore, investigating individual students' development and especially how specific individuals benefited from and contributed to this collective development would be an interesting topic of study. Hence, a further study that explores individual development of students through individual pre-, post-, and ongoing interviews is suggested to provide a complete insight into the learning and development that took place in the classroom. Moreover, measuring the extent of the learning and development by means of quantitative analysis (e.g., pre- and post-tests) would be helpful to have a more holistic and multifaceted understanding of the learning and development that occurred in the classroom community. Furthermore, conducting a longitudinal study would help examine the long-term learning and retainment of the classroom community.

Another potential limitation of the study is that the scope is only limited to direct proportional reasoning. However, many of the essential understandings listed here (e.g., covariation and invariation) are also essential in the development of inverse proportional relationships. Therefore, a study that focuses on designing an instructional sequence and associated local instructional theory for the teaching of inverse proportional relationships and documentation of classroom mathematical practices analysis would contribute to an understanding of the development of a classroom community's inverse proportional relationships in a great extent. Such a study would be a significant contribution to the theory and practice when the scarcity of studies investigating students' understandings of inverse proportional relationships is taken into account.

This study provided evidence that proportional reasoning develops in relation to other concepts such as fractions, rational numbers, linear relationships, and similarity. However, this was not the main focus of this study. Thus, how proportional reasoning is developed within the multiplicative conceptual field might be a significant topic of study. Although a few studies touched upon these issues (e.g., Clark, Berenson, & Cavey, 2003; Hurst, & Cordes, 2019; Lo & Watanabe, 1997; Smith III, 2002; Thompson, & Saldanha, 2003) at the individual level, there is a need to know more about the interplay between those concepts and the development of proportional reasoning as it takes in social context of a classroom.

Lastly, the classroom mathematical practices that emerged in the classroom suggested several revisions to the instructional sequence and the HLT. In particular, a few tasks were suggested to be included in the instructional sequence, and a few tasks were switched places. Moreover, a few changes were made in the phases of the HLT and its components. Therefore, a subsequent design experiment (i.e., Design Experiment 3) can give insight into whether those would be actualized in a further experiment and particularly if those changes would support a classroom community's learning and development of proportional reasoning. Therefore, a follow-up study in which the revised version of the instructional sequence would be tried out is suggested in order to have a more viable and better case instructional sequence.

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APPENDICES

A. METU HUMAN SUBJECTS ETHICS COMMITTEE APPROVAL/ODTÜ İNSAN ARAŞTIRMALARI ETİK KURULU ONAYI

UYGULAMALI ETİK ARAŞTIRMA MERKEZİ
APPLIED ETHICS RESEARCH CENTER

ORTA DOĞU TEKNİK ÜNİVERSİTESİ
MIDDLE EAST TECHNICAL UNIVERSITY

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Sayı: 28620816 / 467

05 ARALIK 2016

Konu: Değerlendirme Sonucu

Gönderilen: Doç. Dr. Mine IŞIKSAL BOSTAN,

Eğitim Fakültesi

Gönderen: ODTÜ İnsan Araştırmaları Etik Kurulu (İAEK)

İlgi: İnsan Araştırmaları Etik Kurulu Başvurusu

Sayın, Doç. Dr. Mine IŞIKSAL BOSTAN, Ar. Gör. Rukiye AYAN

"7. Sınıf Oran ve Orantı Konusunun Öğrenme Rotaları Yaklaşımına Göre Öğretimi" başlıklı araştırmanız İnsan Araştırmaları Kurulu tarafından uygun görülerek gerekli onay **2015-EGT-161** protokol numarası ile **01.01.2017-30.04.2017** tarihleri arasında geçerli olmak üzere verilmiştir.

Bilgilerinize saygılarımla sunarım.

Prof. Dr. Canan SÜMER

İnsan Araştırmaları Etik Kurulu Başkanı

Prof. Dr. Mehmet UTKU

İAEK Üyesi

Prof. Dr. Ayhan SOL

İAEK Üyesi

Prof. Dr. Ayhan Gürbüz DEMİR

İAEK Üyesi

Doç. Dr. Yaşar KONDAKÇI

İAEK Üyesi

Yrd . Doç .Dr. Pınar KAYGAN

İAEK Üyesi

Yrd. Doç. Dr. Emre SELÇUK

İAEK Üyesi

BU BÖLÜM, İLGİLİ BÖLÜMLERİ TEMSİL EDEN İNSAN ARAŞTIRMALARI ETİK ALT KURULU TARAFINDAN DOLDURULACAKTIR.

Protokol No: 2015-EGT-161

İAEK DEĞERLENDİRME SONUCU

Sayın Hakem,

Aşağıda yer alan üç seçenekten birini işaretleyerek değerlendirmenizi tamamlayınız. Lütfen "Revizyon Gereklidir" ve "Ret" değerlendirmeleri için gerekli açıklamaları yapınız.

Değerlendirme Tarihi: 14.12.2015

Ad Soyad:

☒ **Herhangi bir değişikliğe gerek yoktur. Veri toplama/uygulama başlatılabilir.**

☐ **Revizyon gereklidir**

- ☐ Gönüllü Katılım Formu yoktur.
- ☐ Gönüllü Katılım Formu eksiktir.
  Gerekçenizi ayrıntılı olarak açıklayınız:
- ☐ Katılım Sonrası Bilgilendirme Formu yoktur.
- ☐ Katılım Sonrası Bilgilendirme Formu eksiktir.
  Gerekçenizi ayrıntılı olarak açıklayınız:
- ☐ Rahatsızlık kaynağı olabilecek sorular/maddeler ya da prosedürler içerilmektedir.
  Gerekçenizi ayrıntılı olarak açıklayınız:
- ☐ Diğer.
  Gerekçenizi ayrıntılı olarak açıklayınız:

☐ **Ret**

Ret gerekçenizi ayrıntılı olarak açıklayınız:

B. OFFICIAL PERMISSIONS TAKEN FROM THE MINISTRY OF NATIONAL EDUCATION/MEB ARAŞTIRMA İZNİ ONAYI

T.C.
ANKARA VALİLİĞİ
Milli Eğitim Müdürlüğü

ÖĞRENCİ İŞLE...
DAİRESİ BAŞKANL...
Ev. Arş. Md. Saat :

Sayı : 14588481-605.99-E.1787714 16.02.2016
Konu : Araştırma İzni

ORTA DOĞU TEKNİK ÜNİVERSİTESİNE
(Öğrenci İşleri Daire Başkanlığı)

İlgi: a) MEB Yenilik ve Eğitim Teknolojileri Genel Müdürlüğünün 2012/13 nolu Genelgesi.
b) 28/01/2016 tarihli ve 445 sayılı yazınız.

Üniversiteniz İlköğretim Anabilim Dalı doktora öğrencisi Rukiye AYAN'ın **"7. Sınıf Oran ve Orantı Konusunun Öğrenme Rotaları Yaklaşımına Göre Öğretimi"** konulu araştırma kapsamında gözlem, görüşme, ses ve kamera kaydı talebi Müdürlüğümüzce uygun görülmüş ve uygulamanın yapılacağı İlçe Milli Eğitim Müdürlüğüne bilgi verilmiştir.

Görüşme formunun (5 sayfa) araştırmacı tarafından uygulama yapılacak sayıda çoğaltılması ve çalışmanın bitiminde bir örneğinin (cd ortamında) Müdürlüğümüz Strateji Geliştirme (1) Şubesine gönderilmesini arz ederim.

Ali GÜNGÖR
Müdür a.
Şube Müdürü

22-02-2016-3242

Güvenli Elektronik İmzalı
Aslı İle Aynıdır.
16/02/2016

Yaşar SUBAŞI

Konya yolu Başkent Öğretmen Evi arkası Beşevler ANKARA
e-posta: istatistik06@meb.gov.tr

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C. THE INSTRUCTIONAL SEQUENCE/ETKİNLİK DİZİSİ

The final version of the instructional sequence developed in this study together with the suggestions for future uses is presented below. The list of the instructional tasks that comprise the instructional sequence is presented in the table below in English. In the following pages, the list of the instructional tasks and the content of these tasks are provided in Turkish.

Table 1. The tasks in the final version of the instructional sequence

Instructional tasks	Learning goals
1. Let's feed the fish	Linking composite units, iterating linked composites, unitizing and norming, covariation and invariation, within and between measures comparisons, distinguishing rates and ratios, ratio appropriateness (ratio sense), additive and multiplicative reasoning
2. Let's make cakes and fruit punches	Linking composite units, iterating linked composites, unitizing and norming, covariation and invariation, within and between measures comparisons, distinguishing rates and ratios, ratio appropriateness (ratio sense), additive and multiplicative reasoning
3. What do the survey results tell?	Linking composite units, iterating linked composites, unitizing and norming, covariation and invariation, within and between measures comparisons, distinguishing rates and ratios, ratio appropriateness (ratio sense), additive and multiplicative reasoning, informal ratio language
4. Learning ratio and proportion	Linking composite units, iterating linked composites, unitizing and norming, covariation and invariation, within and between measures comparisons, distinguishing rates and ratios, ratio appropriateness (ratio sense), additive and multiplicative reasoning, formal ratio language, symbolic use of ratio and proportion
5. Let's solve problems	Linking composite units, iterating linked composites, unitizing and norming, covariation and invariation, within and between measures comparisons, ratio appropriateness (ratio sense), additive and multiplicative reasoning, formal and informal ratio language, symbolic use of ratio and proportion

6. Representing proportional situations with graphs and equations	Linking composite units, iterating linked composites, unitizing and norming, covariation and invariation, linear relationships and their representations, proportionality
7. Let's explore proportionality and linearity	Linking composite units, iterating linked composites, unitizing and norming, covariation and invariation, linear relationships and their representations, additive and multiplicative reasoning
8. How much have the trees grown?	Linking composite units, iterating linked composites, unitizing and norming, covariation and invariation, ratio appropriateness (ratio sense), relative and absolute change, additive and multiplicative reasoning
9. Do the pictures look alike?	Iterating linked composites, unitizing and norming, covariation and invariation, within and between measures comparisons, ratio appropriateness (ratio sense), additive and multiplicative reasoning, similarity and distortion.
10. Comparing speeds and deciding on best buy	Iterating linked composites, unitizing and norming, covariation and invariation, within and between measures comparisons, ratio appropriateness (ratio sense), additive and multiplicative reasoning, distinguishing rates and ratios
11. Comparing oranginess	Iterating linked composites, unitizing and norming, covariation and invariation, within and between measures comparisons, ratio appropriateness (ratio sense), additive and multiplicative reasoning, distinguishing rates and ratios
12. Comparing coffee strengths	Unitizing and norming, ratio appropriateness (ratio sense), absolute and relative thinking, additive and multiplicative reasoning, qualitative reasoning

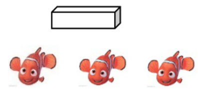
Tablo 2. Öğretimsel etkinlik dizisindeki etkinlikler

Etkinlikler	Öğrenme amaçları
1. Balıkları besleyelim	Birleşik birimleri bağlama, bağlı birleşik birimleri yineleme, birimleme ve biçimlendirme, ortak değişinti (kovaryasyon) ve değişmezlik (invaryasyon), aynı ve farklı ölçüm uzaylarında yapılan kıyaslamalar, oran hissi, toplamsal ve çarpımsal düşünme
2. Tariflerle kek ve meyvu suyu yapalım	Birleşik birimleri bağlama, bağlı birleşik birimleri yineleme, birimleme ve biçimlendirme, ortak değişinti (kovaryasyon) ve değişmezlik (invaryasyon), aynı ve farklı ölçüm uzaylarında yapılan kıyaslamalar, oran hissi, toplamsal ve çarpımsal düşünme
3. Anket sonuçları ne söylüyor?	Birleşik birimleri bağlama, bağlı birleşik birimleri yineleme, birimleme ve biçimlendirme, ortak değişinti (kovaryasyon) ve değişmezlik (invaryasyon), aynı ve farklı ölçüm uzaylarında yapılan kıyaslamalar, oran hissi, toplamsal ve çarpımsal düşünme, informel oran dili
4. Oran ve orantıyı öğrenelim	Birleşik birimleri bağlama, bağlı birleşik birimleri yineleme, birimleme ve biçimlendirme, ortak değişinti (kovaryasyon) ve değişmezlik (invaryasyon), aynı ve farklı ölçüm uzaylarında yapılan kıyaslamalar, oran hissi, toplamsal ve çarpımsal düşünme, formel oran dili, sembolik oran ve orantı gösterimleri
5. Problemler çözelim	Birleşik birimleri bağlama, bağlı birleşik birimleri yineleme, birimleme ve biçimlendirme, ortak değişinti (kovaryasyon) ve değişmezlik (invaryasyon), aynı ve farklı ölçüm uzaylarında yapılan kıyaslamalar, oran hissi, toplamsal ve çarpımsal düşünme, formel ve informel oran dili, sembolik oran ve orantı gösterimleri
6. Orantısal durumları grafikler ve denklemlerle gösterelim	Birleşik birimleri bağlama, bağlı birleşik birimleri yineleme, birimleme ve biçimlendirme, ortak değişinti (kovaryasyon) ve değişmezlik (invaryasyon), doğrusal ilişkiler ve gösterimleri, orantısallık
7. Orantısallık ve doğrusallığı inceleyelim	Birleşik birimleri bağlama, bağlı birleşik birimleri yineleme, birimleme ve biçimlendirme, ortak değişinti (kovaryasyon) ve değişmezlik

	(invaryasyon), doğrusal ilişkiler ve gösterimleri, orantısallık, toplamsal ve çarpımsal düşünme
8. Ağaçlar ne kadar büyüdü?	Birleşik birimleri bağlama, bağlı birleşik birimleri yineleme, birimleme ve biçimlendirme, ortak değişinti (kovaryasyon), oran hissi, mutlak ve göreceli değişim, çarpımsal ve toplamsal düşünme
9. Hangi fotoğraf orijinaline benziyor?	Birleşik birimleri bağlama, bağlı birleşik birimleri yineleme, birimleme ve biçimlendirme, ortak değişinti (kovaryasyon) ve değişmezlik (invaryasyon), aynı ve farklı ölçüm uzaylarında yapılan kıyaslamalar, oran hissi, toplamsal ve çarpımsal düşünme, benzerlik ve çarpıtma
10. Hızları kıyaslama ve en hesaplı ürüne karar verme	Birleşik birimleri bağlama, bağlı birleşik birimleri yineleme, birimleme ve biçimlendirme, ortak değişinti (kovaryasyon) ve değişmezlik (invaryasyon), aynı ve farklı ölçüm uzaylarında yapılan kıyaslamalar, oran hissi, toplamsal ve çarpımsal düşünme
11. Portakal tatlarını kıyaslayalım	Birleşik birimleri bağlama, bağlı birleşik birimleri yineleme, birimleme ve biçimlendirme, ortak değişinti (kovaryasyon) ve değişmezlik (invaryasyon), aynı ve farklı ölçüm uzaylarında yapılan kıyaslamalar, oran hissi, toplamsal ve çarpımsal düşünme
12. Kahvelerin sertliklerini kıyaslayalım	Birimleme ve biçimlendirme, oran hissi, toplamsal ve çarpımsal düşünme, mutlak ve göreceli kıyaslama, nitel muhakeme

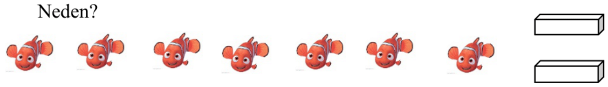
ETKİNLİK 1 - BALIKLARI BESLEYELİM

1. **BÖLÜM**



KURAL: 1 kutu yem ile 3 balık besleniyor.

1. **Yukaridaki kutucukta verilen kurala göre,** aşağıdaki durumda balıkları beslemek için yeterli yem var mıdır? Neden?



Cevap: ____

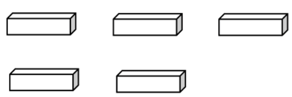
2. Aşağıdaki balıkları beslemek için yeterli yem var mıdır? Neden?

Cevap: ____

3. Aşağıdaki balıkları beslemek için kaç kutu yeme ihtiyaç vardır? Neden?

Cevap: ____

4. 5 kutu yem ile kaç balık beslenebilir? Neden?



Cevap: ____

5. 9 kutu yem ile kaç balık beslenebilir? Neden?

Cevap: ____

6. 10 kutu yem ile kaç balık beslenebilir? Neden?

Cevap: ____

7. 18 kutu yem ile kaç balık beslenebilir? Neden?

Cevap: ____

8. 18 balığı beslemek için kaç kutu yem gerekir? Neden?

Cevap: ____

9. 45 balığı beslemek için kaç kutu yem gerekir? Neden?

Cevap: ____

10. 7 kutu yem 21 balığı beslemek için yeterli midir? Neden?

Cevap: ____

11. 12 kutu yem 36 balığı beslemek için yeterli midir? Neden?

Cevap: ____

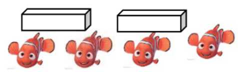
12. 72 balığı beslemek için 25 kutu yem yeterli midir? Neden?

Cevap: ____

13. 25 balığı beslemek için 8 kutu yem yeterli midir? Neden?

Cevap: ____

2. BÖLÜM

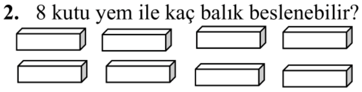


KURAL: 2 kutu yem ile 4 balık besleniyor

1. 12 balığı beslemek için kaç kutu yem gerekir? Neden?

Cevap:____

2. 8 kutu yem ile kaç balık beslenebilir? Neden?



Cevap: ____

3. 14 kutu yem ile kaç balık beslenebilir? Neden?

Cevap: ____

4. 32 balığı beslemek için kaç kutu yem gereklidir? Neden?

Cevap: ____

5. 10 balığı beslemek için kaç kutu yem gereklidir? Neden?

Cevap: ____

6. 42 balığı beslemek için kaç kutu yem gereklidir? Neden?

Cevap: ____

7. 91 kutu yem ile kaç balık beslenebilir? Neden?

Cevap: ____

8. 22 balığı beslemek için 11 kutu yem yeterli midir? Neden?

Cevap: ____

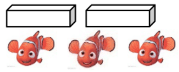
9. 25 balığı beslemek için 13 kutu yem yeterli midir? Neden?

Cevap: ____

10. 50 balığı beslemek için 24 kutu yem yeterli midir? Neden?

Cevap: ____

3. BÖLÜM



KURAL: 2 kutu yem ile 3 balık besleniyor

1. Yukarıdaki kutucukta verilen bilgiye göre aşağıdaki tabloyu doldurunuz.

Yem (Kutu)	2										
Balık	3										

2. 16 kutu yem ile kaç balık beslenebilir? Neden?

Cevap: ____

3. 45 balığı beslemek için kaç kutu yem gereklidir? Neden?

Cevap: ____

4. 28 kutu yem ile kaç balık beslenebilir? Neden?

Cevap: ____

5. 30 balığı beslemek için 20 kutu yem yeterli midir? Neden?

Cevap: ____

6. 18 balığı beslemek için 10 kutu yem yeterli midir? Neden?

Cevap: ____

7. 37 balığı beslemek için 25 kutu yem yeterli midir? Neden?

Cevap: ____

8. Yem sayısındaki değişim ile balık sayısındaki değişim arasında (yatay) nasıl bir ilişki vardır? Neden?

Cevap: ____

9. Yem sayısı ile balık sayısı arasında (dikey) nasıl bir ilişki vardır? Neden?

Cevap: ____

4. BÖLÜM

1. Her bir tabloda verilmeyen değeri bulunuz.

Yem (kutu)	2	1
Balık	6	?

Yem (Kutu)	2	7
Balık	6	?

Yem (Kutu)	2	?
Balık	6	15

Yem (Kutu)	2	?
Balık	6	1

2. Yukarıda verilen bilgilere göre;

i. 21 kutu yem ile kaç balık beslendiğini tablo kullanarak bulunuz.

ii. 63 kutu yem ile kaç balık beslendiğini tablo kullanarak bulunuz.

iii. 33 balığı beslemek için kaç kutu yem gerektiğini tablo kullanarak bulunuz.

iv. 75 balığı beslemek için kaç kutu yem gerektiğini tablo kullanarak bulunuz.

3. Her bir tabloda verilmeyen değeri bulunuz.

Yem (Kutu)	3	6
Balık	5	?

Yem (Kutu)	3	9
Balık	5	?

Yem (Kutu)	3	?
Balık	5	1

Yem (Kutu)	3	?
Balık	5	12

4. Yukarıda verilen bilgilere göre;

i. 36 kutu yem ile kaç balık beslendiğini tablo kullanarak bulunuz.

ii. 135 kutu yem ile kaç balık beslendiğini tablo kullanarak bulunuz.

iii. 20 balığı beslemek için kaç kutu yem gerektiğini tablo kullanarak bulunuz.

iv. 124 balığı beslemek için kaç kutu yem gerektiğini tablo kullanarak bulunuz.

ETKİNLİK 2 - TARİFLERLE KEK VE MEYVE SUYU YAPALIM

1) Aşağıdaki tabloda 8 kişilik bir kek yapmak için gerekli malzemeler listelenmiştir. Buna göre, tabloda boş bırakılan yerleri doldurunuz.

	4 Kişilik	**8 Kişilik**	**12 kişilik**
Yumurta		3 adet	
Yoğurt		1 su bardağı	
Sıvıyağ		1 su bardağı	
Şeker		1 su bardağı	
Un		3 su bardağı	

2) 6 tane portakal ve 4 tane elma sıkılarak özel bir meyve suyu karışımı üretiliyor. Aynı tada sahip farklı miktarlarda karışımlar yapılmak isteniyor. Aşağıda bu durumla ilgili sorulara verilen öğrenci cevaplarını inceleyerek kimin haklı olduğuna karar verelim.

- 60 tane portakal için kaç elma gerekir?

Yavuz:

2 eksik (-2)

6 portakal 4 elma

Elma sayısı portakal sayısından 2 eksik olduğu için her karışımda bu şekilde olmalıdır. Bu yüzden 60 tane portakal için 58 elma gerekir.

Yeşim: Her 6 portakal için 4 elma eklenmesi gerektiğinden 40 elma olmalıdır.

6 portakal	60 portakal
4 elma	40 elma

- 24 tane elma için kaç portakal gerekir?

Aysun:

2 fazla (+2)

4 elma 6 portakal

Portakal sayısı elma sayısında 2 fazla olduğu için her karışımda bu şekilde olmalıdır. Bu yüzden 24 tane elma için 26 portakal gerekir.

Ahmet: Her 4 elma için 6 portakal eklenmesi gerekir. Bu demektir ki karışımdaki her 2 elma için 3 portakal olmalıdır. Bu yüzden 24 elma için 36 portakal olmalıdır.

4 elma	2 elma	24 elma
6 portakal	3 portakal	36 portakal

- 80 adet meyve kullanılarak yapılan karışımda kaç elma ve kaç portakal kullanılır?

- 120 adet meyve kullanılarak yapılan karışımda kaç elma ve kaç portakal kullanılır?

ETKİNLİK 3 - ANKET SONUÇLARI NE SÖYLÜYOR?

1. Ebru okulundaki öğrencilere bir anket uyguluyor ve okula servisle gelen her 3 öğrenciye karşılık 7 öğrencinin okula yürüyerek geldiği sonucuna ulaşıyor. Buna göre, aşağıdaki soruları verilen tabloları kullanarak cevaplayınız.
   - Bu okula servisle gelen 45 öğrenci varsa yürüyerek gelen kaç kişi vardır?

     Cevap: _________________
   - Bu okula yürüyerek gelen 49 öğrenci varsa servisle gelen kaç kişi vardır?

     Cevap: _________________
   - Bu okulda 120 öğrenci varsa kaç kişi okula servisle gelmektedir?

     Cevap: _________________

2. Hasan okulundaki öğrencilere bir anket uyguluyor ve her 8 öğrenciden 5'inin en az bir kardeşi olduğu diğerlerinin ise hiç kardeşinin olmadığı sonucuna ulaşıyor. Buna, göre aşağıdaki soruları birbirinden bağımsız olarak cevaplayınız.
   - Bu okulda en az bir kardeşi olan 65 kişi varsa kardeşi olmayan kaç kişi vardır? Tablo kullanarak bulunuz.
   - Bu okulda kardeşi olmayan 90 kişi varsa okulda toplam kaç kişi vardır? Tablo kullanarak bulunuz.
   - Bu okulda toplam 168 öğrenci varsa kaç kişinin en az bir kardeşi vardır? Tablo kullanarak bulunuz.

3. Deniz okulundaki öğrencilere anket uyguluyor ve okuldaki her 9 öğrenciden 4'ünün Beşiktaş, 3'ünün Galatasaray ve 2'sinin Fenerbahçe taraftarı olduğu sonucuna ulaşıyor. Buna göre, aşağıdaki soruları verilen tabloları kullanarak cevaplayınız.
   - Bu okulda 44 kişi Beşiktaş taraftarı ise Galatasaray ve Fenerbahçe taraftarı kaç öğrenci vardır? Tablo kullanarak bulunuz.
   - Bu okulda 60 kişi Galatasaray taraftarı ise Beşiktaş ve Fenerbahçe taraftarı kaç öğrenci vardır? Tablo kullanarak bulunuz.
   - Bu okulda 70 kişi Fenerbahçe taraftarı ise Beşiktaş ve Galatasaray taraftarı kaç öğrenci vardır? Tablo kullanarak bulunuz.
   - Bu okulda toplam 180 kişi varsa Galatasaray, Beşiktaş ve Fenerbahçe taraftarı kaç öğrenci vardır? Tablo kullanarak bulunuz.

ETKİNLİK 4 - ORAN VE ORANTIYI ÖĞRENELİM

1) 7-A sınıfında 12 kız ve 15 erkek öğrenci vardır. Buna göre:

a) 7-A sınıfındaki kız öğrencilerin sayısının erkek öğrencilerin sayısına oranı:

b) 7-A sınıfındaki erkek öğrencilerin sayısının kız öğrencilerin sayısına oranı:

c) 7-A sınıfındaki kız öğrencilerin sayısının tüm öğrencilerin sayısına oranı:

d) 7-A sınıfındaki erkek öğrencilerin sayısının tüm öğrencilerin sayısına oranı:

2) 7-B sınıfındaki kız öğrencilerin sayısının erkek öğrencilerin sayısına oranının 2:3 olduğu biliniyor. Buna göre aşağıdaki soruları birbirinden bağımsız olarak cevaplayınız.

a) Bu sınıfta 16 kız öğrenci varsa kaç erkek öğrenci vardır?

Kız öğrenci sayısı		
Erkek öğrenci sayısı		

Orantı:

b) Bu sınıfta 18 erkek öğrenci varsa kaç kız öğrenci vardır?

Kız öğrenci sayısı		
Erkek öğrenci sayısı		

Orantı:

c) Aşağıdaki tablodaki kız öğrencilerin sayısının erkek öğrencilerin sayısına oranlarından 7-B sınıfına ait olabilecek olanları yuvarlak içine alınız.

$\frac{8}{12}$	$\frac{5}{15}$	$\frac{18}{27}$	$\frac{14}{15}$
$\frac{22}{33}$	$\frac{10}{20}$	$\frac{12}{18}$	$\frac{9}{16}$

d) Bu sınıfta toplam 35 öğrenci varsa kaç kız ve kaç erkek öğrenci vardır?

Kız öğrenci sayısı: Erkek öğrenci sayısı:

e) Bu sınıftaki kız öğrencilerin sayısının 10'dan fazla olduğu biliniyorsa en az kaç kız öğrenci vardır?

f) Bu sınıftaki öğrencilerin sayısının 24'ten az olduğu biliniyorsa bu sınıfta en fazla kaç öğrenci vardır?

ETKİNLİK 5 - PROBLEMLER ÇÖZELİM

1. 2 kalem 4 TL ediyorsa, 12 TL ile kaç kalem alınabilir?

2. Arda 100 m'yi 15 saniyede koşuyorsa, 500 m'yi kaç saniyede koşar?

3. Bir havuzun $\frac{1}{3}$'i 2 saatte doluyorsa tamamı kaç saatte dolar?

4. Bir araç sabit hızla 4 saatte 320 km yol alabiliyorsa 6 saatte kaç km yol alır?

5. Kadir bilgisayarda bir dakikada 30 kelime yazabiliyor. Kadir'in 390 kelimelik ödevinin yazımını bitirmesi için kaç dakika gerekir?

6. 5 ölçek kırmızı boya ile 3 ölçek beyaz boya karıştırılarak özel bir renk elde ediliyor. Buna göre aynı renkte boya elde etmek isteyen bir kişi 9 ölçek beyaz boyaya karşılık kaç ölçek kırmızı boya kullanmalıdır?

7. Gerçekte 9 km olan bir uzaklık bir haritada 5 cm olarak gösterilmiştir. Buna göre, haritada 20 cm olarak gösterilen uzaklık gerçekte ne kadardır?

8. $\frac{1}{500}$ ölçekli bir haritada iki nokta arasındaki uzaklık 5 cm olduğuna göre bu iki nokta arasındaki gerçek uzaklık kaç m'dir?

9. İki şehir arasındaki uzaklık 600 m'dir. Bu uzaklık $\frac{1}{10000}$ ölçekli bir haritada kaç cm olarak gösterilir?

10. Gökhan ve Halit'in boylarına göre kiloları aşağıdaki tabloda verilmiştir. Buna göre, Gökhan ve Halit'in boy ve kilo değerlerinin orantılı olup olmadığını bulunuz.

Kişi	Boy	Kilo
Gökhan	150 cm	50
Halit	160	60

ETKİNLİK 6 - ORANTISAL DURUMLARI GRAFİKLER VE DENKLEMLERLE GÖSTERELİM

1. Soru- 4 Farklı Versiyon- Gruplara dağıtılacak

1A. Pazarda 1 kg domates 2 TL'dir.

a. Buna göre, alınan miktarlara göre ödenmesi gereken miktarları bularak tabloyu doldurunuz.

Miktar (Kg)						
Fiyat (TL)						
İlişki						

b. 15 kg domates için kaç TL ödemek gerekir?
c. 60 TL'ye kaç kg domates alınabilir?
d. Domates miktarı (kg) ile ödenmesi gereken ücret (TL) arasındaki ilişkinin denklemini yazınız.
e. Domates miktarı (kg) ile ödenmesi gereken ücret (TL) arasındaki ilişkiyi grafikle ifade ediniz (Ek olarak verilen grafik üzerinde).

1B. Pazarda 1 kg elma 2 TL'dir.

a. Buna göre alınan miktarlara göre ödenmesi gereken miktarları bularak tabloyu doldurunuz.

Miktar (Kg)						
Fiyat (TL)						
İlişki						

b. 12 kg elma için kaç TL ödemek gerekir?
c. 50 TL'ye kaç kg elma alınabilir?
d. Tabloda verilen elma miktarı (kg) ile ödenmesi gereken ücret (TL) arasındaki ilişkinin denklemini yazınız.
e. Elma miktarı (kg) ile ödenmesi gereken ücret (TL) arasındaki ilişkiyi grafikle ifade ediniz.

1C. Pazarda 1 kg muz 5 TL'dir.

a. Buna göre alınan miktarlara göre ödenmesi gereken miktarları bularak tabloyu doldurunuz.

Miktar (Kg)						
Fiyat (TL)						
İlişki						

b. 9 kg muz için kaç TL ödemek gerekir?
c. 80 TL'ye kaç kg muz alınabilir?
d. Tabloda verilen muz miktarı (kg) ile ödenmesi gereken ücret (TL) arasındaki ilişkinin denklemini yazınız.
e. Muz miktarı (kg) ile ödenmesi gereken ücret (TL) arasındaki ilişkiyi grafikle ifade ediniz.

1D. Pazarda 1 kg patates 1 TL'dir.

a. Buna göre alınan miktarlara göre ödenmesi gereken miktarları bularak tabloyu doldurunuz.

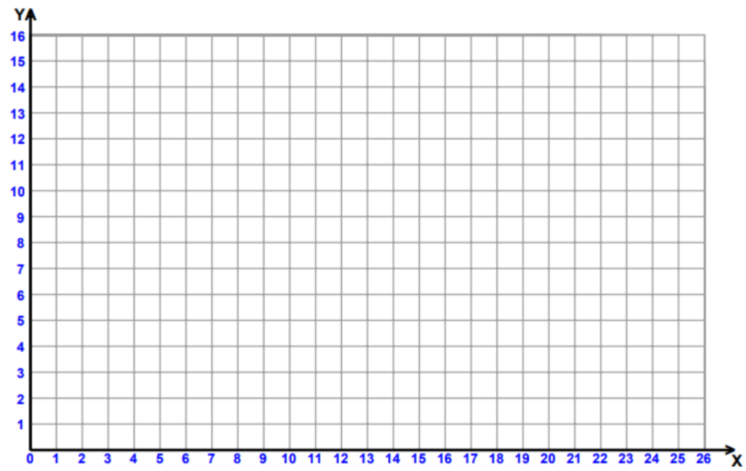
Miktar (Kg)						
Fiyat (TL)						
İlişki						

b. 24 kg patates için kaç TL ödemek gerekir?

c. 55 TL'ye kaç kg patates alınabilir?

d. Tabloda verilen patates miktarı (kg) ile ödenmesi gereken ücret (TL) arasındaki ilişkinin denklemini yazınız.

e. Patates miktarı (kg) ile ödenmesi gereken ücret (TL) arasındaki ilişkiyi grafikle ifade ediniz.



2. Tuncay ve Meryem bayram sonunda harçlıklarını kıyaslamak istemişlerdir. Tuncay'ın parasının Meryem'in parasına oranı $\frac{1}{4}$ ' tür.

a. Buna göre, Tuncay'ın ve Meryem'in paralarının alabileceği farklı değerleri bularak tabloyu doldurunuz.

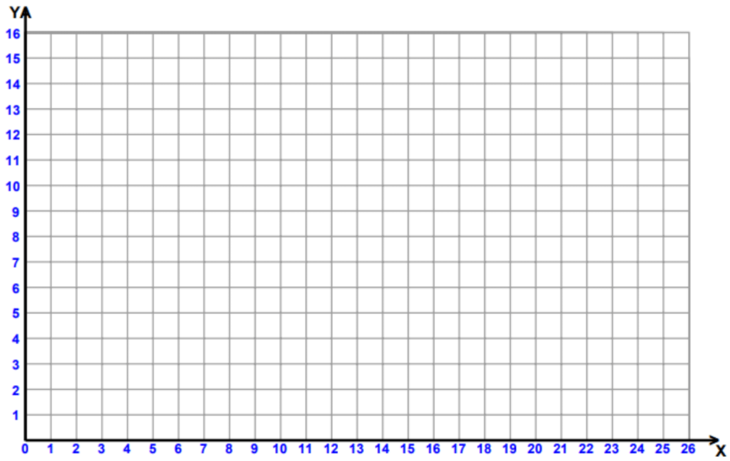
Tuncay'ın parası (TL)										
Meryem'in parası (TL)										
İlişki										

b. Tuncay'ın 50 TL parası varsa Meryem'in kaç TL parası vardır?

c. Meryem'in 64 TL parası varsa Tuncay'ın kaç TL parası vardır?

d. Tuncay'ın parası (TL) ile Meryem'in parası (TL) arasındaki ilişkinin denklemini yazınız.

e. Tuncay'ın parası (TL) ile Meryem'in parası (TL) arasındaki ilişkiyi grafikle ifade ediniz.

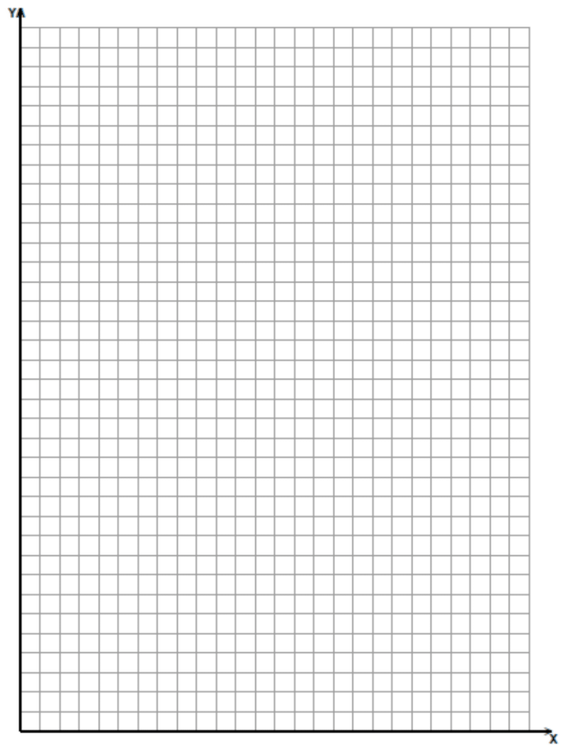


3. Ağırlık, Newton'un genel çekim yasasına göre çekim kuvvetleri sonucu oluşan bir büyüklüktür. Yeryüzünde yerin merkezinden uzaklaştıkça cisimlerin ağırlığı azalır. Bir cismin ağırlığı, bulunduğu yere göre değişir.

Dünya'daki ağırlığı 72 kg olan bir kişinin Ay'daki ağırlığı 12 kg gelmektedir. Buna göre Dünya'daki ağırlıkları verilen kişilerin Ay'daki ağırlıklarını bularak tabloyu doldurunuz.

	Ali	Burcu	Ceyda	Demet	Esra	Fırat	Gökçe	İlişki
Dünya'daki ağırlığı	24	30	36	42	48	54	60	
Ay'daki ağırlığı								

Bir kişinin Dünya'daki ağırlığı ile Ay'daki ağırlığı arasındaki ilişkiyi grafikle ifade ediniz.



4.

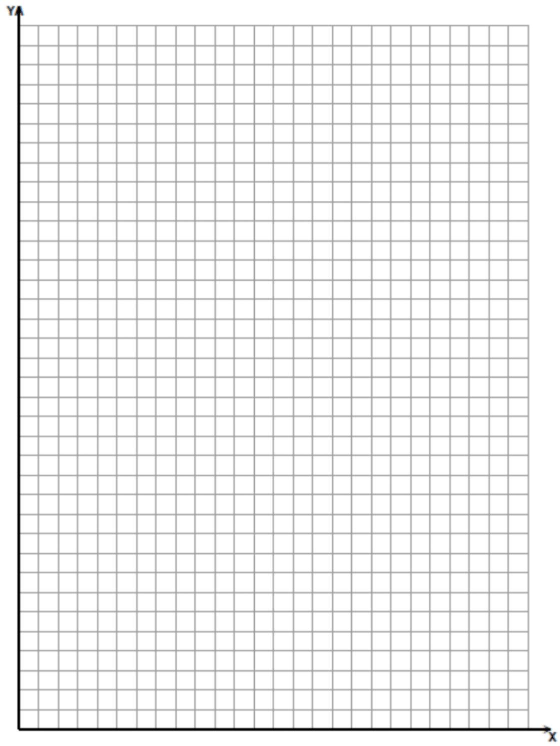
Tablo:

	1	2	3	4	5	x
	4	8	12	16	20	y
İlişki						

a) Yukarıdaki tabloda verilen değerler arasında nasıl bir ilişki vardır?

b) Tabloda verilen değerlere yönelik günlük hayatla ilgili sözel bir durum yazınız.
c) Tabloda verilen ilişkinin denklemini yazınız.

d) Tabloda verilen ilişkiyi grafikle ifade ediniz.


Y
X

ETKİNLİK 7 - ORANTISALLIK VE DOĞRUSALLIĞI İNCELEYELİM

1. Ankara'da taksilerin taksimetre açılış ücreti 3 TL'dir. Yolculuk ücreti için ise kilometre başına 2 TL ödenmesi gerekmektedir. Buna göre aşağıdaki soruları cevaplayınız.

a. Gidilen yola göre ödenmesi gereken fiyatı gösteren tabloyu doldurunuz.

Yol (km)	Başlangıç	1	2	3	4	5	6	7	8	9	10	.....	x
Ücret (TL)													

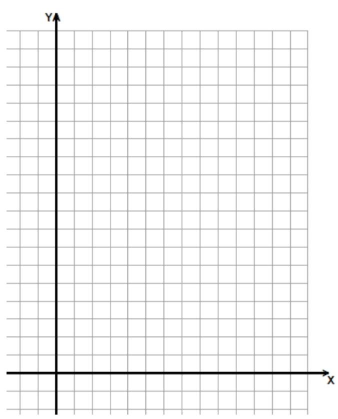
b. 12 km uzaklıktaki bir mesafe için taksiciye kaç TL ödenmesi gereklidir?

c. 40 km uzaklıktaki bir mesafe için taksiciye kaç TL ödenmesi gereklidir?

d. Bu takside 45 TL ile kaç km uzaklığa kadar gidilebilir?

e. Gidilen yola göre taksiciye ödenmesi gereken ücreti gösteren denklemi yazınız.

f. Gidilen yola göre taksiciye ödenmesi gereken fiyatı gösteren denklemin grafiğini çiziniz.



g. Gidilen yol ile taksiciye ödenmesi gereken ücret arasındaki ilişki doğrusal bir ilişki midir? Neden?

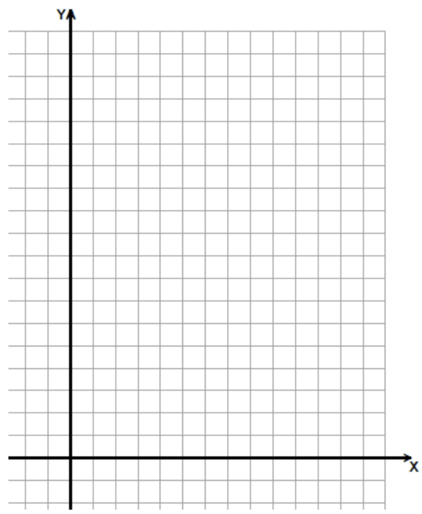
h. Gidilen yol ile taksiciye ödenmesi gereken ücret arasındaki ilişki orantısal bir ilişki midir? Neden?

2. Ilgın'ın 4 yaşına bastığı gün kardeşi Irmak doğmuştur.

a. Buna göre Ilgın ve Irmak'ın yaşları arasındaki ilişkiyi gösteren tabloyu doldurun.

Ilgın'ın yaşı													x
Irmak'ın yaşı													

b. Ilgın 18 yaşına girdiğinde Irmak kaç yaşına girer?
c. Irmak 32 yaşına girdiğinde Ilgın kaç yaşına girer?
d. Ilgın'ın yaşı ile Irmak'ın yaşı arasındaki ilişkinin denklemini yazınız.
e. Ilgın'ın yaşı ile Irmak'ın yaşı arasındaki ilişkinin denklemini belirten grafiği çiziniz.



f. Ilgın'ın yaşı ile Irmak'ın yaşı arasındaki ilişki doğrusal bir ilişki midir? Neden?

g. Ilgın'ın yaşı ile Irmak'ın yaşı arasındaki ilişki orantısal bir ilişki midir? Neden?

3. Bir ceviz fidanının uzunluğu dikildiğinde 20 cm'dir ve her ay sonunda 2 cm uzadığı bilinmektedir. Buna göre aşağıdaki soruları cevaplayınız.

a. Aylara göre ceviz fidanının uzunluğunu gösteren tabloyu doldurunuz.

Aylar	Başlangıç												x
Uzunluk (cm)													

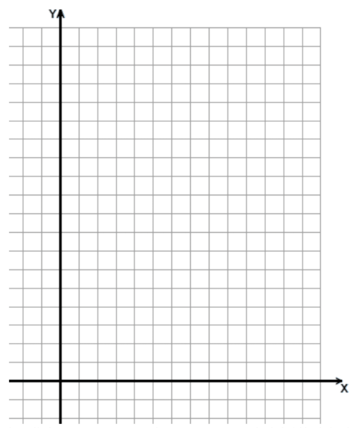
b. Bu ceviz fidanının 16 ay sonundaki uzunluğunu bulunuz.

c. Bu ceviz fidanının 30 ay sonundaki uzunluğunu bulunuz.

d. Kaç ay sonra ceviz fidanının uzunluğu 40 cm olur?

e. Aylara göre ceviz fidanının uzunluğunu belirten denklemi yazınız.

f. Aylara göre ceviz fidanının uzunluğunu belirten denklemin grafiğini çiziniz.



g. Geçen süre (ay olarak) ve ceviz fidanının uzunluğu arasındaki ilişki doğrusal bir ilişki midir? Neden?

h. Geçen süre (ay olarak) ve ceviz fidanının uzunluğu arasındaki ilişki orantısal bir ilişki midir? Neden?

4. Ahmet'in annesi Ahmet'e doğum gününde kumbara almış ve kumbaranın içerisine 10 TL koyarak Ahmet'e hediye etmiştir. Ahmet bundan sonraki her gün kumbarasına 3 TL koymuştur. Buna göre aşağıdaki soruları cevaplayınız.

a. Geçen zamana (gün sayısına) göre kumbarada biriken para miktarını gösteren tabloyu doldurunuz.

Gün sayısı	Başlangıç												x
Para (TL)													

b. 10 gün sonunda Ahmet'in kumbarasında kaç TL birikir?

c. 25 gün sonunda Ahmet'in kumbarasında kaç TL birikir?

d. Geçen zamana (gün sayısına) göre kumbarada biriken para miktarını gösteren denklemi yazınız.

e. Geçen zamana (gün sayısına) göre kumbarada biriken para miktarını gösteren denklemin grafiğini çiziniz.

f. Geçen zaman (gün sayısı) ile kumbarada biriken para miktarı arasındaki ilişki doğrusal bir ilişki midir? Neden?

g. Geçen zaman (gün sayısı) ile kumbarada biriken para miktarı arasındaki ilişki orantısal bir ilişki midir? Neden?

5. Emine ve Öznur yuvarlak bir pist etrafında eşit hızda koşuyorlar. Emine koşuya Öznur'dan önce başlamıştır. Emine bu pist etrafındaki 9. turunu tamamladığında, Öznur aynı pist etrafındaki 3. turunu tamamlamıştır. Buna göre aşağıdaki soruları cevaplayınız.

a. Öznur ve Emine'nin bu pist etrafında aynı anda tamamladıkları tur sayısını gösteren tabloyu doldurunuz.

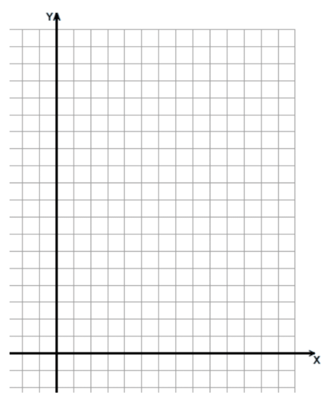
Öznur'un attığı tur sayısı													
Emine'nin attığı tur sayısı													

b. Öznur bu pist etrafındaki 15. turunu tamamladığında, Emine kaçıncı turunu tamamlamış olur?

c. Emine bu pist etrafındaki 27. turunu tamamladığında Öznur kaçıncı turunu tamamlamış olur?

d. Öznur ve Emine'nin bu pist etrafında aynı anda tamamladıkları tur sayısı arasındaki ilişkiyi gösteren denklemi yazınız.

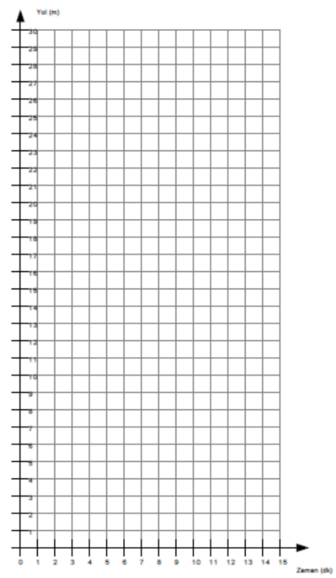
e. Öznur ve Emine'nin bu pist etrafında aynı anda tamamladıkları tur sayısı arasındaki ilişkiyi gösteren denklemin grafiğini çiziniz.



f. Öznur ve Emine'nin bu pist etrafında aynı anda tamamladıkları tur sayısı arasındaki ilişki doğrusal bir ilişki midir? Neden?

g. Geçen zaman (gün sayısı) ile kumbarada biriken para miktarı arasındaki ilişki orantısal bir ilişki midir? Neden?

6. Aşağıdaki soruları verilen grafiklere göre cevaplayınız.



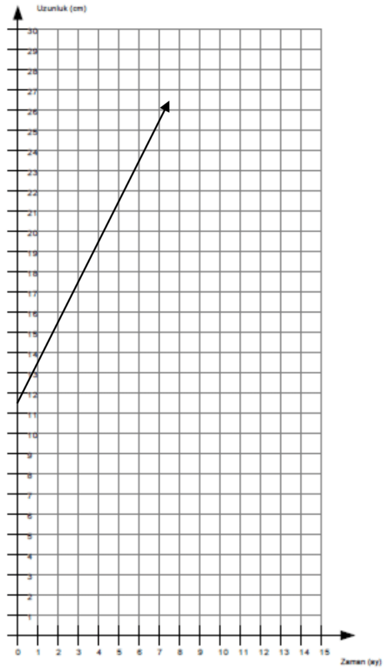
a. Yanda verilen grafikteki değerlerin tablosunu oluşturunuz.

Zaman (dk)	Yol	İlişki

b) Yukarıda verilen grafiğin denklemini yazınız.

c) Verilen grafiğe uygun bir günlük hayat durumu yazınız.

7. Aşağıdaki soruları grafiğe göre cevaplayınız.



a. Yanda verilen grafikteki değerlerin tablosunu oluşturunuz.

Zaman (ay)	Uzunluk (cm)	İlişki

b. Yukarıda verilen grafiğin denklemini yazınız.

c. Verilen grafiğe uygun bir günlük hayat durumu yazınız.

ETKİNLİK 8 - AĞAÇLAR NE KADAR BÜYÜDÜ?

1) Aslı ve kardeşi Arda dedelerini ziyarete gittiklerinde bahçeye fidan diktiler. Aslı'nın fidanı dikildiğinde 16 cm, Arda'nın fidanı dikildiğinde 20 cm'ydi. Aslı ve Arda bir yıl sonra dedelerini ziyarete gittiklerinde Aslı fidanının boyunu 24 cm, Arda fidanının boyunu 28 cm olarak ölçüyor. Buna göre aşağıdaki öğrenci cevaplarını tartışarak hangilerinin doğru olduğuna karar veriniz.

Ömer: Aslı ve Arda'nın fidanları 1 yılda aynı miktarda uzamıştır.

Elif: Aslı ve Arda'nın fidanlarının büyüme hızları aynıdır.

Özlem: Aslı'nın fidanı daha hızlı büyümektedir.

Erdem: Aslı'nın fidanı ilk boyunun $\frac{1}{4}$'i kadar, Arda'nın fidanı ise ilk boyunun $\frac{1}{5}$'i kadar büyümüştür. Bu sebepten Aslı'nın fidanının büyüme hızı daha fazladır.

2) Aslı ve Arda'nın fidanları aynı hızda büyümeye devam ettiklerine göre:
   a. Aslı'nın fidanının uzunluğu 1 yıl sonra kaç cm olur?

   b. Arda'nın fidanının uzunluğu 1 yıl sonra kaç cm olur?

ETKİNLİK 9 - HANGİ FOTOĞRAF ORİJİNALİNE BENZİYOR?

1. Annesi Selin'in fotoğrafını farklı boyutlarda çoğaltmak istiyor. Fotoğrafın orijinal boyutları ve çoğaltılan kopyaların boyutları fotoğrafların yanlarında yazılmıştır. Buna göre:
   - Çoğaltılan fotoğraflardan hangileri orijinal fotoğraf ile benzerdir? Açıklayınız.
   - Çoğaltılan fotoğraflardan hangileri orijinal fotoğraf ile benzer değildir? Açıklayınız.
   - Orijinal fotoğrafın boyutları ile orijinal fotoğrafa benzeyen kopyaların boyutları arasında nasıl bir ilişki vardır?

Ekler:

Orijinal fotoğraf

6 cm

4 cm

Orijinal fotoğrafa benzerliği incelenecek fotoğraflar- 1

9 cm

4 cm

Orijinal fotoğrafa benzerliği incelenecek fotoğraflar- 2

6 cm

6 cm

Orijinal fotoğrafa benzerliği incelenecek fotoğraflar- 3

12 cm

8 cm

Orijinal fotoğrafa benzerliği incelenecek fotoğraflar- 4

15 cm

13 cm

2. Aşağıda boyutları verilen fotoğraflar benzer ise verilmeyen kenar uzunluğunu bulunuz.

8 cm

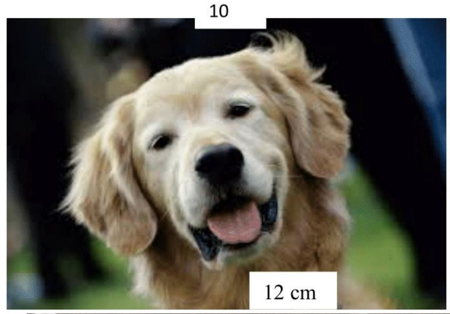
6 cm

?

3 cm

5 cm

4 cm



x

8 cm

4 cm

x

3. Verilen fotoğraf çiftlerinin benzer olup olmadıklarını bulunuz. Sebeplerinizi açıklayınız.

8 cm

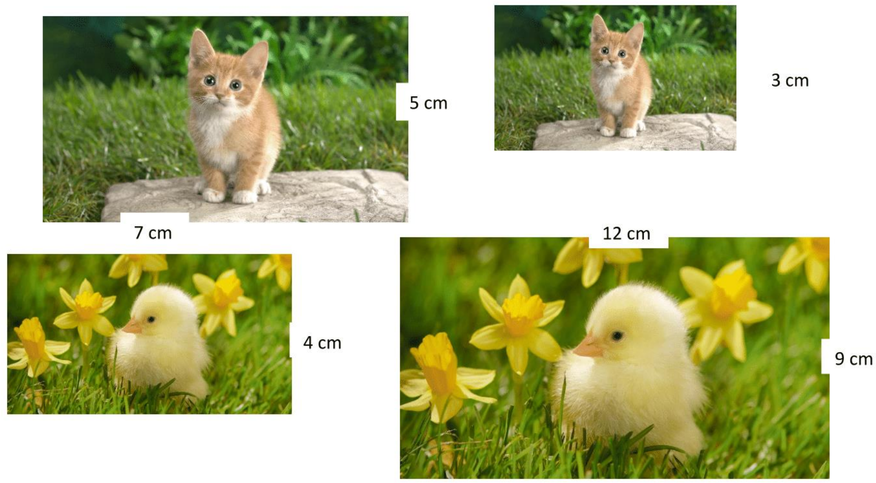
4 cm

6 cm

3 cm

9 cm

7 cm


5 cm
7 cm
3 cm
12 cm
4 cm
9 cm

ETKİNLİK 10 - HIZLARI KIYASLAMA VE EN HESAPLI ÜRÜNE KARAR VERME

1. Bir araç Ankara'dan İstanbul'a seyahat etmek için bazı şehir ve ilçelerden geçmelidir ve bazı kesimlerde yollar bozuk ve bazı kesimlerde trafik yoğunluğu olduğu için araçlar yavaş gitmek durumunda kalmaktadır. Ankara'dan İstanbul'a seyahat etmekte olan bir araç yol üzerinde 3 noktada durmuştur. Bu süreçte zaman ve alınan yol ile ilgili bilgiler aşağıdaki gibi kaydedilmiştir.

	Uzaklık	**Süre**
Ankara-Bolu arası	180 km	2 saat
Bolu-Adapazarı arası	70 km	1 saat
Adapazarı-İstanbul arası	150 km	3 saat

Buna göre aracın ortalama hızı hangi şehirler arasında en azdır? Hangi şehirler arasında en fazladır?

2. Aşağıda 4 arkadaşın yürüdükleri mesafeler ve zamanları verilmiştir.

	Yürünen mesafe	**Süre**
Ahmet	10 km	2 saat
Beren	14 km	4 saat
Ceylan	15 km	5 saat
Derya	24 km	6 saat

Buna göre bu 4 arkadaşı en yavaştan hızlıya doğru sıralayınız.

3. En hesaplı fiyatla alışveriş yapmak isteyen Mehmet Bey 3 farklı markete giderek almak istediği ürünlerin fiyatlarını not alıyor. Aşağıdaki tabloda bazı ürünlerin bu marketlerdeki fiyatları verilmiştir.

	Sevgi Market	**Dost Market**	**Çiçek Market**
Pirinç	3 kg 18 TL	2 kg 14 TL	5 kg 25 TL
Yoğurt	1 kg 4 TL	2 kg 6 TL	4 kg 8 TL
Şeker	5 kg 25 TL	8 kg 36 TL	10 kg 60 TL
Ayçiçek Yağı	3 lt 15 TL	5 lt 30 TL	10 lt 68 TL

a. Pirinç için en kârlı (hesaplı) alışveriş hangi marketten yapılır? Açıklayınız.

b. Yoğurt için en kârlı (hesaplı) alışveriş hangi marketten yapılır? Açıklayınız.

c. Şeker için en kârlı (hesaplı) alışveriş hangi marketten yapılır? Açıklayınız.
d. Ayçiçek yağı için en kârlı (hesaplı) alışveriş hangi marketten yapılır? Açıklayınız.

ETKİNLİK 11 - PORTAKAL TATLARINI KIYASLAYALIM

Aşağıda içerisinde farklı miktarda portakal suyu ve su bulunan sürahiler verilmiştir.

	Sürahi A	Sürahi B	Sürahi C	Sürahi D	Sürahi E
					
Portakal suyu miktarı	3 bardak	3 bardak	4 bardak	5 bardak	2 bardak
Su miktarı	2 bardak	3 bardak	3 bardak	5 bardak	3 bardak

I. BÖLÜM

Buna göre aşağıdaki tablodaki sürahi çiftlerinden hangisindeki karışımın **daha güçlü portakal tadına** sahip olacağına karar veriniz ve cevabınızın gerekçesini yazınız.

Sürahi A ve Sürahi B: Gerekçe:	Sürahi B ve Sürahi D: Gerekçe:
Sürahi A ve Sürahi C: Gerekçe:	Sürahi B ve Sürahi E: Gerekçe:
Sürahi A ve Sürahi D: Gerekçe:	Sürahi C ve Sürahi D: Gerekçe:
Sürahi A ve Sürahi E: Gerekçe:	Sürahi C ve Sürahi E: Gerekçe:
Sürahi B ve Sürahi C: Gerekçe:	Sürahi D ve Sürahi E: Gerekçe:

II. BÖLÜM

İlk bölümde tabloda verilen A, B, C, D ve E sürahilerinin her birine ***1 bardak portakal suyu ve 1 bardak su eklenirse*** sürahilerdeki karışımlardaki portakal tadı ilk duruma göre nasıl olur? ***Aynı, daha fazla, daha az*** portakal tadı cevaplarından birini yazarak cevabınızın gerekçesini yazınız.

A sürahisindeki yeni karışımın tadı ilk durumu göre ....portakal tadına sahip olur.
Gerekçe:
B sürahisindeki yeni karışımın tadı ilk durumu göre ....portakal tadına sahip olur.
Gerekçe:
C sürahisindeki yeni karışımın tadı ilk durumu göre ....portakal tadına sahip olur.
Gerekçe:

D sürahisindeki yeni karışımın tadı ilk durumu göre ....portakal tadına sahip olur.
Gerekçe:
E sürahisindeki yeni karışımın tadı ilk durumu göre ....portakal tadına sahip olur.
Gerekçe:

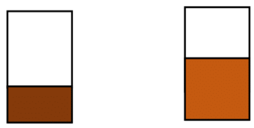
III. BÖLÜM

I. ve II. bölümdeki sorulara verdiğiniz cevaplar doğrultusunda bir karışımın tadının nelere bağlı olduğunu söyleyebilir misiniz?

ETKİNLİK 12 - KAHVELERİN SERTLİKLERİNİ KIYASLAYALIM

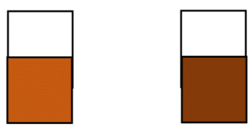
1. Aşağıdaki kaplarda kahve ve süt tozu ile yapılan karışımlar verilmiştir.

a) A kabındaki sütlü kahve karışımının tadı B kabındakine göre daha serttir. A kabına 1 kaşık kahve ve B kabına 1 kaşık süt tozu eklendiğinde hangi kaptaki karışımın tadı daha sert olur? Açıklayınız.



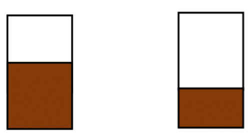
A Kabı B Kabı

b) A kabındaki sütlü kahve karışımının tadı B kabındakine göre daha hafiftir. Her iki kaba da 1 kaşık kahve eklendiğinde hangi kaptaki karışımın tadı daha sert olur? Açıklayınız.



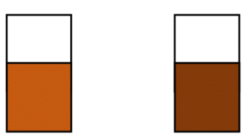
A Kabı B Kabı

c) A kabındaki ve B kabındaki sütlü kahve karışımlarının tadı aynıdır. Her iki kaba da 1 kaşık kahve eklendiğinde hangi kaptaki karışımın tadı daha sert olur? Açıklayınız.



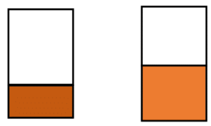
A Kabı B Kabı

d) B kabındaki sütlü kahve karışımının tadı A kabındakine göre daha serttir. A kabına 1 kaşık kahve ve B kabına 1 kaşık süt tozu eklendiğinde hangi kaptaki karışımın tadı daha sert olur? Açıklayınız.



A Kabı B Kabı

e) A kabındaki sütlü kahve karışımının tadı B kabındakine göre daha serttir. A kabına 1 kaşık süt tozu ve B kabına 1 kaşık kahve eklendiğinde hangi kaptaki karışımın tadı daha sert olur? Açıklayınız.



A Kabı B Kabı

2. a ve/veya b değerleri değiştiğinde $\frac{a}{b}$ şeklindeki bir oranın değerindeki değişimin nasıl olacağını bulalım. Aşağıdaki tabloyu cevabınız oranın değeri azalır ise , oranın değeri artar ise , oranın değeri değişmez ise ve oranın değerindeki değişim bilinemez ise **?** sembollerini kullanarak doldurunuz.

$\frac{a}{b}$ oranı için	b değişmez	b artar	b azalır
a değişmez			
a artar			
a azalır			

D. INFORMED CONSENT FORM/GÖNÜLLÜ KATILIM FORMU

ARAŞTIRMAYA GÖNÜLLÜ KATILIM FORMU

Bu çalışma, Orta Doğu Teknik Üniversitesi Eğitim Fakültesi İlköğretim Bölümü araştırma görevlisi Rukiye AYAN ve Orta Doğu Teknik Üniversitesi Eğitim Fakültesi İlköğretim Bölümü öğretim üyesi Doç. Dr. Mine IŞIKSAL BOSTAN tarafından doktora tezi kapsamında yapılan bir çalışmadır. Bu form sizi araştırma koşulları hakkında bilgilendirmek için hazırlanmıştır.

Bu çalışmanın amacı öğrenme rotaları yaklaşımına dayanan 7. sınıf oran ve orantı konusu ile ilgili bir öğretim modülünün geliştirilmesi, bu modülün uygulanması, bu süreçte öğrencilerin ve öğretmenlerin deneyimlerinin araştırılması ve sonuç olarak hazırlanan öğretim modülünün düzenlenerek son haline getirilmesidir.

Çalışma kapsamında siz matematik öğretmenlerimizden Ortaokul Matematik Öğretim Programı'nda öngörülen kazanımlar ve ders saatleri doğrultusunda 24 saatlik bir öğretim sürecini kapsayacak şekilde oran orantı konusu ile ilgili öğretim materyallerinin geliştirilmesine katkıda bulunmanız ve işbirlikli olarak geliştirilen bu materyalleri 7. Sınıf öğrencileri ile gerçekleştirdiğiniz derslerde kullanmanız istenecektir. Bu amaçla sizden ses kaydı alınmak üzere birebir görüşmeler ve video kaydı alınmak üzere derslerinizin gözlemlenmesi talep edilecektir.

Bu çalışmaya katılmak tamamen gönüllülük esasına dayalıdır. Herhangi bir yaptırıma veya cezaya maruz kalmadan çalışmaya katılmayı reddedebilir veya istediğiniz aşamada çalışmayı bırakabilirsiniz. Araştırma esnasında cevap vermek istemediğiniz sorular olursa cevap vermeme hakkına sahipsiniz.

Araştırmaya katılanlardan toplanan veriler tamamen gizli tutulacak, veriler ve kimlik bilgileri herhangi bir şekilde eşleştirilmeyecektir. Katılımcıların isimleri bağımsız bir listede toplanacaktır. Ayrıca toplanan verilere sadece araştırmacılar ulaşabilecektir. Bu araştırmanın sonuçları bilimsel ve profesyonel yayınlarda veya eğitim amaçlı kullanılabilir, fakat katılımcıların kimliği gizli tutulacaktır. Çalışma fiziksel veya ruhsal sağlığınıza zarar verecek hiçbir uygulama içermemektedir.

Bu çalışmanın sonucunda uluslararası ve ulusal alanda 7. sınıf oran ve orantı konusunun öğretiminde kullanılabilecek öğrenme rotaları yaklaşımına dayanan bir öğretim modülünü geliştirilecek olmasından dolayı ulaşabildiğimiz katılımcı sayısı bizim için büyük önem taşımaktadır.

Çalışma hakkında daha fazla bilgi almak için aşağıda bilgileri verilen kişilerle iletişime geçebilirsiniz.

Arş. Gör. Rukiye AYAN	Doç. Dr. Mine IŞIKSAL BOSTAN
ODTÜ Eğitim Fak. İlköğretim Böl.	ODTÜ Eğitim Fak. İlköğretim Böl.
Ofis Tel: 0 312 210 75 07	Ofis Tel : 0 312 210 64 12
e-mail: rayan@metu.edu.tr	e-mail: misiksal@metu.edu.tr

Yukarıdaki bilgileri okudum ve bu çalışmaya tamamen gönüllü olarak katılıyorum.
(Formu doldurup imzaladıktan sonra uygulayıcıya geri veriniz).

İsim Soyad Tarih İmza

E. PARENT PERMISSION FORM/VELİ ONAY FORMU

ORTA DOĞU TEKNİK ÜNİVERSİTESİ
MIDDLE EAST TECHNICAL UNIVERSITY
06800 ANKARA-TURKEY

1956

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Department of Elementary Education

Tel: 90 (312) 210 40 54
Faks:90 (312) 210 79 84

Veli Onay Mektubu

Sayın Veliler, Sevgili Anne-Babalar,

Bu çalışmanın amacı öğrenme rotaları yaklaşımına dayanan 7. sınıf oran ve orantı konusu ile ilgili bir öğretim modülünün geliştirilmesi, bu modülün uygulanması, bu süreçte öğrencilerin ve öğretmenlerin deneyimlerinin araştırılması ve sonuç olarak hazırlanan öğretim modülünün düzenlenerek son haline getirilmesidir. Çalışma kapsamında Ortaokul Matematik Öğretim Programı'nda öngörülen kazanımlar ve ders saatleri doğrultusunda 24 saatlik bir öğretim sürecini kapsayacak şekilde oran orantı konusu ile öğretim materyalleri öğrencinizin öğretmeni ile birlikte geliştirilecek ve bu materyaller konunun öğretiminde öğretmen tarafından kullanılacaktır. Öğrencilerin öğrenmeleri ve öğretmenlerin deneyimlerinin kayıt altına alınması için gerçekleştirilen öğretim süreci boyunca sınıf ortamı araştırmacı tarafından video kamera ile kayıt altına alınacaktır. Bu esnada öğrencilerin yüzleri birebir olarak görüntülenmeyecektir. Bu mektubun yollanış amacı çocuğunuzun bu çalışmaya katılmasını onaylayıp onaylamadığınızı belirtmenizdir.

Bu çalışma, Orta Doğu Teknik Üniversitesi Eğitim Fakültesi İlköğretim Bölümü araştırma görevlisi Rukiye AYAN ve Orta Doğu Teknik Üniversitesi Eğitim Fakültesi İlköğretim Bölümü öğretim üyesi Doç. Dr. Mine IŞIKSAL BOSTAN tarafından yürütülen doktora tezi kapsamında yapılan bir çalışmadır. Çalışma kapsamında çocuklarınızın sınıfında öğretim gerçekleştirilecek ve zaman zaman konu ile ilgili geliştirilen etkinlikler ve başarı testleri uygulanacaktır. Sorular kişisel rahatsızlık verecek herhangi bir ayrıntı içermeyecektir. Ancak, katılım sırasında, katılımcılar sorulardan ya da herhangi başka bir nedenden ötürü rahatsız hissederlerse çalışmayı yarıda bırakıp çıkmakta serbest olacaklardır. Yapılan gözlemlere ve testten alınan sonuçlara göre hazırlanan etkinliklerin öğrencilerin öğrenmelerine katkı sağlayıp sağlamadığı araştırılacaktır. Çalışmaya katılım tamamıyla gönüllülük temelindedir ve çalışma hiçbir aşamasında öğrencilerden kimlik belirleyici hiçbir bilgi istenmemektedir. Öğrencilerin cevapları gizli tutulacak ve sadece bilimsel amaçlar için araştırmacılar tarafından değerlendirilecektir; elde edilecek bilgiler bilimsel yayımlarda kullanılacaktır. Bu çalışmanın sonucunda uluslararası ve ulusal alanda 7. sınıf oran ve orantı konusunun öğretiminde kullanılabilecek öğrenme rotaları yaklaşımına dayanan bir öğretim modülünü geliştirilecek olmasından dolayı ulaşabildiğimiz öğrenci sayısı bizim için büyük önem taşımaktadır.

Çocuğunuzun bu çalışmaya katılmasını istiyorsanız, lütfen aşağıdaki formu doldurunuz. Çalışma hakkında daha fazla bilgi almak için Orta Doğu Teknik Üniversitesi İlköğretim Bölümü Araştırma Görevlisi Rukiye AYAN veya Doç. Dr. Mine IŞIKSAL BOSTAN ile aşağıdaki e-posta adresini kullanarak iletişime geçebilirsiniz.

Saygılarımızla,

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Lütfen bu araştırmaya katılmak konusundaki tercihinizi aşağıdaki seçeneklerden size en uygun gelenin altına imzanızı atarak belirtiniz ve bu formu çocuğunuzla okula geri gönderiniz.

A) Bu araştırmaya tamamen gönüllü olarak katılıyorum ve çocuğum ....’nın da katılımcı olmasına izin veriyorum. Çalışmayı istediğim zaman yarıda kesip bırakabileceğimi biliyorum ve verdiğim bilgilerin bilimsel amaçlı olarak kullanılmasını kabul ediyorum.

Baba Adı-Soyadı.................................. Anne Adı-Soyadı......................................

İmza .... İmza ....

B) Bu çalışmaya katılmayı kabul etmiyorum ve çocuğumun ....’nın da katılımcı olmasına izin vermiyorum.

Baba Adı-Soyadı.................................. Anne Adı-Soyadı......................................

İmza .... İmza ....

F. TURKISH SUMMARY / TÜRKÇE ÖZET

BİR YEDİNCİ SINIFTA MATEMATİKSEL UYGULAMALARIN GELİŞİMİ: ÖĞRENCİLERİN ORANTISAL AKIL YÜRÜTMELERİNİN GELİŞİMİNİN İNCELENMESİ

1. Giriş

Orantısal akıl yürütme en genel anlamıyla ortak değişinti (kovaryasyon) ve çarpımsal kıyaslamaların anlamlandırılması ile ilgili olan bir matematiksel akıl yürütme biçimidir (Cramer ve Post, 1993; English, 2004; Lesh, Post ve Behr, 1988). Özelde ise, orantısal ilişkileri belirleme, ifade etme, inceleme, açıklama ve bu ilişkilerle ilgili iddialar ortaya koymayı içerir (Lamon, 2007). Orantısal akıl yürütme birçok matematik konusunun temelinde yatmaktadır. Örneğin, rasyonel sayılar (Lamon, 2012); kesirler, yüzde, benzerlik, ölçekler, trigonometri (Beswick, 2011); cebir, geometri, problem çözme (Empson, 1999; Fuson ve Abrahamson, 2005); fonksiyonlar, grafik çizimi, denklemler, ölçme (Karplus, Pulos ve Stage, 1983); olasılık ve istatistik (Greenes ve Fendell, 2000) konuları için orantısal akıl yürütme çok önemlidir. Matematik dersine ek olarak, orantısal akıl yürütme fen dersi ve günlük hayattaki durumları anlamak için de büyük önem taşır (Cramer ve Post, 1993; Spinillo ve Bryant, 1999).

Dolayısıyla, orantısal akıl yürütme geniş kapsamlı, birleştirici ve diğer kavramların gelişiminde önemli rol oynayan anahtar bir beceridir. (Amerikan) Ulusal Matematik Öğretmenleri Konseyi ([NCTM], 1989) orantısal akıl yürütmenin 5-8. sınıf aralığında geliştiğini öne sürmüş ve bu sınıf düzeylerinde orantısal akıl yürütmenin gelişimine önem verilmesi gerektiğine vurgu yapmıştır. Benzer şekilde, ülkemizde Millî Eğitim Bakanlığı (MEB, 2013, 2018) Ortaokul Matematik Dersi Öğretim Programı'nda 6. ve 7. sınıfta oran ve orantı konularına ayrılan sürenin çokluğu dikkat çekmektedir.

Bu kadar önemli ve geniş kapsamlı bir kavram olmasına rağmen, birçok çalışmada öğrencilerin orantısal durumlar içeren problemlere yanıt vermede ve oran ve orantı konusunu anlamlandırmada zorlandıkları görülmüştür (Atabaş ve Öner, 2017; Ben-Chaim vd., 1998; Duatepe, Akkuş-Çıkla ve Kayhan, 2005; Fernández, Llinares, van Dooren, De Bock ve Verschaffel, 2012; Harel, Behr, Lesh ve Post, 1994; Hart, 1981, 1984, 1988; Inhelder ve Piaget, 1958; Kahraman, Kul ve İskenderoglu, 2019; Kaplan, İşleyen ve Öztürk, 2011; Kaput ve West, 1994; Karplus vd., 1983; Kayhan, Duatepe ve Akkuş-Çıkla, 2004; Mersin, 2018; Misailidou ve Williams, 2003; Noelting, 1980a, 1980b; Özgün-Koca ve Altay, 2009; Piaget ve Beth, 1966; Piaget ve Inhelder, 1975; Resnick ve Singer, 1993; Steinthorsdottir ve Sriraman, 2009; Tourniaire ve Pulos, 1985, Tourniaire, 1986; van Dooren vd., 2010).

Öğrencilere benzer şekilde, birçok çalışmada, öğretmenlerin de orantısal akıl yürütmenin temel bileşenlerini bilmede ve öğrencilerine etkili ve zengin öğrenme fırsatları sunmada yetersiz oldukları belirtilmiştir (Canada, Gilbert ve Adolphson, 2008; Harel ve Behr, 1995; Hilton ve Hilton, 2019; Hines ve McMahon, 2005; Kastberg, D'Ambrosio ve Lynch-Davis, 2012; Nagar, Weiland, Orrill ve Burke, 2015; Sowder, Armstrong, Lamon, Simon, Sowder ve Thompson, 1998; Simon ve Blume, 1994a, 1994b; Sowder ve Philipp, 1995; Thompson ve Thompson, 1994, 1996; Weiland, Orrill, Nagar, Brown ve Burke, 2020). Birçok çalışmada, öğretmenlerin orantısal akıl yürütme problemleri için çoğunlukla işlemsel stratejiler kullandıkları (Fisher, 1988; Lobato, Orrill, Druken ve Jacobson, 2011; Orrill ve Burke, 2013) ve bilgilerinin yüzeysel ve sınırlı olduğu (Hilton ve Hilton, 2019; Nagar vd., 2015) sonucuna ulaşılmıştır.

Buradan anlaşılacağı üzere, öğrencilerin yaşadıkları zorlukların kaynakları sınıf ortamında gerçekleşen öğretimden kaynaklandığı söylenebilir (Hilton, Hilton, Dole ve Goos, 2016). Birçok çalışmada, oran ve orantı konusunun öğretiminin içler-dışlar çarpımı algoritmasına vurgu yapılarak verildiği ve diğer konulardan bağımsız bir konu olarak öğretildiği sonucuna varılmıştır (Karplus vd., 1983; Lamon, 1995; Lesh vd., 1988). Ancak, birçok çalışma öğrencilerin bu algoritmayı anlamlandırmadığı (Lamon, 1995; Post, Behr ve Lesh, 1988) ve orantısal akıl yürütmeyi geliştirmediği; aksine, sınırlandırdığına (Lesh vd., 1988) dikkate çekmiştir. Sonuç olarak, okullarda gerçekleşen oran ve orantı öğretiminin orantısal akıl yürütmenin temel bileşenlerini geliştirmede yeterli olmadığı; oran ve orantı öğretiminin kalitesinin artırılması gerektiği sonucuna varılabilir.

Çok sayıda önemli çalışma ve program dokümanında oran ve orantı öğretiminin öğrencilerin var olan sezgisel ve informel bilgileri üzerine kurulması gerektiğine dikkat çekilmiştir (Lamon, 1995; NCTM, 2000; Resnick ve Singer, 1993). Diğer yandan, öğrencilerin orantısal akıl yürütmelerinin gelişiminin ortaya konulduğu çalışmalar da öğretim tasarlanırken göz önüne alınmalıdır. Piaget ve arkadaşları, birçok çalışmada çocuklarda 11-12 yaşına kadar orantısal akıl yürütmenin gelişmediğini iddia etmiştir (Inhelder ve Piaget, 1958; Piaget, 1968; Piaget ve Beth, 1966; Piaget ve Inhelder, 1975). Dolayısıyla, alan yazındaki çalışmalar orantısal akıl yürütmenin gelişimi ile ilgili karşıt görüşler öne sürmektedir.

Bu noktada, Simon ve Tzur'un (2004) önerdiği gibi, gelişim süreci belirli olmayan ve öğrencilerin zorlandıkları kavramların gelişiminin desteklenmesi için yeni yollar bulunmalıdır. Bu bağlamda, varsayıma dayalı öğrenme rotalarının öğretimin iyileştirilmesi için kullanılması önerilmektedir. Varsayıma dayalı öğrenme rotaları, Simon (1995) tarafından ortaya atılmış ve "öğrenmenin hangi rotada gerçekleşeceğine yönelik öngörüler" olarak tanımlanmıştır (s. 135). Varsayıma dayalı öğrenme rotaları, öğrenme amaçlarını, öğrenme etkinliklerini ve öğrencilerin bu etkinlikler süresince deneyimleyecekleri düşünme biçimlerini ve öğrenmeyi içerir (Clements ve Sarama, 2004; Simon, 1995). İleriki yıllarda, birçok araştırmacı öğrenme etkinliklerinin varsayıma dayalı öğrenme rotalarının önemli bir bölümünü oluşturduğunu ve bu etkinliklerin öğrencilerin öngörülen rota boyunca ilerlemelerini sağlayacak şekilde sıralanması gerektiğini öne sürmüş ve bu sıralı etkinlikleri etkinlik dizisi olarak tanımlamıştır (Cobb, 1999; Clements ve Sarama, 2004). Bu çalışmada, Stephan ve arkadaşları (2015) tarafından oran ve orantı öğretimi için Gerçekçi Matematik Eğitimi'ne dayalı olarak geliştirilen etkinlik dizisi ve ilgili varsayıma dayalı öğrenme rotası geliştirilmiş, test edilmiş ve düzenlenmiştir. Etkinlik dizisinin uygulanmasında Gerçekçi Matematik Öğretimi Teorisi esas alınmıştır. Sınıf içi uygulamanın analizi Toulmin Argümantasyon modelinin Stephan ve Rasmussen (2002) tarafından uyarlanan sınıf içi matematiksel uygulamalar analizi ile analiz edilmiştir. Bu çalışmanın amaçları ve bu amaçlara yönelik olarak oluşturulan araştırma soruları bir sonraki bölümde sunulmuştur.

1.1. Çalışmanın Amaçları ve Araştırma Soruları

Bu çalışmanın birinci amacı, yedinci sınıfta orantısal akıl yürütmenin öğretilmesi için bir varsayıma dayalı öğrenme rotası ve ilgili etkinlik dizisinin geliştirilmesi, test edilmesi ve düzenlenmesidir. Çalışmanın ikinci amacı, öğrencilerin informel ve formel araçlarla ortaklaşa akıl yürütmelerinin ve bu akıl yürütmenin Gerçekçi Matematik Eğitimi perspektifi doğrultusunda formel araçlarla akıl yürütmeye doğru gelişiminin açıklanmasıdır. Üçüncü amaç ise, öğrencilerin orantısal akıl yürütmeye yönelik fikir ve kavramların ortaklaşa gelişiminin ortaya konulmasıdır (sınıf içi matematiksel uygulamalar analizi). Orantısal akıl yürütme, ters orantı ile ilgili durumların da anlamlandırılmasını içerse de bu çalışmanın kapsamı doğru orantısal ilişkilerle sınırlıdır.

Daha detaylı olarak ele almak gerekirse, çalışmanın ilk iki amacı orantısal akıl yürütmenin öğretimi için bir etkinlik dizisi, varsayıma dayalı öğrenme rotası ve bu etkinlik dizisi ile öğrenme rotasının nasıl etkili olabileceğini açıklayan ilgili yerel öğretimsel teorinin ortaya konulması ile ilgilidir. Bu bağlamda, bu çalışma öğrenme rotaları ve Gerçekçi Matematik Öğretimi perspektiflerinin kritik bileşenlerini bir araya getirerek, oran ve orantı öğretiminin öğrencilerin orantısal akıl yürütme ile ilgili informel ve sezgisel bilgilerinin üzerine inşa edilmesi sürecini ele almaktadır. Bu doğrultuda, orantısal akıl yürütme öğretiminin başlangıç noktalarının ne olması ve informel araç kullanımının nasıl olması gerektiği ile öğrencilerin bu informel bilgilerini ve informel araçları kullanarak oran ve orantıya yönelik formel bilgiye nasıl ulaştıklarının (matematikleştirme) incelenmesi ilk iki amaç kapsamında ele alınmıştır. Üçüncü amaç ise, öngörülen öğrenme rotasının sınıf içinde nasıl gerçekleştiğinin matematiksel uygulamalar analizi ile ortaya konmasına dayalıdır. Bu doğrultuda, etkinlik dizisi ve öğrenme rotasının öğrencilerin ortaklaşa matematikleştirme sürecini desteklemedeki potansiyel güçlükleri ve sınırlılıkları üzerinde durulmaktadır. Çalışmanın araştırma soruları aşağıdaki gibidir:

1. Orantısal akıl yürütmenin öğretimi için ideal bir varsayıma dayalı öğrenme rotası ve ilgili etkinlik dizisi nasıl olmalıdır?

    - Gerçekçi Matematik Eğitimi'ne dayalı olarak orantısal akıl yürütme öğretiminin başlangıç noktaları nedir?

    - Öğrenciler informel bilgi ve araçları kullanarak oran ve orantıya yönelik formel bilgiye nasıl ulaşırlar (matematikleştirme)?

    - Geliştirilen etkinlik dizisi ve varsayıma dayalı öğrenme rotası bu matematikleştirme sürecini nasıl desteklemektedir?

    - Geliştirilen etkinlik dizisi ve öğrenme rotasının öğrencilerin ortaklaşa matematikleştirme sürecini desteklemedeki potansiyel güçlükleri ve sınırlılıkları nelerdir?

    - Geliştirilen varsayıma dayalı öğrenme rotası ve etkinlik dizisi kullanılarak yapılan öğretimde bu konularla ilgili hangi kanıtlar ortaya sunulmaktadır?

2. Geliştirilen etkinlik dizisi ve öğrenme rotası ile yapılan öğretim sürecinde hangi matematiksel uygulamalar (mathematical practices) oluşturulmuştur?

Bu araştırma sorularına yönelik olarak, bu çalışmanın gerçekleştirilmesinin önemine bir sonraki bölümde yer verilmiştir.

1.2. Çalışmanın Önemi

Yukarıda belirtildiği gibi, bu çalışmanın en genel amacı orantısal akıl yürütmenin gelişime yönelik varsayıma dayalı bir öğrenme rotası ve ilgili etkinlik dizisinin geliştirilmesi, test edilmesi ve düzenlenmesidir. Dolayısıyla, bu çalışmanın sonuçlarının uygulamaya ve kurama birçok yönden katkı sağlayacağına inanılmaktadır. Bu katkılar aşağıda belirtilmiştir.

Daha önce bahsedildiği gibi, orantısal akıl yürütme öğrencilerin özellikle matematik ve fendeki başarıları ve günlük hayattaki durumları anlamlandırmaları için önemlidir (Cramer ve Post, 1993). Diğer bir yandan, öğrencilerin ve öğretmenlerin orantısal akıl yürütmede zorlandıkları ve oran ve orantı öğretiminin genellikle işlemsel becerilere odaklandığı ve orantısal akıl yürütmeyi geliştirmediği belirtilmiştir (Hart, 1984; Karplus vd., 1983; Lamon, 1995; Post vd., 1988). Bu sebeplerden dolayı oran ve orantı öğretiminin iyileştirilmesi önem arz etmektedir. Bu çalışma kapsamında oran ve orantı konusu ile ilgili anahtar öğrenmeleri en anlamlı, kapsamlı ve sıralı bir şekilde gerçekleştirmeyi amaçlayan bir etkinlik dizisi geliştirilmesi hedeflenmektedir. Bu sebeplerden dolayı çalışma sonuçlarının okullardaki ve ders kitaplarındaki oran ve orantı konusunun öğretiminin iyileştirilmesine katkı sağlaması beklenmektedir.

Diğer bir yandan, 2013 yılında yenilenen öğretim programlarında oran ve orantı konusuna yeni kazanımlar eklenmiştir (MEB, 2013). Örneğin öğrencilerden gerçek yaşam durumlarını, tabloları ve doğru grafiklerini inceleyerek iki çokluğun orantılı olup olmadığına karar verebilmeleri, doğru orantılı iki çokluk arasındaki ilişkiyi tablo ve denklem olarak ifade etmesi beklenmektedir. Ancak, bu kazanımlar uluslararası alan yazında vurgulanan tüm anahtar öğrenmeleri içermemekte ve var olan kazanımlar belli bir sırada sunulmuş olsa da bu sıralamanın öğrencilerin oran orantı konusunu en anlamlı şekilde öğrenmesini sağlayacak şekilde olup olmadığı ile ilgili ampirik bir bilgi bulunmamaktadır. Buna ek olarak, programın doğası gereği konularla ilgili öğrenme çıktılarına hangi öğrenme materyalleri ile ve nasıl bir yol izlenerek ulaşılması gerektiği program kapsamına dahil edilmemiştir. Diğer yandan, ders kitaplarında ve ek kaynaklarda bulunan öğretim materyalleri ise genellikle işlemsel becerileri ön plana çıkarmakta ve kendi içerisinde belirli bir sıra barındırmamaktadır. Bu durum öğretmenler için de yeni olan ve var olan kazanımlara ulaşmayı sağlayacak öğrenme ortamları sunmada sorun oluşturmaktadır. Bu çalışmada geliştirilecek olan etkinlikler alan yazında rapor edilen çalışma sonuçlarına göre hazırlanarak test edilip öğrencilerin en iyi öğrenmelerini sağlayacak şekilde düzenlenmiştir. Bu bağlamda, hazırlanacak olan materyallerin öğretmenlere oran ve orantı konusunun en iyi, anlamlı ve sıralı bir şekilde öğretilmesine olanak sağlayacak bir kılavuz niteliği taşıyacak olmasından dolayı oran ve orantı konusunun öğretim kalitesinin artırılmasında yardımcı olacağı düşünülmektedir. Bu anlamda çalışmanın hem kurama hem de pratiğe özgün katkı sağlayacağına inanılmaktadır.

Alan yazındaki çalışmalar öğrencilerin yanı sıra, matematik öğretmenlerinin ve aday öğretmenlerin orantısal akıl yürütmelerinin yetersiz olduğu ve bu konuda bilgi eksikliklerinin ve benzer kavram yanılgılarının olduğunu (Canada, Gilbert ve Adolphson, 2008; Harel ve Behr, 1995), bu konuyu öğretmede zorlandıklarını (Behr, Harel, Post ve Lesh, 1992) ve öğrencilerin oran orantı konusundaki düşünüş biçimleri, stratejileri ve gelişimsel süreçleri hakkında yetersiz bilgiye sahip olduklarını (Hines ve McMahon, 2005) göstermiştir. Bu sebeplerden dolayı, öğretmenler öğretim gerçekleştirirken farklı bağlamlarda örnekler sunamamakta ve öğrencilerin farklı stratejiler ve düşünüş biçimleri geliştirmelerine olanak sağlayacak öğrenme ortamı sağlayamamaktadırlar. Buna ek olarak, öğretmenler öğrencilerin oran ve orantı konusunu anlamlı öğrenmelerini sağlamak için ne öğreteceklerini, bunları hangi materyal ve bağlamlarla ve hangi sırada öğreteceklerini belirlemede güçlük yaşamaktadırlar. Bu sebeplerden dolayı, bu çalışma sürecinde geliştirilen varsayıma dayalı öğrenme rotası ve ilgili etkinlik dizisi ve bunların nasıl uygulanması gerektiğini yerel öğretim teorisi sayesinde öğrencilerin orantısal akıl yürütmedeki gelişimlerini anlamalarında yardımcı olması ve bu bağlamda bu etkinliklerin öğretime entegre edilerek öğretmenlerin alan ve pedagojik alan bilgilerinin arttırılmasında önemli potansiyele sahip olduğu düşünülmektedir. Bu anlamda, çalışmanın uygulamaya ve kurama büyük bir katkı sağlayacağı öngörülmektedir.

Diğer bir taraftan, bu çalışmada öğrencilerin orantısal akıl yürütmelerini geliştirmek için hazırlanan etkinliklerde Gerçekçi Matematik Eğitimi (GME) öğretim teorisi temel alınmıştır. Bu bağlamda, öğrencilerin orantısal akıl yürütmelerine ilişkin bilgiler onlara hazır kurallar sistemi olarak verilmemiş; aksine, bu bilgi ve becerilerin matematik problem çözme sürecinde edinmeleri sağlanmıştır (Gravemeijer, 1999). Diğer bir ifade ile çalışmada öğrencilere gerçek yaşam durumları sunularak bağlam içerikli çözüm stratejileri üretmeleri için fırsatlar sunulmuştur (Gravemeijer, 1994). Böylece, matematikleştirme sürecinde öğrencilerin orantısal akıl yürütmeye ilişkin informel bilgilerinden yararlanılarak formel bilgiye geçmeleri desteklenmiştir (Freudenthal, 1973, 1991). Ancak alan yazına bakıldığında, orantısal akıl yürütmenin gelişimini Gerçekçi Matematik Eğitimi Teorisi kapsamında inceleyen çok az sayıda çalışmaya rastlanmıştır. Ayrıca, bu çalışmaların bu projede hedeflenen uzun soluklu, tasarı tabanlı araştırma modeli ile değil kısa süreli mevcut durumu ortaya koymaya yönelik veri toplamaya dayalı çalışmalardır. Bu anlamda mevcut çalışma alan yazınına katkı sağlamak adına büyük önem taşımaktadır. Buna ek olarak, genelde yurtdışındaki çalışmalarda kullanılan bu teorinin farklı kültürlerde uygulanabilirliğine yönelik çalışmalara olan ihtiyaç literatürde belirtilmiştir. Bu çalışmada Gerçekçi Matematik Eğitimi temel alınarak 7. Sınıf programında yer alan kazanımlar, bunlara ek olarak alan yazınından edinilen bulgular doğrultusunda belirlenen anahtar öğrenmeler ile öğrencilerin orantısal akıl yürütmelerinin gelişiminin desteklenmesine yardımcı olacak bir etkinlik dizisi geliştirilmesi hedeflenmektedir. Bu etkinlikler Türkiye bağlamında incelendiğinden

Gerçekçi Matematik Eğitiminin farklı kültürlerde ve farklı bağlamlarda nasıl çalıştığına yönelik sonuçların alan yazına kuramsal bir katkı sağlaması beklenmektedir.

Son olarak, ulusal ve uluslararası alan yazında öğrencilerin orantısal akıl yürütmeye yönelik informel bilgileri, zorlukları ve kavram yanılgıları ile ilgili çok sayıda çalışma bulunmaktadır. Fakat bu çalışmaların sonuçlarına dayanarak hazırlanan bir öğretimin sınıf içerisinde test edilmesine yönelik çalışmalara sık rastlanmamaktadır (Wilson, 2009). Benzer şekilde, orantısal akıl yürütmenin bileşenlerinin neler olduğu, öğrencilerin orantısal akıl yürütme gelişimlerinin nasıl ve hangi öğretimsel etkinliklerle destekleneceğine yönelik sorular cevapsız kalmaktadır (Rresnick ve Singer, 1993). Diğer bir deyişle, orantısal akıl yürütme özelinde de eğitim alanındaki araştırmalar ile pratikteki uygulamalar arasında önemli bir boşluk bulunmaktadır (Lamon, 1993; Misialidou ve Williams, 2003; Resnick ve Singer, 1993). Bu çalışmanın amaçları düşünüldüğünde, bu çalışmanın orantısal akıl yürütmeye yönelik araştırmalar ile öğretim arasındaki boşluğu doldurmaya yönelik olduğu görülebilir.

Son olarak, alan yazında orantısal akıl yürütmeye yönelik varsayıma dayalı öğrenme rotalarını odak noktası olarak alan çalışmalar (Carpenter vd., 1999; Steinthorsdottir ve Sriraman, 2009; Wright, 2014) var olsa da bu çalışmalar bireysel öğrenme rotalarına odaklanmaktadır. Bu sebepten dolayı, hazır olarak öğretmenlerin sınıf içerisinde kullanımına uygun değildir (Daro vd., 2011). Bu çalışmada ise, bir sınıfın ortaklaşa olarak orantısal akıl yürütmelerinin nasıl gerçekleştiği ve bu gelişimin nasıl destekleneceğine odaklanılmaktadır. Bu sayede, bu çalışmada bir sınıf içerisinde orantısal akıl yürütmenin nasıl geliştirildiğine yönelik bir rota ve ilgili öğretimsel etkinlikler öğretmenlere hazır olarak sunulmaktadır.

2. Alanyazın Taraması

Orantısal akıl yürütme en genel anlamıyla kovaryasyon ve çarpımsal kıyaslamaların anlamlandırılması ile ilgili olan bir matematiksel akıl yürütme biçimidir (Cramer ve Post, 1993; English, 2004; Lesh vd., 1988). Dolayısıyla, ezberlenmiş algoritmalarla (içler-dışlar çarpımı vb.) orantısal durumlar içeren problemleri çözmek orantısal akıl yürütme olarak adlandırılamayacağı vurgusu birçok araştırmacı tarafından dile getirilmiştir (Cramer ve Post, 1993; Lesh vd., 1988). Bundan ziyade, orantısal akıl yürütme, iki rasyonel ifade (oran, bölüm veya kesir) arasındaki bütünsel ilişkileri anlamlandırma ve bunlar hakkında akıl yürütmeyi içerir (Lesh vd., 1988). Bu ilişkiler, oranı oluşturan aynı ölçüm uzayına (measure space) ait çokluklar ve farklı ölçüm uzayına ait çokluklar arasındaki ilişkilerdir. Aynı ölçüm uzayına sahip çokluklar arasındaki ilişkileri anlamlandırmak için, "Bu çokluk ikinci durumda kaç katına çıkmış/kaça bölünmüş?" sorusu sorulmalıdır. Farklı ölçüm uzayına ait çokluklar arasındaki ilişkileri anlamlandırmak için ise, "Bu iki çokluk arasındaki (fonksiyonel) ilişki nedir?" sorusuna cevap aranmalıdır. Lamon'a (1994) göre, öğrenciler, hangi stratejiyi kullanırlarsa kullansınlar, orantısal akıl yürütme problemlerini bu iki ilişkiden birini kullanarak çözmektedir.

Bu çalışmada, etkinlik dizisinin tasarlanması ve uygulanmasında Gerçekçi Matematik Eğitimi Teorisi kılavuz olarak kullanıldığı için, orantısal akıl yürütmenin öğretimi için "Öğretici Olgu" ilkesine bağlı kalınmıştır. Bu ilke matematiksel kavramların nasıl oluştuğunu belirleyebilmekle ilgilidir. Bu kapsamda, bir kavramın öğretici olgusu, o kavramla ilgili öğrencilerin muhtemel deneyimleri ve öğrenme fırsatları ile ilgilidir (Freudenthal, 1983). Öğrenciler bu deneyimler ve öğrenme fırsatları aracılığıyla öğrenme sürecine girerler ve süreç boyunca hedeflenen matematiksel fikir sürecin yeniden keşfi ile kazanılır. Bu ilkeden yola çıkarak, Lamon (1995) orantısal akıl yürütme için gerekli olan öğretici deneyimleri ve orantısal akıl yürütmenin kritik bileşenlerini ortaya koyduğu bir teorik çerçeve ortaya koymuştur. Bu çerçeveye göre, orantısal akıl yürütmenin kritik bileşenleri göreceli (relative) ve mutlak (absolute) değişim, ortak değişinti (covariance) ve değişmezlik (invariance) ve oran hissidir (ratio sense). Orantısal akıl yürütmenin öğrenilmesinde etkili olan öğretici (didactical) deneyimler ise, bölümlere ayırma (partitioning), ilişkiler ve birimleştirmedir (unitizing) (Lamon, 1995).

Bu çalışma kapsamında, Lamon (1995) tarafından ortaya konulan bu çerçeve genişletilmiştir. Bu bağlamda, nitel ve nicel muhakeme ile çarpımsal düşünme, orantısal akıl yürütmenin diğer kritik bileşenleri olarak kabul edilmiştir. Ayrıca, birleşik birimleri bağlama ve bağlı birleşik birimleri yineleme de diğer bir önemli öğretici deneyim olarak ele alınmıştır.

Bu çalışmanın amaçları kapsamında öğrencilerin informel bilgi ve stratejilerine yönelik alanyazın taramasının incelenmesi de önemlidir. Bu sayede, geliştirilen öğretimsel tasarının ve etkinlik dizisinin öğrencilerin formel öğretime girmeden önce sahip oldukları bilgiler ve kullandıkları stratejilerin üzerine inşa edilmesi sağlanabilir. Birçok araştırmacı öğrencilerin orantısal akıl yürütmeye yönelik informel bilgilerini incelemiş ve bu bilgilerin genellikle şunlara dayandığını ortaya koymuştur: nitel akıl yürütme (Inhelder ve Piaget, 1958; Piaget, 1968; Piaget ve Beth, 1966; Piaget ve Inhelder, 1975); sayma, eşleştirme, eşit parçalara ayırma (Confrey vd., 2014; Lamon, 1995); gruplama, ritmik sayma, artırma stratejileri, birim faktör yaklaşımı (Kaput ve West, 1994); birleşik birimler oluşturma ve birleşik birimleri yineleme (Battista ve van Auken Borrow, 1995; Lamon, 1994; Steffe, 1988); parça-parça ilişkileri ve yarım imgesi (Spinillo ve Bryant, 1991) ve kovaryasyon (Lamon, 2007; Spinillo ve Bryant, 1999). Birçok çalışmada, 6-11 yaş öğrencilerin bu sezgisel bilgileri kullanarak orantısal durumlar içeren problem hakkında akıl yürütebildikleri sonucuna varılmıştır (Bryant, 1974; Lamon, 1995; Muller, 1977, 1978; Resnick ve Singer, 1993; Spinillo ve Bryant, 1991, 1999).

Buna ek olarak, alanyazın taramasında öğrencilerin sahip oldukları informel stratejilerin çoğunlukla artırma (build-up), kısa yoldan artırma (abbreviated build-up) ve birim faktör stratejilerine dayandığı belirtilmiştir (Ben-Chaim vd., 1998; Kaput ve West, 1994; Tourniaire ve Pulos, 1985). Diğer yandan, öğrencilerin bu stratejileri kullanırken tablo ve tablo-benzeri gösterimleri kendilerine öğretilmeden önce sezgisel olarak kullandıkları ve bu gösterimlerin öğrencilerin orantısal akıl yürütmelerini geliştirdiği belirtilmiştir

(Kenney, Lindquist ve Heffernan, 2002; Middleton ve Van den Heuvel-Panhuizen, 1995; Streefland, 1984, 1985).

Öğrencilerin orantısal akıl yürütmeye yönelik zorlukları incelendiğinde ise, öğrencilerin orantısal akıl yürütme problemlerine cevap verirken birçok zorluk yaşadıkları ve bazı kavram yanılgılarına sahip oldukları belirtilmiştir. Örneğin, bazı çalışmalarda öğrencilerin orantısal ve orantısal olmayan durumları ayırt etmede zorlandıkları ve orantısal durumlar için kullanılan stratejileri orantısal olmayan durumlar için de kullanma eğiliminde oldukları görülmüştür (Ayan ve Isiksal-Bostan, 2018; De Bockverschaffel ve Janssens, 1998; Freudenthal, 1983; Modestou ve Gagatsis, 2007, 2009, 2010; van Dooren, De Bock, Janssens ve Verschaffel, 2007). Diğer bir çalışmada, öğrencilerin problemlerin bağlamına odaklanmadan rastgele işlemler yaptıkları, orantısal durumları oluşturan çokluklardan yalnızca birine odaklandıkları ve diğerlerini görmezden geldikleri ve öznel ve alakasız cevaplar verdikleri görülmüştür (Ben-Chaim vd., 1998).

Bu zorluklara ek olarak, orantısal durumlarla ilgili yaşanılan en büyük zorluk yanlış toplamsal düşünme biçimidir (Atabaş ve Öner, 2017; Ben-Chaimvd., 1998; Duatepe vd.,2005; Fernández vd.,2012; Harel vd.1994; Hart, 1981, 1984, 1988; Inhelder ve Piaget, 1958; Kahraman vd.,2019; Kaplan vd., 2011; Kaput ve West, 1994; Karplus vd., 1983; Kayhan vd., 2004; Mersin, 2018; Misailidou ve Williams, 2003; Noelting, 1980a, 1980b; Özgün-Koca ve Altay, 2009; Piaget ve Beth, 1966; Piaget ve Inhelder, 1975; Resnick ve Singer, 1993; Steinthorsdottir ve Sriraman, 2009; Tourniaire ve Pulos, 1985, Tourniaire, 1986; van Dooren vd., 2010). Dahası, yanlış toplamsal düşünme biçimi, orantısal akıl yürütmenin gelişiminde bir engel olarak görülmektedir (Ayan ve Isiksal Bostan, 2018).

Bu çalışmanın amaçlarından birisi yedinci sınıf düzeyinde bir sınıftaki öğrencilerin orantısal akıl yürütmelerinin gelişiminin desteklenmesi için bir etkinlik dizisi ve varsayıma dayalı öğrenme rotasının geliştirilmesidir. Bu etkinlik dizisi ve varsayıma dayalı öğrenme rotasının geliştirilmesinde ve uygulanmasında Hollandalı matematikçi ve eğitimci olan Hans Freudenthal'ın 1970'li yıllarda ortaya koyduğu Gerçekçi Matematik Eğitimi Teorisi kılavuz alınmıştır. Bu teori somut çözüm stratejilerini formel matematik bilgiye dönüştürebilmede ön plana çıkan bir öğretim teorisidir. Freudenthal (1973) anti-didaktik olarak tanımladığı klasik matematik eğitiminde matematikçilerin sonuç olarak ulaştıkları formel bilginin matematik eğitimcileri tarafından derslerde başlangıç noktası olarak verildiğini, böylece öğrencilerin matematiği ezber ve hazır kurallar bütünü olarak gördüklerini belirtmiştir. Bu düşünceye karşıt olarak, Freudenthal (1968, 1973, 1991) matematiğin hazır kurallar sistemi değil bir insan etkinliği olarak görülmesi görüşünü savunur.

Gerçekçi Matematik Eğitimi'nin temeli matematikleştirmeye dayanır. Öğrenciler öğrenmeleri gereken matematiksel bilgiyi matematik problem çözme sürecinde edinirler. Diğer bir deyişle, matematiğin hazır bir sistem olarak değil bir etkinlik olarak ele alınması gerektiği vurgulanır. Bahsedilen bu matematiksel etkinliklerin temelini ise matematiksel perspektiften düşünmeyi gerektiren matematikleştirme (mathematizing) oluşturur. Freudenthal bahsedilen matematik etkinliklerinin sonucunda formel matematiksel bilgiye ulaşma sürecini matematikleştirme olarak isimlendirmiştir. Farklı bir ifade ile, gerçek hayat durumlarının matematikleştirilmesi sürecinde öğrencilerin informel bilgilerinden yararlanılarak formel bilgiye geçtikleri savunulur. Bu sebepten öğrencilerin öğrenme ortamına getirdikleri informel bilgiler GME'de büyük önem taşır. Ayrıca, öğrencilerin gerçek yaşam durumları içerisinde verilen problemi inceleyerek ve informel bilgilerinden yola çıkarak formel bilgiye ulaşmaları beklenir (Gravemeijer, 1994). Bu süreçte öğrencilerden her şeyi kendi başlarına keşfetmeleri beklenmez; önemli olan öğrencilerin edindikleri bilginin kendilerine ait özel bir bilgi olduğunu ve bu bilgiden kendilerinin sorumlu olduğunu benimsemeleridir (Freudenthal, 1991). Bu noktada modellere de formel matematiği yeniden keşif süreci kapsamında ihtiyaç duyulabilir çünkü modellerin bu geçişi desteklediği belirtilmiştir.

Gerçekçi Matematik Eğitimi Teorisi'ne ek olarak bu çalışmada varsayıma dayalı öğrenme rotasının geliştirilmesinde Simon (1995) tarafından ortaya atılan fikirler benimsenmiştir. Simon (1995) varsayıma dayalı öğrenme rotalarını öğrenmenin gerçekleşmesine yönelik öngörüler olarak tanımlamış ve bunların öğrenmeye yönelik amaç, bu amaca yönelik etkinlikler ve öğrencilerin bu etkinlikler süresince geçirecekleri muhtemel düşünce ve öğrenme süreçlerini içermesi gerektiğini savunmuştur. Zamanla bu tanım değişime uğramış ve farklı araştırmacılar tarafından farklı bileşenler içerdiği belirtilmiştir. Clements ve Sarama (2004) öğrenme rotalarını belirli bir matematik konusunda geliştirilen etkinlikler doğrultusunda öğrencilerin düşünmelerini ve öğrenmelerini gelişimsel bir süreç olarak belirten açıklamalar olarak tanımlamış ve öğrenme rotalarının Simon (1995)'un bahsettiği 3 temel bileşeni içermesi gerektiği düşüncesini devam ettirmiştir.

İleriki yıllarda, Stephan (2015) bir sınıfa ait öğrenme rotaları (classroom learning trajectories) kavramını ortaya atmıştır. Stephan (2015)'a göre bir sınıfa ait öğrenme rotaları bir sınıftaki öğrencilerin birbirleriyle ve öğretmenle etkileşimleri süresince oluşur. Stephan'a göre belli bir konunun öğretiminde sınıf içi tartışmalardan doğan fikirler tüm sınıfça kabul görmelidir. Ayrıca, belirli bir konunun öğretimine yönelik araçlar, imgeler, etkinlikler, fikir paylaşımı ve muhtemel matematiksel söylemler ile muhtemel jest ve metaforlar öğrenme rotalarının bileşenlerini oluşturur (Gravemeijer, Bowers ve Stephan, 2003; Stephan ve Rasmussen, 2002; Stephan ve Akyuz, 2012).

Dolayısıyla, bu çalışmada, yedinci sınıf öğrencilerinin orantısal akıl yürütmelerinin gelişimini desteklemek amacıyla, Gerçekçi Matematik Eğitimi Teorisi'ne dayanan bir sınıfa ait varsayıma dayalı öğrenme rotaları ve ilgili öğretimsel etkinlik dizisi oluşturulmuştur. Bu etkinlik dizisi ve varsayıma dayalı etkinlik dizisinin uygulanması süreci ve veri analizine yönelik detaylar bir sonraki bölümde ele alınmıştır.

3. Yöntem

Bu çalışmanın birinci amacı, yedinci sınıfta orantısal akıl yürütmenin öğretilmesi için bir varsayıma dayalı öğrenme rotası ve ilgili etkinlik dizisinin geliştirilmesi, test edilmesi ve düzenlenmesidir. Çalışmanın ikinci amacı öğrencilerin informel ve formel araçlarla ortaklaşa akıl yürütmelerinin ve bu akıl yürütmenin Gerçekçi Matematik Eğitimi perspektifi doğrultusunda formel araçlarla akıl yürütmeye doğru gelişiminin açıklanmasıdır. Üçüncü amaç ise, öğrencilerin orantısal akıl yürütmeye yönelik fikir ve kavramların ortaklaşa gelişiminin ortaya konulmasıdır (sınıf içi matematiksel uygulamalar analizi).

Bu amaçlara ulaşabilmek için çalışmanın deseni tasarı tabanlı araştırma (Gravemeijer ve Cobb, 2006; van den Akker vd., 2006) modeli olarak belirlenmiştir. Tasarı temelli araştırmanın amacı sadece sınıfta olanları betimlemek değil muhtemel öğrenme süreci ve bu öğrenme sürecini destekleyecek öğretimsel araç, etkinlik, yöntem, sınıf kültürü ve öğretmenin rolü ile ilgili kanıları (conjectures) içeren yerel öğretim teorilerini (Gravemeijer ve Cobb, 2006) desteklemek ve başka durumlardaki öğretime veya tasarıma durum oluşturacak nitelikte teorik altyapı oluşturmaktır. Ayrıca, tasarı tabanlı araştırmanın temelinde öğretimsel etkinliklerin yeniden tasarımı ve test edilmesini içeren döngüsel süreçler yer alır. Bu sebeplerden dolayı, çalışmanın amaçları ile bu araştırma modeli birebir örtüşmektedir. Gravemeijer ve Cobb'a (2006) göre tasarı tabanlı araştırma 3 aşamadan oluşur. Bunlar 1) Uygulama için hazırlık, 2) Sınıf içi uygulama ve 3) Geriye dönük analizlerdir (Gravemeijer ve Cobb, 2006).

3.1. Çalışmanın Bağlamı, Katılımcılar

Bu çalışma kapsamında, İlk olarak MEB'e bağlı Altındağ ilçesindeki bir devlet okulu ve bu okulda çalışan iş birliğine açık, anlamlı öğrenmeye önem veren ve oran ve orantı konusunun öğretiminde sekiz yıllık ve öğretmenlikte 10 yıllık deneyime sahip olan bir öğretmen seçilmiştir. Bu okul genellikle düşük sosyo-ekonomik düzeye sahip öğrencileri barındırmaktadır.

Stephan, McManus, Smith ve Dickey (2015) tarafından geliştirilen etkinlik dizisi ve sınıf içi varsayıma dayalı öğrenme rotası uyarlanarak ve düzenlenerek 2015-2016 öğretim yılında bu öğretmen ve bu öğretmenin bir 7. sınıf öğrencileriyle araştırmanın ilk (pilot) makro döngüsü gerçekleştirilmiştir. İlk uygulama süresinde ve sonrasında yapılan tasarı ekibi (öğretmen, doktora öğrencisi ve danışmanı) toplantılarında ortaya konulan tartışmalara dayanarak bu etkinlik dizisi ve öğrenme rotası yeniden düzenlenerek çalışmanın ikinci makro döngüsü 2016-2017 öğretim yılında aynı öğretmen ve aynı şartları sağlayan başka bir 7. sınıf öğrencileriyle 6 hafta boyunca gerçekleştirilen öğretim kapsamında uygulanmıştır. Her iki döngüde de seçilen sınıftaki öğrencilerin grup çalışmasına ve matematiksel argümantasyon yapma becerileri ve alışkanlıklarına sahip olma şartları aranmıştır.

Öğretmen, tasarı tabanlı bu çalışmanın ilk iki kısmında (uygulama için hazırlık ve sınıf içi uygulama) etkin olarak rol almıştır. Hazırlık aşamasında, araştırmacılarla toplantılara katılarak öğrencilerin muhtemel düşünüş biçimleri, Ortaokul Matematik Öğretim Programı kazanımları ve öğrenme rotaları ile ilgili bilgi ve deneyimlerini paylaşarak etkinlik dizisinin uyarlanmasında ve düzenlenmesinde kritik rol oynamıştır. Öğretmen sınıf içi uygulama kısmında araştırmacı ile her ders öncesi ve sonrasında görüşmüştür. Ders öncesinde dersin amaçları, öğrenme rotasının bileşenleri ve muhtemel öğrenci düşünüşleri gözden geçirilmiştir. Ders sonrasında ise, o günkü uygulamanın başarılı/başarısız yanlarına yönelik görüşlerini bildirmiş ve araştırmacı ile beraber bir sonraki uygulama için yapılması uygun görülen düzenlemelere karar vermiştir. Ayrıca, bu kısımda geliştirilen etkinlik dizisini tasarlandığı gibi uygulamış ve sınıf içerisinde zengin argümantasyon süreçlerinin oluşmasını sağlamıştır.

3.2. Veri Toplama Süreci

Veri toplama sürecinin ilk basamağı Stephan ve arkadaşları tarafından hazırlanan etkinlik dizisinin ve varsayıma dayalı öğrenme rotasının uyarlanması, düzenlenmesi ve geliştirilmesi olmuştur. Çalışmanın ilk döngüsü öncesinde öğretmen ve araştırmacılardan oluşan tasarı ekibi bu süreçte ortaklaşa çalışmışlardır. Çalışmanın ilk döngüsü süresince ve sonrasında tasarı ekibi toplantılarında yapılan görüşmeler doğrultusunda bir sonraki ders ve çalışmanın ikinci döngüsü için gerekli düzenlemeler yapılmıştır. İkinci döngü öncesinde öngörülen varsayıma dayalı öğrenme rotası altı bölümden oluşmaktadır.

Varsayıma dayalı öğrenme rotasının birinci bölümünde öğrenciler "Balıkları besleyelim" bağlamı kapsamında verilen kurala göre balık ve yem resimlerini gruplayarak birleşik birimler oluşturma ve bu birleşik birimleri gruplayarak birbirine bağlarlar. Resimlerle başlayan bu yinelemelerin zamanla sayısal yinelemelere döneceği ve öğrencilerin tablo benzeri gösterimler kullanacağı öngörülmüştür (örn. 1-3, 2-6- 3-9, vb.). Buradan yola çıkarak, bağlamaların ve yinelemelerin daha düzenli bir şekilde yapılabilmesi için uzun oran tablolarının öğrencilere tanıtılması öngörülmüştür. Bu sayede, öğrencilerin artırma ve kısa yoldan artırma stratejilerini tablolar üzerinde etkili ve düzenli bir şekilde kullanmaları ve tablolarda yatay ilişkileri anlamlandırmalarının sağlanması hedeflenmiştir. Kısa yoldan artırma stratejisinin sınıf içi söylemde ortaya çıkmasından sonra ikinci uzun oran tablolarının kısaltılarak kısa oran tablolarına geçiş yapılması planlanmıştır. Bu süreçte, öğrencilerin uzun ve kısa oran tablolarında farklı birimlere sahip çokluklar arasındaki fonksiyonel ilişkileri de incelemeleri ve bu ilişkilerin dikey ilişkiler olarak adlandırılması amaçlanmıştır. Bu şekilde, birinci bölüm kapsamında öğrencilerin uzun ve kısa oran tablolarında ortak değişinti (artırma/kısa yoldan artırma stratejileri) ve değişmezlik (oran/ birim oran) ilişkilerini keşfetmeleri hedeflenmiştir.

Öğrenme rotasının ikinci bölümünde, öğrencilerin keşfettikleri bu ortak değişinti ve değişmezlik ilişkilerinin parça-bütün bağlamlarında anlamlandırılması hedeflenmiştir.

Üçüncü bölümde kısa oran tablolarının dikey kenarlıkları kaldırılarak oran ve orantının sembolik ve formel gösterimine geçiş yapılması öngörülmüştür. Bu bölümde, öğrencilerin kısa oran tablolarında keşfettikleri ortak değişinti (yatay ilişkiler) ve değişmezlik (dikey ilişkiler) ilişkilerini orantının sembolik gösteriminde anlamlandırmaları ve bu ilişkileri kullanarak denk oranlar kurarak orantısal durumlardaki bilinmeyenleri bulmaları hedeflenmiştir.

Öğrenme rotasının dördüncü bölümünde öğrencilerin düşünüşlerinin orantısal durumların tablosal ve sembolik gösterimlerinden grafiksel ve cebirsel gösterimlerine doğru yönlendirilmesi öngörülmüştür. Bu bağlamda, orantısal durumların çoklu gösterimleri (tablo, sayı, grafik, cebirsel) arasındaki ilişkilerin de anlamlandırılması hedeflenmiştir. Ayrıca, doğrusal ilişkilerin farklı gösterimleri ve bu gösterimlerin orantısal ilişkilerin gösterimleri arasındaki ilişkilerin incelenmesi de bu bölümde öngörülen muhtemel matematiksel söylemlerdendir. Beşinci bölümde ise, öğrencilerin daha önceki bölümlerde keşfettikleri aynı ölçüm uzayı içerisinde kısa yoldan artırma stratejileri ile yineleme (yatay ilişkiler) ve farklı ölçüm uzayları içerisinde fonksiyonel ilişkileri kullanarak farklı oranları kıyaslamaları ve hangi oranın daha büyük/küçük olduğuna farklı bağlamlar içerisinde karar vermeleri beklenmektedir. Bu bağlamlardan bazıları benzer şekiller ve farklı miktarlarda bileşen içeren karışımların tatlarının kıyaslanmasıdır. Son olarak, altıncı bölümde öğrencilerin sayılardan bağımsız olarak, sayı içermeyen bir bağlamda, oranın değerinin değişip-değişmeyeceği ve değişecekse hangi yönde değişeceğine yönelik çıkarımlar yapmaları hedeflenmiştir.

Bu varsayıma dayalı öğrenme rotası bir sınıftaki öğrencilerin ortaklaşa bir şekilde orantısal akıl yürütmelerinin gelişiminin genel çerçevesini ortaya koymaktadır. Öngörülen bu öğrenme rotasının geliştirilen etkinlik dizisi boyunca sınıf içerisinde nasıl gerçekleştiğinin incelenmesi, geliştirilen varsayıma dayalı öğrenme rotası ve ilgili etkinlik dizisinin öğrencilerin orantısal akıl yürütmelerinin gelişimine ne derecede katkı sağladığı ve bunlara yapılacak düzenlemelerin belirlenmesi sınıf içi matematiksel uygulamalar analizi ile mümkün olmuştur. Bir sonraki bölümde bu analiz yöntemi detaylı olarak anlatılmıştır.

3.3.Veri Analizi

Bu çalışmanın verileri altı hafta ve 30 saat süren sınıf içi uygulamanın video kayıtları ve tasarı ekibi toplantılarının ve öğretmenle yapılan ders öncesi/sonrası görüşmelerin sesli kayıtlarından oluşmaktadır. Görüşme kayıtları uygulama süresince devam eden analizler kapsamında analiz edilmiş ve öğrenme rotası ve etkinlik dizisine yapılacak düzenlemeleri belirlemek amacıyla kullanılmıştır. Sınıf içi uygulamanın video kayıtları ise tasarı çalışmasının üçüncü kısmı kapsamında (geriye dönük analizler) uygulama yapılan sınıftaki öğrencilerin orantısal akıl yürütmelerinin ortaklaşa gelişimi sürecinin analiz edilmesi için kullanılmıştır. Bu amaç için ikinci makro döngü kapsamında yapılan sınıf içi uygulamanın videoları deşifre edilmiştir. Sınıf içi videolar ve yazılı deşifreleri, Stephan ve Rasmussen (2002) tarafından geliştirilen üç aşamalı Sınıf içi Matematik Uygulamaları Analizi (Classroom Mathematical Practices Analysis) yöntemi ile analiz edilmiştir. İlk aşama kapsamında, her güne ait sınıf içi uygulaması Toulmin'in (1958) argümantasyon modeline göre analiz edilmiştir. Bu bağlamda, tüm sınıf tartışmaları veri, iddia, gerekçe ve destek olarak kodlanmıştır. İlk aşama sonunda elde edilen şey günlere göre sıralandırılmış bir argümantasyon akış şemasıdır. İkinci aşama kapsamında, bu argümantasyon akış şeması günlere göre incelenmiş ve sınıftaki kişiler tarafından daha öncesinde paylaşılan hangi fikirlerin sınıf içerisinde kabul gördüğünü ve ortaklaşa akıl yürütmenin bir parçası olduğunu (taken-as-shared) belirlemek için Stephan ve Rasmussen (2002) tarafından ortaya atılan aşağıdaki iki kriter kullanılmıştır:

1. Bir argüman için yapılan açıklamalarda artık destek ve/veya gerekçelerin görülmemesi (açıkça belirtilmeden üstü kapalı bir şekilde ima edilmesi, herhangi bir öğrencinin bununla ilgili açıklama istememesi)

2. Bir argümanın herhangi bir bileşeni (veri, iddia, gerekçe, destek) ileriki bir argümanda bir diğerinin yerine geçmesi-dolayısıyla fonksiyonunu değiştirmesi- ve herhangi bir öğrencinin bununla ilgili bir açıklama istememesi

Tüm kabul gören fikirlerin belirlenmesinden sonra, analiz yönteminin üçüncü aşaması kapsamında, bu fikirler bir araya getirilerek belirli matematiksel uygulamalar (mathematical practices) etrafında düzenlenmiştir. Sonuç olarak, bu matematiksel uygulamalar sınıftaki öğrencilerin orantısal akıl yürütmelerinin ortaklaşa gelişiminin düzenli bir biçimde sunulmuş halidir (Cobb vd., 2001).

4. Bulgular, Tartışma ve Öneriler

Bu çalışmanın birinci amacı, yedinci sınıfta orantısal akıl yürütmenin öğretilmesi için bir varsayıma dayalı öğrenme rotası ve ilgili etkinlik dizisinin geliştirilmesi, test edilmesi ve düzenlenmesidir. Çalışmanın ikinci amacı öğrencilerin informel ve formel araçlarla ortaklaşa akıl yürütmelerinin ve bu akıl yürütmenin Gerçekçi Matematik Eğitimi perspektifi doğrultusunda formel araçlarla akıl yürütmeye doğru gelişiminin açıklanmasıdır. Üçüncü amaç ise, öğrencilerin orantısal akıl yürütmeye yönelik fikir ve kavramların ortaklaşa gelişiminin ortaya konulmasıdır (sınıf içi matematiksel uygulamalar analizi). Bu amaçlara yönelik bulgular bu bölümde tek tek ele alınmadan bir bütün halinde sınıf içerisindeki öğrencilerin orantısal akıl yürütmelerinin ortaklaşa gelişimi ele alınacaktır. Böylelikle, bu amaçlara yönelik olarak ulaşılan bulgulara yönelik çıkarımlar okuyucular tarafından yapılabilir.

Matematiksel uygulamalar analiz bulgularına göre, altı hafta ve 30 ders saati süren uygulama kapsamında, sınıf içi söylemde beş adet matematiksel uygulama oluşturulmuştur (bkz. Tablo 4.1)

Tablo 4.1. Oluşturulan Matematiksel Uygulamalar

Matematiksel Uygulamalar
Uygulama 1. Bilinmeyen değerleri bulmak için resim ve tablolarla akıl yürütme • Birleşik birimleri bağlama ve bağlı birleşik birimleri yineleme • Erken çarpımsal düşünme- bir ölçek katsayısı kullanarak bağlı birleşik birimleri eşgüdümlü olarak kısa yoldan yineleme • Çarpımsal düşünme- iki farklı ölçüm uzayında bulunan birbirine bağlı iki birim arasındaki çarpımsal ilişkinin değişmezliği (fonksiyonel ilişki) • İki farklı ölçüm uzayında bulunan birbirine bağlı iki birim arasındaki çarpımsal ilişkinin birim oran olarak anlamlandırma ve bilinmeyen değerleri bulmak için referans noktası (benchmark) olarak kullanma • Ortak değişinti ve değişmezlik ilişkilerini bir bütünün parçaları arasındaki ya da bir bütünün parçaları ile arasındaki ilişkilere genişletme
Uygulama 2. Orantısal durumları belirlemek için tablo ve sembollerle akıl yürütme • Oran ve orantıyı çarpımsal olarak yapılandırma ve ortak değişinti ve değişmezlik ilişkilerini sembolik orantı gösterimine genişletme • Ortak değişinti ve çarpımsal düşünceyi kullanarak orantısallığı belirleme
Uygulama 3. Gösterimler arasındaki ilişkileri koordine etme • Birleşik birimleri bağlama ve bağlı birleşik birimleri yineleme • Çarpımsal düşünme- iki farklı ölçüm uzayında bulunan birbirine bağlı iki birim arasındaki çarpımsal ilişkinin değişmezliği (fonksiyonel ilişki) • Değişmez çarpımsal ilişkiyi bir denklemle ifade etme • Orantısal ilişkileri $y = mx$ formundaki doğrusal denklemlerle ve orijinden geçen doğru grafikleriyle gösterme • Orantısal olmayan doğrusal ilişkileri $y = mx + n$ formundaki doğrusal denklemlerle ve orijinden geçmeyen doğru grafikleriyle gösterme
Uygulama 4. Ortak değişinti ve değişmezlik ilişkilerini sürekli değerler içeren bağlamlara genişletme • Bir şekil içerisindeki ya da iki benzer şeklin kenar uzunlukları arasındaki oranları kullanarak benzer şekillerin verilmeyen kenar uzunluğunu bulma • Şekillerin çarpıtılmasının (distortion) anlamlandırılması
Uygulama 5. Oranları kıyaslama ve hangi oranın daha büyük/küçük/eşit olduğuna karar verme • Çoklukları kıyaslamak için denk oranlar oluşturma ve oluşturulan denk oranları kullanarak akıl yürütme

Matematiksel uygulamalar analizinin sonuçları, geliştirilen etkinlik dizisi ve öğrenme rotasının bir sınıftaki öğrencilerin orantısal akıl yürütmelerinin gelişimini desteklemede büyük potansiyele sahip olduğunu göstermiştir. Örneğin, ilk matematiksel uygulama kapsamında, öğrenciler informel araçları (balık ve yem resimleri, gruplama için yuvarlak içine alma ve oklar kullanarak birleşik birimleri bağlama vb.) kullanarak başladıkları süreç içerisinde daha formel modellere (uzun oran tabloları) geçiş yapabilmişlerdir. Bu uygulama kapsamında, bilinmeyen değerleri bulmak için resim ve tablolarla akıl yürütürken, öngörüldüğü gibi, ritmik sayma, eşleştirme, gruplama, artırma ve kısa yoldan artırma stratejileri, birim faktör yaklaşımı, birleşik birimler oluşturma ve birleşik birimleri yineleme ve parça-parça- bütün ilişkilerinden faydalanmışlardır. Bunlar alanyazın taraması bölümünde belirtilen orantısal akıl yürütme için gerekli olan didaktik etkinlikler ve sezgisel bilgi/stratejilerdir. Ayrıca, ilk uygulama kapsamında, yem kutusu sayısı değiştiğinde bu yem kutularıyla beslenebilecek balık sayısının da belirli bir sayıda değiştiği fikri sınıf içerisinde kabul görmüştür. Bu fikir alanyazında ortak değişinti (kovaryasyon, kovaryans) olarak geçen terimle ilgilidir (Carlson vd., 2002; Lamon, 1995, 2007).

Bu ortak değişinti fikri başlangıçta koordineli artırma stratejileri (örn. yem kutusu sayısı 1 arttığında balık sayısı 3 artar) ya da koordineli ritmik sayma (örn. 1-3, 2-6, 3-9, vb.) şeklinde kabul görmüştür. Öğrenciler etkinlik dizisi boyunca ilerlediğinde bu koordineli artırma stratejileri daha etkili ve kısa yollara dönüştürülmeye başlanmıştır. Öğrenciler, tek tek artırma yapmak yerine, aynı ölçüm uzayındaki birimleri kendi içerisinde aynı ölçek katsayısı ile çarparak kısa yoldan artırma stratejilerini geliştirmişlerdir. Bu süreçte, dört değeri barındıran kısa tablolar tanıtılmış ve öğrenciler kısa tablolarda yatay ilişkiler hakkında akıl yürütmüşlerdir. Daha detaylı olarak belirtmek gerekirse, kısa tablodaki yatay ilişki ile (x10 vb.), ilk etapta informel araçlarla (gruplandırma, grupları birbirine bağlayarak yineleme) yaptıkları süreçleri bağdaştırmışlardır. Sonrasında, öğrenciler uzun ve kısa oran tablolarında farklı ölçüm uzaylarındaki çokluklar (yem kutusu ve balık sayısı) arasındaki değişmeyen çarpımsal (fonksiyonel) ilişkiye odaklanmış ve bu ilişkiyi tablolardaki dikey ilişkilerle ilişkilendirilmişlerdir. Tablolardaki bu değişmeyen ilişki ile verilen kurala yönelik olarak bağlama sürecini ve birim oranı ilişkilendirmişler ve birim oranı bilinmeyen değerleri bulmak için bir referans olarak kullanmışlardır. Son olarak da bu ortak değişinti ve değişmezlik ilişkilerini parça-bütün bağlamlarında anlamlandırmışlardır. Dolayısıyla, çalışmanın ilk uygulamaya yönelik bulguları kapsamında öğrenciler Kaput ve West (1994) tarafından ortaya konulan şu üç informel stratejiyi sırasıyla kullanmışlardır: (1) Koordineli artırma stratejileri, (2) Çarpma/bölme işlemleri kullanarak kısa yoldan artırma stratejiler, (3) Birim faktör yaklaşımı.

Dolayısıyla, matematiksel uygulamalar analizi aynı ölçüm uzayı içerisinde çoklukları aynı ölçek katsayısı kullanarak çarpma işlemi yaparak denk oranlar bulmanın resimlerle yapılan gruplama/yineleme ve sayılarla yapılan artırma işlemlerinin kısa yolu olduğunu göstermiştir. Dolayısıyla, kısa yoldan artırma stratejileri (yatay ilişkiler), birçok çalışmadan farklı şekilde (örn. Lamon, 2007; Vermont Mathematics Partnership's Ongoing Assessment Project, 2011; Wright, 2014), bu çalışma kapsamında çarpımsal düşünme olarak düşünülmemiş, bunun yerine, erken çarpımsal düşünme kapsamında ele alınmıştır.

Diğer bir yandan, matematiksel uygulamalar analizine göre, eş oranlar elde etmek için aynı ölçüm uzayında ölçek katsayısı ile yapılan işlemlere yönelik fikirler uygulamanın ilk gününde sınıf içi tartışmada doğal olarak ortaya çıkmış ve öğrenciler tarafından kabul görmüştür. Halbuki, farklı ölçüm uzaylarındaki birimlerin arasındaki değişmez fonksiyonel ilişkiye yönelik fikirlerin ortaya çıkması öğretmenin "yem kutusu sayısı ile balık sayısı arasında nasıl bir ilişki vardır?" sorusu ile ortaya çıkmıştır. Bu sebepten dolayı, bu çalışmanın sonuçları aynı ölçüm uzayı içerisinde bulunan çokluklar arasındaki ilişkilerin farklı ölçüm uzayında bulunan çokluklar arasındaki ilişkilerden daha doğal ve sezgisel olduğunu ortaya koyan çalışmaların (Freudenthal, 1978; Karplus vd., 1983; Noelting, 1980a, 1980b; Vergnaud, 1980) sonuçlarını doğrular niteliktedir. Aynı zamanda, bu çalışmanın sonuçları yukarıda referans verilen çalışmaların farklı ölçüm uzaylarında bulunan çokluklar arasındaki ilişkilerin aynı ölçüm uzayında bulunan çokluklar arasındaki ilişkilerin anlamlandırılmasından bilişsel olarak daha üst düzey beceriler içerdiğine yönelik sonuçlarını da destekler niteliktedir.

İkinci matematiksel uygulama, ilk uygulama kapsamında sınıf içerisinde kabul gören fikirlerin ne kadar güçlü olduğunu göstermiştir. Öğrenciler bu uygulama kapsamında kısa oran tabloların kenarlıklarını kaldırarak oran ve orantının sembolik gösterimine geçiş yapmışlar ve orantının sembolik gösteriminde bir orantıda verilmeyen değeri bulmak için ortak değişinti ve değişmezlik ilişkilerini anlamlandırmışlardır. Ayrıca, kısa oran tablolarındaki yatay ve dikey ilişkilerden yola çıkarak orantının sembolik gösteriminde de yatay ve dikey ilişkiler ile ortak değişinti ve değişmezlik ilişkileri arasında ilişkiler kurmuşlardır.

Üçüncü uygulama kapsamında, öğrencilerin düşünüşleri tablo ve orantı gösterimlerindeki yatay ve dikey ilişkilerden orantısal durumların grafiksel ve cebirsel denklem olarak ifade edilmesine doğru ilerlemiştir. Bu bağlamda, orantılı çokluklar arasındaki ilişkileri $y = mx$ şeklindeki denklemlerle ve orijinden geçen doğrularla ifade etmişler ve bu gösterimler arasındaki ilişkilere değinmişlerdir. Bunun yanı sıra, üçüncü uygulama kapsamında, orantısal olmayan doğrusal ilişkileri de $y = mx + n$ ($n \neq 0$) şeklindeki denklemlerle ve orijinden geçmeyen doğrularla göstermişlerdir. Son olarak, orantısal ilişkiler ile orantısal olmayan doğrusal ilişkiler arasındaki farklılık/benzerlikleri ortaya koymuşlardır.

Dördüncü uygulama kapsamında, öğrenciler yine aynı ölçüm uzayı içerisindeki çokluklar arasındaki ilişkiler ve/veya farklı ölçüm uzaylarındaki çokluklar arasındaki ortak değişinti ve değişmezlik ilişkilerini kullanarak benzer şekillerin bilinmeyen kenar uzunluklarını bulmuşlar ve verilen şekillerin benzer olup olmadıklarına karar vermişler ve şekillerin çarpıtılması ile ilgili iddialarda bulunmuşlardır.

Aynı şekilde, beşinci uygulama kapsamında da öğrenciler yine aynı ölçüm uzayı içerisindeki çokluklar arasındaki ilişkiler ve/veya farklı ölçüm uzaylarındaki çokluklar arasındaki ortak değişinti ve değişmezlik ilişkilerini kullanarak karışımların tatlarının kıyaslanması ile ilgili farklı bağlamlarda verilen oranları kıyaslamışlar ve iki oranın eşit olup olmadığına ve eşit değillerse hangisinin büyük/küçük olduğuna karar vermişlerdir. Örneğin, portakal suyu ve sudan oluşan karışımların tatlarını kıyaslarken her bir durumdaki portakal suyu miktarı ile su miktarının birbirine bağlayarak yinelemeler yapıp aynı tada sahip karışımlar oluşturmuşlardır (örn. 2 bardak portakal suyu-1 bardak su, 4 bardak portakal suyu-2 bardak su, vb.). Daha sonrasında ise verilen karışımdaki portakal suyu ve su miktarları ile kıyaslama yaparak aynı tada sahip olup olmadıklarına karar vermişlerdir. Diğer bir yandan, her bir bardaktaki portakal suyuna karşılık gelen su miktarı (ya da her bir bardaktaki su miktarına karşılık gelen portakal suyu miktarı) ile ilgili gerekçelendirmeler yaparak çarpımsal düşünme ve birim oran ile ilgili fikirler üzerine oranları kıyaslama ile ilgili fikirleri inşa etmişlerdir. Aynı şekilde, oranların kıyaslanması anahtar öğrenmesi kapsamında kısa ve uzun tablolar oluşturarak ve orantının sembolik gösterimi üzerinde dikey ve yatay ilişkileri kullanarak muhakeme yapabilmişlerdir.

Sonuç olarak, bu çalışma da orantısal düşünmenin temelinin, Steffe (1994) ve Battista ve Van Auken Borrow (1995)'in ileri sürdüğü gibi, birleşik birimleri bağlama ve bağlı birleşik birimleri yineleme becerilerinin kazanılmasından geçtiği sonucuna varılabilir. Öğrenciler artırma ve kısaltılmış artırma stratejileri kullanarak orantısal durumları anlamlandırmaya başlamış ve sonraki süreçte farklı ölçüm uzaylarına ait çokluklar arasındaki fonksiyonel ilişkilere odaklanmışlardır. Öğrenciler diğer fikirleri ve matematiksel uygulamaları bu anahtar öğrenmeler üzerine inşa edebilmiş ve her derste bu fikirleri kullanarak gerekçeler ortaya koyabilmişlerdir. Dolayısıyla erken orantısal akıl yürütme becerilerinin geliştirilmesi için bu becerilere önem verilmesi önerilmektedir.

Diğer yandan, matematiksel uygulamalar analizi orantısal akıl yürütmenin kritik bileşenlerinin, Lamon (1995) tarafından belirtildiği gibi, göreceli ve mutlak değişim, ortak değişinti ve değişmezlik ve oran hissinden oluştuğunu doğrular niteliktedir. Bunun yanı sıra, bu çalışma kapsamında eklenen iki bileşenin de (nitel ve nicel muhakeme ile çarpımsal düşünme) orantısal akıl yürütmenin gelişimi için kritik olduğu ortaya konmuştur. Diğer yandan, Lamon'un (1995) teorik çerçevesi kapsamında ortaya konulan orantısal akıl yürütmenin öğrenilmesinde etkili olan öğretici deneyimler de (bölümlere ayırma, ilişkiler ve birimleştirme) öğrencilerin yukarıda bahsedilen bileşenlere ulaşmasında önemli rol oynamıştır. Son olarak, bu çalışma kapsamında ele alınan birleşik birimleri bağlama ve bağlı birleşik birimleri yinelemenin de diğer bir önemli öğretici deneyim olduğu görülmüştür. Oran ve orantı öğretimi yapılırken öğrencilerin var olan bu öğretici deneyimleri üzerine öğretim kurularak orantısal akıl yürütme için kritik olan bu bileşenlere ulaşmalarının sağlanması önerilmektedir.

Sonuç olarak, matematiksel uygulamalar analizi orantısal düşünmenin bireysel olarak gelişimine yönelik olarak yapılan çalışmaların sonuçları ile benzerlik göstermiştir. Buradan yola çıkarak, orantısal akıl yürütmenin sosyal bir ortamda ortaya konulan gelişiminin bireysel gelişime benzerlik gösterdiği sonucuna varılabilir. Yine de bu konunun araştırılması ileride yapılacak bir çalışmanın konusu olarak önerilmektedir.

Ulusal ve uluslararası alanyazındaki çalışmaların sonuçlarında öğrencilerin orantısal akıl yürütmede birçok zorluk yaşadıklarına ve kavram yanılgılarına sahip oldukları belirtilmiştir. Bu zorluk ve kavram yanılgılarının en çok göze çarpanı yanlış toplamsal düşünme biçimidir (Hart, 1981, 1988; Kaput ve West, 1994; Misailidou ve Williams, 2003; Resnick ve Singer, 1993; Steinthorsdottir ve Sriraman, 2009; Tourniaire ve Pulos, 1985; Tourniaire, 1986; van Dooren vd., 2010). Matematiksel uygulamalar analizinde görüldüğü gibi bu çalışma kapsamında yanlış toplamsal düşünme biçimi yalnızca birkaç öğrenci tarafından birkaç kez ortaya atılmıştır. Dahası, bu yanlış düşünme biçimi ortaya atıldığında birçok öğrenci tarafından reddedilmiştir. Dolayısıyla, bu sonuç yukarıda verilen birçok çalışmanın sonuçlarıyla zıt yöndedir. Bunun sebebinin bu çalışma kapsamında orantısal akıl yürütmenin temelinin, toplamsal düşünme yerine, birleşik birimlerin yinelenmesine (artırma ve kısaltılmış artırma stratejileri) dayanan ortak değişinti ve değişmezlik ilişkilerine vurgu yapılarak yapılandırılması olduğu düşünülmektedir. Diğer bir yandan, bu çalışma kapsamında, öğrencilerin bireysel yerine ortaklaşa bir şekilde orantısal akıl yürütmelerinin gelişimine odaklanılmıştır. Bu bağlamda, öğrenciler fikirlerini gerekçeleriyle birlikte sunar ve diğerlerinin fikir ve gerekçelerini analiz eder ve değerlendirir (Brown, 2017). Böylece, fikirlerini düzenleme ve yeniden yapılandırma ve ortak kanılara ulaşma imkânı bulurlar (Whitenack ve Knipping, 2002). Dolayısıyla, yanlış düşünme biçiminin oluşumu önceden engellenmiş olur. Bu kapsamda, bu çalışmanın sonuçları, Karplus, Formisano ve Paulsen'in (1979) de dediği gibi, toplamsal düşünmenin orantısal akıl yürütmenin gelişiminde her zaman yer almayabileceği ve yanlış toplamsal düşünmenin öğretime bağlı olduğunu doğrular niteliktedir.

Son olarak, alanyazında sıklıkla bahsedilen içler-dışlar çarpımı algoritmasının da matematiksel uygulamalar kapsamında ele alınmadığı görülebilir. Bu çalışma kapsamında bazı öğrenciler içler dışlar çarpımını alternatif çözüm yolu olarak sunsalar da bu fikirlerine gerekçe gösterememişlerdir. Bu çalışma kapsamında benimsenen sosyo-matematiksel normlara (Cobb ve Yackel, 1996) göre öğrencilerin fikir ve iddialarına geçerli gerekçeler sunmaları gerekmektedir. Öğrenciler içler dışlar çarpımını kullanarak iddialarda bulunsalar bile bu iddialarına geçerli ve matematiksel gerekçeler sunmakta zorlanmışlardır. Bu sebepten dolayı, içler dışlar çarpımına yönelik fikirler, diğer fikirler gibi doğal olarak ortaya atılmamış ve sınıf içerisinde kabul görmemiştir. Bu sebepten dolayı, bu çalışmanın sonuçları içler dışlar çarpımı gibi algoritmaların öğrenciler tarafından doğal olarak geliştirilmediğini savunan Hart'ın (1984) çalışmalarını destekler niteliktedir. Buradan çıkarılacak sonuç ise, öğrencilerin içler dışlar gibi algoritmaları kullanmadan farklı gösterim ve araçlar kullanarak birçok orantısal durum hakkında akıl yürütebilecekleridir. Bu sonuç, öğrencilerin içler dışlar çarpımı gibi algoritmalara sıklıkla başvurduklarını belirten birçok çalışmanın (Arıcan, 2019; Atabaş ve Öner, 2017; Ben-Chaim vd., 1998; Cramer ve Post, 1993; Cramer vd., 1993; Duatepe vd.2005; Kahraman vd., 2019; Kaplan vd., 2011; Kayhan vd., 2004; Özgün-Koca ve Altay, 2009) sonuçları ile karşıt düşmektedir. Dolayısıyla geliştirilen öğrenme rotası ve etkinlik dizisinin oran ve orantı öğretiminin iyileştirilmesi için önemli potansiyele sahip olduğundan dolayı öğretmenler tarafından derslerine entegre edilmesi önerilmektedir.

Son olarak, matematiksel uygulamalar analizi öğrenme rotası ve etkinlik dizisinde az sayıda da olsa bazı düzenlemeler yapılması gerekliliğini ortaya koymuştur. Örneğin, balık ve yem etkinliğinden (Etkinlik 1) hemen sonra sürekli değerler içeren bir etkinliğin ikinci etkinlik olarak etkinlik dizisine eklenmesi uygun görülmüştür. Öğrenme rotasına yapılan örnek bir değişiklik ise, öğrenme rotasının bazı kısımlarının birleştirilmesi ve öğrenme rotasına bazı yeni kısımlar eklenmesidir. İleriki bir çalışmanın konusu olarak bu değişikliklerin öğrenci öğrenmeleri için potansiyeli araştırılabilir. Etkinlik dizisinin ve öğrenme rotasının önerilen son hali bu çalışmaya ek olarak verilmiştir.

G. CURRICULUM VITAE

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Ph.D.	2014-2020	4.00 / 4.00	Middle East Technical University, Turkey Faculty of Education Elementary Education
M. Sc.	2011-2014	3.91 / 4.00	Middle East Technical University, Turkey Faculty of Education Elementary Science and Mathematics Education
B. Sc.	2005-2010	3.58 / 4.00	Middle East Technical University, Turkey Faculty of Education Elementary Mathematics Education

JOB EXPERIENCE	
March 2020- Present	Instructor of Mathematics Education Faculty of Education, Istanbul Aydin University, Turkey
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Dec. 2017-May 2018	Visiting Scholar (Under the supervision of Professor Michelle Stephan) University of North Carolina Charlotte College of Education, NC, United States of America
Sept. 2010-Sept. 2011	Mathematics Teacher TED Zonguldak College, Turkey

PUBLICATIONS

Doctoral Dissertation

Ayan-Civak, R. (2020). *The evolution of mathematical practices in a seventh-grade classroom: Analyzing students' development of proportional reasoning* (Unpublished doctoral dissertation). Middle East Technical University, Ankara, Turkey.

Master's Thesis

Ayan, R. (2014). *Middle school students' achievement levels, solution strategies, and reasons underlying their incorrect answers in linear and non-linear problems* (Master's thesis, Middle East Technical University, Ankara, Turkey). http://etd.lib.metu.edu.tr/upload/12616878/index.pdf

Published Papers in International Indexed Journals

Ayan, R. & Isiksal-Bostan, M. (2019). Middle school students' proportional reasoning in real life contexts in the domain of geometry and measurement. *International Journal of Mathematical Education in Science and Technology, 50*(1), 65-81. doi: 10.1080/0020739X.2018.1468042

Link: https://www.tandfonline.com/doi/abs/10.1080/0020739X.2018.1468042

Ayan, R, & Isiksal-Bostan, M. (2018). Middle school students' reasoning in nonlinear proportional problems in geometry. *International Journal of Science and Mathematics Education, 16*, 503–518. doi: 10.1007/s10763-016-9777-z

Link: https://link.springer.com/article/10.1007/s10763-016-9777-z

Scientific Projects

Project Title: Contextualized Mathematics Professional Development
Project length: 15 months (2018-2019)

Funding institution: University of North Carolina Charlotte, USA

Role in the project: Researcher

Project leader: Luke Reinke, Assistant Professor of Mathematics Education

Project Title: Development of a hypothetical learning trajectory and related instructional sequence for teaching ratio and proportion in a meaningful, comprehensive, and coherent way Project length: 9 months (2018-2019) Funding institution: The Scientific and Technological Research Council of Turkey Role in the project: Researcher Project leader: Mine Isiksal-Bostan, Professor of Mathematics Education
Project Title: Middle school students' multiplicative reasoning Project length: 12 months (2016-2017) Funding institution: Middle East Technical University, Turkey Role in the project: Researcher Project leader: Mine Isiksal-Bostan, Professor of Mathematics Education
Project Title: The linearity preponderance in middle school students' thinking Project length: 12 months (2013-2014) Funding institution: Middle East Technical University, Turkey Role in the project: Researcher Project leader: Mine Isiksal-Bostan, Professor of Mathematics Education
Research Presentations
Reinke, L., **Ayan, R**., Casto, A., Stephan, M. Grounded in common sense?: Examining contextual references in a ratio unit. Research presentation at the National Council of Teachers of Mathematics Research Conference (April 2019).
Papers in International Conference Proceedings
Ayan, R., Isiksal-Bostan, M., & Stephan, M. (2019, February). A math teacher's participation in a classroom design research: teaching of ratio and proportion. In U. T. Jankvist, M. van den Heuvel-Panhuizen, & M. Veldhuis (Eds.), *Proceedings of the Eleventh Congress of the European Society for Research in Mathematics Education* (pp.3580-3587). Utrecht, the Netherlands: Freudenthal Group & Freudenthal Institute, Utrecht University and ERME.
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Aytekin, E., **Ayan, R**. & Isıksal-Bostan, M. (2016, August). An Investigation of 7th and 8th grade students' raesoning and misconception in ordering decimals. In Csikos, C., Rausch, A., & Szitanyi, J. (Eds.), *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education (PME-40) (*Vol.1, pp. 117). Szeged, Hungary: PME.
Paper Presentations in International Conferences
Casto, A., **Ayan, R**., Reinke, L. Stephan, M. (2018, November). Anchors & Bridges: Teaching Math with Context. Presentation at the Annual meeting of the North Carolina Council for Teachers of Mathematics.
Ayan, R. (2018, September). *Real life contexts in mathematics textbooks: Are they helpful in making mathematics more realistic?* Paper presented at the European Conference on Educational Research (ECER 2018), Bolzano, Italy.

Ayan, R. (2018, September). I*nvestigating teachers' experiences and perceptions on the difficulties in the teaching and learning of ratio and proportion in 7th grade: a phenomenological study.* Paper presented at the European Conference on Educational Research (ECER 2018), Bolzano, Italy.
Isiksal-Bostan, M., & **Ayan, R**. (2018, September). *Development of mathematics teaching efficacy belief: the case of novice middle school teachers.* Paper presented at the European Conference on Educational Research (ECER 2018), Bolzano, Italy.
Ayan, R & Stephan, M. (2018, February). *Proportional reasoning for middle school: beyond cross multiplication*. Paper presented at the 2018 NCCTM Central Region Spring Conference, University of North Carolina Greensboro.
Ayan, R., & Isiksal-Bostan, M. (2016, July). *Middle school students' (mis)interpretations in length to volume relationships.* Paper presented at the 13th International Congress on Mathematical Education (ICME-13), Hamburg, Germany.
Aytekin, E., **Ayan, R.** & Isiksal-Bostan, M. (2016, August). *Middle school students' attitudes towards use of technology in mathematics lesson and grade level differences.* Paper presented at the meeting of the 13th International Congress on Mathematical Education (ICME-13), Hamburg, Germany.
Ayan, R., & Isiksal-Bostan, M. (2013, July*). Linearity preponderance on 7th grade students' solution strategies in length-area problems*. Paper presented at the 37th Conference of the International Group for the Psychology of Mathematics Education.
Ayan, R., Akyüz, D. Işıksal, M., & Çakıroğlu, E. (2013, February). *Preservice teacher's reflective practice: guide for learning about teaching.* Paper presented at the Eighth Congress of European Research in Mathematics Education (CERME 8), Antalya, Turkey.
Poster Presentations in International Conferences
Ayan, R. & Stephan, M. (2018, March). *Proportional reasoning for middle school: beyond cross multiplication*. Poster presented at the 2nd annual MDSK Faculty & Graduate Student Research Symposium, University of North Carolina Charlotte.
Ayan, R., Işıksal-Bostan, M., & Stephan, M. (2017, July). *Structuring of within and between ratios with the help of horizontal and vertical ratio tables*. Poster presented at the 41st Annual Meeting of the International Group for the Psychology of Mathematics Education (PME 41)**.**

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Scholarship for Graduate Students, The Scientific and Technological Research Council of Turkey, 2011-2018.
Performance Award (The most successful student in the Ph.D. program of the Department of Elementary Education) Middle East Technical University, Turkey, 2013-2014.
Performance Award (The most successful student in M.S. program of the Department of Elementary Science and Mathematics Education) Middle East Technical University, Turkey, 2011-2012.
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