ON THE ISOMORPHIC CLASSIFICATION OF THE CARTESIAN PRODUCTS OF KÖTHE SPACES

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ABSTRACT

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In 1973, V. P. Zahariuta formed a method to classify the Cartesian products of locally convex spaces by using the theory of Fredholm operators. In this thesis, we gave modifications done in the method of Zahariuta. Then by using them, we studied the isomorphic classifications of Cartesian products of ℓ^p and ℓ^q type Köthe sequence spaces.

Keywords: Bounded Operators, Riesz-Type Operators, Köthe Spaces, Modifications of Zahariuta's Method, Isomoprhism of Cartesian Products of Köthe Spaces

ÖZ

KÖTHE UZAYLARININ KARTEZYEN ÇARPIMLARININ İZOMORFİK SINIFLANDIRILMASI

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1973'te V. P. Zahariuta, Fredholm operatörlerin teorisini kullanarak, yerel konveks uzayların Kartezyen çarpımlarını sınıflandırmak için bir yöntem oluşturdu. Bu tezde, Zahariuta'nın yönteminde yapılan değişiklikleri verdik. Daha sonra onları kullanarak, ℓ^p and ℓ^q türü Köthe dizi uzaylarının Kartezyen çarpımlarının izomorfik sınıflandırılmasını çalıştık.

Anahtar Kelimeler: Sınırlı Operatörler, Riesz-Türü Operatörler, Köthe Uzayları, Zahariuta'nın Yönteminin Değişiklikleri, Köthe Uzaylarının Kartezyen Çarpımlarının İzomorfizması

To my family

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CHAPTER 1

INTRODUCTION

Köthe spaces are considerable in mathematical analysis since there are important spaces isomorphic to some kind of Köthe spaces. The space of all holomorphic functions on the unit disk is an example of such a space. Another example is the space of rapidly decreasing sequences. Moreover, Köthe spaces are parts of Fréchet spaces, which makes them are worth review because the structure of Fréchet spaces are known in detail.

Zahariuta developed a method by using Fredholm operators to study the isomorphic classification of Cartesian products of locally convex spaces. Then his method was modificated to study some kind of Fréchet spaces (see [2] and see [3]). In this study, we aimed to collect these two studies ([2], [3]) together based on the Köthe spaces.

CHAPTER 2

PRELIMINARIES

The following definitions are mainly taken from [2] and [13].

Definiton 2.1 Let E be a vector space over the field $\mathbb{K}$ of real or complex numbers and let τ be a topology on E. Then we call E as a **topological vector space (linear topological space)** if τ is a linear topology on E; that is, if (x,y) $\rightarrow$ x+y and (λ,x) $\rightarrow$ λx are continuous for all x, y$\in$E and for all $\lambda\in\mathbb{K}$.

Definiton 2.2 Let (E, τ) be a linear topological space. Then we call it as a **locally convex space** if τ is a Hausdorff topology and if there is a neighborhood basis of zero which consists of convex sets in E.

Definiton 2.3 Let (E, τ) be a locally convex space. We call E as a **metrizable** space if we have a metric d on E which gives the topology τ with
O_n = {x$\in$E : $d(x,0) < \frac{1}{n}$} (n$\in\mathbb{N}$) form a neighborhood basis of zero.

Definiton 2.4 Let E be a topological vector space and F be a subspace of E. Then we call F as **complemented** if there is a subspace G so that $E = F \bigoplus G$ (topologically) and $F \bigcap G = \{0\}$. In this case, the projection of E onto G is continuous.

Definiton 2.5 We call a locally convex space as a **Fréchet space** if it is complete and metrizable.

Proposition 2.1 [9] If E is a Fréchet space and if we take any closed subspace F of it, then both F and $E\big/F$ are Fréchet spaces.

Definiton 2.6 Let E be a locally convex space. We call a subset A$\subset$E as **bounded** if for every absolutely convex zero neigborhood O, there is a $\rho > 0$ such that A $\subset$ ρO.

Definiton 2.7 Let T : E $\rightarrow$ F be a linear continous operator between locally convex spaces E and F. Then T is called **bounded** (resp. **compact**) if we have a zero neighborhood O in E with the property that T(O) is bounded (resp. relatively compact) in F. T is called **strictly singular** if we restrict T on any arbitrary infinite dimensional closed subspace of E, the restriction is not an isomorphism. T is called **strictly cosingular** if for any infinite dimensional locally convex space G there does not exist continuous surjective operators f: E $\rightarrow$ G and g: F $\rightarrow$ G such that g $\circ$ T = f.

Note that if E or F is a normed space, then T is bounded.

Notation 2.1 Let E and F be locally convex spaces. We denote
(E, F) $\in$ *B*, (E, F) $\in$ *K*, (E, F) $\in$ *SS*, (E, F) $\in$ *SC*, (E, F) $\in$ *BSS*, (E, F) $\in$ *BSC*
if each continous linear operator T : E $\rightarrow$ F is bounded, compact, strictly singular, strictly cosingular, bounded and strictly singular, bounded and strictly cosingular, respectively.

Note that (as stated in [2]) (E, F) $\in$ *K* gives (E, F) $\in$ *BSS* because if we have a precompact operator, then it is also a strictly singular, bounded operator. But the converse statement, in general, may not be true. As an illustration of this, consider p, q$\in[1, \infty)$, p<q, then the identity mapping from the ℓ^p space to ℓ^q space is bounded and strictly singular but it is not compact ([8]).

Definition 2.8 Let E and F be locally convex spaces. Then if each linear continuous operator T : E $\rightarrow$ E which factors over F (that is, if T = $A_1 \circ A_2$ where A_1 : F $\rightarrow$ E and A_2 : E $\rightarrow$ F are linear continous operators) is bounded (respectively, compact) then we say that (E, F) has the **bounded** (respectively, **compact**) **factorization property** and we write (E, F) $\in$ *BF* (respectively, (E, F) $\in$ *KF*).

Definiton 2.9 Let T: E $\rightarrow$ F be an operator between linear topological spaces E and F. Then T is called a **Fredholm operator (near-isomorphism)** if it is an open map with the property that its kernel T^{-1}(0) is of finite dimension and its range T(E) is closed and is of finite codimension. In this case, E and F are called **nearly isomorphic**. The **index** indT of T is the number given by indT = dim(T^{-1}(0)) - codim(T(E)).

Definiton 2.10 Let T: E $\rightarrow$ E be an operator on a linear topological space E. Then we call T as a **Riesz-type operator** if I_E - T is Fredholm, where I_E denotes the identity operator of E.

Definition 2.11 We call a Fréchet space as a **Montel space** if every bounded subset of it is relatively compact.

Theorem 2.1 (See [12], pp. 50-54) If E is any locally convex space, then
(1) E has a bounded neighborhood if and only if it is normable
(2) if E has a precompact neighborhood (in particular, relatively compact neighborhood), it is of finite dimension.

Definition 2.12 [11] Let E and F be normed spaces with closed unit balls O and U, respectively. Let T: E $\rightarrow$ F be continuous and linear. Then T is said to be a **nuclear** operator if there exist continuous linear forms a_i in the dual E' of E and elements y_i in F such that $\sum_{i=1}^{\infty} P_{O^\circ}(a_i)P_U(y_i) < \infty$ such that T is given by T(x) = $\sum_{i=1}^{\infty} a_i(x)y_i$ for all x in E (here P_A denote the Minkowski functional of the set A).

Definition 2.13 Let E be any locally convex space, O be a zero neighborhood in E and P_O be the Minkowski functional of O. Then the **quotient map** T_O : E $\rightarrow$ E/KerP_O is given by T_O(x) := $[x]_O$:= { y $\in$ E: x - y $\in$ KerP_O }.
Let $\lambda > 0$ and Let O and U be zero neighborhoods in E with $U \subset \lambda$O. Then the **linking map** is the continuous map $T_{O,U}$: E/KerP_U $\rightarrow$ E/KerP_O :: $[x]_U \rightarrow [y]_O$.

Definition 2.14 Let E be a locally convex space. If we have that for every zero neighborhood O in E there is a zero neighborhood U in E and $\lambda > 0$ with
$U \subset \lambda$O such that the linking map $T_{O,U}$: E/KerP_U $\rightarrow$ E/KerP_O is nuclear (respectively, compact), then E is called a **nuclear space** (respectively, **Schwartz space**).

Note that each nuclear space is a Schwartz space because each nuclear operator between Banach spaces is also a compact operator between them (see [11], pp. 52 and [5], pp. 479), and note that if E is Schwartz, each bounded set in E is precompact (see [5], pp. 202).

As stated in [13], the fact that any set in a locally conves space is a compact set if and only if it is precompact and complete implies that if a space is Fréchet and Schwartz, then it is also a Montel space. But Fréchet Montel spaces which are not Schwartz spaces exist (see also [5], pp. 223).

Note also that for locally convex spaces E and F, we have already that (E, F) $\in K$ implies (E, F) $\in B$. However, as given in [13], the converse is true if E is a Schwartz space or if F is a Montel space.

Definition 2.15 Let E be a locally convex space. A system β consisting of bounded subsets in E is called a **fundamental system** or a **basis** of bounded subsets of E if each bounded subset in E is in some element of β.

Proposition 2.2 ([6], pp. 63-64) A metrizable locally convex space is a normable space if it admits a countable fundamental system of bounded sets.

Definition 2.16 ([14], pp. 270-271) Let E be any Fréchet space such that it has an arbirtary increasing fundamental systems of seminorms $(||.||_k)$.
We say that E has the **property** (DN) if $\exists$ k $\forall$ n $\exists$ N_0, L > 0 $\forall$ x $\in$ E such that
$||x||_n^2 \leq L||x||_k||x||_{N_0}$.
We say that E has the **property** $(\bar{\Omega})$ if $\forall$ p $\exists$ q $\forall$ k $\exists$ M > 0 $\forall$ y $\in E'$ such that
$(||y||_q^*)^2 \leq M||y||_k^*||y||_p^*$ where $||y||_p^* := \sup_{|x|_p \leq 1} |y(x)|$.

Definition 2.17 A matrix $(a_{ik})_{i,k\in\mathbb{N}}$ of nonnegative real numbers such that for each i there is k with $a_{ik} > 0$ and for all i, k $a_{ik} \leq a_{i,k+1}$ is called a **Köthe matrix**.

Definition 2.18 Let $(a_{ik})_{i,k\in\mathbb{N}}$ be a Köthe matrix and let x=(x_i) denote a sequence of real numbers.
Then the **Köthe sequence space of order p** with $1 \leq$ p $< \infty$ is defined as
$K^p(a_{ik}) :=$ {x=$(x_i) \in \mathbb{K}^{\mathbb{N}}$: $|x|_k := (\sum_{i=1}^{\infty}(|x_i|a_{ik})^p)^{\frac{1}{p}} < \infty$ for all $k \in \mathbb{N}$}
It is also called the ℓ^p**-Köthe space** given with the matrix $(a_{ik})_{i,k\in\mathbb{N}}$.
The **Köthe sequence space of order** ∞ is defined as
$K^\infty(a_{ik}) :=$ {x=$(x_i) \in \mathbb{K}^{\mathbb{N}}$: $|x|_k^\infty := \sup_i(|x_i|a_{ik}) < \infty$ for all $k \in \mathbb{N}$}.
and the **Köthe sequence space of order zero** is defined as
$c_0(a_{ik}) = K^0(a_{ik}) :=$ {x=$(x_i) \in K^\infty(a_{ik})$: $\lim_{i\to\infty} |x_i|a_{ik} = 0$ for all $k \in \mathbb{N}$}.

Note that, as stated in [2], pp. 57, $K^p(a_{ik})$ is a Fréchet space with the topology produced by the system of seminorms { $|.|_k : k \in \mathbb{N}$ }, and $K^\infty(a_{ik})$ is also Fréchet. Being a closed subspace of $K^\infty(a_{ik})$, $K^0(a_{ik})$ is also a Fréchet space.

The **dual** of $K^p(a_{ik})$ is given by
$(K^p(a_{ik}))'$:= { y = (y_i) : there exists k such that $|y|_k^* := (\sum_{i=1}^{\infty}(\frac{|y_i|}{a_{ik}})^q)^{\frac{1}{q}} < \infty$ } where p and q are conjugate, i.e. $\frac{1}{p}$ + $\frac{1}{q}$ = 1.

Also note that every Köthe space has a natural basis (e_j), where e_j = δ_{ji} (which is equal to 1 if i =j, and equal to 0 otherwise).

Theorem 2.2 ([9], [15]) A Köthe space K(a_{ik}) is a nuclear space (respectively, Schwartz space) if and only if for all p $\in \mathbb{N}$ there is q $\in \mathbb{N}$ such that $(\frac{a_{ip}}{a_{iq}}) \in \ell^1$ (respectively, $(\frac{a_{ip}}{a_{iq}}) \in c_0$).
The Köthe space $K^p(a_{ik})$ is a nuclear space if and only if there exists r for all k there is m with the property that the sum $\sum_{i=1}^{\infty}(\frac{a_{ik}}{a_{im}})^r$ is finite.

Definition 2.19 A subspace is called a **basic subspace** if it is generated by a subsequence of the natural basis.

Definition 2.20 For 1 $\leq$ p $< \infty$, consider the ℓ^p-Köthe space $K^p(a_{ik})$. If (j(i)) is a strictly increasing subsequence of (i), then we say that the Köthe subspace $K^p(a_{j(i)k})$ is a **basic subspace** of $K^p(a_{ik})$. Note that each basic subspace of a Köthe space is a complemented space ([13]).

We know that (see [9], pp. 329) $K^p(a_{ik})$ is not a Montel space if and only if there is an integer k_0 and a subsequence (i_n) of the sequence (i) with for all k there exists C = C(k) $>$ 0 such that for all n $a_{i_n}k \leq Ca_{i_nk_0}$. So we have that:

Proposition 2.3 An ℓ^p-Köthe space is not Montel if and only if it contains a basic subspace which is isomorphic to the space ℓ^p.

Proposition 2.4 ([2], [13]) For 1 $\leq$ p $<$ q $< \infty$, consider two Köthe sequence spaces $K^p(a_{ik})$ and $K^q(b_{ik})$. If $K^p(a_{ik}) \simeq K^q(b_{ik})$, then $K^p(a_{ik})$ and so $K^q(b_{ik})$ are nuclear spaces.

Proof: $K^p(a_{ik})$ is a Schwartz space because each linear continuous operator from ℓ^q to ℓ^p is compact. Since $K^p(a_{ik}) \simeq K^q(b_{ik})$, then we have an isomorphism
T: $K^p(a_{ik}) \to K^q(b_{ik})$. So, for each k find r, m = m(r), A, B such that $|x|_k \leq$ A$|Tx|_r$ $\leq$ B$|x|_m$ for all x $\in K^p(a_{ik})$. Since $K^p(a_{ik})$ is a Schwartz space, we can pick m sufficiently big in order for $\frac{a_{ik}}{a_{im}}$ to converge zero. By reordering the terms of $(\frac{a_{ik}}{a_{im}})$,

we suppose that it is a decreasing sequence.

Firstly, consider the case p $<$ 2. By [8], Vol. 2, pp. 72, we know that ℓ^q space has type s = min(2, q). Then for any n, there is γ_i = 1 or -1 (1$\leq$i$\leq$ n) and there is a constant M with

$\frac{a_{nk}}{a_{nm}} n^{\frac{1}{p}} \leq (\sum_{i=1}^{n} (\frac{a_{ik}}{a_{im}})^p)^{\frac{1}{p}} = |\sum_{i=1}^{n} \gamma_i \frac{e_i}{a_{im}}|_k \leq A| \sum_{i=1}^{n} \gamma_i \frac{Te_i}{a_{im}}|_r \leq$ MB$(\sum_{i=1}^{n} (\frac{|Te_i|_r}{a_{im}})^s)^{\frac{1}{s}} \leq$ MB$n^{\frac{1}{s}}$.

Then, $\frac{a_{nk}}{a_{nm}} \leq$ MB$n^{\frac{1}{s}-\frac{1}{p}}$ = MB$n^{\frac{p-s}{sp}}$. Thus, for any $\beta > \frac{sp}{p-s}$, the sequence $(\frac{a_{ik}}{a_{im}}) \in \ell^\beta$. Hence, Theorem 2.2 implies that $K^p(a_{ik})$ is a nuclear space in the case p $<$ 2.

Now, suppose p $\geq$ 2. Then ℓ^p has cotype max(2, p) = p. So, $K^q(a_{ik})$ is a nuclear space. Consider the isomorphism T: $K^p(a_{ik}) \to K^q(b_{ik})$. Then T^{-1} is also an isomorphism. So, for each k, there are r, m, A, B with $|x|_k \leq A|T^{-1}x|_r \leq B|x|_m$. Again, by reordering the terms of $(\frac{b_{ik}}{b_{im}})$, we suppose that it is a decreasing sequence. For any n, there is γ_i = 1 or -1 (1$\leq$i$\leq$ n) and there is a constant M with

$\frac{b_{nk}}{b_{nm}} n^{\frac{1}{p}} \leq (\sum_{i=1}^{n} (\frac{b_{ik}}{b_{im}})^p)^{\frac{1}{p}} = (\sum_{i=1}^{n} |\frac{e_i}{b_{im}}|_k^p)^{\frac{1}{p}} \leq A(\sum_{i=1}^{n} |\frac{T^{-1}e_i}{b_{im}}|_r^p)^{\frac{1}{p}} \leq MA| \sum_{i=1}^{n} \gamma_i \frac{T^{-1}e_i}{b_{im}}|_r$

$\leq$ MB$| \sum_{i=1}^{n} \gamma_i \frac{e_i}{b_{im}}|_m$ = MB$n^{\frac{1}{q}}$. Then, $\frac{b_{nk}}{b_{nm}} \leq$ MB$n^{\frac{1}{q}-\frac{1}{p}}$ = MB$n^{\frac{p-q}{qp}}$. Thus, for any $\beta > \frac{qp}{p-q}$, the sequence $(\frac{b_{ik}}{b_{im}}) \in \ell^\beta$. Hence, Theorem 2.2 implies that $K^p(a_{ik})$ is a nuclear space in the case p $\geq$ 2.

Definiton 2.21 (See [3]) Let $(a_{ik})_{i,k\in\mathbb{N}}$ be a Köthe matrix.

Then it is called **(d_1)-kind Köthe matrix** if $\exists n_0 \forall k \exists m, A : a_{ik}^2 \leq A a_{in_0} a_{im}$ $(\forall i \in N)$, and is called **(d_2)-kind Köthe matrix** if $\forall k \exists n_0 \forall m \exists B : B a_{in_0}^2 \geq a_{ik} a_{im}$ $(\forall i \in N)$. In this case, the corresponding spaces are called (d_1) and (d_2) **type Köthe spaces**, respectively.

Proposition 2.5 (See [20], [3]). If $K^p(a_{ik})$ is a (d_2)-type Köthe space and $K^q(b_{ik})$ is a (d_1)-type Köthe space, then we have that $(K^p(a_{ik}), K^q(b_{ik})) \in$ *B*.

Proof: In general, by depending on Vogt's results (in [17], Satz 6.2 and Prop. 5.3), since $K^p(a_{ik})$ and $K^q(b_{ik})$ are Fréchet spaces having the conditions ($\bar{\Omega}$) and (DN), respectively, then ($K^p(a_{ik})$, $K^q(b_{ik})$) $\in$ *B* because the previous definition gives that $(d_2) \Rightarrow$ ($\bar{\Omega}$) and $(d_1) \Rightarrow$ (DN).

As a special case, let T: $K^1(a_{ik}) \to K^1(b_{ik})$ be a linear continuous operator which is given by the matrix (t_{ik}). So, for all p there is q and $C(p) > 0$ with

$|(t_{ik})|_p = |Te_k|_p \leq C(p)|e_k|_q$, which means that $\sum_{i=1}^{\infty} |t_{ik}| \frac{b_{ip}}{a_{kq}}$ <C(p)< $+\infty$.

In order to show that T is bounded, we will find some q_0 such that $|Te_k|_p \leq M(p)|e_k|_{q_0}$ holds for all p for some $M(p)$, that is, $\sum_{i=1}^{\infty} |t_{ik}| \frac{b_{ip}}{a_{kq_0}} < M(p) < +\infty$ for all p for some $M(p)$. Since $K^1(b_{ik})$ is a (d_1)-type Köthe space, then $\exists p_1 \, \forall p \, \exists p_2 = p_2(p) \, \exists B(p) > 0$ such that $b_{ip}^2 \leq B(p) b_{ip_1} b_{ip_2}$ for $i \geq i_0(p)$ for some $i_0(p)$. Since $K^1(a_{ik})$ is a (d_2)-Köthe space, for $q = q(p_1) \, \exists q_0 \, \forall q_2 = q_2(p_2) \, \exists A(p_2)$>0 such that $A(p_2) a_{kq_0}^2 \geq a_{kq_1} a_{kq_2}$ for $k \geq k_0(p)$ for some $k_0(p)$. Therefore, by using Hölder's inequality, $\sum_{i=1}^{\infty} |t_{ik}| \frac{b_{ip}}{a_{kq_0}} \leq B(p)^{\frac{1}{2}} A(p_2)^{\frac{1}{2}} \sum_{i=1}^{\infty} (|t_{ik}| \frac{b_{ip_1}}{a_{kq_1}})^{\frac{1}{2}} (|t_{ik}| \frac{b_{ip_2}}{a_{kq_2}})^{\frac{1}{2}} \leq B(p)^{\frac{1}{2}} A(p_2)^{\frac{1}{2}} (\sum_{i=1}^{\infty} |t_{ik}| \frac{b_{ip_1}}{a_{kq_1}})^{\frac{1}{2}}$ $(\sum_{i=1}^{\infty} |t_{ik}| \frac{b_{ip_2}}{a_{kq_2}})^{\frac{1}{2}} \leq B(p)^{\frac{1}{2}} A(p_2)^{\frac{1}{2}} C(p_1)^{\frac{1}{2}} C(p_2)^{\frac{1}{2}} < \infty$ for all p for some $C(p_1), C(p_2)$>0 by continuity. Take $M(p)$ = $B(p)^{\frac{1}{2}} A(p_2)^{\frac{1}{2}} C(p_1)^{\frac{1}{2}} C(p_2)^{\frac{1}{2}}$. So, we get
$\sum_{i=1}^{\infty} |t_{ik}| \frac{b_{ip}}{a_{kq_0}} < M(p) < \infty$ for all p. Hence, $(K^1(a_{ik}), K^1(b_{ik})) \in$ ***B***.

Definition 2.22 A locally convex space E is said to be a **Mackey-complete space** if for each absolutely convex, closed and bounded subset F of it, the linear span sp(F) of F is Banach with the unit ball F.
Note that a locally convex space which is sequentially complete is also Mackey-complete. Also, a Fréchet space is Mackey-complete [2].

The next proposition comes from [18] [19]:
Proposition 2.6 The set of strictly singular and bounded operators between Mackey-complete spaces generates an ideal of Riesz type operators.

Definition 2.23 [5] Suppose that E_i is a topological vector space for each i ∈ I, where I is a directed set by an order relation $\leq$. Suppose that, for every i, k ∈ I with i ≤ k, there is a linear continuous operator $T_{ik} : E_k \to E_i$ with the properties that T_{ii} = I_{E_i}, identity map on E_i, for each i, and T_{ik} = $T_{ij} \circ T_{jk}$ for i, j, k ∈ I with i ≤ j, j ≤ k. Then we call the system $(E_i, T_{ik})_{(I,\leq)}$ as a **projective system** of topological vector spaces and we call the subspace E ⊂ $\prod_{i \in I} E_i$ such that
E = { $(x_i) \in \prod_{i \in I} E_i$: T_{ik} (x_k) = x_i for all i, k ∈ I with i ≤ k } as the **projective limit** of the system $(E_i, T_{ik})_{(I,\leq)}$ and we denote it by E = $proj_i E_i$. We say that the projective limit E = $proj_i E_i$ is **reduced** if the operator T_k : E → E_k has a dense range for each k ∈ I.

Note that any locally convex space E is a dense subspace of a projective limit of Banach spaces. If the space E is also complete, then E is equal to the reduced projective limit since the set of seminorms $\{ |.|_k : k \in \mathbb{N} \}$ on E can be seen as directed by taking max$\{ |.|_{k_1}, |.|_{k_2} \}$ as a seminorm on E [7] .

Remark 2.1 [7] We can see a Köthe space $K^p(a_{ik})$ as a reduced projective limit.
Consider the case $1 \leq$ p $< \infty$. Define I_k := { i $\in \mathbb{N}$: $a_{ik} \neq 0$ } for each k $\in \mathbb{N}$. Since $(a_{ik})_{i,k\in\mathbb{N}}$ is a Köthe matrix, then $a_{ik} \leq a_{i,k+1}$ for all i, k $\in \mathbb{N}$. So, we have that $I_k \subset I_{k+1}$ for all k $\in \mathbb{N}$. Thus, $\mathbb{N} = \bigcup_{k\in\mathbb{N}} I_k$.
Now consider that Ker $|.|_k$ = { x = $(x_i) \in K^p(a_{ik})$: x_i = 0 for all i $\in I_k$ }.
Also, we have that $\ell^p(a_{ik})$ = { x = $(x_i) \in \mathbb{R}^{\mathbb{N}}$: $|xa_{ik}|_p = (\sum\limits_i (a_{ik}|x_i|)^p)^{\frac{1}{p}} < \infty$ }
= { x = $(x_i) \in \mathbb{R}^{I_k}$: $|xa_{ik}|_p = (\sum\limits_i (a_{ik}|x_i|)^p)^{\frac{1}{p}} < \infty$ }. Then for each k $\in \mathbb{N}$, $\mathbb{R}^{I_k}$ is dense in $\ell^p(a_{ik})$ because $\ell^p(a_{ik})$ is a subspace of $\mathbb{R}^{I_k}$. So, $\mathbb{R}^{\mathbb{N}} \subset K^p(a_{ik})$, since
$\mathbb{R}^{\mathbb{N}} \subset K^p(a_{ik})$ / Ker $|.|_k \simeq$ { x = $(x_i) \in K^p(a_{ik})$: x_i = 0 for all i $\in I_k$ } $\subset \ell^p(a_{ik})$.
This shows that $E_k := (K^p(a_{ik})/Ker|.|_k)^C \simeq \ell^p(a_{ik})$ for each k $\in \mathbb{N}$.
Then by completeness, $K^p(a_{ik}) = proj_k E_k$. Since $E_k \simeq \ell^p(a_{ik})$ for each k $\in \mathbb{N}$, we have that $K^p(a_{ik}) = proj_k \ell^p(a_{ik})$.
By using a similar argument, we can see also that $K^0(a_{ik}) = proj_k c_0(a_{ik})$.

Lemma 2.1 [3] Consider the Köthe space K(a_{ik}). If A is a bounded subset of K(a_{ik}), then for any k_0 and $\epsilon > 0$ there is a Banach basic subspace B of K(a_{ik}) such that A lies in B + ϵU_{k_0}, where U_{k_0} is given by U_{k_0} = {x $\in$ K(a_{ik}) : $|x|_{k_0} \leq 1$}.

Proof: We prove the theorem for the Köthe space of order 1.
A is given bounded. So suppose A = {x $\in$K(a_{ik}) : $|x|_k = \sum_i a_{ik}|x_i| \leq \delta_k$ for all k} for some sequence (δ_k) of nonnegative numbers. Then pick $\delta_k \nearrow \infty$. Thus, $\frac{a_{ik}}{\delta_k} \to 0$ for all i.
Choose $\alpha_i = \sum\limits_{k=1}^{\infty} \frac{a_{ik}}{2^k \delta_k}$ for all i.
Then for all x $\in$ A, $\sum\limits_{i=1}^{\infty} \alpha_i |x_i| = \sum\limits_{i=1}^{\infty} (\sum\limits_{k=1}^{\infty} \frac{a_{ik}}{2^k \delta_k})|x_i| = \sum\limits_{k=1}^{\infty} \frac{1}{2^k} (\sum_i \frac{a_{ik}}{\delta_k}|x_i|) \leq 1$.
Now fix $\epsilon > 0$ and define B = $[e_i : \epsilon\alpha_i \leq a_{ik_0}]$ and D = $[e_i : \epsilon\alpha_i > a_{ik_0}]$ where [.] denote the closed linear span of the corresponding vectors. Then B is a Banach space.
Then for any x $\in$ A $\cap$ D,
$|x|_{k_0} = \sum\limits_{i=1}^{\infty} a_{ik_0}|x_i| < \sum\limits_{i=1}^{\infty} \epsilon\alpha_i|x_i| < \epsilon \sum\limits_{i=1}^{\infty} \alpha_i|x_i| < \epsilon$. Hence, A $\in$ B + ϵU_{k_0}.
For p > 1 the proof can be done in a similar way.

Remark 2.2 Assume that A is a compact subset of the Köthe space K(a_{ik}).So, for all k_0 and $\epsilon > 0$, a basic subspace B of finite dimension exists with the property that A lies in B + ϵU_{k_0}.

Theorem 2.3 [3] Suppose that E is a Köthe space and that T: E $\rightarrow$ E is a bounded (respectively, compact) operator. Then there exist complementary basic subspaces X and Y in E such that
(1) X is a Banach (respectively, finite dimensional) space, and
(2) if π_Y is the canonical projection onto Y and i_Y is an embedding into E, then the operator 1_Y - $\pi_Y T i_Y$ is an automorphism of Y.

Proof Let we have a fundamental system of norms in E, denoted by $|.|_p$, where p is in $\mathbb{N}$. T is given a bounded operator. So, there is a k_0 such that T(U_{k_0}) is bounded in E, where U_{k_0} = $\{$x $\in$ E : $|x|_{k_0} \leq 1\}$. Therefore, we have that for all k there is δ_k such that $|Tx|_k \leq \delta_k |x|_{k_0}$. Then with the help of Lemma 2.1 (respectively, Remark 2.2), there is a Banach (respectively, a finite dimensional) basic subspace X with the property that T(U_{k_0}) lies in X + $\frac{1}{2} U_{k_0}$. Now let Y be the basic subspace such that Y is complementary to X. Take P = $\pi_Y T i_Y$. Then P is from Y to Y. Thus, for all x$\in$Y, we have that $|Px|_{k_0} \leq \frac{1}{2}|x|_{k_0}$. Now for any x$\in$Y take the series Sx = $\sum_{i=0}^{\infty} P^i x$. It is a convergent series since, for each k$\in\mathbb{N}$, $|P^i x|_k \leq \delta_k |P^{i-1}x|_{k_0} \leq \delta_k (\frac{1}{2})^{i-1} |x|_{k_0}$ for all i$\in\mathbb{N}$. Thus Sx defines a linear continuous operator from Y to Y, by Banach-Steinhaus Theorem.
Also, (1_Y - P)Sx = S(1_Y - P)x = x. Thus, S is the inverse of 1_Y - P. This shows that 1_Y - P = 1_Y - $\pi_Y T i_Y$ is an automorphism.

CHAPTER 3

MODIFICATIONS OF THE METHOD OF ZAHARIUTA

Notation 3.1 ([2], [13]) Let E be a locally convex space and let s be any integer. Then if s≥0, $E^{(s)}$ denotes a subspace of E with codimension s, and if s<0, it denotes a product of the kind E x F, where the dimension of F is -s.

In [20], by using the Fredholm operator theory, Zahariuta developed a way to classify isomorphically Cartesian products of locally convex spaces. His result is given by:

Theorem 3.1[20] Let E_1, E_2, F_1, F_2 be locally conves spaces with the properties that (E_1, F_2) $\in K$ and (F_1, E_2) $\in K$. Then E_1 x $E_2 \simeq F_1$ x F_2 if and only if there is an integer s such that $F_1 \simeq E_1^{(s)}$ and $F_2 \simeq E_2^{(-s)}$.

We give the mofidications of Zahariuta's method as in [2] and in [3].

Denote an operator T = $(T_{mn}) : E_1 \times E_2 \to F_1 \times F_2$ with its corresponding
2×2 matrix, whose entries are
$T_{11} : E_1 \to F_1, T_{12} : E_2 \to F_1, T_{21} : E_1 \to F_2, T_{22} : E_2 \to F_2$.

Lemma 3.1 (See [2]) Let E_1, E_2, F_1, F_2 be topological vector spaces.
If $E_1 \times E_2 \simeq F_1 \times F_2$ and $E_1 \simeq F_1$, then $E_2 \simeq F_2$.

Proof: Let T = $(T_{mn}) : E_1 \times E_2 \to F_1 \times F_2$ be an isomorphism.
Denote the inverse of T by T^{-1} = M = (M_{mn}).
Then consider $M_{22} : F_2 \to E_2$ and $T_{22} - T_{21}T_{11}^{-1}T_{12} : E_2 \to F_2$.
Denote H = $T_{22} - T_{21}T_{11}^{-1}T_{12}$.

Then consider T ∘ M = I, that is,

$$\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = \begin{bmatrix} T_{11}M_{11} + T_{12}M_{21} & T_{11}M_{12} + T_{12}M_{22} \\ T_{21}M_{11} + T_{22}M_{21} & T_{21}M_{12} + T_{22}M_{22} \end{bmatrix} = \begin{bmatrix} I_{F_1} & 0 \\ 0 & I_{F_2} \end{bmatrix}$$

and consider M ∘ T = I, that is,

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} = \begin{bmatrix} M_{11}T_{11} + M_{12}T_{21} & M_{11}T_{12} + M_{12}T_{22} \\ M_{21}T_{11} + M_{22}T_{21} & M_{21}T_{12} + M_{22}T_{22} \end{bmatrix} = \begin{bmatrix} I_{E_1} & 0 \\ 0 & I_{E_2} \end{bmatrix}$$

$T_{11}M_{12} + T_{12}M_{22} = 0$ implies that
$HM_{22} = T_{22}M_{22} - T_{21}T_{11}^{-1}T_{12}M_{22} = T_{22}M_{22} + T_{21}M_{12} = I_{F_2}$
Similarly, $M_{21}T_{11} + M_{22}T_{21} = 0$ implies that
$M_{22}H = M_{22}T_{22} - M_{22}T_{21}T_{11}^{-1}T_{12} = M_{22}T_{22} + M_{21}T_{12} = I_{E_2}$
Thus, $E_2 \simeq F_2$.

In [2], a modification of the Zahariuta's method (see [17]) is derived by using Riesz type operators instead of compact operators, which is given in the next theorem and we call it the 1^{st} **Modification Theorem**.

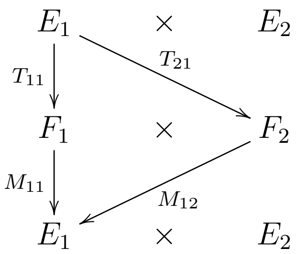
Theorem 3.2 (See [2]) Suppose that E_1, E_2, F_1, F_2 are linear topological spaces with the property that $E_1 \times E_2 \simeq F_1 \times F_2$ and suppose that each operator acting in E_1 and factoring over F_2 is a Riesz type operator. In this case, we have a finite dimensional subspace L_1 in E_1 and complemented subspaces X_1 in E_1 and Y_1 in F_1 such that $E_1 \simeq X_1 \times L_1$, $F_1 \simeq X_1 \times Y_1$ and $Y_1 \times F_2 \simeq L_1 \times E_2$.

Proof: Since $E_1 \times E_2 \simeq F_1 \times F_2$, then there is an isomophism
T = $(T_{mn}) : E_1 \times E_2 \to F_1 \times F_2$. Denote the inverse of T by T^{-1} = M = (M_{mn}). Then T and M are 2×2 matrices with entries T_{mn} and M_{mn} (m, n = 1, 2) such that each of which is an operator acting between factors of the cartesian product, that is

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}, \quad M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix},$$

where $T_{mn} : E_n \longrightarrow F_m$ and $M_{mn} : F_n \longrightarrow E_m$ for m,n = 1, 2.

Now look at the following schema:



Then we get M ∘ T = I, that is

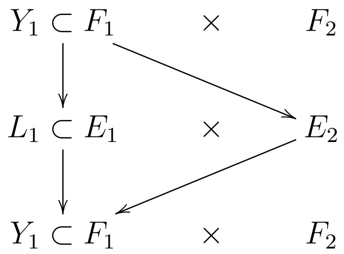
$$\begin{bmatrix} M_{11}T_{11} + M_{12}T_{21} & M_{11}T_{12} + M_{12}T_{22} \\ M_{21}T_{11} + M_{22}T_{21} & M_{21}T_{12} + M_{22}T_{22} \end{bmatrix} = \begin{bmatrix} I_{E_1} & 0 \\ 0 & I_{E_2} \end{bmatrix}$$

So we get $M_{11}T_{11} + M_{12}T_{21} = I_{E_1}$ and $M_{21}T_{12} + M_{22}T_{22} = I_{E_2}$.

Consider $M_{11}T_{11} + M_{12}T_{21} = I_{E_1}$. $M_{11}T_{11} = I_{E_1} - M_{12}T_{21}$ is a Fredholm operator because $M_{12}T_{21}$ is a Riesz type operator factoring over F_2. So if we choose L_1 = ker$M_{11}T_{11}$, then L_1 is a finite dimensional subspace of E_1, and if H = $M_{11}T_{11}(E_1)$, then H is a closed and finite codimensional subspace of E_1. Thus, L_1 and H are complemented in E_1. Take X_1 as a complementary subspace of L_1 in the space E_1 and π_H as the continuous projection onto H. The operator $M_{11}T_{11}|_{X_1} : X_1 \longrightarrow H$ is an isomorphism. So, T_{11} maps X_1 into $T_{11}(X_1) \subset F_1$ isomorphically.
Consider the operator $A = T_{11}(M_{11}T_{11}|_{X_1})^{-1}\pi_H M_{11} : F_1 \longrightarrow F_1$.
$A^2 = T_{11}(M_{11}T_{11}|_{X_1})^{-1}\pi_H M_{11}T_{11}(M_{11}T_{11}|_{X_1})^{-1}\pi_H M_{11}$
$= T_{11}(M_{11}T_{11}|_{X_1})^{-1}\pi_H(M_{11}T_{11}(M_{11}T_{11}|_{X_1})^{-1}\pi_H)M_{11}$
$= T_{11}(M_{11}T_{11}|_{X_1})^{-1}\pi_H M_{11}$
$= A$ because $(M_{11}T_{11}(M_{11}T_{11}|_{X_1})^{-1}\pi_H)$ is the identity operator.
So, $A = T_{11}(M_{11}T_{11}|_{X_1})^{-1}\pi_H M_{11} : F_1 \longrightarrow F_1$ is the continuous projection onto $T_{11}(X_1)$. Thus, $T_{11}(X_1)$ is a complemented subspace of F_1.
Take $Y_1 = A^{-1}(0)$ = ker$A$ as the corresponding complemented subspace. So, we get $E_1 \simeq X_1 \times L_1$, $X_1 \simeq T_{11}(X_1)$, $F_1 = T_{11}(X_1) \bigoplus Y_1 \simeq X_1 \bigoplus Y_1 \simeq X_1 \times Y_1$.
Then $E_1 \times E_2 \simeq F_1 \times F_2$ implies that $X_1 \times L_1 \times E_2 \simeq T_{11}(X_1) \times Y_1 \times F_2$.
By using Lemma 3.1 to $X_1 \times (L_1 \times E_2) \simeq T_{11}(X_1) \times (Y_1 \times F_2)$ we reach the fact that $L_1 \times E_2 \simeq Y_1 \times F_2$.

Corollary 3.1 (See [2]) Suppose that E_1, E_2, F_1, F_2 are linear topological spaces with the property that $E_1 \times E_2 \simeq F_1 \times F_2$, suppose every operator acting in E_1 and factoring over F_2 is a Riesz type operator, and suppose also every operator acting in F_1 and factoring over E_2 is a Riesz type operator. In this case, we have a finite dimensional subspace Y_1 in F_1 and a complemented subspace L_1 in E_1 such that $F_1 \simeq E_1^{(s)}$ and $F_2 \simeq E_2^{(-s)}$, where s = dimL_1 - dimY_1.

Proof: By Theorem 3.2, there exist a finite dimensional subspace L_1 in E_1 and complemented subspaces X_1 in E_1 and Y_1 in F_1 such that $E_1 \simeq X_1 \times L_1$, $F_1 \simeq X_1 \times Y_1$ and $Y_1 \times F_2 \simeq L_1 \times E_2$. Since every operator acting in F_1 factoring over E_2 is Riesz type and since Y_1 is a subspace of F_1, then every operator acting in Y_1 factoring over E_2 is Riesz type.
So, we can apply Theorem 3.2 to $Y_1 \times F_2 \simeq L_1 \times E_2$.



Then we find a finite dimensional subspace Y_3 in Y_1 and complemented subspaces Y_2 in Y_1 and L in L_1 such that $Y_1 \simeq Y_2 \times Y_3$, $L_1 \simeq Y_2 \times L$ and $Y_3 \times F_2 \simeq L \times E_2$. Since Y_2 is a subspace of L_1 and L_1 is finite dimensional, then Y_2 is also finite dimensional. Since $Y_1 \simeq Y_2 \times Y_3$ and since Y_2 and Y_3 are finite dimensional, then Y_1 is also finite dimensional.
Since $E_1 \simeq X_1 \times L_1$, $F_1 \simeq X_1 \times Y_1$ and since L_1 and Y_1 are finite dimensional, we have that $F_1 \simeq E_1^{(s)}$. Also, $Y_1 \times F_2 \simeq L_1 \times E_2$ and again L_1, Y_1 are finite dimensional implies that $F_2 \simeq E_2^{(-s)}$, where s = dimL_1 - dimY_1.

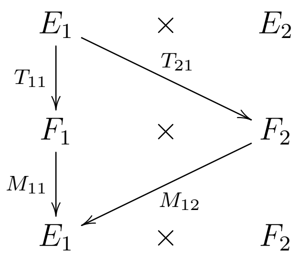
In [3], another modificated case of Zahariuta's method (see [20]) is obtained with the help of boundedness property instead of compactness property. This is given in the next theorem and we call it the 2^{nd} **Modification Theorem**.

Theorem 3.3 (see [3]) Suppose that E_1 is a Köthe space and E_2, F_1, F_2 are any linear topological spaces. If $E_1 \times E_2 \simeq F_1 \times F_2$ and if (E_1, F_2) $\in$ *BF*, then there exist complementary basic subspaces X_1 and Y_1 in E_1 and complementary subspaces X_2 and Y_2 in F_1 such that Y_1 is a Banach space and $X_2 \simeq X_1$, $Y_1 \times E_2 \simeq Y_2 \times F_2$. Furthermore, if (F_1, E_2) $\in$ *BF*, then Y_2 is also a Banach space.

Proof: Since $E_1 \times E_2 \simeq F_1 \times F_2$, then there is an isomophism T = $(T_{mn}) : E_1 \times E_2 \to F_1 \times F_2$. Denote the inverse of T by T^{-1} = M = (M_{mn}). Then T and M are 2×2 matrices with entries T_{mn} and M_{mn} (m, n = 1, 2) such that each of which is an operator acting between factors of the cartesian product, that is

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}, \quad M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

where $T_{mn} : E_n \longrightarrow F_m$ and $M_{mn} : F_n \longrightarrow E_m$ for m,n = 1, 2.
Now look at the following schema:



Then we get M $\circ$ T = I, that is

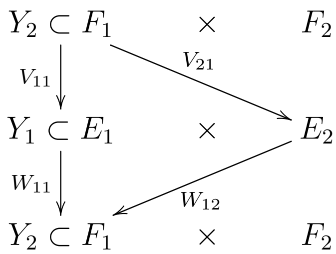
$$\begin{bmatrix} M_{11}T_{11} + M_{12}T_{21} & M_{11}T_{12} + M_{12}T_{22} \\ M_{21}T_{11} + M_{22}T_{21} & M_{21}T_{12} + M_{22}T_{22} \end{bmatrix} = \begin{bmatrix} I_{E_1} & 0 \\ 0 & I_{E_2} \end{bmatrix}$$

So we get $M_{11}T_{11} + M_{12}T_{21} = I_{E_1}$, where $M_{12}T_{21}$ is bounded. Then Theorem 2.3 implies that there are complementary basic subspaces X_1 and Y_1 in E_1 with the property that Y_1 is a Banach space and $\pi_{X_1}M_{11}T_{11}i_{X_1}$ is an automorphism of X_1. Then we have a projection P = $T_{11}(\pi_{X_1}M_{11}T_{11}i_{X_1})^{-1}\pi_{X_1}M_{11}$ on F_1. Now take X_2 = P(F_1) and Y_2 = Ker P = $P^{-1}(0)$. So, $X_2 = T_{11}(X_1)$ and the restriction $T_{11}|_{X_1}$ of T_{11} on X_1 is an isomoprhism of the spaces X_1 and X_2. Then by the Lemma 3.1 we obtain that $Y_1 \times E_2 \simeq Y_2 \times F_2$.
Now suppose also that (F_1, E_2) $\in$ *BF*. Then since $Y_2 \subset F_1$, we have directly that (Y_2, E_2) $\in$ *BF*. Since $Y_1 \times E_2 \simeq Y_2 \times F_2$, then we have an isomorphism V = $(V_{mn}) : Y_2 \times F_2 \to Y_1 \times E_2$. Denote the inverse of V by V^{-1} = W = (W_{mn}).

Then

$$V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}, \quad W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}$$

Similarly, look at the following schema:



where $V_{11} : Y_2 \longrightarrow Y_1$, $V_{12} : F_2 \longrightarrow Y_1$, $V_{21} : Y_2 \longrightarrow E_2$, $V_{22} : F_2 \longrightarrow E_2$
and $W_{11} : Y_1 \longrightarrow Y_2$, $W_{12} : E_2 \longrightarrow Y_2$, $W_{21} : Y_1 \longrightarrow F_2$, $W_{22} : E_2 \longrightarrow F_2$
Then we get W ∘ V = I, that is

$$\begin{bmatrix} W_{11}V_{11} + W_{12}V_{21} & W_{11}V_{12} + W_{12}V_{22} \\ W_{21}V_{11} + W_{22}V_{21} & W_{21}V_{12} + W_{22}V_{22} \end{bmatrix} = \begin{bmatrix} I_{Y_2} & 0 \\ 0 & I_{F_2} \end{bmatrix}$$

So, we get $W_{11}V_{11} + W_{12}V_{21} = I_{Y_2}$. Since the operator $W_{11}V_{11}$ factors through the Banach space Y_1, it is bounded; and since the operator $W_{12}V_{21}$ factors through E_2, it is also bounded. So, I_{Y_2} is bounded. This means that Y_2 is a Banach space.

Remark 3.1 By the proof of Theorem 3.3 and Theorem 2.3, it follows that
(1) if $(E_1, F_2) \in$ *KF*, then we can choose Y_1 finite dimensional, and
(2) moreover, if $(F_1, E_2) \in$ *KF*, then we can also choose Y_2 finite dimensional.
Then we get a known result (see [20], [2]). we gave it as Theorem 3.1.

CHAPTER 4

ISOMORPHISM OF CARTESIAN PRODUCTS OF KÖTHE SPACES

4.1 Applications of the 1^{st} Modification Theorem

To apply Corollary 3.1, we must have the following lemma:

Lemma 4.1 [2] Let E = $proj_k E_k$ and F = $proj_m F_m$ be projective limits of normed spaces with for all k, m (E_k, F_m) $\in$ *SS*. If T : E $\rightarrow$ F is bounded, then it is a strictly singular operator.

Proof: Suppose that the result does not hold; that is, suppose that T : E $\rightarrow$ F is bounded but not strictly singular. So there is an infinite dimensional closed subspace M of E such that the restriction $T|_M$ of T onto M is an isomorphism. Since $T|_M^{-1}$ is continuous, $\forall$ k $\exists$ m(k), A_k such that $|x|_k \leq A_k |Tx|_{m(k)}$ for all x $\in$ M. Also since T is bounded, $\exists$ k_0 $\forall$ m $\exists$ B_m such that $|Tx|_m \leq B_m |x|_{k_0}$ for all x $\in$ E. So we have that $|x|_{k_0} \leq A_{k_0} |Tx|_{m(k_0)} \leq A_{k_0} B_{m(k_0)} |x|_{k_0}$ for all x $\in$ M. So, we can consider T : $E_{k_0} \rightarrow F_{m(k_0)}$ whose restriction to M is an isomorphism. However, we have (E_{k_0}, $F_{m(k_0)}$) $\in$ SS, which is a contradiction. Hence, T : E $\rightarrow$ F is strictly singular.

The next theorem is a generalization of Theorem 2 in [2].

Theorem 4.1 [16] Let p $\neq \tilde{q}$, q $\neq \tilde{p}$, 1 $\leq$ p, q, $\tilde{p}$, $\tilde{q} < \infty$, let (a_{ik}), $(\tilde{a}_{ik})$ be (d_2)-type Köthe matrices and let (b_{ik}), $(\tilde{b}_{ik})$ be (d_1)-type Köthe matrices. Then the following conditions are equivalent:
(1) $K^p(a_{ik}) \times K^q(b_{ik}) \simeq K^{\tilde{p}}(\tilde{a}_{ik}) \times K^{\tilde{q}}(\tilde{b}_{ik})$
(2) there is an integer s such that $K^{\tilde{p}}(\tilde{a}_{ik}) \simeq (K^p(a_{ik}))^{(s)}$ and $K^{\tilde{q}}(\tilde{b}_{ik}) \simeq (K^q(b_{ik}))^{(-s)}$.

Proof: Suppose that $K^p(a_{ik}) \times K^q(b_{ik}) \simeq K^{\tilde{p}}(\tilde{a}_{ik}) \times K^{\tilde{q}}(\tilde{b}_{ik})$.
Proposition 2.5 gives that $(K^p(a_{ik}), K^{\tilde{q}}(\tilde{b}_{ik})) \in$ *B*, $(K^{\tilde{p}}(\tilde{a}_{ik}), K^q(b_{ik})) \in$ *B* because (a_{ik}), $(\tilde{a}_{ik})$ are (d_2)-type Köthe matrices and (b_{ik}), $(\tilde{b}_{ik})$ are (d_1)-type Köthe matrices.
Consider $K^p(a_{ik}) = proj_k \ell^p(a_{ik})$, $K^{\tilde{p}}(\tilde{a}_{ik}) = proj_k \ell^{\tilde{p}}(\tilde{a}_{ik})$, $K^q(b_{ik}) = proj_k \ell^q(b_{ik})$, $K^{\tilde{q}}(\tilde{b}_{ik}) = proj_k \ell^{\tilde{q}}(\tilde{b}_{ik})$.
For p $< \tilde{q}$ we have that $(\ell^p, \ell^{\tilde{q}}) \in$ *SS* and for p $> \tilde{q}$ we have that $(\ell^p, \ell^{\tilde{q}}) \in$ *K* [8]. Also $(\ell^p, \ell^{\tilde{q}}) \in$ *K* implies $(\ell^p, \ell^{\tilde{q}}) \in$ *SS*. So for p $\neq \tilde{q}$ we have that $(\ell^p, \ell^{\tilde{q}}) \in$ *SS*. Thus, by Lemma 4.1, $(K^p(a_{ik}), K^{\tilde{q}}(\tilde{b}_{ik})) \in$ *SS*. Hence, we get $(K^p(a_{ik}), K^{\tilde{q}}(\tilde{b}_{ik})) \in$ *BSS*. In a similar way, we have $(K^{\tilde{p}}(\tilde{a}_{ik}), K^q(b_{ik})) \in$ *BSS*. Since a Fréchet space is Mackey-complete (see [2]) and since Köthe spaces are Fréchet spaces, then Köthe spaces are Mackey-complete. Thus, by Proposition 2.6 and by Corollary 3.1, there is an integer s such that $K^{\tilde{p}}(\tilde{a}_{ik}) \simeq (K^p(a_{ik}))^{(s)}$ and $K^{\tilde{q}}(\tilde{b}_{ik}) \simeq (K^q(b_{ik}))^{(-s)}$.
Conversely, suppose that there is an integer s such that $K^{\tilde{p}}(\tilde{a}_{ik}) \simeq (K^p(a_{ik}))^{(s)}$ and $K^{\tilde{q}}(\tilde{b}_{ik}) \simeq (K^q(b_{ik}))^{(-s)}$.
Because $K^{\tilde{p}}(\tilde{a}_{ik}) \simeq (K^p(a_{ik}))^{(s)}$ we have $K^{\tilde{p}}(\tilde{a}_{ik}) \simeq$ M where M is a subspace of $K^p(a_{ik})$ with the codimension s, and because $K^{\tilde{q}}(\tilde{b}_{ik}) \simeq (K^q(b_{ik}))^{(-s)}$ we have $K^{\tilde{q}}(\tilde{b}_{ik}) \simeq (K^q(b_{ik})) \times$ L where the dimension of L is s. Then there is an s-dimensional subspace $\tilde{L}$ such that $\tilde{L} \simeq$ L and $K^p(a_{ik}) \simeq$ M $\bigoplus \tilde{L}$. Thus, $K^{\tilde{p}}(\tilde{a}_{ik}) \times K^{\tilde{q}}(\tilde{b}_{ik}) \simeq$ M $\times$ $(K^q(b_{ik})) \times$ L $\simeq$ M $\times$ L $\times$ $(K^q(b_{ik})) \simeq K^p(a_{ik}) \times K^q(b_{ik})$.

Note that this result does not hold if p = $\tilde{q}$ or q = $\tilde{p}$.

Similar to Theorem 4.1, we have the next theorem.

Theorem 4.2 [16] Let $1 \leq$ p, $\tilde{p} < \infty$ and (a_{ik}), $(\tilde{a}_{ik})$ be (d_2)-type Köthe matrices and (b_{ik}), $(\tilde{b}_{ik})$ be (d_1)-type Köthe matrices. Then the following conditions are equivalent:
(1) $K^0(a_{ik}) \times K^p(b_{ik}) \simeq K^0(\tilde{a}_{ik}) \times K^{\tilde{p}}(\tilde{b}_{ik})$
(2) there is an integer s such that $K^0(\tilde{a}_{ik}) \simeq (K^0(a_{ik}))^{(s)}$ and $K^{\tilde{p}}(\tilde{b}_{ik}) \simeq (K^p(b_{ik}))^{(-s)}$.

Proof: Suppose that $K^0(a_{ik}) \times K^p(b_{ik}) \simeq K^0(\tilde{a}_{ik}) \times K^{\tilde{p}}(\tilde{b}_{ik})$
Proposition 2.5 gives that $(K^0(a_{ik}), K^{\tilde{p}}(\tilde{b}_{ik})) \in$ *B*, $(K^0(\tilde{a}_{ik}), K^p(b_{ik})) \in$ *B* because (a_{ik}), $(\tilde{a}_{ik})$ are (d_2)-type Köthe matrices and (b_{ik}), $(\tilde{b}_{ik})$ are (d_1)-type Köthe matrices.
Consider $K^0(a_{ik}) = proj_k c_0(a_{ik})$, $K^0(\tilde{a}_{ik}) = proj_k c_0(\tilde{a}_{ik})$, $K^p(b_{ik}) = proj_k \ell^p(b_{ik})$,

$K^{\tilde{p}}(\tilde{b}_{ik}) = proj_k \ell^{\tilde{p}}(\tilde{b}_{ik})$.

For $1 \leq$ p, $\tilde{p} < \infty$ we have that (c_0, $\ell^{\tilde{p}}$) $\in$ *SS* and (c_0, ℓ^p) $\in$ *SS* [8]. Thus, by Lemma 4.1, ($K^0(a_{ik})$, $K^{\tilde{p}}(\tilde{b}_{ik})$) $\in$ *SS*. Hence, we get ($K^0(a_{ik})$, $K^{\tilde{p}}(\tilde{b}_{ik})$) $\in$ *BSS*. In a similar way, we have ($K^0(\tilde{a}_{ik})$, $K^p(b_{ik})$) $\in$ *BSS*. Since a Fréchet space is Mackey-complete (see [2]) and since Köthe spaces are Fréchet spaces, then Köthe spaces are Mackey-complete. Thus, by Proposition 2.6 and by Corollary 3.1, there is an integer s such that $K^0(\tilde{a}_{ik}) \simeq (K^0(a_{ik}))^{(s)}$ and $K^{\tilde{p}}(\tilde{b}_{ik}) \simeq (K^p(b_{ik}))^{(-s)}$.

Conversely, suppose that there is an integer s such that $K^0(\tilde{a}_{ik}) \simeq (K^0(a_{ik}))^{(s)}$ and $K^{\tilde{p}}(\tilde{b}_{ik}) \simeq (K^p(b_{ik}))^{(-s)}$.

Because $K^0(\tilde{a}_{ik}) \simeq (K^0(a_{ik}))^{(s)}$ we have $K^0(\tilde{a}_{ik}) \simeq$ M where M is a subspace of $K^0(a_{ik})$ with the codimension s, and because $K^{\tilde{p}}(\tilde{b}_{ik}) \simeq (K^p(b_{ik}))^{(-s)}$ we have $K^{\tilde{p}}(\tilde{b}_{ik}) \simeq (K^p(b_{ik})) \times$ L where the dimension of L is s. Then there is an s-dimensional subspace $\tilde{L}$ such that $\tilde{L} \simeq$ L and $K^0(a_{ik}) \simeq$ M $\bigoplus \tilde{L}$. Thus, $K^0(\tilde{a}_{ik}) \times K^{\tilde{p}}(\tilde{b}_{ik}) \simeq$ M $\times$ $(K^p(b_{ik})) \times$ L $\simeq$ M $\times$ L $\times$ $(K^p(b_{ik})) \simeq K^0(a_{ik}) \times K^p(b_{ik})$.

4.2 Applications of the 2^{nd} Modification Theorem

Proposition 4.1 [3] Consider an ℓ^p-Köthe space E and consider two complementary subspaces X and Y in E. If Y is a Banach space of infinite dimension, then we have Y $\simeq \ell^p$, and furthermore, X and Y are isomorphic to some basic subspaces of E.

Proof: Consider that E $\simeq$ E $\times$ $\{0\}$ $\simeq$ X $\times$ Y. By Theorem 3.3, there are complementary basic subspaces A and B in E and complementary subspaces X_1 and Y_1 in X with the properties that B is Banach, $X_1 \simeq$ A and B $\simeq Y_1 \times$ Y. We know that any basic Banach subspace with infinite dimension of an ℓ^p Köthe space is isomorphic to ℓ^p. Then we have that B $\simeq \ell^p$. Also, any complemented subspace with infinite dimension of ℓ^p (with p$\in[1, \infty)$) is isomorphic to ℓ^p (by [8], [10]). Thus, Y $\simeq \ell^p$. Then, since B $\simeq \ell^p$, the complemented subspace Y_1 of it is isomorphic to some basic subspace of B and hence X $\simeq$ A $\bigoplus Y_1$ is isomorphic to some basic subspace of E.

This proposition says that if we take any complemented Banach subspace with infinite dimension in an ℓ^p-Köthe space, then it is isomorphic to the ℓ^p space.

As stated in [3], We may take this result as a partial answer to the Pelczynski problem: "Does a complemented subspace of a space with basis have a basis?" Also, we verify the hypothesis of Bessega [1] which is the fact that every complemented subspace of a Köthe space is isomorphic to a basic subspace.

The next theorem includes the case p = q = $\tilde{p}$ = $\tilde{q}$.

Theorem 4.3 [3] Suppose that $E_1 \times E_2 \simeq F_1 \times F_2$ where all of E_1, E_2, F_1, F_2 are non-Montel ℓ^p-Köthe spaces. If E_1, F_1 are (d_2) type and if E_2, F_2 are (d_1) type spaces, then we have that $E_1 \simeq F_1$ and $E_2 \simeq F_2$.

Proof: Proposition 2.5 implies that every linear continuous operator acting in E_1 and factoring over F_2 and every linear continuous operator acting in F_1 and factoring over E_2 are bounded. Then, Theorem 3.3 implies that there exist complementary basic subspaces A and X in E_1 and complementary subspaces B and Y in F_1 such that
B $\simeq$ A, X $\times$ E_2 $\simeq$ Y $\times$ F_2, and X and Y are Banach spaces. So, either X is of finite dimension, or it is isomorphic to the space ℓ^p by the Proposition 4.1. Similarly, either Y is of finite dimension, or it is isomorphic to the space ℓ^p. Then we have that
X $\times$ $\ell^p \simeq \ell^p$ and that Y $\times$ $\ell^p \simeq \ell^p$ because $\ell^p \times \ell^p \simeq \ell^p$. Then by Proposition 2.3, we have that
$E_1 \simeq E_1 \times \ell^p \simeq$ A $\times$ X $\times$ $\ell^p \simeq$ B $\times$ Y $\times$ $\ell^p \simeq F_1 \times \ell^p \simeq F_1$, and
$E_2 \simeq E_2 \times \ell^p \simeq E_2 \times$ X $\times$ $\ell^p \simeq F_2 \times$ Y $\times$ $\ell^p \simeq F_2 \times \ell^p \simeq F_2$.

As stated in [3], this theorem gives an answer to the Question 2 in [4], which is given as "Is it possible to consider stronger linear topological invariants and obtain the condition s_1 + s_2 = 0 without using Riesz theory?"

The following theorem is a generalization of Theorem 4 in [3]. It includes the case p $\neq$ q, p = $\tilde{q}$ and q = $\tilde{p}$.

Theorem 4.4 [3] Let p $\neq$ q. Suppose that $K^p(a_{ik})$, $K^q(\tilde{a}_{ik})$ are (d_2) type non-Montel Köthe spaces and that $K^q(b_{ik})$, $K^p(\tilde{b}_{ik})$ are (d_1) type non-Montel Köthe spaces. Then the following statements are equivalent:
(1) $K^p(a_{ik}) \times K^q(b_{ik}) \simeq K^q(\tilde{a}_{ik}) \times K^p(\tilde{b}_{ik})$
(2) there are complementary submatrices (a'_{ik}), (a''_{ik}), (b'_{ik}), (b''_{ik}), $(\tilde{a}'_{ik})$, $(\tilde{a}''_{ik})$, $(\tilde{b}'_{ik})$, $(\tilde{b}''_{ik})$ of (a_{ik}), (b_{ik}), $(\tilde{a}_{ik})$, $(\tilde{b}_{ik})$, respectively, such that

$K^p(a''_{ik}) \simeq \ell^p$, $K^q(\tilde{a}''_{ik}) \simeq \ell^q$, $K^q(b''_{ik}) \simeq \ell^q$, $K^p(\tilde{b}''_{ik}) \simeq \ell^p$;
$K^p(a'_{ik})$, $K^q(b'_{ik})$, $K^q(\tilde{a}'_{ik})$, $K^p(\tilde{b}'_{ik})$ are nuclear spaces and
$K^p(a'_{ik}) \simeq K^q(\tilde{a}'_{ik})$ and $K^q(b'_{ik}) \simeq K^p(\tilde{b}'_{ik})$.

Proof:
(2) $\Rightarrow$ (1): Suppose (2) holds. Since (a'_{ik}), (a''_{ik}), (b'_{ik}), (b''_{ik}), $(\tilde{a}'_{ik})$, $(\tilde{a}''_{ik})$, $(\tilde{b}'_{ik})$, $(\tilde{b}''_{ik})$ are complementary submatrices of (a_{ik}), (b_{ik}), $(\tilde{a}_{ik})$, $(\tilde{b}_{ik})$, respectively, we have that
$K^p(a_{ik}) \simeq K^p(a'_{ik}) \times K^p(a''_{ik})$ and $K^q(b_{ik}) \simeq K^q(b'_{ik}) \times K^q(b''_{ik})$,
$K^q(\tilde{a}_{ik}) \simeq K^q(\tilde{a}'_{ik}) \times K^q(\tilde{a}''_{ik})$ and $K^p(\tilde{b}_{ik}) \simeq K^p(\tilde{b}'_{ik}) \times K^p(\tilde{b}''_{ik})$.
Then by (2) we have that
$K^p(a_{ik})$ x $K^q(b_{ik}) \simeq K^p(a'_{ik}) \times K^p(a''_{ik}) \times K^q(b'_{ik}) \times K^q(b''_{ik})$
$\simeq K^q(\tilde{a}'_{ik}) \times \ell^p \times K^p(\tilde{b}'_{ik}) \times \ell^q$
$\simeq K^q(\tilde{a}'_{ik}) \times K^p(\tilde{b}''_{ik}) \times K^p(\tilde{b}'_{ik}) \times K^q(\tilde{a}''_{ik})$
$\simeq K^q(\tilde{a}'_{ik}) \times K^q(\tilde{a}''_{ik}) \times K^p(\tilde{b}'_{ik}) \times K^p(\tilde{b}''_{ik})$
$\simeq K^q(\tilde{a}_{ik}) \times K^p(\tilde{b}_{ik})$

(1) $\Rightarrow$ (2): Suppose (1) holds. Then Proposition 2.5 and Theorem 3.3 both imply that there are complementary submatrices (a'_{ik}) and (a''_{ik}) of (a_{ik}) and there are complementary subspaces X and Y in $K^q(\tilde{a}_{ik})$ with the propery that $K^p(a''_{ik})$ and Y are Banach spaces, and $K^p(a'_{ik}) \simeq$ X and $K^p(a''_{ik}) \times K^q(b_{ik}) \simeq$ Y $\times K^p(\tilde{b}_{ik})$.
Then Proposition 4.1 gives that there are complementary submatrices $(\tilde{a}'_{ik})$ and $(\tilde{a}''_{ik})$ of $(\tilde{a}_{ik})$ with X $\simeq K^q(\tilde{a}'_{ik})$, Y $\simeq K^q(\tilde{a}''_{ik})$, and $K^p(a''_{ik})$ is either of finite dimension or isomorphic to the space ℓ^p, and $K^q(\tilde{a}''_{ik})$ is either of finite dimension or isomorphic to the space ℓ^q. Then we have that $K^p(a'_{ik}) \simeq K^q(\tilde{a}'_{ik})$. Thus, Proposition 2.4 implies that $K^p(a'_{ik})$ and $K^q(\tilde{a}'_{ik})$ are nuclear spaces. Now suppose that either $K^p(a''_{ik})$ or $K^q(\tilde{a}''_{ik})$ has finite dimension. Then either $K^p(a_{ik})$ or $K^q(\tilde{a}_{ik})$ is nuclear, and so, a Montel space. This is a contradiction to the assumption of the theorem. Thus, $K^p(a''_{ik})$ and $K^q(\tilde{a}''_{ik})$ has infinite dimension. We get $K^q(b_{ik}) \times \ell^p \simeq K^p(\tilde{b}_{ik}) \times \ell^q$.
In a similar way, there are complementary submatrices (b'_{ik}), (b''_{ik}) of (b_{ik}) and $(\tilde{b}'_{ik})$, $(\tilde{b}''_{ik})$ of $(\tilde{b}_{ik})$ with $K^q(b'_{ik}) \simeq K^p(\tilde{b}'_{ik})$, and $K^q(b''_{ik}) \simeq \ell^q$, and $K^p(\tilde{b}''_{ik}) \simeq \ell^p$, where $K^q(b'_{ik})$ and $K^p(\tilde{b}'_{ik})$ are nuclear spaces.

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