SMOOTH MANIFOLDS WITH INFINITE FUNDAMENTAL GROUP ADMITTING NO REAL PROJECTIVE STRUCTURE

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

HATİCE ÇOBAN

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY
IN
MATHEMATICS

AUGUST 2017

Approval of the thesis:

SMOOTH MANIFOLDS WITH INFINITE FUNDAMENTAL GROUP ADMITTING NO REAL PROJECTIVE STRUCTURE

submitted by **HATİCE ÇOBAN** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy in Mathematics Department, Middle East Technical University** by,

Prof. Dr. Gülbin Dural Ünver
Dean, Graduate School of **Natural and Applied Sciences**

Prof. Dr. Mustafa Korkmaz
Head of Department, **Mathematics**

Prof. Dr. Yıldıray Ozan
Supervisor, **Mathematics Department, METU**

Examining Committee Members:

Prof. Dr. Sergey Finashin
Mathematics Department, METU

Prof. Dr. Yıldıray Ozan
Mathematics Department, METU

Assoc. Prof. Dr. Sinem Onaran
Mathematics Department, Hacettepe University

Assoc. Prof. Dr. Mehmetcik Pamuk
Mathematics Department, METU

Assist. Prof. Dr. Özgün Ünlü
Mathematics Department, Bilkent University

Date:

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Last Name: HATİCE ÇOBAN

Signature :

ABSTRACT

SMOOTH MANIFOLDS WITH INFINITE FUNDAMENTAL GROUP ADMITTING NO REAL PROJECTIVE STRUCTURE

ÇOBAN, Hatice
Ph.D., Department of Mathematics
Supervisor : Prof. Dr. Yıldıray Ozan

August 2017, 71 pages

In this thesis, we construct smooth manifolds with the infinite fundamental group $\mathbb{Z}_2 * \mathbb{Z}_2$, for any dimension $n \geq 4$, admitting no real projective structure. They are first examples of manifolds in higher dimensions with infinite fundamental group admitting no real projective structures. The motivation of our study is the related work of Cooper and Goldman. They proved that $\mathbb{RP}^3 \# \mathbb{RP}^3$ does not admit any real projective structure and this is the first known example in dimension 3.

Keywords: Real projective structures, developing map, holonomy, foliation

ÖZ

REEL PROJEKTİF YAPI KABUL ETMEYEN SONSUZ TEMEL GRUPLU MANİFOLDLAR

ÇOBAN, Hatice
Doktora, Matematik Bölümü
Tez Yöneticisi : Prof. Dr. Yıldıray Ozan

Ağustos 2017 , 71 sayfa

Bu tezde, her $n \geq 4$ için temel grubu $\mathbb{Z}_2 * \mathbb{Z}_2$ olan ve üzerinde projektif yapı olmayan manifoldlar inşa ettik. Bunlar yüksek boyutlardaki sonsuz temel gruplu ilk manifold örnekleridir. Bu çalışma Cooper ve Goldman'ın 2015 tarihli makalesini temel almıştır. Bu makalede Cooper ve Goldman $\mathbb{RP}^3 \# \mathbb{RP}^3$ 3-boyutlu manifoldunun reel projektif yapı taşımadığını göstermişlerdir. Bu 3 boyutta bilinen ilk örnektir.

Anahtar Kelimeler: Reel projektif yapılar, dolanım, yapraklama

To my family

ACKNOWLEDGMENTS

I would like to express my gratitude to my supervisor, Prof. Dr. Yıldıray Ozan, who always had positive attitude that encouraged me during this exhaustive and difficult period. I feel indebted to him for his sincere support, patience and kind guidance throughout my research.

I am indebted to Prof. Dr. Turgut Önder, Prof. Dr. Sergey Finashin, Prof. Dr. Mustafa Korkmaz and Assoc. Prof. Dr. Mehmetcik Pamuk for teaching me geometry and topology through several courses.

I would like to thank Emmanuel Opshtein and Ana Rechtman for being my advisors during my Erasmus period at Strasbourg University.

I am thankful to the administrative staff of the Departments of Mathematics at METU.

I owe many thanks to Şule Yüksel Güngör, who taught me Maple and always stood by me with her patience and lovely friendship.

I am indebted to Ezgi Yavuz, who always understands and supports me even from miles away.

I would like to thank my colleagues and dear friends Tülin Altunöz, Hatice Ünlü, Hanife Varlı, Dr. Adalet Çengel, Asst. Prof. Dr. Arzu Zabun, Merve Seçgin and Elçin Çalışkan. I appreciate them for the inspiring study environment and their lovely friendship.

I also would like to thank my dear friends Tuba Koç and Gonca Tunçbilek.

This work is financially supported by TÜBİTAK-BİDEB National Graduate Scholarship Programme for PhD (2211).

Last but not least, I owe my deepest gratitude to my father Hasan, my mother Kerime and my sisters Hanife and Esra for giving me endless support during my education. I would like to thank to them because of everything they have done for me throughout my whole life.

TABLE OF CONTENTS

LIST OF TABLES

TABLES

LIST OF FIGURES

FIGURES

CHAPTER 1

INTRODUCTION

Projective geometry is the study of geometric properties which are invariant under projective transformations. It arises in many research subjects such as the study of projective varities in algebraic geometry, and the study of differential invariants of the projective transformations in differential geometry ([14], [22]).

Projective geometry is simpler than affine or Euclidean geometries since in projective geometry, there are no parallel lines or right angles, and all nondegenerate conics are equivalent. The group of symmetries of the projective space $PGL(n+1,\mathbb{R})$ is large and thus projective invariants are fewer than others ([17]).

Projective geometry was first systematically studied by Desargues in the 17th century based upon principles of perspective art. However, his work was not well received because he invented and used a lot of new technical terms and also the mathematicians of his age were not ready to comprehend this new form of geometry. After then projective geometry was established by the work of Monge, Poncelet, Chasles and other mathematicians ([6]).

In Felix Klein's Erlanger program of 1872, the geometry is defined to be the study of the properties of a space which are invariant under a group of transformations. Thus, the geometry is a pair (X, G), where X is a manifold and G is a Lie group which acts transitively on the manifold X. This yields a hierarchy of geometries, defined as groups of transformations, where the Euclidean geometry is part of the affine geometry which is itself included in the projective geometry. Although this kind of geometric structures on manifolds were introduced by F. Klein, the study of such geometric structures on manifolds was started by Charles Ehresmann under the name of locally homogeneous structures in [9] (for the details about locally homogeneous spaces, see [3], [27]). In 3-dimension, these locally homogeneous spaces are related with Thurston's Geometrization Conjecture which is proved by Perelman ([19]).

Thurston's Geometrization Conjecture: Every oriented prime closed 3-manifold can be cut along tori, so that the interior of each of the resulting manifolds has a geometric structure with finite volume.

There are eight possible geometries in dimension 3, which are classified by Thurston in the late 1970's ([26], see also [5], [16] and [23]). A three manifold admitting one of Thurston's geometries except the two of them, which are $S^2 \times \mathbb{R}$ and $\mathbb{H} \times \mathbb{R}$ has a real projective structure determined uniquely by this structure. In the remaining two cases, the three manifold also has a real projective structure if the group acting on the manifold preserves the orientation on the $\mathbb{R}$ direction ([8], [16]). On the other hand, by Benoist's work in [2], there are some examples of 3-manifolds with a real projective structure, which are not induced by one of the eight geometries. However, not every closed 3-manifold admits a real projective structure. Cooper and Goldman proved that the connected sum of two copies of real projective 3-space admits no real projective structure ([8]) and it is the first known example in dimension 3.

It is possible to pass from structures modelled on one geometry (X, G) to structures modelled on another geometry (X', G') containing it. For example, every affine structure determines a projective structure by the following embedding of affine geometry in projective geometry

$$(\mathbb{R}^n, Aff(\mathbb{R}^n)) \hookrightarrow (\mathbb{RP}^n, PGL(n+1, \mathbb{R})).$$

Similarly, every hyperbolic structure determines a projective structure

$$(\mathbb{H}^n, PO(n+1)) \hookrightarrow (\mathbb{RP}^n, PGL(n+1, \mathbb{R}))$$

and using the inclusion of projective orthogonal group into projective general linear group

$$PO(n+1) \subset PGL(n+1, \mathbb{R})$$

it can be easily seen that every elliptic geometry structure determines a real projective structure ([12]). As a result, if a manifold does not admit a real projective structure, it does not carry any affine structure or hyperbolic structure. Moreover, every hyperbolic structure has Riemannian metric of constant curvature -1 and every elliptic structure has Riemannian metric of constant curvature 1. Every surface admits a metric of constant curvature and thus every surface admits a real projective structure and the classification of these structures on compact surfaces is known, see [7].

A real projective structure on an n-dimensional smooth manifold M is an $\mathbb{RP}^n$ structure which is an Ehresmann structure modelled on $\mathbb{RP}^n$ with transition functions taking values in the group $PGL(n+1,\mathbb{R})$ of projective transformations of $\mathbb{RP}^n$.

It is well known that any simply connected compact manifold with a real projective structure is a sphere since the developing map (see page 6) must be a covering map. There are several examples of simply connected compact manifolds other than spheres in dimension bigger than 3 (e.g. $\mathbb{CP}^n$) and therefore there are many higher dimensional manifolds that do not admit a real projective structure.

In this thesis, we construct an $n-$manifold, for any $n \geq 4$, with the infinite fundamental group $\mathbb{Z}_2 * \mathbb{Z}_2$ not admitting any real projective structures by generalizing Cooper and Goldman's work in [8].

Let W be an m-dimensional $(m \geq 3)$ closed smooth manifold with $\pi_1(W) \cong \mathbb{Z}_2$ and $\widetilde{W}$ be its universal cover with $\widetilde{W}/ < \tau > \cong W$, where τ is the Deck transformation of the universal cover $\widetilde{W} \longrightarrow W$. Then we let
$M^n = \widetilde{W} \times S^1 / < \sigma >$, where $n = m+1$ and σ is given by

$$\begin{aligned} \sigma : \widetilde{W} \times S^1 &\longrightarrow \widetilde{W} \times S^1, \\ (p,z) &\longmapsto (\tau(p), \bar{z}), \end{aligned}$$

where $\bar{z}$ denotes the complex conjugate of z on $S^1 \subseteq \mathbb{C}$.

Now, our main theorem is below.

Theorem 1.0.1. *Let W be an m-dimensional $(m \geq 3)$ closed smooth manifold with $\pi_1(W) \cong \mathbb{Z}_2$ and $M = \widetilde{W} \times S^1 / < \sigma >$ as above. We assume that:*
1- Either $\widetilde{W}$ is odd dimensional, or
2- $\widetilde{W}$ is even dimensional and it is not the total space of a sphere bundle over a

sphere, where both the base and the fiber are the sphere $S^{m/2}$.
Then the manifold M *does not admit a real projective structure.*

Using a simple observation we give an obstruction to obtain examples of manifolds with the infinite fundamental group $\mathbb{Z}$ admitting no real projective structure.

Theorem 1.0.2. *Let* M^n *be a simply connected manifold which does not admit any immersion into* $\mathbb{R}^{n+1}$. *Then* $M \times S^1$ *does not have any real projective structure.*

In addition, we proved that if a manifold M (with any fundamental group) admits a special contact form consisting of some functions on M then it admits a real projective structure.

Theorem 1.0.3. *Let an odd dimensional manifold* M^{2k-1} *admit a contact form given as below*

$$\alpha = \sum_{i=1}^{k} (f_{2i-1}\, df_{2i} - f_{2i}\, df_{2i-1})$$

for some functions $f_1, ..., f_{2k} \in C^\infty(M)$. *Then* M *admits a real projective structure.*

This thesis is organized as follows:

In Chapter 2, we introduce some preliminary materials such as developing map, holonomy and Ehresmann-Weil-Thurston Principle.

In Chapter 3, we explain the construction of our manifold with the infinite fundamental group $\mathbb{Z}_2 * \mathbb{Z}_2$, for any dimension $n \geq 4$, and then present the main result, Theorem 3.2.1, and prove it.

In Chapter 4, we give some more details and alternative arguments about the proof of the main theorem for the case $M = \mathbb{RP}^4 \# \mathbb{RP}^4$.

In Chapter 5, we give an obstruction to obtain manifolds with the infinite fundamental group $\mathbb{Z}$ admitting no real projective structures, Theorem 5.0.15, and provide some examples.

In Chapter 6, we first give the definition of contact forms and state the result of Ovsienko [18]. Then we state Theorem 6.2.1 and prove it.

CHAPTER 2

PRELIMINARIES

In this chapter, first we define an (X, G) structure on a manifold in the sense of Ehresmann. Let M be a real analytic modelled manifold on X (i.e. M is locally isomorphic to X) and G be a Lie group acting transitively on X. Then we say that M has an (X, G) structure or M is an (X, G) manifold. Therefore, an (X, G) manifold has an underlying real analytic structure (see [13], [14], [15] and [20] for more information about (X, G) structures).

Suppose that M is any (X, G) manifold. Let $\{(U_i,\ \phi_i : U_i \to X)\}$ be an atlas of M with transition functions

$$\gamma_{ij} : \phi_i(U_i \cap U_j) \longrightarrow \phi_j(U_i \cap U_j)$$

such that

$$\gamma_{ij} \circ \phi_i = \phi_j .$$

Consider an analytic continuation (a technique to extend the domain of a given analytic function) of ϕ_1 along a curve α in M beginning in U_1.

On a component of $\alpha \cap U_1$, the analytic continuation of ϕ_1 along α is of the form $\gamma_{12} \circ \phi_1$, where $\gamma_{12} \in G$. The first step is below.

$$\phi_1 : U_1 \longrightarrow V_1 \text{ and } \phi_2 : U_2 \longrightarrow V_2$$

where $V_i \subset X$.

$$\gamma_{12} : \phi_1(U_1 \cap U_2) \longrightarrow \phi_2(U_1 \cap U_2)$$

$$\gamma_{12} \circ \phi_1|_{(U_1 \cap U_2)} = \phi_2|_{(U_1 \cap U_2)}$$

$$\phi_{1|_{(U_1 \cap U_2)}} = \gamma_{12}^{-1} \circ \phi_{2|_{(U_1 \cap U_2)}}.$$

Therefore, inductively, ϕ_1 can be analytically continued along every path to $\bigcup_i U_i$ in M.

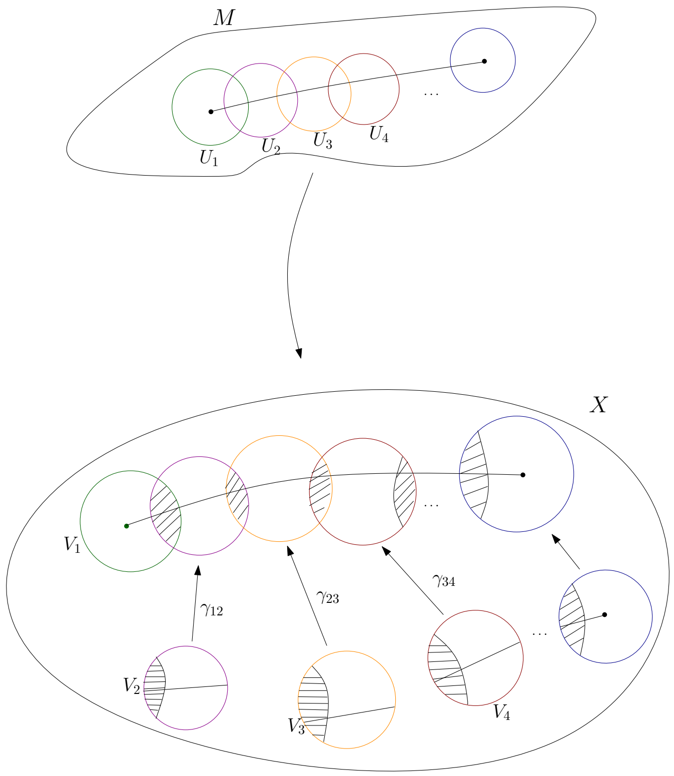


Figure 2.1: An analytic continuation of ϕ_1 along α.

It follows that there is a global analytic continuation of ϕ_1 on the universal cover of M, by using the definition of the universal cover as a quotient space of the paths in M. Now, we define the developing map:

Fix some $p_0 \in M$. Let $\widetilde{M}$ be the homotopy classes of paths starting at p_0, where homotopies fix the end points,

$$\widetilde{M} = \left\{ \alpha : [0,1] \longrightarrow M \mid \alpha(0) = p_0 \right\} / \alpha_1 \sim \alpha_2 ,$$

where two paths α_1 and α_2 are related, $\alpha_1 \sim \alpha_2$, if and only if $\alpha_1(1) = \alpha_2(1)$ and α_1 and α_2 are homotopic relative to the boundary.

Clearly, $\widetilde{M}$ is the universal cover of M. Now, using the above analytic continuation we can define the following map

$$dev : \widetilde{M} \longrightarrow X,$$

which is called a developing map on the uviversal cover of M. It is clearly an immersion. The map dev is unique up to composition with elements of G. It follows from the uniqueness property of dev that, for any covering transformation Γ_α of $\widetilde{M}$ over M, there is an element g_α of G such that

$$dev \circ \Gamma_\alpha = g_\alpha \circ dev.$$

Since

$$dev \circ \Gamma_\alpha \circ \Gamma_\beta = g_\alpha \circ dev \circ \Gamma_\beta = g_\alpha \circ g_\beta \circ dev,$$

we see that the map

$$\begin{aligned} hol : \pi_1(M) &\longrightarrow G, \\ \alpha &\longmapsto g_\alpha \end{aligned}$$

is a homomorphism and it is called the holonomy of M. For more details, see [26].

The pair (dev, hol) is called a developing pair for the (X, G) structure. A real projective structure on M^n is then an $(\mathbb{RP}^n, PGL(n+1, \mathbb{R}))$ structure.

More precisely, M is endowed with a real projective structure if there is an atlas on M with projective coordinate changes. A covering $\{(U_i)\}$ with a family of local diffeomorphisms $\phi_i : U_i \longrightarrow V_i \subset \mathbb{RP}^n$ is called a projective atlas if the local transformations $\phi_j \circ {\phi_i}^{-1} : \phi_i(U_i \cap U_j) \longrightarrow \phi_j(U_i \cap U_j)$ are projective (i.e. they are determined by the action of the group $PGL\big((n+1), \mathbb{R}\big)$).

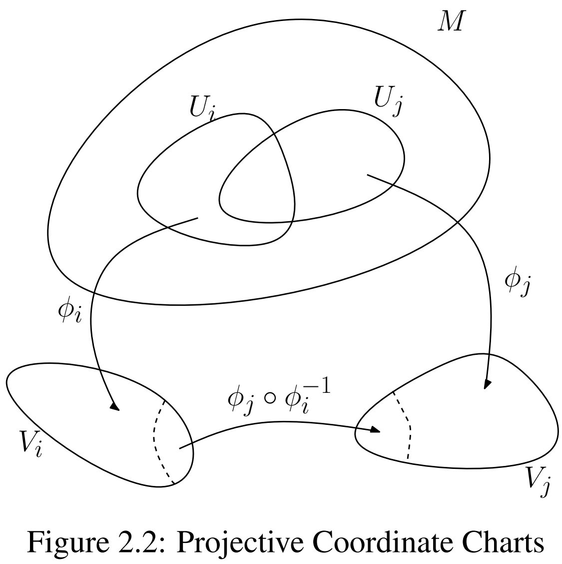


Figure 2.2: Projective Coordinate Charts

Remark 2.0.4. *Let* dev_1 *and* dev_2 *be two developing maps for the same structure. Then they satisfy that* $dev_2 = g \circ dev_1$, *for some* $g \in G$ *and the holonomies are related with each other as follows* $hol_2(\gamma) = ghol_1(\gamma)g^{-1}$, *for any homotopy class* $[\gamma] \in \pi_1(M)$.

Remark 2.0.5 ([9])**.** *Let* M *be a simply connected manifold, possibly noncompact. If* M *admits a real projective structure, then it is unique (up to natural equivalence).*

The following theorem is fundamental to deform Ehresmann structures and it is first studied by Thurston ([24]).

Theorem 2.0.6 (Ehresmann-Weil-Thurston Principle)**.** *Let* M *be an* (X, G) *manifold with holonomy representation* $\varrho : \pi_1(M) \longrightarrow G$. *For* ϱ' *sufficiently close to* ϱ *in the space of representations* $Hom(\pi_1(M), G)$, *there exists an* (X, G) *structure on* M *with holonomy representation* ϱ'.

Corollary 2.0.7. *Let* M *be a closed manifold. The set of holonomy representations of* (X, G) *structures of* M *is open in* $Hom(\pi_1(M), G)$.

The following well known observation is needed in the proof of the main theorem and thus we include it with its proof.

Lemma 2.0.8. *Let* X *and* Y *be Hausdorff spaces and* $f : X \longrightarrow Y$ *be a local homeomorphism. If* X *is compact and* Y *is connected then* f *is a finite sheeted covering map.*

Proof. First, let us show that f is surjective. Local homeomorphisms are open maps, so $A = f(X)$ is an open subset of Y. Since X is compact, $f(X)$ is also compact and Y is Hausdorff, we have $f(X)$ is closed. Thus, $B = Y \setminus f(X)$ is open. If f were not surjective, then $B \neq \emptyset$ and A and B would be a separating pair for Y, contradicting the connectedness of Y. Thus f is surjective.

For each $y \in Y$, since X is compact, there exist finitely many disjoint points, let us say $\{x_1, x_2, ..., x_n\} = f^{-1}(y)$. We can choose disjoint neighborhoods $\{U_1, U_2,$ ..., $U_n\}$ of $\{x_1, x_2, ..., x_n\}$ respectively, using the Hausdorff property of X.

By shrinking the U_i's further, we may assume that each one is homeomorphically mapped onto some open neighborhood V_i of y,

$$f_{|U_i} : U_i \longrightarrow f(U_i) = V_i.$$

Now, let $C = X \setminus (U_1 \cup U_2 \cup ... \cup U_n)$ and set $V = (U_1 \cap U_2 \cap ... \cap U_n) \setminus f(C)$.

Using the assumptions that X is compact, Hausdorff and Y is Hausdorff, we conclude that f is a closed map. C is a closed subset of X and thus $f(C)$ is a closed subset of Y. Hence, V is open and $f^{-1}(y) \cap C = \emptyset$ so that $y \in V$ and it is evenly covered neighborhood of y. Thus, f is a finite sheeted covering space. □

The following two theorems will be needed to study the foliation on the manifold $\widetilde{W}$ and the leaf space $\mathcal{L} \subset \mathbb{RP}^n$ induced by a real projective structure (page 41).

Theorem 2.0.9 (Reeb Stability Theorem)**.** *([21]) Let $\mathcal{F}$ be a codimension one foliation of a compact manifold M^n. Suppose L is a compact leaf of $\mathcal{F}$ such that $\pi_1(L)$ is finite. Then all leaves of $\mathcal{F}$ are diffeomorphic to L (up to two fold covers if there are leaves with one sided tubular neighborhoods). We assume here that if M^n has boundary, then the boundary of M is a union of leaves of $\mathcal{F}$.*

A generalization of the Reeb stability theorem is given by Thurston ([25]) as follows.

Theorem 2.0.10. *Let $\mathcal{F}$ be a codimension one, C^1, transversely oriented foliation of a compact manifold M^n with a compact leaf L such that $H^1(L, \mathbb{R}) = 0$. Then all leaves of $\mathcal{F}$ are diffeomorphic to L, and the leaves of $\mathcal{F}$ are the fibers of a fibration of M^n over S^1 or I, which is an interval. We assume here that if M^n has boundary, then the boundary of M is a union of leaves of $\mathcal{F}$.*

For more details, see [14], [15].

CHAPTER 3

THE MAIN THEOREM AND ITS PROOF

In this section, we establish and prove our main theorem. We construct smooth n-manifolds with the infinite fundamental group $\mathbb{Z}_2 * \mathbb{Z}_2$, for any $n \geq 4$, with no real projective structure.

3.1 Construction of the Manifold

Let W be an m-dimensional ($m \geq 3$) smooth closed manifold with $\pi_1(W) \cong \mathbb{Z}_2$ and say $M = \widetilde{W} \times S^1 \big/ < \sigma >$, where the action is given by

$$\begin{aligned} \sigma : \widetilde{W} \times S^1 &\longrightarrow \widetilde{W} \times S^1, \\ (p, z) &\longmapsto (\tau(p), \bar{z}) \end{aligned}$$

such that S^1 is the unit circle in the complex plane $\mathbb{C}$, $\bar{z}$ is the complex conjugate of z and τ is the Deck transformation of the universal cover $\widetilde{W} \longrightarrow W$ with $\widetilde{W} \big/ < \tau >= W$. The universal cover of M is as follows:

$$\widetilde{M} = \widetilde{W} \times \mathbb{R} \longrightarrow \widetilde{W} \times S^1 \longrightarrow \widetilde{W} \times S^1 \big/ \mathbb{Z}_2 = M.$$

The induced homomorphism by σ on the fundamental group is given below.

$$\begin{aligned} \sigma_\sharp : \pi_1(\widetilde{W} \times S^1) &\longrightarrow \pi_1(\widetilde{W} \times S^1) \\ \mathbb{Z} &\longrightarrow \mathbb{Z} \\ 1 &\longmapsto -1. \end{aligned}$$

Now, we compute $\pi_1(M)$. Consider Figure 3.1 and Figure 3.2.

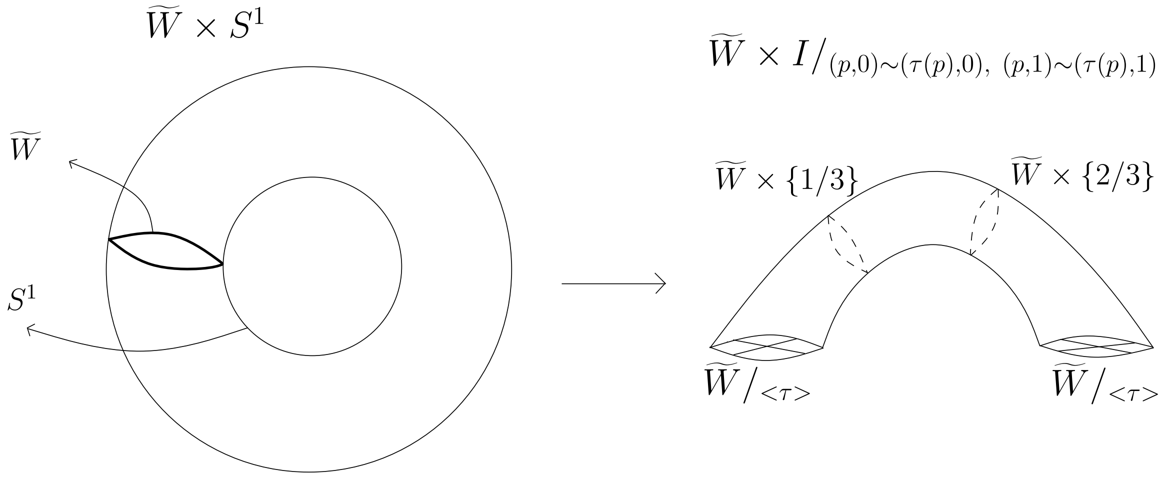


Figure 3.1: The manifold $M = \widetilde{W} \times S^1/ < \sigma >$.

We choose open subsets U and V of M such that $M = U \cup V$ as Figure 3.2.

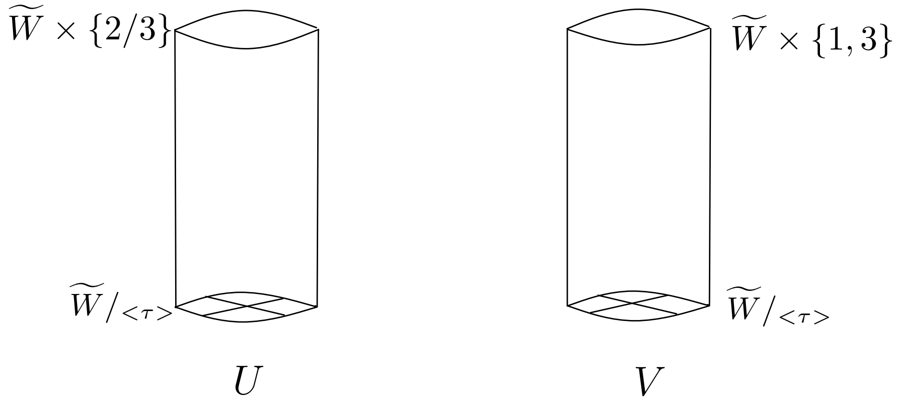


Figure 3.2: Open subsets U and V of M.

Note that both U and V deformation retract to $\widetilde{W}/ < \tau >= W$ and $U \cap V = \widetilde{W} \times (1/3, 2/3)$. By Van Kampen's theorem we have the following diagram (Figure 3.3).

By Figure 3.3, $\pi_1(U \cup V) = \pi_1(U) * \pi_1(V)$ and we have

$$\pi_1(M) = \mathbb{Z}_2 * \mathbb{Z}_2 =< a, b \mid a^2 = 1,\ b^2 = 1 > .$$

By using the presentation of the fundamental group of M, we have a short exact sequence

$$1 \longrightarrow \pi_1(\widetilde{W} \times S^1) \cong \mathbb{Z} \longrightarrow \pi_1(M) \longrightarrow \mathbb{Z}_2 \longrightarrow 1.$$

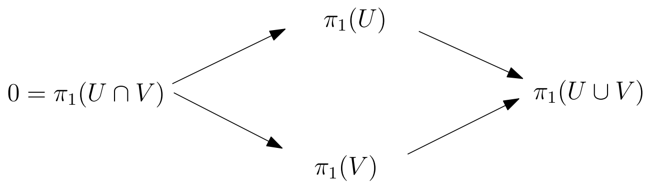


Figure 3.3: The Van Kampen diagram for $M = U \cup V$.

The action of $\mathbb{Z}_2$ on the normal subgroup $\mathbb{Z}$ of $\pi_1(M)$ is multiplicaiton by -1. Therefore, the fundamental group of M has the following presentation:

$$\pi_1(M) \cong \mathbb{Z} \ltimes \mathbb{Z}_2 =< c = ab, a \mid a^2 = 1, \;\; aca = c^{-1} > .$$

3.2 Statement of the Main Theorem

Below, we give our main result.

Theorem 3.2.1. *Let W be an m-dimensional ($m \geq 3$) smooth closed manifold with $\pi_1(W) \cong \mathbb{Z}_2$ and $M = \widetilde{W} \times S^1 \big/ < \sigma >$ as above. We assume that:*
1- Either $\widetilde{W}$ is odd dimensional, or
2- $\widetilde{W}$ is even dimensional and it is not the total space of a sphere bundle over a sphere, where both the base and the fiber are the sphere $S^{m/2}$.
Then the manifold M does not admit a real projective structure.

Remark 3.2.2. *Note that if $m = 2$ and W is a closed surface with $\pi_1(W) \cong \mathbb{Z}_2$ then $W = \mathbb{RP}^2$. Thus, $\widetilde{W} = S^2$ and*

$$M = S^2 \times S^1 \big/ < \sigma > \cong \; \mathbb{RP}^3 \# \mathbb{RP}^3.$$

That is to say our construction does not yield any example other than $\mathbb{RP}^3 \# \mathbb{RP}^3$ in dimension 3.

Similarly, if $m = 3$ and W is a closed 3-manifold with $\pi_1(W) \cong \mathbb{Z}_2$, then by the Elliptization Theorem (cf. Theorem 1.12 in [1]), $W = \mathbb{RP}^3$. Therefore, $\widetilde{W} = S^3$ and thus

$$M = S^3 \times S^1 \big/ < \sigma > \cong \; \mathbb{RP}^4 \# \mathbb{RP}^4 \text{ (see Chapter 4).}$$

For simplicity, we take $n = m + 1$.

We prove Theorem 3.2.1 by contradiction. Therefore, we start with the assumption that M admits a real projective structure. Hence, there exists a developing pair (dev, hol) for M. Before the proof of Theorem 3.2.1, we will prove the following lemma.

Lemma 3.2.3. *The holonomy map* $hol : \pi_1(M) \longrightarrow PGL(n+1, \mathbb{R})$ *is injective.*

Proof. Suppose not. Then the image of the holonomy is a proper quotient of the infinite dihedral group. This implies that it is finite ([28]). Let H be the kernel of the homomorphism

$$hol : \pi_1(M) \longrightarrow PGL(n+1, \mathbb{R}),$$

and $\widetilde{M}' \longrightarrow M$ be the covering space corresponding to the subgroup $H \leq \pi_1(M)$. Hence, the covering map $\widetilde{M}' \longrightarrow M$ is finite, whose total space is immersed into $\mathbb{RP}^n$ by the map $\varphi : \widetilde{M}' \longrightarrow \mathbb{RP}^n$. Here, the developing map descends to φ and the map φ is a covering map since $\widetilde{M}'$ is compact.

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{dev} & \mathbb{RP}^n \\ \downarrow & \nearrow_{\varphi} & \\ \widetilde{M}' & & \\ \downarrow & & \\ M & & \end{array}$$

Thus, $\tilde{M}'$ is a covering space of $\mathbb{RP}^n$. On the other hand, $\pi_1(\widetilde{M}')$ is infinite; therefore, this gives a contradiction since it is also isomorphic to a subgroup of $\pi_1(\mathbb{RP}^n) = \mathbb{Z}_2$. □

3.3 Proof of the Main Theorem

The rest of this chapter is devoted to the proof of Theorem 3.2.1.

Proof. We assume that M admits a real projective structure and thus there exists a developing pair (dev, hol)

$$dev : \widetilde{M} \longrightarrow \mathbb{RP}^n,$$

where $\widetilde{M}$ is the universal cover of M and

$$hol : \pi_1(M) \longrightarrow PGL(n+1,\mathbb{R}),$$

such that for all $\widetilde{m} \in \widetilde{M}$ and $g \in \pi_1(M)$, we have

$$dev(g \cdot \widetilde{m}) = hol(g) \cdot dev(\widetilde{m}).$$

Let $[A]$ and $[B]$ be the images of the generators of the fundamental group $\pi_1(M) \cong \mathbb{Z}_2 * \mathbb{Z}_2 =< a > * < b >$ under the holonomy map, meaning that

$$hol : \pi_1(M) \longrightarrow PGL(n+1,\mathbb{R})$$

$$a \longmapsto hol(a) = [A],$$

$$b \longmapsto hol(b) = [B],$$

where $A, B \in GL(n+1,\mathbb{R})$.

The infinite cyclic normal subgroup $\pi_1(\widetilde{W} \times S^1) \cong \mathbb{Z}$ of $\pi_1(M)$ is generated by the product $c = ab$. For the subgroup $< a, c^n >$ of $\pi_1(M)$, there is an n-sheeted covering space $M^{(n)} \longrightarrow M$ such that the manifold $M^{(n)}$ is diffeomorphic to M. To see this consider the following n-fold cover.

$$\begin{aligned} \widetilde{W} \times S^1 \quad &\longrightarrow \quad \widetilde{W} \times S^1 / < \rho > \cong \widetilde{W} \times S^1, \\ (x, z) \quad &\longmapsto \quad (x, z^n), \end{aligned}$$

and taking the quotient of both sides with σ, we get

$$M \cong \widetilde{W} \times S^1 / < \sigma > \longrightarrow M^{(n)} \cong \widetilde{W} \times S^1 / < \rho, \sigma >,$$

hence we have a diffeomorphism between M and $M^{(n)}$.

Remark 3.3.1. *If a manifold M admits a real projective structure, then any covering space of M admits a real projective structure. In other words, if a covering space of M does not admit a real projective structure then M can not admit a real projective structure.*

In the fundamental group $\pi_1(M)$, $c = ab$ and c^{-1} are conjugate i.e.

$$c^{-1} = (ab)^{-1} = b^{-1}a^{-1} = ba = b(ab)b^{-1} = bcb^{-1}.$$

Let $C = AB$. Then the matrices C and C^{-1} are conjugate and for each eigenvalue λ of C, λ^{-1} is also an eigenvalue of C and the multiplicities of λ and λ^{-1} are equal. By passing to the double cover $M^{(2)}$ of M, we obtain a real projective structure on $M^{(2)}$ for which the matrices A and B are conjugate (this is explained in the proof of Lemma 3.3.2).

Lemma 3.3.2. *It is possible to arrange that C is diagonalizable over real numbers and has positive eigenvalues.*

Proof. First, note that A and B are conjugate on the double cover $M^{(2)}$ of M:

$$c^2 = abab = a'b',$$

where $a' = a$ and $b' = bab$. Therefore, the images of a' and b' are A and BAB, which are conjugate elements. Since $a^2 = 1$ in $\pi_1(M)$ and hol is a homomorphism, $[A]^2 \in PGL(n+1, \mathbb{R})$ is identity. Moreover, after rescaling A we have $A^2 = \pm Id$ and thus A is diagonalizable over complex numbers. If $A^2 = Id$ then the eigenvalues are ± 1. Since we are only interested in $[A]$, we can multiply A with -1 and arrange that the eigenvalue -1 has multiplicity at most $\frac{n+1}{2}$ (if n is odd) and $\frac{n}{2}$ (if n is even).

Depending on the dimension n of M, we have the following cases.

1. If the dimension n is odd, there exist $\frac{n+1}{2} + 1$ possible cases for A. If $A^2 = -Id$ then the corresponding $(n+1) \times (n+1)$ matrix is given below:

$$\text{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & -1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & -1 & 0 \end{bmatrix}.$$

In the remaining cases, $A^2 = Id$ and there are $\frac{n+1}{2}$ possibilities. In the diagonal there exist only ± 1 and all off-diagonal elements are 0. Each

A_i has eigenvalues ± 1 and i represents the number of -1 eigenvalues, where $i \in \{1, 2, ..., (n+1)/2\}$. For example, A_3 is as follows:

$$\mathrm{A}_3 = \begin{bmatrix} -1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & -1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & -1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}.$$

2. If the dimension n is even, there exist $\frac{n}{2}$ possible cases for A. Here, the case $A^2 = -Id$ is not possible since in this case all the eigenvalues of A are $\pm i$ and $n+1$ is odd. For $A^2 = Id$, the possible A_i matrices are as in the odd dimensional case.

Since the matrices A and B are conjugate, there is an element $P \in GL(n+1, \mathbb{R})$ such that $B = PAP^{-1}$. Then $C = AB = APAP^{-1}$. As described in Lemma 2.0.6, we can deform the holonomy by changing P.

Now, define the maps

$$f : GL(n+1, \mathbb{R}) \longrightarrow SL(n+1, \mathbb{R})$$

given by $f(P) = APAP^{-1}$ and

$$g : SL(n+1, \mathbb{R}) \longrightarrow \mathbb{R}^k, \quad \text{given by}$$

$$g(Q) = (\text{trace}(Q), \text{trace}(Q^2), ..., \text{trace}(Q^k)),$$

where k is the number of -1 eigenvalues of A.

Note that these two maps are regular. In other words, they are given by rational functions in the entries of P and Q. Choosing an appropriate P to deform the holonomy map, depending on the number of -1 eigenvalues of A, can be done in cases as follows:

Case 1: If A has only one -1 eigenvalue then the $+1$ eigenspaces of A and B will have an intersection of dimension at least $n-1$ (if the dimension of the manifold is n) for every choice of P. Since $C = AB$, there is an $(n-1)$-dimensional subspace, on which C is identity. Hence, C has eigenvalue 1 with multiplicity at least $(n-1)$. Moreover, by choosing the matrix P as described below one can see that trace$(f(P))$ is a nonconstant function. Since trace$(f(P)) \neq n+1$, there exist two more eigenvalues λ and λ^{-1} of C. Here we can assume $\lambda \neq \pm 1$ by replacing C with C^2 if needed, and clearly $\lambda^{-1} \neq \pm 1$ (replacing C with C^2 can be done replacing M with its diffeomorphic copy $M^{(2)}$). It follows that C has 3 different eigenvalues λ, λ^{-1} and 1.

We can take $P = (a_{ij})_{t\times t}$, where $t = n+1$ as follows:

- If t is even, let

$$a_{k1} = a_{1k} = \begin{cases} 1, & \text{if } k \text{ is odd,} \\ 0, & \text{if } k \text{ is even,} \end{cases}$$

$$a_{kt} = \begin{cases} 0, & \text{if } k \text{ is odd,} \\ 1, & \text{if } k \text{ is even,} \end{cases}$$

$$a_{tk} = \begin{cases} 0, & \text{if } k \text{ is odd,} \\ 1, & \text{if } k \text{ is even and } k \neq 2, \\ x, & \text{if } k = 2, \end{cases}$$

and the core $(t-2)\times(t-2)$ matrix of P

$$\begin{bmatrix} a_{22} & a_{23} & a_{24} & \dots & a_{2(t-1)} \\ a_{32} & a_{33} & a_{34} & \dots & a_{3(t-1)} \\ a_{42} & a_{43} & a_{44} & \dots & a_{4(t-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{(t-1)2} & a_{(t-1)3} & a_{(t-1)4} & \dots & a_{(t-1)(t-1)} \end{bmatrix}$$

is the anti-diagonal matrix, which is as follows:

$$\begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

For instance, if A has only one eigenvalue and $t = 10,$ we choose the matrix $P_{t\times t}$ as below.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

In this case, for any $n \geq 4,$

$$\text{trace}(f(P)) = \frac{\dfrac{t^2-6t+8}{2}x + \dfrac{t^3-10t^2+28t-32}{4}}{\dfrac{t-2}{2}x + \dfrac{t^2-6t+4}{4}}.$$

• If t is odd, let

$$a_{1k} = \begin{cases} 0, & \text{if } k \geq 3 \text{ is odd or } k = 2, \\ 1, & \text{if } k \text{ is even and } k \neq 2 \text{ or } k = 1, \end{cases}$$

$$a_{(k+1)1} = a_{kt} = \begin{cases} 0, & \text{if } k \text{ is odd}, \\ 1, & \text{if } k \text{ is even}, \end{cases}$$

$$a_{tk} = \begin{cases} 0, & \text{if } k \text{ is odd and } k \neq 1, \\ 1, & \text{if } k \text{ is even and } k \neq 2, \\ x, & \text{if } k = 2, \end{cases}$$

and the core $(t-2) \times (t-2)$ matrix is the identity matrix. In this case,

$$\text{trace}(f(P)) = t - 1 + \frac{x + 2t - 6}{x}.$$

For example, for $t = 9$ we choose the matrix $P_{t\times t}$ as below.

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & x & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

Case 2: If A has two -1 eigenvalues, we take P as follows:

• If t is even,

when $k = t/2 \Rightarrow a_{k1} = y$, else (i.e. $k \neq t/2$)

$$a_{k1} = \begin{cases} 1, & \text{if } k \text{ is odd,} \\ 0, & \text{if } k \text{ is even.} \end{cases}$$

Let $a_{12} = y + x,\quad a_{1(t-1)} = y$ and $a_{1k} = 0,$ for $3 \leq k \leq t - 2$,

$a_{t2} = x,\quad a_{t(t-1)} = y - x$ and $a_{tk} = 0,$ for $3 \leq k \leq t - 2$.

When $k = (t/2) + 1 \Rightarrow a_{kt} = x$, else (i.e. $k \neq (t/2) + 1$)

$$a_{kt} = \begin{cases} 0, & \text{if } k \text{ is odd,} \\ 1, & \text{if } k \text{ is even,} \end{cases}$$

and the core matrix $(t - 2) \times (t - 2)$ is the identity matrix. In this case,

$$\begin{aligned} \text{trace}(Q) = t - 4 &- \frac{2(-1 + x)}{1 - x - y + yx - y^2 + x^2} - \frac{(-y - x)(-y + x)}{1 - x - y + yx - y^2 + x^2} \\ &- \frac{(-1 + x)y}{1 - x - y + yx - y^2 + x^2} - \frac{2(y - 1)}{1 - x - y + yx - y^2 + x^2} \\ &- \frac{-x + xy - y^2 + x^2}{1 - x - y + yx - y^2 + x^2} - \frac{-y^2 - y + yx + x^2}{1 - x - y + yx - y^2 + x^2} \\ &- \frac{x(y - 1)}{1 - x - y + yx - y^2 + x^2} - \frac{(y - x)(y + x)}{1 - x - y + yx - y^2 + x^2}, \end{aligned}$$

where $Q = APAP^{-1}$.

Consider the composition below:

$$\mathbb{R}^2 \longrightarrow GL(n+1,\mathbb{R}) \longrightarrow SL(n+1,\mathbb{R}) \longrightarrow \mathbb{R}^2,$$

given by

$$(x,y) \longmapsto P \longmapsto f(P) = APAP^{-1} = Q \longmapsto g(Q) = (\text{trace}(Q), \text{trace}(Q^2))$$

and the Jacobian matrix is given by

$$J = \begin{bmatrix} \dfrac{\partial \, \text{trace}(Q)}{\partial x} & \dfrac{\partial \, \text{trace}(Q)}{\partial y} \\ \dfrac{\partial \, \text{trace}(Q^2)}{\partial x} & \dfrac{\partial \, \text{trace}(Q^2)}{\partial y} \end{bmatrix}.$$

The determinant of the Jacobian at some point is nonzero. For example at the point $(2,3)$, the determinant is -128.

If A has two -1 eigenvalues and for $t = 10$, we choose $P_{t\times t}$ as below.

$$\begin{bmatrix} 1 & y+x & 0 & 0 & 0 & 0 & 0 & 0 & y & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ y & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & x \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & x & 0 & 0 & 0 & 0 & 0 & 0 & y-x & 1 \end{bmatrix}.$$

• If t is odd,

set $k = (t-1)/2$ and $a_{k1} = y$. If $k \neq (t-1)/2$, let

$$a_{k1} = \begin{cases} 1, & \text{if } k \text{ is odd,} \\ 0, & \text{if } k \text{ is even,} \end{cases}$$

$a_{t2} = x, \quad a_{t(t-1)} = y - x, \quad a_{tk} = 0, \quad$ for $3 \leq k \leq t-2$,

$a_{12} = y + x, \quad a_{1(t-1)} = y.$ If $t \neq 5$, take $a_{1((t+3)/2)} = a_{1((t-1)/2)} = 1;$ otherwise, $a_{1k} = 0$, for $3 \leq k \leq t-2$ and if $t = 5$ then $a_{13} = 1$. When

$k = ((t+1)/2)+1$ let $a_{kt} = x$. Otherwise, (i.e. $k \neq ((t+1)/2)+1$)

$$a_{kt} = \begin{cases} 0, & \text{if } k \text{ is odd,} \\ 1, & \text{if } k \text{ is even,} \end{cases}$$

and the core matrix $(t-2) \times (t-2)$ is the identity matrix.

If A has two -1 eigenvalues and $t = 9$ then we choose $P_{t \times t}$ as below.

$$\begin{bmatrix} 1 & y+x & 0 & 1 & 0 & 1 & 0 & y & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ y & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & x \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & x & 0 & 0 & 0 & 0 & 0 & y-x & 0 \end{bmatrix}.$$

If $(t-1)/2$ is even then

$$\begin{aligned} \text{trace}(Q) = t - 6 &- \frac{2y}{1+y+2x+y^2} - \frac{-y-x}{1+y+2x+y^2} - \frac{1+y^2}{1+y+2x+y^2} \\ &- \frac{-1-2x-y^2+yx}{1+y+2x+y^2} - \frac{y(1+x)}{1+y+2x+y^2} - \frac{-y^2}{1+y+2x+y^2} \\ &+ \frac{y+2x}{1+y+2x+y^2} + \frac{1}{1+y+2x+y^2} + \frac{1+y+x+y^2}{1+y+2x+y^2} \\ &+ \frac{1+y+x+yx}{1+y+2x+y^2} - \frac{-x-y^2+yx}{1+y+2x+y^2} - \frac{1+2y+2x}{1+y+2x+y^2} \\ &- \frac{x(-1+y)}{1+y+2x+y^2} + \frac{(y-x)(-1+y)}{1+y+2x+y^2}. \end{aligned}$$

Considering the same map with the case t is even, we get the determinant of the Jacobian matrix at $(2,3)$ is $-\dfrac{1792}{4913}$.

If $(t-1)/2$ is odd then

$$\begin{aligned}\text{trace}(Q) = t - 6 &- \frac{3y}{y^2+2y+2x} - \frac{-y-x}{y^2+2y+2x} - \frac{-2y^2}{y^2+2y+2x} \\ &- \frac{y+x}{y^2+2y+2x} - \frac{-y^2+yx-y-x}{y^2+2y+2x} - \frac{xy+y+x}{y^2+2y+2x} \\ &+ \frac{y^2+y+x}{y^2+2y+2x} + \frac{x}{y^2+2y+2x} + \frac{yx+2x+y}{y^2+2y+2x} \\ &- \frac{y(x-y-1)}{y^2+2y+2x} - \frac{xy}{y^2+2y+2x} + \frac{y(y-x)}{y^2+2y+2x}.\end{aligned}$$

The determinant of the Jacobian at $(2,3)$ is $-\frac{768}{6859}$.

In each case the determinant of the Jacobian is nonzero and thus the image of the map $f \circ g$ contains an open set.

Case 3: If A has more than two -1 eigenvalues, we take P as below.

First, consider the following composition.

$$\mathbb{R}^k \longrightarrow GL(n+1,\mathbb{R}) \longrightarrow SL(n+1,\mathbb{R}) \longrightarrow \mathbb{R}^k,$$

given by

$$(x_1, x_2, ..., x_k) \longmapsto P \longmapsto f(P) = APAP^{-1} = Q \longmapsto g(Q),$$

where $g(Q) = (\text{trace}(Q), \text{trace}(Q^2), ..., \text{trace}(Q^k))$ and k is the number of -1 eigenvalues of A. The Jacobian matrix is given by

$$\mathbf{J} = \begin{bmatrix} \frac{\partial\, \text{trace}(Q)}{\partial x_1} & \frac{\partial\, \text{trace}(Q)}{\partial x_2} & \cdots & \frac{\partial\, \text{trace}(Q)}{\partial x_k} \\ \frac{\partial\, \text{trace}(Q^2)}{\partial x_1} & \frac{\partial\, \text{trace}(Q^2)}{\partial x_2} & \cdots & \frac{\partial\, \text{trace}(Q^2)}{\partial x_k} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial\, \text{trace}(Q^k)}{\partial x_1} & \frac{\partial\, \text{trace}(Q^k)}{\partial x_2} & \cdots & \frac{\partial\, \text{trace}(Q^k)}{\partial x_k} \end{bmatrix}.$$

• If t is even,

let $a_{12} = x_2$, $a_{1(t/2)} = a_{1(t+2)/2} = x_3$, $a_{1(t-1)} = x_1$, $a_{2(t-2)} = x_3$, $a_{(t/2)1} = x_2$, $a_{((t+2)/2)1} = x_3$, $a_{(t/2)t} = x_3$, $a_{((t+2)/2)t} = x_1$, $a_{(t-1)1} = 1$, $a_{t2} = x_3$, $a_{t(t-1)} = x_2$, and all the diagonal elements are 1.

According to the number of -1 eigenvalues of A, we determine the number of different variables $x_i \in \mathbb{R}$, where $3 \leq i \leq k$ and $k = t/2$. In the core matrix, on the antidiagonal there are only x_i's (except x_3) as a pair, which are symmetric with respect to the diagonal. Moreover, the number of some x_i's are more than two conforming to the dimension. In addititon, other entries of P are all 0.

For example, if A has six -1 eigenvalues and $t = 14$ then P is as below.

$$P = \begin{bmatrix} 1 & x_2 & 0 & 0 & 0 & 0 & x_3 & x_3 & 0 & 0 & 0 & 0 & x_1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_3 & x_2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & x_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & x_6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & x_4 & 0 & 0 & 0 & 0 & 0 \\ x_2 & 0 & 0 & 0 & 0 & 0 & 1 & x_1 & 0 & 0 & 0 & 0 & 0 & x_3 \\ x_3 & 0 & 0 & 0 & 0 & 0 & x_1 & 1 & 0 & 0 & 0 & 0 & 0 & x_1 \\ 0 & 0 & 0 & 0 & 0 & x_4 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_6 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_5 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & x_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_2 & 1 \end{bmatrix}.$$

At the point $(2, 3, 4, 5, 6, 7)$ the determinant of the Jacobian is

$$\frac{3203652023}{129225403018523774123952000}.$$

• If t is odd,

let $a_{12} = x_2, \quad a_{1(t+1)/2} = x_3, \quad a_{1(t-1)} = x_1, \quad a_{2(t-2)} = x_3,$
$a_{((t-1)/2)1} = x_2, \quad a_{((t+1)/2)1} = 1, \quad a_{((t+3)/2)1} = x_3, \quad a_{(t-1)1} = 1,$
$a_{((t-1)/2)t} = x_3, \quad a_{((t+3)/2)t} = x_1, \quad a_{t2} = x_3, \quad a_{t(t-1)} = x_2$ and the diagonal elements are all 1.

In the core matrix, on the antidiagonal there are only x_i's (except x_3) as a pair, which are symmetric with respect to the diagonal. Moreover, the number of some

x_i's are more than two conforming to the dimension. In addititon, other entries of P are all 0.

For example, if A has five -1 eigenvalues and $t = 13$ then P is as follows:

$$P = \begin{bmatrix} 1 & x_2 & 0 & 0 & 0 & 0 & x_3 & 0 & 0 & 0 & 0 & x_1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_3 & x_2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & x_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & x_4 & 0 & 0 & 0 & 0 \\ x_2 & 0 & 0 & 0 & 0 & 1 & 0 & x_1 & 0 & 0 & 0 & 0 & x_3 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_3 & 0 & 0 & 0 & 0 & x_1 & 0 & 1 & 0 & 0 & 0 & 0 & x_1 \\ 0 & 0 & 0 & 0 & x_4 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_5 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & x_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_2 & 1 \end{bmatrix}.$$

At the point $(2,3,4,5,6)$ the determinant of the Jacobian is

$$\frac{74929536}{42961619719375}.$$

Case 4: If A has eigenvalues $\pm i$ then both $+i$ eigenspace and $-i$ eigenspace of A are $\dfrac{n+1}{2}$ dimensional. Now, we choose P as in Case 3 with $k = \dfrac{n+1}{2}$ variables.

Note that the calculations above are done with the program Maple.

For each case except Case 1, for a generic P, the dimension of $+1$ eigenspace of C is $n-2k$. We call a matrix P admissible if the number of distinct eigenvalues of C is $2k+1$, which are 1 and some pairs $\lambda_1^{\pm 1}, \lambda_2^{\pm 1}, ..., \lambda_k^{\pm 1}$, $\lambda_i \neq \lambda_j^{\pm 1}$ and $\lambda_i \neq 1$. Let E denote the set of non-admissible matrices P in $GL(n+1, \mathbb{R})$. Below we will show that E is a proper algebraic set in $GL(n+1, \mathbb{R})$ and thus the set of admissible matrices constitutes an open dense subset in $GL(n+1, \mathbb{R})$.

Let T be the set of eigenvalues of C. Since C is conjugate to C^{-1}, there is an involution on T. If we take $P \in E$ then either some $\lambda_i = 1$ or $\lambda_i = \lambda_j^{\pm 1}$, for some $i \neq j$. First, assume that some $\lambda_i = 1$. Without loss of generality, let $\lambda_k = 1$. Then

$$\text{trace}(C) = m + \sum_{i=1}^{k-1} (\lambda_i + \lambda_i^{-1}).$$

In this case, trace(C), trace(C^2), ... , trace(C^k) satisfy an algebraic relation. On the other hand, if some $\lambda_i = \lambda_j^{\pm 1}$, for some $i \neq j$, then again without loss of generality, we may assume that $\lambda_{k-1} = \lambda_k$. Then

$$\text{trace}(C) = m + \sum_{i=1}^{k-1} a_i(\lambda_i + \lambda_i^{-1}),$$

where $a_i = 1$, for $1 \leq i \leq k-2$ and $a_{k-1} = 2$. Hence, again trace(C), trace(C^2), ... , trace(C^k) satisfy an algebraic relation, where $1 \leq k \leq (n+1)/2$ or $1 \leq k \leq n/2$ and $\dim [g \circ f(E)] = k-1$. For example, if all eigenvalues are λ_1 and λ_1^{-1} then

$$\text{trace}(C) = \frac{n+1}{2}(\lambda_1 + \lambda_1^{-1}) \text{ and } \text{trace}(C^2) = \frac{n+1}{2}(\lambda_1^2 + \lambda_1^{-2}).$$

Now, trace(C) and trace(C^2) satisfy the following algebraic relation

$$(\text{trace}(C))^2 = ((n+1)/2)(\text{trace}(C^2)) + (n+1)^2/2.$$

Since the determinant of the Jacobian of $g \circ f$ is nonzero at some points, for example $(2, 3, ..., k+1)$, (as shown on pages 19, 20, 21, 22, 24, 25), the image of the map $g \circ f$ contains an open set and thus E is a closed proper subset of $GL(n+1, \mathbb{R})$. It follows that $GL(n+1, \mathbb{R}) \setminus E$ is open and dense in the Euclidean topology. Therefore, it is possible to perturb P slightly and thus the map hol so that the matrix C is diagonalizable over complex numbers.

With a proper choice of P, it can be arranged that the arguments of complex eigenvalues λ_i of C are rational multiples of π. Moreover, passing to a finite covering space $M^{(n)}$ of M (see page 15), we can suppose all eigenvalues of C are real and by passing to a further double cover these eigenvalues can be assumed to be positive. □

Hence, we have proved the following lemma for the case $A^2 = Id$.

Lemma 3.3.3. *We can arrange that* C_i *corresponding to* A_i *so that its eigenvalues are* $\{\lambda_i^{\pm}\}$ *such that* $\lambda_i > \lambda_{i-1} > ... > \lambda_1 > 1,$ *where* $i \in \{1, ..., [(n+1)/2]\}$ *and the remaining eigenvalues of* C_i *are all* $1.$

When $A^2 = Id,$ the C_i matrices are as follows, depending on the number of -1 eigenvalues of $A_i.$ Namely, the number of -1 eigenvalues of A determine the number of different eigenvalues of $C.$ For example, if the number of -1 eigenvalues of A is 2 then the corresponding C is given by

$$\begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \lambda_1^{-1} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \lambda_2^{-1} & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}.$$

When $A^2 = -Id,$ the corresponding matrix C has two eigenvalues λ_1 and λ_1^{-1} whose multiplicities are both equal to $\dfrac{n+1}{2}.$

Note that in each $C_i,$ the multiplicities of the eigenvalues λ and λ^{-1} are equal since C is conjugate to $C^{-1}.$

Let $N \longrightarrow M$ be the corresponding double cover to the infinite cyclic normal subgroup of $\pi_1(M)$ generated by $c = ab.$ Therefore, $N \cong \widetilde{W} \times S^1$ has a real projective structure via pull-back from M with the same developing map $dev_M = dev_N.$ For this projective structure on $N,$ the image of holonomy is generated by $[C].$

Next, observe that there exists a diagonal subgroup G of $PGL(n+1, \mathbb{R}),$
$\rho : \mathbb{R} \longrightarrow G$ such that $\rho(1) = [C],$ where the group G is given as the unique one parameter subgroup containing the cyclic group H which is generated by C and such that every element of G has real eigenvalues (it is the geodesic starting at Id and passing through the matrix C). Since the group H is normal in $hol(\pi_1(M)),$ G is normalized by $hol(\pi_1(M)).$

Let $z \in gl(n+1, \mathbb{R})$ be an infinitesimal generator of G such that $G = exp(\mathbb{R} \cdot z)$.

There is a flow given by

$$\Phi : \mathbb{RP}^n \times \mathbb{R} \longrightarrow \mathbb{RP}^n$$

on $\mathbb{RP}^n$ generated by G such that

$$\Phi(x, t) = exp(tz) \cdot x,$$

where $t \in \mathbb{R}$ and $x \in \mathbb{RP}^n$.

Let V be the velocity vector field of the flow Φ on $\mathbb{RP}^n$. The zeroes of this vector field are the fixed points of the flow. Since the vector field V is preserved by this flow, V is also preserved by $hol(\pi_1(N))$. Hence, V pulls back via the map dev to a vector field $dev^{-1}(V) = \widetilde{v}$ on $\widetilde{N}$ and it is invariant under the Deck transformations and hence covers a vector filed $\pi(dev^{-1}(V)) = v$ of N by following the diagram below.

$$\begin{array}{ccc} \widetilde{N} & \xrightarrow{dev} & \mathbb{RP}^n \\ \downarrow{\scriptstyle \pi} & & \\ N & & \end{array}$$

The following two lemmas are proved for 3 dimension in [8]. Moreover, the results are still valid in our case and we give the proofs below, for any dimension $n \geq 4$ and for any C_i such that $1 \leq i \leq \dfrac{n+1}{2}$ (n is odd) or $1 \leq i \leq \dfrac{n}{2}$ (n is even).

Lemma 3.3.4. *Let N be the double cover of the manifold M. Then $dev(\widetilde{N})$ contains no source or sink.*

Proof. Let us consider a flowline starting at source and ending at sink. If we reverse the flowline, a source can be changed into a sink. Hence, we can assume that p is a sink in $dev(\widetilde{N})$. Let Q be the $(n-1)$-dimensional projective space which contains the other sink and source points corresponding to the other eigenspaces of C. Then Q is invariant under $hol(\pi_1(N))$.

There exists a decomposition into disjoint subspaces $\mathbb{RP}^n = p \cup \Omega \cup Q$ where $\Omega = S^{n-1} \times \mathbb{R}$ is the basin of attraction for p. Moreover, each of these subspaces is preserved by $hol(\pi_1(N))$. Hence, there is a corresponding decomposition of N

such that

$$N = \pi(dev^{-1}(p)) \cup \pi(dev^{-1}(Q)) \cup \pi(dev^{-1}(\Omega)).$$

Ω has a foliation with concentric spheres centered on p which is preserved by the flow induced by the vector field V and thus by $hol(\pi_1(N))$. This gives a foliation on $\mathbb{RP}^n \setminus p$ by leaves, one of them is $Q = \mathbb{RP}^{n-1}$ and all others are $(n-1)-$spheres. Thus, this induces a foliation on $N \setminus \pi(dev^{-1}(p))$. Since the image $\pi(dev^{-1}(p))$ is not empty, every leaf around it is a small sphere.

Therefore, N has a singular foliation where the singular points are isolated and have a neighborhood foliated by concentric spheres. By Reeb Stability Theorem 2.0.9, if a compact connected n-manifold admits a foliation such that each component of the boundary is a leaf and some leaves are spheres, then the manifold is either $S^{n-1} \times I$ or a punctured $\mathbb{RP}^n$, which have all finite fundamental groups.

Hence, after removing finitely many open balls from $N = \widetilde{W} \times S^1$, we obtain a manifold with infinite fundamental group. However, it is a contradiction. □

Lemma 3.3.5. *The flow given by the vector field v on N is periodic and N is fibered as a product $N = \widetilde{W} \times S^1$ by the flowlines.*

Proof. Let γ be a flowline, including the end points, of V in $\mathbb{RP}^n$ starting at the source $p_{++\ldots+}$ and ending at the sink $p_{--\ldots-}$. Since such flowlines are dense, we can choose γ to intersect with $dev(\widetilde{N})$ at a point. Since the developing map $dev : \widetilde{N} \longrightarrow \mathbb{RP}^n$ is an immersion and both $\widetilde{N}$ and $\mathbb{RP}^n$ are same dimensional, it is also a submersion, i.e. a local homeomorphism. Then the preimage of a 1-dimensional manifold of $\mathbb{RP}^n$ is a 1-dimensional manifold in $\widetilde{N}$. Therefore, $dev^{-1}(\gamma)$ is a nonempty closed manifold of $\widetilde{N}$ since source and sink are not in $dev(\widetilde{N})$ by Lemma 3.3.4. Moreover, $\pi(dev^{-1}(\gamma))$ is a compact nonempty 1-dimensional submanifold in N. Let Γ be a component of this submanifold. If Γ were contractible in N (a circle which bounds a disc) then it would lift to a circle in $\widetilde{N}$ and would be mapped to $\mathbb{RP}^n$ by the map dev. However, this says that immersed image of a circle is a line which is not possible. Hence, $[\Gamma] \neq 0$ in $\pi_1(N)$.

Let $T > 0$ be the period of the closed flowline Γ. Let $U \subset N$ be the union of closed flowlines of period T. U is nonempty and N is connected. Now, we will show that U is both open and closed, so that U is equal to N.

Choose a small disc $D \subset N$ which is transverse to the flow and meeting Γ once. Let $\widetilde{D} \subset \widetilde{N}$ be a lift meeting $\widetilde{\Gamma} \subset \pi^{-1}(\Gamma)$. The union of the flowlines $\widetilde{Y}$ in $\widetilde{N}$ which meets $\widetilde{D}$ maps homeomorphically into a foliated neighborhood of the interior of γ by the map dev.

Let τ be the Deck transformation of $\widetilde{N}$ given by $[\Gamma] \in \pi_1(N)$. Then τ preserves $dev(\widetilde{Y})$ and $\widetilde{\Gamma}$ so preserves $\widetilde{Y}$.

$$Y \doteq \widetilde{Y}/\tau \cong dev(\widetilde{Y})/hol(\Gamma) \cong S^1 \times D^{n-1}$$

is foliated as a product.

Thus Y is a neighborhood of Γ in N foliated as a product by flowlines. This shows that $U \subset N$ is open.

The limit of the flowlines with period T is a closed flowline of period T/m for some integer $m > 0$. But $m = 1$ since the set of flowlines of period T/m is open. Thus $U \subset N$ is closed. □

Let $X = \mathbb{RP}^n \setminus Z$, where Z is the zero set of the vector field V. Then X is foliated by flowlines. Let $\mathcal{L}$ be the leaf space of this foliation on X. G is normalized by $hol(\pi_1(M))$ and thus it acts on $\mathcal{L}$. Since the action of $hol(\pi_1(N))$ on $\mathcal{L}$ is trivial, the action $hol(\pi_1(M))$ on $\mathcal{L}$ is induced by the involution σ (see Section 3.1). Therefore, the holonomy gives an involution on the leaf space $\mathcal{L}$.

Since $dev(\widetilde{N}) \subset X$, where $\widetilde{N}$ is the universal cover of N, there is a map from the leaf space of the induced foliation on $\widetilde{N}$ into $\mathcal{L}$. The leaf space of $\widetilde{N}$ is $\widetilde{W}$ by Lemma 3.3.5. The induced map from the developing map

$$h : \widetilde{W} \longrightarrow \mathcal{L}$$

is a local homeomorphism. Since $dev(\widetilde{N}) \subset \mathbb{RP}^n$ is invariant under $hol(\pi_1(M))$ it follows that $h(\widetilde{W}) \subset \mathcal{L}$ is invariant under the involution τ.

Considering the possible generators $[C_i]$ of the image of the holonomy map, we determine the orbit space $\mathcal{L}_i$, corresponding to C_i, of $X = \mathbb{RP}^n \setminus Z$ in cases below.

3.3.1 The case of even dimension

If the dimension of M is even $n = 2k$ then there are $\frac{n}{2} = k$ possible cases. Namely, the zero set Z for the matrix C_i consists of the disjoint union of $2i$ points and a linear subspace $\mathbb{RP}^{2k-2i}$, where $1 \leq i \leq k$.

For C_1, the zero set consists of one source, one sink and a copy of $\mathbb{RP}^{2k-2}$. Call these elements as p_1: source, $p_2 : \mathbb{RP}^{2k-2}$ and p_3: sink. By taking the boundary of a regular neighborhood of each component of Z in $\mathbb{RP}^{2k}$, we determine the flowlines and the corresponding subspaces of orbits.

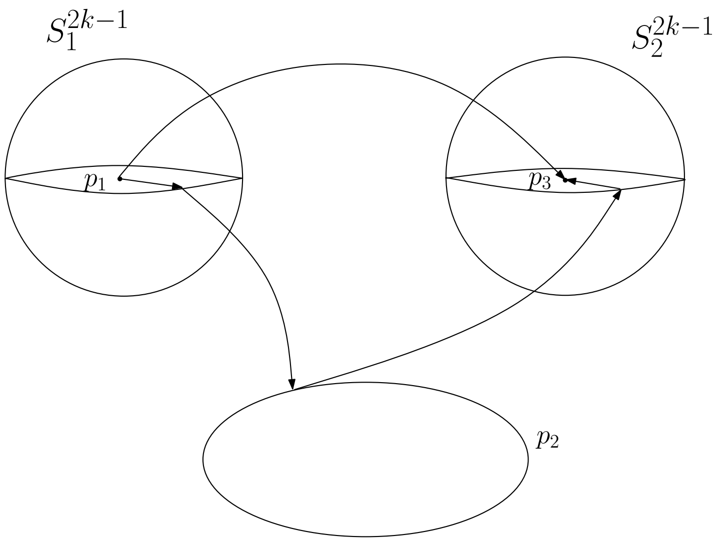


Figure 3.4: Flowlines for C_1 in even dimension.

The corresponding flow for C_1 is given by

$$\left[x_0{\lambda_1}^t : x_1{\lambda_1}^{-t} : x_2 : x_3 : ... : x_n\right].$$

When $x_1 \neq 0$, the flow can be written as

$$\left[\frac{x_0}{x_1}{\lambda_1}^{2t} : 1 : \frac{x_2}{x_1}{\lambda_1}^t : \frac{x_3}{x_1}{\lambda_1}^t : ... : \frac{x_n}{x_1}{\lambda_1}^t\right].$$

Note that as $t \to -\infty$ it flows to its source, which is $\left[0 : 1 : 0 : 0 : ... : 0\right]$.

If $x_0 \neq 0$ then the flow can be written as

$$\left[1 : \frac{x_1}{x_0}\lambda_1{}^{-2t} : \frac{x_2}{x_0}\lambda_1{}^{-t} : \frac{x_3}{x_0}\lambda_1{}^{-t} : ... : \frac{x_n}{x_0}\lambda_1{}^{-t}\right].$$

Similarly, as $t \to +\infty$ it flows to its sink, which is $\left[1 : 0 : 0 : 0 : ... : 0\right]$.

The Euclidean coordinates around the source are as

$$\left(\frac{x_0}{x_1}, \frac{x_2}{x_1}, \frac{x_3}{x_1}, ..., \frac{x_n}{x_1}\right) \in \mathbb{R}^n$$

and around the sink are as

$$\left(\frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{x_3}{x_0}, ..., \frac{x_n}{x_0}\right) \in \mathbb{R}^n.$$

For the flowlines starting at p_1 and leaving S_1 from the northern hemisphere we assume $\frac{x_0}{x_1} > 0$, where $x_1 \neq 0$ and thus $\frac{x_1}{x_0} > 0$, where $x_0 \neq 0$. Therefore, the flowlines go to p_3 from the northern hemisphere of S_2, see Figure 3.4. Similarly, when $\frac{x_0}{x_1} < 0$, for $x_0 \neq 0$ and $x_1 \neq 0$ there is an identification between the southern hemispheres of S_1 and S_2. Hence, the leaf space $\mathcal{L}_1$ is a copy of S^{2k-1} with disjoint two equators. For the spheres with a double equator, we simply use the notation $\mathcal{S}$; therefore, $\mathcal{L}_1 = \mathcal{S}^{2k-1}$.

Note that the table below lists that the subspaces of flowlines starting at some p_i and ending at some other p_j, where $i, j = 1, 2, 3$ of the flowlines. The symbol $\emptyset$ indicates that there is no flowline between the pair of p_i's. In each table, we label source points in the first row and sink points in the first column, respectively.

source/sink	p_1	p_2	p_3
p_1	$\emptyset$	$\emptyset$	$\emptyset$
p_2	S^{2k-2}	$\emptyset$	$\emptyset$
p_3	S^{2k-1}	S^{2k-2}	$\emptyset$

Table 3.1: The subspaces of the leaf space $\mathcal{L}_1$ for even dimensional case.

For the matrix C_2, the flow is given by

$$\left[x_0\lambda_1^t : x_1\lambda_1^{-t} : x_2\lambda_2^t : x_3\lambda_2^{-t} : x_4 : x_5 : ... : x_n\right].$$

For C_2, the zero set consists of p_1, p_2, $p_3 = \mathbb{RP}^{2k-4}$, p_4, p_5, where p_1, p_2, p_4 and p_5 are points and the leaf space is $\mathcal{L}_2 = \mathcal{S}^{2k-1} \cup \mathcal{S}^{2k-3}$.

Similarly, the table below gives the list of subspaces of the space of flowlines for C_2 starting at p_i and ending at p_j, where $i,j = 1, ..., 5$.

source/sink	p_1	p_2	p_3	p_4	p_5
p_1	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
p_2	S^0	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
p_3	$S^{2k-3} - S^0$	S^{2k-4}	$\emptyset$	$\emptyset$	$\emptyset$
p_4	$S^{2k-2} - S^{2k-3}$	$S^{2k-3} - S^{2k-4}$	S^{2k-4}	$\emptyset$	$\emptyset$
p_5	$S^{2k-1} - S^{2k-2}$	$S^{2k-2} - S^{2k-3}$	$S^{2k-3} - S^0$	S^0	$\emptyset$

Table 3.2: The subspaces of the leaf space $\mathcal{L}_2$ for even dimensional case.

Note that in all cases the subspaces above the diagonal in each table are empty and the spheres on the antidiagonal have a double equator.

To understand the topology of the leaf space $\mathcal{L}$ better, we study the following example in detail: Let the manifold M be 6-dimensional and consider the matrix C_3 with distinct eigenvalues as below.

$$\begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_3^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The corresponding leaf space is $\mathcal{L}_3 = \mathcal{S}^5 \cup \mathcal{S}^3 \cup \mathcal{S}^1$, where $\mathcal{S}^5 = \mathcal{S}_1^5$, $\mathcal{S}^3 = \mathcal{S}_2^3$, $\mathcal{S}^1 = \mathcal{S}_3^1$ in Table 3.3. Moreover, the involution τ interchanges the spheres symmetric with respect to the vertical line through $\mathcal{S}_1^5$ in Table 3.3. Indeed, $\tau(S_j^0) = S_{7-j}^0$ and $\tau(\mathcal{S}_j^i) = \mathcal{S}_{7-i-j}^i$.

$$
\begin{array}{ccccccccccc}
 & & & & & \mathcal{S}_1^5 & & & & & \\
 & & & & \swarrow & & \searrow & & & & \\
 & & & & \mathcal{S}_1^4 & & \mathcal{S}_2^4 & & & & \\
 & & & \swarrow & & \searrow\ \swarrow & & \searrow & & & \\
 & & & \mathcal{S}_1^3 & & \mathcal{S}_2^3 & & \mathcal{S}_3^3 & & & \\
 & & \swarrow & & \searrow\ \swarrow & & \searrow\ \swarrow & & \searrow & & \\
 & & \mathcal{S}_1^2 & & \mathcal{S}_2^2 & & \mathcal{S}_3^2 & & \mathcal{S}_4^2 & & \\
 & \swarrow & & \searrow\ \swarrow & & \searrow\ \swarrow & & \searrow\ \swarrow & & \searrow & \\
 & \mathcal{S}_1^1 & & \mathcal{S}_2^1 & & \mathcal{S}_3^1 & & \mathcal{S}_4^1 & & \mathcal{S}_5^1 & \\
\swarrow & & \searrow\ \swarrow & & \searrow\ \swarrow & & \searrow\ \swarrow & & \searrow\ \swarrow & & \searrow \\
S_1^0 & & S_2^0 & & S_3^0 & & S_4^0 & & S_5^0 & & S_6^0
\end{array}
$$

Table 3.3: The leaf space $\mathcal{L}_3 = \mathcal{S}^5 \cup \mathcal{S}^3 \cup \mathcal{S}^1$. Note that all the spheres in the diagram except the ones in the last row have a double equator.

source/sink	p_1	p_2	p_3	p_4	p_5	p_6	p_7
p_1	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
p_2	S^0	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
p_3	$S^1 - S^0$	S^0	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
p_4	$S^2 - S^1$	$S^1 - S^0$	S^0	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
p_5	$S^3 - S^2$	$S^2 - S^1$	$S^1 - S^0$	S^0	$\emptyset$	$\emptyset$	$\emptyset$
p_6	$S^4 - S^3$	$S^3 - S^2$	$S^2 - S^1$	$S^1 - S^0$	S^0	$\emptyset$	$\emptyset$
p_7	$S^5 - S^4$	$S^4 - S^3$	$S^3 - S^2$	$S^2 - S^1$	$S^1 - S^0$	S^0	$\emptyset$

Table 3.4: The subspaces of the leaf space $\mathcal{L}_3$ for dimension 6.

Now, consider the matrix C_2 in dimension 6.

$$
\begin{bmatrix}
\lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda_1^{-1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda_2^{-1} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

such that the zero set consists of $p_1,\ p_2,\ p_3 = \mathbb{RP}^2,\ p_4,\ p_5,$ where $p_1,\ p_2,\ p_4$

and p_5 are points and the corresponding leaf space is $\mathcal{L}_2 = \mathcal{S}^5 \cup \mathcal{S}^3$.

Note the change in the subspaces of the leaf space for the flowlines starting or ending at the $\mathbb{RP}^2$ component of the zero set. For example, in Table 3.5, $p_3 = \mathbb{RP}^2$ and the subspaces of flowlines starting from p_3 or the flowlines ending at p_3 are different from the subspaces corresponding to the flowlines to other p_i's. Compare this with Table 3.4.

source/sink	p_1	p_2	p_3	p_4	p_5
p_1	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
p_2	S^0	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
p_3	$S^3 - S^0$	S^2	$\emptyset$	$\emptyset$	$\emptyset$
p_4	$S^4 - S^3$	$S^3 - S^2$	S^2	$\emptyset$	$\emptyset$
p_5	$S^5 - S^4$	$S^4 - S^3$	$S^3 - S^0$	S^0	$\emptyset$

Table 3.5: The subspaces of the leaf space $\mathcal{L}_2$ for dimension 6.

Then the leaf space $\mathcal{L}_2$ is as follows:

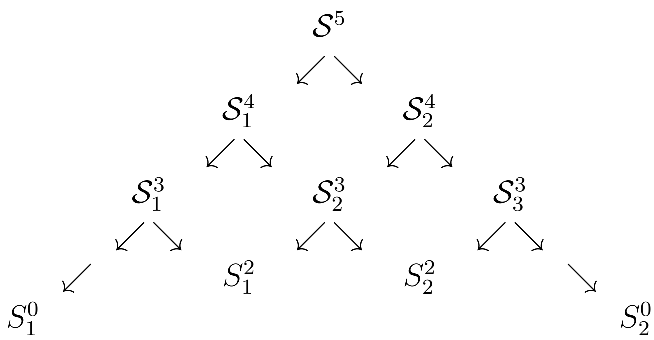


Table 3.6: The leaf space $\mathcal{L}_2 = \mathcal{S}^5 \cup \mathcal{S}^3$.

We have a similar table for each C_i matrix. In general, the leaf space becomes $\mathcal{L}_i = \mathcal{S}^{2k-1} \cup \mathcal{S}^{2k-3} \cup ... \cup \mathcal{S}^{2k-1-2(i-1)}$.

3.3.2 The case of odd dimension

If the dimension of M is odd, let us say $n = 2k-1$ then there are $\dfrac{n+1}{2} + 1 = k+1$ possible cases. We get similar leaf spaces to the subsection 3.3.1, for $1 \le i \le k$. For example, the leaf space corresponding to C_1 is $\mathcal{L}_1 = \mathcal{S}^{2k-2}$, see Table 3.7.

source/sink	p_1	p_2	p_3
p_1	$\emptyset$	$\emptyset$	$\emptyset$
p_2	S^{2k-3}	$\emptyset$	$\emptyset$
p_3	S^{2k-2}	S^{2k-3}	$\emptyset$

Table 3.7: The subspaces of the leaf space $\mathcal{L}_1$ for odd dimensional case.

For C_2, the zero set consists of four points and a copy of $\mathbb{RP}^{2k-5}$, let us call them as p_1, p_2, $p_3 = \mathbb{RP}^{2k-5}$, p_4 and p_5.

The corresponding flow is given by

$$\left[x_0\lambda_1^t : x_1\lambda_1^{-t} : x_2\lambda_2^t : x_3\lambda_2^{-t} : x_4 : x_5 : \ldots : x_{2k-1}\right].$$

source/sink	p_1	p_2	p_3	p_4	p_5
p_1	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
p_2	S^0	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
p_3	$S^{2k-4} - S^0$	S^{2k-5}	$\emptyset$	$\emptyset$	$\emptyset$
p_4	$S^{2k-3} - S^{2k-4}$	$S^{2k-4} - S^{2k-5}$	S^{2k-5}	$\emptyset$	$\emptyset$
p_5	$S^{2k-2} - S^{2k-3}$	$S^{2k-3} - S^{2k-4}$	$S^{2k-4} - S^0$	S^0	$\emptyset$

Table 3.8: The subspaces of the leaf space $\mathcal{L}_2$ for odd dimensional case.

The leaf space corresponding to C_2 is $\mathcal{L}_2 = \mathcal{S}^{2k-2} \cup \mathcal{S}^{2k-4}$.

In general, for C_i the zero set consists of $p_1,\ p_2,\ \ldots,\ p_{i+1} = \mathbb{RP}^{2k-1-2i},\ \ldots,\ p_{2i+1}$ and the leaf space becomes $\mathcal{L}_i = \mathcal{S}^{2k-2} \cup \mathcal{S}^{2k-4} \cup \cdots \cup \mathcal{S}^{2k-2i}$, for $1 \le i < k$ and if $i = k$, $\mathcal{L}_i = \mathcal{S}^{2k-2} \cup \mathcal{S}^{2k-4} \cup \cdots \cup \mathcal{S}^2 \cup S^0$.

We have an immersion induced by the developing map (see page 30)

$$h : \widetilde{W}^{n-1} \longrightarrow \mathcal{L}^{n-1}.$$

In general, the leaf space is $\mathcal{L}_i = \mathcal{S}^{n-1} \cup \mathcal{S}^{n-3} \cup ... \cup \mathcal{S}^{n-1-2(i-1)}$.

The decomposition of $\mathcal{L}$ contains two $(n-1)$-dimensional open discs, which are D_+^{n-1} and D_-^{n-1}, see Figure 3.5.

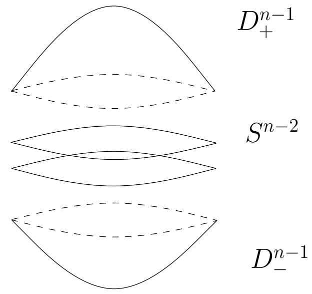


Figure 3.5: The decomposition of $\mathcal{S}^{n-1}$ in $\mathcal{L}$.

Lemma 3.3.6. *Assume that* $D_\pm^{n-1}$ *and the map* h *are as above. Let* $K \subseteq D_+^{n-1}$ *or* $K \subseteq D_-^{n-1}$ *be a closed disc and assume that* $h^{-1}(K) = H$ *is not empty. Then the restriction of* h *to the subspace* H, $h' : H \longrightarrow K$ *is onto and it is a finite sheeted covering.*

Proof. Since $h' : H \longrightarrow K$ is a submersion, H is a submanifold in $\widetilde{W}$ with boundary. Then the induced map $h' : H \longrightarrow K$ is a local homeomorphism. Since local homeomorphisms are open maps, h' is open. The map h' is also closed. To see this, take a closed subset Y of H. Since H is compact, the subset Y is also compact. The image of Y is compact because h' is continuous. Finally, since K is Hausdorff, $h'(Y)$ is closed. Therefore, the map h' is both open and closed. Then $h'(H) = K$ since K is connected. Since H is compact and $h' : H \longrightarrow K$ is a local homeomorphism, where both H and K are Hausdorff, by Lemma 2.0.8, h' is a covering projection. Moreover, it is finite sheeted since H is compact. □

Lemma 3.3.7. *The image* $h(\widetilde{W})$ *contains the top dimensional open discs* D_+^{n-1} *and* D_-^{n-1} *in* $\mathcal{L}$. *Moreover, when we restrict* h *to the preimages of these discs, the map* $h^{-1}(D_\pm^{n-1}) \longrightarrow D_\pm^{n-1}$ *is a finite sheeted covering space.*

Proof. Since the map h is a local homeomorphism, $h(\widetilde{W})$ contains at least one point in one of the $(n-1)$-dimensional open discs in $\mathcal{L}$. By Lemma 3.3.6, if $h(\widetilde{W})$ contains one point of an open disc, it is onto that open disc. Without loss of generality, let us say $h(\widetilde{W})$ contains D_+^{n-1}. Assume that $h(\widetilde{W})$ does not contain any point in D_-^{n-1}. Then $h(\widetilde{W})$ can not contain a point from the equators S^{n-2} of $\mathcal{S}^{n-1}$. Because if the image $h(\widetilde{W})$ contained a point from one of the equators S^{n-2} then the neighborhood of that point would have some points from D_-^{n-1}. Then in this case, $h : \widetilde{W} \longrightarrow D_+^{n-1}$ would be a covering map. Hence, $\widetilde{W}$ would be a disjoint union of open discs, which is a contradiction. In addition, $h_| : h^{-1}(D_\pm^{n-1}) \longrightarrow D_\pm^{n-1}$ is a finite sheeted covering space by the above lemma. □

In fact the above proof implies the following corollary.

Corollary 3.3.8. $\widetilde{W} \setminus h^{-1}(D_\pm^{n-1})$ *is a nonempty* $(n-2)$*-dimensional manifold.*

We will use the following well-known fact repeatedly. Nevertheless, we will provide a proof for the sake of completeness.

Lemma 3.3.9. *Let* L *be an* n*-dimensional connected and simply connected manifold and* $U \subset L$ *be an open ball, where* $n \geq 3$*. Then* $L \setminus U$ *is connected and simply connected.*

Proof. Let L^n be a connected manifold and $U \subseteq L$ be an open ball. Take an open ball V such that $V \subset \bar{V} \subset U$, where $\bar{V}$ is the closure of V, see Figure 3.6.

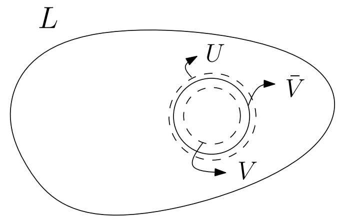


Figure 3.6: Open balls U and V in L.

$L^n \setminus U$ deformation retracts to $L^n \setminus \bar{V}$. We can consider the manifold L as $L = (L \setminus \bar{V}) \cup U$. First, we use Mayer-Vietoris sequence to show that $L \setminus U$ is connected. Consider the following sequence.

$\ldots \to H_1(U \setminus \bar{V}) \to H_1(L \setminus \bar{V}) \oplus H_1(U) \to H_1(L) \to H_0(U \setminus \bar{V}) \to H_0(L \setminus \bar{V}) \oplus H_0(U) \to H_0(L) \to 0.$

Since $U \setminus \bar{V}$ is homotopy equivalet to S^{n-1}, $H_0(L) = \mathbb{Z}$, $H_0(U) = \mathbb{Z}$, $H_0(L \setminus \bar{V}) = \mathbb{Z}^r$ and $H_1(L) = 0$, we have the following exact sequence.

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}^{r+1} \longrightarrow \mathbb{Z} \longrightarrow 0.$$

Hence $r = 1$ and $H_0(L \setminus \bar{V}) = \mathbb{Z}$.

Now, we show that $L \setminus U$ is simply connected by using Van-Kampen's theorem.

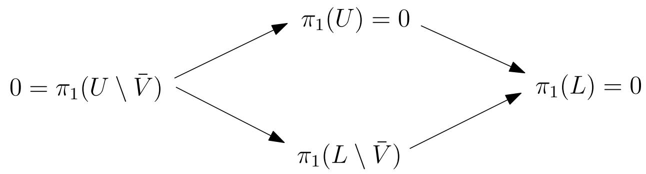


Figure 3.7: The Van Kampen diagram for $L = (L \setminus \bar{V}) \cup U$.

$0 = \pi_1(L) \cong \pi_1(L \setminus \bar{V})$, by Figure 3.7. □

Recall that we have assumed that the manifold M admits a real projective structure and we will show that this assumption yields a contradiction.

To proceed further, we consider the three cases of the leaf space $\mathcal{L}$.

Case 1: Consider the immersion $h : \widetilde{W} \longrightarrow \mathcal{L}_i = \mathcal{S}^{n-1} \cup \mathcal{S}^{n-3} \cup \ldots \cup \mathcal{S}^{n-1-2(i-1)}$, for $n + 1 - 2i \geq 2$.

Now, we remove the top dimensional open discs namely, D_+^{n-1} and D_-^{n-1} from the leaf space $\mathcal{L}$ and their preimages from $\widetilde{W}$. Then the remaining $(n-2)$-dimensional manifold $\mathcal{G}^{n-2} = \widetilde{W} \setminus h^{-1}(D_\pm^{n-1})$ is a closed connected manifold by Lemma 3.3.9 and Corollary 3.3.8 and the map

$$\mathcal{G}^{n-2} \longrightarrow S^{n-2} \cup S^{n-2} \cup \mathcal{S}^{n-3} \cup \ldots \cup \mathcal{S}^{n-1-2(i-1)}$$

is still an immersion. Next, the $(n-2)$-dimensional open discs D^{n-2}'s are removed from $\mathcal{L}$ and their preimages from $\mathcal{G}^{n-2}$ and we get an immersion as follows

$$\mathcal{G}^{n-3} \longrightarrow S^{n-3} \cup S^{n-3} \cup \mathcal{S}^{n-3} \cup \ldots \cup \mathcal{S}^{n-1-2(i-1)}.$$

Here, $\mathcal{G}^{n-3}$ is an $(n-3)$-dimensional manifold since the image of $\mathcal{G}^{n-2}$ should contain points from the equators of S^{n-2}'s.

We continue removing the top dimensional open discs from the leaf space $\mathcal{L}$ and their preimages from the remaining part of $\widetilde{W}$ until we get

$$\mathcal{G}^{n+1-2i} \longrightarrow \mathcal{L}^{n+1-2i} = S^{n+1-2i} \cup S^{n+1-2i} \cup ... \cup S^{n+1-2i} \cup \mathcal{S}^{n+1-2i}.$$

By Lemma 3.3.9, $\mathcal{G}^{n+1-2i}$ is still connected and simply connected as long as $n+1-2i \geq 2$.

Case 2: If the dimension of the manifold is $2k$, for some $k \in \mathbb{Z}$ and $i = k$ then removing cells as above we finally obtain the following immersion

$$\mathcal{G}^2 \longrightarrow \mathcal{L}^2 = S^2 \cup S^2 \cup ... \cup S^2 \cup \mathcal{S}^1.$$

Next, we remove small open discs containing the north and south poles of the 2-dimensional spheres and one of the equators of $\mathcal{S}^1$ then foliate the complement with circles.

Case 3: If the dimension of the manifold is $2k-1$, for some $k \in \mathbb{Z}$ and $i = k$ then the immersion analogously will be

$$\mathcal{G}^2 \longrightarrow \mathcal{L}^2 = S^2 \cup S^2 \cup ... \cup S^2 \cup \mathcal{S}^2 \cup S^0.$$

Then we remove small open discs containing the north and south poles of the 2-dimensional spheres and S^0 then foliate the complement with circles.

Note that in all cases above there are foliations on $\mathcal{L}^{n+1-2i}$ with the spheres S^r's ($r = n-2i$ in Case 1 and $r = 1$ in Case 2 and 3) after removing the small open discs containing the north and south poles of each $(n+1-2i)$-sphere in $\mathcal{L}^{n+1-2i}$. The number of the preimages of these open discs in $\mathcal{G}^{n+1-2i}$ is finite and we remove these open discs from $\mathcal{G}^{n+1-2i}$. Hence, there is also a foliation on $\mathcal{G}^{n+1-2i}$ with r-dimensional manifolds $\mathcal{J}^r$. These r-manifolds must be sphere since the sphere S^r is closed in the leaf space, its preimage is also closed. Furthermore, the preimage of S^r is compact because $\mathcal{G}^{n+1-2i}$ is compact. Hence, the map is a covering by Lemma 2.0.8, $\mathcal{J}^r \longrightarrow S^r$.

For $r = 1$, after removing the preimages of small open discs containing the north and south poles of each sphere in $\mathcal{L}$, the remaining part of $\widetilde{W}$ is foliated by 1-

dimensional manifolds. These 1-dimensional manifolds are circles since $\mathcal{G}^{n+1-2i}$ is compact and $\mathcal{J}^1 \longrightarrow S^1$ is a covering map. Hence, the remaining part of $\widetilde{W}$ is foliated by circles and hence it is an annulus.

By Theorem 2.0.10, for $r > 1$, the foliation on $\mathcal{G}$ is $S^r \times I$, where $I = [-1, 1]$. Hence, the leaf space of this foliation on $\mathcal{G}^{n+1-2i}$ is $I = [-1, 1]$.

On the other hand, the quotient space of the leaf space $\mathcal{L}^{n+1-2i}$ is a non-Hausdorff space which is a union of intervals with one extra origin $I^* = I_1 \cup I_2 \cup ... \cup I_{2i-1} \cup \{0'\}$. The involution interchanges these intervals with each other except the one, which represents the sphere $\mathcal{S}^{n+1-2i}$ and on that sphere it changes the double origins with each other. There is still an immersion $\widetilde{h} : I \longrightarrow I^*$ induced from the immersion h such that $\widetilde{h}(\pm 1)$ are the end points of some of the intervals in I^*. Such an immersion is an embedding, whose image contains one interval with only one copy of the origin. Therefore, this gives a contradiction since in this case the immersed image of $\widetilde{W}$ in $\mathcal{L}$ can not be invariant under involution.

For a better understanding, let us consider the following example: Let the manifold M be 9-dimensional and the leaf space $\mathcal{L} = \mathcal{S}^9 \cup \mathcal{S}^7$.

The induced immersion from the developing map is

$$\widetilde{W}^9 \longrightarrow \mathcal{L} = \mathcal{S}^9 \cup \mathcal{S}^7.$$

After removing the 9-dimensional open discs from $\mathcal{L}$ and their preimages from $\widetilde{W}^9$, we get the following immersion

$$\mathcal{G}^8 \longrightarrow S^8 \cup S^8 \cup \mathcal{S}^7.$$

Here, if the preimage of one of the 8-dimensional sphere $\mathcal{L}$ is empty and the other is not then it contradicts that the image of $\widetilde{W}^9$ is invariant under involution and the proof finishes.

Next, we remove the 8-dimensional open discs from $\mathcal{L}$ and the preimages of these open discs from $\widetilde{W}^8$ then the immersion will be as follows

$$\mathcal{G}^7 \longrightarrow S^7 \cup S^7 \cup \mathcal{S}^7.$$

Note that all the spheres in $\mathcal{L}^7$ are the same dimensional. The leaf space $\mathcal{L}^7$ can be decomposed into disjoint subsets which are the north and south poles of each 7-

dimensional sphere and 6-dimensional spheres foliating the complement with $I^* = I_1 \cup I_2 \cup I_3 \cup \{0'\}$. In addition, $\mathcal{G}^7$ is foliated by the spheres S^6's excluding finitely many open discs which are the preimages of the discs about north and south poles in $\mathcal{L}$ and the leaf space of this foliation is $I = [-1, 1]$ by Theorem 2.0.10.

There is an induced immersion $\widetilde{h} : I \longrightarrow I^*$ from the map $h : \widetilde{W} \longrightarrow \mathcal{L}$.

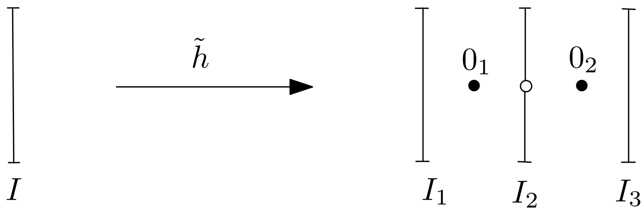


Figure 3.8: The induced immersion $\widetilde{h}$.

In Figure 3.8, the interval I_2, standing for $\mathcal{S}^7$, has two origins 0_1 and 0_2. Moreover, 0_1 is also an origin for I_1 and 0_2 is an origin for I_3. Therefore, each interval in the quotient space of the foliation of the leaf space has a double origin. In addition, the involution interchanges the intervals I_1 and I_3 with ecah other and the origins 0_1 with 0_2. However, such an immersion is an embedding which contains only one interval with one origin. This yields a contradiction. This finishes the proof in the case $A^2 = Id$.

To finish the proof of Theorem 3.2.1, we need to consider also the case where the matrix C_0 corresponds to A, with $A^2 = -Id$. The zero set consists of two disjoint copies of $\mathbb{RP}^{k-1}$'s in $\mathbb{RP}^{2k-1}$, namely $Z = \mathbb{RP}^{k-1} \cup \mathbb{RP}^{k-1}$. Since L_i's are invariant under $hol(\pi_1(N))$, where $L_i = \mathbb{RP}^{k-1}$, $dev^{-1}(L_i)$ is invariant under the covering transformation of the smooth covering map $\pi : \widetilde{N} \longrightarrow N$, for $i = 1, 2$. Therefore, $\alpha_i = \pi(dev^{-1}(L_i))$ is a submanifold of N. Consider the following diagram.

$$\begin{array}{ccc} \tilde{N} = \widetilde{W} \times \mathbb{R} & \xrightarrow{dev} & \mathbb{RP}^{2k-1} \\ \downarrow \pi & & \\ N = \widetilde{W} \times S^1 & & \end{array}$$

Moreover, the zero set of v, the vector field on N, is $\alpha_1 \cup \alpha_2$. Now, we have two cases:

- $\alpha_1 \cup \alpha_2 = \emptyset$. It means that $dev(\widetilde{N})$ does not contain the zero set

$Z = L_1 \cup L_2$. Hence we can use Lemma 3.3.5.

The corresponding flow is given by

$$\left[x_0\lambda_1^t : ... : x_{\frac{n-1}{2}}\lambda_1^t : x_{\frac{n+1}{2}}\lambda_1^{-t} : ... : x_n\lambda_1^{-t}\right].$$

We can choose L_1 as the source and L_2 as the sink and thus all the flowlines start from L_1 and end at L_2. By taking the boundary of a tubular neighborhood of each L_i in $\mathbb{RP}^{2k-1}$, which is $\partial\nu(L_i)$, we determine the leaf space $\mathcal{L}$ since from each point of $\partial\nu(L_i)$ exactly one flowline passes. Therefore, the leaf space is the total space of an S^{k-1} bundle over $\mathbb{RP}^{k-1}$, which is $\mathcal{L} = S^{k-1}\widetilde{\times}\mathbb{RP}^{k-1}$. Now consider the diagram below.

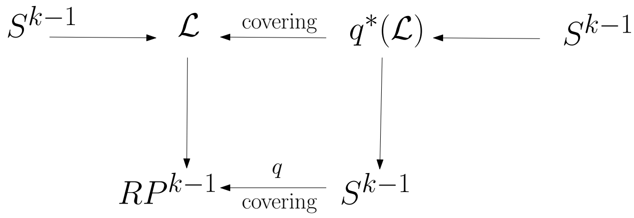


Figure 3.9: $q^*(\mathcal{L})$ is an S^{k-1} bundle over S^{k-1}.

By Figure 3.9, $q^*(\mathcal{L}) = S^{k-1}\widetilde{\times}S^{k-1}$. Since $k \geq 3$, $\pi_1(q^*(\mathcal{L}))$ is trivial. Moreover, since the induced local homeomorphism $\widetilde{W} \longrightarrow \mathcal{L}$ is a covering map by Lemma 2.0.8 and $\pi_1(\widetilde{W})$ is trivial, $\widetilde{W} \cong q^*(\mathcal{L})$.

By the assumption of Theorem 3.2.1, $\widetilde{W}$ is not the total space of a sphere bundle over sphere; therefore, this is a contradiction.

- $\alpha_1 \cup \alpha_2 \neq \emptyset$. In this case we can not use Lemma 3.3.5 and we give an alternative proof below. Suppose α_1 is nonempty. Let ϕ be a flowline, including the end points, of the vector field v and one of the endpoints on α_1. ϕ is a compact one dimensional submanifold of N since its preimage in $\widetilde{N}$ maps into a closed interval in $\mathbb{RP}^{2k-1}$ with one end point in each L_i. Hence, the other end point of ϕ is on α_2, which makes α_2 is also nonempty.

  Next, we will show that α_1 and α_2 are both connected. Let ζ be a component of α_1. Let U be the tubular neighborhood of ζ in N. $dev(\pi^{-1}(\zeta)) \subset L_1$ and they are actually equal. Thus, $dev(\pi^{-1}(U))$ includes a neighborhood of L_1. Hence, U contains the total space Υ of an S^{k-1}

bundle over $\mathbb{RP}^{k-1}$ transverse to the flow and bounds a small neighborhood of ζ. Since U is invariant under the flow, it follows that $U = \Upsilon \times \mathbb{R}$. The boundary of U in N is contained in $\alpha_1 \cup \alpha_2$ and thus α_1 and α_2 are connected and $N = \alpha_1 \cup U \cup \alpha_2$ (this is the analogous argument in [8], p.8).

Note that since the image of dev contains both L_1 and L_2, the map $\alpha_i \longrightarrow L_i$ is a covering map, for $i = 1, 2$ (see Lemma 2.0.8). There are only two possibilities for α_i, which are S^{k-1} and $\mathbb{RP}^{k-1}$.

If $\alpha_i = S^{k-1}$ then the boundary of the neighborhood $\nu(\alpha_i)$ of α_i in N is the total space of an S^{k-1} bundle over S^{k-1}, $\partial\nu(\alpha_i) = S^{k-1}\widetilde{\times}S^{k-1}$. Since $k \geq 3$, the homotopy exact sequence implies that $\pi_1(\nu(\alpha_i))$ is trivial. Now, N can be written as $N = \nu(\alpha_1) \cup \nu(\alpha_2)$, where the neighborhoods $\nu(\alpha_1)$ and $\nu(\alpha_2)$ are glued along their boundaries via a diffeomorphism. By Van Kampen's theorem,

$$\pi_1(N) \cong \pi_1(\nu(\alpha_1)) * \pi_1(\nu(\alpha_2))/K,$$

where K is the normal subgroup corresponding to the kernel of the homomorphism

$$\Phi : \pi_1(\nu(\alpha_1)) * \pi_1(\nu(\alpha_2)) \longrightarrow \pi_1(N).$$

However, since $\pi_1(N) = \pi_1(\widetilde{W} \times S^1) = \mathbb{Z}$, this gives a contradiction.

If $\alpha_i = \mathbb{RP}^{k-1}$ then $\partial\nu(\alpha_i) = S^{k-1}\widetilde{\times}\mathbb{RP}^{k-1}$. Similarly,

$$\pi_1(N) \cong \pi_1(\nu(\alpha_1)) * \pi_1(\nu(\alpha_2))/K$$

and $\pi_1(\nu(\alpha_i)) \cong \mathbb{Z}_2$. Therefore, we get $\mathbb{Z} \cong \mathbb{Z}_2 * \mathbb{Z}_2/K$.

Now, assume $\mathbb{Z}_2 * \mathbb{Z}_2 = < a > * < b >$ has a normal subgroup K and $\pi_1(N) = \mathbb{Z} =< c >$, where $c = ab$. Let $g = abab...aba$ be an element of K. Since $aga = bab...ab$ is also an element of K, their product $gaga$ is in K, i.e. $(ab)^k \in K$, for some $k \in \mathbb{Z}$. Now, consider the element $g' = baba...ab$ in K. Then $bg'b \in K$ and their product $bg'bg'$ is again some power of ab. Hence K contains a power of $c = ab$. Then $K \cap \mathbb{Z} \neq \emptyset$ and this gives a contradiction. Therefore, there is no normal subgroup K of $\mathbb{Z}_2 * \mathbb{Z}_2$ whose quotient is $\mathbb{Z}$.

This final contradiction concludes the proof of Theorem 3.2.1. $\square$

CHAPTER 4

AN ALTERNATIVE PROOF IN DIMENSION 4

In this chapter, our aim is to give some more details and alternative arguments about the proof of the main theorem, Theorem 3.2.1, for the case $M = \mathbb{RP}^4 \# \mathbb{RP}^4$ and the flowlines of $\mathbb{RP}^4 \setminus Z$ such that Z is the zero set in which the vector field $V = 0$ on $\mathbb{RP}^4$. Note that if we take $n = 4$ in our construction (section 3.1), the manifold will be

$$M = \widetilde{W} \times S^1 / < \sigma > = \mathbb{RP}^4 \# \mathbb{RP}^4,$$

by Remark 3.2.2, where $\widetilde{W} = S^3$. Assuming that M admits a real projective structure, we obtain a developing pair (dev, hol) such that

$$dev : \widetilde{M} \longrightarrow \mathbb{RP}^4,$$

and

$$hol : \pi_1(M) \longrightarrow PGL(5, \mathbb{R}).$$

By Van Kampen's theorem, the fundamental group of M is

$$\pi_1(M) = \mathbb{Z}_2 * \mathbb{Z}_2 \cong < a, b \mid a^2 = b^2 = 1 >$$

and it is isomorphic to the infinite dihedral group

$$\mathbb{Z}_2 * \mathbb{Z}_2 = \mathbb{Z} \ltimes \mathbb{Z}_2.$$

Now, we will show that the universal cover of M is $\widetilde{M} = S^3 \times \mathbb{R}$.

Let σ act on $S^3 \times \mathbb{R}$ as the antipodal map on both components given by

$$\begin{aligned} \sigma : S^3 \times \mathbb{R} &\longrightarrow S^3 \times \mathbb{R}, \\ (p, x) &\longmapsto (-p, -x), \end{aligned}$$

and τ act on $S^3 \times \mathbb{R}$ by fixing the first component and as a translation on the second one

$$\begin{aligned} \tau : S^3 \times \mathbb{R} &\longrightarrow S^3 \times \mathbb{R} \\ (p, x) &\longmapsto (p, x+1). \end{aligned}$$

If we take the quotient of $S^3 \times \mathbb{R}$ by $< \tau >$, we get $S^3 \times S^1$, which is the double cover of M.

Let $U = S^1 \setminus \{(-1, 0)\}$ and $V = S^1 \setminus \{(1, 0)\}$ be open sets which cover the circle S^1 then we can write M as follows:

$$(S^3 \times U)/\mathbb{Z}_2 \cup (S^3 \times V)/\mathbb{Z}_2 = (S^3 \times S^1)/\mathbb{Z}_2,$$

$$(\mathbb{RP}^4 - D^4) \cup (\mathbb{RP}^4 - D^4) = \mathbb{RP}^4 \# \mathbb{RP}^4.$$

Below, we explain that $(S^3 \times U)/\mathbb{Z}_2 = \mathbb{RP}^4 - D^4$.

If we paste two 4-dimensional discs D^4 along the two boundaries of $S^3 \times U$ then we get S^4 as shown in the figure below.

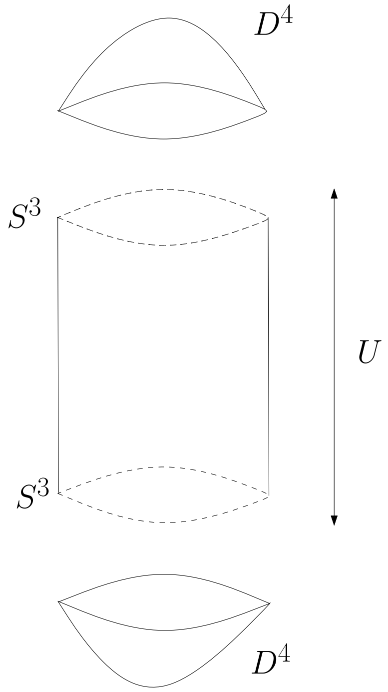


Figure 4.1: Decomposition of the 4-dimensional sphere.

Since

$$((S^3 \times U) \cup 2D^4)/\mathbb{Z}_2 = S^4/\mathbb{Z}_2 = \mathbb{RP}^4,$$

we have

$$(S^3 \times U)/\mathbb{Z}_2 = (S^4 - 2D^4)/\mathbb{Z}_2 = \mathbb{RP}^4 - D^4.$$

Now, as in the proof of Theorem 3.2.1 we determine the possible generators C_i (see Lemma 3.3.3) for the holonomy $hol : \pi_1(N) \to PGL(5, \mathbb{R})$, where $N = S^3 \times S^1$ is the double cover of M.

Similar to the proof of Lemma 3.3.2, we say that $hol(a) = [A]$ and $hol(b) = [B]$, where A and $B \in GL(5, \mathbb{R})$ and also say $C = AB$. Since $a^2 = 1$ and hol is a homomorphism, $[A]^2 \in PGL(5, \mathbb{R})$ is identity. It follows that after rescaling A we have $A^2 = \pm Id$. $A^2 = -Id$ is not possible since the dimension of M is even. If $A^2 = Id$ then the eigenvalues are ± 1. The possible A_i matrices are as below:

$$A_1 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

In the lemma below, which is Lemma 3.3.3 for $n = 4$, we give the possible C_i matrices corresponding to A_i, for $i = 1, 2$.

Lemma 4.0.10. *It is possible to assume that C is one of the following matrices with $\lambda_2 > \lambda_1 > 1$.*

$$C_1 = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & {\lambda_1}^{-1} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad C_2 = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & {\lambda_1}^{-1} & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & 0 & {\lambda_2}^{-1} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Next, observe that there is a 1-parameter diagonal subgroup $G \subset PGL(5, \mathbb{R})$,

$g : \mathbb{R} \longrightarrow G$ such that $g(1) = [C]$. For example $C = C_2$, this subgroup is

$$g_1(t) = \begin{bmatrix} exp(l_1 t) & 0 & 0 & 0 & 0 \\ 0 & exp(-l_1 t) & 0 & 0 & 0 \\ 0 & 0 & exp(l_2 t) & 0 & 0 \\ 0 & 0 & 0 & exp(-l_2 t) & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

where $l_i = log(\lambda_i)$.

This induces a flow on $\mathbb{RP}^4$ (see page 28)

$$\Phi : \mathbb{RP}^4 \times \mathbb{R} \to \mathbb{RP}^4.$$

Let V be the vector field on $\mathbb{RP}^4$, velocity of this flow. The possibilities for the zero set $Z \subset \mathbb{RP}^4$ of V are as follows:

1. For C_1, one projective plane and two points (see page 49 and 50).
2. For C_2, five points (see page 52)).

Let $X = \mathbb{RP}^4 \setminus Z$, then X is foliated by flowlines and let $\mathcal{L}$ be the leaf space of the foliation on X.

Let $\widetilde{N}$ be the universal cover of N. Since $dev(\widetilde{N}) \subset X$ there is a map from the leaf space of the induced foliation on $\widetilde{N}$ into $\mathcal{L}$. The leaf space of $\widetilde{N}$ is S^3 by Lemma 3.3.5. The induced map

$$h : S^3 \longrightarrow \mathcal{L}$$

is a local homeomorphism. Since $dev(\widetilde{N}) \subset \mathbb{RP}^4$ is invariant under $hol(\pi_1(M))$ it follows that $h(S^3) \subset \mathcal{L}$ is invariant under the involution, which is induced by σ.

Our purpose is to determine all possible immersions of S^3 into $\mathcal{L}$ and prove that the image of S^3 can not be invariant under the involution and thus it contradicts that M admits a developing map. Therefore, we will conclude that M has no real projective structure.

The following two lemmas are included in Subsection 3.3.1. Neverthless, we give details of the flowlines and an alternative proof for Lemma 4.0.12 below.

Lemma 4.0.11. C_1 *is not possible.*

Proof. The zero set Z of V is a copy of $\mathbb{RP}^2$ and two points, which are p_+ and p_-. Let us call p_+ as source and p_- as sink. $S_1,\ S_2$ are small unit 3-spheres, which are the boundaries of the neighborhoods of p_+ and p_-, respectively in $\mathbb{RP}^4$, so that the spheres transverse to the flow.

We try to determine the leaf space $\mathcal{L}$. Decompose these $3-$spheres as an equator and the northern and the southern hemispheres.

$$S^3 = D_+ \cup E \cup D_-$$

such that

$$E = S^3 \cap \{x_4 = 0\},$$

$$D_+ = S^3 \cap \{x_4 > 0\},$$

$$D_- = S^3 \cap \{x_4 < 0\}.$$

Moreover, there exists a unique flowline passing through each point of the hemispheres of S_1 and S_2. This is clear from the description of the flowlines given below. Thus, we can identify $D_+ \subset S_1$ with $D_+ \subset S_2$ and $D_- \subset S_1$ with $D_- \subset S_2$. The flow starting from p_+ intersecting with the equator of S_1 goes to $\mathbb{RP}^2$. The flow starting from $\mathbb{RP}^2$ goes to p_- intersecting with the equator of S_2. Through each point of the equators of S_1 and S_2, there passes a unique flowline and therefore there exist two copies of S^2 in $\mathcal{L}$.

The corresponding flow to C_1 is given by

$$\left[x_0{\lambda_1}^t : x_1{\lambda_1}^{-t} : x_2 : x_3 : x_4\right].$$

Now, our purpose is to determine source and sink points.

Case 1: If $x_1 \neq 0$ then the flow is as follows:

$$\left[\frac{x_0}{x_1}{\lambda_1}^{2t} : 1 : \frac{x_2}{x_1}{\lambda_1}^t : \frac{x_3}{x_1}{\lambda_1}^t : \frac{x_4}{x_1}{\lambda_1}^t\right].$$

When $t \to -\infty$, the flow starts at $\left[0 : 1 : 0 : 0 : 0\right]$, which is source.

1. $x_0 \neq 0,$
$$\left[1 : \frac{x_1}{x_0}{\lambda_1}^{-2t} : \frac{x_2}{x_0}{\lambda_1}^{-t} : \frac{x_3}{x_0}{\lambda_1}^{-t} : \frac{x_4}{x_0}{\lambda_1}^{-t}\right].$$
When $t \to +\infty,$ the flow ends at $\left[1 : 0 : 0 : 0 : 0\right],$ which is sink.

The Euclidean coordinates around the source are as
$$\left(\frac{x_0}{x_1}, \frac{x_2}{x_1}, \frac{x_3}{x_1}, \frac{x_4}{x_1}\right) \in \mathbb{R}^4$$
and around the sink as
$$\left(\frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{x_3}{x_0}, \frac{x_4}{x_0}\right) \in \mathbb{R}^4.$$
For the flowlines starting at p_+ and leaving S_1 from the northern hemisphere we assume $\frac{x_0}{x_1} > 0,$ for $x_1 \neq 0.$ Then $\frac{x_1}{x_0} > 0,$ for $x_0 \neq 0$ and the flowlines go to p_- from the northern hemisphere of $S_2.$ The flowlines starting from p_+ and leaving S_1 from northern hemisphere go to p_- from northern hemisphere of $S_2.$ Similarly, the flowlines starting from p_+ and passing through the southern hemisphere of S_1 $(\frac{x_0}{x_1} < 0)$ go to the southern hemisphere of S_2 ending at $p_-.$

2. Now, consider $x_0 = 0,$ i.e. the flowlines do not end at sink, thus the flow is as follows:
$$\left[0 : x_1{\lambda_1}^{-t} : x_2 : x_3 : x_4\right].$$
If $x_2 \neq 0,$ the flow is given by
$$\left[0 : \frac{x_1}{x_2}{\lambda_1}^{-t} : 1 : \frac{x_3}{x_2} : \frac{x_4}{x_2}\right].$$
When $t \to +\infty,$ the flow ends at
$$\left[0 : 0 : 1 : \frac{x_3}{x_2} : \frac{x_4}{x_2}\right],$$
so the flowlines go to $\mathbb{RP}^2.$

If we consider the cases $x_3 \neq 0$ or $x_4 \neq 0$ instead of $x_2 \neq 0,$ the flowlines again end at $\mathbb{RP}^2.$

The flowlines emitting from the source which ends at $\mathbb{RP}^2$ form a subspace homeomorphic to S^2 in $\mathcal{L}.$

Case 2: If $x_1 = 0$, then the flow is as follows:

$$\left[x_0\lambda_1{}^t : 0 : x_2 : x_3 : x_4\right].$$

The cases where $x_2 \neq 0$, $x_3 \neq 0$ or $x_4 \neq 0$, the flows are similar. Hence, we may consider only one of them.

Let us take $x_2 \neq 0$, then the flow is as follows:

$$\left[\frac{x_0}{x_2}\lambda_1{}^t : 0 : 1 : \frac{x_3}{x_2} : \frac{x_4}{x_2}\right].$$

When $t \to -\infty$, the flow starts at

$$\left[0 : 0 : 1 : \frac{x_3}{x_2} : \frac{x_4}{x_2}\right].$$

If $x_0 \neq 0$, the flow is given by

$$\left[1 : 0 : \frac{x_2}{x_0}\lambda_1{}^{-t} : \frac{x_3}{x_0}\lambda_1{}^{-t} : \frac{x_4}{x_0}\lambda_1{}^{-t}\right].$$

When $t \to +\infty$, the flow ends at $\left[1 : 0 : 0 : 0 : 0\right]$.

Thus, the flowlines start from $\mathbb{RP}^2$ and end at sink. Furthermore, the flowlines form a subspace homeomorphic to S^2 in $\mathcal{L}$ as in Case 1.

Therefore, $\mathcal{L}$ is a 3-sphere with a double equator and we simply denote it as $\mathcal{L} = \mathcal{S}^3$. Now, we remove the small open discs containing the north and south poles of $\mathcal{S}^3$ in $\mathcal{L}$ then the complement is foliated by 2-spheres with an extra copy at the origin and the corresponding leaf space of this foliation is $I^* = I \cup \{0'\}$. Here the involution interchanges the origins with each other. In addition, S^3 is foliated by 2-spheres removing the preimages of the discs around the north and south poles in $\mathcal{L}$ and the leaf space of this foliation is $I = [-1, 1]$, by Theorem 2.0.10.

There is an induced immersion $\widetilde{h} : I \longrightarrow I^*$ from the map $h : S^3 \longrightarrow \mathcal{L}$. However, such an immersion is an embedding containing only one origin. Hence, the image can not be invariant under the involution. □

Lemma 4.0.12. *C_2 is not possible.*

Proof. The zero set of V consists of five points. We call them p_{++++}, p_{+++-}, p_{++--}, p_{+---} and p_{----}. We get the points and labelling by considering the following flow

$$\mathbf{g(t)} = \begin{bmatrix} exp(l_1 t) & 0 & 0 & 0 & 0 \\ 0 & exp(-l_1 t) & 0 & 0 & 0 \\ 0 & 0 & exp(l_2 t) & 0 & 0 \\ 0 & 0 & 0 & exp(-l_2 t) & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

such that $l_i = log(\lambda_i)$ and $\lambda_2 > \lambda_1 > 1$.

The labelling denotes the number of attracting and rebelling directions. The number of $(-)$ signs is the number of attracting directions. Hence, the sink is p_{----} and the source is p_{++++}. There is no flowline ending at p_{++++} and similarly, there is no flowline starting at p_{----}. Every flowline starts at a point with a $(+)$ label and ends at a point with a $(-)$ label.

We take the boundary of the tubular neighborhood of each point of Z in $\mathbb{RP}^4$, which is S^3. We need to determine the subspaces corresponding to the flowlines of $\mathcal{L}$ which start at some point (when $t \to -\infty$) and end at some other point (when $t \to +\infty$).

For C_2, the corresponding flow is given by

$$\left[x_0 {\lambda_1}^t : x_1 {\lambda_1}^{-t} : x_2 {\lambda_2}^t : x_3 {\lambda_2}^{-t} : x_4\right].$$

Case 1: If $x_3 \neq 0$ then we consider the flow as follows:

$$\left[\frac{x_0}{x_3}(\lambda_1 \cdot \lambda_2)^t : \frac{x_1}{x_3}\left(\frac{\lambda_2}{\lambda_1}\right)^t : \frac{x_2}{x_3}{\lambda_2}^{2t} : 1 : \frac{x_4}{x_3}{\lambda_2}^t\right].$$

Since $(\lambda_1 \cdot \lambda_2)$, $(\frac{\lambda_2}{\lambda_1})$ and λ_2 are greater than 1, the starting point is p_{++++}, i.e. when $t \to -\infty$, the flow starts at $\left[0 : 0 : 0 : 1 : 0\right]$. In Euclidean coordinates,

$$\left[x_0 : x_1 : x_2 : x_3 : x_4\right] \longrightarrow \left(\frac{x_0}{x_3}, \frac{x_1}{x_3}, \frac{x_2}{x_3}, \frac{x_4}{x_3}\right) \in \mathbb{R}^4$$

and

$$\left[0 : 0 : 0 : 1 : 0\right] \longrightarrow (0, 0, 0, 0).$$

1. If $x_2 \neq 0$ then the flow can be written as

$$\left[\frac{x_0}{x_2}\left(\frac{\lambda_1}{\lambda_2}\right)^t : \frac{x_1}{x_2}(\lambda_1 \cdot \lambda_2)^{-t} : 1 : \frac{x_3}{x_2}{\lambda_2}^{-2t} : \frac{x_4}{x_2}{\lambda_2}^{-t}\right].$$

When $t \to +\infty$, the flow ends at $\left[0 : 0 : 1 : 0 : 0\right]$, which is p_{----}. Through every point in $S^3 - S^2$, which is around p_{----}, there is a unique flowline starting at the source.

2. If $x_2 = 0$ then the flow is given by

$$\left[x_0{\lambda_1}^t : x_1{\lambda_1}^{-t} : 0 : x_3{\lambda_2}^{-t} : x_4\right].$$

If $x_0 \neq 0$ then the flow can be written as

$$\left[1 : \frac{x_1}{x_0}{\lambda_1}^{-2t} : 0 : \frac{x_3}{x_0}(\lambda_1 \cdot \lambda_2)^{-t} : \frac{x_4}{x_0}{\lambda_1}^{-t}\right].$$

When $t \to +\infty$, the flow ends at $\left[1 : 0 : 0 : 0 : 0\right]$, which is p_{+---}.

The flow below

$$\left[1 : \frac{x_1}{x_0}{\lambda_1}^{-2t} : 0 : \frac{x_3}{x_0}(\lambda_1 \cdot \lambda_2)^{-t} : \frac{x_4}{x_0}{\lambda_1}^{-t}\right]$$

consists of 3 parameters in Euclidean coordinates such that $x_3 \neq 0$. Then the corresponding subspace to the flowlines starting at p_{++++} and ending at p_{+---} is $S^2 - S^1$. Hence we can identify both hemispheres of S^2 around p_{++++} with both hemispheres of S^2 around p_{+---}. However, there is no flowline between the two equators of 2-spheres, thus they are not identified.

3. Let $x_2 = 0$ and $x_0 = 0$ then the flow is written as follows:

$$\left[0 : x_1{\lambda_1}^{-t} : 0 : x_3{\lambda_2}^{-t} : x_4\right].$$

If $x_4 \neq 0$ then the flow is given by

$$\left[0 : \frac{x_1}{x_4}{\lambda_1}^{-t} : 0 : \frac{x_3}{x_4}{\lambda_2}^{-t} : 1\right].$$

When $t \to +\infty$, the flow ends at $\left[0 : 0 : 0 : 0 : 1\right]$, which is p_{++--}.

The flowlines from p_{++++} to p_{++--} are through the points of S^1 except its equator since $x_4 \neq 0$. It follows that the corresponding subspace is $S^1 - S^0$.

4. If $x_2 = 0,\ x_0 = 0$ and $x_4 = 0$, then the flow is given by

$$\big[0 : x_1{\lambda_1}^{-t} : 0 : x_3{\lambda_2}^{-t} : 0\big].$$

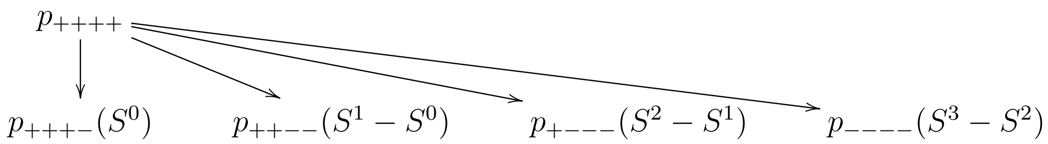
If $x_1 \neq 0$, then the flow can be written as

$$\Big[0 : 1 : 0 : \frac{x_3}{x_1}\big(\frac{\lambda_1}{\lambda_2}\big)^t : 0\Big].$$

When $t \to +\infty$, the flow ends at $\big[0 : 1 : 0 : 0 : 0\big]$, which is p_{+++-}.

The flowlines from p_{++++} to p_{+++-} are only through two points of S^1, namely S^0.

The diagram below lists the flowlines starting from p_{++++} and also the subspaces corresponding to these flowlines.



If $x_3 = 0$, i.e. flowlines do not start at the source point p_{++++} then the flow is

$$\big[x_0{\lambda_1}^t : x_1{\lambda_1}^{-t} : x_2{\lambda_2}^t : 0 : x_4\big].$$

Case 2: Let $x_3 = 0$ and $x_1 \neq 0$, then the flow is given by

$$\Big[\frac{x_0}{x_1}{\lambda_1}^{2t} : 1 : \frac{x_2}{x_1}(\lambda_1 \cdot \lambda_2)^t : 0 : \frac{x_4}{x_1}{\lambda_1}^t\Big].$$

When $t \to -\infty$ the flow starts at $\big[0 : 1 : 0 : 0 : 0\big]$.

The subspace consisting of the flowlines starting from p_{+++-} is S^2. Below, we investigate at which points this flowlines end.

1. If $x_2 \neq 0$, then the flow can be written as

$$\Big[\frac{x_0}{x_2}\big(\frac{\lambda_1}{\lambda_2}\big)^t : \frac{x_1}{x_2}(\lambda_1 \cdot \lambda_2)^{-t} : 1 : 0 : \frac{x_4}{x_2}{\lambda_2}^{-t}\Big].$$

When $t \to +\infty$, the flow ends at $\big[0 : 0 : 1 : 0 : 0\big]$.

Thus, the subspace $S^2 - S^1$ consists of the flowlines starting from p_{+++-} and ending at p_{----}.

2. If $x_2 = 0$, then the flow is $\big[x_0{\lambda_1}^t : x_1{\lambda_1}^{-t} : 0 : 0 : x_4\big]$.

   Let $x_0 \neq 0$, then the flow can be written as

$$\Big[1 : \frac{x_1}{x_0}{\lambda_1}^{-2t} : 0 : 0 : \frac{x_4}{x_0}{\lambda_1}^{-t}\Big].$$

   As $t \to +\infty$ the flow ends at $\big[1 : 0 : 0 : 0 : 0\big]$.

   The subspace $S^1 - S^0$ consists of the flowlines from p_{+++-} to p_{+---}.

3. If $x_2 = 0$ and $x_0 = 0$, then the flow is written as follows:

$$\big[0 : x_1{\lambda_1}^{-t} : 0 : 0 : x_4\big].$$

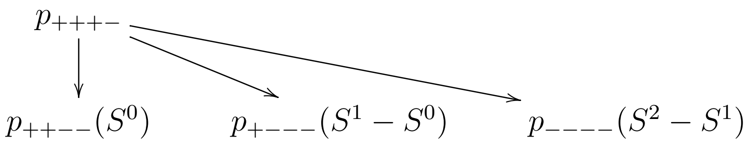
   Let $x_4 \neq 0$, then the flow can be written as

$$\Big[0 : \frac{x_1}{x_4}{\lambda_1}^{-t} : 0 : 0 : 1\Big].$$

   As $t \to +\infty$ the flow ends at $\big[0 : 0 : 0 : 0 : 1\big]$.

   The subspace S^0 consists of the flowlines from p_{+++-} to p_{++--}.

So we have the following diagram.



Case 3 : Consider $x_3 = 0$, $x_1 = 0$ and $x_4 \neq 0$, then the flow is given by

$$\Big[\frac{x_0}{x_4}{\lambda_1}^t : 0 : \frac{x_2}{x_4}{\lambda_2}^t : 0 : 1\Big].$$

As $t \to -\infty$ the flow starts at $\big[0 : 0 : 0 : 0 : 1\big]$.

Thus, the subspace S^1 consists of the flowlines from p_{++--}.

1. If $x_2 \neq 0$ then the flow can be written as

$$\Big[\frac{x_0}{x_2}\big(\frac{\lambda_1}{\lambda_2}\big)^t : 0 : 1 : 0 : \frac{x_4}{x_2}{\lambda_2}^{-t}\Big].$$

   As $t \to +\infty$ the flow ends at $\big[0 : 0 : 1 : 0 : 0\big]$.

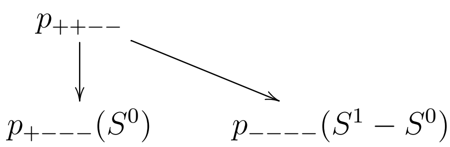
   The subspace $S^1 - S^0$ consists of the flowlines starting from p_{++--} and ending at p_{----}.

2. If $x_2 = 0$ and $x_0 \neq 0$, the flow is given by

$$\Big[1:0:0:0:\frac{x_4}{x_0}{\lambda_1}^{-t}\Big].$$

   As $t \to +\infty$ the flow ends at $\big[1:0:0:0:0\big]$.

   Hence, the subspace S^0 consists of the flowlines from p_{++--} to p_{+---}.



Case 4: If $x_1 = 0$, $x_3 = 0$, $x_4 = 0$ and $x_0 \neq 0$, then the flow is given by

$$\Big[1:0:\frac{x_2}{x_0}\big(\frac{\lambda_2}{\lambda_1}\big)^t:0:0\Big].$$

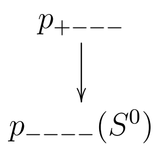
As $t \to -\infty$ the flow starts at $\big[1:0:0:0:0\big]$, which is p_{+---}.

1. If $x_2 \neq 0$, the flow is given by

$$\Big[\frac{x_0}{x_2}\big(\frac{\lambda_1}{\lambda_2}\big)^t:0:1:0:0\Big].$$

   As $t \to +\infty$ the flow ends at $\big[0:0:1:0:0\big]$.

   The subspace S^0 consists of the flowlines from p_{+---} to p_{----}.



Remark 4.0.13. *The flowline starting and ending at the same point is just a point and we call it a constant flowline. We are not interested in constant flowlines in our considerations.*

Hence, the leaf space $\mathcal{L}$ of this foliation of $X = \mathbb{RP}^4 \setminus Z$ can be written as $\mathcal{L} = \mathcal{S}^3 \cup \mathcal{S}^1$.

Now, there is a decomposition of $\mathcal{L}$ into disjoint subsets; four points, such that two of which are $(0,0,0,\pm 1) \subset D_{\pm}$ and the other two points are the extra copy S^0 of $\mathcal{S}^1$, one S^1 and the other subspaces are two spheres which foliate the complement after removing $\mathcal{S}^1$ and the points $(0,0,0,\pm 1)$.

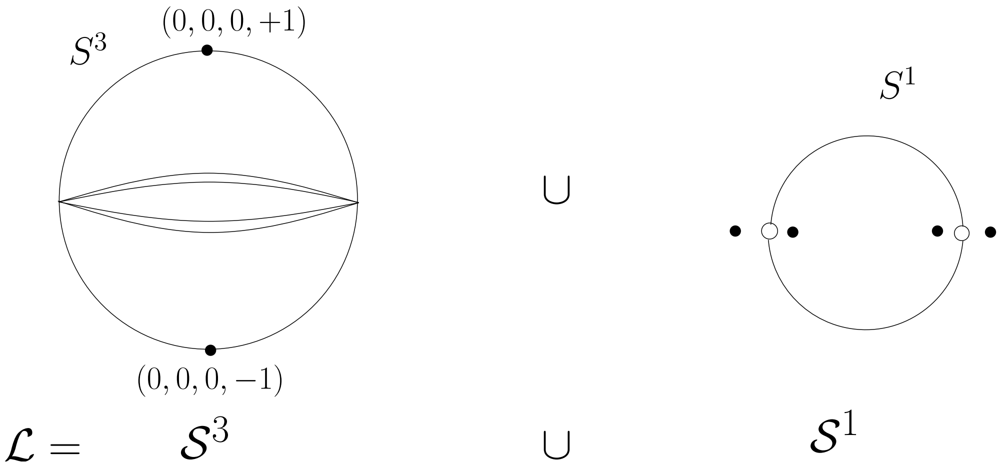


Figure 4.2: Decomposition of the leaf space $\mathcal{L}$.

There is an immersion $h : S^3 \longrightarrow \mathcal{L}$ induced from developing map and since S^3 and $\mathcal{L}$ have same dimension, it is also a submersion. Then the preimages of decomposition of $\mathcal{L}$ give a decomposition of S^3.

If we restrict $h : S^3 \longrightarrow \mathcal{L}$ to the discs of S^3, we get finite sheeted covering projections, which are given as

$$h_| : h^{-1}(D^3_{\pm}) \longrightarrow D^3_{\pm}$$

according to Lemma 2.0.8.

If S^2 is any leaf of the foliation of $\mathcal{L}$ then

$$h_| : h^{-1}(S^2) \longrightarrow S^2$$

is also a covering projection. Thus, $h^{-1}(S^2)$ is a disjoint union of S^2's.

We see that $\mathcal{L}$ has a foliation with S^2 leaves after removing the north pole, the south pole and $\mathcal{S}^1$ from $\mathcal{L}$. Using the decomposition of $\mathcal{L}$ described above (also see Figure 4.2), we get a decomposition of S^3 by taking the preimages of this decomposition of $\mathcal{L}$. Then we use an Euler characteristic argument and arrive at a contradiction in all cases.

Let N be the north pole and S be the south pole.

Since $\mathcal{L}$ and S^3 are locally compact spaces, we can use Euler characteristic with compact supports ([4] page 1).

The compactly supported Euler characteristic of the punctured open disc is

$$\chi_c\big(D^3 \setminus (S^2 \cup N)\big) = \chi_c(D^3) - \chi_c(S^2) - \chi_c(N) = 1 - 2 - 1 = -2,$$

and

$$\chi_c(\mathcal{S}^3) = 2 \cdot \chi_c\big(D^3 \setminus (S^2 \cup N\big) + 2 \cdot \chi_c(N \cup S) + 2 \cdot \chi_c(S^2).$$

Case 1: $h^{-1}(\mathcal{S}^1) = \emptyset$.

n_1: the number of the preimages of D_+.

n_2: the number of the preimages of D_-.

k: the number of the preimages of S^2.

From

$$\chi(S^3) = \chi_c(S^3) = (-2n_1) + (-2n_2) + n_1 + n_2 + 2k,$$

we get

$$2k = n_1 + n_2 \geq 2.$$

This equality says that the number of the open discs is equal to twice the number of spheres.

The possible cases are listed below:

1. $k = 1$ then $n_1 + n_2 = 2$.

   It means that there is one S^2 dividing S^3 into two pieces D_+ and D_-. Then $h(S^3)$ touches one of the equators of $\mathcal{S}^3$, so the image is not invariant under involution. Thus, this case is not possible.

2. $k = 2$ then $n_1 + n_2 = 4$.

   If we remove one sphere from S^3, we have two components which are D_+ and D_-. Then the other sphere is in one of these discs. Without loss of generality, we assume that the sphere is in D_+. Then these two spheres divide S^3 into three connected disc components. However, it is a contradiction since it should be 4.

3. $k \geq 3$ then $n_1 + n_2 \geq 6$.

   In this case, we have a contradiction similar to the case $k = 2$, so this case is also not possible.

Case 2: $h^{-1}(\mathcal{S}^1) \neq \emptyset$.

$$\chi\big(S^3 \setminus h^{-1}(\mathcal{S}^1)\big) < 0.$$

$$-l = \chi\big(S^3 \setminus h^{-1}(\mathcal{S}^1)\big) = (-2n_1) + (-2n_2) + n_1 + n_2 + 2k,$$

where l is the number of the preimages of $\mathcal{S}^1$. The Euler characteristic of S^1 is 0, so S^1 makes no change on the Euler characteristic. One of the double equators S^0 of $\mathcal{S}^1$ decreases the Euler characteristic since we remove it. Finally, we get

$$n_1 + n_2 = 2k + l \quad \text{and} \quad l \geq 1.$$

Now, we consider the below possible cases:

1. $k = 1$ and $l = 1$ then $n_1 + n_2 = 3$.

   Then there are 3 disc connected components, but we have only one S^2 dividing S^3 into two pieces and one point, which is the preimage of S^0. Thus, it is a contradiction.

2. $k \geq 2$ and $l \geq 1$ then $n_1 + n_2 \geq 5$.

   There is again a contradiction with the number of connected disc components and the number of S^2's. This finishes the proof.

□

CHAPTER 5

AN OBSTRUCTION TO THE EXISTENCE OF REAL PROJECTIVE STRUCTURES

In this chapter, we will give an obstruction to obtain examples of manifolds with the infinite fundamental group $\mathbb{Z}$ admitting no real projective structure.

General properties of Pontryagin classes give the following theorem.

Theorem 5.0.14. *If there is an immersion $M^{n-1} \longrightarrow \mathbb{R}^n$, where M is an orientable manifold then the Pontryagin classes $p_i(M^{n-1})$ are all two torsion, for $i \geq 1$.*

Theorem 5.0.15. *Let M^n be a simply connected manifold which does not admit any immersion into $\mathbb{R}^{n+1}$. Then $M \times S^1$ does not have any real projective structure.*

Proof. Assume that $M \times S^1$ admits a real projective structure. Then there exists a developing map such that

$$\begin{array}{ccc} M \times \mathbb{R} & \xrightarrow{dev} & \mathbb{RP}^{n+1} \\ \downarrow & & \\ M \times S^1 & & \end{array}$$

Consider the following diagram.

$$\begin{array}{ccc} & & S^{n+1} \\ & \nearrow & \downarrow \\ M^n & \xrightarrow{dev} & \mathbb{RP}^{n+1} \end{array}$$

Since the map $M \longrightarrow \mathbb{RP}^{n+1}$ is an immersion and the double cover $S^{n+1} \longrightarrow \mathbb{RP}^{n+1}$ is a local diffeomorphism, $M \longrightarrow S^{n+1}$ is also an immersion.

Moreover,

$$M \longrightarrow S^{n+1} \setminus \{p\} = \mathbb{R}^{n+1}$$

is also an immersion where p is a point in S^{n+1}, which is not in the image of M. However, this yields a contradiction. $\square$

Example: Let $M = \mathbb{CP}^2$. The first Pontryagin class of $\mathbb{CP}^2$ is $p_1 = c_1^2 - 2c_2$, where c_i's are Chern classes, for $i = 1, 2$. Then

$$p_1 = c_1^2 - 2c_2 = 9 - 2.3 = 3.$$

Hence p_1 is not a torsion class. By Theorem 5.0.14, there is no immersion $\mathbb{CP}^2 \longrightarrow \mathbb{R}^5$ and it contradicts to the existence of the developing map. Therefore, $\mathbb{CP}^2 \times S^1$ does not have a real projective structure.

Example: Let M^4 be an almost complex manifold such that $\pi_1(M) = \mathbb{Z}_2$. Then the universal cover $\widetilde{M} \longrightarrow M$ has also an almost complex structure via pull back. Then the first Pontryagin class of $\widetilde{M}$ is

$$p_1(\widetilde{M}) = c_1^2 - 2c_2 = c_1^2 - 2\chi(\widetilde{M}).$$

If $p_1(\widetilde{M})$ is zero, by Theorem 5.0.14 there is no real projective structure on $\widetilde{M}$. If it is nonzero, we blow up the universal cover at some pair of points p_i and p_j such that $p_j = \tau(p_i)$, where τ is the Deck trasformation of the cover $\widetilde{M} \longrightarrow M$ and after $2k$-times blowing up we get the manifold $\widetilde{M}\#2k\overline{\mathbb{CP}^2}$. Then $\chi(\widetilde{M}\#2k\overline{\mathbb{CP}^2}) = \chi(\widetilde{M}) + 2k$ and $c_1^2(\widetilde{M}\#2k\overline{\mathbb{CP}^2}) = c_1^2(\widetilde{M}) - 2k$.

$$\begin{aligned} p_1(\widetilde{M}\#2k\overline{\mathbb{CP}^2}) &= c_1^2(\widetilde{M}\#2k\overline{\mathbb{CP}^2}) - 2\chi(\widetilde{M}\#2k\overline{\mathbb{CP}^2}) \\ &= c_1^2(\widetilde{M}) - 2k - 2\chi(\widetilde{M}) - 4k \\ &= p_1(\widetilde{M}) - 6k \end{aligned}$$

Therefore, $p_1(\widetilde{M}\#2k\overline{\mathbb{CP}^2})$ is nonzero (for $k > 1$) if $p_1(\widetilde{M})$ is zero and thus $(\widetilde{M}\#2k\overline{\mathbb{CP}^2}) \times S^1$ admits no real projective structure.

Using the ideas of this chapter we obtain a short cut in the proof of the main theorem (Theorem 3.2.1) in case $\widetilde{W}^{n-1}$ does not admit an immersion into $\mathbb{R}^n$.

Theorem 5.0.16. *Assume that W^{n-1} and M as in Theorem 3.2.1. Assume further that the universal cover $\widetilde{W}$ of W does not admit an immersion into $\mathbb{R}^n$. Then M has no real projective structure.*

Proof. Assume on the contrary that M has a real projective structure. Then the universal cover $\widetilde{W} \times \mathbb{R}$ of M has a real projective structure and thus the developing map $dev : \widetilde{W} \times \mathbb{R} \longrightarrow \mathbb{RP}^n$ provides an immersion of $\widetilde{W}$ into $\mathbb{R}^n$. This finishes the proof. □

Remark 5.0.17. *Note that since S^{n-1} has an immersion into $\mathbb{R}^n$, the above theorem does not imply that $\mathbb{RP}^n \# \mathbb{RP}^n$ can not have a real projective structure.*

CHAPTER 6

PROJECTIVE STRUCTURES AND CONTACT FORMS

6.1 Background

Definition 6.1.1. *Let* M *be a* $(2n+1)$*-dimensional manifold. A contact structure is a maximally non-integrable hyperplane field* $\xi \subset TM$ *such that any local* 1*-form* α *with* $\xi = ker\alpha$ *is required to satisfy*

$$\alpha \wedge (d\alpha)^n \neq 0.$$

Such a 1-form α *is called a contact form and the pair* (M, ξ) *is called a contact manifold.*

For more information about contact structures see [10], [11].

In [18], Ovsienko dealt with the relation between real projective structures and special contact forms on simply connected odd dimensional manifolds. He showed that a locally simply connected projective manifold of odd dimension is contact.

Theorem 6.1.2 (Ovsienko)**.** *Let* M *be a* $(2k-1)$*-dimensional simply connected manifold, where* $k > 1$*. Then* M *admits a real projective structure if and only if there are functions* $f_1, ..., f_{2k} \in C^\infty(M)$ *such that the one form*

$$\alpha = \sum_{i=1}^{k} (f_{2i-1}\, df_{2i} - f_{2i}\, df_{2i-1})$$

is contact.

6.2 Main Result

We proved that if a manifold M (with any fundamental group) admits a special contact form consisting of some functions on M then it admits a real projective structure.

Theorem 6.2.1. *Let an odd dimensional manifold* M^{2k-1} *admit a contact form as given below*

$$\alpha = \sum_{i=1}^{k} (f_{2i-1}\, df_{2i} - f_{2i}\, df_{2i-1})$$

for some functions $f_1, ..., f_{2k} \in C^\infty(M)$. *Then* M *admits a real projective structure.*

Proof. Suppose that there is a contact form α on M as above. Let $\pi : \widetilde{M} \longrightarrow M$ be the universal cover of M. Consider the functions $\hat{f}_1, ..., \hat{f}_{2k} \in C^\infty(\tilde{M})$ such that $\hat{f}_i = f_i \circ \pi$, where $f_1, ..., f_{2k} \in C^\infty(M)$.

Therefore, $\hat{\alpha}$ is a contact form on $\widetilde{M}$ such that

$$\hat{\alpha} = \sum_{i=1}^{k} (\hat{f}_{2i-1}\, d\hat{f}_{2i} - \hat{f}_{2i}\, d\hat{f}_{2i-1}).$$

Let F be the $2k$-dimensional linear space spanned by the functions $\hat{f}_i$,

$$F = < \hat{f}_1, ..., \hat{f}_{2k} > \cong \mathbb{R}^{2k}.$$

To any point $m \in \widetilde{M}$ we assign the subspace $V_m \subset F$ formed by the functions vanishing at the point m such that

$$V_m = \{\hat{f} \in F \mid \hat{f}(m) = 0\}.$$

The dimension of V_m is $(2k-1)$ since the form $\hat{\alpha}$ is nondegenerate.

Then the map $\Phi : \widetilde{M} \longrightarrow \mathbb{RP}^{2k-1}$, given by $\Phi(m) = V_m$ is well defined.

Now, we will show that Φ is a local diffeomorphism.

For any point m, there is a function in F that is nonzero at m. We may assume $\hat{f}_{2k} \neq 0$. Let us say $t_i = \hat{f}_i / \hat{f}_{2k}$, $i = 1, ..., 2k-1$ in the neighborhood of m.

Then the form $\hat{\alpha} \wedge (d\hat{\alpha})^{k-1}$ is proportional to the form $dt_1 \wedge dt_2 \wedge ... \wedge dt_{2k-1} \neq 0$. Hence, the functions $(t_1, ..., t_{2k-1})$ constitute a system of local coordinates on $\widetilde{M}$.

Therefore, Φ is a developing map for some real projective structure on $\widetilde{M}$. Now, we need to say this pojective structure can be pulled down to M.

The real projective structure on $\widetilde{M}$ is given by the functions $\hat{f}_i \in F$. Since the functions have the form $\hat{f}_i = f_i \circ \pi$, the real projective structure on $\widetilde{M}$ descends to a structure on M. We conclude that M admits a real projective structure. $\square$

REFERENCES

[1] F. S. Aschenbrenner, M. and H. Wilton. 3-manifold groups. *ArXiv*, 1205.0202v3.

[2] Y. Benoist. Convexes divisibles. IV. Structure du bord en dimension 3. *Invent. Math.*, 164(2):249–278, 2006.

[3] J.-P. Benzécri. Sur les variétés localement affines et localement projectives. *Bull. Soc. Math. France*, 88:229–332, 1960.

[4] F. Bittner. The universal Euler characteristic for varieties of characteristic zero. *Compos. Math.*, 140(4):1011–1032, 2004.

[5] F. Bonahon. Geometric structures on 3-manifolds. In *Handbook of geometric topology*, pages 93–164. North-Holland, Amsterdam, 2002.

[6] B. Cherowitzo. The rise of projective geometry 2. *Lecture notes*.

[7] S. Choi and W. M. Goldman. The classification of real projective structures on compact surfaces. *Bull. Amer. Math. Soc. (N.S.)*, 34(2):161–171, 1997.

[8] D. Cooper and W. Goldman. A 3-manifold with no real projective structure. *Ann. Fac. Sci. Toulouse Math. (6)*, 24(5):1219–1238, 2015.

[9] C. Ehresmann. Variétes localement projectives. *L' Enseignement Mathématique*, 35:317–333, 1937.

[10] J. B. Etnyre. Introductory lectures on contact geometry. In *Topology and geometry of manifolds (Athens, GA, 2001)*, volume 71 of *Proc. Sympos. Pure Math.*, pages 81–107. Amer. Math. Soc., Providence, RI, 2003.

[11] H. Geiges. *An introduction to contact topology*, volume 109 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2008.

[12] W. M. Goldman. Projective geometry on manifolds. *Lecture Notes*.

[13] W. M. Goldman. Geometric structures on manifolds and varieties of representations. In *Geometry of group representations (Boulder, CO, 1987)*, volume 74 of *Contemp. Math.*, pages 169–198. Amer. Math. Soc., Providence, RI, 1988.

[14] W. M. Goldman. What is... a projective structure? *Notices Amer. Math. Soc.*, 54(1):30–33, 2007.

[15] W. M. Goldman. Locally homogeneous geometric manifolds. In *Proceedings of the International Congress of Mathematicians. Volume II*, pages 717–744. Hindustan Book Agency, New Delhi, 2010.

[16] E. Molnár. The projective interpretation of the eight 3-dimensional homogeneous geometries. *Beiträge Algebra Geom.*, 38(2):261–288, 1997.

[17] V. Ovsienko and S. Tabachnikov. *Projective differential geometry old and new*, volume 165 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2005. From the Schwarzian derivative to the cohomology of diffeomorphism groups.

[18] V. Y. Ovsienko. Projective structures and contact forms. *Funktsional. Anal. i Prilozhen.*, 28(3):47–60, 95, 1994.

[19] J. Porti. Geometrization of three manifolds and Perelman's proof. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM*, 102(1):101–125, 2008.

[20] J. G. Ratcliffe. *Foundations of hyperbolic manifolds*, volume 149 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2006.

[21] G. Reeb. *Sur certaines propriétés topologiques des variétés feuilletées*. Actualités Sci. Ind., no. 1183. Hermann & Cie., Paris, 1952. Publ. Inst. Math. Univ. Strasbourg 11, pp. 5–89, 155–156.

[22] J. Richter-Gebert. *Perspectives on projective geometry*. Springer, Heidelberg, 2011. A guided tour through real and complex geometry.

[23] P. Scott. The geometries of 3-manifolds. *Bull. London Math. Soc.*, 15(2), 1983.

[24] W. Thurston. The geometry and topology of 3-manifolds. 1979.

[25] W. P. Thurston. A generalization of the Reeb stability theorem. *Topology*, 13:347–352, 1974.

[26] W. P. Thurston. *Three-dimensional geometry and topology. Vol. 1*, volume 35 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1997. Edited by Silvio Levy.

[27] J. H. C. Whitehead. Locally homogeneous spaces in differential geometry. *Ann. of Math. (2)*, 33(4):681–687, 1932.

[28] J. S. Wilson. Groups with every proper quotient finite. *Proc. Cambridge Philos. Soc.*, 69:373–391, 1971.

CURRICULUM VITAE

PERSONAL INFORMATION

Surname, Name: Çoban, Hatice
Nationality: Turkish (TC)
Date and Place of Birth: 29/05/1987, Yalova
Marital Status: Single
Phone: 0555 516 99 25
E-mail: haticecoban2905@gmail.com

EDUCATION

Degree	Institution	Year of Graduation
B.S.	Ankara University, Mathematics	2009
High School	Fatih Sultan Mehmet High School	2005

SCHOLARSHIPS

Ph.D scholarship by The Scientific and Technological Research Council of Turkey (TUBITAK) November 2010- August 2015