ON BOUNDED AND UNBOUNDED OPERATORS

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ABSTRACT

ON BOUNDED AND UNBOUNDED OPERATORS

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In this thesis we study on bounded and unbounded operators and obtain some results by considering ℓ-Köthe spaces. As a beginning, we introduce some necessary and sufficient conditions for a Cauchy Product map on a smooth sequence space to be continuous and linear and we consider its transpose. We use the modified version of Zahariuta's method to obtain analogous results for isomorphic classification of Cartesian products of Köthe spaces. We also investigate the SCBS property and show that all separable Fréchet-Hilbert spaces have this property. By the help of this result, we obtain that the bounded perturbation of an automorphism on a separable Fréchet-Hilbert space still takes place up to a complemented Hilbert subspace. We also show that the strong dual of a Fréchet-Hilbert space has the SCBS property. After that, we consider ℓ-Köthe spaces and obtain necessary and sufficient condition for every continuous linear operator from a Fréchet space to ℓ-Köthe space to be bounded. In addition, we obtain a sufficient condition when each continuous linear operator from a Fréchet space X to ℓ-Köthe space $\lambda^{\ell_3}(C)$ that factors over the projective tensor product of ℓ-Köthe spaces $\lambda^{\ell_1}(A)\hat{\otimes}_\pi\lambda^{\ell_2}(B)$ is bounded when $\lambda^{\ell_1}(A)$ and $\lambda^{\ell_2}(B)$ are nuclear. We show that if there exists a continuous linear unbounded operator between ℓ-Köthe spaces, then there exists a continuous unbounded quasi-diagonal operator between them. Using this result, we study in terms of corresponding Köthe matrices when every continuous linear operator between ℓ-Köthe spaces is bounded. As an application, we observe that the existence of an unbounded operator between ℓ-Köthe spaces, under a splitting condition, causes the existence of a common basic subspace.

Keywords: Smooth sequence spaces, strictly singular operators, SCBS property, bounded operators, bounded factorization property, unbounded operators, ℓ-Köthe spaces.

ÖZ

SINIRLI VE SINIRSIZ OPERATÖRLER ÜZERİNE

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Bu tezde, sınırlı ve sınırsız operatörler üzerine çalıştık ve ℓ-Köthe uzaylarını ele alarak bazı sonuçlar elde ettik. Başlangıç olarak düzgün dizi uzayları üzerinde tanımladığımız Cauchy çarpımı dönüşümünün sürekli ve lineer olması için gerek ve yeter şartları elde ettik ve bu dönüşümün transpozunu ele aldık. Zahariuta'nın yönteminin değiştirilmiş versiyonunu kullanarak Köthe uzaylarının kartezyen çarpımlarının izomorfik sınıflandırılmasına ilişkin benzer sonuçlar elde ettik. Ayrıca SCBS özelliğini inceledik ve bütün ayrılabilir Fréchet-Hilbert uzaylarının da bu özelliği sağladığını gösterdik. Bu sonuç yardımıyla, ayrılabilir Fréchet-Hilbert uzaylarında sınırlı operatörlere birim operatörü eklediğimizde, tümlenebilen bir Hilbert altuzayı dışında izomorfizma elde ettik. Ayrıca Fréchet-Hilbert uzaylarının kuvvetli duallerinin de bu özelliği sağladığını gösterdik. Daha sonra, ℓ-Köthe uzaylarını ele aldık ve Fréchet uzayından ℓ-Köthe uzayına tamımlı bütün sürekli lineer operatörlerin sınırlı olması için gerek ve yeter şartı elde ettik. Buna ek olarak, $\lambda^{\ell_1}(A)$ ve $\lambda^{\ell_2}(B)$ nükleer olduğunda, Fréchet uzayı X den ℓ-Köthe uzayı $\lambda^{\ell_3}(C)$ 'e tanımlanan, ℓ-Köthe uzaylarının projektif tensör çarpımları $\lambda^{\ell_1}(A)\hat{\otimes}_\pi\lambda^{\ell_2}(B)$ üzerine faktörizasyonu olan bütün sürekli lineer operatörlerin sınırlı olması için yeter şartı elde ettik. Eğer ℓ-Köthe uzayları arasında tanımlı sürekli lineer sınırsız bir operatör varsa, bu uzaylar arasında sürekli sınırsız yarı-diagonal bir operatörün var olacağını gösterdik. Bu sonucu kullanarak, ℓ-Köthe uzayları arasındaki bütün sürekli lineer operatörlerin sınırlı olması şartını Köthe uzayları cinsinden elde etmeye çalıştık. Uygulama olarak, ℓ-Köthe uzayları arasında sınırsız bir operatörün varlığının, ayırma şartı ile birlikte, ortak temel altuzaylarının mevcut olmasına neden olacağını gözlemledik.

Anahtar Kelimeler: Düzgün dizi uzayları, strictly singular operatörler, SCBS özelliği, sınırlı operatörler, sınırlı faktörizasyon özelliği, sınırsız operatörler, ℓ-Köthe uzayları.

To my family

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TABLE OF CONTENTS

LIST OF NOTATIONS

lts	linear topological space
lcs	locally convex space
FH	Fréchet-Hilbert
SCBS	smallness up to a complemented Banach subspace
iff	if and only if
TFAE	the following are equivalent
vsp	vanishing sequence property
qcnp	quotient countable neighbourhood property

CHAPTER 1

INTRODUCTION

In this thesis, we aim to extend some results in [27], [28] and [8] to ℓ-Köthe spaces by considering bounded and unbounded operators. Also we study Fréchet-Hilbert spaces to investigate isomorphic classification of Cartesian products of Fréchet spaces. Chapter 2 is devoted to give definitions of smooth sequence spaces, Fréchet-Hilbert spaces and necessary explanations for bounded and unbounded operators.

Smooth sequence spaces were introduced by Terzioğlu as a generalization of power series spaces. The space of entire functions on $\mathbb{C}$ (respectively, holomorphic functions on the unit disc) is a well known example of infinite (respectively, finite) power series spaces. In Chapter 3, by considering the Cauchy product of two sequences, we introduce a Cauchy product map and obtain some necessary and sufficient conditions for this map on a nuclear Köthe space $\lambda(A)$ to nuclear G_1-space $\lambda(B)$ to be linear and continuous. In addition, we consider its transpose and find necessary and sufficient conditions for the continuity of this map.

In 1973, Zahariuta developed a method to study the isomorphic classification of Cartesian product of lcs's using the Fredholm operator theory. Zahariuta's method was later modified by using the bounded operators in order to extend its area of applications. By the Fredholm operator theory, isomorphisms of the spaces $X_1 \times Y_1 \cong X_2 \times Y_2$ such that any continuous linear operator from X_1 to Y_2 and from X_2 to Y_1 are compact implies an isomorphism of Cartesian factors up to some finite dimensional subspace. This approach was generalized by using boundedness instead of compactness and obtained the isomorphism up to some complemented Banach subspace [9]. The relation boundedness was studied extensively by Vogt not only for Köthe spaces but also for the general case of Fréchet spaces. Also, it was proved that every continuous linear operator from (d_2)-Köthe space to (d_1)-Köthe space is bounded [33, 9]. A characterization of Fréchet spaces with the bounded factorization property was obtained by Terzioğlu and Zahariuta in the spirit of Vogt's characterization for the boundedness property. Also, in [7], the authors used a modified version of Zahariuta's approach to study the isomorphic classification of Cartesian products of power series spaces by considering Riesz type operators. We highly influenced by this paper and used this approach to obtain analogous results for isomorphic classification of spaces $\lambda^p(A) \times \lambda^q(B)$ where A is a (d_2)-matrix and B is a (d_1)-matrix and some applications related to this result are obtained. In addition, we give an example of an operator which is bounded strictly singular but not compact to explain the importance of this method.

In [9], it was obtained that a bounded perturbation of an automorphism on a Fréchet space with the property SCBS is stable up to some complemented Banach subspace and all Banach valued l^p-Köthe spaces have this property. It was essential there to get the generalized Douady lemma in [33]. In particular, more general form of Köthe, say ℓ-Köthe spaces, are considered in [1] and it was proved that all ℓ-Köthe spaces satisfy the SCBS property. Also, this property does not pass to subspaces or quotients. We investigate the Fréchet-Hilbert spaces and prove that all separable Fréchet-Hilbert spaces have the SCBS property. For that reason, we can show that a bounded perturbation of an automorphism on a separable Fréchet-Hilbert space will be stable up to some complemented Hilbert subspace. In addition, the strong dual of Fréchet-Hilbert spaces also satisfies the SCBS property.

In the last chapter, ℓ-Köthe spaces are considered to extend some results given for Köthe spaces to ℓ-Köthe spaces. We obtain a necessary and sufficient condition for a continuous linear map from a Fréchet space to ℓ-Köthe space to be bounded. Also, we extend some results in [27] to ℓ-Köthe spaces and obtain a sufficient condition for $(X, \lambda^{\ell_1}(A)\hat{\otimes}_\pi\lambda^{\ell_2}(B), \lambda^{\ell_3}(C)) \in \mathcal{BF}$ when $\lambda^{\ell_1}(A)$ and $\lambda^{\ell_2}(B)$ are nuclear, X is a Fréchet space.

In [10] it was proved that the existence of a continuous linear unbounded operator from nuclear l_1-Köthe space to another implies the existence of a continuous unbounded quasi-diagonal operator. Also, if the both Köthe spaces are nuclear, in [21], Nurlu and Terzioğlu proved that the existence of a continuous linear unbounded operator on $\lambda(A)$ to $\lambda(B)$ implies, under some conditions, the existence of a common basic subspaces of $\lambda(A)$ and $\lambda(B)$. Djakov and Ramanujan generalized these results by omitting nuclearity condition [8]. Under the strong influence of this paper, we show that if there is a continuous linear unbounded operator between ℓ-Köthe spaces, then there exists a continuous unbounded quasi-diagonal operator between them. Using this result, we study in terms of corresponding Köthe matrices when every continuous linear operator between ℓ-Köthe spaces is bounded. Therefore we can obtain the boundedness relation for the ℓ-Köthe spaces. After that, we observe the extension of Proposition 3 in [8] to the ℓ-Köthe space case and show that the existence of an unbounded operator between ℓ-Köthe spaces, under a splitting condition, induces the existence of a common basic subspace.

CHAPTER 2

PRELIMINARIES

In this chapter, some definitions and necessary explanations which are used in other chapters will be given.

Our terminology for lcs is standard and we refer to books [12] and [17]. For a lcs X, $\mathcal{U}(X)$ denotes absolutely convex and closed neighborhoods of the origin in X and p_U is the gauge of U. A Fréchet space is a complete metrizable lcs.

2.1 Smooth sequence spaces

Let $A = (a_n^k)_{n,k\in\mathbb{N}}$ be a matrix of real numbers such that $0 \leq a_n^k \leq a_n^{k+1}$ for all n, k and $\sup_k a_n^k > 0$. For $1 \leq p < \infty$ we define the l^p - Köthe space

$$\lambda^p(A) = \left\{ x = (x_n) : \|x\|_k = \left(\sum_n |x_n a_n^k|^p \right)^{\frac{1}{p}} < \infty, \quad \forall k \in \mathbb{N} \right\},$$

the l^∞ - Köthe space

$$\lambda^\infty(A) = \left\{ x = (x_n) : \|x\|_k = \sup_n |x_n| a_n^k < \infty, \quad \forall k \in \mathbb{N} \right\}$$

and the c_0 - Köthe space

$$c_0(A) = \left\{ x \in \lambda^\infty(A) : \lim_{n\to\infty} x_n a_n^k = 0, \quad \forall k \in \mathbb{N} \right\}$$

With the topology generated by the system of seminorms $\{\|.\|_k, k \in \mathbb{N}\}$, these spaces are Fréchet spaces.

We note that $\lambda^p(A)$ (resp. $c_0(A)$) is the projective limit of the Banach space

$$l^p(a^k) = \left\{ x = (x_n) : (x_n a_n^k) \in l^p \right\}$$

$$\left(respectively, \quad c_0(a^k) = \left\{x = (x_n) : (x_n a_n^k) \in c_o\right\}\right)$$

which is the diagonal transformation of $l^p = l^p(I)$ (respectively, $c_0 = c_0(I)$).

The topological dual of l^1 - Köthe space $\lambda(A)$ is isomorphic to the space of all sequences $u = (u_n)$ for which $|u_n| \leq C a_n^k$ for some k and $C > 0$.

It is well known that a Köthe space $\lambda(A)$ associated with the matrix A is nuclear iff $\forall k$, $\exists m$ such that

$$\sum_n \frac{a_n^k}{a_n^m} < +\infty$$

and in this case the fundamental system of seminorms $\|x\|_k = \sum_n |x_n| a_n^k$ can be replaced by the equivalent system of seminorms

$$\|x\|_k = \sup_n |x_n| a_n^k, \quad k \in \mathbb{N}.$$

Smooth sequence spaces were introduced in [24] as a generalization of power series spaces (see also [25]).

Definition 2.1.1 *A Köthe set $A = \{(a_n^k)\}$ is called a G_∞-set and the corresponding Köthe space $\lambda(A)$ a G_∞-space if A satisfies the followings :*

(1) $a_n^1 = 1$, $a_n^k \leq a_{n+1}^k$ for each k and n;

(2) $\forall k\ \exists j$ with $(a_n^k)^2 = O(a_n^j)$

A Köthe set $B = \{(b_n^k)\}$ is called a G_1-set and the corresponding Köthe space $\lambda(B)$ a G_1-space if B satisfies the followings :

(1) $0 < b_{n+1}^k \leq b_n^k < 1$ for each k and n;

(2) $\forall k\ \exists j$ with $b_n^k = O((b_n^j)^2)$

We need the following result [5].

Lemma 2.1.2 *Let $\lambda(A)$ and $\lambda(B)$ be l^1-Köthe spaces. A map $T : \lambda(A) \longrightarrow \lambda(B)$ is continuous linear map iff $\forall k$, $\exists m$ such that*

$$\sup_n \frac{\|Te_n\|_k}{\|e_n\|_m} < +\infty$$

2.2 Compact, strictly singular and bounded operators

Let X and Y be lcs's and $T : X \longrightarrow Y$ be a continuous linear operator. We say that T is bounded (respectively precompact) if $\exists U \in \mathcal{U}(X)$ such that $T(U)$ is bounded (respectively, precompact) in Y. The operator T is strictly singular if the restriction $T|_D$, where D is any closed infinite dimensional subspace of X, is not an isomorphism.

We write $(X,Y) \in \mathcal{B}, (X,Y) \in \mathcal{K}, (X,Y) \in \mathcal{SS}, (X,Y) \in \mathcal{BSS}$ if every continuous linear operator from X into Y is bounded, precompact, strictly singular, bounded and strictly singular, respectively. It was proved that every continuous linear operator from (d_2)-Köthe space to (d_1)-Köthe space is bounded [33, 9]. The relation $\mathcal{B}$ was studied extensively by Vogt [28] by considering Köthe spaces and also Fréchet spaces. In case X or Y is a normed space, it is clear that $(X,Y) \in \mathcal{B}$.

As in [7] we say that (X,Y) has the bounded factorization property and write $(X,Y) \in \mathcal{BF}$ if each linear continuous operator T from X to X that factors over Y (i.e. $T = R_1 R_2$, where $R_2 : X \longrightarrow Y$ and $R_1 : Y \longrightarrow X$ are continuous linear operators) is bounded. Terzioğlu and Zahariuta obtained in [27] a characterization of Fréchet spaces with the property $\mathcal{BF}$ in the spirit of Vogt's characterization for the property $\mathcal{B}$ [28]. In case X is a Schwartz or Y is a Montel the relation $\mathcal{K}$ is equivalent to the relation $\mathcal{B}$.

An operator $T : X \to Y$ between lts's is Fredholm if it is an open mapping with $dim(kerT) < \infty$ and $codim(T(X)) < \infty$ with closed range. An operator $T : X \to X$ between lts's will be called Riesz type operator if $1_X - T$ is a Fredholm operator. A lcs X is called Mackey-complete if for every absolutely convex, closed and bounded subset $B \subset X$ the linear span of B is a Banach space with unit ball X. It is easy to see that every Fréchet space is Mackey-complete. Bounded strictly singular operators from Mackey-complete space to another form an ideal of Riesz type operators (see [30], Theorem 3 and [31], Satz 1).

Zahariuta used the Fredholm operator theory for compact operators to get isomorphic classification of Cartesian products of lcs's [33]. Zahariuta's approach was later modified in [9] by using the bounded operators in order to extend its area of applications. Also, in [7], authors used a modified version to study the isomorphic classification of

Cartesian products of power series spaces by considering Riesz type operators. Our present research is highly influenced by this paper. We used this approach to obtain analogous results for isomorphic classification of spaces $\lambda^p(A) \times \lambda^q(B)$ where A is a (d_2)-matrix and B is a (d_1)-matrix. We could use it since $\lambda^p(A)$ is the projective limit of weighted ℓ_p-spaces [13]. In addition, we obtain some applications for isomorphic classification.

For any linear operator $T : X \longrightarrow Y$ between Fréchet spaces we consider the following operator seminorms

$$\|T\|_{s,t} = \sup \{\|Tx\|_s : \|x\|_t \leq 1\}, \quad s, t \in \mathbb{N}$$

which may take the value $+\infty$. In particular, for any one dimensional operator $T = u \otimes x$, we have

$$\|T\|_{s,t} = \|u\|_t^* \|x\|_s$$

where

$$\|u\|_t^* = sup \{|u(x)| : \|x\|_t \leq 1\}$$

The operator T is continuous iff $\forall k$, $\exists N(k)$ such that

$$\|T\|_{k,N(k)} < \infty,$$

T is bounded iff $\exists N \in \mathbb{N}$ such that $\forall r \in \mathbb{N}$,

$$\|T\|_{r,N} < \infty.$$

We denote by $\mathcal{L}(X, Y)$ the space of all continuous linear operators from X to Y. For Fréchet spaces X and Y, in [28], Vogt proved that $(X, Y) \in \mathcal{B}$ iff for every sequence $N(k)$, $\exists N \in \mathbb{N}$ such that $\forall r \in \mathbb{N}$ we have $k_0 \in \mathbb{N}$ and $C > 0$ with

$$\|T\|_{r,N} \leq C \max_{1 \leq k \leq k_0} \|T\|_{k,N(k)} \tag{2.2.1}$$

for all $T \in \mathcal{L}(X, Y)$.

An operator $T : \lambda^\ell(A) \longrightarrow \lambda^\ell(B)$ is called quasi-diagonal if there exists $k : \mathbb{N} \longrightarrow \mathbb{N}$ and constants m_n such that

$$Te_n = m_n \tilde{e}_{k(n)}, \quad n \in \mathbb{N}$$

Following [15], a pair of Köthe spaces $(\lambda^{\ell}(B), \lambda^{\ell}(A))$ satisfies the condition $\mathcal{S}$ if,

$$\forall p \quad \exists q, k \quad \forall s, l \quad \exists r, C : \frac{b_m^s}{a_n^k} \leq C \max\left\{\frac{b_m^q}{a_n^p}, \frac{b_m^r}{a_n^l}\right\} \tag{2.2.2}$$

2.3 Fréchet-Hilbert spaces

Definition 2.3.1 *A Fréchet space X is called a (FH)-space (Fréchet-Hilbert space) if its topology is generated by a fundamental system of Hilbert seminorms $p_n(x) = \langle x, x\rangle_n^{1/2}$, $(n \in \mathbb{N})$, and X is complete in each seminorm p_n.*

A projective sequence X_n of lcs's is called strict if all the linking maps are open and onto. An inductive sequence X_n of lcs's is called strict if X_n is a topological subspace of X_{n+1}, i.e. the linking maps are injective. If X is the strict inductive limit of the sequence X_n of lcs's, we write $X = s\text{-}\varinjlim X_n$. If X_n are Banach spaces (resp. Hilbert spaces) then X is strict $LB-$spaces (resp. strict $LH-$spaces).

Recall that a Fréchet space X is called a quojection if for every continuous seminorm p on X, $X/Kerp$ is Banach when endowed with the quotient topology. It is easily seen that X is a quojection if it is isomorphic to the projective limit of a sequence of Banach spaces with respect to surjective mappings. Therefore, a quojection is the strict projective limit of Banach spaces. Finite or countable products of Banach spaces are trivial quojections [19, 6]. For a nontrivial quojection, see [18].

Following Nachbin [20], a lcs X is said to have openness condition if,

$$\forall U_1 \in \mathcal{U}(X) \quad \exists U_2 \in \mathcal{U}(X) \quad \forall U_3 \in \mathcal{U}(X) \quad \exists m > 0 : U_2 \subset mU_3 + Kerp_{U_1},$$

that is, $X/KerP_V$ with the quotient topology is normable for every $V \in \mathcal{U}(X)$. Fréchet spaces with the openness condition are exactly quojections [2]. On the other hand, quojections satisfy the openness condition. For a quojection X the strong dual X' is a strict $(LB)-$space.

It is well known that for a Fréchet space X, the quotient spaces $X/Kerp_n$ are Banach spaces in the quotient norms. Accordingly, each Fréchet space is the projective limit of the sequence of its local Banach spaces. In case of FH-spaces, the quotient spaces

$X/Kerp_n$ will be Hilbert spaces. Therefore, a FH-space is the strict projective limit of its local Hilbert spaces.

CHAPTER 3

TRIANGULAR OPERATORS ON SMOOTH SEQUENCE SPACES

If (a_n), (b_n) are two sequences of scalars, then the Cauchy product $(c_n) = (a_n) * (b_n)$ of (a_n) and (b_n) is defined by $c_n = \sum_{k=1}^{n} a_{n+1-k} b_k$.

Now let $\theta = (\theta_n)$ be a fixed sequence of scalars and let $\lambda(A)$, $\lambda(B)$ be two nuclear l^1-Köthe spaces. We define the Cauchy Product map T_θ from $\lambda(A)$ into $\lambda(B)$ by $T_\theta x = \theta * x, \quad x = (x_n) \in \lambda(A)$. An easy computation shows that $T_\theta e_1 = (\theta_1, \theta_2, \theta_3, \ldots)$, $T_\theta e_2 = (0, \theta_1, \theta_2, \ldots)$ and so $T_\theta e_n = (0, 0, \ldots, 0, \theta_1, \theta_2, \ldots)$ where nth place is θ_1. So, $T_\theta : \lambda(A) \longrightarrow \lambda(B)$ can be determined by the lower triangular matrix

$$C = \begin{pmatrix} \theta_1 & 0 & 0 & 0 & \cdots \\ \theta_2 & \theta_1 & 0 & 0 & \cdots \\ \theta_3 & \theta_2 & \theta_1 & 0 & \cdots \\ \vdots & & & \ddots & \end{pmatrix}$$

because for all $x \in A$, we obtain that

$$\begin{aligned} T_\theta x &= T_\theta \left(\sum_{n=1}^{\infty} x_n e_n \right) \\ &= \sum_{n=1}^{\infty} x_n T_\theta(e_n) \\ &= (x_1\theta_1, x_1\theta_2 + x_2\theta_1, x_1\theta_3 + x_2\theta_2 + x_3\theta_1, \ldots) \\ &= C \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{bmatrix} \end{aligned}$$

3.1 The continuity of the Cauchy product map

In this section we introduce some necessary and sufficient conditions for the map T_θ to be linear and continuous.

Theorem 3.1.1 *Let $\lambda(A)$ be a nuclear Köthe space, $\lambda(B)$ be a nuclear G_1-space. Then the Cauchy product map $T_\theta : \lambda(A) \longrightarrow \lambda(B)$ is linear continuous operator iff the following hold:*

i) $\theta \in \lambda(B)$

ii) $\lambda(A) \subset \lambda(B)$

Proof Let $T_\theta : \lambda(A) \longrightarrow \lambda(B)$ be a continuous linear operator.
Note that
$\|T_\theta e_n\|_k = \|(0, 0, ..., 0, \theta_1, \theta_2, \cdots)\|_k = \sup_{j\geq n} |\theta_{j-n+1}| b_j^k$
for $n \in \mathbb{N}$. Clearly $\|e_n\|_m = a_n^m$. So, by Lemma 2.1.2 $\forall k$, $\exists m$, $\exists \rho > 0$ such that

$$\sup_{j\geq n} |\theta_{j-n+1}| b_j^k \leq \rho a_n^m, \quad \forall n \in \mathbb{N}.$$

Choose $j = n$. Then $\forall k$, $\exists m$, $\exists C > 0$ such that

$$b_n^k \leq C a_n^m$$

i.e, $\lambda(A) \subset \lambda(B)$. Since $T_\theta e_1 \in \lambda(B)$, it follows that $\theta \in \lambda(B)$.

Conversely, since B is a G_1-set and by ii and i we have for a given k, there are $m_1(k)$ and $m_2(m_1)$ such that

$$\begin{aligned}
\|T_\theta e_n\|_k &= \sup_{j\geq n} |\theta_{j-n+1}| b_j^k \\
&\leq C_1 \sup_{j\geq n} |\theta_{j-n+1}| (b_j^{m_1})^2 \\
&\leq C_1 \sup_{j\geq n} (|\theta_{j-n+1}| b_j^{m_1})(b_n^{m_1}) \\
&\leq C_2 \sup_{j\geq n} (|\theta_{j-n+1}| b_j^{m_1})(a_n^{m_2}) \\
&\leq C_2 \sup_{j\geq n} (|\theta_{j-n+1}| b_{j-n+1}^{m_1})(a_n^{m_2}) \\
&\leq C a_n^{m_2}.
\end{aligned}$$

Therefore, $\forall k$, $\exists m_2$ such that

$$\sup_n \frac{\|T_\theta e_n\|_k}{\|e_n\|_{m_2}} < \infty$$

that is, T_θ is continuous.

3.2 Transpose of the Cauchy product map

We consider the map $T_\theta{}' : \lambda(A) \longrightarrow \lambda(B)$ which is determined by the matrix C^t (the transpose of C) and try to find necessary and sufficient conditions for the continuity of $T_\theta{}'$.

Theorem 3.2.1 *Let $\lambda(A)$ be a nuclear G_∞-space, $\lambda(B)$ be a nuclear Köthe space. Then, $T_\theta{}' : \lambda(A) \longrightarrow \lambda(B)$ which is given above is linear continuous operator iff the following hold:*

i) $\theta \in \lambda(A)'$

ii) $\lambda(A) \subset \lambda(B)$

Proof The matrix C^t of the operator $T_\theta{}' : \lambda(A) \longrightarrow \lambda(B)$ is the following upper triangular matrix:

$$C^t = \begin{pmatrix} \theta_1 & \theta_2 & \theta_3 & \theta_4 & \cdots \\ 0 & \theta_1 & \theta_2 & \theta_3 & \cdots \\ 0 & 0 & \theta_1 & \theta_2 & \cdots \\ \vdots & & & \ddots & \end{pmatrix}$$

Let $T_\theta{}' : \lambda(A) \longrightarrow \lambda(B)$ be a continuous linear operator.
Note that
$\|T_\theta{}' e_n\|_k = \|(\theta_n, \theta_{n-1}, \cdots, \theta_1, 0, 0, \cdots)\|_k = \sup_{1 \le i \le n} |\theta_{n+1-i}| b_i^k$
for $n \in \mathbb{N}$. So, by Lemma 2.1.2 $\forall k$, $\exists m$, $\exists \mu > 0$ such that

$$\sup_{1 \le i \le n} |\theta_{n+1-i}| b_i^k \le \mu a_n^m, \quad \forall n \in \mathbb{N}.$$

Let $i = 1$. Hence $\exists m$, $\exists C = \frac{\mu}{b_1^k} > 0$ such that

$$|\theta_n| \leq Ca_n^m, \quad \forall n$$

i.e, $\theta \in \lambda(A)'$.

Let $i = n$. Then $\forall k$, $\exists m$ such that

$$b_n^k \leq \frac{\mu}{|\theta_1|} a_n^m$$

i.e,

$$\lambda(A) \subset \lambda(B)$$

On the other hand, since A is a G_∞-set and by (i) and (ii) for a given k, there are m_1 and $m_2(k)$ and $m = max\{m_1, m_2\}$ such that

$$\begin{aligned} \|T_\theta' e_n\|_k &= \sup_{1\leq i\leq n} |\theta_{n-i+1}| b_i^k \\ &\leq C_1 \sup_{1\leq i\leq n} a_{n-i+1}^{m_1} b_i^k \\ &\leq C_1 \sup_{1\leq i\leq n} a_{n-i+1}^{m_1} a_i^{m_2} \\ &\leq C_1 a_n^{m_1} a_n^{m_2} \\ &\leq C_2 (a_n^m)^2. \end{aligned}$$

Since $\lambda(A)$ is G_∞ - space, for this m, $\exists j$ such that

$$\sup_n \frac{(a_n^m)^2}{a_n^j} < \infty$$

Therefore, $\forall k$, $\exists j$ such that

$$\sup_n \frac{\|T_\theta' e_n\|_k}{\|e_n\|_j} < \infty$$

that is, T_θ' is continuous.

It is known that $\mathcal{S}$ is a normal sequence space if whenever $|x_i| < |y_i|$ and $y = (y_i) \in \mathcal{S}$, then $x = (x_i) \in \mathcal{S}$ [14].

Remark Now we write $\theta \in \mathcal{S}$ when the Cauchy product map $T_\theta : \lambda(A) \longrightarrow \lambda(B)$ above is continuous. If $\theta, \eta \in \mathcal{S}$, $\lambda \in \mathcal{K}$, then clearly $T_{\theta+\eta}$ and $T_{\lambda\theta}$ will be continuous since T_θ and T_η are continuous. Hence $\mathcal{S}$ is a vector space.

Now, let $|\theta_i| < |\eta_i|, \forall i, \eta \in \mathcal{S}$. Since T_η is continuous, for all k we find m so that

$$\sup_n \left\{ \sup_{j \geq n} |\theta_{j-n+1}| \frac{b_j^k}{a_n^m} \right\} \leq \sup_n \left\{ \sup_{j \geq n} |\eta_{j-n+1}| \frac{b_j^k}{a_n^m} \right\} < \infty$$

i.e. T_θ is continuous.

Therefore $\theta \in \mathcal{S}$. Hence we obtain that $\mathcal{S}$ is a normal sequence space.

CHAPTER 4

ISOMORPHISMS OF CARTESIAN PRODUCTS OF KÖTHE SPACES

4.1 Modification of Zahariuta's approach

The next theorem is a generalization of Douady's Lemma in [33] obtained by considering Riezs type operators [7].

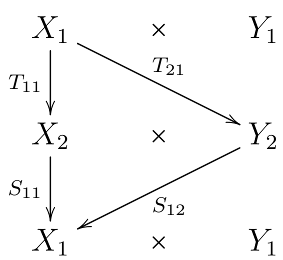
Theorem 4.1.1 *If X_1, X_2, Y_1, Y_2 are lts's such that $X_1 \times Y_1 \cong X_2 \times Y_2$ and each operator acting in X_1 that factors over Y_2 is Riesz type operator then there are complementary subspaces F_1 and C_1 in X_1 and complementary subspaces F_2 and C_2 in X_2 such that F_1 has finite dimension and*

$$C_1 \cong C_2, \quad F_1 \times Y_1 \cong F_2 \times Y_2.$$

Proof Let $T = (T_{ij}) : X_1 \times Y_1 \cong X_2 \times Y_2$ be an isomorphism and let $T^{-1} = S = (S_{ij})$. S and T are 2×2 matrices whose entries are operators acting between factors of the cartesian products, i.e.,

$$\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}, \quad \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$$

Consider the following diagram



We see that $S_{11}T_{11} + S_{12}T_{21} = I_{X_1}$. Since $S_{12}T_{21}$ being an operator on X_1 to X_1 which factors over Y_2 is Riesz type, $I_{X_1} - S_{12}T_{21} = S_{11}T_{11}$ is a Fredholm operator. Therefore $dim(F_1) = dim(ker S_{11}T_{11}) < \infty$ and $G = S_{11}T_{11}(X_1)$ is closed and $codim(G) < \infty$. Let C_1 be a complementary subspace of F_1 in X_1 and π_G be the continuous projection onto G. Obviously the operator $A = (S_{11}T_{11})|_{C_1} : C_1 \longrightarrow G$ is an isomorphism. Set $T_{11}(C_1) = C_2$, then T_{11} maps C_1 into C_2 isomorphicaly. One obtains easily that $P = T_{11}A^{-1}\pi_G S_{11} : X_2 \longrightarrow X_2$ is a projection on C_2. Therefore C_2 is a complemented subspace of X_2. Let $F_2 = P^{-1}(0)$ be the complementary subspace. Since the operator T_{11} acting between first factors of the cartesian products $C_1 \times (F_1 \times Y_1)$ and $C_2 \times (F_2 \times Y_2)$ is isomorphism, then the isomorphism $F_1 \times Y_1 \cong F_2 \times Y_2$ also obtained by Lemma 1 in [7].

For any lcs X and any integer t, the symbol $X^{(t)}$ denotes a $t-$codimensional subspace of X if $t \geq 0$ and a product of the kind $X \times F$, where $dim F = -t$, if $t < 0$.

Corollary 4.1.2 *[7] Retaining the assumptions of the theorem, if each operator acting in X_2 that factors over Y_1 is Riesz type operator, then the subspace F_2 has finite dimension, so $X_2 \cong X_1^{(t)}, Y_2 \cong Y_1^{(-t)}$ with $t = dim F_1 - dim F_2$.*

Finally we need the following lemma [7].

Lemma 4.1.3 *If $X = proj_n X_n$ and $Y = proj_k Y_k$ are projective limits of normed spaces such that $\forall n, k \quad (X_n, Y_k) \in \mathcal{SS}$, then each bounded operator $T : X \longrightarrow Y$ is strictly singular.*

4.2 Köthe space case

We obtain the following:

Theorem 4.2.1 *Let $p \neq \tilde{q}, q \neq \tilde{p}, 1 \leq p, \tilde{p}, q, \tilde{q} < \infty$ and $A, \tilde{A} \in (d_2), B, \tilde{B} \in (d_1)$. Then TFAE:*

(i) $\lambda^p(A) \times \lambda^q(B) \cong \lambda^{\tilde{p}}(\tilde{A}) \times \lambda^{\tilde{q}}(\tilde{B})$

(ii) $\exists t \in \mathbb{Z}$ *such that*

$$\lambda^{\tilde{p}}(\tilde{A}) \cong (\lambda^p(A))^{(t)} \quad \textit{and} \quad \lambda^{\tilde{q}}(\tilde{B}) \cong (\lambda^q(B))^{(-t)}.$$

Proof Since $\lambda^p(A), \lambda^{\tilde{p}}(\tilde{A})$ are (d_2)-Köthe spaces and $\lambda^q(B), \lambda^{\tilde{q}}(\tilde{B})$ are (d_1)-Köthe spaces, by Proposition 1 in [7] we obtain that

$$(\lambda^p(A), \lambda^{\tilde{q}}(\tilde{B})) \in \mathcal{B} \quad \text{and} \quad (\lambda^{\tilde{p}}(\tilde{A}), \lambda^q(B)) \in \mathcal{B}.$$

We can write
$\lambda^p(A) = proj_k \ell_p(a_n^k)$, $\lambda^{\tilde{p}}(\tilde{A}) = proj_k \ell_{\tilde{p}}(\tilde{a}_n^k)$, $\lambda^q(B) = proj_k \ell_q(b_n^k)$ and $\lambda^{\tilde{q}}(\tilde{B}) = proj_k \ell_{\tilde{q}}(\tilde{b}_n^k)$.

$(\ell_p, \ell_{\tilde{q}}) \in \mathcal{SS}$ for $p < \tilde{q}$ and $(\ell_p, \ell_{\tilde{q}}) \in \mathcal{K}$ for $p > \tilde{q}$ (see [16],ch.2,Sec.C). Therefore we obtain that $(\ell_p, \ell_{\tilde{q}}) \in \mathcal{SS}$ for $p \neq \tilde{q}$. Hence, by Lemma 4.1.3, $(\lambda^p(A), \lambda^{\tilde{q}}(\tilde{B})) \in \mathcal{BSS}$. Similarly, $(\lambda^{\tilde{p}}(\tilde{A}), \lambda^q(B)) \in \mathcal{BSS}$. Since bounded strictly singular operators between Mackey-complete spaces form an ideal of Riesz type operators (see [30], Theorem 3 and [31], Satz 1), by Corollary 4.1.2, we obtain the result.

Conversely, since $\lambda^{\tilde{p}}(\tilde{A}) \cong (\lambda^p(A))^{(t)}$ we obtain $\lambda^{\tilde{p}}(\tilde{A}) \cong M$ where M is the t-codimensional subspace of $\lambda^p(A)$. Since $\lambda^{\tilde{q}}(\tilde{B}) \cong (\lambda^q(B))^{(-t)}$, we can write $\lambda^{\tilde{q}}(\tilde{B}) \cong \lambda^q(B) \times L$ where $dim L = t$. Then,

$$\lambda^{\tilde{p}}(\tilde{A}) \times \lambda^{\tilde{q}}(\tilde{B}) \cong M \times \lambda^q(B) \times L \cong M \times L \times \lambda^q(B) \cong \lambda^p(A) \times \lambda^q(B)$$

since $\lambda^p(A) = M \oplus \tilde{L}$ with $dim \tilde{L} = t$ and $L \cong \tilde{L}$. Hence we obtain the result.

We note that this approach does not work in the case where $p = \tilde{q}$ and $q = \tilde{p}$. In [9] an analogous modification is studied by considering bounded operators for this purpose. For the sake of completeness we give it here together with some applications. Similar to Theorem 4.2.1 we obtain also the following.

Theorem 4.2.2 *Let* $1 \leq p, \tilde{p} < \infty$ *and* $A, \tilde{A} \in (d_2), B, \tilde{B} \in (d_1)$. *Then TFAE:*

(i) $c_0(A) \times \lambda^p(B) \cong c_0(\tilde{A}) \times \lambda^{\tilde{p}}(\tilde{B})$

(ii) $\exists t \in \mathbb{Z}$ *such that*

$$c_0(\tilde{A}) \cong (c_0(A))^{(t)} \quad \textit{and} \quad \lambda^{\tilde{p}}(\tilde{B}) \cong (\lambda^p(B))^{(-t)}$$

Proof Since $c_0(A), c_0(\tilde{A})$ are (d_2)-Köthe spaces and $\lambda^p(B), \lambda^{\tilde{p}}(\tilde{B})$ are (d_1)-Köthe spaces, by Proposition 1 in [7] we obtain that

$$(c_0(A), \lambda^{\tilde{p}}(\tilde{B})) \in \mathcal{B} \quad \text{and} \quad (c_0(\tilde{A}), \lambda^p(B)) \in \mathcal{B}.$$

We write

$c_0(A) = proj_k c_0(a_n^k)$, $c_0(\tilde{A}) = proj_k c_0(\tilde{a}_n^k)$, $\lambda^p(B) = proj_k \ell_p(b_n^k)$ and $\lambda^{\tilde{p}}(\tilde{B}) = proj_k \ell_{\tilde{p}}(\tilde{b}_n^k)$.

$(c_0, \ell_{\tilde{p}}) \in \mathcal{SS}$ and $(c_0, \ell_p) \in \mathcal{SS}$ for all $1 \leq p, \tilde{p} < \infty$ (see [16],ch.2,Sec.C). Hence, by Lemma 4.1.3, $(c_0(A), \lambda^{\tilde{p}}(\tilde{B})) \in \mathcal{BSS}$. Also $(c_0(\tilde{A}), \lambda^p(B)) \in \mathcal{BSS}$. Similarly, by Corollary 4.1.2, we obtain the result. One can see the other part of proof similar to the proof of Theorem above.

Now we give an example of an operator which is bounded strictly singular but not compact (see [23]).

Example 4.2.1 *Let X be a Fréchet space defined by an increasing sequence of semi-norms $(|.|_k)$. Then we denote the Fréchet space:*

$$\ell_p[X] := \left\{ \vec{x} = (x_i) : x_i \in X \quad \textit{and} \quad \|\vec{x}\|_k^p = \sum_{i=1}^{\infty} |x_i|_k^p < +\infty, \forall k \right\}$$

Let $1 < p < q < \infty$ and $\lambda(A)$ denote a Montel Köthe space. Take $E = \ell_p[\ell_\infty \times \lambda(A)], F = \ell_q[(\ell_1)^{\mathbb{N}}]$. Since $\lambda(A)$ can be embedded into $(\ell_1)^{\mathbb{N}}$, $(E, F) \notin \mathcal{SS}$ and since ℓ_1 or ℓ_q can be embedded into ℓ_∞, $(F, E) \notin \mathcal{SS}$. Also it is not difficult to see that each continuous linear operator from E to E which factors over F is strictly singular, but not compact. Further each such operator from E to E which factors over F is bounded since any strictly singular operator from a Fréchet space into a complete lcs which admits a continuous norm must be bounded (see [22] or [32]). So we have a Riesz type operator which is not compact.

4.3 Applications of isomorphisms of Cartesian products

In the following theorem, authors obtained a generalization of Douady's Lemma in [33] by using boundedness instead of compactness and obtained the isomorphism of Cartesian factors up to some Banach basic subspace [9].

Theorem 4.3.1 *Suppose X_1 is a Köthe space and X_2, Y_1, Y_2 are topological vector spaces. If $X_1 \times Y_1 \cong X_2 \times Y_2$ and $(X_1, Y_2) \in \mathcal{BF}$, then there are complementary basic subspaces B_1 and C_1 in X_1 and complementary subspaces B_2 and C_2 in X_2 such that B_1 is a Banach space and*

$$C_1 \cong C_2, \qquad B_1 \times Y_1 \cong B_2 \times Y_2.$$

If, in addition, $(X_2, Y_1) \in \mathcal{BF}$, then B_2 is also a Banach space.

Following [26] we say that a lcs X has property (b) if for each neighborhood U, there is a neighborhood V such that for every neighborhood W we have a $\rho > 0$ so that

$$W^\circ \cap X'[U^\circ] \subset \rho V^\circ$$

Here $X'[U^\circ]$ is the linear span of U° in X'. A Fréchet space X satisfies (b) iff X does not satisfy condition (*) of Bellenot and Dubinsky in [2].

From [29] it is known that a Fréchet space X has property (b) iff every continuous linear map into a Köthe space which admits a continuous norm is bounded. Using this fact, we obtain the following:

Theorem 4.3.2 *Suppose X_1, X_2 has property (b) and Y_1, Y_2 are Köthe spaces admitting continuous norms. If $X_1 \times Y_1 \cong X_2 \times Y_2$, then there are complementary basic subspaces B_1, C_1 in Y_1 and B_2, C_2 in Y_2 such that B_1 and B_2 are Banach spaces, $C_1 \cong C_2$, and $B_1 \times X_1 \cong B_2 \times X_2$.*

Proof We have $(X_1, Y_2) \in \mathcal{B}$, $(X_2, Y_1) \in \mathcal{B}$. From Theorem 4.3.1, there are complementary basic subspaces B_1, C_1 in Y_1 and B_2, C_2 in Y_2 such that B_1 and B_2 are Banach spaces, $C_1 \cong C_2$, and $B_1 \times X_1 \cong B_2 \times X_2$.

By [4], a lcs X satisfies the quotient countable neighbourhood property (qcnp) if for each neighborhood U of X, for all neighborhoods (U_n), there is a neighborhood V and $\lambda_n > 0 \quad (n \in \mathbb{N})$ with

$$V \subset \lambda_n U_n + Kerp_U$$

We may restate Theorem 4.3.1 by considering qcnp since a Fréchet space X satisfies the qcnp then $(X, Y) \in \mathcal{B}$ iff Y is a Köthe spaces admitting continuous norms (see [4]). In all applications we used the relation $\mathcal{B}$ instead of the weaker relation $\mathcal{BF}$.

CHAPTER 5

FRÉCHET-HILBERT SPACES AND THE PROPERTY SCBS

A lcs X satisfies the property of smallness up to a complemented Banach subspace (SCBS) if for all $A \in \mathcal{B}(X)$, for all $U \in \mathcal{U}(X)$ and for all $\epsilon > 0$, there are complementary subspaces B and C of X such that B is a Banach space and $A \subset B + \epsilon U \cap C$.

Since normable Fréchet-Hilbert spaces are exactly Hilbert spaces, we consider only nonnormable Fréchet-Hilbert spaces.

5.1 Separable Fréchet-Hilbert spaces

In this section, we consider the separable FH-spaces and obtain some applications. First, we prove the following.

Proposition 5.1.1 *Every separable FH-space X has the SCBS property.*

Proof Every separable FH-space is isomorphic to the space $\omega = \mathbb{R}^{\mathbb{N}}$ of all scalar sequences, or to the space $\ell^2 \times \omega$, or to the space $(\ell^2)^{\mathbb{N}}$. In particular, the spaces ℓ^2_{loc} of locally square summable double sequences and the space $\mathcal{L}^2_{loc}(\mathbb{R})$ isomorphic to the space $(\ell^2)^{\mathbb{N}}$ [34]. We see that these countable product of Banach spaces can be understood as a Banach valued Köthe spaces, that is, k-th row of the Köthe matrix will be $e_1 + e_2 + \cdots + e_k$, and it is known that all Banach valued Köthe spaces have the SCBS property [9].

Now, we have the following result if we follow the steps of the proof in [9].

Theorem 5.1.2 *If X is a separable FH-space and $T : X \longrightarrow X$ is a bounded (respectively, compact) operator, then there are complementary subspaces H and C such that:*
(i) H is Hilbert (respectively, finite-dimensional) space; and
(ii) the operator $I_C - \pi_C T i_C$ is an automorphism on C where π_C and i_C are the canonical projection onto C and embedding into X.

Proof Let $T : X \longrightarrow X$ be a linear bounded (resp. compact) operator. Then, $\exists V \in \mathcal{U}(X)$ such that $T(V)$ is bounded in X, i.e.,

$$\forall U \in \mathcal{U}(X) \quad \exists M_U > 0 : p_U(Tx) \leq M_U p_V(x)$$

Since every separable FH-space has the SCBS property, there are complementary subspaces H and C of X such that H is a Hilbert space and

$$T(V) \subset H + \frac{1}{2} V \cap C.$$

Let $T_1 = \pi_C T i_C : C \longrightarrow C$, Then we obtain

$$p_V(T_1 x) \leq \frac{1}{2} p_V(x), \quad \forall x \in C.$$

We show that $1_C - T_1$ is an automorphism on C. Consider the series

$$Sx = x + T_1 x + T_1^2 x + \cdots + T_1^m x + \cdots, \quad \forall x \in E \tag{5.1.1}$$

It is convergent in C because, $\forall U \in \mathcal{U}(E)$, $m = 1, 2, \ldots$, we obtain

$$p_U(T_1^m x) \leq M_U p_V(T_1^{m-1} x) \leq M_U \left(\frac{1}{2}\right) p_V(T_1^{m-2} x) \leq M_U \left(\frac{1}{2}\right)^{m-1} p_V(x)$$

so by Banach-Steinhaus theorem, (5.1.1) defines a continuous linear operator $S : C \longrightarrow C$. Since $(1_C - T_1)Sx = S(1_C - T_1)x = x$, the operator S is the inverse to the operator $1_C - T_1$.

Now, by the theorem above, we obtain a modification of the generalized Douady lemma in [33], exactly in the same way as in [9].

Theorem 5.1.3 *Suppose X_1 is a separable FH-space and X_2, Y_1, Y_2 are topological vector spaces. If $X_1 \times Y_1 \cong X_2 \times Y_2$ and $(X_1, Y_2) \in \mathcal{BF}$ then there are complementary* *subspaces H_1, C_1 in X_1 and H_2, C_2 in X_2 such that H_1 is a Hilbert spaces, $C_1 \cong C_2$, and $H_1 \times Y_1 \cong H_2 \times Y_2$.*

If in addition $(X_2, Y_1) \in \mathcal{BF}$, then H_2 is a Hilbert space.

A lcs X is said to satisfy the vanishing sequence property (vsp) if given any strongly convergent sequence (x_n), $\exists k \in \mathbb{N}$ such that $x_m = 0$ if $m > k$. Also, a Fréchet space satisfy the vsp iff it has a continuous norm (see [3]). We obtain the following application.

Proposition 5.1.4 *Suppose X_1, X_2 are separable FH-spaces and Y_1, Y_2 are Fréchet spaces admitting continuous norms. If $X_1 \times Y_1 \cong X_2 \times Y_2$, then there are complementary basic subspaces C_1, H_1 in X_1 and C_2, H_2 in X_2 such that H_1 and H_2 are Hilbert spaces, $C_1 \cong C_2$, and $H_1 \times X_1 \cong H_2 \times X_2$.*

Proof Since FH-spaces are reflexive and separable ones are trivial quojections (see [34]), from Corollary 10 in [3], we obtain that $(X_1, Y_2) \in \mathcal{B}$, $(X_2, Y_1) \in \mathcal{B}$. Since X_1 and X_2 satisfies SCBS propery, by Theorem 5.1.3, there are complementary basic subspaces C_1, H_1 in X_1 and C_2, H_2 in X_2 such that H_1 and H_2 are Hilbert spaces, $C_1 \cong C_2$, and $H_1 \times X_1 \cong H_2 \times X_2$.

5.2 Strong dual of Fréchet-Hilbert spaces

It is known that, if X is a strict inductive limit of Banach spaces and $A \in \mathcal{B}(X)$, then A is contained and bounded in some Banach space B from the inductive limit system. Since the limit is strict then B is a subspace of X (see 2.5.13 in [13]).

It is not hard to verify that a Fréchet space X is complete in a seminorm p_n iff the quotient space $X/Kerp_n$ is a Banach space in the associated norm $\tilde{p_n}$. Therefore, in the case of a FH-space X, the quotient spaces $X/Kerp_n$ are Hilbert spaces in the norms $\tilde{p_n}$.

Theorem 5.2.1 *If X is a real FH-space, the strong dual of X has the SCBS.*

Proof Let X be a real FH-space. Then its strong dual $X' = s - \varinjlim H_n$ is a strict (LH)-space where each Hilbert subspace H_n has a topological complement in X' [34]. If $A \in \mathcal{B}(X')$, then A is contained and bounded in some Hilbert space H from the inductive system which is complemented. Hence, X' has the SCBS property. Here, the smallness is trivial since the Hilbert subspace chosen is independent of ϵ and U.

Remark The space of all sequences $\omega = \prod_{k=1}^{\infty} X_k$ where $X_k = \mathbb{R}$ is the real line for all k, is a FH-space [34]. Strong dual X' is the direct sum of Banach spaces dual to X_k, i.e., $X' = \sum_{k=1}^{\infty} X_k{}'$. Locally convex direct sum of Banach spaces has the SCBS property trivially, since every bounded subset of it is contained and bounded in a finite direct sum of its components (which is Banach) from the direct system which is complemented [17].

Corollary 5.2.2 *If X is a real FH-space, then X has openness condition.*

Proof Let X be a real FH-space. Then it is the strict projective limit of a sequence of complemented Hilbert subspaces of it [34]. Therefore, being a quojection, it has the openness condition.

CHAPTER 6

ℓ-KÖTHE SPACES

Following [11], we denote by ℓ a Banach sequence space in which the canonical system (e_n) is an unconditional basis. The norm $\|.\|$ is called monotone if $\|x\| \leq \|y\|$ whenever $|x_n| \leq |y_n|$, $x = (x_n)$, $y = (y_n) \in \ell$, $n \in \mathbb{N}$. Let Λ be the class of such spaces with monotone norm. In particular, $l_p \in \Lambda$ and $c_0 \in \Lambda$. It is known that every Banach space with an unconditional basis has a monotone norm which is equivalent to its original norm. Indeed, it is enough to put

$$\|x\| = \sup_{|\beta_n|\leq 1} \left| \sum_n {e_n}'(x)\beta_n e_n \right|$$

where $|.|$ denotes the original norm, $({e_n}')$ denote the sequence of coefficient functionals.

Let $\ell \in \Lambda$ and $\|.\|$ be a monotone norm in ℓ. If $A = (a_n^k)$ is a Köthe matrix, the ℓ-Köthe space $\lambda^\ell(A)$ is the space of all sequences of scalars (x_n) such that $(x_n a_n^k) \in \ell$ with the topology generated by the seminorms

$$\|(x_n)\|_k = \left\|(x_n a_n^k)\right\|$$

Let us remind that $\|(e_n)\|_k = a_n^k$.

6.1 Bounded operators to ℓ-Köthe spaces

If we follow the steps of the proof of Crone and Robinson Theorem [5], we obtain the following.

Lemma 6.1.1 *$T \in \mathcal{L}(\lambda(A), \lambda^\ell(B))$ iff $\forall m$, $\exists k$ such that*

$$\sup_n \frac{\|Te_n\|_m}{\|e_n\|_k} < +\infty$$

Proof $T \in \mathcal{L}(\lambda(A), \lambda^\ell(B))$ iff $\forall m$, $\exists k$ such that

$$\sup_{x \neq 0, x \in \lambda(A)} \frac{\|Tx\|_m}{\|x\|_k} < +\infty$$

For $x = e_n$, we obtain the result.

Conversely, suppose that $\forall m$, $\exists k$ such that

$$\sup_n \frac{\|Te_n\|_m}{\|e_n\|_k} < +\infty$$

Let $x \in \lambda(A)$.

$$\begin{aligned}
\|Tx\|_m &= \left\| \sum_n x_n T e_n \right\|_m \\
&\leq \sum_n |x_n| \frac{\|Te_n\|_m}{\|e_n\|_k} \|e_n\|_k \\
&\leq \sup_n \frac{\|Te_n\|_m}{\|e_n\|_k} \sum_n |x_n| \, a_n^k \\
&\leq \sup_n \frac{\|Te_n\|_m}{\|e_n\|_k} \|x\|_k
\end{aligned}$$

So, $T \in \mathcal{L}(\lambda(A), \lambda^\ell(B))$.

Notice that when domain is ℓ-Köthe space, we can not use this argument.

Firstly, we consider l^1-Köthe space $\lambda(A)$ and ℓ-Köthe space $\lambda^\ell(B)$ and obtain the following.

Theorem 6.1.2 *TFAE:*

i) $(\lambda(A), \lambda^\ell(B)) \in \mathcal{B}$

ii) for every sequence $N(k)$, $\exists N \in \mathbb{N}$ such that $\forall r \in \mathbb{N}$ we have $k_0 \in \mathbb{N}$ and $C > 0$ with

$$\frac{b_v^r}{a_i^N} \leq C \max_{1 \leq k \leq k_0} \frac{b_v^k}{a_i^{N(k)}}$$

for all $v \in \mathbb{N}$, $i \in \mathbb{N}$.

Proof Suppose that $(\lambda(A), \lambda^\ell(B)) \in \mathcal{B}$. Consider $S : \lambda(A) \longrightarrow \lambda^\ell(B)$ with $S = e_i{}' \otimes e_v$ where $e_i{}'(x) = x_i$ for all $x \in \lambda(A)$. Since S is the operator of rank one, we note that

$$\|S\|_{k,N(k)} = \|e_i^{'}\|_{N(k)}\|e_v\|_k = \frac{b_v^k}{a_i^{N(k)}}$$

Similarly, $\|S\|_{r,N} = \frac{b_v^r}{a_i^N}$. So, the result follows from (2.2.1).

For the converse, let $Te_i = \sum_{v=1}^{\infty} u_{vi}e_v$. Since T is continuous, by Lemma 6.1.1, $\exists N(k)$ such that

$$\begin{aligned}\|T\|_{k,N(k)} &= \sup_{i\in\mathbb{N}} \frac{\|Te_i\|_k}{\|e_i\|_{N(k)}} \\ &= \sup_{i\in\mathbb{N}} \sup_{|\beta_v|\leq 1} \left| \sum_{v=1}^{\infty} u_{vi}\beta_v \frac{b_v^k}{a_i^{N(k)}} e_v \right| < \infty\end{aligned}$$

So we find $N \in \mathbb{N}$ such that

$$\begin{aligned}\|T\|_{r,N} &\leq \sup_{i\in\mathbb{N}} \left\{ \sup_{|\beta_v|\leq 1} \left\| \sum_{v=1}^{\infty} u_{vi}\beta_v \frac{b_v^r}{a_i^N} e_v \right\| \right\} \\ &\leq \sup_{i\in\mathbb{N}} \left\{ \sup_{|\beta_v|\leq 1} \left\| \sum_{v=1}^{\infty} u_{vi}\beta_v \left(C \max_{1\leq k\leq k_0} \frac{b_v^k}{a_i^{N(k)}} \right) e_v \right\| \right\} \\ &\leq C \sum_{k=1}^{k_0} \sup_{i\in\mathbb{N}} \left\{ \sup_{|\beta_v|\leq 1} \left\| \sum_{v=1}^{\infty} u_{vi}\beta_v \frac{b_v^k}{a_i^{N(k)}} e_v \right\| \right\} < \infty\end{aligned}$$

Therefore, T is bounded.

Now, consider the ℓ-Köthe space $\lambda^\ell(A)$ and any Fréchet space X. Then, we obtain the following.

Theorem 6.1.3 *Let X be any Fréchet space. TFAE:*
i) $(X, \lambda^\ell(A)) \in \mathcal{B}$
ii) for every sequence $N(k)$, $\exists N \in \mathbb{N}$ such that $\forall r \in \mathbb{N}$ we have $k_0 \in \mathbb{N}$ and $C > 0$ with

$$a_v^r \|u\|_N^* \leq C \max_{1\leq k\leq k_0} a_v^k \|u\|_{N(k)}^*$$

for all $v \in \mathbb{N}$, $u \in X'$.

Proof Suppose that $(X, \lambda^\ell(A)) \in \mathcal{B}$. Similar to the proof of Theorem 6.1.2, consider the operator of rank one $S = y \otimes e_v$ where $y \in X'$. The result follows from (2.2.1).

For the converse, let $T : X \longrightarrow \lambda^\ell(A)$ be continuous linear operator. Let

$$Tx = \sum_{v=1}^{\infty} e_v'(Tx)e_v = (e_v'T(x)) = (u_v(x)), \quad x \in X$$

where $u_v = e_v' \circ T$.

Then, by continuity we find $N(k)$ such that

$$\sup_{\|x\|_{N(k)}\le 1}\left(\sup_{|\beta_v|\le 1}\left|\sum_v \beta_v u_v(x) a_v^k e_v\right|\right) = M(k)$$

Let $|u_v(x)| = \theta_v u_v(x)$ where $\theta_v = \pm 1$ and $\beta_v\theta_v = \alpha_v$, note that

$$\begin{aligned}
\|Tx\|_r &= \sup_{|\beta_v|\le 1}\left|\sum_v \beta_v u_v(x) a_v^r e_v\right| \\
&\le \sup_{|\beta_v|\le 1}\left|\sum_v \beta_v \left(\frac{|u_v(x)|}{\|x\|_N}\right) a_v^r e_v\right| \|x\|_N \\
&\le \sup_{|\beta_v|\le 1}\left|\sum_v \beta_v \|u_v\|_N^* a_v^r e_v\right| \|x\|_N \\
&\le \sup_{|\beta_v|\le 1}\left|\sum_v \beta_v \left(C \max_{1\le k\le k_0} a_v^k \|u_v\|_{N(k)}^*\right) e_v\right| \|x\|_N \\
&\le C\sum_{k=1}^{k_0} \sup_{|\beta_v|\le 1}\left|\sum_v \beta_v a_v^k \left(\sup_{\|x\|_{N(k)}\le 1} |u_v(x)|\right) e_v\right| \|x\|_N \\
&\le C\sum_{k=1}^{k_0} \sup_{\|x\|_{N(k)}\le 1}\left(\sup_{|\beta_v|\le 1}\left|\sum_v \beta_v a_v^k \theta_v u_v(x) e_v\right|\right) \|x\|_N \\
&\le C\sum_{k=1}^{k_0} \sup_{\|x\|_{N(k)}\le 1}\left(\sup_{|\alpha_v|\le 1}\left|\sum_v \alpha_v a_v^k u_v(x) e_v\right|\right) \|x\|_N \\
&\le \left(C\sum_{k=1}^{k_0} M(k)\right) \|x\|_N
\end{aligned}$$

Hence T is bounded.

6.2 Bounded factorization property for ℓ-Köthe spaces

We need the following theorem [27, Theorem 2.2].

Theorem 6.2.1 *For Fréchet spaces X, Y and Z we have $(X, Y, Z) \in \mathcal{BF}$ iff for every sequence $N(k)$, $\exists N \in \mathbb{N}$ such that $\forall r \in \mathbb{N}$ we have $k_0 = k_0(r) \in \mathbb{N}$ and $C = C(r) > 0$*

so that the following inequality

$$\|T\|_{r,N} \leq C \max_{1\leq k\leq k_0} \left\{\|R\|_{k,N(k)}\right\} \max_{1\leq k\leq k_0} \left\{\|S\|_{k,N(k)}\right\}$$

is satisfied for every $R \in L(Y,Z)$, $S \in L(X,Y)$ where $T = RS$.

The next result is obtained by following the lines of [27, Corollary 3.1].

Theorem 6.2.2 *Let X be a Fréchet space, $\lambda^{\ell}(B), \lambda^{\tilde{\ell}}(C)$ be ℓ-Köthe spaces and $\lambda^{\ell}(B)$ be nuclear. Then $(X, \lambda^{\ell}(B), \lambda^{\tilde{\ell}}(C)) \in \mathcal{BF}$ iff for every sequence $N(k)$, $\exists N \in \mathbb{N}$ such that $\forall r \in \mathbb{N}$ we have $k_0 \in \mathbb{N}$ and $C > 0$ with*

$$c_j^r \|u\|_N^* \leq C \max_{1\leq k\leq k_0} \left\{\|u\|_{N(k)}^* b_i^k\right\} \max_{1\leq k\leq k_0} \left\{\frac{c_j^k}{b_i^{N(k)}}\right\} \tag{6.2.1}$$

for all $i \in \mathbb{N}, j \in \mathbb{N}$ and $u \in X'$.

Proof Let $S = u \otimes e_i$ and $R = e_i{}' \otimes e_j$ where $u \in X'$. Then $RS : X \longrightarrow Z$ is the operator of rank one which sends each $x \in X$ to $u(x)e_j$. If we apply Theorem 6.2.1 we obtain the result.

For sufficiency, we take $S \in L(X, \lambda^{\ell}(B))$, $R \in L(\lambda^{\ell}(B), \lambda^{\tilde{\ell}}(C))$ and $T = RS$. Since $\lambda^{\ell}(B)$ is nuclear, $\exists S(k)$ such that $S(k) > N(k)$ and

$$\sum_i \frac{b_i^{N(k)}}{b_i^{S(k)}} = \theta(k) < \infty, \quad k \in \mathbb{N}.$$

We can write $Sx = \sum_i u_i(x)e_i$ where $u_i = e_i{}' \circ S \in X'$ and $Re_i = \sum_j r_{ji}e_j$. Therefore

$$Tx = \sum_i \sum_j u_i(x) r_{ji} e_j$$

For this $S(k)$ we choose $N \in \mathbb{N}$ such that for each $r \in \mathbb{N}$ we obtain $k \in \mathbb{N}$ and $C > 0$ with

$$c_j^r \|u_i\|_N^* \leq C \max_{1\leq k\leq k_0} \left\{\|u_i\|_{S(k)}^* b_i^k\right\} \max_{1\leq k\leq k_0} \left\{\frac{c_j^k}{b_i^{S(k)}}\right\} \tag{6.2.2}$$

for all $i \in \mathbb{N}, j \in \mathbb{N}, u_i \in X'$.

Since all types of nuclear Köthe spaces determined by one and the same matrix B coincide [11, Corollary 2, p.22], $\lambda(B) = \lambda^{\ell}(B)$ and we have

$$\|S\|_{k,N(k)} = \sup_{\|x\|_{N(k)}\leq 1} \|Sx\|_k = \sup_{\|x\|_{N(k)}\leq 1} \sum_i |u_i(x)|\, b_i^k = \sum_i \|u_i\|_{N(k)}^* b_i^k$$

$$\|R\|_{k,N(k)} = \sup_i \frac{\|Re_i\|_k}{\|e_i\|_{N(k)}} = \sup_i \sup_{|\beta_j|\le 1} \left| \sum_j \frac{r_{ji}\beta_j c_j^k e_j}{b_i^{N(k)}} \right|$$

Therefore, we have

$$\sum_i \|u_i\|_{S(k)}^* b_i^k \le \sum_i \|u_i\|_{N(k)}^* b_i^k = \|S\|_{k,N(k)}$$

and

$$\sum_j \frac{|r_{ji}| c_j^k}{b_i^{S(k)}} \le \theta(k) \sup_j \frac{|r_{ji}| c_j^k}{b_i^{N(k)}} \le \theta(k) \|R\|_{k,N(k)}$$

(see [11, proof of Corollary 2, p.22]) Hence, by (6.2.2) we obtain that

$$\begin{aligned}
\|Tx\|_r &\le \sum_i \sum_j \frac{|u_i(x)|}{\|x\|_N} |r_{ji}| c_j^r \|x\|_N \\
&\le \sum_i \sum_j \|u_i\|_N^* |r_{ji}| c_j^r \|x\|_N \\
&\le \sum_i \sum_j |r_{ji}| \left\{ C \max_{1\le k\le k_0} \left\{ \|u_i\|_{S(k)}^* b_i^k \right\} \max_{1\le k\le k_0} \left\{ \frac{c_j^k}{b_i^{S(k)}} \right\} \right\} \|x\|_N \\
&\le C \sum_{k=1}^{k_0} \sum_i \|u_i\|_{S(k)}^* b_i^k \sum_j \frac{|r_{ji}| c_j^k}{b_i^{S(k)}} \|x\|_N \\
&\le C \sum_{k=1}^{k_0} \sum_i \|u_i\|_{S(k)}^* b_i^k \theta(k) \|R\|_{k,N(k)} \|x\|_N \\
&\le \left(C \sum_{k=1}^{k_0} \theta(k) \|S\|_{k,N(k)} \|R\|_{k,N(k)} \right) \|x\|_N
\end{aligned}$$

Therefore, T is bounded.

Recall that projective tensor product of two l^1-Köthe spaces $\lambda(A)$ and $\lambda(B)$ is isomorphic to $\lambda(D)$ where $d_{vz}^k = a_v^k b_z^k$.

Theorem 6.2.2 enables us to get:

Theorem 6.2.3 *Suppose* $(X, \lambda^{\ell_1}(A)) \in \mathcal{B}$ *and* $(\lambda^{\ell_2}(B), \lambda^{\ell_3}(C)) \in \mathcal{B}$ *where* X *is a Fréchet space,* $\lambda^{\ell_3}(C)$ *is an* ℓ*-Köthe space,* $\lambda^{\ell_1}(A)$ *and* $\lambda^{\ell_2}(B)$ *are nuclear* ℓ*-Köthe spaces. Then* $(X, \lambda^{\ell_1}(A)\hat{\otimes}_\pi \lambda^{\ell_2}(B), \lambda^{\ell_3}(C)) \in \mathcal{BF}$.

Proof Given a non-decreasing sequence $N(k)$. Since $\lambda^{\ell_2}(B)$ is nuclear and $(\lambda^{\ell_2}(B), \lambda^{\ell_3}(C)) \in \mathcal{B}$, we obtain that $\lambda^{\ell_2}(B) = \lambda(B)$ and by Theorem 6.1.2, $\exists n \in \mathbb{N}$ such that $\forall r \in \mathbb{N}$ we have $k_0 = k_0(r) \in \mathbb{N}$ and $C_1 = C_1(r) > 0$ with

$$\frac{c_j^r}{b_i^n} \le C_1 \max_{1\le k\le k_0} \frac{c_j^k}{b_i^{N(k)}}$$

for all $i \in \mathbb{N}$, $j \in \mathbb{N}$.

We then determine $S(k)$ such that $S(k) = N(n)$ if $1 \le k \le n$ and $S(k) > N(k)$ if $n+1 \le k \le s_0$. Since $(X, \lambda^{\ell_1}(A)) \in \mathcal{B}$ by Theorem 6.1.3, for this $S(k)$, we find $m \in \mathbb{N}$ such that $\forall q \in \mathbb{N}$ we have $s_0 = s_0(q)$ and $C_2 = C_2(q)$

$$\begin{aligned} a_v^q \|u\|_m^* &\le C_2 \max_{1\le k\le s_0} a_v^k \|u\|_{S(k)}^* \\ &\le C_2 \max_{n\le k\le s_0} a_v^k \|u\|_{N(k)}^* \end{aligned}$$

Therefore, for this $N(k)$ we have $\tilde{s_0} = s_0(N(k))$ and $\tilde{C_2} = C_2(N(k))$ with

$$\begin{aligned} c_j^r \|u\|_m^* &= \frac{c_j^r}{b_i^n} b_i^n \|u\|_m^* \\ &\le \left\{ C_1 \max_{1\le k\le k_0} \frac{c_j^k}{b_i^{N(k)}} \right\} b_i^n \|u\|_m^* \\ &\le \left\{ C_1 \max_{1\le k\le k_0} \frac{c_j^k}{a_v^{N(k)} b_i^{N(k)}} \right\} a_v^{N(k)} b_i^n \|u\|_m^* \\ &\le \left\{ C_1 \max_{1\le k\le k_0} \frac{c_j^k}{a_v^{N(k)} b_i^{N(k)}} \right\} \left\{ \tilde{C_2} \max_{n\le k\le \tilde{s_0}} a_v^k \|u\|_{N(k)}^* b_i^n \right\} \\ &\le \left\{ C_1 \max_{1\le k\le k_0} \frac{c_j^k}{a_v^{N(k)} b_i^{N(k)}} \right\} \left\{ \tilde{C_2} \max_{1\le k\le \tilde{s_0}} a_v^k b_i^k \|u\|_{N(k)}^* \right\} \end{aligned}$$

Let $s = \max\{k_0, \tilde{s_0}\}$ and $C = C(r) = C_1\tilde{C_2}$. We have proved that $\exists m \in \mathbb{N}$ such that $\forall r \in \mathbb{N}$ we have $s \in \mathbb{N}$ and $C > 0$ with

$$c_j^r \|u\|_m^* \le C \max_{1\le k\le s} \left\{ \frac{c_j^k}{a_v^{N(k)} b_i^{N(k)}} \right\} \max_{1\le k\le s} \left\{ \|u\|_{N(k)}^* a_v^k b_i^k \right\}$$

for all $j \in \mathbb{N}, v \in \mathbb{N}, i \in \mathbb{N}$ and $u \in X'$.

If $\lambda^{\ell_1}(A)$ and $\lambda^{\ell_2}(B)$ are nuclear ℓ-Köthe spaces, then

$$\lambda^{\ell_1}(A)\hat{\otimes}_\pi \lambda^{\ell_2}(B) \cong \lambda(A)\hat{\otimes}_\pi \lambda(B) \cong \lambda(D)$$

is nuclear where $d_{vi}^k = a_v^k b_i^k$ [11, Corollary 2, p.22].

By Theorem 6.2.2 we obtain that $(X, \lambda^{\ell_1}(A)\hat{\otimes}_\pi \lambda^{\ell_2}(B), \lambda^{\ell_3}(C)) \in \mathcal{BF}$.

6.3 Bounded and unbounded operators in ℓ-Köthe spaces

Let $\lambda^\ell(A), \lambda^\ell(B)$ be ℓ-Köthe spaces. As in [8] we obtain the following.

Proposition 6.3.1 *Let $\lambda^\ell(A)$ and $\lambda^\ell(B)$ be ℓ- Köthe spaces. If there is a linear continuous unbounded operator $T : \lambda^\ell(A) \longrightarrow \lambda^\ell(B)$, then there is a continuous unbounded quasi-diagonal operator on $\lambda^\ell(A)$ to $\lambda^\ell(B)$.*

Proof Let $T : \lambda^\ell(A) \longrightarrow \lambda^\ell(B)$ be continuous and unbounded. We may assume wlog that

$$\|Tx\|_k \le \frac{1}{2^k} \|x\|_k, \quad \forall x \in \lambda^\ell(A)$$

$$\sup_n \frac{\|Te_n\|_{k+1}}{\|e_n\|_k} = \infty, \quad k \in \mathbb{N}.$$

Indeed, one may obtain these by using appropriate multipliers and passing to a subsequence of seminorms, if necessary. Let (k_j) be a sequence of integers such that each $k \in \mathbb{N}$ appears in it infinitely many times and choose an increasing subsequence (n_j) such that

$$\frac{\|Te_{n_j}\|_{k_j+1}}{\|e_{n_j}\|_{k_j}} \ge 2^j, \quad \forall j$$

Let us remind that $\|\tilde{e_v}\|_k = b_v^k$ and $\|e_n\|_k = a_n^k$ and let $Te_n = \sum_v \theta_{nv}\tilde{e_v}$. Note that,

$$\begin{aligned}\sup_{|\alpha_v|\le 1}\left|\sum_v \theta_{nv}\alpha_v\left(\sup_k \frac{b_v^k}{a_n^k}\right)\tilde{e_v}\right| &\le \sum_k \left(\frac{b_v^k}{a_n^k}\right)\left(\sup_{|\alpha_v|\le 1}\left|\sum_v \theta_{nv}\alpha_v\tilde{e_v}\right|\right)\\ &\le \sum_k \frac{1}{a_n^k}\sup_{|\alpha_v|\le 1}\left|\sum_v \theta_{nv}\alpha_v b_v^k\tilde{e_v}\right|\\ &\le \sum_k \frac{\|Te_n\|_k}{\|e_n\|_k} \le \sum_k \frac{1}{2^k} \le 1\end{aligned}$$

Therefore we obtain that

$$\sup_{|\alpha_v|\le 1}\left|\sum_v \theta_{n_jv}\alpha_v\left(\sup_k \frac{b_v^k}{a_{n_j}^k}\right)\tilde{e_v}\right| \le 1 \le \frac{1}{2^j}\sup_{|\alpha_v|\le 1}\left|\sum_v \theta_{n_jv}\alpha_v\frac{b_v^{k_j+1}}{a_{n_j}^{k_j}}\tilde{e_v}\right| \qquad (6.3.1)$$

So there is a v_j such that

$$t_j := \sup_k \frac{b_{v_j}^k}{a_{n_j}^k} \le \frac{1}{2^j}\frac{b_{v_j}^{k_j+1}}{a_{n_j}^{k_j}}$$

Otherwise we obtain a contradiction to (6.3.1) by monotonicity of $\|.\|$.

Now, consider the quasi-diagonal operator $D : \lambda^\ell(A) \longrightarrow \lambda^\ell(B)$ defined by

$$De_{n_j} = t_j^{-1} e_{\tilde{v_j}}, \quad j \in \mathbb{N}$$

$$De_n = 0 \quad \text{if} \quad n \neq n_j$$

Let $x = \sum_j x_{n_j} e_{n_j} \in \lambda^\ell(A)$. So, $Dx = \sum_j x_{n_j} t_j^{-1} \tilde{e}_{v_j}$. Since $\left|x_{n_j} t_j^{-1} b^k_{v_j}\right| \leq \left|x_{n_j} a^k_{n_j}\right|$, by monotonicity we obtain that $\left\|\left(x_{n_j} t_j^{-1} b^k_{v_j}\right)\right\| \leq \left\|\left(x_{n_j} a^k_{n_j}\right)\right\|$, i.e.,

$$\|Dx\|_k \leq \|x\|_k \quad \forall k$$

Hence, D is continuous.

Similarly, it is easy to see that D is unbounded since for a fixed k, there is a subsequence (j_m) such that $k_{j_m} = k$, $m \in \mathbb{N}$ and

$$\frac{\|De_{n_{j_m}}\|_{k+1}}{\|e_{n_{j_m}}\|_k} \geq 2^{j_m} \to \infty$$

as $m \to \infty$. This completes the proof.

Proposition 6.3.1 enables us to prove the sufficiency part of the following theorem. Notice that sufficiency can not be obtained directly for a general linear map.

Theorem 6.3.2 *Let $\lambda^\ell(A)$ and $\lambda^\ell(B)$ be ℓ-Köthe spaces. $(\lambda^\ell(A), \lambda^\ell(B)) \in \mathcal{B}$ iff for every sequence $N(k) \uparrow \infty$, $\exists N \in \mathbb{N}$ such that $\forall r \in \mathbb{N}$ we have $k_o \in \mathbb{N}$ and $C > 0$ with*

$$\frac{b^r_v}{a^N_i} \leq C \max_{1 \leq k \leq k_0} \frac{b^k_v}{a^{N(k)}_i}$$

for all $v \in \mathbb{N}, i \in \mathbb{N}$.

Proof Suppose $(\lambda^\ell(A), \lambda^\ell(B)) \in \mathcal{B}$. Consider $T : \lambda^\ell(A) \longrightarrow \lambda^\ell(B)$ with $T = e'_i \otimes e_v$ where $e_i{}'(x) = x_i$ for all $x \in \lambda^\ell(A)$.

Since T is the operator of rank one, we note that

$$\|T\|_{k,N(k)} = \|e_i{}'\|_{N(k)} \|e_v\|_k = \frac{b^k_v}{a^{N(k)}_j}$$

Similarly $\|T\|_{r,N} = \frac{b^r_v}{a^N_i}$. The result follows from (2.2.1).

Conversely we want to show that every continuous linear quasi-diagonal operator is bounded. Let $T : \lambda^\ell(A) \longrightarrow \lambda^\ell(B)$ be a continuous quasi-diagonal operator defined by $T(e_i) = t_i \tilde{e}_{z(i)}$. By continuity, $\exists N(k)$ such that

$$\sup_i \frac{\|Te_i\|_k}{\|e_i\|_{N(k)}} = \sup_i \frac{|t_i|\, b^k_{z(i)}}{a_i^{N(k)}} = C(k) < \infty.$$

Thus for this $N(k)$, $\exists N \in \mathbb{N}$ such that $\forall r \in \mathbb{N}$ we have $k_o \in \mathbb{N}$ and $C > 0$ with

$$\frac{|t_i|\, b^r_{z(i)}}{a_i^N} \leq C \max_{1 \leq k \leq k_0} \frac{|t_i|\, b^k_{z(i)}}{a_i^{N(k)}} \leq C \max_{1 \leq k \leq k_0} C(k).$$

Hence $\|T\|_{r,N} < \infty$, i.e., T is bounded. In view of Proposition 6.3.1, we obtain the result.

$\lambda^\ell(A)$ and $\lambda^\ell(B)$ have a common basic subspace if there is a quasi-diagonal operator $T : X \to Y$ such that $T|_D$, where D is some infinite dimensional basic subspace of X, is an isomorphism. We observe the following extension of Proposition 3 in [8] to the ℓ-Köthe space case. The proof is the same as in [8].

Corollary 6.3.3 *If $(\lambda^\ell(B), \lambda^\ell(A)) \in \mathcal{S}$ and there is a continuous unbounded operator $T : \lambda^\ell(A) \longrightarrow \lambda^\ell(B)$, then $\lambda^\ell(A)$ and $\lambda^\ell(B)$ have a common basic subspace.*

REFERENCES

[1] T. Abdeljawad and M. Yurdakul. The property of smallness up to a complemented Banach subspace. *Publ. Math. Debrecen*, 64:415–425, 2004.

[2] S.F. Bellenot and E. Dubinsky. Fréchet spaces with nuclear Köthe quotients. *Trans. Amer. Math. Soc.*, 273:579–594, 1982.

[3] J. Bonet. On the identity $L(E,F) = LB(E,F)$ for pairs of locally convex spaces E and F. *Proc. Amer. Math. Soc.*, 99(2):249–255, 1987.

[4] J. Bonet and A. Galbis. The identity $L(E,F) = LB(E,F)$, tensor products and inductive limits. *Note Mat.*, 9:195–216, 1989.

[5] L. Crone and W. Robinson. Diagonal maps and diameters in Köthe spaces. *Israel J. Math.*, 20:13–22, 1975.

[6] S. Dierolf and V. B. Moscatelli. A note on quojections. *Funct. Approx. Comment. Math.*, 17:131–138, 1987.

[7] P. B. Djakov, S. Önal, T. Terzioğlu, and M. Yurdakul. Strictly singular operators and isomorphisms of Cartesian products of power series spaces. *Arch. Math.*, 70(1):57–65, 1998.

[8] P. B. Djakov and M.S. Ramanujan. Bounded and unbounded operators between Köthe spaces. *Studia Math.*, 152(1):11–31, 2002.

[9] P. B. Djakov, T. Terzioğlu, M. Yurdakul, and V. Zahariuta. Bounded operators and isomorphisms of Cartesian products of Fréchet spaces. *Michigan Math. J.*, 45:599–610, 1998.

[10] M. M. Dragilev. Riesz classes and multiple regular bases. *Func. Anal. and Func. Theory, Kharkov*, 15:65–77, 1972.

[11] M. M. Dragilev. *Bases in Köthe spaces*. Rostov State University, 1983.

[12] H. Jarchow. *Locally Convex Spaces*. Teubner, Stuttgart, 1981.

[13] H. Junek. *Locally Convex Spaces and Operator Ideals*. B. G. Teubner, Leipzig, 1983.

[14] G. Köthe. *Topological Vector Spaces I*. Springer-Verlag, Berlin, Heidelberg, New York, 1969.

[15] J. Krone and D. Vogt. The splitting relation for Köthe spaces. *Math. Z.*, 190(3):387–400, 1985.

[16] J. Lindenstrauss and L. Tzafriri. *Classical Banach Spaces I, II.* Springer-Verlag, Berlin, Heidelberg, New York, 1977-79.

[17] R. Meise and D. Vogt. *Introduction to Functional Analysis*. Clarendon Press, Oxford, 1997.

[18] V. B. Moscatelli. Fréchet spaces without continuous norms and without bases. *Bull. London Math. Soc.*, 12:63–66, 1980.

[19] V. B. Moscatelli. Strongly nonnorming subspaces and prequojections. *Studia Math.*, 95(3):249–254, 1990.

[20] L. Nachbin. A glance of holomorphic factorization and uniform holomorphy. *North-Holland Math. Studies*, 125:221–245, 1986.

[21] Z. Nurlu and T. Terzioğlu. Consequences of the existence of a noncompact operator between nuclear Köthe spaces. *Manuscripta Math.*, 47:1–12, 1984.

[22] S. Önal and T. Terzioğlu. Concrete subspaces and quotient spaces of locally convex spaces and completing sequences. *Dissertationes Math. (Rozprawy Mat.)*, 318:1–36, 1992.

[23] S. Önal and M. Yurdakul. On sums of complemented subspaces. *'Mathematical Forum', Studies on Mathematical Analysis, Vladikavkaz*, 7:148–152, 2013.

[24] T. Terzioğlu. Die diametrale Dimension von lokalkonvexen Räumen. *Collect. Math.*, 20:49–99, 1969.

[25] T. Terzioğlu. Smooth sequence spaces and associated nuclearity. *Proceedings of the American Mathematical Society*, 37(2):497–504, 1973.

[26] T. Terzioğlu. A note on unbounded linear operators and quotient spaces. *Doğa Mat.*, 10(3):338–344, 1986.

[27] T. Terzioğlu and V. Zahariuta. Bounded factorization property for Fréchet spaces. *Math. Nachr.*, 253:81–91, 2003.

[28] D. Vogt. Fréchetträume, zwischen denen jede stetige lineare Abbildung beschränkt ist. *J. Reine Angew. Math.*, 345:182–200, 1983.

[29] D. Vogt. Remarks on a paper of S. Önal and T. Terzioğlu. *Doğa Mat.*, 15:200–204, 1991.

[30] V. V. Wrobel. Streng singuläre Operatoren in lokalkonvexen Räumen. *Math. Nachr.*, 83:127–142, 1978.

[31] V. V. Wrobel. Strikt singuläre Operatoren in lokalkonvexen Räumen. II, Beschränkte Operatoren. *Math. Nachr.*, 110:205–213, 1983.

[32] M. Yurdakul. A remark on a paper of J. Prada. *Arch. Math.*, 61(4):385–390, 1993.

[33] V. P. Zahariuta. On the isomorphism of cartesian products of locally convex spaces. *Studia Math.*, 46:201–221, 1973.

[34] D. N. Zarnadze. Some topological and geometrical properties of Fréchet-Hilbert spaces. *Russian Acad. Sci. Izv. Math.*, 41:273–288, 1993.

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