USING ULTRA HIGH FREQUENCY DATA IN INTEGRATED VARIANCE ESTIMATION: GATHERING EVIDENCE ON MARKET MICROSTRUCTURE NOISE

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF APPLIED MATHEMATICS
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY
INCI KILICKAYA

IN PARTIAL FULLFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY
IN FINANCIAL MATHEMATICS

APRIL 2017

Approval of the thesis:

USING ULTRA HIGH FREQUENCY DATA IN INTEGRATED VARIANCE ESTIMATION: GATHERING EVIDENCE ON MARKET MICROSTRUCTURE NOISE

submitted by İNCİ KILIÇKAYA in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Department of Financial Mathematics, Middle East Technical University by,

Prof. Dr. Bulent Karasözen
Director, Graduate School of **Applied Mathematics** __________________

Assoc. Prof. Dr. Yeliz Yolcu Okur
Head of Department, **Financial Mathematics** __________________

Assoc. Prof. Dr. Seza Danışoğlu
Supervisor, **Department of Business Administration, METU** __________________

Examining Committee Members

Assoc. Prof. Dr. Seza Danışoğlu
Department of Business Administration, METU __________________

Prof. Dr. Zehra Nuray Güner
Department of Business Administration, METU __________________

Prof. Dr. Aslıhan Salih
Department of Business Administration, TED UNIVERSITY __________________

Assoc. Prof. Dr. Sevtap Kestel
Department of Actuarial Sciences, METU __________________

Assoc. Prof. Dr. Süheyla Özyıldırım
Department of Business Administration, BİLKENT UNIVERSITY __________________

Date: __________________

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Last name: Inci KILIÇKAYA

Signature :

ABSTRACT

USING ULTRA HIGH FREQUENCY DATA IN INTEGRATED VARIANCE ESTIMATION: GATHERING EVIDENCE ON MARKET MICROSTRUCTURE NOISE

Kılıçkaya, İnci
PhD., Department of Financial Mathematics
Supervisor : Assoc. Prof. Dr. Seza Danışoğlu

February 2017, 284 pages

In recent years, as a result of more readily available ultra high frequency data (UHFD), realized volatility (RV) measures became popular in the finance literature since in theory, sampling at increasingly higher frequency should lead to, in the limit, a consistent estimator of integrated return volatility (IV) for Ito-semimartingale asset prices. However, when observed prices are contaminated with an additive market microstructure noise (MMN), an asymptotic bias appears, and, therefore, it becomes necessary to mitigate the effect of MMN in estimation of IV. The success of the available methods in the literature to suppress the MMN effects must be considered only if the empirical evidence backs the assumptions underlying the methods developed for handling MMN. On this issue, we realize that empirical evidence on the MMN structure should be collected taking into account the dimensions of volatility estimation using high frequency data as these dimensions may impair the validity of the methods adopted to handle MMN in the first place. Accordingly, in this Thesis, first we provide a complete discussion of the dimensions of volatility estimation using UHFD. Next, we prove that the formal tests regarding the existence of MMN and the constant variance of MMN increments originally developed under calendar time sampling can also be used under transaction time sampling. Third, we propose a new approach to measure the liquidity of stocks in a high frequency setting. Finally, by using tick data from Borsa İstanbul National Equity Market for a period of 6 months, we show that (i) the data handling procedures as various combinations of cleaning and aggregation methods do not distort UHFD's original traits, (ii) the return dynamics in transaction time are different from those in calendar time, (iii) the RV dynamics are affected by the sampling scheme and liquidity, (iv) the volatility signature plots point to the existence of MMN and suggest a positive relationship between the noise increment and the true price return, valid in all possible dimensions (sampling scheme, liquidity, data handling methods, and session-based or daily calculations), (v) the MMN exhibits statistically significant existence under both CTS and TTS for all stocks, however, the liquidity and the data handling methods matter under TTS in terms of rejection rates of the null hypothesis that the MMN statistically does not exist, (vi) the formal tests on the existence of MMN offer positive correlation between the noise and the efficient price, (vii) the liquidity and the sampling schemes are very influential on the rejection of the null hypothesis that the MMN increments have constant variance independent of the sampling frequency, in particular, under CTS, (assuming an i.i.d MMN with constant variance is proper for frequencies lower than 1 minute but under TTS, this assumption fails especially for liquid stocks), (viii) data handling has suppressive effects under TTS on the rejection percentages regarding the null hypothesis that the MMN increments have constant variance independent of sampling frequency.

Keywords: Integrated Variance, Realized Volatility, Market Microstructure Noise, Sampling Schemes, Data Handling Methods, Liquidity in High Frequency Finance

ÖZ

BİRİKİMLİ VARYANS HESAPLAMASINDA ULTRA YÜKSEK FREKANSLI VERİ KULLANIMI: PİYASA MİKROYAPISINDAN KAYNAKLANAN GÜRÜLTÜ HAKKINDA KANIT TOPLAMA YÖNTEMLERİ

Kılıçkaya, İnci
Doktora, Finansal Matematik Bölümü
Tez Yöneticisi: Doç. Dr. Seza Danışoğlu

Şubat 2017, 284 sayfa

Ultra Yüksek Frekanslı Veri setlerinin (UYFV) yaygınlaşması ile varlık getirilerinin birikimli varyans (BV) tahmininde gerçekleşmiş oynaklık (GO) tipi tahmin edicilerin kullanımı popüler hale gelmiştir çünkü teoride, belli bir zaman aralığında toplanan veri sayısı sonsuza ulaştığında GO, BV'nin tutarlı bir tahmincisidir. Ancak, gözlemlenen varlık fiyatlarının piyasa mikroyapısından kaynaklanan bir gürültü (PMYG) ile kirlenmesi durumunda, asimptotik bir sapma ortaya çıktığından PMYG'nün GO tahmin edicisi üzerindeki etkilerinin azaltılması ihtiyacı doğar. Literatürde PMYG'nün GO tahmin edicisi üzerindeki etkilerinin bastırılması amacıyla çeşitli yöntemler önerilmişse de, bu yöntemler benimsenmeden önce söz konusu yöntemlerin PMYG hakkında dayandığı varsayımların ampirik kanıtlarla desteklenmesi gerekmektedir. Dolayısıyla, GO kullanılarak BV tahmininde, PMYG'nin istatistiksel yapısı hakkında kanıt toplanmalı ancak, kanıt toplanırken BV hesabında UYFV kullanılmasına ilişkin sorun ve boyutlar dikkate alınmalıdır. Bu çerçevede, bu tezde ilk olarak BV hesabında UYFV kullanılmasına ilişkin sorun ve boyutlar hakkında kapsamlı bir tartışma yapılmış, arkasından takvim zamanı altında geliştirilen PMYG'nin varlığına ve veri toplama sıklığından bağımsız olarak farklarının sabit varyansına yönelik istatistiksel testlerin işlem zamanı altında da kullanılabileceği gösterilmiştir. Ek olarak, yüksek frekanslı veri ile yapılan çalışmalarda kullanılmak üzere hisse senetlerini likiditelerine göre sınıflandırmak için yeni bir yöntem önerilmiştir. Son olarak, bu tezde, 6 aylık Borsa İstanbul Ulusal Pazar UYFV'si kullanılarak, (i) hata temizleme ve aynı anlı verileri özetleme tekniklerinin kombinasyonları olarak uygulanan veri hazırlama metotlarının UYFV'nın orijinal özelliklerini bozmadığı, (ii) takvim zamanı altındaki

getiri dinamiklerinin işlem zamanı altında farklı olduğu, (iii) GO tahmin edicisinin dinamiklerinin veri toplama tekniğinden ve likiditeden etkilendiği, (iv) oynaklık imzası grafiklerinin olası tüm boyutlarda (veri hazırlama metotları, veri toplama teknikleri, likidite ve hatta GO'nun günlük veya seanslık hesaplanması) geçerli olmak üzere PMYG'nin varlığı ve gerçek varlık fiyatları ile PMYG arasında pozitif korelasyona dair görsel kanıt sunduğu, (v) istatistiksel test sonuçlarına göre, PMYG'nin varlığının hem işlem hem de takvim zamanında teyit edildiği ancak, takvim zamanı altında likidite ve veri hazırlama metotlarının test sonuçlarını etkilediği, (vi) PMYG'nin varlığını test eden istatistiğin aldığı değerlerin PMYG ile gerçek varlık fiyatları arasında pozitif bir korelasyonun varlığını desteklediği, (viii) PMYG farklarının veri toplama aralığından bağımsız olarak sabit bir varyansı olup olmadığına yönelik istatistiksel test sonuçlarının likidite ve veri toplama tekniklerinden büyük ölçüde etkilendiği, nitekim, takvim zamanı altında sabit varyanslı ve bağımsız ve aynı dağılan PMYG varsayımının 1 dakikadan az veri toplama aralıkları için uygun olduğu ancak, işlem zamanı altında bu tip bir varsayımın özellikle likit hisse senetleri için reddedildiği, (viii) işlem zamanı altında veri hazırlama metotlarının PMYG farklarının sabit varyansına yönelik test sonuçlarını aşağı yönde baskıladığı gösterilmiştir.

Anahtar Kelimeler: Birikimli Varyans, Gerçekleşmiş Oynaklık, Piyasa Mikroyapısından Kaynaklı Gürültü, Veri Hazırlama Metotları, Veri Toplama Teknikleri, Yüksek Frekanslı Finansta Likidite

To my family

ACKNOWLEDGMENTS

I would like to thank my supervisor Assoc. Prof. Dr. Seza Danışoğlu for patiently guiding, motivating, and encouraging me throughout this study.

I am very thankful to Prof. Dr. Zeynep Nuray Güner, Prof. Dr. Aslıhan Salih, Assoc. Prof. Dr. Sevtap Kestel and Assoc. Prof. Dr. Süheyla Özyıldırım for their suggestions and helpful comments.

Though I know, it is not completely possible to express my gratitude, I would like to thank to a few special people.

My greatest debts go to my family: to my admirable mother Nilgün Odabaşıoğlu, my lovely grandmother Güler Odabaşıoğlu, my sister Ipek Altıntaş and my husband Cihan Kılıçkaya for their love, unfailing support and patience.

I acknowledge my debt and express my heartfelt thanks to Nilüfer Çalışkan, Müge Taşçı and Funda Uyar for sharing hardest times, for their endless support and for patiently motivating and listening to me.

Also, the environment of Institute of Applied Mathematics (IAM) itself has an important place in this study. Many thanks to Nejla Erdoğdu for her patience with my endless questions.

To all the other precious friends who are not mentioned by name in this acknowledgement --- dear colleagues from our Institute of Applied Mathematics, dear relatives and other close ones --- I am likewise sincerely committed to heartfelt thanks and I express to all of them my likewise warm gratitude.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

CHAPTER 1

INTRODUCTION

Although the Turkish capital markets have undergone great progress over the last few decades, what is evident regarding the economy as a whole is also evident for the capital markets: the prices of financial securities are very volatile due to macro-economic imbalances as well as domestic factors such as political stability and international factors such as exchange rates. For instance, in a general pattern of cyclical fluctuations, Borsa İstanbul A.Ş. indices that are calculated in order to reflect the price and return performance of all shares exhibit a high degree of volatility. This type of an investment environment is only preferable if the investor is a risk taker. However, investment theory suggests that different investors may have different choices regarding the level of risk to assume and even the same investor may prefer different risk levels at different times. In order to accommodate these different risk preferences, the financial system has to offer means by which investors can manage and adjust the level of risk that they take. The derivative markets and derivative instruments as a means of risk management are one of the best possible ways of achieving this objective. However, benefiting from derivative markets requires measuring return volatilities correctly since volatility is the most crucial and challenging input used in portfolio selection, derivative pricing and risk management, mainly because volatility is a latent variable and is not directly observable. Fortunately, the finance literature offers many parametric and nonparametric volatility models to measure or forecast return volatilities. Some of the models that are developed over the last few decades include ARCH, GARCH, EGARCH, and stochastic volatility specifications and the performance of these models has been studied frequently [51].

Accompanying the introduction of several complex volatility models, one important development in the volatility measurement context has been the advent and availability of “ultra high frequency data” (UHFD), which refers to the data sets including thorough reports of all the financial markets activity information that is available, where “ultra” high frequency data means that it is not possible to dive into finer details than that is provided in these data sets [37]. The basic unit of information contained in UHFD is called the “tick”, which represents a time stamp and a set of information summarizing specifics of the market activity at that time [37].

The availability of UHFD sets is considered to be one of the most groundbreaking changes in the field of volatility measurement and forecasting since such high frequency data not only fostered the development of improved ex-post volatility measurements but also inspired research into their potential value as an information source for longer horizon volatility forecasts [66]. However, due to the complex structures of parametric models, modeling volatility in a high-frequency setting is very challenging. As Andersen et al. [13] put it, volatility models using daily data cannot accommodate high- frequency data whereas parametric models specified directly for intraday data usually cannot capture the daily volatility movements. Therefore, with the advent of UHFD in 1990s, an interest boomed in nonparametric approaches to the estimation of return volatility using high-frequency data. One of the very first of such nonparametric approaches was to use realized volatility measures, which became famous in the late 1990s and early 2000s.

It was first pointed out by Andersen and Bollerslev [8] that squared daily returns provide a poor approximation of actual daily volatility. They suggested that more accurate estimates could be obtained by summing the squared intraday returns. Following this valuable contribution to the finance literature, Andersen et al. [12], Andersen et al. [10], and Barndorff-Nielsen and Shephard [24] were among the pioneers who studied the ‘‘realized’’ volatility (RV) and its relevance in volatility measurement.

The availability of UHFD made RV measures popular in the finance literature during the last two decades because in theory sampling at increasingly higher frequencies should lead to, in the limit, a consistent estimator of the return volatility when asset prices satisfy a certain semimartingale representation. The semimartingale representation of asset prices is adopted widely in the studies because, as explained by Harrison and Pliska [63], with continuous trading allowed, an arbitrage free market is complete (every contingent claim is attainable) if and only if there is a unique probability measure $\mathbb{P}^*$ equivalent to $\mathbb{P}$ under which the discounted asset prices are martingales, and, in this setting, asset prices must satisfy the semimartingale property.

Regarding the semimartingale property of asset prices, one can choose among different specifications but the most popular specification in the finance literature is the Brownian semimartingale representation, which implies that asset prices do not exhibit any discontinuous behavior. In a Brownian semimartingale setting, the log of an asset's price is a real-valued process defined as the solution of a stochastic differential equation such that log of the asset’s price is a function of time, drift, a Brownian motion and return volatility. When return volatility is itself a stochastic process, the main object of interest is the quadratic variation or integrated variance (IV) and it is defined as the amount of variation at a certain point in time accumulated over a finite past time interval. Note that while the asset price can be observed, the volatility is an unobservable latent variable that scales the Brownian process continuously through time [96].

The RV exploits the information in high-frequency returns and estimates volatility by summing the squares of intraday returns sampled at very short intervals [51]. RV per day in this context is calculated as the sum of all squared immediate returns within a day.

In the stochastic processes literature, the sum, RV, is shown to consistently estimate the integral IV and to converge to the true underlying integrated variance when the length of the intraday intervals goes to zero [11], [24].

Barndorff-Nielsen and Shephard [24] prove the consistency of RV and show that its asymptotic distribution is normal. In this context, if asset prices follow a Brownian semimartingale, then the return volatility can be estimated consistently and effortlessly by calculating the RV at the highest frequency possible. However, sampling returns as many times as possible without any further consideration may not be the right approach since there are several aspects of UHFD that should be taken into account in volatility measurement.

The first issue that should be examined when using an UHFD for estimating the IV of asset prices is the fact that UHFD exhibit interesting characteristics such as a tremendous number of ticks per day, temporal spacing of transactions/quotes, strong intraday patterns in the form of diurnal shapes in trade volume and/or returns per fixed amount of time, and the existence of errors and simultaneous entries for the same time stamp. Therefore, before using UHFD in volatility estimation, one should decide on how to handle these aspects before commenting on the results from any volatility estimation attempt.

As a second issue, O'Hara [92] and many other market microstructure researchers claim that the observed asset prices can be decomposed as sum of the unobserved true price and the unobserved aggregate effect of the market microstructure (noise or MMN, henceforth). There are several arguments in the literature about the sources of the MMN but, in general, the MMN is accepted as a combination of factors such as frictions inherent in the trading process, informational effects, and measurement or data recording errors [3]. Many high frequency finance researchers such as Zhou [112], Andersen et al. [11], Andersen et al. [12], Barndorff-Nielsen and Shephard [24], [25], Bandi and Russell [19], Hansen and Lunde [58], Zhang et al. [111] provide abundant mathematical and empirical evidence of a noise contamination in observed asset prices as we increase the sampling frequency.

Contamination of observed prices with market microstructure is a vital concern in volatility estimation via realized type of measures mainly because if there is such a contamination, then the quadratic variation of observed prices calculated over the highest frequency possible does not simply converge to the IV of true prices since an asymptotic bias appears due to the existence of MMN [10], [24], [25]. Accordingly, one would choose to sample at lower frequencies to eradicate the bias due to MMN but this would increase the variance of the total estimation error due to discretization. This is called "the bias-variance trade-off" in the literature. In order to examine how RV deviates from IV as we increase the sampling frequency and to come up with methods to handle those deviations and the bias-variance trade-off (mitigation of MMN effects on RV measures), we first have to make some assumptions regarding the statistical features of market microstructure noise. The most popular assumption in the RV literature states that the MMN is a sequence of independent and identically distributed (i.i.d) random variables with zero mean, constant variance and finite fourth moment, while the MMN and the true prices are orthogonal to each other. Therefore, it is of great importance to mitigate the effect of MMN when we try to estimate the true price volatility of assets using high frequency data, especially considering that high frequency prices include more information compared to data sets of lower frequencies.

The third aspect in volatility estimation using UHFD relates to the asynchronous characteristic of markets. In actual equity markets, transactions take place and quotes arrive asynchronously, leading to transaction and/or quote time series to be observed at discrete and irregularly spaced intervals. This asynchronous characteristic of the stock markets allows us to sample the returns in various ways, i.e., one can follow different sampling schemes for estimating return volatility over a fixed time period. The most common sampling scheme is calendar time sampling (CTS), under which sampling is done at equal intervals in physical time; for instance, sampling at every 1 minute or 10 minutes. However, CTS has one big shortfall: the transactions and/or quotes are irregularly spaced in time and calendar time sampled data needs to be constructed artificially. Alternatively, one can sample prices whenever a transaction takes place, a.k.a. transaction time sampling (TTS). Similarly, if we sample data every time the stock price changes, then the sampling scheme is called tick time sampling (TkTS). Another option is called business time sampling (BTS) where the sampling times are determined to ensure that the IV of all intraday intervals are equal. On this issue, Oomen [94] shows that the mean squared error of the RV can be decreased by sampling returns on a transaction time scale as opposed to the common practice of sampling in calendar time. Hansen and Lunde [61] reveal that MMN is time-dependent and correlated to the unobservable true price under both CTS and TkTS. Likewise, Griffin and Oomen [55] argue that the return dynamics in TTS are different from those in TkTS and the choice of the sampling scheme may have a substantial effect on the properties of realized variance (microstructure noise is highly dependent under TkTS, so the bias correcting method should be decided accordingly). They find that tick time sampling is superior to transaction time sampling in terms of mean squared error, especially when the level of noise, number of ticks, or the arrival frequency of the efficient price moves is low [55]. Accordingly, it is also of great importance to shed light on the influence of the sampling scheme on the statistical properties of realized volatility whenever UHFD are used for IV estimation.

The fourth aspect in measuring realized volatility is the presence of non-trading hours. Realized volatility may underestimate the IV if the RV is calculated by using prices sampled only during trading hours. A number of researchers advocate the upscaling of RV calculated over trading hours to reach the daily RV. Interestingly, only a small number of papers [58], [59], [60], [61], [28], [101] in the RV literature discuss the ways for adjusting the estimator for non-trading hours whereas the majority of studies on the estimation of the quadratic variation of asset prices using high frequency data stay silent about non-trading hours. Regarding this silence, one should acknowledge that there are several questions to be answered before upscaling the RV measures to find the daily RV: Does the existence of non-trading hours only cause a time shift in volatility and is the daily volatility the same regardless of the length of trading hours leading us to observe diurnal shapes in trading volumes and returns due to this fact? Are the trading incentives that accumulate overnight and during the lunch break reflected in the market or limit orders once the markets are open? When the return volatility is u-shape per session, would adjusting $RV_{opentoclose}$ for non-trading hours cause a double-counting of daily volatility? Should we include opening and closing sessions or should we only work with sessions where only continuous auction is allowed? What is the best way to adjust the RV estimators under tick and transaction time sampling schemes? These questions underline the fact that adjusting the IV estimators for non-trading hours is not a straightforward step in the calculation of daily RVs.

Returning back to the issue of the contamination of observed prices with MMN, we remind that the estimation of the IV of the true asset prices using observed prices leads to the estimation of the quadratic variation of the MMN as we increase the sampling frequency, however decreasing sampling frequency causes the variance of the estimation error to increase. This result forces researchers and practitioners to come up with methods to mitigate the bias in IV estimation while taking into account the bias-variance trade-off. The approaches in the literature with respect to handling an additive MMN in the calculation of the quadratic variance of the unobserved efficient/true price can be grouped into 4 categories:

- Adjusting the RV estimator such as using kernel based estimators and subsampled kernel based estimators as in [27] and [28] or employing the two-time-scale or multi-time-scale estimators suggested in [111] and [3].

- Sparse sampling such that a bias-variance tradeoff is attained by choosing a sampling frequency at which MMN is supposed to be not too substantial, as in [11].

- Finding an optimal sampling frequency where the RV calculation is not adjusted but the sampling frequency is optimized such that the mean squared error is minimized or the forecasting performance is maximized as in [20], [21],[55] and [94].

- Pre-whitening of data such as smoothing the intraday returns by fitting a moving average or an autoregressive model as in [10] or [46] or pre-averaging of a certain number of observed prices as in [68].

Such methods to reduce/remove the impact of the noise component in IV estimation have been subject to a great deal of research but all these methods depend on the assumed structure of the MMN. For instance, Awartani et al. [16] draw attention to the fact that many methods to handle the MMN while estimating the IV of the unobserved true returns by using UHFD, including kernel based estimators, subsampling approaches, optimal sampling, and, simple bias correction methods, depend on the assumption that the MMN has an i.i.d and/or constant variance structure. Therefore, the success of the methods used to mitigate the MMN effects must be considered only after gathering empirical evidence from developed and developing markets regarding whether the assumptions underlying the aforementioned methods truly hold. Accordingly, rather than comparing the methods for handling the MMN in the IV estimation with respect to their forecasting performance or some other economical criteria, we delve into gathering evidence regarding the statistical structure of the MMN and aim to answer the question of whether popular assumptions about the statistical features of MMN are adoptable in light of the empirical findings.

We believe that such empirical evidence on the MMN structure should be collected taking into account all the dimensions/aspects of volatility estimation using high frequency data because the validity of methods for handling the MMN in the estimation of the IV may be compromised if all of these issues are not taken into account in the analysis. More specifically, we argue that the MMN evidence can be valid only after addressing the need to detect and clear errors, the need to aggregate simultaneous observations and to interpolate the data under CTS, the need to choose one or more sampling schemes, the need to make assumptions on the MMN structure and the need to consider non-trading hours in the estimation of the daily RV. In the literature, discussions and evidence on the IV estimation using RV only focus on some portion of these dimensions/aspects, such as the effect of sampling scheme on the return and RV dynamics [93], [61], [55], or, the effect of the aggregation methods on the RV calculation [27], or, the effect of cleaning procedures on the RV estimators [3], [27], or, the different ways for scaling RVs over trading hours to reach daily figures [58], [59], [60], [61], [28], [101]. In addition, the literature does not touch the issue of how to examine the existence and the statistical features of the MMN formally under sampling schemes other than CTS. Moreover, all the aforementioned studies use data coming from stock markets of developed economies such as the US or Japan and the literature lacks research on volatility estimation and the MMN structure providing empirical evidence from developing markets. In summary, to the best of our knowledge, none of the published literature on volatility estimation using UHFD

- discusses dimensions/aspects of volatility estimation simultaneously,

- considers how data handling methods in the form of cleaning and aggregation affect the characteristics of UHFD, and, whether the widely accepted outlier handling methods end up overscrubbing or underscrubbing the data,

- examines what happens to the return and RV series dynamics under varying combinations of sampling schemes and data handling methods while controlling for the liquidity of the stock, and,

- examines what happens to the visual and statistical evidence on the existence and/or statistical features of MMN under varying combinations of sampling schemes and data handling methods, and, whether findings about the MMN structure are robust with respect to the liquidity of the stock

at the same time.

Furthermore, we recognize that the liquidity of traded assets is an important issue that is discussed in the finance literature and there are many liquidity definitions and measures that find support in different studies. For instance, a widely accepted definition by Black [35] describes a liquid asset as an asset which can be sold in a short period of time for a price not too different from the price at which the seller would be able to sell if s/he opted to wait longer. Interestingly, when the high frequency finance literature is examined, it is seen that dealing with an asset's liquidity is somewhat problematic in the sense that many of the liquidity indicators/measures fall short when it comes to addressing the existence or the statistical properties of MMN embedded in the observed stock prices, especially if such measures are calculated under different sampling schemes such as CTS. This postulation is accentuated especially when there is a relatively long time lag between two consecutive transactions. As it will be explained in detail in Chapter 2, Section 2.1, in such a case of infrequent trading, the previous tick method is typically used to construct artificial return series, but this, in turn, means that returns are calculated by using pieces of information that belong to distant points in time leading to inflated serial correlations due to long sequences of zero returns [37]. Hence, the previous tick method may work best in IV estimation for very actively traded stocks since we would not want to spur such correlation structures by artificially introducing additional autocorrelation (serial correlation) due to the interpolation method selected. These arguments pave the way for the introduction of a new method to classify stocks with respect to their liquidity (active trading) in a high frequency setting.

In order to realize all these goals, we begin by discussing the dimensions of IV estimation using high frequency data sets in detail in Chapter 2.

Next, in Chapter 3, subject to certain assumptions, we prove that the formal tests developed under CTS by Awartani et al. [16] for determining whether there is any statistically significant asymptotic bias due to the existence of MMN on the RV estimator and whether MMN increments have a constant variance independent of sampling frequency are also applicable under TTS.

In Chapter 4, we first suggest a new approach to classify liquid and illiquid stocks that can be used in a high frequency setting, and then, by applying a grid of the data cleaning methods and different sampling schemes, TTS and CTS in particular, to six stocks that are listed on Borsa Istanbul National Equity Market, we examine what happens to the common characteristic of UHFD, the dynamics of the return and RV series, volatility signature plots and formal tests of the existence and the constant variance of MMN developed by Awartani et al. [16] as we move on the grid while we also look for any significant changes in results due to the liquidity of the sample stocks.

Chapter 5 provides our conclusions.

CHAPTER 2

DIMENSIONS OF VOLATILITY ESTIMATION USING UHFD

2.1. Nature of UHFD and Errors in UHFD Sets

As Brownlees and Gallo [37] put it, the term "financial high/ultra high frequency data" (UHFD) in the literature refers to the data sets including thorough reports of all available financial markets activity information. The term "ultra" also means that it is not possible to dive into finer details than that are provided in these data sets. The basic unit of information contained in the UHFD is called the "tick", which represents a time stamp and a set of information summarizing specifics of the market activity [37]. With the advent and increasing availability of the UHFD, many researchers examined the common properties of such data sets that cause greater complexity in analysis (Andersen and Bollerslev [7], Engle and Russell [48], Dacorogna et al. [45] , Falkenberry [50], Brownlees and Gallo [37], Verousis and Gwilym [106], Engle and Russell [49] among many others). In this context, the literature reports the common properties of UHFD as follows:

- Number of ticks can reach thousands (millions) per day (year),
- The time interval between two consecutive ticks is random (temporal spacing),
- There can be anomalies in the behavior of ticks due to particular market conditions such as openings, closings, trading halts, circuit breakers, etc. (strong intraday patterns),
- The rules and procedures of the institution that records and disseminates UHFD affect the structure and sequence of ticks,
- The data sets can contain wrong ticks such as zero prices or volumes and there is a diversity of possible errors and their causes.

Although using historical high frequency data in finance applications became popular since the 2000's, there is a limited number of studies that address the necessity of detecting errors/outliers while preparing the time series data at hand for further analysis. Furthermore, the literature does not agree on a single definition of what constitutes an unclean data point/error/outlier. For instance, Dacorogna et al. [45] define a data error as *"a piece of quoted data that does not conform to real situation of the market"*, whereas Verousis and Gwilym [106] state that an outlier is an observation which does not reflect the trading process so that the real connection between market participants' behavior and recorded observations are broken. Falkenberry [50] adds that *'the most difficult aspect of cleaning data is the inability to universally define what is unclean'*. In spite of the lack of a common definition, there is consensus in the high frequency finance literature that data errors (outliers) should be defined and removed somehow by a data filter/cleaning algorithm before any computation. Specifically, Dacorogna et al. [45] state that the problem of outliers distorting the reliability of calculations in high frequency setting gets accentuated for finance applications mainly because much of these applications work with returns and the difference operator is quite sensitive to outliers. Dacorogna et al. [45] also underline the fact that professional users may immediately detect erroneous pieces of information and clean the data using their immense practical knowledge, but researchers investigating historical data have a lesser understanding of what constitutes an erroneous tick and why such errors occur. In other words, using UHFD in academic research requires attention with respect to detecting errors and pinpointing the human and/or system failures that give rise to such errors.

In addition to the lack of a common definition, the explanations regarding why bad data/erroneous data/outliers exist change from one researcher to the next. For instance, Falkenberry [50] associates bad data with the asynchronous and voluminous nature of financial data whereas Dacorogna et al. [45] list unintentional (such as typing) and intentional (such as dummy ticks produced just for technical testing) human errors as well as system errors caused by computer systems, their interactions and failures, and they do not make any reference to the trading intensity. Meanwhile, Brownlees and Gallo [37] assert that there are no clear reasons for the existence of erroneous data.

There are a number of studies that mark the first few attempts that underline the importance of treating outliers in a high frequency finance setting. For instance, Dacorogna et al. [44] analyze large amounts of high frequency German mark - US$ quotes by market makers around the world (up to 5000 irregularly spaced prices per day) in order to develop a set of real-time intra-day trading models that give explicit trading recommendations under specific constraints. Likewise, Huang and Stoll [65] examine transaction data encompassing bid-ask quotations, transaction prices, and volumes in order to compare the execution costs for NASDAQ stocks with the execution costs for comparable NYSE stocks. Finally, Zhou [112] concentrates on high frequency exchange rate data for modeling the negative autocorrelation in observed time series and proposes a realized volatility estimator that is suitable for the high frequency setting.

In order to detect outliers and clean the high frequency financial data, in addition to the detection and treatment of obvious errors such as corrected, negative or zero quotes/prices, the initial studies on the subject suggest comparing each quote/price with a median that is calculated by using data points within the close neighborhood of a given trade [112] or deleting trades (quotes) whenever the return calculated by using the previous trade exceeds 10% [65] or 25% [31]. Chung et al. [39] improve on [65] in the sense that the percentage of the return threshold is increased to 50% and the return is calculated in absolute terms in order to make sure that negative returns also trigger outlier detection. Interestingly, although there are several papers published before 2008 which criticize the 10% return criterion and promote alternative criteria such as examining the distance of a data point from a rolling transform, Bandi et al. [21] and Pigorsch et al. [95] continue to adopt the 10% immediate return rule in their studies.

Following such pioneering approaches, Dacorogna et al. [45], Falkenberry [50], Brownlees and Gallo [37], Oomen [94], Gutiérrez and Gregori [56], Verousis and Gwilym [106] and Barndorff-Nielsen et al. [27] also examine outlier handling in high frequency financial data.

The algorithm suggested by Dacorogna et al. [45] concentrates on data from the FX spot and FX derivative markets while the algorithms by Brownlees and Gallo [37], [38] and Barndorff-Nielsen et al. [27] concentrate on data from the stock market and the algorithm by Verousis and Gwilym [106] concentrates on data from the stock option market. These different algorithms share some common characteristics. For instance, each quote/price is compared with a moving threshold that is calculated by using the neighboring data points where a symmetrical number of preceding and following quotes/prices are selected and then the ticks that exceed the threshold are classified and deleted as outliers. More specifically, in order to delete outliers in high frequency stock market data sets, Brownlees and Gallo [37] propose to delete prices whenever the absolute difference between a current price, p_i, and the 10% trimmed sample mean, $\bar{p}_i(k)$, is more than 3 sample standard deviations, $s_i(k)$, of a neighborhood of k observations around the current tick plus a parameter, ξ, that represents a multiple of the minimum allowable price variation for the stock at hand:

$$|p_i - \bar{p}_i(k)| < 3s_i(k) + \xi = \begin{cases} \text{true,} & \text{observation } i \text{ is kept} \\ \text{false,} & \text{observation } i \text{ is removed} \end{cases}$$

Falkenberry [50] states that the higher velocity in trading induces a higher probability of an error in the reported trading data and advocates the use of transaction frequency as a criterion in determining the number of data points to be used in the calculation of a moving transform to which the data point is compared. Likewise, the Brownlees and Gallo [37] approach described above incorporates the trading intensity, γ, in the selection of the number of neighboring data points, k, while calculating the moving average and moving standard deviations. Brownlees and Gallo [37] suggest that inactive trading should lead to a "reasonably small" k so that the window of observations does not contain too distant prices, while active trading should lead to a "reasonably large" k so that the window contains enough observations to produce reasonable estimates of the local characteristics of the price. In order to visually choose the pair of k and γ, they count the number of observations deleted for a grid of parameters k and ξ.

One can also choose not to delete the outliers but replace the outliers with corrected values. This is called the "Search and Modify" approach by Falkenberry [50] who analyses stock market tick data. Falkenberry [50] describes the “Search and Modify” approach as follows. First, a moving transform of the tick is calculated where the number of points to be used in the moving transform calculation is a function of tick frequency in order to adapt the filter to the unique activity levels of various securities. Next, each tick’s distance (in standardized units across different securities) from its moving transform is found. If the tick’s distance exceeds a user-defined threshold, then it is defined as an outlier. Finally, the moving transform replaces the ticks that are classified as outliers.

Some other studies focus on the bid-ask spread in order to detect and delete outliers. For instance, Chordia et al. [41] delete transaction data by using criteria that depends on the quote data. Chordia et al. [41] record an outlier whenever either of the following conditions are met: (i) the Quoted Spread exceeds $5, or, (ii) the Effective Spread to Quoted Spread ratio or the % Effective Spread to % Quoted Spread ratio exceed 4.0, or, (iii) the Quoted Spread to Transaction Price ratio exceeds 0.4. By using this criteria, Chordia et al. [41] end up removing less than 0.02 % of all transaction records. Meanwhile, Benston [30] deletes prices when (i) the effective spread exceeds 20% and the price or bid or ask quote exceeds $5, or, (ii) the effective spread exceeds 20% and the price is less than $5 but is between the bid and ask quotes. Likewise, Hansen and Lunde [61] exclude transaction prices that are more than 1 spread away from the bid and ask quotes and Barndorff-Nielsen et al. [27], [28] remove prices whenever the price is above the ask plus the bid-ask spread or below the bid minus the bid-ask spread. These approaches in general can be summarized as ‘disciplining trade data using quote data’ [27].

Within the context of this Thesis, disciplining trade data using quote data is not applicable for our research because in Borsa İstanbul's National Equity Market, two continuous auction sessions (morning and afternoon) plus an “opening session” prior to each of the sessions to set the opening price for each session, and a “closing session” at the end of the second session to set the closing price of each trading day are held. During the opening and closing sessions, orders are received for a specific period of time, and then the price is set in order to achieve the highest trading volume where trading volume is defined as aggregate price times the amount traded. During these sessions, orders are executed at these single prices and the remaining orders are automatically cancelled. With such an auction mechanism in place, proper quote data are missing for securities included in BIST 100 Index as they are always subject to continuous auction system. In this context, our approach needs to utilize the existing trade information in the detection and deletion of outliers. Barndorff-Nielsen et al. [27], whose cleaning algorithm is also adopted by Koopman and Scharth [74], propose an important step in outlier handling when there is no quote data available. They propose that entries for which the price deviates by more than 10 mean absolute deviations from a rolling centered median of 50 observations (25 preceding, 25 following) should be deleted. They justify the selection of the sample median and mean absolute deviations rather than the sample mean and sample standard deviation by arguing that the first pair is less sensitive to runs of outliers. Verousis and Gwilym [106], on the other hand, argue that the median absolute deviation is more resistant to outliers than the mean absolute deviation and add that if normality cannot be assumed, the median is more efficient than mean. In their study, MAD is defined as the median value of absolute deviations around the median. Verousis and Gwilym [106] first calculate the daily median price, then take the median of the absolute deviations of the prices from that daily median, normalize the MAD, and finally delete a price if (after controlling for the minimum tick, price level and daily price effects) (i) the price is greater than one minimum tick compared to the previous tick, price is higher than a certain level, price is less than 90% or more than 110% of daily average price, simple return with respect to prior price exceeds 10% and/or normalized MAD is less than the standardized price, or, (ii) the price is equal to or less than one minimum tick compared to previous tick, price is lower than a certain level, price is less than 80% or more than 120% of daily average price, simple return with respect to prior price exceeds 20% and/or normalized MAD is less than the standardized price.

Another notable contribution to the data cleaning literature is the algorithm that is designed for transaction time sampling by Oomen [94]. Using IBM transaction data, Oomen [94] investigates the statistical properties of the RV estimator for varying sampling frequencies and sampling schemes. Before applying his methodology to IBM transaction prices, he cleans the data set from obvious errors and data points with time stamps outside of trading hours. He also removes days where trading begins late or the market is closed early. When it is time to detect and treat outliers, he prefers to filter the data for instantaneous price reversals in transaction time. In particular, $p(k)$ (k^{th} transaction price) is deleted if the following two conditions are satisfied simultaneously,

$$|r(k|1)| > c,$$

and

$$|r(k+1|1)| \in [-(1-w)|r(k|1)|, -(1+w)|r(k|1)|],$$

for any $0 < w < 1$, where $r(k|1) = p(k) - p(k-1)$, meaning that, for the k^{th} transaction to be removed, the absolute price change from the $k\text{-}1^{th}$ transaction to the k^{th} transaction should exceed both a threshold set arbitrarily as well as the price changes from the k^{th} transaction to the $k{+}1^{th}$ transaction such that the absolute price reversal is included in the region of $-(1-w)$ and $-(1+w)$ times the price change from the $k\text{-}1^{th}$ transaction to the k^{th} transaction. Based on experimentation, he chooses w as 0.25 and c as eight times a robust interquantile volatility estimate of transaction returns (no further details are provided in the paper).

Compared to Oomen [94], a simpler reversal rule is executed by Bessembinder et al. [32] who suggest a model to reveal the effect of transaction reporting on trade execution costs. In their study, Bessembinder et al. [32] test their model by using a sample of institutional trades in corporate bonds and eliminate “reversal” transactions when a given price exceeds both the preceding and the following prices by at least 15% or is less than both prices by the same magnitude.

Marshall et al. [81] employ a high-frequency data cleaning rule inspired by Brownlees and Gallo [37] with the purpose of guaranteeing that outliers do not affect their findings regarding liquidity commonality in commodity futures markets. For the liquidity measures used in their study, their method estimates an α-trimmed sample mean and standard deviation. Marshall et al. [81] select α as 5%, meaning that the top and bottom 2.5% of the observations are excluded in calculating the trimmed mean and standard deviation, and they remove observations that are outside the trimmed mean plus/minus three standard deviations. Unlike Brownlees and Gallo [37], Marshall et al. [81] neither mention whether the mean and standard deviation are calculated in a rolling k neighborhood or for the whole day, nor incorporate trading intensity in their outlier detection rules.

In another related study, Aït-Sahalia et al. [3] also draw attention to the fact that most of the empirical papers using high frequency data discuss cleaning procedures slightly or do not touch the issue at all. Obviously, raw high frequency financial data sets are preprocessed in order to remove data errors and/or outliers. However, as discussed earlier, cleaning data for evident errors is easy to visualize and implement. The tricky part of the cleaning procedures depend on the definition and treatment of marginal outliers. On this score, like the approach adopted by Oomen [94], Aït-Sahalia et al. [3] define an outlier as a "bounceback":

> *"a log return from one transaction to the next that is both greater in magnitude than an arbitrary cutoff, and is followed immediately by a log return of the same magnitude but of the opposite sign, so that the price returns to its original level before that particular transaction*."

Their analysis is based on a 1% log return cutoff level. Moreover, their paper is the second in the literature after the paper by Barndorff-Nielsen et al. [27] which emphasizes the effects of cleaning procedures on RV estimators. We reviewed many papers on RV but few to none delve into the cleaning procedures or their impact on the RV calculation. Interestingly, in their study on detecting jump and other volatility components in high frequency data using stock market transaction and quotes, Aït-Sahalia and Jacod [4] prefer to clean the data set only for obvious errors. Apart from this, they perform no further cleaning in order to produce "unfiltered transactions".

Rossi [99], who proposes a bond-specific, time-varying friction measure of round-trip liquidity costs, combines two rules: reversals and deviation more than a certain threshold calculated over k neighboring data points. This combination of rules is described by Rossi [99] as follows:

> *"eliminate 50% return reversal, i.e. eliminate a bond price if it is preceded and followed by a price increase or drop of more than 50% and* $|p - med(p,k)| > 5 * MAD(p,k) + g$*, where* g *is a granularity parameter which I set equal to $1, and* $med(p,k)$*, and* $MAD(p,k)$ *are respectively the centered rolling*

median, and median absolute deviations of the price p using k observations (I set $k = 20$).

The common traits of algorithms proposed in the literature are that they are data-specific (authors focus on data from a specific market such as FX or stock market) and data selection rules are arbitrary (10% return threshold, 3 standard deviations threshold etc) [106]. These traits make outlier detection and handling in high frequency finance an art.

In the application of data cleaning algorithms to UHFD, one should also consider the risk of "overscrubbing/underscrubbing" as described by Falkenberry [50]. He states that filtering data too loosely might result in a data set with too many errors and filtering data too tightly might cause the data set's statistical properties to be distorted. He concludes that a proper cleaning algorithm should manage the overscrub/underscrub tradeoff in such a way that the outliers in the user's base unit of analysis (for instance 1 minute) are removed and the resulting time series can support historical backtesting without distorting the real-time properties of the data. Dacorogna et al. [45] discuss the same risk of unwanted side effects caused by cleaning algorithms and propose a general method to test the effects of the algorithm. They argue that the data cleaning application, whichever is chosen, should be implemented twice using two different filters where one filter is weaker in terms of being more tolerant and leading to lesser rejection rates. Comparison of the results from both applications reveals the robustness of the analysis against changes in the cleaning algorithm. Likewise, Jan Wrampelmeyer [108] states that

> "*Removing outliers from the sample is not a meaningful solution since subjective outlier deletion or algorithms as described by Brownlees and Gallo (2006) have the drawback of risking to delete legitimate observations which diminishes the value of the statistical analysis.*"

Even after we complete detecting and deleting outliers/errors in UHFD, there are still other peculiar patterns that emerge in such data sets. Brownlees and Gallo [37], Hansen and Lunde [61], and Barndorff-Nielsen et al. [27] among others suggest some handling methods for such patterns:

a. Simultaneous Observations: In order to demonstrate this pattern, 1 minute of transaction prices for TKCELL stock on the 2nd of January 2012 from 10:10:00 to 10:11:00 are displayed in Figure 2.1. Each square marks a different transaction. Due to the asynchronous nature of trades, simultaneous transactions at different price levels are present in the data. Specifically, 4 transactions at prices ranging from 8.90 to 8.96 TL have the same time stamp (10:10:56). Explanations regarding this phenomenon by Brownlees and Gallo [37] include executions of market orders resulting in more than one transaction report and approximations causing even non-simultaneous trades to be reported as simultaneous.

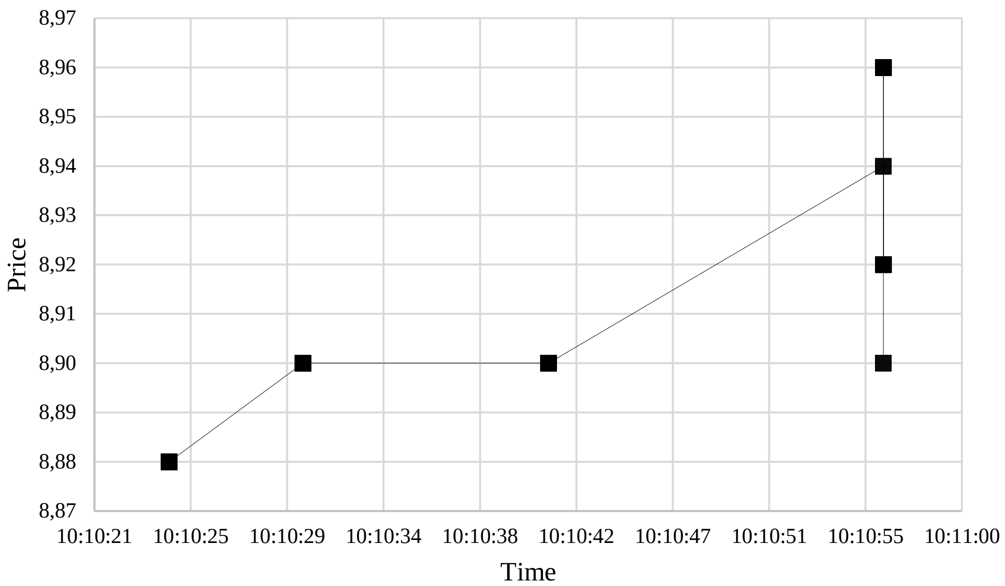


Figure 2.1: TCELL transaction data- 1 minute during the first session on 2nd of January, 2012.

The UHFD literature agrees that simultaneous transaction and/or quotation data is common across many financial markets and since UHFD models necessitate having one observation per time stamp, some form of aggregation needs to be applied. Barndorff-Nielsen et al. [27] suggest that when there is more than one transaction reported per time stamp, either the median price for that time stamp should be used or a unique price should be determined and the volume should be aggregated by using one of the following rules:

- Use the price that has the largest volume.
- Use the volume weighted average price.
- Use the log volume weighted average price.
- Use the number of trades weighted average price.

If multiple transactions have the same time stamp, Brownlees and Gallo [37] propose to use the median price that is less prone to discreteness of prices where Barndorff-Nielsen et al. [27] favor the same approach but only after examining what happens to deleted observation counts, realized volatilities and realized kernels under each of the aggregation methods listed above. Interestingly, the Dacorogna et al. [45], Falkenberry [50] and Verousis and Gwilym [106] studies do not address the subject of handling prices when the data set includes more than one entry per time stamp.

b. Irregularly Spaced Observations: Several of the studies on the subject reveal that UHFD sets may also include transactions with irregular spacing in time. For visualization purposes, Figure 2.2 presents TCELL transaction data over a 10–minute period on an arbitrarily selected date.

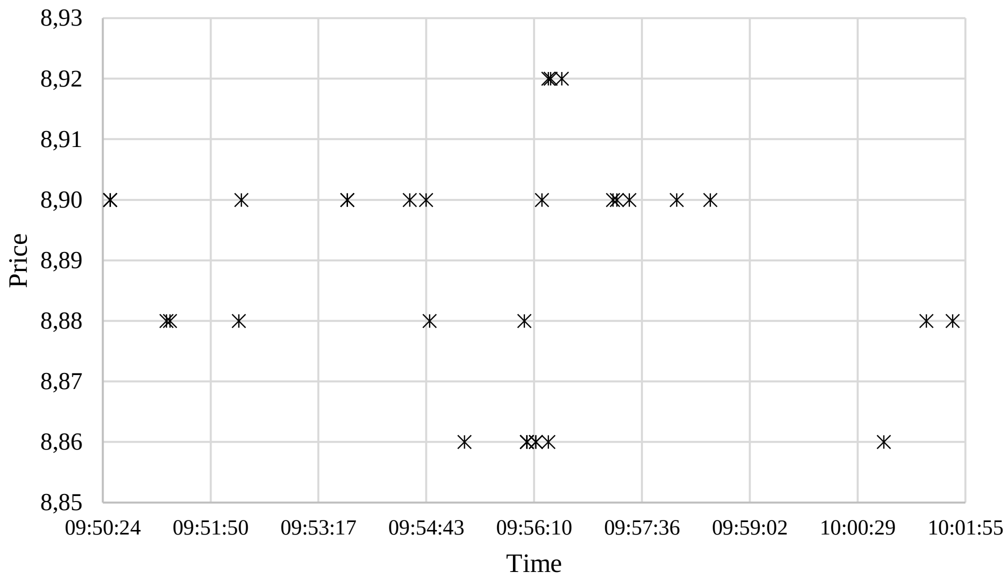


Figure 2.2: TCELL transaction data - 10 minutes during the first session on 2nd of January, 2012.

Irregularly spaced transactions pose a problem since the majority of econometric models require data sets to be regularly spaced in time. Therefore, in order to use such econometric models, one needs to arbitrarily generate a regularly spaced time series by adopting an interpolation rule. The most favorable interpolation methods in the UHFD literature are (i) the previous tick method (if the data point for a time stamp is absent, then use the previously observed transaction [61], [101], [28]), (ii) linear interpolation (if the data point for a time stamp is absent, then take an average of the previous and the next observations with the weights depending on the distance between the time stamps [12]), and, (iii) other interpolation methods such as splines (curve fitting) as in [59] and [73]. Interpolation methods in general need to use information that is not available as of the time of the transaction and Engle and Russell [49] argue that such methods may induce spurious correlations. For instance, using splines with weighted averages causes spurious positive correlation because now the constructed price is a weighted average of the previous and following prices. Additionally, Hansen and Lunde [61] argue that because the quadratic variation of a straight line is zero[1] and since the linear interpolation means fitting straight lines for missing parts, linear interpolation methods will distort quadratic variation (IV) estimations. Hansen and Lunde [61] prefer the previous tick method. On this issue, Brownlees and Gallo [37] direct readers' attention to non-frequently traded stocks. They assert that when there are long periods between two consecutive transactions, the previous tick method will result in using a piece of information that belonged to some considerable time before which, in turn, will lead to inflated serial correlation due to long sequences of zero returns. In light of all the academic debate regarding the methods, the previous tick method is used as the interpolation method in this Thesis since the highly liquid and

[1] *A process X is said to have a finite variation if it has bounded variation over every finite time interval (with probability 1). Such processes are very common including, in particular, all continuously differentiable functions. The quadratic variation exists for all continuous finite variation processes, and is equal to zero.*

actively traded BIST stocks included in the sample are unlikely to suffer from infrequent trading.

2.2. Market Microstructure Contaminates Observed Prices

The term "market microstructure" is first suggested in the seminal paper of the same title by German [53]:

> *We depart from the usual approaches of the theory of exchange by (1) making the assumption of asynchronous, temporally discrete market activities on the part of market agents and (2) adopting a viewpoint which treats the temporal microstructure, i.e., moment to moment aggregate exchange behavior, as an important descriptive aspect of such markets. [p. 257]*

As Hasbrouck [64] puts it, there is no "microstructure manifesto". Accordingly, O'Hara [92] defines market microstructure as the study of the process and outcomes of exchanging assets under explicit rules while Madhavan [79] states that the market microstructure field focuses on how investors' latent or hidden demands are ultimately translated into prices and volumes and Hasbrouck [64] describes market microstructure as the study of the trading mechanisms used for financial securities. All in all, the field of market microstructure studies an undeniable truth: although the traditional finance theories assume frictionless and perfect capital markets, specific trading mechanisms and imperfections of the markets affect the price formation process. Some examples of these trading mechanisms are the existence of specific intermediaries such as stock specialists or order clerks, or the trading taking place at a centralized location such as an organized exchange or at a decentralized location as in the case of over the counter markets [92].

The traditional view of price formation predicts that the intersection of the demand and supply curves determines the price of an asset in equilibrium. O'Hara [92] emphasizes that the beginnings of the market microstructure research stems from the incompetency of the standard economics paradigm in providing answers to how the equilibrium price is actually attained and what coordinates the desires of demanders and suppliers in order for a trade to occur.

O'Hara [92] discusses that before the advent of research on market microstructure, there were two traditional approaches to price formation mechanism. The first one opted for the irrelevance of the price formation process and focused mainly on the analysis of equilibrium and the properties of equilibrium prices and finding market clearing prices without considering how the clearing actually takes place [92]. A classic example of this traditional approach is the rational expectations literature[2]. This

[2] *An easy definition of rational expectations theory is provided by Wikipedia:*

way of modeling expectations was originally proposed by Muth [88] and became widespread when other researchers adopted this assumption to study how economic agents make choices under uncertainty. The rational expectations assumption is used in many modern day macroeconomic models, game theory and applications of the rational choice theory. The rational expectations literature does not concern itself with behavior that is out of equilibrium. O’Hara [92] argues that implicit to this approach is the assumption that the trading mechanism has no effect on the equilibrium price and this assumption is problematic in modeling markets in which traders have differential information.

The second traditional approach to the mechanics of price formation is the assumption of a Walrasian auction setting. In this setting, a Walrasian auctioneer takes no trading position and aggregates the demand for and the supply of an asset in order to set an equilibrium (market clearing) price in a market with perfect competition, perfect information and no transaction costs. Each trader submits his/her demand to the auctioneer and the auctioneer pronounces a possible trading price. After the announcement of this potential trading price, traders calculate their optimal demand at that price and resubmit their new demand to the auctioneer. The auctioneer re-determines the potential price for traders to in order to reflect the changes in the demand schedule at this new price. This process continues until there is no further revision so that the quantity supplied equals quantity demanded. O’Hara [92] argues that this representation of market prices arising from a series of preliminary no-cost auctions where no trading is allowed outside of the equilibrium does not capture the actual process by which prices in financial markets are formed.

One can easily pinpoint how traditional approaches to price formation fail to represent the actual pricing process of financial assets, especially in a market setting such as an organized stock exchange where there are many regulations governing the trading process which is itself assumed to be informationally efficient. O’Hara [92] states that a broad understanding of the securities market design is a prerequisite for the study of price behavior in stock markets. For instance, the most current trading mechanism regulations relevant for the National Equity Market of Borsa İstanbul A.Ş. (BIST) [109] demonstrate how the pricing process in practice may deviate from the traditional economic paradigm's somewhat unrealistic view of the market.

All in all, regardless of the definition of market microstructure and the mechanisms that affect the price formation in stock markets, O’Hara [92] and many other market microstructure researchers claim that the observed asset prices can be decomposed as follows:

$$Y_t = X_t + \epsilon_t,\ 0 \leq t \leq T.$$

Rational expectations states that economic agents' predictions of the future value of economically relevant variables are not systematically wrong in that all errors are random. Equivalently, agents' expectations equal true statistical expected values.

In this equation, Y_t, X_t and ϵ_t represent the observed price, the unobserved true price and the unobserved aggregate effect of market microstructure, respectively. Furthermore, Aït-Sahalia et al. [3] suggest that we can roughly divide the market microstructure effect into three classes. The first class represents the frictions inherent in the trading process such as bid-ask bounces, the discreteness of price changes and rounding, trades occurring on different markets or networks, etc. The second class captures the informational effects such as differences in trade sizes or the informational content of price changes, the gradual response of prices to a block trade, the strategic component of the order flow, inventory control effects, etc. The third class encompasses measurement or data recording errors such as prices entered as zero, misplaced decimal points, etc.

One important note about this so called "market microstructure noise" is that the high frequency finance literature including research by Zhou [112], Andersen et al. [11], Andersen et al. [12], Barndorff-Nielsen and Shephard [24], [25], Bandi and Russell [19], Hansen and Lunde [58], Zhang et al. [111] all provide abundant mathematical and empirical evidence of a noise contamination in observed asset prices as we increase the sampling frequency, which leads to realized volatility to be a biased estimator of the quadratic variance of asset returns. Therefore, it is of great importance to mitigate/cleanse the effect of market microstructure noise when we try to estimate the true price volatility of assets using high frequency data, especially considering the fact that high frequency prices include more information compared to data sets of lower frequencies.

2.3. A Choice for True Asset Prices

During the last decades, many researchers endeavoring to model the stock price behavior increasingly nested in the theory of stochastic processes for describing the uncertainty in financial markets. Although the use of stochastic processes in modeling asset prices dates back to Bachelier [17], accepting that the logarithm of an asset price follows an Itô semimartingale became popular in the late 1960's and early 1970's thanks to the seminal papers by Robert Merton [84], [85], [86] and Black and Scholes [36]. The semimartingale representation of asset prices received wide acceptance because, as explained by Harrison and Pliska [63], with continuous trading allowed, an arbitrage free market is complete (every contingent claim is attainable) if and only if there is a unique probability measure $\mathbb{P}^*$ equivalent to $\mathbb{P}$ under which discounted asset prices are martingales, and, in this setting, asset prices must satisfy the semimartingale property. Specifically, when the log price of an asset is accepted to follow a specific form of semimartingales, i.e. the Itô semimartingale[3] (where elements of the characteristic triple stemming from a Levy-Itô decomposition are absolutely continuous with respect to the Lebesgue measure), the formal statement of the log asset price, denoted by X_t, is given as below:

[3] *Theoretical discussions in [5], [42], [47], [71], [72], [75] and [97] provide a comprehensive review of semimartingale representation of asset prices.*

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dB_s + \int_0^t \int_{\{|x| \leq \varepsilon\}} x(\mu - \varpi)(ds, dx)$$
$$+ \int_0^t \int_{\{|x| > \varepsilon\}} x(\mu)\,(ds, dx).$$

In this equation, t represents the time index up until a finite maturity T, B_t is a real-valued Brownian motion defined on a probability space $(\Omega, F, \mathbb{P})$, μ is the jump measure of the log asset price with the predictable compensator ϖ, μ and ϖ are random positive measures on $\mathbb{R}_+ \times \mathbb{R}$ and $\varpi(dt, dx) = dtF_t(dx)$, $\int_0^t b_s ds$ represents the drift, $\int_0^t \sigma_s dB_s$ represents the continuous part, $\int_0^t \int_{\{|x| \leq \varepsilon\}} x(\mu - \varpi)(ds, dx)$ represents small jumps and finally $\int_0^t \int_{\{|x| > \varepsilon\}} x(\mu)\,(ds, dx)$ represents big jumps. The threshold that distinguishes small jumps from big jumps is arbitrary and a semimartingale generates a finite number of big jumps until a finite maturity with an infinite or a finite number of small jumps. Such a Levy-Itô decomposition of semimartingales shows that log asset prices in an arbitrage free market can be written as the sum of a drift, a continuous local martingale and discontinuous small and big jumps. Aït-Sahalia and Jacod [4] provide a mapping of each of these components to an economic source of risk: the continuous part may represent the part of the asset's total risk that can be hedged, the big jumps may capture the effect of the big news related events, and the small jumps may model price changes that are substantial for a short period of time but not significant over daily or longer sampling intervals. Aït-Sahalia and Jacod [4] further state that this type of small jumps may occur due to the market's inability to absorb large transactions without any price effect. It should be noted that it is still being debated in the finance literature whether stock markets are a good candidate for asset prices to be modeled as discontinuous Itô semimartingales. In fact, some researchers promote Brownian semimartingales where jump components in a regular Itô semimartingale have a size of zero (asset prices do not jump) while others underline the fact that a portion of price changes in financial markets may be too large to be explained by continuous Brownian semimartingales and favor discontinuous Itô semimartingales for modeling log asset prices.

At this point, before commencing with the discussions regarding whether a discontinuous or continuous Itô semimartingale better represents log asset prices, let us remember that the market microstructure literature provides abundant evidence of a noise contamination in observed asset prices as we increase the sampling frequency. Therefore, in a high frequency setting, assuming a proper form Itô semimartingale for the observed log asset prices means choosing a proper semimartingale form for the unobservable true log asset prices and making assumptions regarding the structure of the microstructure noise. With respect to the representation of true log asset prices, the high frequency finance literature's most popular choice (for instance, [112], [8], [29], [80], [24], [25], [82], [60], [61], [111], [73], [2], [19], [20], [77], [14], [28], [43], [6], [23], [78] among many others) has been Brownian semimartingales, i.e., the true log asset prices are accepted to not exhibit any discontinuous behavior. In this Thesis, we also favor the Brownian semimartingale approach mainly because most of the researchers who promote jump-diffusions to model log asset prices ignore the market microstructure noise, take observed prices as true prices and try to explain the big movements in observed asset prices with the existence of jumps in true prices. As Lee and Mykland [76] argue, interpreting the empirical results as evidence of the existence of jumps in financial markets may be erroneous in the sense that since both the true asset prices and the market microstructure noise are unobservable, sharp movements observed in asset prices may be caused by the microstructure noise and not a jump component within the asset's true price. In a setting where the observed prices are accepted to be composed of two unobserved parts, any interestingly large price change should not only be tied to jumps in only one of these parts. The existing literature does not provide a conclusive methodology for identifying the exact source of such price changes. Moreover, there are some practical problems attached to using discontinuous Itô semimartingales: difficulty of estimation as well as dealing with additional dimensions to volatility modeling such as the structure of jump intensity or jump size distribution.

Another aspect to be noted when a researcher models true asset prices as Itô-semimartingales with jumps is the disappearance of market completeness as we introduce jumps. Recall that market completeness and the availability of no arbitrage strategies allow us to price assets using replicating strategies. However, in an incomplete market, some payoffs (contingent claims) cannot be replicated by cash flows from other securities, i.e., in an incomplete market, we cannot be sure that each and every cash flow represented by a security can be replicated by trading in carefully selected other securities. Accordingly, asset pricing with jumps requires us to drop the assumption of market completeness.

In light of above discussions, this Thesis adopts the Brownian semimartingale approach in the modeling of asset prices.

2.4. Assumptions on Market Microstructure Noise

Up until now, in order to generate consistent estimates of asset return volatilities we motivated ourselves to sample returns at intervals converging to zero causing number of returns going to infinity. However, the number of sampled returns during a fixed period of time cannot converge to infinity due to the fact that number of quotations and transactions in an organized market per a fixed period of time (for instance a day) are not infinitely many. Moreover, as stated earlier, observed prices at high frequencies deviate from the efficient theoretical prices as a result of the presence of the MMN.

Recall that number of returns per period $[0, T]$ is n. Then, the i$^{\text{th}}$ return $r_i = X_i - X_{i-1}$ within the period $[0, T]$ can be decomposed as

$$r_i = r_i^* + v_i, \qquad i = 1,2, \cdots, n.$$

where $v_i = \epsilon_i - \epsilon_{i-1}$ and the observed return consists of an efficient return, r_i^*, and an intraday noise increment. As a result, the observed RV can be written as

$$RV = RV^* + 2\sum_{i=1}^{n} r_i^* v_i + \sum_{i=1}^{n} v_i^2,$$

where the last term on the right-hand side (RHS) can be interpreted as the unobservable realized variance of the noise process and the second term is affected from the dependence between the efficient price and the noise [95].

This decomposition of the observed RV implies that the aggregate effect of market microstructure on the properties of the RV estimator is shaped by the assumed structure of the noise process. In this context, the following are examples of the assumptions in the literature regarding noise structure, where first one is the most popular:

Assumption 2.1.

- The microstructure noise, ϵ, has zero mean and is an independent and identically distributed (i.i.d) random variable.

- The noise is independent of the efficient price process.

- The variance of the noise is constant (the intraday noise increment, v, also has constant variance) and the noise has a finite fourth moment.

Under this favorite set of assumptions about noise, conditionally on the efficient returns,

$$\mathbb{E}[RV|r^*] = RV^* + 2n\mathbb{E}[\epsilon^2],$$

and therefore RV for the period $[0, T]$ is a biased estimator of the IV [10], [24], [25]. Moreover, under Assumption 2.1, Zhang et al. [111] show that i.i.d noise introduces a bias into the RV estimator and the asymptotic distribution of RV can be expressed as follows:

$$\frac{(RV - 2n\mathbb{E}[\epsilon^2])}{2\sqrt{n\mathbb{E}[\epsilon^4]}} \xrightarrow{d} N(0,1).$$

Considering the market microstructure of Borsa Istanbul National Stock Market as an example of organized equity markets, we believe assuming that the aggregate effect of microstructure on price process being i.i.d might be improper. Indeed, other researchers also realize how Assumption 2.1 is unrealistic and work under different set of assumptions as given in Assumption 2.2 and Assumption 2.3.

Assumption 2.2.

- The microstructure noise, ϵ, has zero mean and is a strictly stationary stochastic process where joint density might alter depending on the sampling frequency.

- The noise is independent of the efficient price process.

- The variance of the noise increment is allowed to change with sampling frequencies; however, for a specific sampling frequency it is $\mathcal{A} + o(1)$ where $\mathcal{A} > 0$.

Under this partially generalized version of Assumption 2.1, Bandi and Russell [20] prove that as the number of returns converges to infinity, the observed RV converges to infinity as well.

Likewise, Aït-Sahalia et al. [3] examine a similar case where the noise is not i.i.d and adopt the following set of assumptions in estimation of RV:

Assumption 2.3.

- The microstructure noise, ϵ, has zero mean, is stationary, and strong mixing stochastic process, with the mixing coefficients decaying exponentially. In addition, $\mathbb{E}[\epsilon^{4+\kappa}] < \infty$, for some $\kappa > 0$

- The noise is independent of the price process.

Under Assumption 2.3, Aït-Sahalia et al. [3] demonstrate that the RV diverges to infinity linearly in n and for large n, the realized variance may have no connection to true returns.

Even Assumption 2.3 has a potentially problematic component, which says that noise is independent of the efficient price. On this issue, Hansen and Lunde [61] deviate from the existing literature to allow for dependence between true prices and noise where the research setting in [61] includes the following assumption on the MMN:

Assumption 2.4.

- The microstructure noise, ϵ, has zero mean, is covariance stationary such that its autocovariance function is defined by $\pi(s) = \mathbb{E}[\epsilon_t \epsilon_{t+s}]$.

Hansen and Lunde [61] not only show that when the true price follow a Brownian semimartingale form and the MMN satisfies Assumption 2.4, the asymptotic bias on the RV estimator grows linear in number of sampling intervals, but also do they provide evidence of serial dependence in the noise process and correlation with the efficient price for the case of Dow Jones Industrial Average stocks.

All in all, in light of the literature providing abundant mathematical and empirical evidence pointing to the existence of MMN, which may not be i.i.d and be correlated to the true price, we believe the estimation of the IV of true prices must be carried out only after evidence on the MMN structure is gathered so that a proper method to mitigate the MMN effect could be chosen. Accordingly, we examine evidence from Borsa İstanbul National Equity Market while we control for factors such as sampling scheme, liquidity and data handling methods with the aim of deducing robustly whether the MMN visually and statistically exerts presence and whether it exhibits i.i.d behavior.

2.5. Sampling Schemes

In organized equity markets, transactions take place and quotes arrive asynchronously, leading to transaction and/or quote time series to be observed at discrete and irregularly spaced intervals. This asynchronous character of markets allows us to sample the returns in various ways, i.e., one can follow different sampling schemes to estimate the IV over a fixed time period, $[0, T]$.

The most common sampling scheme is calendar time sampling (CTS), under which sampling is done at equal intervals in physical time; i.e., $\delta_{i,n} = \frac{1}{n}$ for all i. Sampling at every 1 or 10 minutes are examples of such a scheme. Even though it is commonly used, CTS has a shortfall: the transactions and/or quotes are irregularly spaced in time and calendar time sampled data need to be constructed artificially. As discussed in Section 2.1, in order to arbitrarily generate a regularly spaced time series, one needs to adopt an interpolation rule such as the previous tick method, the linear interpolation method or the cubic splines method. However, since some interpolation methods in general need to use information that is not available as of the time of the transaction, the researcher must be cautious of spurious correlations induced by the interpolation method selected.

Alternatively, one can sample prices whenever a transaction takes place, which is called transaction time sampling (TTS). If we sample the data everytime the price is changed, the sampling scheme is called tick time sampling (TkTS). Another alternative is called business time sampling (BTS), where the sampling times are determined to ensure that the IV of all intraday intervals are equal; i.e., $IV_{\mathrm{i}} = \frac{IV}{N}$. When these methods are compared, it is seen that an important feature of BTS pops out: under BTS, the observation times become latent, whereas under CTS, TTS, and TkTs they can be observed. Moreover, the BTS depends on the IV, the very unobservable parameter we would like to estimate. Pigorsch et al. [95] argue that since observing data under BTS requires us to estimate the IV before calculating the latent IV, this method is infeasible.

It is of great importance to shed light on the influence of the sampling scheme on the statistical properties of realized variance, if any. The first to contribute in this area is Oomen [93], who examines the following sampling alternatives:

- calendar time sampling,
- transaction time sampling,
- tick time sampling,
- business time sampling.

Oomen [94] develops on Oomen [93] and provides a framework to examine the statistical properties of the RV when data are contaminated with the MMN. Oomen [94] emphasizes that in the absence of the MMN, regardless of the sampling scheme, the plain RV estimator is an unbiased estimator of the IV. His framework, diverting from the standard literature, which prefers diffusion type price processes, takes the efficient price process as a pure jump process (Compound Poisson Process) and adds on a moving average structure to incorporate the microstructure noise. This specification of the observed price models the price as the sum of a finite number of jumps, where the number of transactions is counted by a Poisson Process. Oomen states that, as in the case of the diffusion-based models, the RV is a biased estimator of the jump analogue of the IV under microstructure noise. However, unlike previous results, the bias does not diverge to infinity as the sampling frequency converges to infinity. Oomen derives the closed form expressions for the bias and the mean squared error (MSE) of the RV as functions of model parameters as well as the sampling frequency. It is shown that the MSE of the RV can be decreased by sampling returns on a transaction time scale as opposed to the common practice of sampling in calendar time. This result is shown to be more pronounced when the trading intensity pattern is volatile.

At approximately the same time, in another study, Hansen and Lunde [61] assume that the efficient price follows a continuous diffusion process and use kernel-based estimators to unearth the properties of the MMN. The most notable of these characteristics is the noise being time-dependent and correlated with the unobservable efficient price. Interestingly, their findings are robust under both of CTS and TkTS.

In a later study, Griffin and Oomen [55], propose a new model for transaction prices in order to study the properties of two different time scales, transaction versus tick time. Their results show the finding that the return dynamics in transaction time are different from those in tick time and the choice of the sampling scheme may have a substantial effect on the properties of the RV[4]. They find that tick time sampling is superior to transaction time sampling in terms of the MSE, especially when the level of noise is low and the number of ticks, or the arrival frequency of the efficient price moves are small.

---

[4] *Microstructure noise is highly dependent under TkTS, so bias correcting method should be decided accordingly.*

Finally, in his unpublished master's thesis, Şen [102] introduces a new concept of business intensity and names the sampler as "Optimizable Multiresolution Quadratic Variation Filter". He concludes that his filter is less prone to microstructure effects than any other common sampling method for Turkish Interbank FX market.

All of these aforementioned contributions to the literature underline the fact that the possible technics to sample returns in organized markets due to asynchronous nature of the trading have the potential to influence the return and RV dynamics, which may alter the evidence on the MMN. Consequently in Chapter 4, as we search for the evidence on the existence and the statistical structure of the MMN in Borsa İstanbul National Equity Market, we define sampling schemes as one factor to be controlled for (in addition to data handling methods and liquidity) and compare the evidence acquired under TTS and CTS to conclude if our findings are robust regardless of the sampling schemes we work under.

2.6. Presence of Non-trading Hours

Another problem in measuring the realized volatility is the presence of non-trading hours during a trading day. Organized stock exchanges are open for trading only for certain hours on each weekday. The realized volatility may underestimate the integrated volatility if the realized volatility is calculated by using prices sampled only during the trading hours. In order to avoid this underestimation bias, we may include returns on non-trading hours (overnight and/or lunch break) but such a calculation may cause discretization noise of returns to make realized volatility noisy [104]. In addition, Bannouh et al. [23] state that non-trading hours are not necessarily a source of the MMN in a strict sense, and, therefore in addition to a bias correction, we may need to adjust our estimator for other market microstructure effects such as bid-ask bounce etc. Quote by Bannouh et al. [23] explains this as follows:

> *"For the RV estimator, non-trading increases the variance but does not cause a bias. In contrast, infrequent trading introduces a downward bias in RR estimators as the observed intraday high and low prices are likely to be below and above their 'true' values, respectively."*

Hansen and Lunde study the adjustment of the RV estimates for non-trading hours in 3 different papers [58], [59], [61]. In [58], they present 3 ways of adjusting the RV estimators in order to incorporate the variance over non-trading hours: (i) scaling of $RV_{opentoclose}$ by using a constant scaling factor (the scaling factor is same for each day), (ii) by adding the squared overnight return to $RV_{opentoclose}$ and, (iii) by optimally selecting weights to linearly combine the $RV_{opentoclose}$ and the squared overnight return (by minimizing the MSE as the objective function). Under the scaling approach, each original 6.5-hour variance estimate (before forecasting) is multiplied by a constant factor ϱ defined as

$$\varrho = \frac{\sum_{i=1}^{N}(R_i - \bar{R})^2}{\sum_{i=1}^{N} RV_i},$$

where R_i is the daily log return on the stock/index for day i, N is the total number of days in the sample and $\bar{R} = N^{-1}\sum_{i=1}^{N} R_i$. This procedure ensures that the average of the scaled realized volatility, i.e., $\varrho[X,X]_T$, is equal to the variance of the daily return. Hence, ϱ will inflate the 6,5-hour variance estimate. Unfortunately, [58] makes strong assumptions about the noise such as independence of efficient/true asset price. Therefore, there is a chance that the estimator employed in [58] by Hansen and Lunde may not fit the empirical findings if such findings pointed to a correlation between the noise increment and efficient/true return. Hansen and Lunde [58] state that this estimator is only slightly biased for the IV when the MMN is time dependent and correlated with the efficient returns.

In [59], Hansen and Lunde use a different scaling factor given as follows:

$$\varrho = \frac{n^{-1}\sum_{i=1}^{N} r_i}{\sum_{i=1}^{N} RV_i},$$

where r_i is the daily return. Likewise, in [61], again an upscaling ratio is employed for Zhou's kernel based RV estimator [112]. However, Hansen and Lunde state that the upward scaling as offered in [61] causes the variance of the estimator to become inflated and they switch to the Bartlett kernel in their RV estimation in [58].

Hansen and Lunde [58] discuss the conditions that justify a simple scaling. They state that for the scaled RV to be a proper estimator of the daily volatility, a particular scaling of the $RV_{opentoclose}$ is assumed to be informative about the daily IV and the scaling coefficient is assumed to be estimated consistently by incorporating information from an increasing number of days. In particular, their scaling approach assumes that (a) a fixed proportion of the daily integrated variance occurs during the active period (the validity of this assumption is checked in their empirical analysis), (b) the conditional bias of the $RV_{opentoclose}$ is proportional to the daily IV (they indicate that this requirement is fulfilled whenever the RV measure is unconditionally unbiased and for some of the biased estimators -under additional *"mild suitable"* assumptions such as MMN having constant variance independent of time-, the assumption still holds), and, (c) the squared overnight return is conditionally proportional to the overnight IV. This set of assumptions is problematic when we try to prove the statistical and economic gains of applying MMN adjusted methodologies in estimation of the IV using intraday data, since it is not clear what would happen if we divided several different estimates of the RV by the squared overnight return? More than likely, there would be no single proportion or no single scaling factor. Each RV estimator would have its own scaling factor but this would contradict the initial assumption that volatility calculated by using data from the trading hours is proportional to the volatility of the entire day. We believe that if this proposition holds, then the proportion should not change from one estimator to the next.

Martens [82] also emphasizes the idea that the absence of overnight trading should be reflected in RV measures by adjusting the RV estimator by summing the overnight squared returns with the $RV_{opentoclose}$. He adds that since the square of overnight returns is a noisy estimator, an alternative volatility measure using only intraday returns would be to multiply the vanilla RV by $(1+c)$, where c is the proportion of the sum of all $RV_{opentoclose}$ to the sum of all overnight RVs and it is a positive constant that makes the adjusted RV to be equal to the daily volatility (whereby the correct expected value is attained). Koopman et al. [73] follow Martens [82] in adjusting the vanilla RV to reach a better estimate of the daily IV. Meanwhile, Fleming et al. [51] prefer a dynamic scaling approach in the sense that the adjusting factor ϕ is calculated for each day separately in the following fashion:

$$\phi_i = \frac{\sum_{i=1}^{t} \rho^i r_{t-i}^2}{\sum_{i=1}^{t} \rho^i RV_{t-i}},$$

where $\rho \in (0,1)$ is a factor that manages the weights allocated to the lagged values of the returns and RVs. Unlike Hansen and Lunde [58], [59], Fleming et al. [51] do not lay out the statistical effects and or the assumptions with respect to scaling. They do not show what happens to the consistency, unbiasedness and efficiency of the RV estimator when we scale it using their approach.

Only a small number of papers [58], [59], [60], [61], [28], [101] in the RV literature discuss the methods for adjusting the bias corrected estimator for non-trading hours, whereas the majority of the literature on the estimation of quadratic variation of asset prices using high frequency data stays silent about non-trading hours. Examples of papers that do not address the non-trading hours issue are by Barndorff-Nielsen et al. [27] (kernel based), Barndorff-Nielsen et al. [28] (subsampling), Bandi and Russell [20] (optimal sampling when there is noise), Bandi et al. [22] (optimal sampling frequency in forecasting), Oomen [94] (pure jump process plus MA(q) to analyze the effect of varying sampling schemes and optimal sampling frequencies over realized variance estimator), Griffin and Oomen [55], Gatheral and Oomen [52], Andersen et al. [14] (forecasting in the presence of MMN), Ghysels and Sinko [54] (volatility forecasting and the MMN) etc. The same observation is also mentioned by Brownlees and Gallo [38].

> *"Note that overnight information is not included in these series and this may have a consequence when we use intra-daily information to predict the conditional variance of daily (close-to-close) returns. Gallo (2001) shows that the overnight squared return has a significant impact when used as a predetermined variable in a GARCH for the open-to-close returns. For realized volatility measures, the problem is recognized, among others, by Martens (2002), Fleming, Kirby, and Ostdiek (2003), and Hansen and Lunde (2005)."*

There may be a number of reasons why volatility over non-trading hours is not typically included in the RV literature. It may be that the existence of non-trading hours only causes a time shift in volatility and daily volatility is the same regardless of the length of the trading hours, and as a result a diurnal shape in trading volumes and returns is observed. Alternatively, trading incentives over the night and lunch are already reflected in the market or limit orders given once the market opens. Finally, it may be that when the return volatility displays a U shape per session, adjusting the $RV_{opentoclose}$ for non-trading hours leads to a double counting of the daily volatility.

The first problem that arises when the IV estimates are adjusted for non-trading hours relates to the variety of the RV measures proposed in the literature. Depending on the empirical findings with respect to structure of the MMN, let's say that one chooses subsampling or kernel based estimators that specifically fit the MMN findings. In such a case, it is not clear how the bias corrected estimator can be adjusted for non-trading hours. Moreover, in order to determine the statistical gains by bias correcting the estimators when there is MMN, one should compare the bias corrected and vanilla RV settings with respect to some criteria such as the MSE. Now, if the vanilla RV is adjusted for non-trading hours following Hansen and Lunde [58] and the bias corrected estimator is not, such comparisons are flawed from the very beginning.

The second problem with adjusting the RV estimators for non-trading hours is the existence of opening and closing sessions in BIST. With such a market structure, it is necessary to decide whether the opening and closing sessions should be included in the adjustment process or whether we should only work with sessions where a continuous auction is allowed. There is also a lunch break in BIST. Hansen and Lunde's [58] analogy applies to block trading hours, since there are no lunch breaks at the NYSE during the trading day. Therefore, a simple scaling of the RV as in Hansen and Lunde [58] cannot reflect the daily volatility of returns in BIST.

Another major concern regarding the adjustment of RV estimators for non-trading hours is the effect of sampling schemes. More specifically, [58] and other papers such as [82] and [73] that adopt and adjustment for non-trading hours mainly work under CTS. In addition, the literature about upscaling the $RV_{opentoclose}$ for estimating the daily IV, regardless of whether the scaling factor is constant for the entire sample period [58] or recalculated for each day [51], accepts implicitly or explicitly that the relationship between $RV_{opentoclose}$ and $RV_{overnight}$ can be revealed. We believe that the same analogy may not apply for TTS and/or TkTS.

In light of above discussions, depending on the structure of the MMN and the selected method for correcting the bias, we believe that it is best not to adjust estimators for non-trading hours, particularly if estimation is carried out under different sampling schemes and/or the organized market, from which the data is disseminated, defines a trading day as a combination of continuous auction sessions and single price opening and closing sessions.

CHAPTER 3

TESTING THE STATISTICAL STRUCTURE OF MMN UNDER CTS AND TTS

3.1. Realized Variance formula under CTS and TTS[5].

Let S_t denote the price process of a security and the log price of this security be represented by the process X_t which satisfies the following stochastic differential equation on finite time horizon $t \in [0, T]$:

$$dX_t = \mu_t dt + \sigma_t dB_t. \tag{3.1}$$

The log price X_t of this security is assumed to belong to the Brownian semimartingale family, i.e. X_0 is $\mathcal{F}_0$-measurable, X_t has continuous sample paths, drift μ_t is a locally bounded, predictable continuous process, the continuous stochastic process σ_t that derives the volatility of the log return of the security is square integrable, and B_t denotes the standard Wiener process. We assume that the leverage effect is ruled out, meaning that σ_t is orthogonal to B_t. In this setting, as discussed in the preceding chapters, the parameter of interest for the majority of financial applications is the integrated volatility of log return accumulated over a fixed period of time *T*, usually a trading day, which is written as follows:

$$\int_0^T \sigma_t^2 dt. \tag{3.2}$$

Unfortunately, the integrated volatility accumulated until time T, IV_T, is unobservable and latent, forcing us to estimate it. In doing so, we benefit from the concept of

---

[5] *Interested reader should consult to [71] and Chapter 2 of [97] for further discussions on Brownian motions and quadratic variation of semimartingales, respectively.*

quadratic variation since the quadratic variation process of semimartingale X_t denoted by $[X,X] = ([X,X]_t)_{t\geq 0}$ is defined as

$$[X,X] = X^2 - 2\int X_{-}dX,$$

where X_{-} at time s represents the value as $\lim_{u\to s, u\leq s} X_u$ (Chapter 2, Section 6, p. 58 of [97]). Since we assume that the security price has continuous sample paths, X_{-} at time s equals to X_{+}, i.e., the price process has no jumps.

Theorem 3.1. (Theorem 22 in Chapter 2, Section 6, p. 59 of [97]) The quadratic variation process of a semimartingale is a cádlág (abbreviation of right continuous with left limits in French), increasing and adapted process satisfying the conditions given below:

i) $[X,X]_0 = X_0^2$ and $\Delta[X,X] = (\Delta X)^2$,

ii) If τ shows a finite sequence of stopping times, $0 \leq t_0 \leq t_1 \cdots \leq t_k < \infty$, it is called a random partition (the distance between two consecutive stopping times are random). In this context, if $\tau_n := t_0^n \leq t_1^n \cdots \leq t_{k_n}^n$ is a sequence of random partitions tending to identity, i.e.,

$\lim_n \sup_k t_k^n = \infty$ almost surely (a.s) and $\|\tau_n\| = \sup_k |t_{k+1}^n - t_k^n| \to 0$ a.s.,

then

$$X_0^2 + \sum_i \left(X_{t_{i+1}} - X_{t_i}\right)^2 \to [X,X]_t,$$

where the convergence is in probability. In other words, if the maximum distance between observation times converges to 0 as the number of observation points (stopping times) converges to infinity, then the sum of squared differences of process X with differences taken over consecutive stopping times converge in probability to the quadratic variation of X.

A corollary (Corollary 1 in Chapter 2, Section 6, p. 60 of [97]) to Theorem 3.1 reveals that the quadratic variation of a semimartingale X has paths of finite variation on compact sets and is also a semimartingale. Accordingly,

$$\lim_{n\to\infty} \sum_{t_i} \left(X_{t_{i+1}} - X_{t_i}\right)^2 \xrightarrow{\mathbb{P}} \int_0^T \sigma_t^2 dt, \tag{3.3}$$

showing that the estimation error of the realized variance on $[0, T]$ defined as

$$\sum_{t_i} \left(X_{t_{i+1}} - X_{t_i}\right)^2 - \int_0^T \sigma_t^2 dt\,, \qquad 0 \leq t_i \leq T\,,$$

shrinks to 0 when the number of stopping times (the observation points) increases to infinity and at the same time maximum of sampling intervals converge to zero. This suggests that from a financial applications point of view, the RV calculated over the highest data frequency should give the best possible estimate for the IV both under CTS and TTS because the RHS of the convergence in Equation (3.3) is defined over a sequence of random partitions tending to identity, i.e., the observations times are allowed to be random with CTS being a special case of equidistant observations. In practice, however, trading in organized markets introduces market microstructure frictions to the observed prices, which makes the estimation of return volatility of true prices a challenging task.

3.2. Asymptotic bias of the RV when an MMN exists.

Following the majority of the market microstructure noise literature discussed in the previous chapters, the observed price Y_t is assumed to be contaminated with an additive market microstructure noise. i.e.,

$$Y_t = X_t + \varepsilon_t, \qquad 0 \leq t \leq T, \tag{3.4}$$

where *T* shows a finite horizon, X_t denotes the logarithm of the true/efficient price of the security at time *t* and ε_t represents the logarithm of combined effect of all microstructure noise sources at time *t* including frictions in the market, trading rules, informational asymmetries, bid-ask bounces, non-trading hours etc. The contamination of observed prices with market microstructure noise is a very important assumption in volatility estimation via realized type of measures because if there is such a contamination, then the quadratic variation of observed prices calculated over the highest frequency possible does not simply converge to the IV of the true prices since an asymptotic bias, in addition to the discretization error appear due to existence of the MMN. In order to examine how RV deviates from IV as we increase the sampling frequency and to come up with methods to handle those deviations (mitigation of the market microstructure noise effect on RV measures), we first have to make some assumptions regarding the statistical properties of the MMN. Recall from the preceding chapter that regarding the MMN, the most popular assumptions in the RV literature are as follows:

Assumption 3.1. The market microstructure noise, ε_t, is a sequence of independent and identically distributed (i.i.d) random variables with zero mean, constant variance and finite fourth moment.

Assumption 3.2. Market microstructure noise and true prices are orthogonal to each other for each $t \in [0,T]$.

These assumptions imply that

a) $\mathbb{E}\left[\varepsilon_{t_{i+1}}\right] = \mathbb{E}\left[\varepsilon_{t_i}\right] = 0$ and $\mathbb{E}\left[\varepsilon_{t_{i+1}}\right] \perp \mathbb{E}\left[\varepsilon_{t_i}\right]$ for any $0 \le i \le n$,

b) the increment of the noise also has a constant variance. More specifically, if v_{t_i} represents the noise increment from time t_i to t_{i+1}, $\forall t_i \in [0,T]$, then

$$
\begin{aligned}
&\mathrm{Var}\left[v_{t_i}\right] = \mathbb{E}\left[v_{t_i}^2\right] - \left(\mathbb{E}\left[v_{t_i}\right]\right)^2 \\
&= \mathbb{E}\left[v_{t_i}^2\right] - \left(\mathbb{E}\left[\varepsilon_{t_{i+1}} - \varepsilon_{t_i}\right]\right)^2 \\
&= \mathbb{E}\left[v_{t_i}^2\right] - \left(\mathbb{E}\left[\varepsilon_{t_{i+1}}\right] - \mathbb{E}\left[\varepsilon_{t_i}\right]\right)^2 \\
&= \mathbb{E}\left[v_{t_i}^2\right] \\
&= \mathbb{E}\left[\left(\varepsilon_{t_{i+1}} - \varepsilon_{t_i}\right)^2\right] \\
&= \mathbb{E}\left[\left(\varepsilon_{t_{i+1}}\right)^2 - 2\left(\varepsilon_{t_{i+1}}\right)\left(\varepsilon_{t_i}\right) + \left(\varepsilon_{t_i}\right)^2\right] \\
&= 2(\mathbb{E}[\varepsilon_t{}^2] - \mathbb{E}[\varepsilon_t]\mathbb{E}[\varepsilon_t]) \\
&= 2\mathrm{Var}[\varepsilon_t],
\end{aligned}
\tag{3.5}
$$

where since $\mathrm{Var}[\varepsilon_t]$ is constant, $\mathrm{Var}[v_t]$ constant as well. In order to continue with the examination of the asymptotic bias of the RV estimator when observed prices are contaminated with the MMN, as the next step, we restate[6] the definition of the quadratic variation of a generic semimartingale $\left\{K_{t_i}\right\}_{t_i \in [0,T]}$ relative to a grid (or partition) $\mathcal{G} = \{t_0, t_1, \cdots, t_{n-1}\}, t_0 = 0, t_{n-1} = T$ as follows:

$$
[K,K]_t^{\mathcal{G}} = \sum_{i=0}^{n-2} \left(K_{t_{i+1}} - K_{t_i}\right)^2.
$$

The number of data points in grid $\mathcal{G}$ is denoted by $|\mathcal{G}|$ and equals to n. Let $\Delta(\mathcal{G}) = \max_{1 \le i \le n} (t_{i+1} - t_i)$, then for $n \to \infty$ if $\Delta(\mathcal{G}) \to 0$, for all $t \in [0,T]$ there is a process $[K,K]_t$ so that $[K,K]_t^{\mathcal{G}} \to [K,K]_t$ in probability (Theorem I.4.47 in [70]). This is the same theorem as the one given in Theorem 3.1.

[6] *The asymptotic distribution of the discretization error when there is no MMN under TTS is constructed in [91] and for ease of reading, we prefer following their notation for the rest of the chapter. For this purpose, please see that grid and partition terms are used interchangeably,* $\Delta(\mathcal{G}) = ||\mathcal{G}||$, *condition* $n \to \infty$ *and* $\Delta(\mathcal{G}) \to 0$ *means partition* $\mathcal{G}$ *tends to identity.*

Under these notations, assumptions and definitions, following Zhang et al. [111], Hansen and Lunde [61], Awartani et al. [16] and many other authors, the conditional asymptotic bias of RV on $[0, T]$ calculated using observed prices, Y_t, that are themselves defined as the sum of true prices, X_t, and the aggregate effect of market microstructure, ε_t, is derived as below:

From Equation (3.4), applying the quadratic variation operator to both sides, we get

$$[Y, Y]_T^{\mathcal{G}} = [X, X]_T^{\mathcal{G}} + 2[X, \varepsilon]_T^{\mathcal{G}} + [\varepsilon, \epsilon]_T^{\mathcal{G}}. \tag{3.6}$$

Taking the expectation conditional on the true price process X on both sides yields

$$\mathbb{E}\left[[Y, Y]_T^{\mathcal{G}} \middle| X\right] = \mathbb{E}\left[[X, X]_T^{\mathcal{G}} \middle| X\right] + 2\mathbb{E}\left[[X, \varepsilon]_T^{\mathcal{G}} \middle| X\right] + \mathbb{E}\left[[\varepsilon, \varepsilon]_T^{\mathcal{G}} \middle| X\right]. \tag{3.7}$$

As $n \to \infty$ if $\Delta(\mathcal{G}) \to 0$, $[X, X]_T^{\mathcal{G}} \to \mathrm{IV}_T$ and

$$\begin{aligned}\mathbb{E}\left[[Y, Y]_T^{\mathcal{G}} \middle| X\right] - \mathrm{IV}_T \\ = 2\mathbb{E}\left[\sum_{i=0}^{n-2} \left(X_{t_{i+1}} - X_{t_i}\right)\left(\varepsilon_{t_{i+1}} - \varepsilon_{t_i}\right) \middle| X\right] + \mathbb{E}\left[\sum_{i=0}^{n-2} \left(\varepsilon_{t_{i+1}} - \varepsilon_{t_i}\right)^2 \middle| X\right].\end{aligned}$$

By the linearity of the expectation operator and the independence of the true price and the MMN, where $\mathbb{E}\left[\varepsilon_{t_{i+1}} \middle| X\right] = \mathbb{E}\left[\varepsilon_{t_i} \middle| X\right] = 0$, as $n \to \infty$ and $\Delta(\mathcal{G}) \to 0$

$$\begin{aligned}\mathbb{E}\left[\sum_{i=0}^{n-2} \left(X_{t_{i+1}} - X_{t_i}\right)\left(\varepsilon_{t_{i+1}} - \varepsilon_{t_i}\right) \middle| X\right] &= \mathbb{E}\left[\sum_{i=0}^{n-2} \left(X_{t_{i+1}} \varepsilon_{t_{i+1}}\right) \middle| X\right] + \mathbb{E}\left[\sum_{i=0}^{n-2} \left(X_{t_i} \varepsilon_{t_i}\right) \middle| X\right] \\ &\quad - \mathbb{E}\left[\sum_{i=0}^{n-2} \left(X_{t_{i+1}} \varepsilon_{t_i}\right) \middle| X\right] - \mathbb{E}\left[\sum_{i=0}^{n-2} \left(X_{t_i} \varepsilon_{t_{i+1}}\right) \middle| X\right] \\ &= \sum_{i=0}^{n-2} \mathbb{E}\left[X_{t_{i+1}} \varepsilon_{t_{i+1}} \middle| X\right] + \sum_{i=0}^{n-2} \mathbb{E}\left[X_{t_i} \varepsilon_{t_i} \middle| X\right] \\ &\quad - \sum_{i=0}^{n-2} \mathbb{E}\left[X_{t_{i+1}} \varepsilon_{t_i} \middle| X\right] - \sum_{i=0}^{n-2} \mathbb{E}\left[X_{t_i} \varepsilon_{t_{i+1}} \middle| X\right] \\ &= \sum_{i=0}^{n-2} \mathbb{E}\left[X_{t_{i+1}} \middle| X\right] \mathbb{E}\left[\varepsilon_{t_{i+1}} \middle| X\right]\end{aligned}$$

$$+\sum_{i=0}^{n-2}\mathbb{E}\left[X_{t_i}\middle|X\right]\mathbb{E}\left[\varepsilon_{t_i}\middle|X\right]$$
$$-\sum_{i=0}^{n-2}\mathbb{E}\left[X_{t_{i+1}}\middle|X\right]\mathbb{E}\left[\varepsilon_{t_i}\middle|X\right]$$
$$-\sum_{i=0}^{n-2}\mathbb{E}\left[X_{t_i}\middle|X\right]\mathbb{E}\left[\varepsilon_{t_{i+1}}\middle|X\right]$$
$$= 0,$$

and

$$\begin{aligned}\mathbb{E}\left[\sum_{i=0}^{n-2}\left(\varepsilon_{t_{i+1}}-\varepsilon_{t_i}\right)^2\middle|X\right] &= \sum_{i=0}^{n-2}\mathbb{E}\left[\left(\varepsilon_{t_{i+1}}-\varepsilon_{t_i}\right)^2\middle|X\right] \\ &= \sum_{i=0}^{n-2}\mathbb{E}\left[{\varepsilon_{t_{i+1}}}^2+{\varepsilon_{t_i}}^2-2\varepsilon_{t_{i+1}}\varepsilon_{t_i}\middle|X\right] \\ &= \sum_{i=0}^{n-2}\mathbb{E}\left[{\varepsilon_{t_{i+1}}}^2\middle|X\right] \\ &+\sum_{i=0}^{n-2}\mathbb{E}\left[{\varepsilon_{t_{i+1}}}^2\middle|X\right] \\ &-2\sum_{i=0}^{n-2}\mathbb{E}\left[\varepsilon_{t_{i+1}}\middle|X\right]\mathbb{E}\left[\varepsilon_{t_i}\middle|X\right] \\ &= 2\sum_{i=0}^{n-2}\mathbb{E}[{\varepsilon_t}^2|X] = 2(n-1)\,\mathbb{E}[{\varepsilon_t}^2].\end{aligned} \tag{3.8}$$

Therefore, as $n \to \infty$

$$\mathbb{E}\left[[Y,Y]_T^{\mathcal{G}}\middle|X\right]-\mathrm{IV}_T = 2(n-1)\,\mathbb{E}[{\varepsilon_t}^2] \to \infty \tag{3.9}$$

We would like to underline that if we let the expectation of the MMN to be different than 0, as long as we take the variance of noise as constant, $\mathbb{E}\left[\left(\varepsilon_{t_{i+1}}-\varepsilon_{t_i}\right)^2\right]$ equals to some constant by Equation (3.5) and the asymptotic bias, $\mathbb{E}\left[[Y,Y]_T^{\mathcal{G}}\middle|X\right]-\mathrm{IV}_T$, still explodes to infinity linear in *n* as *n* diverges because now Equation (3.8) becomes $2(n-1)\mathrm{Var}[\varepsilon_t]$.

Awartani et al. [16] point out that even if we take the true price as correlated to the MMN, the asymptotic bias, $2\mathbb{E}\left[[X,\varepsilon]_T^{\mathcal{G}}\middle|X\right]+\mathbb{E}\left[[\varepsilon,\varepsilon]_T^{\mathcal{G}}\middle|X\right]$, is still dominated by $\mathbb{E}\left[[\varepsilon,\varepsilon]_T^{\mathcal{G}}\middle|X\right]$. Likewise, Bandi and Russell [20] state that $[X,\varepsilon]_T$ is stochastically dominated by $[\varepsilon,\varepsilon]_T$. We hereby provide the reasoning of these statements as follows:

A version of Cauchy-Schwarz Inequality,

$$\sum_i a_i b_i \leq \sqrt{\sum a_i^2 \sum b_i^2} \tag{3.10}$$

implies that

$$[X,\varepsilon]_T^2 \leq [X,X]_T[\varepsilon,\varepsilon]_T.$$

Due to the definition of Itô processes and Brownian semimartingales with square integrable σ_t, we know that $[X,X]_T$ is stochastically bounded, i.e. for any $\lambda > 0$, there exists a finite $\mathrm{M} > 0$ such that

$$\mathbb{P}[|[X,X]_T| > \mathrm{M}] < \lambda, \qquad \forall T,$$

which is denoted by $[X,X]_T = O_p(1)$.

As Hansen and Lunde [61] state, r^{tic} variation of the white noise type processes with a constant mean explode as data points in the partition diverges. Recall that, based on our assumptions, the MMN under Assumption 3.1 belongs to the white noise family. The explosion of the quadratic variation of the MMN as *n* goes to infinity can also be deduced from Equation (3.8) because from Equation (3.8), it is evident that under the assumption that MMN has constant variance, $\mathbb{E}\left[[\varepsilon,\varepsilon]_T^{\mathcal{G}}\right] \to \infty$ as n diverges. If the quadratic variation of a stochastic process is stochastically bounded, then the expectation of the aforementioned quadratic variation cannot diverge in *n*. Accordingly, Assumption 3.1 and Equation (3.8) together ensure that as *n* goes to infinity, $[\varepsilon,\varepsilon]_T^{\mathcal{G}}$ explodes.

Therefore, we can write

$$[X,\varepsilon]_T^2 \leq [X,X]_T[\varepsilon,\varepsilon]_T < [\varepsilon,\varepsilon]_T^2,$$

and

$$[X, \varepsilon]_T < [\varepsilon, \varepsilon]_T,$$

in probability. Taking expectations on both sides of the inequality, we get $\mathbb{E}[[X, \varepsilon]_T] < \mathbb{E}[[\varepsilon, \varepsilon]_T]$ and the asymptotic bias is dominated by $[\varepsilon, \varepsilon]_T$ under both TTS and CTS, regardless whether MMN and true price are correlated and/or MMN has a constant mean other than 0.

3.3. Testing the existence of MMN under TTS

Awartani et al. [16] form the following null and alternative hypotheses in order to check whether, under CTS, there is any statistically significant asymptotic bias on RV estimator due to the existence of MMN:

$$H_0: \mathbb{E}\left[\left(\varepsilon_{t_{i+1}} - \varepsilon_{t_i}\right)^2\right] = 0, \tag{3.11}$$

$$H_a: \mathbb{E}\left[\left(\varepsilon_{t_{i+1}} - \varepsilon_{t_i}\right)^2\right] > 0. \tag{3.12}$$

We assert that the same set of hypotheses are also relevant for TTS because under both TTS and CTS, we have proved that if the observed prices are contaminated with the MMN which satisfies Assumption 3.1, then as we increase observation frequencies, the RV, scaled by $(2 \cdot number\ of\ sampling\ intervals))^{-1}$ and calculated over observed prices, estimates more and more the variance of the MMN rather than the quadratic variation of the true price.

Awartani et al. [16] develop a test statistic under CTS to test if we can reject H_0 against H_a implying that the MMN has a statistically significant effect on the RV estimators of the IV at a given sampling frequency. The test statistic $Z_{T,n,h}$ employs the RVs calculated at two artificially selected frequencies, one low and one high, as well as the Realized Quarticity (RQ) calculated at low frequency and is formulated as below:

$$Z_{T,n,h} := \frac{\sqrt{h-1}\left(RV_{T,n} - RV_{T,h}\right)}{\sqrt{\frac{2(h-1)}{3} RQ_{T,h}}}, \tag{3.13}$$

where *h* and *n* stand for the number of observations for the whole estimation period, T, (for instance, the number of observations per day) at low frequency and high frequency, respectively, and

$$RV_{T,n} = \sum_{i=0}^{n-2} \left(Y_{t_{i+1}} - Y_{t_i}\right)^2, \tag{3.14}$$

$$RV_{T,h} = \sum_{i=0}^{h-2} \left(Y_{t_{i+1}} - Y_{t_i}\right)^2, \tag{3.15}$$

$$RQ_{T,h} = \sum_{i=0}^{h-2} \left(Y_{t_{i+1}} - Y_{t_i}\right)^4, \tag{3.16}$$

$$n > h, \qquad \frac{n}{h} \to \infty \text{ as } n, h \to \infty.$$

Beware that the notation in [16] requires to attach each observation with a time index that starts from 1, i.e., in prior and following sections, we take the grid on which observations are positioned in the form $\mathcal{G} = \{t_0, t_1, \cdots, t_{n-1}\}, t_0 = 0, t_{n-1} = T$, where Awartani et al. [16] write same grid in a slightly different form $\mathcal{G} = \{t_1, t_2, \cdots, t_n\}, t_1 = 0, t_n = T$. This is why we have differences in the upper and lower bounds in sigma operators between Equations (3.14), (3.15), (3.16) and calculations suggested in fifth equation in [16]. In addition, the multiplier $(h-1)$ in Equation (3.13) is one less than the multiplier $h = N\bar{T}$ in eight equation in [16]. The reader should keep in mind these notation nuances while reading the following sections henceforth.

3.3.1. Deriving a statistic to test the existence of MMN under CTS

The test statistic $Z_{T,n,h}$ has a standard normal distribution asymptotically and its asymptotic distribution is constructed on two main pillars:

1) Awartani et al. [16] first describe their setting as follows:

- the true price is generated as in Equation (3.1),
- $\int_0^T \sigma_t^4 dt < \infty$,
- the MMN increments have a finite fourth moment on $[0, T]$,
- one can find at least one $\zeta > 0$ such that for $\psi \in (0,1)$, $\liminf_{n\to\infty} (n-1)^{\psi-1}[\varepsilon, \varepsilon]_T > \zeta$ and $\liminf_{h\to\infty} (h-1)^{\psi-1}[\varepsilon, \varepsilon]_T > \zeta$.

Awartani et al. [16] then explain the expansion of the numerator in this setting as below:

$$Z_{T,n,h} = \frac{\sqrt{h-1}\left(RV_{T,n} - IV_T\right)}{\sqrt{\frac{2(h-1)}{3} RQ_{T,h}}} - \frac{\sqrt{h-1}\left(RV_{T,h} - IV_T\right)}{\sqrt{\frac{2(h-1)}{3} RQ_{T,h}}}, \tag{3.17}$$

reveals that the first term on the RHS of Equation (3.17) converges to 0 in probability and asymptotically speaking, the limiting distribution of $Z_{T,n,h}$ is determined by second term on the RHS of Equation (3.17). This argument is built on a result by Jacod and Protter [69], who show that in the absence of the MMN for equidistant observations, the estimation error of the RV_T scaled by the square root of the number of sampling intervals is stochastically bounded, i.e.,

$$\left(RV_{T,n} - IV_T\right) = O_p\left((n-1)^{-1/2}\right), \tag{3.18}$$

$$\left(RV_{T,h} - IV_T\right) = O_p\left((h-1)^{-1/2}\right). \tag{3.19}$$

The Equations (3.18) and (3.19) show that estimation error when the total number of observations equals n needs to grow at least by the square root of $(n-1)$ in order to be bounded asymptotically, and, any smaller amount that scales the estimation error causes it to converge to 0. Since we assumed that $\frac{n}{h} \to \infty$ as $n, h \to \infty$,

$$\lim_{h,n\to\infty} \mathbb{P}\left[\left|\sqrt{h-1}\left(RV_{T,n} - IV_T\right)\right| \geq \gamma\right] = 0, \forall\, \gamma \in (0, \infty),$$

and such convergence in probability is denoted by $\sqrt{h-1}\left(RV_{T,n} - IV_T\right) = o_p(1)$.

2) Since the limiting distribution of the statistic $Z_{T,n,h}$ is driven by the second term on the RHS of Equation (3.17), Awartani et al. [16] benefit from a central limit theorem developed by Jacod and Protter [69] and a result in [24] which shows that the realized quarticity is a consistent estimate of the quadratic variation of the estimation error. The central limit theorem by Jacod and Protter [69] proves that, in the absence of the MMN, the estimation error of the RV with respect to the IV has a limiting mixed normal distribution under CTS, i.e.,

$$\sqrt{h-1}\left(RV_{T,h} - IV_T\right) \xrightarrow{d} N\left(0{,}2T\int_0^T \sigma_s^4 ds\right).$$

By using this central limit theorem combined with the realized quarticity being a consistent estimate of the variance of the estimation error, Awartani et al. [16] deduce that the second term on the RHS of Equation (3.17) is asymptotically standard normal under H_0 defined in (3.11). They also demonstrate that $Z_{T,n,h}$ diverges under H_a, described in (3.12) if for $\gamma > 0$

$$\lim_{n,h\to\infty} \mathbb{P}\left[\frac{(h-1)^{\psi+0.5}}{(n-1)^{\psi+0.5}-(h-1)^{\psi+0.5}} Z_{T,n,h} > \gamma\right] = 1.$$

If the statistic $Z_{T,n,h}$ for a certain calculation horizon, such as one day, is calculated to be negative and we reject the null hypothesis in (3.12) and conclude that the MMN has a statistically significant effect on the RV estimator, then for that day, there should be a negative correlation between the true price X_t, and the aggregate effect of market microstructure, ε_t due to the fact that the quadratic variation of the MMN is always nonnegative. In other words, for $Z_{T,n,h}$ to be negative, the RV calculated at higher frequency should be smaller than the RV calculated at lower frequency. However, we know that the quadratic variation of the MMN is non-decreasing in the number of observation points, and, therefore the only source of negativity of the test statistic $Z_{T,n,h}$ should be a negative correlation between the true price and the MMN which becomes more accentuated as the sampling frequency increases.

3.3.2. Deriving a statistic to test the existence of MMN under TTS

Zhang et al. [111] and Mykland and Zhang [91] show that the two pillars providing the foundation for Awartani et al.'s [16] test statistic also hold under TTS. Based on [111] and [91], we suggest that the same test statistic $Z_{T,n,h}$ can be employed to examine the existence of MMN under TTS. Next, we explain how Awartani et al.'s [16] test statistic $Z_{T,n,h}$ works under TTS by following the notations and proofs in [111] and [91].

Subsection 3.3.2.A: Step 1

Let's take a close look at the estimation error when observations are spaced irregularly in time (TTS) under the null hypothesis H_0 in (3.11). Recall that when the observed prices are not contaminated by the MMN, the estimation error of the RV comes only from the discretization error regardless of the sampling scheme. The discretization error appears because, in practice, the number of observations during a trading day, in other words the number of data points in a grid, is limited. Also, for most financial applications, due to computational challenges, practitioners prefer to estimate the IV using the RV calculated at certain frequencies such as returns at every 3 or 30 transactions. This is called calculating the RV using a subgrid of all available information. More specifically, if $\mathcal{G}$ represents all available trading information, i.e., all transaction prices recorded, then the estimation error, denoted by $\mathcal{Z}_t$, is written as $[Y,Y]_t^{\mathcal{H}} - \mathrm{IV}_t$ and it equals to $[X,X]_t^{\mathcal{H}} - \mathrm{IV}_t$, when there is no MMN over a subgrid $\mathcal{H}, \mathcal{H} \subseteq \mathcal{G}$. Here, the number of data points in grids $\mathcal{H}$ and $\mathcal{G}$ are *h* and *n*, respectively.

Following Mykland and Zhang [91], we make two assumptions in order to simplify the examination of estimation error under TTS:

Assumption 3.3. The true price of the security has no drift and it is local martingale by definition, i.e., the logarithm of the true price of the security satisfies the following equation: for all $t \in [0,T]$

$$X_t = X_0 + \int_0^t \sigma_s dB_s. \tag{3.20}$$

Assumption 3.4. The instantaneous true return volatility,σ_t, is bounded, i.e., there is a nonrandom a such that $\sigma_t^2 \leq a^2$ for all $t \in [0,T]$.

Under Assumptions 3.3 and 3.4, X_t becomes a martingale and we apply Itô's formula to $\left(X_{t_{i+1}} - X_{t_i}\right)^2$. Recall that the Itô's formula states that

$$df(X_t) = f'(X_t)dX_t + \frac{1}{2}f''(X_t)d[X,X]_t. \tag{3.21}$$

Therefore, if $f(X_t) = X_t^2$, $df(X_t) = 2X_t dX_t + \sigma_t^2 dt$. Hence from t_i to t_{i+1},

$$\left(X_{t_{i+1}} - X_{t_i}\right)^2 = \int_{t_i}^{t_{i+1}} 2\left(X_s - X_{t_i}\right)dX_s + \int_{t_i}^{t_{i+1}} \sigma_s^2 ds.$$

If the last time point in the grid does not coincide with the end of the time horizon, as it happens frequently in market data when the last observation on a trading day is before the end of the trading session, then $\max\{t_i\} \neq t$. At this point, following Mykland and Zhang [91], we consider the upper edge of the partition $\mathcal{H}$ and set

$$t_* = \max\{t_i \in \mathcal{H}, t_i \leq t\}.$$

Then,

$$\left(X_t - X_{t_*}\right)^2 = \int_{t_*}^{t} 2\left(X_s - X_{t_*}\right) dX_s + \int_{t_*}^{t} \sigma_s^2 ds, \tag{3.22}$$

and

$$[X, X]_t^{\mathcal{H}} = \sum_{t_{i+1} \leq t} \left(X_{t_{i+1}} - X_{t_i}\right)^2 + \left(X_t - X_{t_*}\right)^2. \tag{3.23}$$

Therefore, the estimation error, $\mathcal{Z}_T$, is rewritten as follows:

$$\mathcal{Z}_t = [X, X]_t^{\mathcal{H}} - \mathrm{IV}_t = \sum_{t_{i+1} \leq t} \left(X_{t_{i+1}} - X_{t_i}\right)^2 + \left(X_t - X_{t_*}\right)^2 - \int_0^t \sigma_s^2 ds. \tag{3.24}$$

Incorporating Equations (3.22) and (3.23) in Equation (3.24) gives,

$$\begin{aligned} \mathcal{Z}_t = & \sum_{t_{i+1} \leq t} 2 \int_{t_i}^{t_{i+1}} \left(X_s - X_{t_i}\right) dX_s + \sum_{t_{i+1} \leq t} \int_{t_i}^{t_{i+1}} \sigma_s^2 ds \\ & + \int_{t_*}^{t} 2\left(X_s - X_{t_*}\right) dX_s \\ & + \int_{t_*}^{t} \sigma_s^2 ds - \int_0^t \sigma_s^2 ds, \end{aligned}$$

where

$$\sum_{t_{i+1} \leq T} \int_{t_i}^{t_{i+1}} \sigma_s^2 ds = \int_0^{t_{i+1}} \sigma_s^2 ds,$$

and

$$\int_0^{t_{i+1}} \sigma_s^2 ds + \int_{t_*}^{t} \sigma_s^2 ds = \int_0^t \sigma_s^2 ds.$$

Hence,

$$\mathcal{Z}_t = \sum_{t_{i+1} \le t} 2 \int_{t_i}^{t_{i+1}} (X_s - X_{t_i}) dX_s + \int_{t_*}^{t} 2(X_s - X_{t_*}) dX_s \,.$$

Since

$$\sum_{t_{i+1} \le t} 2 \int_{t_i}^{t_{i+1}} (X_s - X_{t_i}) dX_s = \int_0^{t_{i+1}} 2(X_s - X_{t_i}) dX_s,$$

and

$$\int_0^{t_{i+1}} 2(X_s - X_{t_i}) dX_s + \int_{t_*}^{t} 2(X_s - X_{t_*}) dX_s = \int_0^{t} 2(X_s - X_{t_i}) dX_s.$$

Then

$$\mathcal{Z}_t = \int_0^{t} 2(X_s - X_{t_i}) dX_s,$$

and the differential form of $\mathcal{Z}_t$ is

$$d\mathcal{Z}_t = 2(X_t - X_{t_i}) dX_t.$$

Accordingly,

$$d[\mathcal{Z}, \mathcal{Z}]_t = 4(X_t - X_{t_i})^2 d[X, X]_t. \tag{3.25}$$

If we apply Itô’s formula to $f(X_t) = (X_t - X_{t_i})^4$, then we get

$$
\begin{aligned}
\left(X_t - X_{t_i}\right)^4 = f(X_0) + \int_0^t 4\left(X_s - X_{t_i}\right)^3 dX_s \\
+ \int_0^t \frac{1}{2} 12\left(X_s - X_{t_i}\right)^2 d[X,X]_s \\
= f(X_0) + \int_0^t 4\left(X_s - X_{t_i}\right)^3 dX_s + \int_0^t \frac{3}{2} d[\mathcal{Z},\mathcal{Z}]_s.
\end{aligned}
\tag{3.26}
$$

Following Mykland and Zhang [91], let's define the realized quarticity of the true price process, $[X,X,X,X]_t$, relative to grid $\mathcal{H}$, as

$$
[X,X,X,X]_t^{\mathcal{H}} := \sum_{t_{i+1} \le t} \left(X_{t_{i+1}} - X_{t_i}\right)^4 + \left(X_t - X_{t_*}\right)^4. \tag{3.27}
$$

By Itô's formula, Equation (3.27) can be written as

$$
\begin{aligned}
d[X,X,X,X]_t^{\mathcal{H}} = \sum_{t_{i+1} \le t} 4\left(X_{t_{i+1}} - X_{t_i}\right)^3 dX_t \\
+ \sum_{t_{i+1} \le t} 6\left(X_{t_{i+1}} - X_{t_i}\right)^2 d[X,X]_t + 4\left(X_t - X_{t_*}\right)^3 dX_t \\
+ 6\left(X_t - X_{t_*}\right)^2 d[X,X]_t.
\end{aligned}
\tag{3.28}
$$

Now, since

$$
\sum_{t_{i+1} \le t} 4\left(X_{t_{i+1}} - X_{t_i}\right)^3 dX_t = \sum_{t_{i+1} \le t} 4 \int_{t_i}^{t_{i+1}} \left(X_s - X_{t_i}\right)^3 dX_s,
$$

and for $t_{i+1} \neq t_*$

$$
4\left(X_t - X_{t_i}\right)^3 dX_t = 4\left(\sum_{t_{i+1} \le t} \int_{t_i}^{t_{i+1}} \left(X_s - X_{t_i}\right)^3 dX_s + \left(X_t - X_{t_*}\right)^3 dX_t\right),
$$

and

$$6\left(X_t - X_{t_i}\right)^2 d[X,X]_t = 6\left(\sum_{t_{i+1}\le t}\int_{t_i}^{t_{i+1}}\left(X_s - X_{t_i}\right)^3 d[X,X]_s + \left(X_t - X_{t_*}\right)^2 d[X,X]_t\right),$$

and by Equation (3.25), Equation (3.28) evolves to

$$\begin{aligned} d[X,X,X,X]_t^{\mathcal{H}} &= 4\left(X_t - X_{t_i}\right)^3 dX_t + 6\left(X_t - X_{t_i}\right)^2 d[X,X]_t \\ &= 4\left(X_t - X_{t_i}\right)^3 dX_t + \frac{3}{2}\, d[\mathcal{Z},\mathcal{Z}]_t. \end{aligned}$$

By defining $\mathcal{J}_t$ as $\sum_{t_{i+1}\le t}\left(X_t - X_{t_i}\right)^3 dX_t$, we get

$$[X,X,X,X]_t^{\mathcal{H}} = 4\mathcal{J}_t + \frac{3}{2}\,[\mathcal{Z},\mathcal{Z}]_t. \tag{3.29}$$

Mykland and Zhang [91] then examine the quadratic variation of the estimation error process, $\mathcal{Z}_t$, written as

$$[\mathcal{Z},\mathcal{Z}]_t = 4\sum_{t_{i+1}\le t}\int_{t_i}^{t_{i+1}}\left(X_s - X_{t_i}\right)^2 d[X,X]_s + 4\int_{t_*}^{t}\left(X_s - X_{t_*}\right)^2 d[X,X]_s, \tag{3.30}$$

to show that ${}^2\!/_3\,(h-1)[X,X,X,X]_t^{\mathcal{H}}$ is a consistent estimate of $(h-1)[\mathcal{Z},\mathcal{Z}]_t$. This result is also stated by Barndorff-Nielsen and Shephard [24] for the approximation of the quadratic variation of the estimation error of RV where returns are sampled equidistantly in time. Mykland and Zhang [91] reach the same result under TTS and prove that $\mathcal{J}_t$ converges to 0 in probability at an order of $(h-1)^{-1}$.

Proposition 3.1. (Proposition 2.17, p.138 in [91]) If the true log price of the security is pure diffusion and the stochastic process σ_t which drives the return volatility is bounded, for a sequence of grids $\mathcal{H}_h = \{0 = t_0 < t_1 < \cdots < t_{h-1} = \mathrm{T}\}$, if as $h \to \infty$, $\Delta(\mathcal{H}_h) = o_p(1)$ and $\sum_{i=0}^{h-1}(t_{i+1}-t_i)^3 = O_p((h-1)^{-2})$, then

$$\sup_{0\le t\le \mathrm{T}}\left|[\mathcal{Z},\mathcal{Z}]_t - {}^2\!/_3\,[X,X,X,X]_t^{\mathcal{H}_h}\right| = o_p((h-1)^{-1}). \tag{3.31}$$

Proof of this proposition is provided in Appendix A.

At this point, we make additional assumptions as in [91].

Assumption 3.5. The observation times t_i in grid $\mathcal{H}_h$ are independent of the true price process X_t.

Assumption 3.6. Over small intervals, the following approximation holds

$$\left(X_t - X_{t_*}\right)^2 \approx [X,X]_t - [X,X]_{t_*}.$$

Under these assumptions, the quadratic variation of estimation error of the RV in Equation (3.30) changes to an approximation as follows:

$$\begin{aligned}[\mathcal{Z},\mathcal{Z}]_t \approx 4 \sum_{t_{i+1} \leq t} \int_{t_i}^{t_{i+1}} \left([X,X]_s - [X,X]_{t_i}\right) d[X,X]_s \\ + 4 \int_{t_*}^{t} \left([X,X]_t - [X,X]_{t_*}\right) d[X,X]_s.\end{aligned}$$

Assumption 3.7. The instantaneous true return variance, σ_t^2, is continuous in mean square, which means that

$$\sup_{0 \leq t-s \leq \alpha} \mathbb{E}(\sigma_t^2 - \sigma_s^2)^2 \to 0 \text{ as } \alpha \to 0.$$

Assumption 3.8. The maximum distance between two consecutive observation times in grid $\mathcal{H}_h$ converges to 0 in probability at an order of $(h-1)^{-1/2}$ as $h \to \infty$.

Assumption 3.9. $\sum_i (t_{i+1} - t_i)^3 = O_p((h-1)^{-2})$.

Assumption 3.10. Asymptotic Quadratic Variation of Time (AQVT) calculated as

$$\mathcal{D}_t := \lim_{h \to \infty} \frac{h-1}{T} \sum_{t_{i+1} \leq t} (t_{i+1} - t_i)^2,$$

and denoted by $\mathcal{D}_t$ exists.

Under Assumptions 3.5 through 3.10, Mykland and Zhang [91] show that

$$[\mathcal{Z},\mathcal{Z}]_t = 2\sum_{t_{i+1}\le t}\left([X,X]_{t_{i+1}} - [X,X]_{t_i}\right)^2 + 2\left([X,X]_t - [X,X]_{t_*}\right)^2 + o_p((h-1)^{-1}). \tag{3.32}$$

Since

$$[X,X]_{t_{i+1}} - [X,X]_{t_i} = \int_{t_i}^{t_{i+1}} \sigma_s^2 ds,$$

$$[X,X]_t - [X,X]_{t_*} = \int_{t_*}^{t} \sigma_s^2 ds.$$

Equation (3.32) can be rewritten as

$$\begin{aligned}
[\mathcal{Z},\mathcal{Z}]_t &= 2\sum_{t_{i+1}\le t}\left(\int_{t_i}^{t_{i+1}} \sigma_s^2 ds\right)^2 + 2\left(\int_{t_*}^{t} \sigma_s^2 ds\right)^2 + o_p((h-1)^{-1}) \\
&= 2\sum_{t_{i+1}\le t}\left((t_{i+1}-t_i)\sigma_{t_i}^2\right)^2 + 2\left((t-t_*)\sigma_{t_*}^2\right)^2 + o_p((h-1)^{-1}) \\
&= 2\sum_{t_{i+1}\le t}(t_{i+1}-t_i)^2\sigma_{t_i}^4 + 2(t-t_*)^2\sigma_{t_*}^4 + o_p((h-1)^{-1}) \\
&= 2\frac{T}{h-1}\int_0^t \sigma_s^4\, d\mathcal{D}_s + o_p((h-1)^{-1}).
\end{aligned}$$

Therefore, as mentioned by Zhang et al. [111] and expressed in Proposition 2.23 in Chapter 2, p.147 of [91], for a fixed period of time $[0,T]$,

$$(h-1)[\mathcal{Z},\mathcal{Z}]_T \xrightarrow{\mathbb{P}} 2T\int_0^t \sigma_s^4\, d\mathcal{D}_s.$$

Subsection 3.3.2.B: Step 2

The next step would be to examine the Central Limit Theorem (CLT) for continuous local martingales by Mykland and Zhang [91]. Before doing so, following Mykland and Zhang [91], it may be helpful to provide some definitions and concepts from probability theory and statistics.

We require a filtration $(\mathfrak{F}_t)$ to which all relevant processes such as the true price process X_t and the instantaneous return volatility σ_t in our setting are adapted. We take the estimation error sequence $\mathcal{Z}_t^h$, which is the error at time t when the total number of data points in the grid is h, as being measurable with respect to a σ-field $\mathfrak{X}$, where $\mathfrak{F}_T \subseteq \mathfrak{X}$.

Now, if $\mathcal{Z}_t^h$ is a sequence of $\mathfrak{X}$-measurable random variables with $\mathfrak{F}_T \subseteq \mathfrak{X}$ and for a $\mathcal{Z}_t$ that is adapted to an extention of $\mathfrak{X}$ such that for all $\mathcal{U} \in \mathfrak{F}_T$ and for all bounded continuous function g, $\mathbb{E}\left[\mathrm{I}_{\mathcal{U}}\mathrm{g}\left(\mathcal{Z}_t^h\right)\right] \to \mathbb{E}[\mathrm{I}_{\mathcal{U}}\mathrm{g}(\mathcal{Z}_t)]$ as $h \to \infty$, then $\mathcal{Z}_t^h$ converges $\mathfrak{F}_T$-stably in law to $\mathcal{Z}_t$ as $h \to \infty$.

We adopt the notation in [91] and stable convergence is denoted by $\overset{\mathcal{L}}{\to}$.

As the number of observations increases, the number of data points in grid $\mathcal{H}$ also increases and, at each frequency, there will be an estimation error. These estimation errors will construct a sequence and estimation errors are continuous martingales. We are interested in such sequences of continuous martingales converging to a limit. Continuous martingales can be interpreted as random variables taking values in the set $\mathbb{C}$ of continuous functions with domain $[0, T]$ and range $(-\infty, \infty)$. For this context, a function g is called a continuous function $\mathbb{C} \to \mathbb{R}$ if $\sup_{0 \le t \le \mathrm{T}} |x_h(t) - x(t)| \to 0$ implies $\mathrm{g}(x_n) \to \mathrm{g}(x)$.

Mykland and Zhang [91] note that the stable convergence of a sequence of random variables $\mathcal{Z}_t^h \overset{\mathcal{L}}{\to} \mathcal{Z}_t$ also implies $\mathcal{Z}_T^h \overset{\mathcal{L}}{\to} \mathcal{Z}_T$ as a random variable.

Limits and the quadratic variation are interchangeable when a sequence of continuous local martingales stably converge to a process (Proposition 2.27, p.151 of [91]).

These definitions and concepts pave the way for CLT for continuous local martingales (Theorem 2.28, p. 152 of [91][7])

Theorem 3.2. If we assume that

- Brownian motions $B_t^1, \cdots, B_t^k$, for some k, generating the filtration $(\mathfrak{F}_t)$ exist,
- $\left(\mathcal{Z}_t^h\right)_{0 \le t \le T}$ is a sequence of continuous local martingales that starts at 0 and is measurable with respect to $(\mathfrak{F}_t)$ for any $t \in [0.T]$,
- A process $\wp_t$ exists such that for any $t \in [0.T]$, $[\mathcal{Z}^h, \mathcal{Z}^h]_t \overset{\mathbb{P}}{\to} \int_0^t \wp_s^2\, ds$,

[7] *The proof of this theorem is summarized in page 152 of [91], but is omitted here as it is beyond the scope of this Thesis.*

- For each $i = 1, \cdots, k$, the quadratic covariation of B_t^k and $\mathcal{Z}_t^h$ converge to 0 in probability,

then, on $[0, T]$, $(\mathcal{Z}_t^h)$ stably converges to a martingale $\mathcal{Z}_t$ that is measurable with respect to filtration $(\mathfrak{F}_t{}^*)$, an extension of $(\mathfrak{F}_t)$. Additionally, a Wiener process B_t^* exists where $(B_t^1, \cdots, B_t^k, B_t^*)$ are all Wiener processes adapted to $(\mathfrak{F}_t{}^*)$. Finally,

$$\mathcal{Z}_t = \int_0^t \mathscr{p}_s \, dB_s^*.$$

A very intriguing application of this CLT for the sequence of local martingales relates to the quadratic variation of the estimation error which comes from observing processes in discrete time while the processes are assumed to be continuous in time. Mykland and Zhang [81, [90], [91] work on this application and prove the following theorem:

Theorem 3.3. When the estimation error process, $(\mathcal{Z}_t^h)_{0 \le t \le T} = [X, X]_t^{\mathcal{H}} - IV_t$, is scaled by the square root of the total number of sampling intervals in the grid $\mathcal{H}$, for fixed $t \in [0, T]$,

$$\sqrt{h-1}\mathcal{Z}_t^h \xrightarrow{\mathcal{L}} N\left(0{,}2T \int_0^t \sigma_s^4 d\mathcal{D}_s\right),$$

under conditions of Theorem 3.2 and Assumptions 3.3-3.5, 3.7-3.10 where AQVT is absolutely continuous.

Subsection 3.3.2.C: Step 3

From the beginning of Subsection 3.3.2 up to this point, the CLT and mixed normality of the estimation error due to discretization under TTS (where observations are randomly scattered in time) are built on the assumption that the true security prices are pure diffusions with no drift observed. This assumption as given in Assumption 3.3 is made by benefiting from the idea of changing measures in asset pricing literature such as Ross [98] or Harrison and Pliska [63] etc., where the studies suggest that discounted asset prices are martingales under a risk-neutral measure that is equivalent to the actual risky probability measure. Mykland and Zhang [90], [91] show that the consistency and the rate of convergence of the RV estimator calculated using grid $\mathcal{H}$ is not affected by the change of measure due to the time period $[0, T]$ being finite. More specifically, Mykland and Zhang [90] deduce that asymptotic variance of estimation error $\mathcal{Z}_t^h$ keeps unaltered even if we change the measure. As a simplification strategy for inference, Mykland and Zhang [90] rewrite dX_t under a probability measure $\mathbb{P}^*$, equivalent to the measure $\mathbb{P}$, where under the measure $\mathbb{P}$ the true prices are observed with the drift and under the measure $\mathbb{P}^*$, the drift disappears, i.e.,

$$dX_t = \sigma_t dB_t^*, X_0 = x_0,$$

where B_t^* is a Brownian motion measurable with respect to measure $\mathbb{P}^*$. In such a case, by Girsanov's Theorem (see Theorem 2.37 in Chapter 2, p.159 of [91]),

$$\log \frac{d\mathbb{P}^*}{d\mathbb{P}} = -\int_0^T \frac{\mu_t}{\sigma_t} dB_t^* - \frac{1}{2}\int_0^T \left(\frac{\mu_t}{\sigma_t}\right)^2 dt,$$

with $dB_t^* = dB_t + \frac{\mu_t}{\sigma_t} dt$.

Mykland and Zhang [90] ask the question of excluding or including the drift in the true price process by offering to carry out the analysis under a risk neutral measure $\mathbb{P}^*$ so that the drift disappears and adjust results back to $\mathbb{P}$ using the likelihood ratio $\frac{d\mathbb{P}^*}{d\mathbb{P}}$. Recall that by the Radon-Nikodym Theorem (Theorem 2.35 in Chapter 2, p.158 of [91]), if the probability measure $\mathbb{P}^*$ is absolutely continuous under $\mathbb{P}$ on a σ-field $\mathcal{A}$, then a random variable $\frac{d\mathbb{P}^*}{d\mathbb{P}}$, adapted to $\mathcal{A}$, exists such that for all events $A \in \mathcal{A}$,

$$\mathbb{P}^*(A) = \mathbb{E}_{\mathbb{P}}\left(\frac{d\mathbb{P}^*}{d\mathbb{P}} \mathbb{I}_A\right).$$

Mykland and Zhang [90], [91] give the example of $\int_0^T \sigma_t^2 dt$ or $\int_0^T \sigma_t^4 dt$ as quantities to be estimated and propose to find an estimator for a quantity by working under $\mathbb{P}^*$ such that an asymptotic convergence in law is found under $\mathbb{P}^*$, then switch to $\mathbb{P}$ relying on measure theoretic equivalence of $\mathbb{P}$ and $\mathbb{P}^*$. Such equivalence of measures ensures that convergence under risk neutral measure $\mathbb{P}^*$ would also hold under the risky measure $\mathbb{P}$. For instance if $\sqrt{h-1}\mathcal{Z}_t^h \xrightarrow{\mathcal{L}} N(a,b)$ under $\mathbb{P}^*$, $\sqrt{h-1}\mathcal{Z}_t^h$ also converges in law under $\mathbb{P}$.

Accordingly, following Mykland and Zhang [91], Aït-Sahalia et al. [3] and many other studies in the lierature, we assume that the true price process, which is continuous in time but can only be observed at discrete times, is observed with no drift.

Similarly, Mykland and Zhang [91] discuss that it is possible to weaken Assumption 3.4 on instantaneous true return volatility. In fact, Theorem 3.3 is shown to be holding even if condition $\sigma_t^2 \leq a^2$ for all $t \in [0,T]$ is substituted with Assumption 3.11 as given below:

Assumption 3.11. The instantaneous return volatility, σ_t, is locally bounded so that for a sequence of stopping times τ_h and a constant $\sigma_{h,t}$, $\mathbb{P}[\tau_h < T] \to 0$ as $h \to \infty$ and $\sigma_t^2 \leq \sigma_{h,t}^2$ for all $t \in [0, \tau_h]$.

All in all, under the null hypothesis H_0 in Equation (3.11), for all $t \in [0, T]$, for contaminated observed prices $Y_t = X_t + \varepsilon_t$, the total estimation error $[Y, Y]_t^{\mathcal{H}} - IV_t$ is equal to the discretization error, $[X, X]_t^{\mathcal{H}} - IV_t$. We assume that we observe Y_t on two different grids (meaning two different frequencies), $\mathcal{H}$ and $\mathcal{G}$, both tending to identity, where $|\mathcal{H}| = h$ and $|\mathcal{G}| = n$, then

$$[Y, Y]_T^{\mathcal{H}} - IV_T = O_p((h-1)^{-1}),$$

$$[Y, Y]_T^{\mathcal{G}} - IV_T = O_p((n-1)^{-1}).$$

As shown in Section 3.3.1, if $\frac{h}{n} \to \infty$ as $h, n \to \infty$, then $\sqrt{h-1}\left([Y, Y]_T^{\mathcal{G}} - IV_T\right) = o_p(1)$.

By this token, $\left(\sqrt{h-1}\left([Y, Y]_T^{\mathcal{G}} - IV_T\right) - \sqrt{h-1}\left([Y, Y]_T^{\mathcal{H}} - IV_T\right)\right)$ is driven by $\sqrt{h-1}\left([Y, Y]_T^{\mathcal{H}} - IV_T\right)$.

Following discussions and proofs in [90], [91], we apply Theorem 3.3 to the estimation error of the RV under conditions and assumptions mentioned in Theorem 3.3 but assume that the true prices have no drift at irregularly spaced times (on grid $\mathcal{H}$ or $\mathcal{G}$) and the process σ_t^2 is locally bounded on $[0, T]$. So, we know that the amount that the RV over observed prices deviates from the IV of the true price converges in law to a mixed normal distribution asymptotically. The asymptotic distributions of the estimation errors on grid $\mathcal{H}$ and grid $\mathcal{G}$ have 0 mean and ${2T}/{h-1} \int_0^T \sigma_s^4 d\mathcal{D}_s$ and ${2T}/{n-1} \int_0^T \sigma_s^4 d\mathcal{D}_s$ variance, respectively. For instance, $\sqrt{h-1}\left([Y, Y]_T^{\mathcal{H}} - IV_T\right) \xrightarrow{\mathcal{L}} N\left(0{,}2T \int_0^T \sigma_s^4 d\mathcal{D}_s\right)$ and

$$\frac{\left(\sqrt{h-1}\left([Y, Y]_T^{\mathcal{G}} - IV_T\right) - \sqrt{h-1}\left([Y, Y]_T^{\mathcal{H}} - IV_T\right)\right)}{\sqrt{2T \int_0^T \sigma_s^4 d\mathcal{D}_s}} \xrightarrow{\mathcal{L}} N(0{,}1). \tag{3.33}$$

In order to estimate the RHS of the convergence in (3.33), we use Proposition 3.1 such that for any grid $\mathcal{H}$, where grid tends to identity, ${2}/{3}\,(h-1)[X, X, X, X]_T^{\mathcal{H}}$ consistently estimates variance of $(h-1)\left([X, X]_T^{\mathcal{H}} - IV_T\right)$ and under the null hypothesis in Equation (3.11), $[X, X]_T^{\mathcal{H}} - IV_T$ is equal to $[Y, Y]_T^{\mathcal{H}} - IV_T$. Hence,

$$\frac{\left(\sqrt{h-1}\left([Y,Y]_T^{\mathcal{G}} - [Y,Y]_T^{\mathcal{H}}\right)\right)}{\sqrt{2/_3\,(h-1)\sum_{i=0}^{h-2}\left(Y_{t_{i+1}} - Y_{t_i}\right)^4}} \xrightarrow{\mathcal{L}} N(0,1).$$

Under H_a in (3.12) when the microstructure noise statistically affects the RV measure, $Z_{T,n,h}$ will diverge because $[Y,Y]_T^{\mathcal{G}}$ will explode at a larger rate compared to $[Y,Y]_T^{\mathcal{H}}$. Remember that as we increase the sampling frequency, $[Y,Y]_T$ on any grid starts to be dominated by the quadratic variation of the microstructure noise. Similar supporting arguments are also presented in Lemma 2 in [61] and Lemma 1 and Proposition 1 in [111]. These studies calculate the total estimation error that is due to both of discretization and MMN. Their results agree that under slightly different conditions compared to conditions of Theorem 3.3, when there is MMN satisfying Assumptions 3.1 and 3.2 and it is related to observed prices through Equation (3.4) on grid $\mathcal{G}$, the asymptotic distribution of the estimation error is mixed normal with mean $2(n-1)\mathbb{E}[\epsilon^2]$ and variance is commanded by $4(n-1)\mathbb{E}[\epsilon^4]$.

In detail, when MMN is independently distributed with finite fourth moment almost surely such that for all $t \in [0,T]$, i.e., $\mathbb{E}[|\Delta\epsilon_t|^4] < \infty$ a.s., and Assumptions 3.2, 3.3, 3.5-3.11 hold, then under alternative hypothesis defined in (3.12), following the discussions by Awartani et al. [16], the test statistic, $Z_{T,n,h}$ diverges if for $\psi \in (0,1)$

$$\frac{(n-h)^{\psi}}{(h-1)^{0.5-\psi}} \to \infty$$

holds. The reasoning is explained below:

1) Under the alternative hypothesis, from Equation (3.6), we can expand the numerator and denominator of the test statistic $Z_{T,n,h}$ as

$$\frac{\left(\sqrt{h-1}\left([X,X]_T^{\mathcal{G}} - [X,X]_T^{\mathcal{H}} + [\epsilon,\epsilon]_T^{\mathcal{G}} - [\epsilon,\epsilon]_T^{\mathcal{H}} + 2[\epsilon,X]_T^{\mathcal{G}} - 2[X,\epsilon]_T^{\mathcal{H}}\right)\right)}{\sqrt{\frac{2(h-1)\left([X,X,X,X]_T^{\mathcal{H}} + [\epsilon,\epsilon,\epsilon,\epsilon]_T^{\mathcal{H}} + A\right)}{3}}},$$

where

$$[X,X,X,X]_T^{\mathcal{H}} := \sum_{i=0}^{h-2}\left(\Delta X_{t_i}\right)^4,$$

$$[\epsilon,\epsilon,\epsilon,\epsilon]_T^{\mathcal{H}} := \sum_{i=0}^{h-2} \left(\Delta\epsilon_{t_i}\right)^4,$$

$$A := \sum_{i=0}^{h-2} 6\left(\Delta X_{t_i}\right)^2\left(\Delta\epsilon_{t_i}\right)^2 + \sum_{i=0}^{h-2} 4\Delta X_{t_i}\left(\Delta\epsilon_{t_i}\right)^3 + \sum_{i=0}^{h-2} 4\left(\Delta X_{t_i}\right)^3\Delta\epsilon_{t_i}.$$

2) As justified in Sections 3.1. and 3.2, under both CTS and TTS, $[X,X]_T$ is stochastically bounded where $[X,\varepsilon]_T < [\varepsilon,\varepsilon]_T$ causing $[Y,Y]_T$ on any grid to be dominated by $[\varepsilon,\varepsilon]_T$ asymptotically. Therefore, regarding the numerator of $Z_{T,n,h}$, $\sqrt{h-1}\left([Y,Y]_T^{\mathcal{G}} - [Y,Y]_T^{\mathcal{H}}\right)$, under the alternative hypothesis, the asymptotic commanding term is $[\epsilon,\epsilon]_T^{\mathcal{G}} - [\epsilon,\epsilon]_T^{\mathcal{H}}$.

3) When a random variable has finite absolute moments of order k, it has absolute moments of orders $1,2,\dots,k-1$ (Chapter 4, Section 21, p. 292 of [34]). By this token, $\mathbb{E}[|\Delta\epsilon_t|^4] < \infty$ implies $\mathbb{E}[|\Delta\epsilon_t|^3] < \infty$, $\mathbb{E}[|\Delta\epsilon_t|^2] < \infty$ and $\mathbb{E}[|\Delta\epsilon_t|] < \infty$. By Markov's Inequality, which states that for any nonnegative random variable x,

$$\mathbb{P}[|x| \geq \gamma] \leq \frac{1}{\gamma^k}\mathbb{E}[|x|^k], \forall\, \gamma \in (0,\infty),$$

$\mathbb{E}[|\Delta\epsilon_t|^4] < \infty$ means stochastic boundedness of $(\Delta\epsilon_t)^4$, $(\Delta\epsilon_t)^3$, $(\Delta\epsilon_t)^2$ and $(\Delta\epsilon_t)^4$, i.e. for any $\lambda_1,\lambda_2,\lambda_3,\lambda_4 > 0$, there exist finite and nonnegative bounds $\mathrm{M}_{(\Delta\epsilon_t)^4}, \mathrm{M}_{(\Delta\epsilon_t)^3}, \mathrm{M}_{(\Delta\epsilon_t)^2}$ and $\mathrm{M}_{(\Delta\epsilon_t)^1}$ such that

$$\mathbb{P}\left[|\Delta\epsilon_t|^1 > \mathrm{M}_{(\Delta\epsilon_t)^1}\right] < \lambda_1, \forall t \in [0,T],$$

$$\mathbb{P}\left[|\Delta\epsilon_t|^2 > \mathrm{M}_{(\Delta\epsilon_t)^2}\right] < \lambda_2, \forall t \in [0,T],$$

$$\mathbb{P}\left[|\Delta\epsilon_t|^3 > \mathrm{M}_{(\Delta\epsilon_t)^3}\right] < \lambda_3, \forall t \in [0,T],$$

$$\mathbb{P}\left[|\Delta\epsilon_t|^4 > \mathrm{M}_{(\Delta\epsilon_t)^4}\right] < \lambda_4, \forall t \in [0,T].$$

Hence, supremum of $[\epsilon,\epsilon]_T^{\mathcal{H}}$, supremum of $[\epsilon,\epsilon,\epsilon,\epsilon]_T^{\mathcal{H}}$ and supremum of $\left[\sum_{i=0}^{h-2}\left(\Delta X_{t_i}\right)^2\left(\Delta\epsilon_{t_i}\right)^2\right]$ should be bounded with

$$(h-1)\mathrm{M}_{(\Delta\epsilon_t)^2},$$

$$(h-1)\mathrm{M}_{(\Delta\epsilon_t)^4},$$

and

$$(h-1)\sup_{t_i\in[0,T]}\left(\int_{t_i}^{t_{i+1}}\sigma_s^2 ds\right)\mathrm{M}_{(\Delta\epsilon_t)^2},$$

respectively.

Moreover, from Appendix C, we conclude that $\mathbb{E}\big[[X,X,X,X]_T^{\mathcal{H}}\big]$, $\mathbb{E}\left[\sum_{i=0}^{h-2}\left(\Delta X_{t_i}\right)^3\Delta\epsilon_{t_i}\right]$ and $\mathbb{E}\left[\sum_{i=0}^{h-2}\left(\Delta\epsilon_{t_i}\right)^3\Delta X_{t_i}\right]$ are all stochastically bounded even if the sequence of MMN is not identically distributed, which by Markov's Inequality causes $[X,X,X,X]_T^{\mathcal{H}}$, $\sum_{i=0}^{h-2}\left(\Delta X_{t_i}\right)^3\Delta\epsilon_{t_i}$ and $\sum_{i=0}^{h-2}\left(\Delta\epsilon_{t_i}\right)^3\Delta X_{t_i}$ to be also $O_p(1)$.

Hence, for $Z_{T,n,h}$ to diverge to plus infinity where $\frac{h}{n}\to\infty$ as $h,n\to\infty$,

$$\frac{(n-h)}{\sqrt{(h-1)}}\frac{\mathrm{M}_{(\Delta\epsilon_t)^2}}{\sqrt{O_p(1)+\mathrm{M}_{(\Delta\epsilon_t)^4}+\sup_{t_i\in[0,T]}\left(\int_{t_i}^{t_{i+1}}\sigma_s^2 ds\right)\mathrm{M}_{(\Delta\epsilon_t)^2}}}, \tag{3.34}$$

must diverge. Considering the fact that the second term on the RHS of (3.34) is finite but can take values in the close neighborhood of 0 depending on $\mathrm{M}_{(\Delta\epsilon_t)^2}, \mathrm{M}_{(\Delta\epsilon_t)^4}$ and $\sup_{t_i\in[0,T]}\left(\int_{t_i}^{t_{i+1}}\sigma_s^2 ds\right)$, for (3.34) to go to infinity, the first term must reach infinity at a suitable rate, which is satisfied when

$$\frac{(n-h)^{\psi}}{(h-1)^{0.5-\psi}}\to\infty$$

for $\psi\in(0,1)$.

Thus, we completed the discussion regarding the test statistic $Z_{T,n,h}$, developed by Awartani et al. [16] to be used for equidistant observations and we demonstrated that $Z_{T,n,h}$ works also for irregularly spaced observations under conditions and assumptions of Theorem 3.3. This postulation holds even if we release Assumption 3.4 such that the instantaneous return volatility is only locally bounded.

3.4. Testing the constant variance of MMN increments under TTS

As it is explained in detail in Chapter 2, many of the robust IV estimators in the literature, such as kernel based estimators or estimators built on subsampling, depend on the validity of the assumption that the MMN has constant variance through time. Likewise, handling the MMN in the estimation of the true price's IV may be carried out via an "optimal sampling" of the returns following Bandi and Russell [19]. However, this method also relies on the assumption that the MMN has constant variance. Therefore, the failure to reject the model with the noise having constant variance would be quite interesting if one plans to employ aforementioned estimators/methods that take the variance of the MMN constant independent of the sampling frequency. To this end, Awartani et al. [16] suggest to test the following set of null and alternative hypotheses:

$$H_0\colon \mathcal{g}_{t,\mathcal{G}} = \mathcal{g}_{t,\mathcal{H}}, \tag{3.35}$$

$$H_a\colon \mathcal{g}_{t,\mathcal{G}} \neq \mathcal{g}_{t,\mathcal{H}}, \tag{3.36}$$

where $\mathcal{g}_{t,\mathcal{G}}$ and $\mathcal{g}_{t,\mathcal{H}}$ denote the variance of MMN increments on grids $\mathcal{H}$ and $\mathcal{G}$, respectively, and are defined as

$$\mathbb{E}\left[\left(\varepsilon_{t_{i+1}} - \varepsilon_{t_i}\right)^2\right] = 2\mathcal{g}_{t,\mathcal{G}}, \forall t_{i+1}, t_i \in \mathcal{G},$$

$$\mathbb{E}\left[\left(\varepsilon_{t_{i+1}} - \varepsilon_{t_i}\right)^2\right] = 2\mathcal{g}_{t,\mathcal{H}}, \forall t_{i+1}, t_i \in \mathcal{H}.$$

The null hypothesis in (3.35) reflects the constant variance of MMN increments through time and the alternative hypothesis in (3.36) is consistent with presence of autocorrelation in MMN.

For the purpose of testing whether the MMN increments have constant variance independent of the sampling frequency, Awartani et al. [16] advocate a test statistic $V_{T,n,h}$, which combines RVs calculated at 3 different sampling frequencies (on 3 different grids), one low, one high and one very very low compared to each other as well as the Realized Quarticity (RQ) calculated at low frequency. This test statistic is defined as follows:

$$V_{T,n,h,l} := \sqrt{h-1}\frac{\frac{\left(RV_{T,n}-RV_{T,l}\right)}{2(n-1)}-\frac{\left(RV_{T,h}-RV_{T,l}\right)}{2(h-1)}}{\sqrt{3\left(\frac{RQ_{T,h}}{2(h-1)^2}-\left(\frac{RV_{T,h}}{2(h-1)}\right)^2\right)}}, \tag{3.37}$$

where *n*, *h* and *l* stand for the number of observations for the whole estimation period , T (for instance the number of observations per day), at high frequency, low frequency and very low frequency, respectively and as before

$$RV_{T,n} = \sum_{i=0}^{n-2}\left(Y_{t_{i+1}}-Y_{t_i}\right)^2,$$

$$RV_{T,h} = \sum_{i=0}^{h-2}\left(Y_{t_{i+1}}-Y_{t_i}\right)^2,$$

$$RQ_{T,h} = \frac{2}{3}\sqrt{h-1}\sum_{i=0}^{h-2}\left(Y_{t_{i+1}}-Y_{t_i}\right)^4,$$

$$n > h > l, \qquad \frac{n}{h} \to \infty, \frac{h}{l} \to \infty \; as \; n, h, l \to \infty.$$

The very low frequency that is included in the test statistic $V_{T,n,h,l}$ represents a frequency at which the consensus in the literature makes it possible to ignore MMN. In this context, Awartani et al. [16] suggest that *l* might be chosen at 20 minutes sampling interval under CTS by referring to sparse sampling literature and the 5-minutes threshold promoted by Andersen et al. [11].

3.4.1. Deriving a statistic to test the constant variance of MMN increments under CTS

Similar to the test statistic $Z_{T,n,h}$, the test statistic $V_{T,n,h,l}$ has a standard normal distribution asymptotically and its asymptotic distribution is built on three main pillars:

1) Awartani et al. [16] demonstrate that under the null hypothesis in (3.35) $\frac{\left(RV_{T,n}-RV_{T,l}\right)}{2n-2}$ and $\frac{\left(RV_{T,h}-RV_{T,l}\right)}{2h-2}$ converge to the same limit in probability, say the limit corresponds to $\mathcal{g}_t$, but the former term converges faster compared to the latter. Thus, asymptotically, it is possible to ignore the term $\frac{(RV_{T,n}-RV_{T,l})}{2n-2}$ and, in limit, as the number of observations at each frequency goes to infinity, Equation (3.37) boils down to

$$\sqrt{h-1}\frac{\mathcal{g}_t-\dfrac{(RV_{T,h}-RV_{T,l})}{2h-2}}{\sqrt{3\left(\dfrac{RQ_{T,h}}{2(h-1)^2}-\left(\dfrac{RV_{T,h}}{2(h-1)}\right)^2\right)}}.$$

2) Awartani et al. [16] next illustrate that if a) the true price is generated as in Equation (3.1), b) $\int_0^T \sigma_s^4\, ds < \infty$, c) MMN satisfies Assumption 3.1, then under the null hypothesis in (3.35), as $n, h, l \to \infty$, $\frac{n}{h} \to \infty, \frac{h}{l} \to \infty$, $V_{T,n,h,l} \xrightarrow{d} N(0,1)$. They also show that when (a) and (b) hold and MMN increments have finite fourth moment, under the alternative hypothesis in (3.36), for any $\lambda > 0$

$$\lim_{n,h,l\to\infty} \mathbb{P}\left[\frac{1}{\sqrt{h-1}}\left|V_{T,n,h,l}\right| > \lambda\right] = 1.$$

In order to prove the asymptotic distribution of $V_{T,n,h,l}$ under the null hypothesis in (3.35), Awartani et al. [16] employ a central limit theorem for the estimation error of RV with respect to IV under TTS, provided in Theorem A1 in [111]. The CLT proposed by Zhang et al. [111] is applicable to both CTS and TTS, making this CLT a breakthrough in IV estimation.

Theorem 3.4. (Theorem A1 in [111]): Let's assume that the true and observed prices are generated as in Equation (3.1) and (3.4), respectively. The observation times are represented by the grid $\mathcal{G}$, tending to identity, where $\mathcal{G} = \{t_0, t_1, \cdots, t_{n-1}\}, t_0 = 0, t_{n-1} = T$, $|\mathcal{G}| = n$ such that transactions take place irregularly in time. Then, under Assumptions 3.1 and 3.2, as the number of observations in the grid $\mathcal{G}$ diverges to infinity, conditional on the true price process

$$\sqrt{n-1}\left(\widehat{\mathbb{E}[\epsilon^2]} - \mathbb{E}[\epsilon^2]\right) \xrightarrow{d} N(0, \mathbb{E}[\epsilon^4]),$$

$$\frac{1}{\sqrt{n-1}}\left([Y,Y]_T^{\mathcal{G}} - [X,X]_T^{\mathcal{G}} - 2(n-1)\mathbb{E}[\epsilon^2]\right) \xrightarrow{d} N(0, 4\mathbb{E}[\epsilon^4]).$$

Explanation of the proof of this theorem is provided here and proof in full version is given in Appendix B.

In summary, the proof is made in two steps: first it is shown that $[X,\epsilon]_T^{\mathcal{G}}$ and $[X,X]_T^{\mathcal{G}}$ are both stochastically bounded (Lemma A.2 in [111]) so that a restatement of Equation (3.6) leads to

$$\begin{aligned}[Y,Y]_T^{\mathcal{G}} &= [X,X]_T^{\mathcal{G}} + 2[X,\epsilon]_T^{\mathcal{G}} + [\epsilon,\epsilon]_T^{\mathcal{G}} \\ &= O_p(1) + [\epsilon,\epsilon]_T^{\mathcal{G}}.\end{aligned}$$

Additionally, we know from Section 3.2 and Equation (3.9) that the asymptotic bias conditional on the true price is $2(n-1)\mathbb{E}[\epsilon^2]$. Hence, the expected estimation error conditional on the true price, $\mathbb{E}\big[[Y,Y]_T^{\mathcal{G}} - [X,X]_T^{\mathcal{G}}\big|X\big] = 2\mathbb{E}\big[[X,\varepsilon]_T^{\mathcal{G}}\big|X\big] + \mathbb{E}\big[[\varepsilon,\varepsilon]_T^{\mathcal{G}}\big|X\big]$ is driven by $\mathbb{E}\big[[\varepsilon,\varepsilon]_T^{\mathcal{G}}\big|X\big]$ since the expected value of a figure that is stochastically bounded cannot diverge while $[\varepsilon,\varepsilon]_T^{\mathcal{G}}$ inflates very erratically as the number of observations goes to infinity.

Therefore, the second step includes investigating the convergence and distribution of $\frac{1}{\sqrt{n-1}}\big([\epsilon,\epsilon]_T^{\mathcal{G}} - \mathbb{E}\big[[\epsilon,\epsilon]_T^{\mathcal{G}}\big|X\big]\big)$. By showing that $\frac{1}{\sqrt{n-1}}\big([\epsilon,\epsilon]_T^{\mathcal{G}} - \mathbb{E}\big[[\epsilon,\epsilon]_T^{\mathcal{G}}\big|X\big]\big)$ can be written as the sum of two independent and asymptotically mixed normal terms and a term that converges to 0 in probability, the proof is complete. However, we differ from Zhang et al. [111] in this step such that while they use CLT for martingale differences given in Theorem 3.2 and Condition 3.1 in Chapter 3, p.58 of [57], we employ Law of Large Numbers (LLN) and CLT for triangular arrays laid out in Theorem 27.2 in Chapter 5, Section 27, p. 352 of [34]. This Theorem 2.27 in [34] is also known as the Lindeberg Theorem and states that if for each n and k, the sequence $x_{1,n},\cdots,x_{k,n}$ represents independently distributed random variables with 0 means and finite variances, then

$$\frac{\sum_{i=1}^{k} x_{i,n}}{\sqrt{\sum_{i=1}^{k}\mathrm{Var}\big[x_{i,n}\big]}} \xrightarrow{d} N(0,1),$$

provided that Lindeberg's condition is satisfied, i.e., for each sequence $x_{1,n},\cdots,x_{k,n}$ for any $\delta > 0$,

$$\frac{1}{\sum_{i=1}^{k}\mathrm{Var}\big[x_{i,n}\big]}\sum_{i=1}^{k}\mathbb{E}\left[x_{i,n}^2\mathbb{I}_{\left\{|x_{i,n}|\geq\delta\sqrt{\sum_{i=1}^{k}\mathrm{Var}[x_{i,n}]}\right\}}\right],$$

converge to 0 as $n\to\infty$.

Concurrently, Theorem 3.2 and Condition 3.1 in Chapter 3, p.58 of [57] assert that if for each n and k, the sequence $S_{1,n}, \cdots, S_{k,n}$ represents a 0 mean, square integrable martingale array with respect to filtration $\mathcal{F}_{k,n}$ with differences $x_{1,n}, \cdots, x_{k,n}$ such that for $\mathfrak{y} < \infty$ a.s. and $\delta > 0$

$$\sum_{i=1}^{k} x_{i,n}^2 \xrightarrow{\mathbb{P}} \mathfrak{y},$$

$$\mathcal{F}_{k,n} \subseteq \mathcal{F}_{k,n+1},$$

$$\sum_{i=1}^{k} \mathbb{E}\left[x_{k,n}^2 \mathbb{I}_{\{|x_{i,n}| \geq \delta\}} \middle| \mathcal{F}_{k-1,n}\right] \to 0,$$

then

$$\sum_{i=1}^{k} x_{i,n} \xrightarrow{\mathcal{L}} Z,$$

where random variable Z has characteristic function $\mathbb{E}\left[\exp\left(-\frac{1}{2}\mathfrak{y}t^2\right)\right]$.

3) Zhang et al. [111] state but do not prove that

$$\widehat{\mathbb{E}[\epsilon^4]} := \frac{1}{2(n-1)}[Y, Y, Y, Y]_T^{\mathcal{G}} - 3\left(\widehat{\mathbb{E}[\epsilon^2]}\right)^2,$$

is a consistent estimator of $\mathbb{E}[\epsilon^4]$ where

$$\widehat{\mathbb{E}[\epsilon^2]} := \frac{1}{2(n-1)}[Y, Y]_T^{\mathcal{G}}.$$

The proof of this postulation is provided in Appendix C.

As the third pillar, Awartani et al. [16] then exploit the estimator with respect to $\mathbb{E}[\epsilon^4]$ suggested by Zhang et al. [111] and propose that

$$\sqrt{h-1}\frac{\mathcal{g}_t - \frac{\left(RV_{T,h} - RV_{T,l}\right)}{2h-2}}{\sqrt{3\left(\frac{RQ_{T,h}}{2(h-1)^2} - \left(\frac{RV_{T,h}}{2(h-1)}\right)^2\right)}} \xrightarrow{d} N(0,1).$$

Accordingly, Awartani et al. [16] comment that if we reject the null hypothesis, we can conclude that the MMN increments do not have constant variance independent of grids (sampling frequencies) so that it is possible to reject at the same time that MMN has an i.i.d structure with constant variance independent of time.

3.4.2. Deriving a statistic to test the constant variance of MMN increments under TTS

We suggest that the same statistic $V_{T,n,h,l}$ can be employed to examine the constant variance of the MMN increment under TTS since three pillars which Awartani et al. [16] built their test statistic on are developed under TTS in the first place. In detail, Proposition 1 in [111] suggests that under similar but slightly different conditions of Theorem 3.3 (true price and observed prices satisfy Equations (3.1) and (3.4), respectively, $|\mu_t|$ and σ_t are bounded above by a constant, Assumptions 3.1, 3.2, 3.4-3.6, 3.8-3.10 hold but the law of MMN is allowed to depend on the number of observations) where grid $\mathcal{G}$ represents observation/transaction times that are scattered irregularly in time, $|\mathcal{G}| = n$, $\mathcal{G}$ tends to identity, and $\mathcal{U}_{total}$ is a quantity that is asymptotically standard normal, conditional on true price

$$\begin{aligned}
[Y,Y]_T^{\mathcal{G}} - IV_T \xrightarrow{\mathcal{L}} & 2(n-1)\mathbb{E}[\epsilon^2] \\
& + \Bigg(4(n-1)\mathbb{E}[\epsilon^4] + 8[X,X]_T^{\mathcal{G}}\mathbb{E}[\epsilon^2] - 2\mathrm{var}[\epsilon^2] \\
& + \frac{2T}{n-1}\int_0^T \sigma_s^4 \, d\mathcal{D}_s\Bigg)\mathcal{U}_{total} + O_p\left((n-1)^{-\frac{1}{4}}(\mathbb{E}[\epsilon^2])^{-\frac{1}{2}}\right) \\
& + o_p\left((n-1)^{-\frac{1}{2}}\right).
\end{aligned} \qquad (3.38)$$

Meanwhile, Hansen and Lunde [61] advocate that again under similar but slightly different conditions of Theorem 3.3 (true price and observed prices satisfy Equations (3.1) and (3.4), respectively, σ_t is a random function that is independent of Brownian motion B_t, $\left|\sigma_t^2 - \sigma_{t+\theta}^2\right| \leq \theta\varepsilon$ for some ε and all t and θ with probability one, Assumption 3.3 holds, $\epsilon_{t_i} \perp \epsilon_{t_j}, t_i \neq t_j$, $\mathbb{E}[\epsilon_t] = 0$, $\mathbb{E}[|\epsilon_t|^2] < \infty$, and $\mathbb{E}[|\epsilon_t|^4] < \infty$, for all t) where grid $\mathcal{G}$ represents observation times that are scattered irregularly in time, $|\mathcal{G}| = n$, $\mathcal{G}$ tends to identity and $\mathcal{U}_{total}$ is a quantity that is asymptotically standard normal

$$
\begin{aligned}
[Y,Y]_T^{\mathcal{G}} - IV_T \xrightarrow{d} & \, 2(n-1)\mathbb{E}[\epsilon^2] \\
& + \left(4(n-1)\mathbb{E}[\epsilon^4] + 8\mathbb{E}[\epsilon^2] \sum_{i=0}^{n-1} \sigma_{t_i}^2 - 2\text{var}[\epsilon^2] \right. \\
& \left. + 2(\mathbb{E}[\epsilon^2])^2 + 2 \sum_{i=0}^{n-1} \sigma_{t_i}^4 \right) \mathcal{U}_{total}.
\end{aligned} \tag{3.39}
$$

Results in [61] and [111] differ depending on the assumptions regarding

- the form of the true price (whether or not the true price has a bounded drift, or whether the instantaneous return volatility σ_t is bounded above or Lipschitz almost surely),
- the MMN (whether it is identically distributed or not).

However, we would like to underline the fact that both of the convergences in (3.38) and (3.39) tell us that as the number of observations tend to infinity, conditional on true price

$$
\frac{\left([Y,Y]_T^{\mathcal{G}} - IV_T - 2(n-1)\mathbb{E}[\epsilon^2]\right)}{\sqrt{4(n-1)\mathbb{E}[\epsilon^4]}} \xrightarrow{d} N(0,1),
$$

since the scaled asymptotic variance of the estimation error, $\frac{\left([Y,Y]_T^{\mathcal{G}} - IV_T\right)}{\sqrt{(n-1)}}$, is completely determined by the estimation error due to noise as the discretization error divided by the number of sampling intervals diminishes to 0.

In this context, we, by following methods offered in proofs of Lemma 1 and Proposition 1 in [111] as well as Lemma 2 in [61], also prove in Appendix D that the estimation error of the RV with respect to the IV is asymptotically mixed normal.

Theorem 3.5. Once we assume that we observe the contaminated prices as in Equation (3.4) on a grid $= \{t_0, t_1, \cdots, t_{n-1}\}, t_0 = 0, t_{n-1} = T,\ |\mathcal{G}| = n,\ \mathcal{G}$ tends to identity, conditions of Theorem 3.2 are satisfied, the Assumptions 3.1, 3.2, 3.3, 3.5-3.11 hold and AQVT is absolutely continuous,

$$
\begin{aligned}
[Y,Y]_T^{\mathcal{G}} - IV_T \xrightarrow{d} & \, 2(n-1)\mathbb{E}[\epsilon^2] \\
& + \left(4(n-2)\mathbb{E}[\epsilon^4] + 8\mathbb{E}[\epsilon^2] \left([X,X]_T^{\mathcal{G}} - O_p(1) \right) \right. \\
& \left. + \frac{2T}{n-1} \int_0^T \sigma_s^4 d\mathcal{D}_s \right) \mathcal{U}_{total},
\end{aligned} \tag{3.40}
$$

where the convergence in law is conditional on the true price and $\mathcal{U}_{total}$ is a quantity that is asymptotically standard normal.

Hence,

$$[Y,Y]_T^{\mathcal{G}} - IV_T - 2(n-1)\mathbb{E}[\epsilon^2] \xrightarrow{d} N\left(0, 4(n-2)\mathbb{E}[\epsilon^4] + 8\mathbb{E}[\epsilon^2]\left([X,X]_T^{\mathcal{G}} - O_p(1)\right) + \frac{2T}{n-1}\int_0^T \sigma_s^4 d\mathcal{D}_s\right)$$
$$\Rightarrow \frac{[Y,Y]_T^{\mathcal{G}} - IV_T - 2(n-1)\mathbb{E}[\epsilon^2]}{2\sqrt{n-1}} \xrightarrow{d} N(0, \mathbb{E}[\epsilon^4]),$$

where $\frac{IV_T}{2\sqrt{(n-1)}} = o_p(1)$, $\frac{8\mathbb{E}[\epsilon^2]\left([X,X]_T^{\mathcal{G}} - O_p(1)\right)}{4(n-1)} = o_p(1)$ and $\frac{2T}{4(n-1)^2}\int_0^T \sigma_s^4 d\mathcal{D}_s = o_p(1)$ as $n \to \infty$.

Therefore, $\mathcal{g}_t$ equaling to $\mathbb{E}[\epsilon^2]$, we have

$$\frac{[Y,Y]_T^{\mathcal{G}}}{2(n-1)} - \mathcal{g}_t \xrightarrow{d} N\left(0, \frac{\mathbb{E}[\epsilon^4]}{(n-1)}\right)$$
$$\Rightarrow \sqrt{n-1}\left(\frac{[Y,Y]_T^{\mathcal{G}}}{2(n-1)} - \mathcal{g}_t\right) \xrightarrow{d} N(0, \mathbb{E}[\epsilon^4]). \tag{3.41}$$

As stated before, inspired by Zhang et al. [111], we have proved in Appendix C that

$$\widehat{\mathbb{E}[\epsilon^4]} := \frac{1}{2(n-1)}[Y,Y,Y,Y]_T^{\mathcal{G}} - 3\left(\frac{1}{2(n-1)}[Y,Y]_T^{\mathcal{G}}\right)^2,$$

is a consistent estimator of $\mathbb{E}[\epsilon^4]$. Therefore, we can estimate the asymptotic distribution in (3.41) as

$$\frac{\sqrt{n-1}\left(\frac{[Y,Y]_T^{\mathcal{G}}}{2(n-1)} - \mathcal{g}_t\right)}{\frac{1}{2(n-1)}[Y,Y,Y,Y]_T^{\mathcal{G}} - 3\left(\frac{1}{2(n-1)}[Y,Y]_T^{\mathcal{G}}\right)^2} \xrightarrow{d} N(0,1).$$

In order to benefit from the above convergence in law, we will manipulate the test statistic $V_{T,n,h,l}$ as defined in Equation (3.37) such that for the observations of contaminated prices on three different grids, $\mathcal{G}, \mathcal{H}$ and $\mathcal{W}$, where all the grids tend to identity, $|\mathcal{G}| = n$ and $|\mathcal{H}| = h$ and $|\mathcal{W}| = l, n > h > l,\ \frac{n}{h} \to \infty, \frac{h}{l} \to \infty$ as $n, h, l \to \infty$ if conditions of Theorem 3.2 are satisfied, the Assumptions 3.1, 3.2, 3.3, 3.5-3.11 hold and AQVT is absolutely continuous, then

$$\begin{aligned} V_{T,n,h,l} &= \sqrt{h-1}\frac{\frac{\left([Y,Y]_T^{\mathcal{G}} - [Y,Y]_T^{\mathcal{W}}\right)}{2(n-1)} - \frac{\left([Y,Y]_T^{\mathcal{H}} - [Y,Y]_T^{\mathcal{W}}\right)}{2(h-1)}}{\sqrt{\frac{[Y,Y,Y,Y]_T^{\mathcal{H}}}{2(h-1)} - 3\left(\frac{[Y,Y]_T^{\mathcal{H}}}{2(h-1)}\right)^2}} \\ &= \sqrt{h-1}\frac{\frac{[Y,Y]_T^{\mathcal{G}}}{2(n-1)} - o_p(1) - \frac{[Y,Y]_T^{\mathcal{H}}}{2(h-1)} + o_p(1)}{\sqrt{\frac{[Y,Y,Y,Y]_T^{\mathcal{H}}}{2(h-1)} - 3\left(\frac{[Y,Y]_T^{\mathcal{H}}}{2(h-1)}\right)^2}}. \end{aligned} \tag{3.42}$$

Adding and subtracting $\mathscr{g}_t$ to and from numerator of Equation (3.42) produces

$$V_{T,n,h,l} = \sqrt{h-1}\frac{\left(\frac{[Y,Y]_T^{\mathcal{G}}}{2(n-1)} - \mathscr{g}_t\right) - \left(\frac{[Y,Y]_T^{\mathcal{H}}}{2(h-1)} - \mathscr{g}_t\right) + o_p(1)}{\sqrt{\frac{[Y,Y,Y,Y]_T^{\mathcal{H}}}{2(h-1)} - 3\left(\frac{[Y,Y]_T^{\mathcal{H}}}{2(h-1)}\right)^2}}. \tag{3.43}$$

Since convergence in (3.41) as $\sqrt{n-1}\left(\frac{[Y,Y]_T^{\mathcal{G}}}{2(n-1)} - \mathscr{g}_t\right) \overset{d}{\to} N(0, \mathbb{E}[\epsilon^4])$ entails that $\sqrt{n-1}\left(\frac{[Y,Y]_T^{\mathcal{G}}}{2(n-1)} - \mathscr{g}_t\right)$ is stochastically bounded, Equation (3.43) develops into

$$V_{T,n,h,l} = \sqrt{h-1}\frac{-\left(\frac{[Y,Y]_T^{\mathcal{H}}}{2(h-1)} - \mathscr{g}_t\right)}{\sqrt{\frac{[Y,Y,Y,Y]_T^{\mathcal{H}}}{2(h-1)} - 3\left(\frac{[Y,Y]_T^{\mathcal{H}}}{2(h-1)}\right)^2}} + o_p(1),$$

which was proved to converge to standard normal distribution asymptotically.

Under the alternative hypothesis in (3.36), when we observe the contaminated prices as in Equation (3.4) on grids $\mathcal{G} = \{t_0, t_1, \cdots, t_{n-1}\}$, $|\mathcal{G}| = n$, $\mathcal{H} = \{t_0, t_1, \cdots, t_{h-1}\}$, $|\mathcal{H}| = h, \mathcal{W} = \{t_0, t_1, \cdots, t_{l-1}\}$, $|\mathcal{W}| = l$, $t_0 = 0, t_{n-1} = t_{h-1} = t_{l-1} = T$, all grids tending to identity, conditions of Theorem 3.2 are satisfied, the Assumptions 3.2, 3.3, 3.5-3.11 hold and AQVT is absolutely continuous, then if the MMN is independently distributed with finite fourth moment almost surely for all $t \in [0, T]$, then following the discussions by Awartani et al. [16], we claim that the absolute value of the test statistic, $V_{T,n,h,l}$ diverges. The reasoning is explained below:

Under the alternative hypothesis, we can organize the test statistic $V_{T,n,h,l}$ in (3.43) as

$$\frac{\sqrt{h-1}\left(\frac{[Y,Y]_T^{\mathcal{G}}}{2(n-1)} - \mathscr{g}_{t,\mathcal{G}} - \frac{[Y,Y]_T^{\mathcal{H}}}{2(h-1)} + \mathscr{g}_{t,\mathcal{H}}\right)}{\sqrt{\frac{[Y,Y,Y,Y]_T^{\mathcal{H}}}{2(h-1)} - 3\left(\frac{[Y,Y]_T^{\mathcal{H}}}{2(h-1)}\right)^2}} + \frac{\sqrt{h-1}\left(\mathscr{g}_{t,\mathcal{G}} - \mathscr{g}_{t,\mathcal{H}}\right)}{\sqrt{\frac{[Y,Y,Y,Y]_T^{\mathcal{H}}}{2(h-1)} - 3\left(\frac{[Y,Y]_T^{\mathcal{H}}}{2(h-1)}\right)^2}}, \tag{3.44}$$

where the first term on the RHS of (3.44) is asymptotically normal because although we broke the i.i.d structure of the MMN by assuming that the MMN increments have different variances on different grids, since we accept that all other conditions of Theorem 3.5 are satisfied and on a single grid the MMN increments have constant variance, the first term on the RHS of (3.44) still has a mixed normal limiting distribution. Furthermore, since we assumed that the MMN has finite fourth moment, the denominator of (3.44), being a consistent estimate of the fourth moment of the MMN, is finite, therefore as $h \to \infty$, the second term on the RHS of (3.44) diverges.

This completes the discussion that $V_{T,n,h,l}$ can be used under TTS to test whether MMN increments have constant variance independent of sampling frequency (grid) and, therefore, is also orthogonal to time.

CHAPTER 4

EMPIRICAL EVIDENCE FROM THE BORSA ISTANBUL NATIONAL STOCK MARKET

As discussed in Chapter 2, in organized stock exchanges, the observed stock prices are subject to distortions due to reasons such as trading rules and regulations, frictions in the markets and informational asymmetries. The aggregate effect of such distortions in observed prices is called the MMN and may cause the observed prices to deviate from the true prices of assets. In Chapter 3, it is shown that if the MMN is additive, then as the sampling frequency for observed prices increases, the bias of the RV explodes while estimating the IV of the true prices. In theory, sampling the observed prices as many times as possible should yield an IV estimate with a smaller total estimation error since the discretization error is expected to diminish with more frequent sampling. However, in the presence of the MMN, the quadratic variation of the observed prices calculated over the highest frequency possible does not simply converge to the IV of the true prices. The existence of such a bias needs to be addressed in the empirical estimation of the IV. There are several methods developed in the literature for mitigating the effect of the MMN on the IV estimate and the success of such methods should be evaluated based on empirical evidence obtained from both developed and developing markets.

We believe that such empirical evidence on the MMN structure should be collected by taking into account the dimensions/aspects of volatility estimation using high frequency data. Such an effort should address issues like the characteristics of UHFD, contamination of the observed prices with the MMN, specification options regarding an asset's true and unobservable price, the need make some assumptions in order to handle the unobservable MMN, the possibility of calculating returns using different sampling schemes, and, finally, the existence of non-trading hours. The existence of such issues may threaten the validity of methods developed to handle the MMN in the estimation of the IV. Typically, studies in the related literature take into account one or a few of these issues while estimating the IV without addressing all of the potential issues simultaneously. In addition, most of the studies use UHFD obtained from developed economies such as the US or Japan. Accordingly, one of the main objectives of the analyses conducted in this Chapter is to take into account all relevant MMN issues simultaneously while using UHFD from an emerging market.

The liquidity of traded assets is an important issue that is discussed in the finance literature and there are many liquidity definitions and measures that find support in different studies. For instance, a widely accepted definition by Black [35] describes a liquid asset as an asset which can be sold in a short period of time for a price not too different from the price at which the seller would be able to sell if s/he opted to wait longer. Interestingly, with respect to the high frequency finance literature, it is seen dealing with an asset's liquidity is somewhat problematic in the sense that many of the liquidity indicators/measures fall short when it comes to addressing the existence or the statistical properties of MMN embedded in the observed stock prices, especially if such measures are calculated under different sampling schemes such as CTS. This postulation is accentuated especially when there is a relatively long time lag between two consecutive transactions. As explained in Chapter 2, Section 2.1, in such a case of infrequent trading, the previous tick method is typically used to construct artificial return series, but this, in turn, means that returns are calculated by using pieces of information that belong to distant points in time leading to inflated serial correlations due to long sequences of zero returns [37]. Hence, the previous tick method may work best in IV estimation for very actively traded stocks since we would not want to spur such correlation structures by artificially introducing additional autocorrelation (serial correlation) due to the interpolation method selected. These arguments pave the way for the introduction of a new method to classify stocks with respect to their liquidity (active trading) in a high frequency setting.

In this Chapter, we first suggest a new liquidity measure for UHFD. Next, we apply a grid of data cleaning methods and different sampling schemes, to a sample of 6 stocks that are listed on Borsa Istanbul's National Equity Market. By applying this methodology, we are able to observe the common characteristics of the UHFD at hand and to understand the dynamics of the return and RV series obtained from these data. We also generate volatility signature plots and run formal tests of the existence of MMN and the constant variance of MMN increments, as suggested by Awartani et al. [16]. By moving across the grid, we also note any significant changes that result from the liquidity levels of the sample stocks.

In accordance with proofs and discussions in Chapter 3, all analyses in this Chapter are carried out under the following assumptions:

- The observed price Y_t is contaminated with an additive market microstructure noise. i.e.,

$$Y_t = X_t + \varepsilon_t, \qquad 0 \leq t \leq T$$

  where T shows a finite horizon, X_t denotes the logarithm of the true/efficient price of a security at time t and ε_t represents the logarithm of the combined effect of all microstructure noise sources at time t.

- Conditions and assumptions of Theorem 3.2 hold,

- The true price of the security is observed with no drift and it is a local martingale such that if S_t denotes the true price process of a security, the log price of this security is represented by the process X_t and it satisfies the following stochastic differential equation over a finite time horizon $t \in [0, T]$:

$$dX_t = \sigma_t dB_t$$

  The log price X_t is assumed to belong to Brownian semimartingale family so that X_0 is $\mathcal{F}_0$-measurable, X_t has continuous sample paths with no drift, the continuous stochastic process σ_t that derives the volatility of log return of the security is locally bounded and continuous in mean square, B_t denotes standard Wiener process, and finally σ_t is orthogonal to B_t.

- The observation times are independent of the true price process X_t, and the maximum distance between two consecutive observation times converges to 0 in probability at an order of

$$(number\ of\ sampling\ intervals - 1)^{-\frac{1}{2}}$$

  as the number of observations tend to infinity.

- For any two consecutive observation times t_{i+1} and t_i, $\sum_i (t_{i+1} - t_i)^3 = O_p((number\ of\ sampling\ intervals - 1)^{-2})$.

- The Asymptotic Quadratic Variation of Time (AQVT) is calculated as

$$\mathcal{D}_t := \lim_{h \to \infty} \frac{h-1}{T} \sum_{t_{i+1} \le t} (t_{i+1} - t_i)^2$$

  $\mathcal{D}_t$ exists and is absolutely continuous.

4.1. Data

The original data set consists of tick by tick transaction data for all BIST 30 Index constituent stocks between January 1, 2010 and December 31, 2014. The data are obtained directly from Borsa Istanbul. When the five year period between 2010 and 2014 is analyzed, it is seen that there are some important political/economic events that may have an influence on the price formation process in Borsa Istanbul:

- 12/09/2010: Constitutional referendum,
- 12/06/2011: General elections,
- 2012: Relatively stable period, closing sessions begin taking place on 02.03.2012, therefore our sample period cannot start before 02.03.2012. We choose 01.07.2012-31.12.2012 as the sample period.
- 2010-2012: Arabian Spring and its effects on Turkey (Dispute between Turkish and Syrian Prime Ministers)
- 2013:Gezi Park Events (June 2013) and corruption allegations about high level government officials and 4 ministers (17-25 December 2013)
- 2014: Local elections on 30/03/2014 and Presidency elections on 10/08/2014

In order to avoid the possibility of these events confounding the price formation process, we have chosen the six-month horizon between July 1 and December 31, 2012 as the sample period. There are a total of 124 trading days in our sample period. The distribution of these days over the sample months is as follows:

Table 4.1: The Number of Trading Days in the Sample Period by Month

July	22
August	20
September	20
October	19
November	22
December	21

In addition, due to the computational burden of our methodology, 6 of the BIST-30 Index constituents are included in the final sample. These stocks are AKBNK (Akbank T.A.Ş. – commercial bank), MIGRS (Migros Ticaret A.Ş. - chain of supermarkets), GARAN (Türkiye Garanti Bankası A.Ş. – commercial bank), ISCTR (Türkiye İş Bankası A.Ş. – commercial bank), NETAS (Netaş Telekomünikasyon A.Ş. - telecommunications) and ARCLK (Arçelik A.Ş. - home appliances). The selection of these particular stocks is random to some extent but the inclusion of financial institutions, manufacturing and telecommunications companies as well as a fast moving consumer products seller is on purpose in order to introduce an acceptable level of diversification to our data set.

Table 4.2 reports descriptive statistics regarding each stock in our sample set. All descriptive statistics in the table reflect transactions data prior to any data handling except for summarizing the opening and closing sessions. The transaction data for the morning and afternoon opening sessions and the afternoon closing session are included in the set as 3 singular transaction entries at 09:49:59, 14:19:59, and 17:30:01, respectively. Log returns and durations are calculated between consecutive transactions starting with the second transaction in each session. Minimum - maximum returns, average – maximum durations and respective standard deviations are computed over the full sample period. None of the log return series in Table 4.2 exhibit normality as revealed by the corresponding Jarque-Bera statistics and skewness-kurtosis figures.

Table 4.2: Descriptive statistics for each stock in the sample set

Descriptive Statistics	**ISCTR**	**GARAN**	**NETAS**	**AKBNK**	**MIGRS**	**ARCLK**
Number of Transactions	589,900	500,347	496,666	316,321	197,044	112,654
Daily Average Number of Transactions	4,757	4,035	4,038	2,551	1,589	909
Min Log Return	-0.0081	-0.0072	-0.0244	-0.0084	-0.0104	-0.0326
Max Log Return	0.0309	0.0084	0.0300	0.0089	0.0103	0.0452
St Dev of Log Returns	0.0015	0.0011	0.0017	0.0012	0.0010	0.0017
Skewness of Log Returns	0.0166	-0.0022	0.0172	0.0009	0.0042	0.2261
Kurtosis of Log Returns	6.1313	5.4295	7.6303	5.3192	7.4163	18.5629
Jarque-Bera Statistic of Log Returns	238,488	122,560	440,826	70,641	159,645	1,124,430
Average Duration Between Transactions (seconds)	4.4609	5.2247	5.2312	8.2558	13.2202	23.3322
Max Duration Between Transactions (seconds)	556	905	518	963	1,217	1,469
St Dev of Durations (seconds)	13.2647	16.1346	15.0965	25.4349	41.8501	59.1598

4.2. A New Measure of Liquidity in a High Frequency Setting

Liquidity in economics and finance literature has several definitions depending on the context and purpose. For instance, macroeconomic liquidity refers to the monetary base controlled by a central bank through its monetary tools such as open market operations or reserve requirements. Alternatively, an asset's liquidity refers to the ease and speed of selling the asset without triggering drastic changes in its price, and, accounting liquidity refers to the ability of a company to fulfill its financial commitments with the liquid assets on hand. Among the different definitions the most relevant to our attempt for uncovering the MMN structure using individual stock data is the concept of "an asset's liquidity". Although there is no single definition of an asset's liquidity, a widely accepted definition by Black [35] describes a liquid asset as an asset which can be sold in a short period of time for a price not too different from the price at which seller would be able to sell if s/he opted to wait longer. Sarr and Lybek [100] add on to this definition and argue that liquid financial assets are identified by low transaction costs, easy trading, prompt settlement, and, a limited effect of large trades on the asset's price. Sarr and Lybek [100] also draw attention to the possible changes in the perception of investors with respect to liquidity due to time and economic developments:

> *"…during periods of stability, the perception of an asset's liquidity may primarily reflect transaction costs. During period of stress and significantly changing fundamentals, prompt price discovery and adjustment to a new equilibrium becomes much more important.[p.5]"*

Sarr and Lybek [100] distinguish between an asset's liquidity and the liquidity of a financial market and state that liquid markets possess five characteristics. These characteristics are <u>tightness</u>, referring to low transaction costs, <u>immediacy</u>, representing the speed at which orders can be executed, <u>depth</u>, showing the existence of abundant orders at below or above the current transaction price, <u>breadth</u>, pointing to the volume and number of orders at each price tick (lower and higher compared to the current price) so that large orders in either direction have a minimal impact on price, and, finally, <u>resiliency</u>, referring to new orders arriving immediately to correct order imbalances [100].

In line with the aforementioned diverse understanding of what constitutes an asset's liquidity, a closer look at the literature on the liquidity of financial assets reveals that several measures/indicators are proposed to gauge the liquidity of stocks. For instance, transaction cost measures focus on costs of trading assets and frictions in secondary markets, volume-based measures assess breadth and depth by looking at the volume of transactions while controlling for price volatility, equilibrium-price based measures identify orderly movements towards equilibrium prices and are used as indicators of resiliency, and market-impact measures try to distinguish between price changes stemming from liquidity such that they are used to comment on resiliency and the speed of price discovery [100].

Unfortunately, none of these definitions/indicators/measures are usable when analyzing the existence of MMN, especially when such analyses are carried out under different sampling schemes. Recall from the discussions in Chapter 2, Sections 2.1 and 2.5 that due to trading data being observed at discrete and irregularly spaced intervals, it is possible to calculate returns from UHFD under different sampling schemes such as CTS (sampling prices every 10 minutes or 5 seconds or 2 hours etc.), TkTS (sampling prices whenever there is a price change), TTS (sampling prices whenever there is a transaction) and BTS (sampling prices so that the integrated volatility (IV) for all sampling periods throughout a day is constant). The most popular one in literature amongst these sampling schemes is the CTS. However, as explained in Section 2.1, the asynchronous nature of trading in stock exchanges makes it necessary to artificially construct a time series where all time stamps of interest (for instance every five seconds) become attached to a transaction price. The liquidity measures proposed in the literature are not designed to accommodate such artificially constructed time series and would result in misleading liquidity assessments if they are used in the high frequency data setting.

When the methods for building the artificial time series under CTS are examined, it is seen that the previous tick method may be more appropriate compared to other linear or nonlinear interpolation methods since methods that employ information which is not available at a particular time may induce spurious correlations [49] and linear interpolation will distort quadratic variation (IV) estimations [61]. On this issue, let’s demonstrate how the previous tick method is implemented. Suppose that a small portion of transaction data for Stock X are given in Table 4.3.

After the application of previous tick method, the artificial calendar sampled time series should look like information reported in Table 4.4 (no quantity or short selling information is provided because the time series is artificially constructed only for prices).

From Table 4.3 and Table 4.4, it is evident that when there is a long time lag between two consecutive transactions, the previous tick method will use a piece of information that belonged to some considerably older time and this may lead to inflated serial correlation due to long sequences of zero returns [37]. Taking into consideration all pros and cons of several interpolation methods, we concluded in Section 2.1 that the previous tick method works best in IV estimation for very liquid and actively traded stocks listed on BIST, as we would not want to spur such correlation structures by artificially introducing additional autocorrelation (serial correlation) due to the interpolation method selected.

Table 4.3: A hypothetical portion of transaction information on a stock

Date (Day/Month/Year)	**Time**	**Ticker**	**Price (TL)**	**Quantity**
02-01-2012	09:50:00	X	8.88	5
02-01-2012	09:50:03	X	8.86	10
02-01-2012	09:50:06	X	8.88	300
02-01-2012	09:50:08	X	8.90	24567
02-01-2012	09:50:11	X	8.92	562

Table 4.4: Previous tick method applied to information on Table 4.3

Date (Day/Month/Year)	**Time**	**Ticker**	**Price (TL)**
02-01-2012	09:50:00	X	8.88
02-01-2012	09:50:01	X	8.88
02-01-2012	09:50:02	X	8.88
02-01-2012	09:50:03	X	8.86
02-01-2012	09:50:04	X	8.86
02-01-2012	09:50:05	X	8.86
02-01-2012	09:50:06	X	8.88
02-01-2012	09:50:07	X	8.88
02-01-2012	09:50:08	X	8.90
02-01-2012	09:50:09	X	8.90
02-01-2012	09:50:10	X	8.90
02-01-2012	09:50:11	X	8.92

This is the point where the frequency of trading in a stock becomes crucial in the MMN analysis under different sampling schemes. The results of visual and/or statistical analyses regarding the existence and statistical features of the MMN should be interpreted keeping in mind that results may be distorted by the artificial correlation induced by the previous tick method. Therefore, any study that attempts to analyze the empirical features of the MMN via using UHFD should take into account the liquidity (how transactions are distributed in time) of the assets under analysis. Unfortunately, not only the aforementioned liquidity measures do not serve properly to distinguish transaction frequencies per each trading day but also the average number of transactions/quotations per session or trading day is insufficient to pinpoint the liquid versus illiquid stocks. We would like to explain this insufficiency as follows: Suppose that over a given trading session we have collected transaction data for Stocks A and B with the same number of transactions taking place in each stock. This would imply that the average number of transactions per session in each stock equals each other. However, if the percentage of simultaneous transactions in Stock A is greater than that of Stock B, then the time gap between transactions in Stock A will be longer compared to Stock B causing the researcher, working under CTS, to fill a greater number of blank time points in the data, and, therefore, increasing serial autocorrelation artificially.

In this context, inspired by the approach adopted by Aït-Sahalia et al. [3], who employ the average duration between two consecutive transactions as an indicator of liquidity, we introduce a new approach to sort stocks with respect to liquidity to be used in UHFD and define the liquidity in a high frequency setting as the number of all time stamps (under calendar time) having at least one transaction entry. In other words, during the sample period, if the number of sessions with no transactions during a given time period, such as 10 minutes, for a stock exceeds the same number for another stock, then the first stock is characterized as being less liquid, which means that this classification of stocks is specific to the sample set at hand. Based on this definition, we analyze the raw data on our sample stocks to find the number of continuous auction sessions for which the maximum duration between two consecutive transactions exceeds a set of arbitrarily selected 300 seconds, 600 seconds, 1200 seconds, 1800 seconds and 3600 seconds (coded in MATLAB). The results are given in Tables 4.5 and 4.6. The findings are in line with what average durations suggest and show that ISCTR, GARAN and NETAS stocks are more liquid compared to the others in the sample. We interpret the liquidity of AKBNK as moderate and classify the ARCLK and MIGRS stocks as illiquid. These classifications are reflected in the Table 4.5 and Table 4.6 with colors, where blue, red and yellow are used to highlight liquid, illiquid and in-between stocks, respectively.

Table 4.5: The number of continuous auction sessions for which the maximum duration between two consecutive transactions exceeds a set of arbitrarily selected thresholds – GARAN, NETAS and ISCTR

	GARAN-Session 1					**NETAS-Session 1**					**ISCTR-Session 1**				
	Thresholds in Seconds					Thresholds in Seconds					Thresholds in Seconds				
Months	300	600	1200	1800	3600	300	600	1200	1800	3600	300	600	1200	1800	3600
July 2012	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0
August 2012	5	0	0	0	0	0	0	0	0	0	1	0	0	0	0
September 2012	6	0	0	0	0	2	0	0	0	0	3	0	0	0	0
October 2012	3	0	0	0	0	2	0	0	0	0	1	0	0	0	0
November 2012	4	0	0	0	0	7	0	0	0	0	1	0	0	0	0
December 2012	7	0	0	0	0	4	0	0	0	0	4	0	0	0	0
	GARAN-Session 2					**NETAS-Session 2**					**ISCTR-Session 2**				
	Thresholds in Seconds					Thresholds in Seconds					Thresholds in Seconds				
Months	300	600	1200	1800	3600	300	600	1200	1800	3600	300	600	1200	1800	3600
July 2012	13	0	0	0	0	10	0	0	0	0	11	0	0	0	0
August 2012	5	1	0	0	0	1	0	0	0	0	1	0	0	0	0
September 2012	3	0	0	0	0	2	0	0	0	0	0	0	0	0	0
October 2012	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0
November 2012	0	0	0	0	0	3	0	0	0	0	0	0	0	0	0
December 2012	4	1	0	0	0	1	0	0	0	0	2	0	0	0	0

Table 4.6: The number of continuous auction sessions for which the maximum duration between two consecutive transactions exceeds a set of arbitrarily selected thresholds – AKBNK, ARCLK and MIGRS

	AKBNK-Session 1					**ARCLK-Session 1**					**MIGRS-Session 1**				
	Thresholds in Seconds					Thresholds in Seconds					Thresholds in Seconds				
Months	300	600	1200	1800	3600	300	600	1200	1800	3600	300	600	1200	1800	3600
July 2012	10	0	0	0	0	19	4	0	0	0	19	6	0	0	0
August 2012	7	0	0	0	0	17	4	0	0	0	15	2	0	0	0
September 2012	19	4	0	0	0	17	7	1	0	0	17	2	0	0	0
October 2012	9	0	0	0	0	13	5	0	0	0	18	4	0	0	0
November 2012	8	0	0	0	0	17	4	0	0	0	22	9	0	0	0
December 2012	13	1	0	0	0	17	4	0	0	0	10	1	0	0	0
	AKBNK-Session 2					**ARCLK-Session 2**					**MIGRS-Session 2**				
	Thresholds in Seconds					Thresholds in Seconds					Thresholds in Seconds				
Months	300	600	1200	1800	3600	300	600	1200	1800	3600	300	600	1200	1800	3600
July 2012	16	0	0	0	0	22	7	0	0	0	21	3	0	0	0
August 2012	7	0	0	0	0	18	4	1	0	0	17	1	0	0	0
September 2012	16	4	0	0	0	17	10	0	0	0	16	4	0	0	0
October 2012	9	0	0	0	0	13	2	0	0	0	18	7	0	0	0
November 2012	8	0	0	0	0	16	5	1	0	0	22	6	0	0	0
December 2012	12	1	0	0	0	17	7	0	0	0	9	0	0	0	0

4.3. Data Handling – A Necessary Step in Analyzing UHFD

4.3.1. Data Handling in the Form of Cleaning and Aggregating

As previously discussed in detail in Chapter 2, Section 2.1, UHFD may include erroneous entries distorting the validity of results coming from any type of analysis conducted with the data set. Moreover, due to recording algorithms of stock exchanges as well as the asynchronous nature of trading, many transactions are attached to a single time stamp. Correspondingly, before commencing with testing the existence of MMN and the validity of popular assumptions on the MMN, we apply data handling methods as combinations of cleaning and aggregation algorithms. In doing so, for each continuous auction session in the sample, the cleaning rules are applied separately. Therefore, an entry from one session is not compared with an entry from the preceding or following sessions.

A comprehensive list of all of the cleaning and aggregation rules that are available in the literature and mentioned in Chapter 2, Section 2.1 are turned into algorithms that are applied to each stock in our sample before carrying out any return calculations under the calendar or transaction time sampling schemes. The summary of these data cleaning and aggregation algorithms in application order is given below:

1) Delete entries with time stamps that lie outside of opening, closing and continuous auction sessions.

2) Delete entries when price or volume is zero or negative.

3) Delete entries when price is not a multiple of the respective price tick.

4) Delete entries that satisfy one of the following criteria of being an outlier:

4.i.) Delete entries if the immediate return (absolute or not – percentage return since last transaction) exceeds an arbitrarily selected threshold (rule proposed by Huang and Stoll [65] and Bessembinder [31] and improved by Chung et al. [39] and later adopted by Bandi et al. [21], Bandi et al. [22] and Pigorsch et al. [95])

4.i.a) Delete the entry if percentage return since last transaction exceeds 10% [65], [21] [22] [95].

4.i.b) Delete the entry if percentage return since last transaction exceeds 25% [31].

4.i.c) Delete the entry if absolute percentage return since last transaction exceeds 50% [39].

4.ii) Delete entries for which the price deviates by more than a threshold from an average of daily or arbitrarily selected neighborhood prices. This approach was first introduced by Zhou [112] and later developed by Dacorogna et al. [45], Falkenberry [50], Brownlees and Gallo [37][8], Verousis and Gwilym [106] and Barndorff-Nielsen et al. [27] among others.

> **4.ii.a)** Delete the entry if the absolute difference between the current price and the 10% trimmed sample mean over a k transaction neighborhood exceeds or equals to 3 (k neighborhood) standard deviations over the same neighborhood plus a granularity parameter λ, which is used to introduce a lower positive bound on price variations accepted as admissible and is equal to a multiple of the relevant tick size for the stock price [37].
>
> **4.ii.b)** Delete the entries for which the price deviates by more than 10 mean absolute deviations from a rolling centered median of 50 observations (25 preceding, 25 following) (proposed by Barndorff-Nielsen et al. [27] and adopted by Koopman and Scharth [74] among others).
>
> **4.ii.c)** Delete the entries for which the price deviates by more than 2.9652 median absolute deviations from a daily median (inspired by Verousis and Gwilym [106]).

4.iii) Delete entries which are bouncebacks/reversals as defined by Aït-Sahalia et al. [3] or Oomen [94] and Bessembinder et al. [32].

> **4.iii.a)** Filter data for instantaneous price reversals in transaction time. For the k^{th} transaction to be removed, (i) the absolute price change from the $k\text{-}1^{th}$ transaction to the k^{th} transaction exceeds a threshold set arbitrarily, and, (ii) the price change from the k^{th} transaction to the $k+1^{th}$ transaction is such that the absolute price reversal is included in the region of –(1-w) and –(1+w) times the price change from the $k\text{-}1^{th}$ transaction to the k^{th} transaction [94].
>
> **4.iii.b)** Delete a particular entry if the log return from one transaction to the next is both greater in magnitude than an arbitrary cutoff and followed immediately by a log return of the same magnitude but of the opposite sign, so that the price returns to its original level before that particular transaction [3].
>
> **4.iii.c)** Eliminate “reversal” transactions, where a given price exceeds both the preceding and following prices by at least 15%, or is less than both prices by the same magnitude [32].

---

[8] This paper of Brownlees and Gallo is cited 136 times, making it quite popular in high frequency literature.

5) If there are multiple entries per second (trades that took place at different or same prices at the same time, i.e. there is more than one entry that have the same time stamp), then aggregate the price and calculate a single price for that time stamp:

5.i) Determine unique prices and aggregate volume. Use the price that has the largest volume.

5.ii) Determine unique prices and aggregate volume. Use the volume weighted average price.

5.iii) Determine unique prices and aggregate volume. Use the logvolume weighted average price.

5.iv) Determine unique prices and aggregate volume. Use the number of trades weighted average price.

5.v) Use the median price.

We implement steps 1, 2 and 3 for the detection of obvious errors, and then select one option under step 4 and one option under step 5 and go through each possible combination of the options. As a result, the number of cleaning and aggregation combinations is 45.

The results from the application of the various options under step 4 are summarized in the Table 4.7.

Table 4.7: The number of deleted entries under each cleaning rule for each stock

	ISCTR	GARAN	NETAS	AKBNK	MIGRS	ARCLK
Number of Data Points In Raw Data Set (Opening and Closing Sessions Summarized as 09:49:59, 14:19:59, and 17:30:01)	589,900	500,347	496,666	316,321	197,044	112,654
Number (Percentage) of Data Points Deleted By Cleaning Rule						
Step 1	0	0	0	0	0	0
Step 2	0	0	0	0	0	0
Step 3	0	0	0	0	0	0
Step 4						
Algorithm 4.i.a	0	0	0	0	0	0
Algorithm 4.i.b	0	0	0	0	0	0
Algorithm 4.i.c	0	0	0	0	0	0
Algorithm 4.ii.a (K=20, L=1)	14 (0.0023%)	22 (0.0044%)	16 (0.0032%)	29 (0.0092%)	27 (0.0137%)	24 (0.0213%)
Algorithm 4.ii.b	54,206 (9.1890%)	14,202 (2.8384%)	16,745 (3.3715%)	14,202 (4.4897%)	9,604 (4.8740%)	2,488 (2.2085%)
Algorithm 4.ii.c	69,804 (11.8332%)	42,466 (8.4873%)	53,330 (10.7376%)	42,466 (13.4250%)	22,035 (11.1828%)	12,306 (10.9237%)
Algorithm 4.iii.a	0	0	0	0	0	0
Algorithm 4.iii.b (c=0.01)	0	0	0	0	0	0
Algorithm 4.iii.c	0	0	0	0	0	0

By examination of Table 4.7, we deduce that only cleaning methods 4.ii.a and 4.ii.b shall be included in the analysis of testing the existence and statistical features of the MMN under different data handling and sampling schemes (recall overscrubbing and underscrubbing risks discussed in Section 2.1.) because:

- 10% price limit per session rule as applied by Borsa Istanbul's for stocks quoted on National Market nullify the error detection by method 4.i.a and 4.i.b and 4.i.c.

- Cleaning method 4.ii.c identifies many entries as errors because the 2.96 MAD (MAD over session) criterion might be too low (MADs are close to 0) and therefore, the errors caught by method 4.ii.c are most probably correct entries that artificially signified as errors. Due to this high probability of artificial success at catching errors, method 4.ii.c is not included in our analysis.

- Cleaning methods 4.iii.a, 4.iii.b and c do not catch significant amount of errors, so application of these rules is practically same as not cleaning the data at all.

4.3.2. Does Data Handling Alter UHFD's Characteristics?

Now that all of the obvious errors, outliers and simultaneous ticks are cleaned and/or aggregated, it is time to discuss what happens to the common characteristics of the UHFD after cleaning and aggregation procedures are applied as we want to make sure that the data handling procedures do not overscrub the UHFD and distort its original characteristics, such as discreteness, irregular temporal spacing, and diurnal patterns.

i) Discreteness: Transaction price changes occurring as multiples of ticks causes price discreteness [20]. Organized exchanges introduce rules such as price limits in the form of price bands and minimum allowed price changes that are called ticks. Borsa Istanbul is no exception. Recall that for any session in National Equity Market of Borsa Istanbul, the price of a stock is allowed to oscillate between 90% and 110% of its base price where the base price is determined using information from previous session. In addition, the smallest price variation that may occur between consecutive trades is 0.01 TL or 0.02 TL or 0.50 TL or 0.10 TL depending on the range of the base price. These trading rules result in prices to assume a small set of possible outcomes [49]. As Engle and Russell [49] underline, such discreteness will affect the characteristics of prices especially when they are small relative to the tick size. In this context, the discreteness of transaction prices holds under CTS and TTS for all data handling methods and for all stocks in our sample because as long as there are price ticks, especially when the ticks are large in size, discreteness is a natural consequence.

ii) Irregular temporal spacing is defined as the arrival of transactions being random in calendar time. Comparing this characteristic under CTS and TTS is meaningless because the random arrival of transactions is relevant only for TTS. Additionally, this characteristic cannot change from one cleaning method to the other or amongst aggregation methods, since errors or transactions recorded with the exact same time stamp are not the cause of irregular temporal spacing. Therefore, this characteristic holds for all data handling methods and for all stocks in our sample.

iii) There can be diurnal patterns in the behavior of stocks due to particular market conditions such as openings, closings, trading halts, circuit breakers, etc. A few of the pioneering researchers who examined patterns of average intraday financial market returns and reported a U-shape pattern in return volatility over the trading day are Woodi McInish and Ord [107], Harris [62], Müller et al. [87] and Baillie and Bollerslev [18], Andersen and Bollerslev [7]. In more recent studies, similar strong periodic patterns in intraday financial data are observed in many dimensions such as trading frequency, trading volume or returns per some predetermined amount of time such as 15 minutes. A set of similar findings was reported for the Borsa İstanbul National Equity Market by Bildik [33].

By examining 15-min, 5-min and 1-min. interval stock returns in the İstanbul Stock Exchange National Equity Market for the period from 1996 to 1999, Bildik [33] finds that stock returns follow a W-shaped pattern over the trading day, where such patterns are tied to the existence of lunch breaks and two continuous auction sessions per trading day. He adds that return volatility is higher at the market openings and exhibits an L-shaped pattern (if we ignore the relative increase in return volatility at the opening of the second session) during both of the sessions. Bildik [33] argues that the relatively higher mean return and standard deviation at the openings of the trading sessions may be explained by the existence of non-trading hours, i.e., information accumulates overnight and during the lunch breaks so that once the market is open, traders immediately take positions in light of the information flow during the non-trading hours.

The existence of such behaviors is very important for us since we argue in Chapter 2, Section 2.6 that such intraday patterns may remove the need for adjusting RVs for non-trading hours. These patterns can be analyzed only under CTS because of their definitions such as the number of trades per x minutes or the absolute return per y seconds. Examples of these aforementioned diurnal patterns over cleaned and aggregated AKBNK transaction data under CTS is provided in Figure F.3.

In order to check the diurnal patterns in all stocks and to find out whether data handling distorts such diurnal patterns, we calculate 10-minute average transaction volumes, 10-minute absolute percentage returns, 10-minute average trade intensities, and, 10-minute average absolute returns for each stock 11 times, i.e., once for the raw data and 10 times for cleaned and aggregated data (there are 10 combinations of cleaning rules 4.ii.a and 4.ii.b and aggregation rules 5.i, 5.ii, 5.iii, 5.iv and 5.v). Results, which are summarized in Table 4.8 and given in detail in Appendix E, suggest that for all the stocks in our sample, there are significant diurnal patterns in returns and trading activity in the form of intensity and volume under CTS and these patterns look exactly the same even after various combinations of cleaning and aggregation methods are applied. This finding suggests that the data handling methods do not distort the naturally occurring diurnal patterns in stock returns.

Table 4.8: Comparison of average transaction volumes, absolute percentage returns, average trade intensities and average absolute returns over 10-minute intervals for each stock under TTS-raw and CTS-clean and aggregated.

TICKER	Average of 10 Minutes Volume	Average of 10 Minutes Trade Intensity	Average of 10 Minutes Absolute Percentage Returns	Average of 10 Minutes Absolute Returns
GARAN	W shape	W shape	We cannot name the pattern as W or U or L.	W shape without last spike at the closing
ISCTR	W shape	W shape	L shape	W shape without last spike at the closing
NETAS	W shape	W shape	W shape without last spike at the closing	W shape without last spike at the closing
AKBNK	W shape	W shape	W shape without last spike at the closing	W shape without last spike at the closing
ARCLK	W shape	W shape	W shape without last spike at the closing	We cannot name the pattern as W or U or L.
MIGRS	W shape	W shape	L shape	W shape without last spike at the closing

4.4. Calculating Returns and RVs - CTS and TTS

Calculating Returns and RV under Calendar Time Sampling:

After the data sets are cleaned of errors and all simultaneous entries are successfully aggregated, we prepare 11 different (1 uncleaned, 10 cleaned and aggregated) artificial transaction time series for each stock in our sample by applying the previous tick method for continuous auction sessions and summarizing opening and closing session information as entries at 09:50:00, 14:20:00 and 17:30:00 if there are no entries in the original data for those time stamps. Due to the existence of non-trading hours in Borsa Istanbul (recall that first and second continuous auction sessions take place between 09:50-12:30 and 14:20-17:30, respectively, with a lunch break), the resulting artificial time series has 9601 entries for the first session and 11401 entries for the second session on each trading day. The total number of entries in a trading day is 21002.

In agreement with the discussions in Chapter 2, Sections 2.1. and 2.6, we pick prices at frequencies appropriate for the analyses in this Chapter while acknowledging the trading halt due to the lunch break. With the first and second session's continuous auction periods corresponding to 9601 and 11401 seconds, respectively, a plausible set of frequencies is given Table 4.9:

Table 4.9: A plausible set of sampling frequencies under CTS

Sampling Interval in Seconds	Sampling Interval in Minutes	Number of Returns During First Session	Number of Returns During Second Session
10	0.17	960	1140
30	0.50	320	380
60	1.00	160	190
150	2.50	64	76
300	5.00	32	38
600	10.00	16	19
900	15.00	10	12
1200	20.00	8	9

Since the majority of empirical research in the RV literature uses UHFD coming from the New York Stock Exchange (NYSE) where trading is carried out without any lunch breaks, there are not many studies that address the non-trading hours over lunch. The existence of a lunch break complicates return sampling. As an illustration of this complication we present the starting and ending time stamps for 15, 20, 30 and 60 minutes sampling. For these frequencies, it is not possible to include the last several minutes of the morning sessions causing the session hours to be shrunk and we cannot benefit from the information (highlighted in grey) that is contained in the trimmed last minutes of each session.

15 Minutes Sampling

First Session		Second Session	
Start	End	Start	End
09:50:00	10:05:00	14:20:00	14:35:00
10:05:00	10:20:00	14:35:00	14:50:00
10:20:00	10:35:00	14:50:00	15:05:00
10:35:00	10:50:00	15:05:00	15:20:00
10:50:00	11:05:00	15:20:00	15:35:00
11:05:00	11:20:00	15:35:00	15:50:00
11:20:00	11:35:00	15:50:00	16:05:00
11:35:00	11:50:00	16:05:00	16:20:00
11:50:00	12:05:00	16:20:00	16:35:00
12:05:00	12:20:00	16:35:00	16:50:00
12:20:00	12:30:00	16:50:00	17:05:00
		17:05:00	17:20:00
		17:20:00	17:30:00

20 Minutes Sampling

First Session		Second Session	
Start	End	Start	End
09:50:00	10:10:00	14:20:00	14:40:00
10:10:00	10:30:00	14:40:00	15:00:00
10:30:00	10:50:00	15:00:00	15:20:00
10:50:00	11:10:00	15:20:00	15:40:00
11:10:00	11:30:00	15:40:00	16:00:00
11:30:00	11:50:00	16:00:00	16:20:00
11:50:00	12:10:00	16:20:00	16:40:00
12:10:00	12:30:00	16:40:00	17:00:00
		17:00:00	17:20:00
		17:20:00	17:30:00

30 Minutes Sampling

First Session		Second Session	
Start	End	Start	End
09:50:00	10:20:00	14:20:00	14:50:00
10:20:00	10:50:00	14:50:00	15:20:00
10:50:00	11:20:00	15:20:00	15:50:00
11:20:00	11:50:00	15:50:00	16:20:00
11:50:00	12:20:00	16:20:00	16:50:00
12:20:00	12:30:00	16:50:00	17:20:00
		17:20:00	17:30:00

60 Minutes Sampling

First Session		Second Session	
Start	End	Start	End
09:50:00	10:50:00	14:20:00	15:20:00
10:50:00	11:50:00	15:20:00	16:20:00
11:50:00	12:30:00	16:20:00	17:20:00
		17:20:00	17:20:00

This complication is discussed by papers that examine the RV of transaction prices in Tokyo Stock Exchange and Hong Kong Exchanges and Clearing Limited. Ishida and Watanabe [67], Chow et al. [40], Masuda and Morimoto [83], Takaishi et al. [105] and Ubukata and Watanabe [103] are the studies that we reference for handling the lunch breaks in the RV calculation.

Ishida and Watanabe [67] apply the ARFIMA-GARCH model to the RV and the continuous sample path variations constructed from high-frequency Nikkei 225 data coming from the Tokyo Stock Exchange (TSE). The TSE sets trading hours as 09:00-15:00 over two sessions with a lunch break between 11:00 and 12:30, where price limit rules apply in each session. Ishida and Watanabe [67] treat each session individually and calculate 23 and 30 five-minute returns for session 1 and 2, respectively. They calculate the RV for each day using all of the five-minute returns from the entire day. Although they consider that it is possible to adjust the RV for non-trading hours by adding squared returns during night and lunch, they state that since the TSE is open only for 4.5 hours during a day, such an adjustment might be "stretch"ing the information available. In summary, they do not take squared returns from closing session 1 to opening session 2 and do not adjust the daily RV for the lunch breaks or the overnight period.

Chow et al. [40] underline the fact that many of the studies on volatility structures using high-frequency financial data are concentrated on developed markets but there is little evidence regarding emerging markets such as Hong Kong. Accordingly, using transaction data, they investigate the statistical properties of the return volatility of shares listed on the Hong Kong Exchanges and Clearing Limited (HKEx). Similar to the approach adopted by Ishida and Watanabe [67], Chow et al. [40] use all available 5 min returns coming from session 1 and session 2 separately to calculate the daily RV. They do not mention lunch breaks or non-trading hours and, therefore, do not make any adjustments to the daily RV calculation for overnight or lunch breaks.

Masuda and Morimoto [83] also carry out an empirical study with data from TSE and treat each session separately while adding the RV from each session to get RV_daily. However, they divert from Ishida and Watanabe [67] in the sense that the adjustment of the RV as the sum of RV_session1 and RV_session2 for non-trading hours is taken as a requirement. They modify Hansen and Lunde's [58] approach and solve for the optimal weights for RV_session1, RV_session2, squared overnight return and squared lunch return. They argue that adding the optimally weighted squared returns overnight and the lunch break onto the RVs from session 1 and session 2 improves forecasting performance.

Takaishi et al. [105] claim that, consistent with a mixture of distributions hypothesis, price returns standardized by realized volatilities on the TSE become approximately Gaussian. In the calculation of the RVs, they acknowledge the complexity induced by the lunch break and in order to avoid the adjustment for non-trading hours, they calculate RVs of each session separately and then divide the returns in each session with the corresponding RV.

More recently, Ubukata and Watanabe [103] investigate whether handling the microstructure noise, the non-trading hours and large jumps in the calculation of realized volatilities would change the pricing performance of options on the Nikkei 225 index. Regarding non-trading hours, they emphasize that adding the squares of overnight and lunchtime returns may yield a noisy RV (due to price discreteness) and opt for scaling RV, as is done by Hansen and Lunde [58], calculated over all available transaction data. They claim that the adjustment for non-trading hours in [58] improves the option pricing performance and if the Hansen–Lunde adjustment is used, other methods (such as kernel or subsampling based estimators) that mitigate noise induced bias are not necessarily needed.

By distilling all these papers, depending on the analysis we carry out, we sample prices at a subset (or full set) of Table 4.9 to calculate stock returns at each frequency and calculate RV_session1 and RV_session2, then sum these figures to find the RV_open to close. For plotting the volatility signature plots and calculating the Average RV, we take the simple average of daily RVs over the whole sample period at each frequency.

Following arguments in Chapter 2, Section 2.6, no adjustment is made to RV_open to close for the existence of non-trading hours since we claim that the existence of diurnal patterns in trading intensity and returns already reflect the volatility accumulated during the non-trading hours.

<u>Calculating Returns and RV under Transaction Time Sampling:</u>

Sampling prices under TTS does not require the artificial construction of time series except for cleaning and aggregating the raw data as well as summarizing the opening and closing sessions at the beginning and end of each session. After the data sets are cleaned of errors and all simultaneous entries are successfully aggregated, we get 11 different (1 uncleaned, 10 cleaned and aggregated) transaction time series for each stock in our sample. Obviously, the asynchronous nature of trading causes entries in the actual transaction data per each session to be different in number. Accordingly, the set of frequencies (in terms of the number of transactions), reported in Table 4.10, are determined to make sure that such frequencies fit the purpose of the analyses in this Chapter rather than ensuring the number of returns from the first and second sessions from different trading days are always equal. The decision with regards to sampling intervals in transactions is made by also taking into consideration the liquidity of stocks defined in Section 4.2. of this Chapter, for instance, the number of returns in a session may be very small for relatively illiquid stocks for the longer intervals such as 100 transactions.

Table 4.10: A plausible set of sampling frequencies under TTS

Sampling Interval in Transactions
3
6
10
15
20
30

Following our discussions on calculating returns and RVs under CTS, depending on the analysis we carry out, we again sample prices at a subset (or full set) of Table 4.10 to calculate returns at each frequency and find the RV_open to close by adding RV_session1 and RV_session2. For the Average RV, we again take the simple average of daily RVs over the whole sample period at each frequency.

In line with the arguments in Chapter 2, Section 2.6, we still do not make any adjustments to the RV_opentoclose for the existence of non-trading hours since we claim that the existence of diurnal patterns in trading intensity and returns may already reflect the volatility accumulated during the non-trading hours.

4.5. Is Temporal Dependence in Returns Distorted By Data Handling Procedures and/or Sampling Schemes?

Literature is abundant with evidence from stock exchanges scattered around the world that points to the existence of first order autocorrelation and volatility clustering in intraday returns. Correspondingly, checking whether the data handling procedures and/or sampling schemes alter the return structure becomes vital before commencing the analysis of the MMN's significance and structure, keeping in mind that such analyses use the RV as an input and any change in the return (consequently the RV) structure due to data handling and/or sampling scheme should be carefully scrutinized before commenting on what happens to the MMN under different data handling or sampling schemes.

In order to shed some light on this question, work by Andersen and Bollerslev [7], Andersen et al. [11] and Engle and Russell [49] leads us to delve into correlograms of returns and durations under different sampling schemes and data handling procedures. We give special importance to these correlograms because an intriguing finding of Griffin and Oomen [55] reveals that while moving from transaction time to tick time, the dependence structure of returns is altered dramatically which, in turn, affects the properties of the RV. In order to illuminate the robustness of this finding across securities, they list the first five autocorrelations of returns in both transaction time and tick time for all DJ30 components and show that in tick time all first and third (second and fourth) autocorrelations are negative (positive) without exception.

By comparing the autocorrelation and partial autocorrelation functions of 60-second and/or 600-seconds[9] absolute returns and log returns under CTS (clean and aggregated and interpolated) as well as absolute returns, log returns and durations in seconds from one transaction to the next under TTS (raw versus clean and aggregated) for each stock in our sample set for December of 2012, we observe that for stocks in our sample:

[9] *Since first order autocorrelation was observed in 10-min returns for all cleaning and aggregation methods under CTS, we do not feel the urgency to check for 1-min returns under CTS for ARCLK, AKBNK, GARAN and ISCTR. We examine 1-min returns in addition to 10-min returns under CTS for MIGRS and NETAS because the 10-min log returns exhibit no autocorrelation at all.*

- There are differences between the ACF and PACF structures of absolute and log returns between 10-minutes CTS and 1 transaction TTS, i.e., transforming 1 transaction sampled data by first cleaning, then aggregating and then interpolating (all needed for CTS) to 600-seconds sampled data distorts the ACF and PACF of log and absolute return series. The absolute return autocorrelation structure is changed under CTS at the 600-seconds sampling interval compared to results under TTS at the 1 transaction interval. Likewise, switching to CTS and calculation returns at 600 seconds suppresses partial autocorrelation figures at several lags of both absolute and log returns.

- Regardless of the cleaning or aggregation methods, volatility clustering is verified in the form of very slow decay in ACF and PACF of absolute returns under TTS and durations between two consecutive transactions- lags are positive and significant up to 20.

- In line with findings by Griffin and Oomen [55], return dynamics in transaction time are different from those in calendar time and the choice of sampling scheme may have a substantial effect on the properties of realized variance.

- In general, comparing data handling combinations to each other, any combination of the cleaning and aggregation methods (compared to other combinations) does not cause any major change in the total and partial correlation structures once we move under a sampling scheme, whether it is either TTS or CTS, regardless of liquidity. However, cleaning and aggregation under TTS yields different PACF structures in the absolute and/or log returns[10] compared to the results produced with raw data.

- Working at different frequencies under CTS distorts the autocorrelation structure of absolute returns and log returns in the same way: returns become less autocorrelated as we sample a smaller number of prices.

4.6. Do Sampling Schemes or Data Handling Methods Change the Empirical Distributions, Correlograms and Stationarity of RV Series?

A number of papers analyze the properties of the RV. Andersen et al. [12] focus on currencies, Andersen et al. [10] examine individual stocks, Ebens [46] studies the Dow Jones Industrial Average, and Areal and Taylor [15] work on stock index futures. The results are interesting such that the RV appears to be lognormally distributed and daily returns standardized by the RV are approximately normal [51]. Moreover, the RV exhibits long-memory dynamics consistent with a fractionally integrated process with

[10] *For AKBNK, GARAN, and NETAS, PACF of log returns is affected. For ARCLK and MIGRS, PACF of both of log and absolute returns are different when data handling methods are applied under TTS. With regards to ISCTR, such an effect of data handling methods is not observed.*

a degree of integration around 0.4, volatility clustering is apparent at as long as the monthly level, and the RV obeys precise scaling laws under temporal aggregation [51].

Accordingly, in addition to assessing the effects (if any) of data handling and/or sampling schemes on return dynamics in the form of temporal dependence, we also inquire whether sampling schemes and/or data handling procedures have a significant impact on the RV dynamics by focusing on the empirical distributions, correlograms and stationarity of the RV series (by session and daily) for each of the 6 stocks in our sample.

For each frequency in a frequency set of 3, 6, 10, 15, 20, 30 transactions and 60, 300, 600 seconds under each sampling scheme (raw-TTS, CTS) and cleaning (4.ii.a, 4.ii.b) -aggregation method (5.i, 5.ii, 5.iii, 5.iv, 5.v) combination, we construct two RV time series, namely session-based RVs and daily RVs. Consequently, the number of RV series per stock becomes 36 (6- TTS–raw, 30-CTS-clean and aggregated). Each daily RV time series has 124 data points, whereas each session-based RV time series is comprised of 248 entries.

For each RV series under each sampling scheme, for each frequency and for each cleaning and aggregation method combination we calculate preliminary statistics, conduct ACF and PACF analyses and lastly check the existence of a unit root wherever autocorrelations exhibit a slow decay.

With regards to preliminary statistics, we analyze the mean, skewness and kurtosis values and Jarque-Bera (JB) test results at the 5% significance level in order to determine the normality or lognormality of the RV series at hand. The preliminary statistics also allow us to determine whether the mean of the session-based and daily RVs become smaller as the sampling interval is lengthened or whether there is an identifiable relationship between the sampling frequency and the change in skewness, kurtosis or the JB statistic values.

By constructing the correlograms with 20 lags, we check for the existence of autocorrelation in our RV series.

In order to test for stationarity, i.e. whether the series moves around a constant mean or diverges as time passes, the Augmented Dickey Fuller (ADF) test is preferred. By a visual inspection of graphs, we do not observe any trend in any of our RV series, and, therefore, the ADF test is run with an intercept and no trend. The number of lags to be used in the stationarity tests is chosen by the Schwarz criterion as it is the default choice suggested by E-views.

Respective sections of Appendix E are summarized in Table 4.11, Table 4.12, Table 4.13 and Table 4.14

Table 4.11: Results on rejection of the null hypothesis that the RV series come from a Gaussian distribution at 5% significance level for various sampling frequencies under TTS-raw and CTS-clean and aggregated for each stock in the sample set

TICKER	**TTS-Raw (3, 6, 10, 15, 20 and 30 transactions)**		**CTS-Clean and Aggregated and Interpolated (1, 5 and 10 minutes) (All cleaning and aggregation combinations)**	
	Session-based	**Daily**	**Session-based**	**Daily**
GARAN	Reject null for all session series at all frequencies	Reject null for all daily series at all frequencies	Reject null for all session series at all frequencies	Reject null for all daily series at 5 and 10 minutes frequencies, i.e. 1 min daily RV series is coming from a normal distribution
ISCTR	Reject null for all session series at all frequencies	Reject null for all daily series at all frequencies	Reject null for all session series at 5 and 10 minutes frequencies, i.e. 1 min session RV series is coming from a normal distribution	Reject null for all daily series at 5 and 10 minutes frequencies, i.e. 1 min daily RV series is coming from a normal distribution
NETAS	Reject null for all session series at all frequencies	Reject null for all daily series at all frequencies	Reject null for all session series at all frequencies	Reject null for all daily series at all frequencies
AKBNK	Reject null for all session series at all frequencies	Reject null for all daily series at 3,6,10,15 and 30 transaction frequencies, i.e. 20 transaction daily RV series is coming from a normal distribution	Reject null for all session series at all frequencies	Reject null for all daily series at 5 and 10 minutes frequencies, i.e. 1 min daily RV series is coming from a normal distribution
ARCLK	Reject null for all session series at all frequencies	Reject null for all session series at all frequencies	Reject null for all session series at all frequencies	Reject null for all daily series at all frequencies
MIGRS	Reject null for all session series at all frequencies	Reject null for all session series at all frequencies	Reject null for all session series at all frequencies	Reject null for all daily series at all frequencies

Table 4.12: Results on rejection of the null hypothesis that the logarithm of RV series come from a Gaussian distribution at 5% significance level for various sampling frequencies under TTS-raw and CTS-clean and aggregated for each stock in the sample set

TICKER	TTS-Raw (3, 6, 10, 15, 20 and 30 transactions)		CTS-Clean and Aggregated and Interpolated (1, 5 and 10 minutes) (All cleaning and aggregation combinations)	
	Session-based	Daily	Session-based	Daily
GARAN	Reject null for all session series at 3, 6, 10, 15 and 30 transactions frequencies, i.e. 20 transactions session RV series is coming from a lognormal distribution	Reject null for all daily series at all frequencies	Reject null for all session series at all frequencies	Warning: Cleaning method matters! Under 4.ii.a for all aggregation methods, reject null for all daily series at 1 min frequency, i.e. 5 and 10 min daily RV series is coming from a lognormal distribution. Under 4.ii.b, only 10 min daily RV series is found to be lognormal.
ISCTR	Reject null for all session series at all frequencies	Reject null for all daily series at all frequencies	Reject null for all session series at 1 and 5 minutes frequencies, i.e. 10 min session RV series is coming from a lognormal distribution	Reject null for all daily series at 1 and 5 minutes frequencies, i.e. 10 min daily RV series is coming from a lognormal distribution
NETAS	Reject null for all session series at all frequencies	Reject null for all daily series at all frequencies	Reject null for all session series at 1 and 10 minutes frequencies, i.e. 5 min session RV series is coming from a lognormal distribution	Reject null for all daily series at 1 and 10 minutes frequencies, i.e. 5 min daily RV series is coming from a lognormal distribution
AKBNK	Reject null for all session series at 6, 10, 15 and 30 transactions frequencies, i.e. 3 and 20 transactions session RV series	Reject null for all daily series at all frequencies	Reject null for all session series at all frequencies	Reject null for all daily series at 1 minute frequency, i.e. 5 and 10 min daily RV series are coming from a lognormal distribution

	are coming from a lognormal distribution			
ARCLK	Reject null for all session series at 3, 6, 10, 15 and 30 transactions frequencies, i.e. 20 transactions session RV series is coming from a lognormal distribution	Reject null for all daily series at all frequencies	Reject null for all session series at 1 minute frequency, i.e. 5 min and 10 min session RV series are coming from a lognormal distribution	Reject null for all daily series at 1 minute frequency, i.e. 5 min and 10 min daily RV series are coming from a lognormal distribution
MIGRS	Reject null for all session series at all frequencies	Reject null for all daily series at 3, 6 and 15 transactions frequencies, i.e. 10, 20 and 30 transactions daily RV series are coming from a lognormal distribution	Reject null for all session series at all frequencies	Reject null for all daily series at all frequencies

Table 4.13: Comparison RV series' correlograms for various sampling frequencies under TTS-raw and CTS-clean and aggregated for each stock in the sample

TICKER	**TTS-Raw (3, 6, 10, 15, 20 and 30 transactions)**		**CTS-Clean and Aggregated and Interpolated (1, 5 and 10 minutes) (All cleaning and aggregation combinations)**	
	Session-based	**Daily**	**Session-based**	**Daily**
GARAN	Correlogram of all session RV series look alike. Generally speaking, ACFs and PACFs of RVs are decaying but not hyperbolically such that total and partial autocorrelations are strong at even lags and weak at odd legs (ACF positive significant up to 13rd at odd lags -up to 20th lag at even lags and PACF positive significant selectively at legs, 1, 2, 4, and 8)	Although correlograms of all daily RVs resemble one another, compared to correlogram of session series, autocorrelation structure of daily RVs looks different. Now, a quick decay with first two lags and lag 4 being positive significant in PACF is evident, while decay in ACF is wave like with significant positive values up to lag 11-13.	ACFs of session RVs change as the sampling frequency changes, such that for increasing frequencies RV series exhibit significant positive total autocorrelation up to higher number of lags. PACFs of session RVs differ slightly from one another depending on the frequency. For frequency 1 min, lags 1, 2, 4 and 6 are significant in PACF, whereas lags 2 and 1, 2 and 6 are significant (on the edge) for 5 min and 10 min frequencies, respectively.	Compared to correlogram of session series, autocorrelation structure of daily RVs looks different. ACFs of daily RVs change as the sampling frequency changes, such that for increasing frequencies RV series exhibit significant positive total autocorrelation up to higher number of lags. PACFs of daily RVs differ slightly from one another depending on the frequency. For frequency 1 min, lags 1, 2 and 3 are significant in PACF, whereas lags 1 and 3 and lags 1, 3, and 6 are significant for 5 min and 10 min frequencies, respectively.
ISCTR	Correlogram of all session RV series look alike. Generally speaking, ACFs and PACFs of RVs are decaying but not hyperbolically such that total and partial autocorrelations are strong at even lags and weak at odd legs (ACF positive significant up to 12th - 14th and PACF positive or negative	Although correlograms of all daily RVs resemble one another, compared to correlogram of session series, autocorrelation structure of daily RVs looks different. Now, a quick decay with first two lags being positive significant in PACF is evident, while decay in ACF starts from significant	ACFs of session RVs change as the sampling frequency changes, such that for increasing frequencies RV series exhibit significant positive total autocorrelation up to higher number of lags. PACFs of session RVs differ slightly from one another depending on the frequency. For frequency 1 min, lags 1, 2, 4 and 6 are significant in PACF, whereas lags 1, 2 and 4 and 1, 2 and 6 are significant (on the	Compared to correlogram of session series, autocorrelation structure of daily RVs looks different. ACFs of daily RVs change as the sampling frequency changes, such that for increasing frequencies RV series exhibit significant positive total autocorrelation up to higher number of lags. PACFs of daily RVs differ slightly from one another depending on the frequency. For frequency 1 min,

	significant selectively at lags, 1, 2, 4, 6 and 11).	positive values, hit 0, then become negative significant where first 7-8 legs are positive significant, legs 8-12 are not significant and legs 13-20 are negative significant.	edge) for 5 min and 10 min frequencies, respectively.	lags 1 and 2 are significant in PACF, whereas lags 1, 2 and 13 and lags 1 and 3 are significant for 5 min and 10 min frequencies, respectively.
NETAS	Correlogram of all session RV series look very much alike. Total autocorrelation is significant up to 20th lag but significance decreases and increases as the lag number converges to 20. Only first two lags and lag 14 are significant in PACF	Although correlograms of all daily RVs resemble one another, compared to correlogram of session series, autocorrelation structure of daily RVs looks different. Now, first 10 lags and first lag are positive significant in ACF and PACF, respectively.	ACFs of session RVs change as the sampling frequency changes, such that for increasing frequencies RV series exhibit significant positive total autocorrelation up to higher number of lags. PACFs of session RVs differ slightly from one another depending on the frequency. For frequency 1 min, lags 1, 2, 4 and 12 are significant in PACF, whereas lags 1, 2, 5 and 14 and lags 1, 2, and 5 are significant for 5 min and 10 min frequencies, respectively.	Compared to correlogram of session series, autocorrelation structure of daily RVs looks different. ACFs of daily RVs change as the sampling frequency changes, such that for increasing frequencies RV series exhibit significant positive total autocorrelation up to higher number of lags. PACFs of daily RVs differ slightly from one another depending on the frequency. For frequency 1 min, lags 1, 2 and 10 are significant in PACF, whereas lags 1, 2, 3 and 7 are significant for both of 5 min and 10 min frequencies.
AKBNK	Correlograms of all session RV series look alike such that even lags are positive significant up to 20th lag with odd lags being insignificant where decreasing the sampling frequencies depresses significance levels. Lags 1, 2, 4, and 6 are positive significant in PACF of all	Although correlograms of all daily RVs resemble one another, compared to correlogram of session series, autocorrelation structure of daily RVs looks different. ACFs exhibit similar shapes where decreasing frequencies cause less number of lags	ACFs of session RVs change as the sampling frequency changes, such that for increasing frequencies RV series exhibit significant positive total autocorrelation up to higher number of lags. PACFs of session RVs differ slightly from one another depending on the frequency. For frequency 1 min, lags 1, 2, 4 and 6 are significant in PACF, whereas lags 1, 2,	Compared to correlogram of session series, autocorrelation structure of daily RVs looks different. ACFs of daily RVs change as the sampling frequency changes, such that for increasing frequencies RV series exhibit significant positive total autocorrelation up to higher number of lags. PACFs of daily RVs differ

	session RV series at all frequencies, where lag 6 drops from the significant lags list at 30 transactions frequency.	being significant with lesser significant levels, i.e. frequency again depresses autocorrelation structure. Generally speaking, for daily RV series, lags 1, 2, 3 and 14 are significant in the PACF.	and 4 and only 2 are significant for 5 min and 10 min frequencies, respectively.	slightly from one another depending on the frequency. For frequency 1 min, lags 1 and 9 are significant in PACF, whereas lags 1 and 2 and lags 1, 2, 3, and 8 are significant for 5 min and 10 min frequencies, respectively.
ARCLK	Correlograms of all session RV series look alike for frequencies 3, 6, 15 and 30. At these frequencies total autocorrelation is significant up to 20th lag but significance decreases and increases as the lag number converges to 20. Meanwhile, at frequency of 10 transactions, session RV series is autocorrelated up to 6th lag, then significance disappears just to emerge at lags 9 and 12. Only first three lags are significant in PACF of all session RV series at all frequencies except 20 transactions frequency. At sampling interval of 20 transactions, no total or partial autocorrelation is detected	Correlograms of all daily RV series look alike for sampling intervals of 3, 6, 15 and 30 transactions but compared to correlograms of session series at 3, 6, 15 and 30 transaction sampling intervals, autocorrelation structure of daily RVs looks different. Now, first 10 lags and lags 1, 2 and 6 are positive significant in ACF and PACF, respectively.	ACFs of session RVs change as the sampling frequency changes, such that for increasing frequencies RV series exhibit significant positive total autocorrelation up to higher number of lags. PACFs of session RVs differ slightly from one another depending on the frequency. For frequency 1 min, lags 1, 2, 3 and 12 are significant in PACF, whereas lags 1, 2, 3, and 11 and lags 1, 2, 5, 12 and 13 are significant for 5 min and 10 min frequencies, respectively.	Compared to correlogram of session series, autocorrelation structure of daily RVs looks different. ACFs of daily RVs change as the sampling frequency changes, such that for increasing frequencies RV series exhibit significant positive total autocorrelation up to higher number of lags. PACFs of daily RVs differ slightly from one another depending on the frequency. For frequency 1 min, lags 1 and 9 are significant in PACF, whereas lags 1 and 2 and lags 1, 2, 3, and 8 are significant for 5 min and 10 min frequencies, respectively.
MIGRS	Correlogram of all session RV series look same, RVs are autocorrelated up to 11th lag and	Although correlograms of all daily RVs resemble one another, compared to correlogram of session series,	ACFs of session RVs change as the sampling frequency changes, such that for increasing frequencies RV series exhibit significant positive total autocorrelation	Compared to correlogram of session series, autocorrelation structure of daily RVs looks different. ACFs of daily RVs change as the sampling frequency

	lags 1, 3, 4 and 10 are significant in PACF.	autocorrelation structure of daily RVs looks different. Now, a quick decay in ACF and PACF is evident, only five lags are significant in ACF and first, second and sixth lags are significant in PACF.	up to higher number of lags. PACFs of session RVs differ slightly from one another depending on the frequency. PACFs of session RVs differ slightly from one another depending on the frequency. For frequency 1 min legs 1, 2, 4 and 13 are significant in PACF, whereas legs 1, 3, 5 and 8 and legs 1, 3, 5, 8, and 9 are significant for 5 min and 10 min frequencies, respectively.	changes, such that for increasing frequencies RV series exhibit significant positive total autocorrelation up to higher number of lags. PACFs of daily RVs differ slightly from one another depending on the frequency. For frequency 1 min, legs 1, 2 and 6 are significant in PACF, whereas legs 1 and 6 and legs 1, 4, 5 and 6 are significant for 5 min and 10 min frequencies, respectively.

Table 4.14: Results on rejection of null hypothesis that the RV series does not have unit root at 5% significance level for various sampling frequencies under TTS-raw and CTS-clean and aggregated for each stock in the sample set

TICKER	**TTS-Raw (3, 6, 10, 15, 20 and 30 transactions)**		**CTS-Clean and Aggregated and Interpolated (1, 5 and 10 minutes) (All cleaning and aggregation combinations)**	
	Session-based	**Daily**	**Session-based**	**Daily**
GARAN	The null of nonstationarity is rejected at 5% significance level for all session RV series at all frequencies.	The null of nonstationarity is rejected at 5% significance level for all daily RV series at all frequencies.	The null of nonstationarity is rejected at 5% significance level for all session RV series at 5 and 10 min frequencies under cleaning method 4.ii.a, i.e. 1 min session RV series is nonstationary under 4.ii.a but becomes stationary under cleaning method 4.ii.b.	The null of nonstationarity is rejected at 5% significance level for all daily RV series at 5 and 10 min frequencies under cleaning method 4.ii.a, i.e. 1 min daily RV series is nonstationary under 4.ii.a but becomes stationary under cleaning method 4.ii.b.
ISCTR	The null of nonstationarity is rejected at 5% significance level for all session RV series at all frequencies.	The null of nonstationarity is rejected at 5% significance level for all daily RV series at all frequencies.	The null of nonstationarity is rejected at 5% significance level for all session RV series at all frequencies.	The null of nonstationarity is rejected at 5% significance level for all daily RV series at all frequencies.
NETAS	The null of nonstationarity is rejected at 5% significance level for all session RV series at all frequencies.	The null of nonstationarity is rejected at 5% significance level for all daily RV series at all frequencies.	The null of nonstationarity is rejected at 5% significance level for all session RV series at all frequencies.	The null of nonstationarity is rejected at 5% significance level for all daily RV series at all frequencies.
AKBNK	The null of nonstationarity is rejected at 5% significance level for all session RV series at all frequencies.	The null of nonstationarity is rejected at 5% significance level for all daily RV series at all frequencies.	The null of nonstationarity is rejected at 5% significance level for all session RV series at 5 and 10 min frequencies under cleaning method 4.ii.a, i.e. 1 min session RV series is nonstationary under 4.ii.a but	The null of nonstationarity is rejected at 5% significance level for all daily RV series at 5 and 10 min frequencies under BOTH of cleaning methods 4.ii.a and 4.ii.b, i.e. 1 min daily RV series is nonstationary under CTS.

			becomes stationary under cleaning method 4.ii.b.	
ARCLK	The null of nonstationarity is rejected at 5% significance level for all session RV series at all frequencies	The null of nonstationarity is rejected at 5% significance level for all daily RV series at all frequencies	The null of nonstationarity is rejected at 5% significance level for all session RV series at all frequencies	Daily 10 min RV series are nonstationary and stationary at 5% significance level for cleaning method 4.ii.a and 4.ii.b, respectively. Daily 5 min RV series are stationary at 5% significance level for all cleaning methods. At 1 min frequency, aggregation method matters such that 1 min daily RV series are not stationary under 5.i and 5.ii aggregation methods while stationary is attained under remaining aggregation techniques combined with cleaning method 4.ii.a at 5% significance level.
MIGRS	The null of nonstationarity is rejected at 5% significance level for all session RV series at all frequencies except 3 transactions frequency, for which the p-value is around 5.5%.	The null of nonstationarity is rejected at 5% significance level for all daily RV series at all frequencies.	The null of nonstationarity is rejected at 5% significance level for all session RV series at 1 and 5 min frequencies, i.e. 10 min session RV series is nonstationary.	The null of nonstationarity is rejected at 5% significance level for all dailly RV series at 1 and 5 min frequencies, i.e. 10 min daily RV series is nonstationary.

By careful examination of summarized and aggregated information reported in Table 4.11, Table 4.12, Table 4.13 and Table 4.14 as well as results in Appendix E, we find that

- Regardless of liquidity, sampling scheme, daily or session-based calculations, cleaning and aggregation methods or frequency, the mean of the RV series becomes smaller as the sampling interval is lengthened.

- Regardless of liquidity or frequency or daily or session calculation, all RV series under raw-TTS are non-normal (except for the AKBNK 20 tr daily RV series). Normality is achieved for only some of the liquid stocks under CTS and only at the highest 1 min frequency.

- Liquidity matters in terms of the RV normality: for illiquid stocks, no RV series, either under raw-TTS or CTS (for all combinations of cleaning and aggregation algorithms), session or daily or at any frequency is normal. Therefore, sampling scheme or cleaning and aggregation or sampling frequency or session-daily calculation do not change the non-normality of RV series if the stock is illiquid.

- However, for liquid stocks, although all the RV series under raw-TTS are non-normal, switching to CTS and increasing frequency and calculating RVs on a daily basis make the RV series more and more normal such that at 1 min frequency, we cannot reject the null hypothesis of the daily 1 min RV sample coming from a normally distributed population at the 5 significance level for ISCTR and GARAN. We also see that for GARAN, daily-session-based calculation changes normality, where such a calculation choice does not affect ISCTR or NETAS.

- The cleaning and aggregation algorithm combinations do not affect the normality if we work under CTS, whether the stock is liquid or not.

- Unlike the results on normality, liquidity turns out to be ineffective on the log normality of the RV series.

- Generally speaking (except for GARAN), session-based-daily choice, frequency and sampling scheme are found to be effective on the log normality of the RV series. For GARAN, and only under CTS-daily calculations, switching between cleaning methods 4.ii.a and 4.ii.b alters the frequencies at which the RV series is lognormal.

- Under CTS, the ACFs of session and daily RVs change as the sampling frequency changes, such that for increasing frequencies, RV series exhibit significant positive total autocorrelation up to a higher number of lags. Such a particular relationship between frequency and decay patterns in ACF is not so obvious under raw-TTS for GARAN, MIGRS and ISCTR (for AKBNK, ARCLK and NETAS, inflating frequencies yields stronger total autocorrelations).

- Under raw-TTS and CTS, regardless of liquidity, although the correlograms of all the daily RVs resemble one another, compared to the correlogram of session-based series, the autocorrelation structure of daily RVs looks different, i.e., under both raw-TTS and CTS, session-based-daily calculation matters as it influences the autocorrelation decay patterns.

- The cleaning and aggregation methods do not affect autocorrelation decay patterns and the structure of the RV series.

- Liquidity does not alter the stationarity of the RV series in a specific way.

- Regardless of liquidity, under raw-TTS, all RV series, either session-based or daily at all frequencies are found to be stationary at the 5% significance level. Non-stationarity becomes a problem only under CTS for some RV series.

- Under CTS, no particular patterns are observed with respect to effect of UHFD dimensions on stationarity of the RV series. Irrespective of liquidity, some stocks for some frequencies for some cleaning and/or aggregation methods and depending on session-based-daily calculation, turn out to be non-stationary at the 5% significance level. In detail, for ISCTR and NETAS, stationarity results are affected from the sampling schemes, frequencies, the cleaning/aggregation methods, or, the session/daily basis choice, while for GARAN, the sampling scheme, frequency and the cleaning methods, for AKBNK, the sampling scheme, frequency, the cleaning methods and session-based/daily basis choice, for ARCLK, the sampling scheme, daily-session-based calculation, frequency and the aggregation method and, lastly, for MIGRS, sampling scheme, daily-session-based calculation and frequency distort the stationarity of the respective RV series.

4.7. Do Sampling Schemes or Data Handling Methods Alter Volatility Signature Plots?

Recall from Chapter 3, Section 3.1 that when the observed price, Y_t, is contaminated by an additive market microstructure noise, as in Equation (3.4) such that

$$Y_t = X_t + \varepsilon_t, \qquad 0 \leq t \leq T$$

where T shows a finite horizon, X_t denotes the logarithm of true/efficient price of the security at time t and ε_t represents the logarithm of the combined effect of all microstructure noise sources at time t, the quadratic variation of observed prices calculated over the highest frequency possible does not converge to the IV of the true prices since an asymptotic bias (quantified in Equation (3.9)) appears due to the existence of MMN. Basically, applying the quadratic variation operator to both sides of Equation (3.4), we get

$$[Y,Y]_t^{\mathcal{G}} = [X,X]_t^{\mathcal{G}} \ + 2[X,\varepsilon]_t^{\mathcal{G}} + [\varepsilon,\epsilon]_t^{\mathcal{G}}$$

Where the last and first terms are always positive so that if the RV becomes smaller with increasing sampling frequency, then such a decrease in the RV should stem from the large enough negativity of the second term which can offset the positive last term. Hansen and Lunde [58] interpret decreasing RVs accompanying increasing sampling returns in the same way. Moreover, Hansen and Lunde [61] state that

> "*…if the correlation is sufficiently negative, the second bias component can more than offset the first component. Thus the total bias may be negative, as is often seen when RV(m) is based on midquotes.*"

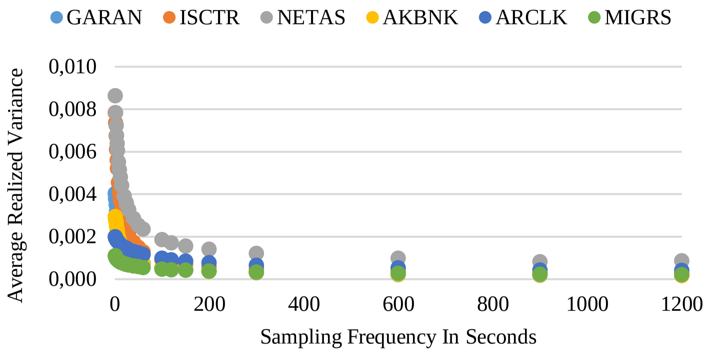
Hansen and Lunde [61] conclude that the negative correlation between the noise and the efficient price could be due to the nonsynchronous revision of the bid and ask prices once efficient prices change. In order to back their argument, they present actual versus bid and ask prices over a 20-minute window and show that the bid-ask spread tends to get wider when prices move up or down.

In addition, we prove in Chapter 3, Section 3.2 that the asymptotic bias is shown to be dominated by $\mathbb{E}[[\varepsilon,\varepsilon]_t|X]$ under both TTS and CTS, regardless of the fact that the MMN and the true price are correlated and/or MMN has a constant mean other than 0.

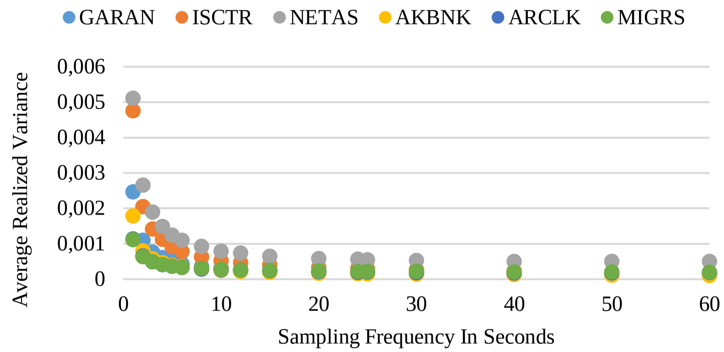
Therefore, it is important to confirm the existence of MMN as well as to test the assumptions regarding the statistical features of the MMN in order to examine how the RV deviates from the IV as we increase the sampling frequency. Remember from Chapter 2, Section 2.4 that the most popular assumptions in the RV literature on the MMN state that the MMN is a sequence of i.i.d random variables with a 0 mean, a constant variance and a finite fourth moment where the MMN and the true prices are orthogonal to each other at each point in time within the time horizon.

One tool to reveal if there is any bias in the RV due to the MMN as well as the frequency at which the MMN becomes evident and the direction of the bias is the visual inspection of volatility signature plots. A volatility signature plot (VSP), a version of which can be traced back to Zhou's work in 1996 [112], is defined as a plot of average realized volatility (Ave RV) against sampling frequencies and is made popular by Andersen et al. [11]. Andersen et al. [11] explain how to derive inferences via volatility signature plots. If Ave RV falls or rises as frequency increases, there is a MMN that kicks in with higher frequencies. A fall in Ave RV as frequency increases is tied to a negative correlation between the efficient returns and the noise. In detail, Andersen et al. [11] state that this negative correlation may be caused by the bid-ask bounce, one of the reasons suggested in the literature for the existence of MMN in stock exchanges such as the NYSE where market making is the usual practice. Andersen et al. [11] interpret a rise in the Ave RV when frequency increases as evidence of the existence of positive correlation between the efficient returns and the noise and state that this positive correlation may be caused by inactive trading, which is another reason for the existence of MMN.

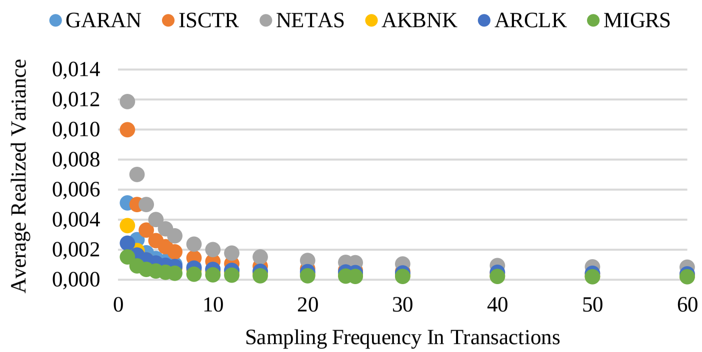
Since VSPs are used as a visual inspection tool for deriving conclusions regarding the existence of MMN and/or correlations (including the direction of correlations) between the true price of an asset and the MMN, we believe that it is crucial to test whether the results driven from VSPs are consistent across sampling schemes and data handling approaches. In this attempt, we analyze VSPs for each stock under various data handling procedures and sampling schemes (CTS and TTS) to conclude about the potential bias problems of the RV type estimators due to the existence and statistical characteristics of the MMN. We plot 3 VSPs, one from session 1 RV series, one from session 2 RV series and one from daily RVs, per each data handling procedure. We compare VSPs produced under CTS for 10 different combination of cleaning rules 4.ii.a and 4.ii.b with aggregation methods 5.i, 5.ii, 5.iii, 5.iv and 5.v and VSPs produced under TTS for raw data and cleaned and aggregated data. At this point, we would like to emphasize that regarding VSPs under TTS, we skip 4.ii.a-5.i-5.ii-5.iii-5.iv-5.v combinations, mainly because the number of cleaned points under 4.ii.a is so small that cleaning makes no real difference. Any possible difference stemming from cleaning may be observed under cleaning method 4.ii.b, which ends up deleting more data points. Moreover, since we compare 4.ii.a and 4.ii.b under CTS, we additionally search for any marginal effect that the cleaning method 4.ii.b has over the cleaning method 4.ii.a. By this logic, the resulting number of VSPs produced per stock is 48. Discussions on these VSPs are given in Appendix E, while we provide Figure 4.1 to summarize our findings regarding VSPs of each stock under TTS and CTS (cleaned and aggregated).



(a) CTS -4.ii.a-5.i



(b) CTS -4.ii.a-5.ii



(c) TTS-raw

Figure 4.1: VSPs for GARAN, ISCTR, NETAS, AKBNK, ARCLK, MIGRS under CTS and TTS

Based on Figure 4.1 as well as the results in Appendix E, we observe that

- The sampling schemes or the cleaning and aggregation techniques do not alter the fact that inflating sampling frequency, either in seconds or in transactions, causes the average realized volatility of return based on transaction prices to boom, irrespective of the liquidity. This observation is valid both for session-based and daily figures.

- Explosion becomes trivial for the sampling intervals that are less than 300 seconds or 15 transactions. These frequencies serve as optimal sampling frequencies at which market microstructure noise dominates the RV of observed prices.

- In all possible dimensions (sampling scheme, liquidity, data handling methods, and session-daily calculation) for all stocks, we find visual proof regarding the existence of market microstructure noise and pointing to a positive relationship between the noise increment and true price return, under both CTS and TTS.

4.8. Do Sampling Schemes or Data Handling Methods Affect Results of Formal Tests on the Statistical Structure of MMN?

In order to search whether there is a MMN effect on observed prices and whether the popular assumptions regarding the MMN structure are backed by empirical data, we employ Awartani et al.'s [16] formal statistical tests of the no noise and noise increments with constant variance assumptions. Their approach depends on the comparison of two or more realized volatilities computed over different frequencies under CTS where the artificial construction of price series ensures that prices are regularly spaced in time. In Chapter 3, as a contribution to the available literature, we provide discussions and proofs that the same tests can be used under TTS where prices are scattered irregularly over time. In line with those arguments and proofs, we examine

a) in Section 4.8.1 whether the formal test results confirm the existence of MMN under CTS and TTS for varying data handling procedures,

b) in Section 4.8.2 whether the formal test results confirm the constant variance of MMN increments under CTS and TTS for varying data handling procedures,

while liquidity in the sense of Section 4.2 is also taken into account.

4.8.1. Formal Tests of the Existence of MMN under Different Sampling Schemes and Data Handling Procedures

Suppose that the observed price Y_t is assumed to be contaminated by an additive market microstructure noise as in Equation (3.4) such that

$$Y_t = X_t + \varepsilon_t, \qquad 0 \leq t \leq T$$

where T shows a finite horizon, X_t denotes the logarithm of true/efficient price of the security at time t and ε_t represents the logarithm of the combined effect of all microstructure noise sources at time t. Moreover, X_t, the log price of an asset, satisfies the Equation (3.1) which gives the following stochastic differential equation on the finite time horizon $t \in [0, T]$:

$$dX_t = \mu_t dt + \sigma_t dB_t$$

where X_0 is $\mathcal{F}_0$-measurable, X_t has continuous sample paths, drift μ_t is a locally bounded, predictable continuous process, the continuous stochastic process σ_t that derives the volatility of log return of the security is square integrable, B_t denotes a standard Wiener process and σ_t is orthogonal to B_t.

In this setting, Awartani et al. [16] propose to check whether, due to the existence of MMN, there is any statistically significant asymptotic bias on the RV estimator under CTS by testing the null hypothesis in Equation (3.11) which asserts that the second moments of all MMN increments equal zero against the alternative hypothesis in Equation (3.12) which claims that the second moments of all MMN increments are greater than zero.

We claim in Chapter 3, Section 3.3 that the same set of hypotheses are also relevant for TTS because under both TTS and CTS, as shown in Chapter 3, Section 3.2, if observed prices are contaminated with a MMN where the MMN is a sequence of i.i.d random variables with a 0 mean and a finite fourth moment while the noise increments have constant variance as we increase observation frequencies, the RV, scaled by $(2 \cdot number\ of\ sampling\ intervals))^{-1}$ and calculated over observed prices, estimates more and more the variance of MMN rather than the quadratic variation of the true price.

Awartani et al. [16] develop a test statistic under CTS to test if we can reject the null hypothesis in Equation (3.11) against the alternative hypothesis in Equation (3.12) so that the MMN has a statistically significant effect on RV estimators of the IV at a given sampling frequency. The test statistic $Z_{T,n,h}$ employs RVs calculated at two artificially selected frequencies, one low and one high, as well as the Realized Quarticity (RQ) calculated at low frequency and is formulated in Equation (3.13) as follows:

$$Z_{T,n,h} := \frac{\sqrt{h-1}\left(RV_{T,n} - RV_{T,h}\right)}{\sqrt{\frac{2(h-1)}{3} RQ_{T,h}}}$$

where *h* and *n* stand for the number of observations for the whole estimation period, *T* (for instance, the number of observations per day) at low frequency and high frequency, respectively, and

$$RV_{T,n} = \sum_{i=0}^{n-2} \left(Y_{t_{i+1}} - Y_{t_i}\right)^2$$

$$RV_{T,h} = \sum_{i=0}^{h-2} \left(Y_{t_{i+1}} - Y_{t_i}\right)^2$$

$$RQ_{T,h} = \sum_{i=0}^{h-2} \left(Y_{t_{i+1}} - Y_{t_i}\right)^4$$

$$n > h, \qquad \frac{n}{h} \to \infty \; as \; n, h \to \infty$$

Awartani et al. [16] prove that the test statistic $Z_{T,n,h}$ has a standard normal distribution asymptotically under the following assumptions:

- the true price is generated as in Equation (3.1),
- $\int_0^T \sigma_t^4 dt < \infty$,
- the MMN increments have a finite fourth moment on $[0, T]$
- there is at least one $\zeta > 0$ such that for $\psi \in (0,1)$, $\liminf_{n\to\infty} (n-1)^{\psi-1}[\varepsilon, \varepsilon]_T > \zeta$ and $\liminf_{h\to\infty} (h-1)^{\psi-1}[\varepsilon, \varepsilon]_T > \zeta$.

We suggest and show in Chapter 3, Section 3.3 that the same test statistic $Z_{T,n,h}$ can be employed to examine the existence of MMN under TTS where

- conditions and assumptions of Theorem 3.2 hold,

- the true price of the security is observed with no drift and it is a local martingale by definition as in Equation (3.20),

- the instantaneous true return variance, σ_t^2, is locally bounded and continuous in mean square,

- the observation times are independent of the true price process X_t, and the maximum distance between two consecutive observation times converges to 0 in probability at an order of $(number\ of\ sampling\ intervals - 1)^{-1/2}$ as the number of observations tend to infinity,

- for any two consecutive observation times t_{i+1} and t_i, $\sum_i (t_{i+1} - t_i)^3 = O_p((number\ of\ sampling\ intervals - 1)^{-2})$,

- Asymptotic Quadratic Variation of Time (AQVT) is calculated as

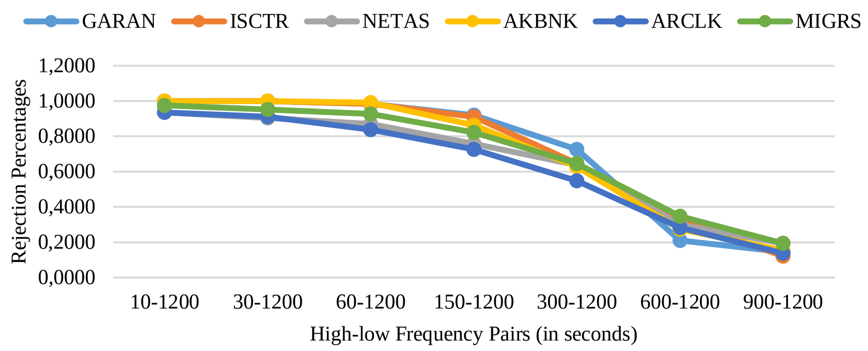
$$\mathcal{D}_t := \lim_{h \to \infty} \frac{h-1}{T} \sum_{t_{i+1} \leq t} (t_{i+1} - t_i)^2$$

  and denoted by $\mathcal{D}_t$ exists and is absolutely continuous.

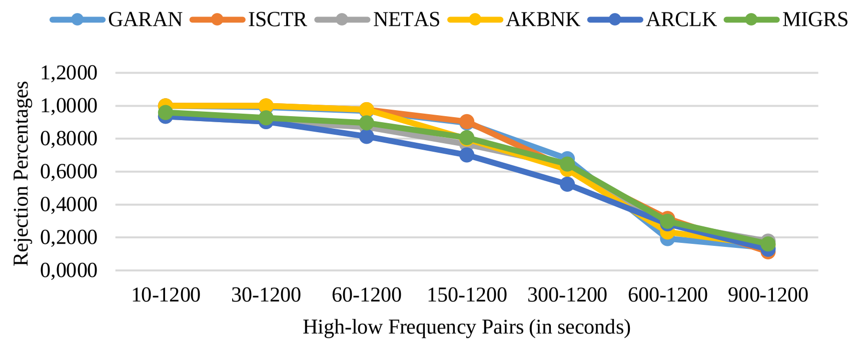
For both TTS and CTS, since bias of the RV estimator is dominated by the expectation of the square of the noise increment, if we reject the null hypothesis in Equation (3.11), this implies that a MMN exerts a statistically significant impact on the realized estimator of the IV.

In this context, we calculate the $Z_{T,n,h}$ statistic testing the null hypothesis in Equation (3.11) by comparing RVs that are calculated over different frequency pairs, under sampling schemes CTS and TTS for raw and cleaned and aggregated data series. The high-low frequency pairs are (60,600) (10,1200), (30, 1200) (60,1200), (150,1200), (300,1200), (600,1200) (900,1200) seconds for CTS and (3,30), (6,30), (10,30), (15,30), and (20,30) transactions for TTS. Regarding data handling methods, we consider test results for 10 different combinations of cleaning methods 4.ii.a and 4.ii.b with aggregation methods 5_i, 5_ii, 5_iii, 5_iv and 5_v once under CTS and once under TTS. For each day in the sample period of 124 days and each frequency pair, we run the aforementioned test at a 5% significance level.

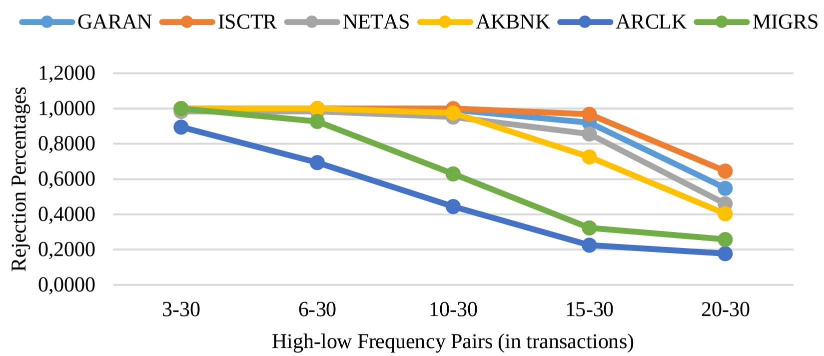
The results for each stock, for each frequency pair, under each data handling method and sampling scheme are given in the respective sections of Appendix E. To summarize, carefully selected examples of these findings comparing each stock for two different data handling procedures under CTS (4.ii.a_5.i versus 4.ii.b_5.i) and TTS (raw versus 4.ii.b_5.i) are reported in Figure 4.2.



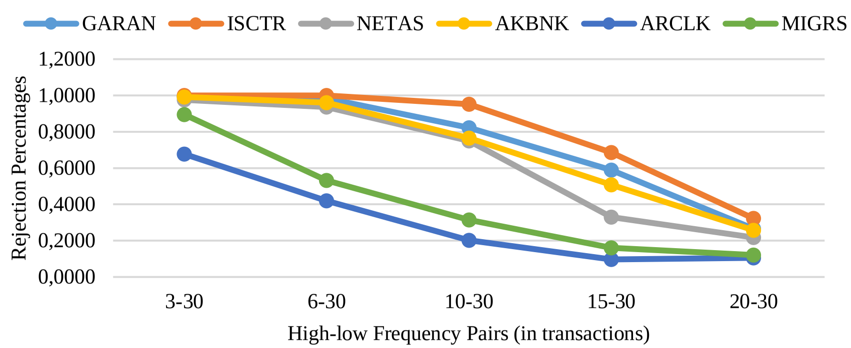
(a) CTS-Clean and Aggregated and Interpolated Data (4.ii.a-5.i)



(b) CTS-Clean and Aggregated and Interpolated Data (4.ii.b-5.i)



(c) TTS-raw



(d) TTS- 4.ii.b-5.i

Figure 4.2: Rejection percentages regarding the null hypothesis that MMN does not have a statistically significant effect on the RV estimator. Sample period: second half of 2012 (a) CTS-Clean and Aggregated and Interpolated Data (4.ii.a-5.i) (b) CTS-Clean and Aggregated and Interpolated Data (4.ii.b-5.i) (c) TTS- raw (d) TTS- 4.ii.b-5.i

Based on Appendix E and Figure 4.2, we learn the following:

- The existence of MMN is verified statistically (i.e. there is a significant decrease in the rejection percentages as we increase the "high frequency" leg of each pair) under both CTS and TTS for all stocks, regardless of the liquidity of the stock or the data handling method.

  - Rejection rate graphs reveal that the MMN starts to accentuate as the sampling frequency converges to 10-15 transactions under TTS, and 250-300 seconds under CTS. These findings are in conformity with those supplied by the VSP analysis. The MMN is felt strongly once we cross over the sampling interval thresholds of 15 transactions or 5 minutes under TTS and CTS, respectively. For higher frequencies, rejection rates turn out to be quite high, and especially after 150 seconds under CTS and 10 transactions under TTS, rejection rates explode.

- The liquidity and the data handling methods matter such that for all stocks in our sample we observe the following:

  - The lower the liquidity, the lower the rejection percentage at all frequency-pairs under TTS (raw or clean and aggregated).

  - Cleaning and aggregating the data do not amend the downward trend in rejection percentages under TTS, but make it steeper.

  - A visual inspection of the test statistic $Z_{T,n,h}$ for several frequency pairs, either under TTS or CTS, reveals that majority of the time the test statistic is positive and outside the 5% standard normal confidence interval, meaning that there is a positive correlation between the noise and efficient price, which is again in conformity with the exploding VSPs.

4.8.2. Formal Tests of the Constant Variance of MMN Increments under Different Sampling Schemes and Data Handling Procedures

In addition to developing a formal test to find out whether the MMN has a statistically significant effect over the RV estimator, Awartani et al. [16] also suggest testing the popular assumption stating that the MMN has constant variance independent of time. Their effort in revealing whether the MMN has constant variance over time is particularly important since, as explained in Chapter 2, methods proposed in the literature for handling the MMN effects while estimating the IV of the true price depend on the validity of this assumption. Hence, before using any of the methods to mitigate the MMN effects on the IV estimators, one has to make sure that this assumption is backed by empirical evidence. For this purpose, as we delve into detail in Chapter 3, Section 3.4, Awartani et al. [16] suggest testing the null hypothesis in Equation (3.45) which states that that the variance of the MMN increments, defined as 2 times the second moment of the increments while the mean of MMN is taken as zero, observed over different grids $\mathcal{H}$ and $\mathcal{G}$ under CTS are equal whereas the alternative hypothesis given in Equation (3.46) claims that such variances over different grids are not equal.

The null hypothesis in (3.35) employs the analogy that if the MMN increments have constant variance through time, then the variance of MMN increments over different grids under CTS should also be equal to each other. Therefore, the alternative hypothesis in Equation (3.36) is consistent with the presence of autocorrelation in the MMN.

As explained in Chapter 3, Section 3.4, for the purpose of testing whether MMN increments have constant variance independent of the sampling frequency, Awartani et al. [16] develop a test statistic $V_{T,n,h}$, that compares RVs calculated at 3 different sampling frequencies (on 3 different grids), one low, one high and one very low compared to each other and requires the Realized Quarticity (RQ) to be calculated at low frequency. $V_{T,n,h}$, is defined as follows:

$$V_{T,n,h,l} := \sqrt{h-1}\frac{\frac{\left(RV_{T,n}-RV_{T,l}\right)}{2(n-1)}-\frac{\left(RV_{T,h}-RV_{T,l}\right)}{2(h-1)}}{\sqrt{3\left(\frac{RQ_{T,h}}{2(h-1)^2}-\left(\frac{RV_{T,h}}{2(h-1)}\right)^2\right)}}$$

where n, h and l stand for the number of observations for the whole estimation period, T (for instance, number of observations per day), at high frequency, low frequency and very low frequency, respectively and as before

$$RV_{T,n} = \sum_{i=0}^{n-2}\left(Y_{t_{i+1}} - Y_{t_i}\right)^2$$

$$RV_{T,h} = \sum_{i=0}^{h-2}\left(Y_{t_{i+1}} - Y_{t_i}\right)^2$$

$$RQ_{T,h} = \frac{2}{3}\sqrt{h-1}\sum_{i=0}^{h-2}\left(Y_{t_{i+1}} - Y_{t_i}\right)^4$$

$$n > h > l, \qquad \frac{n}{h} \to \infty, \frac{h}{l} \to \infty \; as \; n, h, l \to \infty$$

The very low frequency that is employed in the test statistic $V_{T,n,h,l}$ represents a frequency at which we can safely ignore the MMN according to the literature. On this issue, Awartani et al. [16] argue that l might be chosen as a 20-minute sampling interval under CTS following the sparse sampling literature and the 5-minute threshold promoted by Andersen et al. [11].

In addition, Awartani et al. [16] show that if

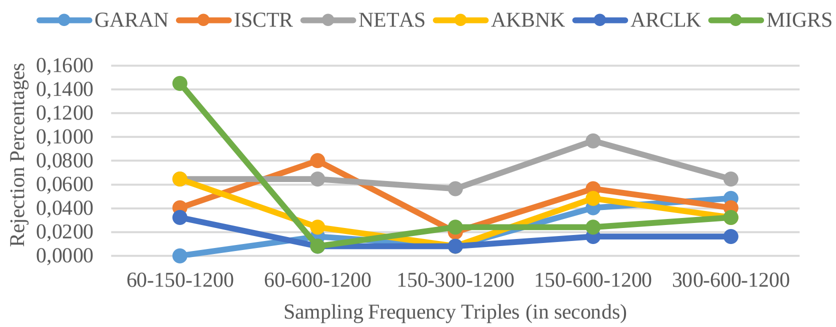
- the true price is generated as in Equation (3.1),
- $\int_0^T \sigma_s^4 \, ds < \infty$,
- the MMN satisfies Assumption 3.1,

then under the null hypothesis in (3.35), as $n, h, l \to \infty$, $\frac{n}{h} \to \infty, \frac{h}{l} \to \infty$, $V_{T,n,h,l} \xrightarrow{d} N(0,1)$.

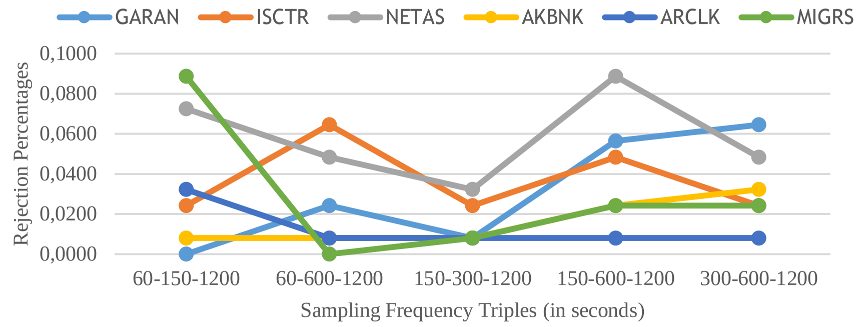
We suggest and show in Chapter 3, Section 3.4 that the same statistic $V_{T,n,h,l}$ can be used to test the constant variance of MMN increments under TTS due to the fact that the three pillars that are used by Awartani et al. [16] for building their test statistic are developed by Zhang et al. [111] under TTS in the first place. Therefore, since the alternative hypothesis is in harmony with the presence of autocorrelation in the MMN, following Awartani et al. [16] we argue in Chapter 3, Section 3.4 that the rejection of the null hypothesis in Equation (3.35) under CTS and/or TTS would provide the empirical evidence that we need to reject the assumption of an i.i.d MMN with a constant variance.

In this context, we calculate the $V_{T,n,h,l}$ statistic testing the null hypothesis in Equation (3.35) by comparing RVs that are calculated over different frequency triples, where various frequency pairs are combined with a sampling interval of 20 minutes, under sampling schemes CTS and TTS for raw and cleaned and aggregated data series. Frequency triples are (3,10,30), (3,15,30), (3,20,30), (6,15,30) (6,20,30) and (10,20,30) transactions under TTS, (60,150,1200), (60,600,1200), (150,300,1200), (150,600,1200) and (300,600,1200) seconds under CTS. Regarding data handling methods, we consider the test results for 10 different combinations of cleaning methods 4.ii.a and 4.ii.b with aggregation methods 5_i, 5_ii, 5_iii, 5_iv and 5_v once under CTS and once under TTS. For each day in the sample period of 124 days and each frequency triple, we run the aforementioned test at a 5% significance level.

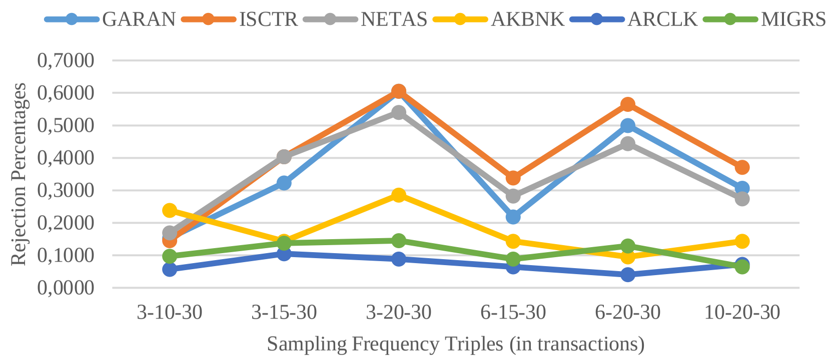
The results for each stock, for each frequency triple, under each data handling method and sampling scheme are given in the respective sections of Appendix E. To summarize, carefully selected examples of these findings comparing each stock for two different data handling procedures under CTS (4.ii.a_5.i versus 4.ii.b_5.i) and TTS (raw versus 4.ii.b_5.i) are reported in Figure 4.3.



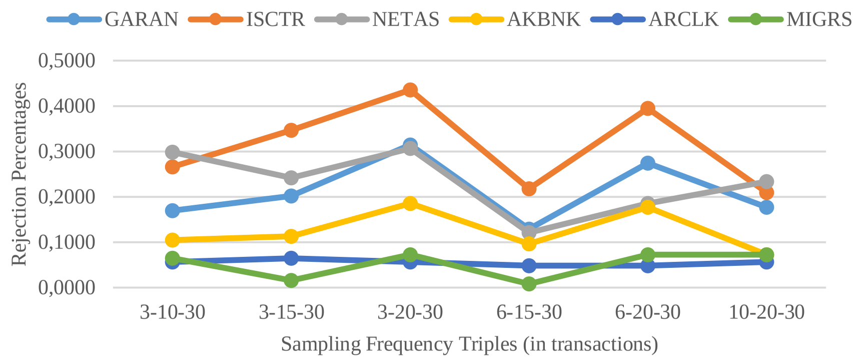
(a) CTS-Clean and Aggregated and Interpolated Data (4.ii.a-5.i)



(b) CTS-Clean and Aggregated and Interpolated Data (4.ii.b-5.i)



(c) TTS-raw



(d) TTS- 4.ii.b-5.i

Figure 4.3: Rejection percentages regarding the null hypothesis that the MMN increments have a constant variance independent of the sampling frequency. Sample period: second half of 2012 (a) CTS-Clean and Aggregated and Interpolated Data (4.ii.a-5.i) (b) CTS-Clean and Aggregated and Interpolated Data (4.ii.b-5.i) (c) TTS- raw (d) TTS-4.ii.b-5.i

Based on Appendix E and Figure 4.3, we learn the following:

- Under CTS, assuming an i.i.d MMN with a constant variance may be appropriate for frequencies lower than 1 min but under TTS, this assumption fails especially for liquid stocks. The finding regarding CTS is in line with results provided by Awartani et al. [16] and Pigorsch et al. [95].

    - Awartani et al. [16] conclude that for ultra-high frequencies, the assumption of an i.i.d noise with a constant variance is not verified by their empirical findings. Likewise, Pigorsch et al. [95] find that the constant variance assumption is rejected only at very high frequencies.

- The liquidity and the sampling schemes are discovered to be very influential on the rejection of the null hypothesis that the MMN increments have a constant variance independent of sampling frequency.

    - For liquid stocks, the assumption of an i.i.d MMN with a constant variance may be appropriate under CTS but under raw-TTS, for more than 50% of the days, the null of the MMN increments having constant variance is rejected for triples with very high frequencies combined with very low. This may be evidence of the i.i.d assumption not holding at frequencies lower than 15 transactions. As liquidity diminishes, rejection percentages also shrink to a point that, for the least liquid stocks ARCLK and MIGRS, the null of the MMN increments having constant variance is only rejected for a maximum of 13% of the days for any frequency triples. This finding implies that the assumption of an i.i.d MMN having constant variance across sampling frequencies may be more appropriate for MIGRS and ARCLK instead of the other more liquid stocks.

- Cleaning algorithms have a suppressive effect on rejection percentages particularly under TTS. For all stocks in our sample, cleaning and aggregating the data shift the rejection rate graphs downwards under TTS.

- The liquidity of the stocks and the data handling methods do not present a particular relationship under CTS: moving across the grid of cleaning and aggregation algorithm combinations does not change the rejection results substantially for any of the stocks in our sample.

CHAPTER 5

CONCLUSION

It is first pointed out by Andersen and Bollerslev [8] that the squared daily returns provide a poor approximation of the actual daily volatility. They suggest that more accurate estimates could be obtained by summing the squared intraday returns. Following this valuable contribution to the finance literature, Andersen et al. [12], Andersen et al. [10], and Barndorff-Nielsen and Shephard [24] are among the pioneers who studied ‘‘realized’’ volatility (RV) and its relevance in volatility measurement. RV exploits the information in high-frequency returns and estimates volatility by summing the squares of intraday returns sampled at very short intervals [51].

Barndorff-Nielsen and Shephard [24] prove the consistency of the RV and show that its asymptotic distribution is normal. In this context, if asset prices follow a Brownian semimartingale, return volatility can be estimated consistently and effortlessly by calculating the RV at the highest possible frequency. However, sampling returns as many times as possible without any further consideration on the characteristics of the high frequency data set may not be the right approach. UHFD should be analyzed first to determine whether there are any effects on the observed asset prices that become more significant as we increase the sampling frequency, whether the Brownian semimartingale representation of asset prices is appropriate, whether there are different ways of sampling returns, and, whether non-trading hours limit the accuracy of volatility estimation. All of these issues constitute the dimensions of volatility estimation using UHFD.

Among the aforementioned concerns of a researcher/practitioner who aims to measure return volatility using UHFD, one issue stands out: the observed prices are contaminated with a noise component which represents the aggregate effect of all market microstructure frictions. If there is such a contamination, then the quadratic variation of the observed prices calculated over the highest frequency possible does not simply converge to the integrated variance of the true prices because an asymptotic bias appears due to the existence of MMN [10], [24], [25]. In order to examine how RV deviates from the IV as we increase the sampling frequency and to come up with methods to handle these deviations (mitigation of the MMN effects on RV measures), we first have to make some assumptions regarding the statistical features of the MMN. The most popular assumption in the RV literature states that MMN is a sequence of i.i.d random variables with zero mean, constant variance and finite fourth moment, while MMN and true prices are orthogonal to each other for each $t \in [0, T]$. Therefore, although it is of great importance to mitigate the effect of the MMN in the estimation of the true price volatility using high frequency data, success of the available methods in literature to suppress the MMN effects must be considered only if empirical evidence from developed or developing markets support the assumptions made by these methods regarding MMN.

We realize that empirical evidence including visual and formal tests on MMN structure should be collected taking into account the dimensions of volatility estimation using high frequency data as these dimensions might result in impairment to validity of methods adopted to handle the MMN in estimation of the IV in the first place. However, none of the available literature on the IV estimation using UHFD takes into account all of these dimensions simultaneously. Besides, the literature does not touch the issue of how to examine existence and statistical features of the MMN under sampling schemes other than CTS formally. Meanwhile, the published literature on the IV estimation using UHFD relies on data coming from stock markets of developed economies such as US or Japan and the literature lacks research on volatility estimation and MMN structure with empirical evidence from developing markets.

Additionally, we recognize that the generally accepted definition of an asset's liquidity -the speed and ease of the sale of a stock at a price not too different from the price at which the seller would be able to get if s/he opted to wait longer- may not be appropriate to use for analyzing the MMN embedded in the observed stock prices, especially if such analyses are carried out under CTS. This observation is underlined particularly when there are long periods between two consecutive transactions and the interpolation method selected causes an artificially introduced additional autocorrelation in returns. Therefore, there is room in the literature about offering alternative methods for measuring the liquidity of assets by taking into account how evenly tick by tick data are distributed in time.

In this framework, In Chapter 2, we elaborate on

1. the characteristics of UHFD sets and the methods to detect and clean errors, ways to aggregate simultaneous observations, approaches on interpolating the data for constructing artificial series if returns are to be calculated based on a fixed frequency over time,

2. observed prices being contaminated by a MMN,

3. how the semimartingale representation of the true/efficient/equilibrium asset prices necessitates implicit decisions regarding the structure of the MMN and whether the asset prices should be taken as continuous or discontinuous,

4. popular assumptions regarding the statistical features of the MMN,

5. the sampling schemes and their relevance to RV estimation, and,

6. how the presence of non-trading hours is reflected in daily RV estimations by various researchers and the types of problems that accompany such approaches.

After providing comparative discussions on the available literature regarding the dimensions of volatility estimation using UHFD in Chapter 2, we show in Chapter 3 that formal tests developed under CTS by Awartani et al. [16] can be used under TTS as well for determining whether there is any statistically significant asymptotic bias in the RV estimator due to the existence of MMN and whether the MMN increments have constant variance independent of the sampling frequency. In our discussions, we benefit from the fact that if the true price of an asset fulfills a Brownian semimartingale specification then

- the RV calculated over the highest data frequency should give the best possible estimate for the IV both under CTS and TTS provided that there is no MMN contamination,

- when observed prices are not contaminated by a MMN, the estimation error of RV only stems from the discretization error that appears as a result of the number of observations per trading day being limited in practice, regardless of the sampling scheme,

- in the presence of an additive MMN and if the MMN is a sequence of i.i.d random variables with a constant mean and a finite fourth moment, its increments having constant variance, while MMN and true price are independent, then the conditional asymptotic bias of RV calculated using observed prices explodes to infinity as the number of observations increases,

- the test statistic developed by Awartani et al. [16] to examine the existence of MMN under CTS also can be employed for the same task under TTS since the two pillars supporting the Awartani et al. [16] test statistic are shown to hold under TTS in [111] and [91],

- the test statistic developed by Awartani et al. [16] to find out if the MMN increments possess a constant variance orthogonal to sampling frequency can also be used for the same purpose under TTS since the three pillars supporting the Awartani et al. [16] test statistic are developed under TTS in the first place.

In the next step, in Chapter 4, we gather evidence from Borsa Istanbul National Equity Market regarding the validity of the most popular assumptions on the market microstructure noise, i.e., whether the aggregate effect of deviations from perfect capital markets lead to i.i.d MMN or whether the unobservable true prices and MMN are independent, under combinations of CTS or TTS with data handling methods and if liquidity is important in terms of the assumed MMN structure. For these analyses, we use tick by tick transaction data for 6 stocks listed on Borsa Istanbul National Equity Market for the sample period of 01.07.2012-31.12.2012, (a total of 124 trading days). The data are retrieved directly from Borsa Istanbul. The 6 stocks are selected with the purpose of having an acceptable level of diversification in the sample.
As explained earlier, if the previous tick method is used to construct return series under CTS then there is a risk of introducing artificial serial correlation into the returns for inactively traded stocks. In order to avoid this problem, in Chapter 4 we propose a new

approach for classifying stocks with respect to their liquidity in the high frequency data setting. Our approach defines the liquidity as the number of all time stamps (under calendar time) having at least one transaction entry. In other words, during the sample period, if the number of sessions with no transactions during a given time period, such as 10 minutes, for a stock exceeds the same number for another stock, then the first stock is characterized as being less liquid. Coherent with our definition, we examine the raw data to find the number of continuous auction sessions in which the maximum duration between two consecutive transactions exceeds a set of arbitrarily selected thresholds. Based on this definition, we categorize three stocks in our sample set as more liquid, one stock as moderately liquid, and two stocks as illiquid. This categorization is relative and specific to the existing sample set in the sense that inclusion of additional stocks with different trading activities has the potential of inducing a different categorization.

Before any return calculations under calendar or transaction time sampling are carried out, we apply data handling methods as combinations of cleaning and aggregation algorithms. With these methods, we address the possibility that the UHFD we use in our analyses may include erroneous entries due to the recording algorithms of the stock exchange and there may be transactions that are attached to a single time stamp due to the asynchronous nature of trading. After covering all the available literature on data handling methods, we design a data cleaning methodology in four steps, where the first three steps aim to detect obvious errors such as zero prices or volumes, and step four, with nine options regarding detection, finds and removes outliers. By considering the number of deleted entries under each option of step 4 and taking into account the overscrubbing and underscrubbing risks, we conclude that only two cleaning methods should be included in the analysis of testing existence and statistical features of MMN under different data handling and sampling schemes. These two methods delete entries for which the price deviates by more than a threshold from an average of daily or arbitrarily selected neighborhood prices. After cleaning, we apply five different aggregation rules, resulting in a total of ten different data handling method combinations.

Now that all obvious errors, outliers and simultaneous ticks are cleaned and/or aggregated, we explore whether there are any changes in the common characteristics of UHFD due to the application of these cleaning and aggregation procedures since we want to make certain that our data handling procedures do not overscrub the UHFD set and distort its original traits such as discreteness, irregular temporal spacing and diurnal patterns. Regarding the discreteness of prices, we observe that it holds under CTS and TTS for all data handling methods and stocks in our sample because as long as there are price ticks, especially when the ticks are large in size, discreteness seems to be a natural occurrence. On irregular temporal spacing, since the errors or transactions recorded at the same second are not the cause of irregular temporal spacing, this characteristic is found to hold for all data handling methods and stocks in our sample as well. In order to check the diurnal patterns in the sample stocks, we calculate average transaction volumes, absolute percentage returns, average trade intensities and average absolute returns over 10-minute intervals for each stock 11 times, i.e., one time for raw data and 10 times for cleaned and aggregated data. Results suggest that, under CTS, all stocks in our sample exhibit significant diurnal patterns in returns, trading intensity and volume, and these patterns look exactly the same regardless of the various combinations of cleaning and aggregation methods applied.

In order to calculate returns and RV series under CTS, we prepare 11 different (one uncleaned, 10 cleaned and aggregated) artificial transaction time series for each stock in our sample. These series are formed by applying the previous tick method for continuous auction sessions and summarizing opening and closing session information as entries at 09:50:00, 14:20:00 and 17:30:00 if there are no entries in the original data for these time stamps. The resulting artificial time series has 9601 entries for the first (morning) sessions and 11401 entries for the second (afternoon) sessions in each trading day. Hence, the total number of entries under CTS in a trading day is 21002.

For calculating returns and RV series under TTS, since sampling prices under TTS does not require the artificial construction of time series, after the data sets are cleaned of errors and all simultaneous entries are successfully aggregated, we produce 11 different (1 uncleaned, 10 cleaned and aggregated) transaction time series for each stock in our sample. In accordance with our discussions on calculating returns and RVs when there are lunch breaks during the trading day, we sample prices at the appropriate frequencies and calculate the RV over trading hours by adding the RV from session 1 and RV from session 2. Following arguments in Chapter 2, Section 2.6, we do not make any adjustments for the non-trading hours since the existence of diurnal patterns in trading intensity and returns imply that volatility accumulated during non-trading hours is reflected in these patterns, regardless of sampling scheme.

The literature provides abundant evidence that points to the existence of first order autocorrelation and volatility clustering in intraday returns. Accordingly, checking to see whether the data handling procedures and/or sampling schemes alter the return structure becomes vital before continuing with the analyses on MMN significance and structure. By comparing autocorrelation and partial autocorrelation functions of 60-second and/or 600-second absolute returns and log returns under CTS (clean and aggregated and interpolated) as well as absolute returns, log returns and durations in seconds from one transaction to the next under TTS (raw versus clean and aggregated) for each stock in our sample for December 2012, we observe that for stocks in our sample

- volatility clustering is verified,
- in line with the findings of Griffin and Oomen [55], return dynamics in transaction time are different from those in calendar time,
- any combination of cleaning and aggregation methods (compared to other combinations) does not cause any major change in total and partial correlation structures once we move under a sampling scheme, either TTS or CTS,
- data handling under TTS yields different PACF structures in absolute and/or log returns compared to results produced with raw data under TTS.

In addition to assessing the effects (if any) of data handling and/or sampling schemes on return dynamics in the form of temporal dependence, we also inquire whether the sampling schemes and/or data handling procedures have a significant impact on RV dynamics. For each sample stocks, each RV series under each sampling scheme, for each frequency and for each cleaning and aggregation method combination, we calculate preliminary statistics and conduct ACF and, PACF analyses and unit root tests. Results show that

- liquidity matters in terms of RV normality such that the sampling scheme or cleaning and aggregation or sampling frequency or session-daily calculation do not change the non-normality of RV series if the stock is illiquid,

- unlike results for normality, liquidity does not have an effect on the log normality of RV series,

- the session-daily choice and the frequency and sampling schemes have an effect on the log normality of RV series,

- autocorrelation decay patterns and structures of RV series are affected by the sampling scheme, regardless of liquidity,

- decreasing the sampling frequency depresses the autocorrelation structure of the RV series under CTS regardless of liquidity or session-daily calculation,

- cleaning and aggregation methods do not affect the autocorrelation decay patterns of the RV series significantly,

- liquidity does not alter the stationarity of the RV series in a specific way,

- regardless of liquidity, under raw-TTS, all RV series, both the session or daily at all frequencies are stationary at the 5% significance level. Non-stationarity becomes a problem only under CTS for some RV series.

In the next step, in order to determine the frequency at which MMN becomes evident as well as the direction of the bias, we compare the VSPs produced under CTS for 10 different combinations of cleaning rules and aggregation methods with the VSPs produced under TTS for raw data and cleaned and aggregated data. In all possible dimensions (sampling scheme, liquidity, data handling methods, and session-daily calculation) for all stocks, we find visual proof regarding the existence of MMN and a positive relationship between the noise increment and the true price return, under both CTS and TTS. More specifically, sampling intervals of 300 seconds under CTS and 15 transactions under TTS appear to be the thresholds at which MMN begins to dominate the RV of the observed prices.

Following the visual inspection via VSPs, we employ formal statistical tests of the no noise and the noise with increment of constant variance assumptions proposed by Awartani et al. [16]. Their approach depends on the comparison of two or more realized volatilities computed over different frequencies under CTS where the artificial construction of price series ensures that prices are regularly spaced in time. In line with the arguments and proofs in Chapter 3 that the same tests can be used under TTS where prices are scattered irregularly in time, we examine the formal test results under CTS and TTS for 10 different combinations of data handling methods compared with raw data in order to confirm the existence of MMN and the constant variance of MMN increments. Test results show that

- MMN exhibits statistically significant existence under both CTS and TTS for all stocks regardless of the data handling methods and liquidity such that MMN starts to accentuate as the sampling interval converges to 10-15 transactions under TTS and 250-300 seconds under CTS (in conformity with the results from VSPs)

- liquidity and data handling methods matter under TTS such that for all stocks in our sample the lower the liquidity, the lower the rejection percentage at all frequency-pairs under TTS (raw or clean and aggregated) and data handling does not change the downward trend in the rejection percentages under TTS, but makes it steeper,

- there is evidence of positive correlation between noise and efficient price, which is again in conformity with exploding VSPs,

- liquidity and sampling schemes are very influential on the rejection of the null hypothesis that the MMN increments have constant variance that is independent of sampling frequency; in particular, under CTS, assuming an i.i.d MMN with constant variance is appropriate for frequencies lower than 1 min but under TTS, this assumption fails especially for liquid stocks,

- data handling has a suppressive effect on the rejection percentages of the null hypothesis that the MMN increments have constant variance that is independent of sampling frequency particularly under TTS while it does not exhibit a particular effect under CTS.

All in all, in this Thesis we

- discuss many dimensions/aspects of volatility estimation,

- consider how data handling methods in the form of cleaning and aggregation affect the characteristics of UHFD and whether the widely accepted outlier handling methods end up overscrubbing or underscrubbing the data,

- examine what happens to return and RV series dynamics under varying combinations of sampling schemes and data handling methods while controlling for liquidity of the stock,

- examine visual and statistical evidence regarding the existence and/or statistical features of MMN under varying combinations of sampling schemes and data handling methods and whether our findings on the MMN structure are robust with respect to the liquidity of the stocks

simultaneously. Our efforts justify our hesitance regarding deriving conclusions on the significance and structure of MMN using empirical evidence without considering as many dimensions of volatility estimation using UHFD as possible because we show that liquidity, sampling scheme and data handling methods have the potential to affect the return and RV dynamics. If these dimensions are not taken into account, then findings regarding the validity of popular assumptions about the statistical features of MMN have the potential of telling different tales in different research settings. Thus, we hope that our findings will serve as additional bricks in the wall of high frequency finance research and future researchers will benefit from the theoretical discussions/proofs made and empirical evidence gathered in this Thesis.

REFERENCES

[1] Aït-Sahalia, Y., Mancini, M., 2008. Out of sample forecasts of quadratic variation. Journal of Econometrics 147(1), 17-33.

[2] Aït-Sahalia, Y., Mykland,, P., Zhang, L., 2005. How often to sample a continuous-time process in the presence of market microstructure noise. Review of Financial Studies 18, 351-416.

[3] Aït-Sahalia, Y., Mykland,, P., Zhang, L., 2011. Ultra high frequency volatility estimation with dependent microstructure noise. Journal of Econometrics 160, 160–175.

[4] Aït-Sahalia, Y., Jacod, J., 2012. Analyzing the spectrum of asset returns: Jump and volatility components in high frequency data. Journal of Econometric Literature 50(4), 1007-1050.

[5] Aït-Sahalia, Y., Jacod, J., 2014. High Frequency Financial Econometrics. Princeton University Press.

[6] Aït-Sahalia, Y., Fan, J., Li, Y., 2013. The leverage effect puzzle: Disentangling sources of bias at high frequency. Journal of Financial Economics 109, 224-249.

[7] Andersen, T.G., Bollerslev, T., 1997. Intraday periodicity and volatility persistence in financial markets. Journal of Empirical Finance 4(2), 115-158.

[8] Andersen, T.G., Bollerslev, T., 1998. Answering the skeptics: Yes, standard volatility models do provide accurate forecasts. International Economic Review 39, 885-905.

[9] Andersen, T.G., Bollerslev, T., Cai, J., 2000. Intraday and interday volatility in the Japanese stock market, Journal of International Financial Markets, Institutions and Money 10(2), 107-130.

[10] Andersen, T.G., Bollerslev, T., Diebold, F.X., Ebens, H., 2001. The distribution of realized stock return volatility. Journal of Financial Economics 61, 43– 76.

[11] Andersen, T.G., Bollerslev, T., Diebold, F.X., Labys, P., 2000. Great realizations. Risk 13, 105-108.

[12] Andersen, T.G., Bollerslev, T., Diebold, F.X., Labys, P., 2001. The distribution of exchange rate volatility. Journal of the American Statistical Association 96, 42– 55.

[13] Andersen, T.G., Bollerslev, T., Diebold, F.X., Labys, P., 2003. Modeling and forecasting realized volatility. Econometrica 71, 579– 625.

[14] Andersen, T. G., Bollerslev, T., Meddahi, N., 2011. Realized volatility forecasting and market microstructure noise. Journal of Econometrics 160, 220-234.

[15] Areal, N., Taylor, S.J., 2002. The realized volatility of FTSE-100 futures prices. Journal of Futures Markets 22, 627–648.

[16] Awartani, B., Corradi, V., Distaso, W., 2009. Assessing market microstructure effects via realized volatility measures with an application to the Dow Jones Industrial Average Stocks. Journal of Business & Economic Statistics 27, 251-265.

[17] Bachelier, L., Etheridge, A., Davis, M. L., 2006. Louis Bachelier's theory of speculation : The origins of modern finance. Princeton University Press.

[18] Baillie, R, Bollerslev. T. 1991. Intra-day and inter-market volatility in foreign exchange rates. Review of Economic Studies 58, 565–585.

[19] Bandi, F. M., Russell, J. R., 2006. Separating microstructure noise from volatility. Journal of Financial Economics 79, 655-692.

[20] Bandi, F. M., Russell, J. R. 2008. Microstructure noise, realized variance, and optimal sampling. Review of Economic Studies 75, 339–369.

[21] Bandi, F. M., Russell, J. R., Yang, C., 2008. Realized volatility forecasting and option pricing. Journal of Econometrics 147, 34-46.

[22] Bandi, F. M., Russell, J. R., Yang, C., 2013. Realized volatility forecasting in the presence of time-varying noise. Journal of Business and Economic Statistics 31(3), 331-345.

[23] Bannouh, K., Martens, M., Dijk D., 2013. Forecasting volatility with the realized range in the presence of noise and non-trading. The North American Journal of Economics and Finance 26, 535-551.

[24] Barndorff-Nielsen, O. E., Shephard, N., 2002a. Econometric analysis of realized volatility and its use in estimating stochastic volatility models. Journal of the Royal Statistical Society Series B 64, 253–280.

[25] Barndorff-Nielsen, O. E., Shephard, N., 2002b. Estimating quadratic variation using realized variance. Journal of Applied Econometrics 17(5), 457-477.

[26] Barndorff-Nielsen, O. E., Shephard, N., 2004. Power and bipower variation with stochastic volatility and jumps. Journal of Financial Econometrics 2, 1-37.

[27] Barndorff-Nielsen, O. E., Hansen, P. R., Lunde, A., Shephard, N., 2009. Realized kernels in practice: Trades and quotes. Econometrics Journal 12, C1-C32.

[28] Barndorff-Nielsen, O. E., Hansen, P. R., Lunde, A., Shephard, N., 2011. Subsampling realized kernels. Journal of Econometrics 160(1), 204–219.

[29] Barucci, E., Reno R., 2002. On measuring volatility of diffusion processes with high frequency data. Economics Letters 74, 371–378.

[30] Benston, G.J., 2007. Did NASDAQ market makers successfully collude to increase spreads? A reeaxamination of evidence from stocks that move from NASDAQ to New York or American stock exchanges. Financial Markets Group, London School of Economics, Special Paper 170. Available at http://fmg.lse.ac.uk or SSRN.

[31] Bessembinder, H., 1997. The degree of price resolution and equity costs. Journal of Financial Economics 45, 9-34.

[32] Bessembinder, H., Maxwell, W., Venkataraman, K., 2006. Market transparency, liquidity externalities, and institutional trading costs in corporate bonds. Journal of Financial Economics 82 (2), 251-288.

[33] Bildik, R., 2001. Intra-day seasonalities on stock returns: Evidence from the Turkish Stock Market. Emerging Markets Review 2, 387-417.

[34] Billingsley, P., 2012. Probability and measure. Wiley.

[35] Black, F., 1971. Towards a fully automated exchange. Financial Analysts Journal 27, 28-35.

[36] Black, F., Scholes, M., 1973. The pricing of options and corporate Liabilities. Journal of Political Economy 81, 637-654.

[37] Brownlees, C. T., Gallo, G. M., 2006. Financial econometric analysis at ultra-high frequency: Data handling concerns. Universita' di Firenze, Dipartimento di Statistica G. Parenti Working Paper No. 2006-3. Available at SSRN: http://ssrn.com/abstract=886204 or http://dx.doi.org/10.2139/ssrn.886204

[38] Brownlees, C. T., Gallo, G. M., 2010. Comparison of volatility measures: A risk management perspective. Journal Of Financial Econometrics 8(1), 29-56.

[39] Chung, K.H., Chuwonganant, C., McCormick, T.D., 2004. Order preferencing and market quality on NASDAQ before and after decimalization. Journal of Financial Economics 71(3), 581–612.

[40] Chow, Y. F., Lam, J. T. K., Yeung, H. S., 2009. Realized volatility of index constituent stocks in Hong Kong. Mathematics and Computers in Simulation 79 (9), 2809-2818.

[41] Chordia, T., Roll, R., Subrahmanyam, A., 2001. Market liquidity and trading activity. Journal of Finance 56(2), 501-530.

[42] Cont, R., Tankov, P., 2004. *Financial Modelling with Jump Processes*. Chapman & Hall

[43] Curci, G., Corsi, F., 2012. Discrete sine transform for multi-scale realized volatility measures. Quantitative Finance 12 (2). 263–279.

[44] Dacorogna, M.M., Müller, U.A., Jost, C., Pictet, O.V., Ward, J.R., 1995. Heterogeneous real-time trading strategies in the foreign exchange market. European Journal of Finance 1, 383–403.

[45] Dacorogna, M.M., Gençay, R., Müller, U.A., Olsen, R.B., Pictet, O.V., 2001. An introduction to high-frequency finance. San Diego: Academic Press.

[46] Ebens, H., 1999. Realized stock index volatility. Unpublished working paper: 420. Department of Economics, Johns Hopkins University.

[47] Elliott, R.J.,Kopp, P.E., 2005. Mathematics of Financial Markets. New York: Springer Finance.

[48] Engle, R., Russell, J. R., 1998. Autoregressive conditional duration: A new model for irregularly spaced transaction data. Econometrica 66 (5), 1127-1162.

[49] Engle, R., Russell, J. R., 2010. Analysis of High Frequency Data. Handbook of Financial Econometrics: Volume 1: Tools and Techniques (Eds. Ait Sahalia, Y., Hansen., L.P.). Amsterdam and Boston: Elsevier, North Holland.

[50] Falkenberry, T. N., 2002. High frequency data filtering. Technical report, Tick Data.

[51] Fleming, J., Kirby, C., Ostdiek, B., 2003. The economic value of volatility timing using ‘‘realized’’ volatility. Journal of Financial Economics 67, 473–509.

[52] Gatheral, J., Oomen, R.C.A, 2010. Zero-intelligence realized variance estimation. Finance and Stochastics 14, 249-283.

[53] German, M., 1976. Market microstructure. Journal of Financial Economics 3(3), 257-275.

[54] Ghysels, E., Sinko A., 2011. Volatility forecasting and microstructure noise. Journal Of Econometrics 160, 257-271.

[55] Griffin, J.E., Oomen, R.C., 2008. Sampling returns for realized variance calculations: Tick time or transaction time? Economic Review 27(1–3), 230–253.

[56] Gutiérrez, J. M. P., Gregori, J. F., 2008. Clustering Techniques applied to outlier detection of financial market series using a moving window filtering algorithm. European Central Bank, Working Paper Series No:948.

[57] Hall, P., Heyde, C.C., 1980. Martingale limit theory and its applications. Academic Press, Boston.

[58] Hansen, P. R., Lunde, A., 2005a. A realized variance for the whole day based on intermittent high-frequency data, Journal of Financial Econometrics 3(4), 525-554.

[59] Hansen, P. R., Lunde, A., 2005b. A forecast comparison of volatility models: Does anything beat a GARCH(1,1)?, Journal of Applied Econometrics 20, 873-889.

[60] Hansen, P. R., Lunde, A., 2006a. Consistent ranking of volatility models. Journal of Econometrics 131, 97-121.

[61] Hansen, P. R., Lunde, A., 2006b. Realized variance and market microstructure noise. Journal of Business & Economic Statistics 24 (2), 127-161.

[62] Harris, L., 1986. A transaction data study of weekly and intradaily patterns in stock returns. Journal of Financial Economics 16(1), 99-117.

[63] Harrison, J.M., Pliska, S.R., 1981. Martingales and stochastic integrals in the theory of continuous trading. Stochastic Processes and their Applications 11(3), 215–260.

[64] Hasbrouck, J., 2007. Empirical market microstructure: The institution economics, and econometrics of securities trading. Oxford University Press.

[65] Huang, R.D., Stoll, H.R., 1996. Dealer versus auction markets: A paired comparison of execution costs on NASDAQ and the NYSE. Journal of Financial Economics 41(3), 313–357.

[66] Hol, E., Koopman S., 2002. Stock index volatility forecasting with high frequency data. Tinbergen Institute Discussion Paper No. 2002-068/4 Index Volatility.

[67] Ishida, I., Watanabe, T., 2009. Modeling and forecasting the volatility of the Nikkei 225 realized volatility using the ARFIMA-GARCH model. Global COE Hi-Stat Discussion Paper Series 032. Center for International Research on the Japanese Economy, University of Tokyo, Hitotsubashi University.

[68] Jacod, J., Li, Y., Mykland, P.A., Podolskij, M., Vetter, M., 2009. Microstructure noise in the continuous case: The pre-averaging approach. Stochastic Processes and their Applications 119(7), 2249-2276.

[69] Jacod, J., Protter, P., 1998. Asymptotic error distributions for the Euler Method for stochastic differential equations. Annals of Probability 26, 267–307.

[70] Jacod., J., Shiryaev, A.N., 2003. Limit Theorems for Stochastic Processes. Springer.

[71] Karatzsas, I., Shreve, S. E., 1991. Brownian Motion and Stochastic Calculus. New York: Springer Verlag.

[72] Kijima, M., 2003. Stochastic Processes with applications to Finance. Chapman& Hall/CRC.

[73] Koopman, S.J., Jungbacker, B., Hol, E., 2005. Forecasting daily variability of S&P 100 stock index using historical, realized and implied volatility measures. Journal of Empirical Finance 12, 445-475.

[74] Koopman, S.J., Scharth, M., 2013. The analysis of stochastic volatility in the presence of daily realized measures. Journal of Financial Econometrics, 11 (1), 76—115.

[75] Kyprianou, A.E., 2006. Introductory lectures on fluctuations of Levy processes with applications. Berlin ; New York : Springer.

[76] Lee S., Mykland, P., 2012. Jumps in equilibrium prices and market microstructure noise. Journal of Econometrics 168(2), 396–406.

[77] Li, Y., Mykland, P., 2007. Are volatility estimators robust with respect to modeling assumptions?, Bernoulli 13(3), 601–622.

[78] Li, Y., Mykland, P., Renault, E., Zhang, L. Zheng, X., 2014. Realized volatility when sampling times are possibly endogenous. Econometric Theory 30, 580–605.

[79] Madhavan, A., 2002. Market microstructure: A practioners guide. Financial Analysts Journal 58(5), 28-43.

[80] Malliavin, P., Mancino, M.E., 2002. Fourier series method for measurement of multivariate volatilities. Finance and Stochastics 6, 49–61.

[81] Marshall, B., Nguyen, N., Visaltanachoti, N., 2012. Commodity liquidity measurement and transaction costs. Review of Financial Studies 25 (2), 599–638.

[82] Martens, M., 2002. Measuring and forecasting S&P 500 index futures volatility using high-frequency data. Journal of Futures Markets 22(6), 497–518.

[83] Masuda, H., Morimoto, T., 2012. Optimal weight for realized variance based on intermittent high-frequency data. Japanese Economic Review [serial online] 63(4), 497-527.

[84] Merton, R.C., 1971. Optimum consumption and portfolio rules in a continuous-time model. Journal of Economic Theory 3, 373-413.

[85] Merton, R.C., 1973. An intertemporal capital asset pricing model. Econometrica 41(5), 867-887.

[86] Merton, R.C., 1973. Theory of rational option pricing. Bell Journal of Economics and Management Science 4, 141-183.

[87] Müller, U., Dacorogna, M., Olsen, R., Pictet, O.V., Schwarz, M., Morgenegg, C., 1990. Statistical study of foreign exchange rates, empirical evidence of a price change scaling law, and intraday analysis. Journal of Banking & Finance 14(6), 1189-1208.

[88] Muth, J. F., 1961. Rational expectations and the theory of price movements. Econometrica 29 (3), 315-335.

[89] Mykland, P., Zhang, L., 2006. Anova for diffusions and Itô Processes. The Annals of Statistics 34(4), 1931-1963.

[90] Mykland, P., Zhang, L., 2009. Inference for continuous semimartingales observed at high frequency. Econometrica 77(5), 1403-1445.

[91] Mykland, P., Zhang, L., 2012. The econometrics of high frequency data. Statistical Methods for Stochastic Differential Equations (eds. Kessler, M., Lindner, A., Sørensen, M.). CRC Press.

[92] O'Hara, M., 1995. Market Microstructure Theory. Cambridge, MA: Blackwell.

[93] Oomen, R. C. A, 2005. Properties of bias-corrected realized variance under alternative sampling schemes. Journal of Financial Econometrics 3, 555–577.

[94] Oomen, R. C. A, 2006. Properties of realized variance under alternative sampling schemes. Journal of Business and Economic Statistics 24, 219–237.

[95] Pigorsch, C., Pigorsch, U., Popov, I., 2011. Volatility estimation based on high frequency data. Handbook of Computational Finance (eds. Jin-Chuan Duan, James E. Gentle and Wolfgang Härdle). New York: Springer.

[96] Poon, S.,H., 2005. A practical guide to forecasting financial volatility, John Wiley and Sons Ltd.

[97] Protter, P., 1990. Stochastic integration and differential equations: A new approach. Springer-Verlag.

[98] Ross, S., 1976. The arbitrage theory of capital asset pricing. Journal of Economic Theory 13, 341-360.

[99] Rossi, M., 2014. Realized volatility, liquidity, and corporate yield spreads. Quarterly Journal Of Finance 4(1), 1-42.

[100] Sarr, A., Lybek, T., 2002. Measuring liquidity in financial markets. IMF Working Paper No: WP/02/232. Available at https://www.imf.org

[101] Shephard, N., Shephard, K., 2010. Realizing the future: Forecasting with high-frequency-based volatility (heavy) models. Journal Of Applied Econometrics 25(2), 197-231.

[102] Şen A., 2009. Optimizable multiresolution quadratic variation filter for high-frequency financial data. Unpublished Master Thesis, Department of Financial Mathematics, Middle East Technical University.

[103] Ubukata, M., Watanabe, T., 2014. Pricing Nikkei 225 options using realized volatility. Japanese Economic Review 65, 431–467.

[104] Takahashi, M., Omori, Y., Watanabe, T., 2009. Estimating stochastic volatility models using daily returns and realized volatility simultaneously. Computational Statistics and Data Analysis 53, 2404-2426.

[105] Takaishi T, Chen T, Zheng Z., 2012. Analysis of realized volatility in two trading sessions of the Japanese stock market. Progress Of Theoretical Physics Supplement [serial online] 194, 43-54.

[106] Verousis, T., Gwilym, O., 2010. An improved algorithm for cleaning ultra high-frequency data. Journal Of Derivatives & Hedge Funds 15(4), 323-340.

[107] Wood, R.A., McInish, T.H., Ord, J.K., 1985. An investigation of transactions data for NYSE stocks. Journal of Finance 40(3), 723-739.

[108] Wrampelmeyer, J., 2011. Ambiguity, illiquidity, and hedge funds: An analysis of recent developments and current research topics in post-crisis financial markets. PhD Dissertation, University of Zurich, Faculty of Economics.

[109] http://www.borsaistanbul.com/en/products-and-markets/markets/equity-market

[110] Zhang, L., 2006. Efficient estimation of stochastic volatility using noisy observations: A multiscale approach. Bernouilli 12, 1019-1043.

[111] Zhang, L., Mykland, P. A., Aït-Sahalia, Y., 2005. A tale of two time scales: Determining integrated volatility with high frequency data. Journal of American Statistical Association 100, 1394-1411.

[112] Zhou, B., 1996. High-frequency data and volatility in foreign-exchange rates. Journal of Business and Economic Statistics 14, 45-52.

APPENDIX A

PROOF OF PROPOSITION 3.1

The proof depends on showing that $\mathcal{J}_t$ defined as $\left(X_t - X_{t_i}\right)^3 dX_t$ converges to 0 in probability at a rate of $1/(h-1)$ where $[X,X,X,X]_t^{\mathcal{H}}$=$4\mathcal{J}_t + \frac{3}{2}\, d[\mathcal{Z},\mathcal{Z}]_t$. To do so, we first focus on quadratic variation of $\mathcal{J}_t$ to benefit from the Burkholder-Davis-Gundy inequality which connotes that there are universal constants c_p and C_p so that for all generic continuous martingales N_t,

$$c_p \|[N,N]_T\|_{p/2}^{1/2} \leq \left\| \sup_{0\leq t\leq \mathrm{T}} |N_t| \right\|_p \leq C_p \|[N,N]_T\|_{p/2}^{1/2},$$

where $\|N_t\| = (\mathbb{E}[|N_t|^p])^{1/p}$, and

$$C_p^2 = q^p \frac{p(p-1)}{2}, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

In this effort, the quadratic variation of $\mathcal{J}_t$ is as follows:

$$\begin{aligned}[\mathcal{J},\mathcal{J}]_t &= \sum_{t_{i+1}\leq t} \left(X_{t_{i+1}} - X_{t_i}\right)^6 d[X,X]_t + \left(X_t - X_{t_*}\right)^6 d[X,X]_t \\ &= \sum_{t_{i+1}\leq t} \int_{t_i}^{t_{i+1}} \left(X_s - X_{t_i}\right)^6 d[X,X]_s + \int_{t_*}^{t} \left(X_s - X_{t_*}\right)^6 d[X,X]_s.\end{aligned}$$

Moreover, application of Itô's formula to $f(X_t) = \left(X_t - X_{t_i}\right)^8$ with $X_0 = 0$, under Assumption 3.3, yields

$$\left(X_t - X_{t_i}\right)^8 = f(X_0) + \int_0^t 8\left(X_s - X_{t_i}\right)^7 dX_s + \int_0^t 28\left(X_s - X_{t_i}\right)^6 d[X,X]_s,$$

where on a grid $\mathcal{H}$, $[X;8]_t^{\mathcal{H}}$ denotes $\left(X_t - X_{t_i}\right)^8$ and

$$\left(X_t - X_{t_i}\right)^8 = \sum_{t_{i+1} \le t} \left(X_{t_{i+1}} - X_{t_i}\right)^8 + \left(X_t - X_{t_*}\right)^8.$$

Hence,

$$[\mathcal{J},\mathcal{J}]_t = \frac{1}{28}[X;8]_t^{\mathcal{H}} - \frac{8}{28}\int_0^t \left(X_s - X_{t_i}\right)^7 dX_s. \qquad \text{(A.1)}$$

By Theorem 20 in Chapter 2, p.56 of [97], because under Assumptions 3.3 and 3.4, X_t is locally square integrable local martingale and $\left(X_s - X_{t_i}\right)^7$ is an adapted process with cádlág paths, the second term on the RHS of the Equation (A.1) is a locally square integrable local martingale.

Additionally, for any stopping time $\tau \le \mathrm{T}$, $[X;8]_\tau^{\mathcal{H}} = \sum_i \left(X_{\tau \wedge t_{i+1}} - X_{\tau \wedge t_i}\right)^8$.

To show that supremum of $|\mathcal{J}_t|$ converge to 0 in probability at an order of $(h-1)^{-1}$, i.e., $\mathbb{P}\left[h \sup_{0 \le t \le \mathrm{T}} |\mathcal{J}_t| > \delta\right] = 0$, for any $\delta > 0$ we need to look at $[\mathcal{J},\mathcal{J}]_t$ and prove that

$$\left\| \sup_{0 \le t \le \mathrm{T}} |\mathcal{J}_t| \right\|_p \le C_p \|[\mathcal{J},\mathcal{J}]_T\|_{p/2}^{1/2}.$$

Let's define τ_h as $\tau_h := \inf\{t \in [0,T]: (h-1)^2 \sum_i (t_{i+1} \wedge t - t_i \wedge t)^4 < \varphi\}$ for any $\varphi > 0$. Then,

$$\mathbb{P}\left[(h-1) \sup_{0 \le t \le \mathrm{T}} |\mathcal{J}_t| > \delta\right] \le \mathbb{P}\left[(h-1) \sup_{0 \le t \le \tau_h} |\mathcal{J}_t| > \delta\right] + \mathbb{P}[\tau_h \ne T].$$

Recall that Chebyshev's inequality requires that for any integrable random variable A and positive r,

$$\mathbb{P}[|A| \geq k] \leq \frac{\mathbb{E}[|A|^r]}{k^r}.$$

For the proof, we take $r = 2$ and write

$$\mathbb{P}\left[(h-1)\sup_{0\leq t\leq \mathrm{T}} |\mathcal{J}_t| > \delta\right] \leq \frac{1}{\delta^2}\mathbb{E}\left[\left((h-1)\sup_{0\leq t\leq \tau_h} |\mathcal{J}_t|\right)^2\right] + \mathbb{P}[\tau_h \neq T].$$

By definition,

$$\left\|\sup_{0\leq t\leq \tau_h} |\mathcal{J}_t|\right\|_2 = \left(\mathbb{E}\left[\left(\sup_{0\leq t\leq \tau_h} |\mathcal{J}_t|\right)^2\right]\right)^{1/2}.$$

Now, we apply the last part of Burkholder-Davis-Gundy inequality

$$\begin{aligned}
\left(\left\|\sup_{0\leq t\leq \tau_h} |\mathcal{J}_t|\right\|_2\right)^2 &\leq \left(C_2 \left\|[\mathcal{J},\mathcal{J}]_{\tau_h}\right\|_1^{\frac{1}{2}}\right)^2 \\
\Rightarrow \mathbb{E}\left[\left(\sup_{0\leq t\leq \tau_h} |\mathcal{J}_t|\right)^2\right] &\leq C_2^2 \mathbb{E}\left[[\mathcal{J},\mathcal{J}]_{\tau_h}\right] \qquad \text{(A.2)} \\
\Rightarrow \mathbb{P}\left[(h-1)\sup_{0\leq t\leq \mathrm{T}} |\mathcal{J}_t| > \delta\right] &\leq \frac{1}{\delta^2}(h-1)^2 C_2^2 \mathbb{E}\left[[\mathcal{J},\mathcal{J}]_{\tau_h}\right] + \mathbb{P}[\tau_h \neq T].
\end{aligned}$$

At this stage, let's examine the term $\mathbb{E}[[\mathcal{J},\mathcal{J}]_\tau]$. Since $\sigma_\tau \perp B_\tau$, $\left(X_\tau - X_{t_i}\right) \perp dX_\tau$, $\mathbb{E}[dB_\tau]$, the true price X_t is a local martingale from Assumption 3.3, and by Theorem 2.13 in Chapter 2, p.129 of [91], which explains that it is possible to write difference of a martingale at different times in terms of difference of quadratic variation of the same martingale at those times,

$$\begin{aligned}
[[\mathcal{J},\mathcal{J}]_\tau] &= \frac{1}{28}\mathbb{E}[[X;8]_\tau^{\mathcal{H}}] - \frac{8}{28}\mathbb{E}\left[\left(X_\tau - X_{t_i}\right)^7 dX_\tau\right] \\
&= \frac{1}{28}\mathbb{E}[[X;8]_\tau^{\mathcal{H}}] - \frac{8}{28}\mathbb{E}\left[\left(X_\tau - X_{t_i}\right)^7\right]\mathbb{E}[dX_\tau] \\
&= \frac{1}{28}\mathbb{E}[[X;8]_\tau^{\mathcal{H}}] - \frac{8}{28}\mathbb{E}\left[\left(X_\tau - X_{t_i}\right)^7\right]\mathbb{E}[\sigma_\tau dB_\tau] \\
&= \frac{1}{28}\mathbb{E}[[X;8]_\tau^{\mathcal{H}}] - \frac{8}{28}\mathbb{E}\left[\left(X_\tau - X_{t_i}\right)^7\right]\mathbb{E}[\sigma_\tau]\mathbb{E}[dB_\tau]
\end{aligned}$$

$$= \frac{1}{28}\mathbb{E}[[X;8]_\tau^{\mathcal{H}}] \tag{A.3}$$
$$= \frac{1}{28}\mathbb{E}\left[\sum_i \left(X_{\tau\wedge t_{i+1}} - X_{\tau\wedge t_i}\right)^8\right]$$
$$= \frac{1}{28}\mathbb{E}\left[\sum_i \left([X,X]_{\tau\wedge t_{i+1}} - [X,X]_{\tau\wedge t_i}\right)^4\right].$$

When we set c_p as $\sqrt[8]{1/28}$, Burkholder-Davis-Gundy inequality yields

$$\sqrt[8]{\frac{1}{28}}\left(\mathbb{E}\left[\sum_i \left([X,X]_{\tau\wedge t_{i+1}} - [X,X]_{\tau\wedge t_i}\right)^4\right]\right)^{\frac{1}{8}}$$
$$\leq \sqrt[8]{\frac{1}{28}}C_8\left(\mathbb{E}\left[\sum_i \left([X,X]_{\tau\wedge t_{i+1}} - [X,X]_{\tau\wedge t_i}\right)^4\right]\right)^{\frac{1}{8}}.$$

Taking the 8th power of both sides gives

$$\frac{1}{28}\mathbb{E}\left[\sum_i \left([X,X]_{\tau\wedge t_{i+1}} - [X,X]_{\tau\wedge t_i}\right)^4\right]$$
$$\leq \frac{1}{28}C_8^8\mathbb{E}\left[\sum_i \left([X,X]_{\tau\wedge t_{i+1}} - [X,X]_{\tau\wedge t_i}\right)^4\right]. \tag{A.4}$$

By the linearity of expectation operator,

$$\mathbb{E}\left[\sum_i \left([X,X]_{\tau\wedge t_{i+1}} - [X,X]_{\tau\wedge t_i}\right)^4\right] = \sum_i \mathbb{E}\left([X,X]_{\tau\wedge t_{i+1}} - [X,X]_{\tau\wedge t_i}\right)^4.$$

By the definition of quadratic variation and Assumption 3.4,

$$\mathbb{E}\left([X,X]_{\tau\wedge t_{i+1}} - [X,X]_{\tau\wedge t_i}\right)^4 = \mathbb{E}\left[\left(\int_{\tau\wedge t_i}^{\tau\wedge t_{i+1}} \sigma_s^2 dB_s\right)^4\right]$$
$$\leq \mathbb{E}[a^8(\tau\wedge t_{i+1} - \tau\wedge t_i)^4].$$

Hence, from Equation (A.3) and Burkholder-Davis-Gundy inequality,

$$
\begin{aligned}
\mathbb{E}[[\mathcal{J},\mathcal{J}]_\tau] &\le \frac{1}{28}C_8^8\mathbb{E}\left[\sum_i\left([X,X]_{\tau\wedge t_{i+1}}-[X,X]_{\tau\wedge t_i}\right)^4\right]\\
&\le \frac{1}{28}C_8^8\sum_i\mathbb{E}\left([X,X]_{\tau\wedge t_{i+1}}-[X,X]_{\tau\wedge t_i}\right)^4\\
&\le \frac{1}{28}C_8^8\sum_i\mathbb{E}[a^8(\tau\wedge t_{i+1}-\tau\wedge t_i)^4]\\
&= \frac{1}{28}C_8^8a^8\sum_i\mathbb{E}[(\tau\wedge t_{i+1}-\tau\wedge t_i)^4]\\
&= \frac{1}{28}C_8^8a^8\mathbb{E}\left[\sum_i(\tau\wedge t_{i+1}-\tau\wedge t_i)^4\right].
\end{aligned}
$$

Since we assumed that τ_h satisfies the condition $\inf\{t\in[0,T]:(h-1)^2\sum_i(t_{i+1}\wedge t-t_i\wedge t)^4<\varphi\}$ for any $\varphi>0$, therefore $\sum_i(\tau\wedge t_{i+1}-\tau\wedge t_i)^4<(h-1)^{-2}\varphi$, Inequality (A.2) is restated as below:

$$
\begin{aligned}
\mathbb{P}\left[(h-1)\sup_{0\le t\le \mathrm{T}}|\mathcal{J}_t|>\delta\right] &\le \frac{1}{\delta^2}(h-1)^2C_2^2\mathbb{E}\left[[\mathcal{J},\mathcal{J}]_{\tau_h}\right]+\mathbb{P}[\tau_h\neq T]\\
&\le \frac{1}{\delta^2}(h-1)^2C_2^2\frac{1}{28}a^8C_8^8(h-1)^{-2}\varphi+\mathbb{P}[\tau_h\neq T]\\
&\le \frac{1}{\delta^2}a^8\frac{1}{28}C_2^2C_8^8\varphi+\mathbb{P}[\tau_h\neq T].
\end{aligned}
$$

It is time we consider $\mathbb{P}[\tau_h\neq T]$ as $h\to\infty$. If we assume that

$$(h-1)^2\sum_i(t_{i+1}\wedge t-t_i\wedge t)^4\le(h-1)^2\Delta(\mathcal{H})\sum_i(t_{i+1}-t_i)^3\xrightarrow{\mathbb{P}}0,$$

then as Mykland and Zhang [91] put it, $\mathbb{P}[\tau_h\neq T]\to 0$ as $h\to\infty$. Accordingly,

$$\mathbb{P}\left[\sup_{0\le t\le \mathrm{T}}|\mathcal{J}_t|>\delta\right]\le\frac{1}{h-1}\frac{1}{\delta^2}a^8\frac{1}{28}C_2^2C_8^8\varphi.$$

$$\mathbb{P}\left[\sup_{0\le t\le \mathrm{T}}|\mathcal{J}_t|>\delta\right]\to 0 \text{ as } h\to\infty.$$

We have proved that supremum of $|\mathcal{J}_t|$ converge to 0 in probability at an order of $(h-1)^{-1}$. Finally from Equation (3.29) and because supremum of absolute value of a stochastic process going to 0 in probability implies that the process itself is also convergent in probability, we have

$$\sup_{0\leq t\leq T}\left|[\mathcal{Z},\mathcal{Z}]_t - {}^{2}\!/_{3}\,[X,X,X,X]_t^{\mathcal{H}_h}\right| = o_p((h-1)^{-1}) \text{ as } h\to\infty.$$

APPENDIX B

PROOF OF THEOREM 3.4

Step 1: Showing that $[\boldsymbol{X}, \boldsymbol{\epsilon}]_{\boldsymbol{T}}^{\mathcal{G}} = \boldsymbol{O_p}(\mathbf{1})$ following Lemma A.2 in [111]

On a full grid $\mathcal{G}$ with $|\mathcal{G}| = n$

$$
\begin{aligned}
[X,\epsilon]_T^{\mathcal{G}} &= \sum_{i=0}^{n-2} (\Delta X_{t_i})(\Delta \epsilon_{t_i}) = \sum_{i=0}^{n-2} (\Delta X_{t_i})(\epsilon_{t_{i+1}} - \epsilon_{t_i}) \\
&= \sum_{i=0}^{n-2} (\Delta X_{t_i})\epsilon_{t_{i+1}} - \sum_{i=0}^{n-2} (\Delta X_{t_i})\epsilon_{t_i} \\
&= \Delta X_{t_0}\epsilon_{t_1} - \Delta X_{t_0}\epsilon_{t_0} + \Delta X_{t_1}\epsilon_{t_2} - \Delta X_{t_1}\epsilon_{t_1} + \cdots \\
&\quad + \Delta X_{t_{n-2}}\epsilon_{t_{n-1}} - \Delta X_{t_{n-2}}\epsilon_{t_{n-2}} \\
&= \epsilon_{t_1}(\Delta X_{t_0} - \Delta X_{t_1}) + \epsilon_{t_2}(\Delta X_{t_1} - \Delta X_{t_2}) \\
&\quad + \epsilon_{t_3}(\Delta X_{t_2} - \Delta X_{t_3}) + \cdots \\
&\quad + \epsilon_{t_{n-2}}(\Delta X_{t_{n-3}} - \Delta X_{t_{n-2}}) - \Delta X_{t_0}\epsilon_{t_0} + \Delta X_{t_{n-2}}\epsilon_{t_{n-1}} \\
&= \sum_{i=1}^{n-2} (\Delta X_{t_{i-1}} - \Delta X_{t_i})\epsilon_{t_i} - \Delta X_{t_0}\epsilon_{t_0} + \Delta X_{t_{n-2}}\epsilon_{t_{n-1}} \\
&= a + b - c,
\end{aligned}
\tag{B.1}
$$

where

$$\Delta X_{t_i} = (X_{t_{i+1}} - X_{t_i}),$$

$$\Delta \epsilon_{t_i} = (\epsilon_{t_{i+1}} - \epsilon_{t_i}),$$

$$a = \sum_{i=1}^{n-2} (\Delta X_{t_{i-1}} - \Delta X_{t_i})\epsilon_{t_i},$$

$$b = \Delta X_{t_{n-2}} \epsilon_{t_{n-1}},$$

$$c = \Delta X_{t_0} \epsilon_{t_0}.$$

Now, under Assumptions 3.1 and 3.2, we calculate $\mathrm{Var}\left[[X,\epsilon]_T^{\mathcal{G}} \middle| X\right] = \mathbb{E}\left[\left([X,\epsilon]_T^{\mathcal{G}}\right)^2 \middle| X\right]$ where $\mathbb{E}\left[[X,\epsilon]_T^{\mathcal{G}} \middle| X\right] = 0$ from the true price and the MMN being orthogonal to each other, i.e., on the grid $\mathcal{G}$, $X_{t_i} \perp \epsilon_{t_i}$, $\Delta X_{t_i} \perp \Delta\epsilon_{t_i}$ for all $t_i \in \mathcal{G}$. Moreover, $\mathbb{E}\left[\epsilon_{t_i} \middle| X\right] = \mathbb{E}[\epsilon_t] = 0$.

$$\mathbb{E}\left[\left([X,\epsilon]_T^{\mathcal{G}}\right)^2 \middle| X\right] = \mathbb{E}[a^2 + b^2 + c^2 - 2ac - 2bc + 2ab|X], \tag{B.2}$$

$$\begin{aligned} \mathbb{E}[b^2|X] =\ & \mathbb{E}\left[\left(\Delta X_{t_{n-2}} \epsilon_{t_{n-1}}\right)^2 \middle| X\right] \\ =\ & \mathbb{E}\left[\left(\Delta X_{t_{n-2}}\right)^2 \middle| X\right] \mathbb{E}\left[\left(\epsilon_{t_{n-1}}\right)^2 \middle| X\right] \\ =\ & \mathbb{E}\left[\left(\Delta X_{t_{n-2}}\right)^2 \middle| X\right] \mathbb{E}[(\epsilon_t)^2], \end{aligned}$$

$$\begin{aligned} \mathbb{E}[c^2|X] =\ & \mathbb{E}\left[\left(\Delta X_{t_0} \epsilon_{t_0}\right)^2 \middle| X\right] \\ =\ & \mathbb{E}\left[\left(\Delta X_{t_0}\right)^2 \middle| X\right] \mathbb{E}\left[\left(\epsilon_{t_0}\right)^2 \middle| X\right] \\ =\ & \mathbb{E}\left[\left(\Delta X_{t_0}\right)^2 \middle| X\right] \mathbb{E}[(\epsilon_t)^2], \end{aligned}$$

$$\begin{aligned} [a^2|X] &= \mathbb{E}\left[\left(\sum_{i=1}^{n-2} \left(\Delta X_{t_{i-1}} - \Delta X_{t_i}\right)\epsilon_{t_i}\right)^2 \middle| X\right] \\ &= \mathbb{E}\left[\begin{matrix} \epsilon_{t_1}^2\left(\Delta X_{t_0} - \Delta X_{t_1}\right)^2 + \cdots + \epsilon_{t_{n-2}}^2\left(\Delta X_{t_{n-3}} - \Delta X_{t_{n-2}}\right)^2 \\ +2\epsilon_{t_1}\left(\Delta X_{t_0} - \Delta X_{t_1}\right)\epsilon_{t_2}\left(\Delta X_{t_1} - \Delta X_{t_2}\right) \\ +2\epsilon_{t_1}\left(\Delta X_{t_0} - \Delta X_{t_1}\right)\epsilon_{t_3}\left(\Delta X_{t_2} - \Delta X_{t_3}\right) \\ \vdots \\ +2\epsilon_{t_1}\left(\Delta X_{t_0} - \Delta X_{t_1}\right)\epsilon_{t_{n-2}}\left(\Delta X_{t_{n-3}} - \Delta X_{t_{n-2}}\right) \\ +2\epsilon_{t_2}\left(\Delta X_{t_1} - \Delta X_{t_2}\right)\epsilon_{t_3}\left(\Delta X_{t_2} - \Delta X_{t_3}\right) \\ \vdots \\ +2\epsilon_{t_2}\left(\Delta X_{t_1} - \Delta X_{t_2}\right)\epsilon_{t_{n-2}}\left(\Delta X_{t_{n-3}} - \Delta X_{t_{n-2}}\right) \\ \vdots \\ +2\epsilon_{t_{n-3}}\left(\Delta X_{t_{n-4}} - \Delta X_{t_{n-3}}\right)\epsilon_{t_{n-2}}\left(\Delta X_{t_{n-3}} - \Delta X_{t_{n-2}}\right) \end{matrix} \middle| X\right] \end{aligned}$$

$$= \mathbb{E}\left[\left.\begin{array}{c} \epsilon_{t_1}^2\left(\Delta X_{t_0} - \Delta X_{t_1}\right)^2 + \cdots + \epsilon_{t_{n-2}}^2\left(\Delta X_{t_{n-3}} - \Delta X_{t_{n-2}}\right)^2 \\ +2\sum_{i=1}^{n-3}\sum_{j=2}^{n-2}\left(\Delta X_{t_{i-1}} - \Delta X_{t_i}\right)\epsilon_{t_i}\left(\Delta X_{t_{j-1}} - \Delta X_{t_j}\right)\epsilon_{t_j} \end{array}\right| X\right], i \neq j.$$

Recall that from i.i.d structure of MMN, if $i \neq j$, $\epsilon_{t_i} \perp \epsilon_{t_j}$. Hence,

$$\begin{aligned}
&\mathbb{E}\left[2\sum_{i=1}^{n-3}\sum_{j=2}^{n-2}\left(\Delta X_{t_{i-1}} - \Delta X_{t_i}\right)\epsilon_{t_i}\left(\Delta X_{t_{j-1}} - \Delta X_{t_j}\right)\epsilon_{t_j}\middle| X\right] \\
&\qquad = 2\sum_{i=1}^{n-3}\sum_{j=2}^{n-2}\mathbb{E}\left[\left(\Delta X_{t_{i-1}} - \Delta X_{t_i}\right)\epsilon_{t_i}\left(\Delta X_{t_{j-1}} - \Delta X_{t_j}\right)\epsilon_{t_j}\middle| X\right] \\
&\qquad = 2\sum_{i=1}^{n-3}\sum_{j=2}^{n-2}\mathbb{E}\left[\epsilon_{t_i}\middle|X\right]\mathbb{E}\left[\Delta X_{t_{i-1}} - \Delta X_{t_i}\middle|X\right]\mathbb{E}\left[\Delta X_{t_{j-1}} - \Delta X_{t_j}\middle| X\right]\mathbb{E}\left[\epsilon_{t_j}\middle| X\right] \\
&\qquad = 0,
\end{aligned}$$

and

$$\mathbb{E}[a^2|X] = \mathbb{E}\left[\epsilon_{t_1}^2\left(\Delta X_{t_0} - \Delta X_{t_1}\right)^2 + \cdots + \epsilon_{t_{n-2}}^2\left(\Delta X_{t_{n-3}} - \Delta X_{t_{n-2}}\right)^2\middle| X\right].$$

Incorporating $\mathbb{E}[a^2|X]$, $\mathbb{E}[b^2|X]$ and $\mathbb{E}[c^2|X]$ into Equation (B.2) gives rise to

$$\begin{aligned}
&\mathbb{E}[a^2 + b^2 + c^2 - 2ac - 2bc + 2ab|X] \\
&\qquad = \mathbb{E}\left[\sum_{i=1}^{n-2}\epsilon_{t_i}^2\left(\Delta X_{t_{i-1}} - \Delta X_{t_i}\right)^2\middle| X\right] \\
&\qquad + \mathbb{E}\left[\left(\Delta X_{t_{n-2}}\right)^2\middle| X\right]\mathbb{E}[(\epsilon_t)^2] \\
&\qquad + \mathbb{E}\left[\left(\Delta X_{t_0}\right)^2\middle| X\right]\mathbb{E}[(\epsilon_t)^2] - 2\mathbb{E}[ac + bc - ab|X].
\end{aligned} \tag{B.3}$$

To calculate the last term on the RHS of Equation (B.3), we focus on $\mathbb{E}[ac|X]$, $\mathbb{E}[ab|X]$, and $\mathbb{E}[bc|X]$.

$$\begin{aligned}
\mathbb{E}[ac|X] &= \mathbb{E}\left[\left(\sum_{i=1}^{n-2}\epsilon_{t_i}\left(\Delta X_{t_{i-1}} - \Delta X_{t_i}\right)\right)\Delta X_{t_0}\epsilon_{t_0}\middle| X\right] \\
&= \mathbb{E}\left[\epsilon_{t_1}\epsilon_{t_0}\Delta X_{t_0}\left(\Delta X_{t_0} - \Delta X_{t_1}\right)\middle|X\right] \\
&\quad + \mathbb{E}\left[\epsilon_{t_2}\epsilon_{t_0}\Delta X_{t_0}\left(\Delta X_{t_1} - \Delta X_{t_0}\right)\middle|X\right] + \cdots \\
&\quad + \mathbb{E}\left[\epsilon_{t_{n-2}}\epsilon_{t_0}\Delta X_{t_0}\left(\Delta X_{t_{n-3}} - \Delta X_{t_{n-2}}\right)\middle|X\right],
\end{aligned} \tag{B.4}$$

where as long as $i \neq j$, $\epsilon_{t_i} \perp \epsilon_{t_j}$ and $\mathbb{E}\left[\epsilon_{t_1}\epsilon_{t_0}\Delta X_{t_0}\left(\Delta X_{t_0} - \Delta X_{t_1}\right)\middle|X\right] = 0$, so each term on the RHS of Equation (B.4) equals to 0. By this token, $\mathbb{E}[ac|X] = 0$, $\mathbb{E}[ab|X] = 0$, and $\mathbb{E}[bc|X] = 0$. Therefore, Equation (B.2) becomes

$$
\begin{aligned}
&\mathbb{E}\left[\left([X,\epsilon]_T^{\mathcal{G}}\right)^2\middle| X\right] \\
&\qquad = \mathbb{E}\left[\sum_{i=1}^{n-2}\epsilon_{t_i}^2\left(\Delta X_{t_{i-1}} - \Delta X_{t_i}\right)^2\middle| X\right] + \mathbb{E}\left[\left(\Delta X_{t_{n-2}}\right)^2\middle| X\right]\mathbb{E}[(\epsilon_t)^2] \\
&\qquad + \mathbb{E}\left[\left(\Delta X_{t_0}\right)^2\middle| X\right]\mathbb{E}[(\epsilon_t)^2] \\
&= \mathbb{E}[(\epsilon)^2]\,\mathbb{E}\left[\left(\Delta X_{t_0}\right)^2 + \left(\Delta X_{t_{n-2}}\right)^2 + \sum_{i=1}^{n-2}\left(\Delta X_{t_{i-1}} - \Delta X_{t_i}\right)^2\middle| X\right] \\
&= \mathbb{E}[(\epsilon)^2]\,\mathbb{E}\left[\begin{array}{c}\left(\Delta X_{t_0}\right)^2 + \left(\Delta X_{t_{n-2}}\right)^2 \\ + \sum_{i=1}^{n-2}\left(\Delta X_{t_{i-1}}\right)^2 + \sum_{i=1}^{n-2}\left(\Delta X_{t_i}\right)^2 - 2\sum_{i=1}^{n-2}\Delta X_{t_{i-1}}\Delta X_{t_i}\end{array}\middle| X\right] \\
&= \mathbb{E}[(\epsilon)^2]\,\mathbb{E}\left[2\sum_{i=0}^{n-2}\left(\Delta X_{t_i}\right)^2 - 2\sum_{i=1}^{n-2}\Delta X_{t_{i-1}}\Delta X_{t_i}\middle| X\right],
\end{aligned}
\tag{B.5}
$$

where

$$\left(\Delta X_{t_{n-2}}\right)^2 + \sum_{i=1}^{n-2}\left(\Delta X_{t_{i-1}}\right)^2 = \sum_{i=0}^{n-2}\left(\Delta X_{t_i}\right)^2,$$

$$\left(\Delta X_{t_0}\right)^2 + \sum_{i=1}^{n-2}\left(\Delta X_{t_i}\right)^2 = \sum_{i=0}^{n-2}\left(\Delta X_{t_i}\right)^2,$$

Since $(\alpha - \beta) \leq |\alpha| + |\beta|$ and for any duo of real numbers α and β, we can rewrite Equation (B.5) as an inequality,

$$
\begin{aligned}
\mathbb{E}\left[\left([X,\epsilon]_T^{\mathcal{G}}\right)^2\middle| X\right] &\leq 2\mathbb{E}[(\epsilon)^2]\,\mathbb{E}\left[\left|\sum_{i=0}^{n-2}\left(\Delta X_{t_i}\right)^2\right| + \left|\sum_{i=1}^{n-2}\Delta X_{t_{i-1}}\Delta X_{t_i}\right|\middle| X\right] \\
&= 2\mathbb{E}[(\epsilon)^2]\left(\mathbb{E}\left[\sum_{i=0}^{n-2}\left(\Delta X_{t_i}\right)^2\middle| X\right] + \left|\sum_{i=1}^{n-2}\Delta X_{t_{i-1}}\Delta X_{t_i}\right|\right).
\end{aligned}
\tag{B.6}
$$

By using the inequality $|\sum_i \alpha_i \beta_i| \leq \sum_i |\alpha_i \beta_i|$ and the interchangeability of expectation and sigma operators, Inequality (B.6) is restated as

$$\begin{aligned}\mathbb{E}\left[\left([X,\epsilon]_T^{\mathcal{G}}\right)^2\middle|X\right] \leq\ & 2\mathbb{E}[(\epsilon)^2]\left(\mathbb{E}\left[\left|\sum_{i=0}^{n-2}\left(\Delta X_{t_i}\right)^2\right| + \sum_{i=1}^{n-2}\left|\Delta X_{t_{i-1}}\right|\left|\Delta X_{t_i}\right|\middle|X\right]\right)\\ &= 2\mathbb{E}[(\epsilon)^2]\left[\sum_{i=0}^{n-2}\left(\Delta X_{t_i}\right)^2\middle|X\right] \\ &+2\mathbb{E}[(\epsilon)^2]\sum_{i=1}^{n-2}\mathbb{E}\left[\left|\Delta X_{t_{i-1}}\right|\left|\Delta X_{t_i}\right|\middle|X\right].\end{aligned} \tag{B.7}$$

From the Cauchy-Schwarz Inequality, which says that for two random variables $\mathcal{S}$ and $\mathcal{Q}$, $|\mathbb{E}[\mathcal{S}\mathcal{Q}]|^2 \leq \mathbb{E}[\mathcal{S}^2]\mathbb{E}[\mathcal{Q}^2]$,

$$\mathbb{E}\left[\left|\Delta X_{t_{i-1}}\right|\left|\Delta X_{t_i}\right|\middle|X\right] \leq \mathbb{E}^{\frac{1}{2}}\left[\left(\Delta X_{t_i}\right)^2\middle|X\right]\mathbb{E}^{\frac{1}{2}}\left[\left(\Delta X_{t_{i-1}}\right)^2\middle|X\right],$$

and Inequality (B.7) becomes

$$\begin{aligned}&\mathbb{E}\left[\left([X,\epsilon]_T^{\mathcal{G}}\right)^2\middle|X\right]\\ &\qquad\leq 2\mathbb{E}[(\epsilon)^2]\left(\mathbb{E}\left[\sum_{i=0}^{n-2}\left(\Delta X_{t_i}\right)^2\middle|X\right]\right.\\ &\qquad\left.+\sum_{i=1}^{n-2}\mathbb{E}^{\frac{1}{2}}\left[\left(\Delta X_{t_{i-1}}\right)^2\middle|X\right]\mathbb{E}^{\frac{1}{2}}\left[\left(\Delta X_{t_i}\right)^2\middle|X\right]\right).\end{aligned}$$

By application of the version of Cauchy-Schwarz Inequality laid out in Inequality (3.10) and $[X,X]_T > 0$, $\left(\Delta X_{t_0}\right)^2 > 0$, $\left(\Delta X_{t_n}\right)^2 > 0$, $\mathbb{E}\left[\sum_{i=0}^{n-2}\left(\Delta X_{t_i}\right)^2\middle|X\right] = [X,X]_T$, we get

$$
\begin{aligned}
\sum_{i=1}^{n-2} \mathbb{E}^{\frac{1}{2}}\left[\left(\Delta X_{t_{i-1}}\right)^2 \middle| X\right] \mathbb{E}^{\frac{1}{2}}\left[\left(\Delta X_{t_i}\right)^2 \middle| X\right] \\
&\leq \sqrt{\left(\sum_{i=1}^{n-2} \mathbb{E}\left[\left(\Delta X_{t_i}\right)^2 \middle| X\right]\right)\left(\sum_{i=1}^{n-2} \mathbb{E}\left[\left(\Delta X_{t_{i-1}}\right)^2 \middle| X\right]\right)} \\
&= \sqrt{\mathbb{E}\left[\sum_{i=0}^{n-2}\left(\Delta X_{t_i}\right)^2 - \left(\Delta X_{t_0}\right)^2 \middle| X\right] \mathbb{E}\left[\sum_{i=0}^{n-2}\left(\Delta X_{t_{i-1}}\right)^2 - \left(\Delta X_{t_{n-2}}\right)^2 \middle| X\right]} \\
&= \sqrt{\left([X,X]_T - \mathbb{E}\left[\left(\Delta X_{t_{n-2}}\right)^2 \middle| X\right]\right)\left([X,X]_T - \mathbb{E}\left[\left(\Delta X_{t_0}\right)^2 \middle| X\right]\right)} \\
&\leq \sqrt{[X,X]_T [X,X]_T}.
\end{aligned}
$$

Then, the final version of Inequality (B.7) is as follows:

$$
\mathbb{E}\left[\left([X,\epsilon]_T^{\mathcal{G}}\right)^2 \middle| X\right] \leq 4\mathbb{E}[(\epsilon)^2][X,X]_T.
$$

From Assumption 3.1, $\mathbb{E}[(\epsilon)^2]$ is constant and from the stochastic boundedness of $[X,X]_T$, mentioned in Section 3.2 before,

$$
\mathbb{E}\left[\left([X,\epsilon]_T^{\mathcal{G}}\right)^2 \middle| X\right] = O_p(1),
$$

We now apply the Markov Inequality to $[X,\epsilon]_T^{\mathcal{G}}$ with

$$
\mathbb{P}\left[\left|[X,\epsilon]_T^{\mathcal{G}}\middle|X\right| \geq \gamma\right] \leq \frac{1}{\gamma^2} \mathbb{E}\left[\left([X,\epsilon]_T^{\mathcal{G}}\right)^2 \middle| X\right], \forall\, \gamma \in (0,\infty). \tag{B.8}
$$

Because $\mathbb{E}\left[\left([X,\epsilon]_T^{\mathcal{G}}\right)^2 \middle| X\right]$ is stochastically bounded, the RHS of Inequality (B.8) equals to a real number ρ. As it is possible to find a γ for any ρ, the definition of stochastic boundedness is satisfied and $[X,\epsilon]_T^{\mathcal{G}} = O_p(1)$.

Step 2: Showing asymptotical convergence and distribution of $\frac{1}{\sqrt{n-1}}\left([\epsilon,\epsilon]_T^{\mathcal{G}} - \mathbb{E}\left[[\epsilon,\epsilon]_T^{\mathcal{G}}\middle|X\right]\right)$

$$
\begin{aligned}
[\epsilon,\epsilon]_T^{\mathcal{G}} - \mathbb{E}\big[[\epsilon,\epsilon]_T^{\mathcal{G}}\big]\big|X &= [\epsilon,\epsilon]_T^{\mathcal{G}} - 2(n-1)\mathbb{E}[\epsilon^2] \\
&= \sum_{i=0}^{n-2}\big(\Delta\epsilon_{t_i}\big)^2 - 2(n-1)\mathbb{E}[\epsilon^2] \\
&= \left(\sum_{i=0}^{n-2}\epsilon_{t_{i+1}}^2 - 2\epsilon_{t_{i+1}}\epsilon_{t_i} + \epsilon_{t_i}^2\right) - 2(n-1)\mathbb{E}[\epsilon^2] \\
&= \epsilon_{t_1}^2 + \epsilon_{t_0}^2 + \cdots + \epsilon_{t_{n-1}}^2 + \epsilon_{t_{n-2}}^2 \\
&\quad -2\left(\sum_{i=0}^{n-2}\epsilon_{t_{i+1}}\epsilon_{t_i}\right) - 2(n-1)\mathbb{E}[\epsilon^2] \\
&= 2\epsilon_{t_1}^2 + \cdots + 2\epsilon_{t_{n-2}}^2 + \epsilon_{t_0}^2 + \epsilon_{t_{n-1}}^2 \\
&\quad -2\left(\sum_{i=0}^{n-2}\epsilon_{t_{i+1}}\epsilon_{t_i}\right) - 2(n-1)\mathbb{E}[\epsilon^2] \\
&= \sum_{i=1}^{n-2}\epsilon_{t_i}^2 + \left(\epsilon_{t_0}^2 + \epsilon_{t_{n-1}}^2\right) - 2\left(\sum_{i=0}^{n-2}\epsilon_{t_{i+1}}\epsilon_{t_i}\right) \\
&\quad -2(n-1)\mathbb{E}[\epsilon^2] \\
&= 2\left(\sum_{i=1}^{n-2}\epsilon_{t_i}^2 - \sum_{i=1}^{n-2}\mathbb{E}[\epsilon^2]\right) + \left(\epsilon_{t_0}^2 - \mathbb{E}[\epsilon^2]\right) \\
&\quad +\left(\epsilon_{t_{n-1}}^2 - \mathbb{E}[\epsilon^2]\right) - 2\left(\sum_{i=0}^{n-2}\epsilon_{t_{i+1}}\epsilon_{t_i}\right) \\
&= 2\left(\sum_{i=1}^{n-2}\epsilon_{t_i}^2 - \mathbb{E}[\epsilon^2]\right) + \left(\epsilon_{t_0}^2 - \mathbb{E}[\epsilon^2]\right) \\
&\quad +\left(\epsilon_{t_{n-1}}^2 - \mathbb{E}[\epsilon^2]\right) - 2\left(\sum_{i=0}^{n-2}\epsilon_{t_{i+1}}\epsilon_{t_i}\right),
\end{aligned}
\tag{B.9}
$$

where

$$
2(n-1)\mathbb{E}[\epsilon^2] = 2\sum_{i=1}^{n-2}\mathbb{E}[\epsilon^2] + 2\mathbb{E}[\epsilon^2].
$$

As stated in Assumption 3.1, we take $\mathbb{E}[\epsilon^4]$ as finite, then by the Markov's Inequality,

$$
\mathbb{P}[|\mathbb{E}[\epsilon^2]| \geq \gamma] \leq \frac{1}{\gamma^2}\mathbb{E}[\epsilon^4], \forall\, \gamma \in (0,\infty),
$$

and for any finite value on the RHS of the above inequality, we can find at least one finite and positive γ, which shows that $\mathbb{E}[\epsilon^2]$ is stochastically bounded. Embedding this fact into Equation (B.9) leads to

$$[\epsilon,\epsilon]_T^{\mathcal{G}} - 2(n-1)\mathbb{E}[\epsilon^2] = 2\left(\sum_{i=1}^{n-2} \epsilon_{t_i}^2 - \mathbb{E}[\epsilon^2]\right) - 2\left(\sum_{i=0}^{n-2} \epsilon_{t_{i+1}}\epsilon_{t_i}\right) + O_p(1), n \geq 3.$$

If we make the following definitions:

$$M_T^{(1)} := \frac{1}{\sqrt{n-2}} \sum_{i=1}^{n-2} \epsilon_{t_i}^2 - \mathbb{E}[\epsilon^2],$$

$$M_T^{(2)} := \frac{1}{\sqrt{n-2}} \sum_{i=0}^{n-2} \epsilon_{t_{i+1}}\epsilon_{t_i},$$

Then, Equation (B.9) is rewritten as

$$[\epsilon,\epsilon]_T^{\mathcal{G}} - 2(n-1)\mathbb{E}[\epsilon^2] = 2\sqrt{n-2}\left(M_T^{(1)} - M_T^{(2)}\right) + O_p(1).$$

If $n = 3$,

$$M_T^{(1)} = \frac{1}{\sqrt{1}}\left(\epsilon_{t_1}^2 - \mathbb{E}[\epsilon^2]\right) = \frac{1}{\sqrt{1}}\left(b_{1,1}\right),$$
$$b_{1,1} := \epsilon_{t_1}^2 - \mathbb{E}[\epsilon^2],$$

$$M_T^{(2)} = \frac{1}{\sqrt{1}}\left(\epsilon_{t_1}\epsilon_{t_0} + \epsilon_{t_2}\epsilon_{t_1}\right) = \frac{1}{\sqrt{1}}\left(c_{1,1} + c_{1,2}\right),$$
$$c_{1,1} := \epsilon_{t_1}\epsilon_{t_0}, c_{1,2} := \epsilon_{t_2}\epsilon_{t_1}.$$

If $n = 4$,

$$M_T^{(1)} = \frac{1}{\sqrt{2}}\left(\epsilon_{t_1}^2 - \mathbb{E}[\epsilon^2] + \epsilon_{t_2}^2 - \mathbb{E}[\epsilon^2]\right) = \frac{1}{\sqrt{2}}\left(b_{2,1} + b_{2,2}\right),$$
$$b_{2,1} := \epsilon_{t_1}^2 - \mathbb{E}[\epsilon^2], b_{2,2} := \epsilon_{t_2}^2 - \mathbb{E}[\epsilon^2],$$

$$M_T^{(2)} = \frac{1}{\sqrt{2}}\left(\epsilon_{t_1}\epsilon_{t_0} + \epsilon_{t_2}\epsilon_{t_1} + \epsilon_{t_3}\epsilon_{t_2}\right) = \frac{1}{\sqrt{2}}\left(c_{2,1} + c_{2,2} + c_{2,3}\right),$$
$$c_{2,1} := \epsilon_{t_1}\epsilon_{t_0}, c_{2,2} := \epsilon_{t_2}\epsilon_{t_1}, c_{2,3} := \epsilon_{t_3}\epsilon_{t_2}.$$

If $n = 5$,

$$M_T^{(1)} = \frac{1}{\sqrt{3}}\left(\epsilon_{t_1}^2 - \mathbb{E}[\epsilon^2] + \epsilon_{t_2}^2 - \mathbb{E}[\epsilon^2] + \epsilon_{t_3}^2 - \mathbb{E}[\epsilon^2]\right) = \frac{1}{\sqrt{3}}\left(b_{3,1} + b_{3,2} + b_{3,3}\right),$$
$$b_{3,1} := \epsilon_{t_1}^2 - \mathbb{E}[\epsilon^2], b_{3,2} := \epsilon_{t_2}^2 - \mathbb{E}[\epsilon^2], b_{3,3} := \epsilon_{t_3}^2 - \mathbb{E}[\epsilon^2],$$

$$M_T^{(2)} = \frac{1}{\sqrt{3}}\left(\epsilon_{t_1}\epsilon_{t_0} + \epsilon_{t_2}\epsilon_{t_1} + \epsilon_{t_3}\epsilon_{t_2} + \epsilon_{t_4}\epsilon_{t_3}\right) = \frac{1}{\sqrt{3}}\left(c_{3,1} + c_{3,2} + c_{3,3} + c_{3,4}\right),$$
$$c_{3,1} := \epsilon_{t_1}\epsilon_{t_0}, c_{3,2} := \epsilon_{t_2}\epsilon_{t_1}, c_{3,3} := \epsilon_{t_3}\epsilon_{t_2}, c_{3,4} := \epsilon_{t_4}\epsilon_{t_3}.$$

Accordingly, as $n \to \infty$, if we organize terms in the form $b_{n-2,i}$ and $c_{n-2,i+1}$ in a triangular way separately, we have two arrays in the following forms

$$\begin{array}{lll} b_{1,1} & & \\ b_{2,1} & b_{2,2} & \\ b_{3,1} & b_{3,2} & b_{3,3} \\ \vdots & & \end{array}$$

$$\begin{array}{llll} c_{1,1} & c_{1,2} & & \\ c_{2,1} & c_{2,2} & c_{2,3} & \\ c_{3,1} & c_{3,2} & c_{3,3} & c_{3,4} \\ \vdots & & & \end{array}$$

Then, for a specific n, $M_T^{(1)}$ and $M_T^{(2)}$ are separate sums of the row $n-2$ in first and second triangular array divided by $\frac{1}{\sqrt{n-2}}$, respectively.

Entries in each row of each triangular array are i.i.d with mean 0 and finite variance, since for $i \neq j, \forall\, t_i, t_j \in \mathcal{G},$ by Assumptions 3.1 and 3.2

$$\epsilon_{t_i}^2 - \mathbb{E}[\epsilon^2] \perp \epsilon_{t_j}^2 - \mathbb{E}[\epsilon^2], \epsilon_{t_i}\epsilon_{t_{i+1}} \perp \epsilon_{t_j}\epsilon_{t_{j+1}}, \epsilon_{t_i} \perp X_{t_i},$$

$$\mathbb{E}\left[\epsilon_{t_i}^2 - \mathbb{E}[\epsilon^2] \middle| X\right] = \mathbb{E}[\epsilon^2] - \mathbb{E}[\epsilon^2] = 0,$$

$$\left[\epsilon_{t_i}\epsilon_{t_{i+1}}\middle|X\right] = \mathbb{E}\left[\epsilon_{t_i}\right]\mathbb{E}\left[\epsilon_{t_{i+1}}\right] = \mathbb{E}[\epsilon^2]\mathbb{E}[\epsilon^2] = 0,$$

$$\begin{aligned}\mathrm{Var}\left[\epsilon_{t_i}^2 - \mathbb{E}[\epsilon^2]\middle|X\right] &= \mathbb{E}\left[\left(\epsilon_{t_i}^2 - \mathbb{E}[\epsilon^2]\right)^2\right] - \left(\mathbb{E}\left[\epsilon_{t_i}^2 - \mathbb{E}[\epsilon^2]\right]\right)^2 = \mathbb{E}\left[\left(\epsilon_{t_i}^2 - \mathbb{E}[\epsilon^2]\right)^2\right] \\ &= \mathbb{E}\left[\epsilon_{t_i}^4 + (\mathbb{E}[\epsilon^2])^2 - 2\epsilon_{t_i}^2\mathbb{E}[\epsilon^2]\right] = \mathbb{E}\left[\epsilon_{t_i}^4\right] - (\mathbb{E}[\epsilon^2])^2 \\ &= \mathrm{Var}[\epsilon^2] < \infty,\end{aligned}$$

$$\begin{aligned}\mathrm{Var}\left[\epsilon_{t_i}\epsilon_{t_{i+1}}\middle|X\right] &= \mathbb{E}\left[\left(\epsilon_{t_i}\epsilon_{t_{i+1}}\right)^2\right] - \left(\mathbb{E}\left[\epsilon_{t_i}\epsilon_{t_{i+1}}\right]\right)^2 = \mathbb{E}\left[\left(\epsilon_{t_i}\epsilon_{t_{i+1}}\right)^2\right] \\ &= \mathbb{E}\left[\epsilon_{t_i}^2\right]\mathbb{E}\left[\epsilon_{t_{i+1}}^2\right] \\ &= (\mathbb{E}[\epsilon^2])^2 < \infty,\end{aligned}$$

In this context, application of LLN and CLT for triangular arrays (Theorem 27.2 in Chapter 5, Section 27, p. 352 of [34]) yields

$$\begin{aligned}&\frac{\mathfrak{Z}_n}{\mathfrak{I}_n} \xrightarrow{d} N(0,1), \\ \Rightarrow\ &\frac{\sum_{i=1}^{n-2} b_{n-2,i}}{\sqrt{(n-2)\mathrm{Var}[\epsilon^2]}} \xrightarrow{d} N(0,1) \Rightarrow \frac{\sqrt{(n-2)}}{\sqrt{(n-2)}}\frac{M_T^{(1)}}{\sqrt{\mathrm{Var}[\epsilon^2]}} \xrightarrow{d} N(0,1), \\ &\Rightarrow M_T^{(1)} \xrightarrow{d} N(0, \mathrm{Var}[\epsilon^2]) \text{ as } n \to \infty,\end{aligned}$$

$$\begin{aligned}&\frac{\mathfrak{K}_n}{\mathfrak{L}_n} \xrightarrow{d} N(0,1), \\ \Rightarrow\ &\frac{\sum_{i=0}^{n-2} c_{n-2,i+1}}{\sqrt{(n-1)(\mathbb{E}[\epsilon^2])^2}} \xrightarrow{d} N(0,1) \Rightarrow \frac{\sqrt{(n-2)}}{\sqrt{(n-1)}}\frac{M_T^{(2)}}{\sqrt{(\mathbb{E}[\epsilon^2])^2}} \xrightarrow{d} N(0,1), \\ &\Rightarrow M_T^{(2)} \xrightarrow{d} N(0, (\mathbb{E}[\epsilon^2])^2) \text{ as } n \to \infty, \frac{\sqrt{(n-2)}}{\sqrt{(n-1)}} \to 1,\end{aligned}$$

where

$$\begin{aligned}\mathfrak{Z}_n &:= \sum_{i=1}^{n-2} b_{n-2,i}, \\ \mathfrak{I}_n^2 := \mathrm{Var}[\mathfrak{Z}_n] &= \sum_{i=1}^{n-2} \mathrm{Var}\left[b_{n-2,i}\right] = \sum_{i=1}^{n-2} \mathrm{Var}[\epsilon^2] = (n-2)\mathrm{Var}[\epsilon^2],\end{aligned}$$

$$\mathfrak{K}_n := \sum_{i=0}^{n-2} c_{n-2,i+1},$$
$$\mathfrak{L}_n^2 := \mathrm{Var}[\mathfrak{K}_n] = \sum_{i=0}^{n-2} \mathrm{Var}\left[c_{n-2,i+1}\right] = \sum_{i=0}^{n-2} (\mathbb{E}[\epsilon^2])^2 = (n-1)(\mathbb{E}[\epsilon^2])^2,$$

required that the Lindeberg's condition is satisfied for each row in each triangular array, i.e., whether or not for any $\delta > 0$,

$$\frac{1}{\mathfrak{J}_n^2} \sum_{i=1}^{n-2} \mathbb{E}\left[b_{n-2,i}^2 \mathbb{I}_{\{|b_{n-2,i}| \geq \delta \mathfrak{J}_n\}} \middle| X\right],$$

and

$$\frac{1}{\mathfrak{L}_n^2} \sum_{i=0}^{n-2} \mathbb{E}\left[c_{n-2,i+1}^2 \mathbb{I}_{\{|c_{n-2,i+1}| \geq \delta \mathfrak{L}_n\}} \middle| X\right]$$

converge to 0 as $n \to \infty$.

Since i.i.d characteristic of MMN under Assumption 3.1 makes $b_{n-2,i}$ and $c_{n-2,i+1}$ i.i.d as well,

$$\begin{aligned}
\sum_{i=1}^{n-2} \mathbb{E}\left[b_{n-2,i}^2 \mathbb{I}_{\{|b_{n-2,i}| \geq \delta \mathfrak{J}_n\}} \middle| X\right] &= \sum_{i=1}^{n-2} \mathbb{E}\left[b_{n-2,i}^2 \mathbb{I}_{\{|b_{n-2,i}| \geq \delta \mathfrak{J}_n\}}\right] \\
&= \mathbb{E}\left[b_{n-2,1}^2 \mathbb{I}_{\{|b_{n-2,1}| \geq \delta \mathfrak{J}_n\}}\right] + \cdots + \mathbb{E}\left[b_{n-2,n-2}^2 \mathbb{I}_{\{|b_{n-2,n-2}| \geq \delta \mathfrak{J}_n\}}\right] \\
&= (n-2)\mathbb{E}\left[b_{n-2,1}^2 \mathbb{I}_{\left\{|b_{n-2,1}| \geq \delta \sqrt{(n-2)\mathrm{Var}[\epsilon^2]}\right\}}\right],
\end{aligned}$$

$$\begin{aligned}
\sum_{i=0}^{n-2} \mathbb{E}\left[c_{n-2,i+1}^2 \mathbb{I}_{\{|c_{n-2,i+1}| \geq \delta \mathfrak{L}_n\}} \middle| X\right] &= \sum_{i=0}^{n-2} \mathbb{E}\left[c_{n-2,i+1}^2 \mathbb{I}_{\{|c_{n-2,i+1}| \geq \delta \mathfrak{L}_n\}}\right] \\
&= \mathbb{E}\left[c_{n-2,1}^2 \mathbb{I}_{\{|c_{n-2,i+1}| \geq \delta \mathfrak{L}_n\}}\right] + \cdots + \mathbb{E}\left[c_{n-2,n-1}^2 \mathbb{I}_{\{|c_{n-2,i+1}| \geq \delta \mathfrak{L}_n\}}\right] \\
&= (n-1)\mathbb{E}\left[c_{n-2,1}^2 \mathbb{I}_{\{|c_{n-2,1}| \geq \delta \mathfrak{L}_n\}}\right].
\end{aligned}$$

Then,

$$
\begin{aligned}
&\lim_{n\to\infty} \frac{1}{\mathfrak{J}_n^2} \sum_{i=1}^{n-2} \mathbb{E}\left[b_{n-2,i}^2 \mathbb{I}_{\{|b_{n-2,i}|>\delta \mathfrak{J}_n\}}\right] \\
&= \lim_{n\to\infty} \frac{1}{(n-2)\mathrm{Var}[\epsilon^2]} \sum_{i=1}^{n-2} \mathbb{E}\left[b_{n-2,i}^2 \mathbb{I}_{\left\{|b_{n-2,i}|>\delta\sqrt{(n-2)\mathrm{Var}[\epsilon^2]}\right\}}\right] \\
&= \lim_{n\to\infty} \frac{1}{(n-2)\mathrm{Var}[\epsilon^2]} (n-2)\mathbb{E}\left[b_{n-2,1}^2 \mathbb{I}_{\left\{|b_{n-2,1}|>\delta\sqrt{(n-2)\mathrm{Var}[\epsilon^2]}\right\}}\right] \\
&= \lim_{n\to\infty} \frac{1}{\mathrm{Var}[\epsilon^2]} \mathbb{E}\left[b_{n-2,1}^2 \mathbb{I}_{\left\{|b_{n-2,1}|>\delta\sqrt{(n-2)\mathrm{Var}[\epsilon^2]}\right\}}\right],
\end{aligned}
\tag{B.10}
$$

$$
\begin{aligned}
&\lim_{n\to\infty} \frac{1}{\mathfrak{L}_n^2} \sum_{i=0}^{n-2} \mathbb{E}\left[c_{n-2,i+1}^2 \mathbb{I}_{\{|c_{n-2,i+1}|\geq\delta \mathfrak{L}_n\}}\right] \\
&= \lim_{n\to\infty} \frac{1}{(n-1)(\mathbb{E}[\epsilon^2])^2} \sum_{i=0}^{n-2} \mathbb{E}\left[c_{n-2,i+1}^2 \mathbb{I}_{\left\{|c_{n-2,1}|\geq\delta\sqrt{(n-1)(\mathbb{E}[\epsilon^2])^2}\right\}}\right] \\
&= \lim_{n\to\infty} \frac{1}{(n-1)(\mathbb{E}[\epsilon^2])^2} (n-1)\mathbb{E}\left[c_{n-2,1}^2 \mathbb{I}_{\left\{|c_{n-2,1}|\geq\delta\sqrt{(n-1)(\mathbb{E}[\epsilon^2])^2}\right\}}\right] \\
&= \lim_{n\to\infty} \frac{1}{(\mathbb{E}[\epsilon^2])^2} \mathbb{E}\left[c_{n-2,1}^2 \mathbb{I}_{\left\{|c_{n-2,1}|\geq\delta\sqrt{(n-1)(\mathbb{E}[\epsilon^2])^2}\right\}}\right].
\end{aligned}
\tag{B.11}
$$

Next, we apply the Dominated Convergence Theorem which allows us to interchange limit and expectation in Equations (B.10) and (B.11)

$$
\lim_{n\to\infty} \frac{1}{\mathfrak{J}_n^2} \sum_{i=1}^{n-2} \mathbb{E}\left[b_{n-2,i}^2 \mathbb{I}_{\{|b_{n-2,i}|\geq\delta \mathfrak{J}_n\}}\right] = \frac{1}{\mathrm{Var}[\epsilon^2]} \mathbb{E}\left[\lim_{n\to\infty} b_{n-2,1}^2 \mathbb{I}_{\left\{|b_{n-2,1}|\geq\delta\sqrt{(n-2)\mathrm{Var}[\epsilon^2]}\right\}}\right],
$$

$$
\begin{aligned}
&\lim_{n\to\infty} \frac{1}{\mathfrak{L}_n^2} \sum_{i=0}^{n-2} \mathbb{E}\left[c_{n-2,i+1}^2 \mathbb{I}_{\{|c_{n-2,i+1}|\geq\delta \mathfrak{L}_n\}}\right] \\
&\qquad\qquad = \frac{1}{(\mathbb{E}[\epsilon^2])^2} \mathbb{E}\left[\lim_{n\to\infty} c_{n-2,1}^2 \mathbb{I}_{\left\{|c_{n-2,1}|\geq\delta\sqrt{(n-1)(\mathbb{E}[\epsilon^2])^2}\right\}}\right].
\end{aligned}
$$

At this moment, we return to Assumption 3.1, and recall that we take second and fourth moment of the MMN finite, so

$$
\mathbb{E}\left[b_{n-2,1}^2 \middle| X\right] = \mathbb{E}\left[b_{n-2,1}^2\right] = \mathbb{E}\left[\left(\epsilon_{t_1}^2 - \mathbb{E}[\epsilon^2]\right)^2\right] = \mathbb{E}\left[\epsilon_{t_i}^4\right] - (\mathbb{E}[\epsilon^2])^2 < \infty,
$$

$$\left[c_{n-2,1}^2 \middle| X\right] = \mathbb{E}\left[c_{n-2,1}^2\right] = \mathbb{E}\left[\left(\epsilon_{t_1}\epsilon_{t_0}\right)^2\right] = (\mathbb{E}[\epsilon^2])^2 < \infty,$$

and

$$\mathbb{E}\left[b_{n-2,1}^2\right] = \mathbb{E}\left[b_{n-2,1}^2 \mathbb{I}_{\left\{|b_{n-2,1}| \geq \delta\sqrt{(n-2)\mathrm{Var}[\epsilon^2]}\right\}}\right] + \mathbb{E}\left[b_{n-2,1}^2 \mathbb{I}_{\left\{|b_{n-2,1}| < \delta\sqrt{(n-2)\mathrm{Var}[\epsilon^2]}\right\}}\right],$$

$$\mathbb{E}\left[c_{n-2,1}^2\right] = \mathbb{E}\left[c_{n-2,1}^2 \mathbb{I}_{\left\{|c_{n-2,1}| \geq \delta\sqrt{(n-1)(\mathbb{E}[\epsilon^2])^2}\right\}}\right] + \mathbb{E}\left[c_{n-2,1}^2 \mathbb{I}_{\left\{|c_{n-2,1}| < \delta\sqrt{(n-1)(\mathbb{E}[\epsilon^2])^2}\right\}}\right],$$

$$\delta\sqrt{(n-2)\mathrm{Var}[\epsilon^2]} \to \infty, \delta\sqrt{(n-1)(\mathbb{E}[\epsilon^2])^2} \to \infty \text{ as } n \to \infty\,.$$

The postulations $\mathbb{E}\left[b_{n-2,1}^2\right] < \infty$ and $\mathbb{E}\left[c_{n-2,1}^2\right] < \infty$ contradict with positive probabilities of the event that $\left|b_{n-2,1}\right| \geq \delta\sqrt{(n-2)\mathrm{Var}[\epsilon^2]}$ or $\left|c_{n-2,1}\right| \geq \delta\sqrt{(n-1)(\mathbb{E}[\epsilon^2])^2}$, because if there is at least one event where $\left|b_{n-2,i}\right| \to \infty$ or $\left|c_{n-2,i}\right| \to \infty$ with positive probability, then $\mathbb{E}\left[b_{n-2,1}^2\right] \to \infty$ or $\mathbb{E}\left[c_{n-2,1}^2\right] \to \infty$ but now Assumption 3.1 is broken.

Hence, Assumption 3.1 ensures that $\mathbb{E}\left[b_{n-2,1}^2\right] < \infty$ and $\mathbb{E}\left[c_{n-2,1}^2\right] < \infty$, which in turn requires both $\mathbb{I}_{\left\{|b_{n-2,i}| \geq \delta\sqrt{(n-2)\mathrm{Var}[\epsilon^2]}\right\}}$ and $\mathbb{I}_{\left\{|c_{n-2,1}| \geq \delta\sqrt{(n-1)(\mathbb{E}[\epsilon^2])^2}\right\}}$ to converge to 0 as $n \to \infty$. Thus, Equations (B.10) and (B.11) should equal to 0, i.e.,

$$\lim_{n\to\infty} \frac{1}{\mathfrak{J}_n^2} \sum_{i=1}^{n-2} \mathbb{E}\left[b_{n-2,i}^2 \mathbb{I}_{\{|b_{n-2,i}| \geq \delta \mathfrak{J}_n\}}\right] = \frac{1}{\mathrm{Var}[\epsilon^2]} \mathbb{E}[0] = 0,$$

$$\lim_{n\to\infty} \frac{1}{\mathfrak{L}_n^2} \sum_{i=0}^{n-2} \mathbb{E}\left[c_{n-2,i+1}^2 \mathbb{I}_{\{|c_{n-2,i+1}| \geq \delta \mathfrak{L}_n\}}\right] = \frac{1}{(\mathbb{E}[\epsilon^2])^2} \mathbb{E}[0] = 0.$$

It follows that, the final job to be done with regards to asymptotic distribution of $[\epsilon,\epsilon]_T^{\mathcal{G}} - \mathbb{E}\left[[\epsilon,\epsilon]_T^{\mathcal{G}} \middle| X\right]$ is to find covariance of $M_T^{(1)}$ and $M_T^{(2)}$.

$$\begin{aligned}\mathrm{Cov}\left[M_T^{(1)}, M_T^{(2)} \middle| X\right] &= \mathbb{E}\left[\left(M_T^{(1)} - \mathbb{E}\left[M_T^{(1)}\right]\right)\left(M_T^{(2)} - \mathbb{E}\left[M_T^{(2)}\right]\right) \middle| X\right] \\ &= \mathbb{E}\left[M_T^{(1)} M_T^{(2)}\right]\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}\left[\left(\frac{1}{\sqrt{n-2}}\sum_{i=1}^{n-2}\epsilon_{t_i}^2 - \mathbb{E}[\epsilon^2]\right)\left(\frac{1}{\sqrt{n-2}}\sum_{i=0}^{n-2}\epsilon_{t_{i+1}}\epsilon_{t_i}\right)\right] \\
&= \frac{1}{n-2}\mathbb{E}\left[\left(\sum_{i=1}^{n-2}\epsilon_{t_i}^2 - \mathbb{E}[\epsilon^2]\right)\left(\epsilon_{t_0}\epsilon_{t_1} + \sum_{i=1}^{n-2}\epsilon_{t_{i+1}}\epsilon_{t_i}\right)\right] \\
&= \frac{1}{n-2}\left(\mathbb{E}\left[\sum_{i=1}^{n-2}\epsilon_{t_0}\epsilon_{t_1}\epsilon_{t_i}^2 - \epsilon_{t_0}\epsilon_{t_1}\mathbb{E}[\epsilon^2]\right] + \mathbb{E}\left[\sum_{i=1}^{n-2}\epsilon_{t_i}^2 - \mathbb{E}[\epsilon^2]\sum_{i=1}^{n-2}\epsilon_{t_{i+1}}\epsilon_{t_i}\right]\right) \\
&= \frac{1}{n-2}\mathbb{E}\left[\left(\sum_{i=1}^{n-2}\epsilon_{t_0}\epsilon_{t_1}\epsilon_{t_i}^2 - \epsilon_{t_0}\epsilon_{t_1}\mathbb{E}[\epsilon^2]\right) + \left(\sum_{i=1}^{n-2}\epsilon_{t_i}^2 - \mathbb{E}[\epsilon^2]\sum_{i=1}^{n-2}\epsilon_{t_{i+1}}\epsilon_{t_i}\right)\right] \\
&= \frac{1}{n-2}\mathbb{E}\left[\left(\epsilon_{t_0}\epsilon_{t_1}^3 + \epsilon_{t_0}\epsilon_{t_1}\epsilon_{t_2}^2 + \cdots + \epsilon_{t_0}\epsilon_{t_1}\epsilon_{t_{n-1}}^2 - (n-2)\epsilon_{t_0}\epsilon_{t_1}\mathbb{E}[\epsilon^2]\right)\right] \\
&\quad + \frac{1}{n-2}\mathbb{E}\begin{bmatrix} \epsilon_{t_1}^2\epsilon_{t_0}\epsilon_{t_1} + \epsilon_{t_1}^2\epsilon_{t_1}\epsilon_{t_2} + \cdots + \epsilon_{t_1}^2\epsilon_{t_{n-2}}\epsilon_{t_{n-1}} \\ +\epsilon_{t_2}^2\epsilon_{t_0}\epsilon_{t_1} + \cdots + \epsilon_{t_2}^2\epsilon_{t_{n-2}}\epsilon_{t_{n-1}} + \cdots \\ +\epsilon_{t_{n-2}}^2\epsilon_{t_0}\epsilon_{t_1} + \epsilon_{t_{n-2}}^2\epsilon_{t_1}\epsilon_{t_2} + \cdots + \epsilon_{t_{n-2}}^3\epsilon_{t_{n-1}} \end{bmatrix} \\
&\quad - \frac{n-2}{n-2}\mathbb{E}\left[\epsilon_{t_0}\epsilon_{t_1}\mathbb{E}[\epsilon^2] + \epsilon_{t_1}\epsilon_{t_2}\mathbb{E}[\epsilon^2] + \cdots + \epsilon_{t_{n-2}}\epsilon_{t_{n-1}}\mathbb{E}[\epsilon^2]\right] \\
&= \frac{1}{n-2}\begin{pmatrix} \mathbb{E}[\epsilon_{t_0}]\mathbb{E}[\epsilon_{t_1}^3] + \mathbb{E}[\epsilon_{t_0}]\mathbb{E}[\epsilon_{t_1}]\mathbb{E}[\epsilon_{t_2}^2] \\ + \cdots + \mathbb{E}[\epsilon_{t_0}]\mathbb{E}[\epsilon_{t_1}]\mathbb{E}[\epsilon_{t_{n-1}}^2] \\ -(n-2)\mathbb{E}[\epsilon_{t_0}]\mathbb{E}[\epsilon_{t_1}]\mathbb{E}[\epsilon^2] \end{pmatrix} \\
&\quad + \frac{1}{n-2}\begin{pmatrix} \mathbb{E}[\epsilon_{t_0}]\mathbb{E}[\epsilon_{t_1}^3] + \mathbb{E}[\epsilon_{t_2}]\mathbb{E}[\epsilon_{t_1}^3] + \\ \cdots + \mathbb{E}[\epsilon_{t_{n-2}}]\mathbb{E}[\epsilon_{t_{n-1}}]\mathbb{E}[\epsilon_{t_1}^2] \\ +\mathbb{E}[\epsilon_{t_0}]\mathbb{E}[\epsilon_{t_1}]\mathbb{E}[\epsilon_{t_2}^2] + \mathbb{E}[\epsilon_{t_1}]\mathbb{E}[\epsilon_{t_2}^3] \\ + \cdots + \mathbb{E}[\epsilon_{t_{n-2}}]\mathbb{E}[\epsilon_{t_{n-1}}]\mathbb{E}[\epsilon_{t_2}^2] + \cdots \\ +\mathbb{E}[\epsilon_{t_0}]\mathbb{E}[\epsilon_{t_1}]\mathbb{E}[\epsilon_{t_{n-2}}^2] + \cdots + \mathbb{E}[\epsilon_{t_{n-1}}]\mathbb{E}[\epsilon_{t_{n-2}}^3] \end{pmatrix} \\
&\quad - \left(\mathbb{E}[\epsilon_{t_0}]\mathbb{E}[\epsilon_{t_1}]\mathbb{E}[\epsilon^2] + \cdots + \mathbb{E}[\epsilon_{t_{n-2}}]\mathbb{E}[\epsilon_{t_{n-1}}]\mathbb{E}[\epsilon^2]\right) \\
&= 0.
\end{aligned}
$$

Thus, we have shown that $[\epsilon,\epsilon]_T^{\mathcal{G}} - \mathbb{E}\left[[\epsilon,\epsilon]_T^{\mathcal{G}}\right]\big|X$ can be written in terms of $M_T^{(1)}$ and $M_T^{(2)}$ that are independent, centered normal asymptotically with variances $\mathrm{Var}[\epsilon^2]$ and $(\mathbb{E}[\epsilon^2])^2$.

Because as $n \to \infty$

$$
\begin{aligned}
\frac{1}{\sqrt{n-1}}\left([\epsilon,\epsilon]_T^{\mathcal{G}} - \mathbb{E}\left[[\epsilon,\epsilon]_T^{\mathcal{G}}\right]\big|X\right) &= \frac{2\sqrt{n-2}\left(M_T^{(1)} - M_T^{(2)}\right) + O_p(1)}{\sqrt{n-1}} \\
&\to 2\left(M_T^{(1)} - M_T^{(2)}\right) + o_p(1)\,,
\end{aligned}
$$

asymptotic distribution of $\frac{1}{\sqrt{n-1}}\left([\epsilon,\epsilon]_T^{\mathcal{G}} - \mathbb{E}\left[[\epsilon,\epsilon]_T^{\mathcal{G}}\right]\middle|X\right)$ is also centered normal while its asymptotic variance is calculated as follows:

$$\begin{aligned}\operatorname{Var}\left[\frac{1}{\sqrt{n-1}}\left([\epsilon,\epsilon]_T^{\mathcal{G}} - \mathbb{E}\left[[\epsilon,\epsilon]_T^{\mathcal{G}}\right]\middle|X\right)\right] &\to \operatorname{Var}\left[2\left(M_T^{(1)} - M_T^{(2)}\right)\right] \\ &= 4\operatorname{Var}\left[M_T^{(1)}\right] + 4\operatorname{Var}\left[M_T^{(2)}\right] - 4\operatorname{Cov}\left[M_T^{(1)}, M_T^{(2)}\right] \\ &= 4\operatorname{Var}[\epsilon^2] + 4(\mathbb{E}[\epsilon^2])^2 = 4\,\mathbb{E}[\epsilon^4] - 4(\mathbb{E}[\epsilon^2])^2 + 4(\mathbb{E}[\epsilon^2])^2 \\ &= 4\,\mathbb{E}[\epsilon^4].\end{aligned}$$

Furthermore, we have proved in Step 1 that $[X,\epsilon]_T^{\mathcal{G}} = O_p(1)$, and Equation (3.6) converts to

$$[Y,Y]_T^{\mathcal{G}} = [X,X]_T^{\mathcal{G}} + [\epsilon,\epsilon]_T^{\mathcal{G}} + O_p(1) = [\epsilon,\epsilon]_T^{\mathcal{G}} + O_p(1).$$

Correspondingly with $\mathbb{E}\left[[\epsilon,\epsilon]_T^{\mathcal{G}}\middle|X\right]$=$2(n-1)\mathbb{E}[\epsilon^2]$,

$$\begin{aligned}[Y,Y]_T^{\mathcal{G}} - 2(n-1)\mathbb{E}[\epsilon^2]\big|X &= \left([\epsilon,\epsilon]_T^{\mathcal{G}} - 2(n-1)\mathbb{E}[\epsilon^2]\right)\big|X + O_p(1) \\ \Rightarrow \frac{\left([Y,Y]_T^{\mathcal{G}} - 2(n-1)\mathbb{E}[\epsilon^2]\big|X\right)}{\sqrt{n-1}} &\xrightarrow[n\to\infty]{} \frac{\left([\epsilon,\epsilon]_T^{\mathcal{G}} - 2(n-1)\mathbb{E}[\epsilon^2]\big|X\right)}{\sqrt{n-1}} + o_p(1),\end{aligned}$$

and

$$\begin{aligned}[Y,Y]_T^{\mathcal{G}} &= [X,X]_T^{\mathcal{G}} + 2[X,\varepsilon]_T^{\mathcal{G}} + [\varepsilon,\epsilon]_T^{\mathcal{G}} = [X,X]_T^{\mathcal{G}} + [\epsilon,\epsilon]_T^{\mathcal{G}} + O_p(1) \\ \Rightarrow [Y,Y]_T^{\mathcal{G}} - [X,X]_T^{\mathcal{G}} - 2(n-1)\mathbb{E}[\epsilon^2] &= [\varepsilon,\epsilon]_T^{\mathcal{G}} - 2(n-1)\mathbb{E}[\epsilon^2] + O_p(1).\end{aligned}$$

Then,

$$\frac{\left([Y,Y]_T^{\mathcal{G}} - [X,X]_T^{\mathcal{G}} - 2(n-1)\mathbb{E}[\epsilon^2]\right)\big|X}{\sqrt{n-1}} \xrightarrow[n\to\infty]{} \frac{\left([\varepsilon,\epsilon]_T^{\mathcal{G}} - 2(n-1)\mathbb{E}[\epsilon^2]\right)\big|X}{\sqrt{n-1}} + o_p(1).$$

From above convergences, we deduce that conditional distribution of $\frac{1}{\sqrt{n-1}}\left([\epsilon,\epsilon]_T^{\mathcal{G}} - 2(n-1)\mathbb{E}[\epsilon^2]\right)$ is also conditional distribution of both of $\frac{1}{\sqrt{n-1}}\left([Y,Y]_T^{\mathcal{G}} - 2(n-1)\mathbb{E}[\epsilon^2]\right)$ and $\frac{1}{\sqrt{n-1}}\left([Y,Y]_T^{\mathcal{G}} - [X,X]_T^{\mathcal{G}} - 2(n-1)\mathbb{E}[\epsilon^2]\right)$, i.e., conditional on true price

$$[Y,Y]_T^{\mathcal{G}} - 2(n-1)\mathbb{E}[\epsilon^2] \xrightarrow{d} N(0,4(n-1)\mathbb{E}[\epsilon^4]),$$

$$\left([Y,Y]_T^{\mathcal{G}} - [X,X]_T^{\mathcal{G}} - 2(n-1)\mathbb{E}[\epsilon^2]\right) \xrightarrow{d} N(0,4(n-1)\mathbb{E}[\epsilon^4]).$$

Let's define $\widehat{\mathbb{E}[\epsilon^2]}$ as $[Y,Y]_T^{\mathcal{G}}/2(n-1)$ like it is suggested by Zhang et al. [111]. Then, $2(n-1)\widehat{\mathbb{E}[\epsilon^2]} = [Y,Y]_T^{\mathcal{G}}$ and conditional on true price

$$2(n-1)\widehat{\mathbb{E}[\epsilon^2]} - 2(n-1)\mathbb{E}[\epsilon^2] \xrightarrow{d} N(0,4(n-1)\mathbb{E}[\epsilon^4])$$
$$\Rightarrow 2(n-1)\left(\widehat{\mathbb{E}[\epsilon^2]} - \mathbb{E}[\epsilon^2]\right) \xrightarrow{d} N(0,4(n-1)\mathbb{E}[\epsilon^4])$$
$$\Rightarrow \sqrt{n-1}\left(\widehat{\mathbb{E}[\epsilon^2]} - \mathbb{E}[\epsilon^2]\right) \xrightarrow{d} N(0,\mathbb{E}[\epsilon^4]).$$

APPENDIX C

AN UNBIASED AND CONSISTENT ESTIMATOR OF THE FOURTH MOMENT OF MMN

Suppose that we observe security prices on a grid $\mathcal{G} = \{t_0, t_1, \cdots, t_{n-1}\}, t_0 = 0, t_{n-1} = T$, where the number of data points in grid $\mathcal{G}$ is denoted by $|\mathcal{G}|$ and equals to n. Let $\Delta(\mathcal{G}) = \max_{1\le i\le n}(t_{i+1} - t_i)$, then for $n \to \infty, \Delta(\mathcal{G}) \to 0$. In this setting,

$$
\begin{aligned}
[Y,Y,Y,Y]_T^{\mathcal{G}} := \sum_{i=0}^{n-2}\left(\Delta Y_{t_i}\right)^4 &= \sum_{i=0}^{n-2}\left(\left(\Delta X_{t_i} + \Delta\epsilon_{t_i}\right)^2\right)^2 \\
&= \sum_{i=0}^{n-2}\left(\left(\Delta X_{t_i}\right)^2 + \left(\Delta\epsilon_{t_i}\right)^2 + 2\Delta X_{t_i}\Delta\epsilon_{t_i}\right)^2 \\
&= \sum_{i=0}^{n-2}\left(\Delta X_{t_i}\right)^4 + \sum_{i=0}^{n-2}\left(\Delta\epsilon_{t_i}\right)^4 \\
&+ \sum_{i=0}^{n-2} 6\left(\Delta X_{t_i}\right)^2\left(\Delta\epsilon_{t_i}\right)^2 \\
&+ \sum_{i=0}^{n-2} 4\Delta X_{t_i}\left(\Delta\epsilon_{t_i}\right)^3 + \sum_{i=0}^{n-2} 4\left(\Delta X_{t_i}\right)^3\Delta\epsilon_{t_i} \\
&= [X,X,X,X]_T^{\mathcal{G}} + [\epsilon,\epsilon,\epsilon,\epsilon]_T^{\mathcal{G}} + A,
\end{aligned}
\tag{C.1}
$$

where

$$
[X,X,X,X]_T^{\mathcal{G}} := \sum_{i=0}^{n-2}\left(\Delta X_{t_i}\right)^4,
$$

$$[\epsilon,\epsilon,\epsilon,\epsilon]_T^{\mathcal{G}} := \sum_{i=0}^{n-2} \left(\Delta\epsilon_{t_i}\right)^4,$$

$$A := \sum_{i=0}^{n-2} 6\left(\Delta X_{t_i}\right)^2\left(\Delta\epsilon_{t_i}\right)^2 + \sum_{i=0}^{n-2} 4\Delta X_{t_i}\left(\Delta\epsilon_{t_i}\right)^3 + \sum_{i=0}^{n-2} 4\left(\Delta X_{t_i}\right)^3\Delta\epsilon_{t_i}.$$

Taking expectation on the both sides of Equation (C.1) conditional on true price yields

$$\mathbb{E}\left[[Y,Y,Y,Y]_T^{\mathcal{G}}\middle|X\right] = \mathbb{E}\left[[X,X,X,X]_T^{\mathcal{G}}\middle|X\right] + \mathbb{E}\left[[\epsilon,\epsilon,\epsilon,\epsilon]_T^{\mathcal{G}}\middle|X\right] + \mathbb{E}[A|X]. \qquad \text{(C.2)}$$

Handling terms in the RHS of Equation (C.2) one by one under Assumptions 3.1, 3.2, 3.3, 3.11 and the null hypothesis that the MMN increments have constant variance gives us

$$\begin{aligned}
\mathbb{E}\left[[X,X,X,X]_T^{\mathcal{G}}\middle|X\right] &= \mathbb{E}\left[\sum_{i=0}^{n-2}\left(\Delta X_{t_i}\right)^4\middle|X\right] \\
&= \mathbb{E}\left[\left([X,X]_T^{\mathcal{G}}\right)^2\middle|X\right] - 2\sum_{\substack{i=0\\ i\neq j}}^{n-2}\sum_{j=0}^{n-2}\mathbb{E}\left[\left(\Delta X_{t_i}\right)^2\left(\Delta X_{t_j}\right)^2\middle|X\right] \\
&= \mathbb{E}\left[\left([X,X]_T^{\mathcal{G}}\right)^2\middle|X\right] - 2\sum_{\substack{i=0\\ i\neq j}}^{n-2}\sum_{j=0}^{n-2}\mathbb{E}\left[\left(\Delta X_{t_i}\right)^2\middle|X\right]\mathbb{E}\left[\left(\Delta X_{t_j}\right)^2\middle|X\right] \\
&= \mathbb{E}\left[\left([X,X]_T^{\mathcal{G}}\right)^2\middle|X\right] \\
&\quad -2\sum_{\substack{i=0\\ i\neq j}}^{n-2}\sum_{j=0}^{n-2}\left(\sigma_{t_{i+1}}^2 t_{i+1} - \sigma_{t_i}^2 t_i\right)\left(\sigma_{t_{j+1}}^2 t_{j+1} - \sigma_{t_j}^2 t_j\right) \\
&= O_p(1) - O_p(1) \\
&= O_p(1),
\end{aligned}$$

where by no leverage effect, the definition of Brownian Motion and Assumptions 3.3 and 3.11, $\Delta X_{t_i} \perp \Delta X_{t_j}$, $\Delta X_{t_i} \perp X_{t_i}$, $\mathbb{E}\left[X_{t_i}\middle|X\right] = 0$, $\mathrm{Var}\left[X_{t_{i+1}}\middle|X\right] = \sigma_{t_{i+1}}^2 t_{i+1}$, $[X,X]_T^{\mathcal{G}} = O_p(1)$ and $O_p(1)O_p(1) = O_p(1)$, so that $\left([X,X]_T^{\mathcal{G}}\right)^2 = O_p(1)$ and $\mathbb{E}\left[\left([X,X]_T^{\mathcal{G}}\right)^2\middle|X\right] = O_p(1)$, while

$$
\begin{aligned}
\left[\left(\Delta X_{t_i}\right)^2 \middle| X\right] &= \mathbb{E}\left[X_{t_{i+1}}^2 \middle| X\right] + \mathbb{E}\left[X_{t_i}^2 \middle| X\right] - 2\mathbb{E}\left[X_{t_i} X_{t_{i+1}} \middle| X\right] \\
&= \mathrm{Var}\left[X_{t_{i+1}} \middle| X\right] + \left(\mathbb{E}\left[X_{t_{i+1}} \middle| X\right]\right)^2 + \mathrm{Var}\left[X_{t_i} \middle| X\right] \\
&\quad + \left(\mathbb{E}\left[X_{t_i} \middle| X\right]\right)^2 - 2\mathbb{E}\left[X_{t_i}\left(\left(X_{t_{i+1}} - X_{t_i}\right) + X_{t_i}\right) \middle| X\right] \\
&= \sigma_{t_{i+1}}^2 t_{i+1} + \sigma_{t_i}^2 t_i - 2\,\mathbb{E}\left[X_{t_i}^2 \middle| X\right] + 2\mathbb{E}\left[X_{t_i}\left(X_{t_{i+1}} - X_{t_i}\right) \middle| X\right] \\
&= \sigma_{t_{i+1}}^2 t_{i+1} + \sigma_{t_i}^2 t_i - 2\,\mathbb{E}\left[X_{t_i}^2 \middle| X\right] \\
&\quad + 2\mathbb{E}\left[X_{t_i} \middle| X\right]\mathbb{E}\left[\left(X_{t_{i+1}} - X_{t_i}\right) \middle| X\right] \\
&= \sigma_{t_{i+1}}^2 t_{i+1} + \sigma_{t_i}^2 t_i - 2\mathbb{E}\left[X_{t_i}^2 \middle| X\right] \\
&= \sigma_{t_{i+1}}^2 t_{i+1} + \sigma_{t_i}^2 t_i - 2\sigma_{t_i}^2 t_i = \sigma_{t_{i+1}}^2 t_{i+1} - \sigma_{t_i}^2 t_i \\
&= O_p(1),
\end{aligned}
\tag{C.3}
$$

because under Assumption 3.1, $\mathbb{E}[|\sigma_t|^2] < \infty$ for all $t \in [0, T]$ and if a random variable has finite absolute moments of order k, then it has absolute moments of orders $1, 2, \ldots, k-1$ (Chapter 4, Section 21, p. 292 of [34]) so that $\mathbb{E}[|\sigma_t|] < \infty$ also holds. First moment of σ_t being finite opens the way for the Markov's Inequality,

$$
\mathbb{P}[|\sigma_t| \geq \gamma] \leq \frac{1}{\gamma}\mathbb{E}[|\sigma_t|], \forall\, \gamma \in (0, \infty),
$$

so that as $\gamma \to \infty$, $\mathbb{P}[|\sigma_t| \geq \gamma] \to 0$. Recall that definition of stochastic boundedness requires that for any $\lambda > 0$, there exists a finite $\mathrm{M} > 0$ such that

$$
\mathbb{P}[|\sigma_t| > \mathrm{M}] < \lambda, \forall t,
$$

Setting $\gamma = \mathrm{M}$ and $\frac{1}{\gamma}\mathbb{E}[|\sigma_t|] = \lambda$, we get $\sigma_t = O_p(1)$ and $\sigma_t^2 = O_p(1)O_p(1) = O_p(1)$ for all $t \in [0, T]$.

Regarding remaining terms on the RHS of Equation (C.2), we have

$$
\begin{aligned}
\left[[\epsilon,\epsilon,\epsilon,\epsilon]_T^{\mathcal{G}}\middle|X\right] &= \mathbb{E}\left[\sum_{i=0}^{n-2}\left(\Delta\epsilon_{t_i}\right)^4\middle|X\right] = \mathbb{E}\left[\sum_{i=0}^{n-2}\left(\Delta\epsilon_{t_i}\right)^4\right] \\
&= \mathbb{E}\left[\sum_{i=0}^{n-2}\epsilon_{t_{i+1}}^4 + \epsilon_{t_i}^4 + 6\epsilon_{t_i}^2\epsilon_{t_{+1}}^2 - 4\epsilon_{t_i}^3\epsilon_{t_{+1}} - 4\epsilon_{t_i}\epsilon_{t_{+1}}^3\right] \\
&= \sum_{i=0}^{n-2}\mathbb{E}\left[\epsilon_{t_{i+1}}^4\right] + \mathbb{E}\left[\epsilon_{t_i}^4\right] + 6\mathbb{E}\left[\epsilon_{t_i}^2\right]\mathbb{E}\left[\epsilon_{t_{+1}}^2\right] - 4\mathbb{E}\left[\epsilon_{t_i}^3\right]\mathbb{E}\left[\epsilon_{t_{i+1}}\right] \\
&- 4\mathbb{E}\left[\epsilon_{t_i}\right]\mathbb{E}\left[\epsilon_{t_{+1}}^3\right] = \sum_{i=0}^{n-2}2\mathbb{E}[\epsilon^4] + 6(\mathbb{E}[\epsilon^2])^2 \\
&= 2(n-1)\mathbb{E}[\epsilon^4] + 6(n-1)(\mathbb{E}[\epsilon^2])^2,
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{i=0}^{n-2}\left(\Delta X_{t_i}\right)^2\left(\Delta\epsilon_{t_i}\right)^2\middle|X\right] \\
&= \mathbb{E}\left[\left(\sum_{i=0}^{n-2}\Delta X_{t_i}\Delta\epsilon_{t_i}\right)^2\middle|X\right] - \mathbb{E}\left[2\sum_{\substack{i=0\\ i\neq j}}^{n-2}\sum_{j=0}^{n-2}\Delta X_{t_i}\Delta\epsilon_{t_i}\Delta X_{t_j}\Delta\epsilon_{t_j}\middle|X\right] \\
&= \mathbb{E}\left[\left([X,\epsilon]_T^{\mathcal{G}}\right)^2\middle|X\right] - 2\sum_{\substack{i=0\\ i\neq j}}^{n-2}\sum_{j=0}^{n-2}\mathbb{E}\left[\Delta X_{t_i}\Delta\epsilon_{t_i}\Delta X_{t_j}\Delta\epsilon_{t_j}\middle|X\right] \\
&= \mathbb{E}\left[\left([X,\epsilon]_T^{\mathcal{G}}\right)^2\middle|X\right] \\
&- 2\sum_{\substack{i=0\\ i\neq j}}^{n-2}\sum_{j=0}^{n-2}\mathbb{E}\left[\Delta X_{t_i}\middle|X\right]\mathbb{E}\left[\Delta\epsilon_{t_i}\middle|X\right]\mathbb{E}\left[\Delta X_{t_j}\middle|X\right]\mathbb{E}\left[\Delta\epsilon_{t_j}\middle|X\right] \\
&= \mathbb{E}\left[\left([X,\epsilon]_T^{\mathcal{G}}\right)^2\middle|X\right] = O_p(1),
\end{aligned}
$$

$$
\mathbb{E}\left[\sum_{i=0}^{n-2}\left(\Delta X_{t_i}\right)^3\Delta\epsilon_{t_i}\middle|X\right] = \sum_{i=0}^{n-2}\mathbb{E}\left[\left(\Delta X_{t_i}\right)^3\middle|X\right]\ \mathbb{E}\left[\Delta\epsilon_{t_i}\middle|X\right] = 0,
$$

$$
\mathbb{E}\left[\sum_{i=0}^{n-2}\left(\Delta\epsilon_{t_i}\right)^3\Delta X_{t_i}\middle|X\right] = \sum_{i=0}^{n-2}\mathbb{E}\left[\left(\Delta\epsilon_{t_i}\right)^3\middle|X\right]\ \mathbb{E}\left[\Delta X_{t_i}\middle|X\right] = 0.
$$

Consequently, Equation (C.2) is found out to be sum of a stochastically bounded term and $\mathbb{E}\left[[\epsilon,\epsilon,\epsilon,\epsilon]_T^{\mathcal{G}}\middle|X\right]$, i.e.,

$$\left[[Y,Y,Y,Y]_T^{\mathcal{G}}\middle|X\right] = 2(n-1)\mathbb{E}[\epsilon^4] + 6(n-1)(\mathbb{E}[\epsilon^2])^2 + O_p(1). \tag{C.4}$$

Adopting the estimators proposed by Zhang et al. [111], let's make the following definitions:

$$\widehat{\mathbb{E}[\epsilon^2]} := \frac{1}{2(n-1)}\left([Y,Y]_T^{\mathcal{G}}\right)^2,$$

$$\widehat{\mathbb{E}[\epsilon^4]} := \frac{1}{2(n-1)}[Y,Y,Y,Y]_T^{\mathcal{G}} - 3\left(\widehat{\mathbb{E}[\epsilon^2]}\right)^2,$$

Then, from Equation (C.4) and above definitions, we obtain

$$\begin{aligned}\mathbb{E}\left[\left|\widehat{\mathbb{E}[\epsilon^4]} - \mathbb{E}[\epsilon^4]\right|\middle|X\right] &= \mathbb{E}\left[\frac{2(n-1)\mathbb{E}[\epsilon^4] + 6(n-1)(\mathbb{E}[\epsilon^2])^2}{2(n-1)}\middle|X\right] \\ &+\mathbb{E}\left[\frac{O_p(1)}{2(n-1)}\middle|X\right] - 3\mathbb{E}\left[\left(\frac{\left([Y,Y]_T^{\mathcal{G}}\right)}{2(n-1)}\right)^2\middle|X\right] - \mathbb{E}[\mathbb{E}[\epsilon^4]|X] \\ &\xrightarrow[n\to\infty]{} 3(\mathbb{E}[\epsilon^2])^2 + o_p(1) - 3\mathbb{E}\left[\left(\frac{[Y,Y]_T^{\mathcal{G}}}{2(n-1)}\right)^2\middle|X\right].\end{aligned} \tag{C.5}$$

Since

$$\mathbb{E}\left[\left([Y,Y]_T^{\mathcal{G}}\right)^2\middle|X\right] = \mathrm{Var}\left[\left([Y,Y]_T^{\mathcal{G}}\right)^2\middle|X\right] + \left(\mathbb{E}\left[[Y,Y]_T^{\mathcal{G}}\middle|X\right]\right)^2,$$

and following Barndorff-Nielsen and Shephard [24], Zhang et al. [111] and Hansen and Lunde [61] we have verified that as $n \to \infty$

$$\mathrm{Var}\left[\left([Y,Y]_T^{\mathcal{G}}\right)^2\middle|X\right] \to 4(n-1)\ \mathbb{E}[\epsilon^4],$$

$$\mathbb{E}\left[\left([Y,Y]_T^{\mathcal{G}}\right)^2\middle|X\right] \to 2(n-1)\ \mathbb{E}[\epsilon^2].$$

Equation (C.5) converts to

$$\left[\left|\widehat{\mathbb{E}[\epsilon^4]} - \mathbb{E}[\epsilon^4]\right| \middle| X\right]$$

$$\begin{aligned}
&\rightarrow 3(\mathbb{E}[\epsilon^2])^2 + o_p(1) \\
&- \frac{3}{4(n-1)^2}\left(4(n-1)\,\mathbb{E}[\epsilon^4] + 4(n-1)^2\,(\mathbb{E}[\epsilon^2])^2\right) \\
&\rightarrow 3(\mathbb{E}[\epsilon^2])^2 + o_p(1) - o_p(1) - 3(\mathbb{E}[\epsilon^2])^2 = o_p(1)
\end{aligned}$$

as $n \rightarrow \infty$. Thus, $\widehat{\mathbb{E}[\epsilon^4]}$ is an unbiased estimator of $\mathbb{E}[\epsilon^4]$. Remember that for an estimator to be consistent, it should converge to the parameter estimated as sample size goes to infinity, i.e., $\widehat{\mathbb{E}[\epsilon^4]}$ is called a consistent estimator of $\mathbb{E}[\epsilon^4]$ if

$$\lim_{n\rightarrow\infty} \mathbb{P}\left[\left|\widehat{\mathbb{E}[\epsilon^4]} - \mathbb{E}[\epsilon^4]\right| \middle| X \geq \gamma\right] = 0, \forall\, \gamma \in (0, \infty).$$

By Markov's Inequality, as $n \rightarrow \infty$

$$\begin{aligned}
&\lim_{n\rightarrow\infty} \mathbb{P}\left[\left|\widehat{\mathbb{E}[\epsilon^4]} - \mathbb{E}[\epsilon^4]\right| \middle| X \geq \gamma\right] \leq \frac{\mathbb{E}\left[\left|\widehat{\mathbb{E}[\epsilon^4]} - \mathbb{E}[\epsilon^4]\right| \middle| X\right]}{\gamma} \\
&\lim_{n\rightarrow\infty} \mathbb{P}\left[\left|\widehat{\mathbb{E}[\epsilon^4]} - \mathbb{E}[\epsilon^4]\right| \middle| X \geq \gamma\right] \leq \frac{o_p(1)}{\gamma} \\
&\Rightarrow \lim_{n\rightarrow\infty} \mathbb{P}\left[\left|\widehat{\mathbb{E}[\epsilon^4]} - \mathbb{E}[\epsilon^4]\right| \middle| X \geq \gamma\right] = 0.
\end{aligned}$$

APPENDIX D

PROOF OF THEOREM 3.5

From Equation (3.6)

$$[Y,Y]_T^{\mathcal{G}} = [X,X]_T^{\mathcal{G}} + 2[X,\varepsilon]_T^{\mathcal{G}} + [\varepsilon,\epsilon]_T^{\mathcal{G}}$$
$$[Y,Y]_T^{\mathcal{G}} - [X,X]_T^{\mathcal{G}} - 2(n-1)\mathbb{E}[\epsilon^2] = 2[X,\varepsilon]_T^{\mathcal{G}} + [\varepsilon,\epsilon]_T^{\mathcal{G}} - 2(n-1)\mathbb{E}[\epsilon^2]. \quad \text{(D.1)}$$

We blend Equations (B.9) and (B.1) in Equation (D.1) so that

$$\begin{aligned}
&[Y,Y]_T^{\mathcal{G}} - [X,X]_T^{\mathcal{G}} - 2(n-1)\mathbb{E}[\epsilon^2] \\
&\quad = 2\left(\sum_{i=1}^{n-2}\left(\Delta X_{t_{i-1}} - \Delta X_{t_i}\right)\epsilon_{t_i} - \Delta X_{t_0}\epsilon_{t_0} + \Delta X_{t_{n-2}}\epsilon_{t_{n-1}}\right) \\
&\quad + 2\left(\sum_{i=1}^{n-2}\epsilon_{t_i}^2 - \mathbb{E}[\epsilon^2]\right) + \left(\epsilon_{t_0}^2 - \mathbb{E}[\epsilon^2]\right) + \left(\epsilon_{t_{n-1}}^2 - \mathbb{E}[\epsilon^2]\right) \\
&\quad - 2\left(\sum_{i=0}^{n-2}\epsilon_{t_{i+1}}\epsilon_{t_i}\right).
\end{aligned}$$

Since $\Delta X_{t_0}\epsilon_{t_0} + \Delta X_{t_{n-2}}\epsilon_{t_{n-1}}$ and $\left(\epsilon_{t_0}^2 - \mathbb{E}[\epsilon^2]\right) + \left(\epsilon_{t_{n-1}}^2 - \mathbb{E}[\epsilon^2]\right)$ are both $O_p(1)$, as $n \to \infty$

$$\begin{aligned}
&\frac{[Y,Y]_T^{\mathcal{G}} - [X,X]_T^{\mathcal{G}} - 2(n-1)\mathbb{E}[\epsilon^2]}{\sqrt{n-2}} \\
&= \frac{2\sum_{i=1}^{n-2}\left(\Delta X_{t_{i-1}} - \Delta X_{t_i}\right)\epsilon_{t_i} + 2\left(\sum_{i=1}^{n-2}\epsilon_{t_i}^2 - \mathbb{E}[\epsilon^2]\right) - 2\left(\sum_{i=0}^{n-2}\epsilon_{t_{i+1}}\epsilon_{t_i}\right)}{\sqrt{n-2}} \quad \text{(D.2)} \\
&+ o_p(1).
\end{aligned}$$

Recall that in proof of the Theorem 3.4 that by defining $M_T^{(1)}$ as $\frac{1}{\sqrt{n-2}}\sum_{i=1}^{n-2}\epsilon_{t_i}^2-\mathbb{E}[\epsilon^2]$ and $M_T^{(2)}$ as $\frac{1}{\sqrt{n-2}}\sum_{i=0}^{n-2}\epsilon_{t_{i+1}}\epsilon_{t_i}$ and applying CLT for triangular arrays after checking for Lindeberg's conditions, we demonstrated that $M_T^{(1)}$ and $M_T^{(2)}$ are asymptotically independent and centered Gaussian with variances $\mathrm{Var}[\epsilon^2]$ and $(\mathbb{E}[\epsilon^2])^2$, respectively. At this point, if we define $M_T^{(3)}$ as $\frac{1}{\sqrt{n-2}}\sum_{i=1}^{n-2}\left(\Delta X_{t_{i-1}}-\Delta X_{t_i}\right)\epsilon_{t_i}$, then Equation (D.2) is affirmed to be

$$\begin{aligned}&[Y,Y]_T^{\mathcal{G}}-[X,X]_T^{\mathcal{G}}-2(n-1)\mathbb{E}[\epsilon^2]\\&\qquad=2\sqrt{n-2}\left(M_T^{(1)}+M_T^{(3)}-M_T^{(2)}\right)+O_p(1).\end{aligned}\tag{D.3}$$

So, to find asymptotic distribution of the LHS of Equation (D.2) conditional on true price, we need to examine applicability of the CLT for triangular arrays or martingale sequences with respect to $M_T^{(3)}$. In this endeavor, let's write

$$M_T^{(3)}:=\frac{1}{\sqrt{n-2}}\sum_{i=1}^{n-2}\left(\Delta X_{t_{i-1}}-\Delta X_{t_i}\right)\epsilon_{t_i}=\frac{1}{\sqrt{n-2}}\sum_{i=1}^{n-2}d_{n-2,i}.$$

If $n=3$,

$$\begin{gathered}M_T^{(3)}=\frac{1}{\sqrt{1}}\left(\Delta X_{t_0}-\Delta X_{t_1}\right)\epsilon_{t_1}=\frac{1}{\sqrt{1}}\left(d_{1,1}\right),\\d_{1,1}:=\left(\Delta X_{t_0}-\Delta X_{t_1}\right)\epsilon_{t_1}.\end{gathered}$$

If $n=4$,

$$\begin{gathered}M_T^{(3)}=\frac{1}{\sqrt{2}}\left(\left(\Delta X_{t_0}-\Delta X_{t_1}\right)\epsilon_{t_1}+\left(\Delta X_{t_1}-\Delta X_{t_2}\right)\epsilon_{t_2}\right)=\frac{1}{\sqrt{2}}\left(d_{2,1}+d_{2,2}\right),\\d_{2,1}:=\left(\Delta X_{t_0}-\Delta X_{t_1}\right)\epsilon_{t_1},d_{2,2}:=\left(\Delta X_{t_1}-\Delta X_{t_2}\right)\epsilon_{t_2}.\end{gathered}$$

If $n=5$,

$$\begin{aligned}M_T^{(3)}&=\frac{1}{\sqrt{3}}\left(\left(\Delta X_{t_0}-\Delta X_{t_1}\right)\epsilon_{t_1}+\left(\Delta X_{t_1}-\Delta X_{t_2}\right)\epsilon_{t_2}+\left(\Delta X_{t_2}-\Delta X_{t_3}\right)\epsilon_{t_3}\right)\\&=\frac{1}{\sqrt{3}}\left(d_{3,1}+d_{3,2}+d_{3,3}\right),\end{aligned}$$

$$d_{3,1} := (\Delta X_{t_0} - \Delta X_{t_1})\epsilon_{t_1}, d_{3,2} := (\Delta X_{t_1} - \Delta X_{t_2})\epsilon_{t_2}, d_{3,3} := (\Delta X_{t_2} - \Delta X_{t_3})\epsilon_{t_3}.$$

If we organize the terms in the form $d_{n-2,i}$ in a triangular fashion as $n \to \infty$, we get the following array

$$\begin{array}{l} d_{1,1} \\ d_{2,1}\ d_{2,2} \\ d_{3,1}\ d_{3,2}\ d_{3,3} \\ \vdots \end{array}$$

See that, for a specific n, $M_T^{(3)}$ is the sum of the row $n-2$ in the above triangular array divided by $\frac{1}{\sqrt{n-2}}$ while entries in each row of the above triangular array are i.i.d with mean 0 and finite variance, since for $i \neq j, \forall\, t_i, t_j \in \mathcal{G}$, by Assumptions 3.1, 3.2 and 3.3**,** with respect to the filtration $\mathcal{F}_i = \sigma\left(\epsilon_{t_j}, j \leq i, X_t, \forall t \in [0,T]\right)$ (ΔX_{t_i} is $\mathcal{F}_{t_i}$ measurable)

$$\epsilon_{t_i} \perp X_{t_i}, \Delta X_{t_i} \perp \Delta X_{t_j}, \epsilon_{t_i} \perp \epsilon_{t_j}, \mathrm{Var}\big[X_{t_i}\big] = \sigma_{t_i}^2 t_i, \mathbb{E}\big[X_{t_i}\big] = 0, \Delta X_{t_i} \perp X_{t_{i+1}},$$
$$\epsilon_{t_i} \perp \mathcal{F}_{t_{i-1}}, \Delta X_{t_i} \perp \mathcal{F}_{t_{i-1}}, \Delta X_{t_i} \perp \mathcal{F}_{t_i},$$

$$\mathbb{E}\big[(\Delta X_{t_{i-1}} - \Delta X_{t_i})\epsilon_{t_i}\big|\mathcal{F}_{i-1}\big] = \mathbb{E}\big[(\Delta X_{t_{i-1}} - \Delta X_{t_i})\big|\mathcal{F}_{i-1}\big]\mathbb{E}[\epsilon] = 0. \quad \text{(D.4)}$$

$$\begin{aligned}
&\mathrm{Var}\big[(\Delta X_{t_{i-1}} - \Delta X_{t_i})\epsilon_{t_i}\big|\mathcal{F}_{i-1}\big] \\
&\quad = \mathbb{E}\left[\left((\Delta X_{t_{i-1}} - \Delta X_{t_i})\epsilon_{t_i}\right)^2\middle|\mathcal{F}_{i-1}\right] - \left(\mathbb{E}\big[(\Delta X_{t_{i-1}} - \Delta X_{t_i})\epsilon_{t_i}\big|\mathcal{F}_{i-1}\big]\right)^2 \\
&\quad = \mathbb{E}\left[\left((\Delta X_{t_{i-1}} - \Delta X_{t_i})\epsilon_{t_i}\right)^2\middle|\mathcal{F}_{i-1}\right] \\
&\quad = \begin{pmatrix} \mathbb{E}\left[\left(\Delta X_{t_{i-1}}\right)^2\middle|\mathcal{F}_{i-1}\right] + \mathbb{E}\left[\left(\Delta X_{t_i}\right)^2\middle|\mathcal{F}_{i-1}\right] \\ -2\mathbb{E}\big[(\Delta X_{t_{i-1}}\Delta X_{t_i})\big|\mathcal{F}_{i-1}\big] \end{pmatrix}\mathbb{E}[\epsilon^2] \\
&\quad = \left(\mathbb{E}\left[\left(\Delta X_{t_{i-1}}\right)^2\middle|\mathcal{F}_{i-1}\right] + \mathbb{E}\left[\left(\Delta X_{t_i}\right)^2\middle|\mathcal{F}_{i-1}\right]\right)\mathbb{E}[\epsilon^2] \\
&\quad = \left(\mathbb{E}\left[\left(\Delta X_{t_{i-1}}\right)^2\right] + \mathbb{E}\left[\left(\Delta X_{t_i}\right)^2\right]\right)\mathbb{E}[\epsilon^2] = O_p(1)\mathbb{E}[\epsilon^2] < \infty, \quad \text{(D.5)}
\end{aligned}$$

where by Equation (C.3), $\mathbb{E}\left[\left(\Delta X_{t_{i-1}}\right)^2\middle|\mathcal{F}_{i-1}\right]$ and $\mathbb{E}\left[\left(\Delta X_{t_i}\right)^2\middle|\mathcal{F}_{i-1}\right]$ are deduced to be stochastically bounded and $\mathbb{E}\big[(\Delta X_{t_{i-1}}\Delta X_{t_i})\big|\mathcal{F}_{i-1}\big]$ equals to 0, because $X_{t_{i+1}} = \left((X_{t_{i+1}} - X_{t_i}) + X_{t_i}\right)$ and

$$
\begin{aligned}
[(X_{t_i}X_{t_{i+1}})|\mathcal{F}_{i-1}] &= \mathbb{E}\left[\left(X_{t_i}\left((X_{t_{i+1}}-X_{t_i})+X_{t_i}\right)\right)\middle|\mathcal{F}_{i-1}\right] \\
&= \mathbb{E}[X_{t_i}^2|\mathcal{F}_{i-1}] + \mathbb{E}\left[\left(X_{t_i}(X_{t_{i+1}}-X_{t_i})\right)\middle|\mathcal{F}_{i-1}\right] \\
&= \mathbb{E}[X_{t_i}^2|\mathcal{F}_{i-1}] + \mathbb{E}[X_{t_i}|\mathcal{F}_{i-1}]\mathbb{E}[(X_{t_{i+1}}-X_{t_i})|\mathcal{F}_{i-1}] \\
&= \mathbb{E}[X_{t_i}^2|\mathcal{F}_{i-1}] \\
&= \mathrm{Var}[X_{t_i}|\mathcal{F}_{i-1}],
\end{aligned}
$$

so that

$$
\begin{aligned}
\mathbb{E}[(\Delta X_{t_{i-1}}\Delta X_{t_i})|\mathcal{F}_{i-1}] &= \mathbb{E}[(X_{t_i}-X_{t_{i-1}})(X_{t_{i+1}}-X_{t_i})|\mathcal{F}_{i-1}] \\
&= \mathbb{E}[(X_{t_i}X_{t_{i+1}})|\mathcal{F}_{i-1}] - \mathbb{E}[(X_{t_{i-1}}X_{t_{i+1}})|\mathcal{F}_{i-1}] - \mathbb{E}[(X_{t_i}^2)|\mathcal{F}_{i-1}] \\
&\quad +\mathbb{E}[(X_{t_{i-1}}X_{t_i})|\mathcal{F}_{i-1}] \\
&= \mathrm{Var}[X_{t_i}|\mathcal{F}_{i-1}] - \mathrm{Var}[X_{t_{i-1}}|\mathcal{F}_{i-1}] - \mathrm{Var}[X_{t_i}|\mathcal{F}_{i-1}] \\
&\quad +\mathrm{Var}[X_{t_{i-1}}|\mathcal{F}_{i-1}] \\
&= 0.
\end{aligned}
$$

In summary, from Equations (D.4) and (D.5), we infer that each sequence of summands $\frac{d_{n-2,i}}{\sqrt{n-2}}$ in each $M_T^{(3)}$ for all $n \geq 3$ is a martingale difference sequence, i.e.,

$$
\mathbb{E}\left[\frac{d_{n-2,i}}{\sqrt{n-2}}\middle|\mathcal{F}_{i-1}\right] = 0.
$$

Additionally, each summand is square integrable, i.e., $\mathbb{E}\left[\left(\frac{d_{n-2,i}}{\sqrt{n-2}}\right)^2\middle|\mathcal{F}_{i-1}\right]$ is finite.

Then, we immediately check for the conditional Lindeberg's condition as given in Condition 3.31, Chapter VIII in [70] to conclude if the CLT for triangular array of martingale sequences is applicable, i.e., whether or not for any $\delta > 0$,

$$
\sum_{i=1}^{n-2} \mathbb{E}\left[\left|\frac{d_{n-2,i}^2}{n-2}\right| \mathbb{I}_{\left\{\left|\frac{d_{n-2,i}}{\sqrt{n-2}}\right| \geq \delta\right\}} \middle| \mathcal{F}_{i-1}\right] \xrightarrow{\mathbb{P}} 0
$$

as $n \to \infty$. With $n \geq 3$ and $d_{n-2,i}^2 \geq 0$ by definition, the convergence in probability given above can be rewritten as

$$\lim_{n\to\infty} \mathbb{P}\left[\left|\sum_{i=1}^{n-2} \mathbb{E}\left[\frac{d_{n-2,i}^2}{n-2} \mathbb{I}_{\left\{\left|\frac{d_{n-2,i}}{\sqrt{n-2}}\right| \geq \delta\right\}} \middle| \mathcal{F}_{i-1}\right] - 0\right| < \gamma\right] = 1, \forall\gamma \in (0,\infty).$$

We benefit from the idea that for proving an arbitrary sequence of random variables $\{\mathfrak{v}_n\}_{n\geq 1}$ converging in probability to a constant 0, i.e., $\lim_{n\to\infty} \mathbb{P}[|\mathfrak{v}_n - 0| < \gamma] = 1, \forall\gamma \in (0,\infty)$, we can examine if $\lim_{n\to\infty} \mathbb{P}[|\mathfrak{y}_n - 0| < \gamma] = 1, \forall\gamma \in (0,\infty)$ for $|\mathfrak{y}_n| > |\mathfrak{v}_n|$, since if $|\mathfrak{y}_n| > |\mathfrak{v}_n|$ and $\lim_{n\to\infty} \mathbb{P}[|\mathfrak{y}_n - 0| < \gamma] = 1$, then $\lim_{n\to\infty} \mathbb{P}[|\mathfrak{v}_n - 0| < \gamma] = 1, \forall\gamma \in (0,\infty)$. From $|\sum_i \mathfrak{y}_i| \leq \sum_i |\mathfrak{y}_i|$

$$\left|\sum_{i=1}^{n-2} \mathbb{E}\left[\frac{d_{n-2,i}^2}{n-2} \mathbb{I}_{\left\{\left|\frac{d_{n-2,i}}{\sqrt{n-2}}\right| \geq \delta\right\}} \middle| \mathcal{F}_{i-1}\right] - 0\right| \leq \sum_{i=1}^{n-2} \left|\mathbb{E}\left[\frac{d_{n-2,i}^2}{n-2} \mathbb{I}_{\left\{\left|\frac{d_{n-2,i}}{\sqrt{n-2}}\right| \geq \delta\right\}} \middle| \mathcal{F}_{i-1}\right] - 0\right|$$
$$= \frac{1}{n-2} \sum_{i=1}^{n-2} \left|\mathbb{E}\left[d_{n-2,i}^2 \mathbb{I}_{\{|d_{n-2,i}| \geq \delta\sqrt{n-2}\}} \middle| \mathcal{F}_{i-1}\right]\right|.$$

From Equations (D.4) and (D.5) and the definition of identity function, we realize that as $n, i \to \infty$

a) $\mathbb{I}_{\{|d_{n-2,i}| \geq \delta\sqrt{n-2}\}} + \mathbb{I}_{\{|d_{n-2,i}| < \delta\sqrt{n-2}\}} = 1,$

b) $\mathbb{E}\left[d_{n-2,i}^2 \middle| \mathcal{F}_{i-1}\right] \geq \mathbb{E}\left[d_{n-2,i}^2 \mathbb{I}_{\{|d_{n-2,i}| \geq \delta\sqrt{n-2}\}} \middle| \mathcal{F}_{i-1}\right],$

c) $\mathbb{E}\left[d_{n-2,i}^2 \middle| \mathcal{F}_{i-1}\right] < \infty,$

d) $\mathbb{E}\left[d_{n-2,i}^2 \mathbb{I}_{\{|d_{n-2,i}| \geq \delta\sqrt{n-2}\}} \middle| \mathcal{F}_{i-1}\right] < \infty.$

Hence, as $n, i \to \infty$, $\delta\sqrt{n-2} \to \infty$ and probability of $|d_{n-2,i}|$ being greater than or equal to $\delta\sqrt{n-2}$ should reach 0 and stay at there, otherwise $\mathbb{E}\left[d_{n-2,i}^2 \middle| \mathcal{F}_{i-1}\right]$ would diverge. Therefore,

$$\sup_i \mathbb{P}\left[\mathbb{I}_{\{|d_{n-2,i}| \geq \delta\sqrt{n-2}\}} = 0\right] \to 1,$$

and

$$\frac{1}{n-2}\sum_{i=1}^{n-2}\left|\mathbb{E}\left[d_{n-2,i}^{2}\mathbb{I}_{\left\{\left|d_{n-2,i}\right|\geq\delta\sqrt{n-2}\right\}}\middle|\mathcal{F}_{i-1}\right]-0\right|\xrightarrow{\mathbb{P}}0,$$

should be true as $n,i\to\infty$, since contribution of

$$\mathbb{E}\left[d_{n-2,i}^{2}\mathbb{I}_{\left\{\left|d_{n-2,i}\right|\geq\delta\sqrt{n-2}\right\}}\middle|\mathcal{F}_{i-1}\right]$$

to

$$\sum_{i=1}^{n-2}\left|\mathbb{E}\left[d_{n-2,i}^{2}\mathbb{I}_{\left\{\left|d_{n-2,i}\right|\geq\delta\sqrt{n-2}\right\}}\middle|\mathcal{F}_{i-1}\right]-0\right|$$

should shrink to 0 as $n,i\to\infty$ such that the growth rate of

$$\sum_{i=1}^{n-2}\left|\mathbb{E}\left[d_{n-2,i}^{2}\mathbb{I}_{\left\{\left|d_{n-2,i}\right|\geq\delta\sqrt{n-2}\right\}}\middle|\mathcal{F}_{i-1}\right]-0\right|$$

is less than growth rate of $\frac{1}{n-2}$.

In this context, the application of the LLN and the CLT for the triangular arrays of martingale difference sequences (Chapter 3, p.58 of [57]) yields conditional on true price

$$\frac{\mathfrak{S}_n}{\mathfrak{T}_n}\xrightarrow{d}N(0,1)$$

$$\Rightarrow\frac{M_T^{(3)}}{\sqrt{\frac{\mathbb{E}[\epsilon^2]}{n-2}2\left([X,X]_T^{\mathcal{G}}-O_p(1)\right)}}\xrightarrow{d}N(0,1)$$

$$\Rightarrow M_T^{(3)}\xrightarrow{d}N\left(0,\frac{2[X,X]_T^{\mathcal{G}}\mathbb{E}[\epsilon^2]-O_p(1)}{n-2}\right),$$

where from Equation (D.5)

$$\mathfrak{S}_n := \frac{1}{\sqrt{n-2}} \sum_{i=1}^{n-2} d_{n-2,i},$$

$$\begin{aligned}
\mathfrak{T}_n^2 := \mathrm{Var}[\mathfrak{S}_n|\mathcal{F}_{i-1}] &= \frac{1}{n-2} \sum_{i=1}^{n-2} \mathrm{Var}\left[d_{n-2,i}\middle|\mathcal{F}_{i-1}\right] \\
&= \frac{1}{n-2} \sum_{i=1}^{n-2} \left(\mathbb{E}\left[\left(\Delta X_{t_{i-1}}\right)^2\right] + \mathbb{E}\left[\left(\Delta X_{t_i}\right)^2\right] \right) \mathbb{E}[\epsilon^2] \\
&= \frac{\mathbb{E}[\epsilon^2]}{n-2} \mathbb{E}\left[\sum_{i=1}^{n-2} \left(\Delta X_{t_{i-1}}\right)^2 + \sum_{i=1}^{n-2} \left(\Delta X_{t_i}\right)^2 \right] \\
&= \frac{\mathbb{E}[\epsilon^2]}{n-2} \mathbb{E}\left[[X,X]_T^{\mathcal{G}} - \left(\Delta X_{t_{n-2}}\right)^2 + [X,X]_T^{\mathcal{G}} - \left(\Delta X_{t_0}\right)^2 \right] \\
&= \frac{\mathbb{E}[\epsilon^2]}{n-2} 2 \left([X,X]_T^{\mathcal{G}} - O_p(1) \right).
\end{aligned}$$

Now, we have illustrated that the LHS of Equation (D.3) divided by $\sqrt{n-2}$, is sum of 3 asymptotically centered Gaussian terms, namely $M_T^{(1)}, M_T^{(2)}$ and $M_T^{(3)}$, with respective conditional variances as $\mathrm{Var}[\epsilon^2]$, $(\mathbb{E}[\epsilon^2])^2$ and $\frac{\mathbb{E}[\epsilon^2]}{n-2} 2 \left([X,X]_T^{\mathcal{G}} - O_p(1) \right)$.

Consequently, the final step in order to find asymptotic conditional distribution of the LHS of Equation (D.3) is to find covariances between $M_T^{(3)}$ and $M_T^{(2)}$ as well as $M_T^{(3)}$ and $M_T^{(1)}$ and calculate the asymptotic conditional variance where by Theorem 3.4 $\mathrm{Cov}\left[M_T^{(1)}, M_T^{(2)}\middle|X\right]$ is already known to be 0.

$$
\begin{aligned}
\operatorname{Cov}\left[M_T^{(3)}, M_T^{(2)} \middle| \mathcal{F}_{i-1}\right] &= \mathbb{E}\left[\left(M_T^{(3)} - \mathbb{E}\left[M_T^{(3)}\right]\right)\left(M_T^{(2)} - \mathbb{E}\left[M_T^{(2)}\right]\right) \middle| \mathcal{F}_{i-1}\right] \\
&= \mathbb{E}\left[M_T^{(3)} M_T^{(2)} \middle| \mathcal{F}_{i-1}\right] \\
&= \frac{1}{n-2}\mathbb{E}\left[\left(\sum_{i=1}^{n-2}\left(\Delta X_{t_{i-1}} - \Delta X_{t_i}\right)\epsilon_{t_i}\right)\left(\sum_{i=0}^{n-2}\epsilon_{t_{i+1}}\epsilon_{t_i}\right) \middle| \mathcal{F}_{i-1}\right] \\
&= \frac{1}{n-2}\mathbb{E}\left[\begin{pmatrix} \left(\Delta X_{t_0} - \Delta X_{t_1}\right)\epsilon_{t_1} + \cdots \\ +\left(\Delta X_{t_{n-2}} - \Delta X_{t_{n-2}}\right)\epsilon_{t_{n-2}} \end{pmatrix}\begin{pmatrix} \epsilon_{t_1}\epsilon_{t_0} + \cdots \\ +\epsilon_{t_{n-1}}\epsilon_{t_{n-2}} \end{pmatrix} \middle| \mathcal{F}_{i-1}\right] \\
&= \frac{1}{n-2}\mathbb{E}\left[\begin{matrix} \left(\Delta X_{t_0} - \Delta X_{t_1}\right)\epsilon_{t_1}\epsilon_{t_1}\epsilon_{t_0} + \cdots \\ +\left(\Delta X_{t_0} - \Delta X_{t_1}\right)\epsilon_{t_1}\epsilon_{t_{n-1}}\epsilon_{t_{n-2}} \\ \vdots \\ \left(\Delta X_{t_{n-2}} - \Delta X_{t_{n-2}}\right)\epsilon_{t_{n-2}}\epsilon_{t_1}\epsilon_{t_0} + \cdots \\ +\left(\Delta X_{t_{n-2}} - \Delta X_{t_{n-2}}\right)\epsilon_{t_{n-2}}\epsilon_{t_{n-1}}\epsilon_{t_{n-2}} \end{matrix} \middle| \mathcal{F}_{i-1}\right] \\
&= 0,
\end{aligned}
$$

and

$$
\begin{aligned}
\operatorname{Cov}\left[M_T^{(3)}, M_T^{(1)} \middle| \mathcal{F}_{i-1}\right] &= \mathbb{E}\left[\left(M_T^{(3)} - \mathbb{E}\left[M_T^{(3)}\right]\right)\left(M_T^{(1)} - \mathbb{E}\left[M_T^{(1)}\right]\right) \middle| \mathcal{F}_{i-1}\right] \\
&= \mathbb{E}\left[M_T^{(3)} M_T^{(1)} \middle| \mathcal{F}_{i-1}\right] \\
&= \frac{1}{n-2}\mathbb{E}\left[\left(\sum_{i=1}^{n-2}\left(\Delta X_{t_{i-1}} - \Delta X_{t_i}\right)\epsilon_{t_i}\right)\left(\sum_{i=0}^{n-2}\epsilon_{t_i}^2 - \mathbb{E}[\epsilon^2]\right) \middle| \mathcal{F}_{i-1}\right] \\
&= \frac{1}{n-2}\mathbb{E}\left[\begin{pmatrix} \left(\Delta X_{t_0} - \Delta X_{t_1}\right)\epsilon_{t_1} + \cdots \\ +\left(\Delta X_{t_{n-2}} - \Delta X_{t_{n-2}}\right)\epsilon_{t_{n-2}} \end{pmatrix}\begin{pmatrix} \epsilon_{t_0}^2 + \cdots + \epsilon_{t_{n-2}}^2 \\ -(n-1)\mathbb{E}[\epsilon^2] \end{pmatrix} \middle| \mathcal{F}_{i-1}\right] \\
&= \frac{1}{n-2}\mathbb{E}\left[\begin{matrix} \left(\Delta X_{t_0} - \Delta X_{t_1}\right)\epsilon_{t_1}\epsilon_{t_0}^2 + \cdots \\ +\left(\Delta X_{t_0} - \Delta X_{t_1}\right)\epsilon_{t_1}\epsilon_{t_{n-2}}^2 \\ -\left(\Delta X_{t_0} - \Delta X_{t_1}\right)\epsilon_{t_1}(n-1)\mathbb{E}[\epsilon^2] \\ \vdots \\ \left(\Delta X_{t_{n-2}} - \Delta X_{t_{n-2}}\right)\epsilon_{t_{n-2}}\epsilon_{t_0}^2 + \cdots \\ +\left(\Delta X_{t_{n-2}} - \Delta X_{t_{n-2}}\right)\epsilon_{t_{n-2}}\epsilon_{t_{n-2}}^2 \\ -\left(\Delta X_{t_{n-2}} - \Delta X_{t_{n-2}}\right)\epsilon_{t_{n-2}}(n-1)\mathbb{E}[\epsilon^2] \end{matrix} \middle| \mathcal{F}_{i-1}\right] \\
&= \frac{1}{n-2}\mathbb{E}\left[\sum_{i=1}^{n-2}\left(\Delta X_{t_{i-1}} - \Delta X_{t_i}\right)\epsilon_{t_i}^3 \middle| \mathcal{F}_{i-1}\right] \\
&= \frac{1}{n-2}\mathbb{E}\left[\epsilon_{t_i}^3 \middle| \mathcal{F}_{i-1}\right]\sum_{i=1}^{n-2}\mathbb{E}\left[\left(\Delta X_{t_{i-1}} - \Delta X_{t_i}\right) \middle| \mathcal{F}_{i-1}\right] \\
&= \frac{1}{n-2}\mathbb{E}[\epsilon^3]\sum_{i=1}^{n-2}\mathbb{E}\left[\left(X_{t_i} - X_{t_{i-1}} + X_{t_i} - X_{t_{i+1}}\right)\right] \\
&= \frac{1}{n-2}\mathbb{E}[\epsilon^3]0
\end{aligned}
$$

$$= 0.$$

Then,

$$\begin{aligned}
\operatorname{Var}&\left[[Y,Y]_t^{\mathcal{G}} - [X,X]_t^{\mathcal{G}} - 2(n-1)\mathbb{E}[\epsilon^2]\right] \\
&= \operatorname{Var}\left[2\sqrt{n-2}\left(M_T^{(1)} - M_T^{(2)} + M_T^{(3)}\right)\right] \\
&= 4(n-2)\left(\operatorname{Var}\left[M_T^{(1)}\right] + \operatorname{Var}\left[M_T^{(2)}\right] + \operatorname{Var}\left[M_T^{(2)}\right] - \operatorname{Cov}\left[M_T^{(1)}, M_T^{(2)}\right]\right. \\
&\quad \left. - \operatorname{Cov}\left[M_T^{(3)}, M_T^{(2)}\right] + \operatorname{Cov}\left[M_T^{(1)}, M_T^{(3)}\right]\right) \\
&= 4(n-2)\left(\operatorname{Var}[\epsilon^2] + (\mathbb{E}[\epsilon^2])^2 + 2\frac{\mathbb{E}[\epsilon^2]}{n-2}\left([X,X]_T^{\mathcal{G}} - O_p(1)\right)\right) \\
&= 4(n-2)\left(\mathbb{E}[\epsilon^4] - (\mathbb{E}[\epsilon^2])^2 + (\mathbb{E}[\epsilon^2])^2\right. \\
&\quad \left. + 2\frac{\mathbb{E}[\epsilon^2]}{n-2}\left([X,X]_T^{\mathcal{G}} - O_p(1)\right)\right) \\
&= 4(n-2)\mathbb{E}[\epsilon^4] + 8\mathbb{E}[\epsilon^2]\left([X,X]_T^{\mathcal{G}} - O_p(1)\right).
\end{aligned}$$

As a conclusion, the asymptotic distribution of $[Y,Y]_T^{\mathcal{G}} - [X,X]_T^{\mathcal{G}} - 2(n-1)\mathbb{E}[\epsilon^2]$ conditional on true price turns out to be also centered normal, i.e., conditional on true price as $n \to \infty$

$$\begin{aligned}
&\left([Y,Y]_T^{\mathcal{G}} - [X,X]_T^{\mathcal{G}} - 2(n-1)\mathbb{E}[\epsilon^2]\right) \\
&\qquad \overset{d}{\to} N\left(0, 4(n-2)\mathbb{E}[\epsilon^4] + 8\mathbb{E}[\epsilon^2]\left([X,X]_T^{\mathcal{G}} - O_p(1)\right)\right)
\end{aligned}$$

or with U_{noise} being a random variable that is asymptotically standard normal,

$$\begin{aligned}
&[Y,Y]_T^{\mathcal{G}} - [X,X]_T^{\mathcal{G}} \\
&\qquad = 2(n-1)\mathbb{E}[\epsilon^2] + \left(4(n-2)\mathbb{E}[\epsilon^4]\right. \\
&\qquad \left. + 8\mathbb{E}[\epsilon^2]\left([X,X]_T^{\mathcal{G}} - O_p(1)\right)\right)^{1/2} U_{noise}.
\end{aligned}$$

Finally, following Zhang et al. [111], to gauge the total estimation error, $[Y,Y]_T^{\mathcal{G}} - IV_T$, stemming from discretization and existence of noise at the same time, we merge the results on asymptotic distribution of discretization error with those on asymptotic distribution of error due to noise. From Theorem 3.3, under conditions explained therein, we can write (stably in law)

$$[X,X]_T^{\mathcal{G}} = IV_T + \left(\frac{2T}{n-1}\int_0^T \sigma_s^4 d\mathcal{D}_s\right)^{1/2} U_{discretization}$$

where $U_{discretization}$ represents another random variable that is also asymptotically standard normal. As Zhang et al. [111] argue, since MMN is taken as orthogonal to the true price process, $U_{discretization}$ is also orthogonal to U_{noise}. Then,

$$\begin{aligned}[Y,Y]_T^{\mathcal{G}} - IV_T = \\ & 2(n-1)\mathbb{E}[\epsilon^2] + \Bigg(4(n-2)\mathbb{E}[\epsilon^4] + 8\mathbb{E}[\epsilon^2]\left([X,X]_T^{\mathcal{G}} - O_p(1)\right) \\ & + \frac{2T}{n-1}\int_0^T \sigma_s^4 d\mathcal{D}_s\Bigg)^{1/2} U_{total}\end{aligned}$$

such that U_{total} denotes a third random variable which is again asymptotically standard normal.

APPENDIX E

CHAPTER 4 RESULTS

AKBNK SUMMARY AND REVIEW OF CHAPTER 4 RESULTS

1) UHFD Characteristics Under Different Sampling Schemes and Error Cleaning and Data Filtering Combinations

a) Irregular Temporal Spacing

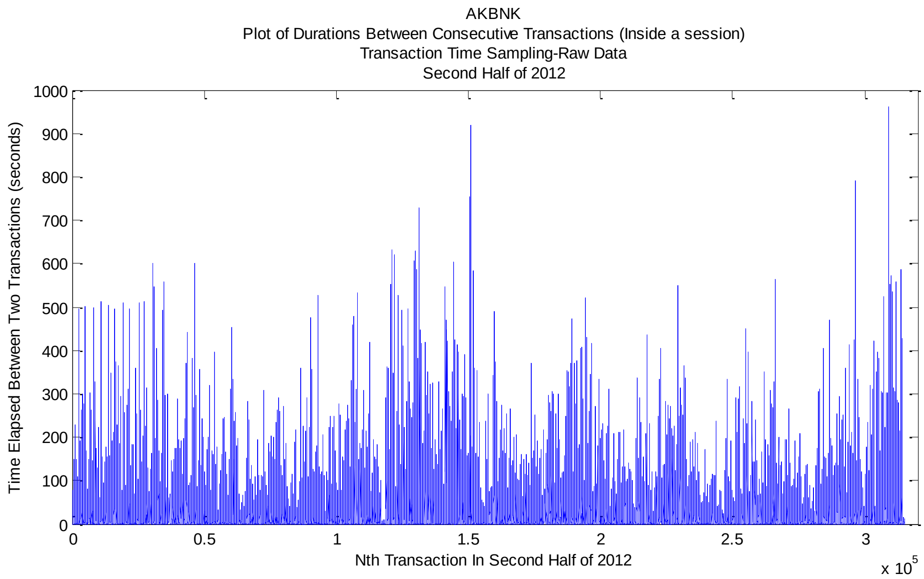


Figure F.1: Plot of durations between consecutive transactions (inside a session) for AKBNK TTS-raw data throughout second half of 2012.

b) **Temporal dependence**: By comparing autocorrelation and partial autocorrelation functions of 600 seconds[11] absolute returns and log returns under CTS (clean and aggregated and interpolated) as well as absolute returns, log

[11] *We also included 1 min returns under CTS for MIGRS and NETAS just because 10 min log returns exhibited no autocorrelation at all.*

returns and durations in seconds from one transaction to the next under TTS (raw versus clean and aggregated) for December of 2012, we see that there are differences between ACF and PACF structure of absolute and log returns between 10 min CTS and 1 transaction TTS, i.e.: transforming 1 transaction sampled data by first cleaning, then aggregating and then interpolating (all needed for CTS) to 600 second sampled data distorts ACF and PACF of return series.

- TTS-Raw-Durations: ACF (slowly decaying positive significant up to 20 lags) and PACF (moderate decay, positive significant up to 10 lags) (shocks persist)

- TTS-Raw-Absolute Returns: ACF (hyperbolic slow decay, positive significant up to 20 lags) and PACF (moderate decay, positive and significant up to 11 lags) (shocks persist)

- TTS-Raw-Log returns: ACF (quick decay, first three lags negative-positive-negative significant) PACF(slower hyperbolic decay, first 14 lags negative significant)

- TTS(4.ii.a and b-5.i-5.ii-5.iii-5.iv-5.v)-Durations: ACF (slowly wave like decaying positive significant up to 20 lags) (shocks persist) and PACF (wave like decay, positive significant up to 20 lags)

- TTS(4.ii.a and b-5.i-5.ii-5.iii-5.iv-5.v)-Absolute returns: ACF (decaying positive and significant up to 20 lags) and PACF (hyperbolic decay, lags up to 10^{th} lag positive significant)

- TTS(4.ii.a and b-5.i-5.ii-5.iii-5.iv-5.v)-Log returns: ACF (quick decay, first three lags negative-positive-negative significant) PACF(slower hyperbolic decay, first 8 lags negative significant)

- CTS-Durations: Meaningless, after interpolation duration from one entry to the next is always 1 second

- CTS (4.ii.a and b-5.i-5.ii-5.iii-5.iv-5.v)-10 min Absolute Returns: ACF (lags 1, 2,3 and 6 positive significant), PACF (lags 1 and 2 are positive significant)

- CTS (4.ii.a and b-5.i-5.ii-5.iii-5.iv-5.v)-10 min Log returns: ACF (only first lag is negative significant) and PACF (first order negative partial autocorrelation)

Under TTS, with raw or clean and aggregated data, there is significant positive autocorrelation up to 20 lags in absolute returns, significant up to third order autocorrelation in log returns and very significant positive autocorrelation up to 20 lags in seconds elapsed between two transactions, thus volatility clustering is verified. Whereas, for 10 min returns under CTS, log returns display first order autocorrelation, which is in conformity with evidence laid out by the finance literature in general, that very short term returns exhibit strong autocorrelation especially on the first lag. Absolute return autocorrelation structure is changed under CTS at 600 seconds sampling interval compared to results under TTS at 1 transaction interval. Likewise, switching to CTS and calculation returns at 600 seconds suppresses partial autocorrelation figures at several lags of both absolute and log returns. Meanwhile, comparing data handling combinations to each other, any combination of cleaning methods and aggregation methods (compared to other combinations) does not cause any major change in total and partial correlation structures once we move under a sampling scheme, it being either TTS or CTS. However, cleaning and aggregation under TTS yield different PACF structures in log returns compared to results produced with raw data.

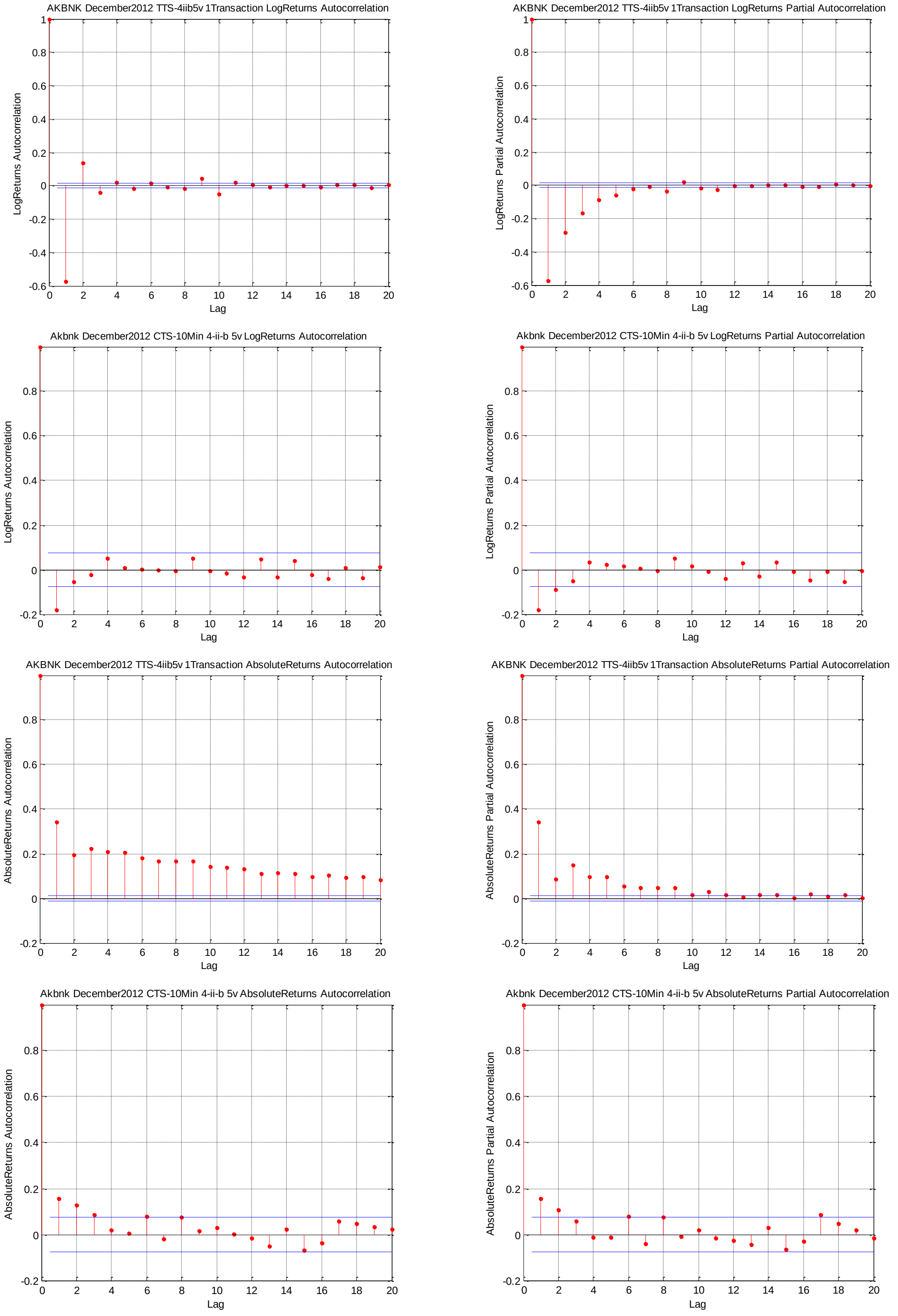


Figure F.2: ACFs and PACFs of logreturn and absolute return series of AKBNK for December 2012 under TTS and CTS

c) Diurnal Patterns: These patterns can be sought only under CTS because of their definitions such as number of trades per x minutes or absolute return per y seconds. For AKBNK case, there are strong W shapes which are persistent across cleaning and aggregation methods in 10 minutes trade volumes and 10 minutes trade intensities throughout days in second half of 2012, whereas patterns in 10 minutes absolute returns and 10 minutes absolute percentage returns are closer to W without last spike at the end of the day[12]. All in all, there are significant diurnal patterns in returns and trading activity in the form of intensity and volume under CTS and these patterns look exactly same when various combinations of cleaning and aggregation methods are applied.

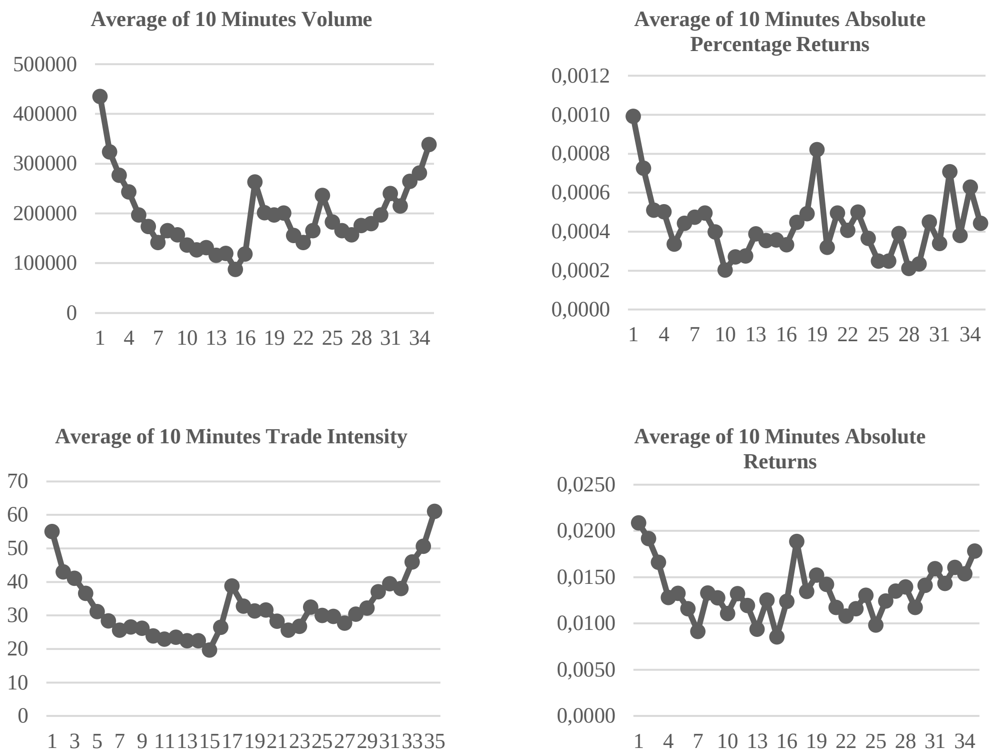
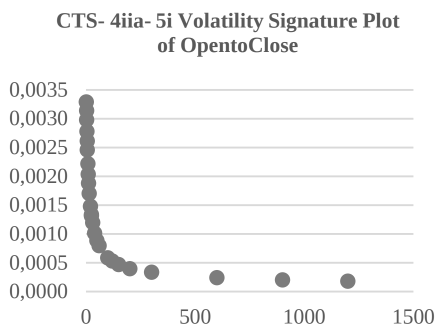


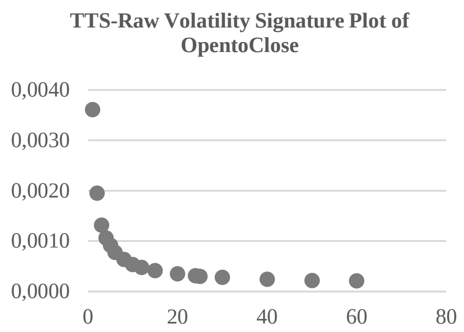
Figure F.3: Diurnal patterns - AKBNK cleaned and aggregated transaction data under CTS

[12] *Unlike the rough L shape in MIGROS and ISCTR or rough W without last spike in NETAS, for 10 min absolute percentage returns.*

2) Visual and Formal Statistical Tests of Existence and Statistical Features of Market Microstructure Noise

a) **<u>VSP:</u>** In line with findings for MIGRS, ISCTR and ARCLK, sampling schemes or cleaning and aggregation techniques do not alter the fact that inflating sampling frequency, either in seconds or in transactions, causes average realized volatility of return on transaction price to boom. Specifically, 6 month VSPs explode as the sampling frequency increases under raw or cleaned TTS as well as under CTS. At this point, we would like to emphasize that for VSPs, we skipped 4.ii.a-5.i-5.ii-5.iii-5.iv-5.v combinations under TTS, mainly because the number of cleaned points under 4.ii.a is so small, cleaning makes no real difference comparing to no cleaning of the data set. Any possible difference might have been observed under cleaning method 4.ii.b, which ended up deleting more data points. Moreover, since we compare 4.ii.a and 4.ii.b under CTS, we additionally search for any marginal effect that cleaning method 4.ii.b has over cleaning method 4.ii.a.





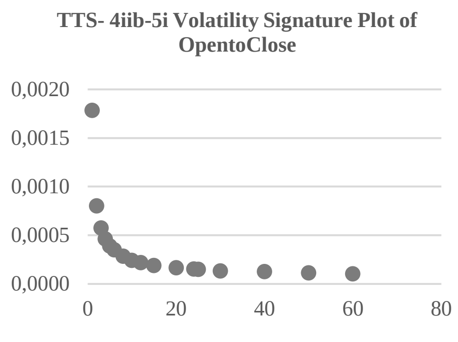


Figure F.4: VSPs for AKBNK over Daily RVs using clean and aggregated data under CTS, raw data under TTS, and clean and aggregated data under TTS.

Explosion becomes trivial for the sampling intervals that are less than 200 seconds or 15 transactions. This observation is valid both for session and daily figures, serving as a visual proof regarding existence of market microstructure noise and pointing to a positive relationship between noise increment and true price return, both under CTS and raw or cleaned TTS, showing that sampling scheme, or cleaning or aggregation do not affect the results.

b) <u>Statistical Tests Regarding Existence and Statistical Features of MMN :</u>

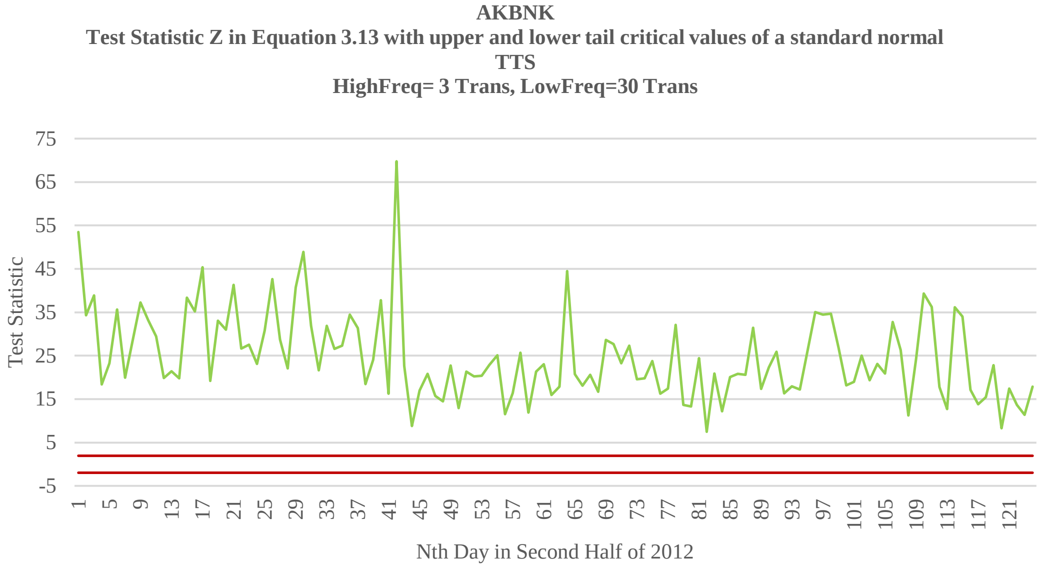
- Existence of MMN is verified statistically under both of CTS and TTS. We calculated $Z_{T,n,h}$ testing null hypothesis in (3.11) by comparing RVs that are calculated over different frequency pairs composed of high-low frequencies, namely (60,600) (10,1200), (30,1200) (60,1200), (150,1200), (300,1200), (600,1200) (900,1200) seconds under CTS and (3,30), (6,30), (10,30), (15,30), and (20,30) transactions under raw-TTS. Recall that bias of the RV estimator is dominated by expectation of square of the noise increment. Therefore, if we reject the null hypothesis, it means that the MMN has statistically significant impact on realized estimator of the IV.

  For each day in the sample period of 124 days and each frequency pair, we run the aforementioned test at 5% significance level. Sample rejection percentages of null hypothesis are 100% under raw-TTS, 99% under clean and aggregated TTS and around 99% under CTS for all cleaning and aggregation method combinations when we compare RVs calculated over 3 and 30 transactions under TTS and 60 and 600 seconds under CTS[13]. As we decrease the sampling frequency at the high frequency leg, rejection percentages of null hypothesis shrink, which is true under both of TTS and CTS. For raw-TTS, the rejection percentages begin with 100% and decrease gradually to 40% as high frequency leg moves toward 20 transactions when low frequency leg is 30 transactions. Cleaning and aggregating the data does not amend the downward trend in rejection percentages under TTS, but make it steeper. For all aggregation choices with cleaning method 4.ii.b applied under TTS, the rejection percentages begin with 99% and decrease gradually to 24% as high frequency leg moves toward 20 transactions. Switching to CTS as well as moving across the grid of cleaning and aggregation combinations do not change the results either. For CTS, the rejection percentages begin with around 100% for 10 to 1200 seconds pair and goes down the hill to 15% as high frequency legs are slowed to 900 seconds.

  Following representative rejection rate graphs reveal that MMN starts to accentuate as the sampling frequency converges to 10-15 transactions under TTS, and 300 seconds under CTS. These findings are in conformity with those supplied by VSP analysis. MMN is felt strongly once we cross over the sampling interval thresholds of 15 transactions or 5 minutes under TTS and CTS, respectively. Moreover, visual inspection of the test statistic $Z_{T,n,h}$ for several frequency pairs either under TTS or CTS reveals that for the majority of the time test statistic is positive and outside 5% standard

[13] *These rejection percentages resemble to those for NETAS, GARAN and ISCTR cases, but are higher than those for ARCLK.*

normal confidence interval, which can be interpreted as a positive correlation between noise and efficient price, again in conformity with exploding VSPs.



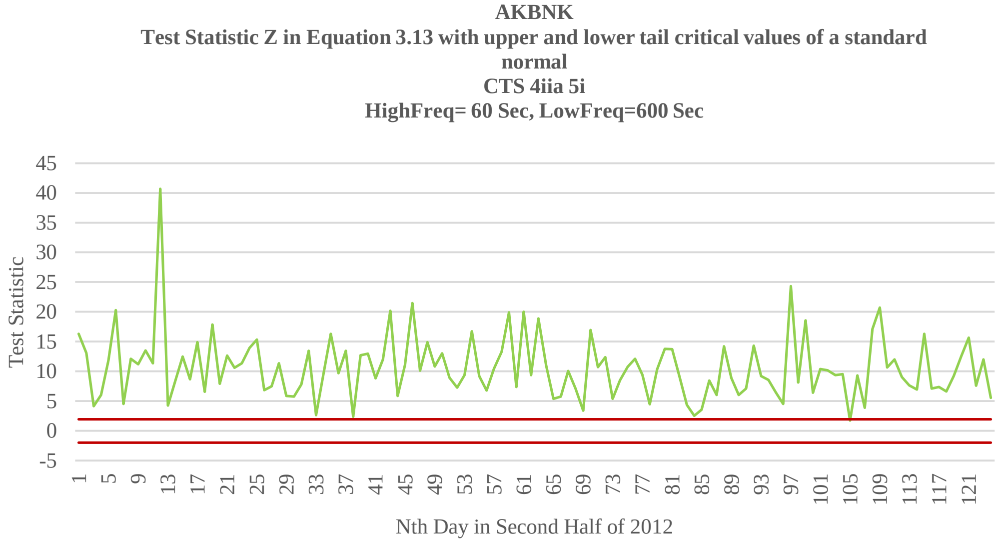
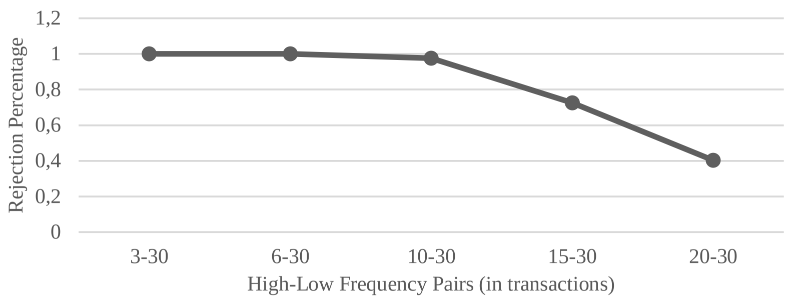
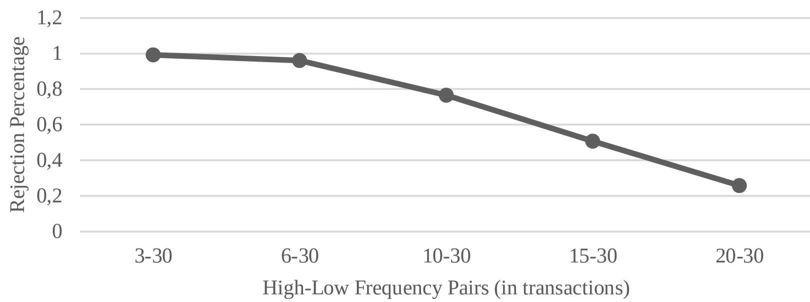


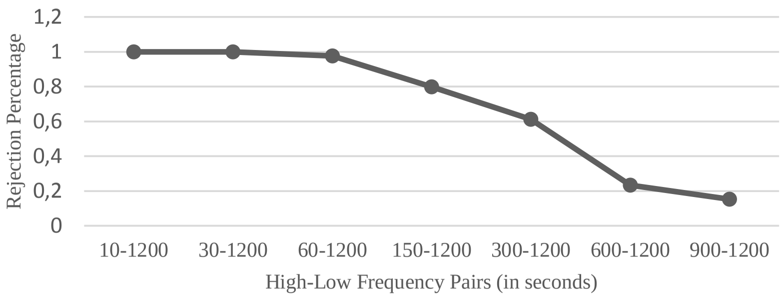
Figure F.5: AKBNK - Plots of $Z_{T,n,h}$ for each day in the sample period with upper and lower tail critical values of standard normal under TTS and CTS.



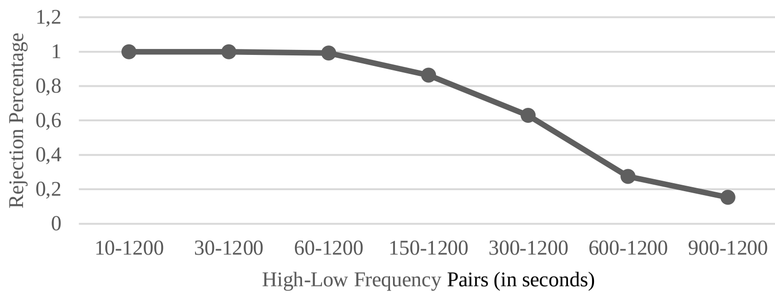
TTS-Raw Data



TTS-Clean and Aggregated Data (4iib-5i)



CTS-Clean and Aggregated Data (4iib-5i)



CTS-Clean and Aggregated Data (4iia-5i)

Figure F.6: AKBNK –Plots of rejection percentages with regards to the null hypothesis that MMN does not have statistically significant effect on RV under TTS and CTS.

- **Summary:** Model of an i.i.d MMN with constant variance might be proper under CTS but not under raw-TTS (cleaned and aggregated TTS), for more than 40% (19%) of the days, null of constant variance is rejected for triples with very high frequencies combined with very low. This might be evidence of i.i.d assumption not holding at frequencies lesser 15 transactions under TTS. Sampling scheme, but not the aggregation method, is discovered to very influential on rejection of null hypothesis that the MMN has variance independent of sampling frequency. Meanwhile, like NETAS and ISCTR cases cleaning algorithms have some suppressive effect on rejection percentages particularly under TTS.

  Awartani et al. [16] derive a test with the idea that if the MMN has constant variance, then noise variances calculated over frequencies 1/M or 1/N should be same independent of M or N chosen. Their null and null hypotheses are as in (3.35) and (3.36).

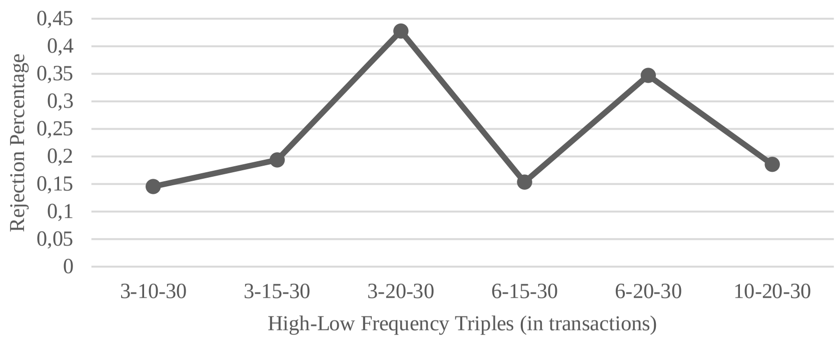
  Since alternative hypothesis is in harmony with the presence of autocorrelation in MMN, by reminding corollary 3 of Hansen and Lunde [61], Awartani et al. [16] interpret the rejection of null hypothesis as a sign of the rejection of the null hypothesis that the MMN is a sequence of i.i.d random variables with constant variance. To test the validity of this null hypothesis, a test statistic compares RV differences using two frequency pairs, where pairs are M,L and N,L. L represents a frequency at which we can ignore the MMN safely, say 20 minutes and M and N are frequencies at which the MMN is considered to be significant. Therefore, the test is build on RVs calculated over frequency triples i.e. for each high frequency pair combined with 20 minutes, we test null hypothesis that E(noise increment square at low frequency)=E(noise increment square at high frequency). If we reject the null hypothesis, it means that the MMN has variance that is NOT independent of sampling frequency, therefore any assumptions regarding i.i.d nature of MMN can be taken as invalidated. Frequency triples are as follows: (3,10,30), (3,15,30), (3,20,30), (6,15,30) (6,20,30) and (10,20,30) transactions under TTS, (60,150,1200), (60,600,1200), (150,300,1200), (150,600,1200) and (300,600,1200) seconds under CTS.

  For each day in the sample period of 124 days and each frequency triple, we run the aforementioned test at 5% significance level. Sample rejection percentages of null hypothesis clearly change from one triple to another and as we clean and aggregate data. Beware that under raw-TTS especially for combinations of frequencies with highest differences between frequent legs, rejection percentages exceed 40%, while they fall to 15% for 3-10-30 triple with lowest distance between first two legs. However, once we clean and aggregate the data, rejection percentages range decline to levels 19-7% depending on the triple[14]. For CTS 4.ii.a and 4.ii.b, constant

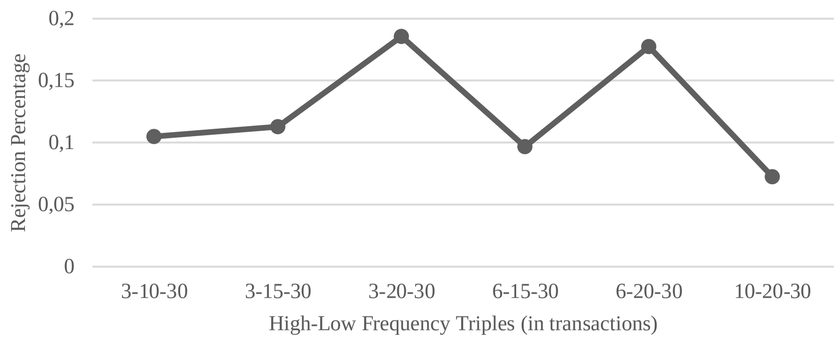
[14] *In a sense, these findings agree with findings for MIGRS and ISCTR cases, where rejection percentages are highest for triples with distant constituents and TTS-raw data; however, MIGRS and*

variance assumption rejection percentages vary between at most 6% and at least 1%, both of which are just a fraction of rejection percentages under TTS-raw or TTS-cleaned. Therefore, unlike ARCLK but like NETAS, GARAN and ISCTR results, sampling scheme is discovered to be influential on rejection of null hypothesis that the MMN has variance independent of sampling frequency. We reject this null hypothesis under TTS, either raw or cleaned and aggregated but cannot reject for any combination of cleaning and aggregation methods under CTS confidently and conclude that i.i.d with constant variance MMN assumption does not reflect the real life structure of the MMN under TTS, whereas under CTS, such an assumption seems to hold for all frequencies. Evidence reveals that aggregation method does not affect rejection percentages and for triples with high frequency legs being close to very slow frequency leg, rejection percentages are severely damaged independent of the sampling scheme.

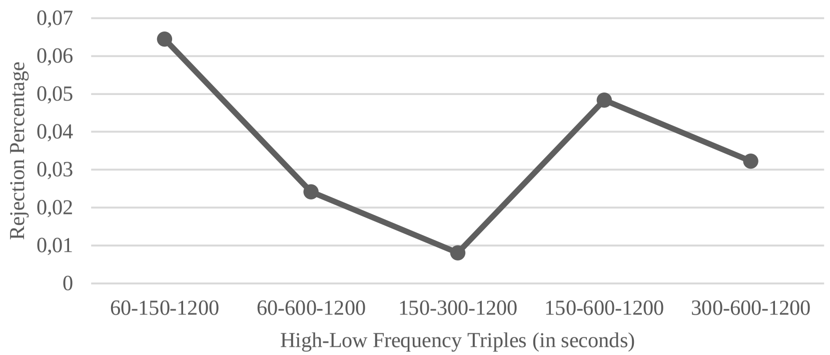
ARCLK rejection percentages are way below those of ISCTR's or NETAS' or AKBNK's rejection percentages.



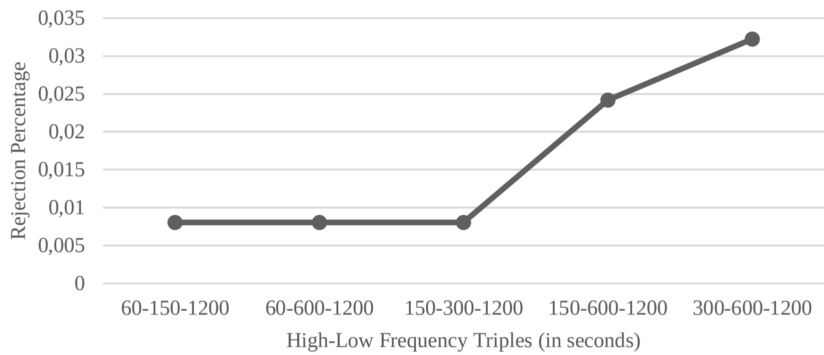
TTS-Raw Data



TTS-Clean and Aggregated Data (4iib-5i)



CTS-Clean and Aggregated Data (4iia-5i)



CTS-Clean and Aggregated Data (4iib-5i)

Figure F.7: AKBNK – Plots of rejection percentages with regards to the null hypothesis that the MMN increments have constant variance independent of sampling frequency under TTS and CTS.

3) RV Analysis

We constructed two RV time series, namely session RVs and daily RVs, for each frequency in a frequency set of 3, 6, 10, 15, 20, 30 transactions and 60, 300, 600 seconds under each sampling scheme (raw-TTS, CTS) and cleaning (4.ii.a, 4.ii.b) - aggregation method (5.i, 5.ii, 5.iii, 5.iv, 5.v) combination. Daily RV time series has 124 data points, whereas session RV time series is constituted of 248 entries. Each time, RV series under each sampling scheme and cleaning and aggregation method combinations is subjected to preliminary statistics, ACF and PACF analysis and lastly unit root is checked where autocorrelation exhibits slow decay.

- The factors that have any effect on RV series' normality-lognormality and autocorrelation structure turn out to be whether the RV is on a session or daily basis, whether it is under raw-TTS or CTS and the frequency at which the RV is calculated. Under raw-TTS, session and daily RV series at all frequencies except daily 20 are found to be nonnormal. Higher frequencies also lead to skewness and kurtosis values to converge to those of normal distribution to such an extent that on the level, daily RV series at 1 min frequency under CTS can be inferred to come from a normally distributed population at 5% significance level. Taking logarithm makes daily RV series at frequencies 10 and 5 minutes normal under CTS, while such a transformation only works in terms of normality for RV session series at 3 and 20 transactions under raw-TTS[15]. All of remaining log RV series, either under raw-TTS or CTS, either raw or cleaned and aggregated, either on a session or daily basis, are not normally distributed as JB statistics and kurtosis-skewness values suggest.

- Decreasing frequencies cause less number of lags being significant with lesser significant levels, i.e. decreasing frequency again depresses autocorrelation structure of RV series regardless of sampling scheme, the cleaning and aggregation methods or session-daily calculation, which is in line with the existence of MMN. In fact, ACFs of session and daily RVs change as the sampling frequency changes, such that for increasing frequencies RV series exhibit significant positive total autocorrelation up to higher number of lags. Calculating RVs on a session basis make the RV series more autocorrelated at higher lags under both of raw-TTS and CTS. Once we are working on a daily or session series at a particular frequency under CTS, cleaning and aggregation methods do not alter RV series' non-normality/normality or autocorrelation structure.

- Sampling scheme, frequency and cleaning methods or session/daily basis choice affects the stationarity results[16]. E-views ADF Test results reveal that we can reject null of unit root at 5%significance level for RV series under raw-TTS at all frequencies[17]; however, switching to CTS and moving between cleaning methods or session or daily RV calculation basis while

[15] *Unlike the case of MIGRS*
[16] *Unlike the findings for ISCTR, NETAS and ARCLK cases.*
[17] *Unlike the case of MIGRS.*

increasing the frequency changes the game such that under cleaning method 4.ii.a, regardless of aggregation method, the null hypotheses that daily or session 1 min RV series have unit root cannot be rejected even weakly. Whereas, adopting cleaning method 4.ii.b makes session 1 min series stationary at 1% significance level orthogonal to aggregation methods[18]. On the contrary, only frequency and daily/session choice matter if we change the tests parameters. ADF test with fixed two lags and an intercept in MATLAB show us that all RV series session or daily, TTS or CTS at 10 min or 5 min frequencies are stationary but with 1 min daily RV series, at 5%significance level, we cannot reject null of unit root.

a) <u>Descriptive statistics by frequency, by sampling scheme and by cleaning and aggregation methods:</u>

- **<u>TTS-Raw:</u>** For all frequencies, the session and daily RV series except daily RV series at 20 transactions frequency are not normally distributed as skewness, kurtosis and JB test statistic values reveal. Nevertheless, JB test gives that the null of daily RV series at 20 transaction frequency coming from a normally distributed population cannot be rejected at 5% significance level. Unlike other stocks in our sample, skewness and kurtosis values of this daily RV series are close to 0 and 3. Mean of the session and daily RVs become smaller as the sampling interval is lengthened, but there is no clear relationship between sampling frequency and change in skewness, kurtosis or JB statistic values, which deviates from the findings for MIGRS and ISCTR[19]. Correlograms of all session RV series look alike such that even lags are positive significant up to 20th lag with odd lags being insignificant where decreasing the sampling frequencies depresses significance levels. Lags 1, 2, 4, and 6 are positive significant in PACF of all session RV series at all frequencies, where lag 6 drops from the significant lags list at 30 transactions frequency. Although correlograms of all daily RVs resemble one another, compared to correlogram of session series, autocorrelation structure of daily RVs looks different. ACFs of daily RV series exhibit similar shapes where decreasing frequencies cause less number of lags being significant with lesser significant levels, i.e. frequency again depresses autocorrelation structure, which is in line with existence of the MMN. Generally speaking, for daily RV series, lags 1, 2, 3 and 14 are significant in the PACF. The change in autocorrelation structure of RV series by looking at session and daily RVs separately calls for stationarity test and accordingly, we checked for unit roots in daily series to see if summing RV from session one and session two to reach daily RV distorts anything in RV stationarities at different frequencies.

[18] *Matlab ADF test with NO INTERCEPT reveals that taking logarithm ensures stationarity at all frequencies under CTS with all cleaning and aggregation methods.*
[19] *For MIGRS and ISCTR, a decrease in skewness, kurtosis and JB statistic was observed as we sample less frequently.*

TTS- Raw-Session-Frequency:3 Transactions

Date: 02/19/16 Time: 18:17
Sample: 1 248
Included observations: 248

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.282	0.282	19.959	0.000
		2	0.430	0.380	66.475	0.000
		3	0.170	-0.016	73.772	0.000
		4	0.377	0.236	109.84	0.000
		5	0.143	-0.024	115.02	0.000
		6	0.393	0.226	154.54	0.000
		7	0.172	0.017	162.13	0.000
		8	0.381	0.151	199.72	0.000
		9	0.112	-0.070	202.99	0.000
		10	0.365	0.144	237.75	0.000
		11	0.131	-0.009	242.27	0.000
		12	0.303	0.022	266.31	0.000
		13	0.052	-0.092	267.02	0.000
		14	0.259	0.019	284.84	0.000
		15	0.043	-0.041	285.34	0.000
		16	0.281	0.078	306.45	0.000
		17	0.051	-0.038	307.13	0.000
		18	0.314	0.119	333.79	0.000
		19	0.055	-0.027	334.61	0.000
		20	0.269	0.053	354.32	0.000

TTS- Raw-Daily-Frequency:3 Transactions

Date: 02/19/16 Time: 18:28
Sample: 1 248
Included observations: 124

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.484	0.484	29.769	0.000
		2	0.410	0.230	51.308	0.000
		3	0.392	0.176	71.120	0.000
		4	0.396	0.162	91.576	0.000
		5	0.380	0.112	110.49	0.000
		6	0.311	0.007	123.30	0.000
		7	0.231	-0.066	130.44	0.000
		8	0.235	0.011	137.88	0.000
		9	0.278	0.093	148.39	0.000
		10	0.229	0.003	155.57	0.000
		11	0.196	0.001	160.89	0.000
		12	0.259	0.126	170.28	0.000
		13	0.177	-0.062	174.70	0.000
		14	0.035	-0.232	174.87	0.000
		15	0.054	-0.052	175.28	0.000
		16	0.071	0.023	176.01	0.000
		17	0.051	-0.014	176.40	0.000
		18	0.060	0.057	176.93	0.000
		19	-0.070	-0.106	177.65	0.000
		20	-0.021	0.023	177.72	0.000

TTS- Raw-Session-Frequency:30 Transactions

Date: 02/19/16 Time: 18:27
Sample: 1 248
Included observations: 248

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.198	0.198	9.8123	0.002
		2	0.422	0.399	54.740	0.000
		3	0.145	0.019	60.027	0.000
		4	0.399	0.262	100.51	0.000
		5	0.141	0.012	105.60	0.000
		6	0.279	0.050	125.58	0.000
		7	0.101	-0.013	128.23	0.000
		8	0.286	0.106	149.32	0.000
		9	0.081	-0.031	151.03	0.000
		10	0.274	0.105	170.58	0.000
		11	0.049	-0.048	171.20	0.000
		12	0.246	0.057	187.09	0.000
		13	0.021	-0.055	187.20	0.000
		14	0.124	-0.088	191.28	0.000
		15	0.039	0.036	191.68	0.000
		16	0.176	0.064	199.99	0.000
		17	-0.035	-0.096	200.32	0.000
		18	0.180	0.108	209.09	0.000
		19	-0.019	-0.032	209.19	0.000
		20	0.125	-0.042	213.42	0.000

TTS- Raw-Daily-Frequency:30 Transactions

Date: 02/19/16 Time: 18:34
Sample: 1 248
Included observations: 124

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.490	0.490	30.540	0.000
		2	0.469	0.301	58.747	0.000
		3	0.298	-0.015	70.236	0.000
		4	0.326	0.125	84.104	0.000
		5	0.287	0.087	94.927	0.000
		6	0.238	-0.016	102.44	0.000
		7	0.131	-0.099	104.74	0.000
		8	0.107	-0.013	106.28	0.000
		9	0.136	0.088	108.78	0.000
		10	0.085	-0.046	109.77	0.000
		11	0.102	0.027	111.23	0.000
		12	0.137	0.130	113.85	0.000
		13	-0.015	-0.203	113.89	0.000
		14	-0.114	-0.239	115.74	0.000
		15	-0.079	0.083	116.63	0.000
		16	-0.071	0.045	117.37	0.000
		17	0.079	0.180	118.27	0.000
		18	-0.031	-0.055	118.41	0.000
		19	-0.110	-0.151	120.20	0.000
		20	-0.099	0.009	121.67	0.000

Figure F.8: AKBNK - Correlograms of session and daily RV series under TTS for different sampling intervals

- **CTS:** The session RV series at all frequencies, plus daily RV series at 10 min and 5 min frequencies are not normally distributed as skewness, kurtosis and JB statistic values reveal. However, JB test produces such a test statistic that we cannot reject the null of normality for the daily RV series at 1 min frequency. This finding is in line with we have found for ISCTR. Like the case under TTS,

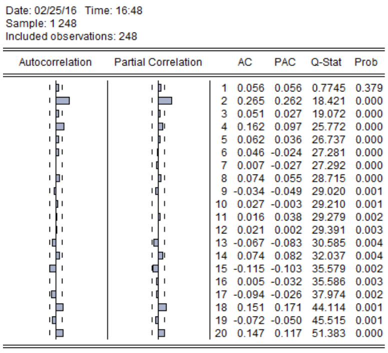
    i. mean of the session and daily RVs become smaller as the sampling interval is lengthened.

- However, contrary to findings for RV series under TTS,

    i. decrease in skewness, kurtosis and JB statistic values is observed as we sample more and more frequently (resembles to MIGRS, ARCLK and ISCTR, deviates from NETAS and GARAN)

    ii. ACFs of session and daily RVs change as the sampling frequency changes, such that for increasing frequencies RV series exhibit significant positive total autocorrelation at higher number of lags with higher significances. Apart from this common trait, the decay patterns in total correlation of daily and session RVs are different, especially obvious at 1 min frequency.

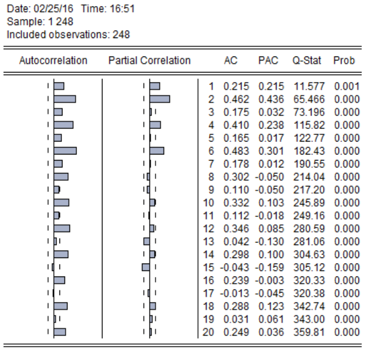
CTS-4.ii.a-5.i-Session 10 min RV series



Date: 02/25/16 Time: 16:48
Sample: 1 248
Included observations: 248

	AC	PAC	Q-Stat	Prob
1	0.056	0.056	0.7745	0.379
2	0.265	0.262	18.421	0.000
3	0.051	0.027	19.072	0.000
4	0.162	0.097	25.772	0.000
5	0.062	0.036	26.737	0.000
6	0.046	-0.024	27.281	0.000
7	0.007	-0.027	27.292	0.000
8	0.074	0.055	28.715	0.000
9	-0.034	-0.049	29.020	0.001
10	0.027	-0.003	29.210	0.001
11	0.016	0.038	29.279	0.002
12	0.021	0.002	29.391	0.003
13	-0.067	-0.083	30.585	0.004
14	0.074	0.082	32.037	0.004
15	-0.115	-0.103	35.579	0.002
16	0.005	-0.032	35.586	0.003
17	-0.094	-0.026	37.974	0.002
18	0.151	0.171	44.114	0.001
19	-0.072	-0.050	45.515	0.001
20	0.147	0.117	51.383	0.000

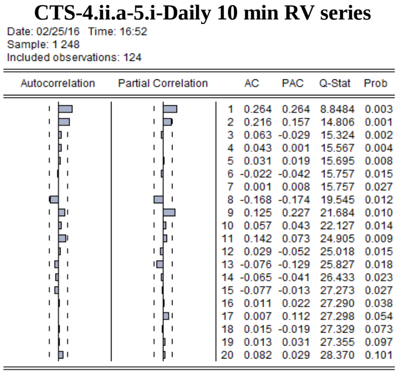
CTS-4.ii.a-5.i-Session 1 min RV series



Date: 02/25/16 Time: 16:51
Sample: 1 248
Included observations: 248

	AC	PAC	Q-Stat	Prob
1	0.215	0.215	11.577	0.001
2	0.462	0.436	65.466	0.000
3	0.175	0.032	73.196	0.000
4	0.410	0.238	115.82	0.000
5	0.165	0.017	122.77	0.000
6	0.483	0.301	182.43	0.000
7	0.178	0.012	190.55	0.000
8	0.302	-0.050	214.04	0.000
9	0.110	-0.050	217.20	0.000
10	0.332	0.103	245.89	0.000
11	0.112	-0.018	249.16	0.000
12	0.346	0.085	280.59	0.000
13	0.042	-0.130	281.06	0.000
14	0.298	0.100	304.63	0.000
15	-0.043	-0.159	305.12	0.000
16	0.239	-0.003	320.33	0.000
17	-0.013	-0.045	320.38	0.000
18	0.288	0.123	342.74	0.000
19	0.031	0.061	343.00	0.000
20	0.249	0.036	359.81	0.000

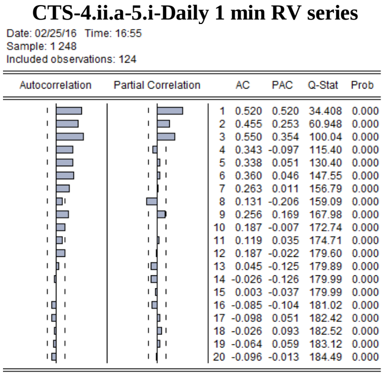
CTS-4.ii.a-5.i-Daily 10 min RV series



Date: 02/25/16 Time: 16:52
Sample: 1 248
Included observations: 124

	AC	PAC	Q-Stat	Prob
1	0.264	0.264	8.8484	0.003
2	0.216	0.157	14.806	0.001
3	0.063	-0.029	15.324	0.002
4	0.043	0.001	15.567	0.004
5	0.031	0.019	15.695	0.008
6	-0.022	-0.042	15.757	0.015
7	0.001	0.008	15.757	0.027
8	-0.168	-0.174	19.545	0.012
9	0.125	0.227	21.684	0.010
10	0.057	0.043	22.127	0.014
11	0.142	0.073	24.905	0.009
12	0.029	-0.052	25.018	0.015
13	-0.076	-0.129	25.827	0.018
14	-0.065	-0.041	26.433	0.023
15	-0.077	-0.013	27.273	0.027
16	0.011	0.022	27.290	0.038
17	0.007	0.112	27.298	0.054
18	0.015	-0.019	27.329	0.073
19	0.013	0.031	27.355	0.097
20	0.082	0.029	28.370	0.101

CTS-4.ii.a-5.i-Daily 1 min RV series



Date: 02/25/16 Time: 16:55
Sample: 1 248
Included observations: 124

	AC	PAC	Q-Stat	Prob
1	0.520	0.520	34.408	0.000
2	0.455	0.253	60.948	0.000
3	0.550	0.354	100.04	0.000
4	0.343	-0.097	115.40	0.000
5	0.338	0.051	130.40	0.000
6	0.360	0.046	147.55	0.000
7	0.263	0.011	156.79	0.000
8	0.131	-0.206	159.09	0.000
9	0.256	0.169	167.98	0.000
10	0.187	-0.007	172.74	0.000
11	0.119	0.035	174.71	0.000
12	0.187	-0.022	179.60	0.000
13	0.045	-0.125	179.89	0.000
14	-0.026	-0.126	179.99	0.000
15	0.003	-0.037	179.99	0.000
16	-0.085	-0.104	181.02	0.000
17	-0.098	0.051	182.42	0.000
18	-0.026	0.093	182.52	0.000
19	-0.064	0.059	183.12	0.000
20	-0.096	-0.013	184.49	0.000

Figure F.9: AKBNK - Correlograms of session and daily RV series under CTS for different sampling intervals

iii. PACFs of session RVs differ slightly from one another depending on the frequency. For frequency 1 min, lags 1, 2, 4 and 6 are significant in PACF, whereas lags 1, 2, and 4 and only 2 are significant for 5 min and 10 min frequencies, respectively.

iv. PACFs of daily RVs differ slightly from one another depending on the frequency. For frequency 1 min, lags 1 and 9 are significant in PACF, whereas lags 1 and 2 and lags 1, 2, 3, and 8 are significant for 5 min and 10 min frequencies, respectively.

- Regardless of the shapes, slow decay in some of the ACFs calls for stationarity tests.
- All of these observations hold under all cleaning methods and aggregation algorithms.

b) Stationarity-Unit root test:

- To test for stationarity and unit root, i.e. if the series move around a constant mean or diverge as time passes, Augmented Dickey Fuller (ADF) Test is preferred. By visual inspection of graphs, no trend is observed in any of our RV series, therefore, ADF Test is run with an intercept and no trend, the number of legs to be involved in the analysis is chosen by Schwarz criterion as it is the default choice suggested by E-views.

- **TTS-Raw-:** In the E-views setting, where number of lags are optimized by E-views according to Schwarz criterion, R-squared values vary in a band of 19-54% (higher for session values). The null of nonstationarity is rejected at 1% significance level for all session and daily series[20] except session series for 3, 6 and 10 transactions sampling intervals, for which the rejection significance level increases to 5%.

- **CTS:** In the E-views setting, where number of lags are optimized by E-views according to Schwarz criterion, R-squared values have a range of 29% to 59%. At 1% significance level, RV series calculated at 5 and 10 min sampling intervals, either session or daily or under any cleaning or aggregation combination, are found to be stationary. The frequency 1 min is intriguing in the sense that under cleaning method 4.ii.a, regardless of aggregation method, the null hypotheses that daily or session RV series have unit root cannot be rejected even weakly. Whereas, adopting cleaning method 4.ii.b makes session 1 min series stationary at 1% significance level. It can be inferred that cleaning methods as well as frequencies matter for stationarity such that increasing frequency to 1 min under CTS and calculating RVs on a daily basis makes the RV series nonstationary.

[20] *In MIGRS analysis, significance level of rejection regarding nonstationarity increases when we switch to daily series. Here, switching between daily or session series does not affect significance level at which we can reject null for half of the frequencies. For the remaining half, switching to session calculations deteriorates rejection significance levels.*

Table F.1: AKBNK - p-values of ADF Test on session and daily RV series under various cleaning and aggregation method combinations for different sampling intervals

Frequency	Session Based /Daily	Cleaning and Aggregation Method Combination									
		4.ii.a-5.i	4.ii.a-5.ii	4.ii.a-5.iii	4.ii.a-5.iv	4.ii.a-5.v	4.ii.b-5.i	4.ii.b-5.ii	4.ii.b-5.iii	4.ii.b-5.iv	4.ii.b-5.v
1min	Sess. Based	0.1111	0.1135	0.1096	0.1165	0.1083	0.0000	0.0000	0.0000	0.0000	0.0000
	Daily	0.1300	0.1352	0.1320	0.1384	0.1308	0.0658	0.0712	0.0687	0.0759	0.0728
5min	Sess. Based	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	Daily	0.0002	0.0002	0.0002	0.0003	0.0003	0.0001	0.0002	0.0002	0.0002	0.0002
10min	Sess. Based	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	Daily	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

ARCLK SUMMARY AND REVIEW OF CHAPTER 4 RESULTS

1) UHFD Characteristics Under Different Sampling Schemes and Error Cleaning and Data Filtering Combinations

a) Irregular Temporal Spacing

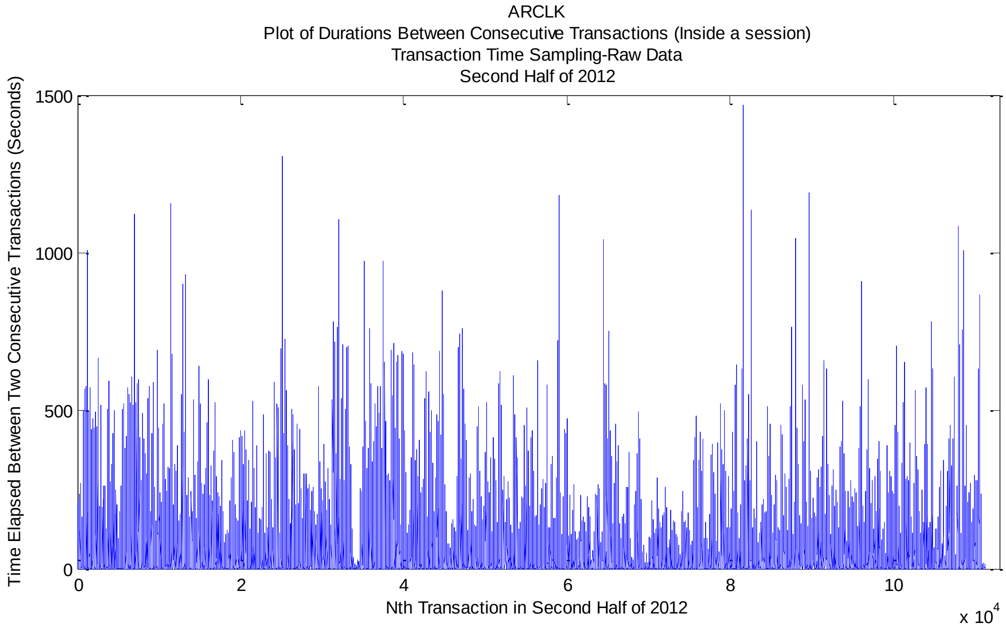


Figure F.10: Plot of durations between consecutive transactions (inside a session) for ARCLK TTS-raw data throughout the second half of 2012.

b) **Temporal dependence**: By comparing autocorrelation and partial autocorrelation functions of 600 seconds[21] absolute returns and log returns under CTS(clean and aggregated and interpolated) as well as absolute returns, log returns and durations in seconds from one transaction to the next under TTS (raw versus clean and aggregated) for December of 2012, we see that there are differences between ACF and PACF structure of absolute and log returns between 10 min CTS and 1 transaction TTS, i.e.: transforming 1 transaction sampled data by first cleaning, then aggregating and then interpolating (all needed for CTS) to 600 second sampled data distorts ACF and PACF of return series.

- TTS-Raw-Durations: ACF (slowly wave like decaying positive significant up to 20 lags) and PACF (wave like decay, positive significant up to 15 lags) (shocks persist)

[21] *Recall that we also included 1 min returns under CTS for MIGRS and NETAS just because 10 min log returns exhibited no autocorrelation at all.*

- TTS-Raw-Absolute Returns: ACF (hyperbolic decay, positive significant up to 20 lags) and PACF (hyperbolic decay, positive and significant up to 9 lags)

- TTS-Raw-Log returns: ACF (quick decay, first two-three lags negative significant) PACF(slower hyperbolic decay, first 14 lags negative significant)

- TTS(4.ii.a and b-5.i-5.ii-5.iii-5.iv-5.v)-Durations: ACF (slowly wave like decaying positive significant up to 20 lags) and PACF (wave like decay, positive significant up to 15 lags)

- TTS(4.ii.a and b-5.i-5.ii-5.iii-5.iv-5.v)-Absolute returns: ACF (decaying positive and significant up to 20 lags) and PACF (quick decay, first lag positive significant, other lags are significant but close to critical value boundaries)[22]

- TTS(4.ii.a and b-5.i-5.ii-5.iii-5.iv-5.v)-Log returns: ACF (quick decay, first lag negative significant) PACF(slower hyperbolic decay, first 6 lags negative significant)

- CTS-Durations: Meaningless, after interpolation duration from one entry to the next is always 1 second.

- CTS (4.ii.a and b-5.i-5.ii-5.iii-5.iv-5.v)-10 min Absolute Returns: 10 min Absolute Returns: ACF (no autocorrelation), PACF(no significant partial autocorrelation) Except 4.ii.b-5.iv. Under this combination, lags 1,2, 4 and 5 and 1, 2 and 4 are positive significant in ACF and PACF, respectively[23].

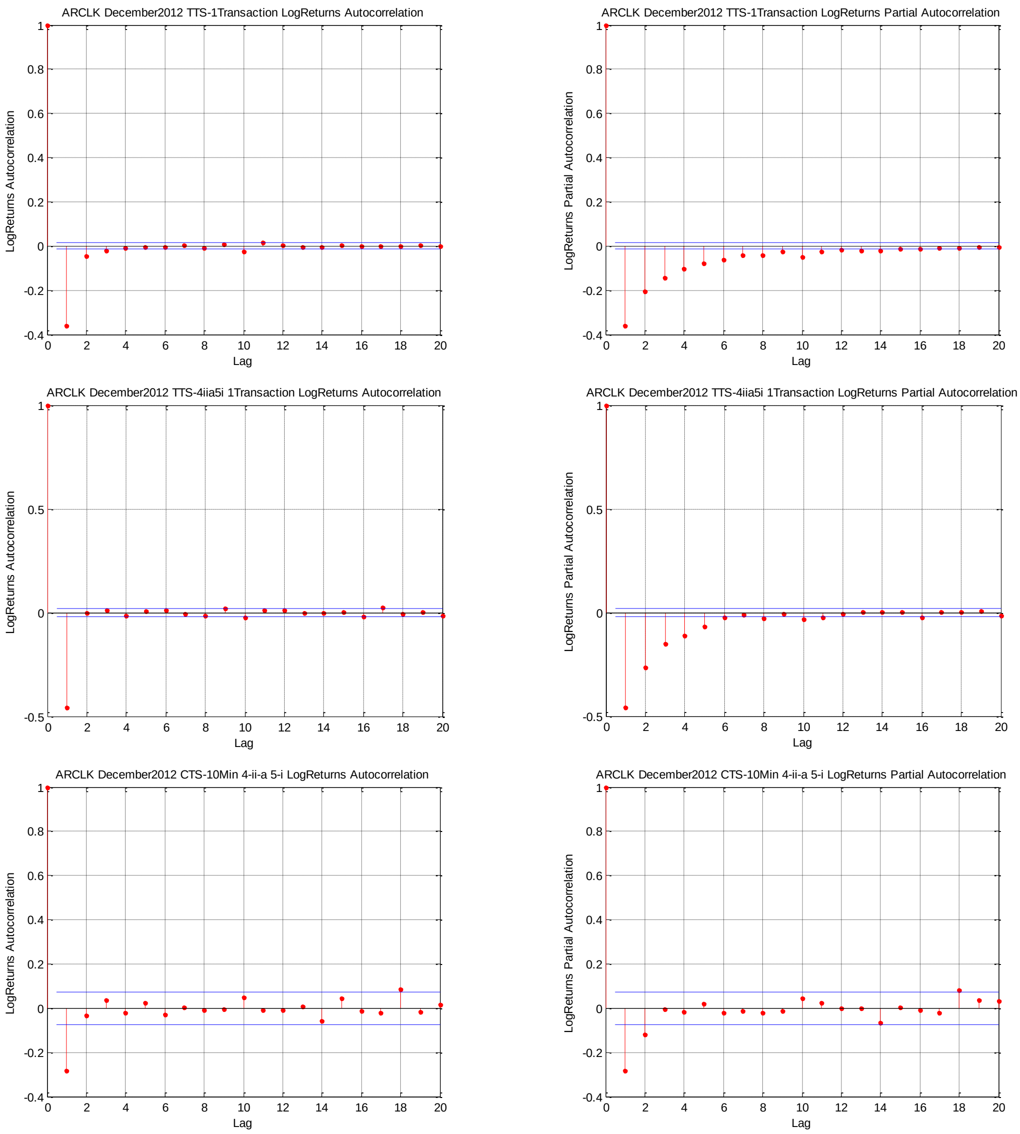
- CTS (4.ii.a and b-5.i-5.ii-5.iii-5.iv-5.v)-10 min Log returns: ACF (only first lag is negative significant) and PACF (first two legs are negative significant)

Under TTS, with raw or clean and aggregated data, there is significant positive autocorrelation up to 20 lags in absolute returns, significant up to third order autocorrelation in log returns and very significant positive autocorrelation up to 20 lags in seconds elapsed between two transactions, thus volatility clustering is verified. Whereas, for 10 min returns under CTS, log returns display first order autocorrelation, which is in conformity with evidence laid out by the finance literature in general, that very short term returns exhibit strong autocorrelation especially on the first lag. Absolute return autocorrelation structure is changed under CTS at 600 seconds sampling interval compared to results under TTS at 1 transaction interval. Likewise, switching to CTS and calculating returns at 600 seconds suppresses partial autocorrelation figures at several lags of both absolute

[22] *Unlike findings on MIGRS, ISCTR and NETAS.*

[23] *For MIGRS, ISCTR and NETAS, aggregation or cleaning did not major differences in absolute returns correlograms. Now, aggregation method 5iv yields different results under 4iib-CTS, compared to other aggregation methods under CTS.*

and log returns. Meanwhile, comparing data handling combinations to each other, any combination of cleaning methods and aggregation methods (compared to other combinations) does not cause any major change in total and partial correlation structures once we move under a sampling scheme, it being either TTS or CTS. Nevertheless, this postulate fails at aggregation method 5.iv, since it yields different results under 4.ii.b-CTS, compared to other aggregation methods. Moreover, cleaning and aggregation under TTS yields different PACF structures in log and absolute returns compared to results produced with raw data.



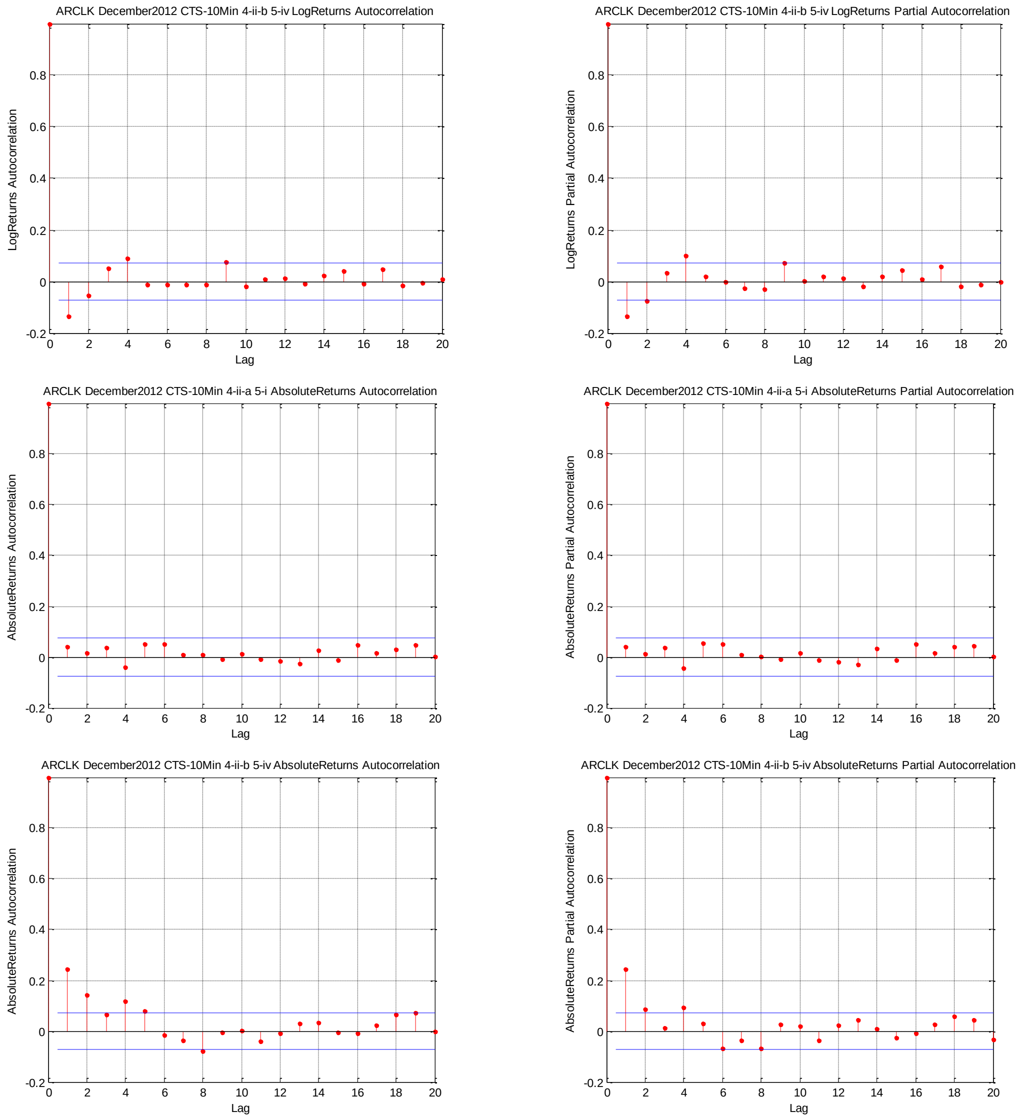


Figure F.11: ACFs and PACFs of logreturn and absolute return series of ARCLK for December 2012 under TTS and CTS.

c) **Diurnal Patterns**: These patterns can be sought only under CTS because of their definitions such as number of trades per x minutes or absolute return per y seconds. For ARCLK case, there are strong W shapes which are persistent across cleaning and aggregation methods in 10 minutes trade volumes and 10 minutes trade intensities throughout days in second half of 2012, whereas patterns in 10 minutes absolute returns are closer to W without last spike at the end of the day[24]. Regarding 10 minutes absolute percentage returns, there are so many churnings throughout the day such that we cannot name the pattern as W or U or L. Nonetheless, the unnamed pattern is same across all cleaning and aggregation

[24] *Unlike the rough L shape in MIGROS and ISCTR or rough W without last spike in NETAS, for 10 min absolute percentage returns.*

methods. All in all, there are significant diurnal patterns in returns and trading activity in the form of intensity and volume under CTS and these patterns look exactly same when various combinations of cleaning and aggregation methods are applied.

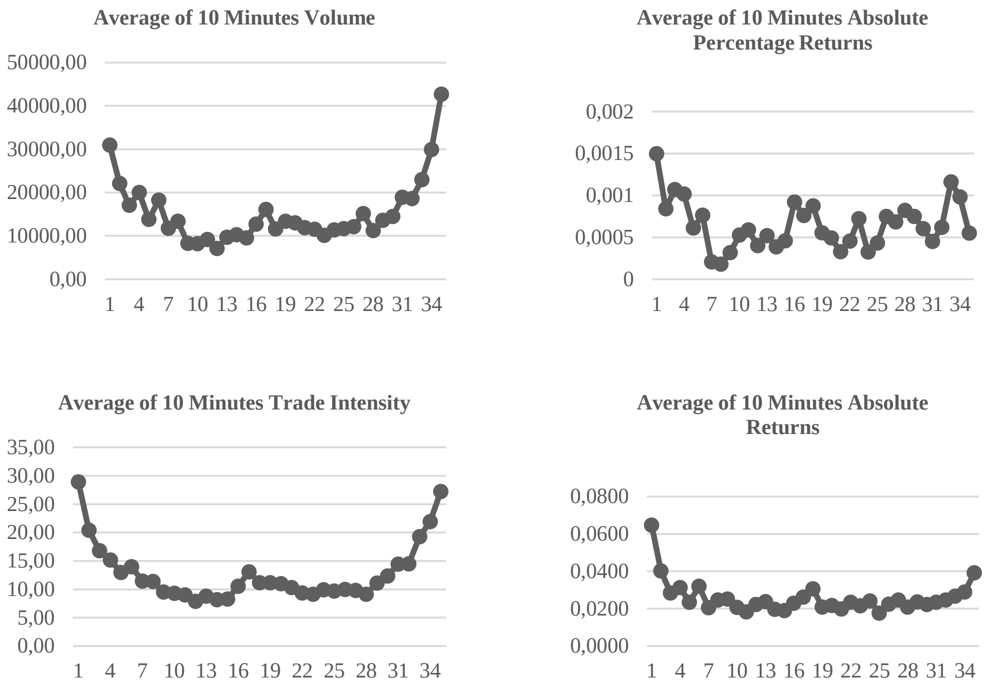


Figure F.12: Diurnal patterns - ARCLK cleaned and aggregated transaction data under CTS

2) Visual and Formal Statistical Tests of Existence and Statistical Features of Market Microstructure Noise

a) **<u>VSP:</u>** In line with findings for MIGRS and ISCTR, sampling schemes or cleaning and aggregation techniques do not alter the fact that inflating sampling frequency, either in seconds or in transactions, causes average realized volatility of return on transaction price to boom. Specifically, 6 month VSPs explode as the sampling frequency increases under raw or cleaned TTS as well as under CTS.

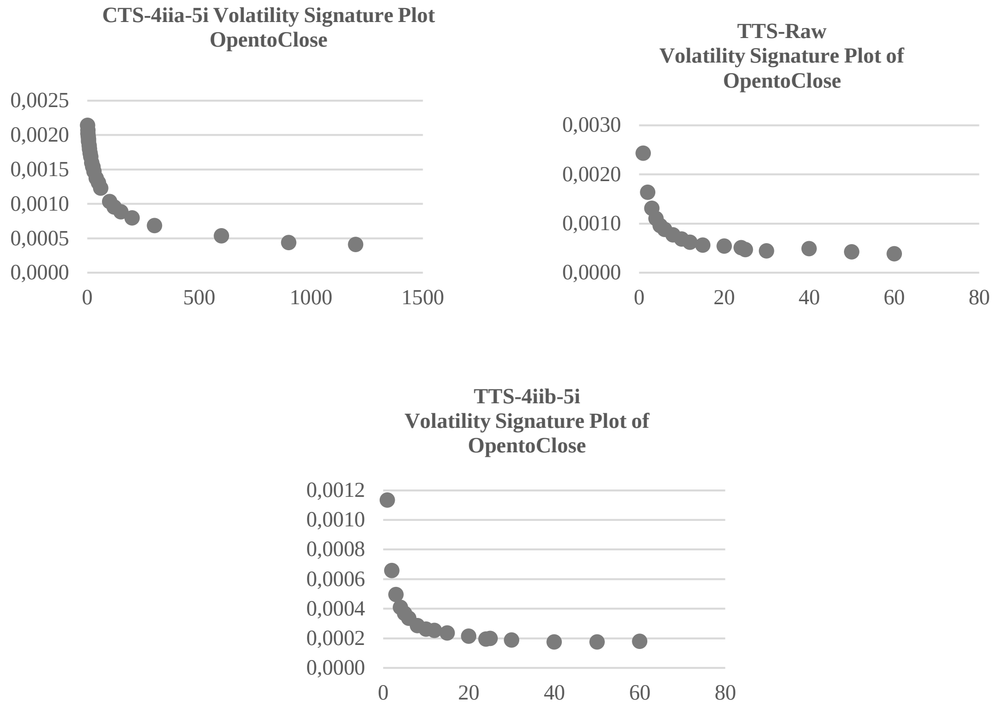


Figure F.13: VSPs of ARCLK over Daily RVs using clean and aggregated data under CTS, raw data under TTS, and clean and aggregated data under TTS.

Explosion becomes trivial for the sampling intervals that are less than 200 seconds or 15 transactions. This observation is valid both for session and daily figures, serving as a visual proof regarding existence of market microstructure noise and pointing to a positive relationship between noise increment and true price return, both under CTS and raw-TTS.

However, for clean and aggregated TTS, first session average RVs in June, August, September and December and second session average RVs in June exhibit a somehow erratic behavior in the sense that rising sampling frequencies does not inflate average RVs hyperbolically. These erratic shapes do not change from one aggregation method to the next under a particular cleaning method. To be more precise, please consider the following VSPs.

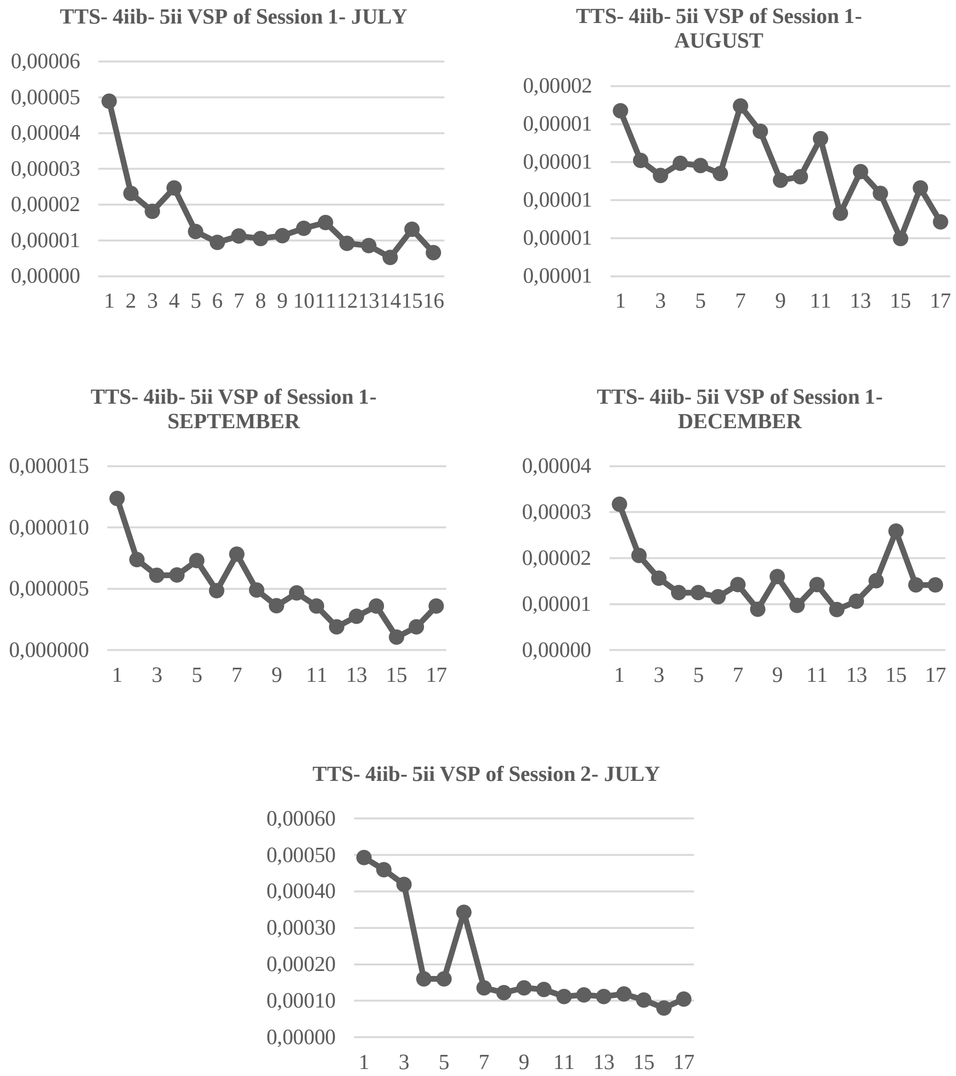


Figure F.14: VSPs of ARCLK over Session RVs using clean and aggregated data under CTS, raw data under TTS, and clean and aggregated data under TTS.

Although, average RV at 3 transactions frequency is the highest one compared to average RVs at other frequencies, postulate being true for all months in the sample period, the shape of the VSPs of session 1 average RVs in August and December are especially hard to comment, since examination of the disclosures by ARCLK throughout second half of 2012 reveals that no specific information that is disclosed to public seems responsible for unexpected VSP patterns in aforementioned months.

Recall that we found another extraordinary pattern in VSP of NETAS for session 1 in June, which caused 6 month daily averages to exhibit a swing for all cleaned and aggregated average RV series under TTS. However, contrary to NETAS case, the erratic shapes in VSPs of ARCLK in several months are smoothed by figures coming from remaining months such that 6 month average of daily or session RVs still increase hyperbolically as the sampling frequency converges to lowest available value under TTS, with all combinations of cleaning method 4.ii.b and aggregation methods 5.i, 5.ii, 5.iii, 5.iv and 5.v.

This piece of information supports our finding that in general, sampling scheme, or cleaning or aggregation do not affect the result that market microstructure becomes dominant after 15 transactions under TTS and 200 seconds under CTS and that the shape of VSPs suggest a positive correlation between noise increment and true price return.

b) <u>Statistical Tests Regarding Existence and Statistical Features of MMN :</u>

- Existence of MMN is verified statistically under both of CTS and TTS. We calculated $Z_{T,n,h}$ testing null hypothesis in (3.11) by comparing RVs that are calculated over different frequency pairs composed of high-low frequencies, namely (60,600) (10,1200), (30,1200) (60,1200), (150,1200), (300,1200), (600,1200) (900,1200) seconds under CTS and (3,30), (6,30), (10,30), (15,30), and (20,30) transactions under raw-TTS. Recall that bias of the RV estimator is dominated by expectation of square of the noise increment. Therefore, if we reject the null hypothesis, it means that the MMN has statistically significant impact on realized estimator of the IV.

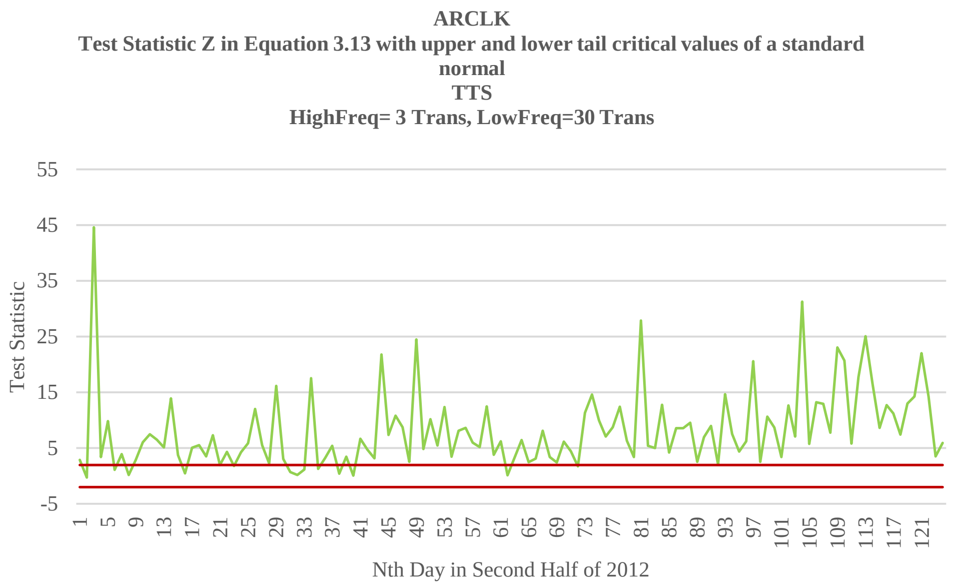
  For each day in the sample period of 124 days and each frequency pair, we run the aforementioned test at 5% significance level. Sample rejection percentages of null hypothesis are 90% under raw-TTS, 68% under clean and aggregated TTS and around 76% under CTS for all cleaning and aggregation method combinations when we compare RVs calculated over 3 and 30 transactions under TTS and 60 and 600 seconds under CTS[25]. As we decrease the sampling frequency at the high frequency leg, rejection percentages of null hypothesis shrink, which is true under both of TTS and CTS. For raw-TTS, the rejection percentages begin with 90% and decrease gradually to 18% as high frequency leg moves toward 20 transactions when low frequency leg is 30 transactions. Cleaning and aggregating the data does not amend the downward trend in rejection percentages under TTS, but make it steeper. For all aggregation choices with cleaning method 4.ii.b applied under TTS, the rejection percentages begin with 68% and decrease gradually to 10% as high frequency leg moves toward 20 transactions. Switching to CTS as well as moving across the grid of cleaning and aggregation combinations do not change the results either. For CTS, the rejection percentages begin with around 94% for 10 to 1200 seconds pair and goes down the hill to 14% as high frequency legs are slowed to 900 seconds.

  Following representative rejection rate graphs reveal that MMN starts to accentuate as the sampling frequency converges to 10-15 transactions under TTS, and 300 seconds under CTS. These findings are in conformity with those supplied by VSP analysis. MMN is felt strongly once we cross

---

[25] *These rejection percentages are significantly lower than those for NETAS and ISCTR cases.*

over the sampling interval thresholds of 15 transactions or 5 minutes under TTS and CTS, respectively. Compared to NETAS and ISCTR findings, although steady increase in ARCLK rejection rates as sampling frequency is increased still suggest accentuating MMN depending on frequency, we have to underline the fact that unlike NETAS and ISCTR results, now, the marginal increase in the rejection rates with inflating sampling frequencies do not fall, i.e., there seems to be a positive linear relationship between rejection rates and sampling frequencies, holding both under CTS and TTS. Moreover, visual inspection of the test statistic $Z_{T,n,h}$ for several frequency pairs either under TTS or CTS reveals that for the majority of the time test statistic is positive and outside 5% st. normal confidence interval, which can be interpreted as a positive correlation between noise and efficient price, again in conformity with exploding VSPs.



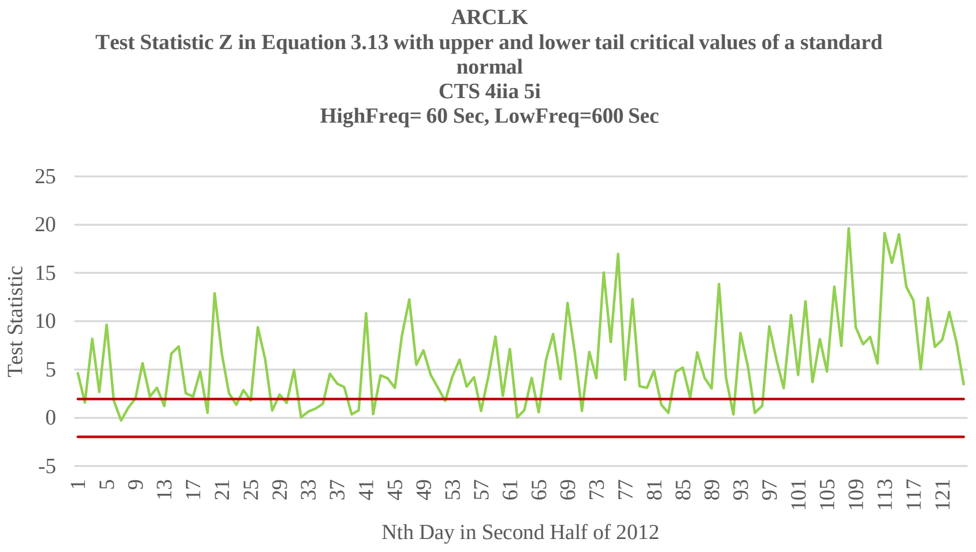
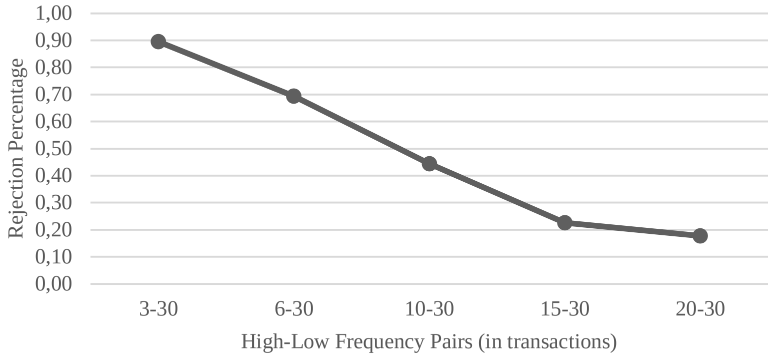
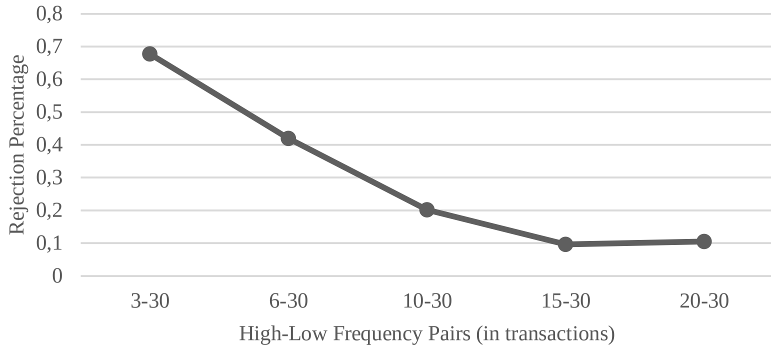


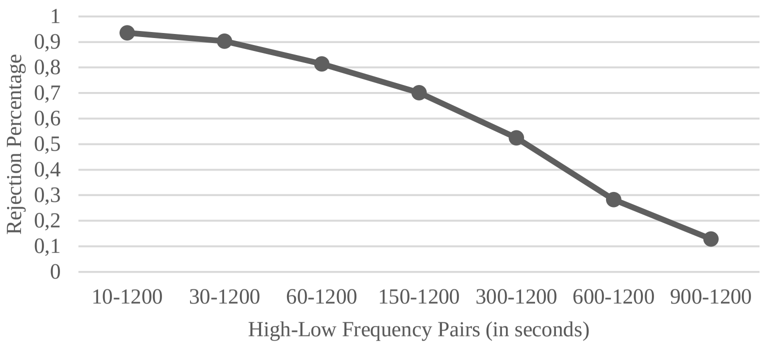
Figure F.15: ARCLK - Plots of $Z_{T,n,h}$ for each day in the sample period with upper and lower tail critical values of standard normal under TTS and CTS.



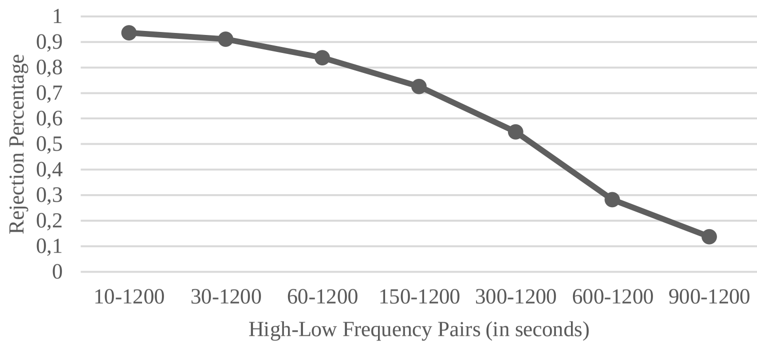
TTS-Raw Data



TTS-Clean and Aggregated Data (4iib-5i)



CTS-Clean and Aggregated Data (4iib-5i)



CTS-Clean and Aggregated Data (4iia-5i)

Figure F.16: ARCLK –Plots of rejection percentages with regards to the null hypothesis that the MMN does not have statistically significant effect on RV under TTS and CTS.

- **Summary:** Model of an i.i.d MMN with constant variance might be proper under both of CTS and TTS, for more than 90% of the days, the null hypothesis of constant variance is not rejected for triples with very high frequencies combined with very low. This might be evidence of i.i.d assumption holding at all frequencies. Sampling schemes or aggregation methods are discovered to be not influential on rejection of the null hypothesis that MMN has variance independent of sampling frequency. Additionally, cleaning algorithms do not have any substantial suppressive effect on rejection percentages unlike NETAS and ISCTR cases.

Awartani et al. [16] derive a test with the idea that if the MMN has constant variance, then noise variances calculated over frequencies 1/M or 1/N should be same independent of M or N chosen. Their null and null hypotheses are as in (3.35) and (3.36).

Since alternative hypothesis is in harmony with the presence of autocorrelation in MMN, by reminding corollary 3 of Hansen and Lunde [61], Awartani et al. [16] interpret the rejection of null hypothesis as a sign of the rejection of the null hypothesis that the MMN is a sequence of i.i.d random variables with constant variance. To test the validity of this null hypothesis, a test statistic compares RV differences using two frequency pairs, where pairs are M,L and N,L. L represents a frequency at which we can ignore the MMN safely, say 20 minutes and M and N are frequencies at which the MMN is considered to be significant. Therefore, the test is build on RVs calculated over frequency triples i.e. for each high frequency pair combined with 20 minutes, we test null hypothesis that E(noise increment square at low frequency)=E(noise increment square at high frequency). If we reject the null hypothesis, it means that the MMN has variance that is NOT independent of sampling frequency, therefore any assumptions regarding i.i.d nature of MMN can be taken as invalidated. Frequency triples are as follows: (3,10,30), (3,15,30), (3,20,30), (6,15,30) (6,20,30) and (10,20,30) transactions under TTS, (60,150,1200), (60,600,1200), (150,300,1200), (150,600,1200) and (300,600,1200) seconds under CTS.

For each day in the sample period of 124 days and each frequency triple, we run the aforementioned test at 5% significance level. Sample rejection percentages of the null hypothesis clearly change from one triple to another and as we clean and aggregate data. Beware that under raw-TTS especially for combinations of frequencies with highest differences between frequent legs, rejection percentages exceed only 10%, while they stagger around 4% for 3-10-30 triple with lowest distance between first two legs. However, once we clean and aggregate the data, rejection percentages range decline to levels 6-4% depending on the triple[26]. For CTS 4.ii.a and 4.ii.b, constant variance assumption rejection percentages vary between at

[26] *In a sense, these findings agree with findings for MIGRS and ISCTR cases, where rejection percentages are highest for triples with distant constituents and TTS-raw data; however, MIGRS and ARCLK rejection percentages are way below those of ISCTR's or NETAS' rejection percentages.*

most 3% and at least 1%, resembling the rejection percentages under cleaned TTS. Therefore, unlike NETAS, AKBNK, GARAN and ISCTR results, sampling scheme is discovered NOT to be influential on rejection of null hypothesis that the MMN has variance independent of sampling frequency. We cannot reject this null hypothesis under either of TTS and CTS confidently and conclude that i.i.d with constant variance MMN assumption might reflect the real life structure of MMN. Evidence reveals that cleaning or aggregation method does not affect rejection percentages substantially.

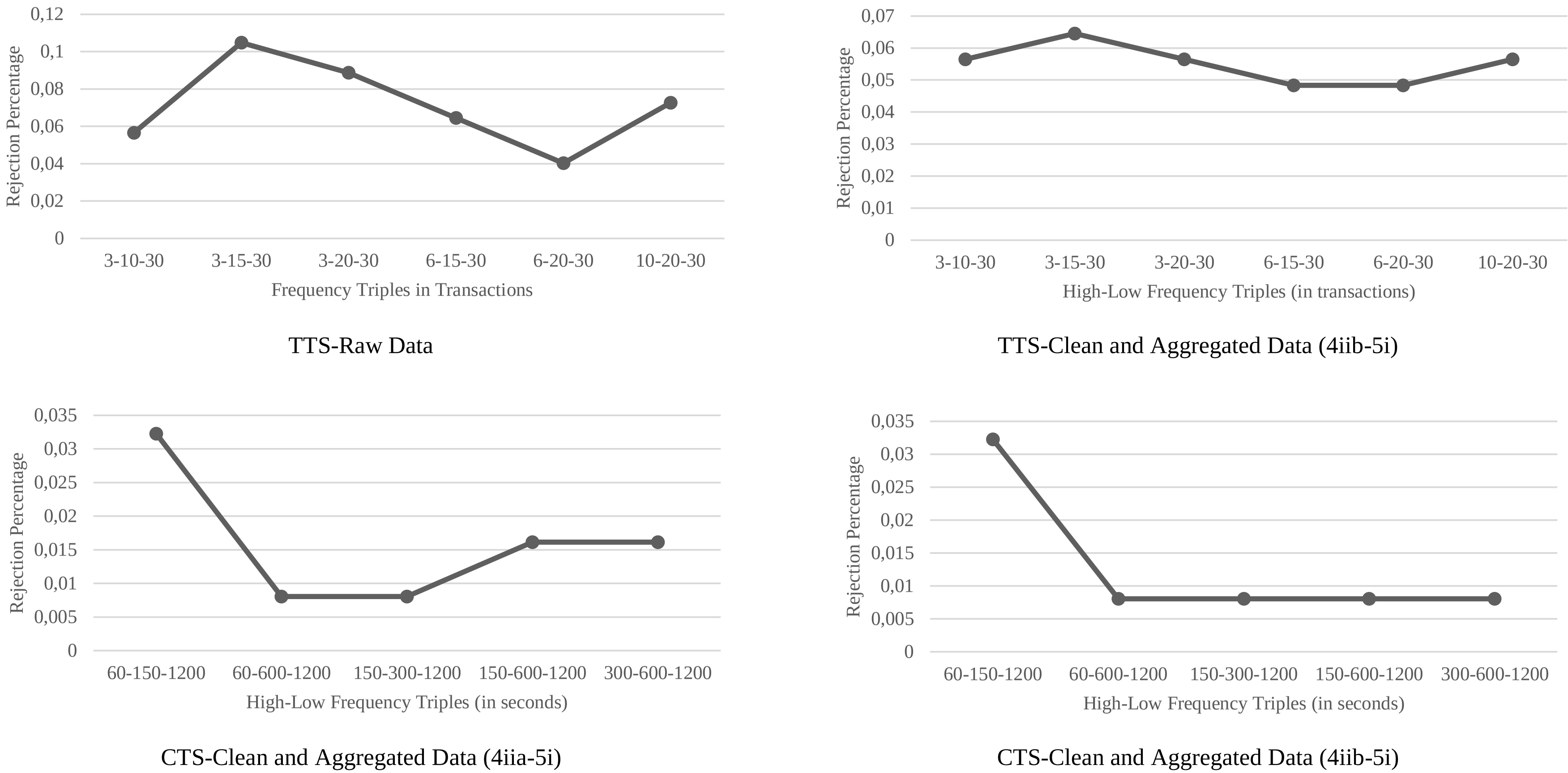


Figure F.17: ARCLK –Plots of rejection percentages with regards to the null hypothesis that the MMN increments have constant variance independent of sampling frequency under TTS and CTS.

3) RV Analysis

We constructed two RV time series, namely session RVs and daily RVs, for each frequency in a frequency set of 3, 6, 10, 15, 20, 30 transactions and 60, 300, 600 seconds under each sampling scheme (raw-TTS, CTS) and cleaning (4.ii.a, 4.ii.b) - aggregation method (5.i, 5.ii, 5.iii, 5.iv, 5.v) combination. Daily RV time series has 124 data points, whereas session RV time series is constituted of 248 entries. Each RV series under each sampling scheme and cleaning and aggregation method combinations is subjected to preliminary statistics, ACF and PACF analysis and lastly unit root is checked where autocorrelation exhibits slow decay.

- The factors that have any effect on RV series' lognormality and autocorrelation structure turn out to be whether the RV is on a session or daily basis, whether it is under raw-TTS or CTS and the frequency at which the RV is calculated. Normality is not affected by any of these factors. All of RV series, either under raw-TTS or CTS, either raw or cleaned and aggregated, either on a session or daily basis, are not normally distributed as JB statistics and kurtosis-skewness values suggest. Taking logarithm makes daily and session RV series at all frequencies normal under raw-TTS except session RV series at 20 transactions frequency, while such a transformation works in terms of normality for 10 and 5 min RV session and daily series under CTS but not for 1 min session or daily series under any cleaning or aggregation combination.

- Specifically, frequency is effective on RV autocorrelation structure under both of raw-TTS and CTS, regardless of the cleaning and aggregation methods. ACFs of all session RV series look alike for frequencies 3, 6, 10, 15 and 30 transactions. Likewise, ACFs of all daily RV series look alike for frequencies 3, 6, 10, 15 and 30 transactions. ACFs of session and daily RVs change as the sampling frequency changes, such that for increasing frequencies RV series exhibit significant positive total autocorrelation up to higher number of legs under CTS. Moreover, calculating RVs on a session basis makes the RV series more autocorrelated at higher lags under both of raw-TTS and CTS.

- Once we are working on a daily or session series at a particular frequency under CTS, cleaning and aggregation methods do not alter RV series' non-normality/normality or autocorrelation structure.

- When it comes to stationarity, the picture changes. E-views ADF Test results reveal that under raw-TTS, all RV series are found to be stationary at 5% significance level, while turning to CTS alters the stationarity at certain frequencies for certain cleaning methods and certain aggregation algorithms. Under CTS, daily/session calculation of RV, the frequency and aggregation method affect ADF test results. For instance, daily 10 min RV series is nonstationary and stationary at 5% significance level for cleaning method 4.ii.a and 4.ii.b, respectively. Moreover, even under a cleaning method, stationarity results might differ from one aggregation method to next, which is obvious for 1 min RV series being not stationary under 5.i and 5.ii aggregation methods while becoming stationary under remaining aggregation techniques combined with cleaning method 4.ii.a[27]. Likewise, if we run ADF test for fixed lag length (2) and intercept in MATLAB, test results leads us to reject null hypothesis of unit root for all session RV series under CTS or raw-TTS, for all daily series under raw-TTS, and daily series under CTS for 1 and 5 min frequencies. MATLAB results also support our finding that aggregation method affect stationarity such that daily RV series at all frequencies are nonstationary at 5% significance level under 4.ii.a-5.i and 4.ii.a-5.ii, while daily RV series at frequencies 5 and 1 minutes become stationary for remaining aggregation method combinations under cleaning method 4.ii.a. Surprisingly, MATLAB results unveil the fact that cleaning method alters stationarity results for daily RV series, because under cleaning method 4.ii.b, all daily RV series regardless of the aggregation method turn out to be stationary at 5% significance level.

a) Descriptive statistics by frequency, by sampling scheme and by cleaning and aggregation methods:

- **TTS-Raw:** For all frequencies, the session and daily RV series are not normally distributed[28] as very high skewness, kurtosis and JB statistic values reveal. Mean of the session and daily RVs become smaller as the sampling interval is lengthened, but there is no clear relationship between the sampling frequency and any change in skewness, kurtosis or JB statistic values, which deviates from the findings for MIGRS and ISCTR[29]. Correlograms of all session RV series look alike for frequencies 3, 6, 15 and 30. At these frequencies total autocorrelation is significant up to 20th lag but significance decreases and increases as the lag number converges to 20. Meanwhile, at frequency of 10 transactions, session RV series is autocorrelated up to 6th lag, then significance disappears just to emerge at lags 9 and 12. Only first three lags are significant in PACF of all session RV series at all frequencies except 20 transactions frequency. At sampling interval of 20 transactions, no total or partial autocorrelation is detected[30]. Compared to correlogram of session series, autocorrelation structure of daily RVs looks different. Correlograms of all daily RV series look alike for sampling intervals of 3, 6, 15 and 30 transactions but compared to correlograms of session series at 3, 6, 15 and 30 transaction sampling intervals, autocorrelation structure of daily RVs looks different. Now, first 10 lags and lags 1, 2 and 6 are positive significant in ACF and PACF, respectively. Differing from the findings for MIGRS, NETAS and ISCTR cases, for both of session and daily RV series under raw-TTS, the lags at which there is or there is not significant autocorrelation changes for sampling intervals 10 and 20 transactions, compared to remaining sampling intervals. The change in autocorrelation structure of RV series by looking at session and daily RVs separately calls for stationarity test

[27] *Matlab ADF test with NO INTERCEPT reveals that taking logarithm ensures stationarity at all frequencies under CTS with all cleaning and aggregation methods.*
[28] *By application of JBTEST in MATLAB, we are not able to reject log normality at all frequencies.*
[29] *For MIGRS and ISCTR, a decrease in skewness, kurtosis and JB statistic was observed as we sample less frequently.*
[30] *Unlike the case of MIGRS.*

and accordingly, we checked for unit roots in daily series to see if summing RV from session one and session two to reach daily RV distorts anything in RV stationarities at different frequencies.

TTS- Raw-Session-Frequency:20 Transactions

Date: 04/25/16 Time: 16:05
Sample: 1 248
Included observations: 248

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.109	0.109	2.9666	0.085
		2	0.077	0.065	4.4419	0.109
		3	0.093	0.079	6.6339	0.085
		4	0.083	0.063	8.3937	0.078
		5	0.064	0.040	9.4335	0.093
		6	0.084	0.061	11.256	0.081
		7	0.071	0.042	12.567	0.083
		8	0.061	0.032	13.536	0.095
		9	0.075	0.045	14.986	0.091
		10	0.049	0.015	15.608	0.111
		11	0.031	0.000	15.859	0.146
		12	0.066	0.039	17.012	0.149
		13	0.019	-0.013	17.106	0.194
		14	0.067	0.045	18.283	0.194
		15	0.052	0.022	19.013	0.213
		16	0.031	0.003	19.274	0.255
		17	0.030	0.005	19.516	0.300
		18	0.045	0.019	20.064	0.329
		19	0.090	0.069	22.278	0.271
		20	0.051	0.019	22.998	0.289

TTS- Raw-Daily-Frequency:20 Transactions

Date: 04/25/16 Time: 16:11
Sample: 1 248
Included observations: 124

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.171	0.171	3.7032	0.054
		2	0.124	0.098	5.6699	0.059
		3	0.143	0.112	8.3129	0.040
		4	0.128	0.083	10.461	0.033
		5	0.097	0.045	11.693	0.039
		6	0.097	0.048	12.947	0.044
		7	0.070	0.017	13.601	0.059
		8	0.075	0.030	14.367	0.073
		9	0.079	0.035	15.226	0.085
		10	0.104	0.062	16.721	0.081
		11	0.021	-0.035	16.780	0.115
		12	0.007	-0.033	16.786	0.158
		13	0.023	-0.007	16.860	0.206
		14	0.012	-0.012	16.880	0.263
		15	-0.060	-0.078	17.395	0.296
		16	-0.048	-0.046	17.733	0.340
		17	-0.044	-0.032	18.020	0.388
		18	-0.061	-0.042	18.574	0.419
		19	-0.022	0.011	18.644	0.480
		20	-0.039	-0.013	18.873	0.530

TTS- Raw-Session-Frequency:30 Transactions

Date: 04/25/16 Time: 16:06
Sample: 1 248
Included observations: 247

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.339	0.339	28.803	0.000
		2	0.267	0.171	46.666	0.000
		3	0.315	0.211	71.607	0.000
		4	0.317	0.169	97.114	0.000
		5	0.297	0.123	119.47	0.000
		6	0.248	0.050	135.20	0.000
		7	0.150	-0.068	140.98	0.000
		8	0.209	0.048	152.21	0.000
		9	0.138	-0.054	157.15	0.000
		10	0.180	0.063	165.58	0.000
		11	0.228	0.116	179.14	0.000
		12	0.240	0.125	194.20	0.000
		13	0.133	-0.038	198.85	0.000
		14	0.278	0.164	219.20	0.000
		15	0.222	0.015	232.25	0.000
		16	0.159	-0.062	238.95	0.000
		17	0.134	-0.062	243.78	0.000
		18	0.185	0.034	253.01	0.000
		19	0.248	0.125	269.66	0.000
		20	0.199	0.037	280.38	0.000

TTS- Raw-Daily-Frequency:30 Transactions

Date: 04/25/16 Time: 16:12
Sample: 1 248
Included observations: 124

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.473	0.473	28.423	0.000
		2	0.500	0.356	60.422	0.000
		3	0.345	0.036	75.780	0.000
		4	0.304	0.021	87.849	0.000
		5	0.270	0.059	97.408	0.000
		6	0.365	0.224	115.01	0.000
		7	0.284	0.019	125.77	0.000
		8	0.282	-0.019	136.49	0.000
		9	0.288	0.084	147.74	0.000
		10	0.289	0.094	159.16	0.000
		11	0.148	-0.167	162.20	0.000
		12	0.205	-0.001	168.08	0.000
		13	0.042	-0.133	168.33	0.000
		14	0.015	-0.121	168.36	0.000
		15	0.018	-0.013	168.40	0.000
		16	-0.005	-0.054	168.41	0.000
		17	0.017	0.043	168.45	0.000
		18	-0.009	-0.060	168.46	0.000
		19	0.015	0.045	168.50	0.000
		20	-0.031	0.020	168.64	0.000

Figure F.18: ARCLK - Correlograms of session and daily RV series under TTS for different sampling intervals

- **<u>CTS:</u>** For all frequencies, the session and daily RV series are not normally distributed as very high skewness, kurtosis and JB statistic values reveal[31]. Like the case under raw-TTS,

    i. mean of the session and daily RVs become smaller as the sampling interval is lengthened.

- However, contrary to findings for RV series under raw-TTS,

    i. decrease in skewness, kurtosis and JB statistic values is observed as we sample more and more frequently (resembles to MIGRS and ISCTR, deviates from NETAS)

---

[31] *Like MIGRS and NETAS, unlike ISCTR.*

ii. ACFs of session and daily RVs change as the sampling frequency changes, such that for increasing frequencies RV series exhibit significant positive total autocorrelation at higher number of lags with higher significances.

CTS-4.ii.a-5.i-Session 10 min RV series

Date: 04/25/16 Time: 16:14
Sample: 1 248
Included observations: 248

	AC	PAC	Q-Stat	Prob
1	0.387	0.387	37.567	0.000
2	0.366	0.254	71.291	0.000
3	0.336	0.166	99.787	0.000
4	0.277	0.071	119.29	0.000
5	0.339	0.166	148.64	0.000
6	0.340	0.138	178.23	0.000
7	0.319	0.085	204.42	0.000
8	0.297	0.046	227.18	0.000
9	0.187	-0.086	236.23	0.000
10	0.170	-0.055	243.74	0.000
11	0.248	0.091	259.78	0.000
12	0.323	0.178	287.26	0.000
13	0.130	-0.169	291.74	0.000
14	0.221	0.042	304.67	0.000
15	0.232	0.111	318.99	0.000
16	0.089	-0.118	321.13	0.000
17	0.180	0.015	329.81	0.000
18	0.231	0.129	344.19	0.000
19	0.205	0.020	355.54	0.000
20	0.176	-0.049	363.92	0.000

CTS-4.ii.a-5.i-Session 1 min RV series

Date: 04/25/16 Time: 16:16
Sample: 1 248
Included observations: 248

	AC	PAC	Q-Stat	Prob
1	0.664	0.664	110.77	0.000
2	0.683	0.432	228.24	0.000
3	0.654	0.241	336.47	0.000
4	0.597	0.058	426.90	0.000
5	0.566	0.020	508.50	0.000
6	0.528	0.001	579.85	0.000
7	0.479	-0.035	638.96	0.000
8	0.490	0.080	700.91	0.000
9	0.430	-0.022	748.78	0.000
10	0.443	0.065	799.80	0.000
11	0.411	0.013	844.08	0.000
12	0.453	0.147	897.91	0.000
13	0.418	0.020	944.03	0.000
14	0.424	0.026	991.74	0.000
15	0.411	-0.006	1036.7	0.000
16	0.399	-0.018	1079.2	0.000
17	0.353	-0.086	1112.6	0.000
18	0.353	-0.019	1146.2	0.000
19	0.302	-0.056	1170.8	0.000
20	0.300	-0.005	1195.2	0.000

CTS-4.ii.a-5.i-Daily 10 min RV series

Date: 04/25/16 Time: 16:18
Sample: 1 248
Included observations: 124

	AC	PAC	Q-Stat	Prob
1	0.589	0.589	44.134	0.000
2	0.505	0.241	76.741	0.000
3	0.502	0.218	109.22	0.000
4	0.426	0.046	132.83	0.000
5	0.318	-0.072	146.13	0.000
6	0.391	0.178	166.38	0.000
7	0.277	-0.096	176.64	0.000
8	0.268	0.057	186.33	0.000
9	0.332	0.139	201.33	0.000
10	0.243	-0.089	209.41	0.000
11	0.151	-0.090	212.56	0.000
12	0.129	-0.109	214.88	0.000
13	0.061	-0.069	215.41	0.000
14	-0.063	-0.168	215.99	0.000
15	0.016	0.091	216.02	0.000
16	-0.112	-0.169	217.84	0.000
17	-0.081	0.099	218.81	0.000
18	-0.069	-0.013	219.51	0.000
19	-0.065	0.021	220.15	0.000
20	-0.082	0.104	221.15	0.000

CTS-4.ii.a-5.i-Daily 1 min RV series

Date: 04/25/16 Time: 16:21
Sample: 1 248
Included observations: 124

	AC	PAC	Q-Stat	Prob
1	0.779	0.779	77.136	0.000
2	0.706	0.251	140.93	0.000
3	0.610	0.006	188.94	0.000
4	0.564	0.081	230.36	0.000
5	0.489	-0.033	261.73	0.000
6	0.528	0.248	298.66	0.000
7	0.489	-0.008	330.57	0.000
8	0.465	-0.023	359.75	0.000
9	0.394	-0.096	380.86	0.000
10	0.341	-0.073	396.83	0.000
11	0.285	0.011	408.06	0.000
12	0.219	-0.127	414.76	0.000
13	0.187	0.007	419.67	0.000
14	0.146	-0.065	422.68	0.000
15	0.136	0.052	425.33	0.000
16	0.140	0.102	428.15	0.000
17	0.146	0.035	431.25	0.000
18	0.116	-0.012	433.24	0.000
19	0.158	0.155	436.95	0.000
20	0.152	0.060	440.44	0.000

Figure F.19: ARCLK - Correlograms of session and daily RV series under CTS for different sampling intervals

iii. PACFs of session RVs differ slightly from one another depending on the frequency. For frequency 1 min, lags 1, 2, 3 and 12 are significant in PACF, whereas lags 1, 2, 3, and 11 and lags 1, 2, 5, 12 and 13 are significant for 5 min and 10 min frequencies, respectively.

iv. PACFs of daily RVs differ slightly from one another depending on the frequency. For frequency 1 min, lags 1, 2 and 6 are significant in PACF, whereas lags 1, 2, and 16 and lags 1, 2, 3, 6 and 14 are significant for 5 min and 10 min frequencies, respectively.

- Regardless of the shapes, slow decay in the ACFs calls for stationarity tests.

- All of these observations hold under all cleaning methods and aggregation algorithms.

b) <u>Stationarity-Unit root test:</u>

- To test for stationarity and unit root, i.e. if the series move around a constant mean or diverge as time passes, Augmented Dickey Fuller (ADF) Test is preferred. By visual inspection of graphs, no trend is observed in any of our RV series, therefore, ADF Test is run with an intercept and no trend, the number of legs to be involved in the analysis is chosen by Schwarz criterion as it is the default choice suggested by E-views.

- **<u>TTS-Raw-:</u>** In the E-views setting, where number of lags are optimized by E-views according to Schwarz criterion, R-squared values vary in a band of 33-45%. The null of nonstationarity is rejected at 1% significance level for all session and daily series[32].

- **<u>CTS:</u>** In the E-views setting, where number of lags are optimized by E-views according to Schwarz criterion, R-squared values have a range of 11% to 40%. At 5% significance level, session RV series at all frequencies under all cleaning and aggregation methods and daily RV series at 5 min under 4.ii.b and 4.ii.a as well as daily RV series at 10 min under 4.ii.b are found to be stationary. However, unlike ISCTR and NETAS but like MIGRS, daily/session calculation of RV, the frequency and aggregation method affects ADF test results. Under CTS, daily 10 min RV series is nonstationary and stationary at 5% significance level for cleaning method 4.ii.a and 4.ii.b, respectively. Moreover, even under a cleaning method, stationarity results might differ from one aggregation method to next, which is obvious for 1 min RV series being not stationary under 5.i and 5.ii aggregation methods while becoming stationary under remaining aggregation methods combined with cleaning method 4.ii.a.

[32] *In MIGRS analysis, significance level of rejection regarding nonstationarity increases when we switch to Daily series. Here, switching between Daily or session series does not affect significance level at which we can reject null.*

Table F.2: ARCLK - p-values of ADF Test on session and daily RV series under various cleaning and aggregation method combinations for different sampling intervals

Frequency	Session Based /Daily	Cleaning and Aggregation Method Combination									
		4.ii.a-5.i	4.ii.a-5.ii	4.ii.a-5.iii	4.ii.a-5.iv	4.ii.a-5.v	4.ii.b-5.i	4.ii.b-5.ii	4.ii.b-5.iii	4.ii.b-5.iv	4.ii.b-5.v
1min	Sess. Based	0.0343	0.0361	0.0353	0.0352	0.0363	0.0420	0.0434	0.0424	0.0418	0.0414
	Daily	0.0608	0.0576	0.0033	0.0033	0.0036	0.0687	0.0650	0.0637	0.0633	0.0645
5min	Sess. Based	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002	0.0002	0.0002	0.0002	0.0002
	Daily	0.0096	0.0097	0.0095	0.0096	0.0109	0.0080	0.0079	0.0076	0.0076	0.0087
10min	Sess. Based	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	Daily	0.0529	0.0609	0.0610	0.0612	0.0622	0.0031	0.0033	0.0032	0.0033	0.0037

GARAN SUMMARY AND REVIEW OF CHAPTER 4 RESULTS

1) UHFD Characteristics Under Different Sampling Schemes and Error Cleaning and Data Filtering Combinations

a) <u>Irregular Temporal Spacing</u>

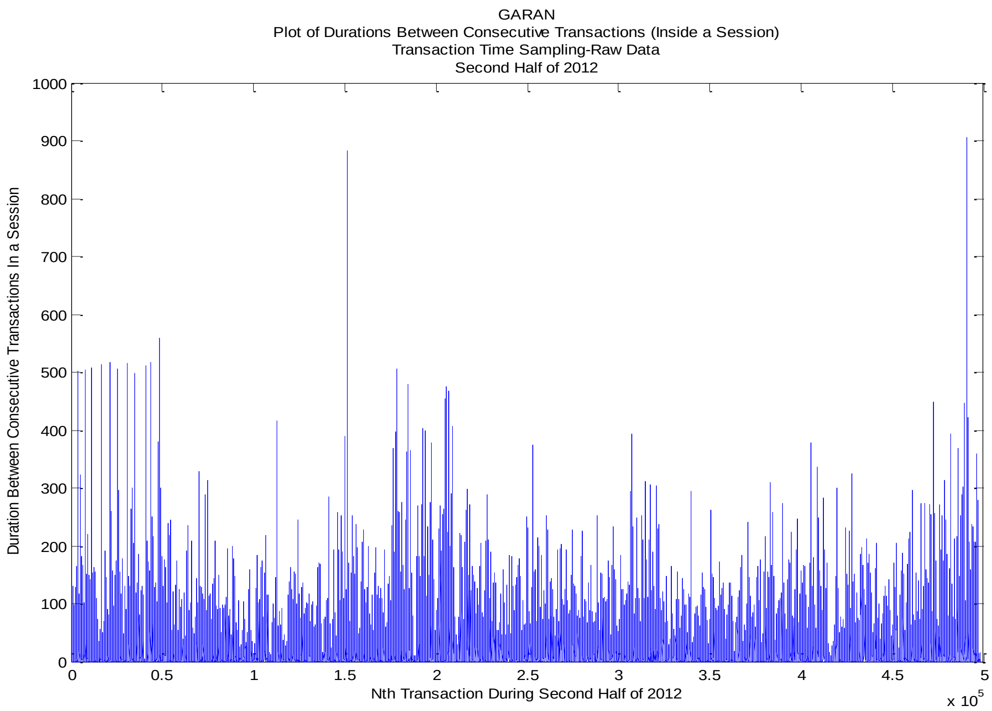


Figure F.20: Plot of durations between consecutive transactions (inside a session) for GARAN TTS-raw data throughout the second half of 2012.

b) **<u>Temporal dependence</u>**: By comparing autocorrelation and partial autocorrelation functions of 600 seconds[33] absolute returns and log returns under CTS (clean and aggregated and interpolated) as well as absolute returns, log returns and durations in seconds from one transaction to the next under TTS (raw versus clean and aggregated) for December of 2012, we see that there are differences between ACF and PACF structure of absolute and log returns between 10 min CTS and 1 transaction TTS, i.e.: transforming 1 transaction sampled data by first cleaning, then aggregating and then interpolating (all needed for CTS) to 600 second sampled data distorts ACF and PACF of return series.

---

[33] *Since first order autocorrelation was observed in 10 min returns under all cleaning and aggregation methods under CTS, we did not feel the urgency to check for 1 min returns under CTS. Recall that we included 1 min returns under CTS for MIGRS just because 10 min log returns exhibited no autocorrelation at all.*

- TTS-Raw-Durations: ACF (very very slowly decaying positive significant up to 20 lags) and PACF (hyperbolic decay, significant up to 11 lags) (shocks persist)

- TTS-Raw-Absolute Returns: ACF (very very slow decay, significant upto 20 lags) and PACF (decaying positive and significant up to 15 lags) (shocks persist)

- TTS-Raw-Log returns: ACF (quick decay, first two lags negative-positive significant) PACF(slower hyperbolic decay, first 14 lags significant)

- TTS (4.ii.a and b-5.i-5.ii-5.iii-5.iv-5.v)-Durations: ACF (very very slowly decaying positive and significant up to 20 lags) and PACF (hyperbolic decaying positive and significant up to 18 to 20 lags) (shocks persist)

- TTS (4.ii.a and b-5.i-5.ii-5.iii-5.iv-5.v)-Absolute returns: ACF (decaying positive and significant up to 20 lags) and PACF (decaying positive and significant up to 12-13 lags)

- TTS (4.ii.a and b-5.i-5.ii-5.iii-5.iv-5.v)-Log returns: ACF (quick decay, first two legs negative-positive significant) PACF(slower hyperbolic decay, first 7-8 lags negative significant)

- CTS-Durations: Meaningless, after interpolation duration from one entry to the next is always 1 second.

- CTS (4.ii.a and b-5.i-5.ii-5.iii-5.iv-5.v)-10 min Absolute Returns: ACF (first lag is positive significant), PACF(positive significant at first lag)

- CTS (4.ii.a and b-5.i-5.ii-5.iii-5.iv-5.v)-10 min Log returns: ACF (only first lag is negative significant) and PACF (first lag is negative significant)

Under TTS, with raw or clean and aggregated data, there is significant positive autocorrelation up to 20 lags in absolute returns, significant up to third order autocorrelation in log returns and very significant positive autocorrelation up to 20 lags in seconds elapsed between two transactions, thus volatility clustering is verified. Whereas, for 10 min returns under CTS, log returns display first order autocorrelation, which is in conformity with evidence laid out by the finance literature in general, that very short term returns exhibit strong autocorrelation especially on the first lag. Absolute return autocorrelation structure is changed under CTS at 600 seconds sampling interval compared to results under TTS at 1 transaction interval. Likewise, switching to CTS and calculation returns at 600 seconds suppresses partial autocorrelation figures at several lags of both absolute and log returns. Meanwhile, comparing data handling combinations to each other, any combination of cleaning methods and aggregation methods (compared to other combinations) does not cause any major change in total and partial correlation structures once we move under a sampling scheme, it being either TTS or CTS. However, cleaning and aggregation under TTS yield different PACF structures in log returns compared to results produced with raw data.

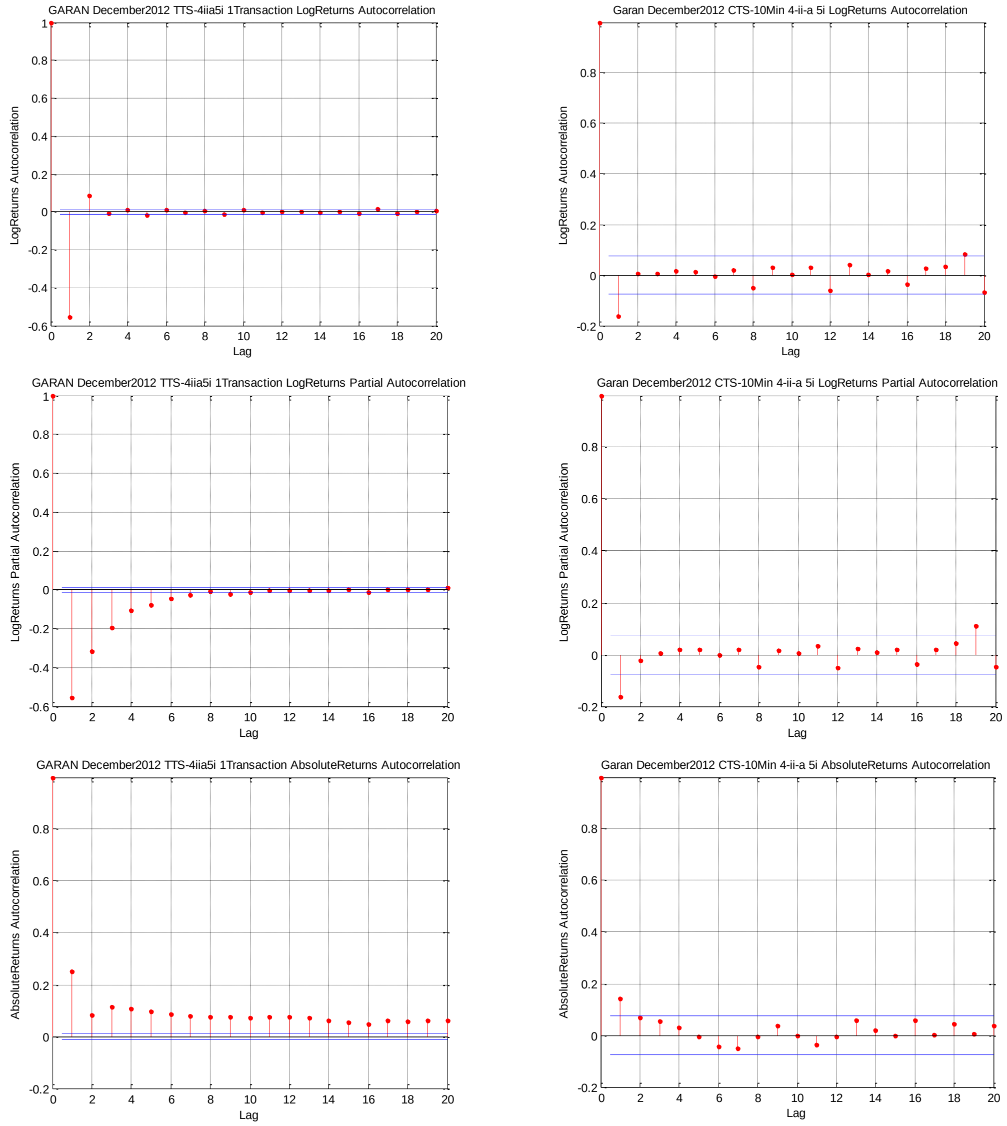


Figure F.21: ACFs and PACFs of logreturn and absolute return series of GARAN for December 2012 under TTS and CTS

c) **<u>Diurnal Patterns</u>**: These patterns can be sought only under CTS because of their definitions such as number of trades per x minutes or absolute return per y seconds. For GARAN case, there are strong W shapes which are persistent across cleaning and aggregation methods in 10 minutes trade volumes and 10 minutes trade intensities throughout days in second half of 2012, whereas patterns in 10 minutes absolute returns are closer to W without last spike at the end of the day. The existence of several spikes around lunch break in 10 min absolute returns is interesting. Likewise, 10 minutes absolute percentage returns exhibit a shape, but although the shape is persistent across cleaning and aggregation methods, it does not resemble to a W or L[34]. All in all, there are significant diurnal patterns in returns and trading activity in the form of intensity and volume under CTS and these patterns look exactly same when various combinations of cleaning and aggregation methods are applied.

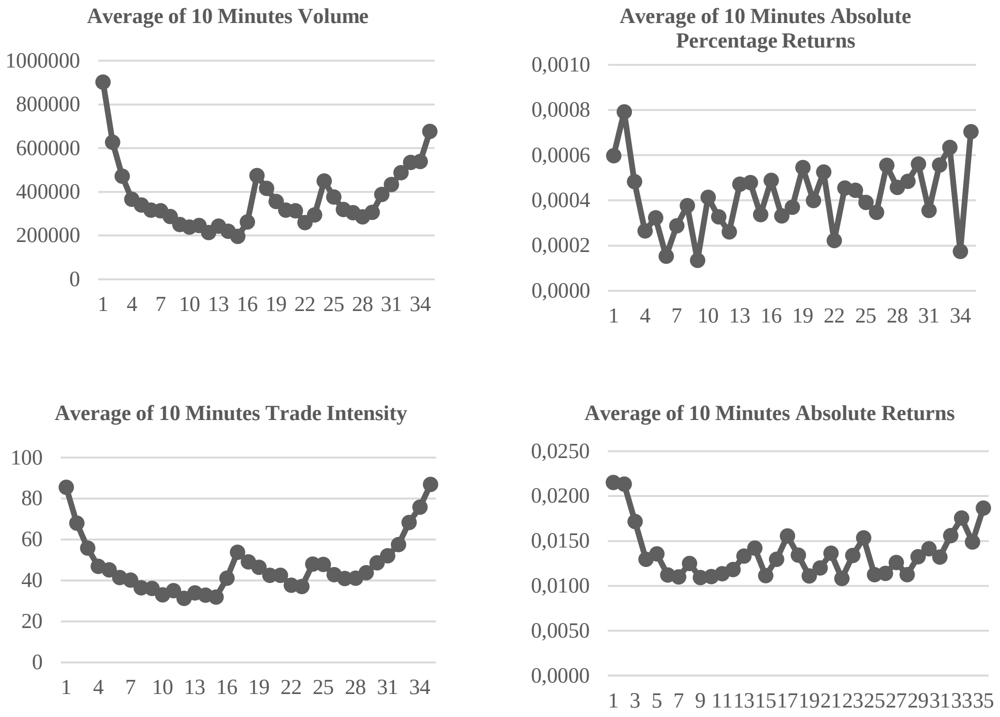


Figure F.22: Diurnal patterns - GARAN cleaned and aggregated transaction data under CTS

2) Visual and Formal Statistical Tests of Existence and Statistical Features of Market Microstructure Noise

a) **VSP:** Regardless of the sampling schemes or cleaning and aggregation techniques combinations, average realized volatility of return on transaction price explode as we increase the sampling frequency, either in seconds or in transactions. Explosion becomes trivial for the sampling intervals that are less than 200 seconds or 15 transactions. This observation is valid both for session and daily figures, serving as a visual proof regarding existence of market microstructure noise and pointing to a positive relationship between noise increment and true price return, both under CTS and TTS even if the data set is cleaned or aggregated. At this point, we would like to emphasize that for VSPs, we skipped 4.ii.a-5.i-5.ii-5.iii-5.iv-5.v combinations under TTS, mainly because

[34] *Unlike the W or L shapes in other stocks.*

the number of cleaned points under 4.ii.a is so small, cleaning makes no real difference comparing to no cleaning of the data set. Any possible difference might have been observed under cleaning method 4.ii.b, which ended up deleting more data points. Moreover, since we compare 4.ii.a and 4.ii.b under CTS, we additionally search for any marginal effect that cleaning method 4.ii.b has over cleaning method 4.ii.a. However, as put forward previously, cleaning or aggregation does not affect the result that market microstructure becomes dominant after 15 transactions under TTS and 200 seconds under CTS and that the shape of VSP suggest a positive correlation between noise increment and true price return.

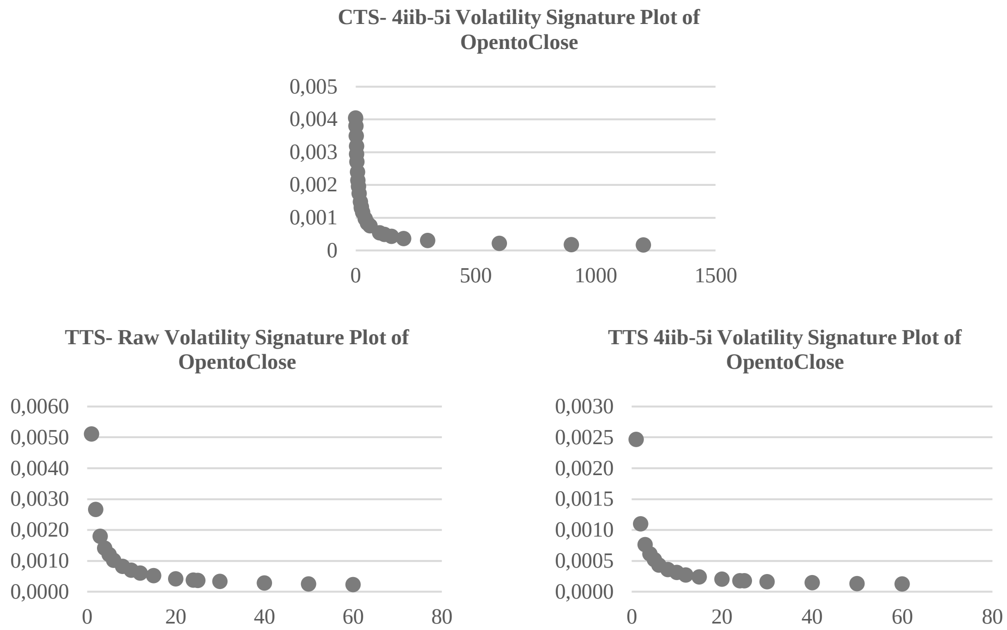


Figure F.23: VSPs of GARAN over Daily RVs using clean and aggregated data under CTS, raw data under TTS, and clean and aggregated data under TTS.

b) <u>Statistical Tests Regarding Existence and Statistical Features of the MMN [35]:</u>

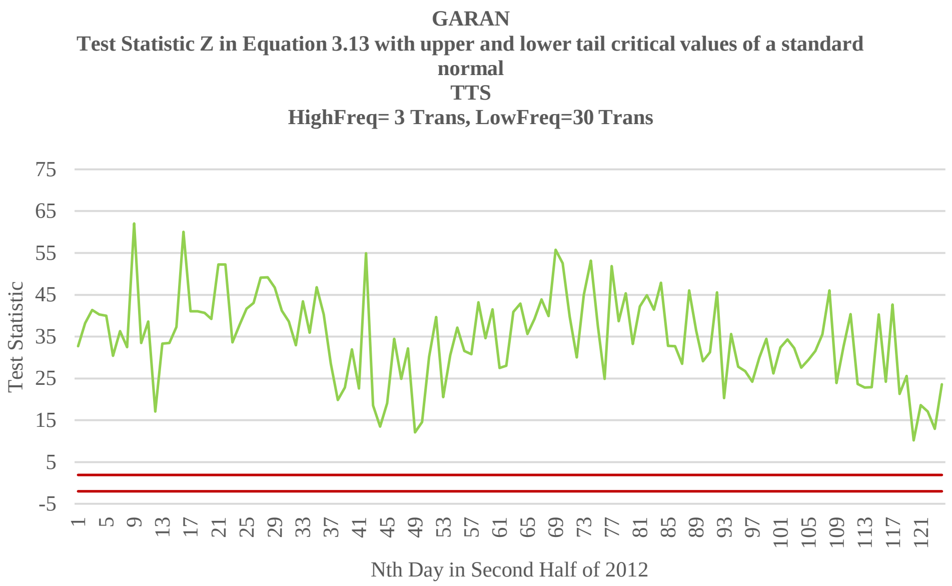
- Existence of the MMN is verified statistically under both of CTS and TTS. We calculated $Z_{T,n,h}$ testing null hypothesis in (3.11) by comparing RVs that are calculated over different frequency pairs composed of high-low frequencies, namely (60,600) (10,1200), (30,1200) (60,1200), (150,1200), (300,1200), (600,1200) (900,1200) seconds under CTS and (3,30), (6,30), (10,30), (15,30), and (20,30) transactions under raw-TTS. Recall that bias of the RV estimator is dominated by expectation of square of the noise

[35] *Findings under this section are very much alike to those for ISBANK.*

increment. Therefore, if we reject the null hypothesis, it means that the MMN has statistically significant impact on realized estimator of the IV.

For each day in the sample period of 124 days and each frequency pair, we run the aforementioned test at 5% significance level. Sample rejection percentages of null hypothesis are 100% under raw-TTS, 99% under clean and aggregated TTS and around 98% under CTS for all cleaning and aggregation method combinations when we compare RVs calculated over 3 and 30 transactions under TTS and 60 and 600 seconds under CTS. As we decrease the sampling frequency at the high frequency leg, rejection percentages of null hypothesis shrink, which is true under both of TTS and CTS. For raw-TTS, the rejection percentages begin with 100% and decrease gradually to 55% as high frequency leg moves toward 20 transactions when low frequency leg is 30 transactions. Cleaning and aggregating the data does not amend the downward trend in rejection percentages under TTS, but make it steeper. For all aggregation choices with cleaning method 4.ii.b applied under TTS, the rejection percentages begin with 99% and decrease gradually to levels around 25% as high frequency leg moves toward 20 transactions. Switching to CTS as well as moving across the grid of cleaning and aggregation combinations do not change the results either. For CTS, the rejection percentages begin with around 100% for 10 to 1200 seconds pair and goes down the hill to 12-15% as high frequency legs are slowed to 900 seconds.

The following representative rejection rate graphs reveal that MMN starts to accentuate as the sampling frequency converges to 10-15 transactions under TTS, and 250-300 seconds under CTS. These findings are in conformity with those supplied by VSP analysis. MMN is felt strongly once we cross over the sampling interval thresholds of 15 transactions or 5 minutes under TTS and CTS, respectively. For higher frequencies, rejection rates turn out to be quite high, especially after 150 seconds under CTS and 10 transactions under TTS, rejection rates become flat in a band of 95-100%. Moreover, visual inspection of the test statistic $Z_{T,n,h}$ for several frequency pairs either under TTS or CTS reveals that for the majority of the time test statistic is positive and outside 5% st. normal confidence interval, meaning there is positive correlation between noise and efficient price, which is again in conformity with exploding VSPs.



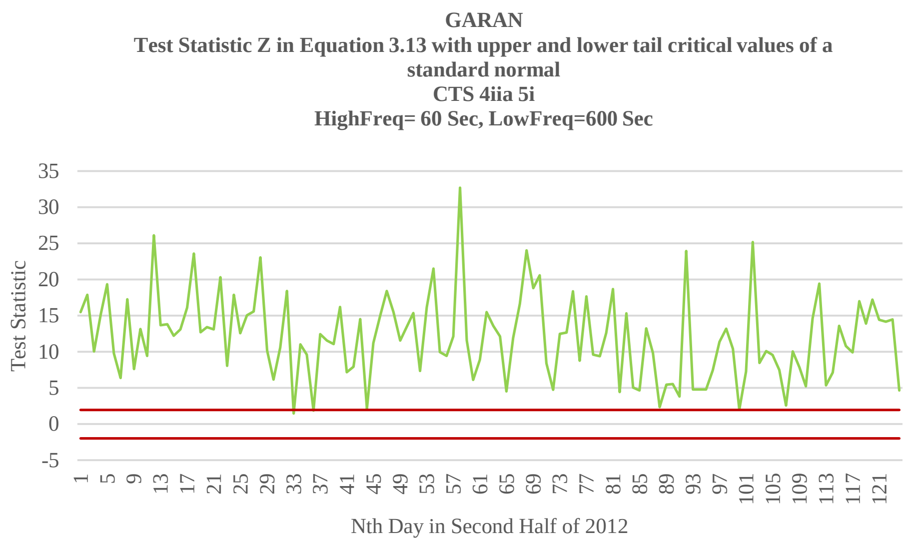


Figure F.24: GARAN - Plots of $Z_{T,n,h}$ for each day in the sample period with upper and lower tail critical values of standard normal under TTS and CTS.

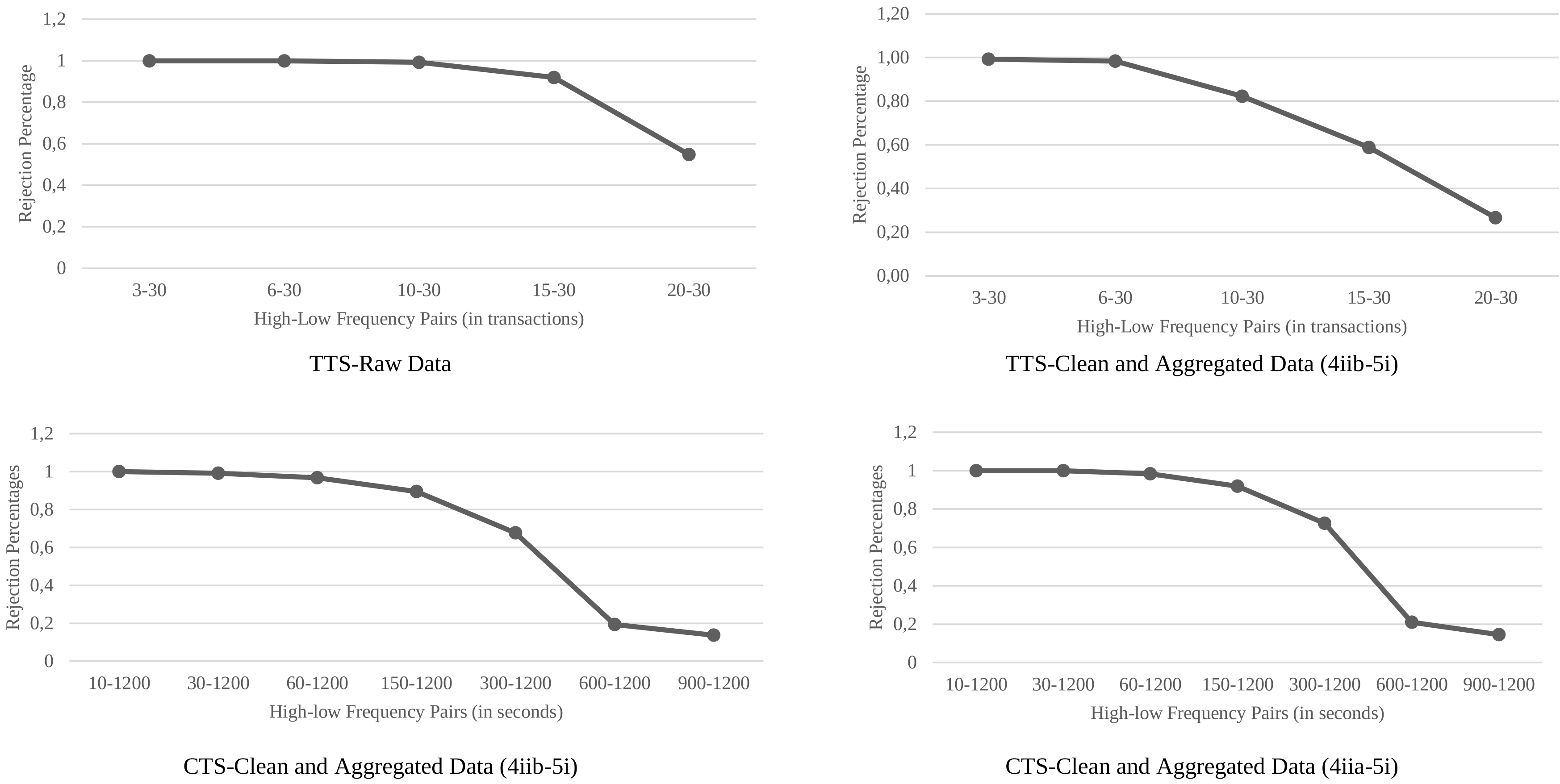


Figure F.25: GARAN –Plots of rejection percentages with regards to the null hypothesis that MMN does not have statistically significant effect on RV under TTS and CTS.

- **Summary:** Model of i.i.d MMN with constant variance might be proper under CTS but under raw-TTS, for more than 50% of the days, null of constant variance is rejected for triples with very high frequencies combined with very low. This might be evidence of i.i.d assumption not holding at frequencies lesser than 15 transactions. Sampling scheme, but not the aggregation method, is discovered to very influential on rejection of null hypothesis that MMN has variance independent of sampling frequency. Meanwhile, cleaning algorithms have some suppressive effect on rejection percentages particularly under TTS.

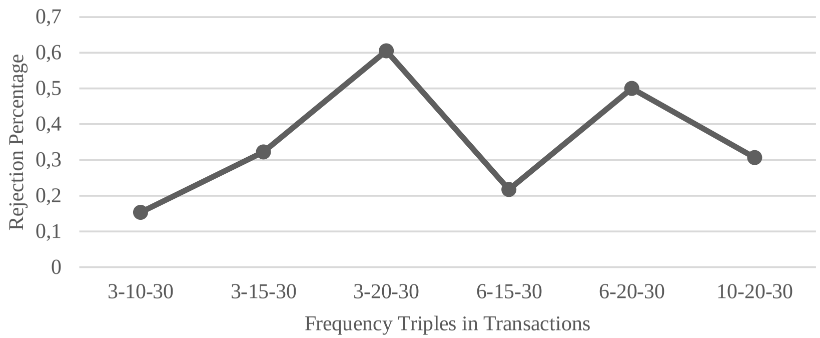
  Awartani et al. [16] derive a test with the idea that if the MMN has constant variance, then noise variances calculated over frequencies 1/M or 1/N should be same independent of M or N chosen. Their null and null hypotheses are as in (3.35) and (3.36).

  Since alternative hypothesis is in harmony with the presence of autocorrelation in MMN, by reminding corollary 3 of Hansen and Lunde [61], Awartani et al. [16] interpret the rejection of null hypothesis as a sign of the rejection of the null hypothesis that the MMN is a sequence of i.i.d random variables with constant variance. To test the validity of this null hypothesis, a test statistic compares RV differences using two frequency pairs, where pairs are M,L and N,L. L represents a frequency at which we can ignore the MMN safely, say 20 minutes and M and N are frequencies at which the MMN is considered to be significant. Therefore, the test is build on RVs calculated over frequency triples i.e. for each high frequency pair combined with 20 minutes, we test null hypothesis that E(noise increment square at low frequency)=E(noise increment square at high frequency). If we reject the null hypothesis, it means that the MMN has variance that is NOT independent of sampling frequency, therefore any assumptions regarding i.i.d nature of MMN can be taken as invalidated. Frequency triples are as follows: (3,10,30), (3,15,30), (3,20,30), (6,15,30) (6,20,30) and (10,20,30) transactions under TTS, (60,150,1200), (60,600,1200), (150,300,1200), (150,600,1200) and (300,600,1200) seconds under CTS.

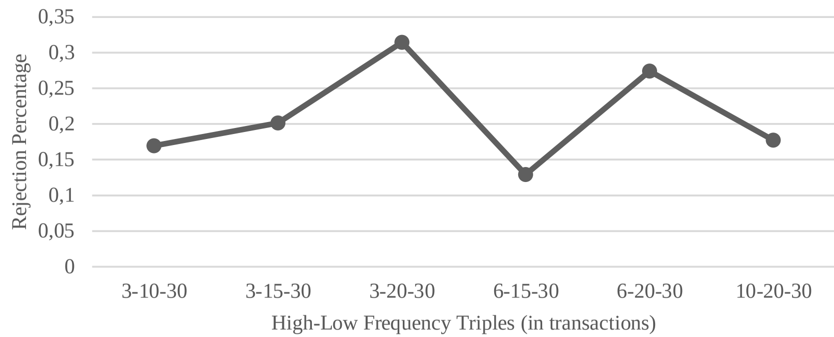
  For each day in the sample period of 124 days and each frequency triple, we run the aforementioned test at 5% significance level. Sample rejection percentages of null hypothesis clearly changes from one triple to another and as we clean and aggregate data. Beware that under raw-TTS especially for combinations of frequencies with highest differences between frequent legs, rejection percentages exceed 60%, while they stagger around 15% for 3-10-30 triple with lowest distance between first two legs. However, once we clean and aggregate the data, rejection percentages, except for 3-10-30 triple, are severed to levels 31% or 12% depending on the triple[36]. Regarding 3-10-30 triple, rejection percentage slightly increases. For CTS

[36] *In a sense, these findings agree with findings for MIGRS case, where rejection percentages are highest for triples with distant constituents and TTS-raw data; however, MIGRS rejection percentages are way below those of ISCTR or GARAN rejection percentages.*

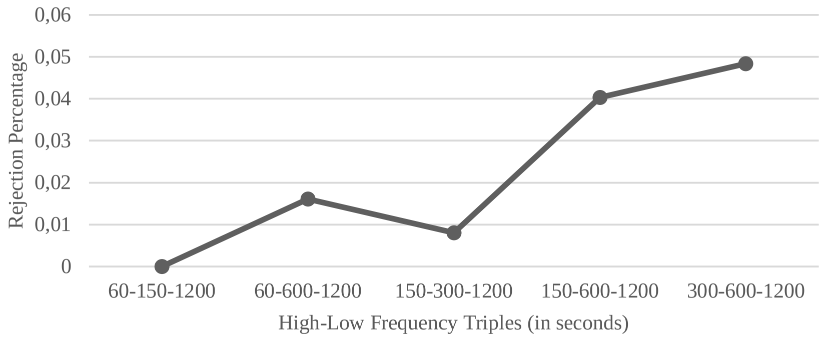
4.ii.a (b), constant variance assumption rejection percentage varies between 0% and 6.4%, both of which are just a fraction of rejection percentages under TTS-raw or TTS-cleaned. Therefore, sampling scheme is discovered to very influential on rejection of null hypothesis that MMN has variance independent of sampling frequency. We can reject this null under TTS confidently and conclude that i.i.d with constant variance MMN assumption does not reflect the real life structure of MMN, whereas under CTS, such an assumption seems to hold especially for frequencies higher than 150 seconds. Evidence reveals that aggregation method does not affect rejection percentages and for triples with high frequency legs being close to very slow frequency leg, rejection percentages are substantially damaged independent of the sampling scheme.



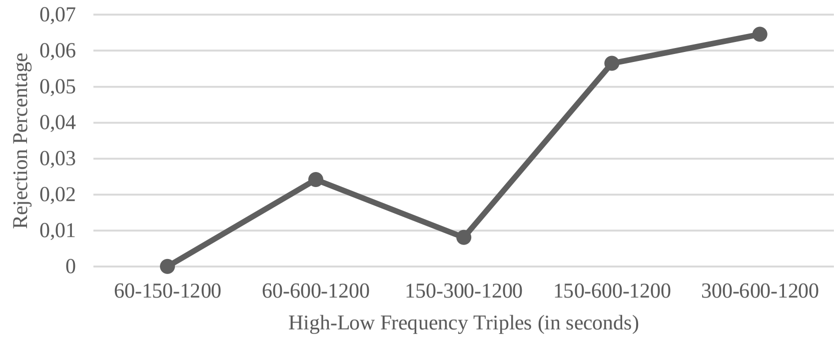
TTS-Raw Data



TTS-Clean and Aggregated Data (4iib-5i)



CTS-Clean and Aggregated Data (4iia-5i)



CTS-Clean and Aggregated Data (4iib-5i)

Figure F.26: GARAN –Plots of rejection percentages with regards to the null hypothesis that the MMN increments have constant variance independent of sampling frequency under TTS and CTS.

3) RV Analysis

We constructed two RV time series, namely session RVs and daily RVs, for each frequency in a frequency set of 3, 6, 10, 15, 20, 30 transactions and 60, 300, 600 seconds under each sampling scheme (raw-TTS, CTS) and cleaning (4.ii.a, 4.ii.b) -aggregation method (5.i, 5.ii, 5.iii, 5.iv, 5.v) combination. Daily RV time series has 124 data points, whereas session RV time series is constituted of 248 entries. Each time RV series under each sampling scheme and cleaning and aggregation method combinations is subjected to preliminary statistics, ACF and PACF analysis and lastly unit root is checked where autocorrelation exhibits slow decay.

- The factors that have any effect on RV series' normality and autocorrelation structure turn out to be whether the RV is on a session or daily basis, whether it is under raw-TTS or CTS and the frequency at which the RV is calculated. For all frequencies, the session and daily RV series are not normally distributed under raw-TTS as skewness, kurtosis and very high JB statistic values reveal. Switching to CTS and increasing frequency and calculating RVs on a daily basis make RV series more and more normal such that at 1 min frequency, we cannot reject null hypothesis of daily 1 min RV sample coming from a normally distributed population at 5 or 1% significance levels. For 5 min and 10 min frequencies with daily calculation and all frequencies with session calculations, under CTS, RV series are not coming from a normally distributed population. Taking logarithm of RV series converts them to normal for all frequencies (except session 20) under raw-TTS and for daily 5 and 10 min frequencies under cleaning method 4.ii.a, but only for daily 10 min under cleaning method 4.ii.b under CTS[37]. All session CTS series are non-lognormal. Therefore, session-daily choice, cleaning method, frequency and sampling scheme are found to be effective on lognormality of RV series.

- Decreasing frequencies cause less number of lags being significant with lesser significant levels, i.e. decreasing frequency again depresses autocorrelation structure of RV series regardless of sampling scheme or session-daily calculation, which is in line with existence of MMN. However, the suppression effect is not too strong under raw-TTS. Unlike findings for AKBNK, ISCTR and NETAS, calculating RVs on a session basis DOES NOT make the RV series more autocorrelated under CTS. Still, session-daily choice alters autocorrelation decay patterns under CTS and raw-TTS.

- Once we are working on a daily or session series at a particular frequency under CTS, cleaning and aggregation methods do not alter RV series' non-normality/normality or autocorrelation structure.

- Sampling scheme, frequency and cleaning methods affects the stationarity results. [38]. E-views ADF Test results reveal that we can reject null of unit root at

[37] *Unlike the case of MIGRS.*
[38] *Unlike the findings for ISCTR, NETAS and ARCLK cases, alike to AKBNK.*

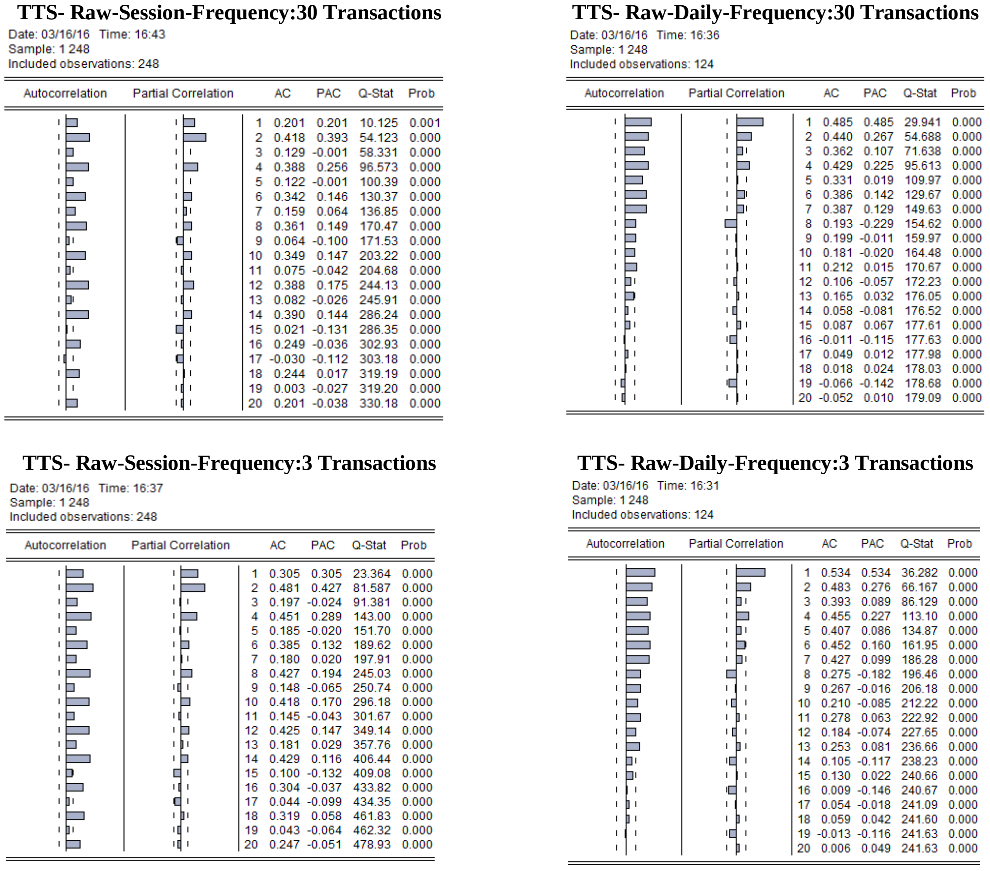
5% significance level for RV series under raw-TTS at all frequencies[39]; however, switching to CTS and moving between cleaning methods or session or daily RV calculation basis while increasing the frequency changes the game such that under cleaning method 4.ii.a, regardless of aggregation method, the null hypotheses that daily or session 1 min RV series have unit root cannot be rejected at 5% significance level. Whereas, adopting cleaning method 4.ii.b makes session and daily 1 min series stationary at 5% significance level orthogonal to aggregation methods[40]. Likewise, if we run ADF test for fixed lag length (2) and intercept in MATLAB, test results leads us not to reject null hypothesis of unit root for 1 min daily RV series under CTS with cleaning method 4.ii.a, where switching to cleaning method 4.ii.b ensures stationarity for RV series at all frequencies, either session or daily. For raw-TTS, results of MATLAB ADF test for 2 lags and an intercept coincide with results from E-Views ADF test, i.e., RV series session or daily at all frequencies are stationary at 5% significance level.

a) <u>Descriptive statistics by frequency, by sampling scheme and by cleaning and aggregation methods:</u>

- **<u>TTS-Raw:</u>** For all frequencies, the session and daily RV series are not normally distributed[41] as skewness, kurtosis and high JB statistic values reveal. Mean of the session and daily RVs become smaller as the sampling interval is lengthened, but there is no clear relationship between sampling frequency and change in skewness, kurtosis or JB statistic values, which deviates from the findings for MIGRS and ISCTR but is in line with findings for AKBNK. Still, normality of the any of these series is out of question. Correlogram of all session RV series look alike but are not exactly same. Generally speaking, ACFs and PACFs of RVs are decaying but not hyperbolically such that total and partial autocorrelations are strong at even lags and weak at odd legs[42] (ACF positive significant up to 13rd at odd lags -up to 20th lag at even lags and PACF positive significant selectively at legs, 1, 2, 4, and 8)[43]. Although correlograms of all daily RVs resemble one another, compared to correlogram of session series, autocorrelation structure of daily RVs looks different. Now, a quick decay with first two lags and lag 4 being positive significant in PACF is evident, while decay in ACF is wave like with significant positive values up to lag 11-13. Unlike case of AKBNK but similar to ISBNK, the decrease in sampling frequency does only have minimal suppression effect over the significance levels and the number of significant lags. The change in autocorrelation structure of RV series by looking at session and daily RVs separately, calls for stationarity test and accordingly, we checked for unit roots in daily series to see if summing RV from session one and session

[39] *Unlike the case of MIGRS.*
[40] *Matlab ADF test with NO INTERCEPT reveals that taking logarithm erases stationarity at all frequencies under CTS with all cleaning and aggregation methods. Unlike ISCTR.*
[41] *By application of JBTEST in MATLAB, we are not able to reject log normality at all frequencies either Daily or session, except session-20 transactions.*
[42] *Like ISCTR.*
[43] *Unlike the case of MIGRS.*

two to reach daily RV distorts anything in RV stationarities at different frequencies.



TTS- Raw-Session-Frequency:30 Transactions

Date: 03/16/16 Time: 16:43
Sample: 1 248
Included observations: 248

Lag	AC	PAC	Q-Stat	Prob
1	0.201	0.201	10.125	0.001
2	0.418	0.393	54.123	0.000
3	0.129	-0.001	58.331	0.000
4	0.388	0.256	96.573	0.000
5	0.122	-0.001	100.39	0.000
6	0.342	0.146	130.37	0.000
7	0.159	0.064	136.85	0.000
8	0.361	0.149	170.47	0.000
9	0.064	-0.100	171.53	0.000
10	0.349	0.147	203.22	0.000
11	0.075	-0.042	204.68	0.000
12	0.388	0.175	244.13	0.000
13	0.082	-0.026	245.91	0.000
14	0.390	0.144	286.24	0.000
15	0.021	-0.131	286.35	0.000
16	0.249	-0.036	302.93	0.000
17	-0.030	-0.112	303.18	0.000
18	0.244	0.017	319.19	0.000
19	0.003	-0.027	319.20	0.000
20	0.201	-0.038	330.18	0.000

TTS- Raw-Daily-Frequency:30 Transactions

Date: 03/16/16 Time: 16:36
Sample: 1 248
Included observations: 124

Lag	AC	PAC	Q-Stat	Prob
1	0.485	0.485	29.941	0.000
2	0.440	0.267	54.688	0.000
3	0.362	0.107	71.638	0.000
4	0.429	0.225	95.613	0.000
5	0.331	0.019	109.97	0.000
6	0.386	0.142	129.67	0.000
7	0.387	0.129	149.63	0.000
8	0.193	-0.229	154.62	0.000
9	0.199	-0.011	159.97	0.000
10	0.181	-0.020	164.48	0.000
11	0.212	0.015	170.67	0.000
12	0.106	-0.057	172.23	0.000
13	0.165	0.032	176.05	0.000
14	0.058	-0.081	176.52	0.000
15	0.087	0.067	177.61	0.000
16	-0.011	-0.115	177.63	0.000
17	0.049	0.012	177.98	0.000
18	0.018	0.024	178.03	0.000
19	-0.066	-0.142	178.68	0.000
20	-0.052	0.010	179.09	0.000

TTS- Raw-Session-Frequency:3 Transactions

Date: 03/16/16 Time: 16:37
Sample: 1 248
Included observations: 248

Lag	AC	PAC	Q-Stat	Prob
1	0.305	0.305	23.364	0.000
2	0.481	0.427	81.587	0.000
3	0.197	-0.024	91.381	0.000
4	0.451	0.289	143.00	0.000
5	0.185	-0.020	151.70	0.000
6	0.385	0.132	189.62	0.000
7	0.180	0.020	197.91	0.000
8	0.427	0.194	245.03	0.000
9	0.148	-0.065	250.74	0.000
10	0.418	0.170	296.18	0.000
11	0.145	-0.043	301.67	0.000
12	0.425	0.147	349.14	0.000
13	0.181	0.029	357.76	0.000
14	0.429	0.116	406.44	0.000
15	0.100	-0.132	409.08	0.000
16	0.304	-0.037	433.82	0.000
17	0.044	-0.099	434.35	0.000
18	0.319	0.058	461.83	0.000
19	0.043	-0.064	462.32	0.000
20	0.247	-0.051	478.93	0.000

TTS- Raw-Daily-Frequency:3 Transactions

Date: 03/16/16 Time: 16:31
Sample: 1 248
Included observations: 124

Lag	AC	PAC	Q-Stat	Prob
1	0.534	0.534	36.282	0.000
2	0.483	0.276	66.167	0.000
3	0.393	0.089	86.129	0.000
4	0.455	0.227	113.10	0.000
5	0.407	0.086	134.87	0.000
6	0.452	0.160	161.95	0.000
7	0.427	0.099	186.28	0.000
8	0.275	-0.182	196.46	0.000
9	0.267	-0.016	206.18	0.000
10	0.210	-0.085	212.22	0.000
11	0.278	0.063	222.92	0.000
12	0.184	-0.074	227.65	0.000
13	0.253	0.081	236.66	0.000
14	0.105	-0.117	238.23	0.000
15	0.130	0.022	240.66	0.000
16	0.009	-0.146	240.67	0.000
17	0.054	-0.018	241.09	0.000
18	0.059	0.042	241.60	0.000
19	-0.013	-0.116	241.63	0.000
20	0.006	0.049	241.63	0.000

Figure F.27: GARAN - Correlograms of session and daily RV series under TTS for different sampling intervals

- **CTS:** For 1 min frequency, we cannot reject null hypothesis that the daily RV series come from a normally distributed population at 5% significance level, whereas rejection of such hypothesis for all remaining RV series (daily or session) at all frequencies is evident by skewness, kurtosis and high JB statistic values. Like the case under raw-TTS,

    i. mean of the session and daily RVs become smaller as the sampling interval is lengthened.

    ii. there is no clear relationship between sampling frequency and change in skewness, kurtosis or JB statistic values, which deviates from the findings for MIGRS, ARCLK and ISCTR but resembles to findings for NETAS.

- However, contrary to findings for RV series under RAW-TTS,

  i. ACFs of session and daily RVs change as the sampling frequency changes, such that for increasing frequencies RV series exhibit significant positive total autocorrelation up to higher number of legs. Apart from this common trait, the decay patterns in total correlation of daily and session RVs are different, especially obvious at 1 min frequency.

CTS-4.ii.a-5.i-Session 10 min RV series

Date: 03/31/16 Time: 13:57
Sample: 1 248
Included observations: 247

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.129	0.129	4.1452	0.042
		2	0.164	0.150	10.916	0.004
		3	-0.014	-0.053	10.965	0.012
		4	0.072	0.057	12.268	0.015
		5	0.047	0.045	12.831	0.025
		6	0.189	0.165	21.968	0.001
		7	0.019	-0.032	22.065	0.002
		8	0.039	-0.013	22.460	0.004
		9	-0.046	-0.040	23.003	0.006
		10	0.100	0.095	25.594	0.004
		11	0.079	0.065	27.223	0.004
		12	0.241	0.177	42.371	0.000
		13	0.090	0.041	44.520	0.000
		14	0.139	0.074	49.608	0.000
		15	-0.047	-0.073	50.182	0.000
		16	0.014	-0.049	50.235	0.000
		17	-0.063	-0.085	51.292	0.000
		18	0.034	-0.031	51.596	0.000
		19	-0.000	0.007	51.596	0.000
		20	0.093	0.076	53.916	0.000

CTS-4.ii.a-5.i-Session 1 min RV series

Date: 03/31/16 Time: 14:00
Sample: 1 248
Included observations: 248

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.175	0.175	7.6481	0.006
		2	0.423	0.405	52.756	0.000
		3	0.148	0.041	58.294	0.000
		4	0.340	0.186	87.658	0.000
		5	0.178	0.077	95.756	0.000
		6	0.423	0.273	141.58	0.000
		7	0.120	-0.048	145.27	0.000
		8	0.272	-0.004	164.40	0.000
		9	0.064	-0.066	165.47	0.000
		10	0.275	0.085	185.15	0.000
		11	0.111	0.017	188.39	0.000
		12	0.331	0.130	217.22	0.000
		13	0.016	-0.113	217.28	0.000
		14	0.237	0.031	232.19	0.000
		15	0.027	-0.012	232.39	0.000
		16	0.242	0.053	248.06	0.000
		17	0.022	-0.070	248.19	0.000
		18	0.280	0.101	269.35	0.000
		19	-0.046	-0.087	269.93	0.000
		20	0.145	-0.059	275.68	0.000

CTS-4.ii.a-5.i-Daily 10 min RV series

Date: 03/31/16 Time: 14:02
Sample: 1 248
Included observations: 124

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.201	0.201	5.1447	0.023
		2	0.055	0.015	5.5294	0.063
		3	0.247	0.243	13.393	0.004
		4	0.015	-0.088	13.423	0.009
		5	0.053	0.070	13.796	0.017
		6	0.307	0.250	26.249	0.000
		7	0.178	0.101	30.476	0.000
		8	-0.071	-0.170	31.154	0.000
		9	0.037	-0.046	31.339	0.000
		10	0.049	0.028	31.669	0.000
		11	-0.077	-0.050	32.494	0.001
		12	0.055	-0.018	32.923	0.001
		13	-0.032	-0.147	33.069	0.002
		14	-0.084	0.025	34.080	0.002
		15	-0.085	-0.059	35.105	0.002
		16	-0.038	0.006	35.315	0.004
		17	-0.136	-0.134	38.026	0.002
		18	-0.147	-0.075	41.232	0.001
		19	-0.003	0.085	41.233	0.002
		20	-0.070	0.028	41.965	0.003

CTS-4.ii.a-5.i-Daily 1 min RV series

Date: 03/31/16 Time: 14:04
Sample: 1 248
Included observations: 124

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.474	0.474	28.496	0.000
		2	0.464	0.310	56.125	0.000
		3	0.510	0.301	89.694	0.000
		4	0.262	-0.154	98.603	0.000
		5	0.330	0.078	112.87	0.000
		6	0.370	0.161	131.01	0.000
		7	0.203	-0.056	136.53	0.000
		8	0.231	-0.050	143.75	0.000
		9	0.220	0.011	150.35	0.000
		10	0.077	-0.079	151.16	0.000
		11	0.126	-0.010	153.35	0.000
		12	0.090	-0.024	154.48	0.000
		13	0.002	-0.041	154.48	0.000
		14	-0.000	-0.090	154.48	0.000
		15	-0.023	-0.015	154.56	0.000
		16	-0.138	-0.115	157.31	0.000
		17	-0.101	-0.026	158.82	0.000
		18	-0.073	0.071	159.60	0.000
		19	-0.129	0.021	162.05	0.000
		20	-0.147	-0.115	165.28	0.000

Figure F.28: GARAN - Correlograms of session and daily RV series under CTS for different sampling intervals

  ii. PACFs of session RVs differ slightly from one another depending on the frequency. For frequency 1 min, lags 1,2,4 and 6 are significant in PACF, whereas lags 2 and 1,2 and 6 are significant (on the edge) for 5 min and 10 min frequencies, respectively.

  iii. PACFs of daily RVs differ slightly from one another depending on the frequency. For frequency 1 min, lags 1,2 and 3 are significant in PACF, whereas lags 1 and 3 and lags 1, 3, and 6 are significant for 5 min and 10 min frequencies, respectively.

- Slow decay in some of the ACFs calls for stationarity tests.
- All of these observations hold under all cleaning methods and aggregation algorithms.

b) **Stationarity-Unit root test:**

- To test for stationarity and unit root, i.e. if the series move around a constant mean or diverge as time passes, Augmented Dickey Fuller (ADF) Test is preferred. By visual inspection of graphs, no trend is observed in any of our RV series, therefore, ADF Test is run with an intercept and no trend, the number of legs to be involved in the analysis is chosen by Schwarz criterion as it is the default choice suggested by E-views.

- **TTS-Raw-:** In the E-views setting, where number of lags are optimized by E-views according to Schwarz criterion, R-squared values vary in a band of 29-55%. The null of nonstationarity is rejected at 1 and 5% significance level for all session and daily series[44].

- **CTS:** In the E-views setting, where number of lags are optimized by E-views according to Schwarz criterion, R-squared values has a wide range of 39% to 57%. At 1% significance level, all RV series, either session or daily and at 5 and 10 min frequencies, are found to be stationary. However, both of session and daily 1 min RV series under cleaning method 4.ii.a turn out to be nonstationary at 5 % significance level. Interestingly, 1 min session or day based series become stationary under cleaning method 4.ii.b.

[44] *In MIGRS analysis, significance level of rejection regarding nonstationarity increases when we switch to Daily series. Here, switching between Daily or session series does not affect p-values of test statistic (like ISCTR)*

Table F.3: GARAN - p-values of ADF Test on session and daily RV series under various cleaning and aggregation method combinations for different sampling intervals

Frequency	Session Based /Daily	Cleaning and Aggregation Method Combination									
		4.ii.a-5.i	4.ii.a-5.ii	4.ii.a-5.iii	4.ii.a-5.iv	4.ii.a-5.v	4.ii.b-5.i	4.ii.b-5.ii	4.ii.b-5.iii	4.ii.b-5.iv	4.ii.b-5.v
1min	Sess. Based	0.0889	0.0947	0.1009	0.0971	0.0971	0.0001	0.0001	0.0001	0.0001	0.0001
	Daily	0.1209	0.1247	0.1293	0.1178	0.1178	0.0293	0.0285	0.0305	0.0341	0.0347
5min	Sess. Based	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	Daily	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
10min	Sess. Based	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	Daily	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

ISCTR SUMMARY AND REVIEW OF CHAPTER 4 RESULTS

1) UHFD Characteristics Under Different Sampling Schemes and Error Cleaning and Data Filtering Combinations

a) <u>Irregular Temporal Spacing</u>

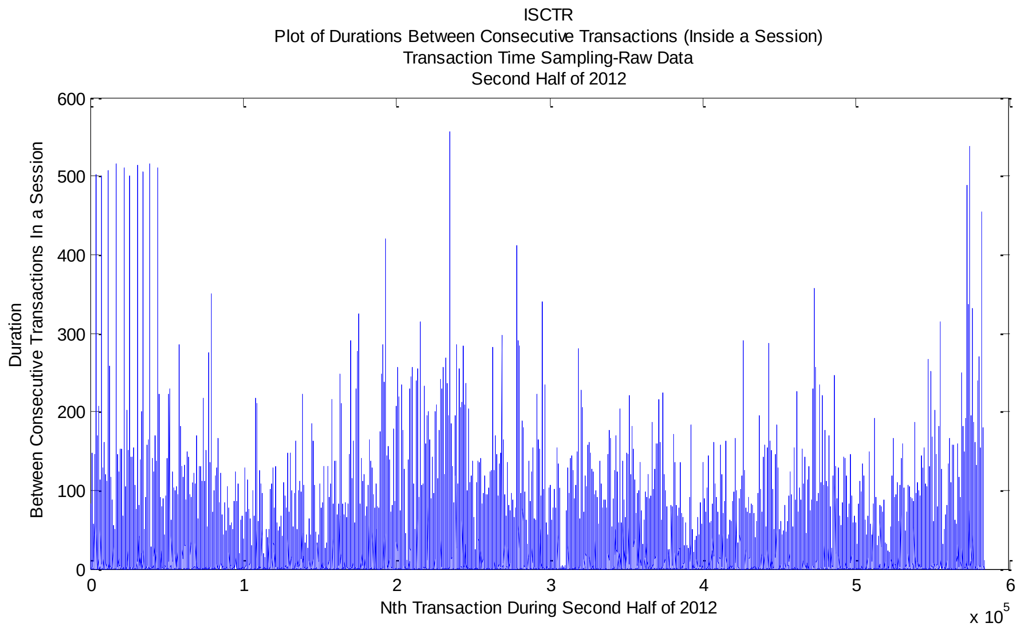


Figure F.29: Plot of durations between consecutive transactions (inside a session) for ISCTR TTS-raw data throughout the second half of 2012.

b) <u>Temporal dependence</u>: By comparing autocorrelation and partial autocorrelation functions of 600 seconds[45] absolute returns and log returns under CTS(clean and aggregated and interpolated) as well as absolute returns, log returns and durations in seconds from one transaction to the next under TTS(raw versus clean and aggregated) for December of 2012, we see that there are differences between ACF and PACF structure of absolute and log returns between 10 min CTS and 1 transaction TTS, i.e.: transforming 1 transaction sampled data by first cleaning, then aggregating and then interpolating (all needed for CTS) to 600 second sampled data distorts ACF and PACF of return series.

- TTS-Raw-Durations: ACF (very very slowly decaying positive significant up to 20 lags) and PACF (hyperbolic decay, significant up to 20 lags) (shocks persist)

---

[45] *Since first order autocorrelation was observed in 10 min returns under all cleaning and aggregation methods under CTS, we did not feel the urgency to check for 1 min returns under CTS. Recall that we included 1 min returns under CTS for MIGRS just because 10 min log returns exhibited no autocorrelation at all.*

- TTS-Raw-Absolute Returns: ACF (very very slow decay) and PACF (decaying positive and significant up to 16-20 lags) (shocks persist)

- TTS-Raw-Log returns: ACF (quick decay, first three lags negative-positive-negative significant) PACF(slower hyperbolic decay, first 14 lags significant)

- TTS(4.ii.a and b-5.i-5.ii-5.iii-5.iv-5.v)-Durations: ACF (very very slowly decaying positive and significant up to 20 lags) and PACF (hyperbolic decaying positive and significant up to 18 to 20 lags) (shocks persist)

- TTS (4.ii.a and b-5.i-5.ii-5.iii-5.iv-5.v)-Absolute returns: ACF (decaying positive and significant up to 20 lags) and PACF (decaying positive and significant up to 18-20 lags) (shocks persist)

- TTS (4.ii.a and b-5.i-5.ii-5.iii-5.iv-5.v)-Log returns: ACF (quick decay, first two-three lags negative-positive-negative significant) PACF(slower hyperbolic decay, first 10-12 lags negative significant)

- CTS-Durations: Meaningless, after interpolation duration from one entry to the next is always 1 second.

- CTS (4.ii.a and b-5.i-5.ii-5.iii-5.iv-5.v)-10 min Absolute Returns: ACF (first 2 lags are positive significant), PACF(positive significant up to 2nd lag)

- CTS (4.ii.a and b-5.i-5.ii-5.iii-5.iv-5.v)-10 min Log returns: ACF (only first lag is negative significant) and PACF (first two legs are negative significant)

Under TTS, with raw or clean and aggregated data, there is significant positive autocorrelation up to 20 lags in absolute returns, significant up to third order autocorrelation in log returns and very significant positive autocorrelation up to 20 lags in seconds elapsed between two transactions, thus volatility clustering is verified. Whereas, for 10 min returns under CTS, log returns display first and second order autocorrelation, which is in conformity with evidence laid out by the finance literature in general, that very short term returns exhibit strong autocorrelation especially on the first lag. Absolute return autocorrelation structure is changed under CTS at 600 seconds sampling interval compared to results under TTS at 1 transaction interval. Likewise, switching to CTS and calculation returns at 600 seconds suppresses partial autocorrelation figures at several lags of both absolute and log returns. Meanwhile, comparing data handling combinations to each other, any combination of cleaning methods and aggregation methods (compared to other combinations) does not cause any major change in total and partial correlation structures once we move under a sampling scheme, it being either TTS or CTS.

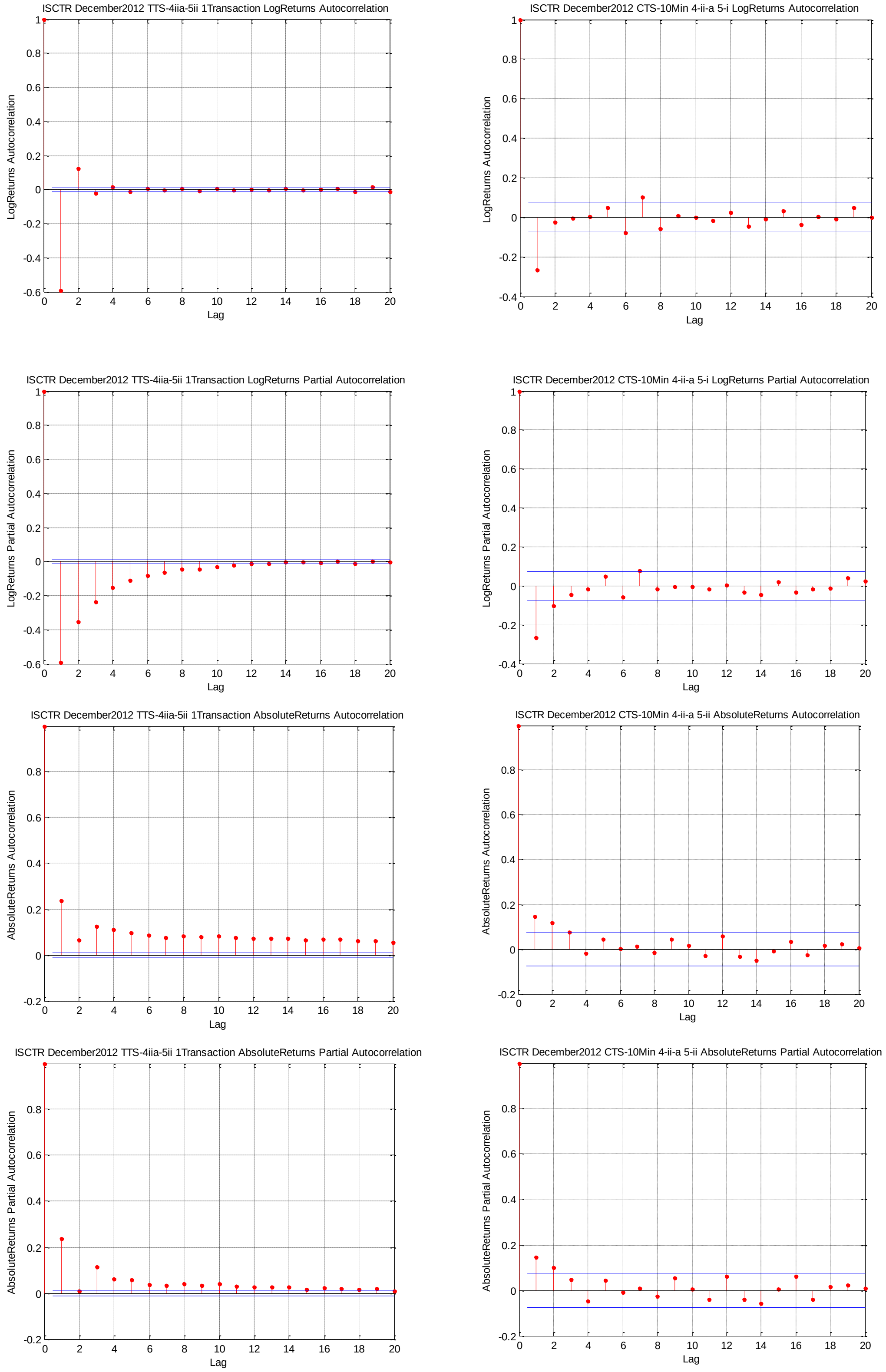


Figure F.30: ACFs and PACFs of logreturn and absolute return series of ISCTR for December 2012 under TTS and CTS

c) Diurnal Patterns: These patterns can be sought only under CTS because of their definitions such as number of trades per x minutes or absolute return per y seconds. For ISCTR case, there are strong W shapes which are persistent across cleaning and aggregation methods in 10 minutes trade volumes and 10 minutes trade intensities throughout days in second half of 2012, whereas patterns in 10 minutes absolute returns are closer to W without last spike at the end of the day. 10 minutes absolute percentage returns strongly exhibit L shape. All in all, there are significant diurnal patterns in returns and trading activity in the form of intensity and volume under CTS and these patterns look exactly same when various combinations of cleaning and aggregation methods are applied.

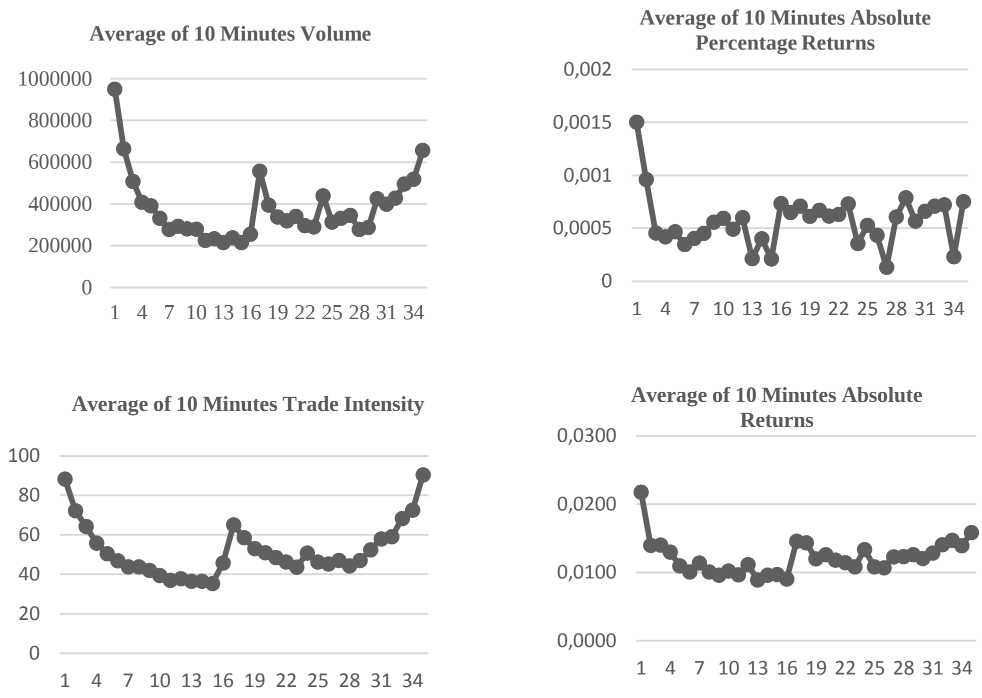


Figure F.31: Diurnal patterns - ISCTR cleaned and aggregated transaction data under CTS

2) Visual and Formal Statistical Tests of Existence and Statistical Features of Market Microstructure Noise

a) VSP: Regardless of the sampling schemes or cleaning and aggregation techniques combinations, average realized volatility of return on transaction price explode as we increase the sampling frequency, either in seconds or in transactions. Explosion becomes trivial for the sampling intervals that are less than 200 seconds or 15 transactions. This observation is valid both for session and daily figures, serving as a visual proof regarding existence of market microstructure noise and pointing to a positive relationship between noise increment and true price return, both under CTS and TTS even if the data set is cleaned or aggregated. At this point, we would like to emphasize that for VSPs, we skipped 4.ii.a-5.i-5.ii-5.iii-5.iv-5.v combinations under TTS, mainly because the number of cleaned points under 4.ii.a is so small, cleaning makes no real difference comparing to no cleaning of the data set. Any possible difference might have been observed under cleaning method 4.ii.b, which ended up deleting more data points. Moreover, since we compare 4.ii.a and 4.ii.b under CTS, we additionally search for any marginal effect that cleaning method 4.ii.b has over cleaning method 4.ii.a. However, as put forward previously, cleaning or aggregation does not affect the result that market microstructure becomes dominant after 15 transactions under TTS and 200 seconds under CTS and that the shape of VSP suggest a positive correlation between noise increment and true price return.

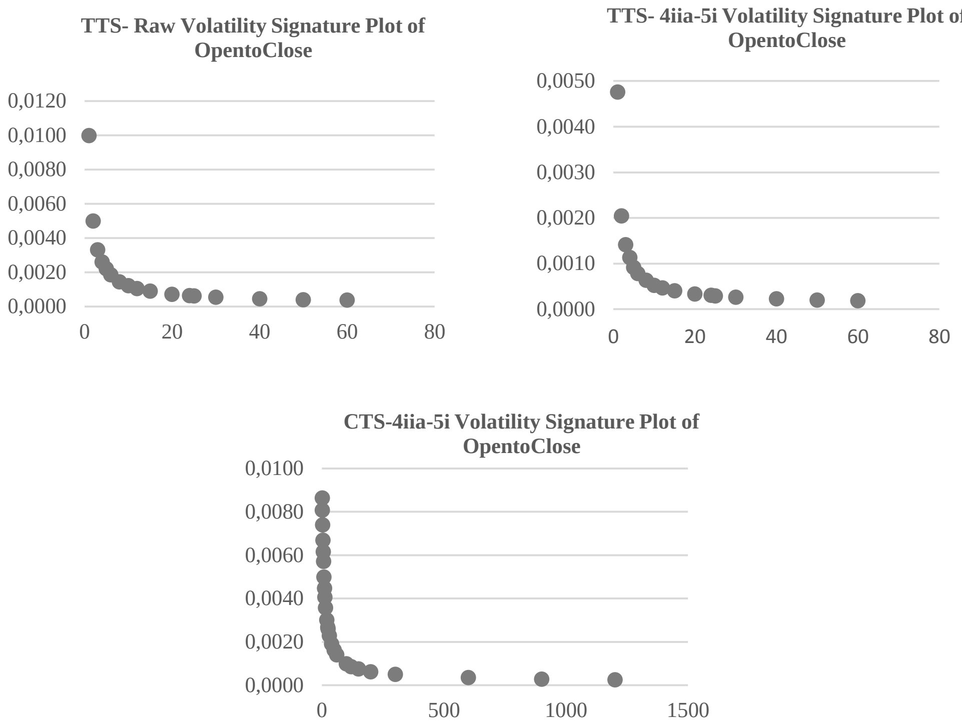


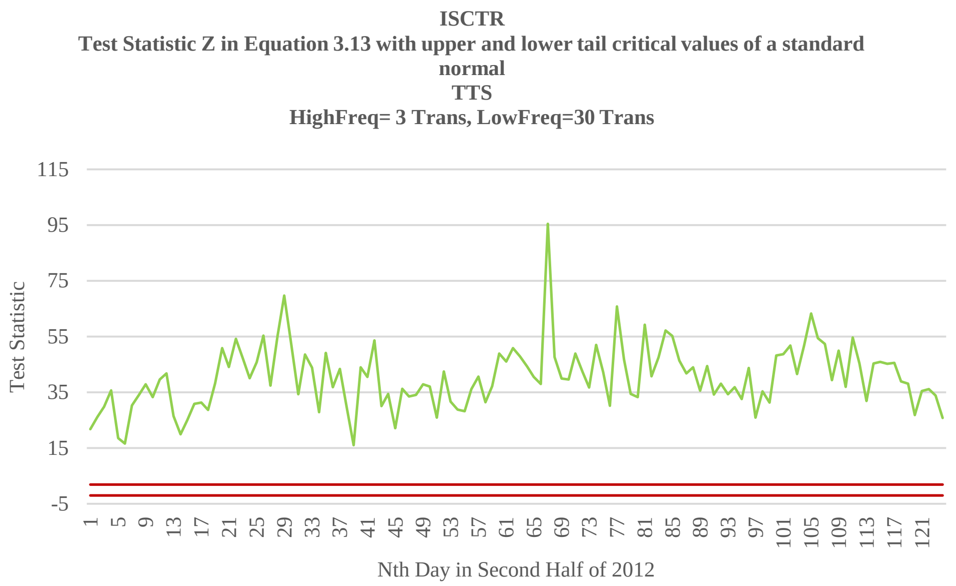
Figure F.32: VSPs of ISCTR over Daily RVs using clean and aggregated data under CTS, raw data under TTS, and clean and aggregated data under TTS.

b) <u>Statistical Tests Regarding Existence and Statistical Features of MMN :</u>

- Existence of MMN is verified statistically under both of CTS and TTS. We calculated $Z_{T,n,h}$ testing null hypothesis in (3.11) by comparing RVs that are calculated over different frequency pairs composed of high-low frequencies, namely (60,600) (10,1200), (30,1200) (60,1200), (150,1200), (300,1200), (600,1200) (900,1200) seconds under CTS and (3,30), (6,30), (10,30), (15,30), and (20,30) transactions under raw-TTS. Recall that bias of the RV estimator is dominated by expectation of square of the noise increment. Therefore, if we reject the null hypothesis, it means that the MMN has statistically significant impact on realized estimator of the IV.

For each day in the sample period of 124 days and each frequency pair, we run the aforementioned test at 5% significance level. Sample rejection percentages of null hypothesis are 100% under raw-TTS, 100% under clean and aggregated TTS and around 98% under CTS for all cleaning and aggregation method combinations when we compare RVs calculated over 3 and 30 transactions under TTS and 60 and 600 seconds under CTS. As we decrease the sampling frequency at the high frequency leg, rejection percentages of null hypothesis shrink, which is true under both of TTS and CTS. For raw-TTS, the rejection percentages begin with 100% and decrease gradually to 65% as high frequency leg moves toward 20 transactions when low frequency leg is 30 transactions. Cleaning and aggregating the data does not amend the downward trend in rejection percentages under TTS, but make it steeper. For all aggregation choices with cleaning method 4.ii.b applied under TTS, the rejection percentages begin with 100% and decrease gradually to 32% as high frequency leg moves toward 20 transactions. Switching to CTS as well as moving across the grid of cleaning and aggregation combinations do not change the results either. For CTS, the rejection percentages begin with around 100% for 10 to 1200 seconds pair and goes down the hill to 12% as high frequency legs are slowed to 900 seconds.

Following representative rejection rate graphs reveal that the MMN starts to accentuate as the sampling frequency converges to 10-15 transactions under TTS, and 250-300 seconds under CTS. These findings are in conformity with those supplied by the VSP analysis. The MMN is felt strongly once we cross over the sampling interval thresholds of 15 transactions or 5 minutes under TTS and CTS, respectively. For higher frequencies, rejection rates turn out to be quite high, especially after 150 seconds under CTS and 10 transactions under TTS, rejection rates become flat in a band of 95-100%. Moreover, visual inspection of the test statistic $Z_{T,n,h}$ for several frequency pairs either under TTS or CTS reveals that for the majority of the time, test statistic is positive and outside 5% st. normal confidence interval, meaning that there is positive correlation between noise and efficient price, which is again in conformity with the exploding VSPs.



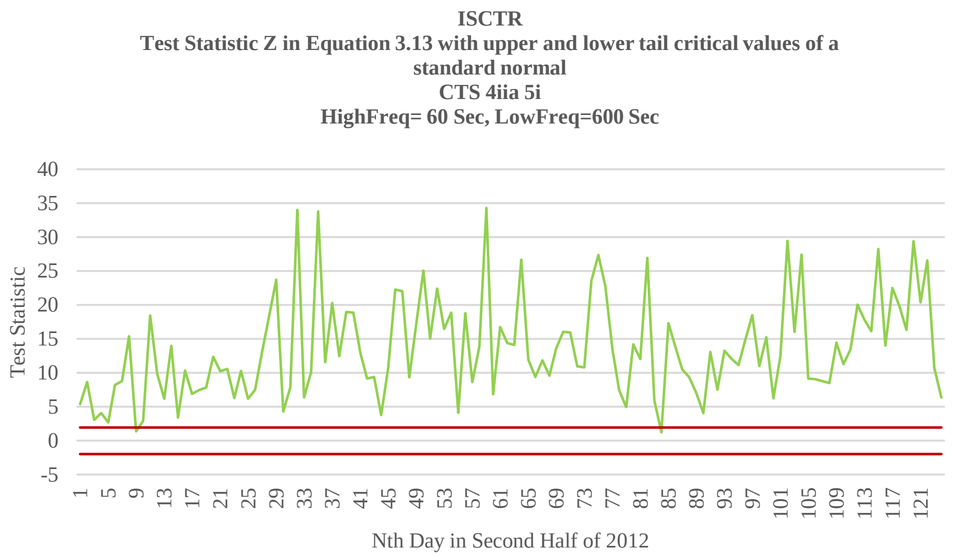
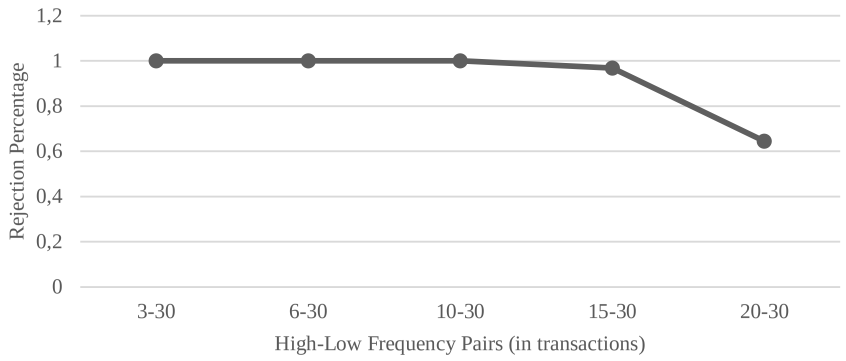
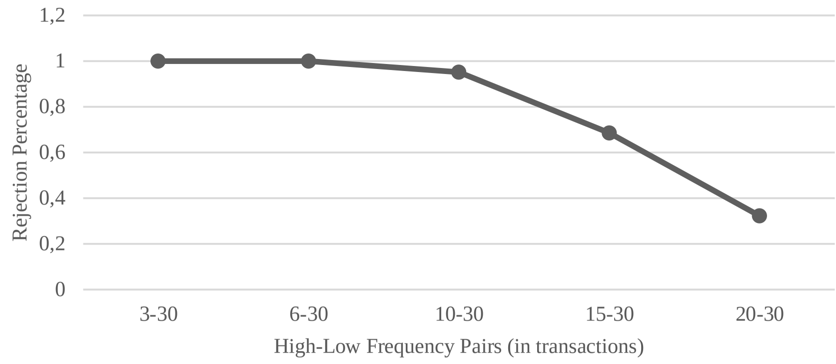


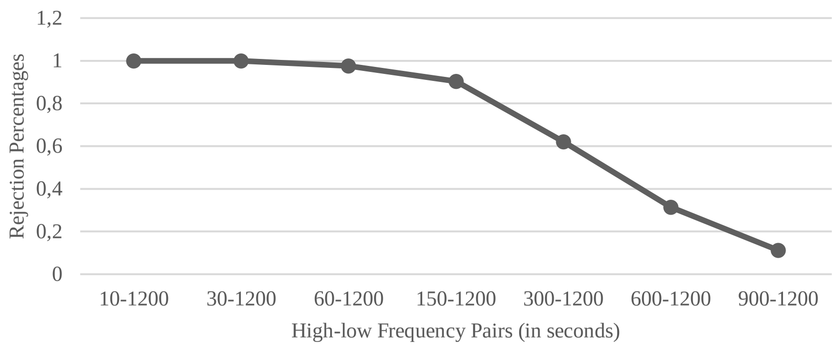
Figure F.33: ISCTR - Plots of $Z_{T,n,h}$ for each day in the sample period with upper and lower tail critical values of standard normal under TTS and CTS.



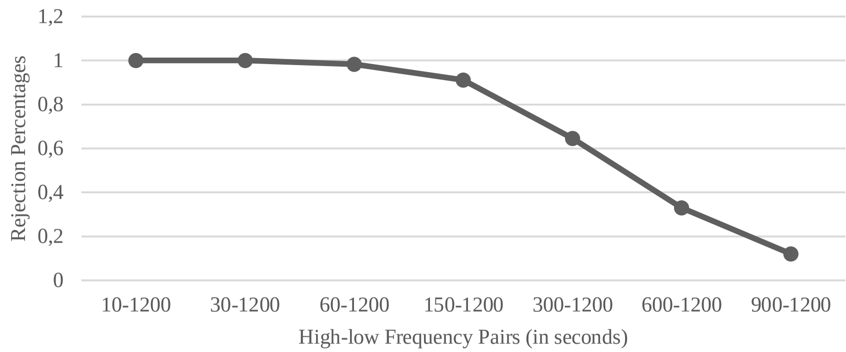
TTS-Raw Data



TTS-Clean and Aggregated Data (4iib-5i)



CTS-Clean and Aggregated Data (4iib-5i)



CTS-Clean and Aggregated Data (4iia-5i)

Figure F.34: ISCTR –Plots of rejection percentages with regards to the null hypothesis that the MMN does not have statistically significant effect on RV under TTS and CTS.

- **Summary:** Model of an i.i.d MMN with constant variance might be proper under CTS but under raw-TTS, for more than 50% of the days, null of constant variance is rejected for triples with very high frequencies combined with very low. This might be evidence of i.i.d assumption not holding at frequencies lesser 15 transactions. Sampling scheme, but not the aggregation method, is discovered to very influential on rejection of null hypothesis that MMN has variance independent of sampling frequency. Meanwhile, cleaning algorithms have some suppressive effect on rejection percentages particularly under TTS.

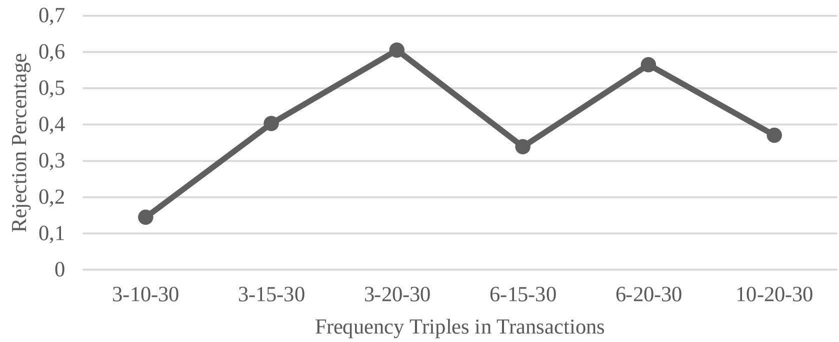
  Awartani et al. [16] derive a test with the idea that if the MMN has constant variance, then noise variances calculated over frequencies 1/M or 1/N should be same independent of M or N chosen. Their null and null hypotheses are as in (3.35) and (3.36).

  Since alternative hypothesis is in harmony with the presence of autocorrelation in MMN, by reminding corollary 3 of Hansen and Lunde [61], Awartani et al. [16] interpret the rejection of null hypothesis as a sign of the rejection of the null hypothesis that the MMN is a sequence of i.i.d random variables with constant variance. To test the validity of this null hypothesis, a test statistic compares RV differences using two frequency pairs, where pairs are M,L and N,L. L represents a frequency at which we can ignore the MMN safely, say 20 minutes and M and N are frequencies at which the MMN is considered to be significant. Therefore, the test is build on RVs calculated over frequency triples i.e. for each high frequency pair combined with 20 minutes, we test null hypothesis that E(noise increment square at low frequency)=E(noise increment square at high frequency). If we reject the null hypothesis, it means that the MMN has variance that is NOT independent of sampling frequency, therefore any assumptions regarding i.i.d nature of MMN can be taken as invalidated. Frequency triples are as follows: (3,10,30), (3,15,30), (3,20,30), (6,15,30) (6,20,30) and (10,20,30) transactions under TTS, (60,150,1200), (60,600,1200), (150,300,1200), (150,600,1200) and (300,600,1200) seconds under CTS.

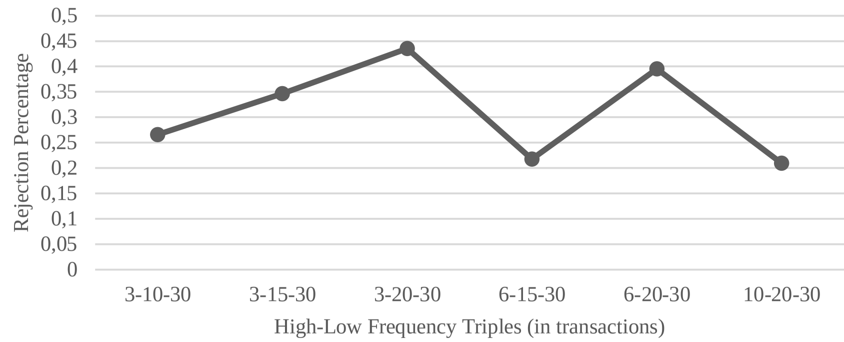
  For each day in the sample period of 124 days and each frequency triple, we run the aforementioned test at 5% significance level. Sample rejection percentages of null hypothesis clearly changes from one triple to another and as we clean and aggregate data. Beware that under raw-TTS especially for combinations of frequencies with highest differences between frequent legs, rejection percentages exceed 50%, while they stagger around 15% for 3-10-30 triple with lowest distance between first two legs. However, once we clean and aggregate the data, rejection percentages decline to levels 40% or 20% depending on the triple[46]. For CTS 4.ii.a (b), constant variance assumption rejection percentage varies between 2% and 8%, both

[46] *In a sense, these findings agree with findings for MIGRS case, where rejection percentages are highest for triples with distant constituents and TTS-raw data; however, MIGRS rejection percentages are way below those of ISCTR rejection percentages.*

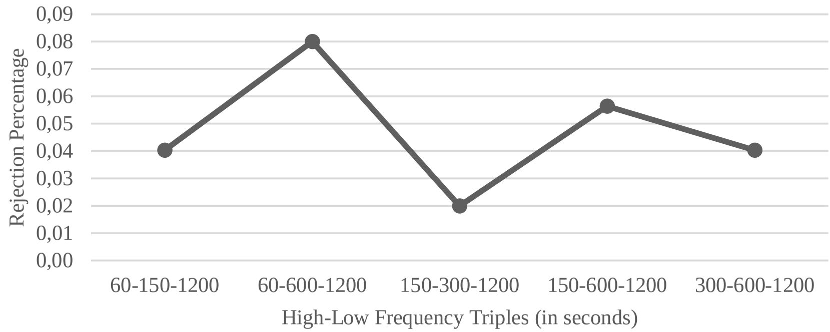
of which are just a fraction of rejection percentages under TTS-raw or TTS-cleaned. Therefore, sampling scheme is discovered to be very influential on rejection of the null hypothesis that the MMN has variance independent of sampling frequency. We can reject this null hypothesis under TTS confidently and conclude that assumption of an i.i.d MMN with constant variance does not reflect the real life structure of the MMN, whereas under CTS, such an assumption seems to hold especially for frequencies lower than 150 seconds. Evidence reveals that aggregation method does not affect rejection percentages and for triples with high frequency legs being close to very slow frequency leg, rejection percentages are severely damaged independent of the sampling scheme.



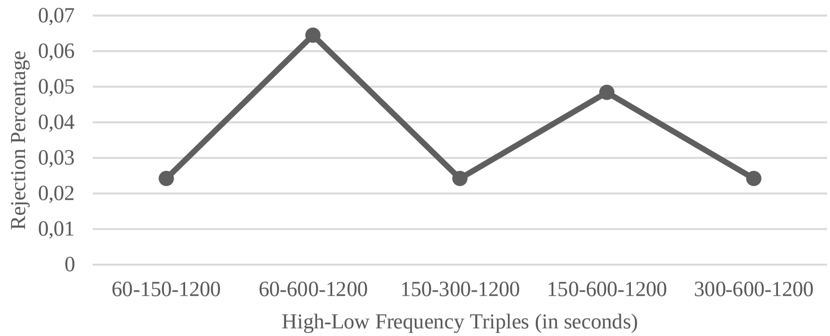
TTS-Raw Data



TTS-Clean and Aggregated Data (4iib-5i)



CTS-Clean and Aggregated Data (4iia-5i)



CTS-Clean and Aggregated Data (4iib-5i)

Figure F.35: ISCTR – Plots of rejection percentages with regards to the null hypothesis that the MMN increments have constant variance independent of sampling frequency under TTS and CTS.

3) RV Analysis

We constructed two RV time series, namely session RVs and daily RVs, for each frequency in a frequency set of 3, 6, 10, 15, 20, 30 transactions and 60, 300, 600 seconds under each sampling scheme (raw-TTS, CTS) and cleaning (4.ii.a, 4.ii.b) -aggregation method (5.i, 5.ii, 5.iii, 5.iv, 5.v) combination. Daily RV time series has 124 data points, whereas session RV time series is constituted of 248 entries. Each RV series under each sampling scheme and cleaning and aggregation method combinations is subjected to preliminary statistics, ACF and PACF analysis and lastly unit root is checked where autocorrelation exhibits slow decay.

- The factors that have any effect on RV series' normality and autocorrelation structure turn out to be whether the RV is on a session or daily basis, whether it is under raw-TTS or CTS and the frequency at which the RV is calculated. Regarding lognormality, frequency and sampling scheme are found to be influential. For all frequencies, the session and daily RV series are not normally distributed under raw-TTS as very high skewness, kurtosis and JB statistic values reveal. Switching to CTS and increasing frequency makes RV series more and more normal such that at 1 min frequency (session or daily), we cannot reject null hypothesis of RV sample coming from a normally distributed population at 5 or 1% significance levels. Taking logarithm of RV series converts them to normal for all frequencies under raw-TTS and for 10 min frequency under CTS[47].

- Decreasing frequencies cause less number of lags being significant with lesser significant levels under CTS, i.e. decreasing frequency again depresses autocorrelation structure of RV series regardless of session-daily calculation, which is in line with the existence of MMN. However, the suppression effect is not evident under raw-TTS. Moreover, calculating RVs on a session basis makes the RV series more autocorrelated under CTS. Regarding raw-TTS, only daily/session calculation of RV is found to have effect on correlogram such that daily RVs at all frequencies have some significant negative autocorrelations at lags that are greater than 14.

- Once we are working on a daily or session series at a particular frequency under CTS, cleaning and aggregation methods do not alter RV series' non-normality/normality or autocorrelation structure.

- Neither sampling schemes, nor frequencies or cleaning/aggregation methods or session/daily basis choice affect the stationarity results as E-views ADF Test results reveal that we can reject null of unit root at 5%significance level for all RV series under TTS or CTS at all frequencies[48]. On the contrary, if we run ADF test for fixed lag length (2) and intercept in MATLAB, test results leads us not to reject null hypothesis of unit root for 1 min and 5 min daily RV series under CTS.

[47] *Unlike the case of MIGRS.*
[48] *Unlike the case of MIGRS.*

a) **Descriptive statistics by frequency, by sampling scheme and by cleaning and aggregation methods:**

- **TTS-Raw:** For all frequencies, the session and daily RV series are not normally distributed[49] as very high skewness, kurtosis and JB statistic values reveal. Mean of the session and daily RVs become smaller as the sampling interval is lengthened, which is accompanied by a decrease in skewness, kurtosis and JB statistic as we sample less frequently. Still, normality of the any of these series is out of question. Correlogram of all session RV series look alike but are not exactly same. Generally speaking, ACFs and PACFs of RVs are decaying but not hyperbolically such that total and partial autocorrelations are strong at even lags and weak at odd legs (ACF positive significant up to 12th - 14th and PACF positive or negative significant selectively at legs, 1, 2, 4, 6 and 11)[50]. Although correlograms of all daily RVs resemble one another, compared to correlogram of session series, autocorrelation structure of daily RVs looks different. Now, a quick decay with first two lags being positive significant in PACF is evident, while decay in ACF starts from significant positive values, hit 0, then become negative significant where first 7-8 legs are positive significant, legs 8-12 are not significant and legs 13-20 are negative significant. The change in autocorrelation structure of RV series by looking at session and daily RVs separately, calls for stationarity test and accordingly, we checked for unit roots in daily series to see if summing RV from session one and session two to reach daily RV distorts anything in RV stationarities at different frequencies.

[49] *By application of JBTEST in MATLAB, we are not able to reject log normality at all frequencies.*
[50] *Unlike the case of MIGRS.*

TTS- Raw-Session-Frequency:3 Transactions

Date: 05/03/16 Time: 14:22
Sample: 1 248
Included observations: 248

	AC	PAC	Q-Stat	Prob
1	0.331	0.331	27.431	0.000
2	0.440	0.372	76.270	0.000
3	0.255	0.053	92.721	0.000
4	0.396	0.224	132.48	0.000
5	0.185	-0.048	141.20	0.000
6	0.367	0.177	175.63	0.000
7	0.188	-0.007	184.72	0.000
8	0.368	0.159	219.77	0.000
9	0.176	-0.023	227.83	0.000
10	0.337	0.095	257.49	0.000
11	0.086	-0.133	259.40	0.000
12	0.203	-0.048	270.28	0.000
13	0.100	0.021	272.90	0.000
14	0.219	0.027	285.64	0.000
15	0.038	-0.062	286.03	0.000
16	0.124	-0.079	290.18	0.000
17	-0.042	-0.110	290.64	0.000
18	0.015	-0.124	290.70	0.000
19	-0.070	-0.011	292.03	0.000
20	0.055	0.051	292.86	0.000

TTS- Raw-Daily-Frequency:3 Transactions

Date: 05/03/16 Time: 14:29
Sample: 1 248
Included observations: 124

	AC	PAC	Q-Stat	Prob
1	0.557	0.557	39.455	0.000
2	0.456	0.212	66.140	0.000
3	0.396	0.118	86.385	0.000
4	0.404	0.156	107.61	0.000
5	0.364	0.062	125.00	0.000
6	0.233	-0.116	132.19	0.000
7	0.224	0.026	138.86	0.000
8	0.077	-0.190	139.66	0.000
9	-0.041	-0.205	139.89	0.000
10	0.006	0.085	139.89	0.000
11	-0.062	-0.073	140.42	0.000
12	-0.117	-0.070	142.32	0.000
13	-0.173	0.009	146.52	0.000
14	-0.186	-0.034	151.42	0.000
15	-0.246	-0.125	160.11	0.000
16	-0.231	0.064	167.82	0.000
17	-0.221	-0.041	174.97	0.000
18	-0.259	-0.105	184.85	0.000
19	-0.252	0.049	194.28	0.000
20	-0.265	-0.073	204.86	0.000

TTS- Raw-Session-Frequency:30 Transactions

Date: 05/03/16 Time: 14:27
Sample: 1 248
Included observations: 248

	AC	PAC	Q-Stat	Prob
1	0.299	0.299	22.436	0.000
2	0.491	0.442	83.315	0.000
3	0.235	0.028	97.256	0.000
4	0.468	0.285	152.89	0.000
5	0.181	-0.056	161.28	0.000
6	0.442	0.207	211.28	0.000
7	0.144	-0.078	216.62	0.000
8	0.370	0.088	252.01	0.000
9	0.140	-0.011	257.10	0.000
10	0.381	0.118	294.95	0.000
11	0.051	-0.142	295.64	0.000
12	0.288	0.018	317.40	0.000
13	0.015	-0.094	317.46	0.000
14	0.216	-0.027	329.78	0.000
15	-0.033	-0.069	330.07	0.000
16	0.179	-0.009	338.63	0.000
17	-0.073	-0.047	340.05	0.000
18	0.107	-0.052	343.14	0.000
19	-0.086	-0.001	345.14	0.000
20	0.083	-0.024	347.02	0.000

TTS- Raw-Daily-Frequency:30 Transactions

Date: 05/03/16 Time: 14:33
Sample: 1 248
Included observations: 124

	AC	PAC	Q-Stat	Prob
1	0.600	0.600	45.743	0.000
2	0.524	0.256	80.929	0.000
3	0.446	0.098	106.63	0.000
4	0.391	0.056	126.56	0.000
5	0.387	0.105	146.18	0.000
6	0.252	-0.128	154.60	0.000
7	0.165	-0.109	158.24	0.000
8	0.092	-0.074	159.39	0.000
9	0.011	-0.096	159.41	0.000
10	-0.001	0.002	159.41	0.000
11	-0.071	-0.048	160.11	0.000
12	-0.087	0.014	161.16	0.000
13	-0.166	-0.097	165.04	0.000
14	-0.244	-0.129	173.48	0.000
15	-0.256	-0.054	182.87	0.000
16	-0.232	0.047	190.67	0.000
17	-0.279	-0.093	202.03	0.000
18	-0.255	0.030	211.63	0.000
19	-0.272	-0.006	222.65	0.000
20	-0.307	-0.115	236.83	0.000

Figure F.36: ISCTR - Correlograms of session and daily RV series under TTS for different sampling intervals

- **<u>CTS:</u>** For 1 min frequency, we cannot reject the null hypothesis that the session and/or daily RV series come from a normally distributed population at 5% significance level, whereas rejection of such hypothesis for session and/or daily RV series at 5 and 10 min frequencies is evident by high skewness, kurtosis and JB statistic values. Like the case under TTS,

    i. mean of the session and daily RVs become smaller as the sampling interval is lengthened.

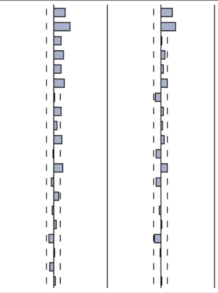
- However, contrary to findings for RV series under RAW-TTS,

    i. decrease in skewness, kurtosis and JB statistic values is observed as we sample more and more frequently, to a point that while session and daily 10 min RV series are not normal, JB tests on session and daily 1 min RV series fail to reject normality at 5% significance level.

ii. ACFs of session and daily RVs change as the sampling frequency changes, such that for increasing frequencies RV series exhibit significant positive total autocorrelation up to higher number of legs. Apart from this common trait, the decay patterns in total correlation of daily and session RVs are different, especially obvious at 1 min frequency.

CTS-4.ii.a-5.i-Session 10 min RV series

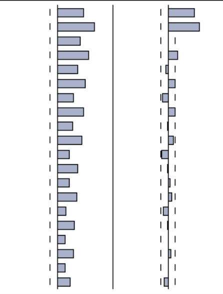
Date: 05/03/16 Time: 14:35
Sample: 1 248
Included observations: 248



Lag	AC	PAC	Q-Stat	Prob
1	0.227	0.227	12.892	0.000
2	0.327	0.291	39.836	0.000
3	0.144	0.030	45.094	0.000
4	0.195	0.084	54.746	0.000
5	0.141	0.057	59.796	0.000
6	0.220	0.129	72.155	0.000
7	0.024	-0.106	72.309	0.000
8	0.152	0.062	78.313	0.000
9	0.071	0.033	79.620	0.000
10	0.156	0.068	85.961	0.000
11	-0.002	-0.097	85.962	0.000
12	0.185	0.134	94.941	0.000
13	-0.045	-0.097	95.464	0.000
14	0.096	0.002	97.922	0.000
15	-0.023	-0.032	98.063	0.000
16	0.062	0.023	99.099	0.000
17	-0.093	-0.109	101.40	0.000
18	0.032	-0.002	101.68	0.000
19	-0.067	0.012	102.89	0.000
20	0.043	0.019	103.40	0.000

CTS-4.ii.a-5.i-Session 1 min RV series

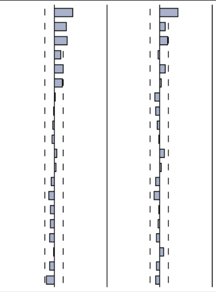
Date: 05/03/16 Time: 14:37
Sample: 1 248
Included observations: 248



Lag	AC	PAC	Q-Stat	Prob
1	0.496	0.496	61.753	0.000
2	0.689	0.588	181.54	0.000
3	0.439	0.013	230.32	0.000
4	0.588	0.186	318.24	0.000
5	0.381	-0.039	355.27	0.000
6	0.531	0.132	427.61	0.000
7	0.309	-0.102	452.13	0.000
8	0.492	0.134	514.62	0.000
9	0.292	-0.016	536.78	0.000
10	0.463	0.094	592.53	0.000
11	0.233	-0.116	606.71	0.000
12	0.385	-0.002	645.65	0.000
13	0.227	0.037	659.20	0.000
14	0.374	0.066	696.35	0.000
15	0.170	-0.096	704.00	0.000
16	0.315	-0.007	730.52	0.000
17	0.147	0.003	736.29	0.000
18	0.302	0.049	760.85	0.000
19	0.145	0.009	766.55	0.000
20	0.239	-0.072	782.03	0.000

CTS-4.ii.a-5.i-Daily 10 min RV series

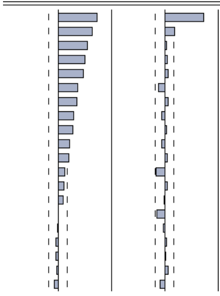
Date: 05/03/16 Time: 14:37
Sample: 1 248
Included observations: 124



Lag	AC	PAC	Q-Stat	Prob
1	0.376	0.376	17.982	0.000
2	0.238	0.112	25.236	0.000
3	0.255	0.157	33.613	0.000
4	0.135	-0.023	35.991	0.000
5	0.179	0.108	40.198	0.000
6	0.162	0.038	43.653	0.000
7	0.031	-0.087	43.784	0.000
8	-0.011	-0.071	43.799	0.000
9	-0.032	-0.046	43.936	0.000
10	-0.040	-0.013	44.153	0.000
11	0.057	0.100	44.608	0.000
12	0.046	0.033	44.908	0.000
13	-0.053	-0.075	45.306	0.000
14	-0.105	-0.103	46.874	0.000
15	-0.079	-0.013	47.761	0.000
16	-0.074	-0.025	48.553	0.000
17	-0.087	-0.056	49.654	0.000
18	-0.002	0.087	49.654	0.000
19	-0.089	-0.053	50.832	0.000
20	-0.146	-0.073	54.035	0.000

CTS-4.ii.a-5.i-Daily 1 min RV series

Date: 05/03/16 Time: 14:39
Sample: 1 248
Included observations: 124



Lag	AC	PAC	Q-Stat	Prob
1	0.759	0.759	73.267	0.000
2	0.663	0.203	129.53	0.000
3	0.574	0.047	172.11	0.000
4	0.517	0.057	206.87	0.000
5	0.485	0.079	237.72	0.000
6	0.391	-0.114	257.95	0.000
7	0.372	0.080	276.44	0.000
8	0.308	-0.055	289.19	0.000
9	0.289	0.043	300.55	0.000
10	0.234	-0.065	308.03	0.000
11	0.215	0.052	314.42	0.000
12	0.137	-0.157	317.06	0.000
13	0.114	0.056	318.90	0.000
14	0.097	-0.008	320.23	0.000
15	0.014	-0.140	320.26	0.000
16	-0.012	-0.022	320.28	0.000
17	-0.046	0.036	320.59	0.000
18	-0.034	0.018	320.76	0.000
19	-0.024	0.075	320.85	0.000
20	-0.067	-0.098	321.52	0.000

Figure F.37: ISCTR - Correlograms of session and daily RV series under CTS for different sampling intervals

i. PACFs of session RVs differ slightly from one another depending on the frequency. For frequency 1 min, lags 1, 2, 4 and 6 are significant in PACF, whereas lags 1, 2 and 4 and 1, 2 and 6 are significant (on the edge) for 5 min and 10 min frequencies, respectively.

ii. PACFs of daily RVs differ slightly from one another depending on the frequency. For frequency 1 min, lags 1 and 2 are significant in PACF, whereas lags 1, 2 and 13 and lags 1 and 3 are significant for 5 min and 10 min frequencies, respectively.

- Slow decay in some of the ACFs calls for stationarity tests.

- All of these observations hold under all cleaning methods and aggregation algorithms.

b) <u>Stationarity-Unit root test:</u>

- To test for stationarity and unit root, i.e. if the series move around a constant mean or diverge as time passes, Augmented Dickey Fuller (ADF) Test is preferred. By visual inspection of graphs, no trend is observed in any of our RV series, therefore, the ADF Test is run with an intercept and no trend, the number of legs to be involved in the analysis is chosen by the Schwarz criterion as it is the default choice suggested by E-views.

- **<u>TTS-Raw-:</u>** In the E-views setting, where number of lags are optimized by E-views according to the Schwarz criterion, R-squared values vary in a band of 46-55%. The null hypothesis of nonstationarity is rejected at 5% significance level for all session and daily series[51].

- **<u>CTS:</u>** In the E-views setting, where number of lags is optimized by E-views according to Schwarz criterion, R-squared values has a wide range of 11% to 54%. At 1% significance level, all RV series, either session or daily and at all frequencies, are found to be stationary.

[51] *In MIGRS analysis, significance level of rejection regarding nonstationarity incereases when we switch to Daily series. Here, switching between Daily or session series does not affect p-values of test statistic.*

Table F.4: ISCTR - p-values of ADF Test on session and daily RV series under various cleaning and aggregation method combinations for different sampling intervals

Frequency	Session Based /Daily	Cleaning and Aggregation Method Combination									
		4.ii.a-5.i	4.ii.a-5.ii	4.ii.a-5.iii	4.ii.a-5.iv	4.ii.a-5.v	4.ii.b-5.i	4.ii.b-5.ii	4.ii.b-5.iii	4.ii.b-5.iv	4.ii.b-5.v
1min	Sess. Based	0.0017	0.0018	0.0018	0.0015	0.0015	0.0003	0.0138	0.0145	0.0002	0.0130
	Daily	0.0018	0.0018	0.0033	0.0017	0.0017	0.0141	0.0166	0.0181	0.0173	0.0162
5min	Sess. Based	0.0004	0.0003	0.0004	0.0004	0.0003	0.0006	0.0006	0.0007	0.0007	0.0006
	Daily	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0006	0.0007	0.0006	0.0006
10min	Sess. Based	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	Daily	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

MIGRS SUMMARY AND REVIEW OF CHAPTER 4 RESULTS

1) UHFD Characteristics Under Different Sampling Schemes and Error Cleaning and Data Filtering Combinations

a) <u>Irregular Temporal Spacing</u>

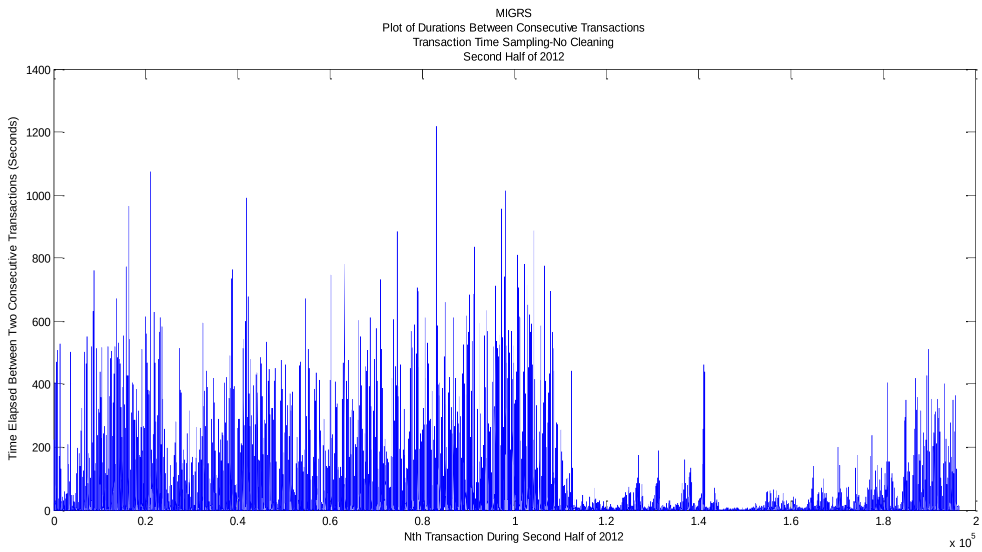


Figure F.38: Plot of durations between consecutive transactions (inside a session) for MIGRS TTS-raw data throughout the second half of 2012.

b) **<u>Temporal dependence</u>**: By comparing autocorrelation and partial autocorrelation functions of 60 and 600 seconds absolute returns and log returns under CTS (clean and aggregated and interpolated) as well as absolute returns, log returns and durations in seconds from one transaction to the next under TTS (raw versus clean and aggregated) for December of 2012, we see that there are differences between ACF and PACF structure of absolute and log returns between 10 min CTS and 1 transaction TTS, i.e.: transforming 1 transaction sampled data by first cleaning, then aggregating and then interpolating (all needed for CTS) to 600 second sampled data distorts ACF and PACF of return series.

- TTS-Raw-Durations: ACF and PACF very very slowly decaying positive and significant up to 20 lags (shocks persist)
- TTS-Raw-Absolute Returns: ACF and PACF decaying positive and significant up to 20 lags (shocks persist)
- TTS-Raw-Log returns: ACF (quick decay, first two lags negative significant) PACF (slower decay, first 10 lags significant)

- TTS(4.ii.a and b-5.i-5.ii-5.iii-5.iv-5.v)-Durations: ACF and PACF very very slowly decaying positive and significant up to 18 to 20 lags (shocks persist)

- TTS (4.ii.a and b-5.i-5.ii-5.iii-5.iv-5.v)-Absolute returns: ACF (decaying positive and significant up to 20 lags) and PACF (decaying positive and significant up to 8-10 lags)

- TTS (4.ii.a and b-5.i-5.ii-5.iii-5.iv-5.v)-Log returns: ACF (quick decay, first lag negative significant) PACF (slower decay, first 5 lags negative significant)

- CTS-Durations: Meaningless, after interpolation duration from one entry to the next is always 1 second.

- CTS (4.ii.a and b-5.i-5.ii-5.iii-5.iv-5.v)-10 min Absolute Returns: ACF (positive decaying, significant up to 10th lag), PACF (quick decay, positive significant up to 3rd lag)

- CTS (4.ii.a and b-5.i-5.ii-5.iii-5.iv-5.v)-10 min Log returns: ACF and PACF (no lag is significant at all)

- CTS (4.ii.a and b-5.i-5.ii-5.iii-5.iv-5.v)-1 min Absolute Returns: ACF (positive slowly decaying, significant up to 20th lag), PACF (slow decay, positive significant up to 12 lags)

- CTS (4.ii.a and b-5.i-5.ii-5.iii-5.iv-5.v)-1 min Log returns: ACF and PACF quick decay, first lag negative significant)

Under TTS, with raw or clean and aggregated data, there is significant positive autocorrelation up to 20 lags in absolute returns, significant up to third order autocorrelation in log returns and very significant positive autocorrelation up to 20 lags in seconds elapsed between two transactions, thus volatility clustering is verified. Whereas, for 10 min returns under CTS, log returns display no autocorrelation at all, which is quite contrary to general consensus in the finance literature, that very short term returns exhibit strong autocorrelation especially on the first lag. Thus, we check for ACF and PACF of 1 min log returns and observe negative first order autocorrelation. Absolute return autocorrelation structure is changed under CTS at 600 seconds sampling interval compared to results under TTS at 1 transaction interval. Likewise, switching to CTS and calculating returns at 600 seconds suppresses partial autocorrelation figures at several lags of both absolute and log returns. Meanwhile, comparing data handling combinations to each other, any combination of cleaning methods and aggregation methods (compared to other combinations) does not cause any major change in total and partial correlation structures once we move under a sampling scheme, it being either TTS or CTS. However, cleaning and aggregation under TTS yield different PACF structures in absolute and log returns compared to results produced with raw data. Under CTS, rather than cleaning and aggregation methods, sampling interval matters in terms of return autocorrelation structure.

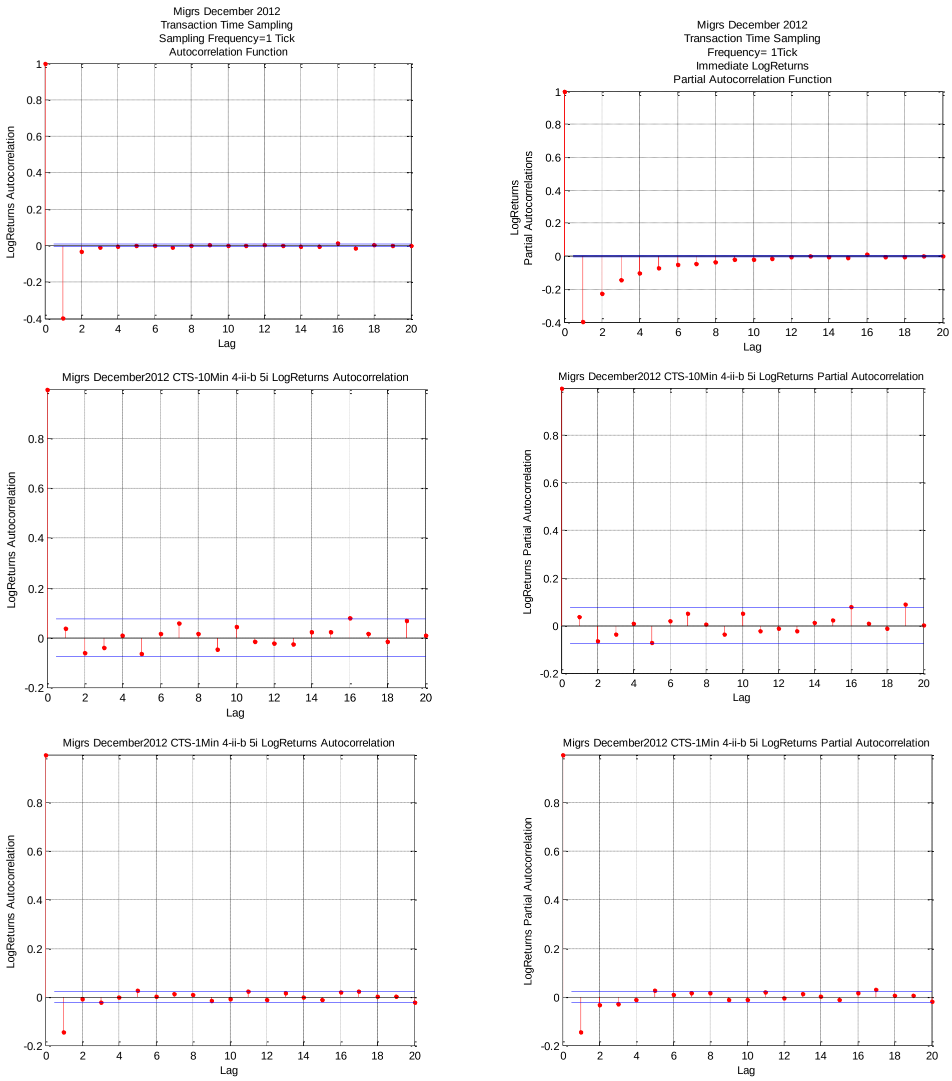


Figure F.39: ACFs and PACFs of logreturn and absolute return series of MIGRS for December 2012 under TTS and CTS

c) **<u>Diurnal Patterns</u>**: These patterns can be sought only under CTS because of their definitions such as number of trades per x minutes or absolute return per y seconds. For MIGRS case, there are strong W shapes which are persistent across cleaning and aggregation methods in 10 minutes trade volumes and 10 minutes trade intensities throughout days in second half of 2012, whereas patterns in 10 minutes absolute returns and 10 minutes absolute percentage returns are closer to W without last spike at the end of the day and a L shape, respectively. All in all, there are significant diurnal patterns in returns and trading activity in the form of intensity and volume under CTS and these patterns look exactly same when various combinations of cleaning and aggregation methods are applied.

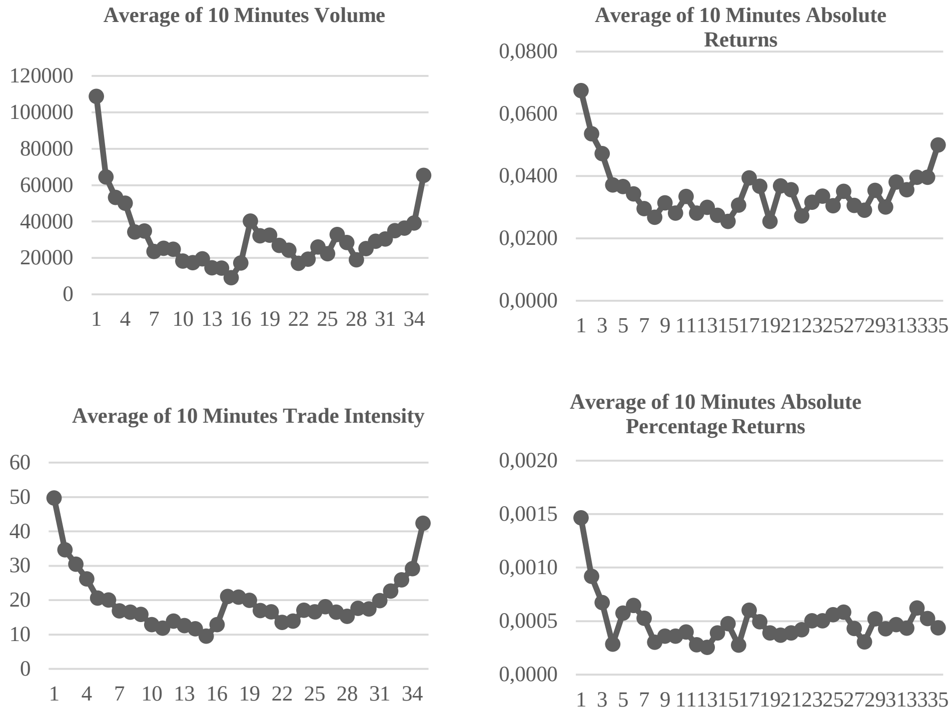


Figure F.40: Diurnal patterns at MIGRS cleaned and aggregated transaction data under CTS

2) Visual and Formal Statistical Tests of Existence and Statistical Features of Market Microstructure Noise

a) **<u>VSP:</u>** Regardless of the sampling schemes or cleaning and aggregation techniques combinations, average realized volatility of return on transaction price explode as we increase the sampling frequency, either in seconds or in transactions. Explosion becomes trivial for the sampling intervals that are less than 200 seconds or 15 transactions. This observation is valid both for session and daily figures, serving as a visual proof regarding existence of market microstructure noise and pointing to a positive relationship between noise increment and true price return, both under CTS and TTS even if the data set is cleaned or aggregated. At this point, we would like to emphasize that for VSPs, we skipped 4.ii.a-5.i-5.ii-5.iii-5.iv-5.v combinations under TTS, mainly because the number of cleaned points under 4.ii.a is so small, cleaning makes no real difference comparing to no cleaning of the data set. Any possible difference might have been observed under cleaning method 4.ii.b, which ended up deleting more data points. Moreover, since we compare 4.ii.a and 4.ii.b under CTS, we additionally search for any marginal effect that cleaning method 4.ii.b has over cleaning method 4.ii.b. However, as put forward previously, cleaning or aggregation does not affect the result that market microstructure becomes dominant after 15 transactions under TTS and 200 seconds under CTS and that the shape of VSP suggest a positive correlation between the noise increment and the true price return.

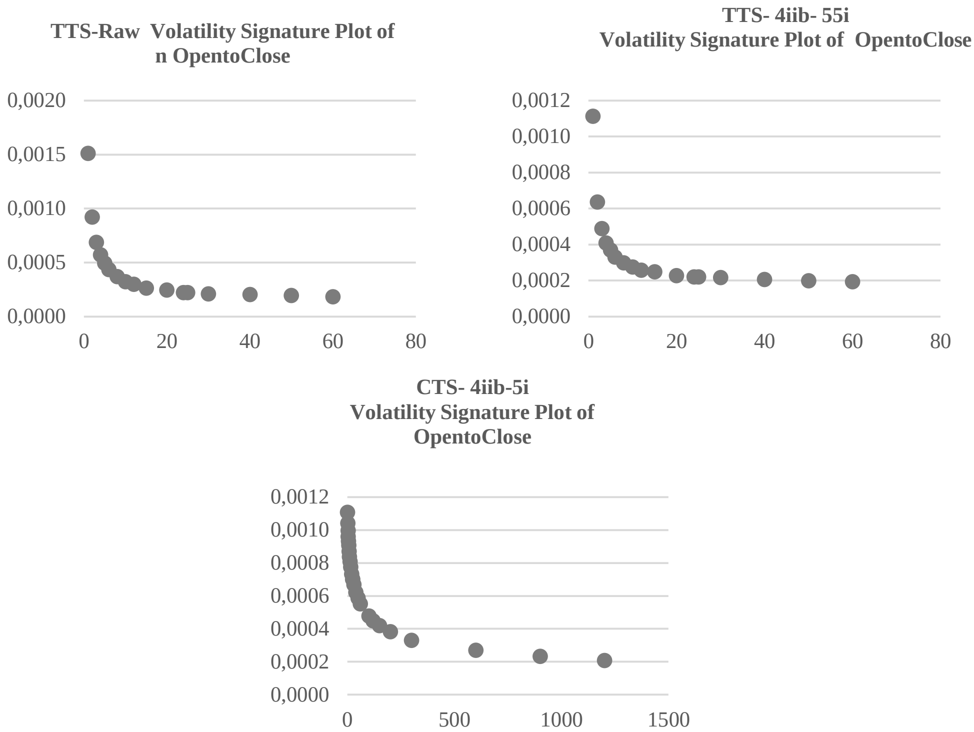


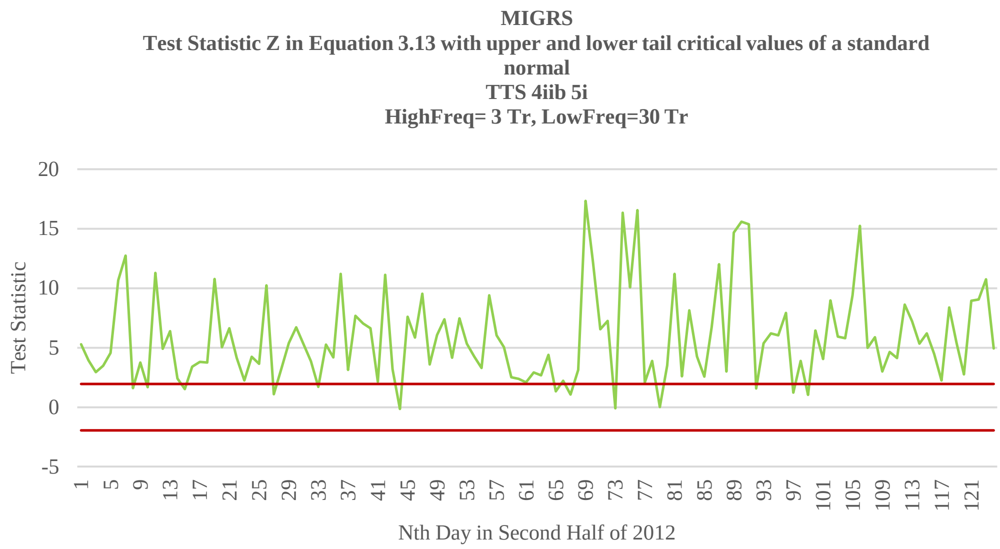
Figure F.41: VSPs of MIGRS over Daily RVs using clean and aggregated data under CTS, raw data under TTS, and clean and aggregated data under TTS.

b) <u>Statistical Tests Regarding Existence and Statistical Features of MMN :</u>

- Existence of MMN is verified statistically under both of CTS and TTS. We calculated $Z_{T,n,h}$ testing null hypothesis in (3.11) by comparing RVs that are calculated over different frequency pairs composed of high-low frequencies, namely (60,600) (10,1200), (30,1200) (60,1200), (150,1200), (300,1200), (600,1200) (900,1200) seconds under CTS and (3,30), (6,30), (10,30), (15,30), and (20,30) transactions under raw-TTS. Recall that bias of the RV estimator is dominated by expectation of square of the noise increment. Therefore, if we reject the null hypothesis, it means that the MMN has statistically significant impact on realized estimator of the IV.

  For each day in the sample period of 124 days and each frequency pair, we run the aforementioned test at 5% significance level. Sample rejection percentages of null hypothesis are 100% under raw-TTS, 90% under clean and aggregated TTS and around 91% under CTS for all cleaning and aggregation method combinations when we compare RVs calculated over 3 and 30 transactions under TTS and 60 and 600 seconds under CTS. As we decrease the sampling frequency at the high frequency leg, rejection percentages of null hypothesis shrink, which is true under both of TTS and CTS. For raw-TTS, the rejection percentages begin with 100% and decrease gradually to 26% as high frequency leg moves toward 20 transactions when lowfrequency leg is 30 transactions. Cleaning and aggregating the data does not amend the downward trend in rejection percentages under TTS. For all aggregation choices with cleaning method 4.ii.b applied under TTS, the rejection percentages begin with 80-90% and decrease gradually to 12% as high frequency leg moves toward 20 transactions. Switching to CTS as well as moving across the grid of CTS cleaning and aggregation combinations do not change the results either. For CTS, the rejection percentages begin with around 98% for 10 to 1200 seconds pair and goes down the slope to 19% as high frequency legs are slowed to 900 seconds.

Following representative rejection rate graphs reveal that the MMN starts to accentuate as the sampling frequency converges to 10-15 transactions under TTS, and 300 seconds under CTS. These findings are in conformity with those supplied by the VSP analysis. The MMN is felt strongly once we cross over the sampling interval thresholds of 15 transactions or 5 minutes under TTS and CTS, respectively. For higher frequencies, the rejection rates turn out to be quite high. Moreover, visual inspection of the test statistic $Z_{T,n,h}$ for several frequency pairs either under TTS or CTS reveals that for the majority of the time the test statistic is positive and outside 5% st. normal confidence interval, meaning that there is positive correlation between noise and efficient price, which is again in conformity with the exploding VSPs.



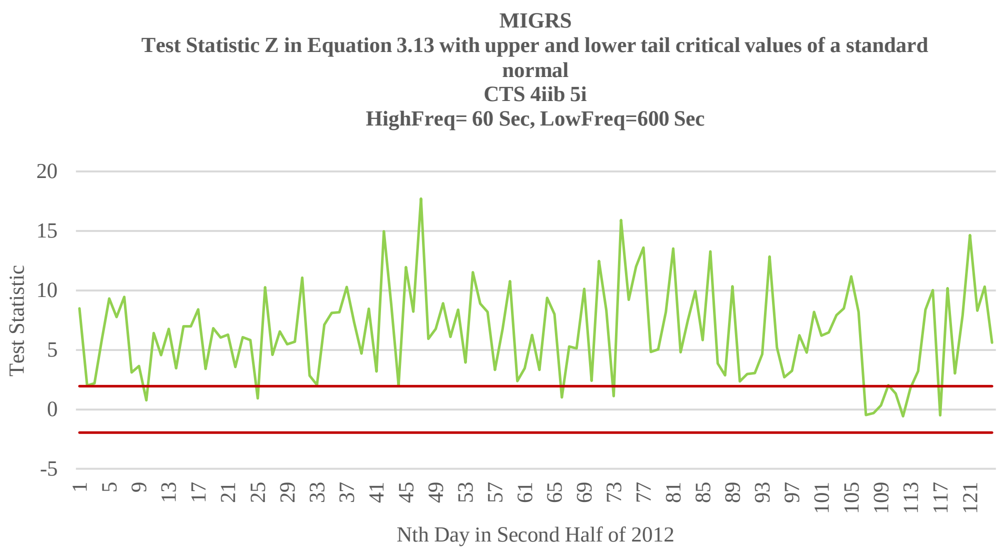
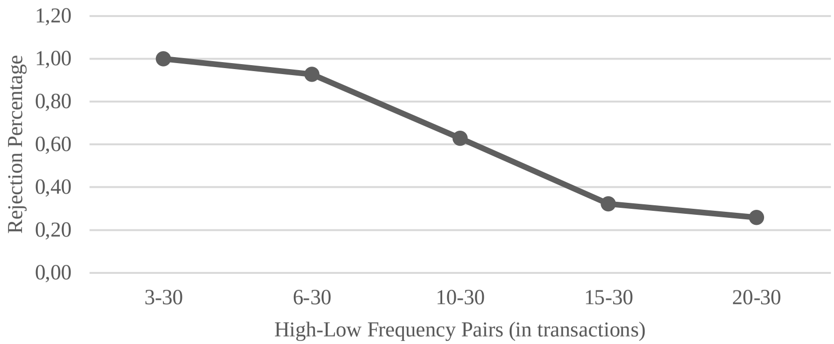
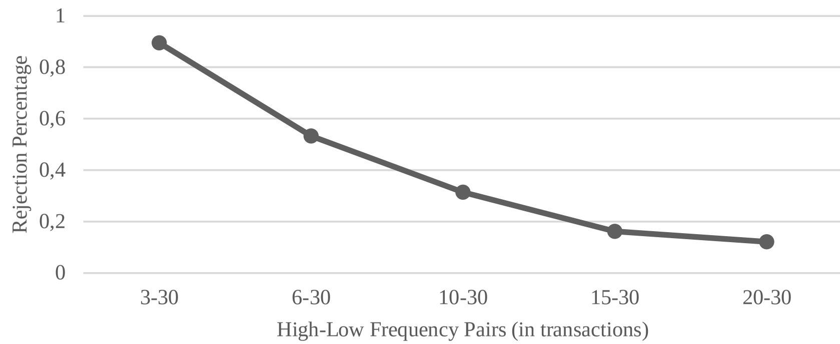


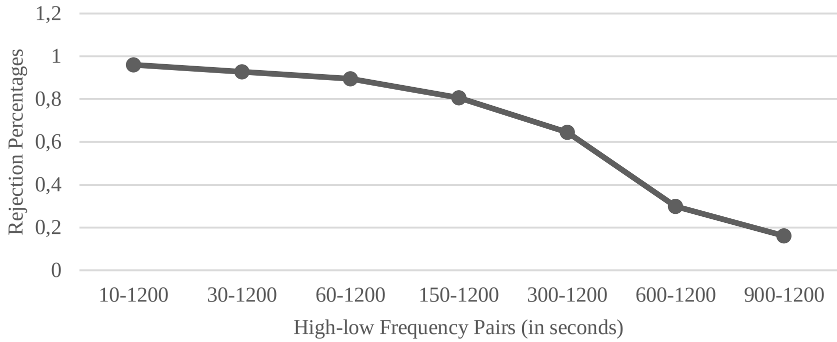
Figure F.42: MIGRS - Plots of $Z_{T,n,h}$ for each day in the sample period with upper and lower tail critical values of standard normal under TTS and CTS.



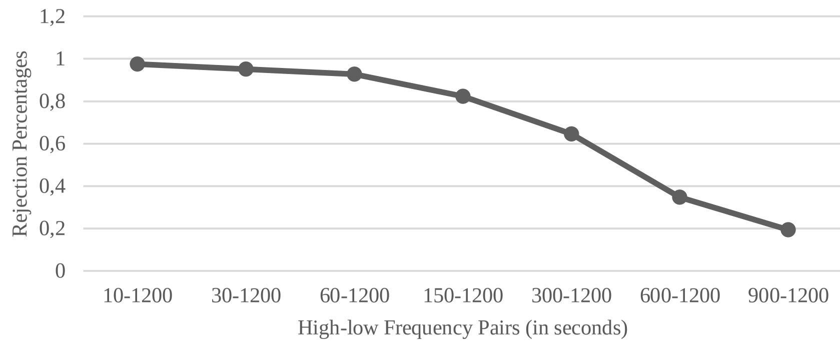
TTS-Raw Data



TTS-Clean and Aggregated Data (4iib-5i)



CTS-Clean and Aggregated Data (4iib-5i)



CTS-Clean and Aggregated Data (4iia-5i)

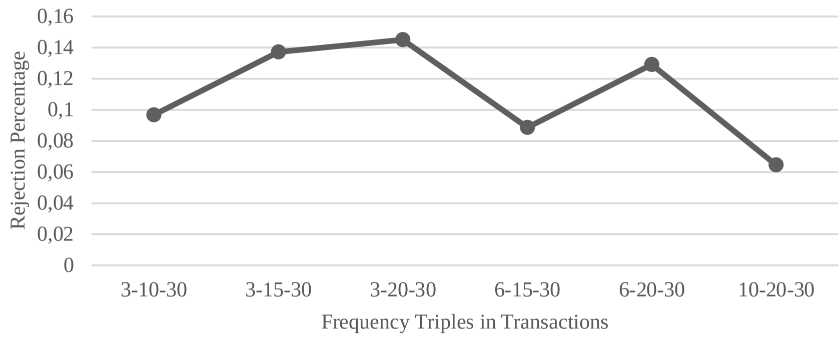
Figure F.43: MIGRS –Plots of rejection percentages with regards to the null hypothesis that the MMN does not have statistically significant effect on RV under TTS and CTS.

- **Summary**: Model of i.i.d MMN with constant variance might be proper under CTS but under raw-TTS, for more than 10% of the days, null of constant variance is rejected for triples with very high frequencies combined with very low. This might be evidence of i.i.d assumption not holding at frequencies lesser 15 transactions.

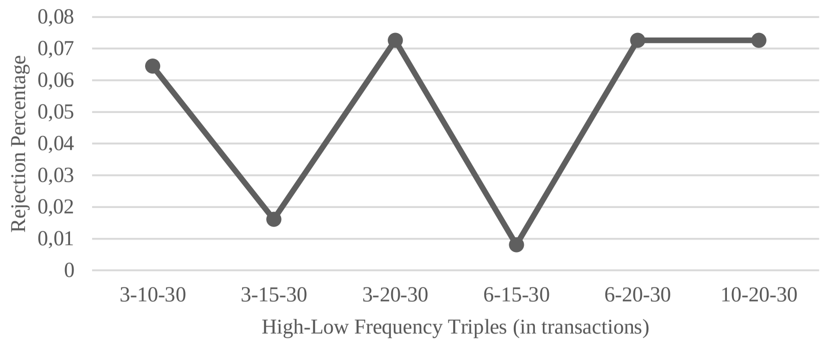
  Awartani et al. [16] derive a test with the idea that if the MMN has constant variance, then noise variances calculated over frequencies 1/M or 1/N should be same independent of M or N chosen. Their null and null hypotheses are as in (3.35) and (3.36).

  Since alternative hypothesis is in harmony with the presence of autocorrelation in the MMN, by reminding corollary 3 of Hansen and Lunde [61], Awartani et al. [16] interpret the rejection of null hypothesis as a sign of the rejection of the null hypothesis that the MMN is a sequence of i.i.d random variables with constant variance. To test the validity of this null hypothesis, a test statistic compares RV differences using two frequency pairs, where pairs are M,L and N,L. L represents a frequency at which we can ignore the MMN safely, say 20 minutes and M and N are frequencies at which the MMN is considered to be significant. Therefore, the test is build on RVs calculated over frequency triples i.e. for each high frequency pair combined with 20 minutes, we test null hypothesis that E(noise increment square at low frequency)=E(noise increment square at high frequency). If we reject the null hypothesis, it means that the MMN has variance that is NOT independent of sampling frequency, therefore any assumptions regarding i.i.d nature of MMN can be taken as invalidated. Frequency triples are as follows: (3,10,30), (3,15,30), (3,20,30), (6,15,30) (6,20,30) and (10,20,30) transactions under TTS, (60,150,1200), (60,600,1200), (150,300,1200), (150,600,1200) and (300,600,1200) seconds under CTS.

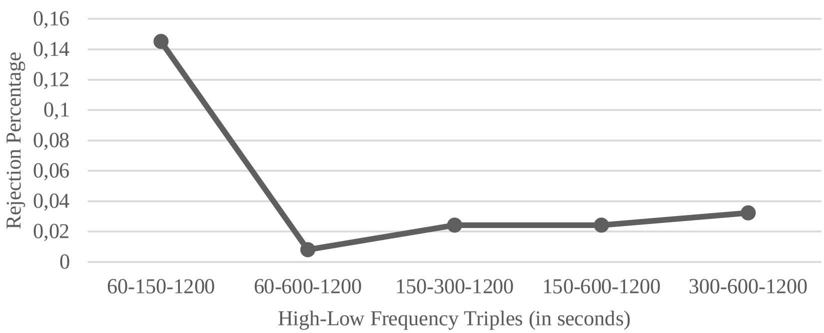
  For each day in the sample period of 124 days and each frequency triple, we run the aforementioned test at 5% significance level. Sample rejection percentages of null hypothesis clearly changes from one triple to another and as we clean and aggregate data. Beware that under raw-TTS especially for combinations of frequencies with highest differences, rejection percentages spike to 14%. However, once we clean and aggregate the data, rejection percentages decline to levels 7% or 1-2% depending on the triple. For CTS 4.ii.a (b), constant variance assumption rejection percentage is 14% (9%) for 60-150-1200 seconds triple, which represents the highest distance between high pair and very low third leg. Evidence reveals that the aggregation method does not affect the rejection percentages and for triples with high frequency legs being close to very slow frequency leg, the rejection percentages are severely damaged.



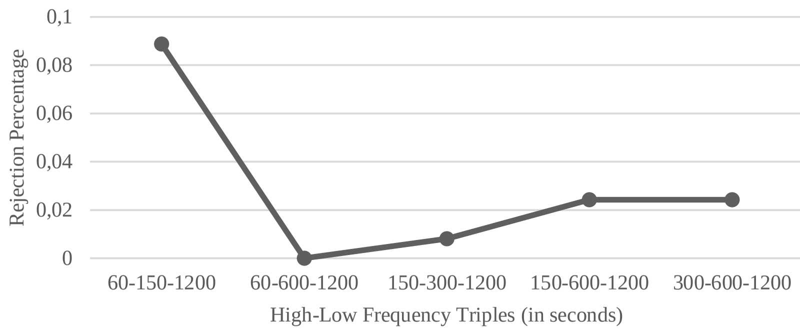
TTS-Raw Data



TTS-Clean and Aggregated Data (4iib-5i)



CTS-Clean and Aggregated Data (4iia-5i)



CTS-Clean and Aggregated Data (4iib-5i)

Figure F.44: MIGRS –Plots of rejection percentages with regards to the null hypothesis that the MMN increments have constant variance independent of sampling frequency under TTS and CTS.

3) RV Analysis

We constructed two RV time series, namely session RVs and daily RVs, for each frequency in a frequency set of 3, 6, 10, 15, 20, 30 transactions and 60, 300, 600 seconds under each sampling scheme (raw-TTS, CTS) and cleaning (4.ii.a, 4.ii.b) -aggregation method (5.i, 5.ii, 5.iii, 5.iv, 5.v) combination. Daily RV time series has 124 data points, whereas session RV time series is constituted of 248 entries. Each time RV series under each sampling scheme and cleaning and aggregation method combinations is subjected to preliminary statistics, ACF and PACF analysis and lastly unit root is checked where autocorrelation exhibits hyperbolic decay.

- The factors that have any effect on RV lognormality and autocorrelation structure turn out to be whether the RV is on a session or daily basis, whether it is under raw-TTS or CTS and the frequency at which the RV is calculated. Normality is affected by no dimension. For all frequencies, the session and daily RV series under raw-TTS or CTS are not normally distributed as very high skewness, kurtosis and JB statistic values reveal. Taking logarithm does not change the non-normality under CTS for any frequency or session/daily calculation choice. However, for log daily RV series at 10, 20 and 30 transaction under raw-TTS, we are not able to reject the null hypothesis of normality.

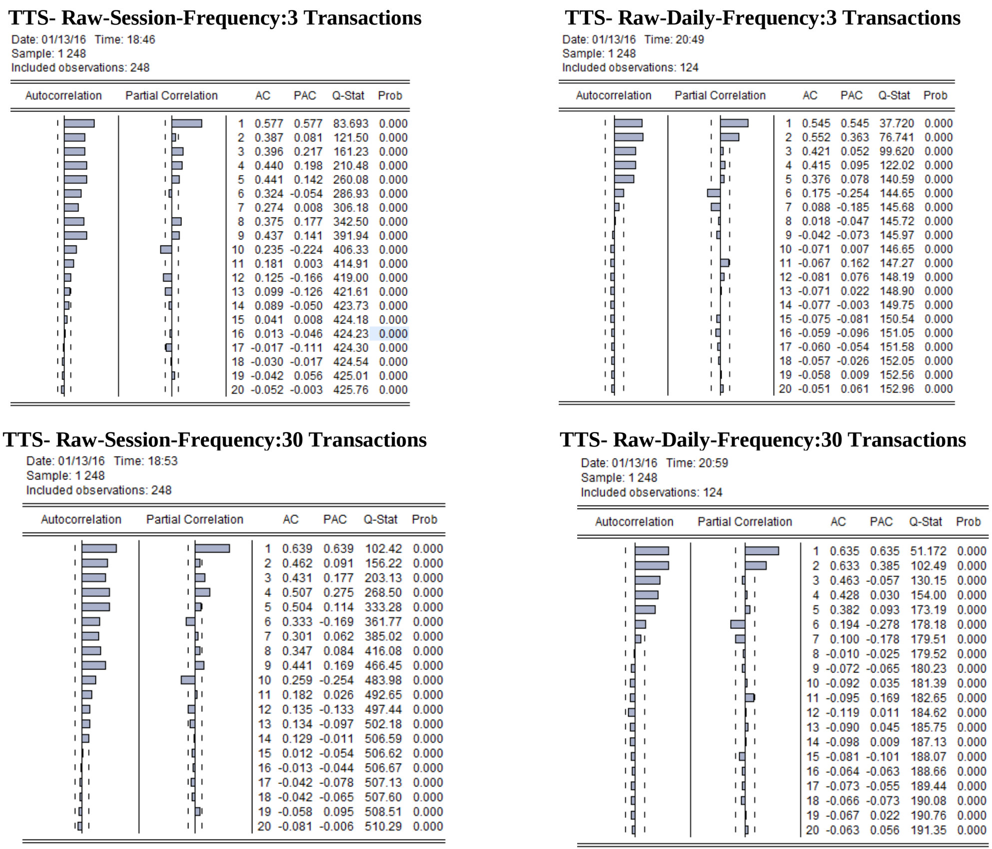
- Decreasing frequency suppresses autocorrelation significance and number of significant lags in ACFs under CTS. The suppression effect is not very evident under raw-TTS. Moreover, calculating RVs on a session basis, makes the RV series more autocorrelated, which holds under both of raw-TTS and CTS.

- If we run ADF test for fixed lag length (2) and intercept in MATLAB, test results leads us to reject the null hypothesis of unit root for all RV series, session or daily, under CTS or raw-TTS at any frequency. However, running stationarity test in E-views with different settings cause different conclusions to be reached. Now, frequency/daily or session combination matters, such that none of the 10 min session RV series is stationary at even 10% significance level under any cleaning and aggregation method combination if we let E-views optimize number of lags to include and the structure of regression according to Schwarz Info Criterion. However, if we choose number of lags as 2, the resulting p-values for all of 10 min RV series decrease to levels less than 5%, with R-squared values decreasing as well.

- Once we are working on a daily or session series at a particular frequency under CTS, the cleaning and aggregation methods do not alter the RV series' non-normality or autocorrelation structure or ADF test results in E-views significantly.

a) **<u>Descriptive statistics by frequency, by sampling scheme and by cleaning and aggregation methods:</u>**

- **<u>TTS-Raw:</u>** For all frequencies, the session and daily RV series are not normally distributed[52] as very high skewness, kurtosis and JB statistic values reveal. Mean of the session and daily RVs become smaller as the sampling interval is lengthened, which is accompanied by a decrease in skewness, kurtosis and JB statistic as we sample less frequently. Still, normality of the any of these series is out of question. Correlogram of all session RV series look same, RVs are autocorrelated up to 11th lag and lags 1, 3, 4 and 10 are significant in PACF. Although correlograms of all daily RVs resemble one another, compared to correlogram of session series, autocorrelation structure of daily RVs looks different. Now, a quick decay in ACF and PACF is evident, only five lags are significant in ACF and first, second and sixth lags are significant in PACF. We checked for unit roots in daily series to see if summing RV from session one and session two to reach daily RV changed anything in RV stationarities at different frequencies.



TTS- Raw-Session-Frequency:3 Transactions

Date: 01/13/16 Time: 18:46
Sample: 1 248
Included observations: 248

Lag	AC	PAC	Q-Stat	Prob
1	0.577	0.577	83.693	0.000
2	0.387	0.081	121.50	0.000
3	0.396	0.217	161.23	0.000
4	0.440	0.198	210.48	0.000
5	0.441	0.142	260.08	0.000
6	0.324	-0.054	286.93	0.000
7	0.274	0.008	306.18	0.000
8	0.375	0.177	342.50	0.000
9	0.437	0.141	391.94	0.000
10	0.235	-0.224	406.33	0.000
11	0.181	0.003	414.91	0.000
12	0.125	-0.166	419.00	0.000
13	0.099	-0.126	421.61	0.000
14	0.089	-0.050	423.73	0.000
15	0.041	0.008	424.18	0.000
16	0.013	-0.046	424.23	0.000
17	-0.017	-0.111	424.30	0.000
18	-0.030	-0.017	424.54	0.000
19	-0.042	0.056	425.01	0.000
20	-0.052	-0.003	425.76	0.000

TTS- Raw-Daily-Frequency:3 Transactions

Date: 01/13/16 Time: 20:49
Sample: 1 248
Included observations: 124

Lag	AC	PAC	Q-Stat	Prob
1	0.545	0.545	37.720	0.000
2	0.552	0.363	76.741	0.000
3	0.421	0.052	99.620	0.000
4	0.415	0.095	122.02	0.000
5	0.376	0.078	140.59	0.000
6	0.175	-0.254	144.65	0.000
7	0.088	-0.185	145.68	0.000
8	0.018	-0.047	145.72	0.000
9	-0.042	-0.073	145.97	0.000
10	-0.071	0.007	146.65	0.000
11	-0.067	0.162	147.27	0.000
12	-0.081	0.076	148.19	0.000
13	-0.071	0.022	148.90	0.000
14	-0.077	-0.003	149.75	0.000
15	-0.075	-0.081	150.54	0.000
16	-0.059	-0.096	151.05	0.000
17	-0.060	-0.054	151.58	0.000
18	-0.057	-0.026	152.05	0.000
19	-0.058	0.009	152.56	0.000
20	-0.051	0.061	152.96	0.000

TTS- Raw-Session-Frequency:30 Transactions

Date: 01/13/16 Time: 18:53
Sample: 1 248
Included observations: 248

Lag	AC	PAC	Q-Stat	Prob
1	0.639	0.639	102.42	0.000
2	0.462	0.091	156.22	0.000
3	0.431	0.177	203.13	0.000
4	0.507	0.275	268.50	0.000
5	0.504	0.114	333.28	0.000
6	0.333	-0.169	361.77	0.000
7	0.301	0.062	385.02	0.000
8	0.347	0.084	416.08	0.000
9	0.441	0.169	466.45	0.000
10	0.259	-0.254	483.98	0.000
11	0.182	0.026	492.65	0.000
12	0.135	-0.133	497.44	0.000
13	0.134	-0.097	502.18	0.000
14	0.129	-0.011	506.59	0.000
15	0.012	-0.054	506.62	0.000
16	-0.013	-0.044	506.67	0.000
17	-0.042	-0.078	507.13	0.000
18	-0.042	-0.065	507.60	0.000
19	-0.058	0.095	508.51	0.000
20	-0.081	-0.006	510.29	0.000

TTS- Raw-Daily-Frequency:30 Transactions

Date: 01/13/16 Time: 20:59
Sample: 1 248
Included observations: 124

Lag	AC	PAC	Q-Stat	Prob
1	0.635	0.635	51.172	0.000
2	0.633	0.385	102.49	0.000
3	0.463	-0.057	130.15	0.000
4	0.428	0.030	154.00	0.000
5	0.382	0.093	173.19	0.000
6	0.194	-0.278	178.18	0.000
7	0.100	-0.178	179.51	0.000
8	-0.010	-0.025	179.52	0.000
9	-0.072	-0.065	180.23	0.000
10	-0.092	0.035	181.39	0.000
11	-0.095	0.169	182.65	0.000
12	-0.119	0.011	184.62	0.000
13	-0.090	0.045	185.75	0.000
14	-0.098	0.009	187.13	0.000
15	-0.081	-0.101	188.07	0.000
16	-0.064	-0.063	188.66	0.000
17	-0.073	-0.055	189.44	0.000
18	-0.066	-0.073	190.08	0.000
19	-0.067	0.022	190.76	0.000
20	-0.063	0.056	191.35	0.000

Figure F.45: MIGRS - Correlograms of session and daily RV series under TTS for different sampling intervals

[52] *Log normality is also rejected at all frequencies.*

- **<u>CTS:</u>** For all frequencies, the session and daily RV series are not normally distributed as very high skewness, kurtosis and JB statistic values reveal. Like the case under RAW-TTS,

    i. mean of the session and daily RVs become smaller as the sampling interval is lengthened.

    ii. ACFs of session and daily RVs change as the sampling frequency changes, such that for increasing frequencies RV series exhibit significant positive total autocorrelation up to higher number of legs. Apart from this common trait, the decay patterns in total correlation of daily and session RVs are different, especially obvious at 1 min frequency.

CTS-4.ii.a-5.i-Session 1 min RV series

Date: 05/18/16 Time: 18:39
Sample: 1 248
Included observations: 248

	AC	PAC	Q-Stat	Prob
1	0.543	0.543	74.145	0.000
2	0.452	0.222	125.57	0.000
3	0.364	0.077	159.02	0.000
4	0.467	0.288	214.44	0.000
5	0.408	0.067	256.86	0.000
6	0.337	-0.022	285.92	0.000
7	0.219	-0.096	298.22	0.000
8	0.359	0.217	331.47	0.000
9	0.349	0.060	363.02	0.000
10	0.281	-0.073	383.57	0.000
11	0.143	-0.108	388.90	0.000
12	0.191	0.041	398.47	0.000
13	0.080	-0.222	400.15	0.000
14	0.170	0.091	407.77	0.000
15	0.010	-0.104	407.80	0.000
16	0.041	-0.003	408.24	0.000
17	-0.035	-0.112	408.56	0.000
18	0.024	0.023	408.71	0.000
19	-0.084	-0.084	410.61	0.000
20	-0.014	0.077	410.66	0.000

CTS-4.ii.a-5.i-Daily 1 min RV series

Date: 05/18/16 Time: 18:46
Sample: 1 248
Included observations: 124

	AC	PAC	Q-Stat	Prob
1	0.635	0.635	51.203	0.000
2	0.589	0.312	95.661	0.000
3	0.448	-0.015	121.53	0.000
4	0.416	0.075	144.08	0.000
5	0.397	0.115	164.76	0.000
6	0.198	-0.282	169.97	0.000
7	0.133	-0.098	172.32	0.000
8	0.017	-0.059	172.36	0.000
9	-0.013	-0.041	172.38	0.000
10	-0.104	-0.108	173.85	0.000
11	-0.091	0.143	175.00	0.000
12	-0.133	-0.002	177.45	0.000
13	-0.099	0.063	178.82	0.000
14	-0.141	-0.058	181.63	0.000
15	-0.133	0.002	184.16	0.000
16	-0.104	-0.018	185.73	0.000
17	-0.115	-0.050	187.67	0.000
18	-0.110	-0.091	189.44	0.000
19	-0.101	0.061	190.97	0.000
20	-0.090	-0.030	192.18	0.000

CTS-4.ii.a-5.i-Session 10 min RV series

Date: 01/13/16 Time: 21:36
Sample: 1 248
Included observations: 248

	AC	PAC	Q-Stat	Prob
1	0.506	0.506	64.376	0.000
2	0.223	-0.046	76.867	0.000
3	0.288	0.260	97.809	0.000
4	0.165	-0.117	104.76	0.000
5	0.222	0.254	117.28	0.000
6	0.205	-0.092	128.10	0.000
7	0.141	0.118	133.23	0.000
8	0.333	0.241	161.95	0.000
9	0.427	0.204	209.15	0.000
10	0.239	-0.117	224.03	0.000
11	0.121	-0.068	227.84	0.000
12	0.058	-0.116	228.73	0.000
13	0.053	0.024	229.47	0.000
14	0.036	-0.152	229.82	0.000
15	-0.009	0.014	229.84	0.000
16	-0.015	-0.117	229.90	0.000
17	-0.036	-0.143	230.26	0.000
18	0.035	0.022	230.58	0.000
19	0.043	0.019	231.07	0.000
20	-0.020	0.065	231.18	0.000

CTS-4.ii.a-5.i-Daily 10 min RV series

Date: 01/13/16 Time: 21:48
Sample: 1 248
Included observations: 124

	AC	PAC	Q-Stat	Prob
1	0.471	0.471	28.190	0.000
2	0.337	0.147	42.704	0.000
3	0.277	0.095	52.590	0.000
4	0.368	0.238	70.248	0.000
5	0.455	0.260	97.451	0.000
6	0.073	-0.393	98.157	0.000
7	0.040	-0.060	98.372	0.000
8	-0.023	-0.109	98.441	0.000
9	-0.001	-0.134	98.441	0.000
10	-0.006	0.007	98.446	0.000
11	-0.071	0.202	99.148	0.000
12	-0.064	-0.008	99.723	0.000
13	-0.081	0.041	100.64	0.000
14	-0.081	-0.019	101.58	0.000
15	-0.045	-0.033	101.86	0.000
16	-0.016	-0.031	101.90	0.000
17	-0.020	0.006	101.96	0.000
18	-0.037	-0.031	102.16	0.000
19	-0.022	0.046	102.23	0.000
20	-0.002	-0.012	102.23	0.000

Figure F.46: MIGRS - Correlograms of session and daily RV series under CTS for different sampling intervals

- However, contrary to findings for RV series under RAW-TTS,

    i. decrease in skewness, kurtosis and JB statistic values is observed as we sample more and more frequently,

ii. PACFs of session RVs differ slightly from one another depending on the frequency. For frequency 1 min legs 1, 2, 4 and 13 are significant in PACF, whereas legs 1, 3, 5 and 8 and legs 1, 3, 5, 8, and 9 are significant for 5 min and 10 min frequencies, respectively.

iii. PACFs of daily RVs differ slightly from one another depending on the frequency. For frequency 1 min, legs 1, 2 and 6 are significant in PACF, whereas legs 1 and 6 and legs 1, 4, 5 and 6 are significant for 5 min and 10 min frequencies, respectively.

- Correlograms of all session and daily RV series under all cleaning and aggregation combinations remind that of an AR (1) process.

- Slow decay in the ACFs calls for stationarity tests. First, second and sixth legs are significant in PACF.

- All of these observations hold under all cleaning methods and aggregation algorithms.

b) <u>Stationarity-Unit root test:</u>

- To test for stationarity and unit root, i.e. if the series move around a constant mean or explode as time passes, Augmented Dickey Fuller (ADF) Test is preferred. By visual inspection of graphs, no trend is observed in any of our RV series, therefore, ADF Test is run with an intercept and no trend, the number of lags to be involved in the analysis is chosen by Schwarz criterion as it is the default choice suggested by E-views.

- **<u>TTS-Raw-:</u>** In the E-views setting, where the number of lags is optimized by E-views according to the Schwarz criterion, R-squared values vary around 30-35%. The null hypothesis of nonstationarity is rejected at 10% significance level for all session series in contrast to p-values of test statistic being around or less than 1% in unit root hypothesis testing in all daily series. Only session RV series calculated at 3 transactions, 20 transactions and 30 transactions are found to be stationary at 5% significance level. Taking logarithm helps with significance levels[53].

P-values of ADF Test –Log RV series

	Frequency					
Sess. Based / Daily	**3tr**	**6tr**	**10tr**	**15tr**	**20tr**	**30tr**
Sess. Based	0.05	0.02	0.03	0.00	0.00	0.00
Daily	0.04	0.03	0.01	0.01	0.00	0.01

[53] *E-views ADF test with INTERCEPT, lags chosen automatically by E-views according to Schwarc info criterion.*

- **CTS:** In the E-views setting, where number of lags are optimized by E-views according to Schwarz criterion, R-squared values have a range of 23% to 47%. At 5% significance level, all RV series, either session or daily and at all frequencies, are found to be stationary except 10 min session row. None of the 10 min session RV series is stationary under any cleaning and aggregation method combination if we let E-views optimize number of lags to include and the structure of regression according to Schwarz Info Criterion. However, if we choose number of lags as 2, the resulting p-values for all of 10 min RV series decrease to levels less than 1%, with R-squared values shrinking as well.

Table F.5: MIGRS - p-values of ADF Test on session and daily RV series under various cleaning and aggregation method combinations for different sampling intervals

Frequency	Session Based /Daily	Cleaning and Aggregation Method Combination									
		4.ii.a-5.i	4.ii.a-5.ii	4.ii.a-5.iii	4.ii.a-5.iv	4.ii.a-5.v	4.ii.b-5.i	4.ii.b-5.ii	4.ii.b-5.iii	4.ii.b-5.iv	4.ii.b-5.v
1min	Sess. Based	0.0046	0.0060	0.0060	0.0062	0.0084	0.0052	0.0070	0.0072	0.0076	0.0104
	Daily	0.0109	0.0143	0.0142	0.0141	0.0186	0.0139	0.0186	0.0186	0.0191	0.0250
5min	Sess. Based	0.0019	0.0019	0.0000	0.0000	0.0000	0.0024	0.0024	0.0020	0.0018	0.0000
	Daily	0.0221	0.0171	0.0164	0.0167	0.0178	0.0237	0.0081	0.0174	0.0177	0.0186
10min	Sess. Based	0.1942	0.1775	0.1744	0.1700	0.1824	0.2083	0.1901	missing	0.1818	0.1941
	Daily	0.0278	0.0237	0.0228	0.0226	0.0244	0.0257	0.0215	0.0208	0.0206	0.0220

NETAS SUMMARY AND REVIEW OF CHAPTER 4 RESULTS

1) UHFD Characteristics Under Different Sampling Schemes and Error Cleaning and Data Filtering Combinations

a) Irregular Temporal Spacing

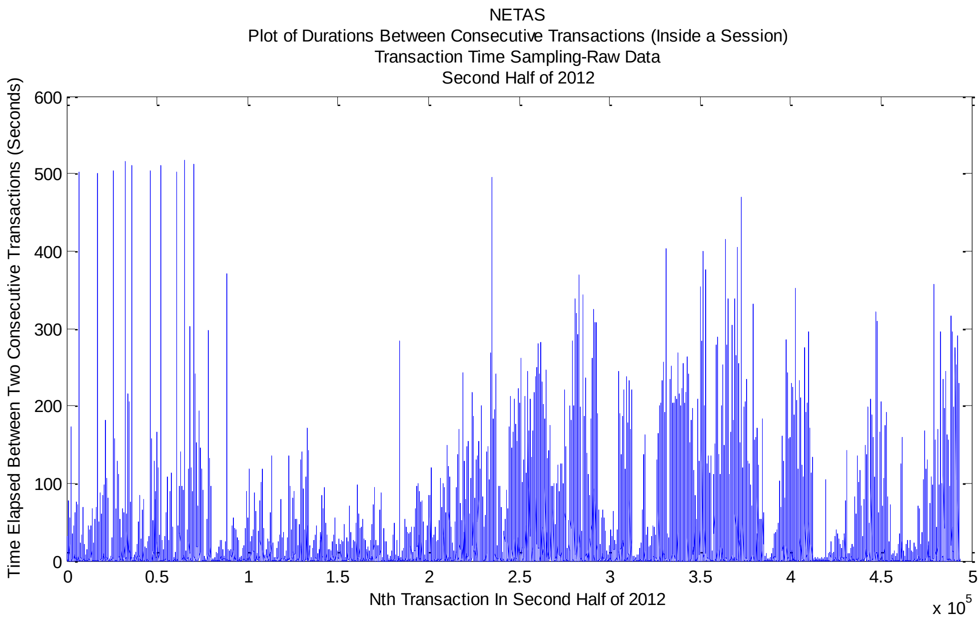


Figure F.47: Plot of durations between consecutive transactions (inside a session) for NETAS TTS-raw data throughout the second half of 2012.

b) **Temporal dependence**: By comparing autocorrelation and partial autocorrelation functions of 60 and 600 seconds[54] absolute returns and log returns under CTS (clean and aggregated and interpolated) as well as absolute returns, log returns and durations in seconds from one transaction to the next under TTS (raw versus clean and aggregated) for December of 2012, we see that there are differences between ACF and PACF structure of absolute and log returns between 10 min CTS and 1 transaction TTS, i.e.: transforming 1 transaction sampled data by first cleaning, then aggregating and then interpolating (all needed for CTS) to 600 second sampled data distorts ACF and PACF of return series.

- TTS-Raw-Durations: ACF (very very slowly decaying positive significant up to 20 lags) and PACF (hyperbolic decay, positive significant up to 20 lags) (shocks persist)

[54] *Recall that we also included 1 min returns under CTS for MIGRS just because 10 min log returns exhibited no autocorrelation at all.*

- TTS-Raw-Absolute Returns: ACF (slow hyperbolic decay, positive significant up to 20 lags) and PACF (decaying positive and significant up to 20 lags) (shocks persist)

- TTS-Raw-Log returns: ACF (quick decay, first two-three lags negative significant) PACF(slower hyperbolic decay, first 12-14 lags negative significant)

- TTS(4.ii.a and b-5.i-5.ii-5.iii-5.iv-5.v)-Durations: ACF (very very slowly decaying positive and significant up to 20 lags) and PACF (hyperbolic decay, positive and significant up to 18 to 20 lags) (shocks persist)

- TTS(4.ii.a and b-5.i-5.ii-5.iii-5.iv-5.v)-Absolute returns: ACF (decaying positive and significant up to 20 lags) and PACF (decaying positive and significant up to 18-20 lags) (shocks persist)

- TTS(4.ii.a and b-5.i-5.ii-5.iii-5.iv-5.v)-Log returns: ACF (quick decay, first lag negative significant, second lag positive and slightly significant) PACF(slower hyperbolic decay, first 8 lags negative significant)

- CTS-Durations: Meaningless, after interpolation duration from one entry to the next is always 1 second.

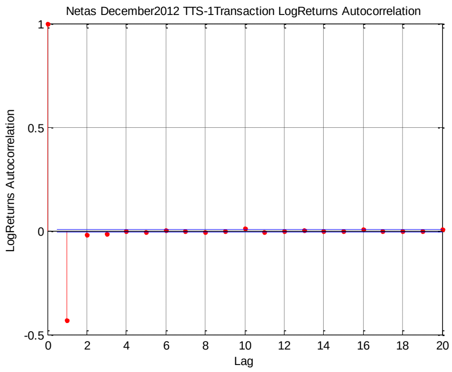
- CTS (4.ii.a and b-5.i-5.ii-5.iii-5.iv-5.v)-10 min Absolute Returns: ACF(wave pattern in decaying positive significance up to 11^{th} lag, while lags 18, 19 and 20 become positive significant again), PACF (decaying, lags 1, 2, 4 and 7 are positive significant)

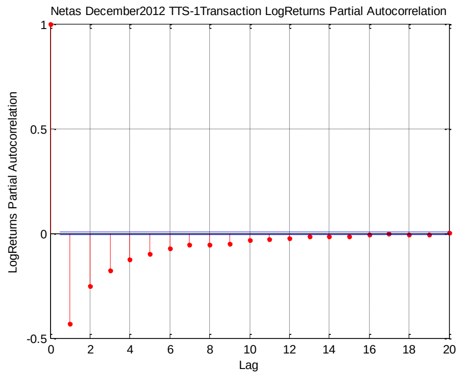
- CTS (4.ii.a and b-5.i-5.ii-5.iii-5.iv-5.v)-10 min Log returns: ACF (lags 1, 6 and 11 are negative significant, significances are on the edge), PACF (lags 1, 6 and 11 are negative significant)

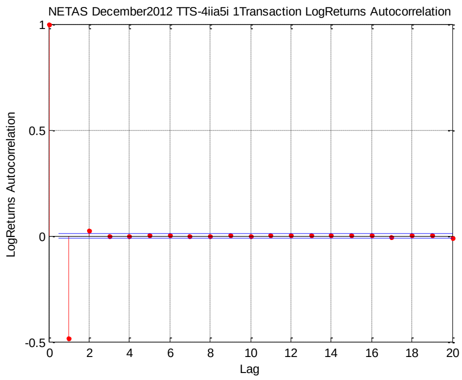
- CTS (4.ii.a and b-5.i-5.ii-5.iii-5.iv-5.v)-1 min Absolute Returns: ACF (wave pattern in decaying positive significance up to 20th lag), PACF (slow decay, positive significant up to 12 lags)

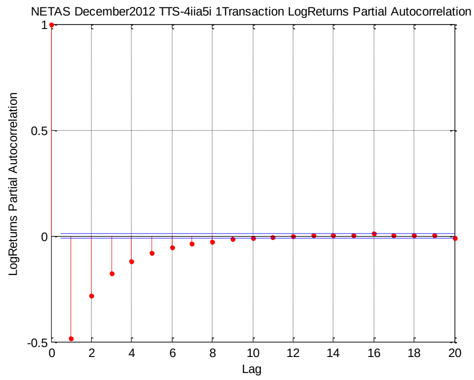
- CTS (4.ii.a and b-5.i-5.ii-5.iii-5.iv-5.v)-1 min Log returns: ACF (first lag negative significant) and PACF (quick decay, first 5 lags negative significant)

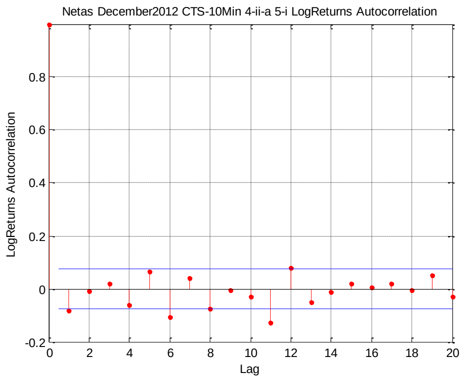
Under TTS, with raw or clean and aggregated data, there is significant positive autocorrelation up to 20 lags in absolute returns, significant up to third order autocorrelation in log returns and very significant positive autocorrelation up to 20 lags in seconds elapsed between two transactions, thus volatility clustering is verified. Whereas, for 10 min returns under CTS, log returns display irregular and hard to comment autocorrelations at lags 1, 6 and 11 with significance levels very close to critical values. Thus, we check for ACF and PACF of 1 min log returns and observe negative first order autocorrelation. Absolute return autocorrelation structure is changed under CTS at 600 seconds sampling interval compared to results under TTS at 1 transaction interval. Likewise, switching to CTS and calculating returns at 600 seconds suppresses partial autocorrelation figures at several lags of both absolute and log returns. Meanwhile, comparing data handling combinations to each other, any combination of cleaning methods and aggregation methods (compared to other combinations) does not cause any major change in total and partial correlation structures once we move under a sampling scheme, it being either TTS or CTS. However, cleaning and aggregation under TTS yield different PACF structures in log returns compared to results produced with raw data. Under CTS, rather than cleaning and aggregation methods, sampling interval matters in terms of return autocorrelation structure. Supporting MIGRS case, working at different frequencies under CTS distorts autocorrelation structure of absolute returns and logreturns same way, returns become less autocorrelated as we sample lesser number of prices.

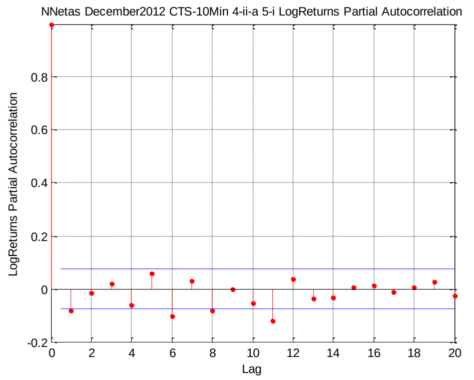












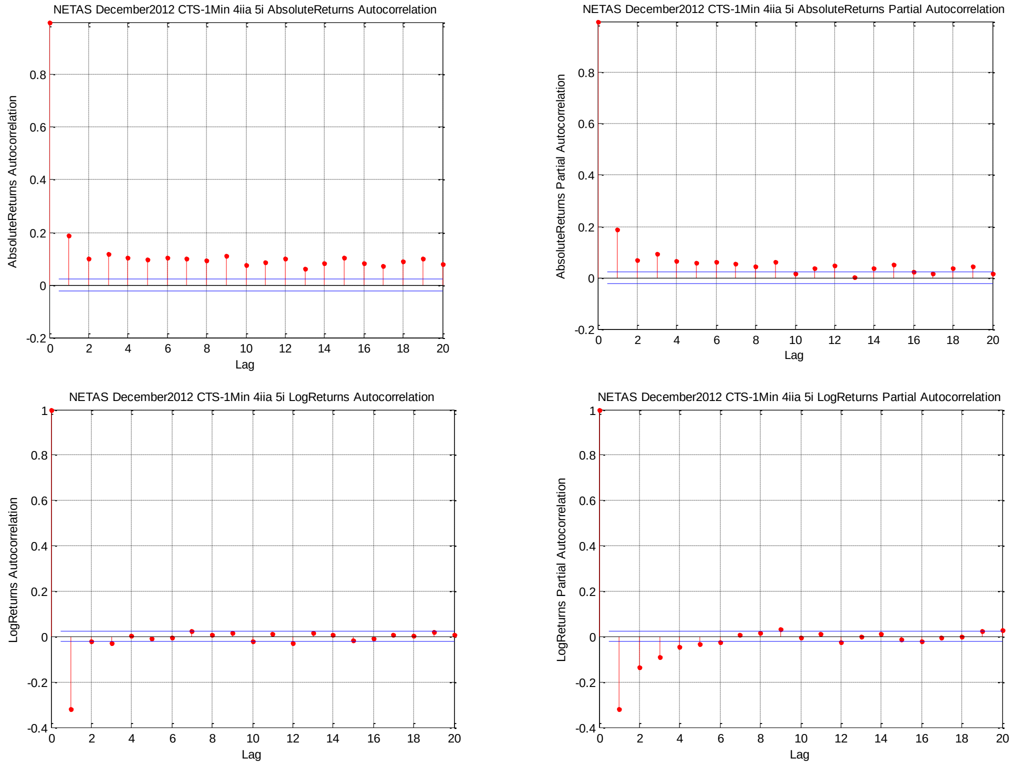


Figure F.48: ACFs and PACFs of logreturn and absolute return series of NETAS for December 2012 under TTS and CTS

c) **Diurnal Patterns**: These patterns can be sought only under CTS because of their definitions such as number of trades per x minutes or absolute return per y seconds. For NETAS case, there are strong W shapes which are persistent across cleaning and aggregation methods in 10 minutes trade volumes and 10 minutes trade intensities throughout days in second half of 2012, whereas patterns in 10 minutes absolute returns and 10 minutes absolute percentage returns are closer to W without last spike at the end of the day[55]. All in all, there are significant diurnal patterns in returns and trading activity in the form of intensity and volume under CTS and these patterns look exactly same when various combinations of cleaning and aggregation methods are applied.

[55] *Unlike the L shape in MIGROS and ISCTR for 10 min absolute percentage returns.*

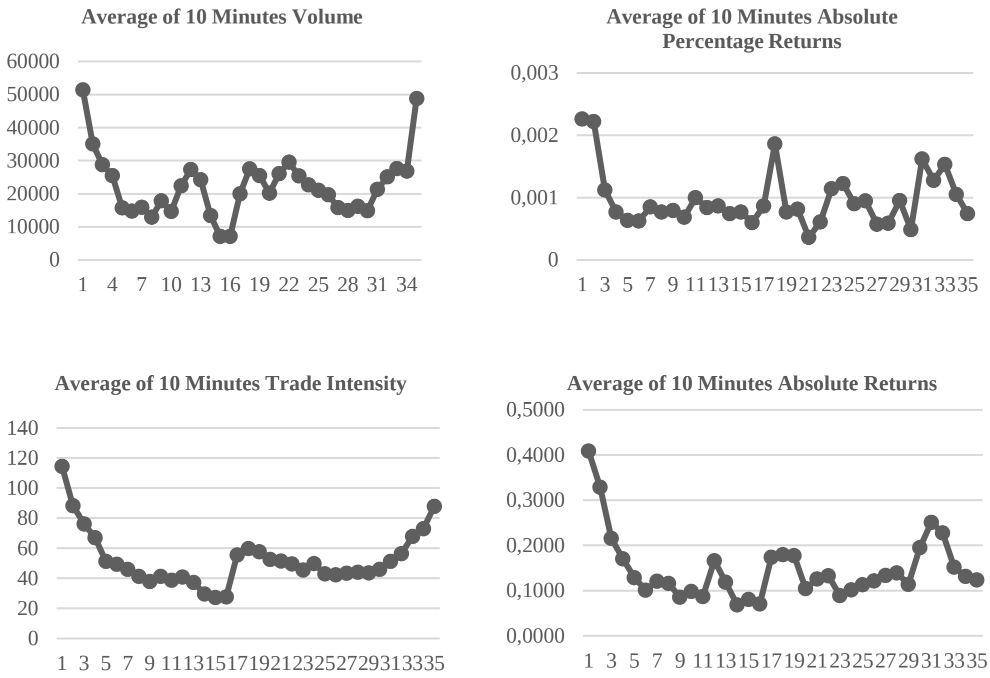


Figure F.49: Diurnal patterns - NETAS cleaned and aggregated transaction data under CTS

2) Visual and Formal Statistical Tests of Existence and Statistical Features of Market Microstructure Noise

a) <u>**VSP:**</u> In line with the findings for MIGRS and ISCTR, sampling schemes or cleaning and aggregation techniques do not alter the fact that inflating sampling frequency, either in seconds or in transactions, causes average realized volatility of return on transaction price to boom. Specifically, 6 month VSPs explode as the sampling frequency increases under raw-TTS as well as under CTS.

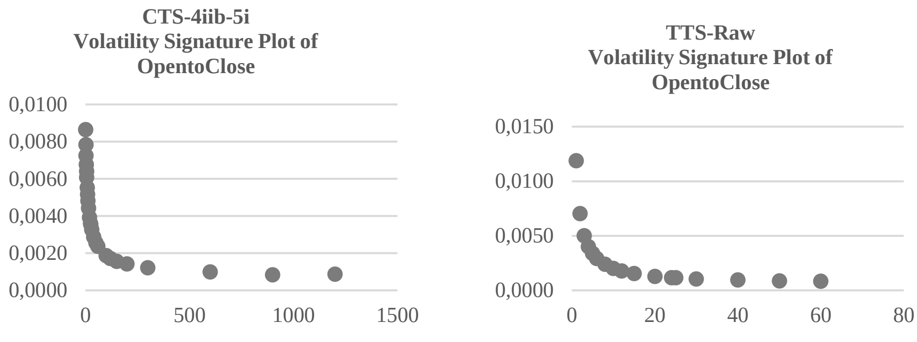


Figure F.50: VSPs of NETAS over Daily RVs using clean and aggregated data under CTS and raw data under TTS.

Explosion becomes trivial for the sampling intervals that are less than 200 seconds or 15 transactions. This observation is valid both for session and daily figures, serving as a visual proof regarding the existence of MMN and pointing to a positive relationship between noise increment and true price return, both under CTS and raw-TTS.

However, for clean and aggregated TTS, and only for first session RVs in June, rising sampling frequencies first deflate then inflate average RVs, leading to a swing in the shape of VSP. This extraordinary pattern causes 6 month averages to exhibit a swing as well for all cleaned and aggregated average RV series under TTS. To be more precise, please consider the following VSPs. The plot on the left is 6 month average of session 1 RVs against sampling frequencies, whereas the same plot is reproduced for 5 months, with June exluded on the right, where both VSPs are drawn under clean and aggregated TTS.

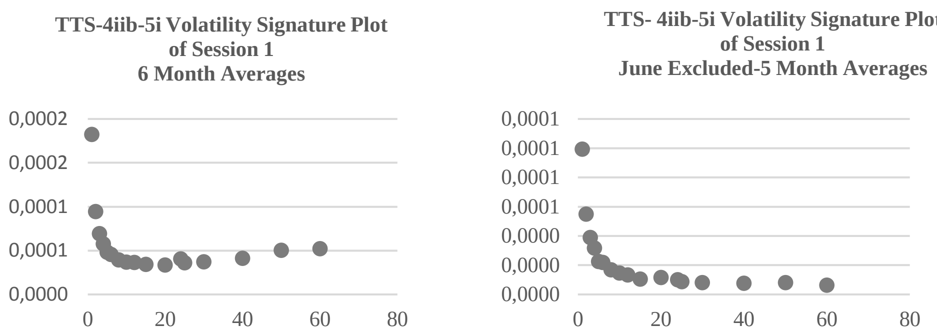


Figure F.51: VSPs of NETAS over Session RVs using clean and aggregated data under CTS and raw data under TTS.

As it is clear from the plots above, data coming from June is responsible for unexpected movement in the VSP of session 1 RV averages. Examination of disclosures of material information throughout June, 2012 by NETAS if there is anything pushing VSPs in a different direction reveals that

i) NETAS applied to Capital Markets Board (CMB) in April, 2012 for new shares to be registered to increase registered capital by 800%, source of the capital raise being internal, i.e., a stock split of 8-for-1 was on the way when June 2012 came.

ii) There had been tremendous extraordinary price movements in NETAS stock before June.

iii) CMB decided to register the free shares on the 19th of June, but at the same time, directed Borsa Istanbul to hold NETAS stock trade until the capital increase was completed. Therefore, on the 20th of June, a trade halt in NETAS stock was active. It ended on the 21st of June.

iv) Even so, the extraordinary price movements in NETAS market dominated the return series for June, 2012.

v) Sampling scheme is not the cause of the swing in VSP, but it is the tool that helps us detect such irregularities in direction of the average RVs as the sampling frequencies change.

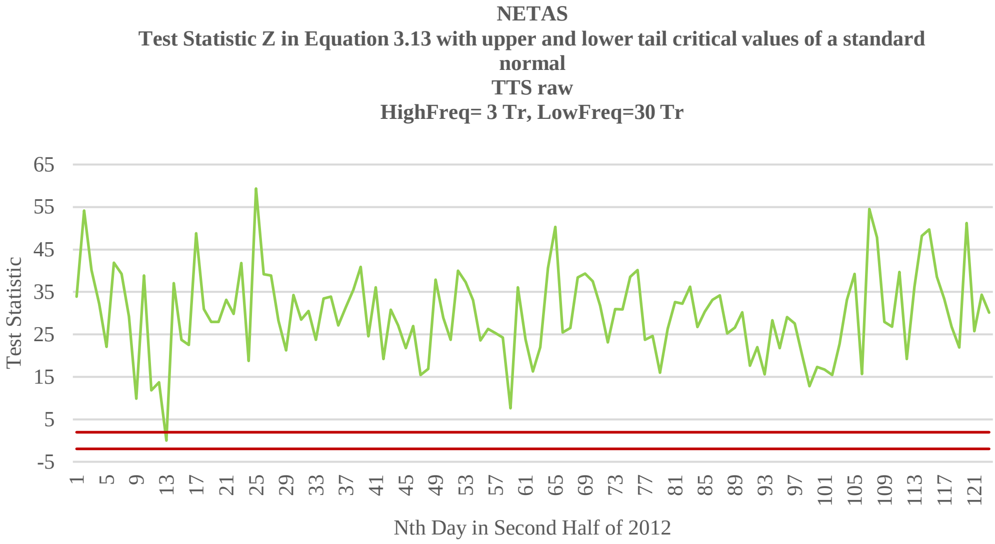
This piece of information supports our finding that in general, (when there is no unexpected information specific to the stock), sampling scheme, or cleaning or aggregation do not affect the result that market microstructure becomes dominant after 15 transactions under TTS and 200 seconds under CTS and that the shape of VSPs suggest a positive correlation between noise increment and true price return.

b) <u>Statistical Tests Regarding Existence and Statistical Features of MMN :</u>

- Existence of MMN is verified statistically under both of CTS and TTS. We calculated $Z_{T,n,h}$ testing null hypothesis in (3.11) by comparing RVs that are calculated over different frequency pairs composed of high-low frequencies, namely (60,600) (10,1200), (30,1200) (60,1200), (150,1200), (300,1200), (600,1200) (900,1200) seconds under CTS and (3,30), (6,30), (10,30), (15,30), and (20,30) transactions under raw-TTS. Recall that bias of the RV estimator is dominated by expectation of square of the noise increment. Therefore, if we reject the null hypothesis, it means that the MMN has statistically significant impact on realized estimator of the IV.

  For each day in the sample period of 123 days (recall that on 20th, the trading halt lasted for whole day) and each frequency pair, we run the aforementioned test at 5% significance level. Sample rejection percentages of null hypothesis are 99% under raw-TTS, 97% under clean and aggregated TTS and around 86% under CTS for all cleaning and aggregation method combinations when we compare RVs calculated over 3 and 30 transactions under TTS and 60 and 600 seconds under CTS. As we decrease the sampling frequency at the high frequency leg, rejection percentages of null hypothesis shrink, which is true under both of TTS and CTS. For raw-TTS, the rejection percentages begin with 99% and decrease gradually to 46% as high frequency leg moves toward 20 transactions when low frequency leg is 30 transactions. Cleaning and aggregating the data does not amend the downward trend in rejection percentages under TTS, but make it steeper. For all aggregation choices with cleaning method 4.ii.b applied under TTS, the rejection percentages begin with 98% and decrease gradually to 20-22% as high frequency leg moves toward 20 transactions. Switching to CTS as well as moving across the grid of cleaning and aggregation combinations do not change the results either. For CTS, the rejection percentages begin with around 93% for 10 to 1200 seconds pair and goes down the hill to 17% as high frequency legs are slowed to 900 seconds.

Following representative rejection rate graphs reveal that the MMN starts to accentuate as the sampling frequency converges to 10-15 transactions under TTS, and 250-300 seconds under CTS. These findings are in conformity with those supplied by the VSP analysis. The MMN is felt strongly once we cross over the sampling interval thresholds of 15 transactions or 5 minutes under TTS and CTS, respectively. For higher frequencies, rejection rates turn out to be quite high, especially after 150 seconds under CTS and 10 transactions under TTS, rejection rates become flat in a band of 95-100%. Moreover, the visual inspection of the test statistic $Z_{T,n,h}$ for several frequency pairs either under TTS or CTS reveals that for the majority of the time the test statistic is positive and outside 5% st. normal confidence interval, meaning that there is positive correlation between the noise and the efficient price, which is again in conformity with the exploding VSPs.



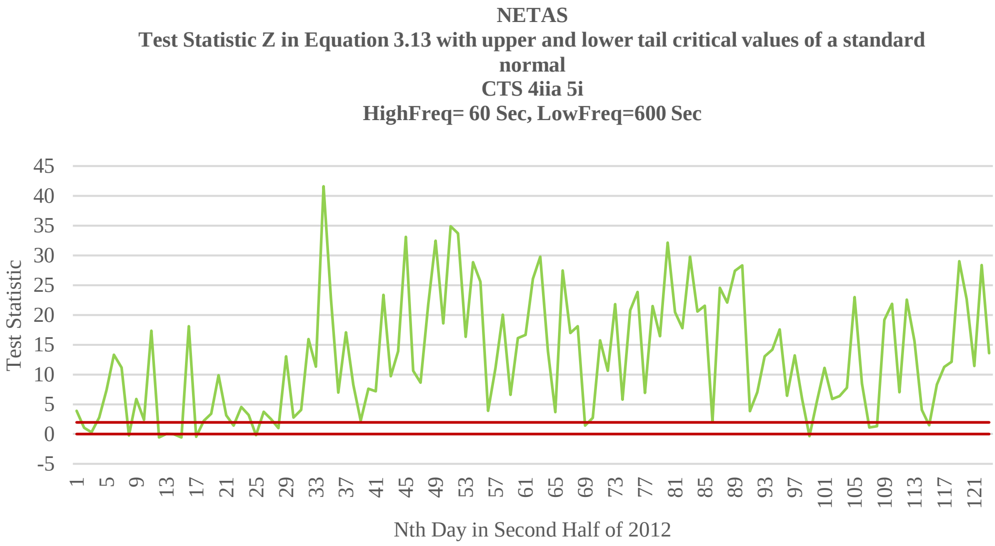


Figure F.52: NETAS - Plots of $Z_{T,n,h}$ for each day in the sample period with upper and lower tail critical values of standard normal under TTS and CTS.

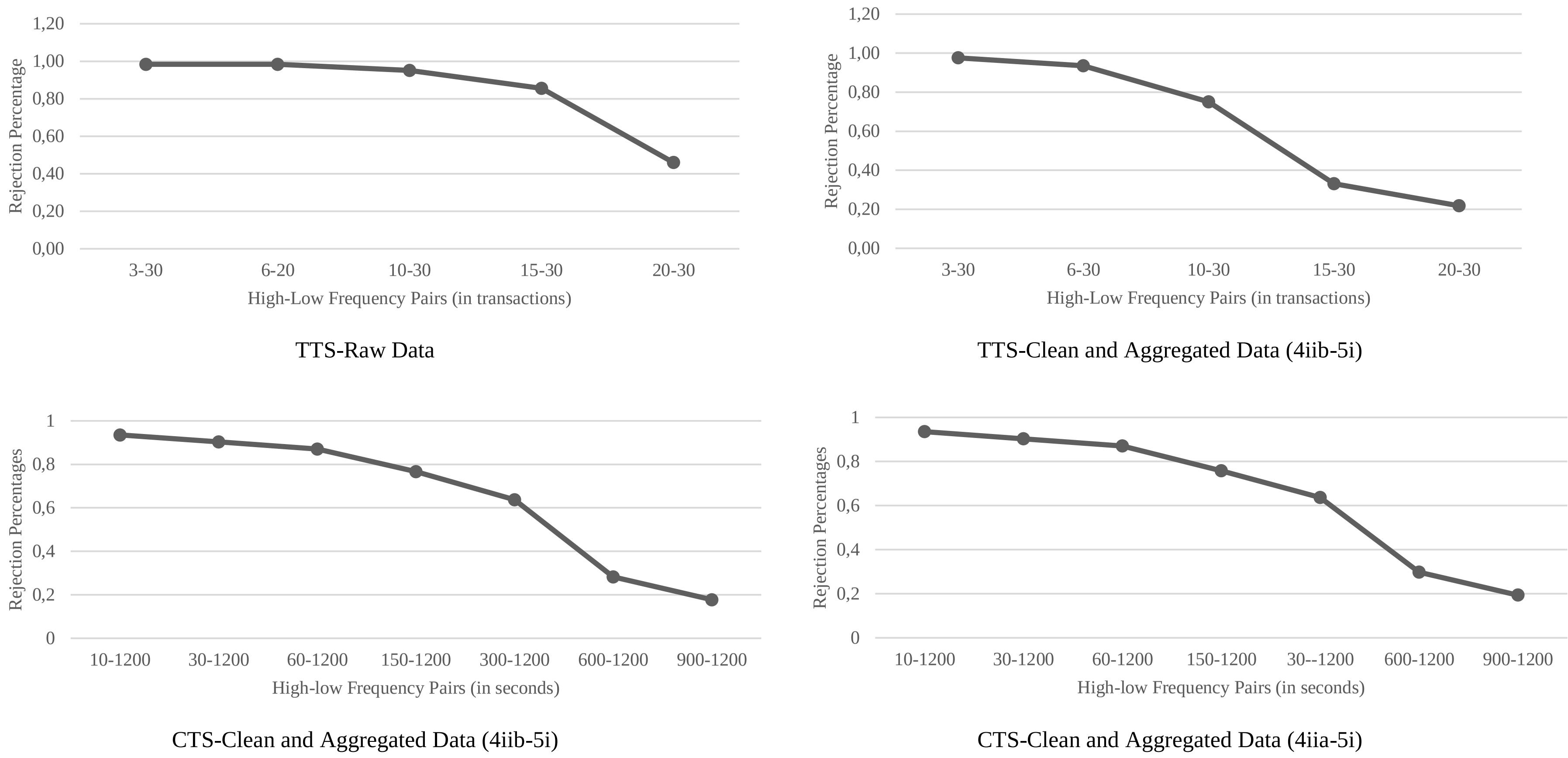


Figure F.53: NETAS –Plots of rejection percentages with regards to the null hypothesis that the MMN does not have statistically significant effect on RV under TTS and CTS.

- **Summary:** Model of an i.i.d MMN with constant variance might be proper under CTS but under raw-TTS, for more than 50% of the days, the null hypothesis of constant variance is rejected for triples with very high frequencies combined with very low. This might be evidence of i.i.d assumption not holding at frequencies lesser than 15 transactions. The sampling scheme, but not the aggregation method, is discovered to very influential on rejection of the null hypothesis that the MMN has variance independent of the sampling frequency. Meanwhile, the cleaning algorithms have some suppressive effect on rejection percentages particularly under TTS[56].

  Awartani et al. [16] derive a test with the idea that if the MMN has constant variance, then noise variances calculated over frequencies 1/M or 1/N should be same independent of M or N chosen. Their null and null hypotheses are as in (3.35) and (3.36).

  Since alternative hypothesis is in harmony with the presence of autocorrelation in MMN, by reminding corollary 3 of Hansen and Lunde [61], Awartani et al. [16] interpret the rejection of null hypothesis as a sign of the rejection of the null hypothesis that the MMN is a sequence of i.i.d random variables with constant variance. To test the validity of this null hypothesis, a test statistic compares RV differences using two frequency pairs, where pairs are M,L and N,L. L represents a frequency at which we can ignore the MMN safely, say 20 minutes and M and N are frequencies at which the MMN is considered to be significant. Therefore, the test is build on RVs calculated over frequency triples i.e. for each high frequency pair combined with 20 minutes, we test null hypothesis that E(noise increment square at low frequency)=E(noise increment square at high frequency). If we reject the null hypothesis, it means that the MMN has variance that is NOT independent of sampling frequency, therefore any assumptions regarding i.i.d nature of MMN can be taken as invalidated. Frequency triples are as follows: (3,10,30), (3,15,30), (3,20,30), (6,15,30) (6,20,30) and (10,20,30) transactions under TTS, (60,150,1200), (60,600,1200), (150,300,1200), (150,600,1200) and (300,600,1200) seconds under CTS.

  For each day in the sample period of 123 days and each frequency triple, we run the aforementioned test at 5% significance level. Sample rejection percentages of the null hypothesis clearly change from one triple to another and as we clean and aggregate data. Beware that under raw-TTS especially for combinations of frequencies with highest differences between frequent legs, rejection percentages exceed 50%, while they stagger around 16% for 3-10-30 triple with lowest distance between first two legs. However, once we clean and aggregate the data, the rejection percentages range

[56] *Same conclusions were made for ISCTR case as well.*

decline to levels 12-40% depending on the triple[57]. For CTS 4.ii.a and 4.ii.b, rejection percentages vary between at most 10% and at least 3%, both of which are just a fraction of rejection percentages under TTS-raw or TTS-cleaned. Therefore, sampling scheme is discovered to be very influential on the rejection of the null hypothesis that the MMN has variance independent of sampling frequency. We can reject this null hypothesis under TTS confidently and conclude that the assumption of an i.i.d MMN with constant variance does not reflect the real life structure of the MMN, whereas under CTS, such an assumption seems to hold especially for frequencies higher than 150 seconds. Evidence reveals that the aggregation method does not affect the rejection percentages and for triples with high frequency legs being close to very slow frequency leg, the rejection percentages are severely damaged independent of the sampling scheme.

[57] *In a sense, these findings agree with findings for MIGRS and ISCTR cases, where rejection percentages are highest for triples with distant constituents and TTS-raw data; however, MIGRS rejection percentages are way below those of ISCTR's or NETAS' rejection percentages.*

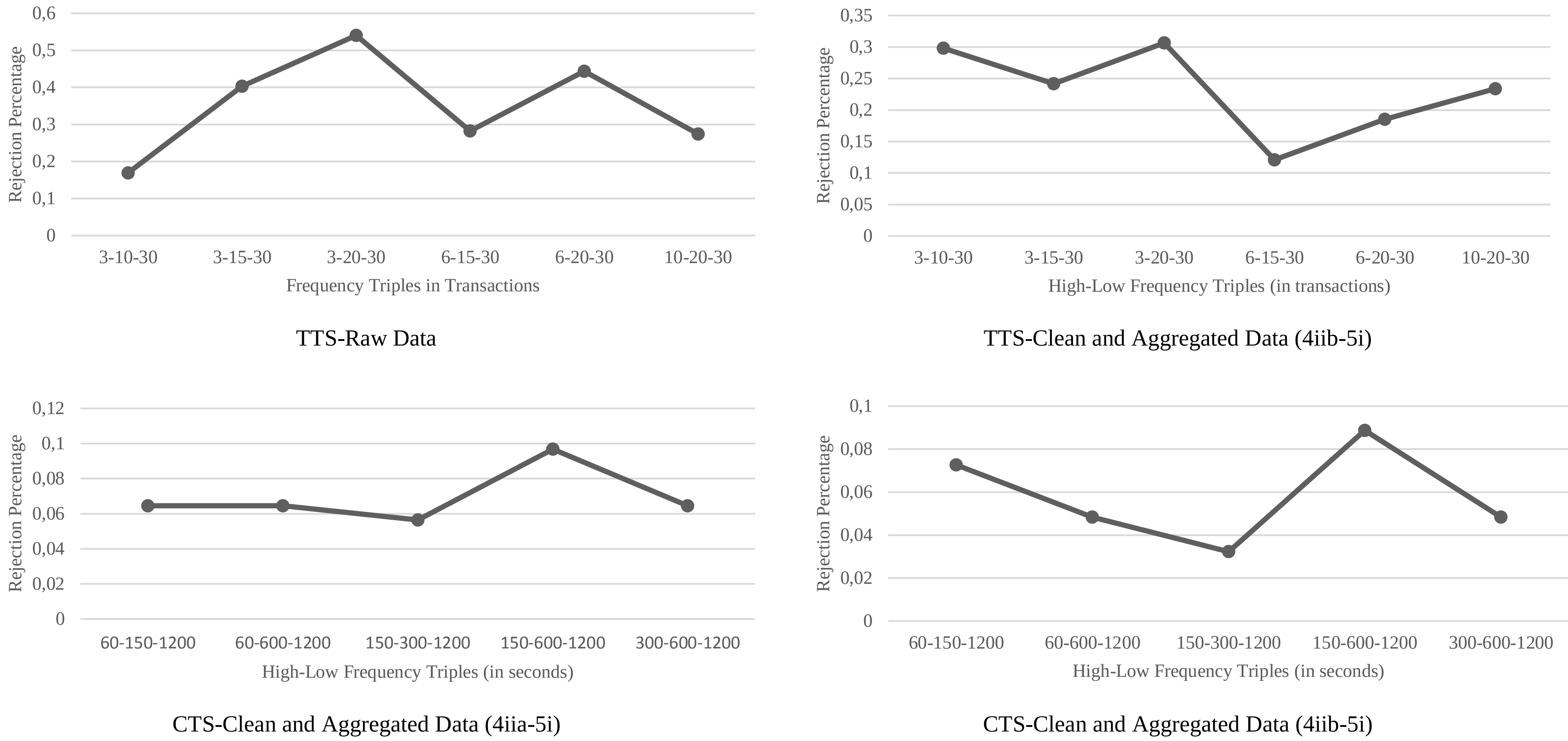


Figure F.54: NETAS – Examples of plots of rejection percentages with regards to the null hypothesis that the MMN increments have constant variance independent of sampling frequency under TTS and CTS.

3) RV Analysis

We constructed two RV time series, namely session RVs and daily RVs, for each frequency in a frequency set of 3, 6, 10, 15, 20, 30 transactions and 60, 300, 600 seconds under each sampling scheme (raw-TTS, CTS) and cleaning (4.ii.a, 4.ii.b) -aggregation method (5.i, 5.ii, 5.iii, 5.iv, 5.v) combination. Daily RV time series has 123 data points, whereas session RV time series is constituted of 246 entries. Each RV series under each sampling scheme and cleaning and aggregation method combinations is subjected to preliminary statistics, ACF and PACF analysis and lastly unit root is checked where autocorrelation exhibits slow decay.

- The factors that have any effect on RV series' lognormality and autocorrelation structure turn out to be whether the RV is on a session or daily basis, whether it is under raw-TTS or CTS and the frequency at which the RV is calculated. Normality is not affected by any of these factors. All of RV series, either under raw-TTS or CTS, either raw or cleaned and aggregated, either on a session or daily basis, are not normally distributed as JB statistics and high kurtosis-skewness values suggest. Taking logarithm makes RV series at all frequencies normal under raw-TTS, while such a transformation only works in terms of normality for 5 min RV session and daily series under CTS[58].

- Decreasing frequencies cause lesser number of lags being significant with lesser significant levels, i.e. decreasing frequency again depresses autocorrelation structure of RV series under CTS but not under TTS regardless of session-daily calculation. Supression effect of decreasing frequency is in line with existence of MMN under CTS. In fact, ACFs of session and daily RVs change as the sampling frequency changes, such that for increasing frequencies RV series exhibit significant positive total autocorrelation up to higher number of lags under CTS. Calculating RVs on a session basis, makes the RV series more autocorrelated, which holds under both of raw-TTS and CTS.

- Once we are working on a daily or session series at a particular frequency under CTS, cleaning and aggregation methods do not alter RV series' non-normality or autocorrelation structure.

- Neither sampling schemes, nor frequencies or cleaning/aggregation methods or session/daily basis choice affects the stationarity results, E-views ADF test results reveal that we can reject null of unit root at 1% significance level for all RV series under raw-TTS or CTS at all frequencies[59]. MATLAB ADF test with fixed two lags and an intercept supports results in E-views that all RV series at hand are stationary for all frequencies, cleaning and aggregation methods, daily/session calculations and sampling schemes.

[58] *Unlike the case of MIGRS*
[59] *Unlike the case of MIGRS.*

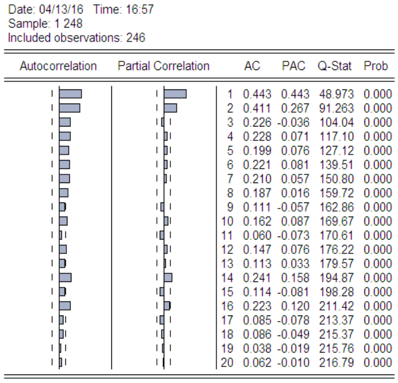
a) <u>Descriptive statistics by frequency, by sampling scheme and by cleaning and aggregation methods:</u>

- **<u>TTS-Raw:</u>** For all frequencies, the session and daily RV series are not normally distributed[60] as very high skewness, kurtosis and JB statistic values reveal. Mean of the session and daily RVs become smaller as the sampling interval is lengthened, but there is no clear relationship between sampling frequency and change in skewness, kurtosis or JB statistic values, which deviates from the findings for MIGRS and ISCTR[61]. Correlogram of all session RV series look very much alike. Total autocorrelation is significant up to 20th lag but significance decreases and increases as the lag number converges to 20. Only first two lags and lag 14 are significant in PACF[62]. Although correlograms of all daily RVs resemble one another, compared to correlogram of session series, autocorrelation structure of daily RVs looks different. Now, first 10 lags and first lag are positive significant in ACF and PACF, respectively. The change in autocorrelation structure of RV series by looking at session and daily RVs separately, calls for stationarity test and accordingly, we checked for unit roots in daily series to see if summing RV from session one and session two to reach daily RV distorts anything in RV stationarities at different frequencies.

---

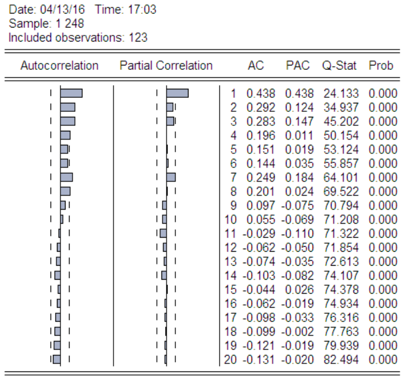
[60] *By application of JBTEST in MATLAB, we are not able to reject log normality at all frequencies.*
[61] *For MIGRS and ISCTR, a decrease in skewness, kurtosis and JB statistic was observed as we sample less frequently.*
[62] *Unlike the case of MIGRS.*

TTS- Raw-Session-Frequency:3 Transactions


Date: 04/13/16 Time: 16:57
Sample: 1 248
Included observations: 246

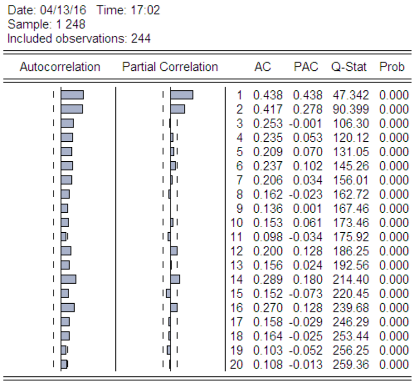
Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.443	0.443	48.973	0.000
		2	0.411	0.267	91.263	0.000
		3	0.226	-0.036	104.04	0.000
		4	0.228	0.071	117.10	0.000
		5	0.199	0.076	127.12	0.000
		6	0.221	0.081	139.51	0.000
		7	0.210	0.057	150.80	0.000
		8	0.187	0.016	159.72	0.000
		9	0.111	-0.057	162.86	0.000
		10	0.162	0.087	169.67	0.000
		11	0.060	-0.073	170.61	0.000
		12	0.147	0.076	176.22	0.000
		13	0.113	0.033	179.57	0.000
		14	0.241	0.158	194.87	0.000
		15	0.114	-0.081	198.28	0.000
		16	0.223	0.120	211.42	0.000
		17	0.085	-0.078	213.37	0.000
		18	0.086	-0.049	215.37	0.000
		19	0.038	-0.019	215.76	0.000
		20	0.062	-0.010	216.79	0.000

TTS- Raw-Daily-Frequency:3 Transactions


Date: 04/13/16 Time: 17:03
Sample: 1 248
Included observations: 123

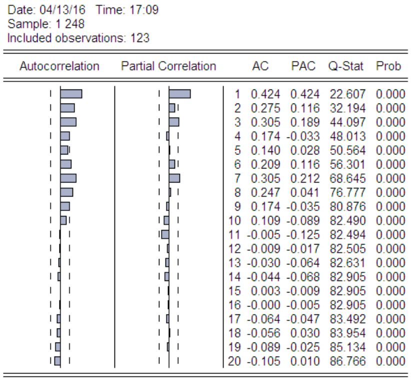
Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.438	0.438	24.133	0.000
		2	0.292	0.124	34.937	0.000
		3	0.283	0.147	45.202	0.000
		4	0.196	0.011	50.154	0.000
		5	0.151	0.019	53.124	0.000
		6	0.144	0.035	55.857	0.000
		7	0.249	0.184	64.101	0.000
		8	0.201	0.024	69.522	0.000
		9	0.097	-0.075	70.794	0.000
		10	0.055	-0.069	71.208	0.000
		11	-0.029	-0.110	71.322	0.000
		12	-0.062	-0.050	71.854	0.000
		13	-0.074	-0.035	72.613	0.000
		14	-0.103	-0.082	74.107	0.000
		15	-0.044	0.026	74.378	0.000
		16	-0.062	-0.019	74.934	0.000
		17	-0.098	-0.033	76.316	0.000
		18	-0.099	-0.002	77.763	0.000
		19	-0.121	-0.019	79.939	0.000
		20	-0.131	-0.020	82.494	0.000

TTS- Raw-Session-Frequency:30 Transactions


Date: 04/13/16 Time: 17:02
Sample: 1 248
Included observations: 244

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.438	0.438	47.342	0.000
		2	0.417	0.278	90.399	0.000
		3	0.253	-0.001	106.30	0.000
		4	0.235	0.053	120.12	0.000
		5	0.209	0.070	131.05	0.000
		6	0.237	0.102	145.26	0.000
		7	0.206	0.034	156.01	0.000
		8	0.162	-0.023	162.72	0.000
		9	0.136	0.001	167.46	0.000
		10	0.153	0.061	173.46	0.000
		11	0.098	-0.034	175.92	0.000
		12	0.200	0.128	186.25	0.000
		13	0.156	0.024	192.56	0.000
		14	0.289	0.180	214.40	0.000
		15	0.152	-0.073	220.45	0.000
		16	0.270	0.128	239.68	0.000
		17	0.158	-0.029	246.29	0.000
		18	0.164	-0.025	253.44	0.000
		19	0.103	-0.052	256.25	0.000
		20	0.108	-0.013	259.36	0.000

TTS- Raw-Daily-Frequency:30 Transactions


Date: 04/13/16 Time: 17:09
Sample: 1 248
Included observations: 123

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.424	0.424	22.607	0.000
		2	0.275	0.116	32.194	0.000
		3	0.305	0.189	44.097	0.000
		4	0.174	-0.033	48.013	0.000
		5	0.140	0.028	50.564	0.000
		6	0.209	0.116	56.301	0.000
		7	0.305	0.212	68.645	0.000
		8	0.247	0.041	76.777	0.000
		9	0.174	-0.035	80.876	0.000
		10	0.109	-0.089	82.490	0.000
		11	-0.005	-0.125	82.494	0.000
		12	-0.009	-0.017	82.505	0.000
		13	-0.030	-0.064	82.631	0.000
		14	-0.044	-0.068	82.905	0.000
		15	0.003	-0.009	82.905	0.000
		16	-0.000	-0.005	82.905	0.000
		17	-0.064	-0.047	83.492	0.000
		18	-0.056	0.030	83.954	0.000
		19	-0.089	-0.025	85.134	0.000
		20	-0.105	0.010	86.766	0.000

Figure F.55: NETAS - Correlograms of session and daily RV series under TTS for different sampling intervals

- **CTS:** For all frequencies, the session and daily RV series are not normally distributed as very high skewness, kurtosis and JB statistic values reveal[63]. Like the case under RAW-TTS,

  i. mean of the session and daily RVs become smaller as the sampling interval is lengthened.

  ii. there is no clear relationship between sampling frequency and change in skewness, kurtosis or JB statistic values, which deviates from the findings for MIGRS and ISCTR.

- However, contrary to findings for RV series under RAW-TTS,

  i. ACFs of session and daily RVs change as the sampling frequency changes, such that for increasing frequencies RV series exhibit significant positive total autocorrelation up to higher number of lags with higher significances. Apart from this common trait, the decay

[63] *Like MIGRS, unlike ISCTR.*

patterns in total correlation of daily and session RVs are different, especially obvious at 1 min frequency.

CTS-4.ii.a-5.i-Session 10 min RV series

Date: 04/13/16 Time: 14:15
Sample: 1 248
Included observations: 246

Lag	AC	PAC	Q-Stat	Prob
1	0.201	0.201	10.029	0.002
2	0.175	0.141	17.716	0.000
3	0.133	0.079	22.151	0.000
4	0.170	0.118	29.435	0.000
5	0.365	0.314	63.123	0.000
6	0.114	-0.027	66.427	0.000
7	0.084	-0.035	68.225	0.000
8	0.095	0.022	70.552	0.000
9	0.128	0.035	74.742	0.000
10	0.148	-0.004	80.396	0.000
11	0.014	-0.065	80.443	0.000
12	0.099	0.077	83.009	0.000
13	0.081	0.023	84.721	0.000
14	0.172	0.104	92.542	0.000
15	0.111	0.026	95.770	0.000
16	0.088	0.059	97.823	0.000
17	0.109	0.020	101.00	0.000
18	0.097	0.007	103.51	0.000
19	0.087	-0.046	105.52	0.000
20	0.110	0.044	108.77	0.000

CTS-4.ii.a-5.i-Session 1 min RV series

Date: 04/13/16 Time: 14:19
Sample: 1 248
Included observations: 246

Lag	AC	PAC	Q-Stat	Prob
1	0.337	0.337	28.249	0.000
2	0.425	0.351	73.348	0.000
3	0.228	0.022	86.444	0.000
4	0.330	0.164	113.96	0.000
5	0.248	0.083	129.47	0.000
6	0.283	0.076	149.81	0.000
7	0.191	-0.004	159.14	0.000
8	0.210	0.021	170.49	0.000
9	0.091	-0.087	172.62	0.000
10	0.193	0.073	182.27	0.000
11	0.035	-0.098	182.60	0.000
12	0.233	0.158	196.71	0.000
13	0.123	0.054	200.67	0.000
14	0.245	0.101	216.48	0.000
15	0.033	-0.122	216.77	0.000
16	0.155	0.026	223.12	0.000
17	0.065	-0.005	224.25	0.000
18	0.099	-0.065	226.86	0.000
19	-0.006	-0.092	226.87	0.000
20	-0.001	-0.099	226.87	0.000

CTS-4.ii.a-5.i-Daily 10 min RV series

Date: 04/13/16 Time: 14:20
Sample: 1 248
Included observations: 123

Lag	AC	PAC	Q-Stat	Prob
1	0.184	0.184	4.2456	0.039
2	0.304	0.280	15.984	0.000
3	0.319	0.256	29.052	0.000
4	0.086	-0.066	30.015	0.000
5	0.178	0.025	34.150	0.000
6	0.081	-0.024	35.021	0.000
7	0.252	0.234	43.453	0.000
8	0.132	0.041	45.771	0.000
9	0.162	0.039	49.311	0.000
10	0.133	-0.062	51.710	0.000
11	0.013	-0.087	51.734	0.000
12	0.012	-0.096	51.754	0.000
13	0.081	0.117	52.666	0.000
14	-0.013	-0.034	52.689	0.000
15	-0.002	-0.051	52.690	0.000
16	0.027	-0.061	52.794	0.000
17	-0.043	-0.041	53.062	0.000
18	-0.057	-0.048	53.530	0.000
19	-0.054	0.018	53.963	0.000
20	-0.089	-0.060	55.143	0.000

CTS-4.ii.a-5.i-Daily 1 min RV series

Date: 04/13/16 Time: 14:23
Sample: 1 248
Included observations: 122

Lag	AC	PAC	Q-Stat	Prob
1	0.510	0.510	32.495	0.000
2	0.446	0.251	57.554	0.000
3	0.365	0.096	74.479	0.000
4	0.263	-0.017	83.364	0.000
5	0.191	-0.028	88.105	0.000
6	0.213	0.094	94.030	0.000
7	0.251	0.141	102.32	0.000
8	0.132	-0.105	104.63	0.000
9	0.126	-0.030	106.76	0.000
10	-0.026	-0.196	106.86	0.000
11	-0.122	-0.148	108.89	0.000
12	-0.152	-0.042	112.07	0.000
13	-0.186	-0.054	116.86	0.000
14	-0.225	-0.091	123.93	0.000
15	-0.137	0.090	126.59	0.000
16	-0.205	-0.104	132.58	0.000
17	-0.256	-0.076	142.00	0.000
18	-0.202	0.064	147.95	0.000
19	-0.235	-0.017	156.10	0.000
20	-0.248	-0.022	165.25	0.000

Figure F.56: NETAS - Correlograms of session and daily RV series under CTS for different sampling intervals

ii. PACFs of session RVs differ slightly from one another depending on the frequency. For frequency 1 min, lags 1, 2, 4 and 12 are significant in PACF, whereas lags 1, 2, 5 and 14 and lags 1,2, and 5 are significant for 5 min and 10 min frequencies, respectively.

iii. PACFs of daily RVs differ slightly from one another depending on the frequency. For frequency 1 min, lags 1, 2 and 10 are significant in PACF, whereas lags 1, 2, 3 and 7 are significant for both of 5 min and 10 min frequencies.

- Slow decay in some of the ACFs calls for stationarity tests.

- All of these observations hold under all cleaning methods and aggregation algorithms.

b) Stationarity-Unit root test:

- To test for stationarity and unit root, i.e. if the series move around a constant mean or diverge as time passes, Augmented Dickey Fuller (ADF) Test is preferred. By visual inspection of the graphs, no trend is observed in any of our RV series, therefore, ADF Test is run with an intercept and no trend, the number of legs to be involved in the analysis is chosen by the Schwarz criterion as it is the default choice suggested by E-views.

- **RAW-TTS-Raw-:** In the E-views setting, where the number of lags is optimized by E-views according to the Schwarz criterion, R-squared values vary in a band of 28-34%. The null hypothesis of nonstationarity is rejected at 1% significance level for all session and daily series[64].

- **CTS:** In the E-views setting, where number of lags are optimized by E-views according to the Schwarz criterion, R-squared values have a range of 25% to 49%. At 1% significance level, all RV series, either session or daily and at all frequencies, are found to be stationary.

[64] *In MIGRS analysis, significance level of rejection regarding nonstationarity increases when we switch to Daily series. Here, switching between Daily or session series does not affect significance level at which we can reject null.*

Table F.6: NETAS - p-values of ADF Test on session and daily RV series under various cleaning and aggregation method combinations for different sampling intervals

Frequency	Session Based /Daily	Cleaning and Aggregation Method Combination									
		4.ii.a-5.i	4.ii.a-5.ii	4.ii.a-5.iii	4.ii.a-5.iv	4.ii.a-5.v	4.ii.b-5.i	4.ii.b-5.ii	4.ii.b-5.iii	4.ii.b-5.iv	4.ii.b-5.v
1min	Sess. Based	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	Daily	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
5min	Sess. Based	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	Daily	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
10min	Sess. Based	0.0041	0.0032	0.0030	0.0029	0.0032	0.0036	0.0028	0.0027	0.0026	0.0028
	Daily	0.0079	0.0070	0.0071	0.0069	0.0067	0.0070	0.0062	0.0063	0.0061	0.0059

CURRICULUM VITAE

PERSONAL INFORMATION

Surname, Name: Kılıçkaya, İnci
Nationality: Turkish
Phone: +90 312 29 8076
E-mail: eseni@spk.gov.tr

EDUCATION

Degree	Institution	Year of Graduation
M.Sc. in Quantitative Finance	London Business School	2010
M.Sc. in Financial Mathematics (CGPA: 3.28/4.00)	Middle East Technical University, Institute of Applied Mathematics	2007
B.Sc. in Business Administration (CGPA: 3.79/4.00)	Middle East Technical University, Faculty of Economic and Administrative Sciences	2001
High School (CGPA: 4.65/5.00)	Karşıyaka Anatolian High School	1997

PROFESSIONAL EXPERIENCE

- **Senior Expert, Department of Auditing and Accounting Standards, Capital Markets Board of Turkey**
  Apr 2008 - Present

  - Representing CMB at Committee 6 on Credit Rating Agencies of IOSCO.
  - Representing CMB at General Colleges on Credit Rating.
  - Running projects on updating Communiqué Regarding Independent Audit Standards in Capital Markets due to changes in International Auditing Standards by IAASB with a view of tailoring auditing standards for the needs of local capital markets.
  - Running projects on redesigning Communiqué on Principles Regarding Rating Activity and Rating Agencies in order to ensure that Turkish capital market regulations are in line with latest European Union regulations and reflect lessons from 2008-2010 financial crisis.
  - Preparing quality control/surveillance guides for international and domestic credit rating agencies as well as corporate governance rating agencies.
  - Examination of the applications for registration of audit firms.
  - Examination of the applications for authorization of local and/or international agencies that offer compliance with corporate governance principles rating and/or credit rating.
  - Supervision/surveillance of the quality of the services rendered by audit firms and all classes of rating agencies authorized by the CMB.
  - Providing guidance and consultancy for international credit rating agencies and international corporate governance rating agencies to enhance their degree of compliance with existing Turkish regulations.
  - Running project on introducing Sustainability rating and Sustainability principles to Turkish Capital Markets with an aim to support Istanbul Stock Exchange's Sustainability Index Project.
  - Undertaking research on corporate governance rating and how existing percentages assigned for each heading should be revised in the light of international exercises and local practices.
  - Review of the quarterly and annual reports, disclosed to the public or presented to the Capital Markets Board, of corporations whose shares are listed on Borsa Istanbul.

- **Expert, Department of Enforcement, Capital Markets Board of Turkey**
  Sep 2005 - Apr 2008

  - Inspection of financial crimes in capital markets such as manipulation, insider trading, unauthorized capital market activities, disseminating misleading information.

- **Expert, Department of Research, Capital Markets Board of Turkey**
  Jul 2004 - Sep 2005

  - Research regarding developments in capital markets.
  - Preparation of presentations and speeches for President of Capital Markets Board of Turkey

- **Assistant Expert, Department of Corporate Finance, Capital Markets Board of Turkey**
  Dec 2001 - Jul 2004

  - Evaluation of applications for initial and secondary public offerings, mergers and acquisitions and adoption of registered capital system.
  - Providing advisory services for companies and investors to help them clarify Capital Market Regulations.
  - Review of the quarterly and annual reports, disclosed to the public or presented to the Capital Markets Board of Turkey, of corporations whose shares are listed on Borsa Istanbul

HONORS AND AWARDS

- **Full Scholarship by the Capital Markets Board of Turkey during 2009-2010**
  For the M.Sc. in Quantitative Finance Programme at the London Business School

- **Best Thesis Award in 2008**
  2006-2007 Academic Year Middle East Technical University-Best Thesis Award with the Thesis Titled "How Does The Stock Market Volatility Change After Inception of Futures Trading: An Empirical Analysis of Istanbul Stock Exchange"

- **Highest Score in Capital Markets Board of Turkey's Employee Selection Exam in 2001**
  Highest score among 300 candidates in an election process including 6 written exams and 1 oral exam

- **Second Highest Grade in Civil Servant Selection Exam in 2001**
  Second highest grade among thousands of test takers

- **Merit Based Scholarship in 1997**
  Scholarship is granted by the Capital Markets Board of Turkey on the basis of University Entrance Exam Score

- **University Entrance Exam Score in %99.9 percentile**
  Placed as 130th in University Entrance Exam among 1.5 million test takers

RESEARCH HISTORY

Manuscripts

- **Unpublished Term Project with Emre Köker, 2010, London Business School**
  Title: “Comparison of Volatility Models Using Intraday and Interday Data”
  Supervisor: Prof. Dr. Michail Chernov, London Business School

- **Unpublished Masters Thesis, 2007, Department of Financial Engineering, Middle East Technical University**
  Title: "How Does The Stock Market Volatility Change After Inception of Futures Trading: An Empirical Analysis of Istanbul Stock Exchange"
  Supervisor: Assoc. Prof. Dr. Seza Danışoğlu, Middle East Technical University

- **Unpublished Research Report, 2005, Department of Research, Capital Markets Board of Turkey**
  Title: “Efficiency Measurement in Developing Stock Markets Using Data Envelopment Analysis”

- **Unpublished Thesis, 2004, Capital Markets Board of Turkey**
  Title: “The Problem of Unauthorized Public Offerings in Turkey: Discussion On Potential Remedies”
  Supervisor: Senior Expert Yıldırım Akar

Conference Presentations

- ASSA Middle East Economic Association 2015 Meeting, January 2015, Boston, USA: “Stock Market Volatility and Futures Trading” Coauthor: Assoc. Prof. Dr. Seza Danışoğlu