ON SOME CONSEQUENCES OF THE ISOMORPHIC CLASSIFICATION OF CARTESIAN PRODUCTS OF LOCALLY CONVEX SPACES

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ABSTRACT

ON SOME CONSEQUENCES OF THE ISOMORPHIC CLASSIFICATION OF CARTESIAN PRODUCTS OF LOCALLY CONVEX SPACES

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This thesis takes its motivation from the theory of isomorphic classification of Cartesian products of locally convex spaces which was introduced by V. P. Zahariuta in 1973. In the case $X_1 \times X_2 \simeq Y_1 \times Y_2$ for locally convex spaces X_i and $Y_i, i = 1, 2$; it is proved that if X_1, Y_2 and Y_1, X_2 are in compact relation in operator sense, it is possible to say that the respective factors of the Cartesian products are also isomorphic, up to their some finite dimensional subspaces. Zahariuta's theory has been comprehensively studied for special classes of locally convex spaces, especially for finite and infinite type power series spaces under a weaker operator relation, namely strictly singular. In this work we give several sufficient conditions for such operator relations, and give a complete characterization in a particular case. We also show that a locally convex space property, called the smallness up to a complemented Banach subspace property, whose definition is one of the consequences of isomorphic classification theory, passes to topological tensor products when the first factor is nuclear. Another result is about Fréchet spaces when there exists a factorized unbounded operator between them. We show that such a triple of Fréchet spaces (X, Z, Y) has a common nuclear Köthe subspace if the range space has a property called (y) which was defined by Önal and Terzioğlu in 1990.

Keywords: Isomorphic classification of Cartesian products, unbounded operators, strictly singular operators, compact operators, smallness up to a complemented Banach subspace property

ÖZ

YEREL KONVEKS UZAYLARIN KARTEZYEN ÇARPIMLARININ İZOMORFİK SINIFLANDIRILMASININ BAZI SONUÇLARI ÜZERİNE

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Bu tez motivasyonunu V. P. Zahariuta tarafından öncülük edilen yerel konveks uzayların Kartezyen çarpımlarının izomorfik sınıflandırılması teorisinden almaktadır. X_i ve Y_i, $i = 1, 2$ yerel konveks uzayları verilmiş olsun. $X_1 \times X_2 \simeq Y_1 \times Y_2$ durumunda çarpan uzayların da sonlu boyutlu birer altuzay hariç izomorfik olabilmesi için X_1, Y_2 ve Y_1, X_2 uzayları arasında operatör teorisi bağlamında bir kompakt bağıntı olması gerektiği ispatlanmıştır. Zahariuta'nın bu teorisi daha sonra sonlu ve sonsuz tipi kuvvet toplamlı uzaylar başta olmak üzere bazı özel yerel konveks uzaylar için daha zayıf operatör bağıntıları-strictly singular-altında detaylı bir şekilde ele alınmıştır. Bu çalışmada söz konusu operatör bağıntılarının varlığı için yeterli koşullar türetilmiş ve belli bir durumda karakterizasyon elde edilmiştir. Bunun dışında izomorfik sınıflandırma teorisinin sonuçlarından biri olarak yerel konveks uzaylar için tanımlanan SCBS (tümlenebilen bir Banach altuzayı dışında yeterince küçük olma) özelliğinin, ilk çarpanın nükleer olması koşuluyla, topolojik tensör çarpımına geçtiği ispatlanmıştır. Bir diğer sonuç ise Fréchet uzayları üzerine olup, iki Fréchet uzayı arasında tanımlı üçüncü bir Fréchet uzayı üzerinden çarpanlarına ayrılan bir sınırsız operatörün varlığına dayanmaktadır. Bu durumun sonucunda bu üç uzayın ortak nükleer Köthe altuzayı olabilmesi için, görüntü uzayında Önal ve Terzioğlu tarafından 1990'da tanımlanan (y) özelliğinin olmasının yeterli olduğu ispatlanmıştır.

Anahtar Kelimeler: Kartezyen çarpımların izomorfik sınıflandırılması, sınırsız operatörler, strictly singular operatörler, kompakt operatörler, SCBS özelliği

To the memory of

Prof. Dr. Tosun Terzioğlu

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LIST OF NOTATIONS

Abbreviations

ICCP	The isomorphic classificaion of Cartesian products
tvs	Topological vector space
lcs	Locally convex space
pss	Power series space
wsc	Weakly sequentially complete (Banach space)
SP	Schur property
DPP	Dunford-Pettis property

General

$\mathscr{U}(X)$	The base of absolutely convex closed neighborhoods of the topological vector space X.
$\mathscr{B}(X)$	The class of bounded subsets of the space X.
$X \simeq Y$	There exists a topological isomorphism between the topological vector spaces (X, τ_1) and (Y, τ_2).
$\theta(X)$	The origin of the topological vector space X.
$V \leq U$	V is a (infinite dimensional closed) subspace of U.
$\mathfrak{L}(X, Y)$	The set of linear continuous operators defined on X into Y.
$\mathfrak{K}(X, Y)$	The set of compact operators defined on X into Y.
$\mathfrak{W}(X, Y)$	The set of weakly compact operators defined on X into Y.
$\mathfrak{V}(X, Y)$	The set of completely continuous operators from X into Y.
$\mathfrak{S}(X, Y)$	The set of strictly singular operators defined on X into Y.
$\mathfrak{B}(X, Y)$	The set of bounded operators defined on X into Y.
$(X, Z, Y) \in \mathfrak{BF}$	The triple (X, Z, Y) has bounded factorization property.
P	A class of Banach spaces satisfying a property P.
$\mathfrak{s}(\mathsf{P})$	The class of locally convex spaces with local Banach spaces each of which belongs to P.

$\mathfrak{s}(\mathsf{P}^{\neg})$	The class of locally convex spaces with local Banach spaces each of which having no infinite dimensional subspaces belonging to P.
ω	The set of all scalar sequences.
$\overline{\text{acx}}(A)$	Absolutely convex closed hull of the set A.
$\overline{\text{co}}(A)$	Closed hull of the set A.

CHAPTER 1

INTRODUCTION

The set of results obtained in this thesis is in connection with the theory of isomorphic classification of Cartesian products (ICCP) of locally convex spaces (lcs) which was initiated by the remarkable note of Zahariuta [70] published in 1973. In that paper he defined and characterized a relation between locally convex spaces X and Y called the relation $\mathfrak{K}$ which means that every continuous linear operator $T : X \to Y$ is compact. It is proved that for lcs's $X = X_1 \times X_2$ and $Y = Y_1 \times Y_2$ with $(X_1, Y_2) \in \mathfrak{K}$ and $(Y_1, X_2) \in \mathfrak{K}$ being isomorphic to each other is equivalent to the case that the factors are near isomorphic, that is, they are isomorphic up to their some finite dimensional subspaces. Namely, he made use of Fredholm operator theory to compose an ICCP of locally convex spaces. In Chapter 2, we introduce some results concerning the relation $\mathfrak{K}$ in the class of Banach spaces. We then briefly mention strictly singular operators on lcs's in the context of operator ideals and emphasize the situation when their class is an operator ideal. The last part of Chapter 2 is devoted to unbounded operators and their factorization in Fréchet spaces. We prove that the existence of an unbounded operator $T : X \to Y$ over a third Fréchet space Z causes the existence of a common nuclear Köthe subspace of the triple (X, Z, Y) when the range space has the property (y), which was introduced by Önal and Terzioğlu [60].

In 1998, Djakov, Önal, Terzioğlu and Yurdakul [20] investigated the ICCP of a special class of lcs's, called finite and infinite type power series spaces (pss). They modified Zahariuta's method to obtain a similar ICCP of pss's with the help of a weaker operator theoretic relation. This relation is based on the type of operators called strictly singular. We denote $(X, Y) \in \mathfrak{S}$ iff every operator $T : X \to Y$ is strictly singular. In

Chapter 3, we give sufficient conditions to obtain this relation under some conditions. First we introduce such conditions in terms of Banach spaces, and then we extend some of them to the class of lcs's via projective limit topologies and Grothendieck space ideals. It is even possible to claim a characterization when we slightly modify the assumptions for $(X, Y) \in \mathfrak{K}$. These results are helpful to extend the ICCP of pss's to general lcs's. In Chapter 3, we also revisit the advances in the ICCP of lcs's. We see the consequences of changing the assumptions on various types of spaces in Zahariuta's theorem. Here we also mention about bounded operators. It is denoted $(X, Y) \in \mathfrak{B}$ when every operator between X and Y is bounded. Referring to the note of Djakov, Terzioğlu, Yurdakul, and Zahariuta [19] which investigate the ICCP of Fréchet spaces up to basic Banach subspaces under the assumption of $(X, Y) \in \mathfrak{B}$, we finally setup the basis of the definition of smallness up to a complemented Banach subspace property (SCBS) which is enjoyed by all Köthe spaces. In Chapter 4, we give its definition and prove that it is stable under topological tensor products, provided that the first factor is nuclear. We also mention the class of ℓ-Köthe spaces as a type of generalized Köthe spaces, in which the canonical basis $\{e_n\}$ is an unconditional one. Their topological tensor products are not known explicitly. However, with the help our result, we deduce that this product has the SCBS property when the first factor is nuclear.

CHAPTER 2

OPERATORS IN LOCALLY CONVEX SPACES

2.1 Preliminaries

In this chapter, we focus on the operator theory of lcs's. Our concentration will be on compact, strictly singular, bounded and unbounded operators and their roles in composing relations between lcs's. We give results concerning compact and unbounded operators. Results on compact operators will be in terms of Banach spaces which are in particular locally convex. These results rest on weak and strong convergence, hereditary properties, some other well-known vector space properties such as Schur property (SP), Dunford-Pettis property (DPP), approximation property and so on. The duality theory of Banach spaces is also used in the proofs. We will continue our discussion of Banach spaces in Chapter 3 in strictly singular operators perspective. We then extend some of these results to the general class by means of projective limit topologies, and Grothendieck space ideals. The results for unbounded operators highly depend on failure of bounded factorization property and continuous norm arguments. There we consider the class of Fréchet spaces. Now let us define the tools we need for the proofs.

A vector space X over the field $\mathbb{K}$ is said to be a topological vector space (tvs) denoted (X, τ) if X is equipped with the Hausdorff topology τ which is compatible with its vector space structure (the maps $+ : X \times X \to X$ and $\cdot : \mathbb{K} \times X \to X$ are continuous). A tvs (X, τ) is said to be locally convex if it has a base of neighborhoods $\mathscr{U} = \{U_\alpha\}$ of the origin consisting of convex sets U_α. Let such $\mathscr{U}$ be a filter-base of absolutely convex absorbent (a subset S is called absorbent if for all $x \in X$ there exists $r \in \mathbb{R}$

such that for all $\alpha \in \mathbb{K}$, $|\alpha| \geq r$ implies $x \in \alpha S$) subsets U_α of a vector space X with $\bigcap\limits_{\alpha} U_\alpha = \theta$. If each set $\rho U_\alpha \in \mathscr{U}$ for $\rho > 0$ when $U_\alpha \in \mathscr{U}$, then a lcs (X, τ) is defined by considering $\mathscr{U}$ as a base of neighborhoods of the origin. Each lcs can be constructed this way. Alternatively, let $\{p_\alpha(x)\}$ be a system of semi-norms on a vector space X, such that for each $x_0 \neq \theta$ there is at least one p_α with $p_\alpha(x_0) \neq 0$. If $U_\alpha = \{x \in X : p_\alpha(x) \leq 1\}$, then the system ρU $(\rho > 0)$ of $U = \bigcap\limits_{\iota=1}^{n} U_{\alpha_\iota}$. This base is composed of absolutely convex open sets. Each lcs can also be constructed this way. A complete metrizable lcs is called a Fréchet space. The metrizable lcs (X, τ) can always be topologized by a system of absolutely convex neighborhoods of $\theta(X)$. This system constitutes a decreasing sequence. The latter is equivalent to the topology generated by the increasing sequence of semi-norms associated to these neighborhoods. A vector space X is called a normed space if its topology is given by a norm, which is a functional satisfying norm axioms. A complete normed space is called a Banach space. For a more detailed description, the reader is referred to [38].

Let U be an absolutely convex closed neighborhood of a lcs X. $N(U) = p_U^{-1}(0)$ is a closed subspace of X, where p_U is the gauge functional of U. Let $X_U := X/p_U^{-1}(0)$ be the quotient space with the norm induced by $p_U(\cdot)$. Its dual is the Banach space $X'[U^\circ] := \bigcup\limits_{n=1}^{\infty} nU^\circ$ (cf. Definition 2.4.2) with the norm defined by U°. If $V \subset \rho U$ for some $\rho > 0$ and $V \in \mathscr{U}(X)$ also, then $N(U) \subset N(V)$. Let $\pi_U : X \to X_U$ be the canonical quotient map. Then for all U, one can find $V \subset U$ such that there exists $\phi_{UV} : X_V \to X_U$ making the following diagram commutative.

$$\begin{array}{ccc} X & \xrightarrow{I_X} & X \\ \Big\downarrow{\scriptstyle \pi_V} & & \Big\downarrow{\scriptstyle \pi_U} \\ \widetilde{X_V} & \xrightarrow[\phi_{UV}]{} & \widetilde{X_U} \end{array}$$

If $\phi_{UV} \in \mathfrak{A}(X_V, X_U)$, for a pre-ideal $\mathfrak{A}$ of operators then X is called a Grothendieck $\mathfrak{A}$-space and we denote $X \in Groth(\mathfrak{A})$. Here, $\widetilde{X_U}$ and $\widetilde{X_V}$ are Banach spaces. This construction may be useful to get a grip on the nuclearity assumption in Theorem 4.2.1. A nuclear space X actually belongs to $Groth(\mathfrak{N})$, where $\mathfrak{N}$ denotes the class of nuclear operators. Almost every class of well-known lcs's is generated by an operator ideal. A non-example is the class of Montel spaces. For necessary conditions for a class of Hausdorff lcs's to be generated by an operator ideal one may read [9, Theorem 1].

Given a family $\{\phi_k : X \to X_k\}$ of linear maps from a tvs X to tvs's X_k, the projective topology induced on X by the family is the weakest topology on X which makes each of the maps ϕ_k continuous. A family $\{X_k, \phi_{km}\}$ where k, m belong to a directed set $\mathfrak{I}$, X_k is a tvs for each $k \in \mathfrak{I}$, $\{\phi_{km} : X_m \to X_k\}$ is a continuous linear map for each pair $k, m \in \mathfrak{I}$ with $k < m$ and $\phi_{km} \circ \phi_{mn} = \phi_{kn}$ whenever $k < m < n$ is called an inverse directed system of tvs's. The projective limit $\varprojlim X_k$ of such a system is the subspace of the Cartesian product $\prod X_k$ consisting of elements $\{x_k\}$ which satisfy $\phi_{km}(x_m) = x_k, \quad k < m$. The projective limit $\varprojlim X_k$ is a closed subspace of $\prod X_k$ and has the projective topology induced by the family of maps $\{\phi_k : \varprojlim X_k \to X_k\}$ where ϕ_k is the inclusion $\varprojlim X_k \to \prod X_k$ followed by the projection on X_k. If X is a lcs, and $\{X_k\}$ are the local Banach spaces for $k \in \mathfrak{I}$ then the canonical mappings are $\phi_n : X \to X_n$, for $n \in \mathfrak{I}$ and $\phi_{nm} : X_n \to X_m$, for $n < m$.

Throughout, unless otherwise stated, a "subspace" always means an infinite dimensional closed subspace, and will be denoted $Y \leq X$.

2.2 Compact operators

Definition 2.2.1 *Let X and Y be lcs's. $T \in \mathfrak{L}(X, Y)$ is called (weakly) compact if there exists a zero neighborhood U of X such that its image $T(U)$ is (weakly) precompact in Y.*

As usual we denote $(X, Y) \in \mathfrak{K}$ (resp. $(X, Y) \in \mathfrak{W}$) when any operator from X to Y is compact (resp. weakly compact). Zahariuta [70, Proposition 1.1] characterized $(X, Y) \in \mathfrak{K}$ in the following sense: Y is a pre-Montel (each bounded subset B of Y is pre-compact) lcs iff $(X, Y) \in \mathfrak{K}$ for each normed space X. Obviously we have $\mathfrak{K}(X, Y) \subset \mathfrak{W}(X, Y)$. The converse is not true in general, unless the domain space has the Schur property (the equivalence of weak and strong convergence) [39, Lemma 9]. Note that (weakly) compact operators have the conjugacy property on Banach spaces, that is, $T' : Y' \to X'$ is (weakly) compact iff $T : X \to Y$ is (weakly) compact. In this section, we introduce some results concerning sufficient conditions for $(X, Y) \in \mathfrak{K}$, where X and Y are Banach spaces. For similar results in the category of lcs's, the reader is referred to [3], [14], [15], [48], and [70, Section III]. To work with Banach spaces, we will need some concepts related to sequences and equivalence of their convergence with respect to different topologies. A Banach space X is said to be weakly sequentially complete (wsc) if weakly Cauchy sequences in X converges weakly. It is called almost reflexive if every bounded sequence $(x_n) \in X$ has a weakly Cauchy subsequence.

Theorem 2.2.2 *Let X' have SP and let Y be wsc. Then, $(X,Y) \in \mathfrak{K}$.*

Proof By [39, Corollary 11], $(X,Y) \in \mathfrak{W}$. Let $T : X \to Y$. Then, the conjugate map $T' : Y' \to X'$ is also weakly compact. Then, T' maps bounded sequences in Y' into the sequences in X' which have weakly convergent subsequences. But X' has SP, so those weakly convergent subsequences converge in norm. In other words, T' is compact. By the conjugacy property, T is also compact. Therefore, $(X,Y) \in \mathfrak{K}$.

Lemma 2.2.3 *[57] A Banach space X is almost reflexive iff $\ell^1 \not\hookrightarrow X$.*

Theorem 2.2.4 *Let X' and Y have SP. Then, $(X,Y) \in \mathfrak{K}$.*

Proof By [42] and [59], $\mathfrak{L}(X,Y)$ has SP. Now suppose there exists $T \in \mathfrak{L}(X,Y)$ which fails to be compact. Since Y has SP, T cannot be almost weakly compact because if so, for every bounded sequence $(x_n) \in X$, there would exist a subsequence $(x_{kn}) \in X$ such that $T(x_{kn})$ is convergent. So there exists a sequence $(\gamma_n) \in X$ with no weak Cauchy subsequence. That implies by Lemma 2.2.3 that $\ell^1 \hookrightarrow X$. However, by [18], X is almost reflexive. Contradiction.

2.3 Strictly singular operators

Definition 2.3.1 *Let X and Y be lcs's. $T \in \mathfrak{L}(X,Y)$ is said to be strictly singular if for any $M \leq X$, the restriction $T|_M$ is not a topological isomorphism.*

The definition above, which is a generalization of the concept of compact operators in Banach spaces, is due to Kato [33]. In this work, Kato also proved that the operator ideal property is preserved. Later, van Dulst first [63] stated the obvious generalization of strictly singular operators acting on the general class of vector spaces and gave characterization on Ptak spaces (or B-complete spaces). Then he pursued his investigation for generalized Hilbert spaces in [64]. Goldberg [24] proved that a linear operator $T : X \to Y$ is stricly singular iff for every $M \leq X$ there exists $N \leq M$ such that the restiction $T|_N$ has a norm not exceeding ε, for any positive ε. The conjugacy property of strictly singular operators acting on Banach spaces is shown to be absent by Goldberg and Thorp [25] with a counterexample. The conditions under which this property exists are investigated by Whitley in [67].

Note that a compact operator $T : X \to Y$ is strictly singular (see [11, Theorem 10.3.2] for a relatively new proof), while the converse is not true in general (the injection map $\iota : \ell^p \hookrightarrow \ell^q, 1 \leq p < q < \infty$). If the pair (X, Y) belongs to the class of Hilbert spaces, then $\mathfrak{S}(X,Y) = \mathfrak{K}(X,Y)$. It is also understood that $\mathfrak{K}(X) = \mathfrak{S}(X)$, if $X = \ell^p$ for $1 \leq p < \infty$ or $X = c_0$. Alternatively, a non-compact operator is non-strictly singular if it fixes an isomorphic copy of ℓ^p, that is, for a bounded linear operator $T : X \to Y$ there exists $M \leq X$ with $M \simeq \ell^p$ for which $\exists \alpha > 0$ so that $\|Tx\| \geq \alpha \|x\|$, $\forall x \in M$. To find the equivalence of strictly singular and compact operators on Hardy and Bergman spaces in terms of composition, Hankel, Toeplitz, or Volterra-type operators is also a popular research area recently [31], [40], [44]. Strictly singular operators also have a role in the theory of invariant subspaces [4], [11], [26], [27]. Fortunately, strictly singular operators are somehow ubiquitiously compact. That is why they are called semi-compact in some resources. Wrobel [68] characterized strictly singular operators on lcs's for the class of B_r-complete spaces. There he used Ptak-type B_r-completeness. To understand that, let X be a barrelled lcs (every barrelled set in X is a neighborhood of the origin), and let $T : Y \to X$ be a linear map where Y is a barrelled space which has a closed graph. If any such T is continuous, X is then called a B_r-complete lcs.

Lemma 2.3.2 *[68, Theorem 1-IV] Let X and Y be B_r-complete lcs's. Then, TFAE*

1. *$T \in \mathfrak{S}(X, Y)$.*

2. *For every $M \leq X$, there exists $N \leq M$ such that $T|_N$ is precompact.*

Now let us investigate strictly singular operators in the operator ideal perspective.

Definition 2.3.3 *An operator ideal $\mathfrak{A}(X,Y) \leq \mathfrak{L}(X,Y)$ satisfies the following conditions:*

1. $I_{\mathbb{K}} \in \mathfrak{A}$.
2. *If* $A, B \in \mathfrak{A}(X,Y)$*, then* $A + B \in \mathfrak{A}(X,Y)$.
3. *If* $C \in \mathfrak{L}(E,X), B \in \mathfrak{A}(X,Y), A \in \mathfrak{L}(Y,F)$*, then* $A \circ B \circ C \in \mathfrak{A}(E,F)$.

$\mathfrak{S}(X,Y)$ is a non-surjective operator ideal, if the pair (X,Y) belongs to the class of Banach spaces. As proved in [17] by construction of a non-strictly singular operator which can be written as the summation of two strictly singular operators, this is not the case when it belongs to the class of general lcs's. Remember that $\mathfrak{K}(X,Y)$ is an operator ideal in lcs's. With the help of the latter, we readily prove the following using Wrobel's characterization.

Theorem 2.3.4 *Let X and Y be B_r-complete lcs's. Then, $\mathfrak{S}(X,Y)$ forms an operator ideal.*

Proof Suppose that $T : X \to Y$ and $S : X \to Y$ are strictly singular operators. Then, for any $M \leq X$, by Lemma 2.3.2, find $N \leq M$ such that $T|_N$ is precompact. Then find $P \leq N$ such that $S|_P$ is precompact. The ideal property of precompact operators on lcs's yields the result.

Let X and Y be lcs's. For $M \leq X$, $\alpha \in I$ and $\beta \in J$, $\omega_{\alpha\beta}(T|_M) := \sup\{q_\beta(Tx) : p_\alpha(x) \leq 1, x \in M\}$. The following is a characterization of strictly singular operators in lcs's.

Lemma 2.3.5 *[46, Theorem 2.1] Let X and Y be lcs's and let $T \in \mathfrak{L}(X,Y)$. Then T is strictly singular iff for any $\varepsilon > 0$, $\beta \in J$ and $M \leq X$ there exists $\alpha_0 \in I$ and there exists $N \leq M$ such that $\omega_{\alpha\beta}(T|_N) \leq \varepsilon$ for all α.*

The following theorem is an extension of [2, Problem 4.5.2] to B_r-complete lcs's by means of Theorem 2.3.4 and Lemma 2.3.5.

Theorem 2.3.6 *Let $X_i, i = 1, 2, \dots n$ and $Y_j, j = 1, 2, \dots m$ be B_r-complete lcs's, and let $\tau : \bigoplus_{i=1}^{n} X_i \to \bigoplus_{j=1}^{m} Y_j$ be continuous. τ can be represented by uniquely determined continuous operators $T_{ji} : X_i \to Y_j$ so that the matrix representation of τ is*

$$\begin{bmatrix} T_{11} & T_{12} & \dots & T_{1n} \\ T_{21} & T_{22} & \dots & T_{2n} \\ \vdots & & \ddots & \vdots \\ T_{n1} & T_{n2} & \dots & T_{nn} \end{bmatrix}$$

Then, τ is strictly singular iff each of T_{ij} is strictly singular for each $i = 1, 2, \dots, n$ and for each $j = 1, 2, \dots, n$.

Proof Let $X := \bigoplus_{i=1}^{n} X_i$ and $Y := \bigoplus_{j=1}^{m} Y_j$, for simplicity of notation, and assume that each T_{ji} is strictly singular. Let $\pi_i : X \to X_i$ be the canonical projection and define $\rho_j : Y_j \to Y$ by $\rho y_j = 0 \oplus 0 \oplus \cdots \oplus 0 \oplus y_j \oplus 0 \oplus \cdots \oplus 0$, for which y_j is the j-th summand. Consider $X \xrightarrow{\pi_i} X_i \xrightarrow{T_{ji}} Y_j \xrightarrow{\rho_j} Y$, and write $\tau_{ji} = \rho_j \circ T_{ji} \circ \pi_i$. Then $\tau_{ji}(x_1 \oplus x_2 \oplus \cdots \oplus x_n) = 0 \oplus 0 \oplus \cdots \oplus 0 \oplus T_{ji}x_i \oplus 0 \oplus \cdots \oplus 0$, where T_{ji} is the j-th summand. By Theorem 2.3.4 and rewriting $\tau = \sum_{i=1}^{n} \sum_{j=1}^{m} \tau_{ji}$, τ is strictly singular.

For the converse, let $\tau \in \mathfrak{S}(X, Y)$, and suppose that the operator T_{ji} is not strictly singular for some i, j. Then by Lemma 2.3.5, for any $M \le X_i$ and for some $\beta \in J$, $\omega_{\alpha\beta}(\tau_N) > \varepsilon$, for all $\alpha \in I$. If we write $\hat{M} := \{0\} \oplus \{0\} \oplus \cdots \oplus M \oplus \{0\} \oplus \cdots \oplus \{0\}$ where M places in the i-th summand, $\hat{M}$ is a vector subspace of X. $\omega_{\alpha\beta}(\tau_{\hat{M}}) > \varepsilon$, for all $\alpha \in I$. But, that contradicts the assumption τ is strictly singular.

We refer the reader to Section 3.2 for sufficient conditions satisfying $(X, Y) \in \mathfrak{S}$ for which we also give a characterization. Concerning the smallest surjective operator ideal $\mathfrak{S}^S(X, Y)$ containing $\mathfrak{S}(X, Y)$, Weis [66] proved that X is almost reflexive iff $(X, Y) \in \mathfrak{S}^S$.

2.4 Factorized unbounded operators

Definition 2.4.1 *A linear operator $T : X \to Y$ between lcs's is called bounded if there exists a neighborhood U of $\theta(X)$ whose image $T(U)$ is a bounded set in Y.*

An operator T is bounded iff it can be factored over a normed space. T is called almost bounded if it can be factored over $\omega \times B$, where ω is the set of all sequences (it admits no continuous norm) and B is a Banach space. We call T unbounded, when it is not necessarily bounded. A triple (X, Z, Y) is said to have the bounded factorization property and it is written $(X, Z, Y) \in \mathfrak{BF}$ if each $T \in \mathfrak{L}(X, Y)$ that factors over Z (that is, $T = R_1 \circ R_2$, where $R_2 \in \mathfrak{L}(X, Z)$ and $R_1 \in \mathfrak{L}(Z, Y)$) is bounded.

Nurlu and Terzioğlu [49] proved that under some conditions, existence of continuous linear unbounded operators between nuclear Köthe spaces causes existence of common basic subspaces. Djakov and Ramanujan [21] sharpened this work by removing nuclearity assumption and using a weaker splitting condition. In [62], it is shown that the existence of an unbounded factorized operator for a triple of Köthe spaces, under some assumptions, implies the existence of a common basic subspace for at least two of the spaces. Concerning the class of general Fréchet spaces, the existence of an unbounded operator in between is studied in [61]. The motivation was there the paper of Bessaga, Pelczynski, and Rolewicz [7] in which it is proved that any non-normable Fréchet space which is not isomorphic to $\omega \times B$ where B is a Banach space, contains a subspace isomorphic to a Köthe space. In the light of [7, Theorem 1] it is proved that the existence of a continuous unbounded linear operator $T : X \to Y$ implies that X and Y have a closed common nuclear subspace when Y has a basis, and admits a continuous norm. When the range space has the property (y), that implies the existence of a common nuclear Köthe quotient as proved in [60]. Combining these two results, when the range space has the property (y), common nuclear Köthe subspace is obtained in [52, Proposition 1]. The aim of this section is to prove the Fréchet space analogue of [62, Proposition 6], that is, under the condition that Y has property (y), and $(X, Z, Y) \notin \mathfrak{BF}$ there is a common nuclear subspace for all three spaces.

Definition 2.4.2 *A lcs X with neighborhood base $\mathscr{U}(X)$ is said to have property (y)*

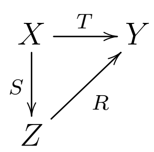
if there is a neighborhood $U_1 \in \mathscr{U}(X)$ such that

$$X' = \bigcup_{U \in \mathscr{U}(X)} \overline{X'[U_1^\circ] \cap U^\circ}^{\langle X, X' \rangle},$$

Condition (y) implies that $X'[U_1^\circ]$ is dense in X' and hence p_{U_1} is a continuous norm on X (See also [50]). Property (y) is equivalent to being locally closed [65, Lemma 2.1], or being isomorphic to a closed subspace of a Köthe space admitting a continuous norm[65, Theorem 2.3].

Theorem 2.4.3 *Let X, Y, Z be Fréchet spaces where Y has property (y). Assume there is a continuous, linear, unbounded operator $T : X \to Y$ which factors through Z such as $T = R \circ S$. Then, there exists a nuclear Köthe subspace $M \leq X$ such that the restriction $T|_M$ and the restriction $R|_{S(M)}$ are isomorphisms.*

Proof Let $T : X \to Y$ be an unbounded operator factoring through Z.



By [52, Proposition 1], there exists a closed nuclear Köthe subspace M of X such that the restriction $T|_M$ is an isomorphism onto $T(M)$. Since T is injective on M, R is injective on $S(M)$ and maps $S(M)$ onto $T(M) = R(S(M))$. Now let $y \in \overline{S(M)}$. So find a sequence $(S(m_n))_{n\in\mathbb{N}}$ in $S(M)$ such that $\lim S(m_n) = y$. R is continuous at y, then $\lim RS(m_n) = \lim T(m_n) = Ry \in \overline{T(M)} = T(M)$, since $T(M)$ is closed. Thus $\lim T(m_n) = T(m) = Ry$ for some $m \in M$. Since T is an isomorphism on M, $\lim T^{-1}T(m_n) = T^{-1}T(m)$, that is, $\lim m_n = m$. S is continuous at m, and that implies $\lim S(m_n) = S(m) = y \in S(M)$. Therefore $S(M)$ is closed. Hence $R : S(M) \to R(S(M))$ is an isomorphism by the Open Mapping Theorem.

Note that this result is consistent with [7, Theorem 1]. Indeed, if Z were isomorphic to $\omega \times B$ for a Banach space B, that is, T were almost bounded, then by [69, Proposition 1] there would be a continuous projection $P : Y \to Y$ such that $P(T(X)) = P(Y) \cong$

ω. Since Y has property (y), it has a continuous norm and therefore the operator T which factors over $\mathbb{K}^n \times B$ must be bounded.

The article [37] is composed of this section.

CHAPTER 3

ISOMORPHIC CLASSIFICATION OF CARTESIAN PRODUCTS OF LOCALLY CONVEX SPACES

Before we start our discussion on the concept of ICCP, let us re-state Zahariuta's result. Let $X^{(i)}$ denote an arbitrary subspace of the lcs X of codimension i (all such subspaces are isomorphic) when $i > 0$; and when $i < 0$ an arbitrary space $X \times Z$, where $\dim Z = -i$. Now let $X = X_1 \times X_2$ and $Y = Y_1 \times Y_2$ be lcs's such that $X \simeq Y$. Suppose $(X_1, Y_2) \in \mathfrak{K}$, $(Y_1, X_2) \in \mathfrak{K}$. Then $X \simeq Y$ iff there exists s such that $Y_1 \simeq X_1^{(s)}$, $Y_2 \simeq X_2^{(-s)}$ [70, Theorem 7.1]. His result highly depends on the assumptions $(X_1, Y_2) \in \mathfrak{K}$, $(Y_1, X_2) \in \mathfrak{K}$. In this chapter we see how weakening these assumptions slightly will affect the theory. As we mentioned before, compactness condition is stronger than that of strict singularity.

3.1 Strictly singular operators and isomorphic classification

Remember that we mentioned Zahariuta [70] made use of Fredholm operator theory to construct a method to classify Cartesian products of lcs's as we explained above. Djakov, Önal, Terzioğlu, and Yurdakul [20] modified Zahariuta's method for ℓ^p-finite and ℓ^q-infinite type pss's. Let $0 \le a_0 \le a_1 \le\nearrow \infty$, $r \in \{0, \infty\}$. Then $\Lambda_r(a) = \{x = (x_j) : \|x\|_t = \sum_j |x_j|_t e^{ta_i} < \infty, \forall t < r\}$ is called a pss, if $r = \infty$ of infinite type, if $r = 0$ of finite type. They proved that $\Lambda_0(a) \times \Lambda_\infty(b) \simeq \Lambda_0(\tilde{a}) \times \Lambda_\infty(\tilde{b})$ is equivalent to $\exists s$ such that $\Lambda_0(\tilde{a}) \simeq (\Lambda_0(a))^{(s)}$ and $\Lambda_\infty(\tilde{b}) \simeq (\Lambda_\infty(b))^{(-s)}$. In the proof, they made use of the fact that $(\Lambda_0(a), \Lambda_\infty(b)) \in \mathfrak{B}$ and by means of projective limit arguments applied through $(\ell^p, \ell^q) \in \mathfrak{S}$ to end up with $(\Lambda_0(a), \Lambda_\infty(\tilde{b})) \in \mathfrak{B}\mathfrak{S}$, and

$(\Lambda_0(\tilde{a}), \Lambda_\infty(b)) \in \mathfrak{BS}$. Those combined with the properties of factorized Riesz-type operators gives the result. Even if they did not assume any operator based relations in advance, it turns out that their selection of a special type of lcs's shows that actually they implicitly did. In this chapter, we focus on $(X,Y) \in \mathfrak{S}$ and $(X,Y) \in \mathfrak{BS}$ in a more general sense as far as the classes of domain and range spaces are concerned. We investigate the sufficient conditions implying these relations. Our starting point will be the class of Banach spaces and sequential arguments on it as we did in Section 2.2. There we extensively use H. Rosenthal's celebrated ℓ^1-Theorem [57], and intersections of operator classes explained in [39, Section II]. Again, with the help of projective limit structure, we extend some of our result to the class of lcs's and topological tensor products. In a particular case, we obtain a characterization (see Theorem 3.2.32). Our ultimate aim here is to enhance the classification in [20].

3.2 Sufficient conditions for $(X,Y) \in \mathfrak{S}$

The most commonly known example for $(X,Y) \in \mathfrak{S}$ in Banach space theory is when we choose $X = \ell^p$ and $Y = \ell^q$ such that $1 \leq p < q < \infty$. This example is non-trivial since $(\ell^p, \ell^q) \notin \mathfrak{K}$. This fact was cruical for the ICCP of pss's [20] since $\Lambda_0(a) = \varprojlim_k \ell^p(\exp(-1/k\, a_n))$ and $\Lambda_\infty(\tilde{b}) = \varprojlim_m \ell^{\tilde{q}}(\exp(k\tilde{b}_n))$. By virtue of Lemma 3.2.26 they reached that $(\Lambda_0(a), \Lambda_\infty(\tilde{b})) \in \mathfrak{S}$. The aim of this section is to generalize this classification by relaxing the choice of lcs's. To do that, we start with finding pairs of Banach spaces $(X,Y) \in \mathfrak{S}$. A pair of Banach spaces (X,Y) is called totally incomparable if there exists no Banach space Z which is isomorphic to a subspace of X and to a subspace of Y.

Theorem 3.2.1 $(X,Y) \in \mathfrak{S}$ *for every totally incomparable pair of Banach spaces* (X,Y).

Proof Suppose $T : X \to Y$ is a non-strictly singular operator. Then, we may find $M \leq X$ to which T restricted is an isomorphism. This means $M \simeq T(M)$. But $T(M) \leq Y$. Since X and Y are totally incomparable, this is impossible.

A Banach space X is said to be very-irreflexive if it contains no reflexive subspace.

It is said to be quasi-reflexive (of order n), if $X''/\pi(X)$ is of finite dimension n. A quasi-reflexive space is reflexive iff it is wsc [12, Theorem 4.4].

Theorem 3.2.2 $(X,Y) \in \mathfrak{S}$ *if*

1. *X is very-irreflexive and Y is reflexive [25].*

2. *X is very-irreflexive, and Y is quasi-reflexive.*

Proof 1. One may read the proof of [25, Theorem b].

2. Suppose there exists a non-strictly singular operator $T \in \mathfrak{L}(X,Y)$. Then, T is an isomorphism when restricted to $M \leq X$. Then, M is quasi-reflexive. However, by [30, Lemma 2] there exists a reflexive $N \leq M$. This contradicts the assumption X is very-irreflexive.

A property P on a Banach space X is called hereditary if it is enjoyed by all of its subspaces. X is said to have nowhere P if it has no subspace having the property P. Up on that, we give a generalized version of Theorem 3.2.2.

Theorem 3.2.3 *Let P be a property of Banach spaces which respects isomorphisms. Then, $(X,Y) \in \mathfrak{S}$ and $(Y,X) \in \mathfrak{S}$ if*

1. *Y has hereditary P,*

2. *X has nowhere P.*

simultaneously.

Proof Let X and Y be Banach spaces satisfying (1) and (2), and for some $M \leq X$ suppose there exists $T : X \to Y$ such that $M \simeq T(M)$. But $T(M)$ inherits P. Hence M has P. This contradicts (2). Now let $S : Y \to X$ be such that $N \simeq S(N) \subseteq X$ for some $N \leq Y$. Since X has nowhere P, $S(N)$ does not enjoy P. This contradicts (1).

Corollary 3.2.4 *Let Y be almost reflexive. For any X which is hereditarily ℓ^1, $(X,Y) \in \mathfrak{S}$.*

Corollary 3.2.5 $(X, \ell^1) \in \mathfrak{S}$ *for any almost reflexive* X.

Proof Let $M \leq X$ on which T has a bounded inverse. So $M \simeq T(M)$ which is a subspace of ℓ^1. By [41, Proposition 2.a.2], $T(M)$ contains a subspace Z with $Z \simeq \ell^1$, so does M. By a remark in [18], ℓ^1 can be embedded into X. Contradiction.

$T \in \mathfrak{L}(X, Y)$ is called completely continuous if for every weakly convergent sequence $(x_n) \in X$, (Tx_n) converges in Y-norm. By [54, Proposition 1.6.3] we also know that a completely continuous operator also maps weakly Cauchy sequences into norm convergent sequences. We denote $(X, Y) \in \mathfrak{V}$ if any such T is completely continuous. It is not hard to prove that $\mathfrak{K}(X, Y) \subset \mathfrak{V}(X, Y)$ [34]. The converse is also true iff the domain space is almost reflexive. A Banach space X is said to have the Dunford-Pettis property (DPP) if $\mathfrak{W}(X, Y) \subseteq \mathfrak{V}(X, Y)$, for any Banach space Y. X is said to have the reciprocal Dunford-Pettis property (rDPP), if $\mathfrak{V}(X, Y) \subseteq \mathfrak{W}(X, Y)$. One may browse [53] for some relations between DPP and SP in connection with the Radon-Nikodym property.

Lemma 3.2.6 *[39, Theorem 1.7] Let* Y *be almost reflexive. Then* $(X, Y) \in \mathfrak{V}$ *implies* $(X, Y) \in \mathfrak{S}$.

Theorem 3.2.7 *Let* X *have the SP and* Y *be almost reflexive. Then,* $(X, Y) \in \mathfrak{S}$.

Proof Any operator T with range Y maps bounded sequences into weakly Cauchy sequences, since Y is almost reflexive. On the other hand, any such T defined on X maps weakly Cauchy sequences into norm convergent sequences by the SP. That implies $(X, Y) \in \mathfrak{V}$. Hence by Lemma 3.2.6 the result follows.

Lemma 3.2.8 *Let* X *have SP. Then, for every* $M \leq X$, $\ell^1 \hookrightarrow M$.

Proof Let X have the SP and suppose there exists $M \leq X$ for which $\ell^1 \not\hookrightarrow M$. Then, any bounded sequence (x_n) in M, has a weakly Cauchy subsequence since M is equivalently almost reflexive. However, M inherits SP. Then the weakly Cauchy subsequence of (x_k) converges in X. Therefore, M is finite dimensional. Contradiction.

Despite the result above, being a hereditarily ℓ^1 space does not imply satisfying the SP. Indeed, some counter-examples are given in [6] and such a subspace of L^1 is constructed in [55]. As mentioned in [39], an almost reflexive Banach space has nowhere SP. Hence, the following is straightforward in the light of Corollary 3.2.5.

Corollary 3.2.9 *Let X have nowhere SP. Then, $(X, \ell^1) \in \mathfrak{S}$.*

Theorem 3.2.10 *Let Y have SP. Then $(X, Y) \in \mathfrak{S}$ for every reflexive X.*

Proof Let $T \in \mathfrak{L}(X, Y)$ have a bounded inverse on some $M \leq X$, that is, $M \simeq T(M)$. Since Y have SP, it also has the hereditary DPP [18]. Hence so does M. But M is reflexive. By [34, Theorem 2.1], reflexive spaces have nowhere DPP. Contradiction.

Corollary 3.2.11 *Let X', Y', Z' have the SP and let W be almost reflexive. Then, $((X\hat{\otimes}_\pi Y)', W\hat{\otimes}_\pi Z) \in \mathfrak{S}$.*

Proof By [39, Corollary 1.6], $(W, Z') \in \mathfrak{K}$ (cf. Section 2.2). So by [23, Theorem 3] we deduce $W\hat{\otimes}_\pi Z$ is almost reflexive. On the other hand, by [58, Theorem 3.3(b)] we reach that $\mathfrak{L}(X, Y')$ has SP. But in [59] it is proved that $\mathfrak{L}(X, Y') \simeq (X\hat{\otimes}_\pi Y)'$. So $(X\hat{\otimes}_\pi Y)'$ has SP. Therefore, Theorem 3.2.7 yields the result.

Lemma 3.2.12 *[18] X' has SP iff X has DPP and X is almost reflexive.*

Corollary 3.2.13 *Let X and Y be almost reflexive, X have DPP. Then, $(X', Y) \in \mathfrak{S}$.*

Proposition 3.2.14 *Let X and Y be almost reflexive and let Y have DPP. Then, $(X\hat{\otimes}_\pi Y, \ell^1) \in \mathfrak{S}$.*

Proof By Lemma 3.2.12, Y' has SP. Then by [39, Corollary 1.6], $(X, Y') \in \mathfrak{K}$. Hence, [23, Theorem 3] yields that $X\hat{\otimes}_\pi Y$ is almost reflexive. So by Corollary 3.2.5, we are done.

In the theorem above, DPP assumption on Y can be replaced by $(x_n \otimes y_n)$ is weakly null whenever (x_n) and (y_n) are weakly null [16, Theorem 5]. A Banach space X

is said to have the approximation property if every compact operator defined on X is the limit of a sequence of finite rank operators.

Proposition 3.2.15 *Let X and Y be reflexive spaces one of which having the approximation property, and let $(X, Y') \in \mathfrak{K}$. Let W and Z be spaces having SP. Then, $(X\hat{\otimes}_\pi Y, W\check{\otimes}_\varepsilon Z) \in \mathfrak{S}$.*

Proof By [59, Theorem 4.21], $X\hat{\otimes}_\pi Y$ is reflexive. By [42], SP respects injective tensor products. So $W\check{\otimes}_\varepsilon Z$ has SP. Then, Theorem 3.2.10 finishes the proof.

Proposition 3.2.16 *Let X have the hereditary DPP and let Y be reflexive. Then, $(X, Y) \in \mathfrak{S}$.*

Proof Let $M \leq X$ on which an arbitrary operator $T : X \to Y$ has a bounded inverse. Then, $M \simeq T(M)$. So, M is reflexive. So M cannot have DPP. Contradiction.

Any operator T defined on any complemented subspace M of each of $C(K)$, $B(S)$, $L_\infty(S, \Sigma, \mu)$, and $L(S, \Sigma, \mu)$ to a reflexive space is strictly singular [39].

Proposition 3.2.17 *Let $(X, Y) \in \mathfrak{W}$ where X has DPP. Then, $(X, Y) \in \mathfrak{S}$.*

Proof Since X has DPP, $(X, Y) \in \mathfrak{V}$. Then, by [39, Theorem 2.3], $(X, Y) \in \mathfrak{S}$.

Corollary 3.2.18 *Let $(X, Y) \in \mathfrak{V}$ where X has the rDPP. Then, $(X, Y) \in \mathfrak{S}$.*

Proof Since X has the rDPP and any $T : X \to Y$ is completely continuous, $T \in \mathfrak{V} \cap \mathfrak{W}$. By [39, Theorem 2.3] , we are done.

A Banach space X is called a Grothendieck space if any weak*-convergent sequence in X' is weakly convergent.

Theorem 3.2.19 *Let X be a Grothendieck space with the DPP, and let Y be separable. Then, $(X, Y) \in \mathfrak{S}$.*

Proof By [47, Theorem 4.9], any such operator $T : X \to Y$ is weakly compact. Since X has the DPP, T is completely continuous. Hence, by Corollary 3.2.17, $(X, Y) \in \mathfrak{S}$.

Example 3.2.20 Let K be a compact Hausdorff space and c_0 cannot be embedded into $C(K)$. Then, $C(K)$ is a Grothendieck space [56]. Hence, $(C(K), c_0) \in \mathfrak{S}$.

Theorem 3.2.21 *Let X be almost reflexive. Then, for any wsc Banach space Y, $(X, Y) \in \mathfrak{W}$.*

Proof Since X is almost reflexive, if (x_n) is a bounded sequence in X, then (Tx_n) has a weakly Cauchy sequence in Y. But Y is wsc, that is, every weakly Cauchy sequence converges weakly in Y. Therefore, T is weakly compact.

Corollary 3.2.22 *Let X be very-irreflexive and almost reflexive (see Example 3.2.23), let Y be wsc. Then, $(X, Y) \in \mathfrak{S}$.*

Proof By Theorem 3.2.21, $(X, Y) \in \mathfrak{W}$. Now let $T : X \to Y$ which has a bounded inverse on $M \leq X$. If (x_n) is a bounded sequence in M, then there exists (Tx_{kn}) a weakly convergent subsequence of (Tx_n) in Y. Hence (x_{kn}) is weakly convergent in M, since T has a bounded inverse on M. Thus, every bounded sequence in M has a weakly convergent subsequence in M. This is equivalent to saying that M is reflexive. Contradiction.

Example 3.2.23 Note that the non-reflexive space c_0 is almost reflexive. Suppose there exists a reflexive subspace E of c_0. Since c_0 fails SP, it is not isomorphic to any subspace of E. But this contradicts [41, Proposition 2.a.2].

Lemma 3.2.24 *Let X have the Dieudonné property (see [28] for definition) and let Y be wsc. Then, $(X, Y) \in \mathfrak{W}$.*

Proof Let $(x_n) \in X$ be weakly Cauchy, and let $T \in \mathfrak{L}(X, Y)$. Then (Tx_n) is weakly Cauchy in Y. Since Y is assumed to be wsc, (Tx_n) converges weakly. But X has the Dieudonné property, so $T \in \mathfrak{W}(X, Y)$.

Theorem 3.2.25 *Let X possess both DPP and Dieudonné properties (e.g. $C(K)$, where K is a compact Hausdorff space [28], [34]), and let Y be wsc. Then, $(X, Y) \in \mathfrak{S}$.*

Proof By Lemma 3.2.24, $(X, Y) \in \mathfrak{W}$. X possesses the DPP, so $(X, Y) \in \mathfrak{V}$. By [39, Theorem 2.3], $(X, Y) \in \mathfrak{S}$.

Now let us turn our attention to general class of lcs's. Let P be a class of Banach spaces having a certain property P. Then, $\mathfrak{s}(\mathsf{P})$ [10] is defined by the set of lcs's X with local Banach spaces $X_U \in \mathsf{P}$ for which X_U is the completion of the normed space obtained by $X/p_U^{-1}(0)$, where U is an absolutely convex closed neighborhood of $\theta(X)$ and p_U its gauge functional. For instance, by $\mathfrak{s}(\mathsf{X})$, we denote the class of lcs's such that each of their local Banach spaces are hereditarily ℓ^1. $\mathfrak{B}\mathfrak{S}(X, Y)$ denotes the intersection of $\mathfrak{B}(X, Y)$ and $\mathfrak{S}(X, Y)$. By virtue of [69, Proposition 1], we know that a strictly singular operator defined on a Fréchet space into a complete lcs cannot be unbounded since existence of such an unbounded operator contradicts with the result in [51]. We see that such an operator is almost bounded or bounded. If the range space admits a continuous norm, then we guarantee boundedness. A lcs X is called locally Rosenthal [8], if it can be written as a "projective limit of Banach spaces each of which contains no isomorpic copy of ℓ^1". A general lcs X with no copies of ℓ^1 need not to be locally Rosenthal. A counterexample might be found in [38]. In addition, one should assume that it is a quasinormable Fréchet space [45].

Lemma 3.2.26 *[20, Lemma 2] Let X and Y be lcs's such that $X = \varprojlim X_k$ and $Y = \varprojlim Y_m$, where $\{X_k\}$ and $\{Y_m\}$ are collections of Banach spaces. If $(X_k, Y_m) \in \mathfrak{S}$ for all m, k then $\mathfrak{B}(X, Y) = \mathfrak{S}(X, Y)$.*

Example 3.2.27 Let $\lambda_1(A) \in (d_2)$, and $\lambda_p(A) \in (d_1)$ as they are defined in [22]. Then, by [70], $(\lambda_1(A), \lambda_p(A)) \in \mathfrak{B}$. It is known that $\lambda_p(A) = \varprojlim \ell^p(a_n)$, for $1 \leq p < \infty$. Since $\ell^p(a_n), 1 < p < \infty$ has no subspace isomorphic to ℓ^1, $(\ell^1, \ell^p) \in \mathfrak{S}$. Then, by Lemma 3.2.26, $(\lambda_1(A), \lambda_p(A)) \in \mathfrak{B}\mathfrak{S}$.

Theorem 3.2.28 *Let X, Y be lcs's where Y is locally Rosenthal, and $X \in \mathfrak{s}(\mathsf{X})$. Then, $\mathfrak{B}(X, Y) = \mathfrak{S}(X, Y)$.*

Proof Since Y is locally Rosenthal, there exists a family of Banach spaces $\{Y_m\}$ each of which does not contain an isomorphic copy of ℓ^1 such that $Y = \varprojlim Y_m$. Because $X \in \mathfrak{s}(\mathsf{X})$, there exists a family of Banach spaces $\{X_k\}$ such that every $M_k \leq X_k$ contains a subspace isomorphic to ℓ^1. By Corollary 3.2.4, any linear operator $T_{mk} : X_k \to Y_m$ is strictly singular. Making use of Lemma 3.2.26, we deduce that every bounded operator $T : X \to Y$ is strictly singular.

By $\mathfrak{s}(\mathsf{V})$, we denote the class of lcs's with local Banach spaces each of which having SP. Notice that $Groth(\mathfrak{V}) \neq \mathfrak{s}(\mathsf{V})$, since the operator ideal $\mathfrak{V}$ on Banach spaces is not idempotent (see [54, pp. 60] for this property).

Corollary 3.2.29 *Let Y be a quasinormable Fréchet space with $\ell^1 \not\hookrightarrow Y$, and let $X \in \mathfrak{s}(\mathsf{V})$. Then, $(X, Y) \in \mathfrak{S}$.*

Proof By [45, Theorem 6], Y is locally Rosenthal. Since $X \in \mathfrak{s}(\mathsf{V})$, by Lemma 3.2.8, $X \in \mathfrak{s}(\mathsf{X})$. Then, by Theorem 3.2.28, we are done.

A lcs X is called infra-Schwartz if each of its local Banach spaces X_k is reflexive. An infra-Schwartz space turns out to be locally Rosenthal, as proved in [8]. Note that by [13], $Groth(\mathfrak{W}) = \mathfrak{s}(\mathsf{W})$, where W denotes the class of reflexive Banach spaces.

Corollary 3.2.30 *Let X be infra-Schwartz lcs, and let $Y \in \mathfrak{s}(\mathsf{V})$. Then, $\mathfrak{B}(X, Y) = \mathfrak{S}(X, Y)$.*

Proof Since X is infra-Schwartz, any of its local Banach spaces X_k is reflexive. The assumption on Y completes the conditions in Theorem 3.2.10. Combined with Proposition 3.2.26, we are done.

Theorem 3.2.31 $(X, Y) \in \mathfrak{S}$ *for a pair of Fréchet spaces if they satisfy*

1. $X \in \mathfrak{s}(\mathsf{P}^{\neg})$.
2. $Y \in \mathfrak{s}(\mathsf{P})$.

Proof Since $X \in \mathfrak{s}(\mathsf{P}^{\neg})$, one may rewrite $X = \varprojlim X_k$, where each X_k has nowhere P. That is, no subspace of X_k has property P. Similarly, $Y = \varprojlim Y_m$ where each Y_m has the hereditary property P. Hence, by Theorem 3.2.3, $(X_k, Y_m) \in \mathfrak{S}$ for every k, m. Applying Lemma 3.2.26, we obtain $(X, Y) \in \mathfrak{S}$.

Theorem 3.2.32 *Let (X, Y, Z) be a triple of Fréchet spaces satisfying the following*

1. *Every subspace of X contains a subspace isomorphic to Z.*

2. *Y has no subspace isomorphic to Z.*

Then, $(X, Y) \in \mathfrak{S}$. Let Y have continuous norm in addition. Then, (2) is also necessary if Y is a Fréchet-Montel space.

Proof The sufficiency part is very similar to the proof of Theorem 3.2.31. For necessity, let Y be a Fréchet space equipped with the topology identified by an increasing sequence of semi-norms (p_k). Suppose $\|\cdot\|$ is a continuous norm on Y. If there exists $c > 0$ and k_0 such that $\|y\| \leq cp_{k_0}, y \in Y$, then p_{k_0} is a norm and so the topology of Y can be defined by a sequence of norms $(p_k), k \geq k_0$. Let X be a Fréchet space and let Y be an (FM)-space admitting a continuous norm. Let any linear operator $T : X \to Y$ be strictly singular. Then, by [69, Proposition 1], it is bounded. Now let there exist $N \leq Y$ which is isomorphic to Z. Then $I|_N : N \to Z$ is bounded, hence compact. Then N is finite dimensional. Contradiction.

Example 3.2.33 Let $\Lambda_0(\alpha)$ and $\Lambda_\infty(\alpha)$ denote pss's of finite type and infinite type, respectively. By [70] we know that no subspace of $\Lambda_\infty(\alpha)$ can be isomorphic to a pss of finite type. Choose α as it is in the proof (b) of [5, Theorem 1] so that any subspace X of $\Lambda_0(\alpha)$ with a basis has a complemented subspace which is isomorphic to a pss of finite type. By Theorem 3.2.31, every continuous operator $T : \Lambda_0(\alpha) \to \Lambda_\infty(\alpha)$ is strictly singular.

This section constitutes the main substance of the paper [35].

3.3 Bounded operators and isomorphic classification

In Section 3.1, we explained how Zahariuta's method was modified to classify Cartesian products of pss's without spoiling the Fredholm operator theory. Given $X_1 \times X_2 \simeq Y_1 \times Y_2$, Zahariuta assumed $(X_1, Y_2) \in \mathfrak{K}$ and $(Y_1, X_2) \in \mathfrak{K}$ to make his method work. It turned out in the previous section that Djakov, Önal, Terzioğlu, and Yurdakul implicitly assumed $(X_1, Y_2) \in \mathfrak{B}\mathfrak{S}$ and $(Y_1, X_2) \in \mathfrak{B}\mathfrak{S}$ to obtain a modified method which makes the respective factors isomorphic up to finite dimensional subspaces. In 1998, Djakov, Terzioğlu, Yurdakul, and Zahariuta [19] weakened the operator relational assumptions on X_i, Y_i even more, namely they started with $(X_1, Y_2) \in \mathfrak{B}$ and $(Y_1, X_2) \in \mathfrak{B}$. However, they discovered, that relaxation caused the failure of Fredholm theory. That is, "isomorphic up to a finite dimensional subspace" argument on the respective factors dissappeared. Despite this failure, the authors still managed to obtain a meaningful consequence in terms of Köthe spaces. In this case the argument on the respective factors becomes "isomorphic up to a basic Banach subspace" with an additional bounded factorization property assumption. In this note they proved that if X_1 is a Köthe space and X_2, Y_1, Y_2 are topological vector spaces satisfying $X_1 \times X_2 \simeq Y_1 \times Y_2$ with $(X_1, Y_2) \in \mathfrak{B}\mathfrak{F}$, then there exist complementary basic subspcaes E and B of X_1 and complementary subspaces F and G of Y_1 such that B is a Banach space and $F \simeq E$, $B \times X_2 \simeq G \times Y_2$. In the same paper, while constructing their method, the authors consider Köthe spaces X whose bounded sets are dominated by a basic Banach subspace of X added to εU_{k_0} for any positive ε, where $U_{k_0} = \{x \in X : |x|_{k_0} \leq 1\}$. At the end, they introduce this property as one which is enjoyed by a class of lcs's larger than the one of Köthe spaces. They named it "smallness up to a complemented Banach subspace property" (SCBS) and used it to generalize some results given for Köthe spaces. In Chapter 4 we investigate SCBS property and its stability under topological tensor products. We consider the case when the first factor is a nuclear Fréchet space. This condition causes the equivalence of topologies of projective and injective tensor products.

CHAPTER 4

TOPOLOGICAL TENSOR PRODUCTS OF LOCALLY CONVEX SPACES

In this chapter, we see the definition of smallness up to a complemented Banach subspace property (SCBS) which arises from the isomorphic classification theory of Cartesian products of lcs's. The question is whether the SCBS property respects topological tensor products. We have an affirmative answer for that in case X or Y is nuclear. Our proof takes its power from Jarchow's lemma [32]. There it is proved that given Fréchet spaces X (nuclear) and Y with complemented subspaces E and F. Then $E\hat{\otimes}_\pi F$ is a complemented subspace of $X\hat{\otimes}_\pi Y$. After introducing our result, we also mention projective tensor products of generalized Köthe spaces as a consequence of our result.

4.1 The SCBS property

Definition 4.1.1 *A Fréchet space X with fundamental system of semi-norms $(|\cdot|_k)$ is said to have smallness up to a complemented Banach subspace (SCBS) property if for each bounded subset Ω of X, for any k_0 and for every $\varepsilon > 0$ there exist complementary subspaces B and E in X such that B is a Banach space, and $\Omega \subset B + \varepsilon U_{k_0} \cap E$.*

This property was introduced by Djakov, Terzioğlu, Yurdakul and Zahariuta [19] in connection with their investigations of the isomorphic classification of Cartesian products of Fréchet spaces. They also proved that Köthe spaces have the SCBS property. The characterization of Fréchet spaces having this property is put forward as an open problem. In [1] it is proved that certain generalized Köthe echelon spaces (ℓ-Köthe spaces), some quasinormable Fréchet spaces, and strong duals of some asymptotically normable Fréchet spaces have this property, while the reflexive, quasinormable, non-Montel, primary Fréchet space $\ell^p_+ := \bigcap_{q>p} \ell^q$ defined in [43], which has no infinite dimensional Banach subspace, fails it. Hence it is mentioned that SCBS property is neither hereditary nor passes to quotients.

4.2 Topological tensor products of Fréchet spaces with SCBS property

By $\mathscr{B}(X)$, we denote the family of all bounded subsets of the Fréchet space X. $\overline{\text{acx}}(A)$ and $\overline{\text{co}}(A)$ represents the absolutely convex closed hull, and closed convex hull of a set A, respectively. If $X_i, i = 1,2$ are Fréchet spaces with fundamental systems of seminorms $|\cdot|_{k_i}$, $X_1\hat{\otimes}_\pi X_2$ is a Fréchet space with fundamental system of seminorms $|\cdot|_\pi$ by the fact that projective tensor product preserves metrizability [32, Corollary 15.1.4]. As it is proved in [32, pp. 324], for any $(U,V) \in X_1 \times X_2$, $\overline{\text{co}}(U\hat{\otimes}_\pi V)$ is absorbent in $X_1\hat{\otimes}_\pi X_2$.

Theorem 4.2.1 *Let $X_i, i = 1,2$ be Fréchet spaces with the smallness up to a complemented Banach space property, where X_1 is nuclear. Then, $X_1\hat{\otimes}_\pi X_2$ has the smallness up to a complemented Banach space property.*

Proof Let $X_i, i = 1,2$ be Fréchet spaces having the SCBS property with fundamental systems of seminorms $|\cdot|_{k_i}$, where X_1 is nuclear. Let $\varepsilon > 0$, and let $\pi_0 := \pi_{U_{k_0},V_{k_0}} = \pi_{U_{k_0}}\pi_{V_{k_0}}$ [32, Proposition 15.1.5] be the gauge functional of $\overline{\text{acx}}(U_{k_0}\hat{\otimes}_\pi V_{k_0})$, where $U_{k_0} \in \mathscr{U}(X_1)$ and $V_{k_0} \in \mathscr{U}(X_2)$ for a fixed k_0. Take any $\Omega \in \mathscr{B}(X_1\hat{\otimes}_\pi X_2)$. Since X_1 is nuclear, by [32, Theorem 21.5.8] there exist $\Omega_i \in \mathscr{B}(X_i), i = 1,2$ such that $\Omega \subset \overline{\text{acx}}(\Omega_1\hat{\otimes}_\pi\Omega_2)$. Since, by assumption, X_i have the SCBS property find complementary subspaces B_i, E_i for $X_i, i = 1,2$ such that B_i are Banach spaces satisfying $\Omega_1 \subset B_1 + \varepsilon U_{k_0} \cap E_1$, and $\Omega_2 \subset B_2 + \varepsilon V_{k_0} \cap E_2$.

Take any $z \in \overline{\text{acx}}(\Omega_1\hat{\otimes}_\pi\Omega_2)$ which might be represented by $z = x \otimes y$ where $x \in \Omega_1$ and $y \in \Omega_2$. Let us rewrite $x = b_1 + e_1$ for which $b_1 \in B_1$ and $e_1 \in E_1$ such that $\pi_{U_{k_0}}(e_1) < \sqrt{\varepsilon}$. Formulate y similarly as $y = b_2+e_2$ with $\pi_{V_{k_0}}(e_2) < \sqrt{\varepsilon}$. Then, $b_1\otimes b_2 \in$

$B_1\hat{\otimes}_\pi B_2$ since B_i are Banach spaces. Consider $\pi_{U_{k_0},V_{k_0}} = \pi_{U_{k_0}}(e_1)\pi_{V_{k_0}}(e_2) < \varepsilon$. Thus $e_1 \otimes e_2 \in \varepsilon \cdot \overline{\text{acx}}(U_{k_0}\hat{\otimes}_\pi V_{k_0}) \cap E_1\hat{\otimes}_\pi E_2$. Hence for any bounded subset $\Omega \subset X_1\hat{\otimes}_\pi X_2$,

$$\Omega \subset \overline{\text{acx}}(\Omega_1\hat{\otimes}_\pi\Omega_2) \subset B_1\hat{\otimes}_\pi B_2 + \varepsilon \cdot \overline{\text{acx}}(U_{k_0}\hat{\otimes}_\pi V_{k_0}) \cap E_1\hat{\otimes}_\pi E_2. \tag{4.2.1}$$

Consider now $X_1\hat{\otimes}_\pi X_2$. By assumption, the direct sum $X_i = B_i \oplus E_i$ is topological. Therefore, by [32, Proposition 15.2.3(b)], $B_1\hat{\otimes}_\pi B_2$ and $E_1\hat{\otimes}_\pi E_2$ should also be complementary in $X_1\hat{\otimes}_\pi X_2$. Combining with 4.2.1, this is equivalent to saying that $X_1\hat{\otimes}_\pi X_2$ has SCBS property.

Theorem 4.2.1 is also valid for injective tensor products of Fréchet spaces by the characterization of Grothendieck for nuclearity: X_1 or X_2 is nuclear iff $X_1\hat{\otimes}_\pi X_2 \simeq X_1\check{\otimes}_\varepsilon X_2$ [29].

4.3 ℓ-Köthe spaces

Let ℓ denote the class of all spaces in which the canonical basis $\{e_n\}$ is an unconditional one. For the Köthe matrix A, $K^\ell(A)$ is then defined to be

$$K^\ell(A) = \{x = (x_n) : xa = (x_n a_n) \in \ell, \forall a \in A\}.$$

Equipped with the semi-norms $\|x\|_a = \|xa\|$, for $a \in A$, we obtain a complete lcs $(K^\ell(A), \|\cdot\|)$. In addition, if A is assumed to be countable, then $K^\ell(A)$ is a Fréchet space.

Corollary 4.3.1 *Let K be given by*

$$K := K^\ell(A)\hat{\otimes}_\pi K^{\tilde{\ell}}(B)$$

where $K^\ell(A)$ is nuclear. Then, K has the SCBS property.

Proof As proved in [1], ℓ-Köthe spaces have SCBS property. Since $K^\ell(A)$ is nuclear, by Theorem 4.2.1, K has SCBS property.

Note that Corollary 4.3.1 is not trivial, since it is not known whether the topological tensor product of ℓ-Köthe spaces is again an ℓ-Köthe space. By means of Theorem 4.2.1, [19, Theorem 7] can be restated as follows:

Corollary 4.3.2 *Let $X := X_1 \hat{\otimes}_\pi X_2$ as in Theorem 4.2.1 and let $T \in \mathfrak{B}(X)$. Then there exist complementary subspaces B and E of X such that*

1. *B is a Banach space,*
2. *if π_E and i_E are the canonical projection onto E and embedding into X, respectively, then the operator $1_E - \pi_E T i_E$ is an automoprhism on E.*

This chapter might also be read from [36].

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PUBLICATIONS

1. E. Kızgut, M. H. Yurdakul, *On pairs of locally convex spaces between which all operators are strictly singular*, arXiv:1412.5761.

2. E. Kızgut, M. H. Yurdakul, *On projective tensor products of Fréchet spaces with smallness up to a complemented Banach subspace property*, arXiv:1504.03254.

3. E. Kızgut, M. Yurdakul, *On the existence of a factorized unbounded operator between Fréchet spaces*, arXiv:1605.06251.