A SOLUTION TO THE KNOWABILITY PARADOX AND THE PARADOX OF IDEALIZATION IN MODAL EPISTEMIC LANGUAGES

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ABSTRACT

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Human beings are endowed with *finite* cognitive capacities so that there are forever unknown truths. This fact is stated by *non-omniscience* thesis (NO). On the other hand many philosophers, especially semantic anti-realists, hold that all truths (even the unknown ones) are knowable, and this is stated by the *knowability principle* (KP). The so-called *Knowability Paradox* consists in the derivation of a contradiction from the conjunction of (NO) and (KP). We shall show that the derivation of such a contradiction can be blocked by interpreting (NO) as the thesis that there are truths forever unknown to *actual* agents. We further provide a solution to the so-called *Paradox of Idealization* which consists in the derivation of

a contradiction from the following, initially plausible, premises. First, thesis (FU) stating that there are feasibly unknowable truths in the sense of truths knowable only by *idealized agents*, second, thesis (NI) stating that there are *no* idealized agents, and third, above mentioned thesis (KP). We show that by interpreting (NI) as stating that no *actual* agent is idealized, the derivation of contradiction from the conjunction of (FU), (NI), and (KP) is blocked.

Keywords: The Knowability Paradox, The Paradox of Idealization,
Modal Epistemic Languages, Possible World Semantics.

ÖZ

KİPSEL-BİLGİSEL DİLLERDE BİLİNEBİLİRLİK PARADOKSU VE İDEALLEŞTİRME PARADOKSUNUN BİR ÇÖZÜMÜ

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İnsanlar *sonlu* bilgi edinme yetenekleriyle donatılmıştır, dolayısıyla hiçbir zaman bilinmeyen doğrular olacaktır. Bu olgu, *bilinmeyen doğruların varlığı* savı (NO) ile ifade edilmektedir. Öte yandan birçok felsefeci, özellikle anlambilimsel antirealistler, bilinmeyen doğrular da dahil olmak üzere, tüm doğruların bilinebileceğini savunurlar. Bu sav *bilinebilirlik ilkesi* (KP) ile ifade edilir. *Bilinebilirlik Paradoksu* olarak bilinen paradoks, (NO) ve (KP)'den çelişki türetilmesi sonucu olarak ortaya çıkar. Bu paradoksun çözümü olarak (NO) savını, *gerçek* özneler tarafından hiçbir zaman bilinmeyecek doğruların varolduğu iddiası olarak yorumlayacağız. Böyle bir yorumlama yoluyla, (NO) ile (KP)'den bu tür bir çelişkinin türetilmesinin engellenebileceğini göstereceğiz. Ayrıca *İdealleştirme Paradoksu* olarak bilinen paradoksa da bir çözüm getiriyoruz. Bu paradoks, ilk bakışta kabul edilebilir üç öncülden bir çelişkinin türetilmesi sonucu olarak ortaya çıkar. Birinci öncül, sadece *idealleşmiş özneler* tarafından bilinebilen doğruların varlığını dile getiren (FU) savıdır. İkinci öncül, idealleşmiş özne *olmadığını* dile getiren (NI) savıdır. Üçüncü öncül ise yukarıda bahsedilen *bilinebilirlik ilkesi* (KP)'dir. Bu paradoksun çözümü olarak (NI) savını, hiçbir *gerçek* öznenin idealleşmediği şeklinde yorumlayacağız. Böyle bir yorumlama yoluyla (FU), (NI) ve (KP)'den bu tür bir çelişkinin türetilmesinin engellenebileceğini göstereceğiz.

Anahtar Kelimeler: Bilinebilirlik Paradoksu, İdealleştirme Paradoksu,
Kipsel-Bilgisel Diller, Olanaklı-Dünyalar Anlambilimi.

to the memory of my mother, until the last moment
to my father, from the beginning

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Lastly, as the old philosopher said,

Travailler Monsieur Schubert, travailler…

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CHAPTER 1

INTRODUCTION

Human beings are *not* omniscient. There are, and will be, true propositions which we do not know due to our limited cognitive capacities. This fact is stated by non-omniscience thesis, (NO) for short. Let q be a particular unknown truth. Then we can make the following reasoning. First, the proposition

(1) q and we do not know q

is true since q is an unknown truth. We can show that (1) is *unknowable*. Indeed assume by *reductio,* that we know proposition (1). Hence we assert

(2) We know that: q and we do not know q.

Proposition (2) entails (in epistemic logic) the following two propositions

(3) We know q

and

(4) We know that we do not know q.

From (3) we infer (in epistemic logic)

(5) We do not know *q*.

We see that the initial proposition (2) entails both "we know *q*" and "we do not know *q*" which constitute a contradiction. It follows that proposition (1) is indeed unknowable by virtue of its form and thus independently of the content of proposition *q*.

The *knowability paradox* consists in the derivation of contradiction from the thesis that there are unknown truths and the so-called *knowability principle* (KP) claiming that all truths are knowable. The derivation proceeds as follows. Given that there are unknown truths, let *q* be such a truth. Hence "*q* and *q* is unknown" is true. Then it follows from the knowability principle that the unknown truth "*q* and *q* is unknown" is known. But since all unknown truths are unknowable, "*q* and *q* is unknown" is not known. We have thus derived a contradiction.

The knowability paradox has given rise to a living and enduring debate, and to different proposals of solution. This paradox is considered by semantically realist philosophers as a vindication of their thesis that there are unknowable truths, whereas verificationist or semantically anti-realists who claim that all truths are knowable, attempt to hinder the derivation of contradiction by revising the knowability principle. These solutions are based on intuitionistic logic. The reason is as follows. Anti-realists give independent reasons for rejecting classical logic in favor of an intuitionistic logic in order to protect anti-realism from the paradox.[1] On the other hand, lately proposed solutions are generally based on restriction strategies

[1] Salerno (2010), p. 3.

concerning the anti-realist knowability principle. We use classical modal logic with epistemic operators in our thesis. For this reason, we exclude in our survey those solutions involving logical revisions. These logical revisions are mainly based on intuitionistic logic, according to which anti-realist knowability principle is best expressed in intuitionistic logic. The aim of such revisions is to show that no contradiction is derivable from the conjunction of (NO) and (KP) in intuitionistic logic.

The purpose of this thesis is to provide a solution to the knowability paradox and the paradox of idealization in the frame of modal epistemic languages.

In Chapter 2, we give a brief history of the knowability paradox, which was first formulated by Frederic Fitch in 1963. The formal derivation based on his Theorem 4 and Theorem 5 is considered to be a refutation of verificationist and anti-realist theses that endorse the view that all truths are knowable.

In Chapter 3, we will prove the Fitch's theorems which have substantial importance in formalization of the knowability paradox. We will then formalize the knowability paradox. In the last part of this chapter, we will construct a square of opposition which we call *the Knowability Paradox Square of Opposition*. This square of opposition is based on principles and theses which constitute the knowability paradox.

In Chapter 4, we give a survey of various proposals of solutions which are related with our frame of modal epistemic languages. The simplest of these solutions is the one based on restricting (KP) to non-epistemic propositions. The first proposal of solution related to our thesis is stated by

Kvanvig. We will also give the important objections stated by Williamson to Kvanvig's solution, and Kvanvig's reply to these objections. The second proposal of solution is stated by Brogaard and Salerno which is based on Stanley and Szabo's theory of quantifier domain restriction. The third proposal of solution is Kennedy's model theoretic solution.

In Chapter 5, we will briefly restate the knowability paradox and paradox of idealization with respect to the truth theory for first-order modal epistemic languages. This chapter of the thesis is identical with Section 8 of Grünberg & Grünberg's unpublished manuscript *Meaning Theory Precedes Truth Theory*. The above mentioned Section 8 has been written by the author of this thesis jointly with Teo Grünberg and David Grünberg. We will show that the derivation of a contradiction consists in the knowability paradox can be blocked by interpreting (NO) as the thesis that there are truths forever unknown to *actual* agents. We will further provide a solution to the so-called *Paradox of Idealization* which is stated by Florio and Murzi. This paradox is a generalized version of the knowability paradox. The paradox of idealization consists in the derivation of a contradiction from the following, initially plausible, premises. First, thesis (FU) stating that there are feasibly unknowable truths in the sense of truths knowable only by *idealized agents*, second, thesis (NI) stating that there are *no* idealized agents, and third, above mentioned thesis (KP). We will show that by interpreting (NI) as stating that no *actual* agent is idealized, the derivation of contradiction from the conjunction of (FU), (NI), and (KP) is blocked. It should be noted that our solution to the paradox of idealization is the first and only proposal of a solution in the literature up to now.

CHAPTER 2

A BRIEF HISTORY OF THE KNOWABILITY PARADOX

The knowability paradox was first formulated by Fitch in his unpublished paper in 1945. This paper was not published due to an anonymous referee report, but it is mentioned in the fifth footnote of Fitch's 1963 paper: "this earlier paper contained some of the ideas of the present paper." [2]

In this chapter, we will give a brief history of the knowability paradox. In the first part of this chapter, we will analyze Fitch's paper and its contribution to the knowability paradox. Then, we will explicate the importance of the anonymous referee report. In the second part, we will turn to rediscovery of the knowability paradox and its related important implications.

[2] Fitch (1963), p. 138.

2.1. Frederic Fitch's 1963 Paper and the Referee Reports

In his 1963 paper, Fitch provides a logical analysis of value concepts and considers these concepts as classes of propositions. These classes of propositions are striving (for), doing, believing, knowing, and proving. 'Knowing' has central importance through the Fitch's paper.

In the first half of his paper, Fitch defines above mentioned classes of propositions, which are factive and closed with respect to conjunction elimination. Factivity is defined as "truth class" in Fitch's paper.

> [A] class of propositions will be said to be a *truth class* if (necessarily) every member of it is true. If α is a truth class, this fact about α can be expressed in logical symbolism by the formula $(p)\ [(\alpha p) \strictif p]$. [3]

Fitch defines closure with respect to conjunction elimination for classes of propositions as follows: "if (necessarily) where the conjunction of two propositions is in the class so are the two propositions themselves." [4] Fitch then proves the following six theorems concerning above mentioned classes of propositions. Factivity and closure with respect to conjunction elimination have main importance in the proofs of these theorems. Formal derivations of these theorems will be given in the Chapter 3 of this thesis. For this reason, only statements of these theorems will be mentioned in this chapter.

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[3] *Ibid*, p. 138. Factivity is formally defined as $\Box(Op \rightarrow p)$, where O is any operator that application of it implies truth.

[4] *Ibid*, p. 136. Closure with respect to conjunction elimination, which is also known as conjunction distribution, is formally defined as $\Box(O(p \wedge q) \rightarrow (Op \wedge Oq))$. An operator O is said to conjunction distributive if it distributes over its conjuncts.

First two theorems are about truth class of propositions.

> Theorem 1. If α is a truth class which is closed with respect to conjunction elimination, then the proposition, $[p \wedge \square(\alpha p)]$, which asserts that p is true but not a member of α (where p is any proposition), is itself necessarily not a member of α.[5]

Theorem 1 is formalized as $\neg\Diamond O(p \wedge \neg Op)$, where '$\Diamond$' is the modal operator 'it is possible that'.[6]

> Theorem 2. If α is a truth class which is closed with respect to conjunction elimination, and if p is any true proposition which is not a member of α, then the proposition, $[p \wedge \square(\alpha p)]$, is a true proposition which is necessarily not a member of α.[7]

Theorem 2 briefly states that for any truth class, which is factive and closed under conjunction elimination, the proposition of the form $(p \wedge \neg (\alpha p))$ is necessarily not a member of that truth class. Theorem 2 can thus be formalized as $(p \wedge \neg Op) \rightarrow \neg\Diamond O(p \wedge \neg Op)$. In other words, it shows the un-O-ability of the conjunction of the from $(p \wedge \neg Op)$.[8] An instance of this theorem by introducing a symbol 'K' in place of 'O', $(p \wedge \neg Kp) \rightarrow \neg\Diamond K(p \wedge \neg Kp)$ is obtained. The expression 'Kp' can be read in the following two alternative ways.

(i) The proposition p is known

or

(ii) It is known that p

[5] *Ibid*, p. 138.
[6] Salerno (2009b), p. 31.
[7] Fitch (1963), p. 138.
[8] Salerno (2009b), p. 32.

In case (i) 'K' is a one-place predicate applicable to propositions and 'p' is a variable ranging over propositions. In case (ii) 'K' is a sentential operator and 'p' is a substitutional variable whose substitution class consists of the sentences of a language L. [9]

Theorem 2 states that if the conjunction of the form $(p \land \neg Kp)$ is assumed, then the consequent is unknowability of the antecedent. A sentence of this form as well as proposition expressed by $(p \land \neg Kp)$ is called *the Fitch conjunction*. In the historical development of the knowability paradox, the first published version of the Fitch conjunction was already given by Hintikka in his seminal work, *Knowledge and Belief* in 1962. [10] In his book, Hintikka establishes the foundations of epistemic logic and provides a solution for Moore's paradox (in Hintikka's terminology Moore's Problem): the paradoxical sentence "p but a does not believe that p". Hintikka argues about doxastic indefensibility and formalizes the Moore sentence as $Ba(p \land \neg Bap)$, where 'B' is for 'believes that'. Then, Hintikka considers the sentence of the form "b believes that: p but a does not believe that p". Hintikka formalizes the sentence accordingly $Bb(p \land \neg Bap)$. At this point, Hintikka states that unless a is identical with b, $Bb(p \land \neg Bap)$ is doxastically defensible. [11]

In the following part of his book, Hintikka turns to epistemic indefensibility. What is surprising is when he turns to section called "an analogue of Moore's problem for notion of knowledge." Hintikka states that the same proof of doxastic indefensibility carries for epistemic indefensibility, if 'B' is replaced by 'K' without any further change. Hence,

[9] See Kripke (1976).
[10] Hintikka (1962).
[11] *Ibid*, p. 68.

if the Moore sentence is modified, the Fitch-type conjunction is obtained as follows: *p* but *a* does not know that *p*. Therefore, Hintikka has given the conjunction of the form ($p \wedge \neg Kap$) independently of Fitch.[12] Hintikka considers that knowing that conjunction, i.e. *a* knows that: *p* but *a* does not know that *p*, is epistemically indefensible in the sense that $Ka(p \wedge \neg Kap)$ is inconsistent. This inconsistency directly corresponds to Fitch's Theorem 1 and Theorem 2. Unfortunately, Hintikka does not further elaborate the implications of this conjunction. Timothy Williamson also explains the related account of Moore's paradox with *K* operator in terms of assertion. Williamson argues that Hintikka's concept of epistemic indefensibility for the Fitch conjunction is unmotivated unless only knowledge warrants assertion.

> What is wrong can be easily understood on the hypothesis that only knowledge warrants assertion. For then to have warrant to assert the conjunction 'A and I do not know A' is to know that A and one does not know A. But one cannot know that A and one does not know A. [13]

The rest of the theorems are special cases and consequences of the first two theorems.

> Theorem 3. If an agent is all-powerful in the sense that for each situation that is the case, it is logically possible that that situation was brought about by that agent, then whatever is the case was brought about (done) by that agent. [14]

Theorem 3 is formally stated as $\forall p\ (p \wedge \Diamond \mathrm{a}\mathfrak{B}p) \rightarrow \forall p\ (p \wedge a\mathfrak{B}p)$ [15], where '$\mathfrak{B}$' is factive and conjunction distributive operator "brought it about that".

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[12] *Ibid*, p. 79 (41).

[13] Williamson (2000), p. 253 and n. 7.

[14] Fitch (1963), p. 138.

[15] Salerno (2009b), p. 32. In order to avoid any confusion with '*B*' for "believes that", we preferred letter '$\mathfrak{B}$' for "brought it about that".

Fitch proves Theorem 4 and Theorem 5 for knowledge operator K. The following two theorems are also called “knowability proofs”. These two theorems lead the knowability paradox. However, Fitch does not elaborate the implications of these theorems. Hence, he does not show that these theorems give rise to paradox.

> Theorem 4. For each agent who is not omniscient, there is a true proposition which that agent cannot know. [16]

Theorem 4 is formally stated as $\exists p\ (p \wedge \neg aKp) \rightarrow \exists p\ (p \wedge \neg\Diamond aKp)$. [17] Fitch attributes Theorem 4 to an *anonymous* referee in a footnote. There were various speculations regarding the identity of the anonymous referee, but this has not much to say about the paradox itself. The enigma of the referee was not solved until 2005. Today, it is known that the anonymous referee was Alonzo Church, by the help of transcription of Church’s *trademark vertical handwriting*.[18] The Theorem 4 directly contradicts to anti-realist knowability principle (KP), i.e. all truths are knowable. However, Fitch did not point out the implication of this theorem in his paper. This is not so surprising since Fitch did not have a purpose of refuting any truth theories; rather he only aimed to provide an analysis and definitions for value concepts.

[16] Fitch (1963), p. 138.

[17] Salerno (2009b), p. 32.

[18] For further details about the interesting history of the revealing of the identity of the referee see Salerno (2009b), pp. 34-37. For the referee reports issued by Church, see Church (2009).

> Theorem 5. If there is some true proposition which nobody knows (or has known or will know) to be true, then there is a true proposition which nobody can know to be true. [19]

Theorem 5 is formally stated as $\exists p\ (p \wedge \forall a\ \neg aKp) \rightarrow \exists p\ (p \wedge \forall a\ \neg\Diamond aKp)$.[20] Theorem 5 states that if there is an unknown truth, then there is a logically unknowable truth. Theorem 5 strengthens Theorem 4 in a way that it generalizes for all agents in both antecedent and consequent part. In the proof lines, Fitch also noted that the proof of the Theorem 5 is similar to the proof of Theorem 4. [21] Theorem 5 is usually reformulated in the same way with Theorem 4 as $\exists p\ (p \wedge \neg Kp) \rightarrow \exists p\ (p \wedge \neg\Diamond Kp)$ by dropping out '$\forall a$'. The paradox is implicit in the core of Fitch's Theorem 4 and Theorem 5. The contrapositive of these theorems constitutes the so-called *knowability paradox* [22]:

The Knowability Paradox $\quad \forall p\ (p \rightarrow \Diamond Kp) \rightarrow \forall p\ (p \rightarrow Kp)$.

The knowability paradox briefly states that if all truths are knowable then all truths are known. Accordingly, the paradox has shown that the plausible anti-realist claim regarding knowability of truths turns out to be an unacceptable result that all truths are known, in other words *omniscience thesis* (O). [23]

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[19] Fitch (1963), p. 139.
[20] Salerno (2009b), p. 32.
[21] Fitch (1963), p. 139.
[22] Salerno (2009a), p. 1.
[23] Salerno (2010), p. 2.

Theorem 6 is the last of Fitch's theorems.

> Theorem 6. If there is some true proposition about proving that nobody has ever proved or ever will prove, then there is some true proposition about proving that nobody can prove. [24]

Theorem 6 is formally stated as $\exists p\ (p \wedge \forall a\ \neg aPp) \rightarrow \exists p\ (p \wedge \forall a\ \neg\Diamond aPp)$ [25], where '*P*' is read as 'proves that'. This theorem similarly follows from Theorem 5 by replacing the factive and conjunction distributive operator 'proves that' in place of 'knows that'.

In the second half of his paper, Fitch gives various definitions of value concepts by using the relation of partial causation. These definitions are listed as follows: a definition of doing in terms of striving (D1), a definition of knowing in terms of believing (D2), a definition of ability to do in terms of striving (D3), a definition of obligation to do in terms of doing (D4), a definition of desire in terms of believing and striving (D5), and finally a definition of concept of *value* in terms of knowing and striving (D6).[26] The relation between the first half and the second half of Fitch's paper is not very clear. Yet, Salerno perfectly elucidates this relation:

> Fitch published the proof in 1963 to avert a kind of "conditional fallacy" that threatened his informed-desire analysis of value…The existence of unknowable truths ultimately explains why [Fitch] restricts the propositional variables to knowable propositions. For an unknowable truth provides for an impossible antecedent in Fitch's counterfactual, and ultimately trivializes the analysis. [27]

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[24] Fitch (1963), p. 139.
[25] Salerno (2009b), p. 33.
[26] Fitch (1963), pp. 140-141. The corresponding definition numbers are given in bracelets.
[27] Brogaard and Salerno (2013).

Following Salerno's interpretation, it is quite possible that what Church's referee report to Fitch's 1945 paper tried to show is that the existence of unknowable true propositions trivializes the Fitch's definition of value concepts.[28]

In the referee reports, Church indeed gives the first formulation of the knowability paradox:

> Then it may plausibly be maintained that if *a* is not omniscient there is always a true proposition which it is empirically impossible for *a* to know at time t. For let *k* be a true proposition which is unknown to *a* at time t, and let *k'* be the proposition that *k* is true but unknown to *a* at time t. Then *k'* is true. But it would seem that if *a* knows *k'* at time t, then *a* must know *k* at time t, and must also know that he does not know *k* at time t. By Def.2, this is a contradiction. [29]

Church's argument illustrates the mistake in Fitch's analysis. It becomes clear then, why Church's referee report is very important in shaping Fitch's 1963 paper, especially his Theorem 4. Consequently, after the identity of anonymous referee was revealed, the paradox is also named as 'the Church-Fitch Paradox of Knowability'.

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[28] Church wrote two referee reports in response to Fitch's 1945 paper "A definition of Values". The letters between Ernest Nagel, who was the editor of *Journal of Symbolic Logic*, and Alonzo Church tell us that Fitch has withdrawn his paper due to "a defect in [his] definition of value". Therefore, Fitch's 1945 paper is missing. See Salerno (2009), p. 45.

[29] Church (2009), p. 14. Def. 2 is Fitch's definition of knowledge. Salerno hypothesizes Def.2 from the content of Church's referee report: "*a*'s knowing at time *t* that *p* strictly implies *p*: $a\mathrm{KN}tp \prec p$." See Salerno (2009a), p. 19.

2.2. Rediscovery of the Knowability Paradox

Fitch's result of Theorem 4 and Theorem 5 did not take much attention until the end of 1970s. The implication and paradoxical results of these theorems were first reintroduced into discussion by Hart and McGinn in 1976. Hart and McGinn grounded an argument on Fitch's Theorem 4. They considered that the derivation based on this theorem is a refutation of verificationism, which states that all meaningful statements and truths are knowable.

> Axiom 5 [$p \rightarrow \Diamond Kp$] is a apparently weak thesis of idealism or verificationism; a transcendental idealist like Kant m[i]ght thus have accepted axiom 5. But in the presence of obvious truths [$p \rightarrow Kp$] deducible from [axiom] 5. [$p \rightarrow Kp$] is obviously false and is an objectionably strong thesis of idealism…Therefore Axiom 5 is false; there are truths which absolutely cannot be known. [30]

At the time Hart and McGinn's paper was published, many philosophers were not convinced that the derivation based on Fitch's theorem disproves antirealism. Indeed, Hart and McGinn's presentation of derivation based on Fitch's theorems considered to be fallacious thus concluded to be a paradox.[31]

Mackie (1980) and Routley (1981) considered that the derivation from Fitch's Theorem 4 is complicated. Initially, they reject such a simple refutation of anti-realism and verificationism. However, they also agreed

[30] Hart and McGinn (1976), p. 206. Hart was also the first person to draw attention on identity of the anonymous referee. He states that Fitch's knowability theorems and its results are as "unjustly neglected logical gem". See Hart (1979), pp. 164-165.
[31] See Mackie (1980), p. 90.

that Fitch's paradox poses serious threat to verificationism. Mackie argues that the derivation of the paradox is valid; hence he admits that this is a serious threat to verificationism. He replies Hart and McGinn's argument by suggesting a reformulation of the verificationist knowability principle. Mackie explains that the paradoxical result is due to truth entailing property of the factive operators. These operators can be used to construct such self refuting expressions. Therefore, Mackie's solution states that verificationism does not have to maintain antirealist conception of truth that verification entails truth. Rather, verificationism can use the concept of confirmation, which in his sense does not have to entail truth.[32] Routley also takes the result of knowability paradox as the necessary limits of human knowledge. [33]

The knowability paradox is commonly viewed as a threat against truth theories that rely on the statement that all truths are knowable. If the semantic antirealist or verificationist knowability principle is accepted, then by existence of an unknown proposition, this principle will result in omniscience thesis (O), i.e. all truths are known. Hence, the major implication of the knowability paradox has been widely used as an argument against anti-realism. The earlier solutions are generally based on reformulation of the anti-realist and verificationist knowability principle.

> Conceived historically, [approach on restricting the knowability principle] makes quite a bit of sense, for Fitch's proof has been seen to be a threat to anti-realist conceptions of truth, and early discussion by Hart, Mackie, and others focused on the threat to such conceptions, especially in the context of verificationist theories.[34]

[32] See Mackie (1980), pp. 91-92.
[33] See Routley (1981).
[34] Kvanvig (2006), p. 154.

The threat of the knowability paradox comprises not only semantic anti-realism and verificationism, but also a wide range of philosophical theories from epistemology to philosophy of religion. Indeed many forms of these theories are under threat of the knowability paradox, including some versions of ethical expressivism, logical empiricism, Putnam's internal realism, Peircian pragmatism, Kantian transcendental idealism, and Berkeleyian idealism.[35] Since then, the knowability paradox has a long debate of attempts for solutions according to each philosophical theory.

On the other hand, as we argued in Chapter 1, the most important implication of the knowability paradox is that the paradox threatens the logical distinction between actuality and possibility.

> [T]he paradox threatens the logical distinction between actual and possible knowledge in the domain of truth…This result is seriously disturbing, for it is no more plausible to assume that there is no such distinction between known truths and knowable truths than between empirical truths and empirical possibilities.[36]

This implication is independent of above stated philosophical theories. Moreover, it also poses challenge for opponent philosophical theories of anti-realism, such as realism. This is, perhaps, the most problematic implication of the knowability paradox.

[35] See Kvanvig (2006) for comprehensive discussions about threats of the knowability paradox towards various philosophical theories. See also Hand (2003) and Salerno (2010).

[36] Kvanvig (2006), p. 2.

CHAPTER 3

FORMALIZATION OF THE KNOWABILITY PARADOX AND THE KNOWABILITY PARADOX SQUARE OF OPPOSITION

In this chapter, we shall give the formal derivations of Fitch’s theorems. We use modal epistemic logic through the formal derivations of Fitch’s theorems and the knowability paradox. For the sake of clarity, in the first part, we will explain the rules for derivation. These rules are Factivity of *K* Operator, Conjunction Distribution of *K* Operator, Rule of Necessitation, Exchange of Modalities Rule, Quantifier Negation Rule, and Double Negation Elimination Rule. We will also argue in part whether arguments for the denial of these rules are legitimate. In the second part, we will provide a formalization of the knowability paradox. In the third part, we will construct a square of opposition which is based on principles and theses involved in the knowability paradox.

3.1. Rules of Modal Epistemic Logic

Factivity of *K* Operator

(Fact) $\vdash Kp \rightarrow p$

Factivity of knowledge is the ***T*** axiom of modal epistemic logic. The principle is also known as knowledge implies truth (KIT).[37] It has been argued that if the truth implication character of knowledge operator is denied, then the knowability paradox could not be derived.[38] However, Kvanvig states that

> It is hard to find in the history of epistemology reasons for questioning [Knowledge Implies Truth principle]. The only arguments for a rejection of this principle derive from ordinary language considerations, in which people sometimes use the term 'knowledge' and its cognates in way that do not require truth. [39]

Even so, the knowability paradox can still be derived, when factive operators are replaced with non-factive operators such as 'it is rationally believed that'. Stalnaker gives excellent justificatory remark for factivity of knowledge:

> Just as necessity is truth in all possible worlds, so knowledge is truth in all *epistemically* possible worlds. The assumption is that to have knowledge is to have a capacity to locate the actual world in logical space, to exclude certain possibilities from the candidates for actuality. [40]

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[37] See Kvanvig (2006), p. 89.
[38] See Mackie (1980), p. 92; Kvanvig (2006), pp. 89-92.
[39] Kvanvig (2006), p. 89.
[40] Stalnaker (2006), p. 171.

Hence, it is concluded from the above debate that there is no problem in the truth implication character of *K* operator.

Conjunction Distribution of *K*

(K-Dist) $\vdash K(p \wedge q) \leftrightarrow (Kp \wedge Kq)$

Conjunction distribution of *K* over its conjuncts is a theorem of modal epistemic logic. It states that knowledge operator distributes over its conjuncts and closed under conjunction elimination. In other words, knowledge of a conjunction entails knowledge of its conjuncts. There are arguments that deny the conjunction distribution. The issue is a little bit more complicated concerning the distributivity character of *K* over its conjuncts. There are epistemological theories, such as skepticism, denying the conjunction distribution of *K* operator. Nozick, for instance, argues that knowing of a conjunction does not entail knowing the conjuncts.[41] A solution to the knowability paradox could have been proposed in a way that if *K* operator fails to distribute over its conjuncts. However, there is not any independent motivation to support the denial of this principle; rather there are only implications of general theory of skepticism. Moreover, denying distributivity principle require further explanation of the difference between cases in which distribution is acceptable and those in which it is not.[42] Hence, skeptical arguments for questioning distributivity of knowledge are unattainable. Williamson also shows that there are versions

[41] Nozick (1981), Ch. 3.
[42] Kvanvig criticizes Nozick's account on denying the (K-Dist). See Kvanvig (2006), pp. 96-114.

of the knowability paradox that does not require conjunction distribution.[43] Again, it is concluded that from the above discussions that denying conjunction distribution of K is not a proper solution to the knowability paradox.

Rule of Necessitation

$$\text{(RN)} \qquad \text{If } \vdash p, \text{ then} \vdash \Box p$$

Rule of Necessitation is a basic inference rule of modal logic. It states that for the basic modal logical system **K**, if p is a theorem, then $\Box p$ is also a theorem.

Exchange of Modalities Rule

$$\text{(ER)} \qquad \Box\neg p \leftrightarrow \neg\Diamond p$$

Exchange of Modalities Rule (ER) is the definition of possibility in terms of necessity. This is also a theorem of modal logic in the basic system **K**. Just as the quantifiers $\forall$ (all) and $\exists$ (some) are defined as ($\forall p\ p \leftrightarrow \neg\exists p\ \neg p$) in predicate logic, modalities necessity ($\Box$) and possibility ($\Diamond$) are defined in the same manner in first order modal logic.[44]

[43] See Williamson (1993) and Williamson (2000), pp. 275-277 and pp. 281-282. Williamson argues about Nozick's account of the distribution over conjunction in length.
[44] Garson (2013), p. 20.

Quantifier Negation Rule

(QN) $\neg\forall p\ (p) \leftrightarrow \exists p \neg\ (p)$

Double Negation Elimination Rule

(DN) $\neg\neg\, p \leftrightarrow p$

3.2. Proofs of Fitch's Theorems

Theorem 1. $\neg\Diamond O(p \wedge \neg Op)$

Proof of $\neg\Diamond K(p \wedge \neg Kp)$

1.	$\neg\Diamond K(p \wedge \neg Kp)$	Assertion
2.	$\neg K(p \wedge \neg Kp)$	Assertion
3.	$K(p \wedge \neg Kp)$	Assumption of (ID) [45]
4.	$Kp \wedge K\neg Kp$	3, (K-Dist)
5.	Kp	4
6.	$K\neg Kp$	4
7.	$\neg Kp$	6, (Fact)
8.	$\bot$	5, 7
7.	$\neg K(p \wedge \neg Kp)$	3-8
8.	$\Box\neg K(p \wedge \neg Kp)$	7, (RN)
9.	$\neg\Diamond K(p \wedge \neg Kp)$	8, (ER)

[45] (ID) is short for Indirect Derivation.

The above derivation is an instance of Fitch's Theorem 1, by replacing K operator in place of operator O. Fitch originally defined his Theorem 1 for any operator O, which is factive and closed under conjunction elimination. For the sake of clarity and consistency within the context of the knowability paradox, it is more convenient to show the proof of Theorem 1 with K operator.

<u>Theorem 2.</u> $(p \wedge \neg Op) \rightarrow \neg\Diamond O(p \wedge \neg Op)$

Proof of $(p \wedge \neg Kp) \rightarrow \neg\Diamond K(p \wedge \neg Kp)$

1.	$(p \wedge \neg Kp) \rightarrow \neg\Diamond K(p \wedge \neg Kp)$	Assertion
2.	$(p \wedge \neg Kp)$	Assumption of (CD) [46]
3.	$\neg\Diamond K(p \wedge \neg Kp)$	Theorem 1

We shall again use 'K' instead of 'O' for the same reasons.

<u>Theorem 3.</u> $\forall p\ (p \rightarrow \Diamond a\mathfrak{B}p) \rightarrow \forall p\ (p \rightarrow a\mathfrak{B}p)$

Proof.

1.	$\forall p\ (p \rightarrow \Diamond a\mathfrak{B}p) \rightarrow \forall p\ (p \rightarrow a\mathfrak{B}p)$	Assertion
2.	$\forall p\ (p \rightarrow \Diamond a\mathfrak{B}p)$	Assumption of (CD)
3.	$(q \wedge \neg a\mathfrak{B}q) \rightarrow \Diamond a\mathfrak{B}\ (q \wedge \neg a\mathfrak{B}q)$	2, UI
4.	$\neg\Diamond a\mathfrak{B}(q \wedge \neg a\mathfrak{B}q)$	by Theorem 1 [47]
5.	$\neg\ (q \wedge \neg a\mathfrak{B}q)$	3, 4, MT

[46] (CD) is short for Conditional Derivation.

[47] Note that we can obtain an instantiated by version of Theorem 1 by replacing operator '$\mathfrak{B}$' in place of operator 'O', since '$\mathfrak{B}$' is factive and closed under conjunction elimination.

6.	$\exists p \neg (p \land a\mathfrak{B}p)$	4, EG
7.	$\forall p\ (p \rightarrow a\mathfrak{B}p)$	6, Logical Equiv.

Theorem 3 is itself a special operator theorem, where '$\mathfrak{B}$' stands for 'brought it about that'. The operator $\mathfrak{B}$ is also a truth class, which is factive and closed under conjunction elimination. Therefore, the rules for K operator are also applicable for operator $\mathfrak{B}$.

Theorem 4 and Theorem 5

We shall state below a proof for both Theorem 4 and Theorem 5 with the reformulated formalization ($\exists p\ (p \land \neg Kp) \rightarrow \exists p\ (p \land \neg\Diamond Kp)$) of these theorems, as given in Chapter 2.

Proof

1.	$\exists p\ (p \land \neg Kp) \rightarrow \exists p\ (p \land \neg\Diamond Kp)$	Assertion
2.	$\exists p\ (p \land \neg Kp)$	Assumption of (CD)
3.	$(\overline{p} \land \neg K\overline{p})$	2, EI
4.	$\exists p\ (p \land \neg\Diamond Kp)$	Assertion
5.	$\neg\exists p\ (p \land \neg\Diamond Kp)$	Assumption of (ID)
6.	$\forall p\ \neg(p \land \neg\Diamond Kp)$	5, (QN)
7.	$\neg((\overline{p} \land \neg K\overline{p}) \land \neg\Diamond K(\overline{p} \land \neg K\overline{p}))$	6, UI
8.	$(\overline{p} \land \neg K\overline{p}) \rightarrow \Diamond K(\overline{p} \land \neg K\overline{p})$	7, Logical Equiv.
9.	$\Diamond K(\overline{p} \land \neg K\overline{p})$	3, 8, MP
10	$\neg\Diamond K(\overline{p} \land \neg K\overline{p})$	Theorem 1
11.	$\perp$	9, 10
12.	$\exists p\ (p \land \neg\Diamond Kp)$	5-11, (DN)

Theorem 6 $\forall p\ (p \wedge \forall a\ \neg aPp) \rightarrow \forall p\ (p \wedge \forall a\ \neg\Diamond aPp)$

The proof of Theorem 6 is similar to that of Theorem 5, simply by replacing '*P*' operator in place of '*K*'. Since *P* operator is also factive and conjunction distributive, there is no need to repeat the same proof.

3.3. Formalization of the Knowability Paradox

In this section, we shall give two formalizations of the knowability paradox. The first one is derived independently of Fitch's theorems and the second one is derived from Fitch's Theorem 4 (or Theorem 5) by contraposition.

The Knowability Paradox

The knowability paradox is widely derived independently of Fitch's theorems. Formalization of the knowability paradox consists in the derivation of a contradiction from the conjunction of the following two initially plausible premises.

1. (NO) Some true propositions are unknown,

2. (KP) All true propositions are knowable.

The proof consist in showing the inconsistency of (NO) and (KP), hence (KP) results in unacceptable claim that all truths are known, in other words omniscience thesis (O). The following derivation is adopted from Kennedy.[48]

The Knowability Paradox: $\forall p\,(p \rightarrow \Diamond Kp) \rightarrow \forall p\,(p \rightarrow Kp)$

1.	$\exists p\,(p \wedge \neg Kp)$	(NO)
2.	$\forall p\,(p \rightarrow \Diamond Kp)$	(KP)
3.	$\bar{p} \wedge \neg K\bar{p}$	1, EI
4.	$(\bar{p} \wedge \neg K\bar{p}) \rightarrow \Diamond K(\bar{p} \wedge \neg K\bar{p})$	2, UI
5.	$\neg K(\bar{p} \wedge \neg K\bar{p})$	Assertion
6.	$K(\bar{p} \wedge \neg K\bar{p})$	Assumption of (ID)
7.	$K\bar{p} \wedge K\neg K\bar{p}$	6, (K-Dist)
8.	$K\bar{p}$	7
9.	$K\neg K\bar{p}$	7
10.	$\neg K\bar{p}$	9, (Fact)
11.	$\perp$	8, 10
12.	$\neg K(\bar{p} \wedge \neg K\bar{p})$	6-11
13.	$\Box\neg K(\bar{p} \wedge \neg K\bar{p})$	12, (RN)
14.	$\neg\Diamond K(\bar{p} \wedge \neg K\bar{p})$	13, (ER)
15.	$\neg\,(\bar{p} \wedge \neg K\bar{p})$	4, 15, MT
16.	$\bar{p} \rightarrow \neg\neg K\bar{p}$	15, Logical Equivalence
17.	$\bar{p} \rightarrow K\bar{p}$	16, (DN)
18.	$\forall p\,(p \rightarrow Kp)$	17, Rule of Generalization
	Q.E.D.	

[48] See Kennedy (2013), Sec. 2. Kvanvig also provides a formal derivation of the knowability paradox by using second order logic; see Kvanvig (2006), pp. 12-13.

Here, we shall make a remark of a solution which is based on intuitionistic logic. Michael Dummett argues that the knowability paradox is due to the commitment to classical logic underlying epistemic logic. According to Dummett, anti-realism is best described in intuitionistic logic. Hence, the knowability paradox is not a refutation of anti-realism, but rather a refutation of a certain kind of anti-realism which is based on classical logic. For this reason Dummett and other anti-realists prefer non-classical intuitionistic logic to classical logic. [49]

The earliest suggestion of treating the knowability paradox was stated by Williamson. [50] Among the steps of the derivation of the knowability paradox, the inference from step 16 to step 17 is not valid in intuitionistic logic. The reason is that intuitionistic logic does not validate the rule of double negation elimination rule (DN), i.e. ($\neg\neg\ p \rightarrow p$), which is valid in classical logic.[51] In intuitionistic logic, the sentence "there is *no* truth that is *not* known", i.e. $\neg\exists p\ (p \wedge \neg\ Kp)$ does not entail "all truths are known", i.e. $\forall p\ \ (p \rightarrow Kp)$. Furthermore, Williamson points out that in intuitionistic logic, quantifier negation rule ($\neg\forall p\ (p) \rightarrow \exists p\neg\ (p)$) is not unrestrictedly valid.[52] Hence, the intuitionistic anti-realist can express non-omniscience

[49] See Williamson (1982), Williamson (2000), Salerno (2009a), Dummett (2009).

[50] See Williamson (1982).

[51] The motivation for rejecting the rule of double negation elimination is stated briefly in Dummet (2009), p. 52:

> It follows that '$\neg\neg$KA' means 'There is an obstacle in principle to our being able to deny that A will ever be known', in other words 'The possibility that A will come to be known always remains open'. That this holds good for every true proposition A is precisely what the justificationist believes. This is the principle expressed by [$p \rightarrow \neg\neg Kp$]; and [$p \rightarrow\ \neg\neg Kp$] captures the relation which the justificationist believes to obtain between truth and knowledge.

[52] See Brogaard and Salerno (2013) who refers to Williamson (1992).

thesis as "*not all truths are known*", i.e. $\neg\forall p\ (p \rightarrow Kp)$. However, she may *not* accept "there is an unknown truth". It is the latter which leads to the knowability paradox.

As was indicated in Chapter 1, we will rather concentrate on the solutions to the knowability paradox within the perspective of classical logic.

The Knowability Paradox (by Contraposition)

As we have stated in Chapter 2, the knowability paradox is also stated in the form of contraposition of Fitch's Theorem 5. [53]

$\forall p\ (p \rightarrow \Diamond Kp) \rightarrow \forall p\ (p \rightarrow Kp)$

1.	$\exists p\ (p \wedge \neg Kp) \rightarrow \exists p\ (p \wedge \neg\Diamond Kp)$	Fitch's Theorem 5
2.	$\forall p\ (p \rightarrow \Diamond Kp) \rightarrow \forall p\ (p \rightarrow Kp)$	1, Contraposition

[53] See Salerno (2009a), p. 1.

3.4. The Knowability Paradox Square of Opposition

The knowability paradox refers to following two pairs of contradictory principles or theses. These principles and theses constitute a square of opposition, which we call *the Knowability Paradox Square of Opposition*.[54] This square of opposition will clarify the logical relationships between these principles and theses. In this part, we shall first introduce these principles and theses, and then explain the logical relationships between these principles and theses.

1.1	$\exists p\ (p \wedge \neg\Diamond Kp)$	Unknowability Principle (UKP)
1.2	$\forall p\ (p \rightarrow \Diamond Kp)$	The Knowability Principle (KP)
2.1	$\exists p\ (p \wedge \neg Kp)$	Non-Omniscience Thesis (NO)
2.2	$\forall p\ (p \rightarrow Kp)$	Omniscience Thesis (O)

Non-omniscience thesis (NO) states that some truths are not known. Since the existence of the propositions of this form is an undeniable fact, non-omniscience thesis (NO) unanimously accepted. Unknowability principle (UKP) states that some truths are not known. (UKP) is accepted by semantic realists or anti-verificationists. The knowability principle (KP) states that all truths are knowable. (KP) is accepted by semantic anti-realists or verificationists. Omniscience thesis (O) states that all truths are known. It is obvious that human beings are limited in cognitive capacities. Hence, omniscience thesis (O) is unanimously rejected. However, it should be noted that, according to some supernatural and theistic approaches, omniscience thesis (O) may be valid, in principle. For God (or some

[54] As far as we know, no one has mentioned previously such a square of opposition.

supernatural being) have capability to appoint that all agents can know all truths. [55]

The Knowability Paradox Square

(KP) $\forall p\ (p \rightarrow \Diamond Kp)$ or WVER [56]	(NO) $\exists p\ (p \wedge \neg Kp)$
(O) $\forall p\ (p \rightarrow Kp)$ or SVER [57]	(UKP) $\exists p\ (p \wedge \neg\Diamond Kp)$

The above square is a square of opposition in the sense that it describes the following six relationships.

(1) (KP) and (NO) are contraries i.e. (KP) → ¬ (NO) is valid.

(2) (O) and (UKP) are subcontraries i.e. (O) ∨ (UKP) is valid.

(3) (O) is subaltern of (KP) i.e. (KP) → (O) is valid.

(4) (UKP) is subaltern of (NO) i.e. (NO) → (UKP) is valid.

(5) (KP) and (UKP) are contradictories i.e. (KP) ↔ ¬ (UKP) is valid.

(6) (NO) and (O) are contradictories i.e. (NO) ↔ ¬ (O) is valid.

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[55] Kvanvig argues that the knowability paradox also threatens the Christian theistic approach. Kvanvig gives a related argument against omniscience thesis in Christian metaphysical doctrine of incarnation. Although the doctrine of incarnation is not shared with any epistemic truth theories, the knowability paradox also threatens the incarnation doctrine. See Kvanvig (2006) and Kvanvig (2010).

[56] See Williamson (2000), pp. 220-1. WVER is short for "weak verificationism".

[57] See *ibid*., pp. 220-1. SVER is short for "strong verificationism".

Anti-verificationists claim that the knowability paradox, i.e. the proof of (KP) → (O), constitutes a refutation of the knowability principle (KP) since omniscience thesis (O) is unanimously rejected. On the other hand, verificationists attempt to preserve the knowability principle by revising it in different ways. A revision of the knowability principle is successful in case the revised form does no more give rise to paradox. Each successful revision is considered as a particular solution of the knowability paradox.

We shall state below the formal derivations of above stated six relationships regarding the knowability paradox square of opposition.

(1) (KP) → ¬ (NO) $\forall p\, (p \rightarrow \Diamond Kp) \rightarrow \neg\exists p\, (p \wedge \neg Kp)$

(2) (O) ∨ (UKP) $\forall p\, (p \rightarrow Kp) \vee \exists p\, (p \wedge \neg\Diamond Kp)$

(3) (KP) → (O) $\forall p\, (p \rightarrow \Diamond Kp) \rightarrow \forall p\, (p \rightarrow Kp)$ the Knowability Paradox

(4) (NO) → (UKP) $\exists p\, (p \wedge \neg Kp) \rightarrow \exists p\, (p \wedge \neg\Diamond Kp)$ Fitch's Theorem 5

(5) (KP) ↔ ¬ (UKP) $\forall p\, (p \rightarrow \Diamond Kp) \leftrightarrow \neg\exists p\, (p \wedge \neg\Diamond Kp)$

(6) (NO) ↔ ¬ (O) $\exists p\, (p \wedge \neg Kp) \leftrightarrow \neg\forall p\, (p \rightarrow Kp)$

Proof of (1)

The proof of relationship (1) is trivial since this relationship is a modified version of the Knowability Paradox.

$\forall p\ (p \rightarrow \Diamond Kp) \rightarrow \neg\exists p\ (p \wedge \neg Kp)$

1.	$\forall p\ (p \rightarrow \Diamond Kp) \rightarrow \neg\exists p\ (p \wedge \neg Kp)$	Assertion
2.	$\forall p\ (p \rightarrow \Diamond Kp)$	Assumption of (CD)
3.	$(\bar{p} \wedge \neg K\bar{p}) \rightarrow \Diamond K(\bar{p} \wedge \neg K\bar{p})$	2, EI
4.	$\neg\Diamond K(\bar{p} \wedge \neg K\bar{p})$	Theorem 1
5.	$\neg(\bar{p} \wedge \neg K\bar{p})$	3,4, MT
6.	$(\bar{p} \rightarrow \mathrm{K}\bar{p})$	5, Logical Equivalence
7.	$\forall p\ (p \rightarrow \mathrm{K}p)$	6, Rule of Generalization
	Q.E.D.	

Proof of (2)

$\forall p\ (p \rightarrow Kp) \vee \exists p\ (p \wedge \neg\Diamond Kp)$

1.	$\forall p\ (p \rightarrow Kp) \vee \exists p\ (p \wedge \neg\Diamond Kp)$	Assertion
2.	$\neg(\forall p\ (p \rightarrow Kp) \vee \exists p\ (p \wedge \neg\Diamond Kp))$	Assumption of (ID)
3.	$\neg\forall p\ (p \rightarrow Kp) \wedge \neg\exists p\ (p \wedge \neg\Diamond Kp)$	2, Logical Equiv.
4.	$\exists p\ \neg(p \rightarrow Kp) \wedge \forall p\ \neg(p \wedge \neg\Diamond Kp))$	3, (QN)
5.	$\exists p\ \neg(p \rightarrow Kp)$	4
6.	$\neg\ (\bar{p} \rightarrow K\bar{p})$	5, EI
7.	$\forall p\ \neg(p \wedge \neg\Diamond Kp)$	4
8.	$\neg((\bar{p} \wedge \neg K\bar{p}) \wedge \neg\Diamond K(\bar{p} \wedge \neg K\bar{p}))$	7, UI
9.	$\neg(\bar{p} \wedge \neg K\bar{p}) \vee \neg\neg\Diamond K(\bar{p} \wedge \neg K\bar{p}))$	8, Logical Equiv.
10.	$\neg\Diamond K(\bar{p} \wedge \neg K\bar{p})$	Theorem 1
11.	$\neg(\bar{p} \wedge \neg K\bar{p})$	9, 10, MTP
12.	$(\bar{p} \rightarrow \mathrm{K}\bar{p})$	11, Logical Equiv.
13.	$\perp$	6, 12
14.	$\forall p\ (p \rightarrow Kp) \vee \exists p\ (p \wedge \neg\Diamond Kp)$	2-13, (DN)
	Q.E.D.	

Proof of (3) and (4)

Relationships (3) and (4) have been proved previously under formalization of the knowability paradox and proof of Fitch's Theorem 5, respectively.

Proof of (5)

This proof is trivial by simply expressing the instantiated form of the knowability principle, i.e. ($p \rightarrow \Diamond Kp$), in the form of its logically equivalent conjunction of the form $\neg(p \wedge \neg\Diamond Kp)$, and then applying quantifier exchange rule (QN).

$$\forall p\ (p \rightarrow \Diamond Kp) \leftrightarrow \neg\exists p\ (p \wedge \neg\Diamond Kp)$$

1.	$\forall p\ (p \rightarrow \Diamond Kp)$	
2.	$p \rightarrow \Diamond Kp$	1, UI
3.	$\neg(p \wedge \neg\Diamond Kp)$	2, Logical Equivalence
4.	$\forall p\ \neg(p \wedge \neg\Diamond Kp)$	3, Rule of Generalization
5.	$\neg\exists p\ (p \wedge \neg\Diamond Kp)$	4, (QN)

Q.E.D.

Proof of (6)

$$\exists p\ (p \wedge \neg Kp) \leftrightarrow \neg\forall p\ (p \rightarrow Kp)$$

The proof of (6) is similar to that of (5). It is trivial by expressing the instantiated form of the non-omniscience thesis, i.e. ($p \wedge \neg Kp$), in the form of conditional $\neg\ (p \rightarrow Kp)$, and then applying quantifier exchange rule.

CHAPTER 4

SURVEY OF PROPOSALS OF SOLUTIONS TO THE KNOWABILITY PARADOX AND RELATED DEBATES

In this chapter, we will survey various proposals of solutions. As we stated in Chapter 1, our frame is modal epistemic languages. The simplest of these solutions is the one based on restricting (KP) to non-epistemic propositions. The first proposal of solution related to our thesis is stated by Kvanvig. We will also give the important objections stated by Williamson, and Kvanvig's reply to these objections. The second proposal of solution is stated by Brogaard and Salerno which is based on Stanley and Szabo's theory of quantifier domain restriction. The third proposal of solution is Kennedy's model theoretic solution.

4.1. Solution Based on Restriction to Non-epistemic Propositions

The simplest solution of the knowability paradox consist in restricting in (KP) the range (or substitution class) of '*p*' to *non-epistemic* propositions (or sentences) i.e. ones that do not contain any epistemic predicate (or operator).[58] We also assume that the *KK*-thesis, i.e. $Kp \rightarrow KKp$, is invalid.[59]

The knowability principle takes the following restricted form (KP*)

(KP*) $\forall p\ (p \rightarrow \Diamond Kp)$

where the substitution class of variable '*p*' is restricted to non-epistemic sentences of an object language. Note that (KP*) and the *KK*-thesis entail the following:

(1) $\forall p\ (p \rightarrow \Diamond Kp)$

(2) $\forall p\ (KKp \rightarrow \Diamond KKKp)$

.

.

.

where *p* stands for non-epistemic sentences. The invalidity of the *KK*-thesis blocks the derivation of (1), (2), …

Although this solution is correct, it is *not* wholly satisfactory for the reason that it restricts too much the scope of the knowability principle.

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[58] This solution is mentioned in Rabinowicz and Segerberg (1994).

[59] Concerning the invalidity of the *KK*-thesis, see Williamson (1994) and Williamson (2000).

4.2. Kvanvig's Propositional Solution

Kvanvig (1995) begins his proposal of solution by making a distinction between sentences in language and propositions. Kvanvig makes a further distinction between sentences having the same meaning (linguistic character) and sentences expressing the same proposition (content). On the basis of these distinctions sentences in a language can be divided into

i. Indexical sentences
ii. Non-indexical sentences.

Standard indexical sentences containing indexicals or demonstratives are like "I am here now", "It is raining", etc. These indexical sentences are context sensitive in the sense that they express different propositions in different contexts, although these sentences have the same meaning. Similarly, on the indexical understanding of quantifiers, quantified sentences express different propositions in different contexts, because the quantifier is implicitly restricted to a domain. [60]

Kvanvig uses this point concerning extensional contexts to make a similar claim in modal contexts. Kvanvig treats quantified sentences as modally indexical. "If the very same sentence with the very same [linguistic] meaning is asserted in a different possible world with a different domain, a different proposition is expressed."[61] As a result, Kvanvig concludes that quantified sentences are modally indexical. He remarks that this conclusion does not remove the distinction between indexical and non-indexical

[60] Kvanvig (1995), p. 16.
[61] *Ibid*., p. 16.

sentences, but “the theory does imply, however, that there is much more indexicality in language, even in formal language, than might have been thought.” [62]

Kvanvig uses the above stated modal indexicality theory to provide a solution to the knowability paradox. Kvanvig states that there is a mistake in substitution in the derivation of the knowability paradox. The mistake is in the substitution of the Fitch conjunction, formalized by Kvanvig as ‘T*q* & ¬∃y∃s *KyTqs*’, into the knowability principle (KP), formalized as ‘∀*p* (T*p* → ◊∃x∃t *KxTpt*)’. Substitution is legitimate, only if the formula is modally non-indexical. “Otherwise” according to Kvanvig “the unknown proposition expressed by that formula in the actual world may not be the expressed value of that formula in the modal context in question.” [63] The substitutional context of the knowability principle is modal, because the consequent of the conditional is bound by the possibility operator ‘◊’. For this reason, the substitution is illegitimate. Therefore, the set of premises of the knowability paradox, i.e. {(NO), (KP)}, is *not* inconsistent.

Kvanvig, further reinforces his solution to the knowability paradox by reinterpreting non-omniscience thesis (NO). The motivation behind this maneuver is that the introduction of modally unrestricted quantifiers will result in inconsistency. In other words, if both (KP) and (NO) are

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[62] *Ibid*., p. 17. Kvanvig gives two negative aspects of his view. *First*, “if quantified sentences are indexical, the Fregean hope for a logically perfect language must be given up when intensional operators are added to standard first-order theory.” *Second*, the rule of necessitation has to be restricted. “Roughly, the rule of necessitation can be used only on formulas that express the same proposition in any context; on formulas that are rigid in the sense that some take proper names to be rigid.” See *Ibid*., p. 17.

[63] *Ibid*., p. 18.

represented by using modally unrestricted quantifiers, the paradox will still be derived.

(KP) is represented by unrestricted quantifiers as '$\forall p\ (p \rightarrow \exists x \exists t Kxpt)$', by dropping '◊'. This says that any truth is known at some possible time by some possible being. The crucial point Kvanvig states on the other hand is: "(NO) cannot be represented as before as the claim $\exists p(p\ \& \neg\exists x \exists t KxTpt)$."[64] For it will still lead to paradox when instantiated in (KP). Kvanvig restates (NO) as follows:

(NO) $\exists p\ (p\ \&\ \neg\exists x \exists t\ @x\ \&\ @t\ \&\ Kxpt)$

where 'x' and 't' range respectively over possible person and times, '$@x$' says that 'x actually exists' and '$@t$' says that 't actually exists'. Kvanvig states that there are two ways to achieve this result for '$@x$'.

First way: Treat @ as rigid designator. [65]
Second way: Treat @ as picking out the actual world indexicality. [66]

In the first way, '$@x$' expresses the same proposition in each world, hence it is rigid. In the second way, it would express a different proposition in each different possible world. On the basis of this difference, "to avoid the problem with the original proof, the only way to interpret '@' is as a rigid name [1-place predicate]." [67] In the first case there is a reference to the

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[64] *Ibid*., p. 18-19.
[65] 'at w^*, $E!x$' or '$@E!x$' is rigid. at w, '$a(v_i)$ means that $a_i \in D_{w^*}$.
More generally: at w, β means that b → at w, $b \in D_{w^*}$.
[66] '@x'≡'E!x' is indexical.
[67] *Ibid*., p. 19.

actual world regardless of the modal context in which it appears. With this interpretation, substitution of (NO) into (KP) is not paradoxical. [68]

> All that follows from the fact that some instance of [KP] is true is that there is a world in which the following is true:
>
> $\exists x \exists t\ K(p\ \&\ \neg\exists x \exists t\ @x\ \&\ @t\ \&\ Kxpt)$.
>
> [It] only says that some possible being knows both that p is true and that no being in @ knows that p is true. [69]

By this way, it is possible to know the Fitch conjunction. There is no contradiction in supposing that some possible being at some possible time knows that p is true but never known by an actual being at an actual time. As a result, Kvanvig's indexical theory of quantifiers provides a solution to the knowability paradox.

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[68] Kennedy verifies this derivation. Kennedy prefers 'act' in place of '@'. "We must assume something of the form '$q \wedge \neg K_y q$', '$q \wedge \neg$ (act(y) $\wedge K_y q$)' or '$q \wedge \neg\exists y$ (act(y) $\wedge K_y q$)'. If we try the first one, we obtain:

1. $q \wedge \neg K_y q$	Hypothesis
2. $\exists x\ K_x(q \wedge \neg K_y q)$	1, [KP] and MP
3. $K_x(q \wedge \neg K_y q)$	Hypothesis
4. $K_x q \wedge K_x(\neg K_y q)$	3, [K-Dist]
5. $K_x(\neg K_y q)$	4, [Conjunction Elimination]
6. $\neg K_y q$	5, [Fact]
7. $K_x q$	4, [Conjunction Elimination]

The formulas at step 6 and 7 are not mutually inconsistent…most importantly this argument will lead nowhere since the only way to get a Fitch-like contradiction is to have $y = x$, and this is a non-starter since it would violate the most basic of all substitution rules." Kennedy (2013), Sec. 3.3.

[69] Kvanvig (1995), p. 19. Note that here '@' stands for the actual world.

4.2.1. Williamson's Objection to Kvanvig's Solution

Williamson (2000) initially agrees with Kvanvig in that substitution in the scope of modal operators is legitimate only if the substituents are rigid designators. He accepts that universal instantiation does not allow the substitution of non-rigid designators in place of the scope of the modal operators. Williamson objects to Kvanvig's diagnosis that the Fitch conjunction is non-rigid. [70]

In order to determine whether the Fitch conjunction is a rigid designator, it must be well understood what the sentences that designate with respect to possible worlds. Williamson gives two approaches. The first one is Fregean approach. A sentence is called rigid in this approach, if it has the same truth value in every world.[71] This approach is too restrictive because only rigid designators are instantiated in propositional quantification. The second approach considers sentences as designating propositions with respect to possible worlds. In this approach, "a sentence is a rigid designator if it designates the same proposition with respect to every world, even if that proposition varies in truth-value from world to world."[72] We can formally define this approach as follows:

Definition σ is a rigid designator $\leftrightarrow \Sigma p\, \forall w$ (at w, σ means that p) [73]

In this sense, the Fitch conjunction is rigid for the propositional variable '*p*'. Williamson's motivation is that "if the designation of the quantifier is a

[70] Williamson (2000), p. 286.
[71] *Ibid*., p. 286.
[72] *Ibid*., p. 286.
[73] Note that 'Σ' stands for substitutional existential quantifier.

component of the proposition designated by the whole formula, and it is contingent what beings or times there are, then the formula will designate different propositions with respect to different worlds."[74] The important thing to consider is whether the Fitch conjunction designates different propositions with respect to different possible worlds.

> The comparison with non-rigid definite descriptions does not help the suggestion that [the Fitch conjunction] is non-rigid in the relevant sense. Intuitively, a sentence like 'The number of the planets is less than fifty', as uttered in this context with the definite description understood non-rigidly, designates the same proposition with respect to all circumstances. [75]

Williamson considers the above-stated considerations as a crucial flaw in Kvanvig's solution. Kvanvig does not have any ground to suppose the rigidity of the Fitch conjunction that varies its reference in a fixed context, and thus designates different propositions with respect to worlds evaluation. [76]

> Kvanvig's mistake is like that of confusing 'I am eating something that could have been a cake' with 'I could have been eating a cake'; the first but not the second entails that I am eating. Similarly, even read with possibilist quantifiers, SVER [i.e. $\forall p\ (p \rightarrow Kp)$] but not WVER [i.e. $\forall p\ (p \rightarrow \Diamond Kp)$] entails that every truth is known (not necessarily now). [77]

To conclude, the Fitch conjunction is indeed a rigid designator, since it expresses the same proposition in every possible world although the truth value of this conjunction may change with respect to possible worlds. However, change in truth value is not related with rigidity. As a result,

[74] *Ibid*., p. 287.
[75] *Ibid*., p. 287.
[76] *Ibid*., p. 288.
[77] *Ibid*., p. 289.

Williamson concludes that there is no illegitimate universal instantiation in the formal derivation of the knowability paradox.

4.2.2. Kvanvig's Reply to Williamson's Objection and Neo-Russelian View of Proposition

Kvanvig (2006) mainly aims to defend his solution against Williamson's objections in his book. Kvanvig explains his indexical theory of quantification in terms of neo-Russelian view (or theory) of propositions.[78] Indeed, Kvanvig holds his previous proposal of solution to the knowability paradox, i.e. in order to universal substitution be legitimate in the knowability principle, substituents have to be indexical. Otherwise, the substitution will result in the paradox that all truths are known. [79]

On neo-Russelian view of propositions, the domains of quantifiers are implicit in the proposition expressed by the quantified sentence. [80] Therefore, it has been justly argued that modal indexicality is a type of non-rigidity.

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[78] See Kvanvig (2006), pp. 169-170.

[79] There is a difference in the formalized version of the non-omniscience thesis in Kvanvig (2006) and Kvanvig (1995). In Kvanvig ((2006), pp. 163-164), non-omniscience thesis is expressed in somewhat abbreviated form by using only one actuality operator @:

$$\exists p\ (p\ \&\ \neg\exists x\exists t\ @Kxpt),$$

where '@*Kxpt*' is short for 'knowledge of *p* by *x* at *t* is actual.'

[80] Brogaard and Salerno (2007), pp. 21-22.

Kvanvig defends himself against Williamson's charge of confusing rigidity with indexicality. [81] Although, Kvanvig avoids using the term modal indexicality as proper explanation of his neo-Russelian view of propositions, Kvanvig uses this term throughout his book. He points that his solution is not based on analogy between modal indexicality and rigid designation, although these two terms may have analogical uses in context.

> I do describe the view using the term ''modally indexical'', but to infer from that terminology that I think of the view in terms of a kind of indexicality is akin to accusing someone who speaks of former Senators as positing a new kind of Senator. Moreover, the relationship between sentences and propositions is not one of designation, as Williamson's preferred language of rigid designation would have it: sentences don't refer to or designate anything. [82]

Kvanvig ends that his solution to the knowability paradox does not conclude that whether anti-realist knowability principle is justified.

> [T]he solution only needs to demonstrate that the logical distinction between known and knowable truth does not collapse on the basis of Fitch's proof. Such a solution may not rescue anti-realism from the paradox, if it preserves the logical distinction in a way that leaves the anti-realist with no resources for explaining how an unknown truth can be known. The resolution of the paradox is one thing, however, and the sustainability of anti-realism in the face of the paradox is another. [83]

[81] See Kvanvig (2006), pp. 169-170.
[82] *Ibid*., p. 173
[83] *Ibid*., p. 165. For further discussion on the prospects for antirealism, see *ibid*., pp. 165-168.

4.3. Brogaard and Salerno's Restriction Strategy

Brogaard and Salerno (2007) offer a new solution to the knowability paradox which is partly based on Kvanvig's approach. However, they do not reformulate any of the premises (NO) and (KP) leading to the paradox. Rather, they reinterpret the quantified expressions in modal contexts, and they apply the quantifier domain restriction strategy in order to block the derivation in the knowability paradox. Their strategy is based on Stanley and Szabo's theory of quantifier domain restriction.

Brogaard and Salerno state that Kvanvig's diagnosis of the problem in the knowability paradox is that there is a problem of illegitimate substitution in the knowability principle (KP). However, they consider this approach very erroneous. They present their view on two points.

First, Brogaard and Salerno analyze Williamson's criticism regarding Kvanvig's use of modal indexicality and rigidity. They disagree with Williamson's criticisms and they conclude that the Fitch conjunction is non-rigid in the sense of Kvanvig's view of neo-Russelian propositions. Therefore, they agree with Kvanvig's view that modally indexical sentences indeed express different propositions in different worlds, thus these expressions are non-rigid.[84] *Second*, Brogaard and Salerno disagree with Kvanvig that substitution is legitimate only if the substitutient is a rigid designator.[85] In their account, Kvanvig's neo-Russelian approach is not sufficient to block the universal instantiation in (KP). They subsequently analyze Kvanvig's term of 'object-dependent' as follows:

[84] Brogaard and Salerno (2007), pp. 19-21.

[85] *Ibid*., p. 22.

> [I]f the proposition expressed by the Fitch conjunction is in fact object-dependent in Kvanvig's neo-Russellian fashion, then substitution is legitimate…But suppose Kvanvig is wrong to think that the Fitch conjunction expresses an object-dependent proposition. Is substitution then valid? The answer is yes. For if the Fitch-conjunction does not express an object-dependent proposition, then the proposition expressed by it contains a traditional existential quantifier for which the substitution issue does not arise. [86]

From the above stated analysis, Brogaard and Salerno conclude that the substitution is legitimate whether Fitch conjunction is object-dependent or not. For this reason, there is no illegitimate substitution of the Fitch conjunction in (KP). [87]

Brogaard and Salerno aim to provide a stronger approach in order to block the substitution. They presume that indexicality of the quantifiers are implicit in the Fitch conjunction and the knowability principle (KP). With this reasoning, if the Fitch conjunction and (KP) is expressed in terms of quantifier domains, then the derivation of the knowability paradox is blocked. [88] Stanley and Szabo's motivation for their domain restriction strategy can be best expressed by following quotation : "the existence of domain variables comes from the fact that domain variables seem required to account for apparent binding relations with quantifiers." [89]

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[86] *Ibid*., p. 23.
[87] *Ibid*., p. 23.
[88] *Ibid*., p. 23.
[89] Brogaard and Salerno refer this quotation to Stanley and Szabo (2002), p. 368.

From the above stated strategy, '$\Diamond \mathrm{K}p$' can be expressed by associated domain variable as follows:

p is known at some world by <someone, f(i)> knows that *p*

where 'i' is bound by the higher quantifier 'some world' and 'f' is assigned a value by context. [90] The knowability principle '$p \rightarrow \Diamond \mathrm{K}p$' can then be expressed as follows:

for all propositions *p*, if *p* is true, then *p* is known at some world by <someone, f(j)>

where 'j' is bound by higher quantifier possible world and j > i. [91] The Fitch conjunction 'p & $\neg Kp$' can also be expressed as follows:

p and it is not the case that *p* is known by <someone, f(i)>

Knowing that the Fitch conjunction (or the instantiated form of the Fitch's Theorem 1 in Chapter 3) can be explicitly stated by domain variables as follows: [92]

$$\neg\Diamond K_{\mathrm{i}}\, p\ (p \,\&\, \neg K_{\mathrm{i}}\, p)$$

Now, by substitution of the Fitch conjunction in place of '*p*' in (KP), we set the following:

$$(p \,\&\, \neg K_{\mathrm{i}}\, p) \rightarrow \neg\Diamond K_{\mathrm{j}}\, p\ (p \,\&\, \neg K_{\mathrm{i}}\, p)$$

where 'j' is a domain variable bound by possibility operator and j > i. [93]

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[90] Brogaard and Salerno (2007), p. 25.
[91] *Ibid*., p. 25.
[92] *Ibid*., p. 25.

At this point, Brogaard and Salerno state that the application of modus tollens is invalid, because the corresponding theorem of unknowability of the Fitch conjunction, i.e. $\neg\Diamond K_i p(p \ \& \ \neg K_i\ p)$, is not equivalent to the consequent of the above stated instantiated form of (KP). Since j > i, the derivation of the knowability paradox is blocked.

> One could legitimately substitute the claim 'p and it is not the case that someone knows that p' (with no domain variable associated with 'someone') for 'p' in the knowability principle. A contradiction would result, but the substitution would be legitimate…[The derivation] is blocked as a result of domain variable associated with the quantifiers implicit in the concept of knowledge, rather than as a result of illicit substitution.[94]

[93] *Ibid*., p. 25.
[94] *Ibid*., pp. 26-27.

4.4. Kennedy’s Model Theoretic Solution

Kennedy (2013) diagnoses the problem in the knowability paradox as follows: “The problem in this paradox is not so much in the thesis of verificationism itself, but rather in the *way* in which a contradiction is derived from it.” [95]

Kennedy agrees with Kvanvig that there is illegitimate universal instantiation, due to substitution of non-rigid expressions into modal contexts. According to Kennedy, Brogaard and Salerno are only critical of specific use of Kvanvig’s modal indexicality and context-dependence, but they fail to show that how these concepts are not involved in the paradox.[96] Kennedy agrees with Kvanvig’s analysis that non-rigidity of ‘*K*’ will be enough to block the derivation. According to Kennedy, Brogaard and Salerno’s solution is not different form that of Kvanvig in the sense that both of these two solutions present differences between domains of actual and non-actual (possible) agents, which are bound and quantified implicitly by *K* operator.[97] He further criticizes the Brogaard and Salerno’s approach as follows:

> A solution to Fitch’s paradox, first and foremost, must explain why [(O)] can be derived from [(KP)] using seemingly unassailable logical principles. An explanation might be that [(KP] does not portray verificationism correctly but that the logical principles are correct…or, it might be that [(KP)] is correct but that some logical principle is wrong. However, there is just no way of maintaining that both [(KP)] and the

[95] Kennedy (2013), Sec. 1.
[96] *Ibid*., Sec. 3.4.
[97] *Ibid*., Sec. 3.4.

> logical principles are correct, as Brogaard & Salerno maintain, while claiming that the paradox is avoided. [98]

Kennedy agrees with Kvanvig's solution and states his solution regarding to derivation of contradiction in the knowability paradox. Kennedy gives the further steps of proving the consistency of (KP) and (NO) by way of constructing a model satisfying both of these assumptions. [99]

[98] *Ibid*., Sec. 3.4.
[99] See Kennedy (2013), Sec. 4.2 and Appendix.

CHAPTER 5

PROPOSAL AND DEFENSE OF A NEW SOLUTION

In this chapter, we will briefly restate the knowability paradox and the paradox of idealization with respect to the truth theory for first-order modal epistemic language L adopted from Grünberg & Grünberg. We will then provide a solution to the knowability paradox and the paradox of idealization.

5.1. Truth Theory for First-Order Modal Epistemic Language L and Related Axioms

Our proposed solution of the knowability paradox and the analogous paradox of idealization is formulated in the framework of a truth theory for modal epistemic language L. Language L is an extention of first-order (quantificational) modal language L_3 given by Grünberg & Grünberg. The vocabulary of L_3 consists of rigid individual constants $\ulcorner d_i \urcorner$, i = 1,…,m_{in}, predicate constants $\ulcorner F_i \urcorner$, $i = 1,\ldots,m_{pr}$, logical *constants* $\ulcorner = \urcorner$, $\ulcorner \neg \urcorner$, $\ulcorner \wedge \urcorner$, $\ulcorner \Box \urcorner$, $\ulcorner \forall \urcorner$ and the individual variables v_1, v_2, v_3, … where $v_1 = \ulcorner x' \urcorner$, $v_2 = \ulcorner x'' \urcorner$, $v_2 = \ulcorner x''' \urcorner$,... It is assumed that each 'd_i' is rigid and 'F_i' is extensional.[100] The new language L results from L_3 by adjunction on the one hand of a set of unstructured atomic sentences abbreviated respectively as $\ulcorner S_1 \urcorner,\ldots,\ulcorner S_n \urcorner$, and on the other hand of a 2-ary epistemic operator 'K'. 'K' is used in L in such way that for any term β and formula φ in L, $\ulcorner \beta K \varphi \urcorner$ is a formula in L which is read as $\ulcorner$agent β knows that φ$\urcorner$.

A truth theory for a language with modal operators presupposes an intended *frame* of the form $<W, w^*, R, D>$ where W is the set of possible worlds, w^* is the actual world, R is an accessibility relation such that $R \subseteq W \mathrm{x} W$, and D is a function from W into the power set of the set of intended possible individuals. We write D_w in place of $D(w)$. D_w is the set of possible agents existing in possible world $w \in W$. We assume that for all $w \in W$, every member of D_w is a person endowed with cognitive capacities at a given time. We call the members of D_w *agents*. The same person β may have different cognitive capacities at different times so that $\ulcorner \beta K \varphi \urcorner$could have different truth values at different times. In order to secure a unique truth value to sentences of the form $\ulcorner \beta K \varphi \urcorner$ we construe the notion of an agent as consisting of a *person at a time*. From now on we shall write $\ulcorner K_\beta \varphi \urcorner$ as short for the formula $\ulcorner \beta K \varphi \urcorner$.

A truth theory for language L is called a *truth-from-satisfaction-and-meaning-theory* with respect to frame $<W, w^*, R, D>$, or $T_L^{TSM}(<W, w^*, R, D>)$ for short. Such a truth theory is formulated in a *metalanguage ML* for talking about *object language L*. *ML* is a regimented fragment of English

[100] See Grünberg & Grünberg, Ch. III, Sec. 6.3.

augmented with logico-mathematical, syntactic, and semantic symbols. [101] In particular, *ML* contains, besides referential quantification signs '∀', '∃', also the substitutional universal and existential quantification signs, viz. 'Π' and 'Σ'. Truth theory T_L^{TSM}(<*W*, *w**, *R*, *D*>) is an extention of the truth theory $T_{L_3}^{TSM}$(<*W*, *w**, *R*, *D*>) for first-order modal language L_3. [102] The axioms of T_L^{TSM}(<*W*, *w**, *R*, *D*>) are prefixed with sentential operators of the form 'at *w*'. The intuitive meaning of 'at *w*' can be expressed as follows

(at *w*, *p*) iff (if *w* were actual then it would be the case that *p*) [103]

The prefixed axioms of theory T_L^{TSM}(<*W*, *w**, *R*, *D*>) are as follows.

(w^*-$Ax.S_i$) at w^*, ($\ulcorner S_i \urcorner$ means that S_i) $i = 1, \ldots, m_{sn}$

(w-$Ax.v_i$) at w, v_i means that a_i

where 'a_i' is an individual variables in *ML* corresponding to 'v_i' in the sense that 'a_i' and 'v_i' range over the same domain. 'a_i' is the translation of 'v_i' into metalanguage *ML*.

(w^*-$Ax.d_i$) at w^*, $\ulcorner v_i = d_k \urcorner$ means that $a_l = d_k$, $k = 1,\ldots,m_{im}$.

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[101] Concerning metalanguage *ML* see Grünberg & Grünberg, Ch. I, Sec. 1.1.

[102] The axioms of truth theory $T_{L_3}^{TSM}$ <*W*, *w**, *R*, *D*> are given in Grünberg & Grünberg, Ch. III, Sec. 7.2.

[103] Concerning the sentential operators of the form 'at *w*', see Grünberg & Grünberg, Ch. I, Sec. 2.3.

(w^*-$Ax.F_i$) (at w^*, β_1 means that $b_1 \wedge \ldots$ at w^*, β_n means that b_n) $\rightarrow$
at w^*, $\ulcorner F_i^n(\beta_1,\ldots, \beta_n \urcorner$ means that $\ulcorner F_i^n(b_1,\ldots, b_n)\urcorner$.

where '$F_i^{w^*}$' stands for 'F_i', $i = 1,\ldots, m_{pr}$.

(w-$Ax.id$) (at w, β_1 means that $b_1 \wedge$ at w, β_2 means that b_2) $\rightarrow$
at w, $\ulcorner \beta_1 = \beta_2 \urcorner$ means that $\ulcorner b_1 = b_2 \urcorner$.

(w-$Ax.ng$) (at w, φ means that p) $\rightarrow$ (at w, $\ulcorner \neg\varphi \urcorner$ means that $\neg p$)

(w-$Ax.cnj$) ((at w, φ means that p) $\wedge$ (at w, ψ means that q)) $\rightarrow$
($\ulcorner \varphi \wedge \psi \urcorner$ means that $p \wedge q$)

(w-$Ax.uq$) (at w, φ means that p) $\rightarrow$
(at w, $\ulcorner \forall v_i\varphi \urcorner$ means that $\forall a_i\, (a_i \in D_w \rightarrow p)$

(w-$Ax.nec$) (at w, φ means that p) $\rightarrow$
(at w, $\ulcorner \Box\varphi \urcorner$ means that $\forall w'\, (wRw' \rightarrow$ (at w', p))

(w-$Ax.K$) ((at w, β means b) $\wedge$ (at w, φ means p)) $\rightarrow$
(at w, $\ulcorner K_\beta\varphi \urcorner$ means that ($b \in D_w \wedge K_b p$))

(w-SM) (at w, $\varphi(v_i,\ldots, v_n)$ means that $p(a_{i_1},\ldots, a_{i_n})$) $\rightarrow$
$\forall s$ (at w, s satisfies $\varphi \leftrightarrow$ at w, $p(s(i_1),\ldots, s(i_n))$)

where 'satisfies' expresses the *absolute* notion of satisfaction.

$(w\text{-}TM)$ $\quad$ (at w, σ means that p) $\rightarrow$ ((at w, σ is true) $\leftrightarrow$ at w, p)

On the basis of above axioms we can prove the following propositions (w-eq), and (w-pos) concerning respectively quantification sign '∃' and modal operator '◊':

$(w\text{- }eq)$ $\quad$ (at w, φ means that p) $\rightarrow$
(at w, $\ulcorner \exists v_i \varphi \urcorner$ means that $\exists a_i (a_i \in D_w \wedge p)$.

$(w\text{-}pos)$ $\quad$ (at w, φ means that p) $\rightarrow$
(at w, $\ulcorner \Diamond\varphi \urcorner$ means that $\exists w' (wRw' \wedge$ at w', p))

The relative notions of 'satisfies-at' and 'true at' are defined as follows.

Definition 1 (S_{at}) $\quad$ (s satisfies-at-w $\varphi(v_{i_1},\ldots,v_{i_n})$) $\leftrightarrow$
Σp ((at w^*, $\varphi(v_{i_1},\ldots,v_{i_n})$ means that $p(a_{i_1},\ldots,a_{i_n})$) $\wedge$
(at w, s satisfies $p(s(i_1),\ldots, s(i_n))$)

where s is *sequence* in the sense of a function from the set of positive integers into $\bigcup_{w \in W} D_w$, and 'p' is a substitutional variable whose substitution class consists of sentences in *ML*.

Definition 2 (T_{at}) $\quad$ σ is true at $w \leftrightarrow \Sigma p$ (at w^*, σ means that p) $\wedge$ (at w, p)).

One can prove the following propositions:

Proposition 1 ($S_{at}M$) (at $w*$, ($\varphi(v_{i_1},\ldots,v_{i_k})$ means that $p(a_{i_1},\ldots,a_{i_1})) \rightarrow$
$\forall w \forall s$ (s satisfies-at-w φ $(v_{i_1},\ldots,v_{i_n}) \leftrightarrow$
(at w, $p(s(i_1),\ldots s(i_n)))$)

Proposition 2 ($T_{at}M$) (at $w*$, (σ means that p)) $\rightarrow$
Σw (σ is true at $w \leftrightarrow$ at w, p)

Proposition 3 (T^{**}) (σ is true at w) $\leftrightarrow \forall s$ (s satisfies-at-w σ)

The truth theory T_L^{TSM}<*W*, $w*$, *R*, *D*> results from theory $T_{L_3}^{TSM}$<*W*, $w*$, *R*, *D*> by adjunction of the following new axioms. [104]

We assume that the metalanguage *ML* for object language *L* includes *L* as a sublanguage. It follows that both the unstructured atomic sentences 'S_i' and the epistemic operator '*K*' contained in *L* belong also to *ML*. This allows us to use in axioms of the form (w-Ax.*K*) the operator '*K*' itself in the interpretation of $\ulcorner K_\beta\varphi \urcorner$.

[104] See Grünberg & Grünberg, Ch. III, Sec. 7.2.

One can show that truth theory T_L^{TSM}(<W, w*, R, D>) is complete.[105] In the sense the following proposition is provable in this theory

(ΣM) ∀σ Σp (σ means that p)

where σ ranges over all sentences in L.

5.2. The Knowability Paradox

We shall formulate in truth theory T_L^{TSM}<W, w*, R, D> a solution of the so-called Church-Fitch paradox of knowability, or the *Knowability Paradox* for short. We will briefly restate the knowability paradox with respect to first order modal epistemic language L.

[105] Concerning the completeness of truth theories see Grünberg & Grünberg, Ch. I, Sec. 1.7, Definition 4.

The Knowability Paradox consists in the derivation of a contradiction from the conjunction of the following initially plausible premises. [106]

1. The *non-omniscience thesis*:

(NO) Some true propositions are unknown,

and

2. The *knowability principle*:

(KP) All true propositions are knowable.

The derivation of a contradiction from the conjunction of (NO) and (KP) can be formulated in an extension $\bar{L}$ of modal epistemic language L which results by adjunction of substitutional quantification signs $\ulcorner\Pi\urcorner$, $\ulcorner\Sigma\urcorner$, substitutional variables $\ulcorner S\urcorner$, $\ulcorner S'\urcorner$, $\ulcorner S''\urcorner$,…, and place holders $\ulcorner\bar{S}\urcorner$, $\ulcorner\bar{S}_1\urcorner$, $\ulcorner\bar{S}_2\urcorner$, …, for sentences in L. In language $\bar{L}$ the non-omniscience thesis (NO) is formulated by sentence

$$(\mathrm{NO})_{\bar{L}} \qquad \Sigma S\,(S \wedge \neg\exists v_1\, K_{v_1} S)$$

and the knowability principle (KP) is formulated by sentence

$$(\mathrm{KP})_{\bar{L}} \qquad \Pi S\,(S \rightarrow \Diamond\exists v_1\, K_{v_1} S)$$

where ‘S’ is a substitutional variable in $\bar{L}$ whose substitution class consists of the sentences in language L.

[106] Kennedy remarks that “a paradox is when an unexpected consequence follows form assumptions initially believed to be valid.” See Kennedy (2013), Sec.1.

The inconsistency of the set of sentences $\{(\mathrm{NO})_{\bar{L}}, (\mathrm{KP})_{\bar{L}}\}$ is established on the basis of the unknowability of sentences of the form:

$$S \wedge \neg\exists v_1\, K_{v_1} S$$

where 'S' is place holder for that sentences in $\bar{L}$. A sentence of this form, as well the proposition expressed by, is called the Fitch conjunction, as was stated in Chapter 2 of this thesis. The unknowability of the Fitch conjunction can be expressed by the following sentence $(\mathrm{UFC})_{\bar{L}}$:

$$(\mathrm{UFC})_{\bar{L}} \qquad \Pi S\, \neg\Diamond\exists v_1\, K_{v_1} (S \wedge \neg\exists v_1\, K_{v_1} S).$$

The proof of $(\mathrm{UFC})_{\bar{L}}$ can be formulated in $\bar{L}$ as follows.

1.	$\exists \mathrm{v}_1\, K_{v_1}(S \wedge \neg\exists v_1\, K_{v_1} S)$	Assumption
2.	$K_{v_\mathrm{i}}(S \wedge \neg\exists v_1\, K_{v_1} S)$	1, EI
3.	$K_{v_\mathrm{i}} S \wedge K_{v_\mathrm{i}} \neg\exists v_1\, K_{v_1} S$	2, (K-Dist)
4.	$K_{v_\mathrm{i}} \neg\exists v_1 K_{v_1} S$	3
5.	$\neg\exists v_1\, K_{v_1} S$	4, (Fact)
6.	$\forall v_1\, \neg K_{v_1} S$	5, (QN)
7.	$\neg K_{v_\mathrm{i}} S$	6, UI
8.	$K_{v_\mathrm{i}} S$	3
9.	$\perp$	7, 8
10.	$\neg\exists v_1\, K_{v_1}(S \wedge \neg\exists v_1\, K_{v_1} S)$	1-9
11.	$\Box\neg\exists v_1\, K_{v_1}(S \wedge \neg\exists v_1\, K_{v_1} S)$	10, (RN)
12.	$\Pi S\, \neg\Diamond\exists v_1\, K_{v_1}(S \wedge \neg\exists v_1\, K_{v_1} S)$	

11, Rule of Generalization

By means of $(\text{UFC})_{\bar{L}}$ one can easily prove the inconsistency of the set of sentences $\{(\text{NO})_{\bar{L}}, (\text{KP})_{\bar{L}}\}$ as follows

1.	$\Sigma S\ (S \wedge \neg\exists v_1\, K_{v_1} S)$	$(\text{NO})_{\bar{L}}$
2.	$\Pi S\ (S \rightarrow \Diamond\exists v_1\, K_{v_1} S)$,	$(\text{KP})_{\bar{L}}$
3.	$(\bar{S} \wedge \neg\exists v_1\, K_{v_1} \bar{S})$	$(\text{NO})_{\bar{L}}$, EI

where '$\bar{S}$' stands for an arbitrary sentence in $\bar{L}$.

4.	$(\bar{S} \wedge \neg\exists v_1\, K_{v_1} \bar{S}) \rightarrow \Diamond\exists v_1\, K_{v_1}(\bar{S} \wedge \neg\exists v_1\, K_{v_1} S)$	2, UI
5.	$\neg\Diamond\exists v_1\, K_{v_1}(\bar{S} \wedge \neg\exists v_1\, K_{v_1} \bar{S})$	$(\text{UFC})_{\bar{L}}$, UI
6.	$\neg\ (\bar{S} \wedge \neg\exists v_1\, K_{v_1} \bar{S})$	4, 5, MT
7.	$\bot$	3, 6

It follows from above derivation that

$$\{(\text{NO})_{\bar{L}}, (\text{KP})_{\bar{L}}\} \ \vdash \ \bot$$

i.e., that set $\{(\text{NO})_{\bar{L}}, (\text{KP})_{\bar{L}}\}$ is inconsistent. Hence the sentence in $\bar{L}$ of the form

$$(\text{KP})_{\bar{L}} \ \vdash \ \neg\ (\text{NO})_{\bar{L}}$$

is valid. The latter is equivalent to the seemingly paradoxical sentence

$$\Pi S\ (S \rightarrow \Diamond\exists v_1\, K_{v_1} S) \rightarrow \Pi S\ (S \rightarrow \exists v_1\, K_{v_1} S).$$

The above stated sentence is paradoxical because it seems to express a proposition which, in Kvanvig's words "threatens the logical distinction between actual and possible knowledge in the domain of truth." [107]

Kvanvig remarks that a sentence containing a quantifier may express different proposition in different possible world since the quantifier's domain may differ from world to world. Hence the proposition expressed by a quantified sentence may change when substituted for a variable in the scope of a modal operator.[108] In our terminology, we can say that the *proposition* expressed by a sentence σ at a world w is stated by the right hand side of the w-M-sentence corresponding to σ. Indeed, we adopt the following notational abbreviation (NA) concerning the term 'proposition' from Grünberg & Grünberg [109] :

(NA) (at w, σ expresses the proposition that p) $\leftrightarrow$
(at w, σ means that p).

where σ ranges over the sentences of language L.

As already mentioned the propositions in question are unreified fine grained ones. They should be distinguished from Carnapian intentions (i.e. functions from possible worlds to truth values) which are reified rough grained propositions.

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[107] See Kvanvig (2006), p. 2.
[108] See Kvanvig (1995), Sec. 4, pp. 16-19 and Kvanvig (2006), Ch. 6, esp. pp. 155-158.
[109] See Grünberg & Grünberg, Ch. I, Sec. 1.3.

In the frame of truth theory (T_L^{TSM}<W, w^*, R, D>), we shall show by means of the following derivation from assumptions $(\text{NO})_{\bar{L}}$ and $(\text{KP})_{\bar{L}}$ that the two occurrence of the Fitch conjunction in the instance of sentence $(\text{KP})_{\bar{L}}$ express different propositions. It follows, in agreement with Kvanvig's view, that the proof of the inconsistence of set $\{(\text{NO})_{\bar{L}}, (\text{KP})_{\bar{L}}\}$ does not show that (NO) and (KP) are incompatible because the latter one refer to propositions rather than two sentences. Let us now construct the derivation in question.

1.	$\ulcorner \Sigma S\ (S \wedge \neg\exists v_1\ K_{v_1} S)\urcorner$ is true	$(\text{NO})_{\bar{L}}$
2.	$\ulcorner \Pi S\ (S \rightarrow \Diamond\exists v_1\ K_{v_1} S)\urcorner$ is true	$(\text{KP})_{\bar{L}}$
3.	$\ulcorner \bar{S} \wedge \neg\exists v_1\ K_{v_1} \bar{S}\urcorner$ is true	1, EI

Let us use $\ulcorner S_1 \urcorner$ as short for the Fitch conjunction $\ulcorner \bar{S} \wedge \neg\exists v_1\ K_{v_1} \bar{S}\urcorner$. Then the derivation is completed as follows

4.	$\ulcorner \bar{S}_1 \rightarrow \Diamond\exists v_1\ K_{v_1} \bar{S}_1)\urcorner$ is true	2, (UI)
5.	$\bar{S}_1$ is true	3
6.	$\bar{S}_1$ is true $\rightarrow \ulcorner \Diamond\exists v_1\ K_{v_1} \bar{S}_1\urcorner$ is true	4
7.	$\ulcorner \Diamond\exists v_1\ K_{v_1} \bar{S}_1\urcorner$ is true	5, 6, MP
8.	at w^*, $\bar{S}$ means that $\bar{q}$	(ΣM) for L

where '$\bar{q}$' stands for a non-semantic sentence in *ML*.

9. at w^*, $\bar{S}_1$ means that $\bar{q} \wedge \neg\exists a_1\ (a_1 \in D_{w^*} \wedge K_{a_1} \bar{q}\ )$

8, by (w^*-*Ax.cnj*), (w^*-*Ax.ng*), (w^*-*eq*), (w^*-*Ax.K*), since $\ulcorner \bar{S}_1 \urcorner$ stands for $\ulcorner \bar{S} \wedge \neg\exists v_1\ K_{v_1} \bar{S}\urcorner$.

We shall use '$\bar{q}_1$' as short for '$\bar{q} \wedge \neg\exists a_1 (a_1 \in D_{w*} \wedge K_{a_1}\bar{q}$)'.

10. at $w*$, $\ulcorner \Diamond\exists v_1\, K_{v_1}\bar{S} \urcorner$ means that

$\exists w\, (w*Rw \wedge$ at w, $\exists a_1\, (a_1 \in D_w \wedge$

at w, $K_{a_1}(\bar{q} \wedge \neg\exists a_1\, (a_1 \in D_w \wedge K_{a_1}\bar{q}))))$,

9, by ($w*$-*pos*), ($w*$-*Ax.cnj*), ($w*$-*eq*), (w-*eq*), (w-*Ax.cnj*), (w-*Ax.K*).

We shall use '$\bar{q}_1^w$' as short for '$\bar{q} \wedge \neg\exists a_1 (a_1 \in D_w \wedge K_{a_1}\bar{q}$)'.

11. at $w*$, $\ulcorner \Diamond\exists v_1\, K_{v_1}\bar{S}_1 \urcorner$ means that

$\exists w\, (w*Rw \wedge$ at w, $\exists a_1\, (a_1 \in D_w \wedge K_{a_1}\bar{q}_2))$ 10

12. $\exists w\, (w*Rw \wedge$ at w, $\exists a_1\, (a_1 \in D_w \wedge K_{a_1}\bar{q}_2))$

7, 10, ($w*$-*TM*), MP

13. $w*Rw \wedge$ at $\bar{w}$, $\exists a_1\, (a_1 \in D_{\bar{w}} \wedge\ K_{a_1}\bar{q}_2)$ 12, EI

14. at $\bar{w}$, $(a_1 \in D_{\bar{w}} \wedge\ K_{a_1}\bar{q}_2)$ 13, EI

15. $a_1 \in D_{\bar{w}} \wedge\ K_{a_1}(\bar{q}_1 \wedge \neg\exists a_1\, (a_1 \in\ D_{\bar{w}} \wedge K_{a_1}\bar{q}_1)$ 14

16. $(a_1 \in D_{\bar{w}} \wedge\ K_{a_1}\bar{q}_1) \wedge K_{a_1}(\neg\exists a_1\, (a_1 \in\ D_{\bar{w}} \wedge K_{a_1}\bar{q}_1))$

15, (K-Dist)

17. $a_1 \in D_{\bar{w}} \wedge\ K_{a_1}\bar{q}_1$ 16

18. $K_{a_1}(\neg\exists a_1\, (a_1 \in D_{\bar{w}} \wedge K_{a_1}\bar{q}_1))$ 16

19. $\neg\exists a_1\, (a_1 \in D_{\bar{w}} \wedge K_{a_1}\bar{q}_1)$ 18, (Fact)

20. $\neg\, (a_1 \in D_{\bar{w}} \wedge K_{a_1}\bar{q}_1)$ 19, (QN), UI

21. $\bot$ 17, 20

Above derivation shows that the propositions stated respectively by sentences $(\mathrm{NO})_{\bar{L}}$ and $(\mathrm{KP})_{\bar{L}}$ are indeed incompatible. But this fact does not show that (NO) and (KP) are themselves incompatible. Indeed consider line 4 of above derivation which follows from line 2 by universal instantiation (UI). The proposition stated by the occurrence of the Fitch conjunction $\ulcorner \bar{S}_1 \urcorner$ on the left handside of the conditional 4 is '$\bar{q} \wedge \neg\exists a_1$ $(a_1 \in D_{w^*} \wedge K_{a_1}\bar{q})$' as seem in line 9. The latter was abbreviated by '$\bar{q}_1$'. On the other hand the proposition stated by the occurrence of the Fitch conjunction $\ulcorner \bar{S}_1 \urcorner$ on the right side of line 4 is *not* '$\bar{q}_1$' but rather '$\bar{q} \wedge \neg\exists a_1$ $(a \in D_w \wedge K_{a_1}\bar{q})$' as seem in line 10. We have abbreviated the latter by '$\bar{q}_1^w$'.

5.3. A Solution to the Knowability Paradox

We have shown in Section 5.2 above that the inconsistency of the set of sentence in language $\bar{L}$ $\{(\mathrm{NO})_{\bar{L}}, (\mathrm{KP})_{\bar{L}}\}$, i.e.

$$\{\ulcorner \Sigma S\,(S \wedge \neg\exists a_1\, K_{a_1}S) \urcorner, \ulcorner \Pi S\,(\, S \rightarrow \Diamond\exists a_1\, K_{a_1}S) \urcorner\}$$

results from the fact that the two occurrences of a Fitch conjunction in an instance of $(\mathrm{KP})_{\bar{L}}$ express different propositions. Kvanvig secures identity of the propositions expressed by the two occurrences of the Fitch conjunction in question by revising non-omniscience thesis (NO). Kvanvig (1995) proposes a solution of the knowability paradox by way of *quantifier domain restriction* for the epistemic operator '*Kx*' within non-omniscience thesis (NO). For this purpose he revises (NO) by introducing an actuality predicate in the scope of quantifier '$\exists x$'. We shall call this predicate *Kvanvig's actuality predicate*.

The revised form of (NO) is formulated by Kennedy as follows:

(KNO) $\exists p\ (p \wedge \neg\exists x\ (\mathrm{act}(x) \wedge Kxp))$

where (KNO) stands for '*Kvanvig's* (NO)', 'x' ranges over possible agents, and 'act' is a 1-place predicate sign expressing actuality predicate. 'act (x)' is read as 'x is actual'.[110] The condition 'act(x)' in (KNO) provides quantifier domain restriction to the unrestricted quantifier '$\exists x$'. Note that it is (KNO) rather than '$\exists p\ (p \wedge \neg\exists x\ Kxp)$' which formulates faithfully the intended meaning of (NO). Indeed the latter is *not* intended to imply the existence of truths unknown even to merely possible agents.

Kvanvig revises also the knowability principle (KP). The revised form, as formulated by Kennedy is as follows

(Kver) $\forall p\ (p \rightarrow \exists x\ Kxp)$

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[110] Concerning (KNO) see Kennedy (2013), Sec.3.3. As was stated Section 4.3., Kvanvig's original formulation is as follows:

$$\exists p\ (p\ \&\ \neg\exists x\exists t\ @x\ \&\ @t\ \&\ Kxpt)$$

where 'x' and 't' range respectively over possible person and times, and '$@x$' says that 'x actually exists.' See Kvanvig (1995), p. 19(2). We follow Kennedy (2013) in using the predicate sign 'act' in place of Kvanvig's '@'. Kvanvig himself has later replaced (KNO) by the following sentence.

$$\exists p\ (p\ \&\ \neg\exists x\exists t\ @Kxpt)$$

where '@' is the actuality *operator* (rather than the predicate sign) and '$Kxpt$' is short for 'person x knows that p at time t'. See Kvanvig (2006), p. 163(2). In our own formalization of the solution of the knowability paradox we shall adopt Kvanvig's original form (KNO) of the non-omniscience thesis. However we keep to the original formulation '$\Pi S\ (S \rightarrow \Diamond KS)$' of (KP) rather than Kvanvig's because the latter omits the modal operator '$\Diamond$'.

where (Kver) stands for '*Kvanvig's verificationism*' and 'x' ranges over *possible* agents besides the actual ones.[111] As shown by both Kvanvig and Kennedy, the use of (KNO) and (Kver) as the respective formulations of the knowability principle and the non-omniscience thesis block the derivation of the knowability paradox. However as rightfully remarked by Kennedy, "[b]locking a specific derivation of a contradiction from the assumption [(KP)] and (NO) is not the same as proving that these assumptions are consistent."[112]

Our aim is to formalize a solution to the knowability paradox in the frame of our truth theory. For this purpose we shall use an extension $\bar{L}^*$ of the modal epistemic language $\bar{L}$ which results by adjunction of the (1-place) actuality predicate sign 'act'. 'act' is assumed to be persistently rigid in the sense that it has the same extension D_{w*} in *all* possible world $w*$. The truth theory for the extended language $\bar{L}^*$ in question is $T_{\bar{L}^*}^{TSM}(<W, w*, R, D>)$. This truth theory results from the corresponding theory for $\bar{L}^*$ by adjunction of the following axiom for the predicate sign $\ulcorner$act$\urcorner$.

$$(w\text{-}Ax.\text{act}) \qquad (\text{at } w, \beta \text{ means } b) \rightarrow (\text{at } w, \ulcorner \text{act}(\beta) \urcorner \text{means that } b \in D_{w*})$$

[111] Concerning (KVER) see Kennedy (2013), Sec. 3.3. Kvanvig's own formulation of the knowability principle in both (1995), p. 18(1) and (2005), p. 163(1) is as follows

$$\forall p\ (p \rightarrow \exists x \exists t\ Kxpt)$$

We see that Kvanvig omits the modal operator '◊' in his formulation of (KP).
[112] See Kennedy (2013), Sec. 4.

We formulate in language $\bar{L}^*$ the non-omniscience thesis and the principle of knowability respective by the following sentences $(\mathrm{NO})_{\bar{L}^*}$ and $(\mathrm{KP})_{\bar{L}^*}$.

$(\mathrm{NO})_{\bar{L}^*}$ $\quad \Sigma S\ (S \wedge \neg\exists v_1\ (\mathrm{act}\ (v_1) \wedge K_{v_1}S))$,

$(\mathrm{KP})_{\bar{L}^*}$ $\quad \Pi S\ (S \rightarrow \Diamond\exists v_1\ K_{v_1}S)$.

Note that $(\mathrm{NO})_{\bar{L}^*}$ agrees with Kvanvig's formulation in (1995) but differs from sentence $(\mathrm{NO})_{\bar{L}}$ in Section 5.2 above, since the latter does not contain the predicate sign 'act'. On the other hand $(\mathrm{KP})_{\bar{L}^*}$ is identical to sentence $(\mathrm{KP})_{\bar{L}}$ in Section 5.2 but differs from Kvanvig's formulation in both (1995) and (2006) which does *not* contain the modal operator '◊'. Our formalization has the advantage of preserving intact the original formulation of the knowability principle (KP), in agreement with Brogaard and Salerno. [113]

[113] Brogaard and Salerno give a solution to the knowability paradox which is analogous to that of Kvanvig. The former differs from the latter mainly on two points: (i) "the substitution of the Fitch conjunction into the knowability principle is legitimate." (Brogaard and Salerno (2007), p. 23) (ii) They preserve the modal operator '◊' in the formulation of the knowability principle (KP) (*ibid*, p. 27). On the other hand Kennedy's interpretation of Brogaard and Salerno's conception of the knowability principle and of a Fitch conjunction can be respectively formulated as follows.

(KP) $\quad \Pi p\ ((\mathrm{at}\ w_0, p) \rightarrow \exists w\exists a\ (\mathrm{at}\ D_w \wedge \mathrm{at}\ w, K_a p)$

Fitch Conjunction: $p \wedge \neg\exists a\ (\mathrm{at}\ D_{w*} \wedge K_a p)$

See Kennedy (2013), Sec. 3.4 (6) and (7). On the basis of Kennedy's interpretation, we can say that our solution coincides, in essence, with Brogaard and Salerno's.

Let us show now how the derivation of a contradiction from assumptions $(\mathrm{NO})_{\bar{L}^*}$ and $(\mathrm{KP})_{\bar{L}^*}$ is blocked. Consider the following sentence $\ulcorner \bar{S}_1{}^* \urcorner$ which we call a *rigid Fitch conjunction*:

$$(\bar{S}_1{}^*) \qquad \bar{S} \wedge \neg\exists v_1 (\mathrm{act}\,(v_1) \wedge K_{v_1}\bar{S})$$

Contrary to the Fitch conjunction $\ulcorner \bar{S} \wedge \neg\exists v_1 K_{v_1}\bar{S} \urcorner$, the above sentence $\ulcorner \bar{S}_1{}^* \urcorner$ is *not* unknowable. Indeed one can construct following derivation starting with the following sentence expressing that $\ulcorner \bar{S}_1{}^* \urcorner$ is known.

1.	$\exists v_2\, K_{v_2}(\bar{S} \wedge \neg\exists v_1 (\mathrm{act}\,(v_1) \wedge K_{v_1}\bar{S}))$	Assumption
2.	$K_{v_i}(\bar{S} \wedge \neg\exists v_1 (\mathrm{act}\,(v_1) \wedge K_{v_1}\bar{S}))$	1, EI
3.	$K_{v_i}\bar{S} \wedge K_{v_i} (\neg\exists v_1 (\mathrm{act}\,(v_1) \wedge K_{v_1}\bar{S}))$	2, (K-Dist)
4.	$K_{v_i}\bar{S}$	3
5.	$K_{v_i}(\neg\exists v_1 (\mathrm{act}\,(v_1) \wedge K_{v_1}\bar{S}))$	3
6.	$\forall v_1 \neg (\mathrm{act}\,(v_1) \wedge K_{v_1}\bar{S})$	5, (Fact), (QN)
7.	$\neg (\mathrm{act}\,(v_i) \wedge K_{v_1}\bar{S}))$	6, UI
8.	$\mathrm{act}\,(\bar{v}_i) \rightarrow \neg K_{v_i}\bar{S}$	7

We see that line 4 and 8 of above derivation jointly imply a contradiction just in case the condition $\ulcorner \mathrm{act}\,(v_i) \urcorner$ is satisfied. Since non-actual possible agents are admitted, one cannot prove by means of above derivation that the assumption at the top of the derivation is inconsistent. Hence one cannot show by means of above derivation that the rigid Fitch conjunction sentence $\ulcorner \bar{S}_1{}^* \urcorner$ is unknowable. [114]

[114] Note that our above argument for showing that the paradox is blocked differs from that of Kvanvig (1995) only by the presence of the modal operator '◊' in $(\mathrm{NO})_{\bar{L}^*}$.

In order to show on the semantic level that the knowability paradox is blocked we shall now construct in the frame of truth theory $T_{\bar{L}^*}^{TSM}$(<W, w*, R, D>), an intuitively plausible model satisfying the following assumption $\ulcorner \bar{S}_2^* \urcorner$

$(\bar{S}_2^*)$ $\Diamond \exists v_2\, K_{v_2} \bar{S}_1^*$

For this purpose we first construct following derivation

1.	at w*, $\ulcorner \Diamond \exists v_2\, K_{v_2} \bar{S}_1^* \urcorner$ is true	Assumption
2.	at w*, $\ulcorner \bar{S}_2^* \urcorner$ is true	1
3.	at w*, $\bar{S}$ means that $\bar{q}$	(ΣM) for EI $\bar{L}^*$
4.	at w*, $\ulcorner \bar{S}_2^* \urcorner$ means that $\exists w\ (w^*Rw \wedge \exists a_2\ (a_2 \in D_w \wedge$ at w, $K_{a_2}(\bar{q} \wedge \neg\exists a_1\ (a_1 \in D_{w^*} \wedge K_{a_1}\bar{q}))))$	3
5.	$\exists w\ (w^*Rw \wedge \exists a_2\ (a_2 \in D_w \wedge$ at w, $K_{a_2}(\bar{q} \wedge \neg\exists a_1\ (a_1 \in D_{w^*} \wedge K_{a_1}\bar{q}))))$	1, 4, (w-TM), MP

Line 5 of above derivation is the interpretation, hence the truth condition, of sentence $\ulcorner \bar{S}_2^* \urcorner$. On the basis of this interpretation one can construct a model $\mathcal{M}_{\bar{q}}$ of the form <W, w*, R, D, V> which satisfies the sentence $\ulcorner \bar{S}_2^* \urcorner$ in question. The intuitive plausibility of such a model constitutes a semantic justification of the above mentioned way of blocking the knowability paradox.

In order to construct model $\mathcal{M}_{\bar{q}}$, we take the propositions that $\bar{q}$ to be "either [the proposition] that [Professor Timothy Williamson's] office contains an even number of books at noon on 11 October 1999 (time t) or [the proposition that] it does not." Assuming that nobody counts these books the propositions that $\bar{q}$ is a knowable but actually unknown truth.[115] Given that proposition that $\bar{q}$, we can conceive of a man-like, though invisible, possible agent $\bar{a}$ who at time t counts the books in Williamson's office and also observes that nobody else did it. It follows that $\bar{a}$ knows that $\bar{q}$, and also that he knows that no actual agent knows that $\bar{q}$. We can assume that the possible agents $\bar{a}$ exists in some possible world $\bar{w}$ which is accessible from the actual world w^*, i.e. $\bar{w} \in W$, w^*Rw, and $\bar{a} \in D_{\bar{w}}$. Let $\bar{q}_1^*$ be short for

$$\bar{q} \land \neg\exists a_1 (a_1 \in D_{w^*} \land K_{a_1}\bar{q}).$$

We define then set $\mathfrak{K}_{\bar{q}}$ as follows

Definition 1 $\quad \mathfrak{K}_{\bar{q}} = \{w \in W: w^*Rw \land (\text{at } w, K_{\bar{a}}\bar{q}_1^*)\}$

It can be shown that $\bar{w} \in \mathfrak{K}_{\bar{q}}$ but $w^* \notin \mathfrak{K}_{\bar{q}}$. We shall stipulate that $\bar{w}$ is a member of $\mathfrak{K}_{\bar{q}}$ which is maximally similar to the actual world w^*. Since $w^* \notin \mathfrak{K}_{\bar{q}}$, it follows that $\bar{w} \neq w^*$. We can now define model $\mathcal{M}_{\bar{q}}$, qua simplent model satisfying sentence $\ulcorner \bar{S}_2{}^* \urcorner$, as follows

Definition 2 $\quad \mathcal{M}_{\bar{q}} = \langle W, w^*, R, D, V \rangle$

where W, w^*, R, D, V satisfy following conditions

[115] We adopt this example from Williamson (2000), p. 472 who gives it as evidence for the truth of (NO).

(i) $W = \{w^*, \bar{w}, \bar{\bar{w}}\}$

where $w^* \neq \bar{w} \neq \bar{\bar{w}}$.

(ii) $R = \{<w^*, w^*>, <\bar{w}, \bar{w}>, <\bar{\bar{w}}, \bar{\bar{w}}>, <w^*, \bar{w}>, <\bar{w}, \bar{\bar{w}}>, <w^*, \bar{\bar{w}}>\}$

(R is reflexive and transitive, hence S4.)

(iii) $D = \{<w^*, D_{w^*}>, <\bar{w}, D_{\bar{w}}>, <\bar{\bar{w}}, D_{\bar{\bar{w}}}>\}$

where $D_{\bar{w}} = D_{\bar{\bar{w}}} = D_{w^*} \cup \{\bar{a}\}$

(iv) at w^*, $(\bar{q} \wedge \neg\exists a_1 (a_1 \in D_{w^*} \wedge K_{a_1}\bar{q}))$

(v) at $\bar{w}$, $\bar{q}$

(vi) at $\bar{w}$, $(K_{\bar{a}}(\bar{q} \wedge \neg\exists a_1 (a_1 \in D_{w^*} \wedge K_{a_1}\bar{q})))$

(vii) at $\bar{\bar{w}}$, $\neg\bar{q}$

By virtue of (i), (v), and (vii) the proposition that $\bar{q}$ is contingent.

(viii) V is an intended valuation function for language L^*. In particular, $V(\ulcorner d_i \urcorner, w) = d_i$, $i = 1, \ldots, m$. $V(\ulcorner act \urcorner, w) = D_{w^*}$, for all $w \in W$. Note that $V(\ulcorner \bar{S} \urcorner, w) = 1 \leftrightarrow \Sigma p$ ((at w^*, $\ulcorner \bar{S} \urcorner$ means that p) $\wedge$ (at w, p)). Since at w^*, $\bar{S}$ means that $\bar{q}$, it follows that $V(\ulcorner \bar{S} \urcorner, w) = 1 \leftrightarrow$ at w, $\bar{q}$.

Note that conditions (iv) - (vi) merely describe above mentioned properties of the proposition that $\bar{q}$ and of the agent $\bar{a}$. Hence these conditions are intuitively plausible. The remaining conditions are simplificatory assumption. It follows that the existence of model $\mathcal{M}_{\bar{q}}$ is itself intuitively plausible. Note that the condition that $\overline{w}$ should be a member of set $\mathfrak{K}_{\bar{q}}$ which is maximally similar to w* is trivially satisfied, since $\mathfrak{K}_{\bar{q}}$ = $\{\overline{w}\}$ in case of model $\mathcal{M}_{\bar{q}}$. On the basis of above conditions, we can construct the following derivation

1.	$\bar{a} \in D_{\overline{w}} \wedge$ at w, $K_{\bar{a}}\, \bar{q}_1^*$	(iii), (iv)
2.	$\exists a_2\, (a_2 \in D_{\overline{w}} \wedge$ at w, $K_{a_2}\, \bar{q}_1^*)$	1, EG
3.	$w^*R\overline{w} \wedge$ at w, $\exists a_2\, (a_2 \in D_{\overline{w}} \wedge K_{a_2}\, \bar{q}_1^*)$	2, (ii)
4.	$\exists w\, (w^*Rw \wedge \exists a_2\, (a_2 \in D_w \wedge$ at w, $K_{a_2}\, \bar{q}_1^*))$	3, EG

We see that line 4 of this derivation is identical to line 5 of the previous derivation. The displayed sentence in the latter states the truth condition of sentence $\ulcorner \bar{S}_2{}^* \urcorner$, i.e. sentence $\ulcorner \Diamond \exists v_2\, K_{v_2} \bar{S}_1{}^* \urcorner$ of language $\bar{L}^*$. It follows that model $\mathcal{M}_{\bar{q}}$ satisfies this sentence. Hence $\ulcorner \bar{S}_2{}^* \urcorner$ is satisfiable and, therefore, consistent. Sentence $\ulcorner \bar{S}_2{}^* \urcorner$ expresses that the truth of the rigid Fitch conjunction $\ulcorner \bar{S}_1{}^* \urcorner$ is knowable. Since the knowability paradox results from the unknowability of the truth of Fitch conjunctions, the satisfiability of these sentences leads to the dissolution of the knowability paradox.

5.4. The Paradox of Idealization

We have seen in Section 5.3 above that the original knowability paradox results from the inconsistency of the non-omniscience thesis (NO) with the knowability principle (KP). Florio and Murzi devise a new paradox of knowability which they call the *paradox of idealization*.[116] The latter paradox results from the inconsistency of the thesis of *moderate anti-realism* with (KP). Florio and Murzi define moderate anti-realism as the view that “there are feasibly unknowable truths, i.e. truths that because of their complexity or of the complexity of their proofs, can only be known by agents whose capacities finitely exceed ours.”[117] They call such agents *idealized* ones and assume that there is no idealized agent.[118]

We shall reformulate the paradox of idealization in our framework and provide a solution which is analogous to the one of the knowability paradox. For this purpose we use an extension $\bar{L}^{\dagger}$ of language $\bar{L}$ resulting from adjunction of the *idealization predicate* sign $\ulcorner I \urcorner$. For any term β in $\bar{L}^{\dagger}$, $\ulcorner I(\beta) \urcorner$ is read as ‘β is an *idealized agent*’.

[116] See Florio and Murzi (2009), Sec. 9, and Murzi (2010), Sec. 8. As far as we know, up to now no solution to the paradox of idealization has been published.
[117] Florio and Murzi (2009), p. 464.
[118] Florio and Murzi (2009), p. 465.

The moderate antirealist view is expressed in language $\bar{L}^{\dagger}$ by the following two thesis $(\mathrm{FU})_{\bar{L}^{\dagger}}$ and $(\mathrm{NI})_{\bar{L}^{\dagger}}$.

$(\mathrm{FU})_{\bar{L}^{\dagger}}$ There are feasibly unknowable truths.

$$\Sigma S\ (S \wedge \Box\forall v_1\ (K_{v_1} S \rightarrow I\,(v_1)))$$ [119]

$(\mathrm{NI})_{\bar{L}^{\dagger}}$ There is no idealized agent.

$$\neg\exists v_1\ I\,(v_1)$$ [120]

The paradox of idealization results from the inconsistency of the following set $\mathfrak{K}_{\bar{L}^{\dagger}}$ of sentence in $\bar{L}^{\dagger}$.

$$\mathfrak{K}_{\bar{L}^{\dagger}} = \{(\mathrm{FU})_{\bar{L}^{\dagger}}, (\mathrm{NI})_{\bar{L}^{\dagger}}, (\mathrm{KP})_{\bar{L}^{\dagger}}\}$$

here $(\mathrm{KP})_{\bar{L}^{\dagger}}$ is identical to $(\mathrm{KP})_{\bar{L}}$, hence it has the following form

$(\mathrm{KP})_{\bar{L}^{\dagger}}$ $\Pi S\ (S \rightarrow \Diamond\exists v_1\ K_{v_1} S)$

The proof of the inconsistency of set $\mathfrak{K}_{\bar{L}^{\dagger}}$ is based on the unknowability of the truth of sentences of the form $\ulcorner S \wedge \neg\exists v_1\ I\,(v_1)\urcorner$ where $\ulcorner S\urcorner$ expresses a feasibly unknowable truth. Let us call such sentences *Florio-Murzi*

[119] The original formulation is $\exists\varphi\ (\varphi \wedge \Box\forall x_1\ (K_x\,\varphi \rightarrow Ix))$. See Florio and Murzi (2009), p. 464(2).

[120] The original formulation is $\neg\exists x\ Ix$. See Florio and Murzi (2009), p. 465(3).

conjunction in analogy to the Fitch conjunction of the form $\ulcorner S \wedge \neg\exists v_1\, K_{v_1} S)\urcorner$. Hence we make the following definitions.

Definition 1 $\ulcorner S \urcorner$ is *feasibly unknowable* ↔
$\ulcorner \Box\forall v_1\, (K_{v_1} S \rightarrow I\,(v_1))\urcorner$ is true.

Definition 2 $\ulcorner S \wedge \neg\exists v_1\, K_{v_1} S\urcorner$ is a *Florio-Murzi conjunction* ↔
($\ulcorner S \urcorner$ is feasibly unknowable).

Florio and Murzi show that if $\ulcorner S \urcorner$ is feasibly unknowable the $\ulcorner S \wedge \neg\exists v_1\, I(v_1)\urcorner$ is unknowable *simpliciter*. [121] The unknowability of Florio-Murzi conjunction can be expressed by the following sentence $(\mathrm{UFM})_{\bar{L}^{\dagger}}$

$$(\mathrm{UFM})_{\bar{L}^{\dagger}} \qquad \Pi S\, (\Box\forall v_1\, (K_{v_1} S \rightarrow I\,(v_1)) \rightarrow \neg\Diamond\exists v_1\, K_{v_1}(S \wedge \neg\exists v_1\, I(v_1)))$$

[121] The unknowability of Florio-Murzi conjunction is proved by their authors as follows. "Let *q* be one…feasibly unknowable truth…Assume that *q* & ¬∃*xIx* is knowable. Then there is a world *w* where some agent [*a*] knows *q* & ¬∃*xIx*. By [moderate anti-realist thesis]… *a* is idealized. However since *a* knows *q* & ¬∃*xIx*, by distributivity and factivity *q* & ¬∃*xIx* is true at *w*. Hence *a* cannot be an idealized agent. Contradiction. Therefore *q* & ¬∃*xIx* is unknowable". Florio and Murzi (2009), p. 465. Aversion of same proof is given in Murzi (2010), pp. 278-279.

Proof of (UFM)$_{\bar{L}^{\dagger}}$

1. $\forall v_I\, (K_{v_1} S \rightarrow I\,(v_1))$ Assumption
2. $\exists v_I\, K_{v_1}(S \wedge \neg\exists v_I\, I\,(v_1))$ Assumption
3. $K_{v_i}(S \wedge \neg\exists v_I\, I\,(v_1))$ 2, EI
4. $K_{v_i} S \wedge K_{v_i}(\neg\exists v_I\, I\,(v_1))$ 3, (K-Dist)
5. $K_{v_i} S$ 4
6. $K_{v_i}(\neg\exists v_I\, I\,(v_1))$ 4
7. $\forall v_I \neg\, I\,(v_1)$ 6, (Fact), (QN)
8. $\neg\, I\,(v_i)$ 7, UI
9. $I\,(v_i)$ 1, UI, 5, MP
10. $\bot$ 8, 9
11. $\neg\exists v_I\; K_{v_1}(S \wedge \neg\exists v_I\, I\,(v_1))$ 2-10
12. $\forall v_I\, (K_{v_1} S \rightarrow I\,(v_1)) \rightarrow \neg\exists v_I\, K_{v_1}(S \wedge \neg\exists v_I\, I\,(v_1))$
1-11, (CD)
13. $\Box(\forall v_I\, (K_{v_1} S \rightarrow I\,(v_1)) \rightarrow \neg\exists v_I\, K_{v_1}(S \wedge \neg\exists v_I\, I\,(v_1)))$
12, (RN)
14. $\Box(\forall v_I\, (K_{v_1} S \rightarrow I\,(v_1)) \rightarrow \Box\neg\exists v_I\, K_{v_1}(S \wedge \neg\exists v_I\, I\,(v_1)))$
13, modal axiom *K*
15. $\Box(\forall v_I\, (K_{v_1} S \rightarrow I\,(v_1)) \rightarrow \neg\Diamond\exists v_I\, K_{v_1}(S \wedge \neg\exists v_I\, I\,(v_1)))$
14
16. $\Pi S\, (\Box(\forall v_I\, (K_{v_1} S \rightarrow I\,(v_1)) \rightarrow \neg\Diamond\exists v_I\, K_{v_1}(S \wedge \neg\exists v_I\, I\,(v_1))))$
15, Rule of Generalization

Q.E.D.

By means of $(\mathrm{UFM})_{\bar{L}^\dagger}$, the inconsistency of set $\mathfrak{K}_{\bar{L}^\dagger}$ can be proved as follows.

1.	$\bar{S} \wedge \Box\forall v_1 (K_{v_1}\bar{S} \rightarrow I(v_1))$	$(\mathrm{FU})_{\bar{L}^\dagger}$, EI
2.	$\neg\exists v_1 I(v_1)$	$(\mathrm{NI})_{\bar{L}^\dagger}$
3.	$(\bar{S} \wedge \neg\exists v_1 I(v_1)) \rightarrow \Diamond\exists v_1 K_{v_1}(\bar{S} \wedge \neg\exists v_1 I(v_1))$	$(\mathrm{KP})_{\bar{L}^\dagger}$, UI
4.	$\Box\forall v_1 (K_{v_1}\bar{S} \rightarrow I(v_1))$	1
5.	$\neg \Diamond\exists v_1 K_{v_1}(\bar{S} \wedge \neg\exists v_1 I(v_1))$	4, $(\mathrm{UFM})_{\bar{L}^\dagger}$, MP
6.	$\neg (\bar{S} \wedge \neg\exists v_1 I(v_1))$	3, 5, MT
7.	$\bar{S}$	1
8.	$\bar{S} \wedge \neg\exists v_1 I(v_1)$	2, 7
9.	$\perp$	6, 8

Q.E.D.

5.5. A Solution to the Paradox of Idealization

We shall propose a solution of the paradox of idealization which is analogous to the solution of the knowability paradox stated in Section 5.3 above. For this purpose let us turn back to the proof of inconsistency of set $\mathfrak{K}_{\bar{L}^\dagger}$ at the end of Section 5.4 above. We see that line 3 contains two different occurrences of the Florio-Murzi conjunction $\ulcorner \bar{S} \wedge \neg\exists v_1 I(v_1) \urcorner$. We shall show below that these two occurrences do not express the same proposition. Note that the right-hand side occurrence of $\ulcorner \bar{S} \wedge \neg\exists v_1 I(v_1) \urcorner$ in line 3 is in the scope of modal operator $\ulcorner \Diamond \urcorner$ so that the quantifier domain of $\ulcorner \exists v_1 \urcorner$ is different from its domain in the left-hand side of line 3.

In order to state the propositions expressed respectively by the lines of the proof of inconsistency of set $\mathfrak{K}_{\bar{L}^{\dagger}}$, we shall use truth theory $T_{\bar{L}^{\dagger}}^{TSM}$(<*W*, *w**, *R*, *D*, *V* >) for language $\bar{L}^{\dagger}$. This truth theory results from the corresponding truth theory for language $\bar{L}$ (mentioned in Section 5.1) by adjunction of following axiom (*w*-*Ax.I*).

(*w*-*Ax.I*) (at w, β means b) → (at w, (⌈I (β)⌉ means that $b \in D_I$))

where D_I is a non-empty subset of $\bigcup_{w \in W} D_w$ such that $D_I \cap D_{w^*} = \emptyset$. Note that '$D_I$' is a rigid designators in metalanguage *ML* for object language $\bar{L}^{\dagger}$. In the frame of truth theory $T_{\bar{L}^{\dagger}}^{TSM}$(<*W*, *w**, *R*, *D*, *V* >), we construct the following semantic derivation corresponding to the proof of inconsistency of set $\mathfrak{K}_{\bar{L}^{\dagger}}$.

1.	at w^*, ⌈$\Sigma S\ (S \wedge \Box\forall v_1\ (K_{v_1} S \rightarrow I\ (v_1)))$⌉ is true	$(\mathrm{FU})_{\bar{L}^{\dagger}}$
2.	at w^*, ⌈$\bar{S} \wedge \Box\forall v_1\ (K_{v_1} \bar{S} \rightarrow I\ (v_1)))$⌉ is true	1, EI
3.	at w^*, $\bar{S}$ means that $\bar{q}$	(ΣM) for $\bar{L}^{\dagger}$
4.	at w^*, ⌈$\bar{S} \wedge \neg\exists v_1 I\ (v_1)$⌉ means that	
	$\bar{q} \wedge \neg\exists a_1\ (a_1 \in D_{w^*} \wedge a_1 \in D_I)$	3

Let ⌈$\bar{S}_1^{FM}$⌉ be short for ⌈$\bar{S} \wedge \neg\exists v_1\ I\ (v_1)$⌉ and '$\bar{q}_1^{FM}$' for '$\bar{q} \wedge \neg\exists a_1 (a_1 \in D_I)$'. Then the derivation proceeds as follows

5.	at w^*, ⌈$\bar{S}_1^{FM} * \rightarrow \Diamond\exists v_1\ K_{v_1} \bar{S}_1^{FM}$⌉ is true	$(\mathrm{KP})_{\bar{L}^{\dagger}}$, UI
6.	at w^*, (⌈$\bar{S}$⌉ is true)	2
7.	at w^*, (⌈$\neg\exists v_1 I\ (v_1)$⌉ is true)	$(\mathrm{NI})_{\bar{L}^{\dagger}}$

8. at w^*, ($\ulcorner \bar{S} \wedge \neg\exists v_1 I(v_1)\urcorner$ is true) 6, 7

9. at w^*, ($\ulcorner \bar{S}_1^{FM} \urcorner$ is true) 8

10. at w^*, ($\ulcorner \Diamond\exists v_1 K_{v_1} \bar{S}_1^{FM} \urcorner$ is true) 5, 9, MP

11. at w^*, $\ulcorner \Diamond\exists v_1 K_{v_1} \bar{S}_1^{FM} \urcorner$ means that

$\exists w\ (w^*Rw \wedge \exists a_1\ (a_1 \in D_w \wedge \text{at } w, K_{a_1}(\bar{q} \wedge \neg\exists a_1\ (a_1 \in D_w \wedge a_1 \in D_I))))$

Let '$\bar{q}_{1,w}^{FM}$' be short for '$\bar{q} \wedge \neg\exists a_1(a_1 \in D_w \wedge a_1 \in D_I)$'. Then we can say that the left-hand side occurrence of $\ulcorner \bar{S}_1^{FM} \urcorner$ in line 5 above expresses the proposition that $\bar{q}_1^{FM}$ whereas the right-hand side occurrence expresses the proposition that $\bar{q}_{1,w}^{FM}$ which may be different from the former one. Indeed whenever $D_w \neq D$, the proposition that $\bar{q}_{1,w}^{FM}$ is different from the proposition that $\bar{q}_1^{FM}$. Let further $\ulcorner \bar{S}_2^{FM} \urcorner$ be short for $\ulcorner \Diamond\exists v_1 K_{v_1} \bar{S}_1^{FM} \urcorner$. Then the derivation continues in the following way.

12. at w^*, $\ulcorner \bar{S}_2^{FM} \urcorner$ means that

$\exists w\ (w^*Rw \wedge \exists a_1\ (a_1 \in D_w \wedge \text{at } w, K_{a_1} \bar{q}_{1,w}^{FM}))$ 11

13. $\exists w\ (w^*Rw \wedge \exists a_1\ (a_1 \in D_w \wedge \text{at } w, K_{a_1} \bar{q}_{1,w}^{FM}))$

10, 12, (w^*-TM), MP

14. $w^*R\bar{w} \wedge (a_1 \in D_w \wedge \text{at } \bar{w}, K_{a_1} \bar{q}_2^{FM})$ 13, EI

15. at $\bar{w}$, $K_{a_1} \bar{q}_2^{FM}$ 14

16. at $\bar{w}$, $K_{a_1}(\bar{q} \wedge \neg\exists a_1\ (a_1 \in D_{\bar{w}} \wedge I(a_1)))$ 14

17. at $\bar{w}$, $K_{a_1} \bar{q} \wedge K_{a_1}(\neg\exists a_1\ (a_1 \in D_{\bar{w}} \wedge I(a_1))$

16, (K-Dist)

18. at $\bar{w}$, $K_{a_1} \bar{q}$ 17

19. at w^*, $\ulcorner \Box\forall v_1\ (K_{v_1} \bar{S} \rightarrow I(v_1))\urcorner$ means that

$\forall w\ (w^*Rw \rightarrow \forall a_1\ (a_1 \in D_w \rightarrow \text{at } w, (K_{a_1} \bar{q} \rightarrow I(a_1)))$ 3

20. $\forall w\ (w^*Rw \rightarrow \forall a_1\ (a_1 \in D_w \rightarrow \text{at } w, (K_{a_1}\bar{q} \rightarrow I\ (a_1)))$

19, (w^*-TM)

21. $(w^*Rw \wedge (a_1 \in D_{\bar{w}}) \rightarrow \text{at } \bar{w}, (K_{a_1}\bar{q} \rightarrow I\ (a_1)))$ 20, UI

22. $w^*Rw \wedge a_1 \in D_{\bar{w}}$ 14

23. at $\bar{w}$, $K_{a_1}\bar{q} \rightarrow I\ (a_1)$ 21, 22, MP

24. $I\ (a_1)$ 18, 23, MP

25. at $\bar{w}$, $K_{a_1}(\neg\exists a_1\ (a_1 \in D_{\bar{w}} \wedge I\ (a_1))$ 17

26. at $\bar{w}$, $\neg\exists a_1\ (a_1 \in D_{\bar{w}} \wedge I\ (a_1))$ 25, (Fact)

27. $\forall a_1\ (a_1 \in D_w \rightarrow \neg I\ (a_1))$ 26, (QN)

28. $a_1 \in D_{\bar{w}}$

29. $\neg\ I\ (a_1)$ 27, UI, 28

30. $\bot$ 24, 29

Q.E.D.

Above derivation constitute a semantic proof in metalanguage *ML* of the inconsistency of set $\mathfrak{K}_{\bar{L}^\dagger} = \{(\text{FU})_{\bar{L}^\dagger}, (\text{NI})_{\bar{L}^\dagger}, (\text{KP})_{\bar{L}^\dagger}\}$. However neither the previous proof in $\bar{L}^\dagger$ nor the semantic proof in *ML* of the inconsistency of $\mathfrak{K}_{\bar{L}^\dagger}$ really establish the paradox of idealization. The paradox would be established only if the two occurrences of the Florio-Murzi conjunction in the instance of $(\text{KP})_{\bar{L}^\dagger}$ were expressing the same proposition. We shall show below that once the latter condition is secured the paradox does not arise anymore. But we need then to revise the formulation of thesis $(\text{NI})_{\bar{L}^\dagger}$, in analogy to revised form of thesis $(\text{NO})_{\bar{L}^*}$ in the solution of the knowability paradox.

In order to formulate our solution of the paradox of idealization, we shall use a language $\bar{L}^{\dagger *}$ which is an extension of modal epistemic language $\bar{L}$

resulting by adjunction of idealization predicate sign $\ulcorner I \urcorner$ and actuality predicate sign $\ulcorner$act$\urcorner$. The truth theory $T^{TSM}_{\bar{L}^{\dagger *}}$ (<W, w*, R, D, V >) for language $\bar{L}^{\dagger *}$ results from the corresponding truth theory for $\bar{L}^{\dagger}$ by adjunction of the following axioms

(w-Ax-I)　　(at w, β means b) → (at w, $\ulcorner I$ (β)$\urcorner$ means that $b \in D_I$).

(w-Ax-act)　　at w, β means b → at w, $\ulcorner$act (β)$\urcorner$ means that $b \in D_{w^*}$.

We formulate in $\bar{L}^{\dagger *}$ the revised form of (NI), viz. the thesis (NI)$_{\bar{L}^{\dagger *}}$ as follows.

(NI)$_{\bar{L}^{\dagger *}}$　　$\neg\exists v_1$ (act (v_1) $\wedge$ I (v_1))

One can see that (NI)$_{\bar{L}^{\dagger *}}$ is analogous to Kvanvig's revised form (KNO) of non-omniscience thesis (NO) mentioned in the Section 5.3. above.

Let us define now a set $\mathfrak{K}_{\bar{L}^{\dagger *}}$ of sentences in $\bar{L}^{\dagger *}$ as follows

Definition 1　　$\mathfrak{K}_{\bar{L}^{\dagger *}}$ = {(FU)$_{\bar{L}^{\dagger *}}$, (NI)$_{\bar{L}^{\dagger *}}$, (KP)$_{\bar{L}^{\dagger *}}$}.

where (FU)$_{\bar{L}^{\dagger *}}$ and (KP)$_{\bar{L}^{\dagger *}}$ are respectively identical with (FU)$_{\bar{L}^{\dagger}}$ and (KP)$_{\bar{L}^{\dagger}}$.

We shall show below that one cannot derive from set $\mathfrak{K}_{\bar{L}^{\dagger *}}$ a contradiction in a way similar to the derivation of contradiction from set $\mathfrak{K}_{\bar{L}^{\dagger}}$. But this fact does not constitute in itself a solution of the paradox of idealization since the switch from (NI)$_{\bar{L}^{\dagger}}$ to (NI)$_{\bar{L}^{\dagger *}}$ might be an *ad hoc* device of explaining away the paradox. Hence we are under the obligation of justifying our use of $(\mathrm{NI})_{L^{\dagger *}}$ independently of its contribution to the solution of the paradox. For this purpose we shall argue that sentence $\ulcorner\neg\exists v_1\,(\mathrm{act}(v_1) \wedge I\,(v_1))\urcorner$ expresses the *same* proposition as the one expressed by sentence $\ulcorner\neg\exists v_1\, I\,(v_1)\urcorner$ as used by Florio and Murzi themselves. These authors consider two alternative interpretations of the sentence in question "depending of how anti-realists define the notion of an idealized agent. If an agent counts as idealized just in case her cognitive capacities finitely exceeds those of *any actual epistemic agent*, then [$\ulcorner\neg\exists v_1\, I\,(v_1)\urcorner$] is indeed an a priori truth…On the other hand, anti-realist might take [$\ulcorner\neg\exists v_1\, I\,(v_1)\urcorner$] to be an empirical claim…defining [$\ulcorner I(v_1)\urcorner$] in terms of human cognitive capacities." [122]

Our claim is that, in both of above mentioned interpretations, the proposition expressed by sentence $\ulcorner\neg\exists v_1\, I\,(v_1)\urcorner$ as occurring in a Florio-Murzi conjunction cannot consist in the denial of *possible* agents. Otherwise this proposition would be false in both alternative interpretations. Indeed provided possible worlds inhabited by possible individuals are admitted at all, one is committed to admit possible idealized agents since the latter notion is a self-consistent one in both alternative definitions of this notion. It follows that unless the quantifier domain is restricted to D_{w*} (the set of actual agents), $\ulcorner\neg\exists v_1\, I\,(v_1)\urcorner$ is false, contrary to Florio and Murzi's claim. Indeed $\ulcorner\neg\exists v_1\, I\,(v_1)\urcorner$ is claimed to be true a priori in the first alternative and true a posteriori in the second one. Hence one is compelled to interpret the sentence in question as expressing the proposition that there are no *actural* idealized agents. The latter proposition

[122] Florio and Murzi (2009), p. 467. Concerning the definition of the notion of an idealized agent (ideal cognizer) in terms of cognitive capacities see Tennant (1997), pp. 143-144.

is expressed by $(\mathrm{NI})_{\bar{L}^{\dagger *}}$, i.e. $\ulcorner \neg\exists v_1 (\mathrm{act}(v_1) \wedge I(v_1))\urcorner$. We have thus provided a non ad hoc reason for using $(\mathrm{NI})_{\bar{L}^{\dagger *}}$ in place of $(\mathrm{NI})_{\bar{L}^{\dagger}}$ since the former is equivalent to the latter one.

We have now arrived at a stage in which we can formulate our solution to the paradox of idealization, which as already stated, is analogous to the solution of the knowability paradox. We have seen that the derivation of a contradiction from set $\mathfrak{K}_{\bar{L}^{\dagger}}$ involves the instantiation of $(\mathrm{KP})_{\bar{L}^{\dagger}}$ with the Florio-Murzi conjunction $\ulcorner \bar{S} \wedge \neg\exists v_1 I(v_1)\urcorner$. We shall show below that in case $(\mathrm{KP})_{\bar{L}^{\dagger *}}$ (which is identical with $(\mathrm{KP})_{\bar{L}^{\dagger}}$) is instantiated by $\ulcorner \bar{S} \wedge \neg\exists v_1 (\mathrm{act}(v_1) \wedge I(v_1))\urcorner$, which we call a *rigid Florio-Murzi conjunction*, one *cannot* derive a contradiction from set $\mathfrak{K}_{\bar{L}^{\dagger *}}$ is a way similar to the derivation of contradiction from set $\mathfrak{K}_{\bar{L}^{\dagger}}$. Indeed the derivation of contradiction from $\mathfrak{K}_{\bar{L}^{\dagger}}$ is based on unknowability of the Florio-Murzi conjunction. But we can show that the rigid Florio-Murzi conjunction (just as the rigid Fitch conjunction) is *not* unknowable. For this purpose we construct the following derivation.

1.	$\Sigma S\ (S \wedge \Box\forall v_1 (K_{v_1} S \rightarrow I(v_1)))$	$(\mathrm{FU})_{\bar{L}^{\dagger *}}$
2.	$\bar{S} \wedge \Box\forall v_1 (K_{v_1} \bar{S} \rightarrow I(v_1))$	1, EI
3.	$\exists v_2\, K_{v_2}(\bar{S} \wedge \neg\exists v_1 (\mathrm{act}(v_1) \wedge I(v_1))$	Assumption
4.	$K_{v_i}(\bar{S} \wedge \neg\exists v_1 (\mathrm{act}\,(v_1) \wedge I(v_1)))$	3, EI
5.	$K_{v_i}\bar{S} \wedge K_{v_i}(\neg\exists v_1 (\mathrm{act}\,(v_1) \wedge I(\bar{v}_1)))$	4, (K-Dist)
6.	$K_{v_i}\bar{S}$	5
7.	$K_{v_i}(\neg\exists v_1 (\mathrm{act}\,(v_1) \wedge I(v_1)))$	5
8.	$\neg\exists v_1 (\mathrm{act}\,(v_1) \wedge I(v_1))$	7, (Fact)
9.	$\forall v_1 \neg (\mathrm{act}\,(v_1) \wedge I(v_1)))$	8, (QN)

10. $\neg$ (act (v_i) $\wedge$ I (v_i)) 9, UI

11. act (v_i) $\rightarrow$ $\neg I$ (v_i) 10

12. $K_{v_i}\bar{S} \rightarrow I$ (v_i)

2, modal axiom I, UI

13. I (v_i) 6, 12, MP

We see that lines 11 and 13 jointly implies a contradiction just in case condition 'act(v_i)' is satisfied. Since non-actual possible agents are admitted, one cannot prove by means of this derivation that a rigid Florio-Murzi conjunction, and ultimately the set $\mathfrak{K}_{\bar{L}^{\dagger *}}$, is inconsistent. Hence the paradox of idealization is blocked.

In order to show on the semantic level that the paradox is blocked, we construct in the frame of truth theory $T^{TSM}_{\bar{L}^{\dagger *}}$ (<W, w*, R, D, V >) the following derivation from the assumption that a true rigid Florio-Murzi conjunction is knowable.

1. at w*, $\ulcorner \Sigma S\ (S \wedge \Box\forall v_2\ (K_{v_2}S \rightarrow I\ (v_1)))\urcorner$ is true (FU)$_{\bar{L}^{\dagger *}}$
2. at w*, $\ulcorner \Diamond\exists v_2 K_{v_2}\bar{S}^{FM*}_1 \urcorner$ is true Assumption
3. at w*, $\ulcorner \bar{S} \wedge \Box\forall v_2\ (K_{v_2}\bar{S} \rightarrow I\ (v_1))\urcorner$ is true 1, EI
4. at w*, $\ulcorner \Box\forall v_2\ (K_{v_2}\bar{S} \rightarrow I\ (v_1))\urcorner$ is true 3
5. at w*, $\bar{S}^{FM*}_2$ is true 2
6. at w*, $\bar{S}$ means that $\bar{q}$ (ΣM) for $\bar{L}^{\dagger *}$
7. at w*, $\bar{S}^{FM*}_2$ means that

$\exists w\ (w^*Rw \wedge \exists a_2(a_2{\in}D_w \wedge \text{at } w, K_{a_2}(\bar{q} \wedge \neg\exists a_1(a_1{\in}D_{w^*} \wedge a_1{\in}D_I))))$

6

8. $\exists w\ (w^*Rw \wedge \exists a_2(a_2{\in}D_w \wedge \text{at } w, K_{a_2}(\bar{q} \wedge \neg\exists a_1(a_1{\in}D_{w^*} \wedge a_1{\in}D_I))))$

2, 7, (w-TM), MP

9. at w^*, $\ulcorner \Box\forall v_2\,(K_{v_2}\bar{S} \rightarrow I\,(v_1)))\urcorner$ means that

$\forall w\,(w^*Rw \rightarrow \forall a_2\,(a_2{\in}D_w \rightarrow \text{at } w,\,(K_{a_2}\bar{q} \rightarrow a_2{\in}D_I)))$

10. $\forall w\,(w^*Rw \rightarrow \forall a_2\,(a_2{\in}D_w \rightarrow \text{at } w,\,(K_{a_2}\bar{q} \rightarrow a_2{\in}D_I)))$ 4, 9, (w-*TM*)

11. $w^*R\overline{w} \wedge \exists a_2(a_2{\in}D_{\overline{w}} \wedge \text{at } \overline{w},\,K_{a_2}(\bar{q} \wedge \neg\exists a_1\,(a_1{\in}D_{w^*} \wedge a_1{\in}D_I)))$ 8, EI

12. $w^*R\overline{w} \rightarrow \forall a_2(a_2{\in}D_{\overline{w}} \rightarrow \text{at } \overline{w},\,(K_{a_2}\bar{q} \rightarrow a_2{\in}D_I)))$ 10, EI

13. $w^*R\overline{w}$

14. $\forall a_2\,(a_2{\in}D_{\overline{w}} \rightarrow \text{at } \overline{w},\,(K_{a_2}\bar{q} \rightarrow a_2{\in}D_I))$ 12, 13, MP

15. $\bar{a}{\in}D_{\overline{w}} \wedge \text{at } \overline{w},\,K_{\bar{a}}\,(\bar{q} \wedge \neg\exists a_1(a_1{\in}D_{w^*} \wedge a_1{\in}D_I))$ 11, EI

16. $\bar{a}{\in}D_{\overline{w}} \rightarrow \text{at } \overline{w},\,(K_{\bar{a}}\,\bar{q} \rightarrow \bar{a}{\in}D_I)$ 14, UI

17. $\bar{a}{\in}D_{\overline{w}}$ 15

18. at $\overline{w}$, $(K_{\bar{a}}\,\bar{q} \rightarrow \bar{a}{\in}D_I)$ 16, 17, MP

19. at $\overline{w}$, $K_{\bar{a}}\,(\bar{q} \wedge \neg\exists a_1(a_1{\in}D_{w^*} \wedge a_1{\in}D_I))$ 15

20. at $\overline{w}$, $(K_{\bar{a}}\,\bar{q} \wedge K_{\bar{a}}\,(\neg\exists a_1(a_1{\in}D_{w^*} \wedge a_1{\in}D_I)))$ 19

21. at $\overline{w}$, $K_{\bar{a}}\,\bar{q}$ 19

22. at $\overline{w}$, $K_{\bar{a}}\,(\neg\exists a_1(a_1{\in}D_{w^*} \wedge a_1{\in}D_I))$ 20

23. at $\overline{w}$, $\bar{a}{\in}D_I$ 18, 21, MP

24. $\bar{a}{\in}D_I$ 23

25. $\forall a_1\,(a_1{\in}D_{w^*} \rightarrow a_1 \notin D_I)$ 22, (QN)

26. $\bar{a}{\in}D_{w^*} \rightarrow \bar{a} \notin D_I$ 25, UI

We see that lines 24, 26 jointly imply a contradiction just in case $\bar{a} \in D_{w^*}$ holds. Given that non-actual possible agents are admitted, above derivation does not lead to a contradiction so that the paradox of idealization is blocked also on the semantic level. On the basis of above derivation, we can show that the two occurrences of rigid Florio-Murzi conjunction $\ulcorner \bar{S}_1^{FM*} \urcorner$ is an instance of $(\mathrm{KP})_{\bar{L}^{\dagger *}}$ express the same proposition, viz. the proposition that '$\bar{q} \wedge \neg\exists a_1 (a_1 \in D_{w*} \wedge a_1 \in D_I)$', or the proposition that '$\bar{q}_1^{FM*}$' for short. Indeed the following derivation is valid

1. at w^*, $\ulcorner \Pi S\, (S \rightarrow \Diamond\exists v_1\, K_{v_1} S) \urcorner$ is true

Assumption of $(\mathrm{KP})_{\bar{L}^{\dagger *}}$

2. at w^*, $\ulcorner \bar{S}_1^{FM*} \rightarrow \Diamond\exists v_1\, K_{v_1} \bar{S}_1^{FM*} \urcorner$ is true 1, UI
3. at w^*, $\bar{q}_1^{FM*} \rightarrow \exists w\, (w^*Rw \wedge \exists a_1\, K_{a_1} \bar{q}_1^{FM*})$ 2

Finally we shall construct an intuitively plausible model $\mathcal{M}_{\bar{q}}^{FM*}$ which satisfies $\ulcorner \bar{S}_2^{FM*} \urcorner$. For this purpose we take the proposition that $\bar{q}$ to be a true mathematical proposition which is decidable but unknowable to actual agents, due to its high complexity. For this special purpose we suppose that one of the unstructured sentence in $\bar{L}^{\dagger *}$ is the sentence $\ulcorner \bar{S} \urcorner$ which expresses the proposition that $\bar{q}$ in question. We adopt this example from Murzi (2010, pp. 277-278) who uses it for illustrating the paradox of idealization. We define model $\mathcal{M}_{\bar{q}}^{FM*}$ as follows.

Definition $\mathcal{M}_{\bar{q}}^{FM*} = \langle W, w^*, R, D, V \rangle$

where W, w^*, R, D, V satisfy following definitory condition.

(i) $W = \{w^*, \overline{w}\}$

where $\overline{w} \neq w^*$.

(ii) $R = \{<w^*, w^*>, <w^*, \overline{w}>, <\overline{w}, \overline{w}>\}$

R is reflexive and transitive, hence S4.

We conceive of a possible world $\overline{w}$ inhabited by an *idealized agent* $\bar{a}$ who isendowed with the computational capacities required for establishing the truth of the proposition that $\bar{q}$. We *can* assume that $\bar{a}$ knows that $\bar{q}$ and also knows that no actual agent is an idealized one. Indeed, if the notion of an idealized agent is defined in terms of cognitive capacities, then we can assume that, by virtue of these capacities, agent $\bar{a}$ knows that $\bar{q}$, and also knows that no actual agent has the cognitive capacities in question. On the other hand if the notion of an idealized agent is defined as one whose cognitive capacities exceeds those of any actual agent, then knowledge that there are no idealized agents is a priori so that we can again assume that $\bar{a}$ has such a knowledge. We assume that $\bar{a}$ is the unique idealized agent. Then the following conditions concerning $\bar{a}$ hold.

(iii) $\bar{a} \notin D_{w^*}$

(iv) $D_I = \{\bar{a}\}$

$D_I \cap D_{w^*} = \emptyset$ holds by virtue of (iii) and (iv).

(v) $D = \{<w^*, D_{w^*}>, <\overline{w}, D_{\overline{w}}>\}$

where $D_{\overline{w}} = D_{w^*} \cup \{\bar{a}\}$

(vi) at w^*, $\bar{q}$

(vii) at $\overline{w}$, $\overline{q}$

(viii) at $\overline{w}$, $K_{\overline{a}}\,\overline{q}$

By virtue of (i), (vi), and (vii) the proposition that $\overline{q}$ is necessary.

(ix) at $\overline{w}$, $K_{\overline{a}}\,(\neg\exists a_1\ (a_1 \in D_{w*} \rightarrow a_1 \in D_I))$

(x) $\forall a_1\,(a_1 \in D_{w*} \rightarrow$ (at w^*, $\neg K_{a_1}\overline{q}$))

(xi) V is an intended valuation function for language $\overline{L}^{\dagger *}$.
In particular, $V(\ulcorner d_i \urcorner, w) = d_i$, $i = 1,\ldots, m$, $V(\ulcorner I \urcorner, w) = D_I$,
$V(\ulcorner \text{act} \urcorner, w) = D_{w*}$ for every $w \in W$, $V(\ulcorner \overline{S} \urcorner, w^*) = V(\ulcorner \overline{S} \urcorner, \overline{w}) = 1$.

Let us show that model $\mathcal{M}_{\overline{q}}^{FM*}$ satisfies the rigid Florio-Murzi conjunction $\ulcorner \overline{S}_2^{FM*} \urcorner$. For this purpose we shall show that the definitory conditions of model $\mathcal{M}_{\overline{q}}^{FM*}$ jointly imply the following propositions.

(I) $\exists w\ (w^*Rw \wedge \exists a_2\,(a_2 \in D_w \wedge$ at w, $(K_{a_2}(\overline{q} \wedge \neg\exists a_1$
$(a_1 \in D_{w*} \wedge a_1 \in D_I)))))$

and

(II) $\forall w\ (w^*Rw \rightarrow \forall a_2\,(a_2 \in D_w \rightarrow$ at w, $(K_{a_2}\overline{q} \rightarrow a_2 \in D_I)))$

The proposition stated by (I) is expressed by sentence $\ulcorner \overline{S}_2^{FM*} \urcorner$ and the proposition stated by (II) is expressed by sentence $\ulcorner \Box\forall v_1(K_{v_2}\overline{q} \rightarrow I(v_1)) \urcorner$.

Note (I) and (II) are stated respectively in lines 8 and 10 of the mentioned derivation (consisting of 26 lines).

Proof of (I)

1. $w^*R\overline{w}$ by (ii)

2. at $\overline{w}$, $K_{\overline{a}}\overline{q}$ by (viii)

3. at $\overline{w}$, $K_{\overline{a}}(\neg\exists a_1\ (a_1 \in D_{w^*} \wedge a_1 \in D_I))$ by (ix)

4. at $\overline{w}$, $K_{\overline{a}}(\overline{q} \wedge \neg\exists a_1\ (a_1 \in D_{w^*} \wedge a_1 \in D_I))$ 2, 3

5. $\exists a_2\ (a_2 \in D_{\overline{w}} \wedge$ at $\overline{w}$, $K_{a_2}(\overline{q} \wedge \neg\exists a_1\ (a_1 \in D_{w^*} \wedge a_1 \in D_I)))$

4, EG

6. $w^*R\overline{w} \wedge \exists a_2\ (a_2 \in D_{\overline{w}} \wedge$ at $\overline{w}$, $K_{a_2}(\overline{q} \wedge \neg\exists a_1(a_1 \in D_{w^*} \wedge a_1 \in D_I)))$

1, 5

7. $\exists w(w^*R\overline{w} \wedge \exists a_2(a_2 \in D_{\overline{w}} \wedge$ at $\overline{w}$, $K_{a_2}(\overline{q} \wedge \neg\exists a_1\ (a_1 \in D_{w^*} \wedge a_1 \in D_I))))$

6, EG

Q.E.D.

Proof of (II)

1. $\forall a_2\ (a_2 \in D_{w^*} \rightarrow$ at w^*, $(K_{a_2}\overline{q} \rightarrow a_2 \in D_I))$

by (iii), (iv), (x)

2. $\forall a_2\ (a_2 \in D_{\overline{w}} \rightarrow$ at $\overline{w}$, $(K_{a_2}\overline{q} \rightarrow a_2 \in D_I))$

by (iv), (viii)

3. $\forall w\ (w \in D_W \rightarrow \forall a_2\ (a_2 \in D_w \rightarrow$ at w, $(K_{a_2}\overline{q} \rightarrow a_2 \in D_I)))$

2, by (i), (v), UG

4. $\forall w\ (w^*Rw \rightarrow \forall a_2\ (a_2 \in D_w \rightarrow$ at w, $(K_{a_2}\overline{q} \rightarrow a_2 \in D_I)))$

3, by (ii)

Q.E.D.

As already stated, the proposition stated by (I) is the one expressed by the sentence $\ulcorner \bar{S}_2^{FM*} \urcorner$ of the form $\ulcorner \Diamond \exists v_1\, \bar{S}_1^{FM*} \urcorner$. On the other hand the proposition stated by (II) is the one expressed by sentence $\ulcorner \Box \forall v_1 (K_{v_2} \bar{S} \rightarrow \neg I(v_1)) \urcorner$. The truth of the latter sentence ensures that sentence $\ulcorner \bar{S}_1^{FM*} \urcorner$ is a rigid *Florio-Murzi conjunction*. As shown above the definitory conditions of model $\mathcal{M}_{\bar{q}}^{FM*}$ imply (I) and (II). Sentence (I) is metalanguage ML states the truth condition of sentence $\ulcorner \bar{S}_2^{FM*} \urcorner$. Since (II) states the truth condition of $\ulcorner \Box \forall v_1 (K_{v_2} \bar{S} \rightarrow \neg I(v_1)) \urcorner$, model $\mathcal{M}_{\bar{q}}^{FM*}$ satisfies also the latter sentence. Hence sentence $\ulcorner \bar{S}_2^{FM*} \urcorner$ satisfied by $\mathcal{M}_{\bar{q}}^{FM*}$ expresses a proposition to the effect that the truth of a rigid Florio-Murzi conjunction is knowable. The satisfiability of a rigid Florio-Murzi conjunction leads to the dissolution of the paradox of idealization.

CHAPTER 6

CONCLUSION

In this thesis, we have studied the knowability paradox and the paradox of idealization in the frame of modal epistemic languages. The so-called knowability paradox consists in the derivation of a contradiction from the conjunction of (NO) and (KP). We first made the statement of the knowability paradox, and we mentioned its philosophical importance and its implications. We do not aim to save any verificationist or anti-realist theory, rather as we have argued that the result of these paradoxes primarily threatens the necessary logical distinction between knowable truths and known truths. Then, we made a survey of the proposed solutions and of the debates with respect to our frame. We have also made a critical evaluation of our solution with respect to previously proposed solutions. Finally, we have formulated a satisfactory solution to the knowability paradox and the paradox of idealization in the truth theory for first-order modal epistemic languages. We have shown that the derivation of a contradiction consists in the knowability paradox can be blocked by interpreting (NO) as the thesis that there are truths forever unknown to *actual* agents. We have further provided a solution to the paradox of idealization which consists in the derivation of a contradiction from the following, initially plausible, premises. First, thesis (FU) stating that there are feasibly unknowable truths in the sense of truths knowable only by *idealized agents*, second, thesis (NI) stating that there are *no* idealized agents, and third, the above mentioned thesis (KP). We have shown that by interpreting (NI) as stating that no *actual* agent is idealized, the derivation of contradiction from the conjunction of (FU), (NI), and (KP) is blocked. Our solution to the paradox of idealization is the first and only proposal of a solution in the literature up to now.

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APPENDIX

TURKISH SUMMARY

KİPSEL-BİLGİSEL DİLLERDE BİLİNEBİLİRLİK PARADOKSU VE İDEALLEŞTİRME PARADOKSUNUN BİR ÇÖZÜMÜ

İnsanların bilgi edinme yetenekleri sonludur. Dolayısıyla, hiçbir zaman bilinmeyen doğrular olacaktır. Bu olgu *bilinmeyen doğruların varlığı* savı (NO) ile ifade edilmektedir. Şimdi, aşağıdaki gibi bir önermeye göz atalım:

(1) *p* doğrudur ve *p*'nin doğru olduğu bilinmemektedir.

Bu yapıdaki önermelerin varlığı reddedilmez bir gerçektir. Fakat bu önermeler *bilinemez*dir. Gerçekten de dolaylı türetim yöntemiyle, yukarıdaki önermeyi bildiğimizi varsayalım. Başka bir deyişle (2) numaralı önermeyi ifade etmiş oluruz:

(2) Bilinmektedir ki: *p* doğrudur ve *p*'nin doğru olduğu bilinmemektedir.

(2) numaralı önerme bilgisel mantıkta aşağıdaki iki önerme gibi ifade edilebilir:

(3) *p*'nin doğru olduğu bilinmektedir

ve

(4) Bilinmektedir ki: *p*'nin doğru olduğu bilinmemektedir.

(3) numaralı önermeden bilgisel mantıkta aşağıdaki sonuç çıkarılmaktadır.

(5) *p*'nin doğru olduğu bilinmemektedir.

Görülmektedir ki, (2) numaralı önerme "*p*'nin doğru olduğu bilinmektedir" ve "*p*'nin doğru olduğu bilinmemektedir" önermelerini içerdiği için bir çelişkidir. Bu nedenle, dolaylı türetim yönteminin sonucundan dolayı (1) numaralı önermenin çelişki olduğu görülür. Gerçekten de (1) numaralı önerme yapısı itibariyle içeriğinden bağımsız olarak bilinemeyen bir önermedir. *Bilinebilinirlik Paradoksu* olarak bilinen paradoks, bilinmeyen önermenin varlığının, hiçbir zaman bilinemeyecek önermelerin varlığını gerektirdiğini gösteren mantıksal bir çelişkinin türetilmesi sonucu ortaya çıkar. Bu paradoks şöyle türetilebilir. Yukarıda gösterdiğimiz gibi bilinmeyen doğruların varlığını ele alalım. *p* böyle bir önerme olsun. Bu durumda (1) numaralı önerme doğrudur. Fakat yukarıda gösterildiği gibi (1) numaralı önerme, doğruluğu hiçbir zaman bilinemeyen bir önermedir. Öyleyse (1) doğrudur fakat doğruluğu bilinemeyen bir önermedir. Yukarıda belirtilen türden önerme ile *bilinebilirlik ilkesi* (KP)'den bir çelişki türetilir. Bu ise bir paradokstur.

Bilinebilirlik paradoksu birçok felsefeci, özellikle realistler, tarafından bilinemeyen doğruların varlığının bir kanıtı olarak ele alınmaktadır. Öte yandan, özellikle anlam bilimsel realistler, bilinmeyen doğrular da dahil olmak üzere, tüm doğruların bilinebileceğini savunurlar. Bilinebilirlik paradoksu bilgi teorisinde önemli bir tartışmayı oluşturmaktadır.

Bu tezde ayrıca *İdealleştirme Paradoksu* olarak da bilinen paradoksu da inceleyerek bir çözüm getiriyoruz. Bu paradoks bilinebilirlik paradoksunu bir çeşitlemesi olarak ortaya konmuştur. *İdealleştirme paradoksu* ilk bakışta kabul edilebilir üç öncülden bir çelişkinin türetilmesi sonucu olarak ortaya çıkar. Birinci öncül, sadece *idealleşmiş özneler* tarafından bilinebilen doğruların varlığını dile getiren (FU) savıdır. İkinci öncül, idealleşmiş özne *olmadığını* dile getiren (NI) savıdır. Üçüncü öncül ise yukarıda bahsedilen *bilinebilirlik ilkesi* (KP)'dir.

Bu tez çalışmasındaki amacımız *bilinebilirlik paradoksu* ve *idealleştirme paradoksunu* kipsel bilgisel diller çerçevesinde incelemek ve bu çerçevede tatmin edici bir çözüm sunmaktır. İlk olarak bu paradoksların tarihsel arka planını ve ortaya çıkışını inceleyeceğiz. Ardından, paradoksun detaylı mantıksal analizini yapacağız. Sonrasında çözümümüzü ortaya koyacağız.

Birinci bölümde, ilk kez 1963 yılında Frederic Fitch'in bir makalesinde ortaya konulan *bilinebilirlik paradoksunun* kısa bir tarihçesini anlatacağız. Fitch'in 4 ve 5 numaralı teoremlerinden türetilen mantıksal çıkarım, bütün doğruların bilinebileceğini savunan anti-realist bilinebilirlik ilkesine karşı kuvvetli bir sav olarak ele alınmıştır. Ardından Florio ve Murzi (2009) tarafından ortaya konulan *idealleştirme paradoksunu* ele alacağız. Bu paradoks, anti-realist doğruluk kuramına karşı daha kuvvetli bir karşıt sav oluşturmak amacıyla ortaya atılan bir paradokstur.

İkinci bölümde paradoksun kipsel bilgi mantığında bir ispatını sunacağız. Ardından *bilinebilirlik paradoksunda* yer alan savların ve ilkelerin oluşturduğu karşıtlık karesini sunacağız. Bildiğimiz kadarıyla literatürde daha önce bu tür bir karşıtlık karesiden bahsedilmemiştir.
Üçüncü bölümde çerçevesini çizmiş olduğumuz kipsel bilgisel diller içinde literatürde belirtilmiş çözüm önerilerini ve bu çözüm önerilerine getirilen eleştirileri inceleyeceğiz.

Dördüncü bölümde, öncelikle *bilinebilirlik paradoksu* ve *idealleştirme paradoksunu* kipsel bilgisel diller çerçevesinde ifade edeceğiz. Ardından bu iki paradoksun çözümlerini sunacağız. *Bilinebilirlik paradoksunun* çözümü olarak (NO) savını, *gerçek* özneler tarafından hiçbir zaman bilinmeyecek doğruların varolduğu iddiası olarak yorumlayacağız. Böyle bir yorumlama yoluyla, (NO) ile (KP)'den bu tür bir çelişkinin türetilmesinin engellenebileceğini göstereceğiz. *İdealleştirme Paradoksunun* çözümü olarak (NI) savını, hiçbir *gerçek* öznenin idealleşmediği şeklinde yorumlayacağız. Böyle bir yorumlama yoluyla (FU), (NI) ve (KP)'den bu tür bir çelişkinin türetilmesinin engellenebileceğini göstereceğiz.

I. Bilinebilirlik Paradoksunun ve İdealleştirme Paradoksunun Kısa Bir Tarihçesi

Bilinebilirlik Paradoksu, ilk olarak Frederic Fitch'in 1945 yılında anonim bir hakem raporu nedeniyle yayınlanmayan makalesinde ortaya konmuştur. *Journal of Symbolic Logic* dergisinin editörü olan Ernest Nagel ile Alonzo Church arasındaki mektuplar göstermektedir ki Fitch makalesini geri çekmiştir. Bu makale Fitch'in 1963 yılındaki makalesinin beşinci dipnotunda bahsedilmektedir.

1963 yılındaki makalesinde Fitch *değer kavramlarının* mantıksal bir analizini ortaya koymaktadır. Fitch bu kavramları *önerme sınıfı* olarak ele almaktadır. *Önerme sınıfı* olarak tanımlanan en önemli kavramlar şunlardır: *yapmak, inanmak, bilmek, kanıtlamak.* '*Bilmek*' Fitch'in makalesi boyunca merkezi bir öneme sahiptir.

Fitch makalesinin ilk yarısında yukarıdaki önerme sınıflarını olgusal ve tümel evetlemeli yasal biçim olarak ele almaktadır. Fitch olgusallığı *doğruluk öbeği* olarak tanımlamaktadır. Başka bir deyişle, zorunlu olarak bir önerme öbeğinin bileşenlerinin her birisi doğru ise bu önerme öbeğine doğruluk öbeği denir. Tümel evetlemeli yasal biçim ise şöyle tanımlanabilir. Eğer bir önermenin iki bileşeninin tümel evetlemesi bu önerme sınıfının ayrı ayrı iki önermesinin doğruluk değerini taşımaktaysa, buna tümel evetlemeli yasal biçim denir. Bu kavram aynı zamanda bilgi operatorü *K*'nın bileşenleri üzerine dağılma özelliği olarak da adlandırılır.

Bilinebilirlik paradoksu ilk olarak Fitch'in makalesinde yer alan 4 ve 5 numaralı teoremlerden türetilmiştir. Bu teoremler her ne kadar Fitch tarafından ifade edilmiş olsalar da, paradoksa yol açan sonuçları Fitch tarafından ifade edilmemiş, başka bir deyişle önemsenmemiştir.

Fitch makalesinde ‘*K*’ harfiyle kısaltılan bilgisel bir operatör olan ‘bilme’ değişmezini kullanarak aşağıdaki iki teoremi kanıtlamaktadır.

> Teorem 4: Sınırsız bilgi kapasitesine sahip olmayan her bir özne için, bu öznenin bilmeyeceği doğru bir önerme vardır.

Teorem 4 mantıksal sembolleştirmede şöyle ifade edilir: $\exists p\ (p \wedge \neg Kp) \rightarrow \exists p\ (p \wedge \neg\Diamond Kp)$. Fitch dipnotunda, bu teoremini daha sonra adının Alanzo Church olduğu öğrenilecek olan anonim bir hakeme atfetmektedir. Bu sebepten ötürü, özellikle hakemin adının ortaya çıkmasından sonra *bilinebilirlik paradoksu* ‘Church-Fitch *Bilinebilirlik Paradoksu*’ olarak da adlandırılmaktadır.

Bilinebilirlik paradoksu sıklıkla Fitch’in 5 numaralı teoremi ile tanımlanmaktadır.

> Teorem 5: Eğer doğruluğu hiç kimsenin bilmediği (ya da gelecekte bilemeyeceği) herhangi bir doğru önerme varsa, doğruluğu hiç kimse tarafından hiçbir zaman bilinemeyecek doğru bir önerme vardır.

Teorem 5 sıklıkla Teorem 4’e benzer bir şekilde sembolleştirilir. Teorem 5 açıkça göstermektedir ki bilinmeyen doğru önermelerin varlığı, hiçbir zaman doğru olduğu bilinemeyecek önermelerin varlığını gerektirir.

Fitch’in 4 ve 5 numaralı teoremlerinden türetilen mantıksal çıkarım, bütün doğruların bilinebileceğini savunan anti-realist bilinebilirlik ilkesine karşı kuvvetli bir sav olarak ele alınmıştır. Bu çerçevede bilinebilirlik paradoksu sadece yukarıda belirtilen anti-realist ve doğrulamacı doğruluk teorilerine değil aynı zamanda mantıksal pozitivizm, Kantçı aşkınsal idealizm ve Berkeleyci idealizm gibi pek çok teoriyi de etkisi altında bırakmaktadır.

Bilinebilirlik paradoksu, Fitch'in makalesinin yayınlanmasından sonra felsefi literatürde ilk kez Hart ve McGinn (1976) tarafından tartışıldı. Hart (1979), Fitch'in 4 numaralı teoremi ve ilgili mantıksal türetimin anti-realist bilinebilirlik ilkesini yanlışladığını belirtti.

Mackie (1980) ve Routley (1981) daha temkinli davranarak ilk bakışta bu mantıksal türetimin bir yanlışlama değil, paradoks olduğu sonucuna vardılar. Ancak, Hart tarafından ortaya konulan bu sonucun büyük bir soruna yol açtığını da kabullendiler. Bu iki felsefeci, anti-realist *bilinebilirlik ilkesini* çeşitli biçimlerde yeniden ortaya koymak suretiyle paradokstan kaçmanın yolunu araştırdılar.

İdealleştirme paradoksu Florio ve Murzi (2009) tarafından anti-realist doğruluk kuramına karşı daha kuramına karşı daha kuvvetli bir karşıt sav oluşturmak amacıyla ortaya atılan bir paradokstur. Bu paradoks esas olarak yukarıda bahsettiğimiz bilinebilirlik paradoksunun mantıksal bir çeşitlemesidir.

II. Bilinebilirlik Paradoksunun Mantıksal Türetimi

Bilinebilirlik paradoksu Fitch'in teoremlerinden bağımsız olarak ifade edilebilir. Bu sembolleştirmeye göre bilinebilir doğru önermelerin varlığı, tüm doğru önermelerin bilindiğini gerektirir. Açıkça görülmektedir ki bu kabul edilemez sonuç oldukça tutarlı olan (KP) ve (NO) savlarının birlikte türetilmesinden ortaya çıkar.

Aşağıda kipsel bilgisel mantık kurallarını kullanarak bilinebilirlik paradoksunun türetilmesini göstereceğiz.

1.	$\exists p\ (p \wedge \neg Kp)$	(NO)
2.	$\forall p\ (p \rightarrow \Diamond Kp)$	(KP)
3.	$\bar{p} \wedge \neg K\bar{p}$	1, Tikel Özelleme Kuralı
4.	$(\bar{p} \wedge \neg K\bar{p}) \rightarrow \Diamond K(\bar{p} \wedge \neg K\bar{p})$	2, Tümel Özelleme Kuralı
5.	$\neg K(\bar{p} \wedge \neg K\bar{p})$	Kanıtlanacak Önerme
6.	$K(\bar{p} \wedge \neg K\bar{p})$	Varsayım (Dolaylı Türetim)
7.	$K\bar{p} \wedge K\neg K\bar{p}$	6, *K* Operatörünün Dağılma Ö.
8.	$K\bar{p}$	7
9.	$K\neg K\bar{p}$	7
10.	$\neg K\bar{p}$	9, Olgusallık
11.	$\perp$	8, 10, Çelişki
12.	$\neg K(\bar{p} \wedge \neg K\bar{p})$	6-11, Kanıtlanan Önerme 2
13.	$\Box\neg K(\bar{p} \wedge \neg K\bar{p})$	12, Zorunluluk Kuralı
14.	$\neg\Diamond K(\bar{p} \wedge \neg K\bar{p})$	13, Kipsellik Eşliği
15.	$\neg\ (\bar{p} \wedge \neg K\bar{p})$	4, 15, Modüs Tollens
16.	$\bar{p} \rightarrow\ \neg\neg K\bar{p}$	15, Mantıksal Eşitlik
17.	$\bar{p} \rightarrow\ K\bar{p}$	16, Çift Değilleme Kuralı
18.	$\forall p\ (p \rightarrow Kp)$	17, Tümel Genelleme

Bilinebilirlik paradoksu, günümüzde sıklıkla ifade edildiği biçimiyle Fitch'in 5 numaralı teoreminin *tamdevriği* olarak türetilir.

Bilinebilirlik Paradoksu $\forall p\,(p \rightarrow \Diamond Kp) \rightarrow \forall p\,(p \rightarrow Kp)$

1.	$\exists p\,(p \wedge \neg Kp) \rightarrow \exists p\,(p \wedge \neg\Diamond Kp)$	Teorem 5
2.	$\forall p\,(p \rightarrow \Diamond Kp) \rightarrow \forall p\,(p \rightarrow Kp)$	1, *Tamdevriklik*

III. Bilinebilirlik Paradoksu Karşıtlık Karesi

Bilinebilirlik paradoksu aşağıdaki gibi ifade edilen birbirileriyle çelişkili olan ilkeler veya savlardan oluşmaktadır.

1.	$\exists p\,(p \wedge \neg\Diamond Kp)$	*Bilinemezlik İlkesi* (UKP)
2.	$\forall p\,(p \rightarrow \Diamond Kp)$	*Bilinebilirlik İlkesi* (KP)
3.	$\exists p\,(p \wedge \neg Kp)$	*Bilinmeyen Doğruların Varlığı* Savı (NO)
4.	$\forall p\,(p \rightarrow Kp)$	*Kadir-i Mutlaklık* Savı (O)

Bilinmeyen doğruların varlığı savı evrensel olarak kabul edilmektir. *Bilinebilirlik ilkesi* ise daha önce bahsedildiği üzere tüm doğruların bilinebileceğini savunur. Bu ilke anlambilimsel anti-realistler tarafından savunulur. *Bilinemezlik ilkesi* anlambilimsel realistler tarafından kabul edilir. Bu ilke bazı doğruların hiçbir zaman bilinemeyeceğini savunur. Her doğrunun bilindiğini savunan *Kadir-i Mutlaklık* savı ise insanların sonlu bilgisel kapasiteye sahip olması nedeniyle evrensel olarak reddedilmektedir. Bu savlar ve ilkeler aşağıdaki karşıtlık karesini oluşturmaktadır. Bu tür bir karşıtlık karesi ilk kez bu tezde dile getirilmiştir.

Bilinebilirlik Paradoksu Karşıtlık Karesi

(KP) $\forall p\,(p \rightarrow \Diamond Kp)$	(NO) $\exists p\,(p \land \neg Kp)$
(O) $\forall p\,(p \rightarrow Kp)$	(UKP) $\exists p\,(p \land \neg \Diamond Kp)$

Bilinebilirlik paradoksu karşıtlık karesi, tüm karşıtlık karesi önermelerinin sağladığı aşağıdaki karşılıklı ilişkisel koşulları sağlamaktadır.

(1) (KP) ve (NO) karşıtlığıdır. Öyle ki (KP) → ¬ (NO) geçerlidir.

(2) (O) ve (UKP) altkarşıtlığıdır. Öyle ki (O) v (UKP) geçerlidir.

(3) (O), (KP)'nin altıklığıdır. Öyle ki (KP) → (O) geçerlidir. (Bu sembolleştirme Bilinebilirlik Paradoksudur.)

(4) (UKP), (NO)'nun altıklığıdır. Öyle ki (NO) → (UKP) geçerlidir. (Bu sembolleştirme Fitch'in 5. Teoremidir.)

(5) (KP) ve (UKP) çelişkidir. Öyle ki (KP) ↔ ¬ (UKP) geçerlidir.

(6) (NO) ve (O) çelişkidir. Öyle ki (NO) ↔ ¬ (O) geçerlidir.

Anlambilimsel realistler *bilinebilirlik paradoksunun*, *bilinebilirlik ilkesinin* mantıksal olarak değillemesi olan *kadir-i mutlaklık* ilkesini ispatladığını savunmaktadır. Bu türden savunulamaz bir sonuca yol açması nedeniyle, *bilinebilirlik ilkesinin* savunulamayacağını ve reddedilmesi zorunlu olduğunu ifade ederler. Öte yandan anlambilimsel anti-realistler *bilinebilirlik ilkesini* korumak için, bu ilkenin çeşitli biçimlerde yeniden ifade etmeye çalışarak bu tür bir çelişkiden kurtulmaya çalışırlar. İfade edilen yeni biçim, paradoksun yol açtığı bir çelişkiyle sonuçlanmadığı takdirde bu yeniden ifade ediliş biçimi tutarlıdır. Ancak bilinebilirlik ilkesinin bu türden yeniden ifade edilişi biçimleri özel çözümler olarak ele alınmalıdır. Biz de bu noktadan sonra çeşitli çözüm önerilerinden bahsedeceğiz.

IV. Bilinebilirlik Paradoksuna Önerilen Çeşitli Çözüm Önerileri

Bu bölümde çerçevesini çizmiş olduğumuz kipsel bilgisel diller içinde literatürde belirtilmiş çözüm önerilerini ve bu çözüm önerilerine getirilen eleştiri inceleyeceğiz.

Bilinebilirlik paradoksunun en temel çözümü olarak *Bilinebilirlik İlkesini* bilgisel (epistemik) olmayan önermelere sınırlandırılmasına dayanan çözümdür. Şimdi bu çözümün, bilinebilirlik paradoksunu nasıl sağladığını çözdüğünü inceleyeceğiz.

Bilgisel (epistemik) olmayan önermeler, sembolleştirilmiş biçimlerinden herhangi bir bilgisel (epistemik) yüklemi veya operatörü içermeyen önermelerdir. Bilinebilirlik ilkesi (KP) bu tür önermelere dayalı bir kısıtlama yoluyla, "bilme değişkeni" olan *K*'nın ikame sınıfını bu tür önermelerden oluşmayacak şekilde ifade edilmesiyle *bilinebilirlik paradoksu* çözülmüş olur. Böyle bir çözümleme yoluyla yeni *Sınırlandırılmış Bilinebilirlik İlkesi* (KP*), değişken *p* önermesinin ikame sınıfı bilgisel (epistemik) olmayan önermeler olmak üzere, aşağıdaki gibi mantıksal sembolleştirme biçiminde ifade edilir.

(KP*) $\forall p \, (p \rightarrow \Diamond Kp)$

Bu çözüm ilkece doğru olmakla birlikte, bilinebilirlik ilkesinin kapsamını çok fazla kısıtladığı için tamamen tatmin edici değildir.

İkinci çözüm öncesi literatürde geniş bir yer tutan J.L Kvanvig (1996) tarafından ortaya konulan çözüm önerisidir. Kvanvig öncelikle paradoksun mantıksal türetiminde gerçekleşen *Bilinebilirlik İlkesi* (KP)'nin tümel evetlemesi adımının hatalı olduğunu belirtmiştir. Kvanvig, kipsel içeriklerde böyle tür bir tümel evetleme kuralı uygulamanın tümel evetlenen önermenin ancak *kesin belirleyici* olması koşulunda geçerli olduğunu belirtir. Kvanvig tümel evetlenen önerme olan bilinmeyen doğruların varlığı savı (NO)'nun *kesin belirleyici* (rigid designator) değildir. Böylece paradoksun türetiminin ilk adımı olan tümel evetleme adımını engellenerek, paradoksun türetimi engellenmiştir. Bu çözüme en önemli itirazları Williamson (2000) ortaya koymuştur. Williamson'a göre *Bilinmeyen Doğruların Varlığı* Savı (NO) *kesin belirleyici*dir. Kvanvig (2006) Williamson'un eleştirilere karşı, paradoksun çözümünü neo-Russellcı önerme kuramı bağlamında savunmuştur. Bu çözüm önerisine göre *Bilinmeyen Doğruların Varlığı* Savı (NO), *gerçek* özneler tarafından hiçbir zaman bilinmeyecek bazı doğruların varolduğu şeklinde yeniden ifade etmiştir. Böylece paradoksun mantıksal türetiminden ortaya çıkan çelişki engellenmiştir.

Üçüncü çözüm önerisi ise Brogaard ve Salerno (2007) tarafından ortaya konulan çözüm önerisidir. Bu çözüm Stanley ve Szabo (2000) tarafından ortaya konulan *bağıntı önalanı kısıtlaması* kuramına bağlı olarak sunulmuştur. Bu kısıtlamaya göre bilinebilirlik paradoksunu oluşturan (NO) ve (KP)'nun mantıksal türetiminden herhangi bir çelişki elde edilemeyeceği gösterilmiştir. Son olarak Kennedy (2013) tarafından ortaya konulan *model-kuramsal* çözüm önerisine göre (NO) ve (KP)'den herhangi bir çelişkinin ortaya çıkmayacağı gösterilmiştir.

IV. B*ilinebilirlik Paradoksu* ve *İdealleştirme Paradoksunun* Kipsel Bilgisel Dillerde Bir Çözümü

Özetimizin başında, insanların *sonlu* bilgi edinme yetenekleriyle donatıldığını ve dolayısıyla hiçbir zaman bilinmeyen doğrular olacağını belirttik. Bu olgu, *bilinmeyen doğruların varlığı* savı (NO) ile ifade edilmektedir. Birçok felsefeci, özellikle anlambilimsel anti-realistler, bilinmeyen doğrular da dahil olmak üzere, tüm doğruların bilinebileceğini savunurlar. Bu tez *bilinebilirlik ilkesi* (KP) ile ifade edilir. *Bilinebilirlik Paradoksu* olarak bilinen paradoks, (NO) ve (KP)'den çelişki türetilmesi sonucu olarak ortaya çıkar. Bu paradoksun çözümü olarak (NO) savını, *gerçek* özneler tarafından hiçbir zaman bilinmeyecek doğruların varolduğu iddiası olarak yorumlayacağız. Böyle bir yorumlama yoluyla, (NO) ile (KP)'den bu tür bir çelişkinin türetilmesinin engellenebileceğini göstereceğiz.

İdealleştirme Paradoksu olarak da bilinen paradoksa ise aşağıda belirttiğimiz gibi bir çözüm getiriyoruz. Hatırlanacağı gibi bu paradoks, ilk bakışta kabul edilebilir üç öncülden bir çelişkinin türetilmesi sonucu olarak ortaya çıkmaktadır. Birinci öncül, sadece *idealleşmiş özneler* tarafından bilinebilen doğruların varlığını dile getiren (FU) savıdır. İkinci öncül, idealleşmiş özne *olmadığını* dile getiren (NI) savıdır. Üçüncü öncül ise yukarıda bahsedilen *bilinebilirlik ilkesi* (KP)'dir. Bu paradoksun çözümü olarak (NI) savını, hiçbir *gerçek* öznenin idealleşmediği şeklinde yorumlayacağız. Böyle bir yorumlama yoluyla (FU), (NI) ve (KP)'den bu tür bir çelişkinin türetilmesinin engellenebileceğini göstereceğiz.

TEZ FOTOKOPİSİ İZİN FORMU

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YAZARIN

Soyadı : Akçelik

Adı : Oğuz

Bölümü : Felsefe Bölümü

TEZİN ADI (İngilizce) A SOLUTION TO THE KNOWABILITY PARADOX AND THE PARADOX OF IDEALIZATION IN MODAL EPISTEMIC LANGUAGES

TEZİN TÜRÜ Yüksek Lisans ☒ Doktora ☐

1. Tezimin tamamından kaynak gösterilmek şartıyla fotokopi alınabilir. ☐

2. Tezimin içindekiler sayfası, özet, indeks sayfalarından ve/veya bir bölümünden kaynak gösterilmek şartıyla fotokopi alınabilir. ☐

3. Tezimden bir (1) yıl süreyle fotokopi alınamaz. ☐

TEZİN KÜTÜPHANEYE TESLİM TARİHİ