BLACK HOLE COLLISIONS AT THE SPEED OF LIGHT

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ABSTRACT

BLACK HOLE COLLISIONS AT THE SPEED OF LIGHT

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The main purpose of this work is to study the collision of two black holes and the energy loss due to the gravitational waves emitted during this collision in the framework of general relativity. For this purpose we first study plane wave geometries and their collisions. More realistic collisions are the pp-wave collisions. As an analytic treatment of this problem, we investigate the head-on collision of two ultra-relativistic black holes. Treating the problem perturbatively, we extract the news function to compute how much energy is radiated in gravitational waves during the process. We show that the news function vanishes for the solutions obtained meaning that there is no mass-loss at the order of approximation.

Keywords: Colliding Waves, Black Holes, Gravitational Radiation

ÖZ

IŞIK HIZINDA KARA DELİK ÇARPIŞMALARI

Şentürk, Çetin
Doktora, Fizik Bölümü
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Bu tezin ana amacı, iki kara deliğin çarpışmasını ve bu çarpışma süresince yayınlanan kütleçekim dalgalarından dolayı sistemde kaybolan enerjiyi genel görelilik çerçevesinde incelemektir. Bu amaç için, ilk olarak düzlem dalga geometrilerini ve bunların çarpışmalarını gözönüne alıyoruz. Fiziksel açıdan gerçekçi olan çarpışmalar "*pp*-wave" çarpışmalarıdır. Böyle bir çarpışma için analitik bir çalışma olarak, ışık hızında ilerleyen iki kara deliğin kafa kafaya çarpışmasını inceliyoruz. Problemi bir tür yaklaşıklık hesabıyla ele alarak, çarpışmada açığa çıkan çekim dalgalarının enerjisini bulabilmek amacıyla, elde ettiğimiz çözümler için "news" fonksiyonunu hesaplıyoruz ve sıfır olduğunu gösteriyoruz. Bir başka deyişle, uygulanan yaklaşıklık çerçevesinde, çarpışmadan sonra kütle kaybı olmadığı sonucuna varıyoruz.

Anahtar Kelimeler: Çarpışan Dalgalar, Kara Delikler, Kütleçekimsel Işıma

“Memmed” emmime...

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CHAPTER 1

INTRODUCTION

Black hole collisions at the speed of light have attracted much attention recently. There are mainly two reasons for this interest. First of all, they enable us to calculate the quantum scattering amplitudes at very high energies comparable to the Planck energy $M_{pl} \sim 10^{19} GeV$, where gravitational effects dominate [1, 2]. It is well-known that the quantum effects of gravity come into play at the Planck energy. If we collide two particles at about Planck energy, so much energy, and therefore mass, becomes concentrated in a very small region of space that gravitational forces overwhelm all others. Therefore, in collision processes at above the Planck energy semi-classical approximation can be applied by studying the classical space-times such as black hole collisions. 't Hooft [1] proposed that at energies of the order of or higher than the Planck scale, an elementary particle may be represented by its accompanying gravitational field which is, due to special relativistic effects, concentrated in the vicinity of a plane containing the particle and orthogonal to the direction of motion. For massless particles this field becomes a gravitational shock wave for which the Riemann tensor is proportional to the Dirac delta function on this plane. That shock wave solution is already known in the literature. Aichelburg and Sexl [3] obtained the gravitational field of a massless particle by boosting the geometry of a single black hole to the speed of light. In [1], 't Hooft calculated the two-particle scattering amplitude by solving quantum mechanical equations for one particle moving in the external shock wave geometry produced by the other particle which is taken to be massless. So the trans-Planckian scattering of point particles may well be described by black hole scattering. Also, it has long been believed that head-on collisions at the center of mass energy beyond the Planck scale will produce black holes because of the huge concentrated energy involved in the interactions. These black holes, after their formation, are expected to settle down to a stationary state by radiating gravitational waves, and evaporate via Hawking radiation. Recently, motivated by the brane-world scenarios [4, 5], this issue of trans-Planckian scattering and subsequent black hole formation has gained renewed interest. In these scenarios, it is proposed that if our space is a 3-brane situated in a large [4] or warped [5] extra dimensional space, the Planck energy could be as low as the electroweak scale (TeV). This opens up the possibility of observing black hole formation in particle accelerators such as the Large Hadron Collider (LHC) at CERN [6]. An important signature in this search is the energy loss due to gravitational radiation.

On the other hand, these black hole collisions may be a testing ground for the cosmic censorship hypothesis which claims that singularities in nature must be hidden behind event horizons (see, for example, [7]). When black holes collide, due to their rapidly changing accelerations at the moment of collision, gravitational radiation is produced. In [8], Hawking, using the area theorem which states that the area of a future event horizon of a black hole can never decrease, put an upper limit on the gravitational radiation emitted in a collision process of two black holes. For two equal-mass Schwarzschild black holes, that limit is 29% of the initial energy. Later, Penrose [9] considered the head-on collision of two ultra-relativistic black holes described by the Aichelburg-Sexl shock waves [3], and found an apparent horizon (a closed two-dimensional spacelike surface on which the outgoing null geodesics have zero divergence [7]) on the union of the incoming plane shocks. Then, if one assumes that cosmic censorship holds, this information can be used to put an upper bound on the amount of radiation emitted in the collision. This bound is the same as the one obtained by Hawking, namely, a maximum of 29% of the initial energy should be radiated away. In both of these calculations, cosmic censorship hypothesis is assumed to hold, so in a collision process too much radiation than the above bound would be a direct indication that this hypothesis must be wrong. Giving analytical solutions which describe the full dynamics of the spacetime after the collision seems to be unlikely because of the highly nonlinear interaction of the black holes. Therefore, to clarify the structure of the spacetime after the collision, some perturbative or numerical methods must be employed. In [10], D'Eath and Payne studied this problem perturbatively (summarized in [11]), and by calculating the Bondi's news function (see [12]), which describes the emission of gravitational radiation, they gave an estimate of 16.4% for the efficiency of gravitational wave generation in head-on collisions. This is well below the upper bound (29%) calculated by Hawking and Penrose. All these mean that new and more refined methods are necessary to accurately describe black hole collisions and the resulting gravitational wave generation.

In this thesis, we investigate the classical formation of a black hole in a head-on collision of two ultra-relativistic black holes. In particular, we are interested in the total gravitational energy radiated in this process. We treat the problem in the standard fashion, namely, we describe the incoming particles by two Aichelburg-Sexl shock waves, and consider the head-on collision of these shocks in the center of mass frame. By two different perturbative methods, we give approximate solutions which may describe the interaction region of these waves. To study the efficiency of the gravitational radiation emitted in the collision, we apply the method of Bondi and extract the news function which enables us to calculate the mass-loss of the system. We found that the news function vanishes for the solutions we give, which brings out the conclusion that the system does not radiate at the order of approximations. Of course, it is possible that the Bondi's procedure may not be useful here, so it may be necessary to invoke some other methods to evaluate the gravitational radiation problem.

The layout of the thesis is as follows:

In Chapter 2, we first review the plane-fronted gravitational waves and their interactions in general relativity.

In Chapter 3, we construct the shock wave solutions both in classical electrodynamics and in general relativity by the well-known boosting procedure.

In Chapter 4, we review three different methods which enable us to estimate the efficiency of the total gravitational radiation in high-energy particle collisions.

In Chapter 5, to understand the problem of collision and how to calculate the news function, we give a perturbation treatment with respect to the energies of the colliding shock waves.

In Chapter 6, we formulate the problem in terms of the proper time and rapidity.

CHAPTER 2

PLANE WAVES AND THEIR COLLISIONS IN GENERAL RELATIVITY

The study of the collision and subsequent interaction of gravitational and electromagnetic plane waves is of great importance in general relativity. This is due to the fact that, being a highly non-linear theory, general relativity predicts that there will be a non-linear interaction between such waves when they collide. According to the theory, all forms of energy produce curvature in spacetime, so any field propagating in spacetime has an associated gravitational field, which is actually the curvature of the spacetime structure. As a result, when two waves collide, they do not simply pass through each other; they interact non-linearly through their accompanying gravitational fields. Because of this non-linear interaction, in general, in almost all collision processes there occur singularities in spacetime after the collision of the waves. These may be real curvature singularities or just Killing-Cauchy horizons depending on the types of the colliding waves. Colliding plane wave spacetimes have been investigated in great detail and many exact solutions describing the geometries after the collision have been obtained so far [13].

In this section, we very briefly review the plane waves and their collisions in classical general relativity. For much more detailed discussions, see [13, 14].

2.1 *pp*-WAVE SPACETIMES

Plane-fronted waves with parallel rays (*pp*-waves) constitute a well-known class of exact solutions to Einstein's equations [14, 15, 16]. These spacetimes are defined by the property that they admit a covariantly constant null vector field k_μ, i.e.,

$$\nabla_\mu k_\nu = 0, \quad k_\mu k^\mu = 0. \tag{2.1}$$

As is showed in Appendix A, in a suitable coordinate system $x^\mu = (u, v, X, Y)$, the metric of such spacetimes can be written as

$$ds^2 = 2dudv + H(u, X, Y)du^2 - dX^2 - dY^2, \tag{2.2}$$

where

$$u = \frac{1}{\sqrt{2}}(t - Z), \quad v = \frac{1}{\sqrt{2}}(t + Z) \tag{2.3}$$

are the double null coordinates, and X and Y are the spacelike coordinates in the transverse space $u = const.$ with Euclidean geometry. The metric function $H(u, X, Y)$ is called the profile function which characterizes the nature of the wave, and since it is independent of the coordinate v, the spacetime (2.2) has the Killing vector $k = \partial/\partial v$ representing the propagation direction of the wave. Therefore, the metric (2.2) describes a plane-fronted wave with arbitrary profile which propagates along the Z-direction in the background Minkowski specetime.

The only non-zero components of the Riemann tensor, the Ricci tensor and the Weyl tensor are, respectively,

$$\begin{aligned} R_{uiuj} &= -\frac{1}{2}\partial_i\partial_j H(u, X^k), \\ R_{uu} &= \frac{1}{2}\nabla_\perp^2 H(u, X^k), \\ C_{uiuj} &= -\frac{1}{2}\partial_i\partial_j H(u, X^k) + \frac{1}{4}\delta_{ij}\nabla_\perp^2 H(u, X^k), \end{aligned} \tag{2.4}$$

where $\partial_i \equiv \partial/\partial X^i$, $\nabla_\perp^2 \equiv \sum_i \partial_i^2$ and $X^i = (X, Y)$. Moreover, the Ricci scalar vanishes

$$R = 0. \tag{2.5}$$

Actually, all scalar curvature invariants are zero for the pp-wave spacetime (2.2), which is guaranteed by the existence of a covariantly constant null vector field [17, 18].

Therefore the vacuum field equations for the spacetime (2.2) reduce to the two-dimensional Laplace equation in the transverse coordinates:

$$\nabla_\perp^2 H = 0. \tag{2.6}$$

This equation means that any harmonic function H of X and Y, whatever its dependence on u may be, describes a pure gravitational wave, and since it is a linear differential equation, distinct solutions with different wave profiles may be simply superposed as long as they propagate along the same direction.

A particular subclass of pp-waves are the *plane* waves which are defined to be those for which the profile function $H(u, X, Y)$ is quadratic in the transverse coordinates X and Y, that is,

$$H(u, X^k) = \sum_{i,j} h_{ij}(u) X^i X^j \tag{2.7}$$

where the symmetric tensor $h_{ij}(u)$ contains the information about the polarization and amplitude of the wave. Thus the field components (2.4), being dependent only on the second derivatives of the function $H(u, X, Y)$ with respect to X and Y, are the same, i.e. functions of u alone, at every point on the transverse plane. This, in a sense, reflects the extra plane symmetry of the wave; the wave is of infinite extent in all directions in the plane. Plane gravitational waves of infinite extent are of course unphysical, but nevertheless, they may be considered as realistic approximations to real waves within finite regions and at large distances from their sources.

For a plane wave (2.2) with the profile (2.7), the non-zero field components (2.4) become

$$\begin{aligned} R_{uiuj} &= -h_{ij}, \\ R_{uu} &= \mathrm{Tr}(h), \\ C_{uiuj} &= -h_{ij} + \frac{1}{2}\delta_{ij}\mathrm{Tr}(h), \end{aligned} \tag{2.8}$$

where $\mathrm{Tr}(h) = \sum_i h_{ii}$. It can be seen directly from the Ricci tensor component that the trace of the symmetric matrix $h_{ij}(u)$ is related to the other fields (scalar, electromagnetic, etc.) present in the spacetime, so $\mathrm{Tr}(h) \neq 0$ implies that the wave is not a pure gravity wave. For a pure vacuum gravitational plane wave $\mathrm{Tr}(h) = 0$, of course. On the other hand, since the Weyl tensor describes the free gravitational field in spacetime that does not come from the Ricci part of the Riemann tensor, vanishing of it represents a wave directly generated by the matter fields present in the spacetime (a pure electromagnetic plane wave for instance). This condition implies that the matrix $h_{ij}(u)$ is purely diagonal, namely $h_{ij} = \delta_{ij}\mathrm{Tr}(h)/2$.

There is a further restricted class of pp-waves which are called the *homogeneous* plane waves. They are obtained by taking out the u dependency of the matrix $h_{ij}(u)$ in (2.7), i.e.

$$H(u, X^k) = \sum_{i,j} \mu_{ij} X^i X^j \tag{2.9}$$

where μ_{ij} is a constant symmetric matrix.

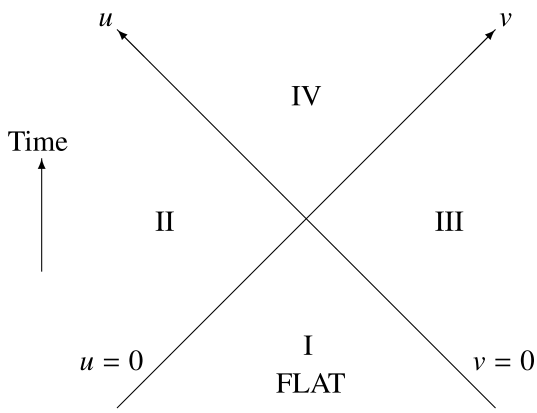

Figure 2.1: The spacetime diagram representing the collision of two plane waves. Two space-like coordinates have been suppressed.

2.2 COLLIDING PLANE WAVES

The importance of the gravitational interaction of waves in general relativity drives from the fact that it manifests the richness and the non-linearity of the theory explicitly. However, in the most general case in which the colliding waves are *pp*-waves, it is too difficult to analyse the problem analytically. Therefore, in order to formulate the problem explicitly and to obtain exact solutions, it is convenient to consider the interaction of plane waves which have more symmetry than *pp*-waves. The first exact solutions which describe the collision of two pure gravitational plane waves with collinear polarization was given by Khan and Penrose [19], and Szekeres [20]. They showed that the collisions of such waves with step or impulsive profiles always produce a curvature singularity in the spacetime after the collision. This singularity is due to the focusing effect of the colliding waves on one another. For a detailed discussion of the exact solutions and the singularities produced, see [13].

The colliding plane wave spacetimes are most conveniently represented as in Figure (2.1) in which the waves approach each other from exactly opposite spatial directions and collide head-on at a point in the figure. The head-on collision assumption is not a restriction in the discussion actually. For the most general case of the collision in arbitrary directions, it is always possible to make a Lorentz transformation to a reference frame in which the collision appears to be head-on.

As represented in Figure (2.1), the colliding plane wave spacetime is divided by two null hypersurfaces $u = 0$ and $v = 0$ into four distinct regions. The region I is the background

Minkowski spacetime on which the collision occurs, regions II and III contain the approaching plane waves, and the region IV is the region where the two waves interact. Since the approaching waves have plane symmetry in the directions of the transverse plane, we assume that the interaction region also retains this symmetry, and so the whole spacetime possesses two commuting spacelike Killing vector fields ∂_x and ∂_y.

In this way, the collision problem has been set up as a characteristic initial value problem with the initial data prescribed on the null boundaries $u = 0$ and $v = 0$ for the interaction region IV. The problem is well-posed and according to the work of Penrose [21], given arbitrary waves in regions II and III, a unique solution exists in the interaction region IV at least in the neighborhood of the boundaries of regions II and III. However, this approach is not always practical in obtaining analytic solutions in region IV. Therefore, it is often more appropriate to take the opposite approach; namely, first find a solution to the field equations in the interaction region IV, and then extend it back to determine the incoming waves (data) which would give rise to that solution. This is accomplished by the substitution

$$u \to u\theta(u), \quad v \to v\theta(v) \tag{2.10}$$

in the solution obtained in the interaction region. Here $\theta(u)$ is the Heaviside step function defined by

$$\theta(u) = \begin{cases} 1 & \text{if } u \geq 0, \\ 0 & \text{if } u < 0. \end{cases} \tag{2.11}$$

It is also necessary to satisfy the junction conditions across the null hypersurfaces $u = 0$ and $v = 0$. In the colliding wave problems, the appropriate junction conditions are those of Lichnerowicz [22] or O'Brien-Synge [23] conditions. The Lichnerowicz conditions require that the metric and its first derivatives be continuous across the junctions. The O'Brien-Synge conditions are weaker than the Lichnerowicz conditions and they allow some derivatives of the metric to be discontinuous across the junctions (as in the Khan-Penrose solution [19] for colliding plane impulsive gravitational waves, and the Bell-Szekeres solution [24] and its higher-dimensional generalization [25] for colliding plane electromagnetic waves in Einstein-Maxwell theory). Therefore, the Lichnerowicz conditions exclude impulsive waves which have Dirac delta function profiles.

CHAPTER 3

SHOCK WAVE GEOMETRIES

As explained in the introduction, shock waves are relevant in collision processes involving energies of the order or higher than the Planck scale. Later, we will consider the collision of two ultra-relativistic black holes which are represented by two shock waves in studying the gravitational radiation emitted in such collisions. So in this chapter we discuss these geometries by first reviewing the boosting procedure to obtain simplest shock waves in both classical electrodynamics and general relativity.

3.1 BOOSTING THE COULOMB FIELD

In this section we mainly follow [26], we expand the discussion there by giving explicit calculations.

We are interested in finding the electromagnetic field of a point charge which is moving with the speed of light along the x-axis in Minkowski spacetime. This can be achieved by first boosting to a reference frame in which the particle is seen to be moving with a constant speed v and then taking the ultra-relativistic limit $v \to c$. At the end we will see that the field of the charge is a plane wave with a Dirac delta function profile, i.e. a plane impulsive electromagnetic wave.

Let us begin with the relativistic notation for the electrodynamic quantities. Throughout this section we shall use the Gaussian units. First it is convenient to parametrize the spacetime with the coordinates

$$x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z) = (ct, \mathbf{x}), \tag{3.1}$$

where the Greek index μ takes the values 0,1,2,3. Then the line element in these coordinates becomes

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = c^2 dt^2 - dx^2 - dy^2 - dz^2 \tag{3.2}$$

with

$$\eta_{\mu\nu} = diag[1, -1, -1, -1] \tag{3.3}$$

being the metric tensor of the Minkowski spacetime. Here we used the notation $diag[\ ]$ to represent a 4×4 diagonal matrix. So, raising or lowering indices with the Minkowski metric (3.3) just amounts to changing the signs of the spatial components of the relevant 4-vector, that is, for the 4-coordinates (3.1)

$$x_\mu = \eta_{\mu\nu} x^\nu = (ct, -\mathbf{x}). \tag{3.4}$$

Similarly, we can define

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = (\partial_0, \partial_1, \partial_2, \partial_3) = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right) \tag{3.5}$$

giving the d'Alembertian operator

$$\Box = \partial_\mu \partial^\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^{\mathbf{2}}. \tag{3.6}$$

The 4-vector potential can be defined in a similar way as

$$A^\mu = (\phi, \mathbf{A}), \tag{3.7}$$

where ϕ and $\mathbf{A}$ are the scalar and vector potentials, respectively. In terms of this 4-potential, the electromagnetic field tensor is defined by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{3.8}$$

and has components

$$F_{0i} = \partial_0 A_i - \partial_i A_0 = -\left(\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} + \nabla\phi\right)_i = E_i, \tag{3.9}$$

and

$$F_{ij} = \partial_i A_j - \partial_j A_i = -\varepsilon_{ijk}(\nabla \times \mathbf{A})_k = -\varepsilon_{ijk} B_k, \tag{3.10}$$

which are the electric and magnetic fields, respectively. Here the latin index i runs over the values 1,2,3 and ε_{ijk} is the totally antisymmetric Levi-Civita symbol defined by

$$\varepsilon_{ijk} = \begin{cases} +1, & \text{if } (ijk) \text{ is an even permutation of } (123), \\ -1, & \text{if } (ijk) \text{ is an odd permutation of } (123), \\ 0, & \text{otherwise.} \end{cases} \tag{3.11}$$

With the definition of the field tensor (3.8), the homogeneous (source-free) Maxwell equations are automatically satisfied, while the inhomogeneous ones can be written as

$$\partial_\nu F^{\mu\nu} = \frac{4\pi}{c} J^\mu \tag{3.12}$$

with the current density 4-vector

$$J^\mu = (c\rho, \mathbf{J}), \tag{3.13}$$

which satisfies the continuity equation

$$\partial_\mu J^\mu = \frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{J} = 0. \tag{3.14}$$

It is also obvious from (3.8) that the potentials are not uniquely determined, namely, we could always change the 4-potential as such

$$A'_\mu = A_\mu + \partial_\mu\chi, \tag{3.15}$$

without changing the field tensor $F_{\mu\nu}$. Here χ is an arbitrary scalar function.

After this summary, we can now calculate the field of a moving charge with a constant speed v. Let us consider a point charged particle q which is at rest in some reference frame $\bar{S}$, so in this frame there is only the Coulomb field of the charge:

$$\bar{\phi} = \frac{q}{\bar{r}}, \quad \bar{r} = \sqrt{\bar{x}^2 + \bar{y}^2 + \bar{z}^2}. \tag{3.16}$$

Since the vector potential is zero in $\bar{S}$, we write the 4-potential of the particle

$$\bar{A}^\mu = (\bar{\phi}, 0, 0, 0) = \bar{\phi}\delta^\mu_0, \quad \bar{A}_\mu = (\bar{\phi}, 0, 0, 0) = \bar{\phi}\delta^0_\mu, \tag{3.17}$$

where

$$\delta^\mu_\nu = \begin{cases} 1, & \text{if } \mu = \nu \\ 0, & \text{if } \mu \neq \nu \end{cases} \tag{3.18}$$

is the Kronecker delta. Therefore we initially have

$$\begin{aligned} \bar{F}_{\mu\nu} &= \bar{\partial}_\mu\bar{A}_\nu - \bar{\partial}_\nu\bar{A}_\mu = (\bar{\partial}_\mu\bar{\phi})\delta^0_\nu - (\bar{\partial}_\nu\bar{\phi})\delta^0_\mu \\ &= \frac{q}{\bar{r}^3}\left(\bar{x}\delta^0_\mu\delta^1_\nu + \bar{y}\delta^0_\mu\delta^2_\nu + \bar{z}\delta^0_\mu\delta^3_\nu - \bar{x}\delta^0_\nu\delta^1_\mu - \bar{y}\delta^0_\nu\delta^2_\mu - \bar{z}\delta^0_\nu\delta^3_\mu\right), \end{aligned} \tag{3.19}$$

where in the second line we put

$$\bar{\partial}_\mu\bar{\phi} = \frac{q}{\bar{r}^3}\left(-\bar{x}\delta^1_\mu - \bar{y}\delta^2_\mu - \bar{z}\delta^3_\mu\right),$$

using the scalar potential (3.16). Then it follows from (3.9), (3.10) and (3.19) that in $\bar{S}$

$$\bar{\mathbf{E}} = \frac{q}{\bar{r}^3}\bar{\mathbf{r}}, \quad \bar{\mathbf{B}} = 0. \tag{3.20}$$

Now we can perform a Lorentz transformation to a reference frame S in which the particle appears to be moving along the positive x-direction with a constant speed v. This means that the frame S is moving along the negative x-direction with respect to the frame $\bar{S}$ with the same speed v. The transformation is

$$x^\mu = \frac{\partial x^\mu}{\partial \bar{x}^\nu}\bar{x}^\nu, \tag{3.21}$$

or explicitly

$$x^0 = \gamma(\bar{x}^0 + \beta\bar{x}), \quad x = \gamma(\bar{x} + \beta\bar{x}^0), \quad y = \bar{y}, \quad z = \bar{z}, \tag{3.22}$$

with

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}, \quad \beta = \frac{v}{c}. \tag{3.23}$$

Then the 4-potential (3.17) transforms as

$$A^\mu = \frac{\partial x^\mu}{\partial \bar{x}^\nu}\bar{A}^\nu = \bar{\phi}\frac{\partial x^\mu}{\partial \bar{x}^0} = \gamma\bar{\phi}(1, \beta, 0, 0) = \frac{q}{R}(\delta^\mu_0 + \beta\delta^\mu_1) \tag{3.24}$$

where we have used the inverse Lorentz transformations to put $\bar{x} = \gamma(x - \beta x^0)$ in (3.16) and defined

$$R \equiv [(x - \beta x^0)^2 + \gamma^{-2}(y^2 + z^2)]^{1/2}. \tag{3.25}$$

It is now straightforward to evaluate the field tensor $F_{\mu\nu}$ in S:

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu = q\left[\partial_\mu\left(\frac{1}{R}\right)(\delta^0_\nu - \beta\delta^1_\nu) - \partial_\nu\left(\frac{1}{R}\right)(\delta^0_\mu - \beta\delta^1_\mu)\right] \\ &= \frac{q(1-\beta^2)}{R^3}\Big[(x-\beta x^0)\delta^0_\mu\delta^1_\nu + y\delta^0_\mu\delta^2_\nu + z\delta^0_\mu\delta^3_\nu - \beta y\delta^1_\mu\delta^2_\nu - \beta z\delta^1_\mu\delta^3_\nu \\ &\qquad - (x-\beta x^0)\delta^1_\mu\delta^0_\nu - y\delta^2_\mu\delta^0_\nu - z\delta^3_\mu\delta^0_\nu + \beta y\delta^2_\mu\delta^1_\nu + \beta z\delta^3_\mu\delta^1_\nu\Big], \end{aligned} \tag{3.26}$$

since from (3.25)

$$\partial_\mu\left(\frac{1}{R}\right) = \frac{1}{R^3}\left[\beta(x-\beta x^0)\delta^0_\mu - (x-\beta x^0)\delta^1_\mu - (1-\beta^2)(y\delta^2_\mu + z\delta^3_\mu)\right].$$

Therefore, the electric and magnetic fields of a moving charge in the frame S are the following

$$E_1 = \frac{q(1-\beta^2)(x-\beta x^0)}{R^3}, \quad E_2 = \frac{q(1-\beta^2)y}{R^3}, \quad E_3 = \frac{q(1-\beta^2)z}{R^3}, \tag{3.27}$$

$$B_1 = 0, \quad B_2 = -\frac{q(1-\beta^2)\beta z}{R^3}, \quad B_3 = \frac{q(1-\beta^2)\beta y}{R^3}. \tag{3.28}$$

From these expressions, it is seen that when $\beta = 0$ there is only the electric field of the static charge, however for $\beta \neq 0$ both the electric and magnetic fields are present, and as β increases the component of the electric field along the direction of motion becomes smaller and smaller. Now we will see that in the ultra-relativistic limit, i.e. $\beta \to 1$, the parallel component of the electric field becomes exactly zero and the only nonzero components are the ones perpendicular to the direction of motion.

To take the ultra-relativistic limit, first we need to observe that

$$\frac{1-\beta^2}{R^3} = \frac{\gamma^{-2}}{R^3} = \frac{1}{y^2+z^2}\frac{\partial}{\partial x}\left(\frac{x-\beta x^0}{R}\right), \tag{3.29}$$

where R is defined in (3.25). Then

$$\lim_{\beta\to 1}\frac{1-\beta^2}{R^3} = \frac{1}{y^2+z^2}\frac{\partial}{\partial x}\left(\frac{x-x^0}{|x-x^0|}\right) = \frac{1}{y^2+z^2}\frac{\partial}{\partial x}[2\theta(x-x^0)-1] = \frac{2\delta(x-x^0)}{y^2+z^2}, \tag{3.30}$$

with the help of the Heaviside theta $\theta(x - x^0)$ and the Dirac delta $\delta(x - x^0)$ functions. Then the limit of the field tensor (3.26) is

$$\begin{aligned}\mathcal{F}_{\mu\nu} \equiv \lim_{\beta\to 1} F_{\mu\nu} \;=\; & \frac{2q\delta(x-x^0)}{y^2+z^2}\Big[y\delta^0_\mu\delta^2_\nu + z\delta^0_\mu\delta^3_\nu - y\delta^1_\mu\delta^2_\nu - z\delta^1_\mu\delta^3_\nu \\ & - y\delta^2_\mu\delta^0_\nu - z\delta^3_\mu\delta^0_\nu + y\delta^2_\mu\delta^1_\nu + z\delta^3_\mu\delta^1_\nu\Big],\end{aligned} \tag{3.31}$$

where we have used that $(x - x^0)\delta(x - x^0) = 0$. In terms of the electric and magnetic fields, we get

$$\mathcal{E}_1 = 0, \quad \mathcal{E}_2 = \frac{2q\delta(x-x^0)y}{y^2+z^2}, \quad \mathcal{E}_3 = \frac{2q\delta(x-x^0)z}{y^2+z^2}, \tag{3.32}$$

$$\mathcal{B}_1 = 0, \quad \mathcal{B}_2 = -\frac{2q\delta(x-x^0)z}{y^2+z^2}, \quad \mathcal{B}_3 = \frac{2q\delta(x-x^0)y}{y^2+z^2}, \tag{3.33}$$

(these can also be obtained directly from (3.27) and (3.28) using (3.30)). These show that the field is an electromagnetic plane wave propagating along the x-axis with a delta function profile, i.e. an impulsive plane wave.

We can also show that the field tensor (3.31) satisfies the Maxwell equations with a null source current density:

$$\begin{aligned}
\partial^\nu \mathcal{F}_{\mu\nu} &= q\Bigg\{\delta(x-x^0)\left[\partial_2\left(\frac{2y}{y^2+z^2}\right)+\partial_3\left(\frac{2z}{y^2+z^2}\right)\right](-\delta^0_\mu+\delta^1_\mu) \\
&\quad -\left(\frac{2y}{y^2+z^2}\delta^2_\mu+\frac{2z}{y^2+z^2}\delta^3_\mu\right)\Big(\partial_0[\delta(x-x^0)]+\partial_1[\delta(x-x^0)]\Big)\Bigg\} \\
&= q\delta(x-x^0)\left\{\frac{\partial^2}{\partial y^2}[\ln(y^2+z^2)]+\frac{\partial^2}{\partial z^2}[\ln(y^2+z^2)]\right\}(-\delta^0_\mu+\delta^1_\mu) \\
&= q\delta(x-x^0)\nabla^2_\perp[\ln(y^2+z^2)](-\delta^0_\mu+\delta^1_\mu) \\
&= 4\pi q\delta(x-x^0)\delta(y)\delta(z)(-\delta^0_\mu+\delta^1_\mu) \\
&\equiv \frac{4\pi}{c}\mathcal{J}_\mu. \qquad (3.34)
\end{aligned}$$

Here in the second equality we used

$$f(x)\frac{\partial}{\partial x}[\delta(x-x^0)] = -\left(\frac{\partial f(x)}{\partial x}\right)_{x=x^0}\delta(x-x^0) \qquad (3.35)$$

for some arbitrary function $f(x)$. In the third line, we have defined

$$\nabla^2_\perp \equiv \frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}, \qquad (3.36)$$

and in the fourth we have used

$$\nabla^2_\perp[\ln(y^2+z^2)] = 4\pi\delta(y)\delta(z), \qquad (3.37)$$

which is just the two dimensional Poisson equation with the Greens' function $\ln(y^2+z^2)$ for a point source. We have also made the identification in the last line

$$\mathcal{J}_\mu \equiv qc\delta(x-x^0)\delta(y)\delta(z)(-\delta^0_\mu+\delta^1_\mu). \qquad (3.38)$$

This is the current density which represents a point particle propagating along the x-axis with the speed of light.

To find the 4-potential $\mathcal{A}_\mu$ that leads to (3.31), we proceed as follows:

$$\begin{aligned}
\mathcal{F}_{\mu\nu} &= \frac{2q\delta(x-x^0)}{y^2+z^2}\Big[y\delta^0_\mu\delta^2_\nu+z\delta^0_\mu\delta^3_\nu-y\delta^1_\mu\delta^2_\nu-z\delta^1_\mu\delta^3_\nu \\
&\quad -y\delta^2_\mu\delta^0_\nu-z\delta^3_\mu\delta^0_\nu+y\delta^2_\mu\delta^1_\nu+z\delta^3_\mu\delta^1_\nu\Big] \\
&= q\delta(x-x^0)\Big\{\Big(\delta^2_\nu\partial_2[\ln(y^2+z^2)]+\delta^3_\nu\partial_3[\ln(y^2+z^2)]\Big)(\delta^0_\mu-\delta^1_\mu) \\
&\quad -\Big(\delta^2_\mu\partial_2[\ln(y^2+z^2)]+\delta^3_\mu\partial_3[\ln(y^2+z^2)]\Big)(\delta^0_\nu-\delta^1_\nu)\Big\} \\
&= \Big(\delta^0_\mu\partial_0+\delta^1_\mu\partial_1+\delta^2_\mu\partial_2+\delta^3_\mu\partial_3\Big)\Big[q\ln(y^2+z^2)\delta(x-x^0)(-\delta^0_\nu+\delta^1_\nu)\Big] \\
&\quad -\Big(\delta^0_\nu\partial_0+\delta^1_\nu\partial_1+\delta^2_\nu\partial_2+\delta^3_\nu\partial_3\Big)\Big[q\ln(y^2+z^2)\delta(x-x^0)(-\delta^0_\mu+\delta^1_\mu)\Big] \\
&= \partial_\mu\mathcal{A}_\nu-\partial_\nu\mathcal{A}_\mu, \qquad (3.39)
\end{aligned}$$

where in the third equality, using (3.35) again, we put

$$\partial_\mu = \delta^0_\nu \partial_0 + \delta^1_\nu \partial_1 + \delta^2_\nu \partial_2 + \delta^3_\nu \partial_3. \tag{3.40}$$

Therefore the 4-potential $\mathcal{A}_\mu$ of a light-like charged particle is

$$\mathcal{A}_\mu = q \ln(y^2 + z^2)\delta(x - x^0)(-\delta^0_\mu + \delta^1_\mu). \tag{3.41}$$

On the other hand, if we try to calculate this potential by taking the ultra-relativistic limit of (3.24), then we get

$$\lim_{\beta \to 1} A^\mu = \frac{q}{|x - x^0|}(\delta^\mu_0 + \delta^\mu_1) \tag{3.42}$$

which is meaningless when $x - x^0 = 0$. It can be shown that this potential gives a vanishing field tensor for $x - x^0 \neq 0$ which is consistent with (3.31). However, the $x - x^0 = 0$ behavior is unclear in this gauge, so we perform a gauge transformation in order to be able to investigate the behavior at $x - x^0 = 0$. In the next section, when we discuss the gravitational analogue of this situation, we will see that this gauge freedom corresponds to a clever coordinate transformation there.

We perform the following transformation to the original 4-potential (3.17) before the Lorentz boost (3.22)

$$\tilde{A}^\mu = \bar{A}^\mu + \partial^\mu \chi = \bar{\phi}\delta^\mu_0 - \frac{q}{\sqrt{\bar{x}^2 + 1}}\delta^\mu_1 \tag{3.43}$$

with $\chi = q \sinh^{-1} \bar{x}$. Of course, $\tilde{A}$ yields the same electric and magnetic fields (3.20). After the Lorentz boost (3.22), (3.43) becomes

$$A^\mu = \frac{\partial x^\mu}{\partial \bar{x}^\nu}\tilde{A}^\nu = \frac{q(\delta^\mu_0 + \beta\delta^\mu_1)}{[(x - \beta x^0)^2 + \gamma^{-2}(y^2 + z^2)]^{1/2}} - \frac{q(\beta\delta^\mu_0 + \delta^\mu_1)}{[(x - \beta x^0)^2 + \gamma^{-2}]^{1/2}}. \tag{3.44}$$

This can also be written as

$$\begin{aligned}
A^\mu &= q\left\{\frac{1}{[(x - \beta x^0)^2 + \gamma^{-2}(y^2 + z^2)]^{1/2}} - \frac{1}{[(x - \beta x^0)^2 + \gamma^{-2}]^{1/2}}\right\}(\delta^\mu_0 + \beta\delta^\mu_1) \\
&\qquad + \frac{q(1 - \beta)}{[(x - \beta x^0)^2 + \gamma^{-2}]^{1/2}}(\delta^\mu_0 - \delta^\mu_1) \\
&= q\frac{\partial}{\partial x}\left[\ln\left(\frac{x - \beta x^0 + [(x - \beta x^0)^2 + \gamma^{-2}(y^2 + z^2)]^{1/2}}{x - \beta x^0 + [(x - \beta x^0)^2 + \gamma^{-2}]^{1/2}}\right)\right](\delta^\mu_0 + \beta\delta^\mu_1) \\
&\qquad + q\frac{1 - \beta^2}{[(x - \beta x^0)^2 + \gamma^{-2}]^{3/2}}\frac{(x - \beta x^0)^2 + \gamma^{-2}}{1 + \beta}(\delta^\mu_0 - \delta^\mu_1) \\
&= q\frac{\partial}{\partial x}\left[\ln\left(\frac{x - \beta x^0 + [(x - \beta x^0)^2 + \gamma^{-2}(y^2 + z^2)]^{1/2}}{x - \beta x^0 + [(x - \beta x^0)^2 + \gamma^{-2}]^{1/2}}\right)\right](\delta^\mu_0 + \beta\delta^\mu_1) \\
&\qquad + q\frac{\partial}{\partial x}\left[\frac{x - \beta x^0}{(x - \beta x^0)^2 + \gamma^{-2}}\right]\frac{(x - \beta x^0)^2 + \gamma^{-2}}{1 + \beta}(\delta^\mu_0 - \delta^\mu_1),
\end{aligned} \tag{3.45}$$

where in the last line we did the same trick as in (3.29). Now using

$$\lim_{\beta\to 1}\ln\left(\frac{x-\beta x^0+[(x-\beta x^0)^2+\gamma^{-2}(y^2+z^2)]^{1/2}}{x-\beta x^0+[(x-\beta x^0)^2+\gamma^{-2}]^{1/2}}\right)=[1-\theta(x-x^0)]\ln(y^2+z^2), \tag{3.46}$$

which is proved in Appendix B, we obtain

$$\begin{aligned}\mathcal{A}^\mu \equiv \lim_{\beta\to 1} A^\mu &= q\frac{\partial}{\partial x}[1-\theta(x-x^0)]\ln(y^2+z^2)(\delta^\mu_0+\delta^\mu_1) \\ &\quad + q\frac{\partial}{\partial x}[2\theta(x-x^0)-1]\frac{(x-x^0)^2}{2}(\delta^\mu_0-\delta^\mu_1) \\ &= -q\delta(x-x^0)\ln(y^2+z^2)(\delta^\mu_0+\delta^\mu_1)\end{aligned} \tag{3.47}$$

since $f(x-x^0)\delta(x-x^0)=0$. This is the same potential as (3.41) considering the covariant index.

3.2 AICHELBURG-SEXL LIMIT

In the previous section we have obtained the electromagnetic field of a null charged particle, i.e. of a point charge moving with the speed of light, and we have seen that the field has a delta function profile which means that all the field is compressed onto the transverse plane orthogonal to the direction of motion. The field is therefore an impulsive electromagnetic field. The gravitational analogue of this situation, i.e. a point mass moving with the speed of light, is studied by Aichelburg and Sexl [3]. They boosted the Schwarzschild metric which may describe the gravitational field of a point mass to a reference frame in which the mass appears to be moving with a constant speed and showed that in the ultra-relativistic limit the gravitational field of the point mass becomes a plane impulsive gravitational wave.

In this section we review this solution keeping the discussion in parallel with the previous section.

The gravitational field of a static spherically symmetric mass in general relativity is described by the well-known Schwarzschild solution which we write here in the coordinates $\tilde{x}^\mu=(c\bar{t},\tilde{r},\theta,\phi)$ (we keep all the relevant universal constants throughout this section):

$$ds^2=\left(1-\frac{2m}{\tilde{r}}\right)c^2d\bar{t}^2-\left(1-\frac{2m}{\tilde{r}}\right)^{-1}d\tilde{r}^2-\tilde{r}^2(d\theta^2+\sin^2\theta d\phi^2), \tag{3.48}$$

where $m\equiv GM/c^2$, with M being the mass of the source, is called the geometric mass. Defining a new radial coordinate by

$$\tilde{r}=\bar{r}\left(1+\frac{m}{2\bar{r}}\right)^2$$

and performing the transformation

$$\bar{x} = \bar{r}\sin\theta\cos\phi, \quad \bar{y} = \bar{r}\sin\theta\sin\phi, \quad \bar{z} = \bar{r}\cos\theta,$$

we can bring the metric (3.48) into the isotropic form

$$ds^2 = \frac{(1-A)^2}{(1+A)^2}c^2 d\bar{t}^2 - (1+A)^4(d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2), \tag{3.49}$$

where we have identified

$$A \equiv \frac{m}{2\bar{r}} = \frac{GM}{2c^2\bar{r}}, \quad \bar{r} = (\bar{x}^2 + \bar{y}^2 + \bar{z}^2)^{1/2}, \tag{3.50}$$

as in [3]. Now we perform the same Lorentz transformation in (3.22) with $x^0 = ct$:

$$ct = \gamma(c\bar{t} + \beta\bar{x}), \quad x = \gamma(\bar{x} + \beta c\bar{t}), \quad y = \bar{y}, \quad z = \bar{z}. \tag{3.51}$$

This is the transformation to a reference frame of an observer who sees the mass moving along the x-axis with the speed v. Then the metric (3.49) becomes

$$\begin{aligned} ds^2 &= \frac{(1-A)^2}{(1+A)^2}\gamma^2(cdt - \beta dx)^2 - (1+A)^4\left[\gamma^2(dx - \beta cdt)^2 + dy^2 + dz^2\right] \\ &= -\left[(1+A)^4 - \frac{(1-A)^2}{(1+A)^2}\right]\gamma^2(cdt - \beta dx)^2 \\ &\qquad + (1+A)^4\left[\gamma^2(cdt - \beta dx)^2 - \gamma^2(dx - \beta cdt)^2 - dy^2 - dz^2\right] \\ &= -\left[(1+A)^4 - \frac{(1-A)^2}{(1+A)^2}\right]\gamma^2(cdt - \beta dx)^2 + (1+A)^4\left[c^2dt^2 - dx^2 - dy^2 - dz^2\right] \end{aligned} \tag{3.52}$$

with, from (3.50),

$$A = \frac{GM/c^2}{2[\gamma^2(x-\beta ct)^2 + y^2 + z^2]^{1/2}} = \frac{(GE/c^4)\gamma^{-2}}{2[(x-\beta ct)^2 + \gamma^{-2}(y^2+z^2)]^{1/2}} = \frac{\mu\gamma^{-2}}{2R}, \tag{3.53}$$

where $E = \gamma Mc^2$ is the total energy of the particle, $\mu \equiv GE/c^4$ and

$$R \equiv [(x-\beta ct)^2 + \gamma^{-2}(y^2+z^2)]^{1/2}. \tag{3.54}$$

In the second line of (3.52), we added and subtracted the quantity $(1+A)^4\gamma^2(cdt - \beta dx)^2$.

In taking the limit $\beta \to 1$, since $\gamma \to \infty$, the total energy $E = \gamma Mc^2$ of the particle diverges, so to keep the energy (or μ through $\mu \equiv GE/c^4$) constant it is necessary to put the rest mass $M \to 0$ in this limit. But before that, let us write the factor of the first term of (3.52) as follows

$$\left[(1+A)^4 - \frac{(1-A)^2}{(1+A)^2}\right]\gamma^2 = \frac{\left(8A + 14A^2 + 20A^3 + 15A^4 + 6A^5 + A^6\right)\gamma^2}{(1+A)^2}. \tag{3.55}$$

Now with the help of (3.29) and (3.30), we see that

$$\begin{aligned}
\lim_{\beta\to 1} A &= \lim_{\beta\to 1} \frac{\mu\gamma^{-2}}{2R} = \lim_{\beta\to 1} \frac{\mu}{2}\frac{\gamma^{-2}}{R^3}R^2 = \lim_{\beta\to 1} \frac{\mu}{2(y^2+z^2)}\left[\frac{\partial}{\partial x}\left(\frac{x-\beta ct}{R}\right)\right]R^2 \\
&= \frac{\mu}{2(y^2+z^2)}\frac{\partial}{\partial x}[2\theta(x-ct)-1](x-ct)^2 \\
&= \frac{\mu}{(y^2+z^2)}(x-ct)^2\delta(x-ct) \\
&= 0 \qquad (3.56)
\end{aligned}$$

for all values of t, x, y and z, but

$$\lim_{\beta\to 1} A\gamma^2 = \frac{\mu}{2|x-ct|} \qquad (3.57)$$

for $x \neq ct$ and so

$$\lim_{\beta\to 1} A^n\gamma^2 = \begin{cases} \frac{\mu}{2|x-ct|}, & \text{for } n=1 \\ 0, & \text{for } n>1. \end{cases} \qquad (3.58)$$

Then, the ultra-relativistic limit of the metric (3.52) for $x \neq ct$ is

$$\lim_{\beta\to 1} ds^2 = -\frac{4\mu}{|x-ct|}(cdt-dx)^2 + c^2dt^2 - dx^2 - dy^2 - dz^2. \qquad (3.59)$$

But this is just the Minkowski spacetime in different coordinates. Indeed, with the null coordinates

$$u = \frac{1}{\sqrt{2}}(ct-x), \quad v = \frac{1}{\sqrt{2}}(ct+x), \qquad (3.60)$$

it takes the same form as the pp-wave metric (2.2), and from (2.4)

$$R_{\mu\nu\alpha\beta} = 0. \qquad (3.61)$$

The metric (3.59) is the analogue of (3.42). In the charged particle case in Section 3.1, we have performed the gauge transformation (3.43) in order to investigate the behavior of the 4-potential at $x = ct$. The corresponding thing here is to perform a coordinate transformation which enables us to see the gravitational field of the massless particle at $x = ct$ more apparently.

The appropriate coordinate transformation is the following [3]

$$\begin{aligned}
x' - \beta ct' &= x - \beta ct \\
x' + \beta ct' &= x + \beta ct - 4\mu \ln\left[\sqrt{(x-\beta ct)^2 + \gamma^{-2}} - (x-ct)\right] \qquad (3.62)
\end{aligned}$$

which leaves the coordinates y and z, and the function A invariant. This brings the metric (3.52) into the following form

$$\begin{aligned} ds^2 &= cdt'^2 - dx'^2 - dy'^2 - dz'^2 \\ &\quad - 4\mu \left\{ \frac{1}{[(x'-\beta ct')^2 + \gamma^{-2}(y'^2+z'^2)]^{1/2}} - \frac{1}{[(x'-\beta ct')^2 + \gamma^{-2}]^{1/2}} \right\} (cdt' - \beta dx')^2 \\ &\quad + O((1-\beta)) \\ &= cdt'^2 - dx'^2 - dy'^2 - dz'^2 \\ &\quad - 4\mu \frac{\partial}{\partial x'} \left[\ln \left(\frac{x'-\beta ct' + [(x'-\beta ct')^2 + \gamma^{-2}(y'^2+z'^2)]^{1/2}}{x'-\beta ct' + [(x'-\beta ct')^2 + \gamma^{-2}]^{1/2}} \right) \right] (cdt' - \beta dx')^2 \\ &\quad + O((1-\beta)). \end{aligned} \tag{3.63}$$

Now using the relation (3.46), we can take the ultra-relativistic limit, which yields

$$ds'^2 \equiv \lim_{\beta\to 1} ds^2 = cdt'^2 - dx'^2 - dy'^2 - dz'^2 + 4\mu \ln(y'^2 + z'^2)\delta(x' - ct')(cdt' - dx')^2. \tag{3.64}$$

Defining the null coordinates

$$u' = ct' - x', \quad v' = ct' + x', \tag{3.65}$$

we can write

$$ds'^2 = du'dv' + 4\mu \ln(y'^2 + z'^2)\delta(u')du'^2 - dy'^2 - dz'^2. \tag{3.66}$$

But if we define the null coordinates as in (2.3), i.e.

$$u' = \frac{1}{\sqrt{2}}(ct' - x'), \quad v' = \frac{1}{\sqrt{2}}(ct' + x'), \tag{3.67}$$

then using $\delta(au) = \delta(u)/|a|$ we get

$$ds'^2 = 2du'dv' + 4\bar{\mu} \ln(y'^2 + z'^2)\delta(u')du'^2 - dy'^2 - dz'^2 \tag{3.68}$$

with $\bar{\mu} \equiv \sqrt{2}\mu = \sqrt{2}GE/c^4$. So the energy parameter μ of the particle should be scaled appropriately depending on the definition of the null coordinates.

The metric (3.68), or (3.66), describes the gravitational field of a massless particle moving along the x-axis with the speed of light in Minkowski spacetime. It has the form of a gravitational plane-fronted wave (pp-wave) which we have discussed in Chapter 2. The profile function of the wave is

$$H(u', y', z') = 4\bar{\mu} \ln(y'^2 + z'^2)\delta(u') \tag{3.69}$$

and so with the help of the expressions (2.4) we can calculate

$$\begin{aligned}
R_{u'y'u'y'} &= -R_{u'z'u'z'} = 4\bar{\mu}\frac{y'^2 - z'^2}{(y'^2+z'^2)^2}\delta(u'),\\
R_{u'y'u'z'} &= 8\bar{\mu}\frac{y'z'}{(y'^2+z'^2)^2}\delta(u'),\\
R_{u'u'} &= 8\pi\bar{\mu}\delta(u')\delta(y')\delta(z'),\\
C_{u'y'u'y'} &= 4\bar{\mu}\frac{y'^2 - z'^2}{(y'^2+z'^2)^2}\delta(u') + 4\pi\bar{\mu}\delta(u')\delta(y')\delta(z'),\\
C_{u'z'u'z'} &= -4\bar{\mu}\frac{y'^2 - z'^2}{(y'^2+z'^2)^2}\delta(u') + 4\pi\bar{\mu}\delta(u')\delta(y')\delta(z'),\\
C_{u'y'u'z'} &= 8\bar{\mu}\frac{y'z'}{(y'^2+z'^2)^2}\delta(u'),
\end{aligned} \tag{3.70}$$

where we have used (3.37). These are the physical quantities like the electromagnetic field tensor (3.31) in Section 3.1. Due to the Dirac delta function the curvature represented by the Riemann tensor is concentrated only on the null plane $u' = 0$ and moves along the x-axis with the speed of light. Therefore this solution describes an impulsive gravitational plane wave, or a *shock* wave, generated by a null point particle propagating in Minkowski spacetime. Indeed, using Einstein's field equations we can show that the spacetime described by the metric (3.68) is sourced by a null particle: Since the metric is in the pp-wave form, the Ricci scalar is zero and therefore the Einstein tensor is identical to the Ricci tensor given in (3.70), so the field equations read

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = R_{u'u'}\delta^{u'}_{\mu}\delta^{u'}_{\nu} = 8\pi\bar{\mu}\delta(u')\delta(y')\delta(z')\delta^{u'}_{\mu}\delta^{u'}_{\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{3.71}$$

which gives

$$T_{\mu\nu} = \frac{c^4\bar{\mu}}{G}\delta(u')\delta(y')\delta(z')\delta^{u'}_{\mu}\delta^{u'}_{\nu} = \sqrt{2}E\delta(u')\delta(y')\delta(z')\delta^{u'}_{\mu}\delta^{u'}_{\nu}, \tag{3.72}$$

where we remember $\bar{\mu} \equiv \sqrt{2}\mu = \sqrt{2}GE/c^4$. Again, the factor $\sqrt{2}$ comes from the definition of the null coordinates (3.67). This is the energy-momentum tensor of a null point particle moving along the x-axis, which can also be obtained directly from transforming the energy-momentum tensor of a static particle under the Lorentz transformations (3.51) and then taking the ultra-relativistic limit $\beta \to 1$, keeping the energy of the particle constant.

3.3 IMPULSIVE WAVES

In this section we generalize the Aichelburg-Sexl solution of the previous section to arbitrary shock wave geometries.

Shock waves belong to the class of pp-waves (2.2) that we have discussed in Chapter 2 with the profile function

$$H(u, X, Y) = f(X, Y)\delta(u) \tag{3.73}$$

for some arbitrary function $f(X, Y)$, so the metric of a shock wave can be written as

$$ds^2 = 2dudv + f(X, Y)\delta(u)du^2 - dX^2 - dY^2 \tag{3.74}$$

in terms of the null coordinates,

$$u = \frac{1}{\sqrt{2}}(t - Z), \quad v = \frac{1}{\sqrt{2}}(t + Z). \tag{3.75}$$

In this section we put $c = G = 1$. The metric (3.74) represents an impulsive plane-fronted wave traveling along the Z-axis with arbitrary profile $f(X, Y)$. For this metric the field components (2.4) becomes

$$\begin{aligned} R_{uiuj} &= -\frac{1}{2}(\partial_i\partial_j f)\delta(u), \\ R_{uu} &= \frac{1}{2}(\nabla_\perp^2 f)\delta(u), \\ C_{uiuj} &= -\frac{1}{2}(\partial_i\partial_j f)\delta(u) + \frac{1}{4}\delta_{ij}(\nabla_\perp^2 f)\delta(u), \end{aligned} \tag{3.76}$$

where, as before, $\partial_i = (\partial/\partial X, \partial/\partial Y)$ and $\nabla_\perp^2 \equiv \sum_i \partial_i^2$. Then, the form of the profile function is determined by the Einstein's field equations which reduce to a Poisson equation in the transverse space:

$$\frac{1}{2}(\nabla_\perp^2 f)\delta(u) = 8\pi T_{uu}. \tag{3.77}$$

Here we should recall that in geometrized units ($c = G = 1$) the energy-momentum tensor has units $[T_{\mu\nu}] = L^{-2}$, and so from (3.77) $[f] = L$ since $[\delta(u)] = L^{-1}$.

Let us give some examples.

Impulsive vacuum pp-waves: In vacuum, $T_{\mu\nu} = 0$, so (3.77) reduces to the Laplace equation in the transverse coordinates

$$\nabla_\perp^2 f = 0. \tag{3.78}$$

Therefore, in this case any harmonic function $f(X, Y)$ describes a gravitational shock wave with the metric (3.74). This is a subclass of vacuum pp-waves discussed in Chapter 2.

Null point particle: If the source is a null spinless particle represented by the energy-momentum tensor

$$T_{uu} = \bar{\mu}\delta(u)\delta(X)\delta(Y), \tag{3.79}$$

where $\bar{\mu}$, being the energy of the particle, has units $[\bar{\mu}] = L$, then (3.77) becomes

$$\nabla_{\perp}^2 f = 16\pi\bar{\mu}\delta(X)\delta(Y), \tag{3.80}$$

from which, with the help of the Greens' function in (3.37),

$$f(X, Y) = 4\bar{\mu} \ln\left(\frac{X^2 + Y^2}{\ell^2}\right), \tag{3.81}$$

where ℓ is an arbitrary integration constant having the units of length. This is just the Aichelburg-Sexl shock wave solution derived in the previous section.

Null shell of matter: Instead of a point particle, it is also possible to consider a null infinite homogeneous shell of matter which has the energy-momentum tensor

$$T_{uu} = \sigma\delta(u), \tag{3.82}$$

where $\sigma = const.$ is the surface energy density having units $[\sigma] = L^{-1}$. Therefore, the field equation (3.77) becomes

$$\nabla_{\perp}^2 f = 16\pi\sigma \tag{3.83}$$

which yields

$$f(X, Y) = 4\pi\sigma\left(X^2 + Y^2\right). \tag{3.84}$$

This solution represents a gravitational shock wave produced by a planar shell of null matter with constant energy density, and was first given by Dray and 't Hooft [27].

CHAPTER 4

THE EFFICIENCY OF THE GRAVITATIONAL RADIATION EMITTED IN HIGH-ENERGY COLLISIONS

In this chapter we review and compare three different methods for estimating the efficiency of the total gravitational energy emitted in high-energy collisions.

4.1 HAWKING'S AREA THEOREM

In 1971, Hawking [8] found an upper limit for the efficiency of gravitational radiation emitted in a collision process of two black holes.

In this section we will review Hawking's reasoning for the case of two colliding black holes, each of which is neutral and non-rotating, i.e. a Schwarzschild black hole, and show that the upper bound on the energy radiated away is 29% of the initial energy of the holes.

If the cosmic censorship hypothesis holds, Hawking's area theorem states that

Theorem: During the evolution of any black hole, the area of the event horizon can never decrease, but only increase or stay the same.

For example, if two black holes collide and unite, the surface area of the event horizon of the final black hole must be greater than the sum of the surface areas of the initial black holes. Let us call the areas of the initial black holes A_1 and A_2, and the area of the final black hole A_f. Then, according to the area theorem,

$$A_f \geq A_1 + A_2. \tag{4.1}$$

Now remembering the well-known relation between the area of the event horizon of a black hole and its mass

$$A = 4\pi r^2 = 16\pi m^2, \tag{4.2}$$

where $r = 2m$ is the radius of the horizon and m is the geometric mass of the black hole (see Section 3.2), we can write (4.1) as

$$m_f^2 \geq m_1^2 + m_2^2. \tag{4.3}$$

Assuming the masses of the black holes to be equal, namely $m_1 = m_2 = m$, we get

$$m_f \geq \sqrt{2}m = \frac{\sqrt{2}}{2}(2m) \simeq 0.71(2m). \tag{4.4}$$

This is the lower bound for the mass of the final black hole. It cannot be less than 71% of the initial total mass. Therefore, the upper bound for the energy radiated should be

$$m_{rad} = 2m - m_f \leq \left(1 - \frac{\sqrt{2}}{2}\right)2m \simeq 0.29(2m), \tag{4.5}$$

which means that a maximum of 29% of the initial energy could be radiated in the collision.

Note that in this derivation there is no assumption on the velocities of the colliding black holes. It is therefore valid in general. However, in the next section, using the trapped surface method, we will show that this bound also comes out in the collision of two black holes moving with the speed of light.

4.2 TRAPPED SURFACE METHOD

Black hole formation and outgoing gravitational radiation in high-energy collisions can also be studied by just proving the existence of a closed trapped surface in the future of the collision geometry. There is no need to know the solution beyond the collision. This method was first applied by Penrose [9] to the head-on, i.e. zero impact parameter $b = 0$, collision of two ultra-relativistic black holes, and later extended by Eardley and Giddings [28] to non-zero impact parameters $b > 0$. Modeling the spacetime as the union of two Aichelburg-Sexl shock waves, Penrose was able to find a closed trapped surface at the moment of the collision, and give an upper limit of about 29% of the initial energy of the spacetime radiated in gravitational waves.

A closed trapped surface is defined as a compact two-dimensional spacelike surface whose outgoing normal null geodesic congruence has negative expansion everywhere on the surface.

Physically, what this means is that there is a closed surface whose normal light rays (in both future-directed directions) converge, so are trapped by gravity. In the special case when the expansion is zero, the surface is said to be marginally trapped, and the outermost marginally trapped surface is called the apparent horizon.

The existence of a marginally trapped surface in a spacetime implies the existence of a singularity in the future. Since an apparent horizon is the outermost marginally trapped surface, this also implies the existence of an apparent horizon outside the marginally trapped surface in the spacetime. If the cosmic censorship conjecture holds and the null energy condition is satisfied, an apparent horizon in turn implies the presence of an event horizon exterior to it, or coincident with it in a stationary spacetime (for a formal proof see [7]). Therefore, the existence of an apparent horizon is a sufficient condition for the black hole formation, and the area of the apparent horizon can be used to put a lower bound on the area of the produced black hole which then provides a lower bound on the mass of the black hole.

Let us apply this procedure to the collision of two general shock waves. As we have discussed in Section 3.3, a shock wave moving in the v direction can be given by the metric (here in polar coordinates on the transverse plane)

$$ds^2 = 2d\bar{u}d\bar{v} + f(\bar{\rho})\delta(\bar{u})d\bar{u}^2 - d\bar{\rho}^2 - \bar{\rho}^2 d\bar{\phi}^2, \tag{4.6}$$

where $f(\bar{\rho})$ is the wave profile, and for a Aichelburg-Sexl wave of Section 3.2 it is

$$f(\bar{\rho}) = 4\bar{\mu}\ln\bar{\rho}^2 \tag{4.7}$$

with $\bar{\mu} = \sqrt{2}E$. The delta function in (4.6) indicates that the coordinates $\bar{x}^\mu = (\bar{u}, \bar{v}, \bar{\rho}, \bar{\phi})$ are discontinuous at $\bar{u} = 0$. Later, we will be interested in the null geodesics crossing the hypersurface $\bar{u} = 0$ to calculate the expansion. It is therefore convenient to pass to the continuous and smooth coordinates $x^\mu = (u, v, \rho, \phi)$ via the coordinate transformation (see Appendix C)

$$\bar{u} = u, \quad \bar{v} = v - \frac{1}{2}\theta(u)f(\rho) + \frac{1}{8}u\theta(u)\left[f'(\rho)\right]^2, \quad \bar{\rho} = \rho - \frac{1}{2}u\theta(u)f'(\rho), \quad \bar{\phi} = \phi. \tag{4.8}$$

The metric in these coordinates takes the form

$$ds^2 = 2dudv - \left[1 - \frac{1}{2}u\theta(u)f''\right]^2 d\rho^2 - \left[1 - \frac{1}{2}u\theta(u)\frac{f'}{\rho}\right]^2 \rho^2 d\phi^2. \tag{4.9}$$

For a Aichelburg-Sexl shock wave, since $f(\bar{\rho}) = 4\bar{\mu}\ln\bar{\rho}^2$, (4.9) becomes

$$ds^2 = 2dudv - \left[1 + 4\bar{\mu}\frac{u\theta(u)}{\rho^2}\right]^2 d\rho^2 - \left[1 - 4\bar{\mu}\frac{u\theta(u)}{\rho^2}\right]^2 \rho^2 d\phi^2. \tag{4.10}$$

Now we can set up the collision of two such identical Aichelburg-Sexl shock waves as follows. The spacetime structure is as in Figure 2.1. Region I is the flat background, region II contains one of the approaching shocks described by the continuous metric (4.10), and region III contains the opposite shock described again by (4.10) but with u and v interchanged. Region IV is the interaction region after the collision. Because the shocks propagate at the speed of light, they do not interact with each other before the collision (by causality). Then we can construct the precollision metric, which describes the geometry outside the future light cone of the collision event (regions I, II, and III), simply by combining the metrics of the right and left shocks:

$$\begin{aligned} ds^2 &= 2dudv - \left[1 + 8\bar{\mu}\frac{u\theta(u) + v\theta(v)}{\rho^2} + 16\bar{\mu}^2\frac{u^2\theta(u) + v^2\theta(v)}{\rho^4}\right]d\rho^2 \\ &\quad - \left[1 - 8\bar{\mu}\frac{u\theta(u) + v\theta(v)}{\rho^2} + 16\bar{\mu}^2\frac{u^2\theta(u) + v^2\theta(v)}{\rho^4}\right]\rho^2 d\phi^2, \end{aligned} \tag{4.11}$$

where $\bar{\mu} = \sqrt{2}E$.

Nonlinearities in the field equations prevent us from obtaining the metric in the interaction region $u > 0$, $v > 0$. However, it is still possible to confirm the black hole formation by proving the existence of an apparent horizon on some slice in regions I, II, and III. We first need to find the null geodesics normal to some surface in this slice, and then impose that their expansion is zero.

We will look for the apparent horizon on the slice $v \leq 0 = u$ and $u \leq 0 = v$ of Figure 2.1. Let us assume that the apparent horizon is given by the surface S which is made up of the union of two pieces $S = S_1 \cup S_2$. The first piece S_1 lies in the null hypersurface $v \leq 0 = u$, while the second piece S_2 lies in $u \leq 0 = v$. These surfaces $S_{1,2}$ are connected on a common one-dimensional boundary C which lies in the intersection $u = v = 0$. Because the system is axially symmetric, the surfaces $S_{1,2}$ are given by functions of ρ only. Therefore, S is composed of

$$\begin{aligned} &S_1 : \{v = -\Psi_1(\rho),\ u = 0\},\ \text{with } \Psi_1 = 0 \text{ on } C, \\ &S_2 : \{u = -\Psi_2(\rho),\ v = 0\},\ \text{with } \Psi_2 = 0 \text{ on } C. \end{aligned} \tag{4.12}$$

Since S_1 and S_2 lie in the regions $v < 0$ and $u < 0$ respectively, it should also be that $\Psi_1(\rho) > 0$ and $\Psi_2(\rho) > 0$. These two functions $\Psi_1(\rho)$ and $\Psi_2(\rho)$ have to be determined by imposing the condition that the null geodesic congruence orthogonal to the surface they define has zero expansion. Because we are considering two identical Aichelburg-Sexl shock waves in our setup, we can simply put $\Psi_1 = \Psi_2 = \Psi$. Also, since we are working in the center of mass frame, by the left-right symmetry we can consider only the surface S_1 in $v \leq 0 = u$. The metric in the neighborhood of $v \leq 0 = u$ is given by (4.9) (or explicitly by (4.10)). In the barred coordinates $\bar{x}^\mu = (\bar{u}, \bar{v}, \bar{\rho}, \bar{\phi})$ of the metric (4.6), the surface S_1 is located at, from (4.8),

$$\bar{v} = -\Psi(\bar{\rho}) - \frac{1}{2} f(\bar{\rho}) \equiv -h(\bar{\rho}), \quad \bar{u} = 0, \tag{4.13}$$

since $\bar{\rho} = \rho$ on $u = 0$, and $\theta(0) = 1$. For this surface to define an apparent horizon, it should be continuous and smooth at all values of $\bar{\rho}$.

Now we need to calculate the normal null geodesic congruence (affinely parameterized) passing through the surface $S_1 : \{v = -\Psi_1(\rho),\ u = 0\}$. We follow the review section of [29]. The tangent vector ${\xi_1}^\mu$ of the congruence is

$${\xi_1}^\mu = \frac{dx^\mu}{d\lambda} = (\dot{u}, \dot{v}, \dot{\rho}, \dot{\phi}), \tag{4.14}$$

since we have the coordinates $x^\mu = (u, v, \rho, \phi)$ in the spacetime of the metric (4.9). Here λ is the affine parameter. Since the surface S_1 lies in the null hypersurface $u = 0$, it is convenient to define the coordinates $y^a = (\rho, \phi)$ on S_1. Then the tangent generators $e^\mu_a = \frac{\partial x^\mu}{\partial y^a}$ of the surface (see [30]) become

$$e^\mu_\rho = \frac{\partial x^\mu}{\partial \rho} = (0, -\Psi_1', 1, 0),$$
$$e^\mu_\phi = \frac{\partial x^\mu}{\partial \phi} = (0, 0, 0, 1), \tag{4.15}$$

where the prime denotes derivative with respect to ρ. We have to impose the conditions that the geodesics are null, and normal to the generators of the surface, i.e.

$${\xi_1}_\mu {\xi_1}^\mu = 0, \quad {\xi_1}_\mu e^\mu_a = 0. \tag{4.16}$$

We can also impose the normalization condition

$${\xi_1}_\mu k^\mu = 1, \tag{4.17}$$

where $k_\mu = \partial_\mu u = \delta^0_\mu$ is the normal vector to the null hypersurface $u = 0$ and defines the auxiliary null vector field (see [30]). These conditions (4.16) and (4.17) together with the metric (4.9) enable us to calculate the tangent vector ξ^μ as

$${\xi_1}^\mu = \left(1, \frac{\Psi_1'^2}{2}, -\Psi_1', 0\right). \tag{4.18}$$

This is the null normal vector to the surface S_1 on $u = 0$, and satisfies the geodesics equation in its affinely parameterized form, i.e.

$$\xi_1{}^{\mu}\nabla_{\mu}\xi_{1\nu} = 0, \tag{4.19}$$

with $\lambda = u$ being the affine parameter.

Similarly, the tangent vector $\xi_2{}^{\mu}$ of the null geodesic congruence which is normal to the surface $S_2 : \{u = -\Psi_2(\rho),\ v = 0\}$ is

$$\xi_2{}^{\mu} = \left(\frac{\Psi_2'^2}{2}, 1, -\Psi_2', 0\right). \tag{4.20}$$

The outer null normal to the apparent horizon S must be continuous across the intersection C at $\rho = \rho_c$. Otherwise, there would be a delta function in the expansion. This means that the tangent vectors $\xi_1{}^{\mu}$ and $\xi_2{}^{\mu}$ should be parallel at $\rho = \rho_c$, and so the condition $\xi_1{}^{u}\xi_2{}^{v} = \xi_1{}^{v}\xi_2{}^{u}$ holds. From (4.18) and (4.20) this is equivalent to

$$\Psi_1'(\rho_c)\Psi_2'(\rho_c) = 2. \tag{4.21}$$

As we said before, our shock waves are identical, i.e. $\Psi_1(\rho) = \Psi_2(\rho) = \Psi(\rho)$, so this condition becomes

$$\Psi'^2(\rho_c) = 2. \tag{4.22}$$

Then, due to the left-right symmetry of the system, considering the left part only we can calculate the expansion with the help of (4.9) and (4.18) (see [30] for the definition of expansion):

$$\theta|_{u=0} = \nabla_{\mu}\xi_1{}^{\mu} = \frac{1}{\sqrt{-g}}(\sqrt{-g}\xi_1{}^{\mu})_{,\mu} = -\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho}\right)\left(\Psi + \frac{f}{2}\right), \tag{4.23}$$

where $f(\rho) = 4\bar{\mu}\ln\rho^2$. As we discussed in (4.13), the function $\Psi(\rho) + f(\rho)/2$ defines the apparent horizon in the barred coordinates ($\bar{\rho} = \rho$ on $u = \bar{u} = 0$), so it should be continuous and smooth at all ρ. Since the apparent horizon is defined as the surface for which the expansion is zero, from (4.23) we get

$$\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho}\right)\left(\Psi + \frac{f}{2}\right) = 0 \tag{4.24}$$

which has the general solution

$$\Psi + \frac{f}{2} = A + B\ln\rho, \tag{4.25}$$

where A and B are two integration constants. Since we are interested in the solution without singularities, we put $B = 0$, and using the condition $\Psi(\rho_c) = 0$ we obtain

$$\Psi(\rho) = -4\bar{\mu}\ln\frac{\rho}{\rho_c}. \tag{4.26}$$

This function describes the shape of the apparent horizon surface in the continuous coordinates $x^\mu = (u, v, \rho, \phi)$. The condition (4.22) determines the radius of the boundary circle C:

$$\Psi'^2(\rho_c) = 2 \quad \Rightarrow \quad \rho_c = 2\sqrt{2}\bar{\mu}. \tag{4.27}$$

Therefore, in the barred coordinates $\bar{x}^\mu = (\bar{u}, \bar{v}, \bar{\rho}, \bar{\phi})$ the apparent horizon consists of two discs

$$\begin{aligned} D_1 &: \left\{ \bar{v} = -\Psi(\bar{\rho}) - \frac{1}{2} f(\bar{\rho}) = -4\bar{\mu} \ln \rho_c,\ \bar{u} = -\frac{1}{2}\theta(v) f + \frac{1}{8} v\theta(v) f'^2 \right\}, \\ D_2 &: \left\{ \bar{u} = -\Psi(\bar{\rho}) - \frac{1}{2} f(\bar{\rho}) = -4\bar{\mu} \ln \rho_c,\ \bar{v} = -\frac{1}{2}\theta(u) f + \frac{1}{8} u\theta(u) f'^2 \right\}, \end{aligned} \tag{4.28}$$

since $\bar{\rho} = \rho$ on $u = 0$ and $v = 0$. These two discs intersect each other at

$$D_1 \cap D_2 : \{\bar{u} = \bar{v} = -4\bar{\mu} \ln \rho_c,\ \rho = \rho_c\}, \tag{4.29}$$

where the boundary circle C lies in the barred coordinates. It can be seen from (4.6) that the metric on these $\bar{u} = \bar{v} = const.$ discs is just the flat disc metric, i.e.

$$ds_D^2 = -d\bar{\rho}^2 - \bar{\rho}^2 d\bar{\phi}^2. \tag{4.30}$$

This means that the area of the apparent horizon in the barred coordinates is just the area of two flat discs of radius $\bar{\rho}_c = \rho_c$ given by (4.27), namely,

$$A(S) = 2A_D = 2\pi\rho_c^2 = 16\pi\bar{\mu}^2. \tag{4.31}$$

As we mentioned in the beginning of this section, this puts a lower limit on the area of the event horizon which either lies outside the apparent horizon or coincides with it. So remembering the area of the event horizon of a black hole given in (4.2) it can be found that

$$A_b = 4\pi r_b^2 \geq A(S) = 16\pi\bar{\mu}^2 \quad \Rightarrow \quad r_b \geq 2\bar{\mu}, \tag{4.32}$$

from which we get

$$m_b = \frac{r_b}{2} \geq \frac{1}{2}(2\bar{\mu}) = \frac{\sqrt{2}}{2}(2E) \simeq 0.71(2E), \tag{4.33}$$

where we put $\bar{\mu} = \sqrt{2}E$. This is the same bound on the mass of the final black hole as (4.4) obtained by the Hawking's area argument in the previous section. Therefore, both Hawking's and Penrose's methods predict an efficiency of about 29% for the gravitational radiation emitted in such collision processes.

4.3 INSTANTANEOUS COLLISION METHOD

In this section, we review a different method which enables us to obtain quantitative results on the gravitational energy released in the collision processes of particles. Whenever two particles collide, due to the changes in their momenta, gravitational radiation is released into the spacetime. In general, the treatment of the gravitational radiation in general relativity is a difficult problem due to the nonlinearity of the field equations. However, it is sometimes enough to work with the linearized form of the Einstein field equations in the weak field limit in order to gain some insight on the weak radiation emitted in the collision of two particles. This method was first applied in [31, 32].

The formalism is reviewed in Appendix D. We have obtained there the radiation spectrum in linearized gravity, which is given by

$$\frac{d^2E}{d\Omega d\omega} = \frac{G\omega^2}{2\pi^2}\left[T^*_{\mu\nu}(\omega,\mathbf{k})T^{\mu\nu}(\omega,\mathbf{k}) - \frac{1}{2}|T(\omega,\mathbf{k})|^2\right], \tag{4.34}$$

where $T_{\mu\nu}(\omega,\mathbf{k})$ is the Fourier transform of the energy-momentum tensor of the source. For a system of colliding free particles, we also obtained in the appendix that

$$T^{\mu\nu}(\omega,\mathbf{k}) = -i\sum_n\left[\frac{P^\mu_n P^\nu_n}{E_n(\omega - \mathbf{k}\cdot\mathbf{v}_n)} - \frac{P'^\mu_n P'^\nu_n}{E'_n(\omega - \mathbf{k}\cdot\mathbf{v}'_n)}\right]. \tag{4.35}$$

Here $\mathbf{k} = \omega\hat{\mathbf{x}}$ is the wave vector which represents the direction of the wave, and E_n and P^μ_n are the energy and 4-momentum of the nth particle defined by

$$E_n = \gamma_n M_n, \quad \gamma_n = \frac{1}{\sqrt{1-v_n^2}}, \quad P^\mu_n = (E_n, \mathbf{P}_n) = E_n(1, \mathbf{v}_n). \tag{4.36}$$

Let us consider the particular case in which two particles collide head-on to form a final particle at rest in the center of mass frame. One of the particle has mass M_1 and Lorentz factor γ_1, and the other particle has the corresponding quantities M_2 and γ_2. Without loss of generality, we may orient the axes so that the motion occurs in the (y,z) plane and the z-axis is the radiation direction, i.e. $\mathbf{k} = \omega\hat{\mathbf{z}}$. Then we have

$$\begin{aligned}
P^\mu_1 &= \gamma_1 M_1(1, 0, v_1\sin\theta, v_1\cos\theta),\\
P^\mu_2 &= \gamma_2 M_2(1, 0, -v_2\sin\theta, -v_2\cos\theta),\\
P'^\mu_1 &= E'_1(1,0,0,0) = \gamma_1 M_1(1,0,0,0),\\
P'^\mu_2 &= E'_2(1,0,0,0) = \gamma_1 M_1(1,0,0,0),
\end{aligned} \tag{4.37}$$

where θ is the angle between $\mathbf{v}_1$ and $\mathbf{k}$. Here the momenta satisfy the relativistic relation

$$P_\mu P^\mu = E^2 - \mathbf{P}^2 = M^2. \tag{4.38}$$

Also conservation of momentum requires that $\gamma_1 M_1 v_1 = \gamma_2 M_2 v_2$. Substituting (4.37) in (4.35) and then using (4.34), we find

$$\frac{d^2E}{d\Omega d\omega} = \frac{G}{4\pi^2}\frac{\gamma_1^2 M_1^2 v_1^2 (v_1 + v_2)^2 \sin^4\theta}{(1 - v_1\cos\theta)^2(1 - v_2\cos\theta)^2}. \tag{4.39}$$

Now consider the special case in which $M_1 = M_2 = M$ and $v_1 = v_2 = v$. Then (4.39) becomes

$$\begin{aligned}\frac{d^2E}{d\Omega d\omega} &= \frac{G}{\pi^2}\frac{\gamma^2 M^2 v^4 \sin^4\theta}{(1 - v^2\cos^2\theta)^2} \\ &= \frac{G}{\pi^2}\frac{E^2 P^4 \sin^4\theta}{(E^2 - P^2\cos^2\theta)^2} \\ &= \frac{G}{\pi^2}\frac{E^2 P^4 \sin^4\theta}{(M^2 + P^2\sin^2\theta)^2},\end{aligned} \tag{4.40}$$

where we have used $E = \gamma M$, $P = Ev$ and $E^2 = M^2 + P^2$. This form of the energy distribution is very useful because the ultra-relativistic limit $v \to 1$ can be taken immediately. Therefore, for the collision of two massless ($M = 0$) particles, the distribution of the radiation is isotropic and very simple: From (4.40)

$$\frac{d^2E}{d\Omega d\omega} = \frac{GE^2}{\pi^2}. \tag{4.41}$$

If we now try to calculate the total emitted energy by integrating (4.41) with respect to ω from 0 to ∞, we get a result that diverges like $\int^\infty d\omega$. This stems from our implicit assumption that the collision occurs instantaneously at $t = 0$ (see Appendix D). In actuality, however, it must take place in a finite time interval $\triangle t$, and so the ω-integral above should be cut off at some ω_c of order $1/\triangle t$. Thus, the total energy radiated away is

$$E_T = \frac{4GE^2\omega_c}{\pi}. \tag{4.42}$$

Assuming a spherical black hole is produced by the collision, we can calculate the efficiency of the total radiated energy from (4.42). Since the effective timescale for the process is $\triangle t \sim r_b$, where $r_b = 2GM_b$ is the horizon radius of the black hole with mass $M_b \sim 2E$ in units $c = 1$, the cutoff frequency ω_c will go as the inverse of the black hole radius r_b; that is,

$$\omega_c \sim \frac{1}{\triangle t} \sim \frac{1}{r_b} = \frac{1}{4GE}. \tag{4.43}$$

Then the efficiency for the gravitational radiation is

$$\epsilon = \frac{E_T}{2E} \sim \frac{1}{2\pi} \simeq 0.16, \tag{4.44}$$

where $2E$ is the total initial energy. Therefore, the instantaneous collision method predicts that 16% of the initial energy is radiated away in the collision process. This result is in agreement with the estimate of D'Eath and Payne [10], but says that Hawking's area theorem and the trapped surface method, which give a 29% efficiency, overestimate the total energy emitted in the collision.

CHAPTER 5

BLACK HOLE COLLISIONS AT THE SPEED OF LIGHT

In this chapter, we consider the collision of two shock waves produced by two null particles and try to extract the total energy in gravitational waves using the Bondi news function. In order to understand how to extract the news function in a collision process, we will perform a perturbation expansion with respect to the energy parameters of the particles which are assumed to be equal and small. In the next chapter we will apply a different method.

5.1 SETUP

As we have obtained in Chapter 3, the gravitational field of a null particle is given by the metric (3.68). Here we write it in the coordinates $x^\mu = (U, V, X, Y)$, i.e.

$$ds^2 = 2dUdV + 4\bar{\mu}\ln(X^2 + Y^2)\delta(U)dU^2 - dX^2 - dY^2, \tag{5.1}$$

where $\bar{\mu} = \sqrt{2}E$ is the parameter related to the energy of the particle, and U and V are the null coordinates defined as in (2.3),

$$U = \frac{1}{\sqrt{2}}(t - Z), \quad V = \frac{1}{\sqrt{2}}(t + Z). \tag{5.2}$$

Throughout this chapter we work in units $c = G = 1$. This metric is an exact solution to the full Einstein equations with the source being a point particle moving with the speed of light along the Z-axis. As we have mentioned earlier, this form of the metric is not suitable for the collision of such particles. This is due to the fact that the Dirac delta function in (5.1) makes the metric discontinuous at $U = 0$, which means that the discussion of the null geodesics through the hypersurface $U = 0$ is not easy in these coordinates. Therefore, it is necessary to transform (5.1) into the Rosen coordinates which is always continuous. This is achieved by the following discontinuous coordinate transformation (see Appendix C):

$$\begin{aligned} U &= u, \\ V &= v - 4\bar{\mu}\ln\rho\theta(u) + 8\bar{\mu}^2\frac{u\theta(u)}{\rho^2}, \\ X &= \left[1 - 4\bar{\mu}\frac{u\theta(u)}{\rho^2}\right]\rho\cos\phi, \\ Y &= \left[1 - 4\bar{\mu}\frac{u\theta(u)}{\rho^2}\right]\rho\sin\phi, \end{aligned} \tag{5.3}$$

which brings the metric (5.1) into the form [33]

$$ds^2 = 2dudv - \left[1 + 4\bar{\mu}\frac{u\theta(u)}{\rho^2}\right]^2 d\rho^2 - \left[1 - 4\bar{\mu}\frac{u\theta(u)}{\rho^2}\right]^2 \rho^2 d\phi^2. \tag{5.4}$$

This form of the metric is now continuous across the null hypersurface $u = 0$.

In order to study the collision of two null particles described by the metric (5.4), we construct the problem as in Figure 2.1 in Section 2.2. We assume that the collision takes place in four-dimensional Minkowski spacetime which is the region I in Figure 2.1, and the regions II and III in the figure contain the approaching particles represented in the coordinates (5.4). Region IV is the interaction region. Therefore, we assume, from (5.4),

$$ds^2 = 2dudv - \left[1 + 4\bar{\mu}_1\frac{u\theta(u)}{\rho^2}\right]^2 d\rho^2 - \left[1 - 4\bar{\mu}_1\frac{u\theta(u)}{\rho^2}\right]^2 \rho^2 d\phi^2 \tag{5.5}$$

represents the right-moving shock wave which propagates along the v direction in Figure 2.1. Here $\bar{\mu}_1 = \sqrt{2}E_1$ is the energy of the null particle. This metric describes globally both region I and region II. The opposite left-moving shock wave which propagates along the u direction in the figure can be given by the same metric (5.5), but with u and v interchanged, i.e.

$$ds^2 = 2dudv - \left[1 + 4\bar{\mu}_2\frac{v\theta(v)}{\rho^2}\right]^2 d\rho^2 - \left[1 - 4\bar{\mu}_2\frac{v\theta(v)}{\rho^2}\right]^2 \rho^2 d\phi^2 \tag{5.6}$$

with the energy $\bar{\mu}_2 = \sqrt{2}E_2$. Now this describes regions I and III globally. We can simply superpose these two metrics to write down the precollision line element which describes all three initial regions I, II, and III globally, that is,

$$\begin{aligned} ds^2 &= 2dudv - \left[1 + 8\frac{\bar{\mu}_1 u\theta(u) + \bar{\mu}_2 v\theta(v)}{\rho^2} + 16\frac{\bar{\mu}_1^2 u^2\theta(u) + \bar{\mu}_2^2 v^2\theta(v)}{\rho^4}\right] d\rho^2 \\ &\quad - \left[1 - 8\frac{\bar{\mu}_1 u\theta(u) + \bar{\mu}_2 v\theta(v)}{\rho^2} + 16\frac{\bar{\mu}_1^2 u^2\theta(u) + \bar{\mu}_2^2 v^2\theta(v)}{\rho^4}\right] \rho^2 d\phi^2. \end{aligned} \tag{5.7}$$

Since the metric has to be continuous across the null boundaries of region IV, all corrections to this metric have to be proportional to $uv\theta(u)\theta(v)$ in these coordinates.

5.2 HEAD-ON COLLISION

In general, the coordinates u and v are no longer null after the collision. This is due to the fact that the gravitational field produced by each particle focuses one another after the collision. It seems impossible to give an exact solution to the full non-linear Einstein field equations which describe the geometry of the interaction region. Therefore we treat the problem perturbatively and try to extract information about the energy emitted in gravitational waves.

From now on, we will assume that the particles have the same energy, namely $\bar{\mu}_1 = \bar{\mu}_2 = \bar{\mu}$, and give the following metric ansatz for the interaction region:

$$\begin{aligned} ds^2 &= [1 + K(u, v, \rho)]2dudv + L(u, v, \rho)(du^2 + dv^2) \\ &\quad -[1 + H(u, v, \rho)]\left[1 + 8\bar{\mu}\frac{u+v}{\rho^2} + 16\bar{\mu}^2\frac{u^2+v^2}{\rho^4}\right]d\rho^2 \\ &\quad -[1 + M(u, v, \rho)]\left[1 - 8\bar{\mu}\frac{u+v}{\rho^2} + 16\bar{\mu}^2\frac{u^2+v^2}{\rho^4}\right]\rho^2 d\phi^2 \end{aligned} \tag{5.8}$$

with the conditions

$$\begin{aligned} &K(0, v, \rho) = K(u, 0, \rho) = K(0, 0, \rho) = 0, \\ &L(0, v, \rho) = L(u, 0, \rho) = L(0, 0, \rho) = 0, \\ &H(0, v, \rho) = H(u, 0, \rho) = H(0, 0, \rho) = 0, \\ &M(0, v, \rho) = M(u, 0, \rho) = M(0, 0, \rho) = 0 \end{aligned} \tag{5.9}$$

to reduce to the initial regions. We have to determine the metric functions K, L, H, M by solving the vacuum field equations. As already mentioned, however, full analytical solution is not possible, so we try to solve the field equations perturbatively with respect to the energy parameter $\bar{\mu}$. Therefore, we assume for the functions K, L, H, M the expansions

$$K(u, v, \rho) = K_1(u, v, \rho)\bar{\mu} + K_2(u, v, \rho)\bar{\mu}^2 + O(\bar{\mu}^3), \quad \text{etc.} \tag{5.10}$$

We need to solve the vacuum field equations $R_{\mu\nu} = 0$ in the interaction region. However, in order to be able to solve the field equations, we will take $L(u, v, \rho) = 0$, and since the spacetime is asymptotically flat, we assume a power series ansatz in the negative powers of ρ for the coefficients in (5.10), that is,

$$K_i(u, v, \rho) = \sum_{\alpha=0}^{\infty} \rho^{-\alpha} k_i^{(\alpha)}(u, v), \quad (i = 1, 2, \ldots), \quad \text{etc.} \tag{5.11}$$

with the conditions (5.9):

$$k_i^{(\alpha)}(0,v) = k_i^{(\alpha)}(u,0) = k_i^{(\alpha)}(0,0) = 0, \quad \text{etc.} \tag{5.12}$$

Then we can determine the coefficients from the field equations easily. Using the Penrose prescription (2.10), we can write the solution which describes the spacetime globally as

$$\begin{aligned} ds^2 &= [1 + K(u,v,\rho)]2dudv \\ &\quad -[1 + H(u,v,\rho)]\left[1 + 8\bar{\mu}\frac{u\theta(u) + v\theta(v)}{\rho^2} + 16\bar{\mu}^2\frac{u^2\theta(u) + v^2\theta(v)}{\rho^4}\right]d\rho^2 \\ &\quad -[1 + M(u,v,\rho)]\left[1 - 8\bar{\mu}\frac{u\theta(u) + v\theta(v)}{\rho^2} + 16\bar{\mu}^2\frac{u^2\theta(u) + v^2\theta(v)}{\rho^4}\right]\rho^2 d\phi^2, \end{aligned} \tag{5.13}$$

where, from (5.10),

$$\begin{aligned} K(u,v,\rho) &= \left[-\frac{a}{\rho^4}uv - \frac{2a}{\rho^6}u^2v^2 - \frac{4a}{\rho^8}u^3v^3 + O(\rho^{-10})\right]\theta(u)\theta(v)\bar{\mu}^2 + O(\bar{\mu}^3), \\ H(u,v,\rho) &= \left[\frac{2a}{\rho^4}uv - \frac{3a}{\rho^6}u^2v^2 - \frac{16a}{3\rho^8}u^3v^3 + O(\rho^{-10})\right]\theta(u)\theta(v)\bar{\mu}^2 + O(\bar{\mu}^3), \\ M(u,v,\rho) &= \left[\frac{2a}{\rho^4}uv + \frac{3a}{\rho^6}u^2v^2 + \frac{16a}{3\rho^8}u^3v^3 + O(\rho^{-10})\right]\theta(u)\theta(v)\bar{\mu}^2 + O(\bar{\mu}^3) \end{aligned} \tag{5.14}$$

with $a = 32$.

5.3 BONDI PROBLEM

Since we are interested in the gravitational radiation emitted in the collision process of two ultrarelativistic black holes constructed in the previous section, here we briefly review the Bondi problem to extract the news function. For more detailed discussions see [12, 34, 35].

Consider a bounded isolated system which is axially symmetric and non-rotating. The spacetime produced by such a source would have a Killing vector field $\partial/\partial\phi$ representing the axial symmetry of the source, and the reflection symmetry $\phi \to -\phi$ preventing the source from rotating around the symmetry axis. We also assume that the spacetime is asymptotically flat. In order to investigate the radiation more properly, it is convenient to introduce the so-called radiation coordinates

$$x^\mu = (x^0, x^1, x^2, x^3) = (u, r, \theta, \phi), \tag{5.15}$$

where u is called the retarded time and labels a family of non-intersecting null hypersurfaces in the spacetime, r is a radial parameter along the null rays which generate the null hypersurface

$u = const.$, and θ and ϕ are the usual spherical polar angles defined on each 2-sphere $u = const.$, $r \to \infty$. The radial coordinate r is called the luminosity distance, because it is chosen such that

$$g_{22}g_{33} = r^4 \sin^2\theta \tag{5.16}$$

exactly and the 2-surfaces $u = const.$ and $r = const.$ have the usual surface area of a 2-sphere, namely, $4\pi r^2$. In such a coordinate system, using the symmetry assumptions (axial symmetry and reflection symmetry), we can write the spacetime metric as

$$ds^2 = g_{00}du^2 + 2g_{01}dudr + 2g_{02}dud\theta - r^2\left(e^{2\gamma}d\theta^2 + e^{-2\gamma}\sin^2\theta d\phi^2\right), \tag{5.17}$$

where all the metric functions g_{00}, g_{01}, g_{02} and γ are functions of u, r and θ. For the metric to be regular on the symmetry axis $\theta = 0, \pi$, the functions

$$g_{00}, \quad g_{01}, \quad \frac{g_{02}}{\sin\theta}, \quad \frac{\gamma}{\sin^2\theta} \tag{5.18}$$

must be regular as $\sin\theta \to 0$. For the general form (5.17) of the metric, the field equations are extremely complicated. Indeed, there is no exact solution which describes a time-dependent asymptotically flat axially symmetric spacetime. Therefore, we carry out an asymptotic analysis to investigate the gravitational radiation at infinity. For this purpose, since the spacetime is asymptotically flat, i.e.

$$\lim_{r\to\infty} g_{\mu\nu} = \eta_{\mu\nu}, \tag{5.19}$$

we assume a series expansion in powers of r^{-1} for each metric function in (5.17). For instance, to the required order, if we assume

$$\gamma = \frac{c(u,\theta)}{r} + O(r^{-2}), \tag{5.20}$$

the vacuum field equations lead to

$$\begin{aligned}
g_{00} &= 1 - \frac{2M}{r} + O(r^{-2}),\\
g_{01} &= 1 - \frac{c^2}{2r} + O(r^{-3}),\\
g_{02} &= -(c_\theta + 2c\cot\theta) + O(r^{-1}),\\
g_{22} &= -r^2 - 2cr + O(1),\\
g_{33} &= -r^2\sin^2\theta + 2cr\sin^2\theta + O(1),
\end{aligned} \tag{5.21}$$

where M and c are related to each other by the relation

$$M_u = -c_u^2 + \frac{1}{2}(c_{\theta\theta} + 3c_\theta\cot\theta - 2c)_u. \tag{5.22}$$

Here the function $c_u(u,\theta)$ is called the "news function". It contains all the information about the source, and if it is given, then the time evolution of the system is completely determined. The other function $M(u,\theta)$ is intimately related to the mass of the system, so it is called the "mass aspect". In fact, it can be shown that the quantity

$$m(u) = \frac{1}{2}\int_0^\pi M(u,\theta)\sin\theta d\theta \tag{5.23}$$

determines the total mass of the system as a function of the retarded time u at null infinity and is called the "Bondi mass". We can also show that

$$m_u = -\frac{1}{2}\int_0^\pi c_u^2 \sin\theta d\theta, \tag{5.24}$$

which is the "Bondi mass-loss" formula. The negative sign on the right hand side indicates that the system radiates out energy during periods for which $c_u \neq 0$.

Thus, if the news function c_u vanishes for a system, then there is no radiation and the Bondi mass (5.23) remains constant. But if there is a non-zero news function, then there is radiation and the system loses mass through the relation (5.24).

The news function also appears in the asymptotic expansions of the Riemann tensor components, i.e. to order r^{-3}

$$\begin{aligned} R &= \frac{N}{r} + \frac{III}{r^2} + O(r^{-3}), \\ N &= -c_{uu}, \quad III \sim (c_u \sin^2\theta)_{\theta\theta}. \end{aligned} \tag{5.25}$$

Therefore, although $c_u \neq 0$ expresses the fact that the mass aspect (5.22) of a given solution is undergoing a change, in order for the Riemann tensor to be of a radiative character it should be that $c_{uu} \neq 0$ also.

5.4 NEWS FOR THE COLLISION

In Section 5.2, we have found an approximate solution given by (5.13) and (5.14), which describes the collision of two shock waves produced by two ultra-relativistic particles. In order to study the mass-loss of the system as a result of the gravitational radiation, as described in the previous section, we need to find the news function for this collision process. For this purpose, first we need to bring the metric (5.13) into the Bondi form (5.17) to extract the news function.

Let us rewrite the metric (5.13) for the interaction region as

$$ds^2 = 2A^2(u,v,\rho)dudv - Q^2(u,v,\rho)d\rho^2 - P^2(u,v,\rho)\rho^2 d\phi^2, \tag{5.26}$$

where

$$\begin{aligned}
&A^2(u,v,\rho) \equiv 1 + K(u,v,\rho),\\
&Q^2(u,v,\rho) \equiv [1 + H(u,v,\rho)]\left[1 + 8\bar{\mu}\frac{u+v}{\rho^2} + 16\bar{\mu}^2\frac{u^2+v^2}{\rho^4}\right],\\
&P^2(u,v,\rho) \equiv [1 + M(u,v,\rho)]\left[1 - 8\bar{\mu}\frac{u+v}{\rho^2} + 16\bar{\mu}^2\frac{u^2+v^2}{\rho^4}\right],
\end{aligned} \tag{5.27}$$

with

$$\begin{aligned}
K(u,v,\rho) &= \left[-\frac{a}{\rho^4}uv - \frac{2a}{\rho^6}u^2v^2 - \frac{4a}{\rho^8}u^3v^3 + O(\rho^{-10})\right]\bar{\mu}^2 + O(\bar{\mu}^3),\\
H(u,v,\rho) &= \left[\frac{2a}{\rho^4}uv - \frac{3a}{\rho^6}u^2v^2 - \frac{16a}{3\rho^8}u^3v^3 + O(\rho^{-10})\right]\bar{\mu}^2 + O(\bar{\mu}^3),\\
M(u,v,\rho) &= \left[\frac{2a}{\rho^4}uv + \frac{3a}{\rho^6}u^2v^2 + \frac{16a}{3\rho^8}u^3v^3 + O(\rho^{-10})\right]\bar{\mu}^2 + O(\bar{\mu}^3).
\end{aligned} \tag{5.28}$$

Remembering the definition of the null coordinates,

$$u = \frac{1}{\sqrt{2}}(t - z), \quad v = \frac{1}{\sqrt{2}}(t + z), \tag{5.29}$$

we can write the metric (5.26) in cylindrical coordinates

$$ds^2 = A^2(dt^2 - dz^2) - Q^2 d\rho^2 - P^2\rho^2 d\phi^2. \tag{5.30}$$

Now if we pass from cylindrical coordinates $\{\rho, z, \phi\}$ to spherical coordinates $\{r, \theta, \phi\}$ by means of the transformation

$$\rho = r\sin\theta, \quad z = r\cos\theta, \quad \phi = \phi, \tag{5.31}$$

and define the retarded time

$$\tilde{u} = t - r, \tag{5.32}$$

we obtain, after some rearrangements,

$$\begin{aligned}
ds^2 = \;& A^2 d\tilde{u}^2 + 2A^2 d\tilde{u}dr + (A^2 - Q^2)\sin^2\theta dr^2\\
&+2(A^2 - Q^2)r\sin\theta\cos\theta dr d\theta\\
&-(Q^2\cos^2\theta + A^2\sin^2\theta)r^2 d\theta^2 - P^2 r^2\sin^2\theta d\phi^2.
\end{aligned} \tag{5.33}$$

Of course, we need to express A^2, Q^2 and P^2, which are functions of u, v, ρ in terms of the variables $\tilde{u}$, r, θ also. From (5.29), (5.31) and (5.32), we can write

$$\begin{aligned}
u &= \frac{1}{\sqrt{2}}[\tilde{u} + (1 - \cos\theta)r], \\
v &= \frac{1}{\sqrt{2}}[\tilde{u} + (1 + \cos\theta)r], \\
u + v &= \sqrt{2}(\tilde{u} + r), \\
uv &= \frac{1}{2}(\tilde{u}^2 + 2\tilde{u}r + r^2 \sin^2\theta).
\end{aligned} \tag{5.34}$$

Now using these relations in (5.27) and (5.28), we find up to $O(\bar{\mu}^3)$

$$\begin{aligned}
A^2 &= 1 - \frac{a}{2r^4 \sin^4\theta}(\tilde{u}^2 + 2\tilde{u}r + r^2 \sin^2\theta)\bar{\mu}^2 + O(r^{-6}), \\
Q^2 &= 1 + \frac{8\sqrt{2}}{r^2 \sin^2\theta}(\tilde{u} + r)\bar{\mu} \\
&\quad + \left[\frac{32}{r^4 \sin^4\theta}(\tilde{u} + r)^2 + \frac{a}{r^4 \sin^4\theta}(\tilde{u}^2 + 2\tilde{u}r + r^2 \sin^2\theta)\right]\bar{\mu}^2 + O(r^{-6}), \\
P^2 &= 1 - \frac{8\sqrt{2}}{r^2 \sin^2\theta}(\tilde{u} + r)\bar{\mu} \\
&\quad + \left[\frac{32}{r^4 \sin^4\theta}(\tilde{u} + r)^2 + \frac{a}{r^4 \sin^4\theta}(\tilde{u}^2 + 2\tilde{u}r + r^2 \sin^2\theta)\right]\bar{\mu}^2 + O(r^{-6}),
\end{aligned} \tag{5.35}$$

where $a = 32$.

In order to bring the metric (5.33) to Bondi's form, we need to find a coordinate system $\bar{x}^\mu = (\bar{u}, \bar{r}, \bar{\theta}, \phi)$ such that from (5.21)

$$\begin{aligned}
\bar{g}_{00} &= 1 + O(\bar{r}^{-1}), \\
\bar{g}_{01} &= 1 + O(\bar{r}^{-1}), \\
\bar{g}_{02} &= O(1), \\
\bar{g}_{22} &= -\bar{r}^2 + O(\bar{r}), \\
\bar{g}_{33} &= -\bar{r}^2 \sin^2\bar{\theta} + O(\bar{r}),
\end{aligned} \tag{5.36}$$

and also

$$\bar{g}_{11} = O(\bar{r}^{-2}), \quad \bar{g}_{12} = O(\bar{r}^{-1}), \quad \bar{g}_{22}\bar{g}_{33} = \bar{r}^4 \sin^2\bar{\theta} + O(\bar{r}^{-3}). \tag{5.37}$$

To satisfy these requirements, especially $\bar{g}_{11} = O(\bar{r}^{-2})$, the necessary transformation from $x^\mu = (\tilde{u}, r, \theta, \phi)$ to $\bar{x}^\mu = (\bar{u}, \bar{r}, \bar{\theta}, \phi)$ should include logarithmic terms as well, and may be expanded in powers of $\bar{r}^{-1}$:

$$\begin{aligned}
\tilde{u} &= f_0(\bar{u},\bar{\theta}) + t_1(\bar{u},\bar{\theta})\ln\bar{r} + t_2(\bar{u},\bar{\theta})\frac{\ln\bar{r}}{\bar{r}} + \frac{f_1(\bar{u},\bar{\theta})}{\bar{r}} + \cdots, \\
r &= q(\bar{u},\bar{\theta})\bar{r} + g_0(\bar{u},\bar{\theta}) + p_1(\bar{u},\bar{\theta})\ln\bar{r} + p_2(\bar{u},\bar{\theta})\frac{\ln\bar{r}}{\bar{r}} + \frac{g_1(\bar{u},\bar{\theta})}{\bar{r}} + \cdots, \\
\theta &= h_0(\bar{u},\bar{\theta}) + w_1(\bar{u},\bar{\theta})\frac{\ln\bar{r}}{\bar{r}} + \frac{h_1(\bar{u},\bar{\theta})}{\bar{r}} + \cdots.
\end{aligned} \tag{5.38}$$

With the requirements (5.36) and (5.37), the transformation law of the metric tensor

$$\bar{g}_{\mu\nu} = \frac{\partial x^\alpha}{\partial\bar{x}^\mu}\frac{\partial x^\beta}{\partial\bar{x}^\nu}g_{\alpha\beta} \tag{5.39}$$

determines the functions f, t, q, g, p, h, w. The old metric components $g_{\mu\nu}$ can be read off from (5.33) with (5.35). Under the transformation (5.38), they are (up to $O(\bar{\mu}^3)$)

$$\begin{aligned}
g_{00} = g_{01} &= 1 - \frac{a\bar{\mu}^2}{2q^2\bar{r}^2\sin^2 h_0} + O(\bar{r}^{-3}), \\
g_{11} &= -\frac{8\sqrt{2}\bar{\mu}}{q\bar{r}} + 8\sqrt{2}\bar{\mu}(p_1 - t_1)\frac{\ln\bar{r}}{q^2\bar{r}^2} \\
&\quad + \frac{16\sqrt{2}(g_0 - f_0)\bar{\mu} - (3a - 32 + 64\csc^2 h_0)\bar{\mu}^2}{2q^2\bar{r}^2} + O(\bar{r}^{-3}), \\
g_{12} &= -8\sqrt{2}\bar{\mu}\cot h_0 + 8\sqrt{2}\bar{\mu}\left(\frac{qw_1}{\sin^2 h_0} - t_1\cot h_0\right)\frac{\ln\bar{r}}{q\bar{r}} \\
&\quad + \frac{16\sqrt{2}(qh_1\csc^2 h_0 - f_0\cot h_0)\bar{\mu} - \cot h_0(3a - 32 + 64\csc^2 h_0)\bar{\mu}^2}{2q\bar{r}} + O(\bar{r}^{-2}), \\
g_{22} &= -q^2\bar{r}^2 - 2qp_1\bar{r}\ln\bar{r} - (2g_0 + 8\sqrt{2}\bar{\mu}\cot^2 h_0)q\bar{r} + O(1), \\
g_{33} &= -q^2\bar{r}^2\sin^2 h_0 - 2q\sin^2 h_0(p_1 + qw_1\cot h_0)\bar{r}\ln\bar{r} \\
&\quad + 2q\left(\frac{4\sqrt{2}\bar{\mu}}{\sin^2 h_0} - qh_1\cot h_0 - g_0\right)\bar{r}\sin^2 h_0 + O(1).
\end{aligned} \tag{5.40}$$

Now the new metric components $\bar{g}_{\mu\nu}$, subject to the restrictions (5.36) and (5.37), can be calculated from the transformation law (5.39). From the leading order terms of $\bar{g}_{00}$ and $\bar{g}_{02}$, we immediately find that

$$h_{0\bar{u}} = 0, \quad q_{\bar{u}} = 0. \tag{5.41}$$

For the condition $\bar{g}_{11} = O(\bar{r}^{-2})$ to be satisfied,

$$t_1 = 4\sqrt{2}\bar{\mu}. \tag{5.42}$$

The coefficient of $\ln\bar{r}$ in $\bar{g}_{12}$ vanishes if

$$w_1 = -\frac{t_{1\bar{\theta}}}{qh_{0\bar{\theta}}} = 0. \tag{5.43}$$

In order to eliminate the term proportional to $\bar{r}\ln\bar{r}$ in $\bar{g}_{22}$, we should have

$$p_1 = -qw_1 \cot h_0 = 0. \tag{5.44}$$

With the help of these, the other conditions lead to the following equations

$$q^2 h_{0\bar{\theta}}^2 = 1, \tag{5.45}$$

$$q^2 \sin^2 h_0 = \sin^2 \bar{\theta}, \tag{5.46}$$

$$qf_{0\bar{\theta}} - 8\sqrt{2}\bar{\mu} q h_{0\bar{\theta}} \cot h_0 + q^2 h_{0\bar{\theta}} h_1 - 4\sqrt{2}\bar{\mu} q_{\bar{\theta}} = 0, \tag{5.47}$$

$$f_{0\bar{u}}^2 + 2 f_{0\bar{u}} g_{0\bar{u}} - q^2 h_{1\bar{u}}^2 = 1, \tag{5.48}$$

$$q f_{0\bar{u}} = 1, \tag{5.49}$$

$$q_{\bar{\theta}} f_{0\bar{u}} - q^2 h_{0\bar{\theta}} h_{1\bar{u}} = 0. \tag{5.50}$$

Also, we can extract the metric function $c(\bar{u}, \bar{\theta})$ from $\bar{g}_{33}$ or $\bar{g}_{22}$ comparing with the forms in (5.21); that is, we find from $\bar{g}_{33}$

$$c = -h_1 \cot h_0 - \frac{g_0}{q} + \frac{4\sqrt{2}\bar{\mu}}{q \sin^2 h_0} \tag{5.51}$$

using (5.46), or from $\bar{g}_{22}$

$$\begin{aligned} c \;\; = \;\; & 4\sqrt{2}\bar{\mu} q h_{0\bar{\theta}}^2 \cot^2 h_0 + q h_{0\bar{\theta}}^2 g_0 + q^2 h_{0\bar{\theta}} h_{1\bar{\theta}} \\ & - q_{\bar{\theta}} f_{0\bar{\theta}} + 8\sqrt{2}\bar{\mu} q_{\bar{\theta}} h_{0\bar{\theta}} \cot h_0 + 4\sqrt{2}\bar{\mu} \frac{q_{\bar{\theta}}^2}{q}. \end{aligned} \tag{5.52}$$

The equations (5.45) and (5.46) can be solved for q explicitly:

$$q = \left(\frac{1}{2\chi} - \frac{\chi}{2} \right) \cos\bar{\theta} + \frac{1}{2\chi} + \frac{\chi}{2}, \tag{5.53}$$

where χ is a constant of integration. There is no need to determine all the functions appearing in the transformation (5.38). We are only interested in the mass-loss of the system which can be calculated from the equation (5.24), so we need to find the news function $c_{\bar{u}}$ only. Taking the derivative of (5.52) with respect to the retarded time $\bar{u}$ and using (5.41), we get

$$c_{\bar{u}} = q h_{0\bar{\theta}}^2 g_{0\bar{u}} + q^2 h_{0\bar{\theta}} h_{1\bar{\theta}\bar{u}} - q_{\bar{\theta}} f_{0\bar{\theta}\bar{u}}. \tag{5.54}$$

Now using (5.45), (5.48), (5.49) and (5.50), it is possible to express (5.54) in terms of the function q and its derivatives:

$$c_{\bar{u}} = \frac{1}{2} - \frac{1 + q_{\bar{\theta}}^2}{2q^2} + \frac{q_{\bar{\theta}\bar{\theta}}}{q}. \tag{5.55}$$

Then putting (5.53) into this expression yields

$$c_{\bar{u}} = 0. \tag{5.56}$$

This means that there is no mass-loss, according to the formula (5.24), at this order of approximation. However, this cannot be correct. As we argued in the previous chapters, there should be mass-loss due to the emission of gravitational waves in axi-symmetric particle collisions. The above result may stem from our perturbation treatment with respect to the energy parameter $\bar{\mu}$; it is actually a dimensionful parameter (dimension of length) and so probably not a good parameter to do perturbation with. Also, there should be contributions to the solutions (5.14) of the order $\bar{\mu}$.

In the next chapter, we give a more physical construction of the problem.

CHAPTER 6

GRUMILLER AND ROMATSCHKE APPROACH

In this chapter, we formulate the collision problem discussed in the previous chapter in a different manner. We will use Milne coordinates in the interaction region of the shock waves. This approach has been first applied by Grumiller and Romatschke [36] in the context of AdS/CFT correspondence.

6.1 HEAD-ON COLLISION

The construction of the problem is the same as in Section 5.1. One of the shock waves propagating along the v-direction has the usual continuous form

$$ds^2 = 2dudv - \left[1 + 4\bar{\mu}_1 \frac{u\theta(u)}{\rho^2}\right]^2 d\rho^2 - \left[1 - 4\bar{\mu}_1 \frac{u\theta(u)}{\rho^2}\right]^2 \rho^2 d\phi^2 \tag{6.1}$$

with $\bar{\mu}_1 = \sqrt{2}E_1$ being the energy of the wave. Similarly, the second shock with energy $\bar{\mu}_2 = \sqrt{2}E_2$ propagating along the u-direction has

$$ds^2 = 2dudv - \left[1 + 4\bar{\mu}_2 \frac{v\theta(v)}{\rho^2}\right]^2 d\rho^2 - \left[1 - 4\bar{\mu}_2 \frac{v\theta(v)}{\rho^2}\right]^2 \rho^2 d\phi^2. \tag{6.2}$$

Then, we can construct the precollision metric by simply superposing these two waves (6.1) and (6.2):

$$\begin{aligned} ds^2 &= 2dudv - \left[1 + 8\frac{\bar{\mu}_1 u\theta(u) + \bar{\mu}_2 v\theta(v)}{\rho^2} + 16\frac{\bar{\mu}_1^2 u^2\theta(u) + \bar{\mu}_2^2 v^2\theta(v)}{\rho^4}\right] d\rho^2 \\ &\quad - \left[1 - 8\frac{\bar{\mu}_1 u\theta(u) + \bar{\mu}_2 v\theta(v)}{\rho^2} + 16\frac{\bar{\mu}_1^2 u^2\theta(u) + \bar{\mu}_2^2 v^2\theta(v)}{\rho^4}\right] \rho^2 d\phi^2. \end{aligned} \tag{6.3}$$

This metric describes globally the regions I, II, and III of Figure 2.1.

To study the geometry of the interaction region IV, we use the coordinates of proper time and spacetime rapidity defined by

$$\tau = \sqrt{2uv}, \quad \eta = \frac{1}{2}\ln\frac{v}{u}. \tag{6.4}$$

In these coordinates, the hypersurfaces $\{u = 0\}$, $\{v = 0\}$ and $\{u = 0\} \cup \{v = 0\}$ bounding the region IV in Figure 2.1 become $\eta = +\infty$, $\eta = -\infty$ and $\tau = 0$, respectively. With this transformation, we can write

$$\begin{aligned}
&2dudv = d\tau^2 - \tau^2 d\eta^2, \\
&\bar{\mu}_1 u + \bar{\mu}_2 v = \frac{1}{2}\sqrt{2\bar{\mu}_1\bar{\mu}_2}\sqrt{2uv}\left[\sqrt{\frac{\bar{\mu}_1}{\bar{\mu}_2}}\sqrt{\frac{u}{v}} + \sqrt{\frac{\bar{\mu}_2}{\bar{\mu}_1}}\sqrt{\frac{v}{u}}\right] = \bar{\mu}\tau\cosh(Y-\eta), \\
&\bar{\mu}_1^2 u^2 + \bar{\mu}_2^2 v^2 = (\bar{\mu}_1 u + \bar{\mu}_2 v)^2 - 2\bar{\mu}_1\bar{\mu}_2 uv = \frac{1}{2}\bar{\mu}^2\tau^2\cosh[2(Y-\eta)],
\end{aligned} \tag{6.5}$$

where we have defined

$$\bar{\mu} \equiv \sqrt{2\bar{\mu}_1\bar{\mu}_2}, \quad Y \equiv \frac{1}{2}\ln\frac{\bar{\mu}_1}{\bar{\mu}_2}. \tag{6.6}$$

Then, considering the form of the precollision metric (6.3), we can make the following general ansatz for the line element after the collision:

$$\begin{aligned}
ds^2 \;=\;& [1 + K(\tau,\eta,\rho)]d\tau^2 - [1 + L(\tau,\eta,\rho)]\tau^2 d\eta^2 + 2N(\tau,\eta,\rho)\tau d\tau d\eta \\
&-[1 + H(\tau,\eta,\rho)]\left\{1 + 8\bar{\mu}\frac{\tau\cosh(Y-\eta)}{\rho^2} + 8\bar{\mu}^2\frac{\tau^2\cosh[2(Y-\eta)]}{\rho^4}\right\}d\rho^2 \\
&-[1 + M(\tau,\eta,\rho)]\left\{1 - 8\bar{\mu}\frac{\tau\cosh(Y-\eta)}{\rho^2} + 8\bar{\mu}^2\frac{\tau^2\cosh[2(Y-\eta)]}{\rho^4}\right\}\rho^2 d\phi^2,
\end{aligned} \tag{6.7}$$

where the functions K, L, N, H, M should vanish at $\tau = 0$ and as $\eta \to \pm\infty$ in order for the ansatz (6.7) to reduce to the precollision metric (6.3). These functions have to be determined by solving the vacuum field equations for the metric (6.7). However, it seems impossible to give a full analytical solution which describes the interaction region of the waves. Therefore, we restrict ourselves to the regime of early times $\tau \ll 1$, i.e. times just after the collision. This enables us to assume power series ansatz in τ for the functions K, L, N, H, M:

$$K(\tau,\eta,\rho) = \sum_{\alpha=0}^{\infty}\tau^\alpha k_\alpha(\eta,\rho), \quad \text{etc.} \tag{6.8}$$

whose coefficients can be determined by solving the field equations order by order in τ.

It should be remembered that the energies of the black holes are $\bar{\mu}_1 = \sqrt{2}E_1$ and $\bar{\mu}_2 = \sqrt{2}E_2$, so that eventually to obtain correct results one should put, from (6.6),

$$\bar{\mu} = 2\sqrt{E_1E_2}, \quad Y = \frac{1}{2}\ln\frac{E_1}{E_2}. \tag{6.9}$$

However, from now on, we will assume that the energies of the black holes are equal, i.e. $E_1 = E_2 = E$, then

$$\bar{\mu} = 2E, \quad Y = 0, \tag{6.10}$$

and for this identical construction, since the spacetime will be symmetrical with respect to $\eta \to -\eta$, we will take $N(\tau, \eta, \rho) = 0$ (no cross term in the metric (6.7)).

It is possible to give the following solution

$$\begin{aligned}
K(\tau,\eta,\rho) &= \frac{c_1\bar{\mu}^2}{\rho^4}\tau^2 + \frac{8(c_1+20)\bar{\mu}^3\cosh(\eta)}{3\rho^6}\tau^3 + \left[\frac{c_2\bar{\mu}^2}{\rho^6} + \frac{c_3\bar{\mu}^4}{\rho^8} + \frac{32\bar{\mu}^4\cosh(2\eta)}{\rho^8}\right]\tau^4 + O(\tau^5) \\
L(\tau,\eta,\rho) &= \frac{(c_1-16)\bar{\mu}^2}{3\rho^4}\tau^2 + \frac{8(c_1+20)\bar{\mu}^3\cosh(\eta)}{9\rho^6}\tau^3 \\
&\quad + \left[\frac{(c_2-32)\bar{\mu}^2}{5\rho^6} - \frac{(352+160c_1+c_1^2-9c_3)\bar{\mu}^4}{45\rho^8} - \frac{32\bar{\mu}^4\cosh(2\eta)}{5\rho^8}\right]\tau^4 + O(\tau^5) \\
H(\tau,\eta,\rho) &= \frac{16\bar{\mu}^2}{\rho^4}\tau^2 + \frac{4(c_1-52)\bar{\mu}^3\cosh(\eta)}{3\rho^6}\tau^3 \\
&\quad + \left[\frac{(4+5c_1)\bar{\mu}^2}{3\rho^6} + \frac{8(452+c_1)\bar{\mu}^4}{9\rho^8} - \frac{8(304-c_1)\bar{\mu}^4\cosh(2\eta)}{9\rho^8}\right]\tau^4 + O(\tau^5) \\
M(\tau,\eta,\rho) &= \frac{16\bar{\mu}^2}{\rho^4}\tau^2 - \frac{4(c_1-52)\bar{\mu}^3\cosh(\eta)}{3\rho^6}\tau^3 \\
&\quad + \left[\frac{(28-c_1)\bar{\mu}^2}{3\rho^6} + \frac{8(412+c_1)\bar{\mu}^4}{9\rho^8} - \frac{8(224-5c_1)\bar{\mu}^4\cosh(2\eta)}{9\rho^8}\right]\tau^4 + O(\tau^5),
\end{aligned} \tag{6.11}$$

where c_1, c_2, c_3 are free integration constants. Thus we have found that the interaction region IV of the colliding shock waves can be described by the metric

$$ds^2 = A^2(\tau,\eta,\rho)d\tau^2 - B^2(\tau,\eta,\rho)\tau^2 d\eta^2 - Q^2(\tau,\eta,\rho)d\rho^2 - P^2(\tau,\eta,\rho)\rho^2 d\phi^2, \tag{6.12}$$

where

$$\begin{aligned}
A^2(\tau,\eta,\rho) &\equiv 1 + K(\tau,\eta,\rho), \\
B^2(\tau,\eta,\rho) &\equiv 1 + L(\tau,\eta,\rho), \\
Q^2(\tau,\eta,\rho) &\equiv [1 + H(\tau,\eta,\rho)]\left[1 + 8\bar{\mu}\frac{\tau\cosh(\eta)}{\rho^2} + 8\bar{\mu}^2\frac{\tau^2\cosh(2\eta)}{\rho^4}\right], \\
P^2(\tau,\eta,\rho) &\equiv [1 + M(\tau,\eta,\rho)]\left[1 - 8\bar{\mu}\frac{\tau\cosh(\eta)}{\rho^2} + 8\bar{\mu}^2\frac{\tau^2\cosh(2\eta)}{\rho^4}\right]
\end{aligned} \tag{6.13}$$

with the functions K, L, H, M given in (6.11).

6.2 NEWS FOR THE COLLISION

Having obtained the solution, we can now study the gravitational radiation produced in the collision by the method of Bondi which we have described in Section (5.3). Our aim is to extract the news function from which we can calculate the mass-loss of the system.

Using the definitions (6.4), we can recast the metric (6.12) in terms of the null coordinates u, v which are defined by (5.29). Then, as we did in section (5.4), passing from cylindrical coordinates $\{\rho, z, \phi\}$ to spherical coordinates $\{r, \theta, \phi\}$ by means of (5.31) and defining the retarded time (5.32), we can rewrite the metric (6.12) as

$$\begin{aligned} ds^2 \;=\; & W^{-1}[A^2(\tilde{u}+r)^2 - B^2 r^2\cos^2\theta]d\tilde{u}^2 \\ & +2W^{-1}\{A^2(\tilde{u}+r)^2 + [B^2\tilde{u} - A^2(\tilde{u}+r)]r\cos^2\theta\}d\tilde{u}dr \\ & +2W^{-1}(A^2 - B^2)(\tilde{u}+r)r^2\cos\theta\sin\theta d\tilde{u}d\theta \\ & +W^{-1}\{A^2r^2\cos^4\theta + (A^2 - Q^2\sin^2\theta)(\tilde{u}+r)^2 \\ & \qquad - [2A^2(\tilde{u}+r)r^2 + B^2\tilde{u}^2 - Q^2r^2\sin^2\theta]\cos^2\theta\}dr^2 \\ & +2W^{-1}\{[A^2r + B^2\tilde{u}^2 - Q^2(\tilde{u}+r)](\tilde{u}+r) \\ & \qquad - (A^2 - Q^2)r^2\cos^2\theta\}r\cos\theta\sin\theta drd\theta \\ & -\{Q^2\cos^2\theta - W^{-1}[A^2r^2\cos^2\theta - B^2(\tilde{u}+r)^2]\sin^2\theta\}r^2d\theta^2 \\ & -P^2r^2\sin^2\theta d\phi^2, \end{aligned} \tag{6.14}$$

where $W \equiv (\tilde{u}+r)^2 - r^2\cos^2\theta$, and A^2, B^2, Q^2, P^2 are functions of τ, η, ρ defined in (6.13) and should be expressed here in terms of the coordinates $\tilde{u}$, r, θ.

In order to bring the metric (6.14) into the Bondi form asymptotically, we need the transformation (5.38), which we reproduce here

$$\begin{aligned} \tilde{u} &= f_0(\bar{u},\bar{\theta}) + t_1(\bar{u},\bar{\theta})\ln\bar{r} + t_2(\bar{u},\bar{\theta})\frac{\ln\bar{r}}{\bar{r}} + \frac{f_1(\bar{u},\bar{\theta})}{\bar{r}} + \cdots, \\ r &= q(\bar{u},\bar{\theta})\bar{r} + g_0(\bar{u},\bar{\theta}) + p_1(\bar{u},\bar{\theta})\ln\bar{r} + p_2(\bar{u},\bar{\theta})\frac{\ln\bar{r}}{\bar{r}} + \frac{g_1(\bar{u},\bar{\theta})}{\bar{r}} + \cdots, \\ \theta &= h_0(\bar{u},\bar{\theta}) + w_1(\bar{u},\bar{\theta})\frac{\ln\bar{r}}{\bar{r}} + \frac{h_1(\bar{u},\bar{\theta})}{\bar{r}} + \cdots. \end{aligned} \tag{6.15}$$

Under this transformation, the metric tensor transforms as

$$\bar{g}_{\mu\nu} = \frac{\partial x^\alpha}{\partial \bar{x}^\mu}\frac{\partial x^\beta}{\partial \bar{x}^\nu}g_{\alpha\beta} \tag{6.16}$$

with the old metric components $g_{\mu\nu}$ to be read off from (6.14). The new metric components $\bar{g}_{\mu\nu}$ are subject to the restrictions (5.36) and (5.37) which lead to the following equations

$$t_1 = 4\bar{\mu}, \tag{6.17}$$

$$h_{0\bar{u}} = q_{\bar{u}} = w_1 = p_1 = 0, \tag{6.18}$$

$$q^2 h_{0\bar{\theta}}^2 = 1, \tag{6.19}$$

$$q^2 \sin^2 h_0 = \sin^2 \bar{\theta}, \tag{6.20}$$

$$q f_{0\bar{\theta}} - 8\bar{\mu} q h_{0\bar{\theta}} \cot h_0 + q^2 h_{0\bar{\theta}} h_1 - 4\bar{\mu} q_{\bar{\theta}} = 0, \tag{6.21}$$

$$f_{0\bar{u}}^2 + 2 f_{0\bar{u}} g_{0\bar{u}} - q^2 h_{1\bar{u}}^2 = 1, \tag{6.22}$$

$$q f_{0\bar{u}} = 1, \tag{6.23}$$

$$q_{\bar{\theta}} f_{0\bar{u}} - q^2 h_{0\bar{\theta}} h_{1\bar{u}} = 0. \tag{6.24}$$

Also, the metric function $c(\bar{u}, \bar{\theta})$ can be extracted from $\bar{g}_{33}$ as

$$c = -h_1 \cot h_0 - \frac{g_0}{q} + \frac{4\bar{\mu}}{q \sin^2 h_0}, \tag{6.25}$$

or from $\bar{g}_{22}$ as

$$\begin{aligned} c \quad = \quad & 4\bar{\mu} q h_{0\bar{\theta}}^2 \cot^2 h_0 + q h_{0\bar{\theta}}^2 g_0 + q^2 h_{0\bar{\theta}} h_{1\bar{\theta}} \\ & - q_{\bar{\theta}} f_{0\bar{\theta}} + 8\bar{\mu} q_{\bar{\theta}} h_{0\bar{\theta}} \cot h_0 + 4\bar{\mu} \frac{q_{\bar{\theta}}^2}{q}. \end{aligned} \tag{6.26}$$

These are actually the same equations as the ones obtained in Section (5.4), considering the energy parameter $\bar{\mu} = 2E$ here and $\bar{\mu} = \sqrt{2}E$ there. Therefore, the conclusion is the same, namely, (6.19) and (6.21) can be solved for q, which yields (5.53) again

$$q = \left(\frac{1}{2\chi} - \frac{\chi}{2}\right) \cos \bar{\theta} + \frac{1}{2\chi} + \frac{\chi}{2} \tag{6.27}$$

with χ being a constant of integration, and expressing the news function $c_{\bar{u}}$ in terms of q and its derivatives by means of the other equations, we get

$$c_{\bar{u}} = \frac{1}{2} - \frac{1 + q_{\bar{\theta}}^2}{2q^2} + \frac{q_{\bar{\theta}\bar{\theta}}}{q} = 0. \tag{6.28}$$

Thus we have again found that there is no mass-loss.

CHAPTER 7

CONCLUSION

In this thesis, we have studied the head-on collision of two identical ultra-relativistic black holes described by Aichelburg-Sexl shock waves. Since it is not easy to give an exact solution to the field equations in the interaction region, we treated the problem perturbatively. First, assuming the energy parameters of the waves to be small, we gave a series solution which may describe the interaction of the waves approximately. Secondly, formulating the problem in terms of proper time and rapidity, we studied the early-time dynamics of the collision spacetime and gave a power series solution in proper time. In this thesis, we used the method of Bondi which renders possible to calculate the efficiency of the gravitational radiation produced by the collision by means of the news function. We calculated the news function for the solutions that we have given, and we found that it is zero at the order of approximations in either case, which means that there is no mass-loss. This result implies that we need to apply some other methods to evaluate the emitted radiation in the process. This will be our future work.

APPENDIX A

pp-WAVE SPACETIMES

Spacetimes admitting a covariantly constant null vector field k_μ are called plane-fronted gravitational waves with parallel rays (pp-waves) [14, 15, 16]. That is, a pp-wave is characterized by the requirements

$$k_{\nu;\mu} \equiv \nabla_\mu k_\nu = 0, \quad k_\mu k^\mu = 0. \tag{A.1}$$

The condition of being covariantly constant also implies that

$$R^\beta{}_{\alpha\mu\nu} k_\beta = k_{\alpha;\mu;\nu} - k_{\alpha;\nu;\mu} = 0, \tag{A.2}$$

and the vector field satisfies the geodesic equation; i.e.

$$k^\nu k_{\mu;\nu} = 0, \tag{A.3}$$

where $k^\mu = \frac{dx^\mu}{dv}$, v being an affine parameter, is the tangent vector to the geodesics. Therefore the vector field k^μ defines a null geodesic congruence. On the other hand, using the nullness, of the field we can also deduce the following relation

$$k^\nu k_{\nu;\mu} = \frac{1}{2}(k^\nu k_\nu)_{;\mu} = 0. \tag{A.4}$$

Now adding and subtracting (A.3) and (A.4) among each other, we get the following two equations

$$k_{(\mu;\nu)} = 0, \tag{A.5}$$

$$k_{[\mu;\nu]} = 0, \tag{A.6}$$

where the round brackets denote symmetrization, while the square brackets denote anti-symmetrization. The first of these equations (A.5) is the Killing equation, which can also be written as

$$g_{\mu\nu,\alpha} k^\alpha + g_{\alpha\nu} k^\alpha{}_{,\mu} + g_{\mu\alpha} k^\alpha{}_{,\nu} = 0, \tag{A.7}$$

so a covariantly constant null vector field is automatically a Killing vector field. On the other hand, (A.6) says that

$$k_{\mu;\nu} - k_{\nu;\mu} = k_{\mu,\nu} - k_{\nu,\mu} = 0, \tag{A.8}$$

and this implies that the vector field k_μ can be written as the gradient of a function $u(x^\alpha)$:

$$k_\mu = u_{,\mu}. \tag{A.9}$$

Now if we choose our coordinates as $x^\mu = (u, v, x^i)$ with $i = 1, 2$, then from (A.9) and $k^\mu = \frac{dx^\mu}{dv}$

$$k_\mu = \delta^0_\mu, \quad k^\mu = \delta^\mu_1. \tag{A.10}$$

Since this is a Killing vector, (A.7) reduces to

$$\frac{\partial g_{\mu\nu}}{\partial v} = 0, \tag{A.11}$$

which means that the metric is independent of the coordinate v, and from (A.10) we also have

$$g_{\mu\nu}k^\nu = k_\mu \quad \Rightarrow \quad g_{\mu 1} = \delta^0_\mu. \tag{A.12}$$

There are no further constraints, and thus the most general form of a spacetime metric admitting a covariantly constant null vector field, i.e. a pp-wave spacetime, can be written in the coordinates $x^\mu = (u, v, x^i)$ as

$$\begin{aligned} ds^2 &= g_{\mu\nu}(u, x^i)dx^\mu dx^\nu \\ &= g_{0\nu}dudx^\nu + g_{1\nu}dvdx^\nu + g_{i\nu}dx^i dx^\nu \\ &= g_{00}du^2 + g_{01}dudv + g_{0i}dudx^i + g_{10}dvdu + g_{i0}dx^i du + g_{i1}dx^i dv + g_{ij}dx^i dx^j \\ &= 2dudv + g_{00}(u, x^i)du^2 + 2g_{0i}(u, x^i)dudx^i + g_{ij}(u, x^i)dx^i dx^j. \end{aligned} \tag{A.13}$$

The metric functions $g_{00}(u, x^i)$, $g_{0i}(u, x^i)$ and $g_{ij}(u, x^i)$ are constrained by the field equations. Just for simplicity, the most useful pp-waves generally considered in the literature are the ones for which $g_{0i} = 0$ and $g_{ij} = \delta_{ij}$; that is,

$$ds^2 = 2dudv + g_{00}(u, x^i)du^2 + \delta_{ij}dx^i dx^j \tag{A.14}$$

which is the form that we have discussed in Chapter 2.

APPENDIX B

PROOF OF THE RELATION (3.46)

In this appendix, we will prove the relation (3.46) which is essential for carrying out the ultrarelativistic limit $\beta \to 1$.

Let us define

$$I \equiv \ln\left(\frac{x-\beta x^0 + [(x-\beta x^0)^2 + \gamma^{-2}(y^2+z^2)]^{1/2}}{x-\beta x^0 + [(x-\beta x^0)^2+\gamma^{-2}]^{1/2}}\right), \quad \gamma^{-2} = 1-\beta^2 \tag{B.1}$$

the limit of which is

$$\lim_{\beta\to 1} I = \ln\left(\frac{x-x^0+|x-x^0|}{x-x^0+|x-x^0|}\right) = \begin{cases} 0, & \text{for } x-x^0>0, \\ \infty-\infty \text{ (indeterminate!)}, & \text{for } x-x^0<0. \end{cases} \tag{B.2}$$

For the case $x-x^0<0$, apply the L'Hospital rule:

$$\begin{aligned}
\lim_{\substack{\beta\to 1\\ x-x^0<0}} I &= \lim_{\substack{\beta\to 1\\ x-x^0<0}} \ln\left[\frac{\frac{\partial}{\partial\beta}\left(x-\beta x^0+[(x-\beta x^0)^2+\gamma^{-2}(y^2+z^2)]^{1/2}\right)}{\frac{\partial}{\partial\beta}\left(x-\beta x^0+[(x-\beta x^0)^2+\gamma^{-2}]^{1/2}\right)}\right] \\
&= \lim_{\substack{\beta\to 1\\ x-x^0<0}} \ln\left[\frac{-x^0-\frac{(x-\beta x^0)x^0+\beta(y^2+z^2)}{[(x-\beta x^0)^2+\gamma^{-2}(y^2+z^2)]^{1/2}}}{-x^0-\frac{(x-\beta x^0)x^0+\beta}{[(x-\beta x^0)^2+\gamma^{-2}]^{1/2}}}\right] \\
&= \ln\left[\frac{-x^0+\frac{(x-x^0)x^0+(y^2+z^2)}{x-x^0}}{-x^0+\frac{(x-x^0)x^0+1}{x-x^0}}\right] \\
&= \ln(y^2+z^2).
\end{aligned} \tag{B.3}$$

Therefore

$$\begin{aligned}
\lim_{\beta\to 1} I &= \begin{cases} 0, & \text{for } x-x^0>0, \\ \ln(y^2+z^2), & \text{for } x-x^0<0, \end{cases} \\
&= [1-\theta(x-x^0)]\ln(y^2+z^2)
\end{aligned} \tag{B.4}$$

which is (3.46). Also,

$$\begin{aligned}
\lim_{\beta\to 1}\frac{\partial I}{\partial x} &= \lim_{\beta\to 1}\left\{[(x-\beta x^0)^2+\gamma^{-2}(y^2+z^2)]^{-1/2} - [(x-\beta x^0)^2+\gamma^{-2}]^{-1/2}\right\} \\
&= -\delta(x-x^0)\ln(y^2+z^2).
\end{aligned} \tag{B.5}$$

APPENDIX C

CONTINUOUS COORDINATES FOR SHOCK WAVES

In this appendix, we give a general coordinate transformation for any shock wave metric which brings the metric into an explicitly continuous form.

Motivated by the form (3.74), we can write the following general shock wave metric with plane polar coordinates on the transverse plane:

$$ds^2 = \sigma d\bar{u}d\bar{v} + f(\bar{\rho})\delta(\bar{u})d\bar{u}^2 + \lambda\left(d\bar{\rho}^2 + \bar{\rho}^2 d\bar{\phi}^2\right), \tag{C.1}$$

where $f(\bar{\rho})$ is the profile function which describes the wave, and σ and λ are just numerical constants introduced for taking into account the signature of the metric and the definition of the null coordinates $(\bar{u}, \bar{v})$. Evidently, due to the appearance of the delta function $\delta(\bar{u})$ in the metric, the coordinate system is discontinuous across the null coordinate $\bar{u}$. However, it is sometimes more convenient to work with the form in which the metric is explicitly continuous (when studying the geodesics in the spacetime, for example). It is therefore necessary to bring the metric (C.1) into continuous form by an appropriate coordinate transformation which is in its general form

$$\bar{u} = u, \quad \bar{v} = v + \alpha\theta(u)f(\rho) + \beta u\theta(u)\left[f'(\rho)\right]^2, \quad \bar{\rho} = \rho + \gamma u\theta(u)f'(\rho), \quad \bar{\phi} = \phi, \tag{C.2}$$

where α, β, γ are again numerical constants, and the prime denotes derivative with respect to the argument. The differentials of the coordinates (C.2) are

$$\begin{aligned} d\bar{u} &= du, \\ d\bar{v} &= dv + \alpha\delta(u)f du + \alpha\theta(u)f' d\rho \\ &\qquad + \beta\theta(u)f'^2 du + 2\beta u\theta(u)f' f'' d\rho, \\ d\bar{\rho} &= d\rho + \gamma\theta(u)f' du + \gamma u\theta(u)f'' d\rho, \\ d\bar{\phi} &= d\phi. \end{aligned} \tag{C.3}$$

Now putting these in (C.1), the metric becomes

$$\begin{aligned} ds^2 &= \sigma dudv + \underbrace{\sigma\alpha\delta(u) f du^2}_{1} + \underbrace{\sigma\alpha\theta(u) f' dud\rho}_{2} \\ &\quad \underbrace{+\sigma\beta\theta(u) f'^2 du^2}_{3} + \underbrace{2\sigma\beta u\theta(u) f' f'' dud\rho}_{4} + \underbrace{f(\bar{\rho})\delta(u) du^2}_{1} \\ &\quad +\lambda d\rho^2 + \underbrace{2\lambda\gamma\theta(u) f' dud\rho}_{2} + 2\lambda\gamma u\theta(u) f'' d\rho^2 \\ &\quad + \underbrace{\lambda\gamma^2\theta(u) f'^2 du^2}_{3} + \underbrace{2\lambda\gamma^2 u\theta(u) f' f'' dud\rho}_{4} + \lambda\gamma^2 u^2\theta(u) f''^2 d\rho^2 \\ &\quad +\lambda\left[1 + \gamma u\theta(u)\frac{f'}{\rho}\right]^2 \rho^2 d\phi^2, \end{aligned} \tag{C.4}$$

where we have used the property $f(u)\delta(u) = f(0)\delta(u)$. To cancel the same-numbered terms, the numerical constants should be related to each other as

$$\alpha = -\frac{1}{\sigma}, \quad \gamma = \frac{1}{2\lambda}, \quad \beta = -\frac{1}{4\sigma\lambda}. \tag{C.5}$$

Therefore, we get

$$ds^2 = \sigma dudv + \lambda\left[1 + \gamma u\theta(u) f''\right]^2 d\rho^2 + \lambda\left[1 + \gamma u\theta(u)\frac{f'}{\rho}\right]^2 \rho^2 d\phi^2. \tag{C.6}$$

As an application, consider the Aichelburg-Sexl shock wave metric (3.68). Here in the barred coordinates it takes the form

$$ds^2 = 2d\bar{u}d\bar{v} + 4\bar{\mu}\ln\bar{\rho}^2\delta(\bar{u})d\bar{u}^2 - \left(d\bar{\rho}^2 + \bar{\rho}^2 d\bar{\phi}^2\right), \tag{C.7}$$

from which we can identify

$$f(\bar{\rho}) = 4\bar{\mu}\ln\bar{\rho}^2, \quad \sigma = 2, \quad \lambda = -1. \tag{C.8}$$

So the transformation is, from (C.2),

$$\bar{u} = u, \quad \bar{v} = v - 4\bar{\mu}\ln\rho\theta(u) + 8\bar{\mu}^2\frac{u\theta(u)}{\rho^2}, \quad \bar{\rho} = \rho - 4\bar{\mu}\frac{u\theta(u)}{\rho}, \quad \bar{\phi} = \phi, \tag{C.9}$$

which brings the metric (C.7) into the following continuous form

$$ds^2 = 2dudv - \left[1 + 4\bar{\mu}\frac{u\theta(u)}{\rho^2}\right]^2 d\rho^2 - \left[1 - 4\bar{\mu}\frac{u\theta(u)}{\rho^2}\right]^2 \rho^2 d\phi^2. \tag{C.10}$$

APPENDIX D

ENERGY SPECTRUM OF GRAVITATIONAL WAVES IN LINEARIZED GRAVITY

We are interested in the generation of gravitational radiation by sources. This in general requires the consideration of the full non-linear Einstein's field equations coupled to matter. But the non-linearity of the equations makes the treatment difficult, so it is necessary to study the weak field limit of the theory in which the spacetime is nearly Minkowskian and the self-interaction of the gravitational field is neglected.

We assume that the spacetime is asymptotically flat and the metric can be decomposed into the flat Minkowski metric plus a small perturbation as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{D.1}$$

where $\eta_{\mu\nu} = (1, -1, -1, -1)$ and $|h_{\mu\nu}| \ll 1$. We have set the speed of light $c = 1$. To first order in h, the Einstein field equations become

$$\Box h_{\mu\nu} = -16\pi G S_{\mu\nu} \tag{D.2}$$

in Lorenz gauge (see [31] for details). Here $\Box = \partial_t^2 - \partial_x^2 - \partial_y^2 - \partial_z^2$ is the flat space d'Alembertian, G is the gravitational constant, and

$$S_{\mu\nu} \equiv T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} T. \tag{D.3}$$

(D.2) can be solved by using the method of Green's function and the solution is

$$h_{\mu\nu}(t, \mathbf{x}) = 4G \int d^3x' \frac{S_{\mu\nu}(t - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}. \tag{D.4}$$

This is the gravitational potential at the point $(t, \mathbf{x})$ produced by the sources $S_{\mu\nu}$ at the point $(t - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}')$. The time argument $t - |\mathbf{x} - \mathbf{x}'|$ is called the "retarded time" and shows that gravitational effects propagate with the speed of light.

In order to investigate the gravitational radiation which is an oscillatory phenomenon, it is convenient to work with the Fourier transform technique. We use the following conventions: Given a function of spacetime $\phi(t, \mathbf{x})$, the Fourier transform and its inverse are defined by

$$\begin{aligned} \phi(\omega, \mathbf{x}) &= \int dt\, e^{i\omega t} \phi(t, \mathbf{x}), \\ \phi(t, \mathbf{x}) &= \int \frac{d\omega}{2\pi}\, e^{-i\omega t} \phi(\omega, \mathbf{x}) \end{aligned} \quad \text{(D.5)}$$

with respect to time, and

$$\begin{aligned} \phi(t, \mathbf{k}) &= \int d^3x\, e^{-i\mathbf{k}\cdot\mathbf{x}} \phi(t, \mathbf{x}), \\ \phi(t, \mathbf{x}) &= \int \frac{d^3k}{(2\pi)^3}\, e^{i\mathbf{k}\cdot\mathbf{x}} \phi(t, \mathbf{k}) \end{aligned} \quad \text{(D.6)}$$

with respect to space. With these definitions, the field $h_{\mu\nu}(t, \mathbf{x})$ in (D.4) can be written as

$$h_{\mu\nu}(t, \mathbf{x}) = 4G \int d^3x' \int \frac{d\omega}{2\pi}\, e^{-i\omega(t-|\mathbf{x}-\mathbf{x}'|)}\, \frac{S_{\mu\nu}(\omega, \mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} + c.c., \quad \text{(D.7)}$$

where "*c.c.*" means the "complex conjugate" of the preceding term which is necessary to make the whole expression real, and

$$S_{\mu\nu}(\omega, \mathbf{x}) \equiv T_{\mu\nu}(\omega, \mathbf{x}) - \frac{1}{2}\eta_{\mu\nu} T(\omega, \mathbf{x}). \quad \text{(D.8)}$$

Now we suppose that the source is isolated and fairly far away. Therefore, the distances $r \equiv |\mathbf{x}|$, at which we observe the radiation, are much larger than the dimension $R \equiv |\mathbf{x}'|$ of the source, and also much larger than ωR^2 and $1/\omega$. Hence we may approximate

$$\begin{aligned} |\mathbf{x}-\mathbf{x}'| &= [(\mathbf{x}-\mathbf{x}')\cdot(\mathbf{x}-\mathbf{x}')]^{1/2}, \\ &= (|\mathbf{x}|^2 - 2\mathbf{x}\cdot\mathbf{x}' + |\mathbf{x}'|^2)^{1/2}, \\ &= (r^2 - 2\mathbf{x}\cdot\mathbf{x}' + R^2)^{1/2}, \\ &= r\left(1 - 2\frac{\mathbf{x}}{r}\cdot\frac{\mathbf{x}'}{r} + \frac{R^2}{r^2}\right)^{1/2}, \\ &\simeq r - \hat{\mathbf{x}}\cdot\mathbf{x}', \end{aligned}$$

where $\hat{\mathbf{x}} \equiv \mathbf{x}/r$. Also,

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} \simeq \frac{1}{r}.$$

This enables us to write the field (D.7) as

$$\begin{aligned} h_{\mu\nu}(t, \mathbf{x}) &= \frac{4G}{r} \int \frac{d\omega}{2\pi}\, e^{-i\omega(t-r)} \int d^3x'\, S_{\mu\nu}(\omega, \mathbf{x}')\, e^{-i\omega\hat{\mathbf{x}}\cdot\mathbf{x}'} + c.c. \\ &= \frac{4G}{r} \int \frac{d\omega}{2\pi}\, e^{-i\omega(t-r)}\, S_{\mu\nu}(\omega, \mathbf{k}) + c.c., \end{aligned} \quad \text{(D.9)}$$

where we put the wave 4-vector

$$k^{\mu} = (k^0, \mathbf{k}) = (\omega, \omega\hat{\mathbf{x}}), \tag{D.10}$$

and then used (D.6). Since $r\omega$ is assumed to be large, (D.9) looks just like an integral over ω of the individual plane waves,

$$h_{\mu\nu}(t, \mathbf{x}) = \int d\omega\; e_{\mu\nu}(\omega, \mathbf{x})\; e^{-ik_{\mu}x^{\mu}} + c.c. \tag{D.11}$$

with $k_{\mu}x^{\mu} = \omega t - \omega r$ and

$$e_{\mu\nu}(\omega, \mathbf{x}) \equiv \frac{4G}{2\pi r} S_{\mu\nu}(\omega, \mathbf{k}) = \frac{4G}{2\pi r}\left[T_{\mu\nu}(\omega, \mathbf{k}) - \frac{1}{2}\eta_{\mu\nu}T(\omega, \mathbf{k})\right] \tag{D.12}$$

being the "polarization tensor".

Now using the standard expression for the energy density of the radiation field (see [31]), we can obtain the distribution of the energy emitted in a direction $\hat{\mathbf{k}}$:

$$\begin{aligned}
\frac{d^2E}{d\Omega d\omega} &= \frac{r^2(\mathbf{k}\cdot\hat{\mathbf{x}})k^0}{8G}\left[e^{*}_{\mu\nu}(\omega, \mathbf{x})e^{\mu\nu}(\omega, \mathbf{x}) - \frac{1}{2}|e^{\mu}{}_{\mu}(\omega, \mathbf{x})|^2\right] \\
&= \frac{G\omega^2}{2\pi^2}\left[S^{*}_{\mu\nu}(\omega, \mathbf{k})S^{\mu\nu}(\omega, \mathbf{k}) - \frac{1}{2}|S(\omega, \mathbf{k})|^2\right] \\
&= \frac{G\omega^2}{2\pi^2}\left[T^{*}_{\mu\nu}(\omega, \mathbf{k})T^{\mu\nu}(\omega, \mathbf{k}) - \frac{1}{2}|T(\omega, \mathbf{k})|^2\right],
\end{aligned} \tag{D.13}$$

where we have inserted (D.10) and (D.12). This is the gravitational energy flux per unit solid angle and per unit frequency in a direction $\hat{\mathbf{k}}$. Thus the problem is solved once we have calculated the Fourier transform of the energy-momentum tensor of the source.

In Section 4.3, we are interested in the total energy radiated in a collision process of two particles. To calculate the energy distribution (D.13) in this case, we need to know the energy-momentum tensor of two moving particles and its Fourier transform. However, we first want to give a general formula describing the energy-momentum tensor of a system of free particles moving with arbitrary constant velocities (see [31]).

The energy-momentum tensor of a moving particle can be obtained from the energy-momentum tensor of a static particle by boosting the particle to a frame moving with constant velocity v. The only non-zero component of the energy-momentum tensor of a particle of mass M at rest in a frame with coordinates $\bar{x}^{\mu} = (\bar{t}, \bar{x}, \bar{y}, \bar{z})$ is

$$\bar{T}_{\bar{t}\bar{t}} = M\delta(\bar{x})\delta(\bar{y})\delta(\bar{z}). \tag{D.14}$$

Under the Lorentz transformation (3.51) with $c = 1$, i.e.

$$t = \gamma(\bar{t} + v\bar{x}), \quad x = \gamma(\bar{x} + v\bar{t}), \quad y = \bar{y}, \quad z = \bar{z}, \tag{D.15}$$

the energy-momentum tensor transforms according to the rule

$$T^{\mu\nu} = \frac{\partial x^\mu}{\partial \bar{x}^\alpha}\frac{\partial x^\nu}{\partial \bar{x}^\beta}\bar{T}^{\alpha\beta} \tag{D.16}$$

which yields the following non-zero components:

$$\begin{aligned} T^{tt} &= \gamma M \delta(x - vt)\delta(y)\delta(z), \\ T^{tx} &= \gamma M v \delta(x - vt)\delta(y)\delta(z), \\ T^{xx} &= \gamma M v^2 \delta(x - vt)\delta(y)\delta(z), \end{aligned} \tag{D.17}$$

with the help of the property $\delta(au) = \delta(u)/|a|$. These are the components of the energy-momentum tensor of a particle of mass M moving along the x-axis with the constant speed v. All these can be written more compactly by using the relativistic notation:

$$T^{\mu\nu} = \frac{P^\mu P^\nu}{E}\delta(x - vt)\delta(y)\delta(z), \tag{D.18}$$

where $E = \gamma M$ is the energy and $P^\mu = (E, P^x, 0, 0) = \gamma M(1, v, 0, 0)$ is the 4-momentum of the particle. Now we can easily generalize (D.18) for an arbitrary number of particles mowing in arbitrary directions with arbitrary velocities, namely

$$T^{\mu\nu} = \sum_n \frac{P_n^\mu P_n^\nu}{E_n}\delta^3(\mathbf{x} - \mathbf{v}_n t), \tag{D.19}$$

where

$$E_n = \frac{M_n}{\sqrt{1 - v_n^2}}, \quad P_n^\mu = (E_n, \mathbf{P}_n) = E_n(1, \mathbf{v}_n) \tag{D.20}$$

are the energy and 4-momentum of the nth particle.

Let us now consider a system of n free particles that are initially moving with constant velocities $\mathbf{v}_n$. We assume that they collide at the origin at $t = 0$, and scatter one another to different directions with different velocities $\mathbf{v}'_n$. Then the total energy-momentum tensor is

$$T^{\mu\nu}(t, \mathbf{x}) = \sum_n \frac{P_n^\mu P_n^\nu}{E_n}\delta^3(\mathbf{x} - \mathbf{v}_n t)\theta(-t) + \sum_n \frac{P_n'^\mu P_n'^\nu}{E'_n}\delta^3(\mathbf{x} - \mathbf{v}'_n t)\theta(t) \tag{D.21}$$

where $\theta(t)$ is the step function. Here the first term represents the particles before the collision, while the second term represents them after the collision. There is an abrupt change in the quantities E_n and P_n^μ at $t = 0$ due to the collision. In order to calculate the distribution of the emitted energy, we need to find the Fourier transform of the energy-momentum tensor (D.21). This can be done by remembering the well-known integral representations of the functions $\theta(s)$ and $\delta^3(\mathbf{x}-\mathbf{x}')$ given by

$$\begin{aligned}\theta(s) &= -i\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{e^{i\omega s}}{\omega - i\varepsilon},\\ \delta^3(\mathbf{x}-\mathbf{x}') &= \int\frac{d^3k}{(2\pi)^3}\,e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}.\end{aligned} \tag{D.22}$$

Then (D.21) becomes

$$\begin{aligned}T^{\mu\nu}(t,\mathbf{x}) \;=\; -i\int\frac{d^3k}{(2\pi)^3}\int\frac{d\omega}{2\pi}\sum_n\Bigg[&\frac{P_n^\mu P_n^\nu}{E_n}\,e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{v}_n t)}\frac{e^{-i\omega t}}{\omega - i\varepsilon}\\ &+\frac{P_n'^\mu P_n'^\nu}{E_n'}\,e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{v}_n' t)}\frac{e^{i\omega t}}{\omega - i\varepsilon}\Bigg].\end{aligned} \tag{D.23}$$

Changing the variables

$$\omega \to \omega - \mathbf{k}\cdot\mathbf{v}_n$$

in the first term, and

$$\omega \to -(\omega - \mathbf{k}\cdot\mathbf{v}_n')$$

in the second term, we get

$$\begin{aligned}T^{\mu\nu}(t,\mathbf{x}) &= -i\int\frac{d^3k}{(2\pi)^3}\int\frac{d\omega}{2\pi}e^{-i\omega t}\sum_n\Bigg[\frac{P_n^\mu P_n^\nu}{E_n}\frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{\omega - \mathbf{k}\cdot\mathbf{v}_n - i\varepsilon}\\ &\qquad -\frac{P_n'^\mu P_n'^\nu}{E_n'}\frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{\omega - \mathbf{k}\cdot\mathbf{v}_n' + i\varepsilon}\Bigg]\\ &= \int\frac{d\omega}{2\pi}\,e^{-i\omega t}\,T^{\mu\nu}(\omega,\mathbf{x}),\end{aligned} \tag{D.24}$$

where we have used (D.5). So we have

$$\begin{aligned}T^{\mu\nu}(\omega,\mathbf{x}) &= -i\int\frac{d^3k}{(2\pi)^3}\,e^{i\mathbf{k}\cdot\mathbf{x}}\sum_n\Bigg[\frac{P_n^\mu P_n^\nu}{E_n(\omega - \mathbf{k}\cdot\mathbf{v}_n - i\varepsilon)}\\ &\qquad -\frac{P_n'^\mu P_n'^\nu}{E_n'(\omega - \mathbf{k}\cdot\mathbf{v}_n' + i\varepsilon)}\Bigg]\\ &= \int\frac{d^3k}{(2\pi)^3}\,e^{i\mathbf{k}\cdot\mathbf{x}}\,T^{\mu\nu}(\omega,\mathbf{k})\end{aligned} \tag{D.25}$$

using (D.6). Therefore, we find

$$T^{\mu\nu}(\omega,\mathbf{k}) = -i\sum_n\Bigg[\frac{P_n^\mu P_n^\nu}{E_n(\omega - \mathbf{k}\cdot\mathbf{v}_n - i\varepsilon)} - \frac{P_n'^\mu P_n'^\nu}{E_n'(\omega - \mathbf{k}\cdot\mathbf{v}_n' + i\varepsilon)}\Bigg]. \tag{D.26}$$

We can drop $\mp i\varepsilon$ in the denominator because $\omega - \mathbf{k}\cdot\mathbf{v}_n$ cannot vanish if $\omega = |\mathbf{k}|$ and $|\mathbf{v}_n| < 1$. For the case of particles traveling at the speed of light, there seems to be a singularity, but it is actually spurious (see [31]). Thus we finally write

$$T^{\mu\nu}(\omega, \mathbf{k}) = -i \sum_n \left[\frac{P_n^\mu P_n^\nu}{E_n(\omega - \mathbf{k} \cdot \mathbf{v}_n)} - \frac{P_n'^\mu P_n'^\nu}{E_n'(\omega - \mathbf{k} \cdot \mathbf{v}_n')} \right]. \tag{D.27}$$

This is the Fourier transform of the energy-momentum tensor of a system of n free particles, which is exploited in Section 4.3 to obtain the radiation spectrum for the special case of two colliding particles.

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