PROLONGATION STRUCTURES, BÄCKLUND TRANSFORMATIONS AND PAINLEVÉ ANALYSIS OF NONLINEAR EVOLUTION EQUATIONS

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ABSTRACT

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The Wahlquist-Estabrook prolongation technique and the Painlevé analysis, used for testing the integrability of nonlinear evolution equations, are considered and applied both to the Drinfel'd-Sokolov system of equations, indeed known to be one of the coupled Korteweg-de Vries (KdV) systems, and Kersten-Krasil'shchik coupled KdV-mKdV equations. Some new Bäcklund transformations for the Drinfel'd-Sokolov system of equations are also found.

Keywords: Prolongation, Bäcklund, Painlevé, Drinfel'd-Sokolov, Kersten-Krasil'shchik.

ÖZ

DOĞRUSAL OLMAYAN EVRİM DENKLEMLERİNİN UZATMA YAPILARI, BÄCKLUND DÖNÜŞÜMLERİ VE PAINLEVÉ ANALİZİ

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Doğrusal olmayan evrim denklemlerinin integre edilebilirliğini test etmede kullanılan Wahlquist-Estabrook uzatma tekniği ve Painlevé analizi incelendi ve Korteweg-de Vries (KdV) tipi denklem çiftlerinden biri olarak bilinen Drinfel'd-Sokolov sistemi ile Kersten-Krasil'shchik bağımlı KdV-mKdV denklemlerine uygulandı. Ayrıca Drinfel'd-Sokolov sisteminin bazı yeni Bäcklund dönüşümleri bulundu.

Anahtar Sözcükler: Uzatma, Bäcklund, Painlevé, Drinfel'd-Sokolov, Kersten-Krasil'shchik.

...TO MY FAMILY

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CHAPTER 1

INTRODUCTION

It is well-known that the mathematical modeling of a great variety of phenomena leads to certain nonlinear evolution equations. For example in classical field theories such as general relativity and nonabelian gauge theories the equations of motion are nonlinear. Other than these areas nonlinear evolution equations are also encountered in branches of physics and engineering such as fluid dynamics, laser physics, protein dynamics, plasma physics etc [1]. These differential equations are usually solved by the use of approximation techniques and numerical methods. However, although some valuable information can be obtained from approximate solutions, the range of applicability and the usefulness of these solutions increase the interest in closed-form solutions and in methods for generating solutions of nonlinear partial differential equations. In fact any analytic result is always preferable to a numerical computation. Furthermore the analytic approach often provides global knowledge of the solution, whereas the numerical approach is always local and hence mostly insufficient. Therefore it is important to classify nonlinear partial differential equations according to their integrability properties.

Historically one of the major developments in the domain of integrability was initiated by the famous observation of John Scott Russell in 1834 [2]. Approximately sixty years later two Dutch mathematicians Korteweg and de Vries derived their famous equation, which is now known as the Korteweg-de Vries (KdV) equation, using Russell's observations [3]. Their equation is a nonlinear evolution equation describing the propagation of long, one-dimensional, small amplitude, surface gravity waves in a shallow water channel[1]. From the mathematical point of view it is the simplest nonclassical partial differential equation since it has the minimum number of independent variables, the simplest (linear and unmixed) and the fewest number of third-order terms (the lowest-order which is required not to be treated classically), the lowest-order (also the simplest and fewest number) term involving differentiation with respect to an other independent variable and the simplest additional term (uu_x) to make the whole equation nonlinear [4]. Although the KdV equation is such an important equation both from physical and mathematical points of view, it had been ignored for many years and had to wait until 1965 to gain its own fame. The reason that makes this equation famous is that some methods for generating solutions, which are applicable to a large class of equations exhibiting some important properties, have been derived from the study of it and its properties.

In 1965 Zabusky and Kruskal [5] obtained the KdV equation when they took the continuum limit of the Fermi-Pasta-Ulam problem [6] and noticed a very remarkable fact about the interaction properties of the localized wave solutions: After interaction these waves regained their initial amplitude and velocity and

[1] Here "small" and "long" mean in comparison to the depth of the channel.

the only remaining effect of interaction was a phase shift. Because of this strange particle-like property they called the solutions *solitons*. Soon after Miura, Lax, Kruskal and his coworkers significantly contributed to the development of the subject [7, 8, 9, 10, 11]. Then, Zakharov and Shabat [12] showed that this important property was not only particular to the KdV equation by explicitly demonstrating that the nonlinear Schrödinger equation also possessed this property. Following the ideas starting from the observations of Kruskal Ablowitz *et al* [13] solved the sine-Gordon equation. In a short time it was shown that many other equations indeed had this property. In this way a new research area in the domain of integrability had been opened.

During the past three decades these and many other developments in Applied Mathematics and Mathematical Physics show that the completely integrable systems of nonlinear partial differential equations have rich mathematical structures such as the existence of Lax pairs, Miura maps, Bäcklund transformations, infinitely many local conservation laws, bi-Hamiltonian structures and recursion operators.

In addition to these it is also well-known in the field of Mathematical Physics that the integrable equations can be solved by the inverse scattering method [13]. The main idea of this method is that, once the Lax pair for a nonlinear partial differential equation is given, one can find a method to construct the soliton solutions [5]. However, before attempting to solve an equation one usually needs to know whether the equation is integrable or not. The obstacles appeared in the construction of the inverse scattering transform technique and make this information more important. Thus there is a strong need for some machinery to test the integrability of nonlinear partial differential equations. In fact there are several approaches for this need such as numerical tests, the search of generalized symmetries and constants of motion, Hirota's direct method, prolongation structures, Painlevé analysis etc [1]. Although most of these methods are basically indicators rather than being decisive tests, two of them are really very powerful and indeed used not only for testing integrability but also in the construction of analytical solutions.

One of these effective methods is the prolongation structure technique of Wahlquist and Estabrook (W-E) [14] which attempts to construct a linear spectral problem associated with the nonlinear equations under consideration. This technique was originally formulated in the framework of differential forms and Cartan's exterior differential systems. Subsequently Dodd and Fordy [15, 16] made the method algorithmic. The other one is the Painlevé test, which uses the Painlevé Property in order to test the integrability of the nonlinear partial differential equations. This method was introduced into the domain of integrability in the sense that is used now by Ablowitz *et al* [17]. Both of these methods can well lead to Bäcklund transformations, which are the relations among solutions of the nonlinear differential equations under consideration.

In this thesis firstly in Chapter 2 and Chapter 3 we give the mathematical preliminaries and the theoretical background for the prolongation structures, the Bäcklund transformations and the Painlevé analysis, which are used subsequently.

Each of these subjects are discussed in the following manner: Firstly we briefly present the historical development. Then we describe the motivation and the main ideas behind. Whenever possible we intend to give basic algorithms in order to handle the analysis more easily. Finally each discussion is finalized by applying the techniques to the KdV equation as an example since this important equation can be considered as a prototype for generating all of these methods used in the field of integrability.

Next in Chapter 4 we apply the W-E prolongation method to a system of nonlinear evolution equations given by Drinfel'd and Sokolov [18]. After showing that this system of equations has a nontrivial prolongation structure, we try to close the prolongation algebra. The closure process gives rise to the $s\ell(4,C)$ algebra which is used in deriving the scattering problem for the system of equations under investigation [19]. We also give new nontrivial Bäcklund transformations and some explicit solutions.

Finally in Chapter 5 we investigate the classical part of one of the supersymmetric extensions of the KdV equation given by Kersten and Krasil'shchik [20]. After giving a brief motivation for this system of equations, we discuss the integrability of it in terms of singularity analysis. Having shown that the system passes the Painlevé test, we find a nontrivial prolongation structure and hence a Lax pair for this system [21] and show that the Lax pair found is indeed nontrivial.

CHAPTER 2

WAHLQUIST-ESTABROOK PROLONGATION METHOD AND BÄCKLUND TRANSFORMATIONS

The techniques for generating the corresponding hierarchy of polynomial flows for a given nondegenerate eigenvalue problem are both easy and completely systematic [8, 22, 23, 24, 25]. However, finding an appropriate linear eigenvalue problem for a given arbitrary nonlinear differential equation is very difficult. In fact, in the absence of any completely systematic method, most linear eigenvalue problems of physically interesting equations were found by *ad hoc* procedures [16]. The reasonably systematic[1] method of finding the eigenvalue problem or Bäcklund transformation for a given arbitrary equation is the W-E prolongation method.

In this Chapter the W-E prolongation method, originally formulated in the framework of differential forms and Cartan's exterior differential systems, is investigated. Although the basic process of constructing a linear spectral problem is algebraic, we still prefer to use the language of differential forms defined on jet-bundles which provides a natural geometric framework for studying differential equations [26]. Even analytically this notation is superior for any treatment of differential equations [14]. The basic idea of jet-bundles is to consider the

[1] Systematic up to a certain degree.

dependent variables and the derivatives of dependent variables with respect to independent ones as additional, algebraically independent variables. An exact introduction to the theory of jet-bundles can be found in [26, 27] and an informal description mainly in local coordinates is given in Appendix A. The symmetries, conservation laws and the Bäcklund problem for a differential equation can be well understood by using jet-bundles. For example Bäcklund transformations require the manipulation of the partial derivatives of dependent variables and repeated shifts of point of view about which variables depend upon which. Clearly the concepts involved are considerably simplified by the help of jet-bundle formalism.

In the first section we describe the W-E prolongation method, which attempts to construct a linear spectral problem associated with the nonlinear equations under consideration. The success of the method comes from the fact that a successful application not only implies complete integrability of the equation but also gives us a chance to integrate it. On the other hand the failure of the method strongly suggests that the equation is not integrable; however, it does not guarantee nonintegrability [1]. In the second section an algebraic procedure for closing off the free Lie algebra is discussed. The main theme in this section is centered around the identification of the nilpotent and the semisimple elements and their embedding into a simple Lie algebra. When such an element is one of the generators of our algebra, the calculations are greatly simplified. In this section the scaling symmetries, which are very important in the construction of the eigenvalue problem, are also discussed. The third section is devoted to Bäcklund transformations which can be defined as the transformations between the integral manifolds of differential ideals. In this section Bäcklund transformations are discussed both from the classical and differential geometric points of view. A comparison between the two procedures for finding Bäcklund transformations, namely the Wahlquist-Estabrook procedure and the Rogers-Shadwick procedure [28], where the latter is in fact the generalization of the former, is also given. Finally in the last section the prolongation algebra and the Bäcklund transformation of the KdV equation is given.

2.1 The Wahlquist-Estabrook Prolongation Method

The starting point of the W-E prolongation method is to use Cartan's geometric theory of partial differential equations which consists of expressing the differential equations in terms of differential forms on jet-bundles. Actually a system of partial differential equations with any number of independent and dependent variables and involving partial derivatives of any order can be written as an exterior differential system [29]. The criteria for the equivalence of a given set of partial differential equations with a closed set of differential forms on jet-bundles has been deeply discussed by Cartan [30, 31]. In fact partial differential equations and exterior differential systems with an independence condition are essentially the same object [29]. We investigate the exterior systems firstly by giving some definitions.

Let N be a manifold. A collection of differential forms on N is said to be an *ideal*, denoted by I, if the following conditions are satisfied:

- For α_1 and $\alpha_2 \in I$,

$$\phi_1\alpha_1 + \phi_2\alpha_2 \in I\,,$$

  where ϕ_1 and ϕ_2 are functions on N.

- For $\alpha \in I$ and η any arbitrary form,

$$\alpha \wedge \eta \in I\,,$$

  where $\wedge$ denotes the exterior product (antisymmetric tensor product).

- For any $\alpha \in I$,

$$d\alpha \in I.$$

This is the closure condition and is necessary for closed ideals. In the literature closed ideals are called *differential ideals.*

An *integral manifold* of the system I is a pair (M, f), where M is a submanifold of N and $f : M \to N$ is a differentiable mapping such that

$$f^*(\alpha) = 0\,, \qquad \forall \alpha \in I, \tag{2.1}$$

for which f^* is the pull-back map. The terms *integral submanifold* or *solution manifold* are also used for integral manifolds.

An *exterior differential system* is a pair (N, I), where N is a manifold and I is a differential ideal. Usually it is denoted only by I. If additionally there is a differential p-form[2] Ω, then the pair (I, Ω) is called an exterior differential

[2] p denotes number of independent variables.

system with an independence condition. An integral manifold of (I, Ω) is an integral manifold of I with the additional condition $f^*\Omega \neq 0$ satisfied. Note that, while for an exterior differential system we do not specify any independent variables, we choose independent variables for exterior differential systems with an independence condition.

In order to represent a partial differential equation as a system of forms firstly new variables, which can be used to write the differential equation as a set of first-order differential equations, are defined. Next N is described to be a manifold the local coordinates of which consist of all independent and dependent variables in the partial differential equation and auxiliary variables introduced in the previous step. By taking the exterior derivative of these coordinates the basis forms can be adopted in order to define the forms α_i on N. These forms generate a differential ideal. Then it is required that any integral manifold (solution manifold) M, which consists of independent variables, annuls this set of forms and the restriction (or sectioning) of initial set of forms α_i on N to M gives us the partial differential equation from which we started, that is on the solution manifold (2.1) gives our partial differential equation in the form of exterior differential equations. It is seen that, if $f^*(\alpha_i) = 0$, then for any form η_i on N we have

$$f^*\Big(\sum_i \eta_i \wedge \alpha_i\Big) = 0 \qquad \text{and} \qquad f^*(d\alpha_i) = 0. \tag{2.2}$$

Hence any form in I generated by α_i vanishes when restricted to the solution manifold M. Therefore it is the whole ideal I generated by the α_i that represents the partial differential equation [32].

It is possible to construct different exterior systems which generate the same ideal. Two exterior systems, α_i and α_i', can be considered as algebraically equivalent if they generate the same ideal [32]. Obviously algebraically equivalent systems have the same solutions since they represent the same partial differential equation. So there arises the question of which one to choose. A natural answer is that we may choose the smallest exterior differential system.

To assert complete equivalence between the forms and the differential equations we require that the set of forms must be closed. This requirement is necessary since on a solution manifold, if we are to have $f^*(\alpha_i) = 0$ which gives back the differential equation, we must also have $d(f^*(\alpha_i)) = f^*(d\alpha_i) = 0$ (since the pull-back map and the exterior derivative commute) and the equations $f^*(d\alpha_i) = 0$ cannot impose any additional conditions. Hence we need to have $d\alpha_i \in I$ which implies that the exterior derivatives of all the forms must be contained in the ring of forms generated by the set

$$d\alpha_i = \sum_{j=1} \eta_{ij} \wedge \alpha_j \,, \tag{2.3}$$

where the summation runs from 1 to the number of forms in the initial set and η_{ij} is some set of 1-forms. This condition also says that all the integrability conditions of the first-order partial differential equations, which we found at the beginning by defining new variables, are satisfied.

To make these points clear we give an example. Consider the KdV equation

$$u_t + u_{xxx} + 12uu_x = 0. \tag{2.4}$$

Defining the variables,

$$z \equiv u_x \qquad \text{and} \qquad p \equiv z_x = u_{xx}, \tag{2.5}$$

we can write (2.4) as a first-order equation

$$u_t + p_x + 12uz = 0. \tag{2.6}$$

Now (2.5) together with (2.6) constitute a set of first-order partial differential equations. We then define the manifold N with coordinates (x, t, u, z, p) and express (2.5) and (2.6) by the following set of differential forms,

$$\begin{aligned} \alpha_1 &= du \wedge dt - z dx \wedge dt, \\ \alpha_2 &= dz \wedge dt - p dx \wedge dt, \\ \alpha_3 &= -du \wedge dx + dp \wedge dt + 12uz dx \wedge dt. \end{aligned} \tag{2.7}$$

Restriction of these forms to an integral manifold M with coordinates (x, t) annuls all of the forms because on a solution manifold M we have

$$du = u_x dx + u_t dt \tag{2.8}$$

and similarly for z and p. By using (2.1) we have the sectioned forms[3]

$$\begin{aligned} \tilde{\alpha}_1 &= (u_x - z) dx \wedge dt = 0, \\ \tilde{\alpha}_2 &= (z_x - p) dx \wedge dt = 0, \\ \tilde{\alpha}_3 &= (u_t + p_x + 12uz) dx \wedge dt = 0. \end{aligned} \tag{2.9}$$

On the integral manifold M with coordinates (x, t) the independence condition is $\Omega = dx \wedge dt$ and, since we have $f^*\Omega \neq 0$ on M, (2.9) implies the KdV equation.

[3] The sectioned forms are denoted by a tilde.

We also note that, since the exterior derivative of (2.7), viz

$$\begin{aligned} d\alpha_1 &= dx \wedge \alpha_2, \\ d\alpha_2 &= dx \wedge \alpha_3, \\ d\alpha_3 &= -12 dx \wedge (z\alpha_1 + u\alpha_2), \end{aligned} \tag{2.10}$$

is in the form of (2.3), the initial set of forms in (2.7) constitutes a closed ideal.

In this thesis we consider only those equations of evolution type with two independent variables x and t which can be written as

$$u^\gamma_t = u^\gamma_{nx} - K^\gamma(u^\beta, u^\beta_x, ..., u^\beta_{(n-1)x}), \tag{2.11}$$

where K is an arbitrary function depending on u^β and x-derivatives of u^β. Clearly the KdV equation is in this form. By considering the above example and some others given in [16] we can generalize that equations of evolution type may be expressed by the following set of differential forms

$$\begin{aligned} \alpha^\beta_i &= du^\beta_{(i-1)x} \wedge dt - u^\beta_{ix} dx \wedge dt, \qquad i = 1, 2, ..., n-1, \\ \alpha^\beta_n &= du^\beta \wedge dx + du^\beta_{(n-1)x} \wedge dt + K^\beta dx \wedge dt. \end{aligned} \tag{2.12}$$

After one has written the differential equation as a differential ideal, the next idea in the W-E prolongation method is to search for the existence of 1-forms. The motivation comes from the existence of conservation laws, which describe quantities that remain invariant during the evolution of a partial differential equation. It is well known that integrable nonlinear evolution equations possess an infinite set of conservation laws [8, 33, 34]. Thus it is natural to expect the existence of various 1-forms which lead to conservation laws.

Any differential equation in the form of (2.11) has local conservation laws of the form

$$\frac{\partial f^k}{\partial t} + \frac{\partial g^k}{\partial x} = 0\,, \qquad k = 1, 2, \ldots\,, \tag{2.13}$$

where f^k and g^k are local expressions depending on the jet variables. In fact they depend on the independent variables, dependent variables and the x-derivatives of the dependent variables since the t-derivatives of the dependent variables can be eliminated by using (2.11).

One of the important properties of these conservation laws is that, they lead to conserved quantities with appropriate boundary conditions. For example integrating (2.13) with respect to x we get

$$\frac{d}{dt}\int_{X_A}^{X_B} f^k\, dx + \left[g^k\right]_{X_A}^{X_B} = 0\,. \tag{2.14}$$

Under asymptotic boundary conditions in which the dependent variables are rapidly vanishing as $x \to \pm\infty$ or under periodic boundary conditions with $(X_B - X_A)$ an integer multiple of the period, the term $\left[g^k\right]_{X_A}^{X_B}$ vanishes. Hence (2.13) gives rise to the constant of motion $\int_{X_A}^{X_B} f^k\, dx$. In fact these conserved quantities can only be defined up to exact x-derivatives since (2.13) is invariant under the following transformation

$$f^k \to f^k + \Lambda^k_x\,, \qquad g^k \to g^k - \Lambda^k_t\,. \tag{2.15}$$

Another property of conservation laws is their close relation with potential functions. A function ϕ of dependent and independent variables is called a *potential function* if it is a constant when the differential equation is satisfied. Thus cross derivative integrability condition implies that every conservation law of the form (2.13) defines a potential function ϕ^k, where

$$\frac{\partial \phi^k}{\partial t} = g^k\,, \qquad \frac{\partial \phi^k}{\partial x} = -f^k\,, \tag{2.16}$$

such that

$$d\phi^k = -f^k dx + g^k dt \tag{2.17}$$

is an exact differential.

In the language of differential forms the above discussion about conservation laws corresponds to the existence of exact 2-forms contained in the ring of the α_is that generate the closed ideals. Therefore we can find 2-forms

$$\beta^k = \sum_{i=1} h_i{}^k \, \alpha_i \tag{2.18}$$

satisfying $d\beta^k = 0$, the condition for exactness. In (2.18) $h_i{}^k$ are arbitrary functions of jet variables and the summation runs from 1 to the number of the forms in the initial set that generates the ideal. In fact $d\beta^k = 0$ can be considered as the integrability condition for the existence of 1-forms, say ω^k, such that

$$\beta^k = d\omega^k\,. \tag{2.19}$$

Since the double exterior derivative of any differential form vanishes, (2.19) conversely implies that $d\beta^k = 0$.

For each exact 2-form $d\omega^k$ the associated conservation law is obtained by using Stoke's theorem

$$\oint_{\partial M_1} \omega^k = \int_{M_1} d\omega^k\,, \tag{2.20}$$

where M_1 is any two-dimensional manifold and ∂M_1 is its closed one-dimensional boundary. If M_1 is chosen to be the solution manifold, on which the restricted exact 2-forms $d\omega^k$ are annulled, then (2.20) gives the conservation laws for ω^k.

In general the 1-forms ω^k can be taken in the form of (2.17)

$$\omega^k = F^k dx + G^k dt\,. \tag{2.21}$$

Consider the KdV equation in (2.4). It has an infinite set of local conservation laws [1] all of which are in the form of (2.13). Actually the KdV equation itself can be written as one of those conservation laws

$$u_t + (p + 6u^2)_x = 0\,, \tag{2.22}$$

from which we can deduce its simplest potential,

$$\phi_x = -u\,, \qquad \phi_t = 6u^2 + p\,. \tag{2.23}$$

Now in accordance with (2.22) take the 1-form ω as

$$\omega = udx - (p + 6u^2)dt\,. \tag{2.24}$$

The exterior derivative of ω,

$$d\omega = \beta = -\alpha_3 - 12u\alpha_1\,, \tag{2.25}$$

shows that the 2-form β is in the ring of the original set of forms α_i which are defined in (2.7) and the exterior derivative of β,

$$d\beta = -d\alpha_3 - 12du \wedge \alpha_1 - 12ud\alpha_1, \tag{2.26}$$

vanishes identically. This can be shown by using (2.7) and (2.10).

As we are motivated by the conservation laws, the core idea in the method is the prolongation of the ideal I generated by α_i. The prolongation process can be summarized as follows.

Consider a differential ideal I on a manifold N and a differential ideal I' on a manifold N' with a projection map,

$$\pi : N' \to N , \tag{2.27}$$

such that I' is constructed by lifting the generators α_i of I to N' and by adding new generators ω^k. In fact this is equivalent to saying that I' is generated by the set $(\pi^* \alpha_i , \omega^k)$. Since I is constructed from a differential equation with two independent variables, the set of forms ω^k in this case is a set of 1-forms in the form of (2.21). In order to have all the integrability conditions satisfied I' generated by the set $(\pi^* \alpha_i , \omega^k)$ is also required to be closed [14]. However, since I' is constructed from I which is known to be closed, it is enough to require that $d\omega^k$ be in I'. Furthermore we have the freedom to add any exact set of 1-forms to ω^k, say dy^k, where y^k are arbitrary scalar functions since they do not change the closure relation. Thus the manifold N is enlarged to a fibre bundle $N' = N \times Y$, where $Y \subset R^m$, by adding the coordinates y^k , $k = 1, \dots$ m. Hence ω^k can be written as

$$\omega^k = dy^k + F^k dx + G^k dt . \tag{2.28}$$

This process of introducing new variables and creating a larger closed ideal is known as the prolongation of the original ideal I.

Any integral manifold M of I' with $f' : M \to N'$ then annuls all the forms that generate I', that is, when N' is restricted to M, we have

$$f'^{*}(\pi^{*}\alpha_i) = 0\,,$$
$$f'^{*}(\omega^k) = 0\,. \tag{2.29}$$

Therefore $(M\,, \pi\circ f')$ is an integral manifold of I' when additional equations given by $f'^{*}\omega^k = 0$ are solved [32, 35]. Furthermore the maximal dimensional regular integral manifolds of I and I' are the same [32, 35, 36]. Thus, if I is a completely integrable system, its prolongation I' is also completely integrable [32, 37].

We now investigate the closure relation for a set of 1-forms ω^k in (2.28) explicitly. Let F^k and G^k in (2.28) depend on the independent, the dependent and the x-derivatives of the dependent variables. The closure relation,

$$d\omega^k = \sum_{i=1}^{n} {h_i}^k \alpha_i\,, \tag{2.30}$$

where ${h_i}^k$ are arbitrary functions and n is the number of the forms α_i, provides the most convenient method to search for such 1-forms. Expanding $d\omega^k$ we have

$$dF^k \wedge dx + dG^k \wedge dt = \sum_{i=1}^{n} {h_i}^k \alpha_i \tag{2.31}$$

or

$$\sum_{\mu} \left(\frac{\partial F^k}{\partial S^\mu} dS^\mu \wedge dx + \frac{\partial G^k}{\partial S^\mu} dS^\mu \wedge dt \right) - \sum_{i=1}^{n} {h_i}^k \alpha_i = 0\,, \tag{2.32}$$

where S^μ is the set of variables upon which F^k and G^k depend. Solving (2.32), that is equating the coefficients of independent 2-forms to zero, we are left with coupled first-order linear equations for F^k and G^k. Then each independent solution of these equations leads to different ω^ks which are considered as the conservation laws for the ideal I.

Prolongation of an ideal I can be done in two different ways. In the first way by requiring (2.28) to be in the ring of the initial set of forms, that is using (2.30), one may search for ω and a *prolongation variable* y. If there exists one, it is possible to add that ω to α_i and prolong the ideal I. Then one may think that F^k and G^k depend on y as well as the other variables upon which they depended before and search for new ωs and new ys. However, this time the closure relation becomes

$$d\omega^k = \sum_{i=1}^{n} h_i{}^k \alpha_i + \eta^k \wedge \omega \,, \tag{2.33}$$

where η^k is some 1-form. This is the closure relation for the first prolonged ideal. If one is successful in finding new ωs, it may be possible to continue in like fashion to prolong the ideal further.

On the other hand in the second way it is possible to think from the beginning that F^k and G^k in (2.28) depend on all the prolongation variables y^k as well as the other variables upon which they depended before. In this case to search for new ω^ks one has to modify the closure relation (2.30) to be

$$d\omega^k - \sum_{i=1}^{n} h_i{}^k \alpha_i - \sum_{i=1}^{m} \eta_i{}^k \wedge \omega_i = 0 \,, \tag{2.34}$$

where m is the number of prolongation variables and $\eta_i{}^k$ is some set of 1-forms. Then this equation can be used, just as (2.30), to find some set of partial differential equations for F^k and G^k, but this time there exist some nonlinear terms in these equations such as

$$\sum_i \left(G^i \frac{\partial F^k}{\partial y^i} - F^i \frac{\partial G^k}{\partial y^i} \right) dx \wedge dt \,. \tag{2.35}$$

We note an important consequence of the existence of two different ways for prolonging the ideal. In (2.23) and (2.24) what we call potential is indeed the prolongation variable y. Actually the prolongation variables y^k appearing in the first way of prolonging the ideal are similar to the potential defined in (2.23) since the functions F^k and G^k do not depend on the newest prolongation variable y^k. Thus it is also natural to refer to these variables as *potentials.* In fact the term potential denotes an integral variable which can be defined by a quadrature. However, this is not always the case. We see that in the second way of prolonging the ideal we may have 1-forms ω^k in which F^k and G^k depend on all of the prolongation variables. In this case we call those prolongation variables *pseudopotentials.* They denote integral variables which cannot be defined by quadratures but those that can only be defined by an integrable set of first-order differential equations. The nonlinear terms in (2.35) are clearly essential for these variables and pseudopotentials cannot be found by using the first way in which we have only linear equations.

Till now we have discussed the basic ideas in order to construct the W-E prolongation method. Indeed most of the discussions above constitute some of the basic steps in the method. Now we give a brief description of the method in the form of a recipe.

Suppose we are given a differential equation. The first step in the W-E prolongation method may be thought of as the representation of this differential equation as a differential ideal. For this purpose the original differential equation is written as a system of first-order equations by defining new variables. Then this set of first-order equations is represented as differential forms. Details of the representation of the differential equation as differential forms were given at the beginning of this section. Even in this first step, for an arbitrary equation, there may be some problems, but, since we are dealing with only equations in the form of (2.11), no such problems occur.

The second step is the extension of this ideal by adding a system of 1-forms ω^k, which depend upon the jet variables used in the construction of the differential forms and some new variables y^k, termed prolongation variables. The prolongation of the ideal is done by using the second way discussed above. Then the requirement that the prolonged ideal must be closed under exterior differentiation gives some set of differential equations for F^k and G^k. We have some nonlinear terms in these equations. Fortunately all of these nonlinear terms have always some commutatorlike structure as in (2.35) and are almost always solvable. The major part of the job in this step is to integrate these equations to find the dependence of F^k and G^k on the jet variables and y^k. Since the set of equations for F^k and G^k are overdetermined, some constraints on the constants of integration naturally arise. Due to the fact that the nonlinear terms are always in a commutatorlike structure, these constraints are in the form of commutation relations between a set of vectors, X_i, depending only on the prolongation variables y^k.

It can be said that the integration constants, X_i, generate a free Lie algebra with constraints. Actually for the equations in the form of (2.11) the constants of integration of F^k generate the free Lie algebra and the constants of integration of G^k are elements of this free Lie algebra [16] (see also Appendix B). Since the constraints are not strong enough for obtaining all the commutation relations, in general this Lie algebra is incomplete and possibly infinite-dimensional which in fact implies the existence of infinite number of possible prolongation variables and associated conservation laws. Thus the next step in the W-E prolongation method is to close this algebra, that is to find the finite-dimensional algebra consistent with the commutation relations. Unfortunately there is no general rule to attain this process. However, a strategy can be formed by the help of the general theory of Lie algebras.

The last step can be considered as finding a representation for the resulting finite-dimensional Lie algebra. In order to obtain a linear scattering problem a linear representation must be found.

On a solution manifold M all of the generators of the prolonged ideal I' are annihilated. Thus the ω^ks in (2.28), which are also generators of I', are annihilated to give

$$dy^k = -F^k dx - G^k dt \,. \tag{2.36}$$

Hence

$$y_x{}^k = -F^k \qquad \text{and} \qquad y_t{}^k = -G^k \,. \tag{2.37}$$

Now, without loss of generality, F^k and G^k can be assumed to be linear in y^k. Thus, when we write $y = (y^1, y^2, \ldots)^T$ and F and G for the matrices $F_j{}^k$ and $G_j{}^k$, where[4]

$$F^k = F_j{}^k y^j \qquad \text{and} \qquad G^k = G_j{}^k y^j \,, \tag{2.38}$$

[4] The Einstein summation convention is used for repeated indices.

the linear scattering problem is achieved

$$\begin{aligned} y_x &= -Fy\,, \\ y_t &= -Gy\,. \end{aligned} \tag{2.39}$$

In this section the first two steps in the recipe have been fully discussed and the last two steps, which are the closure of the prolongation algebra and the finding of the representation of the finite-dimensional Lie algebra, are left to the next section. In the next section, by the help of Lie algebras, an algorithm for the third step initally given by Dodd and Fordy [16] is presented.

2.2 Lie Algebras

As we discussed in the previous section, in the third step of the W-E prolongation method we are faced with a set of vectors, X_i, which satisfy certain commutator relations. In fact they constitute a free Lie algebra with constraints and usually the whole multiplication table cannot be constructed. Therefore it is necessary to close the algebra. For this purpose a nontrivial homomorphism of this set into a finite-dimensional Lie algebra needs be found. However, this is the most difficult part of the W-E prolongation method since there is no systematic way of finding such a homomorphism for every example that arises. In most of the cases *ad hoc* procedures are used in order to close the algebra.

It is known that in almost all examples the scattering problem is governed by a simple Lie algebra [16]. So by the use of this fact it is possible to simplify the problem and to achieve almost an algorithm. Firstly we consider some of the basic concepts of Lie algebras that are useful in the following discussions.

A vector space L over a field F, with an operation $L \times L \to L$, denoted by $(x, y) \to [x, y]$ and called the bracket or commutator of x and y, is called a *Lie algebra* over F if the following properties are satisfied:

i) The bracket operation is bilinear,

ii) $[X, Y] = -[Y, X] \quad (\forall X, Y \in L)$,

iii) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \quad (\forall X, Y, Z \in L)$.

The last condition is called the *Jacobi identity.*

A subspace I of a Lie algebra L is called an *ideal* of L if $X \in L$ and $Y \in I$ together imply $[X, Y] \in I$. By using the Jacobi identity it can be shown that the center $Z(L) = \left\{ Z \in L \mid [X, Z] = 0 \;\; \forall X \in L \right\}$ is actually an ideal. If L has no ideals except itself and 0, it is called *simple.* Clearly, if L is simple, then $Z(L) = 0$. There exists a unique maximal solvable ideal, called the *radical* of L. If the radical of L is 0, then L is called *semisimple.* Since simple algebras are nonsolvable, any simple Lie algebra is semisimple.

A linear map which preserves the commutation operation is called a *Lie algebra homomorphism.* A representation of a Lie algebra, L, is a homomorphism $\phi : L \to g\ell(V)$, where $g\ell(V)$ is a general Lie algebra and V is a vector space over F. For example the linear map $X \to \mathrm{ad}X, \quad X \in L$, is called the *adjoint representation* where $\mathrm{ad}X(Y) = [X, Y]$. Here L itself is considered as the vector space of the representation.

An element $e \in L$ is said to be *nilpotent* if $(\text{ad}e)^n = 0$ for some $n \geq 1$. If $n = 1$, then e is in the center of L. In matrix representation nilpotent elements are conjugate to triangular matrices. An element $X \in L$ is called *semisimple* if $\text{ad}X$ is diagonalizable.

Having discussed the basic concepts in a Lie algebra we now show that a linear scattering problem is governed by a simple Lie algebra. Of course there is the possibility of determining it from a solvable Lie algebra, but this case is not taken into consideration.

We suppose that a linear scattering problem is given as[5]

$$\Psi_x = M\Psi, \qquad \Psi_t = N\Psi, \tag{2.40}$$

where M and N are elements of a Lie algebra L. The integrability condition gives

$$M_t - N_x + [M, N] = 0\,. \tag{2.41}$$

By using Levy's decomposition [38], which states that every Lie algebra is the semidirect sum of its radical and a semisimple subalgebra, it is possible to write L as

$$L = s \oplus_s r\,, \tag{2.42}$$

where s is the semisimple subalgebra and r is the radical. Clearly it is seen that the quotient algebra L/r is semisimple since r is L's radical. Also L satisfies the following commutation relations,

$$[s, s] \subset s, \quad [s, r] \subset r, \quad [r, r] \subset r\,. \tag{2.43}$$

[5] We follow [16] for the subsequent discussion.

Since, as a vector space, L is the direct sum of s and r, there exists a basis of L such that M and N can be decomposed as

$$M = M_s + M_r \quad \text{and} \quad N = N_s + N_r \tag{2.44}$$

and the integrability condition, (2.41), takes the form

$$M_{st} - N_{sx} + [M_s, N_s] = 0\,, \tag{2.45}$$

$$M_{rt} - N_{rx} + [M_s, N_r] + [M_r, N_s] + [M_r, N_r] = 0\,, \tag{2.46}$$

where (2.43) is used.

It is seen that in (2.45) there is no dependence on the variables taken from the radical of the algebra whereas in (2.46) there is coupling. As we noted above, (2.46) is not taken into consideration since it is related to solvable algebras. Thus assuming there is no such coupling our whole attention is given to the semisimple Lie algebras.

However, a further restriction to simple Lie algebras is obvious. Using the theorem [38], which states that any semisimple Lie algebra is the direct sum of a number of simple ideals, it is possible to think that there exists a basis of s such that

$$M_s = \sum_i M_{s_i}\,, \qquad N_s = \sum_i N_{s_i}\,. \tag{2.47}$$

Then (2.45) takes the form

$$M_{s_it} - N_{s_ix} + [M_{s_i}, N_{s_i}] = 0\,, \tag{2.48}$$

which states that all the decoupled equations are determined by a simple Lie algebra.

However, since most of the theory is true for semisimple Lie algebras, we work with them. Our main task is to represent the vectors X_i by matrices and to find a complete multiplication table. To do so the vectors X_i are embedded in a semisimple Lie algebra with a known matrix representation. Since any similarity transformation gives an equivalent embedding, there is no unique way of doing this. Thus the main interest is given to conjugacy classes of elements and there is the freedom to choose a representative matrix for at least one element and sometimes two or more simultaneously. If this or these elements coincide with the generators of the prolongation algebra, that is the integration constants of F^k (see Appendix B), this procedure gains a particular success.

In fact it is known that every complex semisimple Lie algebra contains at least one (and usually many) subalgebra, that is isomorphic to $s\ell(2,C)$ [38]. Thus certain elements of the prolongation algebra are identified with basis elements of $s\ell(2,C)$ and then this copy of $s\ell(2,C)$, if not the whole prolongation algebra, has to be embedded into a larger semisimple Lie algebra. If this larger semisimple Lie algebra is $s\ell(n+1,C)$, where $n \geq 2$, the above discussion is easier to apply.

At the moment we give the basis elements of $s\ell(n+1,C)$. Let h be the abelian subalgebra of all diagonal elements of $s\ell(n+1,C)$; if $a_1,\ldots,a_{n+1} \in C$, $\mathrm{diag}(a_1,\ldots,a_{n+1})$ denotes the diagonal matrix with $a_1,\ldots,a_{n+1}$ as its diagonal entries. We write E_{ij} for the matrix the ijth entry of which is 1, $1 \leq i,j \leq n+1$ and all the remaining entries are 0. Then the matrices

$$E_{ii} - E_{i+1,i+1} \quad (1 \leq i \leq n)\,, \quad E_{ij} \ (i \neq j,\ 1 \leq i,j \leq n+1) \tag{2.49}$$

form a basis for $s\ell(n+1,C)$ [39].

Thus the basis for $s\ell(2,C)$ is

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{2.50}$$

which satisfies the following commutation relations,

$$[h, e_\pm] = \pm 2 e_\pm \,, \qquad [e_+, e_-] = h\,. \tag{2.51}$$

In [40] the elements in (2.50) are termed *neutral*, *nil-positive* and *nil-negative* respectively. In fact any three elements satisfying (2.51) are referred as such. From now our main interest is given to embedding the nilpotent elements into copies of $s\ell(2,C)$. The following theorem and the corollary is used for this purpose.

Theorem 2.2.1 *Every nilpotent element of a complex semisimple Lie algebra can be embedded into a copy of* $s\ell(2,C)$.

Corollary 2.2.1 *Let* $e \in L$ *be nilpotent,* $e \neq 0$*; then* X *and* e *are respectively the neutral and nil-positive elements if and only if the following conditions are satisfied:*

(i) X *is the range of ade,*

(ii) $[X, e] = 2e$.

Furthermore, if X *and* e *satisfy these conditions, then the nil-negative element is uniquely defined.*

The proof of these are given in [38]. It is, of course, also possible to state an analogous corollary for nil-negative elements. Then the nil-positive element is uniquely determined.

After the embedding of the nilpotent elements in a copy of $s\ell(2, C)$ it is necessary to determine if it is the whole prolongation algebra. If not, then it has to be embedded in a larger semisimple Lie algebra, for example $s\ell(n+1, C)$, $n \geq 2$. In order to do this we may equate the nilpotent elements to the linear combinations of basis elements of that larger algebra. However, this way seems to be very long and is unnecessary. Instead it is possible to write these nilpotent elements in a canonical form by using the irreducible $(n+1)$-dimensional representation of $s\ell(2, C)$, which is known [40, 41], and by using the following theorem (for a proof see [39, 41]):

Theorem 2.2.2 (Weyl) *Let $\phi : L \to g\ell(V)$ be a representation of a semisimple Lie algebra L, $V \neq 0$. Then ϕ is completely reducible.*

This means that there exists a basis of V such that the representation of L takes a block diagonal form and each block is an irreducible representation of L.

The concepts given above can be applied to the prolongation algebras. For this purpose the nilpotent and semisimple elements have to be identified. We give the following theorem:

Theorem 2.2.3 *Let X and Y be any two elements of L such that $[Y, [Y, X]] = 0$. Then $[Y, X]$ is nilpotent. In particular, if $[X, Y] = \alpha Y$, $\alpha \neq 0$, then Y is nilpotent.*

The proof of this theorem is given in [40]. By the help of Corollary 2.2.1 this theorem yields an immediate result.

Corollary 2.2.2 *Let X and Y be any two elements of L such that $[X, Y] = \alpha Y$, $\alpha \neq 0$ and $X \in$ range adY. Then it is possible to identify Y with $e_{\pm}$ and X with $\pm\frac{1}{2}\alpha h$.*

Therefore to close and find a representation of a given prolongation algebra the following strategy can be used [16]:

1. Locate the elements of the center of the algebra. Assuming the algebra to be simple equate these elements to zero.

2. Locate a nilpotent and a semisimple element.

3. Embed these into a simple Lie algebra L.

4. Express the remaining elements of the prolongation algebra as linear combinations of a suitable basis of L.

5. Use the fundamental representation of L to generate a linear scattering problem.

Before closing this section we note that scaling symmetries play the major role for having a scattering parameter. Any transformation which leaves the original system of equations invariant is called a symmetry of the system of differential equations. Furthermore, if it is required that the 1-forms ω^k in (2.28) be invariant under such a transformation, then the coefficients of F and G are induced as a symmetry, that is, the elements X_i must be invariant under some transformations. We only consider the scaling symmetry of the equations. In particular a scaling symmetry of the form

$$x \to \lambda^{-1} x\,, \quad t \to \lambda^{-n} t\,, \quad u_i \to \lambda^{m_i} u_i \tag{2.52}$$

induces

$$F \to \lambda F\,, \qquad G \to \lambda^n G\,, \tag{2.53}$$

which in fact induces a scaling symmetry of the elements X_i.

It is also useful to note that the basis elements of $s\ell(2, C)$ have the following scaling symmetry

$$e_- \to \lambda^{-1} e_-\,, \quad h \to h\,, \quad e_+ \to \lambda e_+\,. \tag{2.54}$$

2.3 Bäcklund Transformations

Bäcklund transformations were first developed around 1880 by considering transformations between surfaces [42, 43]. More precisely they were developed as a generalization of contact transformations and used in related theories of differential geometry and differential equations. For example one of the earliest Bäcklund transformations was for the sine-Gordon equation,

$$u_{xt} = \sin u\,,$$

which originally arose in differential geometry to describe surfaces with a constant negative Gaussian curvature. At the beginning of the twentieth century the subject was subsequently developed by Goursat [44] and Clairin [45]. Later

Lamb [46] used these ideas to construct the Bäcklund transformations for certain nonlinear evolution equations. In 1970s the main interest in Bäcklund transformations was focused on the connection between the integral surfaces of certain nonlinear partial differential equations. They have played an important role in finding the solutions of nonlinear partial differential equations since then. In this section we discuss Bäcklund transformations both from the classical and the differential geometric points of view.

Firstly we give the classical definition of Bäcklund transformations. Following Rogers and Shadwick [28] we describe the problem in R^3. Let

$$u = u(x_1, x_2) \quad \text{and} \quad u' = u'(x'_1, x'_2) \tag{2.55}$$

represent two smooth surfaces Λ and Λ', respectively, in R^3. A set of four relations

$$B_i\left(x_1, x_2, x'_1, x'_2, u, u', \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial u'}{\partial x'_1}, \frac{\partial u'}{\partial x'_2}\right) = 0\,, \quad i = 1, \ldots, 4\,, \tag{2.56}$$

which connect the surface elements

$$\left\{x_1, x_2, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}\right\} \quad \text{and} \quad \left\{x'_1, x'_2, u', \frac{\partial u'}{\partial x'_1}, \frac{\partial u'}{\partial x'_2}\right\} \tag{2.57}$$

of Λ and Λ' respectively, is called a *Bäcklund transformation.*

Since we are only interested in Bäcklund transformations which connect the integral surfaces of partial differential equations, we can consider (2.55) to represent these integral surfaces. In addition consider the following explicit form of (2.56)

$$\frac{\partial u'}{\partial x'_i} = B'_i\left(x_1, x_2, u, u', \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}\right)\,, \qquad i = 1, 2\,, \tag{2.58}$$

and

$$\frac{\partial u}{\partial x_i} = B_i \left(x_1', x_2', u, u', \frac{\partial u'}{\partial x_1'}, \frac{\partial u'}{\partial x_2'} \right), \qquad i = 1, 2, \tag{2.59}$$

together with

$$x_i' = X_i \left(x_1, x_2, u, u', \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2} \right), \qquad i = 1, 2. \tag{2.60}$$

In order that these relations transform a surface $u = u(x_1, x_2)$ with surface element $\left\{x_1, x_2, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}\right\}$ to a surface $u' = u'(x_1', x_2')$ with surface element $\left\{x_1', x_2', u', \frac{\partial u'}{\partial x_1'}, \frac{\partial u'}{\partial x_2'}\right\}$ it is required that the relations,

$$\begin{aligned} du - B_1 dx_1 - B_2 dx_2 &= 0, \\ du' - B_1' dx_1' - B_2' dx_2' &= 0, \end{aligned} \tag{2.61}$$

be integrable. From the mixed derivative integrability conditions

$$\frac{\partial^2 u}{\partial x_1 \partial x_2} = \frac{\partial^2 u}{\partial x_2 \partial x_1} \tag{2.62}$$

and

$$\frac{\partial^2 u'}{\partial x_1' \partial x_2'} = \frac{\partial^2 u'}{\partial x_2' \partial x_1'}, \tag{2.63}$$

we obtain

$$\begin{aligned} \frac{\partial B_1}{\partial x_2} - \frac{\partial B_2}{\partial x_1} &= 0, \\ \frac{\partial B_1'}{\partial x_2'} - \frac{\partial B_2'}{\partial x_1'} &= 0, \end{aligned} \tag{2.64}$$

which guarantee the integrability of (2.61).

In particular suppose that two uncoupled partial differential equations, in two independent variables $x_1 = x$ and $x_2 = t$, for the two functions u and u' are expressed as

$$P(u) = 0 \qquad \text{and} \qquad Q(u') = 0\,, \tag{2.65}$$

where P and Q are two operators, which are in general nonlinear. Let (2.56) be four relations between the functions u and u'. Then (2.56) is a Bäcklund transformation if it is integrable for u' when $P(u) = 0$ and, if the resulting u' is a solution of $Q(u') = 0$, vice versa [47]. If $P = Q$, so that u and u' satisfy the same equation, then $B_i = 0$ is called an *auto-Bäcklund transformation*. Obviously it is clear that, only if the relations in (2.56) are somewhat simpler than the original equations in (2.65), then the method of Bäcklund transformation is considered to be useful for finding the solutions of the desired equation in (2.65). In this connection Bäcklund transformations have two important applications:

- They may be used to generate different solutions for the same differential equation.

- They may be used to link a differential equation to another differential equation the properties of which are well known.

We give a very simple example. One of the auto-Bäcklund transformations is the Cauchy-Riemann relation

$$u_x = u'_t\,, \qquad u_t = -u'_x\,, \tag{2.66}$$

for Laplace's equation

$$u_{xx} + u_{tt} = 0\,, \qquad u'_{xx} + u'_{tt} = 0\,. \tag{2.67}$$

Thus, if $u'(x,t) = xt$ is a solution of Laplace's equation, then $u(x,t)$ can be determined from

$$u_x = x\,, \qquad u_t = -t\,, \tag{2.68}$$

which are obtained by using the Bäcklund transformation in (2.66), and so $u(x,t) = \frac{1}{2}(x^2 - t^2)$ is another solution of Laplace's equation.

From a geometric point of view, that is in terms of differential forms, Bäcklund transformations are described as follows:

Let I and $\tilde{I}$ be two differential ideals on manifolds N and $\tilde{N}$ respectively with the same independent variable manifold M. Consider another differential ideal I' defined on N' as a prolongation of the differential ideals I and $\tilde{I}$ with a pair of projection maps

$$\begin{aligned} \pi &: N' \to N\,, \\ \tilde{\pi} &: N' \to \tilde{N}\,. \end{aligned} \tag{2.69}$$

Since I' is the prolongation of I and $\tilde{I}$, the projection maps π and $\tilde{\pi}$ must satisfy

$$\pi^*(I) \subset I' \qquad \text{and} \qquad \tilde{\pi}^*(\tilde{I}) \subset I'\,. \tag{2.70}$$

These data are said to define a Bäcklund transformation between I and $\tilde{I}$ [48]. The projection maps π and $\tilde{\pi}$ are actually homomorphisms from I' to I and $\tilde{I}$.

We concentrate on the correspondence between integral submanifolds of I and $\tilde{I}$ with these data. Let (M, f') be the integral manifold of I' with $f' : M \to N'$. Then $(M, f = \pi \circ f')$ is an integral manifold of I with $f : M \to N$ and $(M, \tilde{f} = \tilde{\pi} \circ f')$ is an integral manifold of $\tilde{I}$ with $\tilde{f} : M \to \tilde{N}$. Thus we have established some sort of correspondence between integral manifolds (M, f) and $(M, \tilde{f})$ of I and $\tilde{I}$.

Another way to describe Bäcklund transformations in terms of prolongation could be as follows:

Suppose that I is a differential ideal on N with an integral manifold (M, f) where $f : M \to N$ such that

$$f^*(I) = 0 \tag{2.71}$$

and I' is its prolongation[6] defined on the fibre space $N' = N \times Y$ $(Y \subset R^m)$ with a projection map $\pi : N \times Y \to N$. The generators of I' are $\{\pi^* \alpha_i, \omega^k\}$. If (M, f') is an integral manifold of I', then

$$\begin{aligned} f'^*(\pi^* \alpha_i) &= 0\,, \\ f'^*(\omega^k) &= 0\,. \end{aligned} \tag{2.72}$$

A diffeomorphism

$$\tau : N \times Y \to N \times Y \tag{2.73}$$

is a Bäcklund symmetry if $\tau^*(\pi^* \alpha_i) \subset I'$ is satisfied [48]. This is a Bäcklund transformation in the sense that, given a solution manifold (M, f), another solution manifold may be generated via τ. Indeed τ is symmetry since it maps solutions into solutions. More precisely the map $\pi \circ \tau \circ f' : M \to N$ defines a new solution manifold $(M, \pi \circ \tau \circ f')$ by τ from f' since

$$\begin{aligned} (\pi \circ \tau \circ f')^*(\alpha_i) &= f'^* \circ \tau^* \circ \pi^*(\alpha_i) = f'^* \circ \tau^*(\pi^* \alpha_i) \\ &= f'^*(\pi^* \alpha_i, \omega^k) = 0 \end{aligned} \tag{2.74}$$

where in the last equality we used (2.72).

[6] Recall that the details of the prolongation of an ideal were given in the first Section of this Chapter.

The discussions above about the Bäcklund correspondence between I and $\tilde{I}$ can be shown in the following diagram,

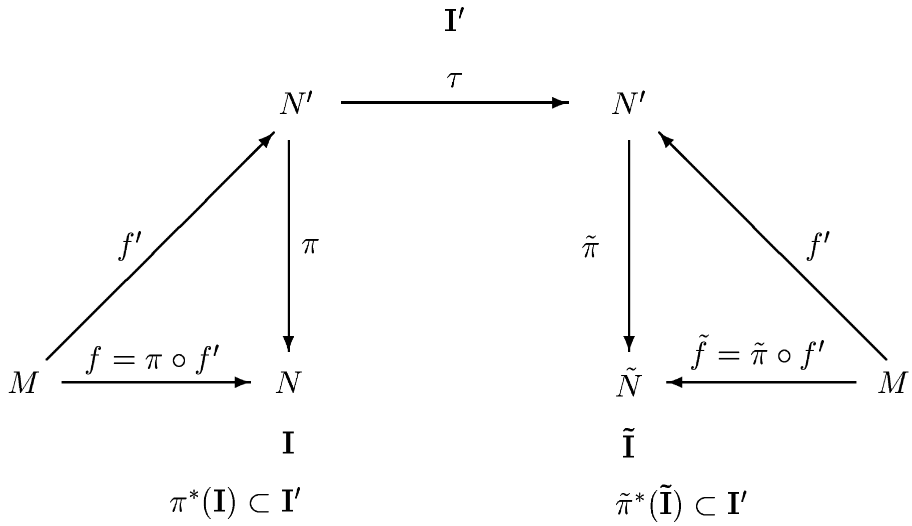

Figure 2.1: The Bäcklund correspondence between I and $\tilde{I}$.

where I' is the augmentation of the ideals I and $\tilde{I}$. In Fig. (2.1) I' is projected into I and $\tilde{I}$ and then this sets up a correspondence between the integral manifolds of I and $\tilde{I}$ which is called a *Bäcklund transformation*. In many cases I and $\tilde{I}$ may be isomorphic which implies an *auto-Bäcklund transformation*.

In practice one efficient way to find Bäcklund transformations is firstly to search for a prolongation structure and for the associated pseudopotentials y^k. Then the new variables in $\tilde{I}$ (usually just the dependent variables) are assumed to be functions of the old variables in I and of the y^k and these new variables are required to satisfy the appropriate equations by the help of the equations for y^k. From this point of view the pseudopotentials can be interpreted as the difference of two solutions related by the Bäcklund transformations [49]. Actually this way of searching for Bäcklund transformations was firstly used by Wahlquist and Estabrook [14]. Later their ideas were generalized by Rogers and Shadwick and their coworkers in [28] and [50] by using the jet-bundle formalism. In fact the procedure developed by Wahlquist and Estabrook has a natural formalism within a jet-bundle context. However, in their method only a subideal is used. On the other hand in the generalized method of [28] and [50] the whole contact ideal is used. Although the method of [28] and [50] provides a more general framework, the W-E method requires fewer variables and is very efficient. Thus we use the W-E method for searching Bäcklund transformations.

In the literature Bäcklund transformations such as (2.58) are sometimes called *Bäcklund maps* and the name 'Bäcklund transformation' is reserved for the correspondence between systems of partial differential equations induced by Bäcklund maps. In this connection a Bäcklund map is defined as a transformation of the dependent variables in a system of differential equations in which the first derivatives of the new variables are given in terms of the new variables themselves as well as of the old variables and their derivatives. Then the original system of differential equations appears as a system of integrability conditions for the Bäcklund map.

Before passing to the next section in which the prolongation structures and Bäcklund transformations of the KdV equation are fully discussed we want to mention one of the most interesting results of Bäcklund transformations. They lead to a simple superposition formula, known as the theorem of permutability, by which multisoliton solutions may be constructed from single-soliton solutions by purely algebraic means.

2.4 An Example: The Korteweg-de Vries Equation

2.4.1 Prolongation Structure of the Korteweg-de Vries Equation

Consider the KdV equation

$$u_t + u_{xxx} + 12uu_x = 0\,. \tag{2.75}$$

We showed that (2.7) constitutes a closed ideal I for this equation. Following the W-E prolongation method, we extend this ideal I by adding to it the system of 1-forms

$$\omega^k = dy^k + F^k dx + G^k dt \qquad k = 1, \ldots, N\,, \tag{2.76}$$

where F^k and G^k are functions of (u, z, p, y^k). Then we require that this set of 1-forms be closed. Thus writing (2.34) explicitly and equating the coefficients of various independent 2-forms to zero gives us some set of partial differential equations for F^k and G^k. Dropping the indices for simplicity, we have

$$\begin{aligned} F_z = F_p = F_u + G_p &= 0\,, \\ zG_u + pG_z - 12uzG_p + [G, F] &= 0\,, \end{aligned} \tag{2.77}$$

where[7]

$$[G, F] = G^i \frac{\partial F}{\partial y^i} - F^i \frac{\partial G}{\partial y^i}\,. \tag{2.78}$$

We now integrate the equations in (2.77). Immediately we have

$$\begin{aligned} F &= F(u, y)\,, \\ G &= -pF_u + H(u, z, y)\,, \end{aligned} \tag{2.79}$$

[7] The Einstein summation convention is used for repeated indices.

where H is the constant of integration[8]. Then putting G into the second equation in (2.77) we get

$$p\Big\{ - zF_{uu} + H_z - [F_u, F]\Big\} + zH_u + 12uzF_u + [H, F] = 0\,. \tag{2.80}$$

Since all of the functions in this equation are independent of p, the coefficient of p must be zero. Thus we get

$$H = \frac{1}{2}z^2 F_{uu} + z[F_u, F] + A(u, y)\,, \tag{2.81}$$

where A is the constant of integration . Then putting this form of H in (2.80) and equating the various coefficients of z to zero gives us the following relations

$$F_{uuu} = 0\,, \tag{2.82}$$

$$[F_{uu}, F] = 0\,, \tag{2.83}$$

$$A_u + 12uF_u + [[F_u, F], F] = 0\,, \tag{2.84}$$

$$[A, F] = 0\,. \tag{2.85}$$

From (2.82) we get

$$F = 2X_1 + 2uX_2 + 3u^2 X_3\,, \tag{2.86}$$

where X_1, X_2 and X_3 are constants of integration depending only on y. The commutator in (2.84) gives us the freedom to define new elements such as

$$[X_1, X_2] = -X_7\,, \quad [X_1, X_7] = X_5\,, \quad [X_2, X_7] = X_6\,. \tag{2.87}$$

With the use of (2.84) and (2.81), H can be written as

$$H = 3z^2 X_3 + 4zX_7 - 24u^3 X_3 - 12u^2 X_2 + 4u^2 X_6 + 8uX_5 + 8X_4 \tag{2.88}$$

[8] Although H is the function of integration we prefer to call it constant of integration. This terminology is used throughout the whole thesis.

and hence from (2.79) G becomes

$$G = -2(p+6u^2)X_2 + 3(z^2 - 8u^3 - 2up)X_3 + 8X_4 + 8uX_5 + 4u^2X_6 + 4zX_7\,. \quad (2.89)$$

The conditions (2.83) and (2.85) impose the following restrictions on the elements X_i

$$\begin{aligned} [X_1, X_3] = [X_2, X_3] = [X_2, X_6] = [X_1, X_4] &= 0\,, \\ [X_1, X_5] + [X_2, X_4] = 0\,, \quad [X_3, X_4] + [X_1, X_6] + X_7 &= 0\,. \end{aligned} \quad (2.90)$$

Using the Jacobi identities we obtain further relations:

$$\begin{aligned} [X_3, X_5] = [X_3, X_6] = [X_3, X_7] &= 0\,, \\ [X_2, X_5] = [X_1, X_6]\,,\ [X_6, X_7] = X_6\,,\ [X_1, X_6] + [X_5, X_6] &= 0\,. \end{aligned} \quad (2.91)$$

We now want to find a finite-dimensional representation of this prolongation algebra. Following the strategy given in Section 2.2 firstly we locate the elements in the center of the algebra. Since X_3 commutes with all the generators of the algebra, it is in the center of the algebra, that is,

$$[X_3, X_1] = [X_3, X_2] = [X_3, X_3] = 0 \quad \Rightarrow X_3 = 0\,. \quad (2.92)$$

Clearly equating X_3 to zero comes from the fact that we assume the algebra is simple. Also from (2.87), (2.90) and (2.92) we have

$$[X_1, X_6 - X_2] = [X_2, X_6 - X_2] = 0 \quad (2.93)$$

which implies $X_6 = X_2$. Thus we have reduced the number of elements in our algebra. Next we have to locate the nilpotent element. From (2.87) and (2.93) we have

$$[X_2, X_7] = X_2\,, \qquad [X_2, X_1] = X_7\,, \tag{2.94}$$

and thus by using Corollary 2.2.2 we say that X_2 is nilpotent and X_7 is neutral since the second commutator in (2.94) implies that $X_7 \in$ range of adX_2.

We now consider the scaling symmetry, which in fact is needed to obtain the eigenvalue problem. The KdV equation is invariant under the following scaling

$$x \to \lambda^{-1} x\,, \qquad t \to \lambda^{-3} t\,, \qquad u \to \lambda^2 u\,. \tag{2.95}$$

Also by requiring that the 1-forms ω^k be invariant under this transformation we have

$$X_1 \to \lambda X_1\,, \quad X_2 \to \lambda^{-1} X_2\,, \quad X_4 \to \lambda^3 X_4\,, \quad X_5 \to \lambda X_5\,, \quad X_7 \to X_7\,. \tag{2.96}$$

Together with (2.96) and (2.54), (2.94) says that we can identify

$$X_2 = e_- \quad \text{and} \quad X_7 = \frac{1}{2} h\,. \tag{2.97}$$

The solution of

$$[X_1, X_2] = -X_7 \tag{2.98}$$

gives

$$X_1 = -\frac{1}{2} e_+ + Z\,, \qquad Z \in \text{Ker ad}e_-\,, \tag{2.99}$$

where e_-, e_+ and h are basis elements of $s\ell(2, C)$. In fact (2.98) gives $X_1 = -\frac{1}{2} e_+$, but we can always add an element which is in the kernel of ade_- since it does not change the commutator in (2.98). By considering the scaling properties and remembering that we are embedding the prolongation algebra into a copy of $s\ell(2, C)$ we can choose $Z = \lambda^2 e_-$.

Then $[X_1, X_7] = X_5$ implies that

$$X_5 = \frac{1}{2}e_+ + \lambda^2 e_- \tag{2.100}$$

and

$$[X_2, X_4] = -[X_1, X_5] = \lambda^2 h \tag{2.101}$$

implies that

$$X_4 = -\lambda^2 e_+ + C\,, \qquad C \in \text{ Ker ad}e_-\,. \tag{2.102}$$

Using $[X_1, X_4] = 0$ one can find C. Hence

$$X_4 = -\lambda^2 e_+ + 2\lambda^4 e_-\,. \tag{2.103}$$

Using the matrix representation of e_-, e_+ and h in (2.50) we have the following:

$$\begin{aligned} X_1 &= \begin{pmatrix} 0 & -\frac{1}{2} \\ \lambda^2 & 0 \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \\ X_4 &= \begin{pmatrix} 0 & -\lambda^2 \\ 2\lambda^4 & 0 \end{pmatrix}, \qquad X_5 = \begin{pmatrix} 0 & \frac{1}{2} \\ \lambda^2 & 0 \end{pmatrix}, \\ X_7 &= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}. \end{aligned} \tag{2.104}$$

Putting these into (2.86) and (2.89) we form the matrices F and G and by using (2.39) we get the linear scattering problem

$$\begin{aligned} \begin{pmatrix} y^1 \\ y^2 \end{pmatrix}_x &= \begin{pmatrix} 0 & -2u - 2\lambda^2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y^1 \\ y^2 \end{pmatrix}, \\ \begin{pmatrix} y^1 \\ y^2 \end{pmatrix}_t &= \begin{pmatrix} -2z & 2p + 8u^2 - 8\lambda^2 u - 16\lambda^4 \\ 8\lambda^2 - 4u & 2z \end{pmatrix} \begin{pmatrix} y^1 \\ y^2 \end{pmatrix}. \end{aligned} \tag{2.105}$$

Hence the scalar Lax equation can be written as

$$L\Psi = (\partial^2 + 2u)\Psi = -2\lambda^2\Psi \tag{2.106}$$

with the corresponding time evolution of Ψ

$$\Psi_t = -4(\partial^3 + 3u\partial + \frac{3}{2}u_x)\Psi\,, \tag{2.107}$$

where $\Psi = y^2$ and ∂ represents the x-derivative.

2.4.2 Bäcklund Transformations for the Korteweg-de Vries Equation

In this Subsection we study the Bäcklund transformations for the KdV equation within a prolongation scheme. Assuming that one particular solution of the prolonged ideal $I'\{\pi^*\alpha_i, \omega^k\}$ is known we search for another solution. As was described in Section 2.3, this new solution, say u', depends upon all of the variables in the space of the prolonged ideal, that is, $u' = u'(u, z, p, y^k)$ and similarly z' and p' are defined in the same way. Then in order to have u' be a solution we introduce this *ansatz* into the set of forms in (2.7)

$$\begin{aligned} {\alpha_1}' &= du' \wedge dt - z'dx \wedge dt, \\ {\alpha_2}' &= dz' \wedge dt - p'dx \wedge dt, \\ {\alpha_3}' &= -du' \wedge dx + dp' \wedge dt + 12u'z'dx \wedge dt \end{aligned} \tag{2.108}$$

and demand that this set of forms be in the ring of the prolonged ideal.

Another way is to substitute the new solution u' into the KdV equation. Then, after ignoring the collection of the terms in which u and u' satisfy the KdV equation, we can equate the coefficients of the various independent variables to zero.

Although the calculations in these two ways are straightforward, they are very tedious. However, Harrison [51] suggests that one can use the ratios of the prolongation variables instead of using all of them separately. Of course this choice of variables considerably simplifies the calculations, but they are still tedious. If we define $\gamma = \frac{y^1}{y^2}$ as the ratio of the prolongation variables, the result of the calculations is that

$$u' = -u - \gamma^2 - 2\lambda^2 \tag{2.109}$$

is always another solution. Actually (2.109) is a Bäcklund transformation in the sense that it relates the old solution of the KdV equation to its new solution.

It is also possible to find a Bäcklund transformation in a much simpler way than the ways discussed above by the help of the scattering equations in (2.105), more precisely by the help of the Lax equation (2.106) and the time evolution equation (2.107). In order to do that firstly we define

$$\gamma = \frac{y^1}{y^2} \tag{2.110}$$

as the ratio of the prolongation variables. Then using (2.105) we get

$$\gamma_x = -2u - 2\lambda^2 - \gamma^2 \tag{2.111}$$

and

$$\gamma_t = -4z\gamma - (8\lambda^2 - 4u)\gamma^2 + 2p + 8u^2 - 8\lambda^2 u - 16\lambda^4\,. \tag{2.112}$$

Since the time evolution of Ψ in (2.107) does not depend on the scattering parameter λ^2, we can eliminate it from (2.112) and have

$$\gamma_t = -4z\gamma - 4\gamma_x\gamma^2 + 2p - 12\gamma_x u - 4{\gamma_x}^2 \,. \tag{2.113}$$

Checking the compatibility condition $\gamma_{xt} = \gamma_{tx}$ we see that it gives back the KdV equation. It is also possible to eliminate u, z and p from (2.111) and (2.113) in order to get

$$\gamma_t = -\gamma_{xxx} + 12\lambda^2\gamma_x + 6\gamma^2\gamma_x \,. \tag{2.114}$$

Actually (2.114) is the modified-KdV (mKdV) equation when $\lambda = 0$. Therefore (2.111) is the Miura transformation between the KdV and the mKdV equation when $\lambda = 0$. In this sense (2.111) is the Bäcklund transformation relating the solutions of the KdV equation to solutions of the mKdV equation. Now it is trivial to see that, if γ is a solution of (2.114), then $-\gamma$ is also a solution since (2.114) is an odd equation. Correspondingly we can find a new solution u' for the KdV equation such that

$$-\gamma_x = -2u' - 2\lambda^2 - \gamma^2 \,, \tag{2.115}$$

$$-\gamma_t = 4z'\gamma + 4\gamma_x\gamma^2 + 2p' + 12\gamma_x u' - 4{\gamma_x}^2 \,. \tag{2.116}$$

Subtracting (2.115) from (2.111) we get $\gamma_x = u' - u$, or $\gamma \equiv v' - v$ with $v_x \equiv u$. It is seen that, as we mentioned in Section 2.3, pseudopotentials can be interpreted as the difference of two solutions related by a Bäcklund transformation. Substituting these relations back into (2.111) and (2.113) we then get the auto-Bäcklund transformation for KdV equation

$$(v' + v)_x = -2\lambda^2 - (v' - v)^2 , \tag{2.117}$$

$$\begin{aligned}(v' - v)_t &= -4v_{xx}(v' - v) - 4(v' - v)_x(v' - v)^2 + 2v_{xxx} \\ &\quad -12(v' - v)_x v_x - 4[(v' - v)_x]^2 , \end{aligned} \tag{2.118}$$

where (2.117), which is in the same form with (2.109), describes the space-part and (2.118) the time-part.

From the Bäcklund transformation we can construct the soliton solutions of the KdV equation. Since soliton solutions correspond to the discrete spectrum of the eigenvalue equation, that is (2.106), we must choose $\lambda = i\kappa$, where κ is real. Only then for (2.106) can we have bound states and hence soliton solutions. For a known solution we take the trivial solution (seed solution) $u = 0$ of the KdV equation. Then integrating (2.117) we have

$$v'(x,t) = \sqrt{2}\kappa \tanh[\sqrt{2}\kappa(x + f(t))] , \tag{2.119}$$

where $f(t)$ is an arbitrary function of t. Using (2.118) we find that

$$f(t) = -8\kappa^2 t - x_0 , \tag{2.120}$$

where x_0 is a constant of integration. Hence v' is written as

$$v'(x,t) = \sqrt{2}\kappa \tanh[\sqrt{2}\kappa(x - x_0 - 8\kappa^2 t)] \tag{2.121}$$

from which the single soliton solution of the KdV equation is found by just taking the space-derivative

$$u'(x,t) = 2\kappa^2 \mathrm{sech}^2[\sqrt{2}\kappa(x - x_0 - 8\kappa^2 t)] . \tag{2.122}$$

We note that (2.122) is valid only if $|v'| < \sqrt{2}\kappa$. On the other hand, if $|v'| > \sqrt{2}\kappa$, then we have the singular solution

$$u'(x,t) = -2\kappa^2 \text{cosech}^2[\sqrt{2}\kappa(x - x_0 - 8\kappa^2 t)]\,. \tag{2.123}$$

CHAPTER 3

PAINLEVÉ ANALYSIS

The types of singularities exhibited by the solutions of systems of differential equations in the complex domain are very important. In fact it is possible to identify many different classes of integrable systems on the basis of their analytic structures. Historically the first work in this direction was started in the late 19th century by classifying ordinary differential equations on the basis of their singularities [52, 53, 54, 55, 56]. In 1884 Fuchs showed that among all possible first-order ordinary differential equations in the form

$$\frac{dw}{dz} = F(w,z) = \frac{P(w,z)}{Q(w,z)}\,, \tag{3.1}$$

where P and Q are polynomials in w with coefficients analytic in z, only the generalized Riccati equations,

$$\frac{dw}{dz} = p_2(z)w^2 + p_1(z)w + p_0(z)\,, \tag{3.2}$$

have no movable critical points [52, 53, 54]. After a short time Painlevé proved that only the movable singularities of the solutions for first-order equations of the form

$$F\left(\frac{dw}{dz}, w, z\right) = 0\,, \tag{3.3}$$

with F a polynomial in dw/dz and w and analytic in z, are poles [52, 53].

Having been influenced by the work of Fuchs and Painlevé, Kovalevskaya made the next significant contribution while studying the integrability of a rotating rigid body. In particular she examined the possible connection between integrability and the presence of only poles in the solutions of the equations of the spinning top. This was the first application of singularity analysis to a physical problem [56].

Shortly after this the next very important result was discovered by Painlevé and his coworkers by examining second-order equations of the form

$$\frac{d^2w}{dz^2} = F\left(\frac{dw}{dz}, w, z\right) , \tag{3.4}$$

where F is a polynomial in dw/dz, w and locally analytic in z. They showed that among all possible equations of the above form there are only fifty canonical equations with the property of having no movable critical points [52, 53, 54, 55, 57]. This property is known as the "Painlevé Property" and ordinary differential equations having this property are called *Painlevé-type equations.* Painlevé and his coworkers further showed that of the fifty equations forty-four were either integrable in terms of known functions (such as elliptic functions and functions that are solutions of linear equations) or were reducible to one of six new nonlinear differential equations which are known as the *Painlevé transcendents* [52, 53, 54, 55].

After the classification of first- and second-order ordinary differential equations, several attempts were made to classify higher-order equations [58, 59, 60]. However, due to many reasons [55, 56] the complete classification of higher-order ordinary differential equations with the Painlevé Property has not yet been given.

Having reached its final point for that period of time (about mid 1910s), the research in this area slowed down as if it had been stopped and almost no significant results (except that the works of Bureau) had appeared until mid 1970s. At the end of the 1970s, starting with the work of Ablowitz, Segur and Ramani [17, 61, 62] on completely integrable nonlinear evolution equations, singularity analysis gained a new interest and a new area of research was opened in which the possible connection between the Painlevé Property and integrability could be investigated. Indeed what they observed was that every reduction of an integrable partial differential equation solvable by inverse scattering transform leads to an ordinary differential equation with the Painlevé Property. Since then their algorithm, which actually is a generalization of the ideas of Kovalevskaya, has been used to identify integrable cases of both ordinary and partial differential equations. However, since it is not always possible to know all reductions of a partial differential equation, there was a strong need for a test of singularity analysis which could be applied directly to partial differential equations. For this reason in 1983 Weiss, Tabor and Carnevale [63] introduced the Painlevé Property for partial differential equations as a method of applying the principles of singularity analysis to a given partial differential equation without having to reduce it to an ordinary differential equation. All these ideas, which might be considered as a resurrection of the Painlevé analysis, have found many applications in the theory of completely integrable systems as well as in many branches of Physics[1,2] and

[1] Such as Statistical Physics, Plasma Physics, Quantum Gravity, General Relativity, Nonlinear Optics etc.

[2] The fact that physical problems were responsible for the birth of this appearingly new theory answers not only why it has found many applications in Physics but also the question

this renewal of interest has never decreased from then.

In this Chapter firstly we describe the Painlevé Property for ordinary differential equations. In order to do that we start with a discussion of singularities of ordinary differential equations. Then we give the main steps of the so-called Painlevé Test. In the second section we mainly concentrate on the Painlevé Property for partial differential equations. For this purpose firstly we describe the two notions of characteristic and noncharacteristic hypersurfaces and then introduce the main steps of the Painlevé Test for partial differential equations. Finally we conclude the Section and the Chapter by applying the singularity analysis to the KdV equation as an example.

3.1 Painlevé Analysis for Ordinary Differential Equations

3.1.1 Singularities of Ordinary Differential Equations

The main theme in the Painlevé analysis is to express the general solution of a differential equation as a Taylor series or more generally a Laurent series expansion. Because of the fact that the domain of validity of these expansions are limited by the singularities, it is appropriate to investigate the Painlevé analysis starting with a discussion of the singularities [64]. Before proceeding we recall that any Taylor or Laurent series, those of which are valid in some interval on the real line, are indeed valid inside a disk. Thus any differential equation and solutions of it should be considered in the complex plane, even if their variables are real [64]. Therefore all of the subsequent discussions are done in the complex

of why it was developed by physicists rather than mathematicians.

plane.

There exist four structures of singularities of solutions of ordinary differential equations which can be classified in two categories. The first classification is the distinction between movable or fixed singularities of the solutions of differential equations and the second classification is the distinction between the critical and noncritical singular points. We now investigate these four different types of singularities.

If the singular point of the solution of an ordinary differential equation is determined only by the equation itself and does not depend on the constant(s) of integration, then it is called a *fixed singular point.* On the contrary, if its location in the complex plane depends on the constant(s) of integration, then it is called a *movable singular point.*

It is well-known [52, 53] that linear ordinary differential equations can only have fixed singularities, that is, the only singularities of solutions of linear ordinary differential equation are those of the coefficients in the equation. For example consider the following equation

$$\frac{dw}{dz} + \frac{w}{z^2} = 0\,, \tag{3.5}$$

the solution of which is $w = ce^{1/z}$. It has a fixed singularity (isolated essential singularity) in its general solution at $z = 0$.

The point at infinity is to be considered as fixed. Sometimes linear ordinary differential equations have zeros which depend on constants of integration and they are called *movable zeros* [64].

In contrast to linear equations nonlinear differential equations can have both fixed and movable singularities [52, 53]. For example consider the very simple nonlinear differential equation

$$\frac{dw}{dz} + w^2 = 0\,, \tag{3.6}$$

the general solution of which is

$$w(z) = \frac{1}{z - z_0}\,\cdot \tag{3.7}$$

Here z_0, the constant of integration (of value $-w^{-1}(0)$), denotes the location of the singularity of the solution. Hence the general solution has a movable singularity. In fact it has a movable pole.

Having discussed the first classification we pass to the second one. However, before giving the definition of a critical point it is appropriate to recall that a function can be viewed as a bijective map which maps a given object onto an image. The objects are chosen from a set called the *domain* and the images are the elements of a set called the *range*. From this definition it is clear that a function is characterized by its single-valuedness [64].

Since the notion of a critical point is directly related to multivaluedness, it is preferable to give its definition in the following way: Any singular point of a bijective map is called a *critical point* if at least two determinations (branches) of that map can be permuted [64]. Thus any singularity, other than a pole of whatever order, should be considered as a critical point. Of course such a point is an obstacle for a map to be a function.

In order to remove this difficulty it is desirable to define from this multivalued map a single-valued one, in fact a function. Actually there exist two ways to do this. The first way is to prevent local turns around critical points by cutting some lines in the domain of the map and the second way is to extend the domain by cutting and pasting several copies of sheets, called *Riemann sheets*.

It is possible to have different combinations of these four different types of singularities. In fact there exist four different structures of them. They are named as: movable critical singularity, movable noncritical singularity, fixed critical singularity and fixed noncritical singularity. However, among these four only one of them is an obstacle for a solution of an ordinary differential equation to define a function. This combination consists of the presence of singularities at the same time movable and critical. Indeed, in such a case, it is not possible to know either where to make cuts or where to paste the Riemann sheets. Hence it is impossible to define a function [64].

3.1.2 Painlevé Test for Ordinary Differential Equations

After discussing the singularities of solutions of ordinary differential equations and stating that they have to be free from movable critical points in order to define a function, hence be a solution, it is now appropriate to give the definition of the Painlevé Property for ordinary differential equations.

An ordinary differential equation in the complex domain has the Painlevé Property if the only movable singularities of its general solution are poles [65]. Of course this definition excludes movable essential singularities as well as movable branch points from the general solution of the differential equation. From this definition it is also clear that the crucial importance in the analysis is to determine whether the solution has movable critical points or not and actually for this purpose there exists a very effective test, called the *Painlevé Test* (in the literature sometimes it is also called the singular point analysis). However, this test is not sensitive to movable essential singularities, the property of which prevents it to be the sufficient condition as well. Hence the Painlevé analysis should be investigated in two parts:

- generation of necessary conditions (Painlevé Test) for the absence of movable critical singularities in the general solution and
- explicit proof of sufficiency by expressing the general solution as a finite expression of a finite number of elementary functions such as solutions of linear equations, elliptic functions etc.

In order to say that an ordinary differential equation has the Painlevé Property it is required that both the necessary and sufficient conditions be satisfied. If only the necessary condition is satisfied, then it is said that the equation passes the Painlevé Test. Forgetting about the sufficiency conditions we now discuss the algorithm for the singular point analysis as described by Ablowitz *et al* [62] in 1980.

The main idea of the algorithm is to express the general solution of an nth-order ordinary differential equation of the form

$$\frac{d^n w}{dz^n} = F\left(\frac{d^{n-1} w}{dz^{n-1}}, \ldots, w, z\right), \tag{3.8}$$

where F is analytic in z and a polynomial in $d^{n-1}w/dz^{n-1}, \ldots, w$, as a Laurent series expansion

$$w(z) = (z - z_0)^{\alpha} \sum_{j=0}^{\infty} a_j (z - z_0)^j \tag{3.9}$$

and to determine the single-valuedness and self-consistency of this expansion.

The algorithm consists of three steps;

1. dealing with the dominant behaviors,

2. finding the resonances and

3. finding the constants of integration,

each of which can be described as follows:

Step 1) Dominant behaviors: To determine α and a_0 there is no need to substitute the full expansion in (3.9) into (3.8). Instead it is better to substitute the following ansatz

$$w(z) = a_0 (z - z_0)^{\alpha}\,, \qquad a_0 \neq 0\,, \tag{3.10}$$

as $z \to z_0$, where z_0 is arbitrary. Then for certain values of α two or more terms in (3.8) balance each other, while the rest can be ignored because of having higher powers of $(z - z_0)$. These most singular terms are called *dominant* or *leading terms*. After they are balanced, it is usually possible to determine a_0.

To be able to say that (3.8) passes the Painlevé Test, α must be a negative integer. If it were not an integer, z_0 would be an algebraic branch point for the dominant terms. Hence the equation under investigation would not pass the Painlevé Test.

In this first step, usually for systems of differential equations (but not just for them only), there may exist more than one dominant behavior. Then it is important to find and examine all of them separately. If any of the possible αs is not an integer, then it should be concluded that the equation does not pass the Painlevé test. On the other hand, if one of those branches (possible αs) has the Laurent series expansion (3.9) with n constants of integration ($(n-1)$ arbitrary coefficients a_j together with the arbitrariness of the location z_0 of the singularity), then that branch is referred to as *generic*, corresponding to the general solution of (3.8).

Step 2) Resonances: In the previous step it was noted that besides the arbitrariness of z_0 there might still be $(n-1)$ arbitrary coefficients in (3.9). The powers of $(z - z_0)$ at which these arbitrary coefficients appear are termed *resonances*. Following the arguments of the previous step, again there is no need to use the full series in order to determine the resonances. Instead a simple analysis can be performed by substituting

$$w(z) = a_0(z - z_0)^\alpha + p(z - z_0)^{\alpha+r}\,, \tag{3.11}$$

where a_0 and α have been determined from **Step 1**, into the simplified equation that keeps only the dominant terms of (3.8). After making this substitution it is possible to set up a linear equation in p to determine the powers, r, at which p is arbitrary. This linear equation in p can be written in the form

$$Q(r)p(z - z_0)^{\beta} = 0\,, \qquad \beta \geq \alpha + r - n\,, \tag{3.12}$$

where $Q(r)$ is a polynomial in r and n denotes the order of (3.8). If one of the dominant terms is the highest derivative in (3.8), then $\beta = \alpha + r - n$ and $Q(r)$ is a polynomial of order n. If this be not the case, then $\beta > \alpha + r - n$ and the order of the polynomial $Q(r)$ is less than n. In fact its order then equals the order of the highest derivative among the dominant terms.

The roots of $Q(r)$ are indeed the resonances since they are responsible for having arbitrary p, which in fact indicates the entrance of free constants into (3.9). It should be mentioned that $r = -1$ is always a root[3] of $Q(r)$ and it is associated with the arbitrariness of the location, z_0, of the singularity. Other than -1 all negative roots[4] should be ignored since they violate that $(z - z_0)^{\alpha}$ is the dominant term. Clearly any positive nonintegral root immediately causes the test to stop since they give rise to movable algebraic branch singularities or worse.

Any branch (admissible values of α) having $Q(r)$ with $(n-1)$ distinct positive integer roots in addition to $r = -1$ is called a generic branch. The presence of negative integer resonances implies that the corresponding branch is nongeneric. Having at least one generic branch indicates that there are no movable algebraic singularities in the general solution. However, there might still be movable logarithmic singularities and their absence should be

[3] See [66] for more discussions about why $r = -1$ is always a root.
[4] See [66, 67, 68, 69, 70, 71] for discussions about negative resonances.

further checked (see the next step).

Step 3) The constants of integration: In the previous step of resonance analysis only the coefficients in (3.9) which should be arbitrary were determined. Now in this step their arbitrariness has to be verified by checking the full recursion relations, that is, the series in (3.9) up to the last resonance has to be substituted into the original equation (3.8). After this substitution the recursion relation, given by

$$Q(j)a_j - R_j(a_0, a_1, \ldots, a_{j-1}, z_0) = 0\,, \tag{3.13}$$

has to vanish identically. If j is smaller than the first resonance, then (3.13) determines the a_j. On the other hand, if j is equal to the first resonance ($Q(j = r_1) = 0$), then in order to guarantee that a_{r_1} is arbitrary the so called *compatibility condition*, $R_{j=a_{r_1}} = 0$, has to be satisfied. If this be the case, it is possible to proceed to determine the other coefficients until the next resonance is reached. At each resonance it is required that all the compatibility conditions be satisfied.

However, it may turn out that for some resonance, r, the compatibility condition is not satisfied, indicating that the corresponding coefficient a_j is not arbitrary. In this case the series in (3.9) has to be generalized in such a way to recapture an arbitrary coefficient at that resonance. Actually this is done by introducing logarithmic terms into the expansion such as $c_r \ln(z - z_0)(z - z_0)^{\alpha+r}$. Then the constant c_r is determined in such a way that the corresponding compatibility condition is satisfied, hence ensuring that the coefficient a_r is indeed arbitrary. Clearly in this case the series in (3.9) (actually the solution) is no longer single-valued and it is impossible to talk about the Painlevé Property.

In summary it can be said that by following the Ablowitz-Ramani-Segur (ARS) algorithm it is possible to conclude whether or not a given ordinary differential equation has algebraic branch points or logarithmic singularities. Hence in a very effective way it is possible to test whether a given equation satisfies the necessary conditions in order to have the Painlevé Property.

Before concluding this Subsection we note that although the dominant behavior in the first step of the ARS algorithm is determined as $z \to z_0$, this is not the only possibility for a general analysis of the leading order behavior since it can also be determined as $z \to \infty$ [66]. This ambiguity in the determination of the dominant terms can only be solved by an analysis of the next to the leading order behavior [66]. For example being the next power in the analysis greater than that of the singularity implies that the analysis should be studied in the neighborhood of the singularity. On the other hand being it lesser implies the studying of the analysis in the asymptotic region. In fact this is nothing but the existence of two types of Laurent series one of which is an increasing series of the form

$$w = \sum_{j=0}^{\infty} a_j (z - z_0)^{j+\alpha}, \qquad \alpha < 0 \tag{3.14}$$

and the other being a decreasing series of the form

$$w = \sum_{j=0}^{\infty} a_j (z - z_0)^{-j+\alpha}. \tag{3.15}$$

This second possibility of having the representation of the solution as a decreasing series was not observed by Ablowitz *et al* [17] (see [62, 66]) (because of this reason we discussed the ARS algorithm in its original form). However, it was first introduced into the literature by Lemmer and Leach in 1993 [67] and thereafter studied in [66, 68, 69]. The series in (3.14) is called the *Right Painlevé Series* and the one in (3.15) the *Left Painlevé Series* [69]. Of course there exist some differences between these two approaches such as:

- In order to have the Left Painlevé Series we require that all of the terms in the equation are dominant.

- Since in the case of a decreasing series we are determining the behavior of the leading order term as $z \to \infty$, the constant z_0 loses its importance. Therefore we need n arbitrary constants in this case as oppose to the situation in the ARS algorithm where we need $n-1$ (together with z_0 there are n arbitrary constants) additional arbitrary constants.

- Those terms which are most singular in the ARS algorithm become the least singular in the case of the Left Painlevé Series.

After mentioning the Left Painlevé Series and stating some of the differences between the two approaches we do not give the detailed analysis of having the Left Painlevé Series. However, these are fully discussed in [66, 67, 68, 69].

Having discussed the Painlevé analysis for ordinary differential equations we can now generalize it to partial differential equations and this is done in the next Section.

3.2 Painlevé Analysis for Partial Differential Equations

As was stated in the very beginning of this Chapter, the Painlevé analysis for partial differential equations was started by the work of Ablowitz, Ramani and Segur [17, 61, 62]. Indeed their work was remarkable in the sense that a connection between integrability and the Painlevé Property had been noted since the work of Kovalevskaya. They realized that the reductions of many integrable equations always seemed to lead to ordinary differential equations with the Painlevé Property and in fact it was this observation that lead them to conjecture that "every ordinary differential equation obtained by an exact reduction of a nonlinear partial differential equation solvable by inverse scattering transform method, has the Painlevé property" [62].

Of course the transformation of variables is allowed by this conjecture and it may well be the case that a given equation can pass the test only after some transformations. The main idea of this conjecture is to test a partial differential equation for integrability in the sense that an inverse scattering transform exists by testing all of its reductions to ordinary differential equations for the Painlevé Property.

However, the reduction of the partial differential equation before application of the algorithm is one of the major drawbacks of the ARS approach since in some cases the reductions are just too trivial to give interesting results. Moreover, it is not always possible to know what all these reductions are [72]. Because of these reasons there exists a strong need for a test which can be directly applied to partial differential equations without reducing them to ordinary differential equations.

In this section the Painlevé Test for partial differential equations, introduced by Weiss, Tabor and Carnevale (WTC) [63], is discussed. For this purpose firstly characteristic and noncharacteristic hypersurfaces are described. Then the test for partial differential equations is given. Finally this section concludes with the application of this test to the KdV equation as an example.

3.2.1 Characteristic and Noncharacteristic Surfaces

After Weiss, Tabor and Carnevale introduced the Painlevé test for partial differential equations, Ward [73] strongly suggested that the singularity manifold must be a noncharacteristic one. From then this fact has been assumed by all the Painlevé practitioners. Therefore, before describing the test, it is better to discuss the two notions of characteristic and noncharacteristic surfaces for partial differential equations. In order to have simple expressions the discussion is restricted to first-order systems. However, it is always possible to generalize it to systems of any order.

Although it is possible to solve for the derivatives of highest order in ordinary differential equations, it may not be so in partial differential equations since there are many derivatives of highest order. However, it may be possible to solve for the highest order derivative with respect to one of the independent variables. Consider the following system

$$F^i(x^1, \ldots, x^p, u^1, \ldots, u^n, u^1_{x^1}, \ldots, u^n_{x^p}) = 0\,, \quad i = 1, \ldots, n\,. \tag{3.16}$$

Since $u^i_{x^j}$, $j = 1, \ldots, p$ are jet variables, it is clear that F^i depends on $p + n + pn$ variables. Suppose that the point $[x^j_0]$, where $[x^j_0]$ denotes the p-tuple $(x^1_0, \ldots, x^p_0)$, satisfies (3.16). If the Jacobian $\left[\left|\frac{\partial F^i}{\partial u^j_{x^1}}\right|_{[x^j]=[x^j_0]} = 0\right]$, the hyperplane $x^1 = x^1_0$ is said to be *characteristic* at the point $[x^j] = [x^j_0]$. On the other hand if the Jacobian $\left[\left|\frac{\partial F^i}{\partial u^j_{x^1}}\right|_{[x^j]=[x^j_0]} \neq 0\right]$, the hyperplane $x^1 = x^1_0$ is said to be *noncharacteristic* at the point $[x^j] = [x^j_0]$.

If the hyperplane $x^1 = x^1_0$ is noncharacteristic, then by the implicit function theorem it is possible to solve (3.16) in some neighborhood of $[x^j_0]$ for

$$u^i_{x^1} = G^i(x^1, \ldots, x^p, u^1, \ldots, u^n, u^1_{x^2}, \ldots, u^n_{x^p}) , \quad i = 1, \ldots, n . \tag{3.17}$$

Now instead of the hyperplane $x^1 = x^1_0$ consider a general hyperplane

$$\phi^1(x^1, \ldots, x^p) = \gamma_1(x^1 - x^1_0) + \gamma_2(x^2 - x^2_0) + \cdots + \gamma_p(x^p - x^p_0) = 0 , \tag{3.18}$$

and introduce new coordinates, $\xi^1, \ldots, \xi^p$, in the form

$$\xi^i = \phi^i(x^1, \ldots, x^p) , \qquad i = 1, \ldots, p , \tag{3.19}$$

where $\phi^2, \ldots, \phi^p$ are chosen arbitrarily, that is, they are linear functions of $[x^j]$. If the Jacobian $\left|\frac{\partial \xi^i}{\partial x^j}\right| \neq 0$, then by the implicit function theorem it is possible to invert (3.19) in order to find $[x^j]$ in terms of $[\xi^i]$, which can be expressed in the form

$$x^j = \zeta^j(\xi^1, \ldots, \xi^p) , \qquad j = 1, \ldots, p . \tag{3.20}$$

Hence it is possible to replace the coordinates x^j by new coordinates ξ^j and by the use of the chain rule it is easy to express the x-derivatives in terms of the

ξ-derivatives which can be written in the form

$$\frac{\partial u^i}{\partial x^j} = \sum_{k=1}^{p} \frac{\partial u^i}{\partial \xi^k}\frac{\partial \xi^k}{\partial x^j}\,, \qquad \begin{cases} i = 1, \ldots, n\,, \\ j = 1, \ldots, p\,. \end{cases} \tag{3.21}$$

Then replacing the coordinates x^j by new coordinates ξ^j and substituting (3.21) into (3.16) one can obtain a new system of differential equations,

$$\tilde{F}^i(\xi^1, \ldots, \xi^p, u^1, \ldots, u^n, u^1_{\xi^1}, \ldots, u^n_{\xi^p}) = 0\,, \quad i = 1, \ldots, n\,. \tag{3.22}$$

In this new system vanishing or nonvanishing situations of the Jacobian $\left|\frac{\partial \tilde{F}^i}{\partial u^j_{\xi^1}}\right|$ determine whether $\xi^1 = 0$ is characteristic or not which in fact determines whether ϕ^1 in (3.18) is characteristic or not. It is also useful to express the entries of the above Jacobian in terms of the old coordinates

$$\frac{\partial \tilde{F}^i}{\partial u^j_{\xi^1}} = \sum_{k=1}^{p} \frac{\partial F^i}{\partial u^j_{x^k}}\frac{\partial u^j_{x^k}}{\partial u^j_{\xi^1}}\,, \qquad i, j = 1, \ldots, n\,. \tag{3.23}$$

Then using (3.21) these entries can be expressed as,

$$\frac{\partial \tilde{F}^i}{\partial u^j_{\xi^1}} = \sum_{k=1}^{p} \frac{\partial F^i}{\partial u^j_{x^k}}\frac{\partial \xi^1}{\partial x^k}\,, \qquad i, j = 1, \ldots, n\,. \tag{3.24}$$

In general an hypersurface $\phi(x^1, \ldots, x^p) = 0$ is said to be characteristic or noncharacteristic at a given point according to whether the tangent hyperplane at that point is characteristic or not [74].

As an example consider the following simple system,

$$F^1(x, y, u, v, u_x, u_y, v_x, v_y) = 0\,,$$

$$F^2(x, y, u, v, u_x, u_y, v_x, v_y) = 0\,. \tag{3.25}$$

In order to decide whether a hypersurface $\phi(x, y) = 0$ is characteristic or not, evaluate the following determinant

$$\begin{vmatrix} F^1_{u_x}\phi_x + F^1_{u_y}\phi_y & F^1_{v_x}\phi_x + F^1_{v_y}\phi_y \\ F^2_{u_x}\phi_x + F^2_{u_y}\phi_y & F^2_{v_x}\phi_x + F^2_{v_y}\phi_y \end{vmatrix} \tag{3.26}$$

at a given point (x_0, y_0). If the determinant vanishes, then $\phi(x, y) = 0$ is a characteristic surface. If not, it is a noncharacteristic surface.

Having discussed the two notions of characteristics and noncharacteristics, we now turn to the Painlevé analysis for partial differential equations. As we said before, in 1984 Ward stressed that the singularity manifold, used in the Laurent series expansion, must be noncharacteristic. In fact this must be the case since on a characteristic manifold any type of singularity may propagate [55]. In order to see this we consider the linear wave equation,

$$u_{tt} = u_{xx} \, , \tag{3.27}$$

which has a general solution

$$u(x, t) = f(x + t) + g(x - t), \tag{3.28}$$

where f and g are arbitrary functions. It is manifestly obvious that any type of singularity may be possessed by the arbitrary functions f and g. Therefore the general solution also has the same singularities on the characteristic manifolds. Since the characteristic manifolds are determined by the partial differential equation itself and not by the particular solution, they can be considered as the analogous of fixed singularities in ordinary differential equations [55].

3.2.2 Painlevé Test for Partial Differential Equations

As in the case of ordinary differential equations the Painlevé analysis for partial differential equations consists of two parts:

- generation of necessary conditions (Painlevé Test) for the absence of movable critical singularities in the general solution,
- explicit proof of sufficiency by finding the transformation that linearizes the partial differential equation or by finding the Bäcklund transformation from which analytic solutions can be constructed.

The two parts of the analysis should be investigated separately and, unless the sufficiency conditions are satisfied, it is impossible to say that the partial differential equation has the Painlevé Property. The first part of the analysis consists of a test which is very similar to the algorithm that we described in Subsection 3.1.2. For the second part the truncation procedure, which uses only the singular part of the Laurent series, may give constructive results. In addition to this, any method, which explicitly finds the Lax pair or Bäcklund transformation, surely serves for the purpose of proving the sufficiency conditions.

In this Subsection we do not consider the second part. However, in Section 5.2 the sufficiency condition is explicitly proved by finding the Lax pair of the system of nonlinear partial differential equations under investigation. We now describe the Painlevé Test for partial differential equations.

In contrast to the case of ordinary differential equations the singularities of analytic functions of several complex variables cannot be isolated. Actually for a function of n complex variables they lie on analytic manifolds of real dimension $2n-2$. These manifolds are determined by conditions of the form

$$\phi(z_1, \ldots, z_n) = 0\,, \tag{3.29}$$

where ϕ is an analytic function of $(z_1, \ldots, z_n)$ in a neighborhood of the manifold [72] and is named as a *singularity manifold*. Thus there appears a way of generalizing the concept of a Laurent series for functions of one complex variable to functions of many complex variables by using such singularity manifolds. All of these considerations lead to the following definition: A partial differential equation is said to pass the Painlevé Test if solutions of it are single-valued in the neighborhood of noncharacteristic, movable singularity manifolds [55].

In order to test whether a given equation passes the Painlevé Test, Weiss *et al* [63] proposed a generalized Laurent expansion of the form

$$u(z_1 \ldots, z_n) \equiv u(\mathbf{z}) = \phi^{\alpha}(\mathbf{z}) \sum_{j=0}^{\infty} u_j(\mathbf{z}) \phi^j(\mathbf{z})\,, \tag{3.30}$$

where $u_j(\mathbf{z})$ are analytic functions of $\mathbf{z} = (z_1, \ldots, z_n)$ with $u_0(\mathbf{z}) \neq 0$, in the neighborhood of a noncharacteristic, movable singularity manifold defined by (3.29) as a general solution of the differential equation. If the validity of this expansion can be demonstrated, then $u(\mathbf{z})$ is claimed to be single-valued about the arbitrary movable singularity manifold (3.29) and hence to be the general solution of the differential equation.

The method, introduced by Weiss *et al*, is analogous to the algorithm described in Subsection 3.1.2 and similarly to that case this method also consists of three main steps. However, since these were fully discussed before, we do not discuss them again. Instead we summarize the general procedure without referring specifically to the steps.

Substitution of (3.30) into the given differential equation determines the admissible values of α and the recursion relations for u_j in the form

$$(j+1)(j-\beta_2)\cdots(j-\beta_N)u_j = F_j(u_0, u_1, \ldots, u_{n-1}, \phi, x)\,, \tag{3.31}$$

where N is the order of the equation and F_j is some function the arguments of which are the coefficients in the expansion together with ϕ and x. It is clear that, since the u_j are functions of $(z_1, \ldots, z_n)$, these relations are in the form of coupled partial differential equations. They define u_j unless $j = \beta_i$, for some i, being an integer between 1 and N. These values of j at which arbitrary functions may enter are called *resonances*. The resonance $j = -1$ is always present and it is associated with the arbitrariness of the singularity manifold $\phi = 0$. For each positive resonance the condition $F_{\beta_i} = 0$ must be identically satisfied. These conditions are called the *compatibility conditions*. The satisfaction of these conditions guarantees that the corresponding coefficient is arbitrary. On the other hand, if one of them is not satisfied, then it is enough to conclude that the test fails. However, in such cases the logarithmic Psi series may be used in order to regain the arbitrariness of the corresponding coefficients.

In order to conclude that a given partial differential equation passes the test, it is required that α be a negative integer and there exist at least one branch (admissible values of α) with $N-1$ positive resonances and all the recursion relations be compatible (in all possible branches), that is, in all possible branches the compatibility conditions at the resonances must be satisfied. Having $N-1$ positive resonances together with -1, the series contains the requisite number of arbitrary functions as required by the Cauchy-Kovalevskaya theorem and thus corresponds to the general solution of the partial differential equation.

Since the recursion relations for u_j are systems of coupled partial differential equations, the analysis becomes more difficult as we proceed to deal with higher-order terms. However, this difficulty can be solved by using Kruskal's gauge[5], which in fact is an application of the implicit function theorem. This is the simplest choice for the test, but it cannot be used to obtain the Lax pair or particular solutions.

Having discussed the Painlevé test for partial differential equations, we conclude this Chapter by applying this test to the KdV equation in the next Subsection.

3.2.3 Painlevé Test for the Korteweg-de Vries Equation

We apply the Painlevé Test to the KdV equation, given in (2.4). Assuming that the generalized Laurent series

$$u(x,t) = \phi^{\alpha}(x,t) \sum_{j=0}^{\infty} u_j(x,t)\phi^j(x,t)\,, \tag{3.32}$$

where $\phi(x,t)$ and $u_j(x,t)$ are analytic functions of (x,t) near the noncharacteristic, movable singularity manifold defined by $\phi(x,t)=0$ ($\phi_x^3 \neq 0$), is the general

[5] In two real variables it is given as $\phi(x,t) = x - \psi(t)$. Then, u_j becomes a function of the variable t only.

solution of (2.4), we substitute (3.32) into (2.4) and try to find the admissible values of α and resonances r together with the full recursion relations. However, since our aim is only to test (2.4), it is better to use Kruskal's gauge, which significantly simplifies the calculations by expressing x in terms of t. Moreover we also follow the analogous steps of the algorithm described in Subsection 3.1.2, which simplifies the test even further.

Following the algorithm step by step we firstly substitute $u_0\phi^\alpha$, where u_0 is a function of t only and $\phi(x,t) = x - \psi(t)$ (Kruskal's gauge), into (2.4). It is immediately seen that the highest order term (u_{xxx}) and the nonlinear term ($12uu_x$) are the leading order terms in (2.4). Balancing these dominant terms we find that $\alpha = -2$ and $u_0 = -1$. Next we have to find the resonances. For this reason we introduce

$$u = -\phi^{-2} + u_j\phi^{-2+j} \tag{3.33}$$

into the reduced equation (the KdV equation which keeps only the dominant terms) and to the leading order in u_j (linear in u_j) we get

$$u_j[(j+1)(j-4)(j-6)]\phi^{-5+j} = 0\,. \tag{3.34}$$

It is seen that arbitrary functions can enter the Laurent series expansion at the points $j = -1, 4$ and 6. Indeed these points are the resonances for the KdV equation. Finally we must substitute the full series (3.32) into the original equation (2.4) and find the full set of recursion relations the compatibility conditions of which are checked at the resonances. In order to do this, we firstly find the derivatives of u,

$$\begin{aligned}
u_t &= \sum_{j=0}^{\infty}\left\{u_{j,t}\phi^{-2+j} - u_j\psi_t(j-2)\phi^{-3+j}\right\}\,, \\
u_x &= \sum_{j=0}^{\infty} u_j(j-2)\phi^{-3+j}\,, \\
u_{xxx} &= \sum_{j=0}^{\infty} u_j(j-2)(j-3)(j-4)\phi^{-5+j}\,.
\end{aligned} \tag{3.35}$$

Then inserting these derivatives into (2.4) we get

$$\sum_{j=0}^{\infty}\left\{u_{j,t}\phi^{-2+j} - u_j\psi_t(j-2)\phi^{-3+j}\right\} + \sum_{j=0}^{\infty} u_j(j-2)(j-3)(j-4)\phi^{-5+j} + 12\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} u_j u_k(k-2)\phi^{-5+j+k} = 0\,, \tag{3.36}$$

which can also be written as

$$\begin{aligned}
&\sum_{j=3}^{\infty}\Big\{u_{(j-3),t} - u_{j-2}\psi_t(j-4) + u_j(j-2)(j-3)(j-4) + 24u_j \\
&\qquad -12u_j(j-2) + 12\sum_{k=1}^{j-1} u_{j-k}u_k(k-2)\Big\}\phi^{-5+j} \\
&-24u_0(1+u_0)\phi^{-5} - 6u_1(6u_0+1)\phi^{-4} + 2(u_0\psi_t - 12u_2u_0 - 6u_1^2)\phi^{-3} = 0\,.
\end{aligned} \tag{3.37}$$

Hence the recursion relations take the form

$$\begin{aligned}
(j+1)(j-4)(j-6)u_j &= -u_{(j-3),t} + (j-4)\psi_t u_{j-2} \\
&\quad -12\sum_{k=1}^{j-1} u_{j-k}u_k(k-2)
\end{aligned} \tag{3.38}$$

from which (also by the help of (3.37)) we find that at

$$\begin{aligned}
&j=0: u_0 = -1\,, \quad j=1: u_1 = 0\,, \quad j=2: u_2 = \frac{\psi_t}{12}\,, \\
&\qquad j=3: u_3 = 0\,, \quad j=5: u_5 = \frac{\psi_{tt}}{72}\,.
\end{aligned} \tag{3.39}$$

Furthermore at $j = 4$ and $j = 6$ (that is at the resonances) we get the compatibility conditions which can be written in the following form by using (3.39)

$$\begin{aligned} j &= 4 : 0 \cdot u_4 = 0 \,, \\ j &= 6 : 0 \cdot u_6 = 24 u_4 \psi_t - 24 u_2 u_4 \,. \end{aligned} \tag{3.40}$$

It is clear that two of the compatibility conditions are satisfied. Even if we did not use Kruskal's gauge, the compatibility conditions would still be satisfied, but in a much more complicated way.

Since all the requirements of the Painlevé Test are satisfied, we conclude that the KdV equation passes the test successfully. Hence it must be expected to possess a Lax pair. Although by the help of a well-known procedure, which uses only the singular part of the Laurent series, it is possible to find the Lax pair or Bäcklund transformations of the KdV equation, we do not consider this procedure here. However, in Chapter 2 we explicitly found the Lax pair and Bäcklund transformation of the KdV equation by using other methods. Thus we must conclude that the KdV equation has the Painlevé property.

CHAPTER 4

DRINFEL'D-SOKOLOV SYSTEM OF EQUATIONS

Integrable systems appear not one at a time, but in big families which are called *hierarchies*. Firstly the KdV hierarchy was invented and then infinitely many generalized KdV hierarchies were found. They were unified to a single one large Kadomtsev-Petviashvili (KP) hierarchy [75]. Very recently it was shown by Gürses and Karasu [76] that the system of equations

$$
\begin{aligned}
u_t &= -u_{xxx} + 6uu_x + 6v_x\,, \\
v_t &= 2v_{xxx} - 6uv_x\,,
\end{aligned}
\tag{4.1}
$$

admits a recursion operator and a bi-Hamiltonian structure. Therefore it has infinitely many constants of motion. The system (4.1) belongs to a class of equation which is called a *quasipolynomial flow* [16]. The Lax pair for this system was first given by Drinfel'd and Sokolov [18] and later by Bogoyavlenskii [77]. Under scaling transformations this system of equations reduces to a special case of the KP hierarchy which was shown by Hirota and Satsuma [78, 79]. They also gave the one soliton solutions. Recently auto-Bäcklund transformations and certain analytical solutions were obtained by Tian and Gao [80] via the Painlevé analysis. In this Chapter we use the prolongation method to derive the linear scattering problem for the system (4.1). We also obtain Bäcklund transformations by using pseudopotentials.

4.1 Prolongation Structure of the Drinfel'd-Sokolov System of Equations

When we introduce the variables

$$p = u_x\,, \quad q = v_x\,, \quad r = p_x\,, \quad s = q_x\,, \tag{4.2}$$

the system of equations (4.1) can be represented by the set of 2-forms

$$\begin{aligned}
\alpha_1 &= du \wedge dt - pdx \wedge dt\,, \\
\alpha_2 &= dp \wedge dt - rdx \wedge dt\,, \\
\alpha_3 &= dv \wedge dt - qdx \wedge dt\,, \\
\alpha_4 &= dq \wedge dt - sdx \wedge dt\,, \\
\alpha_5 &= du \wedge dx - dr \wedge dt + 6(up + q)dx \wedge dt\,, \\
\alpha_6 &= dv \wedge dx + 2ds \wedge dt - 6uqdx \wedge dt\,,
\end{aligned} \tag{4.3}$$

which constitutes a closed ideal I.

Following the W-E prolongation method we extend the ideal I by adding to it the system of 1-forms,

$$\omega^k = dy^k + F^k dx + G^k dt\,, \quad k = 1, \ldots, N\,, \tag{4.4}$$

where F^k and G^k are functions of (u, v, p, q, r, s, y^k) and we require that this set of 1-forms be closed. Thus writing (2.34) explicitly and equating the coefficients of various independent 2-forms to zero gives us some set of partial differential equations for F^k and G^k. Dropping the indices for simplicity we have

$$F_p = F_q = F_r = F_s = 0\,,$$

$$G_s = 2F_v\,, \quad G_r = -F_u\,,$$

$$pG_u + qG_v + rG_p + sG_q - 6(up+q)F_u + 6uqF_v - [F,G] = 0\,, \tag{4.5}$$

where the commutator is defined in (2.78).

Now we have to integrate these equations. Since the calculations are long and straightforward, we do not show them here. In obtaining the result we equate the coefficients of terms quadratic in F to zero. Indeed we can do this since, as we saw in Subsection 2.4.1, they are in the center of the algebra and can be set to zero. We find the following

$$\begin{aligned} F &= X_1 + uX_2 + vX_3\,, \\ G &= X_0 + (-r + 3u^2 + 6v)X_2 + 2(s - 3uv)X_3 - pX_4 \\ &\quad -uX_5 - \frac{1}{2}u^2X_6 + 2qX_7 + 2vX_8 + v^2X_9 + 2uvX_{10}\,, \end{aligned} \tag{4.6}$$

where X_0, X_1, X_2 and X_3 are constants of integration depending on y^k only. The remaining elements are

$$\begin{aligned} [X_1, X_2] &= X_4\,, \quad [X_1, X_3] = X_7\,, \quad [X_1, X_4] = X_5\,, \\ [X_1, X_7] &= X_8\,, \quad [X_2, X_4] = X_6\,, \quad [X_2, X_7] = X_{10}\,, \\ [X_3, X_7] &= X_9\,. \end{aligned} \tag{4.7}$$

The integrability conditions impose the following restrictions on the X_i,

$$\begin{aligned} [X_2, X_3] = [X_2, X_6] = [X_3, X_9] = [X_1, X_0] &= 0\,, \\ [X_3, X_4] = 6X_3 - 2X_{10}\,, \quad [X_2, X_9] + 2[X_3, X_{10}] &= 0\,, \\ [X_2, X_{10}] - \frac{1}{4}[X_3, X_6] &= 0\,, \end{aligned}$$

$$
\begin{aligned}
[X_1, X_6] + 2[X_2, X_5] - 6X_4 &= 0\,, \\
[X_1, X_9] + 2[X_3, X_8] &= 0\,, \\
[X_1, X_5] - [X_2, X_0] &= 0\,, \\
2[X_1, X_8] + [X_3, X_0] + 6X_4 &= 0\,, \\
[X_1, X_{10}] + [X_2, X_8] - \frac{1}{2}[X_3, X_5] - 3X_7 &= 0\,. \qquad (4.8)
\end{aligned}
$$

Using the Jacobi identities we obtain the further relations

$$
\begin{aligned}
[X_1, X_9] = [X_2, X_9] = [X_3, X_8] = [X_3, X_{10}] &= 0\,, \\
[X_2, X_{10}] = [X_3, X_6] &= 0\,, \\
[X_1, X_{10}] - [X_4, X_7] - [X_2, X_8] &= 0\,, \\
[X_1, X_6] = [X_2, X_5] &= 2X_4\,, \\
[X_3, X_5] + 2[X_7, X_4] + 2[X_1, X_{10}] &= 6X_7\,. \qquad (4.9)
\end{aligned}
$$

Now to find a finite-dimensional representation of this prolongation algebra we follow the strategy given in Section 2.2. Firstly we reduce the number of elements. By using (4.7), (4.8), (4.9) and the Jacobi identities we get

$$
\begin{aligned}
[X_1, X_9] = [X_2, X_9] = [X_3, X_9] = 0 \quad \Rightarrow X_9 &= 0\,, \\
[X_3, X_4] = X_{10}\,, \quad [X_3, X_4] = 6X_3 - 2X_{10} \quad \Rightarrow X_{10} &= 2X_3\,, \\
[X_6, X_2] = [X_6, X_3] = 0\,, \quad [X_6, X_4] = 2X_6 \quad \Rightarrow X_6 &= 2X_2\,. \qquad (4.10)
\end{aligned}
$$

Next we locate the nilpotent and the neutral elements. Together with corollary 2.2.2, (4.7) and (4.10) give

$$
[X_2, X_4] = 2X_2 \qquad \Longrightarrow X_2 \ \text{nilpotent}\,,
$$

$$[X_2, X_1] \;=\; -X_4 \qquad \Longrightarrow X_4 \;\text{ neutral}\,. \tag{4.11}$$

We note that the Drinfel'd-Sokolov system of equations has the following scaling symmetry

$$x \to \lambda^{-1} x\,, \quad t \to \lambda^{-3} t\,, \quad u \to \lambda^2 u\,, \quad v \to \lambda^4 v\,, \tag{4.12}$$

which implies that the elements X_i must have the following transformations

$$\begin{aligned} X_0 &\to \lambda^3 X_0\,, \qquad X_1 \to \lambda X_1\,, \quad X_2 \to \lambda^{-1} X_2\,, \\ X_3 &\to \lambda^{-3} X_3\,, \quad X_4 \to X_4\,, \qquad X_5 \to \lambda X_5\,, \\ X_7 &\to \lambda^{-2} X_7\,, \quad X_8 \to \lambda^{-1} X_8\,. \end{aligned} \tag{4.13}$$

By using (4.13) and the scaling symmetries of the basis elements of $s\ell(n+1, C)$ we try to embed this prolongation algebra into $s\ell(n+1, C)$. Starting from the $n = 1$ case we identify

$$X_2 = e_-\,, \qquad X_4 = h\,. \tag{4.14}$$

Thus the solution of $[X_2, X_1] = -X_4$ gives

$$X_1 = e_+ + \lambda^2 e_-\,, \tag{4.15}$$

where the constant λ^2 is chosen to take account of the scaling properties. Then $[X_1, X_4] = X_5$ implies

$$X_5 = -2e_+ + 2\lambda^2 e_- \tag{4.16}$$

and $[X_2, X_3] = [X_3, X_7] = 0$ implies

$$X_3 = A e_-\,, \qquad X_7 = B e_-\,, \tag{4.17}$$

where A and B are some constants. Since $[X_1, X_3] = X_7$, A and B must be zero. So together with $[X_1, X_7] = X_8$ we have

$$X_3 = X_7 = X_8 = 0\,. \tag{4.18}$$

Finally the relation

$$\begin{aligned} [X_2, X_0] &= [X_1, X_5] \\ &= 4\lambda^2 h \end{aligned} \tag{4.19}$$

implies

$$X_0 = -4\lambda^2 e_+ + C\,, \qquad C \in \text{Ker ad}e_- \tag{4.20}$$

and $[X_1, X_0] = 0$ determines C. The result is

$$X_0 = -4\lambda^2 e_+ - 4\lambda^4 e_-\,. \tag{4.21}$$

However, with these choices of elements X_i we see that (4.6) is reduced to (2.86) and (2.89). Thus $s\ell(2, C)$ cannot be the whole algebra. Similarly after some calculations $s\ell(3, C)$ is also dismissed. The simplest nontrivial closure is in terms of $s\ell(4, C)$.

We choose X_2 and X_4 as follows,

$$X_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad X_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{4.22}$$

Then a solution of $[X_1, X_2] = X_4$ is

$$X_1 = \begin{pmatrix} A & 1 & G & 0 \\ B & A & F & G \\ D & 0 & -A & 1 \\ E & D & C & -A \end{pmatrix}, \tag{4.23}$$

where, of course, we add the general form of the element which is in the kernel of $\mathrm{ad}X_2$ to the solution of $[X_1, X_2] = X_4$. Also the most general form of X_3, which satisfies the relation $[X_3, X_4] = 2X_3$, is

$$X_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \alpha & 0 & \delta & 0 \\ 0 & 0 & 0 & 0 \\ \gamma & 0 & \beta & 0 \end{pmatrix}. \tag{4.24}$$

Then, together with $[X_1, X_3] = X_7$, $[X_4, X_7] = 0$ implies that

$$\left.\begin{array}{l} \gamma G + \delta D = 0 \\ \gamma G - \delta D = 0 \end{array}\right\} \Rightarrow \gamma G = 0\,, \quad \delta D = 0\,,$$

$$2\delta A + \beta G - \alpha G = 0\,,$$

$$(\alpha - \beta) D = 2\gamma A\,. \tag{4.25}$$

Considering the scaling properties, we choose $G = 0$ and $\delta = 0$. Moreover $[X_3, X_7] = 0$ implies that $\alpha = \beta = 0$. Thus assuming $\gamma \neq 0$ we find $A = 0$. Also $[X_2, X_8] = 2X_7$ gives $F = 2$. Then from the relations $X_1 \to \lambda X_1$, $X_0 \to \lambda^3 X_0$ and $[X_0, X_1] = 0$ we conclude that $X_0 = \eta {X_1}^3$ which together with $[X_2, X_0] = [X_1, X_5]$ gives $B = D = 0$ and $\eta = -4$. Finally $[X_1, X_8] + \frac{1}{2}[X_3, X_0] = -3X_4$ gives $C = 0$ and $\gamma = -\frac{1}{2}$.

So we find a representation for all of our elements including an arbitrary parameter E. We choose it as $\frac{1}{2}\lambda^4$ by considering scaling properties. Thus the representations of the matrices are

$$X_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2}\lambda^4 & 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$X_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \end{pmatrix}, \quad X_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$X_5 = \begin{pmatrix} 0 & -2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -2 \\ \lambda^4 & 0 & 0 & 0 \end{pmatrix}, \quad X_7 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \end{pmatrix},$$

$$X_8 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad X_0 = \begin{pmatrix} 0 & 1 & 0 & -8 \\ -4\lambda^4 & 0 & 0 & 0 \\ 0 & -2\lambda^4 & 0 & 0 \\ 0 & 0 & -4\lambda^4 & 0 \end{pmatrix}. \quad (4.26)$$

Putting these into (4.6) we form the matrices F and G and then by using (2.39) we get the linear scattering problem

$$\begin{pmatrix} y^1 \\ y^2 \\ y^3 \\ y^4 \end{pmatrix}_x = \begin{pmatrix} 0 & -u & 0 & -\frac{1}{2}\lambda^4 + \frac{1}{2}v \\ -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & -u \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} y^1 \\ y^2 \\ y^3 \\ y^4 \end{pmatrix},$$

$$\begin{pmatrix} y^1 \\ y^2 \\ y^3 \\ y^4 \end{pmatrix}_t = \begin{pmatrix} p & \Gamma & q & \lambda^4 u + s - uv \\ -2u & -p & 2\lambda^4 - 2v & -q \\ 0 & 4u & p & \Gamma \\ 8 & 0 & -2u & -p \end{pmatrix} \begin{pmatrix} y^1 \\ y^2 \\ y^3 \\ y^4 \end{pmatrix}, \qquad (4.27)$$

where $\Gamma = 4\lambda^4 - 4v + r - 2u^2$. Hence the scalar Lax equation can be written as

$$L\Psi = (\partial^4 - 2u\partial^2 - 2u_x\partial - u_{xx} + u^2 + v)\Psi = \lambda^4\Psi \qquad (4.28)$$

with the corresponding time evolution of Ψ

$$\Psi_t = (-4\partial^3 + 6u\partial + 3u_x)\Psi\,, \qquad (4.29)$$

where $\Psi = y^4$ and ∂ represents the x-derivative.

4.2 Bäcklund Transformations for the Drinfel'd-Sokolov System of Equations

Within the prolongation scheme Bäcklund transformations can be derived by assuming the new solution variables to be functions of old ones and the ratios of pseudopotentials [51]. For this purpose we define new variables

$$\alpha = \frac{y^1}{y^4}, \quad \beta = \frac{y^2}{y^4}, \quad \gamma = \frac{y^1}{y^4}. \qquad (4.30)$$

By using (4.27) we can find the equations satisfied by α, β and γ,

$$\begin{aligned}
\alpha_x &= \alpha\gamma - u\beta + \frac{1}{2}(v - \lambda^4)\,, \\
\beta_x &= -\alpha + \beta\gamma\,, \\
\gamma_x &= \gamma^2 - 2\beta - u\,, \\
\alpha_t &= -8\alpha^2 + 2(p + 2u\gamma)\alpha + (4\lambda^4 - 4v + r - 2u^2)\beta + q\gamma \\
&\quad +(\lambda^4 u + s - uv)\,, \\
\beta_t &= -2u\alpha + 2(u\gamma - 4\alpha)\beta + 2(\lambda^4 - v)\gamma - q\,, \\
\gamma_t &= 2u\gamma^2 + 2(p - 4\alpha)\gamma + 4u\beta + (4\lambda^4 - 4v + r - 2u^2)\,. \qquad (4.31)
\end{aligned}$$

The compatibility conditions $\alpha_{xt} = \alpha_{tx}$ and $\gamma_{xt} = \gamma_{tx}$ hold if u and v satisfy (4.1) while $\beta_{xt} = \beta_{tx}$ holds automatically. It is possible to check that the function β satisfies the following equation:

$$\beta_t - 2\beta_{xxx} + 6(u + 2\gamma_x)\beta_x = 0\,. \qquad (4.32)$$

This means that

$$\begin{aligned}
\tilde{u} &= u + 2\gamma_x\,, \\
\tilde{v} &= c_1\beta + c_2 \qquad (4.33)
\end{aligned}$$

are the new solutions of (4.1) if

$$(4\beta^2 + c_1\beta)_x = 0\,, \qquad (4.34)$$

where c_1 and c_2 are constants. We note that the same results for $\tilde{u}$ and $\tilde{v}$ can be obtained if one follows the step (3) of [51]. By setting $y^4 = \Psi$ and using (4.27) and (4.30) we obtain

$$\begin{aligned}\tilde{u} &= u - 2\frac{\Psi_{xx}}{\Psi} + 2\frac{{\Psi_x}^2}{\Psi^2}\,,\\ \tilde{v} &= c_1\left(-\frac{u}{2} + \frac{\Psi_{xx}}{\Psi}\right) + c_2\,.\end{aligned} \tag{4.35}$$

Here u is a known solution of (4.1) and Ψ, being the solution of (4.28) and (4.29), satisfies the condition

$$\left\{\frac{1}{2\Psi^2}[\Psi_{xx} - u\Psi][2(\Psi_{xx} - u\Psi) + c_1\Psi\right\}_x = 0\,, \tag{4.36}$$

which is equivalent to (4.34).

Next we consider the simple case $u = v = 0$ as the known solution of (4.1) and find a new solution. With this choice we find that

$$\Psi = d_1 e^{-\lambda(4\lambda^2 t - x)} + d_2 e^{\lambda(4\lambda^2 t - x)} + d_3 e^{i\lambda(4\lambda^2 t + x)} + d_4 e^{-i\lambda(4\lambda^2 t + x)} \tag{4.37}$$

is a solution of (4.28) and (4.29), where d_1, d_2, d_3 and d_4 are constants. Substituting (4.37) into condition (4.36) we obtain two sets of solutions for Ψ for which $\{d_1 = d_2 = 0, c_1 = 4\lambda^2\}$ and $\{d_3 = d_4 = 0, c_1 = -4\lambda^2\}$.

The respective solutions for $\tilde{u}$ are

$$\begin{aligned}\tilde{u}_1 &= \frac{8 d_3 d_4 \lambda^2 e^{2i\lambda(4\lambda^2 t + x)}}{[d_4 + d_3 e^{2i\lambda(4\lambda^2 t + x)}]^2}\,,\\ \tilde{u}_2 &= \frac{-8 d_1 d_2 \lambda^2 e^{2\lambda(4\lambda^2 t - x)}}{[d_1 + d_2 e^{2\lambda(4\lambda^2 t - x)}]^2}\,,\end{aligned} \tag{4.38}$$

where in both cases $\tilde{v} = c_2 - 2\lambda^4 =$ constant. Thus starting from a trivial background we obtain the one soliton solution of (4.1) with $\tilde{v} =$ constant.

In order to find more general Bäcklund transformations for the system (4.1), we assume that the new solutions $\tilde{U}$ and $\tilde{V}$ are functions of old variables u, v, p, q, r, s

and α, β, γ which are the ratios of prolongation variables satisfying (4.31). After some straightforward but long calculations we obtain the following results:

$$\begin{aligned} \tilde{U} &= \frac{\Omega}{2} - \Delta - \frac{2}{\gamma}(\alpha + 2\beta\gamma)\,, \\ \tilde{V} &= \Theta_{xx} - \frac{\Delta}{\gamma}\Theta_x - \Theta^2\,, \end{aligned} \tag{4.39}$$

where

$$\begin{aligned} \Omega &= -\frac{1}{\gamma^2}(p\gamma + u^2 + 4\beta u + 4\beta^2 - 4\alpha\gamma)\,, \\ \Delta &= 2\beta - 2\gamma^2 + u\,, \\ \Theta &= \frac{1}{4\gamma^2}[-u^2 + 2(\gamma^2 - 2\beta)u - 4(\alpha\gamma + \beta^2)] \end{aligned} \tag{4.40}$$

and Θ must satisfy the condition

$$\Theta_t = 2\Theta_{xxx} + 3\Omega\Theta_x\,. \tag{4.41}$$

Thus, if any solutions (u, v) to the Drinfel'd-Sokolov system of equations are known and if α, β, γ are solutions of (4.31) satisfying the condition (4.41), then $\tilde{U}$ and $\tilde{V}$ are the new solutions of (4.1).

If we take the trivial seed solutions $u = v = 0$, then we get

$$\begin{aligned} \tilde{U} &= -\frac{2\beta^2}{\gamma^2} + 2(\gamma^2 - 2\beta)\,, \\ \tilde{V} &= -\frac{1}{\gamma^4}(17\beta^4 + 8\beta^3\gamma^2 - 10\beta^2\alpha\gamma + \alpha^2\gamma^2) + \left(\frac{\lambda^4}{2} + \frac{c_0}{18}\right) \end{aligned} \tag{4.42}$$

with the condition

$$\beta[-56\beta^4 + 48\beta^2\gamma\alpha - 3\beta\gamma^2\lambda^4 - 8\gamma^2\alpha^2 + \gamma^4\lambda^4] = 0\,. \tag{4.43}$$

By setting $y^4 = \Psi$ one can write these expressions in terms of the solutions of the Lax equations (4.28) and (4.29) and see that our results are more general than those given by Tian and Gao [80].

As an example we consider the simple case, $u = v = 0$, as the known solutions of (4.1) and we obtain the following two sets of explicit solutions:

$$\begin{aligned} \tilde{U} &= \frac{2c_1^2}{(c_1 x + c_2)^2}, \\ \tilde{V} &= \frac{c_0}{18} \end{aligned} \tag{4.44}$$

and

$$\begin{aligned} \tilde{U} &= \frac{30(c_1 + x^2)\{4x[5c_1[2(24t - x^3) - 3c_1 x] - (c_2 + 3x^5)] + 15(x^2 + c_1)^3]\}}{\{5c_1[2(24t - x^3) - 3c_1 x] - (c_2 + 3x^5)\}^2}, \\ \tilde{V} &= -\frac{120x[c_2 + 3x^5 + 15c_1^2 x - 10c_1(24t - x^3)]^3}{\{5c_1[2(24t - x^3) - 3c_1 x] - (c_2 + 3x^5)\}^4}, \end{aligned} \tag{4.45}$$

where c_0, c_1 and c_2 are constants.

As we observed, there are no solitary wave solutions belonging to the class (4.42). Very recently all the special solutions of (4.1) were obtained by Karasu and Sakovich [81].

CHAPTER 5

INTEGRABILITY OF KERSTEN-KRASIL'SHCHIK COUPLED KDV-MKDV EQUATIONS: SINGULARITY ANALYSIS AND LAX PAIR

A symmetry of a system of differential equations is a transformation acting on the independent and dependent variables of the system with the property of transforming solutions of the system to solutions. Symmetries play an important role both from the physical and the mathematical points of view. For example they lead to conservation laws, which are mathematical formulations of the conservation of physical quantities such as energy and momentum. Indeed invariance under time translations implies conservation of energy and invariance under space translations implies conservation of momentum. In Physics particles and fields are described in terms of two classes: bosons and fermions. Classical symmetry principles do not mix these two classes. However, supersymmetry was introduced into Physics in the middle of the 1970s in order to exchange bosons and fermions [82].

From the point of view of nonlinear partial differential equations, supersymmetry enlarges the notion of integrability. Over the last two decades integrable supersymmetric differential equations, with very rich properties, have attracted much attention both in the field of Mathematical Physics and Soliton Theory. The supersymmetric extension of a classical differential equation (the differential equation for a bosonic or a commuting field) refers to a system of coupled differential equations for a bosonic and a fermionic field. Of course it is required that this system of coupled differential equations reduce to the original bosonic equation in the limit of a zero fermionic field. A fermionic field is described in terms of an anticommuting field. However, supersymmetry requires more than just the coupling of a bosonic field to a fermionic field. It also refers to the existence of a transformation relating these two fields which leaves the system of equations invariant [83]. For example the system of equations introduced by Kupershmidt [84] is a superintegrable system in the sense that it couples the bosonic field to the fermionic field, but it is not supersymmetric since it is not invariant under the superspace translation which is a supersymmetry transformation.

Naturally supersymmetric extensions of bosonic equations started with the extension of the KdV equation. The $N = 1, 2$ integrable supersymmetric versions of the KdV equation have been found in the past two decades [83, 85, 86, 87, 88, 89, 90, 91]. Here N refers to the number of supersymmetries, that is the number of odd fields. Since then many other equations have been supersymmetrized.

One of the important properties of supersymmetric extensions of known integrable bosonic systems is that these extensions can generate some new integrable bosonic systems in their zero fermionic field limits which generalize the initial ones. In fact generalization of new integrable bosonic systems is found if the number of supersymmetries, N, is greater than one [92]. In this connection one of the superextensions of the KdV equation, the $N = 2$, $a = 1$ case, where a is a parameter, was reduced to a bosonic system in the zero fermionic field limit by Kersten and Krasil'shchik and was later named after them as the Kersten-Krasil'shchik coupled KdV-mKdV equation. The integrability of this new system was proven by Kersten and Krasil'shchik [20] by finding a recursion operator and an infinite number of symmetries.

In this Chapter firstly we discuss the $N = 1$ and $N = 2$ superextensions of the KdV equation and show that the $N = 2$, $a = 1$ superextension of the KdV equation reduces to the Kersten-Krasil'shchik coupled KdV-mKdV equation when we let the odd variables vanish. Then in the second Section we investigate the integrability of this new bosonic system in terms of singularity analysis, show that it passes the Painlevé Test and hence is integrable. Having proven the integrability of the system in the third Section we search for the Lax pair by using the W-E prolongation method and the Dodd-Fordy algorithm and discuss the removability of the spectral parameter by using gauge invariant techniques.

5.1 Supersymmetric Extensions of the KdV Equation

The supersymmetric extension of the bosonic equations requires two things:

i) The introduction of fermionic fields which are described by anticommuting fields. For example an anticommuting field, γ, satisfying $\{\gamma(x), \gamma(y)\} = 0$, so that $\gamma^2(x) = 0$.

ii) A symmetry transformation which maps anticommuting fields to the usual commuting fields and vice versa.

For the first part it is natural to think that both independent variables x and t can be extended independently to include the anticommuting fields. However, since we only consider the evolution equations in the form of (2.11), the timelike supersymmetric extensions give trivial results. Hence, it is enough to consider only spacelike supersymmetric extension in space. In order to do that the superspace formalism is used. This means that the space variable x is extended to a doublet (x, θ), hence forming a superspace, where θ is an anticommuting variable, which satisfies $\theta^2 = 0$. Then the usual commuting fields $u(x, t)$ are replaced by superfields $U(x, \theta, t)$. Since $\theta^2 = 0$, these superfields have a very simple Taylor expansion in terms of θ,

$$U(x, \theta, t) = u(x, t) + \theta\gamma(x, t)\,, \tag{5.1}$$

where $u(x, t)$ and $\gamma(x, t)$ are called the component fields, $\gamma(x, t)$ is termed the *superpartner* of $u(x, t)$ and vice versa [93]. These superfields provide a compact description of supersymmetry [94]. Note that $U(x, \theta, t)$ is a bosonic superfield and hence has the same statistics as $u(x, t)$. The final thing that is needed in the superspace formalism is the superderivative

$$D = \theta\partial_x + \partial_\theta \tag{5.2}$$

with the property $D^2 = \partial_x$.

For the second part it is enough to consider translations in superspace which are actually supersymmetry transformations. Under the superspace translations such as $x \to x-\eta\theta$ and $\theta \to \theta+\eta$, where η is a constant anticommuting parameter, the superfield $U(x,\theta,t)$ changes as

$$U(x,\theta,t) \to U'(x,\theta,t) \equiv U(x,\theta,t) + \delta_\eta U(x,\theta,t)\,, \tag{5.3}$$

where $U'(x,\theta,t) \equiv U(x-\eta\theta,\theta+\eta,t)$. From the Taylor expansion,

$$U(x-\eta\theta,\theta+\eta,t) = U(x,\theta,t) - \eta\theta\partial_x U(x,\theta,t) + \eta\partial_\theta U(x,\theta,t)\,, \tag{5.4}$$

it is possible to see that $Q = \partial_\theta - \theta\partial_x$ generates the supersymmetry transformation. Hence $U(x,\theta,t)$ transforms as $\delta_\eta U(x,\theta,t) = \eta Q U(x,\theta,t)$ or, in component form, as $\delta_\eta u(x,t) = \eta\gamma(x,t)$ and $\delta_\eta\gamma(x,t) = \eta u_x(x,t)$. This mapping between the bosonic and fermionic fields is called a *supersymmetry transformation.*

Having discussed the general properties of superextensions of classical equations, we turn to the KdV equation. Before proceeding we note that the constant factor 12 in (2.4) can be rescaled by the transformation $u \to Au$ to be any nonzero constant. Hence in order to reach the desired equations we choose $A = -1/2$ and write the KdV equation as

$$u_t = -u_{xxx} + 6uu_x\,. \tag{5.5}$$

In order to find a supersymmetric extension firstly replace $u(x,t)$ by a fermionic superfield as

$$\Phi(x,\theta,t) = \theta u(x,t) + \xi(x,t)\,, \tag{5.6}$$

where $\xi(x,t)$ is an odd field. It is also possible to use the bosonic superfield, but it gives trivial results. The KdV equation is invariant under the following scaling transformations

$$x \to \lambda^{-1} x, \quad t \to \lambda^{-3} t, \quad u \to \lambda^2 u\,, \tag{5.7}$$

which implies that deg $\partial_x = 1$ and deg $u = 2$. Thus $D^2 = \partial_x$ implies that deg $D = \frac{1}{2}$ and hence deg $\theta = -\frac{1}{2}$. In order to have a homogeneous superfield the fermionic field $\xi(x,t)$ has degree $\frac{3}{2}$.

After introducing the fermionic superfield it is not difficult to multiply each term of the KdV equation by θ and write the result in terms of the superfield. Actually θu_t is replaced by Φ_t and θu_{xxx} is replaced by $D^6\Phi$. However, the nonlinear term $\theta(6uu_x)$ can be replaced by either $3D^2(\Phi D\Phi)$ or $6D\Phi D^2\Phi$ since both of them are equal to the nonlinear term in the absence of the $\xi(x,t)$ field. Thus there is no unique extension for the nonlinear term and the best way to overcome this problem is to consider a linear combination of all the possible terms, that is the nonlinear term can be replaced by $c\Phi D^3\Phi + (6-c)D\Phi D^2\Phi$, where c is a free constant. Then the most general one-parameter family of superextensions of the KdV equation can be written as

$$\Phi_t = -D^6\Phi + cD^2(\Phi D\Phi) + (6-2c)D\Phi D^2\Phi\,. \tag{5.8}$$

In terms of the component fields this is equivalent to

$$\begin{aligned} u_t &= -u_{xxx} + 6uu_x - c\xi\xi_{xx}\,, \\ \xi_t &= -\xi_{xxx} + (6-c)\xi_x u + c\xi u_x\,. \end{aligned} \tag{5.9}$$

This system represents the most general, $N = 1$, supersymmetric extension of the KdV equation and is integrable only if $c = 3$ [83]. Moreover, this system is invariant under the supersymmetric transformations $\delta_\eta u = \eta \xi_x$ and $\delta_\eta \xi = \eta u$.

Now it is possible to extend x to a triplet (x, θ_1, θ_2) by adding one more odd variable to our superspace with the property, ${\theta_1}^2 = {\theta_2}^2 = 0$, $\theta_1\theta_2 + \theta_2\theta_1 = 0$. In this enlarged superspace with two anticommuting variables and associated fermionic fields the KdV equation can be once more supersymmetrized. For this purpose two superderivatives

$$D_1 = \partial_{\theta_1} + \theta_1 \partial_x, \quad D_2 = \partial_{\theta_2} + \theta_2 \partial_x \,, \tag{5.10}$$

are defined with the property $D_1^2 = D_2^2 = \partial_x$ and $\{D_1, D_2\} = 0$.

The $N = 2$ supersymmetric extension of the KdV equation is obtained by taking an even homogeneous field Θ

$$\Theta = \omega + \theta_1 \psi + \theta_2 \phi + \theta_2 \theta_1 u \,, \tag{5.11}$$

where the even field Θ and the even and odd component fields depend on x, θ_1, θ_2 and t. Note that another bosonic field ω is introduced in order to equate the number of bosonic and fermionic fields. Again, if we consider the scaling symmetries of the KdV equation, the homogeneous even field Θ has the following degrees: deg $\Theta = 1$, deg $u = 2$, deg $\omega = 1$, deg $\psi = \frac{3}{2}$, deg $\phi = \frac{3}{2}$, deg $\theta_1 = -\frac{1}{2}$, deg $\theta_2 = -\frac{1}{2}$. The most general evolution equation for Θ, which reduces to the KdV equation in the absence of the odd fields ψ and ϕ, gives the $N = 2$ superextension of the KdV equation with a free parameter a,

$$\Theta_t = \partial_x \left(-\partial_x^2 \Theta + 3\Theta D_1 D_2 \Theta + \frac{a-1}{2} D_1 D_2 \Theta^2 + a\Theta^3 \right) . \tag{5.12}$$

In terms of the component fields we have

$$
\begin{aligned}
u_t &= -u_{xxx} + 6uu_x - 3\phi\phi_{xx} - 3\psi\psi_{xx} - 3a\omega_x\omega_{xx} - (a+2)\omega\omega_{xxx} \\
&\quad +3au_x\omega^2 + 6au\omega\omega_x + 6a\omega_x\psi\phi + 6a\omega\psi_x\phi + 6a\omega\psi\phi_x \,, \\
\omega_t &= -\omega_{xxx} + 3a\omega^2\omega_x + (a+2)u_x\omega + (a+2)u\omega_x + (a-1)\psi_x\phi \\
&\quad +(a-1)\psi\phi_x \,, \\
\phi_t &= -\phi_{xxx} + 3u_x\phi + 3u\phi_x + 6a\omega\omega_x\phi + 3a\omega^2\phi_x - (a+2)\omega_x\psi_x \\
&\quad -(a+2)\omega\psi_{xx} - (a-1)\omega_{xx}\psi - (a-1)\omega_x\psi_x \,, \\
\psi_t &= -\psi_{xxx} + 3u_x\psi + 3u\psi_x + 6a\omega\omega_x\psi + 3a\omega^2\psi_x + (a+2)\omega_x\phi_x \\
&\quad +(a+2)\omega\phi_{xx} + (a-1)\omega_{xx}\phi + (a-1)\omega_x\phi_x \,.
\end{aligned} \tag{5.13}
$$

It has been shown that this system is integrable for only special values of a, which are -2, 1, 4 [83, 95]. However, among the three integrable cases the most complicated and perhaps the most interesting one is the $a = 1$ case [95].

The bosonic limit of (5.13) for the $a = 1$ case gives rise to the Kersten-Krasil'shchik coupled KdV-mKdV equations [20]

$$
\begin{aligned}
u_t &= -u_{xxx} + 6uu_x - 3\omega\omega_{xxx} - 3\omega_x\omega_{xx} + 3u_x\omega^2 + 6u\omega\omega_x \,, \\
\omega_t &= -\omega_{xxx} + 3\omega^2\omega_x + 3u\omega_x + 3u_x\omega \,.
\end{aligned} \tag{5.14}
$$

This system of even equations can be considered as a sort of coupling between the KdV and the mKdV equations. Actually for $\omega = 0$ the system of equations reduces to the KdV equation for u, viz

$$u_t = -u_{xxx} + 6uu_x\,, \tag{5.15}$$

and for $u = 0$ it gives the mKdV equation for ω, viz

$$\omega_t = -\omega_{xxx} + 3\omega^2\omega_x\,. \tag{5.16}$$

However, note that for $u = 0$ the system gives not only the mKdV equation for ω but also an ordinary differential equation in ω, which is $(\omega\omega_{xx})_x = 0$.

5.2 Singularity Analysis of Kersten-Krasil'shchik Coupled KdV-mKdV Equations

We study the integrability of (5.14) following the Weiss-Kruskal algorithm of singularity analysis [63, 96]. The algorithm is well known and widely used. Therefore we omit unessential computational details.

Firstly we find that a hypersurface $\varphi(x,t) = 0$ is noncharacteristic for the system (5.14) if $\varphi_x \neq 0$. Thus we set $\varphi_x = 1$ without loss of generality. Then substituting the expansions

$$\begin{aligned} u &= u_0(t)\varphi^{\alpha} + \cdots + u_r(t)\varphi^{r+\alpha} + \cdots\,, \\ \omega &= \omega_0(t)\varphi^{\beta} + \cdots + \omega_r(t)\varphi^{r+\beta} + \cdots \end{aligned} \tag{5.17}$$

into (5.14) we get the following branches, that is the admissible choices of α, β, u_0 and ω_0 together with the positions r of resonances, at which arbitrary functions can enter into the expansions,

$$\alpha = -2\,, \quad \beta = -1\,, \quad u_0 = 1\,, \quad \omega_0 = \pm i\,,$$

$$
\begin{aligned}
r &= -1, 1, 2, 3, 4, 6\,; && (5.18)\\
\alpha &= -2\,, \quad \beta = -1\,, \quad u_0 = 2\,, \quad \omega_0 = \pm 2i\,, \\
r &= -2, -1, 3, 3, 4, 8\,; && (5.19)\\
\alpha &= -2\,, \quad \beta = 2\,, \quad u_0 = 2\,, \quad \forall \omega_0(t)\,, \\
r &= -4, -1, 0, 1, 4, 6\,; && (5.20)\\
\alpha &= -2\,, \quad \beta = 3\,, \quad u_0 = 2\,, \quad \forall \omega_0(t)\,, \\
r &= -5, -1, -1, 0, 4, 6\,; && (5.21)
\end{aligned}
$$

besides the ones that correspond to the Taylor expansions governed by the Cauchy-Kovalevskaya theorem.

The branch (5.18) is generic, that is, the expansions (5.17) describe the behavior of a generic solution near its singularity. The nongeneric branches (5.19), (5.20) and (5.21) correspond to singularities of special solutions. The branches (5.19) and (5.20) admit the following interpretation in the spirit of [97]: (5.19) describes the collision of two generic poles (5.18) with the same sign of ω_0, whereas (5.20) describes collision of two generic poles (5.18) with opposite signs of ω_0. The branch (5.21) corresponds to (5.20) with $\omega_0 \to 0$.

Next we find from (5.14) the recursion relations for the coefficients $u_n(t)$ and $\omega_n(t)$ $(n = 0, 1, 2, \ldots)$ of the expansions (5.17) separately for each of the branches and check the consistency of those recursion relations at the resonances. The recursion relations are found to be consistent. Therefore the expansions (5.17) of the solutions of (5.14) are free from logarithmic terms. We conclude that the system (5.14) passes the Painlevé Test for integrability successfully and must be expected to possess a Lax pair.

5.3 Prolongation Structure of Kersten-Krasil'shchik Coupled KdV-mKdV Equations

By introducing the variables

$$p \equiv u_x\,, \quad q \equiv \omega_x\,, \quad r \equiv p_x\,, \quad s \equiv q_x\,, \tag{5.22}$$

the system of equations (5.14) can be represented by the set of 2-forms

$$\begin{aligned}
\alpha_1 &= du \wedge dt - p dx \wedge dt\,, \\
\alpha_2 &= dp \wedge dt - r dx \wedge dt\,, \\
\alpha_3 &= d\omega \wedge dt - q dx \wedge dt\,, \\
\alpha_4 &= dq \wedge dt - s dx \wedge dt\,, \\
\alpha_5 &= du \wedge dx - dr \wedge dt - 3\omega ds \wedge dt + (6up - 3qs + 3p\omega^2 + 6\omega uq) dx \wedge dt\,, \\
\alpha_6 &= d\omega \wedge dx - ds \wedge dt + (3\omega^2 q + 3uq + 3p\omega) dx \wedge dt\,,
\end{aligned} \tag{5.23}$$

which constitutes a closed ideal I such that $dI \subset I$.

We extend the ideal I by adding to it the system of 1-forms

$$W^k = dy^k + F^k dx + G^k dt\,, \qquad k = 1, \ldots, N, \tag{5.24}$$

where y^k are prolongation variables and F^k and G^k, which are assumed in the form $F^k = F_j{}^k y^j$, $G^k = G_j{}^k y^j$, are functions of $(u, \omega, p, q, r, s, y^k)$. The extended ideal must be closed under exterior differentiation. This requirement gives the set of partial differential equations for F^k and G^k. Dropping the indices for simplicity we have

$$F_p = F_q = F_r = F_s = 0\,, \quad F_u = -G_r\,, \quad 3\omega F_u + F_\omega = -G_s\,,$$

$$pG_u + qG_\omega + rG_p + sG_q - (6up - 3qs + 3p\omega^2 + 6u\omega q)F_u$$

$$-(3\omega^2 q + 3uq + 3p\omega)F_\omega - [F, G] = 0\,, \tag{5.25}$$

where the commutator is defined in (2.78). Next integrating (5.25) we find

$$F = \left(u\omega - \frac{1}{2}\omega^3\right) X_1 + \frac{1}{2}\omega^2 X_2 + uX_3 + \omega X_4 + X_5\,, \tag{5.26}$$

where X_1, X_2, X_3, X_4 and X_5 are constants of integration depending upon y^k only. It is immediately seen that X_1 is in the center of the prolongation algebra. Hence we equate X_1 to zero and find G to be

$$G = (-r - \omega s - q^2 + 2u^2 - \omega^4 - \omega^2 u)X_3 - (s - \omega^3 - 3u\omega)X_4$$

$$-(p + \omega q)X_6 - u\omega X_7 - \left(\frac{1}{2}\omega^2 + u\right) X_8 - qX_9 - \frac{1}{2}\omega^2 X_{10}$$

$$-\omega X_{11} + X_0\,, \tag{5.27}$$

where X_0 is a constant of integration depending on y^k only. The remaining elements are

$$X_6 = [X_5, X_3]\,, \quad X_7 = [X_4, X_6]\,, \quad X_8 = [X_5, X_6]\,,$$

$$X_9 = [X_5, X_4]\,, \quad X_{10} = [X_4, X_9]\,, \quad X_{11} = [X_5, X_9]\,. \tag{5.28}$$

The integrability conditions impose the following restrictions on X_i $(i = 0, \ldots, 11)$

$$[X_2, X_3] = 0\,, \quad [X_5, X_0] = 0\,, \quad [X_3, [X_3, X_6]] = 0\,,$$

$$[X_2, [X_4, X_3]] = 0\,, \quad [X_3, [X_4, X_3]] = 0\,, \quad [X_3, [X_4, [X_4, X_3]]] = 0\,,$$

$$[[X_4, [X_4, X_3]], [X_3, X_6]] = 0\,, \quad 2X_6 + [X_5, X_2] = 0\,,$$

$$4[X_4, X_3] + [X_4, X_2] = 0\,, \quad [X_3, X_0] - [X_5, X_8] = 0\,,$$

$$[X_4, X_0] - [X_5, X_{11}] = 0\,, \quad 3X_6 - \frac{1}{2}[X_5, [X_3, X_6]] - [X_3, X_8] = 0\,,$$

$$3X_2 - 3[X_4, [X_4, X_3]] - [X_2, X_6] + [X_3, X_6] = 0\,,$$

$$[X_3, X_9] - X_7 - 2[X_5, [X_4, X_3]] = 0\,,$$

$$[X_2, X_0] - 2[X_4, X_{11}] - [X_5, X_8] - [X_5, X_{10}] = 0\,,$$

$$[X_2, [X_5, [X_4, X_3]]] + [X_2, X_7] + \frac{1}{2}[X_2, [X_2, X_9]] = 0\,,$$

$$3X_9 - [X_3, X_{11}] - [X_4, X_8] - 2[X_5, [X_5, [X_4, X_3]]] - [X_5, X_7] = 0\,,$$

$$[X_3, [X_5, [X_4, X_3]]] + [X_3, X_7] + \frac{1}{2}[X_4, [X_3, X_6]] = 0\,,$$

$$X_9 - \frac{1}{2}\left([X_2, X_{11}] + [X_4, X_8] + [X_4, X_{10}]\right) - \frac{1}{3}[X_5, [X_5, [X_4, X_3]]]$$

$$-\frac{1}{3}[X_5, X_7] - \frac{1}{6}[X_5, [X_2, X_9]] = 0\,,$$

$$\frac{1}{2}[X_2, X_5] + \frac{1}{4}([X_2, X_8] + [X_2, X_{10}]) + \frac{1}{3}[X_4, [X_5, [X_4, X_3]]]$$

$$+\frac{1}{3}[X_4, X_7] + \frac{1}{6}[X_4, [X_2, X_9]] = 0\,,$$

$$3X_6 - \frac{1}{2}([X_2, X_8] + [X_3, X_8] + [X_3, X_{10}]) - [X_4, [X_5, [X_4, X_3]]]$$

$$-[X_4, X_7] - 2[X_5, [X_4, [X_4, X_3]]] - 2[X_5, [X_3, X_6]] = 0\,,$$

$$8[X_4, X_3] + \frac{1}{4}[X_2, [X_2, X_9]] - \frac{1}{6}[X_3, [X_2, X_9]]$$

$$-2[X_4, [X_4, [X_4, X_3]]] - \frac{11}{6}[X_4, [X_3, X_6]] = 0\,. \qquad (5.29)$$

Using the Jacobi identities we obtain the further relations:

$$[X_4, X_3] = 0\,, \quad [X_2, X_7] = 0\,, \quad [X_3, X_7] = 0\,,$$

$$[X_3, X_{10}] = 0\,, \quad [X_4, X_7] = 0\,, \quad [X_5, X_7] = X_9\,,$$

$$[X_2, [X_2, X_9]] = 0\,, \quad [X_4, [X_3, X_6]] = 0\,,$$

$$[X_2, X_6] + 2[X_3, X_6] = 0\,, \qquad [X_4, X_{11}] - [X_5, X_{10}] = 0\,,$$

$$[X_5, [X_3, X_6]] - [X_3, X_8] = 0\,, \qquad [X_2, X_8] - [X_5, [X_2, X_6]] = 0\,,$$

$$[X_5, [X_4, X_3]] + [X_3, X_9] - X_7 = 0\,,$$

$$[X_2, X_9] - 4[X_5, [X_4, X_3]] + 2X_7 = 0\,,$$

$$[X_2, [X_5, [X_4, X_3]]] + 2[[X_4, X_3], X_6] = 0\,,$$

$$[X_3, [X_5, [X_4, X_3]]] - [[X_4, X_3], X_6] = 0\,,$$

$$[X_3, [X_2, X_9]] - [X_2, [X_3, X_9]] = 0\,,$$

$$[X_5, X_8] + [X_5, X_{10}] = 0 \,. \qquad (5.30)$$

In order to find the Lie algebra generated by F and the matrix representations of the elements $X_i\,(i = 0, \ldots, 11)$ of this algebra, we follow the strategy given in Section 2.2. Firstly we reduce the number of elements. By using (5.28), (5.29) and (5.30) we get

$$X_2 = -2X_3 \,. \qquad (5.31)$$

Next we locate the nilpotent and the neutral elements. The equations (5.28) and (5.29) together with (5.31) give that

$$\begin{aligned} [X_3, X_6] &= 2X_3 \,, \\ [X_5, X_3] &= X_6 \,. \end{aligned} \qquad (5.32)$$

Hence by using corollary 2.2.2 we identify X_3 as the nilpotent and X_6 as the neutral element.

We note that the system of equations in (5.14) has the following scaling symmetry

$$x \to \lambda^{-1}x \,, \quad t \to \lambda^{-3}t \,, \quad u \to \lambda^2 u \,, \quad \omega \to \lambda\omega \,, \qquad (5.33)$$

which implies that the elements X_i must satisfy

$$X_0 \to \lambda^3 X_0 \,, \quad X_3 \to \lambda^{-1}X_3 \,, \quad X_4 \to X_4 \,, \quad X_5 \to \lambda X_5 \,,$$

$$X_6 \to X_6 \,, \quad X_7 \to X_7 \,, \quad X_8 \to \lambda X_8 \,, \quad X_9 \to \lambda X_9 \,,$$

$$X_{10} \to \lambda X_{10} \,, \quad X_{11} \to \lambda^2 X_{11} \,. \qquad (5.34)$$

By using (5.34) and the scaling symmetries of the basis elements of $s\ell(n+1,C)$ we try to embed this prolongation algebra into $s\ell(n+1,C)$. Starting from the case $n=1$ we identify

$$X_3 = e_- \,, \qquad X_6 = h \,. \tag{5.35}$$

In general it is possible to write the other elements as linear combinations of the basis elements of $s\ell(2,C)$ such as

$$X_5 = A_1\lambda^2 e_- + A_2 e_+ + A_3\lambda h \,, \tag{5.36}$$

where $A_i\,(i=1,2,3)$ are arbitrary constants. Then from various commutation relations we get

$$X_4 = X_7 = X_9 = X_{10} = X_{11} = 0 \,, \tag{5.37}$$

together with $A_2 = 1$ and $A_3 = 0$. If we introduce the explicit matrix forms of the basis elements of $s\ell(2,C)$, given in (2.50), together with the conditions $X_4 = 0$ and $X_2 = -2X_3$, (5.26) gives the explicit matrix form of F as

$$F = \begin{pmatrix} 0 & 1 \\ u - \omega^2 + \lambda^2 & 0 \end{pmatrix} , \tag{5.38}$$

where we have chosen $A_1 = 1$. However, this form of F implies that there is only one equation and not a system of equations. Thus $s\ell(2,C)$ cannot be the whole algebra. The simplest nontrivial closure is in terms of $s\ell(3,C)$. We take

$$X_3 = e_{-\alpha_1} \,, \quad X_6 = h_1 \,, \tag{5.39}$$

where we use the standard Cartan-Weyl basis [41] of A_2. The centralizer of $e_{-\alpha_1}$, is spanned by $\{e_{-\alpha_1},\, e_{-\alpha_1-\alpha_2},\, e_{\alpha_2},\, h_1+2h_2\}$ so that $[X_5, X_3] = X_6$ gives

$$X_5 = e_{\alpha_1} + C_1 e_{\alpha_2} + C_2\lambda(h_1+2h_2) + C_3\lambda^2 e_{-\alpha_1} + C_4\lambda^3 e_{-\alpha_1-\alpha_2} \,, \tag{5.40}$$

where $C_i\,(i = 1,2,3,4)$ are arbitrary constants and the scaling symmetry has been taken into account. Then we take the most general form of X_4 as

$$X_4 = D_1 h_1 + D_2 h_2 + D_3 \lambda^{-1} e_{\alpha_1} + D_4 \lambda^{-1} e_{\alpha_2} + D_5 \lambda e_{-\alpha_1} + D_6 \lambda e_{-\alpha_2}$$
$$+ D_7 \lambda^2 e_{-\alpha_1-\alpha_2} + D_8 \lambda^2 e_{\alpha_1+\alpha_2}\,, \quad (5.41)$$

where $D_i\,(i = 1,\ldots,8)$ are arbitrary constants, and using various commutation relations we find the other elements

$$\begin{aligned}
X_0 &= -36{C_2}^3 \lambda^3 (h_1 + 2h_2) - 4C_3 \lambda^2 e_{\alpha_1} - 4{C_3}^2 \lambda^4 e_{-\alpha_1}\,, \\
X_7 &= D_4 \lambda^{-1} e_{\alpha_2} + D_7 \lambda^2 e_{-\alpha_1-\alpha_2}\,, \\
X_8 &= -2e_{\alpha_1} + 2C_3 \lambda^2 e_{-\alpha_1}\,, \\
X_9 &= D_4 \lambda^{-1} e_{\alpha_1+\alpha_2} - D_7 \lambda^2 e_{-\alpha_2} + 3C_2 D_4 e_{\alpha_2} - 3C_2 D_7 \lambda^3 e_{-\alpha_1-\alpha_2}\,, \\
X_{10} &= -D_4 D_7 \lambda (h_1 + 2h_2) - 6C_2 D_4 D_7 \lambda^2 e_{-\alpha_1}\,, \\
X_{11} &= (9{C_2}^2 D_4 + C_3 D_4) \lambda e_{\alpha_2} + 6C_2 D_7 \lambda^3 e_{-\alpha_2} + 6C_2 D_4 e_{\alpha_1+\alpha_2} \\
&\quad + (9{C_2}^2 D_7 + C_3 D_7) \lambda^4 e_{-\alpha_1-\alpha_2} \qquad (5.42)
\end{aligned}$$

with the following conditions on the C_i and D_i

$$C_1 = C_4 = D_3 = D_5 = D_6 = D_8 = 0\,, \quad D_1 D_4 = 0\,, \quad D_1 D_7 = 0\,,$$
$$C_4 D_1 = 0\,, \quad D_1 C_1 = 0\,, \quad D_4 D_7 = 6C_2\,, \quad D_2 = 2D_1\,, \quad C_3 = 9{C_2}^2\,. \quad (5.43)$$

We choose $D_1 = 0$ and $C_2 = D_4 = 1$. So that, $X_7 = X_4$ and $X_0 = -36\lambda^2 X_5$.

Then we obtain the matrix representations of the generators X_i as

$$X_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad X_4 = \begin{pmatrix} 0 & 0 & 0 \\ -\lambda^{-1} & 0 & 0 \\ 0 & 6\lambda^2 & 0 \end{pmatrix},$$

$$X_5 = \begin{pmatrix} -\lambda & 0 & 1 \\ 0 & 2\lambda & 0 \\ 9\lambda^2 & 0 & -\lambda \end{pmatrix}, \qquad X_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$X_8 = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 18\lambda^2 & 0 & 0 \end{pmatrix}, \quad X_9 = \begin{pmatrix} 0 & 6\lambda^2 & 0 \\ -3 & 0 & \lambda^{-1} \\ 0 & -18\lambda^3 & 0 \end{pmatrix},$$

$$X_{10} = \begin{pmatrix} 6\lambda & 0 & 0 \\ 0 & -12\lambda & 0 \\ -36\lambda^2 & 0 & 6\lambda \end{pmatrix}, \quad X_{11} = \begin{pmatrix} 0 & -36\lambda^3 & 0 \\ -18\lambda & 0 & 6 \\ 0 & 108\lambda^4 & 0 \end{pmatrix}. \qquad (5.44)$$

By substituting the matrix representations of the elements of the prolongation algebra into equations (5.26) and (5.27) and calling $X = -F^\dagger$, $T = -G^\dagger$ and $\Psi = y^\dagger$ we can construct the Lax pair,

$$\Psi_x = X\Psi\,, \qquad \Psi_t = T\Psi\,, \qquad (5.45)$$

for the system (5.14) with the following matrices X and T(given in component form):

$$X = \begin{pmatrix} \lambda & \omega\lambda^{-1} & \omega^2 - u - 9\lambda^2 \\ 0 & -2\lambda & -6\omega\lambda^2 \\ -1 & 0 & \lambda \end{pmatrix}, \tag{5.46}$$

$$\begin{aligned}
T_{11} &= p + \omega q + 3\lambda\omega^2 - 36\lambda^3\,, \\
T_{12} &= (\omega^3 + 2u\omega - s)\lambda^{-1} - 3q - 18\lambda\omega\,, \\
T_{13} &= r + \omega s + q^2 - 2u^2 + \omega^4 + \omega^2 u - 9\lambda^2\omega^2 + 18\lambda^2 u + 324\lambda^4\,, \\
T_{21} &= 6q\lambda^2 - 36\lambda^3\omega\,, \\
T_{22} &= -6\lambda\omega^2 + 72\lambda^3\,, \\
T_{23} &= 6(s - \omega^3 - 2u\omega)\lambda^2 - 18q\lambda^3 + 108\lambda^4\omega\,, \\
T_{31} &= -\omega^2 - 2u + 36\lambda^2\,, \\
T_{32} &= q\lambda^{-1} + 6\omega\,, \\
T_{33} &= -p - \omega q + 3\lambda\omega^2 - 36\lambda^3\,.
\end{aligned} \tag{5.47}$$

Since the system (5.14) defines a polynomial flow in accordance with the definition given in [16], the matrices X and T should be expected to be polynomial in the spectral parameter λ. Thus the forms of X and T are unusual in the sense of the dependence on λ. However, it is possible to obtain equivalent matrices by the gauge transformation,

$$X' = SXS^{-1}\,, \qquad T' = STS^{-1}\,, \tag{5.48}$$

where

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & \lambda^{-1} & 0 \end{pmatrix} . \tag{5.49}$$

The result is

$$X' = \begin{pmatrix} \lambda & u - \omega^2 + 9\lambda^2 & \omega \\ 1 & \lambda & 0 \\ 0 & 6\lambda\omega & -2\lambda \end{pmatrix} , \tag{5.50}$$

$$\begin{aligned}
T'_{11} &= p + \omega q + 3\lambda\omega^2 - 36\lambda^3 , \\
T'_{12} &= -r - \omega s - q^2 + 2u^2 - \omega^4 - \omega^2 u + 9\lambda^2\omega^2 - 18\lambda^2 u - 324\lambda^4 , \\
T'_{13} &= \omega^3 + 2u\omega - s - 3q\lambda - 18\lambda^2\omega , \\
T'_{21} &= \omega^2 + 2u - 36\lambda^2 , \\
T'_{22} &= -p - \omega q + 3\lambda\omega^2 - 36\lambda^3 , \\
T'_{23} &= -q - 6\omega\lambda , \\
T'_{31} &= 6q\lambda - 36\lambda^2\omega , \\
T'_{32} &= -6(s - \omega^3 - 2u\omega)\lambda + 18q\lambda^2 - 108\lambda^3\omega , \\
T'_{33} &= -6\lambda\omega^2 + 72\lambda^3 .
\end{aligned} \tag{5.51}$$

The matrices X' and T' give us exactly the spectral problem for the KdV equation when $\omega = 0$, but they do not reduce to the one for mKdV equation when $u = 0$. This result should be expected because the Kersten-Krasil'shchik system, when $u = 0$, gives not only the mKdV equation, as stated in [20], but also an ordinary differential equation in ω, which is $(\omega\omega_{xx})_x = 0$. Finally we note that the Lax pair obtained from (2.39) with (5.50) and (5.51) is a true Lax pair since the parameter λ cannot be removed from X' and T' by a gauge transformation as can be proven by a gauge-invariant technique [98] (see also Appendix C).

CHAPTER 6

CONCLUSION

In this thesis firstly the W-E prolongation method was studied. It was seen that a systematic way for seeking pseudopotentials leads to some overdetermined set of first-order nonlinear partial differential equations the integrability conditions of which give rise to a set of commutator relations with constraints. This set of commutator relations constitutes a free Lie algebra. For the closure of the prolongation algebra process the main interest centers around identifying nilpotent elements and embedding of those into a simple Lie algebra the representation of which leads to an associated linear eigenvalue equation. In particular we rederived the linear scattering problem for Drinfel'd-Sokolov system of equations by using the prolongation algorithm. We also found the auto-Bäcklund transformations and some exact solutions of these equations. This system can be integrated by the method of inverse scattering problem associated with the fourth-order Lax operator, L, which was developed by Iwasaki [99]. It is known that the most general Bäcklund transformation would be the one which utilizes the infinite-dimensional algebra and not all finite algebras give rise to Bäcklund transformations. Without seeking whether the incomplete algebra found in Chapter 4 is finite- or infinite-dimensional we used a finite-dimensional representation of the prolongation algebra and derived nontrivial Bäcklund transformations. Thus the methods given in [16] and [51] are quite useful from the practical point of view for the systems of nonlinear partial differential equations. It is also worth mentioning that a close connection between some stationary flows associated with fourth-order Lax operators and generalizations of some integrable Hamiltonian systems with quartic potentials is known [100]. Equations (4.28) and (4.29) can be considered in this context.

Next in this thesis the Painlevé analysis was discussed for both ordinary and partial differential equations and in particular the Painlevé Test for partial differential equations, introduced by WTC, was applied to the classical part of one of the supersymmetric extensions of the KdV equation, namely the Kersten-Krasil'shchik coupled KdV-mKdV equations. We showed that this system passes the Painlevé Test successfully. Thus we naturally expected that this system possess a Lax pair. Indeed by using the Dodd-Fordy algorithm of the W-E prolongation technique we found a 3×3 matrix spectral problem for the Kersten and Krasil'shchik system and hence proved its integrability. The Lax pair that we found is unusual in the sense of the dependence on the spectral parameter λ. For this reason we obtained an equivalent Lax pair by using gauge transformations. We also showed that the Lax pair obtained is a true Lax pair since the parameter λ cannot be removed by a gauge transformation, as can be proven by a gauge-invariant technique. More recently the solitary wave and doubly periodic wave solutions for this system were obtained [101].

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APPENDIX A

JET-BUNDLES

In this Appendix an informal description of the jet-bundle formalism, which can be considered as a geometric approach to systems of partial differential equations, is given mainly in local coordinates. We begin by giving some definitions.

Two differentiable manifolds M and N together with a continuous surjective map $\pi : N \to M$ constitute a bundle, which is denoted by the triple (N, M, π). Here the manifold N is called the *total space* and the manifold M is called the *base space*. Most of the time either the total space, N, or the projection, π, is used as a shorthand notation for the bundle [26, 27]. If the total space, N, of the bundle is homeomorphic to $M \times U$, where U is a manifold called the *fibre*, then the bundle is called a *trivial-bundle*. Sometimes the total space as a whole is not homeomorphic to the product manifold, but each point of it has a neighborhood which looks like a product manifold. Such a locally trivial bundle is called a *fibre-bundle*. For each point $a \in M$ the subset $\pi^{-1}(a)$ of N is called the *fibre over a*. A *section* or a *cross-section* of a fibre bundle is a smooth map $s : M \to M \times U$ which satisfies $\pi s = id_M$. Clearly $s(a)$ is an element of the fibre at a.

Before giving the definition of a jet-bundle it is useful to mention two notations for jet variables. The first one is the multi-index notation. A multi-index σ is an unordered p-tuple $[\sigma_1, \sigma_2, \ldots, \sigma_p]$ of integers, indicating which derivatives are being taken. The order of such a multi-index σ is given by the sum of the $\sigma_i : |\sigma| = \sigma_1 + \ldots + \sigma_p$ and indicates how many derivatives are being taken [102]. If we choose the independent variables $x^i, i = 1, \ldots, p$, as coordinates of the base manifold and dependent variables $u^\alpha, \alpha = i = 1, \ldots, q$ as coordinates of the fibre manifold, then the jet variable u^α_σ is denoted as

$$u^\alpha_\sigma = \frac{\partial^{|\sigma|} u^\alpha}{\partial (x^1)^{\sigma_1} \cdots \partial (x^p)^{\sigma_p}} \cdot \tag{A.1}$$

The second one is the well-known repeated index notation. In this notation the jet variable $u^\alpha_{ij\ldots}$ is understood to be the derivative with respect to the $x^i, x^j, \ldots$. However, this notation should not be confused with the multi-index one. The derivatives in the multi-index notation are always denoted by Greek letters whereas in repeated-index notation they are denoted by Latin letters. It is also possible to use the repeated-index notation for the multi-index σ. For example, if we choose $p = 2$ and label the coordinates as x and t , then the third order multi-indices will be xxx, xxt, xtt and ttt.

Now we construct the jet-bundles over a fibred-manifold. Firstly consider a fibre-bundle $\pi : M \times U \to M$ with the independent variables $x^i, i = 1, \ldots, p$, as coordinates of the base manifold and the dependent variables $u^\alpha, \alpha = 1, \ldots, q$, as coordinates of the fibre-manifold. Over this bundle it is possible to construct further bundles. For example by adding the first-order partial derivatives u^α_i

as new fibre coordinates the first-order jet-bundle is constructed over the fibre-bundle $M \times U$. Similarly by adding the jet variables $u^{\alpha}{}_{\sigma}$, which are obtained by taking the various partial derivatives of the dependent variables with respect to the independent variables, as the fibre coordinates representing all the derivatives up to order n the nth-order jet-bundle is constructed over the fibre-bundle $M \times U$. In order to give a definition of a jet-bundle it is useful to introduce a set of Euclidean spaces U_i , the coordinates of which are $u^{\alpha}{}_{\sigma}$ with $|\sigma| = i$. Then the space $M \times U \times U_1$ is called the *first-order jet-bundle* over the space $M \times U$. Furthermore, denoting the Cartesian product of Euclidean spaces as

$$U^{(n)} = U \times U_1 \times \ldots \times U_n \,, \tag{A.2}$$

the space $M \times U^{(n)}$ is called the *nth-order jet-bundle over* $M \times U$.

It is easy to compute the dimension of this space. The number of different partial derivatives of order n for a single dependent variable is equal to

$$\dim U_n = \binom{p+n-1}{p-1} . \tag{A.3}$$

Therefore the number of partial derivatives of q dependent variables up to order n (including the zeroth order one) is given by,

$$\dim U^{(n)} = q \sum_{i=0}^{n} \binom{p+i-1}{p-1} = q \binom{p+n}{n} . \tag{A.4}$$

Finally the dimension of the nth order jet space is

$$\dim\,(M \times U^{(n)}) = p + q \binom{p+n}{n} . \tag{A.5}$$

As an example consider the case $p = 2$ and $q = 1$ and denote the independent variables by x and t and the dependent variable by u. Then the zeroth-order jet-bundle $M \times U$ has coordinates (x, t, u), the first-order jet-bundle, $M \times U \times U_1$, has coordinates (x, t, u, u_x, u_t) and the second-order jet-bundle, $M \times U^{(2)}$, has coordinates $(x, t, u, u_x, u_t, u_{xt}, u_{xx}, u_{tt})$.

Alternatively it is also possible to define jet-bundles by using directly the n-jets of functions. The n-jet of a function consists of all the equivalence classes of functions which have the same Taylor series expansion up to order n. More precisely they are defined as follows:

Let M and U be two differentiable manifolds and let $C^\infty(M, U)$ denote the collection of C^∞ maps $f : M \to U$. Two maps $f, g \in C^\infty(M, U)$ are said to agree to order n at $x \in M$ if there are coordinate charts around $x \in M$ and $f(x) = g(x) \in U$ in which they have the same Taylor expansion up to and including order n. This agreement is independent of the coordinates chosen and it is an equivalence relation. The equivalence class of maps which agree with f to order n at $x \in M$ is called the *n-jet of f at x* and is denoted by ${J^n}_x f$ [50]. If $\{x^i\}$ are local coordinates around $x \in M$ and $\{u^\alpha\}$ around $f(x) \in U$, then ${J^n}_x f$ is determined by x^i, u^α and ${u^\alpha}_\sigma$, where $u^\alpha = f^\alpha(x)$ is the presentation of f. The *nth-order jet-bundle* of M and U, denoted by $J^n(M, U)$, is the set of all n-jets, ${J^n}_x f$ with n fixed, $x \in M$ and $f \in C^\infty(M, U)$. Actually the spaces of n-jets at each point are the fibres of the nth-order jet-bundle.

Before closing this appendix we very briefly mention the advantages of the geometric approach to systems of differential equations. Consider a system of nonlinear differential equations of order n:

$$\begin{aligned} &F_1(x^i, u^\alpha, u^\alpha_\sigma) = 0\,, \\ &\vdots \\ &F_r(x^i, u^\alpha, u^\alpha_\sigma) = 0\,, \end{aligned} \tag{A.6}$$

where F_i are smooth functions of independent, dependent and partial derivatives of the dependent variables. The geometric approach to this system is to treat the equations, F_i, not as conditions on the dependent variables, but on the nth order Taylor expansions of these dependent variables. In fact this means that the system of equations have to be studied on nth-order jet spaces. In this connection the system of equations in (A.6) determines a surface ξ which is a fibred submanifold of nth-order jet-bundle. This definition has the following advantages:

- It does not distinguish between a single and a system of equations.
- The differential equation is reduced to an algebraic equation in the nth-order jet-bundle.
- The defined submanifold being a geometric object is independent of a particular set of equations or a particular coordinate system.

APPENDIX B

GENERATION OF THE PROLONGATION ALGEBRA BY THE CONSTANTS OF INTEGRATION OF F

Throughout this thesis we repeatedly mentioned that the prolongation algebra is generated by the constants of integration of F and the constants of integration of G are elements of this algebra. This fact is almost obvious except that the element X_0, which is a constant of integration of G, seems not to be an element of the prolongation algebra. In this appendix we show that it is also an element of the prolongation algebra.

In all of the examples of this thesis we see that the constants of integration of G which are different than those of F are determined by some commutation relations between the constants of integration of F except one, which is named as X_0. Thus X_i, $i = 0, \ldots, n$ generates the prolongation algebra, L, where n denotes the number of constants of integration of F. However, we can as well expect that X_i, $i = 1, \ldots, n$ generates a subalgebra of L. We denote it by L_F and express F and G as

$$F = \sum_{i=1}^{n} F_i X_i \qquad\qquad G = X_0 + G_F \,, \tag{B.1}$$

where G_F is an element of L_F. In order to show that L_F is indeed equal to L firstly we obtain the following relation by using the compatibility condition of the linear scattering problem in (2.39)

$$F_t - G_x + [G, F] = 0\,, \tag{B.2}$$

where F_t and G_x denote the total derivatives with respect to t and x, respectively. Then we substitute (B.1) into (B.2) and have

$$F_t - G_{Fx} + [G_F, F] = [F, X_0]\,, \tag{B.3}$$

from which we immediately see that $[F, X_0] \in L_F$ since the left-hand side of (B.3) is an element of L_F. We further conclude that $L_F \subset L$ is an ideal since $[X_i, X_0] \in L_F$, $\forall X_i$ of L_F. Therefore assuming L is simple we equate L_F to L and have $X_0 \in L_F$, which proves that the constants of integration of F generate the prolongation algebra.

In fact this means that G could be completely determined by F, a well-known fact [8, 22, 23, 24] which states that various time evolutions of the eigenfunctions are completely determined by the eigenvalue problem [16].

APPENDIX C

GAUGE-INVARIANCE OF THE MATRICES F AND T

In Chapter 5 we mentioned that it is possible to find equivalent matrices F' and G' for the linear scattering problem (5.45) in which we used $X = -F^\dagger$ and $T = -G^\dagger$. We also stated that the Lax pair in (5.50) and (5.51) is a true one since the spectral parameter cannot be gauged out by performing any gauge transformation. In this appendix we discuss these points following the ideas in [98, 103, 104].

The compatibility condition for the linear scattering problem in (2.39) is written as

$$F_t - G_x + [G, F] = 0\,, \tag{C.1}$$

where F_t and G_x denote the total derivatives with respect to t and x, respectively. In the literature (C.1) is called the *zero-curvature representation* due to its geometric interpretation [98, 105] and denoted by the matrix Z. It was first introduced by Zakharov and Shabat [25]. If there exists a nonremovable scattering parameter, then this representation usually leads to Bäcklund Transformations and infinite series of conservation laws. Actually it can also be interpreted as the linear representation of the prolongation algebra.

The following transformation of the matrices F and G

$$F' = SFS^{-1} - S_x S^{-1}\,, \qquad G' = SGS^{-1} - S_t S^{-1} \tag{C.2}$$

in which S is any nondegenerate ($\det S \neq 0$) matrix depending on the jet-variables forms another zero-curvature representation[1]. The mapping $(F, G) \to (F', G')$ is called a *gauge transformation* [104]. It is easily seen that the gauge transformations in (C.2) cause the following transformation $Z' = SZS^{-1}$ of the zero curvature representation which indeed implies the equivalence of the representations. Thus the matrices F and G are gauge equivalent to the matrices F' and G' [104]. Moreover if y in (2.39) is a linear pseudopotential for F and G, then Sy is the linear pseudopotential for F' and G'. Because of the existence of this equivalence between the representations, the zero-curvature representation, hence, the linear representation of the prolongation algebra, should be studied by the gauge-invariant methods [103].

The other important property of the gauge-equivalent technique is the selection of the true Lax pairs among all of the possible alternatives. It is well-known [54, 105] that the linear scattering problems such as (2.39) contain a parameter in the matrices used in the construction of the scattering problem. This parameter is called a *spectral parameter*. It is believed that only in the integrable equations this parameter becomes an essential one [98], that is, it is not possible to gauge out it by performing any gauge transformation. Because of this reason those Lax

[1] S_x and S_t should be understood as total derivatives.

pairs whose spectral parameter cannot be removed are called the *true Lax pairs.*

In Chapter 5 we saw that it is not possible to find any transformation matrix S which removes the parameter λ from the matrices F and G. Thus the Lax pair that we found is the true Lax pair for the Kersten-Krasil'shchik coupled KdV-mKdV equations. We also note that in order to find the gauge-equivalent problem we searched for the transformation matrices those of which have only constant entries. The reason of such restriction is just to simplify the calculations. For example we performed the following gauge transformations

$$F' = SFS^{-1}\,, \qquad G' = SGS^{-1}\,. \tag{C.3}$$

VITA

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