UNCERTAINTY MODELLING AND STABILITY ANALYSIS FOR 2-WAY FUZZY ADAPTIVE SYSTEMS

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EVREN GÜRKAN

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Prof. Dr. Tayfur Öztürk
Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Doctor of Philosophy.

Prof. Dr. Mübeccel Demirekler
Head of Department

This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Doctor of Philosophy.

Prof. Dr. Stephen P. Banks
Co-Supervisor

Assoc. Prof. Dr. Aydan M. Erkmen
Supervisor

Examining Committee Members

Prof. Dr. Erol Kocaoğlan

Prof. Dr. Stephen P. Banks

Prof. Dr. Ömer Morgül

Assoc. Prof. Dr. Aydan M. Erkmen

Assoc. Prof. Dr. İsmet Erkmen

ABSTRACT

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Gürkan, Evren
Ph. D., Department of Electrical and Electronics Engineering
Supervisor: Assoc. Prof. Dr. Aydan M. Erkmen
Co-Supervisor: Prof. Dr. Stephen P. Banks

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A novel fuzzy system named as 2-way fuzzy system is developed by combining the intuitionistic fuzzy set theory with the fuzzy systems theory. The developed system is used in modelling and minimizing uncertainty and inconsistency. Uncertainty is the width of the interval introduced by the independent assignment of membership and nonmembership functions of the intuitionistic fuzzy sets; and inconsistency is the violation of the consistency inequality in this assignment. The uncertainty and inconsistency is reduced through a 2 phase training. An evaluation of the degree of reduction of inconsistency is carried out at the end of the first phase of training by forming the shadowed set patterns of the membership and nonmembership functions. The system is further trained for a second phase in order to reduce uncertainty.

There are three different methods developed for the stability analysis of fuzzy systems. The first method is based on the approximating sequences technique, and the design turns into an optimal control problem. In the second analysis, describing function of a 2-way fuzzy system is evaluated analytically, and a systematic design approach is developed using describing function technique. The last analysis technique employs the Lie algebra theory in the stability analysis of Takagi-Sugeno fuzzy systems. The theoretical results are simulated on an application system, which is a flexible-joint robot arm system.

Keywords: Inconsistency modelling and evaluation, uncertainty reduction, intuitionistic fuzzy sets, shadowed sets, 2-way fuzzy systems, Takagi-Sugeno type fuzzy systems, stability analysis, equivalent linearization, Lie algebra theory, describing function analysis

ÖZ

BELİRSİZLİK MODELLEMESİ VE 2-YÖNLÜ BULANIK ADAPTİF SİSTEMLER İÇİN KARARLILIK ANALİZİ

Gürkan, Evren
Doktora, Elektrik ve Elektronik Mühendisliği Bölümü
Tez Yöneticisi: Assoc. Prof. Dr. Aydan M. Erkmen
Ortak Tez Yöneticisi: Prof. Dr. Stephen P. Banks

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Sezgisel bulanık küme teorisi, bulanık sistem teorisiyle birleştirilerek, 2-yönlü bulanık sistem olarak adlandırılan yeni bir sistem geliştirilmiştir. Geliştirilen bu sistem, belirsizlik ve tutarsızlık modellenmesinde ve bunların azaltılmasında kullanılmıştır. Belirsizlik, sezgisel bulanık kümelerin üyelik ve üyesizlik fonksiyonlarının bağımsız olarak atanmasıyla ortaya çıkan aralığın genişliğidir. Tutarsızlık, bu atamadaki tutarlılık eşitsizliğinin sağlanmamasıyla ortaya çıkmaktadır. Belirsizlik ve tutarsızlık 2 fazlı eğitme yoluyla azaltılmaktadır. Tutarsızlıktaki azalmanın derecesi, birinci fazın sonunda üyelik ve üyesizlik fonksiyonlarının gölgelendirilmiş kümelerinin oluşturulmasıyla değerlendirilmiştir. Sistem, belirsizliğin azaltılması için bir defa daha eğitilmiştir.

Bulanık sistemlerin kararlılık analizleri üç değişik metod kullanılarak geliştirilmiştir. Bu metodlardan ilki, yaklaşık seriler tekniğine dayanmaktadır ve tasarım bir optimal kontrol problemine dönüşmüştür. İkinci analizde, 2-yönlü bulanık sistemlerin genelleştirilmiş aktarım işlevi analitik olarak hesaplanmıştır, ve genelleştirilmiş aktarım işlevi tekniğine dayanılarak sistematik bir tasarım yaklaşımı geliştirilmiştir. Son analiz tekniği, Takagi-Sugeno tipi bulanık sistemlerin kararlılık analizinde Lie cebiri teorisini kullanmaktadır. Teorik sonuçlar, esnek eklemli robot kol üzerine uygulanmıştır.

Anahtar Kelimeler: Tutarsızlık modellenmesi ve değerlendirilmesi, belirsizlik azaltılması, sezgisel bulanık kümeler, gölgelendirilmiş kümeler, 2-yönlü bulanık sistemler, Takagi-Sugeno tipi bulanık sistemler kararlılık analizi, eşdeğer doğrusallama, Lie cebir teorisi, genelleştirilmiş aktarım işlevi analizi

To İlke, Işın, Tayyar, Necla

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CHAPTER 1

INTRODUCTION

1.1 Motivation, Objective and Goals

Engineering systems have grown in complexity becoming more biologically inspired, mostly mimicking human ways in their dynamics. Thus, modelling of uncertainty has become an important issue in handling complexity of increasingly natural engineering systems, rendering prediction of future states more important but more difficult as well. More specifically, the problem of adequacy of uncertainty models, tracking, modifying and/or adapting to the varying levels of process uncertainty has become more pronounced ([1], [2]). In recent literature, uncertainty has been treated in its frequential nature, where statistics have been the abundantly used tool to model uncertainty. Vagueness and imprecision are the recently emphasized characteristics of uncertainty, handled by the leading approaches based on extensions of probability theory, which are evidential theory, possibility theory, fuzzy sets and fuzzy measures, rough set theory and interval arithmetic.

Our objective in this thesis is to design a fuzzy control system that is capable of handling inconsistency and uncertainty. Towards this objective, the necessary goal we had to undertake is to generate an adaptive system that learns to minimize vagueness represented as an uncertainty interval with fuzzy bounds. As a further extension to the existing work in the literature, inconsistency types have to be analyzed and modelled giving a second dimension to the uncertainty interval. Our adaptive system has then to be capable of minimizing the inconsistent and uncertain $2D$ surface so as to project it optimally to $1D$, where an uncertainty interval solely exists with full consistency. The system then minimizes this 1-dimensional uncertainty. The last step of our work then covers the development of methodologies for a stability analysis, suitable to the intelligent adaptive system dynamics, handling inconsistency and uncertainty. This analysis should be able to dictate the design parameters for a stable intelligent adaptive system that can therefore be used to control a complex physical plant.

The balance of this thesis puts the major focus on an important gap of the literature that is to model and handle inconsistency through a stable and adaptive intelligent system. The major impact to the scientific field within the existing literature is achieved by our objective, which is that of generating a learning system that not only adapts to and modifies the vagueness nature of uncertainty, but also detects, typifies and decreases any inconsistency content. Its stability analysis is conducted by our new approaches developed to suit the plant dynamics as well as the modification of vagueness and inconsistencies. In this thesis, we develop a novel approach to minimize uncertainty interval and inconsistency through an adaptation by learning that combines intuitionistic fuzzy set theory with Takagi-Sugeno fuzzy control, and we name our novel fuzzy system based on this approach as a 2-way fuzzy adaptive control system. Intuitionistic fuzzy sets are used to model an interval valued distribution of information in the adaptive control architecture with Necessity at the lower bound as the degree of membership functions and Possibility at the upper bound as the complement of the degree of nonmembership functions. Uncertainty is modelled as the width of this interval. A width of zero is at the basis of a deterministic control free of uncertainty. Consistency is represented as a complementarity constraint on the assignment of the membership and nonmembership functions, which is that the sum of the two functions should be less than or equal to unity. However, in many control problems, this inequality constraint is not satisfied giving rise to inconsistency.

Our system is capable of reducing uncertainty and inconsistency through a two phase training. In the first phase, the aim is to reduce inconsistency, thus diminishing as much as possible this second dimension to yield only the dimension coming from the vagueness of uncertainty. The resultant system is a 2-way fuzzy adaptive system with a minimum degree of inconsistency projected optimally within the dimension generated by the uncertainty width. The purpose of the second phase of training is, on the other hand, to reduce this uncertainty width introduced by the definition of the membership and non-membership functions rendered fully consistent by the first phase of training. The resultant system is thus a system without inconsistency and uncertainty, which is the classical one-way fuzzy adaptive system known in the literature. We further develop in our thesis work a novel method for typifying the inconsistency handled in our system by a classification based on shadowed set theory, which is then applied after the first phase of training to our system.

Our aim is to use the developed 2-way fuzzy adaptive system as a controller to nonlinear systems, so it becomes important to have a systematic stability analysis for the design of such controllers. From the literature, it is well known that fuzzy systems suffer from the lack of systematic methods for stability analysis ([3],[4],[5][6],[7],[8]). The available approaches are based on Lyapunov based methods ([3],[4],[5],[9],[10],[11],[12],[13]), design of adaptive fuzzy controllers based on Lyapunov's method ([14],[8],[15],[16],[17],[18],[19]), or the application of robust stability theory ([20],[7],[17],[21],[22]). Describing function is also used to predict stable and unstable limit cycles ([23],[24],[25]). Most of the stability analysis techniques in the literature consider systems with predominantly second-order dynamics [5] without any consideration of an uncertainty interval and especially without having at all the concerns of inconsistencies.

Our 2-way fuzzy adaptive system is of Takagi-Sugeno (T-S) type, which is generally analyzed together with the nonlinear plant that it is controlling, represented as a set of linear subsystems ([9],[11],[26],[27],[12]). The stability analysis of such systems are mainly carried out using Lyapunov method, and suffers from the difficult of finding the existence of a common positive-definite matrix P for all the subsystems in the consequent parts of the rules. Frequency domain based methods such as describing function analysis are recently emerging, but works in that area still remain far from being systematic, and are mainly experimental, and not analytical in nature [25].

This major short coming of the literature on stability analysis in providing systematic ways of designing stable T-S fuzzy controllers, and our need to be systematic in the design of our 2-way fuzzy adaptive controller remaining stable with analytical predictability from its design, despite the complexities generated by uncertainty and inconsistencies led us to develop a systematic way in designing stable fuzzy controllers. In this thesis work, we propose three different methods that apply to systems with orders higher than two, thus, not presuming predominantly second-order dynamics. Our proposed methodologies are all analytical, which make the design process easier and more robust than any experimental method. Each methodology is applicable to a particular representation of the plant to be controlled. The first method aims at finding an optimal controller based on approximating sequences technique. The method assumes that the system can be represented in a pseudo-linear form. The second analysis applies to systems with periodic inputs and outputs. Revolute actuators such as robot manipulators are good examples for such oscillatory behaviors. For such systems, we develop a systematic procedure for the design of a stable 2-way fuzzy controller using describing function method, where we use the additivity property of the fuzzy systems. The third methodology uses Lie algebra theory for the stability analysis of fuzzy systems that can be represented as a T-S fuzzy system. The describing function based analysis can be carried out when the input to the system is periodic, where as the application of the third method requires the system to be modelled as a T-S fuzzy system. In case where these conditions are not satisfied, the design based on approximating sequences technique should be used. We carry out demonstrations of all our contributions on the control of a flexible robot, a nonlinear plant which provides enough complexity incorporating uncertainty and inconsistency that would normally render the design of a classical controller infeasible.

1.2 Contributions

The major contributions of this thesis can be summarized as follows:

- A novel 2-way fuzzy adaptive system is developed and used in the modelling of uncertainty and inconsistency.
- Inconsistency handled by the 2-way fuzzy adaptive system is classified according to a novel evaluation approach based on shadowed sets.
- A stable optimal fuzzy controller design is achieved using approximating sequences technique.
- A systematic design procedure for a stable 2-way fuzzy control system is developed for periodically activated uncertain systems, based on describing function analysis.
- Stability theory is developed for Takagi-Sugeno type fuzzy systems using Lie algebra theory.

1.3 Outline of the Thesis

Chapter 2 gives a detailed survey on works related to the objective of this thesis together with that on the application field of our results, that is: uncertainty modelling, stability of fuzzy systems and flexible robot systems. We then present an overview of the necessary mathematical background with the structure of fuzzy systems, Lie algebra theory, describing function and approximating sequences techniques. We present our novel 2-way fuzzy adaptive system in Chapter 3. We also discuss the training procedure and the shadowed set based typification of inconsistencies handled in the 2-way fuzzy adaptive system. The three novel stability analysis methods are developed in Chapter 4. Chapter 5, gives application examples illustrating the application of the theoretical results obtained in Chapter 3 and 4 to a flexible-joint robot arm. Chapter 6 concludes the thesis, providing also notes on future works.

CHAPTER 2

SURVEY AND MATHEMATICAL BACKGROUND

In this chapter, a literature survey on fuzzy systems stability analysis is given as well as a survey on flexible robot systems, since in the application chapter (Chapter 5) the example system is chosen as a flexible-joint robot arm system. This thesis emphasis that complex uncertain systems such as flexible robot manipulators can only be systematically controlled if the inherent uncertainty dynamics are suitable modelled. We thus find it necessary to give in this chapter a detailed survey on modelling uncertainty. The mathematical background necessary for the development of the different methodologies introduced in this thesis is reviewed here for the technical support in understanding our contributions.

2.1 Fuzzy Systems and Stability

Fuzzy systems have been used as controllers in many applications due to their ability to use all sources of information from human experts, either numerical or linguistic. The most important issue in using fuzzy systems as controllers is the stability of these systems; thus finding systematic ways for this stability analysis has been a major field of interest. Sugeno [3] and Kandel [28] review the existing stability analysis techniques in their papers. The methods can be grouped as Lyapunov methods, robust stability analysis, adaptive fuzzy controller design, and frequency domain methods.

In [3], fuzzy systems are grouped into three according to the structure of the consequent parts of the rules in the rule base. Type I systems are fuzzy systems where the consequent linguistic variable is represented by a fuzzy set, type II systems have singleton consequent parts and type III systems are Takagi-Sugeno type fuzzy systems where the consequents are linearized subsystems of the original nonlinear system. It is indicated that type II systems are special cases of both type I and type III systems. The stability analysis in this reference is applied to type II systems, and the system is found to be a piecewise polytopic affine system. The asymptotic stability is analyzed using Lyapunov approach. Kandel *et al* [28] apply Popov's technique to a single-input single-output system for stability analysis. They also classify important papers on fuzzy control stability and present a comparison table for different stability analysis techniques.

The method proposed in Yi *et al* [4] is based on variable structure system theory, and the stability condition is derived using Lyapunov theorem. In [5], the asymptotic stability is also analyzed with Lyapunov theorem. The system analyzed is a decoupled fuzzy controller with single input, where the decoupling is achieved using signed distance method. Kolodziej *et al* [6] introduce the concept of bounded linguistic variable augmenting the universe of discourse by defining stable and unstable regions. The stability analysis is performed by the modified definitions of the energetic and equilibrium point stability criteria.

Robust stability analysis is carried out in [20], [17], [7], [12], [21] and [22]. Apart from the first two papers in the above reference list, the fuzzy system generally analyzed is T-S type fuzzy system, while reference [17] considers an adaptive fuzzy system. Fuh *et al* [20] transform the fuzzy control system into a Lur'e system with uncertainties or nonlinearities. Lyapunov's direct method is used to guarantee the stability of the perturbed Lur'e system, and the bounds on allowable uncertainties or nonlinearities are given by a robustness measure.

Describing function method is used in [23], [24] and [25]. Kim *et al* [23] derive analytical expressions for the describing functions of a fuzzy system with single input, and a fuzzy system with two inputs, where the second input is the derivative of the first. The existence of the limit-cycle of the fuzzy control system is predicted using the describing function analysis. In [25], the describing function method is used to analyze the behavior of PD and PI fuzzy logic controllers. The existence of stable and unstable limit cycles are predicted. Describing function analysis of a T-S fuzzy system is done in [24], where the describing function is evaluated experimentally. The existence of multiple equilibria and of limits cycles are examined. A semiglobal stabilization of nonlinear systems in the presence of internal dynamics is presented in [29]. The system is decomposed into two subsystems with fast and slow dynamics using a composite model-based control scheme based on fuzzy control. The sufficient stability conditions for the design of fuzzy controllers are derived using singular perturbation methods.

The major work on the design and stability analysis of 1-way fuzzy adaptive controllers is developed by Wang [14], where the controllers are examined under two categories: direct adaptive fuzzy controllers and indirect adaptive fuzzy controllers. The definitions are given as: If an adaptive fuzzy controller uses fuzzy logic systems as controllers, it is called a direct adaptive fuzzy controller and it can incorporate fuzzy control rules directly into itself. If an adaptive fuzzy controller uses fuzzy logic systems as a model of the plant, it is called an indirect adaptive fuzzy controller and it can incorporate fuzzy descriptions of the plant directly into itself. There is also a further classification of the fuzzy adaptive systems according to the systems being linear or nonlinear in their adjustable parameters. First-type adaptive fuzzy controller is an adaptive fuzzy controller, which is linear in its adjustable parameters whereas second-type is nonlinear in its adjustable parameters. The stability analysis of these controllers are based on Lyapunov's theorem.

Tang *et al* [18] develop a 1-way fuzzy adaptive controller based on the work of Wang [14], where the adaptivity law is obtained by Lyapunov synthesis approach. They modify the method such that it does not require specific assignment of membership functions. In [8], tracking control of an unknown nonlinear dynamical system is performed using discrete-time adaptive fuzzy control. The design algorithm and the stability proof is based on the basis vectors of the fuzzy system instead of the ones for the plant, and ε-modification is used for the adaptation of the parameters of the fuzzy system. A stable 1-way fuzzy adaptive controller is designed for a single-input single-output unknown nonlinear system in [15]. Some of the states of the system are not available, so a high gain observer is used. The fuzzy controller design is based on the approach in [14] and there is an additional robust compensator to deal with fuzzy approximation error. For tracking performance, H^{∞} analysis is carried out, and for parameter adaptation, Lyapunov approach is employed. The boundedness of parameters are guaranteed by defining a projection operator.

A model identification by a 1-way fuzzy adaptive controller is designed in [16], where the scaling factors are adjusted on line. The system has three components: a fuzzy logic based controller, a system identification unit, and a controller synthesis unit. The identification unit uses recursive least squares algorithm for parameter estimation, and the scalar factors of the fuzzy controller are determined in the controller synthesis unit by pole placement. The aim is to place the scaling factors of the fuzzy controller such that a desired closed loop performance is obtained. The fuzzy system in [17] is expressed as a series of radial basis function expansion. The parameters to be adjusted are defined as connection weights, variances and centers, and concave/convex optimization technique is used for tuning of these parameters. Global stability analysis is performed using Lyapunov theorem. Chai *et al* . [19] design a stable fuzzy direct adaptive control scheme in their work. First, they design an optimal controller for a system with known mathematical model, then they approximate this optimal controller with a fuzzy logic system. A fuzzy sliding controller is added to the fuzzy controller to compensate for uncertainties. The global asymptotic stability is again achieved in Lyapunov sense.

The stability analysis of T-S type fuzzy systems is also mainly achieved using Lyapunov's direct method. Tanaka and Sugeno [9] have used a Takagi-Sugeno (T-S) fuzzy model, where the consequent parts of the rules form a set of linear models, and have examined the stability of such systems in terms of Lyapunov's direct method generating a sufficient condition for stability in terms of the existence of a common positive-definite matrix P for all the subsystems in the consequent parts of the rules. Cao *et al* [10] have used the same model as in [9] together with a feedback control law for each linear subsystem, and using uncertain linear system theory, have determined a condition to guarantee the global stability of the closed-loop system. This global stability analysis is also based on the Lyapunov's method, but is less conservative than in the analysis of [9] since they have relaxed the condition on finding a common P matrix.

Wang *et al* [11] have modelled a nonlinear plant using T-S fuzzy model and designed a controller using parallel distributed compensation scheme. The stability analysis in their work is based on the Lyapunov theorem; and they have turned the problem of finding the common P matrix into a linear matrix inequality (LMI) problem and used convex programming techniques for the solution. Park *et al* [13] have developed a variety of LMI-based controller design methods, where the stability analysis depends on finding a common P matrix as in [11].

In [12], stability analysis is carried out for a robust fuzzy feedback linearization regulator using T-S fuzzy model. It is assumed that the uncertainties are structured with known bounds, and the analysis is also based on LMI theory. The system is transformed into a Lur'e system and the stability analysis done by using Lyapunov theorem. Liao *et al* [30] implement the T-S fuzzy controller by a parallel distributed structure. A minimum transition matrix set is defined such that it contains the transition matrices of the closed-loop system. l_∞ Lyapunov function method is defined over the transition matrix set for stability analysis.

Zak [26] has proposed a Lyapunov based method for the design of state feedback controllers that guarantee global stability for the systems modelled by T-S fuzzy systems. Thathachar *et al* [27] have shown the equivalence of stability properties of the fuzzy systems and linear time invariant switching systems, and carried out the stability analysis based on Lyapunov method. Leung *et al* [7] design a fuzzy controller for uncertain nonlinear systems that are modelled as T-S fuzzy systems. The stability of the system is guaranteed by finding the areas where the system is robust. Robust stability analysis is used in [21] and [22] to overcome the difficulty of finding the common positive-definite matrix solution of Lyapunov equations.

Kiriakidis *et al* [31] have analyzed the T-S system with offset terms as a perturbed linear system, and derived a sufficient condition on the robust stability of the system against nonlinear perturbations to guarantee quadratic stability. The structures of T-S fuzzy PI and PD controllers are examined in [32]. Small gain theorem is used to find a sufficient global BIBO stability criterion for nonlinear systems controlled by these fuzzy controllers. The local stability is also examined. Wong *et al* [33] develop a T-S fuzzy system with each rule having two consequent parts: a numerator and a denominator. The overall closed-loop system is designed like a linear system, so the difficulty in finding the common positive-definite matrix is eliminated.

2.2 Flexible Robots

Robotic systems are nonlinear in nature and contain parametric uncertainties in their dynamics, their sensing and control mechanisms when operating in unstructured environments. Due to the need of having faster, lighter and more precise robots handling heavy payloads, flexible robots [34], [35], [36] have drawn increasing attention in recent years. The flexibility is introduced by the use of lighter links, under dynamic loading, which can no longer be ignored in the modelling and control of such systems.

Nicosia and Tomei [37] have developed a dynamic output feedback tracking controller for flexible-joint robots. The controller requires only the measurements of position and speed of each link. Tracking of any bounded reference is guaranteed in this work. The dynamic part of the controller consists of a reduced-order observer, which is estimating the position and velocity of each motor rotors and guarantees the tracking of any bounded reference.

Ge [38] has applied an adaptive controller that is based on singular perturbation theory. The method uses only the position and velocity feedback. Motor tracking error is modelled as a fast variable instead of joint elastic forces with an assumption of weak elasticity. The adaptive controller has been extended to an infinite time interval control. Spong [39] has discussed an adaptive control scheme based on singular perturbation methods. The control uses a composite strategy with a fast feedback control for stabilizing a boundary layer system together with a slow control, which is based on a quasi-steady-state system.

In case of insufficient knowledge or unknown parameters, iterative learning schemes and fuzzy controllers are applied to flexible robot systems [40], [41], [34]. An iterative learning scheme developed to overcome the problem of insufficient knowledge about robot dynamics and joint flexibility [41] guarantees bounded convergence and the computations are done off-line using link position, velocity and acceleration tracking errors.

Another iterative scheme uses contraction mapping theorem for the set-point regulation problem of a flexible-joint robot arm with unknown parameters [40]. The method guarantees the convergence to an arbitrarily small neighborhood of the equilibrium point. The developed controller is based only on position measurements. It is used for flexible robots with uncertain parameters and only a set of possible values of these uncertainties is known.

Choi *et al* [42] have made an extension to this control scheme proposed by Ailon such that it applies to a full model of a robot manipulator. In order to handle the uncertainties in system parameters, a modified fuzzy PI controller is designed with position and velocity in the feedback loop [34], capable of handling nonlinearity and uncertainty in the flexible-joint robot arm system.

2.3 Modelling Uncertainty

Engineering has to handle different types and degrees of uncertainty when dealing with all systems from design to implementation. Thus, modelling uncertainty has become an important issue in engineering systems. Sources of uncertainty are generally classified in two levels: the empirical level where uncertainty arises due to measurement errors and due to limits of the measurement devices; and the cognitive level where uncertainty results from the vagueness and ambiguity present in natural language [43].

Two tendencies exist in modelling uncertainty: explicit or implicit. If the tendency is implicit, the engineering models opt for approximating uncertain phenomenon in a deterministic manner by including sufficient amount of slack in the model or by waiting to see whether the uncertainty disappears in time [44]. Whatever the sources, avoiding uncertainty or ignoring it results in heavy and possibly critical loss of information and inevitable loss of richness of information since the uncertain part of the system may contain useful data that enrich knowledge for better performance of a system in terms of sensing, perception, action or control. Consequently, an important aspect of engineering is to develop ways of modelling uncertainty, explicitly.

Uncertainty representations show a wide spectrum of variety according to the developed tools. Statistics has been the most frequently used tool to model uncertainty. However in many cases, statistical approaches have revealed to be insufficient since they require the frequential aspect of uncertainty which seldomly appears in engineering systems, and, thus, different number of other methodologies needed to be developed to overcome this insufficiency. Among the various modelling methods suggested in the literature, the leading approaches are based on extensions of probability theory, which are evidential theory, possibility theory, fuzzy set theory and fuzzy measures and rough set theory.

In this section, the focus is on the facets of uncertainty (Section 2.3.1) and the developed modelling tools encountered in the literature (Section 2.3.2) together with their different applications that are mainly based on reducing uncertainty (Section 2.3.3).

2.3.1 Facets of Uncertainty

In order to discuss different aspects of uncertainty, its definition should first be expressed universally. However this task of generating a proper and general definition of uncertainty proved to be rather difficult. The definition chosen by Zimmermann [44] is the one that can be termed as the most universally accepted: "Uncertainty implies that in a certain situation a person does not dispose about information which quantitatively and qualitatively is appropriate to describe, prescribe or predict deterministically and numerically a system, its behavior or other characteristics."

Causes of uncertainty have also helped to differentiate the different facets of uncertainty. Among these causes are the lack of information or its incompleteness, the redundancy of information, conflicting data or evidence, ambiguity and imprecision in data boundaries [44]. These causes are explained briefly in what follows:

Klir and Folger [1] have identified two categories of uncertainty, namely vagueness and ambiguity, based on different definitions. Vagueness arises when it is not possible to have sharp boundaries in the domain of interest. On the other hand, in the case of one-to-many relations, uncertainty is named as ambiguity due to the increased span of alternatives. There is also a more specific classification where vagueness is called fuzziness due to the gradual changes in the boundary elements of a set or a proposition and ambiguity is further subdivided into: nonspecificity or imprecision, and discord or strife. Nonspecificity or imprecision is related with the sizes of the sets of alternatives of the studied problem. As the sizes of the sets become larger, the characterization becomes less specific. Discord or strife corresponds to the conflicts among the various sets of alternatives. Moreover, many authors have introduced variants to these basic concepts specifically geared towards their own application problems. In

Section 2.3.2 the basic formalisms in uncertainty modelling are discussed while related variants will be exposed in Section 2.3.3.

Lack of information occur where there is insufficient or incomplete information in order to be able to describe the problem deterministically. Measurement errors and the limited capacity of measurement devices are other causes of lack of information or imprecision. Belief is subjective in nature and represents experts opinion, so it can be considered as one of the causes due to the imprecision in natural language.

On one hand, redundancy of information may have elements supporting each other as in associative reasoning such as Dempster-Shafer theory reviewed in Section 2.3.2.1. On the other hand, many redundant elements can introduce conflicting information which is a different aspect of uncertainty. Conflict has been analyzed as a nonassociative measure in belief functions or a measure of their dissonance [45]. Conflicting knowledge in intuition is proposed as a violation of an inequality constraint defining intuitionistic fuzzy sets (Section 2.3.2.1). This violation has to be subsequently restored in order to handle any engineering approaches based on conventionally defined intuitionistic fuzzy sets. Restoring the violation, which means eliminating conflict, is another challenge in fuzzy control [46, 47]. One major approach is to maintain rule bases by compensating for rule interactions. Rule base maintenance generally does not provide a synthesis method for eliminating detected interactions but is only an analysis approach.

2.3.2 Uncertainty Modelling Tools

Statistics and probability theory have been the only tools to model uncertainty until the 60's since the primary aspect of uncertainty was considered as its frequential nature within a large sized sample set. However with pressing needs towards the engineering of intelligent systems, more anthropomorphic types of uncertainty models necessitated their inclusion in system models. The different facets of uncertainty as introduced in Section 2.3.1 motivated the emergence of further theories: possibility theory, with more emphasis on fuzzy set theory, intuitionistic set theory, rough set theory and evidence theory. It is important to choose a proper tool satisfying modelling purposes. The choice depends on the cause of uncertainty, quantity and quality of available information and the needs of the final observer of the system. It should be emphasized that there is no single tool that fits every problem.

The theories mentioned above are classified as uncertainty formalizations (Section 2.3.2.1). Within each formalization, measures of uncertainty (Section 2.3.2.2) are developed: such as for nonspecificity, fuzziness and discord.

2.3.2.1 Theories Related to Formalization of Uncertainty Types

Probability Theory

Reasoning is a major part of an engineering act. Within this act, knowledge is processed in order to support reasoning, making room for assumption and simplifications leaving many facts unknown, undetermined or crudely summarized. Frequential nature of uncertainty is modelled using tools of probability theory where all propositions in the universe do not contain doubtful descriptions and are completely defined leaving room only for doubt in their occurrences, so that each proposition is assigned a numerical measure of uncertainty as a function of probabilities and these measures are combined according to uniform syntactic principles, the way truth values are combined in logic [48]. In the excellent introduction [48] the various approaches to the management of uncertainty are classified into two major categories: extensional and intensional.

Extensional systems are known as production systems, or rule-based systems. As in classical logic, uncertainty is treated as a generalized truth value attached to formula and computed as a function of the uncertainties of its subformula. They suffer from three main limitations: i) improper treatment of correlated sources of evidence (due to their inability to recognize the common origin of information), ii) improper handling of bidirectional inferences (to avoid reasoning cycles), and iii) difficulties in retracting conclusions (due to principle of modularity). Pearl captures the limitations of extensional systems by referring to a "basic struggle between procedural modularity and semantic coherence". Such systems lend themselves readily to computations but lack the robustness of measure-based approaches.

Intensional systems are also known as declarative or model-based. Uncertainty is attached to subsets of "possible worlds". While extensional systems are "computationally convenient but semantically sloppy", intentional systems are "semantically clear but computationally clumsy". In intensional systems, rules denote elastic constraints about the world. In the Bayesian formalism, the rule $A \Longrightarrow^m B$ is interpreted as a conditional probability $P(B|A) = m$ stating that among all propositions satisfying A, those that also satisfy B are m percent in majority. It is also assumed that all other elements in the knowledge base are irrelevant to B and can therefore be ignored. In the Dempster-Shafer formalism, $A \Longrightarrow^m B$ asserts that the set of worlds in which A and $\neg B$ (complement of B) hold simultaneously has low likelihood and should be eliminated with probability m.

Evidential Theory or Dempster-Shafer Theory of Evidence

Bayesian methods begin drawing inferences when the underlying probabilistic model is complete. In causal modelling, the conditional probabilities of the values of each variable, given the factors perceived as causes of those values, must be determined. Rather than completing the model, the Dempster-Shafer (DS) theory [45] computes probability or plausibility indices. Partially specified models can be used to represent qualitative relationships of compatibility among the propositions involved. These qualitative relationships are then used as a logic for assembling proofs that lead from evidence to conclusions. The stronger the evidence, the more likely it is that a complete proof will be assembled. The theory estimates how close the evidence is to forcing the truth of the hypothesis, instead of estimating how close the hypothesis is to being true. As such, it restrains itself to only providing partial answers rather than full answers to probabilistic queries. If a large body of knowledge has been acquired but a few parameters are missing, Bayesian methods presume some reasonable values for the missing parameters. DS theory does not. Thus although Bayesian methods have the capability to tolerate total ignorance, they lack the flexibility to accommodate partial information. The DS theory computes logical entailment rather than conditional probabilities. As such, it is conceptually related to logical inference, deductive databases, logic programming, truth maintenance systems, and incidence calculus.

Here, the basic concepts and definitions from the theory of belief functions ([49],[50] are briefly reviewed.

Frame of discernment: Let a frame of discernment F be the set of all possible values of some numerical or symbolic variable x. Let $b(q)$ denote the degree of belief that the true value of x is in the subset q of F, and in no smaller subset of q. It is convenient to visualize x as a mass of weight $b(q)$ which is confined to q, but can move anywhere inside q.

Basic Probability Assignments: A function $b : 2^F \to [0, 1]$ is called a basic probability assignment (bpa) if: $b(\phi) = 0$, $0 < b(q_i) \leq 1$, $\sum_i b(q_i) = 1$ where ϕ is the empty set, q_i $(i = 1, \ldots, n)$ are subsets of F, and 2^F denotes the set of all subsets of F. $b(q_i)$ is a measure of the belief committed exactly to q_i and to no proper subset of q_i.

Focal Elements: The subsets $q_1, \ldots, q_n$ are called focal elements. The set $Q = \{q_1, \ldots, q_n\}$ of all the focal elements is called the core. A subset of F is a focal element if it is non-empty and if the bpa assigned to it is non-zero. There are no other restrictions on Q. In particular, one of the focal elements can be the entire frame of discernment. Also, the focal elements need not be disjoint, and their union need not cover the entire frame of discernment.

Belief Functions: A belief function $X = [\{Q\}; \{b\}]$ consists of a core, and a set of bpa's assigned to its focal elements. The definition of a belief function establishes a one-to-one correspondence between subsets of F and logical propositions. Thus the notions of conjunction, disjunction, implication, and negation are equivalent to the set-theoretic notions of intersection, union, inclusion, and complementation. A belief function corresponds to the intuitive idea that a portion of one's belief can be committed to set unions but need not be fully committed to any one set or its complement.

Special Cases:

(i) A simple support function consists of a core $Q = \{q, F\}$ where q is any proper non-empty subset of F, and a bpa such that $b(q) + b(F) = 1$.

(ii) A vacuous belief function given by $b(F) = 1$ and $b(q_i) = 0$ for all q_i other than F. This function describes total ignorance, since no portion of one's belief is committed to any proper subset of F. Thus, the true value of x can be anywhere inside F.

(iii) When the core consists of all the singletons in F, the belief function is equivalent to a Bayesian probability density function.

An important aspect of the Dempster-Shafer theory is its ability to differentiate between the dual concepts of total belief and plausibility on one hand, and disbelief and lack of belief on the other hand. These four concepts are defined below:

(i) Total Belief: The total belief $B(q_i) = \sum_i b(q_i)$ is the sum of the bpa's of all proper subsets q_j of q_i. It corresponds to the sum of all the masses that are confined to propositions q_i, and reflects the weight of evidence confirming the truth of proposition q_i.

(ii) Plausibility: The total belief is generally smaller than the plausibility $P(q_i) = 1 - B(F - q_i)$ where $F - q_i$ is the complement of q_i. The plausibility corresponds to the sum of all masses that may enter q_i and reflects the absence of evidence disconfirming proposition q_i. One's beliefs about a proposition q_i are not fully described by one's degree of total belief $B(q_i)$, since $B(q_i)$ does not reveal to what extent one doubts q_i. A degree of doubt on q_i is to what extent one belief in its complement $\neg q_i$: $Dou(q_i) = B(\neg q_i) = B(F - q_i)$. The expression of the amount one fails to doubt q_i, that is to say that the extent to which one find A credible or plausible is more attractive from the point of view of uncertainty measures and forms an upper bound for total beliefs.

(iii) Disbelief: Similarly, the disbelief $D(q_i) = B(F - q_i)$ is the total belief in the complement of q_i, i.e. the sum of all masses that cannot enter q_i. It reflects the weight of the evidence disconfirming q_i.

(iv) Lack of Belief: The disbelief is generally smaller than the lack of belief $L(q_i) = 1 - B(q_i)$ which is the sum of all masses that cannot be confined in q_i, and reflects the absence of evidence confirming q_i.

Dempster-Shafer Rule of Combination: Dempster-Shafer rule [45] is the basic mechanism for the combination of several belief functions. It can be interpreted as the generation of a consensus, and has been shown [51] to reduce the entropy (defined in 2.3.2.2) of a belief system. Consider two belief functions $X = [\{Q\}; \{A\}]$ and $Y = [\{R\}; \{B\}]$ based on independent bodies of evidence. Their orthogonal sum $Z = [\{S\}; \{C\}]$ is obtained by applying Dempster's rule of combination as follows: $Z = [\{S\}; \{C\}] = [\{Q\}; \{A\}] \oplus [\{R\}; \{B\}]$ where

$$c(s_k) = \frac{\sum\limits_{s_k = q_i \cap r_j} \sum a(q_i) b(r_j)}{1 - \sum\limits_{q_i \cap r_j = \emptyset} \sum a(q_i) b(r_j)} \tag{2.1}$$

This rule, which is associative and commutative, allows the sequential combination of multiple bodies of evidence. In the denominator of $c(s_k)$, the amount $K = \sum\limits_{q_i \cap r_j = \emptyset} \sum a(q_i) b(r_j)$ is of particular importance since in this expression, $a(q_i)b(r_j)$ commits probability to disjoint (contradictory) subsets q_i and r_j. The greater the number of such instances and the greater the degree of probability (belief) that are conflictingly committed in each instance, the greater the total probability K that has to be eliminated or reduced. Since the renormalizing constant in $c(s_k)$: $N = 1/(1 - K)$ is increasing with K, it naturally serve as a measure of the extent of the conflict.

Actually, the most useful measure of conflict between $a(q_i)$ and $b(r_j)$ is the quantity $\log N = \log(1/(1 - K)) = -\log(1 - K)$, which is called the weight of conflict between the two belief functions and is denoted by $Con(a(q), b(r))$; $N \geq 1$ so $0 \leq \log N \leq \infty$. If $a(q)$ and $b(r)$ do not conflict at all then $K = 0$

and $Con(a(q), b(r)) = 0$. If $a(q)$ and $b(r)$ flatly contradict each other so that their orthogonal sum do not exist, then $K = 1$ and $Con(a(q), b(r)) = \infty$. Unlike Bayesian theory, the theory of belief functions distinguishes between the notions of belief and plausibility, as well as those of disbelief and lack of belief. This is due to the property that, in general, $B(q_i) + B(F - q_i) > 1$.

The Shafer-Dempster theory is particularly well suited for applications where uncertainty is due to the incomplete but not necessarily random nature of the evidence. Since it deals with set functions rather than point functions, it also allows the representation of uncertain knowledge at multiple levels of granularity (coarsening or refining), thus facilitating reasoning with a hierarchy of hypotheses. An obvious hierarchy of hypotheses occurs with consonant belief functions at the basis of possibility definitions.

Possibility Measures

Possibilistic measure is a special focus of evidence theory that increases the resemblances to fuzzy set theory and where the focal elements of the body of evidence are nested. The associated belief and plausibility functions are called consonant. They have the following property: $B(Y \cap Z) = min[B(Y), B(Z)]$ $\forall Y, Z$ and $P(Y \cup Z) = max[P(Y), P(Z)]$ $\forall Y, Z$, where we find the similarity to norm and co-norm operations of AND and OR operations (intersection, union) with fuzzy sets.

The consonant belief and plausibility measures are the primary elements of possibilistic measures and are named as necessity and possibility measures respectively. Necessity and possibility measures are denoted as η and π respectively and have the following relationship: $\eta(A) = 1 - \pi(\bar{A})$ [1]. These measures are used to model nonspecificity and conflict.

Fuzzy Set Theory

Fuzzy sets are generalizations of crisp sets. The membership of a data element to a fuzzy set is gradual rather than abrupt as it is in crisp sets. This gradual change is expressed by a membership function. The value of the function for an element represents the degree of membership of the element to the given set. The membership function μ_A by which a fuzzy set A is defined has the form: $\mu_A : X \rightarrow [0,1]$ where X is the universal set. The standard operations of fuzzy set theory are [52]: Complement: $\mu_{\bar{A}}(x) = 1 - \mu_A(x)$, Union: $\mu_{A \cup B}(x) = max[\mu_A(x), \mu_B(x)]$, Intersection: $\mu_{A \cap B}(x) = min[\mu_A(x), \mu_B(x)]$.

The gradual characteristics of the membership function forms imprecise boundaries for fuzzy sets. Due to this fact, fuzzy sets are used to model vagueness. In fact, as stated before, fuzziness and vagueness are used to define the same concept. Besides vagueness, fuzzy set theory is also used to model imprecision. Fuzzy sets imbedded in a system generates a fuzzy system in which the state variables are processed based on fuzzy operations. The power of fuzzy systems lies in their ability to represent expressions in natural language. They are able to use experts' linguistic information in the form of IF-THEN rules.

Intuitionistic Fuzzy Set Theory

Human judgement generally is based on an opinion in favor of a possibility for some condition of existence as well as on an opinion of disbelief in other conditions strongly supporting nonexistence. Such a human based intuitive judgement and not a statistical judgement can naturally be represented as an uncertainty interval between two distributions: that of membership and that of nonmembership.

An intuitionistic fuzzy set A [53], for a given underlying set E is generated for expressing such an uncertainty interval and is represented by a pair $\{\mu_A, \upsilon_A\}$ of functions mapping $E \rightarrow [0,1]$. For $x \in E$ where $\mu_A(x)$ gives the degree of membership to A, $\upsilon_A(x)$ gives the degree of nonmembership with the restriction: $\mu_A(x) + \upsilon_A(x) \leq 1$ stressing consistency in intuition. Thus, degrees generated by these two functions are nonantagonistic. Ordinary fuzzy sets are special cases of intuitionistic fuzzy sets with $\upsilon_A(x) = 1 - \mu_A(x)$, where this equation formalizes an antagonistic behavior.

Intuitionistic knowledge of a human expert that executes a control action through an engineered system is, in a vast majority, interval valued with a degree of necessity (or "belief") in a certain control as a lower bound together with a degree of possibility (or "doubt in disbelief": $\{1-disbelief\}$) in the same parameter as an upper bound. Thus, expert knowledge representation becomes interval valued with an amount of uncertainty that is measured by the interval width. Interval valued fuzzy sets have been introduced as the mathematical formalism of intuitionistic fuzzy sets that model such an uncertainty interval as a degree of membership at the lower bound and as the complement of a degree of nonmembership at the upper bound subject to the consistency inequality stated above. Inconsistency or conflict may frequently occur in human judgement and appears in this formation as a violation of the inequality. In this thesis, we explore the adaptation capability and learning ability of fuzzy control system in restoring the inconsistency and reducing the interval valued intuitive uncertainty so that at the end of such a process classical fuzzy adaptive systems may be used [46].

Rough Set Theory

The main concern in the development of rough set theory is to be able to analyze imprecise, uncertain or incomplete information expressed in terms of data gathered by experience. Rough set theory is a mathematical approach to model vagueness. The main representation of rough sets consists of an approximation space, and lower and upper approximations of a set. The approximation space is a classification of the domain of interest into disjoint categories. When objects of the same categories are considered, their memberships to an arbitrary subset of the domain may not be definable. This introduces the terms lower and upper approximations. The objects, which are known with certainty to belong to the subset of interest form the lower approximation, and the objects, which possibly belong to the subset, describe the upper approximation. The elements in the difference of these two approximations form the boundary region, which is the uncertainty region of a rough set [54]. The formal definitions of these concepts are given as follows:

The lower approximation of a set X is described by the domain U objects x, which are known "with certainty" to belong to the subset of interest with respect to the attribute B. $\underline{B}(X) = \{x \in U : B(x) \subseteq X\}$

The upper approximation of a set X containing objects x which "possibly" belong to the subset of interest with respect to the attribute B. $\bar{B}(X) = \{x \in U : B(x) \cap X \neq \emptyset\}$

The boundary region of a rough set is a region of uncertainty where the set of elements of that region are not known to be inside or outside the set "with certainty" with respect to the attribute B. $BN_B(X) = \bar{B}(X) - \underline{B}(X)$

Kaygısız and Erkmen have developed a new approach using rough set modelling to the domains of attraction of nonlinear systems obtained by cell mapping. The stability domain is represented as a rough set where fully stable cells determine the lower approximation of the domain, and possibly stable cells its rough boundary. The totality of these cells forms an upper approximation to the rough stability domain. The boundary of this domain is smoothed, minimizing the inherent stability uncertainty of the region using a reinforcement learning technique [55].

2.3.2.2 Uncertainty Measures

The first measures of uncertainty have been defined during the development of the propositional usage of set theory and during the increasing trend in probabilistic approaches such as in information theory. These measures are namely Hartley's measure and Shannon's entropy. Hartley's measure is in the form: $H(A) = \log A^\sharp$ where $\sharp$ denotes the cardinality of set A. Shannon's entropy is given by: $S(p(x)|x \in X) = -\sum_{x \in X} p(x) \log p(x)$ where $(p(x)|x \in X)$ is a probability distribution on a finite set X.

Hartley's measure is based on classical set theory and it is a measure for nonspecificity. This measure is derived from the following fact that if there is an alternative of interests known to belong to a particular set of alternatives, then the question is to identify them within the given set. This situation results in the need of a measure representing the amount of information needed to remove the uncertainty associated with a set of alternatives. This was the motivation behind the derivation of Hartley's measure. When it is generalized using fuzzy set theory and possibility theory, the measure takes the name of U-uncertainty, which is again a measure of nonspecificity and expressed as: $U(F) = \int_0^1 \log({}^{\alpha}F)^{\sharp} d\alpha$ where $({}^{\alpha}F)^{\sharp}$ denotes the cardinality of the α-cut of F.

Some existing uncertainty measures are given in what follows.

Entropy in Information Theory

The essential mathematical and statistical nature of information theory has been emphasized by Fisher, Shannon and Wiener [56]. Information theory has its mathematical roots in the concept of disorder or entropy in thermodynamics and statistical mechanics [57]. It brings a measure-based approach into the evaluation of cost. The information cost measured by the entropy formalism has been developed by analogy to the concept of energy. For information sources with different cost scales, Katona and Tusnady [58], and, Csiszar et al. [59] have introduced the concept of entropy rate with respect to a stochastic cost scale, and established the "Principle of Conversion of Entropy" for a wide class of encoding procedures. Aczel and Daroczy [56] provide an excellent in-depth analysis of the properties carried by entropic measures of information.

Cross-entropy

Kullback [57] introduces a new measure of information for a set of continuum cardinality for which Shannon's entropy is a limiting case. The concept of relative measure is developed into a cross-entropy formulation as follows: The logarithm of the likelihood ratio $\log[f_1(x)/f_2(x)]$, where $f_i(x)$ is the generalized probability density unique to set probability measure α_i for hypothesis H_i, is defined as the information in a variable x for discrimination in favor of hypothesis 1 against hypothesis 2. Good [60] describes it as the weight of evidence for hypothesis 1 given x.

A measure of divergence between hypotheses 1 and 2, based on the notion of cross-entropy has been investigated within the following formulation [57],[61]: $JD(1,2) = \int (f_1(x) - f_2(x)) \log(\frac{f_1(x)}{f_2(x)}) d\alpha(x) = I(1:2) + I(2:1)$ where $I(i:j) = \int f_i(x) \log(\frac{f_i(x)}{f_j(x)}) d\alpha_i(x)$, $JD(1,2)$ is a measure of the difficulty of discriminating between propositions 1 and 2.

The divergence $JD(1,2)$ has all the properties of a distance metric as defined in the topology except the triangle inequality property, and is therefore not called distance. The information measures $I(1:2)$ and $I(2:1)$ represent directed-divergences, measuring preference in $I(i:j)$ for proposition i over j.

For pattern classification, the notion of "closest" hypothesis suggests the minimization of divergence. Directed divergences have also been called relative entropy or cross-entropy. The principle of minimum cross-entropy has been investigated in Shore and Johnson [62], and, Shore [63] and has been previously applied to clustering [64].

Since divergence is a metric, it incorporates the property of a dissimilarity measure. Other dissimilarity measures have been developed, especially for assessing the diversity within populations. They include diversity coefficients and dissimilarity coefficients [65]. The latter is based on the Jensen Difference: $J_{ij} = H_{ij} - \frac{1}{2}(H_i + H_j)$ where H_{ij} measures the difference between two hypotheses such that: $H_{ij} = \int d(X_1, X_2) p_i(dX_1) p_j(dX_2)$ where X_1 and X_2 are two individuals, d is a non-negative, symmetric difference function and p_i, p_j are probability density functions.

A unified approach for constructing cross-entropy and dissimilarity measure between probability distributions is investigated in Rao and Kayak [66]. There, the formalism of quadratic entropy is introduced as a new metric and its properties are discussed around the concept of the Jensen Difference.

Evidential Entropy Measures

The concept of entropy has been extended to the framework of the DS theory of evidence as a measure of uncertainty [67]. Various aspects of uncertainty within the framework of belief and plausibility measures have been analyzed by Klir and Folger [1]. They include the following:

(i) Dissonance: Conflict or dissonance in evidence is encountered whenever nonzero degrees of evidence are allocated to disjoint subsets of the frame of discernment. Dissonance is derived as: $E(b) = -\sum_i b(q_i) \log P(q_i)$ where q_i is a focal element and $P(q_i)$ is the plausibility of q_i.

(ii) Conflict: Since plausibility and belief measures are duals, i.e.: $P(q_i) = 1 - B(F - q_i)$ a measure of confusion may be derived from dissonance by replacing the plausibility by its dual: $C(b) = -\sum_i b(q_i) \log(B(q_i))$ where $B(q_i)$ is the total belief. Since $B(q_i) \geq b(q_i)$, an upper bound for confusion is: $C(b) \leq -\sum_i b(q_i) \log(b(q_i))$.

The term on the right hand side is the belief entropy in the fractal model of uncertainty derived by Erkmen in [50]. When $b(.)$ represents a probability measure, then $b(q_i) = P(q_i) = B(q_i)$ for all focal elements. Consequently, the confusion measure and the dissonance measure becomes Shannon's entropy.

(iii) Non-specificity: Yager [67] has introduced a measure of specificity associated with a possibility distribution. Klir and Folger [1] have analyzed several properties of a measure of non-specificity defined by: $V(b) = \sum_i b(q_i) \log(q_i)^\sharp$ where $\sharp$ denotes the cardinality of the set q_i.

Measures (i) and (ii) are generalizations of Shannon's entropy in the framework of the theory of evidence. The measure of non-specificity is a generalization of the Hartley measure: $HI(q_i) = \log(q_i)^\sharp$.

Measure of fuzziness on the other hand is a function assigning a nonnegative real number that expresses the degree of vagueness existing for the membership knowledge of elements of a set lying near or on the boundary of a set. There are two ways in defining uncertainty in terms of its vagueness expressed by measures of fuzziness. In the first one, the metric distance between the membership grade functions of the fuzzy set and its nearest crisp set is used. The second and more practical method is to look at the distinction between the fuzzy set and its complement. When the difference is less, then the set is considered fuzzier [43].

2.3.3 Tendency-Based Variants in Uncertainty Models

Many variants have been developed around the different facets of uncertainty based on many problem oriented methods.

Tzouvaras [68] models vague predicates and vague partitions using non-standard set of integers where the main purpose of the developed model is to capture all basic features of the notion of vagueness. It is stated that the use of generalized (nonstandard) integers is successful as a modelling tool but not feasible when applied to concrete vague situations. The definition of vagueness used in the referred paper is: " Vagueness is deficiency of meaning. As such, it is to be distinguished from generality, undecidability and ambiguity". The outcome of this deficiency is seen as truth-value gaps. The paper attempts to describe a kind of structure for vagueness. An example of a vague predicate is given as: "n is small". Then definitions of a measurable predicate and a measurable vague predicate are given. The examples for measurable and nonmeasurable vague predicates are: for the former case, tall, far, heavy, etc., and for the latter, nice, ugly, happy and etc. The definitions of measurable equivalence and vague measurable equivalence are also given in the paper.

Raha and Ray [69] present an approach for default reasoning that is based on fuzzy logic. Possibility theory is used as a framework for modelling vague default rules. Vague default is represented by a vague statement augmented with partial truth. It is stated that a significant part of the belief about the world is uncertain and incomplete, so defaults are needed to fill the gaps due to the uncertainty in the knowledge base. Human have the ability to reason with good results even in the case of incomplete knowledge. In order to have such an ability, reasoning systems should be modified to include facilities for handling such imprecision. In this referred work, it is suggested to use fuzzy set theory. Their understanding of imprecision is in terms of vague statements, which are due to deficiency of meaning. Fuzzy sets are used for a semantic representation of a vague expression and fuzzy operators for the manipulation of such expressions. They also propose an approach based on the theory of possibility for the representation and manipulation of uncertain and imprecise default knowledge.

The work by Denœux [70] presents an approach for representation and manipulation of imprecise degrees of belief in the framework of evidence theory. Interval-valued and fuzzy-valued belief structures are introduced. The application of the approach to decision making under uncertainty and classification of fuzzy data are discussed. The reason for probability theory not being a universal model of uncertainty is given as its unreasonable requirement for modelling precision for example in the structuring of the universe of hypotheses. In the paper of the above reference, imprecise belief masses are assigned to imprecise propositions and an extension of the transferable belief model allowing imprecision in the specification of degrees of belief is presented.

Pedrycz [71] has developed the theory of shadowed sets that are induced by fuzzy sets. It is again a potential tool for modelling vagueness. It has been shown that shadowed sets reveal conceptual and algorithmic relationships between rough sets and fuzzy sets. The dilemma of excessive precision in describing imprecise phenomenon is mentioned for the case of fuzzy sets. It is stated that, although fuzzy sets are regarded as formal devices to capture, represent and process vagueness, they suffer from the mentioned dilemma. The problem arises due to the fact that once the membership functions are defined, the concept is defined very precisely. In this referred work, shadowed sets are proposed for modelling vagueness, since they do not have precise numerical membership values but rely on basic concepts of truth values and an entire unit interval perceived as a zone of uncertainty. Such a zone is also constructed by introducing a fuzzy distance between object pairs for robot motion planning

[72].

Chàvez [73] adopts a standard approach to model vagueness, which is to use second order probability on a first-order one and analyzes the value of reducing vagueness. His aim is to measure the effects of vagueness in decision making where probability and utility assessment are central issues in model construction. Probabilities being generally subjective, uncertainty about probability, is stated as: when humans are asked to express their beliefs about a probability assessment, there happens to be further uncertainty introduced to the system.

Probability distributions in the representation of uncertainty about probabilities have been the focus of diverse technical discussions among theoreticians [74]. There are two different opinions exist on this matter. One group claims that to be uncertain about a probability violates the subjectivists' assumption that is individuals can develop unique and precise probability judgements. The other opinion supports possibility of uncertainty about probabilities and many authors find this concept potentially useful [73],[74],[75],[76]. While, if uncertainty is due to cognitive imprecision, this type is not consistent with the axioms of subjective probability.

Propagation of imprecise probabilities in a Bayesian network is the basic architecture for reducing imprecision iteratively [75]. Here, imprecision is handled through the use of second order probability distributions. Dirichlet distributions are used to express uncertainty about probabilities. The use of these distributions transforms the problem of how to propagate point probabilities in a Bayesian network into how to propagate Dirichlet distributions. Belief networks are another iterative process for reasoning under uncertainty. The sensitivity of belief networks on diagnostic performance under imprecision in the representation of numerical or point probabilities is the focus of [76].

Truth-qualified fuzzy propositions can also represent imprecise and uncertain information. Fuzzy sets can be used to express imprecise information and fuzzy truth values to represent uncertainty since they are capable of expressing the possibility of the degree of truth of a fuzzy proposition. An inference mechanism for fuzzy propositions with fuzzy truth values is developed in [77] resulting in a hybrid approach that brings possibilistic reasoning and fuzzy reasoning together to reason under uncertainty and imprecision. Fuzzy rules and facts having fuzzy truth values are transformed into a set of uncertain classical propositions with necessity and possibility measures on which a possibilistic reasoning is performed. Probabilistic uncertainty may also be included in fuzzy systems modelling and [78] incorporates it with a technique based on Demspter-Shafer theory of evidence.

2.4 Mathematical Background

2.4.1 Fuzzy Systems

The well known architecture of a fuzzy system includes a fuzzifier transforming the numerically described world into semantics (Fig.2.1). Fuzzy systems are in fact fuzzy expert systems drawing inference through an engine based on a fuzzy rule base. Drawn conclusions are defuzzified before being applied to the numerically driven system.

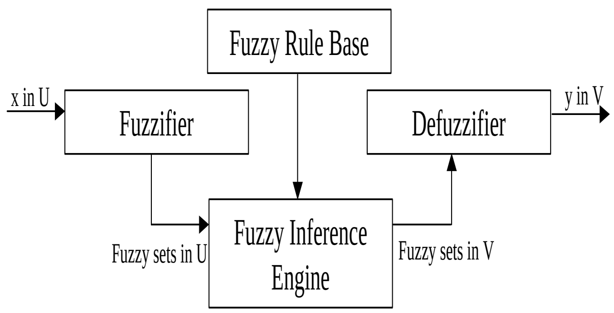


Figure 2.1: Basic configuration of fuzzy logic system with fuzzifier and defuzzifier

Different choices of fuzzifier, fuzzy inference engine and defuzzifier result in different fuzzy logic systems with varying performances. Fuzzy Rule Base consists of a collection of fuzzy IF-THEN rules that determine a mapping from fuzzy sets in the input space $U \subset R^n$ to that in the output space $V \subset R$ based on fuzzy logic principles. These rules are generally represented as:

$$R^{(l)} : \text{IF } x_1 \text{ is } F_1^l \text{ and ... and } x_n \text{ is } F_n^l, \text{ THEN } y \text{ is } G^l. \tag{2.2}$$

where F_i^l and G^l are the respective antecedent and consequent fuzzy sets, $x = (x_1, ..., x_n)^T \in U$ and $y \in V$ are respectively input and output linguistic variables, $l = 1, 2, ..., M$ represents the rule number and $R^l : A \rightarrow B$ refers to the specific rule [14].

The fuzzifier performs a mapping from a crisp point $x = (x_1, ..., x_n)^T \in U$ into a fuzzy set $A^{'}$ in U. Two choices of this mapping are the singleton fuzzifier and the nonsingleton fuzzifier. In our example, we apply the singleton fuzzifier which is defined as: $A^{'}$ is a fuzzy singleton with support x, that is, $\mu_{A^{'}}(x^{'}) = 1$ for $x^{'} = x$ and $\mu_{A^{'}}(x^{'}) = 0$ for all other $x^{'} \in U$ with $x^{'} \neq x$. The reason is that the use of singleton fuzzifier produces a simple fuzzy logic system, and nonsingleton fuzzifiers work better in a noisy environment. Since we do not incorporate noise in our example systems, we are not using nonsingleton fuzzifiers [14].

The AND logic used to compare fuzzy memberships $\mu_{F_i}(x_i)$ in the antecedent A for a particular application is the product operation such that:

$$\mu_{F_1^l \times ... \times F_n^l}(x) = \mu_{F_1^l}(x_1) ... \mu_{F_n^l}(x_n) \tag{2.3}$$

The AND operation can also be performed by using minimum operation instead of product operation, and we are using the product operation in order to be able to represent the system algebraically.

In classical fuzzy systems, the consequent of a rule undergoes a disambiguation, since plants in cascade with such systems can most generally use only nonambiguous, one-valued parameters. The disambiguation value $y_l \in R$ can be taken as the maximum of membership distribution $\mu_G(x)$ of the consequent. This value $y_l = max(\mu_G(x))$ is inserted in $f(x)$ given as a closed form in Equation 2.4. This is a defuzzifier which performs a mapping from consequent fuzzy sets B^i in V to a crisp point $y \in V$. Three choices of this mapping exist which are the maximum defuzzifier, the center average defuzzifier and the modified center average defuzzifier. We make use of the center average defuzzifier defined as:

$$y = \frac{\sum_{l=1}^{M} y_l(\mu_{B^l}(y_l))}{\sum_{l=1}^{M}(\mu_{B^l}(y_l))} = f(x) \tag{2.4}$$

The reason for the choice of center average defuzzifier is that the center average defuzzifier outperforms the maximum defuzzifier, and it produces simpler systems than the ones using modified center average defuzzifier resulting in faster training [14].

This defuzzifier is a weighted average of the y's and the weights $\mu_{B^l}(y)$ determined by

$$\mu_{B^l}(y) = sup_{x \in U}[\mu_{F_1^l \times \ldots \times F_n^l \to G^l}(x, y) * \mu_{A'}(x)] \tag{2.5}$$

and do not take the shape of $\mu_{G^l}(y)$ into consideration.

A rule in a fuzzy system is represented as $R : (\mu_{F_1^l \times \ldots \times F_n^l}(x), y_l)$ where μ_{G^l} achieves its maximum at y_l. We also consider the same structure of rule in our 2-way fuzzy adaptive methodology.

From the product-inference rule defined in Equation 2.3 and Equation 2.5, $\mu_{B^l}(y_l)$ is found to be:

$$\mu_{B^l}(y_l) = sup_{x' \in U}[\prod_{i=1}^{n} \mu_{F_i^l}(x_i^{'})\mu_{G^l}(y_l)\mu_{A'}(x^{'})] \tag{2.6}$$

Since singleton fuzzifier is used, $\mu_{A'}(x^{'}) = 1$ for $x^{'} = x$ and $\mu_{A'}(x^{'}) = 0$ for all other $x^{'} \in U$, and the sup is achieved at $x^{'} = x$. Thus, we can simplify Equation 2.6 into:

$$\mu_{B^l}(y_l) = \prod_{i=1}^{n} \mu_{F_i^l}(x_i) \tag{2.7}$$

When Equation 2.7 is substituted into Equation 2.4, the form of a fuzzy logic system with center average defuzzifier, product-inference rule and singleton fuzzifier as given below is obtained:

$$f(x) = \frac{\sum_{l=1}^{M} y_l (\prod_{i=1}^{n} \mu_{F_i^l}(x_i))}{\sum_{l=1}^{M} (\prod_{i=1}^{n} \mu_{F_i^l}(x_i))} = \frac{a}{b} \tag{2.8}$$

where y_l is the point at which μ_{G^l} achieves its maximum value, and is the disambiguation value of memberships assigned to consequents of rules.

The closed form of the system given by Equation 2.8 can be represented as a three-layer feedforward network (Fig.2.2), where the first layer generates $z_l = \prod_{i=1}^{n} \mu_{F_i^l}(x_i)$. The second layer weighs z_l by y_l and generates the numerator a and denominator b of the equation, separately. The third layer computes $f(x)$ as the ratio of a to b. This layered system is the classical adaptive system introduced by Wang [14] that we call 1-way fuzzy adaptive since it applies to memberships only and does not consider the membership/nonmembership interval reasoning. This fuzzy system is called fuzzy adaptive system when it is equipped with a training algorithm for adjusting the parameters of the system.

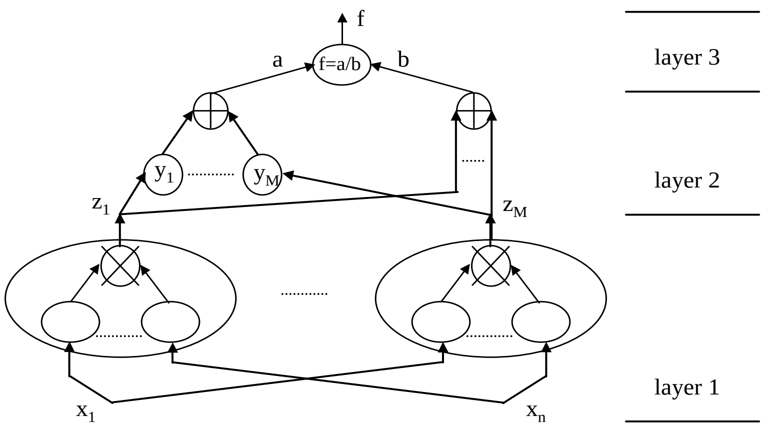


Figure 2.2: Network representation of the fuzzy logic systems

2.4.2 Takagi-Sugeno (T-S) Fuzzy Systems

The Takagi-Sugeno (T-S) fuzzy system used to model a nonlinear system is also a fuzzy rule based system with the following typical rule structure:

$$R^{(l)} : \text{IF } x_1 \text{ is } F_1^l \text{ and ... and } x_n \text{ is } F_n^l, \text{ THEN } \dot{x}(t) = A_l x(t) + B_l u(t), \quad (2.9)$$

where F_i^l (i=1,2,...,n) are the antecedent fuzzy sets, $x(t) = [x_1, x_2, ..., x_n]^T$ are the state variables, and u(t) is the control input. A_l and B_l are respectively the state and control input matrices of the linearized subsystem around the operation point of rule l.

The closed form of the output of the T-S fuzzy system with center average defuzzifier, product-inference rule and singleton fuzzifier is:

$$\dot{x}(t) = \frac{\sum_{l=1}^{M}(A_l x(t) + B_l u(t))(\prod_{i=1}^{n} \mu_{F_i^l}(x_i))}{\sum_{l=1}^{M}(\prod_{i=1}^{n} \mu_{F_i^l}(x_i))} \quad (2.10)$$

where $\mu_{F_i^l}(x_i)$'s are the membership functions assigned to the fuzzy sets in the antecedent parts of the rules, and M is the rule number. Equation 2.10 can be written in the following form as:

$$\dot{x}(t) = [\frac{\sum_{l=1}^{M}(A_l w_l(x(t))}{\sum_{l=1}^{M} w_l(x(t))}]x(t) + [\frac{\sum_{l=1}^{M}(B_l w_l(x(t))}{\sum_{l=1}^{M} w_l(x(t))}]u(t) \quad (2.11)$$

Here, $w_l(x(t)) = \prod_{i=1}^{n} \mu_{F_i^l}(x_i(t))$. For a simpler representation, $\psi_l(x)$ is defined to be $\psi_l(x) = \frac{w_l(x(t))}{\sum_{l=1}^{M} w_l(x(t))}$. Then, Equation 2.11 takes the form:

$$\dot{x}(t) = (\sum_{l=1}^{M} \psi_l(x(t))A_l)x(t) + (\sum_{l=1}^{M} \psi_l(x(t))B_l)u(t) \quad (2.12)$$

We are going to use the closed form in Equation 2.12 in the development of the stability theory using the Lie algebras introduced in Chapter 4.

2.4.3 Lie Algebra

In this section, basic definitions and results from Lie algebra theory are introduced without the proofs. The proofs can be found in [79]. We use Lie algebra theory in the stability analysis of T-S fuzzy systems, which is a novelty in the literature.

Definition 1: A Lie algebra g is a vector space which has a bilinear product map [.,.]: $g \times g$ satisfying:

(i) $[X, Y] = -[Y, X]$ for all $X, Y \in g$

(ii) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ for all $X, Y, Z \in g$

If $[X, Y] = 0$ for all $X, Y \in g$, then g is called an Abelian Lie algebra. Two elements X, Y satisfying $[X, Y] = 0$ are said to be commuting.

Definition 2: A subspace h of g is said to be a subspace if $[h, h] \subseteq h$, i.e. $X, Y \in h$ implies that $[X, Y] \in h$.

Definition 3: An ideal h in a Lie algebra g is a subspace such that $[h, g] \subseteq h$, where $[h, g]$ denotes the subspace spanned by the set of all elements of the form $[X, Y], X \in h, Y \in g$. An ideal h in g is minimal if $\{0\}$ is the only ideal of g contained in h.

$h_1 + h_2$ denotes the subspace spanned by all elements of the form $X + Y$, $X \in h_1$, $Y \in h_2$, for any subsets $h_1, h_2 \subseteq g$. If $h \in g$ is an ideal, then g/h denotes the quotient Lie algebra which is the quotient of the vector spaces g and h with brackets $[\overline{X}, \overline{Y}] = \overline{[X, Y]}, X, Y \in g$, where $\overline{X}$ is the coset of X. The projection map $g \to g/h$ is a homomorphism of Lie algebras with kernel h. A homomorphism of Lie algebras is a homomorphism of the underlying vector spaces which preserves the brackets.

Definition 4: A Lie algebra g is said to be simple if g and $\{0\}$ are the only ideals of g.

Definition 5: If $g = g_1 \oplus g_2 \oplus \ldots \oplus g_k$ (vector space direct sum) and each g_h is an ideal, then g is called the direct sum of $g_1, \ldots, g_k$. For $i \neq j$, $g_i \cap g_j = [g_i, g_j] = \{0\}$.

Definition 6: The ideal $\mathcal{D}g = [g, g]$ of a Lie algebra g is called the derived algebra of g. The derived series of g is:

$g \supseteq \mathcal{D}g \supseteq \mathcal{D}^2 g \supseteq \ldots \supseteq \mathcal{D}^n g \supseteq \ldots,$

where $\mathcal{D}^n g = \mathcal{D}(\mathcal{D}^{n-1} g)$. Each term in the series is ideal.

Definition 7: If $\mathcal{D}^k g = \{0\}$ for some $k > 0$, then g is said to be a solvable Lie algebra.

Definition 8: If g does not contain any solvable ideal apart from $\{0\}$, then g is said to be semisimple.

Theorem 1: Every semisimple Lie algebra is the direct sum of all its minimal ideals.

Theorem 2: Every Lie algebra g has a unique maximal solvable ideal r called the radical of g. Then g/r is semisimple.

Another important class of Lie algebras is the nilpotent class.

Definition 9: If g is a Lie algebra, let $C^{(0)}g = g$, $C^{(1)}g = [g, C^{(0)}g], \ldots,$ $C^{(n+1)}g = [g, C^{(n)}g], \ldots$. Then, all $C^{(n)}g$ $(n = 0, 1, 2, \ldots)$ are ideals of g. The descending central series is obtained as:

$C^{(0)}g \supseteq C^{(1)}g \supseteq \ldots \supseteq C^{(n)}g \supseteq \ldots$

If $C^{(k)}g = 0$ for some $k > 0$, then g is called nilpotent. It can be proved that $\mathcal{D}^n g \subseteq C^{(n)}g$ for each n, so if g is nilpotent, then it is solvable. In fact, g is solvable if and only if $\mathcal{D}g$ is nilpotent.

The adjoint map "ad" is an important linear operator acting on any Lie algebra g, and defined for each $X \in g$ as:

$(adX)Y = [X, Y]$.

Using this map, a geometric structure on a Lie algebra is defined in terms of a symmetric bilinear form $(.,.)$ called the Killing form. The Killing form is defined to be:

$(X, Y) = Tr(adX adY)$

Theorem 3: A Lie algebra g is solvable if and only if $(X, X) = 0$ for all $X \in \mathcal{D}g$.

Theorem 4: (Cartan's criterion) A Lie algebra g is semisimple if and only if the Killing form of g is nondegenerate, i.e. $(X, Y) = 0$ for all $Y \in g$ implies that $X = 0$.

Next, the decomposition of Lie algebras is introduced. As in the case of the decomposition of a vector space into the generalized eigenspaces of any given linear operator, any nilpotent linear Lie algebra h acting on a vector space V defines a decomposition of V in the following way.

For any given linear function $\alpha : h \to \mathcal{C}$ define the set $V^\alpha = \{v \in V : [H - \alpha(H)I]^k v = 0$, for some $k > 0$ and all $H \in h\}$, that is V^α is the generalized eigenspace for all $H \in h$ with eigenvalue $\alpha(H)$. If $V^\alpha \neq \emptyset$, it is said that α is a weight or a root of h in V and V^α is a weight (root) subspace of V. Then,

$$V = \bigoplus_{\alpha \in \Delta} V^\alpha \tag{2.13}$$

where Δ is the set of all weights of h in V.

If g is a Lie algebra and h is a nilpotent subalgebra, then $adh = \{adH : H \in h\}$ is a nilpotent linear Lie algebra acting on g; so if Equation 2.13 is applied with $V = g$, and h replaced by adh, the following decomposition of g is obtained:

$$g = \bigoplus_{\alpha \in \Delta} g^\alpha \tag{2.14}$$

where

$g^\alpha = \{G \in g : [adH - \alpha(adH)I]^k G = 0$, for some $k > 0$ and all $H \in h\}$.

Definition 10: If $h = g^0$, then h is called a Cartan subalgebra of g.

It can be shown that every Lie algebra has a Cartan subalgebra and each such subalgebra is a maximal nilpotent subalgebra. Any two Cartan subalgebras are conjugate under a certain group of automorphisms of the algebra.

In the case of a semisimple Lie algebra, the root space decomposition 2.14 takes the form:

$g = h \oplus \bigoplus_{\alpha \in \Sigma} g^\alpha$

where Σ is the set of non-zero roots of h in g, and the Cartan subalgebra h is a maximal Abelian subalgebra of g. The Killing form $(.,.)$ is nondegenerate on h, each g^α is one-dimensional for $\alpha \neq 0$ and there are ($dimh$) (dim:dimension) linearly independent roots.

Any Lie algebra g can be written in the form:

$$g = r + m, r \cap m = \emptyset \tag{2.15}$$

where r is solvable and m is semisimple. This is called a Levi decomposition of g. However, this decomposition is not a direct sum, so it is not unique. Each Lie algebra has many Levi decompositions.

Two Levi subalgebras m_1 and m_2 are related by

$$R^{-1} m_1 R = m_2 \tag{2.16}$$

where $R = exp(adS), S \in [r, g]$.

A Levi subalgebra m is semisimple, since $m \cong g/r$ and so m can be decomposed in terms of a Cartan subalgebra h_m:

$$m = h_m \oplus \sum_{\varphi \in \Sigma} m^{\varphi} \tag{2.17}$$

where Σ is the set of nonzero roots with respect to h_m. Cartan subalgebras are not unique, but any two Cartan subalgebras h_1 and h_2 of m are conjugate under the group of automorphisms of m generated by $exp(adX)$ where $X \in m$ and adX is nilpotent. Thus,

$$h_1 = \sigma^{-1}(X) h_2 \sigma(X) \tag{2.18}$$

for some $X \in m$ with adX nilpotent, where $\sigma(X) = exp(adX)$. Note that a Cartan subalgebra is a maximal Abelian subalgebra of m. Combining 2.15 and 2.17, any Lie algebra g may be written in the form:

$$g = r + m = r + (h_m \oplus \sum_{\varphi \in \Sigma} m^{\varphi}) \tag{2.19}$$

where h_m is an Abelian Cartan subalgebra.

2.4.4 Describing Function

A stable controller design methodology with periodic input and output can be developed based on describing functions. We expand this theory to a multi input 2-way fuzzy adaptive stable controller design in Chapter 4, so here we review the describing function formulation of a single-input single-output 1-way fuzzy system of [23]. The use of such an analysis technique introduces a systematic way for designing a multi-input fuzzy controller. The fuzzy system has the following rule structure:

$$R^{(i)} : \text{IF } x \text{ is } \mu_i, \text{ THEN } u \text{ is } u_i. \tag{2.20}$$

where x and u are the input and output variables respectively, μ_i is the membership function corresponding to the i^{th} rule, and u_i is the output fuzzy set, which is a singleton in this case.

The closed form of the fuzzy system with singleton fuzzifier, product inference rule and center average defuzzifier is:

$$u = f(x) = \sum_i \Omega_i(x) u_i \tag{2.21}$$

where

$$\Omega_i = \frac{\mu_i(x)}{\sum_r \mu_r(x)} \tag{2.22}$$

There are certain assumptions on the structure of the fuzzy system such that it is possible to find an analytical expression for the describing function. These assumptions are summarized below [23]:

1. The antecedent membership functions are triangular functions distributed completely, consistently and symmetrically with respect to the origin. The functions are of the form:

$$\mu_i(x) = \begin{cases} \dfrac{x - \phi_{i-1}}{\phi_i - \phi_{i-1}}, & \phi_{i-1} \le x < \phi_i \\ \dfrac{x - \phi_{i+1}}{\phi_i - \phi_{i+1}}, & \phi_i \le x < \phi_{i+1} \\ 0 & otherwise \end{cases} \tag{2.23}$$

where $\phi_{-i} = -\phi_i$.

2. Consequent parts have the odd symmetry condition:

$$u_{-i} = -u_i \tag{2.24}$$

which make them sign dependent. This assumption in control system makes the control much more reactive to deviations from the desired values.

3. When $\phi_n \leq x < \phi_{n+1}$, only two rules are fired, and:

$$\sum_{k=1}^{M} \mu_k(x) = \mu_n(x) + \mu_{n+1}(x) = 1 \tag{2.25}$$

So:

$$\Omega_i(x) = \frac{\mu_i(x)}{\sum_{k=1}^{M} \mu_k(x)} = \mu_i(x) \tag{2.26}$$

It is assumed that only two consecutive memberships overlap.

4. $\Omega_{-i}(-x) = \Omega_i(x)$, since $\Omega_{-i}(-x) = \underbrace{\mu_{-i}(-x)}_{\mu_i(x)} = \Omega_i(x)$ determining evenness of membership functions.

By assumption 2, it is guaranteed that the fuzzy system given by Equation 2.21 is odd in x, since Ω_i's are even in x as indicated by assumption 4. Before the derivation of the describing function, the following lemmas are given in [23] in order to simplify the proof of the theorem for the describing function.

Lemma 1: If the fuzzy system satisfies the above four assumptions, then it is also odd in time when the input is $x = A \sin wt$.

Proof: The fuzzy system is odd in x and $x = A \sin wt$ is also odd in time t, then the output of the fuzzy system is odd in t.

Lemma 2: If the fuzzy system satisfies the above four assumptions, and $x = A \sin wt$, then $u(t) = u((\pi/w) - t)$.

Proof: For $\forall t_1 \in R$, let $x_1 = A \sin wt_1$. For $t_2 = (\pi/w - t_1)$, $x_2 \equiv x(t_2) = A \sin wt_2 = A \sin w(\frac{\pi}{w} - t_1) = A \sin wt_1 = x_1$. So, $u(t_1) = u(t_2) = u(\pi/w - t_1)$.

Lemma 3: If the fuzzy system satisfies the above four assumptions, and $\phi_k \leq x < \phi_{k+1}$, then the output u of the system is:

$$u = \sum_i \Omega_i(x) u_i = \frac{\Delta u_k}{\Delta \phi_k} x + \frac{1}{\Delta \phi_k}(\phi_{k+1} u_k - \phi_k u_{k+1}) \tag{2.27}$$

where $\Delta \phi_k \equiv \phi_{k+1} - \phi_k$ and $\Delta u_k \equiv u_{k+1} - u_k$.

Proof: When $\phi_k \leq x < \phi_{k+1}$, only two rules are fired and

$$
\begin{aligned}
u &= \sum_i \Omega_i(x)u_i = \Omega_k(x) + \Omega_{k+1}(x)u_{k+1} \\
&= (-\frac{x}{\Delta\phi_k} + \frac{\phi_{k+1}}{\Delta\phi_k})u_k + (\frac{x}{\Delta\phi_k} - \frac{\phi_k}{\Delta\phi_k})u_{k+1} \\
&= \frac{\Delta u_k}{\Delta\phi_k}x + \frac{1}{\Delta\phi_k}(\phi_{k+1}u_k - \phi_k u_{k+1})
\end{aligned}
\tag{2.28}
$$

The describing function of the fuzzy system satisfying the above assumptions and lemmas are given in the following theorem.

Theorem: The describing function of the fuzzy system given by the Equation 2.21 and satisfying the four assumptions is a real number that only depends on the amplitude of the input sinusoid and is independent of the input frequency w, and is of the form:

$$
\begin{aligned}
&N(A,w) = N(A) = \frac{b_1}{A} = \frac{4}{\pi A}\sum_{i=0}^{d}\{\frac{\Delta u_i A}{2\Delta\phi_i}((\delta_{i+1} - \sin\delta_{i+1}\cos\delta_{i+1}) \\
&-(\delta_i - \sin\delta_i\cos\delta_i)) + \frac{1}{\Delta\phi_i}(\phi_i u_{i+1} - \phi_{i+1}u_i)(\cos\delta_{i+1} - \cos\delta_i)\}
\end{aligned}
\tag{2.29}
$$

where d satisfies $\phi_d \leq A < \phi_{d+1}$, $d > 0$, and varies with A; $\{\delta_i\}$ are defined to be the angles where the input sinusoid $x = A\sin\delta$ intersects the centers $\{\phi_i\}$'s of membership functions. For $\{\delta_i\}$'s, we have:

$$
\begin{aligned}
&\delta_0 \equiv 0 \\
&\delta_i \equiv \sin^{-1}(\frac{\phi_i}{A}), (i = 1,\ldots,d, 0 < \delta_i < \frac{\pi}{2}) \\
&\delta_{d+1} \equiv \frac{\pi}{2}
\end{aligned}
\tag{2.30}
$$

The definition of δ is shown in Fig.2.3 schematically.

Proof: The describing function of $u(t) \equiv u(x = A\sin wt)$ is calculated as: $N(A,w) = \frac{1}{A}(b_1 + ja_1)$, where $a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} u(t)\cos(wt)dwt$ and $b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} u(t)\sin(wt)dwt$. For the fuzzy system:

$$
\begin{aligned}
&a_1 = \tfrac{1}{\pi}\textstyle\int_{-\pi}^{\pi} u(t)\cos(wt)dwt = 0 \text{ by Lemma 1} \\
&b_1 = \tfrac{1}{\pi}\textstyle\int_{-\pi}^{\pi} u(t)\sin(wt)dwt \\
&= \tfrac{2}{\pi}\textstyle\int_0^{\pi} u(t)\sin(wt)dwt \text{ by Lemma 1} \\
&= \tfrac{2}{\pi}(\textstyle\int_0^{\pi/2} u(t)\sin(wt)dwt + \int_{\pi/2}^{\pi} u(t)\sin(wt)dwt) \\
&= \tfrac{2}{\pi}(\textstyle\int_0^{\pi/2} u(t)\sin(wt)dwt + \int_{\pi/2}^{0} u(\frac{\pi}{w} - k)\sin(\pi - wk)d(-wk)) \\
&= \tfrac{4}{\pi}\textstyle\int_0^{\pi/2} u(t)\sin(wt)dwt \text{ by Lemma 2}
\end{aligned}
\tag{2.31}
$$

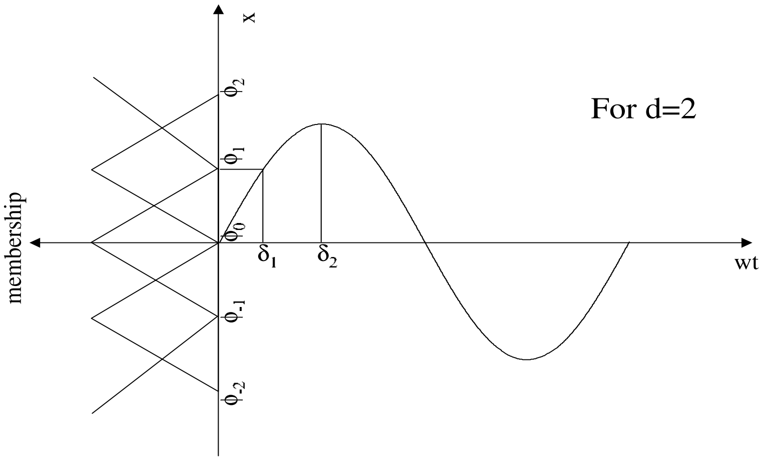


Figure 2.3: Definition of δ

Using the variables δ_i's in b_1, we have:

$$b_1 = \frac{4}{\pi}(\int_{\delta_0}^{\delta_1} u(t)\sin(wt)dwt + \int_{\delta_1}^{\delta_2} u(t)\sin(wt)dwt + \ldots + \int_{\delta_d}^{\delta_{d+1}} u(t)\sin(wt)dwt)$$
$$= \frac{4}{\pi}\sum_{i=0}^{d}\int_{\delta_i}^{\delta_{i+1}} u(t)\sin(wt)dwt$$
$$= \frac{4}{\pi}\sum_{i=0}^{d}\int_{\delta_i}^{\delta_{i+1}}\{\frac{\Delta u_i}{\Delta\phi_i}A\sin(wt) + \frac{1}{\Delta\phi_i}(\phi_{i+1}u_i - \phi_i u_{i+1})\}\sin(wt)dwt \text{ by Lemma 3}$$
$$= \frac{4}{\pi}\sum_{i=0}^{d}\{\frac{\Delta u_i A}{2\Delta\phi_i}((\delta_{i+1} - \sin\delta_{i+1}\cos\delta_{i+1})$$
$$-(\delta_i - \sin\delta_i\cos\delta_i)) + \frac{1}{\Delta\phi_i}(\phi_i u_{i+1} - \phi_{i+1}u_i)(\cos\delta_{i+1} - \cos\delta_i)\} \tag{2.32}$$

This concludes the derivation of the describing function for a 1-way fuzzy system.

2.4.5 Approximating Sequences

We are going to use the approximating sequences technique in the design of an optimal fuzzy controller in Chapter 4, so in this section, this technique of [80] is introduced. When the plant together with a fuzzy controller can be put in the required form we can apply the approximating sequences technique, and this is another novelty we have introduced to the literature.

The aim of the method is to solve the optimal control problem for "pseudo-linear" systems of the form:

$$\dot{x} = A(x)x + B(x)u \tag{2.33}$$

together with a quadratic cost function:

$$J = x^T(t_f)F(t_f) + \int_0^{t_f}(x^TQx + u^TRu)dt \tag{2.34}$$

The proposed method introduces a sequence of time-varying linear-quadratic approximations to the system defined by Equation 2.33. The basic assumption in this method is the local Lipschitz continuity.

The sequence of approximations to the problem of minimizing the cost 2.34 subject to 2.33 is given as:

For k=0,

$$\begin{aligned} \dot{x}^{[0]} &= A(x_0)x^{[0]} + B(x_0)u^{[0]}, x^{[0]}(0) = x_0 \\ J^{[0]} &= x^{[0]T}(t_f)Fx^{[0]}(t_f) + \textstyle\int_0^{t_f}(x^{[0]T}Qx^{[0]} + u^{[0]T}Ru^{[0]})dt \end{aligned} \tag{2.35}$$

and for $k \geq 1$,

$$\begin{aligned} \dot{x}^{[k]} &= A(x^{[k-1]}(t))x^{[k]} + B(x^{[k-1]}(t))u^{[k]}, x^{[k]}(0) = x_0 \\ J^{[k]} &= x^{[k]T}(t_f)Fx^{[k]}(t_f) + \textstyle\int_0^{t_f}(x^{[k]T}Qx^{[k]} + u^{[k]T}Ru^{[k]})dt \end{aligned} \tag{2.36}$$

It is seen that the approximations are linear, time-varying and quadratic, so the optimal control is of the form:

$$u^{[k]} = -R^{-1}B^T(x^{[k-1]}(t))P^{[k]}x^{[k]}(t) \tag{2.37}$$

where $P^{[k]}$ is the solution to the Riccati equation:

$$\begin{aligned} &\dot{P}^{[k]}(t) = -Q - P^{[k]}A(x^{[k-1]}(t)) - A(x^{[k-1]}(t))^TP^{[k]} \\ &+P^{[k]}B(x^{[k-1]}(t))R^{-1}B^T(x^{[k-1]}(t))P^{[k]} \\ &P^{[k]}(t_f) = F \end{aligned} \tag{2.38}$$

Then, the k^{th} dynamical system is:

$$\dot{x}^{[k]} = A(x^{[k-1]}(t))x^{[k]} - B(x^{[k-1]}(t))R^{-1}B^T(x^{[k-1]}(t))P^{[k]}x^{[k]}(t) \tag{2.39}$$

The proof of convergence can be found in [80].

CHAPTER 3

MODELLING INCONSISTENCY: 2-WAY FUZZY SYSTEMS

In this chapter, we introduce our novel 2-way fuzzy adaptive system. We describe the system architecture for the fuzzy system together with the training procedure, which is at the basis of the adaptivity. The methodology of our proposed system models uncertainty and inconsistency using intuitionistic fuzzy sets and generate the novel architecture 2-way fuzzy adaptive system. The 2-way fuzzy adaptive system uses intuitionistic fuzzy sets that model intuitive uncertainty in place of classical fuzzy sets. Generally, the design of a controller involves 2 types of information: 1) absolutely necessary information for the control action and 2) information on parameters possibly important for control.

Necessity and possibility create an interval-valued representation of control information. It is such an interval valued distribution of information, that intuitionistic fuzzy sets model in our proposed adaptive control architecture. Necessity forms the lower bound, and possibility the upper bound of an interval valued information representation with uncertainty modelled as the width of this interval. Intuitionistic fuzzy sets bring flexibility into the system since it is possible to assign control upper bounds (nonmembership functions) independent from control lower bounds (the membership functions). There is only a consistency constraint on this assignment, which requires that the sum of the two functions should be less than or equal to unity. However, this inequality is often violated during control activities rendering knowledge representation inconsistent.

Beside the development of the novel 2-way fuzzy adaptive control architecture, this chapter also investigates the effect of inconsistency and uncertainty in the 2-way fuzzy adaptive system. Our system is subject to two phases of training (2 pass learning). The first phase is found to be highly sensitive to inconsistency and therefore it aims at reducing this inconsistency. The resultant system is a 2-way fuzzy adaptive system with a minimum degree of inconsistency. The main sensitivity of the second phase of training resides on the width of the uncertainty interval. Therefore, this phase is found to reduce the uncertainty width introduced by the definition of the membership and nonmembership functions. The resultant system is a one-way fuzzy adaptive system without uncertainty.

In this chapter, we also develop a method based on shadowed set theory for the evaluation of the inconsistency handling of our 2-way fuzzy system. The inconsistency is reduced by phase 1 of the training and we want to have a measure based understanding of the "more or less consistency" obtained after phase 1 based on shape characteristics of membership and nonmembership functions involved. Therefore we have to track how initially fully inconsistent membership and nonmembership changes after the inconsistency adjustment of phase 1 of our 2-way fuzzy adaptive control system.

We classify this change into different types of inconsistencies based on shadowed sets and generate a correlation between classes or types of inconsistency with membership/nonmembership overlaps based on an index of fuzziness. A global index is then assigned to the type of inconsistency by simply adding both indexes of fuzziness for membership and nonmembership. Our evaluation is based on this correlation mapping of the different patterns (types) of inconsistency to the different global indices of fuzziness of the input membership and nonmembership values.

3.1 Our Proposed System Architecture

Our 2-way fuzzy adaptive controller is also a 3-layer feedforward network as in Fig.2.2. However, it makes use of both the necessity representations and the possibility representations of the aforementioned interval valued fuzzy sets that are reviewed in Chapter 2 (Section 2.3.2.1). Thus, it takes into account the membership/nonmembership interval reasoning. This structural difference with the 1-way fuzzy adaptive controller necessitates a different description and implementation of the fuzzy rule base. Structuring the fuzzy rule is based upon: 1) sources that generate rules and 2) definition of the support functions for F_i^l and G^l in Equation 2.2.

In the 1-way fuzzy adaptive controller of Wang [14], two solutions for these problems are suggested: Concerning the first issue, the rules can be obtained either by asking human experts, or using training algorithms based on measured data. Suggestions for defining support functions are to ask the experts to specify the fuzzy membership functions, and tune them by using numerical data. For this second issue, the functional forms of $\mu_{F_i^l}$ and μ_{G^l} should be specified first. Once the functional forms are fixed, the parameters of these functions should be determined by using measured data, which is a tuning done through the use of 1 level training algorithm. Our approach brings a major modification to this second issue.

The major structural difference in our approach arises from the definition of support that uses necessity measures (membership functions) and possibility measures, which are fuzzy complements of nonmembership functions. In one-way fuzzy adaptive systems, it is enough to define membership functions for F_i^l and G^l in Equation 2.2. In our 2-way fuzzy adaptive systems, the membership and nonmembership functions $\mu(x)$ and $\upsilon(x)$ are defined independently with or without complying with the inequality: $\mu(x) + \upsilon(x) \leq 1$ that determines consistency. There a support to a proposition in a rule is defined as an interval spanned by two distributions that may as well be inconsistent: membership distribution at the lower bound and the complement of a nonmembership distribution at the upper bound.

The process difference that our approach brings, is also due to the interval valued support assigned to propositions of a rule as an expert knowledge representation. Memberships are tuned in a first level training, where learning is done in the same way as in a 1-way fuzzy adaptive controller. However, the uncertainty measured as the width of the support interval requires a further training (second level training) so as to adjust optimally the parameters of the nonmembership functions. The optimality aim of this second level training is to minimize the interval width which is the uncertainty so that $\mu(x) + \upsilon(x)$ tends towards unity making the resultant system one-way fuzzy (uncertainty width is zero).

The novelty in the approach is also to use the designed adaptive fuzzy system in order to model inconsistency. This is achieved through the assignment of inconsistent membership and nonmembership functions, that is $\mu(x) + \upsilon(x) \geq 1$. The assignment of inconsistent membership and nonmembership functions introduces another phase of training to the system.

As a result of the introduction of inconsistency, the control system becomes primarily highly sensitive to this degree of inconsistency and bears a lesser but still important sensitivity to uncertainty measured as the interval width. Consequently, the system we developed becomes subject to two phases of training. In the first phase, the aim is to reduce the inconsistency present in the system. After this first phase of training, the system becomes a 2-way fuzzy adaptive system with minimum inconsistency. The system is then subject to another phase of training to reduce the uncertainty interval width as mentioned previously. In all the phases, there are two levels of training: for the adjustment of the parameters of the membership and for the adjustment of the parameters of the nonmembership functions. The training procedure is discussed in detail in the following section.

3.2 Training Procedure

The proposed system is subject to two phases of training as mentioned before and each phase has two levels of training. The back-propagation training algorithm is used for all levels of training of the 2-way fuzzy adaptive control. The back propagation algorithm provides adaptation to memberships and nonmemberships through the adjustment of their parameters using numerical information given as input-output pair (x, d), $x \in U \subset R^n$, $d \in V \subset R$ (U: Input universe of discourse, V: Output universe of discourse). The purpose of the training is to minimize the error of the form:

$$e = \frac{1}{2}[f(x) - d]^2 \tag{3.1}$$

where $f(x)$ is defined in closed form in Equation 2.8, by adjusting membership/nonmembership parameters.

Assume that the membership and nonmembership functions are assigned to be Gaussian functions as given ($\mu_{F_i^l}(x_i) = exp(-(\frac{x_i - x_{il}}{\sigma_{il}})^2)$). The parameters to be adjusted are the parameters included in the formulation $f(x)$ of the adaptive fuzzy system. They are: the mean x_{il} and the variance σ_{il} of the membership/nonmembership distributions in antecedents of the rules and y_l, the maximum of membership/nonmembership distribution in the consequents of the same rules.

The well known update equation for the back propagation training algorithm [81] is the steepest descent:

$$p(k+1) = p(k) + \Delta p(k) = p(k) - \gamma \frac{\partial e}{\partial p}|_k \tag{3.2}$$

where $\Delta p(k)$ is generally of the form:

$$\Delta p(k) = -\gamma \frac{\partial e}{\partial p} + \eta \Delta p(k-1) \tag{3.3}$$

Here, p denotes the parameters to be trained, so $p \in \{y_l, x_{il}, \sigma_{il}\}$, $l = 1, 2, \ldots, M$ for our case, $k = 0, 1, 2, \ldots$, $i = 1, 2, \ldots, n$ and η is a positive constant called the momentum term which relates to the memory of the system from previous learning states. Since our system is contaminated with uncertainty and inconsistency, and previous learning states have more degrees of uncertainty and inconsistency, we opt not to remember the past states and set $\eta = 0$. Then, our update equations turn out to be in the form of steepest descent equation:

$$p(k+1) = p(k) - \gamma \frac{\partial e}{\partial p}|_k \tag{3.4}$$

where γ is a constant step size.

The training algorithm as described in this chapter is schematically detailed in Fig.3.1. We see that, first f is computed forward along the network. Then, the error e is back-propagated to train individually the parameters y_l, x_{il}, σ_{il} of the system using the steepest descent Equation 3.4.

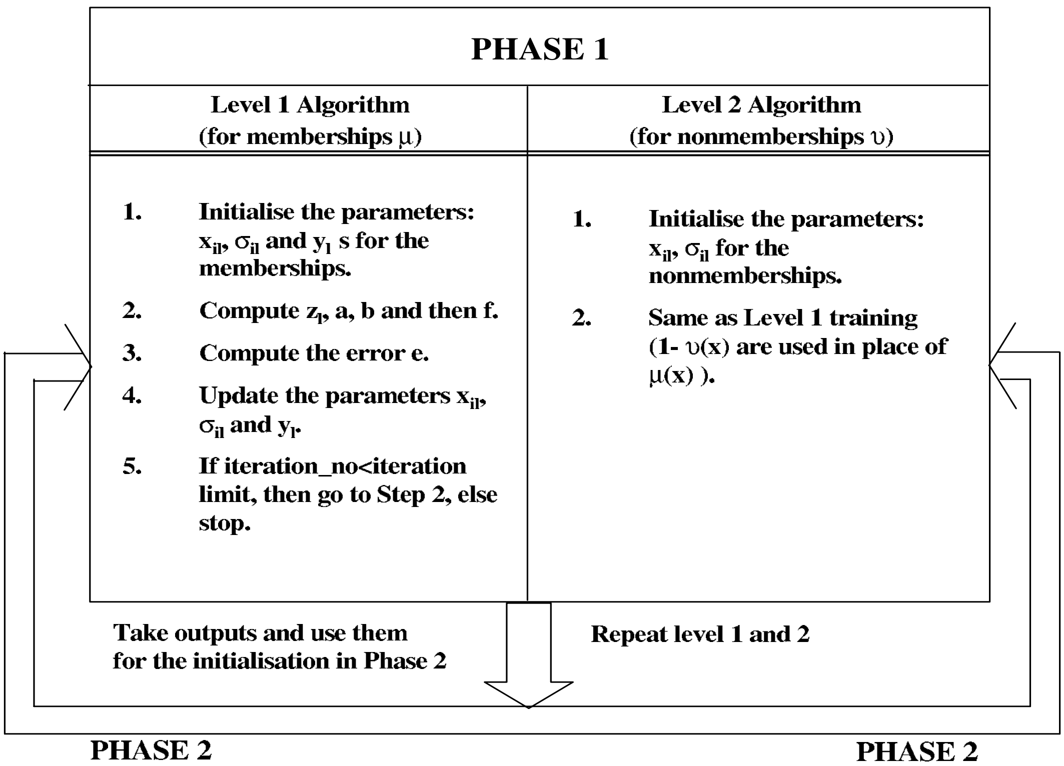


Figure 3.1: Training Algorithm

In all the levels, the above procedure is used. The first level is for the adjustment of the parameters of the membership functions, whereas the second level training is for the adjustment of the parameters of the nonmembership functions. The system to be trained is in the form of Equation 2.8. But in the second level training, the membership functions which are necessity values in Equation 2.8 are replaced by the complement of nonmemberships which are possibility values, (1-nonmembership). The training architecture in this second level taking into consideration possibility values, is again a three-layer feedforward network to which error propagation is again applied as in the first level. But here error undergoes a constrained minimization subject to the constraint $1 - \upsilon_A(x) \rightarrow \mu_A(x)$ which has to be met, upon convergence of the second level training iterations rendering the system as close to fully consistent as possible. The closeness degree at convergence requires a classification of inconsistency based on their evaluation.

3.3 Shadowed Set Evaluation of Inconsistency

In our approach, shadowed sets are used as tools to evaluate inconsistency in our system. Overlapping of shadowed sets obtained from membership and nonmembership functions define an indecisiveness region that we use to characterize the type of inconsistency (Section 3.3.2). Section 3.3.1 presents a quick preview of the theory underlying our approach.

3.3.1 Shadowed Sets

The theory of shadowed sets is developed by Pedrycz [71] in order to overcome the problem of excessive precision in describing imprecise information. That is to say, when vagueness is modelled using fuzzy sets, membership functions are used, which are specific and exact functions in numerical form. This generates a debate on representing imprecise knowledge with precise functions. However, in the case of shadowed sets, there is not a precise numerical representation. This property of shadowed sets allows a computationally simple representation of vagueness within the assignment of fuzzy sets by a rough and less precise construct based on α-cuts.

Shadowed sets can be induced from fuzzy sets such that: the membership values that are considered to be high enough (above a threshold α) are elevated to 1 and those which are considered to be low (below a threshold $1-\alpha$) are reduced to zero, while the intermediate values (between α and $1-\alpha$) are taken as undefined and called a shadow. The formal definition of a shadowed set is given as: Let A be a fuzzy set. Then S is called a shadowed set and is defined as: $S : A \rightarrow \{0, 1, [0, 1]\}$. The values where S is 1 are called the core and the values forming the region $[0, 1]$ are called the shadow of S. An example of forming a shadowed set from a fuzzy set is given in Fig.3.2.

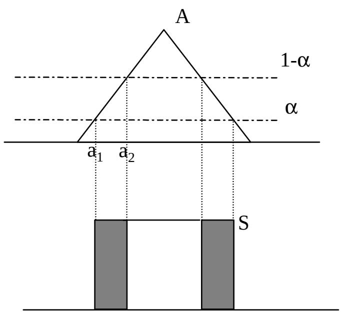


Figure 3.2: Shadowed Set S induced from Fuzzy Set A

A numerical method is to be developed for the selection of the threshold value, α as in Fig.3.2, in order to balance the vagueness since all the intermediate values are defined as indefinite. This balance is needed to ensure that the definite and indefinite regions defined in a shadowed set are balanced properly. The suggested method in [71] is to balance the areas that are below the membership function. The related formulation is:

$$H = \left| \int_{-\infty}^{a_1} A(x)dx + \int_{a_2}^{\infty} (1 - A(x))dx - \int_{a_1}^{a_2} A(x)dx \right|$$

such that for $\alpha \in [0, 0.5)$ $H(\alpha) = 0$.

When the threshold value α is chosen using the above formulation, the values of the fuzzy set that are reduced to zero in the shadowed set and that are increased to one are balanced with values between 0 and 1 assigned as the shadow.

The basic operations on shadowed sets [71] are summarized in Table 3.1.

Table 3.1: Basic Operations on Shadowed Sets

Union

	0	**1**	**[0,1]**
0	0	1	[0,1]
1	1	1	1
[0,1]	[0,1]	1	[0,1]

Intersection

	0	**1**	**[0,1]**
0	0	0	0
1	0	1	[0,1]
[0,1]	0	[0,1]	[0,1]

Complement

0	1
1	0
[0,1]	[0,1]

Operations on shadowed sets exhibit commutativity, associativity, idempotency, distributivity, boundary conditions and involution. When the shadowed set operations are extended to higher dimensional constructs, shadowed relations are obtained. An example of such a case can be given as: $(A \times B)(x, y) = min(A(x), B(y))$ where a shadowed relation is given based on operation on two shadowed sets $A(x)$ and $B(y)$.

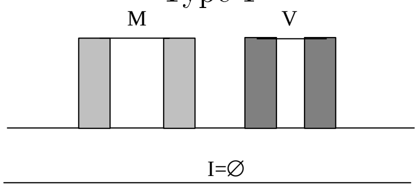
3.3.2 Inconsistency Types Characterized by Shadowed Sets

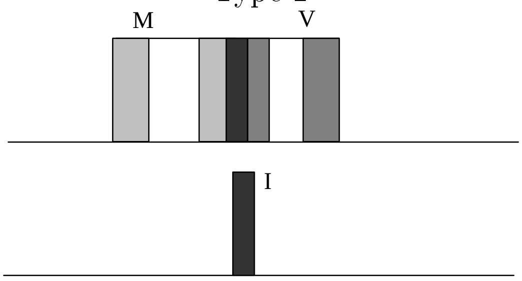
In our approach, we use shadowed sets to evaluate inconsistency in the system. The shadowed sets are formed after the first phase of training, which is the one highly sensitive to inconsistency. Our evaluation of inconsistency is based on a concept of indecisiveness region that we develop out of induced shadowed sets of membership $\mu(x)$ and nonmembership $v(x)$. This indecisiveness region is obtained as the intersection of the induced shadowed sets and is used to characterize the type of inconsistency.

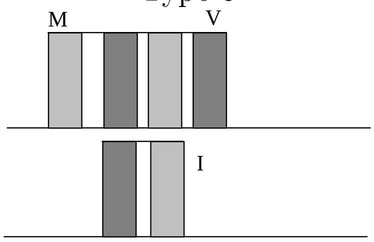
Consider the membership ($\mu(x)$) and nonmembership ($v(x)$) functions and their induced shadowed sets, $M(x)$ and $V(x)$ respectively. In generating the shadowed sets, the balance of vagueness mentioned in Section 3.3.1 is also taken into account in determining the value of the threshold α in each distribution function. The shadowed sets can overlap in 4 different forms which are shown in Table 3.2 (I: Indecisiveness region ($M \cap V$) M: Induced shadowed set for membership V: Induced shadowed set for nonmembership).

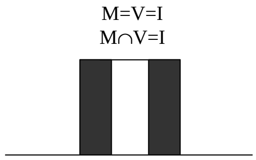
We generate the correspondence between the type of inconsistency obtained

Table 3.2: Types of the Inconsistency Characterized by Shadowed Sets

<table>
<tr><th>Type of Inconsistency</th><th>Global Index of Fuzziness</th></tr>
<tr><td>Type 1
</td><td>$0 < c < 0.5$:

M and V are distinct and there is no intersection region I. This type corresponds to a fully consistent assignment of membership and nonmembership functions.</td></tr>
<tr><td>Type 2
</td><td>$0.5 < c < 1$:

M and V are just overlapping and there is a thin intersection region I. This introduces a certain level of inconsistency for this type of assignment since the values falling into the intersection region are members of both membership and nonmembership functions shadowed sets and this creates a thin indecisiveness region.</td></tr>
<tr><td>Type 3
</td><td>$c > 1$:

M and V have major overlap. The overlap of two shadowed sets introduces a larger intersection region representing a major level of inconsistency leading to a large portion of indecisiveness.</td></tr>
<tr><td>Type 4
</td><td>$c \gg 1$:

M and V fully overlap: full inconsistency.</td></tr>
</table>

at the output of phase 1 of the 2-way adaptive system that tries to reduce inconsistency, and a global index of fuzziness obtained from membership and nonmembership distributions. The measure of fuzziness adapted for this purpose is $f = \sigma/m$ where σ is the variance and m is the mean of the related Gaussian function. In the case of two fuzzy distributions, which are the membership and nonmembership functions for our case, the sum of each individual fuzzy measure is used as a combinational measure $c = \sum_i \frac{\sigma_i}{m_i}\ i \in \{\mu(x), \upsilon(x)\}$.

We cluster membership and nonmembership functions of each rule according to their size of overlap (size of indecisiveness region). Three clusters are obtained where the remaining cases not being clustered are grouped as a 4^{th} cluster. When each cluster is examined according to the fuzziness number c, we find that each cluster is characterized by an interval of c values. Therefore, the shadowed set patterns clusters are found to point to some correlation between inconsistency type (indecisiveness region) and c. We see that values $0 < c < 0.5$ guarantee the characterization of type 1 inconsistency (cluster #1) and as c tends to 1 when $0.5 < c < 1$ the induced shadowed sets get closer to each other and begin overlapping generating a type 2 inconsistency (cluster #2). As $c > 1$, type 3 inconsistency (cluster #3) can lead to even type 4 for large values of c ($c >> 1$).

CHAPTER 4

STABILITY ANALYSIS

There are three stability analysis methods developed in this chapter for fuzzy systems. The first method is based on approximating sequences technique, and aims at designing an optimal fuzzy controller. The second method uses describing function technique to find a stability condition for the 2-way fuzzy adaptive system excited by periodic inputs, and the last method is developed using Lie algebra theory and analyzes the stability of T-S type fuzzy systems. The design of the optimal controller based on the equivalent linearization of approximation techniques can be used in case the other system representations are not available for the plant to be controlled.

4.1 Equivalent Linearization

In this section, we introduce our method of designing an optimal 2-way fuzzy adaptive controller. The method is based on the approximating sequences technique reviewed in Chapter 2.

The 2-way fuzzy adaptive system is applied as a controller to a pseudo-linear system of the form:

$$\dot{x} = A(x)x + B(x)u \tag{4.1}$$

where in the place of u, we have the 2-way fuzzy adaptive system given by the equations:

$$u = f(x) = \sum_{l} \Omega_l(x) y_l \tag{4.2}$$

and

$$\Omega_l = \frac{\prod_{i=1}^{n} \mu_{il}(x_i)}{\sum_{l=1}^{M} \prod_{i=1}^{n} \mu_{il}(x_i)} \tag{4.3}$$

In the equations, μ_{il}'s are the membership/(1-nonmembership) functions, y_l's are the maximum values of the consequent membership functions, M is the rule number, and n is the number of inputs to the fuzzy controller.

When we substitute Equation 4.2 into the Equation 4.1, we obtain:

$$\dot{x} = A(x)x + B(x)\sum_l \Omega_l(x)y_l \tag{4.4}$$

which can be expanded as:

$$\dot{x} = A(x)x + \underbrace{B(x)\begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 \\ \Omega_1(x,\rho) & \Omega_2(x,\rho) & \dots & \Omega_M(x,\rho) \end{bmatrix}}_{B(x,\rho)} \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_M \end{bmatrix}}_{y} \tag{4.5}$$

Here, ρ is the parameter vector containing the parameters of the membership functions. For example, for Gaussian membership functions, components of ρ are defined by the means and variances for each membership function in each rule. The design of an optimal control problem heavily rely on the design of an optimal u that for our case turns into the design of optimal y vector. We consider the system in Equation 4.5 together with the cost function $J = x^T(t_f)F(t_f) + \int_0^{t_f}(x^TQx + u^TRu)dt$ defined in Section 2.4.5 Chapter 2, where u in our case is the y vector. We consider the approximating sequences of this system, and at each approximation the controller is designed such that the cost function is minimized. When the system converges, the controller also converges to the optimal controller [80].

The design for our 2-way fuzzy adaptive system is carried out for both membership and nonmembership cases, where the system equation 4.5 is the same in both cases with the only difference coming from the definition of the

Ω_i functions: We use first the membership functions in the calculation of Ω_l, and then 1-nonmembership functions. The result is an optimal 2-way fuzzy controller.

4.2 Stability Analysis using Describing Function Method

We analyze the stability of our 2-way fuzzy adaptive system using the describing function method. First, we generalize the describing function method of Kim *et al* [23] that was summarized in Section 2.4.4, to multi-input single output fuzzy systems. In order to do this generalization, we derive an additivity property for the fuzzy systems. Then, we calculate the describing function of the 2-way fuzzy system to be used in the stability analysis from which we finally deduce a stability condition for our 2-way fuzzy adaptive controller.

4.2.1 Additively Decomposable Fuzzy Systems

The describing function of a fuzzy system with more than two inputs can be found based on the additivity property of fuzzy systems that if formulated properly would reduce the multi-input single-output fuzzy system to single-input single-output fuzzy systems. To this end, we needed to extend the theory of additivity of fuzzy systems in Cuesta et al. [24] to our fuzzy system. We extend the theory to higher degrees, and especially consider the case of $n = 4$ since we will apply our methodology to the specific example of a 4-dimensional state space of a flexible-joint robot arm in Chapter 5.

The rule structure of the fuzzy system is:

$$\begin{aligned} &R^{(ijk...l)} : \text{IF } x_1 \text{ is } \mu_{1i} \text{ and } x_2 \text{ is } \mu_{2j} \text{ and ... and } x_n \text{ is } \mu_{nl}, \\ &\text{THEN } y \text{ is } y_{ijk...l} \end{aligned} \tag{4.6}$$

where μ's are the antecedent membership functions, $x = (x_1, ..., x_n)^T$ and y are respectively input and output linguistic variables. For this fuzzy system to be additively decomposable, it should satisfy the following property [24]:

$$\begin{aligned} & f(x) = f(x_1, x_2, \ldots, x_n) = \\ & f(x_1, 0, \ldots, 0) + f(0, x_2, \ldots, 0) + \ldots + f(0, 0, \ldots, x_n) \end{aligned} \tag{4.7}$$

In the subsequent paragraph, we consider $f_i(x_i) = f(0, \ldots, x_i, \ldots, 0)$ in order to simplify the notation.

The assumptions on the membership functions for the system to be decomposable are given in Cuesta's work [24] as:

1. $\mu_{qp}(x_q = 0) = 1$, $\mu_{qi}(x_q = 0) = 0$, $i \neq p$, $i = 1, \ldots, M$ and $q = 1, \ldots, n$.
2. $\sum_{i=1}^{M} \mu_{qi}(x_q) = 1$, $\forall_{x_q}$, $q = 1, \ldots, n$.

We use the triangular membership functions of Mamdani:

$$\mu_{qi}(x_q) = \begin{cases} \dfrac{x_q - \phi_{qi-1}}{\phi_{qi} - \phi_{qi-1}}, & \phi_{qi-1} \leq x_q < \phi_{qi} \\ \dfrac{x_q - \phi_{qi+1}}{\phi_{qi} - \phi_{qi+1}}, & \phi_{qi} \leq x_q < \phi_{qi+1} \\ 0 & otherwise \end{cases} \tag{4.8}$$

where $\phi_{-qi} = -\phi_{qi}$. The reason for choosing these type of membership functions is that they also satisfy the assumptions in the calculation of the describing function, which is reviewed in Chapter 2.

The fuzzy controller for $n = 4$ is represented by the equations:

$$f(x) = \sum_i \sum_j \sum_k \sum_l \Omega_{ijkl} y_{ijkl}, \tag{4.9}$$

$$\Omega_{ijkl} = \frac{\mu_{1i}(x_1)\mu_{2j}(x_2)\mu_{3k}(x_3)\mu_{4l}(x_4)}{\sum_p \sum_r \sum_s \sum_t \mu_{1p}(x_1)\mu_{2r}(x_2)\mu_{3s}(x_3)\mu_{4t}(x_4)} \tag{4.10}$$

To each input x_1, x_2, x_3 and x_4 of Equation 4.10, we assign the triangular memberships defined as in Equation 4.8. These memberships satisfy the assumptions (1) and (2) of additivity, so:

$$\sum_p \sum_r \sum_s \sum_t \mu_{1p}(x_1)\mu_{2r}(x_2)\mu_{3s}(x_3)\mu_{4t}(x_4) = 1 \tag{4.11}$$

which then naturally leads to:

$$\Omega_{ijkl} = \mu_{1i}(x_1)\mu_{2j}(x_2)\mu_{3k}(x_3)\mu_{4l}(x_4) \tag{4.12}$$

When $\phi_{1a} \leq x_1 < \phi_{1a+1}$, two consequent rules are fired for x_1 with memberships $\mu_{1a}(x_1)$ and $\mu_{1a+1}(x_1)$. The same applies for the other inputs: for $\phi_{2b} \leq x_2 < \phi_{2b+1}$ with $\mu_{2b}(x_2)$ and $\mu_{2b+1}(x_2)$, for $\phi_{3c} \leq x_3 < \phi_{3c+1}$ with $\mu_{3c}(x_3)$ and $\mu_{3c+1}(x_3)$ and for $\phi_{4d} \leq x_4 < \phi_{4d+1}$ with $\mu_{4d}(x_4)$ and $\mu_{4d+1}(x_4)$. As a total, there are $2^4 = 16$ rules fired, some examples of which are:

$$\begin{aligned}
&R^{(abcd)} : \text{IF } x_1 \text{ is } \mu_{1a} \text{ and } x_2 \text{ is } \mu_{2b} \text{ and } x_3 \text{ is } \mu_{3c} \\
&\text{and } x_4 \text{ is } \mu_{4d}, \text{ THEN } y \text{ is } y_{abcd}. \\
&R^{(a+1bc+1d)} : \text{IF } x_1 \text{ is } \mu_{1a+1} \text{ and } x_2 \text{ is } \mu_{2b} \text{ and } x_3 \text{ is } \mu_{3c+1} \\
&\ \text{and } x_4 \text{ is } \mu_{4d}, \text{ THEN } y \text{ is } y_{a+1bc+1d}. \\
&R^{(a+1b+1c+1d+1)} : \text{IF } x_1 \text{ is } \mu_{1a+1} \text{ and } x_2 \text{ is } \mu_{2b+1} \text{ and } x_3 \text{ is } \mu_{3c+1} \\
&\text{and } x_4 \text{ is } \mu_{4d+1}, \text{ THEN } y \text{ is } y_{a+1b+1c+1d+1}.
\end{aligned} \tag{4.13}$$

Consider the short notation of μ_{ia} instead of $\mu_{ia}(x_i)$ in what follows, in order to simplify the arguments. Such firing of the rules yields the corresponding fuzzy system:

$$\begin{aligned}
&f(x) = \mu_{1a}\mu_{2b}\mu_{3c}\mu_{4d}y_{abcd} + \mu_{1a}\mu_{2b}\mu_{3c}\mu_{4d+1}y_{abcd+1} + \\
&\mu_{1a}\mu_{2b}\mu_{3c+1}\mu_{4d}y_{abc+1d} + \ldots + \mu_{1a+1}\mu_{2b}\mu_{3c+1}\mu_{4d}y_{a+1bc+1d} + \\
&+ \ldots + \mu_{1a+1}\mu_{2b+1}\mu_{3c+1}\mu_{4d}y_{a+1b+1c+1d} \\
&+ \mu_{1a+1}\mu_{2b+1}\mu_{3c+1}\mu_{4d+1}y_{a+1b+1c+1d+1}
\end{aligned} \tag{4.14}$$

If this system satisfies an adequate condition for additivity, the decomposed system should have four single-input single-output systems of the form:

$$\begin{aligned}
f(x_1, 0, 0, 0) = f_1(x_1) = \mu_{1a}y_{afgh} + \mu_{1a+1}y_{a+1fgh} \\
f(0, x_2, 0, 0) = f_2(x_2) = \mu_{2b}y_{ebgh} + \mu_{2b+1}y_{eb+1gh} \\
f(0, 0, x_3, 0) = f_3(x_3) = \mu_{3c}y_{efch} + \mu_{3c+1}y_{efc+1h} \\
f(0, 0, 0, x_4) = f_4(x_4) = \mu_{4d}y_{efgd} + \mu_{4d+1}y_{efgd+1}
\end{aligned} \tag{4.15}$$

We derive the condition under which $f_1(x_1)+f_2(x_2)+f_3(x_3)+f_4(x_4) = f(x)$ is satisfied. First we multiply the above equations by $(\mu_{2b} + \mu_{2b+1})(\mu_{3c} + \mu_{3c+1})(\mu_{4d} + \mu_{4d+1})$, $(\mu_{1a} + \mu_{1a+1})(\mu_{3c} + \mu_{3c+1})(\mu_{4d} + \mu_{4d+1})$, $(\mu_{1a} + \mu_{1a+1})(\mu_{2b} + \mu_{2b+1})(\mu_{4d} + \mu_{4d+1})$, and $(\mu_{1a} + \mu_{1a+1})(\mu_{2b} + \mu_{2b+1})(\mu_{3c} + \mu_{3c+1})$ respectively. All these four terms are equal to 1 for the membership assignments of Equation

4.8. The equations in 4.15 then become:

$$
\begin{aligned}
&f_1(x_1) = \mu_{1a}\mu_{2b}\mu_{3c}\mu_{4d}y_{afgh} + \mu_{1a}\mu_{2b}\mu_{3c}\mu_{4d+1}y_{afgh} + \ldots + \\
&\mu_{1a+1}\mu_{2b}\mu_{3c}\mu_{4d}y_{a+1fgh} + \ldots + \mu_{1a+1}\mu_{2b+1}\mu_{3c+1}\mu_{4d+1}y_{a+1fgh} \\
\\
&f_2(x_2) = \mu_{1a}\mu_{2b}\mu_{3c}\mu_{4d}y_{ebgh} + \mu_{1a}\mu_{2b}\mu_{3c}\mu_{4d+1}y_{ebgh} + \ldots + \\
&\mu_{1a}\mu_{2b+1}\mu_{3c}\mu_{4d}y_{eb+1gh} + \ldots + \mu_{1a+1}\mu_{2b+1}\mu_{3c+1}\mu_{4d+1}y_{eb+1gh} \\
\\
&f_3(x_3) = \mu_{1a}\mu_{2b}\mu_{3c}\mu_{4d}y_{efch} + \mu_{1a}\mu_{2b}\mu_{3c}\mu_{4d+1}y_{efch} + \ldots + \\
&\mu_{1a}\mu_{2b}\mu_{3c+1}\mu_{4d}y_{efc+1h} + \ldots + \mu_{1a+1}\mu_{2b+1}\mu_{3c+1}\mu_{4d+1}y_{efc+1h} \\
\\
&f_4(x_4) = \mu_{1a}\mu_{2b}\mu_{3c}\mu_{4d}y_{efgd} + \mu_{1a}\mu_{2b}\mu_{3c}\mu_{4d+1}y_{efgd+1} + \ldots + \\
&\mu_{1a+1}\mu_{2b}\mu_{3c}\mu_{4d}y_{efgd} + \ldots + \mu_{1a+1}\mu_{2b+1}\mu_{3c+1}\mu_{4d+1}y_{efgd+1}
\end{aligned}
\tag{4.16}
$$

Then, we add the above equations (Equation 4.16) and carry out a term by term comparison with Equation 4.14. From the comparison of the first terms, we deduce that if $y_{afgh} + y_{ebgh} + y_{efch} + y_{efgd} = y_{abcd}$, the first terms become equal. If we do this comparison for the rest of the terms, we derive all the constraints under which the system is additively decomposable. These constraints are summarized in Table 4.1.

4.2.2 Describing Function of a 2-Way Fuzzy System

In this subsection, we derive an analytical expression for the describing function of our 2-way fuzzy system. The derivation is carried out for a single-input single-output fuzzy system, since in the design of a fuzzy controller, we are able to decompose a multi-input single-output fuzzy system into single-input single-output fuzzy systems using the additivity property introduced in the previous subsection, when the derived conditions for such decomposability are met.

The describing function of a 2-way fuzzy system has two components: the describing function of the system with membership functions, and the describing function of the system with 1-nonmemberships. The one for the system

Table 4.1: Constraints on y's for additive decomposability

1	$y_{abcd} = y_{afgh} + y_{ebgh} + y_{efch} + y_{efgd}$
2	$y_{abcd+1} = y_{afgh} + y_{ebgh} + y_{efch} + y_{efgd+1}$
3	$y_{abc+1d} = y_{afgh} + y_{ebgh} + y_{efc+1h} + y_{efgd}$
4	$y_{abc+1d+1} = y_{afgh} + y_{ebgh} + y_{efc+1h} + y_{efgd+1}$
5	$y_{ab+1cd} = y_{afgh} + y_{eb+1gh} + y_{efch} + y_{efgd}$
6	$y_{ab+1cd+1} = y_{afgh} + y_{eb+1gh} + y_{efch} + y_{efgd+1}$
7	$y_{ab+1c+1d} = y_{afgh} + y_{eb+1gh} + y_{efc+1h} + y_{efgd}$
8	$y_{ab+1c+1d+1} = y_{afgh} + y_{eb+1gh} + y_{efc+1h} + y_{efgd+1}$
9	$y_{a+1bcd} = y_{a+1fgh} + y_{ebgh} + y_{efch} + y_{efgd}$
10	$y_{a+1bcd+1} = y_{a+1fgh} + y_{ebgh} + y_{efch} + y_{efgd+1}$
11	$y_{a+1bc+1d} = y_{a+1fgh} + y_{ebgh} + y_{efc+1h} + y_{efgd}$
12	$y_{a+1bc+1d+1} = y_{a+1fgh} + y_{ebgh} + y_{efc+1h} + y_{efgd+1}$
13	$y_{a+1b+1cd} = y_{a+1fgh} + y_{eb+1gh} + y_{efch} + y_{efgd}$
14	$y_{a+1b+1cd+1} = y_{a+1fgh} + y_{eb+1gh} + y_{efch} + y_{efgd+1}$
15	$y_{a+1b+1c+1d} = y_{a+1fgh} + y_{eb+1gh} + y_{efc+1h} + y_{efgd}$
16	$y_{a+1b+1c+1d+1} = y_{a+1fgh} + y_{eb+1gh} + y_{efc+1h} + y_{efgd+1}$

with membership functions is the same as that of a 1-way fuzzy system reviewed in Chapter 2 Section 2.4.4, which is:

$$\begin{aligned} N(A,w) = N(A) = \frac{b_1}{A} = \frac{4}{\pi A}\sum_{i=0}^{d}\{\frac{\Delta u_i A}{2\Delta\phi_i}((\delta_{i+1} - \sin\delta_{i+1}\cos\delta_{i+1}) \\ -(\delta_i - \sin\delta_i\cos\delta_i)) + \frac{1}{\Delta\phi_i}(\phi_i u_{i+1} - \phi_{i+1}u_i)(\cos\delta_{i+1} - \cos\delta_i)\} \end{aligned} \tag{4.17}$$

where d satisfies $\phi_d \le A < \phi_{d+1}$, $d > 0$, and varies with A; $\{\delta_i\}$ are defined to be the angles where the input sinusoid $x = A\sin\delta$ intersects the centers $\{\phi_i\}$'s of membership functions. For $\{\delta_i\}$'s, we have:

$$\begin{aligned} &\delta_0 \equiv 0 \\ &\delta_i \equiv \sin^{-1}(\frac{\phi_i}{A}), (i = 1, \ldots, d, 0 < \delta_i < \frac{\pi}{2}) \\ &\delta_{d+1} \equiv \frac{\pi}{2} \end{aligned} \tag{4.18}$$

Here as part of our thesis work, we derive the expression for the system with 1-nonmemberships, together with the necessary and suitable modification of the assumptions stated for the 1-way fuzzy system case, so as to adapt to our case. The closed form of the 2-way fuzzy adaptive system dealing with nonmembership is the same as 1-way fuzzy system apart from the definition of membership functions. In this case, we have $1 - \upsilon(x)$ (1-nonmembership functions), and the closed form of the system after substituting this specific plausibility expression becomes:

$$u = f(x) = \sum_{l} \Omega_l(x) u_l \tag{4.19}$$

where

$$\Omega_l(x) = \frac{1 - \upsilon_l(x)}{\sum_{k=1}^{ruleno} 1 - \upsilon_k(x)} \tag{4.20}$$

The assumptions are:

1. The antecedent nonmembership functions are triangular functions distributed completely, consistently and symmetrically with respect to the origin. The functions are of the form:

$$\upsilon_i(x) = \begin{cases} \dfrac{x - \alpha_{i-1}}{\alpha_i - \alpha_{i-1}}, & \alpha_{i-1} \leq x < \alpha_i \\ \dfrac{x - \alpha_{i+1}}{\alpha_i - \alpha_{i+1}}, & \alpha_i \leq x < \alpha_{i+1} \\ 0 & otherwise \end{cases} \tag{4.21}$$

where $\alpha_{-i} = -\alpha_i$.

2. Consequent parts have the odd condition:

$$u_{-l} = -u_l \tag{4.22}$$

3. When $\alpha_n \leq x < \alpha_{n+1}$, only two rules are fired, and:

$$\sum_{k=1}^{ruleno} 1 - \upsilon_k(x) = 1 - \upsilon_n(x) + 1 - \upsilon_{n+1}(x) = 2 - \underbrace{(\upsilon_n(x) + \upsilon_{n+1}(x))}_{1} = 1 \tag{4.23}$$

for the triangular nonmembership functions of Equation 4.21.

So:

$$\Omega_l(x) = \frac{1 - \upsilon_l(x)}{\sum_{k=1}^{ruleno} 1 - \upsilon_k(x)} = 1 - \upsilon_l(x) \tag{4.24}$$

4. $\Omega_{-l}(-x) = \Omega_l(x)$, since $\Omega_{-l}(-x) = 1 - \underbrace{\upsilon_{-l}(-x)}_{\upsilon_l(x)} = \Omega_l(x)$.

The fuzzy logic system with nonmembership functions satisfying the above assumptions also satisfies the following lemmas:

Lemma 1: If the fuzzy system satisfies the above four assumptions, then it is also odd in time when the input is $x = A\sin wt$.

Lemma 2: If the fuzzy system satisfies the above four assumptions, and $x = A\sin wt$, then $u(t) = u((\pi/w) - t)$.

Lemma 3: If the fuzzy system satisfies the above four assumptions, and $\alpha_k \leq x < \alpha_{k+1}$, then the output u of the system is:

$$u = \sum_l \Omega_l(x)u_l = (u_k + u_{k+1}) - \{\frac{\Delta u_k}{\Delta\alpha_k}x + \frac{1}{\Delta\alpha_k}(\alpha_{k+1}u_k - \alpha_k u_{k+1})\} \quad (4.25)$$

where $\Delta\alpha_k \equiv \alpha_{k+1} - \alpha_k$ and $\Delta u_k \equiv u_{k+1} - u_k$.

The proofs of the lemmas are similar to the proofs in the 1-way fuzzy system case introduced in [23], so they are omitted here.

The describing function of the fuzzy system with nonmembership functions satisfying the above four assumptions, and the three lemmas is given in the following theorem.

Theorem: The describing function of the fuzzy system given by Equation 4.19 that satisfies the four assumptions is a real number, which depends only on the amplitude A of the input sinusoid, and is in the following form:

$$\begin{aligned}
&\bar{N}(A,w) = \bar{N}(A) = \frac{b_1}{A} = \frac{4}{\pi A}\sum_{i=0}^{d}((u_i + u_{i+1})(\cos\delta_i - \cos\delta_{i+1})) - \\
&\{\frac{\Delta u_i A}{2\Delta\alpha_i}((\delta_{i+1} - \sin\delta_{i+1}\cos\delta_{i+1}) - (\delta_i - \sin\delta_i\cos\delta_i)) \\
&+\frac{1}{\Delta\alpha_i}(\alpha_i u_{i+1} - \alpha_{i+1}u_i)(\cos\delta_{i+1} - \cos\delta_i)\}
\end{aligned} \quad (4.26)$$

where α_i's are the centers of the triangular nonmembership functions, d satisfies $\alpha_d \leq A < \alpha_{d+1}$, $d > 0$, and varies with A; $\{\delta_i\}$ are defined to be the angles where the input sinusoid $x = A\sin\delta$ intersects the centers $\{\alpha_i\}$'s of nonmembership functions. For $\{\delta_i\}$'s, we have:

$$\begin{aligned}
&\delta_0 \equiv 0 \\
&\delta_i \equiv \sin^{-1}(\frac{\alpha_i}{A}), (i = 1, \ldots, d, 0 < \delta_i < \frac{\pi}{2}) \\
&\delta_{d+1} \equiv \frac{\pi}{2}
\end{aligned} \quad (4.27)$$

Proof: The describing function of $u(t) \equiv u(x = A\sin wt)$ is calculated as: $\bar{N}(A,w) = \frac{1}{A}(b_1 + ja_1)$, where $a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} u(t)\cos(wt)dwt$ and $b_1 =$

$\frac{1}{\pi}\int_{-\pi}^{\pi} u(t)\sin(wt)dwt$. For the fuzzy system:

$$
\begin{aligned}
a_1 &= \tfrac{1}{\pi}\textstyle\int_{-\pi}^{\pi} u(t)\cos(wt)dwt = 0 \text{ by Lemma 1} \\
b_1 &= \tfrac{1}{\pi}\textstyle\int_{-\pi}^{\pi} u(t)\sin(wt)dwt \\
&= \tfrac{2}{\pi}\textstyle\int_{0}^{\pi} u(t)\sin(wt)dwt \text{ by Lemma 1} \\
&= \tfrac{2}{\pi}(\textstyle\int_{0}^{\pi/2} u(t)\sin(wt)dwt + \int_{\pi/2}^{\pi} u(t)\sin(wt)dwt) \\
&= \tfrac{2}{\pi}(\textstyle\int_{0}^{\pi/2} u(t)\sin(wt)dwt + \int_{\pi/2}^{0} u(\frac{\pi}{w}-k)\sin(\pi - wk)d(-wk)) \\
&= \tfrac{4}{\pi}\textstyle\int_{0}^{\pi/2} u(t)\sin(wt)dwt \text{ by Lemma 2}
\end{aligned}
\tag{4.28}
$$

Using the variables δ_i's in b_1, we have:

$$
\begin{aligned}
& b_1 = \tfrac{4}{\pi}(\textstyle\int_{\delta_0}^{\delta_1} u(t)\sin(wt)dwt + \int_{\delta_1}^{\delta_2} u(t)\sin(wt)dwt + \ldots + \int_{\delta_d}^{\delta_{d+1}} u(t)\sin(wt)dwt) \\
& = \tfrac{4}{\pi}\textstyle\sum_{i=0}^{d}\int_{\delta_i}^{\delta_{i+1}} u(t)\sin(wt)dwt \\
& = \tfrac{4}{\pi}\textstyle\sum_{i=0}^{d}\int_{\delta_i}^{\delta_{i+1}}\{(u_i + u_{i+1}) - (\frac{\Delta u_i}{\Delta\alpha_k}A\sin(wt) + \frac{1}{\Delta\alpha_k}(\alpha_{k+1}u_k - \alpha_k u_{k+1}))\} \\
& \sin(wt)dwt \text{ by Lemma 3} \\
& = \tfrac{4}{\pi}\textstyle\sum_{i=0}^{d}\{((u_i + u_{i+1})(\cos\delta_i - \cos\delta_{i+1})) - \\
& \{\tfrac{\Delta u_i A}{2\Delta\alpha_i}(\delta_{i+1} + \delta_i - \sin\delta_{i+1}\cos\delta_{i+1} + \sin\delta_i\cos\delta_i) + \\
& \tfrac{1}{\Delta\alpha_i}(\alpha_i u_{i+1} - \alpha_{i+1}u_i)(\cos\delta_{i+1} - \cos\delta_i)\}\}
\end{aligned}
\tag{4.29}
$$

The describing function of the 2-way fuzzy system is given by $\{N(A), \bar{N}(A)\}$.

4.2.3 Stability Analysis of 2-Way Fuzzy Adaptive Systems

We use the interval valued describing function derived in the previous section for the stability analysis of our 2-way fuzzy adaptive system. In this approach, the describing function of the fuzzy system is considered in cascade with a linear plant with transfer function $G(s) = \dfrac{n(s)}{d(s)}$, which naturally has to have a low-pass property for minimal sensitivity to higher order errors. This well known property is crucial in the application of describing function analysis, since this analysis is based on the fundamental harmonic of the output of the fuzzy system. In our application example, we will then have to verify the low-pass nature of the robotic system in hand and how much error is introduced by considering it as such.

The characteristic equation of the feedback system with the fuzzy controller replaced by the interval valued describing function $\{N(A), \bar{N}(A)\}$, in cascade with the linear plant $G(s)$ is: $C(s) = 1 + \{N(A), \bar{N}(A)\}G(s)$. This yields the interval valued characteristic equation $[C_1, C_2]$ where:

$$\begin{aligned} C_1(s) &= 1 + N(A)G(s) = d(s) + N(A)n(s) \\ C_2(s) &= 1 + \bar{N}(A)G(s) = d(s) + \bar{N}(A)n(s) \end{aligned} \tag{4.30}$$

The C_1 and C_2 in the above equation are both interval polynomials, since $N(A)$ and $\bar{N}(A)$ are real and interval-valued that depend on A. For the stability analysis of the C_i interval polynomials, determining themselves the characteristic interval valued equation, we use the Kharitonov's theorem for real polynomials [82].

Theorem: Let $I(s)$ be the set of real polynomials of degree n of the form $\delta(s) = \delta_0 + \delta_1 s + \delta_2 s^2 + \delta_3 s^3 + \ldots + \delta_n s^n$, where the coefficients lie within given ranges, $\delta_0 \in [x_0, y_0]$, $\delta_1 \in [x_1, y_1]$,..., $\delta_n \in [x_n, y_n]$.

Every polynomial in the family $I(s)$ is Hurwitz if and only if the following four extreme polynomials are Hurwitz:

$$\begin{aligned} K_1(s) &= x_0 + x_1 s + y_2 s^2 + y_3 s^3 + x_4 s^4 + x_5 s^5 + y_6 s^6 + \ldots \\ K_2(s) &= x_0 + y_1 s + y_2 s^2 + x_3 s^3 + x_4 s^4 + y_5 s^5 + y_6 s^6 + \ldots \\ K_3(s) &= y_0 + x_1 s + x_2 s^2 + y_3 s^3 + y_4 s^4 + x_5 s^5 + x_6 s^6 + \ldots \\ K_4(s) &= y_0 + y_1 s + x_2 s^2 + x_3 s^3 + y_4 s^4 + y_5 s^5 + x_6 s^6 + \ldots \end{aligned} \tag{4.31}$$

The proof of the theorem can be found in [82].

For our system in Equation 4.30, we need to check the Kharitonov polynomials for each characteristic equation C_1 and C_2, with $N(A) \in [N_{min}, N_{max}]$ and $\bar{N}(A) \in [\bar{N}_{min}, \bar{N}_{max}]$. If both polynomials are found to be Hurwitz, then we conclude that our system is stable. The reader may refer to Chapter 5 Section 5.4 to find an illustrative example on how the Kharitonov check is carried out on a specific application system in order to ensure the fuzzy controller stability.

4.3 Lie Algebra

This section introduces our approach for the stability analysis of fuzzy control systems where rules have consequents that can be represented in the form of Equation 2.9 that is $R^{(l)}$: IF x_1 is F_1^l and ... and x_n is F_n^l, THEN $\dot{x}(t) = A_l x(t) + B_l u(t)$ (Section 2.4.2, Chapter 2). In order to handle such control system dynamics, we consider the Lie algebra L_A generated by the A matrices of the linear subsystems of the rules, i.e. by $\{A_1, A_2, \ldots, A_M\}$.

In the first subsection, we first start with a simpler case where all the A_i matrices commute, that is when they form an Abelian Lie algebra L_A. Then, we generalize the results to noncommuting system matrices.

4.3.1 Commuting Fuzzy Systems

First, we assume not only that L_A is Abelian, so that all the A_i's commute, but also that all the A_i's are diagonalizable. This is not a particularly strong assumption as it may seem, since such matrices are generic, and the results easily generalize.

There exists a common diagonalizing matrix P for the singularity transformation of A_i: $P^{-1}A_iP = \Lambda_i$, where $\Lambda_i = diag(\lambda_1^i, \ldots, \lambda_n^i)$, λ_j^i being the eigenvalues of A_i. Hence, considering $y = P^{-1}x$, we transform Equation 2.12

$$\dot{x}(t) = (\sum_{l=1}^{M} \psi_l(x(t))A_l)x(t) + (\sum_{l=1}^{M} \psi_l(x(t))B_l)u(t) \tag{4.32}$$

into

$$\dot{y}(t) = (\sum_{l=1}^{M} \psi_l(Py(t))\Lambda_l)y(t) + (\sum_{l=1}^{M} \psi_l(Py(t))P^{-1}B_l)u(t) \tag{4.33}$$

i.e. in a simplified representation:

$$\dot{y}_i(t) = \alpha_i y_i(t) + \beta_i u(t), 1 \leq i \leq n \tag{4.34}$$

where

$$\alpha_i = \sum_{l=1}^{M} \psi_l(Py(t))\lambda_i^l, \beta_i = \sum_{l=1}^{M} \psi_l(Py(t))(P^{-1}B_l)_i \tag{4.35}$$

The subscript $(.)_i$ denotes the i^{th} element of the corresponding vector represented by (.).

To achieve stability for the general system of the form 4.34, our approach is to generate a control u which satisfies the inequalities:

$$y_i(t)\dot{y}_i(t) = \alpha_i y_i^2(t) + \beta_i u(t) y_i(t) \leq -\frac{1}{2}\varepsilon y_i^2(t), 1 \leq i \leq n \tag{4.36}$$

for some $\varepsilon > 0$.

The reason in our selection of the control is the same condition as for the convergence rule of a sliding mode controller, where $\frac{1}{2}\varepsilon y_i^2(t)$ acts as a Lyapunov function:

If $y_i(t) = 0$, then the inequality in Equation 4.36 is obviously satisfied, since it is the trivial solution for the inequality where we can choose any u. If $y_i(t) \neq 0$, for α_i real, solving:

$$\frac{1}{2}\frac{d}{dt}y_i^2(t) = y_i(t)\dot{y}_i(t) = \alpha_i y_i^2(t) + \beta_i u(t) y_i(t) \leq -\frac{1}{2}\varepsilon y_i^2(t) \tag{4.37}$$

then yields,

$$y_i^2(t) \leq e^{-\varepsilon t} y_i^2(0) \tag{4.38}$$

which shows that y_i's are bounded decaying functions, so are stable.

The stability condition for $y_i \neq 0$ for real α_i's can be obtained similarly as in the case of sliding mode control, by dividing Equation 4.37 to y_i as:

$$\left\{ \begin{array}{l} \alpha_i y_i(t) + \beta_i u(t) \leq -\frac{1}{2}\varepsilon y_i(t) \text{ if } y_i(t) > 0 \\ \alpha_i y_i(t) + \beta_i u(t) \geq -\frac{1}{2}\varepsilon y_i(t) \text{ if } y_i(t) < 0 \end{array} \right. \tag{4.39}$$

for $1 \leq i \leq n$.

Now, let us derive similar conditions for the case of complex constants α_i.

Remark: If some α_i's are complex, then we write:

$$y_i(t) = \xi_i(t) + i\eta_i(t), \alpha_i = a_i + ib_i \tag{4.40}$$

Then, Equation 4.34 becomes:

$$\begin{aligned} \dot{\xi}_i &= a_i\xi_i - b_i\eta_i + (Re(\beta_i))u(t), \\ \dot{\eta}_i &= b_i\xi_i + a_i\eta_i + (Im(\beta_i))u(t) \end{aligned} \tag{4.41}$$

This time we choose u so that:

$$\begin{aligned} a_i\xi_i^2 - b_i\xi_i\eta_i + \xi_i(Re(\beta_i))u(t) &\leq -\tfrac{1}{2}\varepsilon\xi_i^2, \\ a_i\eta_i^2 + b_i\xi_i\eta_i + \eta_i(Im(\beta_i))u(t) &\leq -\tfrac{1}{2}\varepsilon\eta_i^2 \end{aligned} \tag{4.42}$$

The above results both for real and complex α_i cases lead to the following theorem:

Theorem 1: The system in Equation 4.32 with Abelian Lie algebra L_A is stabilizable (in the case of real eigenvalues-with a similar condition in the complex case) if the inequalities in Equation 4.39 (Equation 4.42 for complex variables case) are solvable for u.

However, the inequalities in Equation 4.39 (Equation 4.42) are difficult to be solved, in general. But, we can find a simple, but more conservative, condition for solution in the following way. Define the sets:

$$\begin{aligned} S_1 &= \{y \in \mathcal{R}^n : \textstyle\sum_{i=1}^n \alpha_i y_i^2 < 0\}, \\ S_2 &= \{y \in \mathcal{R}^n : \textstyle\sum_{i=1}^n \beta_i y_i = 0\} \end{aligned} \tag{4.43}$$

and suppose that $S_2(\varepsilon) \subset S_1$ where

$$S_2(\varepsilon) = \{y \in \mathcal{R}^n : d(y, S_2) \leq \varepsilon\} \tag{4.44}$$

for some $\varepsilon > 0$. Now, consider the Lyapunov function:

$$V = \frac{1}{2}\sum_{i=1}^n y_i^2 \tag{4.45}$$

Then, we have:

$$\dot{V} = \sum_{i=1}^n y_i\dot{y}_i = \sum_{i=1}^n \alpha_i y_i^2 + (\sum_{i=1}^n \beta_i y_i)u \tag{4.46}$$

and we choose the control:

$$u = \{ \begin{array}{l} \frac{-2\sum_{i=1}^n \alpha_i y_i^2}{\sum_{i=1}^n \beta_i y_i} \text{ if } y \in \mathcal{R}^n \setminus S_1 \\ 0 \text{ if } y \in S_1 \end{array} \tag{4.47}$$

This choice of control guarantees $\dot{V}$ to be negative, since when $y \in S_1$ the first term in Equation 4.46 is negative, so we can set $u = 0$. When $y \in \mathcal{R}^n \setminus S_1$, the control u in the above equation results in $\dot{V} = -\sum_{i=1}^{n} \alpha_i y_i^2$, and this is guaranteed to be negative since $y \in \mathcal{R}^n \setminus S_1$.

Here, we will give two simple examples to illustrate how the above results apply.

Example 1

We consider a simple numerical example, where we assume that all the pairs $(\Lambda_l, P^{-1}B_l)$ in Equation 4.33 are uniformly stabilizable in the sense that there exists a vector K independent of l such that $\Lambda_l - P^{-1}B_l K$ is a stable matrix for each rule l (rule l). The only constraint is that the pairs should be uniformly stabilizable, but not necessarily controllable.

In this example, there are three rules with the following structure:

$$\begin{aligned} &R^{(l)} : \text{IF } x_1 \text{ is } F_1^l \text{ and } x_2 \text{ is } F_2^l, \text{ THEN} \\ &\dot{x}(t) = A_l x(t) + B_l u(t), \text{ for } l = 1, 2, 3 \end{aligned} \tag{4.48}$$

where

$$A_1 = \begin{bmatrix} -2.8 & 3.6 \\ -2.4 & 3.8 \end{bmatrix}, b_1 = \begin{bmatrix} -1.2 \\ -1.6 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -6.2 & 8.4 \\ -5.6 & 9.2 \end{bmatrix}, b_2 = \begin{bmatrix} -0.6 \\ -0.8 \end{bmatrix} \tag{4.49}$$

$$A_3 = \begin{bmatrix} -4.3 & 3.6 \\ -2.4 & 2.3 \end{bmatrix}, b_3 = \begin{bmatrix} -0.3 \\ -0.4 \end{bmatrix}$$

Here, F_i^l are the fuzzy sets in the antecedent parts of the rules. It is easy to check that the matrices A_1, A_2, A_3 commute and are diagonalized by the matrix:

$$P = \begin{bmatrix} -0.8944 & -0.6 \\ -0.4472 & -0.8 \end{bmatrix}$$

The diagonalized matrices and transformed vectors $P^{-1}b_l$ are:

$$\Lambda_1 = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}, P^{-1}b_1 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$\Lambda_2 = \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix}, P^{-1}b_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{4.50}$$

$$\Lambda_3 = \begin{bmatrix} -2.5 & 0 \\ 0 & 0.5 \end{bmatrix}, P^{-1}b_3 = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}$$

If we choose the control to be $u = -Ky$ where $y = P^{-1}x$ and $K = \begin{bmatrix} 0 & 6 \end{bmatrix}$, the subsystems become stable. Since we are able to find a control u to stabilize all the subsystems, we conclude that the overall system is stable.

Example 2

In Example 1, it was easy to find a control that would stabilize all the subsystems. If this is not the case, we have to apply the control in Equation 4.47. We use the membership functions given in Fig.4.1 for all the states of the system, i.e. for x_1 and x_2.

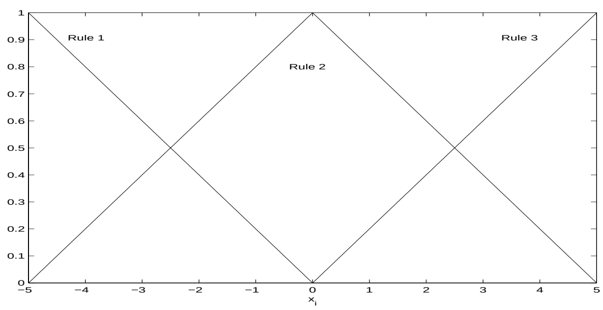


Figure 4.1: Membership functions for Example 2

When we apply the control in the form of Equation 4.47 to the system given by the diagonalized system matrices of Example 1, we obtain a stable system as shown in Fig.4.2 together with the control input u.

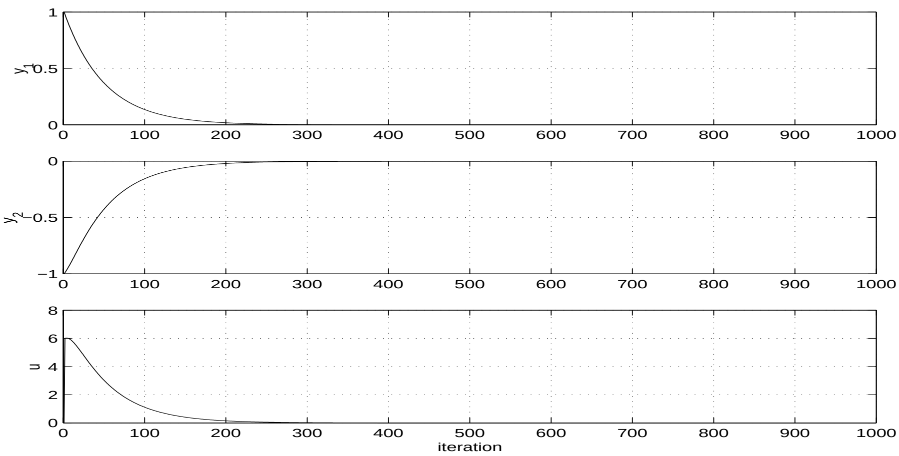


Figure 4.2: Stabilized states and control input for Example 2

4.3.2 Noncommuting Fuzzy Systems: General Case

In this section, we generalize our results to the more generalized noncommuting case, i.e. where L_A generated by the linear subsystem matrices $\{A_1, A_2, \ldots, A_M\}$ of the rules ($R^{(l)}$: IF x_1 is F_1^l and ... and x_n is F_n^l, THEN $\dot{x}(t) = A_l x(t) + B_l u(t)$) is not Abelian.

First, we assume that there is an Abelian Lie algebra $L_{\tilde{A}}$ such that the error in approximation of the noncommuting system by this commuting Lie algebra is small. Then, we will generalize the commuting case results using Levi decomposition introduced in the Appendix.

Now, suppose that the Lie algebra L_A is not Abelian and consider the error of approximating the original set of matrices by the new commuting set of matrices $\{\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_M\}$, and we aim at minimizing such an error:

$$\zeta = max_{1 \leq l \leq M} \|A_l - \tilde{A}_l\| \tag{4.51}$$

Then, if we combine this with our system given by Equation 4.32, we get:

$$\begin{aligned} \dot{x}(t) = (\sum_{l=1}^{M} \psi_l(x(t))\tilde{A}_l)x(t) + (\sum_{l=1}^{M} \psi_l(x(t))B_l)u(t) \\ +(\sum_{l=1}^{M} \psi_l(x(t))(A_l - \tilde{A}_l))x(t) \end{aligned} \tag{4.52}$$

Hence, if $\tilde{P}$ is a common diagonalizing matrix for the set $\{\tilde{A}_1, \ldots, \tilde{A}_M\}$, then we have:

$$\tilde{P}^{-1}\tilde{A}_l\tilde{P} = \tilde{\Lambda}_l = diag(\tilde{\lambda}_1^l, \dots, \tilde{\lambda}_n^l) \tag{4.53}$$

and with $y = \tilde{P}^{-1}x$

$$\begin{aligned} \dot{y}(t) = (\sum_{l=1}^{M} \psi_l(\tilde{P}y(t))\tilde{\Lambda}_l)y(t) + (\sum_{l=1}^{M} \psi_l(\tilde{P}y(t))\tilde{P}^{-1}B_l)u(t) \\ +(\sum_{l=1}^{M} \psi_l(\tilde{P}y(t))\tilde{P}^{-1}(A_l - \tilde{A}_l)\tilde{P})y(t) \end{aligned} \tag{4.54}$$

i.e.,

$$\dot{y}_i(t) = \tilde{\alpha}_i y_i(t) + \tilde{\beta}_i u(t) + \{[\sum_{k=1}^{M} \psi_k(\tilde{P}y(t))\tilde{P}^{-1}(A_k - \tilde{A}_k)\tilde{P}]y(t)\}_i, 1 \leq i \leq n \tag{4.55}$$

where

$$\tilde{\alpha}_i = \sum_{l=1}^{M} \psi_l(\tilde{P}y(t))\tilde{\lambda}_i^l, \tilde{\beta}_i = \sum_{l=1}^{M} \psi_l(\tilde{P}y(t))(\tilde{P}^{-1}B_l)_i \tag{4.56}$$

As before with the same motivation as for the commuting case, we choose a control u such that:

$$\sum_i (\tilde{\alpha}_i y_i^2(t) + \tilde{\beta}_i u(t) y_i(t)) \leq -\frac{1}{2}\varepsilon \sum_i y_i^2(t) \tag{4.57}$$

We have in the case of real eigenvalues:

$$\begin{aligned} &\tfrac{1}{2}\tfrac{d}{dt}\sum_i y_i^2(t) = \sum_i y_i(t)\dot{y}_i(t) \\ &= \sum_i (\tilde{\alpha}_i y_i^2(t) + \tilde{\beta}_i u(t) y_i(t)) + \\ &\sum_i \{[\sum_{k=1}^{M} \psi_k(\tilde{P}y(t))\tilde{P}^{-1}(A_k - \tilde{A}_k)\tilde{P}]y(t)\}_i y_i(t) \\ &\leq -\tfrac{1}{2}\varepsilon \sum_i y_i^2(t) + \sum_i \{[\sum_{k=1}^{M} \psi_k(\tilde{P}y(t))\tilde{P}^{-1}(A_k - \tilde{A}_k)\tilde{P}]y(t)\}_i y_i(t) \end{aligned} \tag{4.58}$$

so,

$$\frac{1}{2}\frac{d}{dt}\|y(t)\|^2 \leq -\frac{1}{2}\varepsilon\|y(t)\|^2 + \|y(t)\|^2\|\tilde{P}^{-1}\| \cdot \|\tilde{P}\| \cdot \zeta \tag{4.59}$$

We therefore have stability if

$$\|\tilde{P}^{-1}\| \cdot \|\tilde{P}\| \cdot \zeta < \frac{1}{2}\varepsilon \tag{4.60}$$

Remark: If some $\tilde{\alpha}_i$'s are complex, then expressing:

$$y_i(t) = \xi_i(t) + i\eta_i(t), \tilde{\alpha}_i = \tilde{a}_i + i\tilde{b}_i \tag{4.61}$$

we have:

$$
\begin{aligned}
&\sum_i(\tilde{a}_i\xi_i^2 - \tilde{b}_i\xi_i\eta_i + \xi_i Re(\tilde{\beta}_i)u(t)+ \\
&\sum_i Re\{[\sum_{k=1}^{M}\psi_k(\tilde{P}y(t))\tilde{P}^{-1}(A_k-\tilde{A}_k)\tilde{P}]y(t)\}_i\xi_i(t) \leq -\tfrac{1}{2}\varepsilon\sum_i \xi_i^2(t) \\
&+\sum_i Re\{[\sum_{k=1}^{M}\psi_k(\tilde{P}y(t))\tilde{P}^{-1}(A_k-\tilde{A}_k)\tilde{P}]y(t)\}_i\xi_i(t), \\
&\sum_i(\tilde{a}_i\eta_i^2 + \tilde{b}_i\xi_i\eta_i + \eta_i Im(\tilde{\beta}_i)u(t)+ \\
&\sum_i Im\{[\sum_{k=1}^{M}\psi_k(\tilde{P}y(t))\tilde{P}^{-1}(A_k-\tilde{A}_k)\tilde{P}]y(t)\}_i\eta_i(t) \leq -\tfrac{1}{2}\varepsilon\sum_i \eta_i^2(t) \\
&+\sum_i Im\{[\sum_{k=1}^{M}\psi_k(\tilde{P}y(t))\tilde{P}^{-1}(A_k-\tilde{A}_k)\tilde{P}]y(t)\}_i\eta_i(t),
\end{aligned}
\tag{4.62}
$$

So,

$$
\begin{aligned}
&\sum_i(\tilde{a}_i\xi_i^2 - \tilde{b}_i\xi_i\eta_i + \xi_i Re(\tilde{\beta}_i)u(t)+ \\
&\sum_i Re\{[\sum_{k=1}^{M}\psi_k(\tilde{P}y(t))\tilde{P}^{-1}(A_k-\tilde{A}_k)\tilde{P}]y(t)\}_i\xi_i(t) \leq -\tfrac{1}{2}\varepsilon\|\xi\|^2+ \\
&\|\xi\|^2\|\tilde{P}^{-1}\|\|\tilde{P}\|\zeta \\
&\sum_i(\tilde{a}_i\eta_i^2 + \tilde{b}_i\xi_i\eta_i + \eta_i Im(\tilde{\beta}_i)u(t)+ \\
&\sum_i Im\{[\sum_{k=1}^{M}\psi_k(\tilde{P}y(t))\tilde{P}^{-1}(A_k-\tilde{A}_k)\tilde{P}]y(t)\}_i\eta_i(t) \leq -\tfrac{1}{2}\varepsilon\|\eta\|^2+ \\
&\|\eta\|^2\|\tilde{P}^{-1}\|\|\tilde{P}\|\zeta
\end{aligned}
\tag{4.63}
$$

We again have stability if

$$
\|\tilde{P}^{-1}\| \cdot \|\tilde{P}\| \cdot \zeta < \frac{1}{2}\varepsilon \tag{4.64}
$$

The above results form the proof and thus lead to the following theorem.

Theorem 2: The system in Equation 4.32 with Lie algebra L_A is stabilizable (in the case of real eigenvalues - with a similar condition in the complex case) if there is an Abelian Lie algebra $L_{\tilde{A}}$ such that inequalities 4.57 are solvable for u and the inequality 4.60 (4.64 for the complex case) is satisfied where $\zeta = max_{1\leq l\leq M}\|A_l - \tilde{A}_l\|$.

Theorem 2 provides the stabilization criterion in terms of the approximating Abelian Lie algebra $L_{\tilde{A}}$, however we need to establish a criterion for the original system with noncommuting A's. For developing such a criterion for the original system, carrying out from Theorem 2, we consider the general noncommuting case using the Levi decomposition given by $g = r + m = r + (h_m \oplus \sum_{\varphi \in \Sigma} m^{\varphi})$, $r \cap m = \emptyset$. For any choice of the Levi subalgebra m and any choice of Cartan subalgebra h_m of m, the subsystem matrices A_l's of Equation 4.32 can be written as:

$$A_l = A_{l(r)} + \bar{A}_{l(h_m)} + \sum_{\varphi \in \Sigma} A_{l(m^{\varphi})} \tag{4.65}$$

where $A_{l(r)} \in r$, $\bar{A}_{i(h_m)} \in h_m$ and $A_{i(m^{\varphi})} \in m^{\varphi}$ $(\varphi \in \Sigma)$. Note that the set of matrices $\{\bar{A}_{1(h_m)}, \ldots, \bar{A}_{M(h_m)}\}$ is commutative.

If we now apply Theorem 2 with $\tilde{A}_l = \bar{A}_{l(h_m)}$, then we immediately obtain the following theorem.

Theorem 3: The system in Equation 4.32 with Lie algebra L_A is stabilizable (in the case of real eigenvalues and with a similar condition in the complex case) if there is a Levi and Cartan decomposition of L_A such that inequalities 4.57 are solvable for u and the inequality 4.60 (4.64 for the complex case) is satisfied where $\zeta = max_{1 \le l \le M} \|A_l - \bar{A}_{l(h_m)}\|$ and $\tilde{P}$ diagonalizes the Cartan subalgebra of L_A.

CHAPTER 5

RESULTS (APPLICATION EXAMPLES)

The theories developed in the previous Chapters 3 and 4, are applied to example systems in this chapter. The first example is the application of our 2-way fuzzy adaptive system to the modelling of an unknown nonlinear function. The results demonstrate the uncertainty and inconsistency handling of our 2-way fuzzy system. In this first example, we also demonstrate the application of shadow set evaluation of inconsistency typifying them according to their characteristics. The system in the second example is chosen to be a flexible-joint robot arm system, which is also modelled by the 2-way fuzzy adaptive system, and the inconsistency handling is also evaluated for this new plant.

We design fuzzy controllers for the flexible-joint robot arm system in the last three examples using our three different techniques developed in Chapter 4. That is to say, in Example 3, we design an optimal fuzzy controller using the equivalent linearization of approximating sequences. The design of a 2-way fuzzy controller using describing function method is demonstrated in the fourth example, and the last example illustrates the use of Lie algebra based stability results for the stable controller design, where the robotic system dynamics is put into a T-S fuzzy representation.

5.1 Nonlinear Function Identification

In this first application example, we demonstrate the modelling of an unknown nonlinear system using our 2-way fuzzy adaptive system, and illustrate the uncertainty and inconsistency reduction through training. We also evaluate the inconsistencies of our fuzzy system by extracting and classifying their characteristics using shadowed set theory.

The example system is selected from the problems in [14] for comparative analysis of the reduction rates of inconsistency and uncertainty of the 2-way fuzzy adaptive system compared to the classical 1-way fuzzy adaptive systems that cannot handle disbelief uncertainties and inconsistencies. In this analysis, we demonstrate how our 2-way fuzzy adaptive system slowly approaches its 1-way counterpart as inconsistency and uncertainty are reduced by multiple phases of learning. The application problem we consider here is a system identification problem, where the adaptive system is expected to match an unknown nonlinear function as training progresses toward completing the identification process. The difference equation of the plant to be identified is:

$$y(k+1) = 0.3y(k) + 0.6y(k-1) + g[u(k)] \tag{5.1}$$

where the unknown function g to be identified is given as:

$$g(u) = 0.6\sin(\pi u) + 0.3\sin(3\pi u) + 0.1\sin(5\pi u) \tag{5.2}$$

and the input u is taken to as $u(k) = \sin(2\pi k/250)$, for verification purposes.

In our case, $g[*]$ is identified as $\hat{g}[*]$ by the fuzzy adaptive system described in Chapter 3. The closed form of the 2-way fuzzy adaptive system is as introduced previously:

$$f(x) = \frac{\sum_{l=1}^{M} y_l(\prod_{i=1}^{n} \mu_{il}(x_i))}{\sum_{l=1}^{M}(\prod_{i=1}^{n} \mu_{il}(x_i))} = \frac{a}{b} \tag{5.3}$$

The system parameters are adjusted through training in order to minimize the error between the model output $\hat{g}(u)$ and the desired response $g(u)$ (the desired response is obtained using Equations 5.1 and 5.2). The rule number M is taken to be 40 in the simulations. We use Gaussian membership and nonmembership functions of the form $\mu_{il} = exp(-(\frac{x_i - x_{il}}{\sigma_{il}})^2)$, so the three adjustable parameters for each rule are the consequent y_l, the mean x_{il}, and the variance σ_{il}. Thus, the fuzzy logic system given by Equation 5.3 has 40x3=120 adjustable parameters.

For the initialization of these adjustable parameters, on-line initial parameter-choosing method is used, where the training algorithm has a delayed start and does not trigger for the first M points. The parameters are set as follows: $x_{il}(M) = u_i(l)$ and $y_l(M) = g(\bar{u}(l))$, where $i = 0, 1, , n$, $l = 0, 1, , M$, $\bar{u}(l) = (u_1(l), \ldots, u_n(l))^T$ is the input to both original system and identified model, and $g(\bar{u}(l))$ is the desired output of the fuzzy logic system learning to identify $\hat{g}[*]$. σ_{il}'s are initialized using: $\sigma_{il}(M) = [max(u_i(l) : l = 1, 2, \ldots, M) - min(u_i(l) : l = 1, 2, \ldots, M)]/2M$, training starting at iteration time $M + 1$.

The update equations used in the training phase for the three adjustable parameters, y_l, x_{il}, and σ_{il}, are already given in Chapter 3 Section 3.2 as $p(k+1) = p(k) + \Delta p(k) = p(k) - \gamma \frac{\partial e}{\partial p}|_k$. In the example system, there is only one input namely u, so $n = 1$ in Equation 5.3. Using the error equation $e = \frac{1}{2}[f(x) - d]^2$ (d: desired output) and the update equations, and taking the partial derivatives $\frac{\partial e}{\partial y_l}$, $\frac{\partial e}{\partial x_{il}}$ and $\frac{\partial e}{\partial \sigma_{il}}$ the following update equations are obtained.

$$\begin{aligned}
y_l(k+1) &= y_l(k) - \gamma \tfrac{f-d}{b} z_l \\
x_{il}(k+1) &= x_{il}(k) - \gamma \tfrac{f-d}{b}(y_l - f) z_l \tfrac{2(x_i - x_{il}(k))}{\sigma_{il}^2} \\
\sigma_{il}(k+1) &= \sigma_{il}(k) - \gamma \tfrac{f-d}{b}(y_l - f) z_l \tfrac{2(x_i - x_{il}(k))^2}{\sigma_{il}^3}
\end{aligned} \tag{5.4}$$

where $z_l = \prod_{i=1}^{n} \mu_{il}(x_i)$, $l = 0, 1, \ldots, M$ and $k = 0, 1, 2, \ldots$.

In the first phase of the training algorithm, there are two levels of training as mentioned in Chapter 3 Section 3.2. The first level of training is carried out for the adjustment of the parameters of the membership functions. So in the simulations the membership functions are used. The second level is for the adjustment of the parameters of the nonmembership functions. In order to achieve this, the complements of the nonmembership functions (1-nonmembership's) are used in the update equations, and the $\mu_{il}(x_i)$ terms in Equations 5.4 become $1 - \mu_{il}(x_i)$.

The simulation results for three adaptive fuzzy systems, namely the 1-way

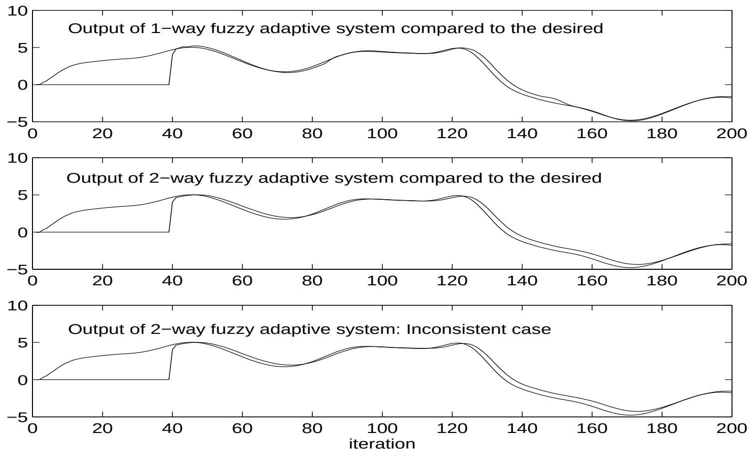


Figure 5.1: Simulation Results

fuzzy adaptive system, where the 2-way fuzzy adaptive system converges with fully reduced uncertainty, the 2-way fuzzy adaptive system without any inconsistency (or fully reduced inconsistency) and the 2-way fuzzy adaptive system incorporating inconsistency and uncertainty are given in Fig.5.1 together with the corresponding error curves in Fig.5.2. The training is carried out for 200 iteration steps. Since the example system is not of high nonlinearity, the 1-way fuzzy adaptive system trained with memberships (supports) (1^{st} graph) is able to track the desired response very closely. The introduction of nonmemberships (2^{nd} graph) that are not optimized but only satisfying consistency generates a slightly larger tracking error in ascending portions. This shows that learning attempting to reduce uncertainty is still slower than the error dynamics in the system, leading to a slightly larger error than the 1-way fuzzy adaptive system. The third tracking graph attests to the extreme success of our system in reducing inconsistencies. The 2-way inconsistent system is turned by training into a 100% 2-way fuzzy adaptive system in terms of performance. The inconsistency is thus fully reduced.

We also evaluate the inconsistency handling of the system by forming the shadowed set patterns of the fuzzy system after the first phase of training. Since the initial membership and nonmembership functions of phase 1 of training in the 2-way adaptive approach are taken to be fully overlapping to rep-

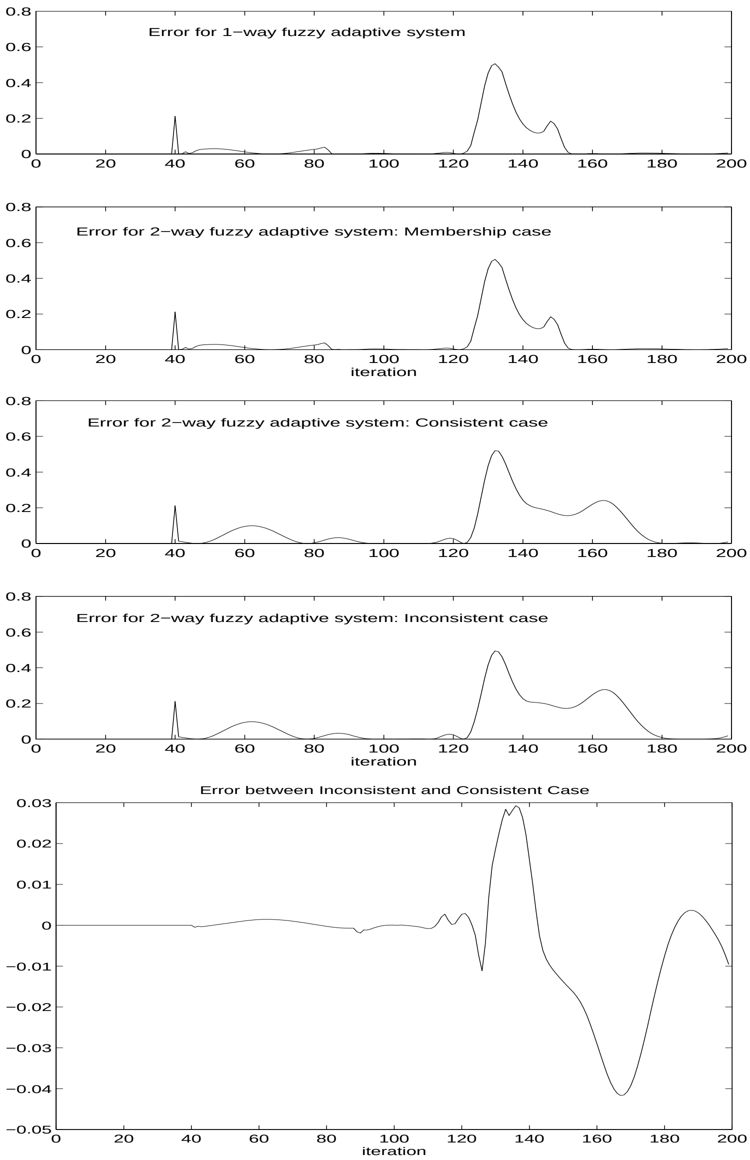


Figure 5.2: Error curves

resent full inconsistency of type 4, their induced shadowed sets cover the full range of support for both membership and nonmembership functions and represent total indecisiveness on this entire region. After training, we see that the shadowed set pattern of the system changes as the system adjusts the parameters of the membership and nonmembership functions.

Inconsistency types that we consider in this thesis work are correlated with how the membership and nonmembership functions are assigned. In order to unveil this correlation, we now generate the correspondence between the type of

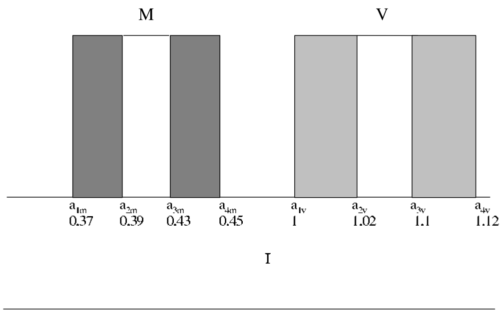


Figure 5.3: Shadowed Set Pattern for Rule 13

inconsistency obtained at the output of phase 1 of the 2-way adaptive system that tries to reduce inconsistency, and a global index of fuzziness obtained from membership and nonmembership functions altogether. The measure of fuzziness that we adopt and further modify for this purpose is $f = \sigma/m$ where σ is the variance and m is the mean of the related Gaussian function. In the case of two fuzzy sets assigned as it is in our 2-way fuzzy adaptive system, namely membership and nonmembership functions, the sum of each individual fuzzy measure is used as a combinational measure c representing the global index of fuzziness for both assignments.

When the shadowed set patterns for each rule in the fuzzy rule base of our application are examined, it is seen that there exists a correlation between inconsistency type and c. We find that values $0 < c < 0.5$ guarantee the characterization of type 1 inconsistency and as c tends to 1 within the interval $0.5 < c < 1$ the induced shadowed sets get closer to each other and begin overlapping generating a type 2 inconsistency. As $c \gg 1$, type 3 inconsistency dominates that can even lead to type 4 inconsistency for large values of c. We demonstrate these results by three examples, where fully inconsistent membership and nonmembership functions reduced to type 1, type 2 or type 3 inconsistencies depending on the value of c.

For Rule 13 of our example $c = 0.1034$ and the corresponding pattern is given in Fig.5.3. It can be seen that this figure matches with the pattern demonstrated under type 1 inconsistency in Section 3.3.2 of Chapter 3.

Rule 36 has a value of c=0.7758 and the pattern shown in Fig.5.4. This

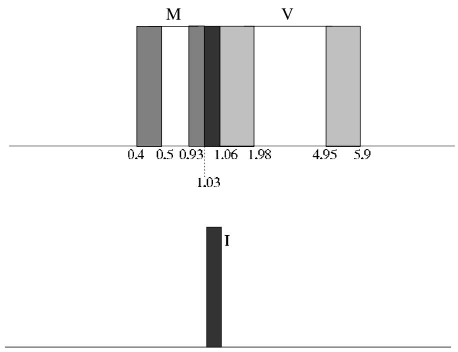


Figure 5.4: Shadowed Set Pattern for Rule 36

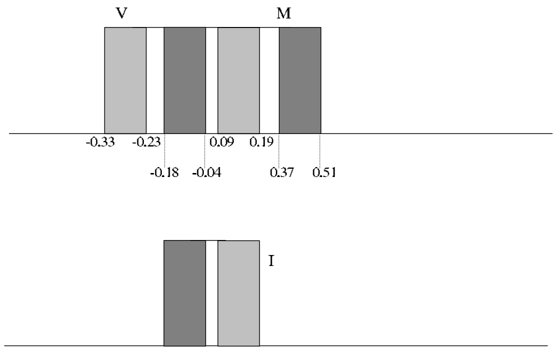


Figure 5.5: Shadowed Set Pattern for Rule 10

case is classified under type 2 inconsistency. We see that c is close to 1 and there happens to be a thin intersection region of the induced shadowed sets showing the emergence of a thin region of indecisiveness.

Rule 10 has a pattern that matches with type 3 inconsistency with a c value of $3.9485 \gg 1$ and can be displayed in the Fig.5.5.

As a result of the evaluation of inconsistency we deduce that our 2-way adaptive system is able to reduce, through the two levels of training of phase 1 of the 2-way adaptive control system, an initial full inconsistency of type 4 to either inconsistency of type 2 or of type 1 (inconsistency is fully eliminated), according to the global index of fuzziness c. The only case where phase 1 training fail in fully reducing inconsistency by only reducing a type 4 inconsistency to type 3 inconsistency is for rule 10, where $c = 3.9485 \gg 1$. This is due to the wide spread of the corresponding membership and nonmembership functions. Moreover, in the design of a 2-way fuzzy adaptive controller, the global index of fuzziness c can be used in the initial assignment of consistent membership and nonmembership functions.

5.2 Identification of the Model of Flexible Robot Arm using 2-Way Fuzzy Adaptive System

In this example application, we apply the 2-way fuzzy adaptive system for the identification of a flexible robot arm to be used in its model based control.

5.2.1 System Model

The flexible-joint robot arm system used in this paper, is shown in Fig.5.6.

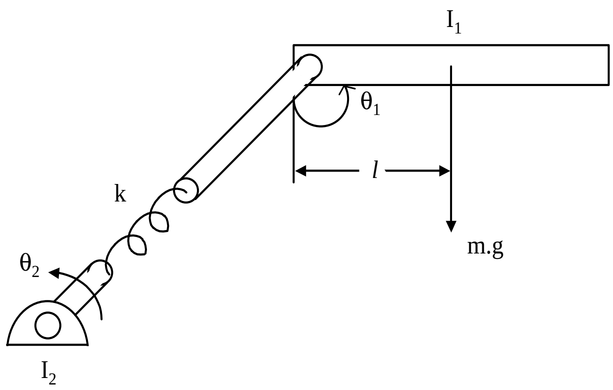


Figure 5.6: Flexible-joint robot arm system

The model is described by the following equations [34]:

$$I_1\ddot{\theta_1} + mglsin(\theta_1) + k(\theta_1 - \theta_2) = 0 \tag{5.5}$$

$$I_2\ddot{\theta_2} + k(\theta_2 - \theta_1) = u \tag{5.6}$$

In the equations, u is the torque input, I_1 the link inertia, I_2 the motor inertia, m the mass, g the gravity constant, l the link length, k the stiffness, θ_1 joint 1 angular position, and θ_2 joint 2 angular position.

We adjust the system parameters with the 2-way adaptive system through its training in order to minimize the error between the model output and the

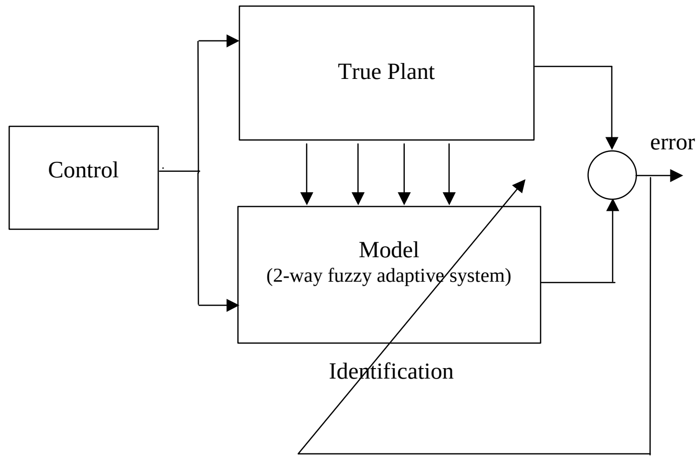
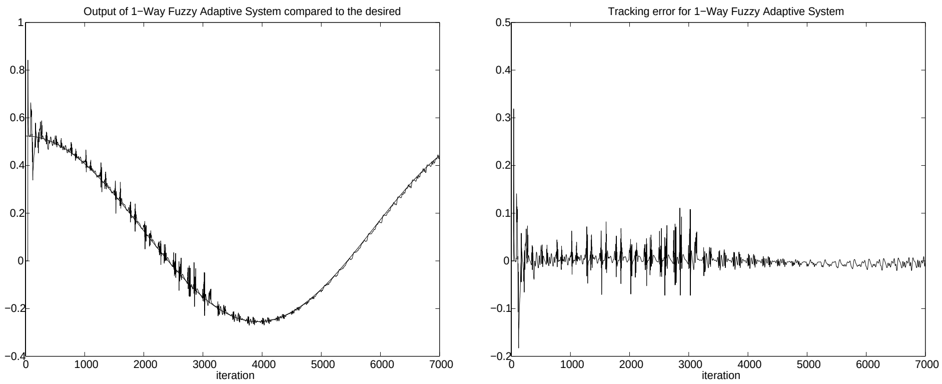


Figure 5.7: Identification Process

desired response. The desired response is obtained using Equations 5.5 and 5.6. The flexible-joint robot arm system is simulated by applying Euler's method to the above equations for the given control input u. In parallel, the 2-way fuzzy adaptive system performs an identification using the system output as the desired signal d (Fig.5.7).

The rule number M in Equation 5.3 is taken to be 40 in our simulations, which is further varied to $M = 10$ and $M = 25$. The three adjustable parameters for each rule are y_l, x_{il}, and σ_{il}. So the fuzzy logic system given by the Equation 5.3 has again $40 \times 3 = 120$ adjustable parameters initially for the 40 rules. For the initialization of the adjustable parameters, on-line initial parameter-choosing method introduced in Example 1 is used because this method is proven to be effective for faster convergence [14]. The update equations used in the training phase for the three adjustable parameters, y_l, x_{il}, and σ_{il}, are the same as the previous example and are given in Equations 5.4.

The first phase of the training has in itself two levels of training as mentioned before: the first level of training is carried out for the adjustment of the parameters of the membership functions. In the second level, the adjustment of the parameters of the nonmembership functions are achieved using the complements of the nonmembership functions (1-nonmembership's) in the update



(a) (b)

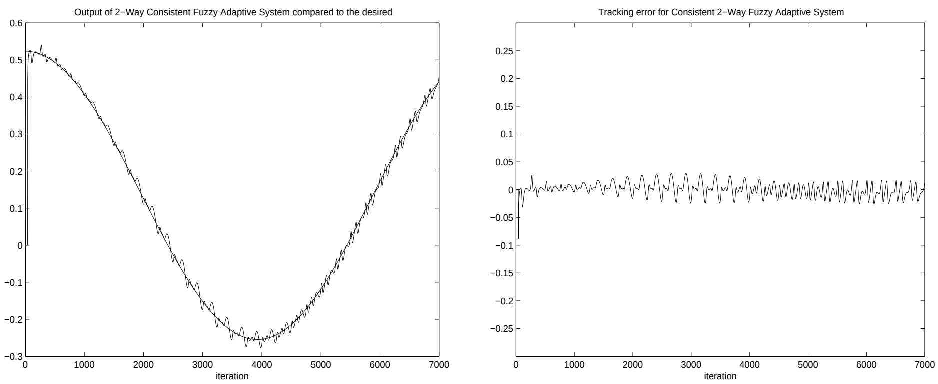
Figure 5.8: Output of 1-way fuzzy adaptive system and Tracking Error for M=40

equations.

5.2.2 Elimination of Uncertainty and Reduction of Inconsistencies

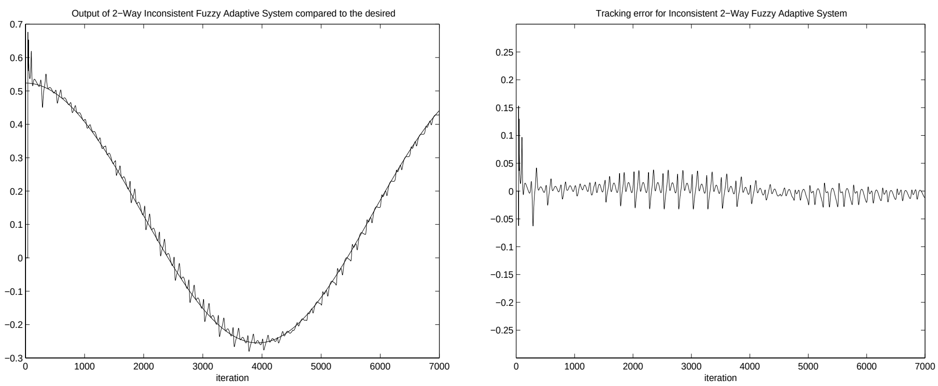
The system parameters are taken to be $m = 0.0234kg$, $g = 9.81m/s^2$, $l = 0.2m$, $k = 0.015N.m/rad$, $bs = 0.007N.m.s/rad$, $I1 = m * l^2$, $I2 = 6.24 * 10^{-2}kg.m^2$ in the simulations and the input u is chosen to be $u(k) = \sin(2\pi k/250)$. The simulation results for three adaptive fuzzy systems, namely 1-way fuzzy adaptive system, 2-way fuzzy adaptive system without any inconsistency and 2-way fuzzy adaptive system incorporating inconsistency in the assignment of membership and nonmembership functions are given in Figures 5.8, 5.9, and 5.10 below. The solid smooth line in the figures is the desired output obtained from simulations using the Euler's method. The error curves for each case are also given together with the output curves (see Figures b in 5.8, 5.9 and 5.10).

The performance of the system after the first phase of training is given in Fig.5.10. As expected the system reduces inconsistency after training based on the tracking performance of the system in Fig.5.10 when compared to that in Fig.5.9. Both performances are close to each other within a tolerance of $0.038rad$. The identification and thus tracking control activity exhibited in Fig.5.10 is of lower frequency (30Hz) than that in Fig.5.9 (36Hz). This is an



(a) (b)

Figure 5.9: Output of 2-way fuzzy adaptive system with consistent membership assignment and Tracking Error for M=40



(a) (b)

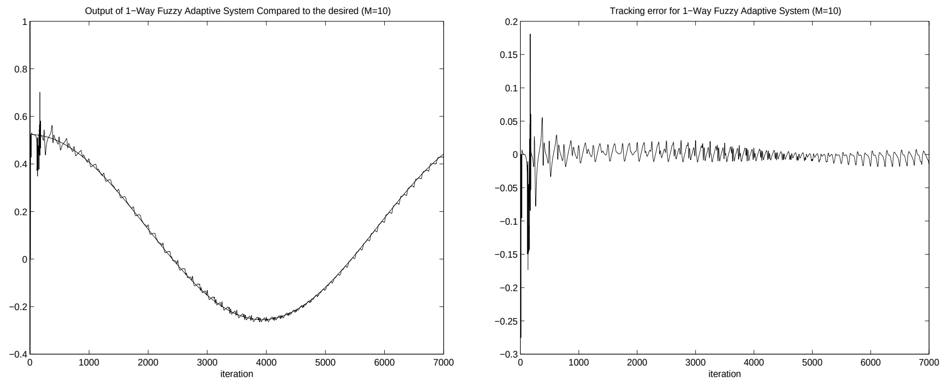
Figure 5.10: Output of 2-way fuzzy adaptive system with inconsistent membership assignment and Tracking Error for M=40

extremely desired property for tracking control with flexible structures since high frequency control activity may easily trigger the unmodelled modes in the system. We then conclude from this reduction of frequency that, the system is able to reduce inconsistencies present in the system since frequency is inconsistency driven. The resultant system is further trained to reduce the uncertainty due to the interval width introduced by the independent assignment of membership and nonmembership values. The system performance after this second phase of training is given in Fig.5.8. The system obtained after the two phases of training is close to a 1-way fuzzy adaptive system despite the amount of uncertainty and inconsistency it had to overcome, and its performance is the best among the others.

Our proposed 2-way fuzzy adaptive system thus cannot only compensate for inconsistencies through its learning capability, but also considerably reduces the intuitionistic uncertainty interval once inconsistencies have all been minimized leading to a performance close to that of a 1-way fuzzy adaptive system. Now let us focus on the inconsistency handling process of our architecture and proceed with its evaluation.

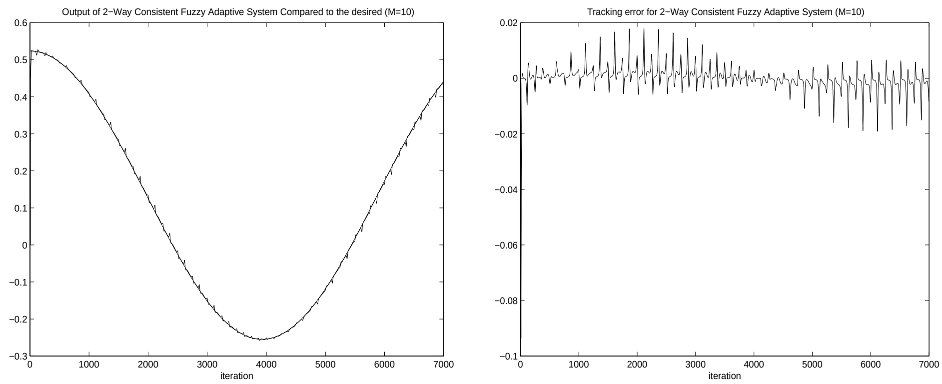
We have also simulated our system for $M = 10$ and $M = 25$ number of rules. The results are given in Figures 5.11, 5.12 and 5.13 for the case where $M = 10$, and in Figures 5.14, 5.15 and 5.16 for $M = 25$. In the case where $M = 25$, it is seen that 1-way fuzzy adaptive system does not give the best result and our proposed 2-way system outperforms the 1-way fuzzy adaptive system with a superior tracking performance. The reason for such a behavior is the selection of the rule number, which is not optimal in every case.

Fig.5.17 shows one typical case of how the system has generated consistent membership and nonmembership functions from totally inconsistent membership and nonmembership assignments. In the first graph, the initial assignments of the functions are shown. They are both the same Gaussian functions with mean 0.1253 and variance 0.012, thus fully overlapping (type 4 inconsistency). After the first phase of training, the functions are separated (type 1



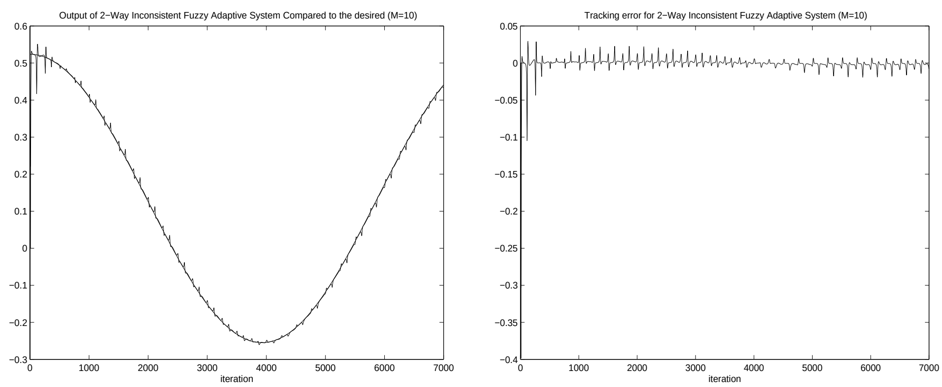
(a) (b)

Figure 5.11: Output of 1-way fuzzy adaptive system and Tracking Error for M=10



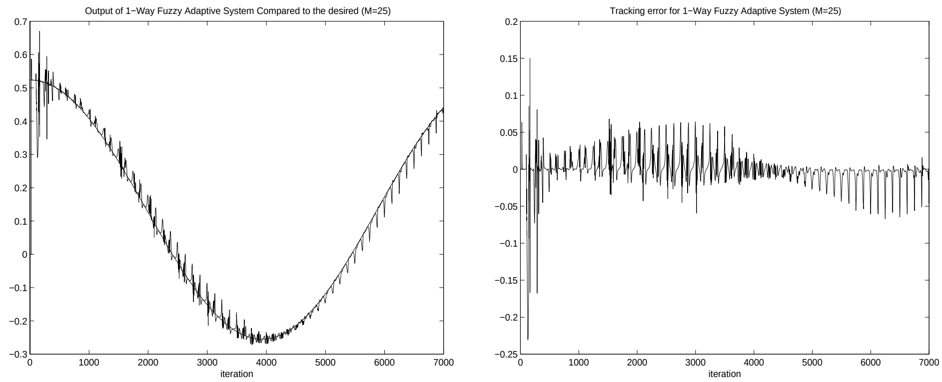
(a) (b)

Figure 5.12: Output of 2-way fuzzy adaptive system with consistent membership assignment and Tracking Error for M=10



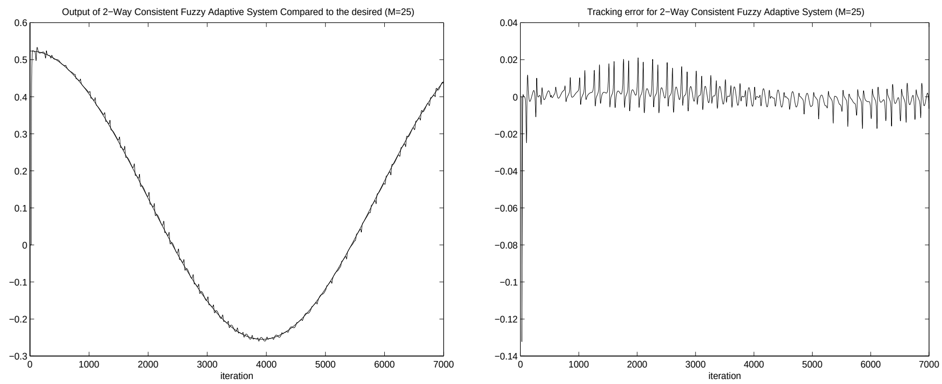
(a) (b)

Figure 5.13: Output of 2-way fuzzy adaptive system with inconsistent membership assignment and Tracking Error for M=10



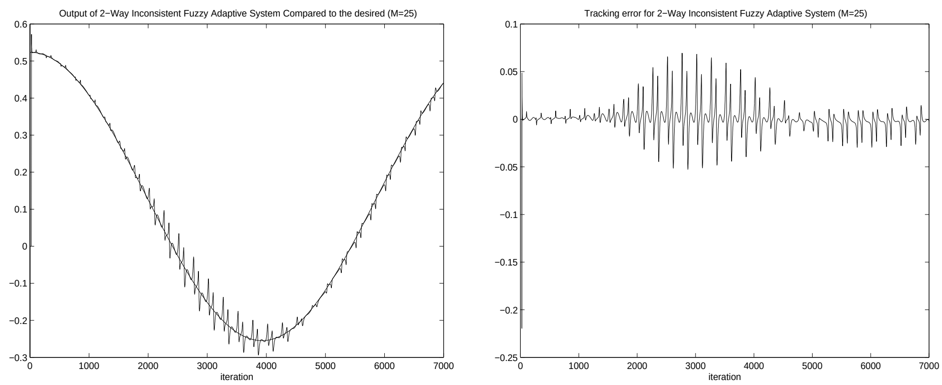
(a) (b)

Figure 5.14: Output of 1-way fuzzy adaptive system and Tracking Error for M=25



(a) (b)

Figure 5.15: Output of 2-way fuzzy adaptive system with consistent membership assignment and Tracking Error for M=25



(a) (b)

Figure 5.16: Output of 2-way fuzzy adaptive system with inconsistent membership assignment and Tracking Error for M=25

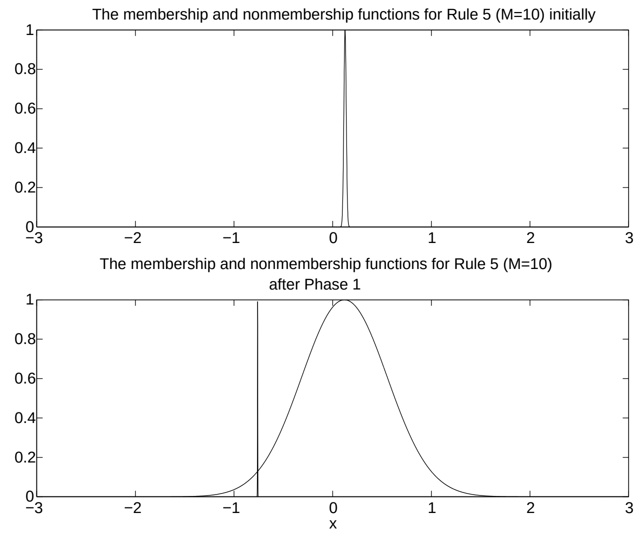


Figure 5.17: The membership and nonmembership functions: Initially and after phase 1 training

inconsistency= fully consistent) and the final values for the means and the variances become: for the membership $mean = -0.7608$ and $variance = 0.0015$, and for nonmembership $mean = 0.1191$ and $variance = 0.6146$.

5.2.3 Shadowed Set Evaluation Results for Inconsistency Minimization

We conduct here the performance evaluation for inconsistency typification in the model based control of a flexible robot arm and form after the first phase of training the shadowed set patterns of the membership/nonmemberships in the antecedents of each rule. The initial assignment of the membership and nonmembership functions are done in a way that their corresponding shadowed sets fully overlap, giving rise to a type 4 inconsistency. It is expected that if the fuzziness index c for each rule falls within the corresponding ranges of the clusters defined in the previous example (example system in Section 5.1), the shadowed set pattern at the output of the training may contain type 1 or type 2 inconsistency guaranteeing its reduction down from type 4. In our example with 40 rules ($M = 40$), it is seen that, at the output of phase 1 training, when the combinational measure c is very high for some rules such as for rule

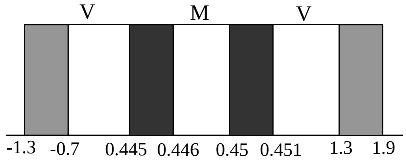


Figure 5.18: Shadowed Set Pattern for Rule 19

1: $c = 52.39$, for rule 3: $c = 119.9$, for rule 19 $c = 962.15$. With $c >> 1$, the corresponding shadowed set patterns for such rules are characterized as type 4 (Fig.5.18); thus type 4 inconsistency has lead to type 4 inconsistency after training. However, further analysis shows that within this type, inconsistency has been reduced, but not in an amount enough to be able to cause a switch in inconsistency type. We will consider here typical result samples from our application.

Let us note that our rules are of the form $R : (\mu_{F_1^l \times \ldots \times F_n^l}(x), y_l)$ where $y_l = max(\mu_{G^l})$ and implication from antecedent to consequent is taken as the product operation in the computation of $f(x)$.

If we take as an example rule 19 at the output of a single phase training, represented by $R_{19} : (M, V, y_l = 0.3860)$ with $c = 962.15$, (threshold for shadowed set M is $\alpha = 0.3354$ and for V, $\alpha = 0.3393$) the shadow set pattern is in the form of Fig.5.18.

This pattern is classified as type 4 according to the value of c even though type 4 in fact corresponds to fully overlapping shadowed sets. We see that after training the fully overlapping shadowed sets given initially have been separated, leading to a reduction of inconsistency, which, still, was not enough to change its type. So inconsistency reduction has been done within the cluster corresponding to type 4 without being able to be effective enough to decrease its type number. This is due to the fuzziness index, which assumes a very high value. The reason for not being able to reduce the <u>type</u> of uncertainty is the high variance of the membership and nonmembership functions that is also reflected in the high valued fuzziness index.

In the case of rule 40 $R_{40} : (M, V, y_l = 0.3507)$, $c = 5.44$ (threshold for shadowed set M is $\alpha = 0.3354$ and for V $\alpha = 0.3372$) which is not a value

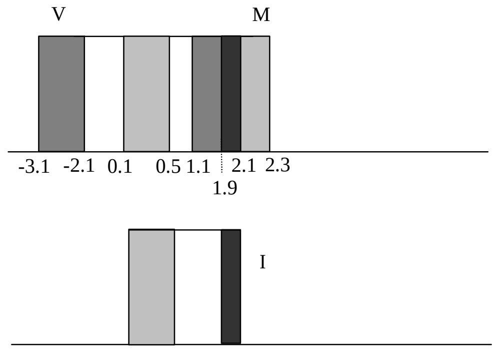


Figure 5.19: Shadowed Set Pattern for Rule 40

as large as in the other rules. We observe here a reduction in the type of inconsistency changing it from type 4 to type 3. The corresponding output pattern is given in Fig.5.19.

We see that as the combinational index c is decreasing and getting closer to one, the inconsistency type decrease is powerful enough to reduce type numbers, thus showing that inconsistency is truly substantially decreasing.

In the simulations for $M = 10$ and $M = 25$, it is seen that the combinational index c is much greater than 1 leading to similar results as discussed for the case with 40 rules ($M = 40$). Some example results are summarized in Table 5.1, where α represents the threshold value used in forming the shadowed set patterns of the membership and nonmembership functions.

Table 5.1: Examples for Shadowed Evaluation Results

M	**Rule No.**	c	**Type of Incon.**	**Threshold α**
10	Rule 3	9.172	4	For M: 0.3354 For V: 0.5551
10	Rule 4	11.01	4	For M: 0.3354 For V: 0.5411
25	Rule 2	54.97	4	For M: 0.3354 For V: 0.6423
25	Rule 18	17.39	4	For M: 0.3354 For V: 0.4501

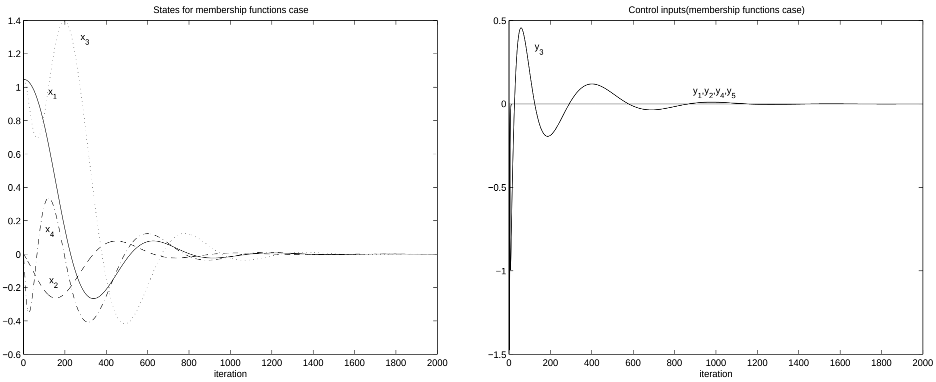
When triggered by fully inconsistent belief (membership) and disbelief (nonmembership) functions, the 2-way fuzzy adaptive system in its first training phase, always reduces inconsistency. But when variances of the functions are high, meaning high dispersion in knowledge, leading to high c values, inconsistency is reduced within one type cluster without being substantial enough to also reduce its type number. For more focused knowledge leading to low variance in membership and nonmembership functions, the values of c are in the interval of $[0.5, 1]$ or tends toward 1. The 2-way fuzzy adaptive system is then able to reduce full inconsistency of type 4 to type 2 or to type 1 (which is the fully consistent case).

This fuzzy measure-based evaluation taking into account membership/ nonmembership variances provides a valuable empirically found and verified prediction measure for the expected efficiency of the 2-way fuzzy adaptive system in reducing inconsistency.

5.3 2-Way Fuzzy Controller Design Using Equivalent Linearization Method

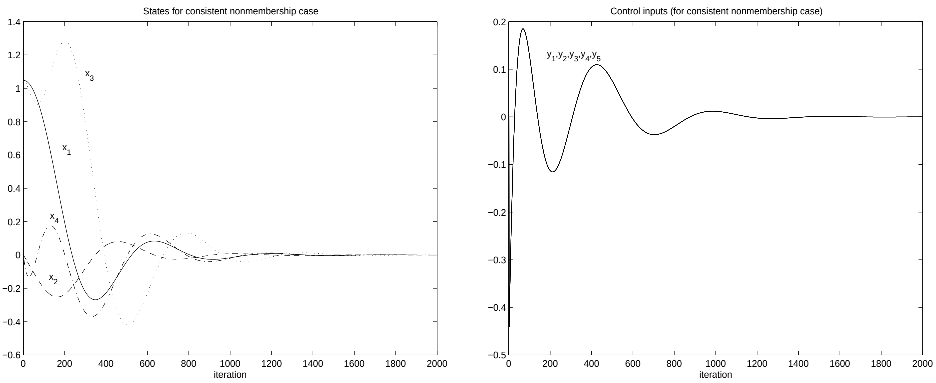
In this section, we deal with the stable design of a 2-way fuzzy adaptive system acting as a controller for the flexible-joint robot arm. The 2-way fuzzy controller design is turned into an optimal controller design problem using approximating sequences technique, and it is applied to the control of the flexible-joint robot arm system. First we need to represent the system with controller as a pseudo-linear system. The states equations of the controlled system turn into: $\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ (-\frac{mgl\sin x_1}{I_1 x_1} - \frac{k}{I_1}) & 0 & \frac{k}{I_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{I_2} & 0 & -\frac{k}{I_2} & 0 \end{bmatrix}}_{A(x)} x +$

$\underbrace{\begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \frac{1}{I_2}\Omega_1(x,m,\sigma) & \dots & \dots & \frac{1}{I_2}\Omega_M(x,m,\sigma) \end{bmatrix}}_{B(x,m,\sigma)} y$ where the membership and



(a) (b)

Figure 5.20: (a) States and (b) Control Inputs for 1-Way Fuzzy System

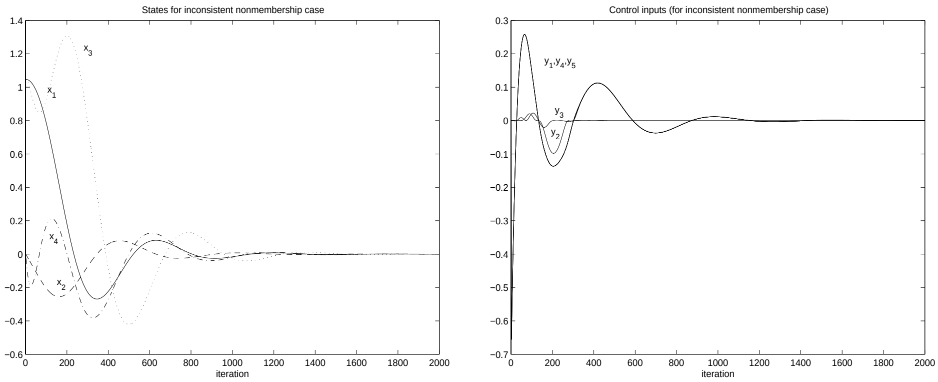


(a) (b)

Figure 5.21: (a) States and (b) Control Inputs for 2-Way Fuzzy System with Consistent Membership Assignment

nonmembership functions are chosen to be in the most general case Gaussian, and (m, σ) are the means and the variances of these functions. We solve the optimal control problem using the theory on approximating sequences introduced in Chapter 2. The simulation results are given in Figures 5.20, 5.21 and 5.22.

When we examine the results, the states in the case of inconsistent nonmembership functions are close to the states in the consistent case, so we conclude that the system has the ability to handle inconsistency. The best results are obtained in the case where membership functions are used.



(a) (b)

Figure 5.22: (a) States and (b) Control Inputs for 2-Way Fuzzy System with Inconsistent Membership Assignment

5.4 2-Way Fuzzy Controller Design Using Describing Function Method

In this example, we design a fuzzy controller for the flexible-joint robot arm system introduced in the second example application. The state equations of the system are again:

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\tfrac{mgl}{I_1}\sin(x_1) + \tfrac{k}{I_1}(x_3 - x_1) \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= \tfrac{k}{I_2}(x_1 - x_3) + \tfrac{u}{I_2}
\end{aligned} \tag{5.7}$$

where $x_1 = \theta_1$, $x_2 = \dot{\theta}_1$, $x_3 = \theta_2$ and $x_4 = \dot{\theta}_2$.

In order to be able to apply the stability analysis derived in Chapter 4 Section 4.2, we need a linearized form of the flexible-joint robot arm system. We use the input-state linearization of the system introduced in [83]. The transformed states are as follows:

$$\begin{aligned}
z_1 &= x_1 \\
z_2 &= x_2 \\
z_3 &= -\tfrac{mgl}{I_1}\sin x_1 - \tfrac{k}{I_1}(x_1 - x_3) \\
z_4 &= -\tfrac{mgl}{I_1}x_2\cos x_1 - \tfrac{k}{I_1}(x_2 - x_4)
\end{aligned} \tag{5.8}$$

The corresponding input transformation is:

$$u = \frac{I_1 I_2}{k}(v - a(x)) \tag{5.9}$$

where $a(x) = \frac{mgl}{I_1}\sin x_1(x_2^2 + \frac{mgl}{I_1}\cos x_1 + \frac{k}{I_1}) + \frac{k}{I_1}(x_1 - x_3)(\frac{k}{I_1} + \frac{k}{I_2} + \frac{mgl}{I_1}\cos x_1)$. Then, the linear state equation are found to be:

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ \dot{z}_3 &= z_4 \\ \dot{z}_4 &= v \end{aligned} \tag{5.10}$$

The transfer function of this linearized system is $G(s) = \dfrac{1}{s^4}$. The degree of this system is $n = 4$, so there are four inputs to our fuzzy controller. In order to be able to find an analytical expression for the describing function of our fuzzy controller, we use additivity property reviewed in Chapter 4 Section 4.2.1. The system parameters y_{ijkl} of the fuzzy controller $f(x) = \sum_i \sum_j \sum_k \sum_l \Omega_{ijkl} y_{ijkl}$ are assigned such that the fuzzy system is additively decomposable, so the output of the fuzzy controller is:

$$\begin{aligned} &v = f(z) = f(z_1, z_2, z_3, z_4) = f(z_1, 0, 0, 0) + f(0, z_2, 0, 0) + f(0, 0, z_3, 0) \\ &+f(0, 0, 0, z_4) = f_1(z_1) + f_2(z_2) + f_3(z_3) + f_4(z_4) \end{aligned} \tag{5.11}$$

If the describing functions of $f(z)$ are $\{N(A), \bar{N}(A)\}$, then under the light of Equation 5.11, the describing functions become $N(A) = N_1(A) + N_2(A)s + N_3(A)s^2 + N_4(A)s^3$, and $\bar{N}(A) = \bar{N}_1(A) + \bar{N}_2(A)s + \bar{N}_3(A)s^2 + \bar{N}_4(A)s^3$, where $\{N_1(A), \bar{N}_1(A)\}$ are the describing functions of the system with input z_1, calculated using Equations 4.17 and 4.26. In the expression of $N(A)$, $N_i(A)$ corresponds to the local describing function with respect to $z_i = s^i z_1$, that is the reason why the terms $N_i(A)$ are multiplied by s^i.

The characteristic equation of the closed loop system is calculated as:

$$C_1(s) = s^4 + N_4 s^3 + N_3 s^2 + N_2 s + N_1 \tag{5.12}$$

for the fuzzy system with membership functions and

$$C_2(s) = s^4 + \bar{N}_4 s^3 + \bar{N}_3 s^2 + \bar{N}_2 s + \bar{N}_1 \tag{5.13}$$

for the fuzzy system with nonmembership functions. For notational simplicity, we write N_i instead of $N_i(A)$ where $i = 1, \ldots, 4$.

The ranges for N_i's are taken as: $N_1(A) \in [a_1, b_1]$, $N_2(A) \in [a_2, b_2]$, $N_3(A) \in [a_3, b_3]$, and $N_4(A) \in [a_4, b_4]$, and for $\bar{N}_i$'s: $\bar{N}_1(A) \in [c_1, d_1]$, $\bar{N}_2(A) \in [c_2, d_2]$, $\bar{N}_3(A) \in [c_3, d_3]$, and $\bar{N}_4(A) \in [c_4, d_4]$. Let us apply the upper bounds and the lower bounds of each $N_i(A)$ within the expressions of C_1 and C_2. For stability, we have to check the Kharitonov polynomials of C_1 and C_2, and since our system is of degree 4, we only need to check K_3 and K_4 of Equation 4.31 [82]. For $C_1(s)$:

$$\begin{aligned} K_3(s) &= b_1 + a_2 s + a_3 s^2 + b_4 s^3 + s^4 \\ K_4(s) &= b_1 + b_2 s + a_3 s^2 + a_4 s^3 + s^4 \end{aligned} \tag{5.14}$$

and for $C_2(s)$:

$$\begin{aligned} K_3(s) &= d_1 + c_2 s + c_3 s^2 + d_4 s^3 + s^4 \\ K_4(s) &= d_1 + d_2 s + c_3 s^2 + c_4 s^3 + s^4 \end{aligned} \tag{5.15}$$

Since there are too many parameters to be adjusted, we fix the ranges for the first three describing functions and we only solve for the range of $N_4(A)$ and $\bar{N}_4(A)$ such that the Kharitonov's theorem is satisfied, i.e. we solve for a_4, b_4, c_4 and d_4.

We take five rules for each system $f_1(z_1)$, $f_2(z_2)$, $f_3(z_3)$ and $f_4(z_4)$, and the parameters for the first three systems are given in Table 5.2, where ϕ's represent the centers.

For the parameters in Table 5.2, the corresponding ranges are fixed for N_i's: $N_1 \in [0.6366, 1.3831]$, $N_2 \in [3.6, 5.0930]$, and $N_3 \in [8.4506, 15.9155]$, and for $\bar{N}_i$'s: $\bar{N}_1 \in [0.1739, 1.7608]$, $\bar{N}_2 \in [1.3916, 4.3835]$ and $\bar{N}_3 \in [4.3487, 10.0923]$.

We use these ranges in Equations 5.14 and 5.15 to solve for the ranges of N_4 and $\bar{N}_4$, which are found to be $N_4 \in (0.6148, 21.5611)$ and $\bar{N}_4 \in (1.1249, 3.0798)$. In order to have a stable controller, we need to assign the parameters for $f_4(z_4)$ such that its describing functions $\{N_4, \bar{N}_4\}$ fall in the calculated ranges for stability.

We take the same ϕ_i's for f_4 as in the other f_i's. Then, we find the range of y_i's so that the controller is stable. The contour plots for minimum and

Table 5.2: Fuzzy Controller Parameters

For f_1	$\phi_{-2} = -\pi$ $\phi_{-1} = -0.5\pi$ $\phi_0 = 0$ $\phi_1 = 0.5\pi$ $\phi_2 = \pi$	$y_{-2} = -5$ $y_{-1} = -1$ $y_0 = 0$ $y_1 = 1$ $y_2 = 5$
For f_2	$\phi_{-2} = -\pi$ $\phi_{-1} = -0.5\pi$ $\phi_0 = 0$ $\phi_1 = 0.5\pi$ $\phi_2 = \pi$	$y_{-2} = -10$ $y_{-1} = -8$ $y_0 = 0$ $y_1 = 8$ $y_2 = 10$
For f_3	$\phi_{-2} = -\pi$ $\phi_{-1} = -0.5\pi$ $\phi_0 = 0$ $\phi_1 = 0.5\pi$ $\phi_2 = \pi$	$y_{-2} = -20$ $y_{-1} = -25$ $y_0 = 0$ $y_1 = 25$ $y_2 = 20$

maximum of N_4 are shown in Fig.5.23 and Fig.5.24 respectively. In these plots, the contours represent the value of N_4.

The contour plots for minimum and maximum of $\bar{N}_4$ are given in Fig.5.25 and Fig.5.26 respectively, where the contours represent the value of $\bar{N}_4$.

From the four plots (Fig.5.23, Fig.5.24,Fig.5.25 and Fig.5.26), we see that if we choose $y_1 = y_2 = 1$, the system is unstable, since N_4 and $\bar{N}_4$ are out of the stability range, $N_4 \in (0.6148, 21.5611)$ and $\bar{N}_4 \in (1.1249, 3.0798)$. When we apply the controller with these settings for f_4 and with the settings in Table 5.2, the result is represented in Fig.5.27. Since we have chosen the parameters of the system outside the stable region, the states of the closed loop system become unstable (Fig.5.27).

If we choose $y_1 = 5$ and $y_2 = 8$ such that $N_4 \in (0.6148, 21.5611)$ and $\bar{N}_4 \in (1.1249, 3.0798)$ from Figures 5.23, 5.24, 5.25 and 5.26, wee see that N_4 and $\bar{N}_4$ are both in the stability range and the stable system states together with the corresponding control input are shown in Fig.5.28 and Fig.5.29 respectively. As can be seen from the figures, the closed loop system states are stable, together with a bounded input. This agrees with the theoretical results stating that if the parameters are chosen from the stability range for N_4 and $\bar{N}_4$, the closed

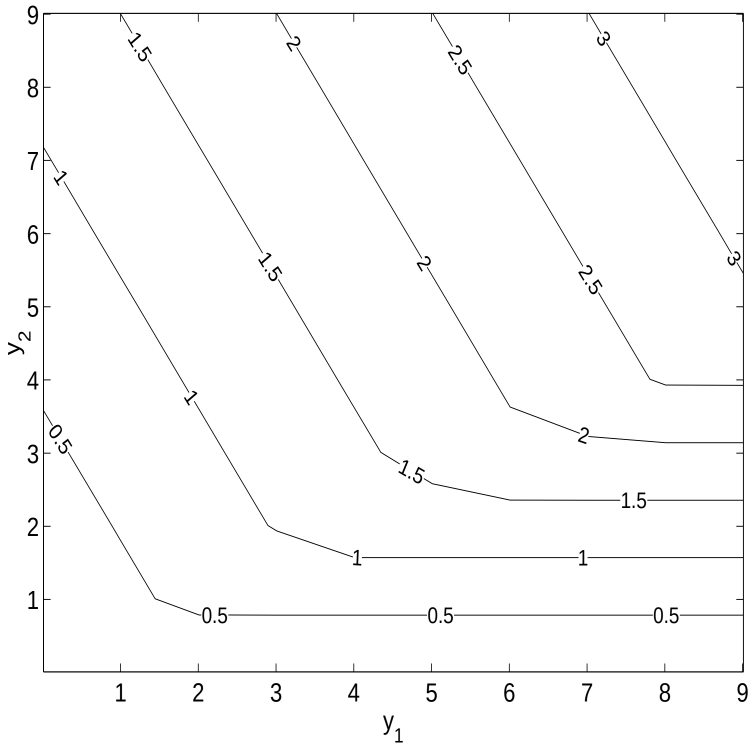


Figure 5.23: Contour Plot for Minimum N_4

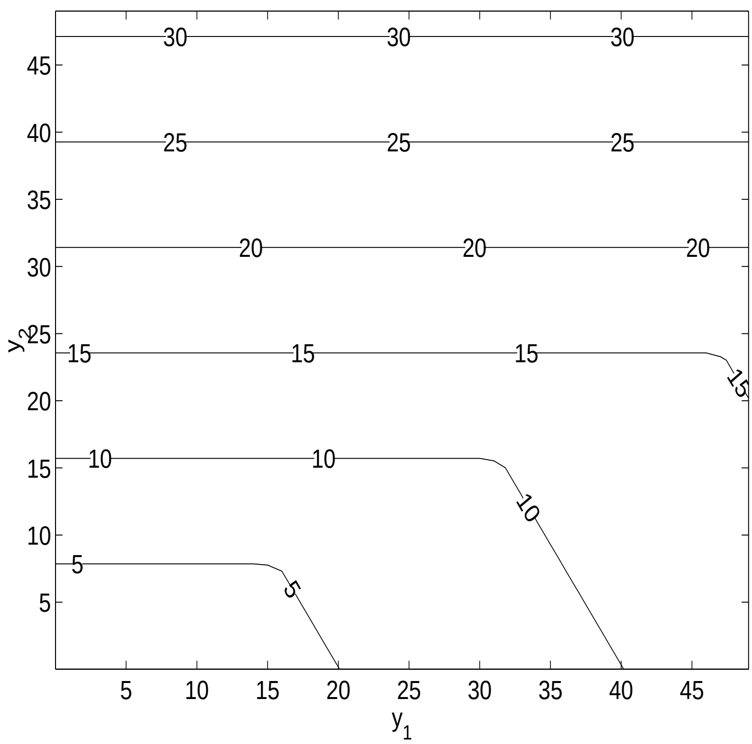


Figure 5.24: Contour Plot for Maximum N_4

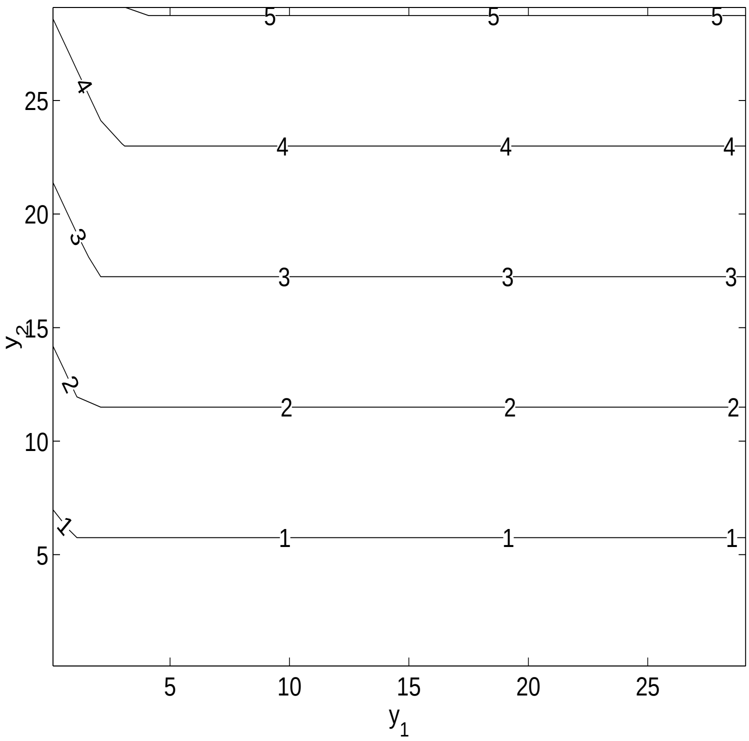


Figure 5.25: Contour Plot for Minimum $\bar{N}_4$

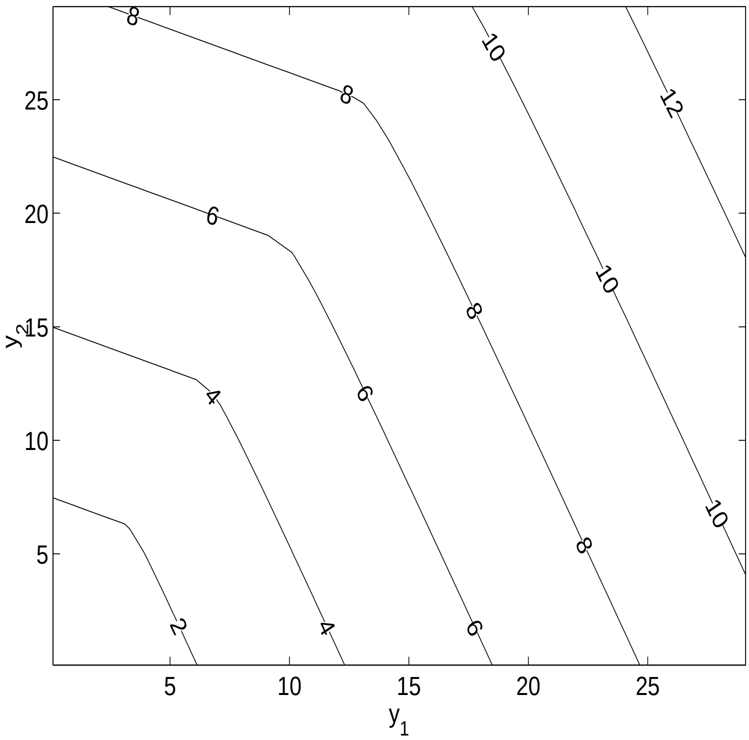


Figure 5.26: Contour Plot for Maximum $\bar{N}_4$

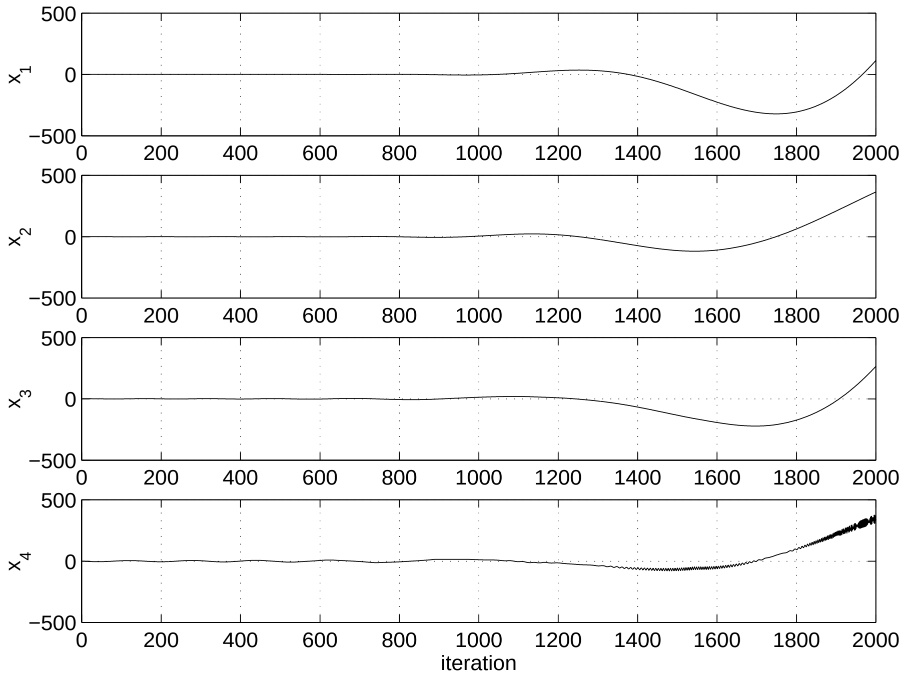


Figure 5.27: Unstable System States

loop system becomes stable.

5.5 Fuzzy Controller Design based on Lie Algebra Theory

In this section, we apply the Lie algebra based analysis to the design of the 2-way fuzzy adaptive system controller for a flexible-joint robot arm system. First, we have to represent the system as a T-S fuzzy system, modelling our system with rules having linear subsystems as consequents (with the rule structure $R^{(l)}$: IF x_1 is F_1^l and ... and x_n is F_n^l, THEN $\dot{x}(t) = A_l x(t) + B_l u(t)$). We then discuss a systematic way on how to find a Levi decomposition of the Lie algebra L_A generated by the A matrices of the linear subsystems of this model, since our example does not have commuting system matrices A_l's (Section 4.3 of Chapter 4). In this decomposition, the matrices in the semisimple part should form a stabilizable pair with the B_l matrices of the system. Finally, we design the controller so that the system is stable.

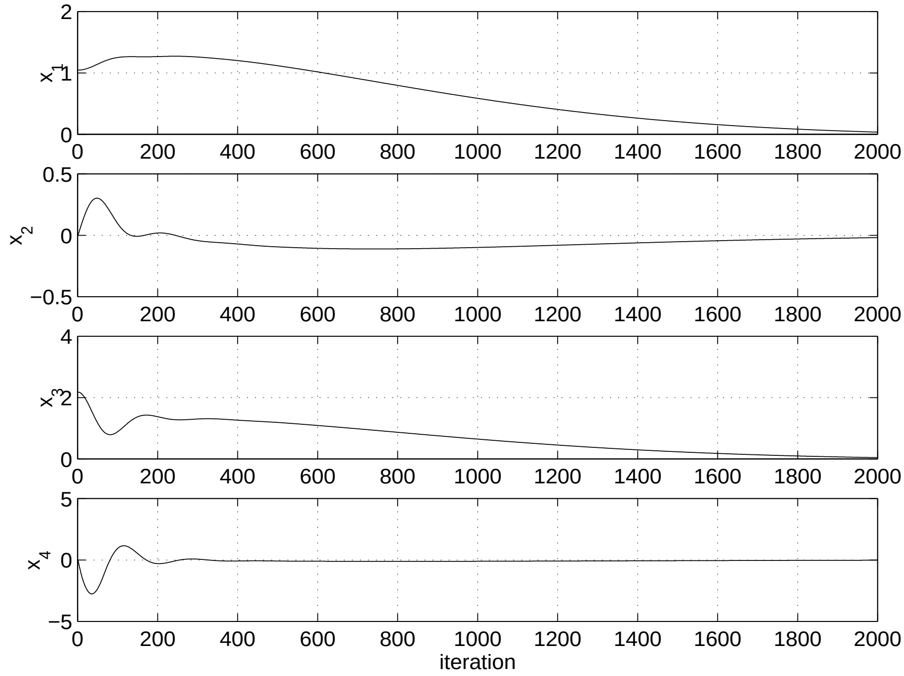


Figure 5.28: Stable System States

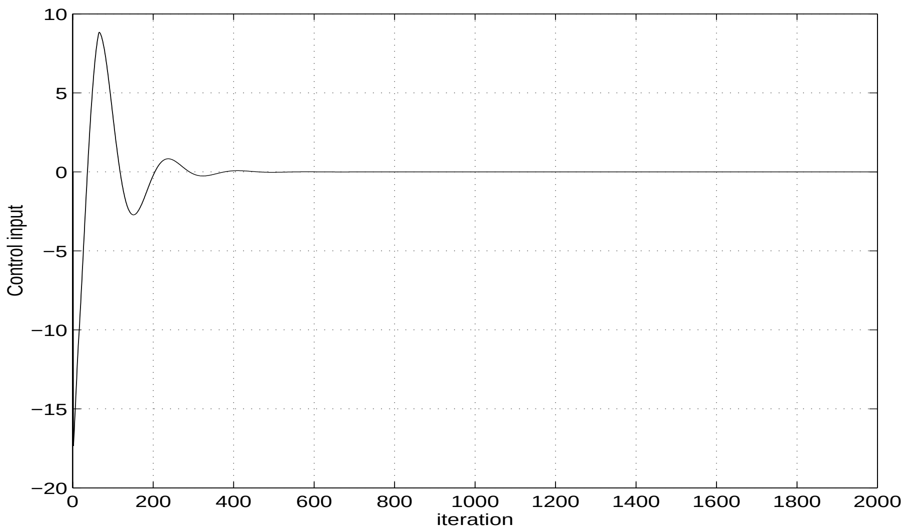


Figure 5.29: Control Input

5.5.1 T-S Representation of the System

First, we put the system given by the state equations 5.7 into the $\dot{x} = A(x)x + B(x)u$ form needed in the consequents of the rules in a T-S fuzzy system representation. Let us remember the robotic system equations:

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\tfrac{mgl}{I_1}\sin(x_1) + \tfrac{k}{I_1}(x_3 - x_1) \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= \tfrac{k}{I_2}(x_1 - x_3) + \tfrac{u}{I_2}
\end{aligned} \tag{5.16}$$

In matrix form we have:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{mgl}{I_1}\frac{\sin(x_1)}{x_1} - \frac{k}{I_1} & 0 & \frac{k}{I_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{I_2} & 0 & -\frac{k}{I_2} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{I_2} \end{bmatrix} u \tag{5.17}$$

where $x = [x_1, x_2, x_3, x_4]^T$.

The only nonlinear term in $A(x)$ is $sinc(x_1) = \frac{\sin(x_1)}{x_1}$, so we need to linearize it in the T-S representation. We use the exact value of the *sinc* function at the operation point of the rules. According to this, the rules acquire a linear form around the point of operation:

$R^{(l)}$: IF x_1 is F_1^l and ... and x_4 is F_4^l, THEN

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{mgl}{I_1}\frac{\sin(x_1)}{x_1}\Big|_{x_1^l} - \frac{k}{I_1} & 0 & \frac{k}{I_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{I_2} & 0 & -\frac{k}{I_2} & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{I_2} \end{bmatrix} u(t) \tag{5.18}$$

If we give numerical examples for the rules, we have:

$R^{(1)}$: IF x_1 is around 0 and ... and x_4 is F_4^1, THEN

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{mgl}{I_1} - \frac{k}{I_1} & 0 & \frac{k}{I_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{I_2} & 0 & -\frac{k}{I_2} & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{I_2} \end{bmatrix} u(t) \tag{5.19}$$

$R^{(2)}$: IF x_1 is around $\frac{\pi}{2}$ and ... and x_4 is F_4^2, THEN

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{mgl}{I_1}\frac{2}{\pi} - \frac{k}{I_1} & 0 & \frac{k}{I_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{I_2} & 0 & -\frac{k}{I_2} & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{I_2} \end{bmatrix} u(t) \tag{5.20}$$

As the number of rules increase, the accuracy of modelling the original system by this T-S fuzzy system increases.

The system parameters are taken to be $m = 0.01kg$, $I_1 = I_2 = 1kg.m^2$, $k = 0.05N.m/rad$, $l = 1m$ and $g = 9.81m/s^2$ for illustrative purposes, so the system matrices become:

$$A_l = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -0.0981\frac{\sin(x_1)}{x_1}|_{x_1^l} - 0.05 & 0 & 0.05 & 0 \\ 0 & 0 & 0 & 1 \\ 0.05 & 0 & -0.05 & 0 \end{bmatrix}, B_l = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{5.21}$$

If we set $c_l = -0.0981\frac{\sin(x_1)}{x_1}|_{x_1^l} - 0.05$, then $A_l = \begin{bmatrix} 0 & 1 & 0 & 0 \\ c_l & 0 & 0.05 & 0 \\ 0 & 0 & 0 & 1 \\ 0.05 & 0 & -0.05 & 0 \end{bmatrix}$.

Next, we define how to decompose the Lie algebra generated by these A_l matrices.

5.5.1.1 Decomposition Procedure

In order to decompose the A_l matrix into the semisimple and solvable bits, we need to look at the constant matrices A_{l1} and A_{l2} that would generate A_l. The choice of these matrices forms the first step of the procedure. It should be noted that the choice of A_{l1} and A_{l2} is not unique.

We start with an arbitrary decomposition of the A_l matrix as follows:

$$A_l = c_l \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{A_{l1}} + \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0.05 & 0 \\ 0 & 0 & 0 & 1 \\ 0.05 & 0 & -0.05 & 0 \end{bmatrix}}_{A_{l2}} \tag{5.22}$$

We now consider the Lie algebra L_A generated by the matrices $\{A_{l1}, A_{l2}\}$. In order to find the Levi decomposition of L_A, we need to separate the semisimple bit and the rest will form the solvable part. From the stability point of view, the semisimple bit can be considered as the controllable part and the solvable bit as the uncontrollable part, so we want the solvable bit to be small enough to be able to stabilize the system by stabilizing the semisimple bit.

The procedure for this decomposition is based on the Cartan's criterion (see Section 2.4.3), and follow the four steps described below.

Step 1: Form the basis for the Lie Algebra: In order to do this, first find the basis elements for L_A generated by $\{A_{l1}, A_{l2}\}$. Then, find the commutators ($[X, Y] = XY - YX$ is the commutator of two matrices X, Y) of the basis elements, and form the set of matrices composed of the basis elements and the commutators. Finally, find the basis elements of this new set of matrices, which form the Lie algebra basis.

Step 2: Obtain the Killing form.

Step 3: Obtain the Killing matrix, K. The Killing matrix K is found by the Killing form equality $kf = xKy^T$, where $x = [x_1, x_2, \ldots, x_n]$, $y = [y_1, y_2, \ldots, y_n]$, n is the dimension of the Lie algebra L_A and k_{ij}'s are the coefficients of the terms $x_i y_j$ in the Killing form such that $k_{ij} = k_{ji}$. Diagonalize the Killing matrix K, i.e. $\Lambda_K = P^{-1}KP$, where P is the modal matrix. Transform the basis elements by the same P. The semisimple bit is taken to be the transformed basis elements corresponding to the large eigenvalues of the matrix K. This is deduced from the fact that eigenvectors corresponding to eigenvalues that are close to zero result in a degenerate Killing form, meaning that the corresponding basis elements belong to the solvable part.

Step 4: Decompose A_l with respect to the transformed basis. This gives the semisimple part (that corresponds to the basis elements for the semisimple part found in Step 3) and the solvable part (that corresponds to the rest of the basis elements).

When we apply this procedure to our example system, we obtain the following computational results:

Step 1: The Lie algebra basis for A_{l1}, A_{l2} is:

$$L_{basis} = \{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0.05 & 0 \\ 0 & 0 & 0 & 1 \\ 0.05 & 0 & -0.05 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$
$$\begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & -0.05 & 0 \\ 0 & 0 & 0 & 0 \\ -0.05 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -0.15 & 0 \\ 0 & 0 & 0 & 0.05 \\ -0.05 & 0 & 0 & 0 \\ 0 & 0.15 & 0 & 0 \end{bmatrix},$$
$$\begin{bmatrix} 0 & -4 & 0 & 0 \\ 0 & 0 & -0.05 & 0 \\ 0 & 0 & 0 & 0 \\ -0.05 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0.2 \\ -0.005 & 0 & 0.0025 & 0 \\ 0 & 0.2 & 0 & 0 \\ 0.0025 & 0 & -0.015 & 0 \end{bmatrix},$$
$$\begin{bmatrix} 0 & 0 & 0.15 & 0 \\ 0 & 0 & 0 & 0.05 \\ -0.05 & 0 & 0 & 0 \\ 0 & -0.15 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0.1 \\ 0.005 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0.015 & 0 \end{bmatrix},$$
$$\begin{bmatrix} -0.015 & 0 & 0.0125 & 0 \\ 0 & 0.015 & 0 & -0.0025 \\ 0.0025 & 0 & -0.025 & 0 \\ 0 & -0.0125 & 0 & 0.025 \end{bmatrix} \} \tag{5.23}$$

Step 2: The Killing form for this example is:

$$\begin{aligned}
&kf = 12y_4x_1 - 24y_6x_1 + 0.18y_5x_5 - 0.18y_4x_7 - 0.18y_9x_6 + 0.18y_9x_2 \\
&-0.18y_7x_4 + 0.003y_7x_9 + 0.012y_7x_7 - 0.18y_8x_8 + 0.18y_{10}x_3 \\
&+0.01095y_{10}x_{10} - 0.003y_{10}x_8 + 0.003y_9x_7 - 0.18y_6x_9 + 6y_2x_1 \\
&-0.6y_2x_2 + 12y_3x_3 - 0.012y_{10}x_5 - 0.012y_5x_{10} + 6y_1x_2 + 12y_1x_4 \\
&-24y_1x_6 - 0.003y_8x_{10} + 0.18y_3x_{10} + 0.18y_2x_9
\end{aligned} \tag{5.24}$$

Step 3: The Killing matrix is:

$$K_M = \begin{bmatrix} k_1 & k_2 & k_3 & k_4 & k_5 & k_6 & k_7 & k_8 & k_9 & k_{10} \end{bmatrix} \tag{5.25}$$

where $k_1 = (0, 6, 0, 12, 0, -24, 0, 0, 0, 0)^T$, $k_2 = (6, -0.6, 0, 0, 0, 0, 0, 0, 0.18, 0)^T$, $k_3 = (0, 0, 12, 0, 0, 0, 0, 0, 0, 0.18)^T$, $k_4 = (12, 0, 0, 0, 0, 0, -0.18, 0, 0, 0)^T$, $k_5 = (0, 0, 0, 0, 0.18, 0, 0, 0, 0, -0.012)^T$, $k_6 = (-24, 0, 0, 0, 0, 0, 0, 0, -0.18, 0)^T$, $k_7 = (0, 0, 0, -0.18, 0, 0, 0.012, 0, 0.003, 0)^T$, $k_8 = (0, 0, 0, 0, 0, 0, 0, -0.18, 0, -0.003)^T$, $k_9 = (0, 0.18, 0, 0, 0, -0.18, 0.003, 0, 0, 0)^T$, and $k_{10} = (0, 0, 0.18, 0, -0.012, 0, 0, -3 \times 10^{-3}, 0, 12 \times 10^{-3})^T$. When we diagonalize K_M, we get:

$$\begin{gathered}
\Lambda_{K_M} = diag(-0.1678, 0.0232, 0.1906, -0.6054, 0.18, \\
0.0075, 0.1808, -27.5109, 27.4823, 12.0027)
\end{gathered} \tag{5.26}$$

The modal matrix P is:

$$P = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 & p_9 & p_{10} \end{bmatrix} \tag{5.27}$$

$p_1 = (6 \times 10^{-4}, 0.11, 0, 0.64, 0, 0.35, 0.63, 0, 0.24, 0)^T$, $p_2 = (6.6 \times 10^{-3}, -0.19, 0, -0.04, 0, -0.07, 0.44, 0, -0.87, 0)^T$, $p_3 = (-4 \times 10^{-4}, 0.08, 0, -0.62, 0, -0.29, 0.63, 0, 0.36, 0)^T$, $p_4 = (6 \times 10^{-3}, 0.95, 0, -0.13, 0, 0.17, -0.04, 0, -0.23, 0)^T$, $p_5 = (0, 0, 2 \times 10^{-4}, 0, -5 \times 10^{-4}, 0, 0, -1, 0, -0.02)^T$, $p_6 = (0, 0, 0.01, 0, -0.07, 0, 0, 0.02, 0, -0.99)^T$, $p_7 = (0, 0, 1 \times 10^{-3}, 0, 1, 0, 0, 6 \times 10^{-4}, 0, -0.07)^T$, $p_8 = (0.71, -0.16, 0, -0.31, 0, 0.62, -2 \times 10^{-3}, 0, 5 \times 10^{-3}, 0)^T$, $p_9 = (0.71, 0.15, 0, 0.31, 0, -0.62, 0, 0, 5 \times 10^{-3}, 0)^T$, $p_{10} = (0, 0, -1, 0, 0, 0, 0, 0, 0, -0.01)^T$.

When we compare the eigenvalues, we conclude that apart from the last three, the others can be considered as 0, so we take the last three transformed basis elements as the semisimple part. The three basis elements after transformation become:

$$T_{basis} = \{ \begin{bmatrix} 0 & -3.2412 & 0 & 0.0001 \\ 0.7069 & 0 & -0.0233 & 0 \\ 0 & 0.0001 & 0 & -0.1576 \\ -0.0233 & 0 & 0.0080 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 3.2395 & 0 & 0.0001 \\ 0.7073 & 0 & 0.023 & 0 \\ 0 & 0.0001 & 0 & 0.1511 \\ 0.023 & 0 & -0.0075 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 1.0001 & 0 & -0.0002 & 0 \\ 0 & -1.0001 & 0 & 0 \\ 0 & 0 & 0.0004 & 0 \\ 0 & 0.0002 & 0 & -0.0004 \end{bmatrix}$$

The transformation of all the basis elements is performed by: $New_{basis} = Pe^T$, where $e = [e_1, e_2, \ldots, e_{10}]$ are the basis elements. In T_{basis}, the last three of the transformed basis elements are taken.

For stability, each of the three matrices should form a stabilizable pair with the B_l in Equation 5.21. However, we see that the last matrix violates this condition, i.e. it does not form a stabilizable pair with B_l (controllability matrix for this pair is $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -0.0004 & 0 & 0 \end{bmatrix}$ which is not full rank). To solve this problem, one way is to find a similarity transformation so that the result is a stabilizable set of matrices, but this is not an easy task.

Another way is to start with a different decomposition of A_l than the one in Equation 5.22. We have used some intuition in the decomposition of the system, and obtained the following decomposition of the system into:

$$A_l = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0.05 & 0 \\ 0 & -0.05 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}}_{A_{l1}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ c_l + 1 & 0 & 0 & 0 \\ 0 & 0.05 & 0 & 0 \\ 0.05 & 0 & 0.95 & 0 \end{bmatrix}}_{A_{l2}} \tag{5.28}$$

The first part of this matrix decomposition, i.e. A_{l1}, is certainly one dimensional and therefore Abelian, and also a simple algebra. We can think of this one-dimensional Abelian Lie algebra as a kind of 'degenerate' semisimple algebra (we cannot directly say that it is semisimple because one-dimensional algebras are not semisimple). It is Abelian, so the Cartan subalgebra is the whole algebra and the roots are all zero. Thus, A_{l1} is the semisimple part of A_l.

The second part, A_{l2}, generates a solvable Lie algebra for different c_l's, since they are lower triangular matrices. Since we have found a direct decomposition of the system into semisimple and solvable bits, we do not need to apply the decomposition steps here. This concludes our decomposition of A_l, and we see that A_{l1} forms a stabilizable pair with B_l, so we can stabilize our system.

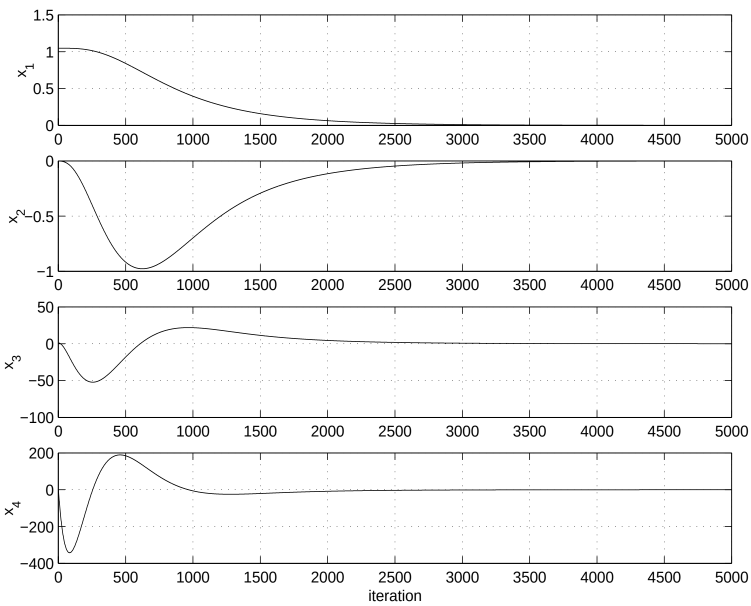


Figure 5.30: The states of the controlled system

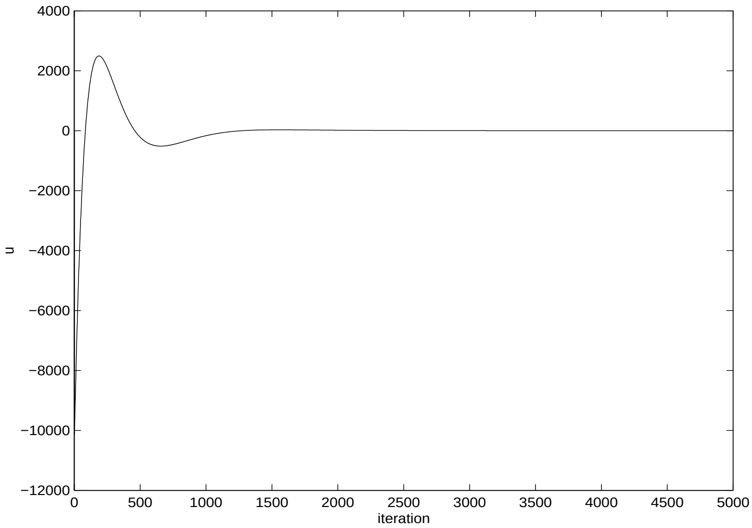


Figure 5.31: The control input

We use pole placement for the design of a controller that stabilizes each linear subsystem in the T-S model. Since the semisimple part of each rule are equal to each other, which is A_{l1}, the same control is used for every rule. This satisfies the criterion in Equation 4.57, which is $\sum_i(\tilde{\alpha}_i y_i^2(t) + \tilde{\beta}_i u(t) y_i(t)) \leq -\frac{1}{2}\varepsilon \sum_i y_i^2(t)$.

The controller is of the form: $u = -Kx(t)$, $K = \begin{bmatrix} 9520 & 9599 & 148 & 20 \end{bmatrix}$. When this control is applied to the original system, we see that we can stabilize the system. Fig.5.30 shows an example of stabilized states for initial condition $(\pi/3, 0, \pi/3, 0)$.

The control input is shown in Fig.5.31.

CHAPTER 6

CONCLUSIONS AND FUTURE WORK

6.1 Concluding Remarks

In this thesis, we have developed a novel 2-way fuzzy adaptive system being able to minimize vagueness and inconsistencies based on intuitionistic fuzzy sets in place of classical fuzzy sets. The two way characteristic comes from the property of our system to handle not only inconsistencies by rendering the system fully consistent in a first learning phase, and also minimizing uncertainty in a second phase, but also to be able to handle interval valued vagueness with membership and nonmembership assignments. Inconsistencies naturally arise from the independent assignments of these membership and nonmembership functions. We had to also analyze the type of these inconsistencies and look at the performance of their reduction based on their types. We call this an evaluation and develop a novel approach for this evaluation of inconsistency based on shadowed sets. Then, came the actual dynamical role of our 2-way fuzzy adaptive system coupled to a physical plant. We applied the 2-way fuzzy adaptive system as a nonlinear system identifier and analyzed its performance in matching the nonlinear function of the real plant. During identification, the parameters of the 2-way fuzzy system have been adjusted adaptively, and their results have proven that the fuzzy system is capable of reducing uncertainty and inconsistency through training. We carried out in the performance analysis the evaluation of inconsistencies after the first training phase for inconsistency reduction. The results have shown that fully inconsistent membership and nonmembership functions have been reduced to consistent ones.

The real plant has also been chosen as a flexible-joint robot arm system identified by the proposed 2-way system and its inconsistency reduction based on shadowed sets proved to have a high correlation with the overlapping degree of membership and nonmemberships. This has been tied to a fuzziness measure c, where high values of the combinational index c used in classification of inconsistency types led to inconsistency not being able to switch its type but still be reduced within the same type, and for small values of c around 1 inconsistencies was reduced lowering their type until fully consistent case. Therefore, the subjective assignment of belief and disbelief in terms of membership and nonmembership, and their fuzzy support (span) as well as their degree of disagreement is a crucial initial condition for the inconsistency that determines if it can be fully or partially reduced by learning.

Prediction and moreover, assurance of stability of the 2-way fuzzy adaptive system has been achieved using 3 complementary approaches. First method we develop is based on the approximating sequences technique, where the design of the fuzzy controller is turned into an optimal control problem applied to the control of a flexible-joint robot arm system. Our second approach uses a describing function technique to develop a systematic design procedure for the 2-way fuzzy controller. We have calculated an analytical expression for the describing function of the 2-way fuzzy system using the additivity property for which decomposability conditions have been defined. The fuzzy controller driving the flexible-joint robot arm system considered previously prove to guarantee stability within the stability regions we determine prior to design in the parameter space. The simulation results for unstable and stable cases agreed with the theoretical results.

If the plant in question, here the flexible-joint robot arm, can be represented as a T-S type fuzzy system, stability conditions for the design of 2-way fuzzy adaptive controller were generated using Lie algebra theory. Stability conditions for the most general case where we have noncommuting system matrices in T-S form were derived after their development for commuting matrices. Our basic idea is to decompose the Lie algebra generated by the system matrices into the semisimple and solvable parts, such that the solvable part, which can be considered as the uncontrollable part, is small compared to the semisimple part, where the controller is then designed for the semisimple part. For this analysis, we had to also develop a decomposition procedure.

The major impact of this thesis to the literature is that to our knowledge, it is the first time that the intuitionistic fuzzy sets are used within a fuzzy control architecture modelling uncertainty and inconsistency. The design of a fuzzy system using approximating sequences technique is also a novel approach introduced in the framework of this study. The design of a stable 2-way fuzzy controller using describing function method and the stability analysis of the T-S fuzzy systems based on Lie algebra are the other novelties of our work.

6.2 Future Work

The thesis also forms the basis for new research areas. The existing stability analysis techniques in literature can be adapted such that they can be used to analyze the stability of the 2-way fuzzy system. Based on the Lie algebra analysis, new adaption laws can be derived to be used in the design of adaptive fuzzy systems. The decomposition of the Lie algebra is not a unique decomposition, and it is not a trivial process. This decomposition procedure can be modified into such a procedure that the result is an optimum decomposition. This is not an easy task to accomplish, but it will be a major contribution to the literature. In the stability analysis using Lie algebra, the case where A_i's are Jordan form can be considered, and the analysis results can be extended for this case. The empirical results used in inconsistency evaluation can be made analytical by the derivation of a rigorous correlation between assignment of membership and nonmembership functions and the global index used.

REFERENCES

[1] G. Klir and T. Folger, *Fuzzy Sets, Uncertainty, and Information.* Prentice Hall, 1988.

[2] B. M. Ayyub and M. M. Gupta, *Uncertainty Analysis in Engineering and Sciences: Fuzzy Logic, Statistics, and Neural Network Approach.* Kluwer Academic Publishers, 1997.

[3] M. Sugeno, "On stability of fuzzy systems expressed by fuzzy rules with singleton consequents," *IEEE Transactions on Fuzzy Systems*, vol. 7, pp. 201–224, April 1999.

[4] S. Y. Yi and M. J. Chung, "Systematic design and stability analysis of a fuzzy logic controller," *Fuzzy Sets and Systems*, vol. 72, pp. 271–298, 1995.

[5] S. Y. Chen, F. M. Yu, and H. Y. Chung, "Decoupled fuzzy controller design with single-input fuzzy logic," *Fuzzy Sets and Systems*, vol. 129, pp. 335–342, 2002.

[6] C. Kolodziej and R. Priemer, "Stability analysis of fuzzy systems," *Journal of the Franklin Institute*, vol. 336, pp. 851–873, 1999.

[7] F. H. F. Leung, H. K. Lam, and P. K. S. Tam, "Design of fuzzy controllers for uncertain nonlinear systems using stability and robustness analyses," *Systems & Control Letters*, vol. 35, pp. 237–243, 1998.

[8] S. Jagannathan, M. W. Vandegrift, and F. L. Lewis, "Adaptive fuzzy logic control of discrete-time dynamical systems," *Automatica*, vol. 36, pp. 229–241, 2000.

[9] K. Tanaka and M. Sugeno, "Stability analysis and design of fuzzy control systems," *Fuzzy Sets and Systems*, vol. 45, pp. 135–156, 1992.

[10] S. G. Cao, N. W. Rees, and G. Feng, "Analysis and design of fuzzy control systems using dynamic fuzzy-state space models," *IEEE Transactions on Fuzzy Systems*, vol. 7, no. 2, pp. 192–200, 1999.

[11] H. O. Wang, K. Tanaka, and M. F. Griffin, "An approach to fuzzy control of nonlinear systems: Stability and design," *IEEE Transactions on Fuzzy Systems*, vol. 4, no. 1, pp. 14–23, 1996.

[12] C. W. Park, “Lmi-based robust stability analysis for fuzzy feedback linearization regulators with its applications,” *Information Sciences*, vol. 152, pp. 287–301, 2003.

[13] J. Park, J. Kim, and D. Park, “Lmi-based design of stabilizing controllers for nonlinear systems described by takagi-sugeno fuzzy model,” *Fuzzy Sets and Systems*, vol. 122, pp. 73–82, 2001.

[14] L. X. Wang, *Adaptive Fuzzy Systems and Control: Design and Stability Analysis.* Prentice Hall, 1994.

[15] S. C. Tong, T. Wang, and J. T. Tang, “Fuzzy adaptive output tracking control of nonlinear systems,” *Fuzzy Sets and Systems*, vol. 111, pp. 169–182, 2000.

[16] D. I. Sagias, E. N. Sarafis, C. I. Siettos, and G. V. Bafas, “Design of a model identification fuzzy adaptive controller and stability analysis of nonlinear processes,” *Fuzzy Sets and Systems*, vol. 121, pp. 169–179, 2001.

[17] H. Han and C. Y. Su, “Robust fuzzy control of nonlinear systems using shape-adaptive radial basis functions,” *Fuzzy Sets and Systems*, vol. 125, pp. 23–38, 2002.

[18] Y. Tang, N. Zhang, and Y. Li, “Stable fuzzy adaptive control for a class of nonlinear systems,” *Fuzzy Sets and Systems*, vol. 104, pp. 279–288, 1999.

[19] T. Chai and S. Tong, “Fuzzy direct adaptive control for a class of nonlinear systems,” *Fuzzy Sets and Systems*, vol. 103, pp. 379–387, 1999.

[20] C. C. Fuh and P. C. Tung, “Robust stability analysis of fuzzy control systems,” *Fuzzy Sets and Systems*, vol. 88, pp. 289–298, 1997.

[21] L. Luoh, “New stability analysis of t-s fuzzy system with robust approach,” *Mathematics and Computers in Simulation*, vol. 59, pp. 335–340, 2002.

[22] J. H. Chou and S. H. Chen, “Stability analysis of the discrete takagi-sugeno fuzzy model with time-varying consequent uncertainties,” *Fuzzy Sets and Systems*, vol. 118, pp. 271–279, 2001.

[23] E. Kim, H. Lee, and M. Park, “Limit-cycle prediction of a fuzzy control system based on describing function method,” *IEEE Transactions on Fuzzy Systems*, vol. 8, no. 1, pp. 11–22, 2000.

[24] F. Cuesta, F. Gordillo, J. Aracil, and A. Ollero, “Stability analysis of nonlinear multivariable takagi-sugeno fuzzy control systems,” *IEEE Transactions on Fuzzy Systems*, vol. 7, no. 5, pp. 508–520, 1999.

[25] J. Aracil and F. Gordillo, “Describing function method for stability analysis of pd and pi fuzzy controllers,” *Fuzzy Sets and Systems, in press*, 2003.

[26] S. H. Zak, "Stabilizing fuzzy system models using linear controllers," *IEEE Transactions on Fuzzy Systems*, vol. 7, no. 2, pp. 236–240, 1999.

[27] M. A. L. Thathachar and P. Viswanath, "On the stability of fuzzy systems," *IEEE Transactions on Fuzzy Systems*, vol. 5, pp. 145–151, February 1997.

[28] A. Kandel, Y. Luo, and Y. Q. Zhang, "Stability analysis of fuzzy control systems," *Computers and Structures*, vol. 105, pp. 33–48, 1999.

[29] C. I. Siettos and G. V. Bafas, "Semiglobal stabilization of nonlinear systems using fuzzy control and singular perturbation methods," *Fuzzy Sets and Systems*, vol. 129, pp. 275–294, 2002.

[30] D. Z. Liao and L. F. Yeung, "l_∞-based stability criteria and its applications on flc systems," *Fuzzy Sets and Systems, in press*, 2002.

[31] K. Kiriakidis, A. Grivas, and A. Tzes, "Quadratic stability analysis of the takagi-sugeno fuzzy model," *Fuzzy Sets and Systems*, vol. 98, pp. 1–14, 1998.

[32] Y. Ding, H. Ying, and S. Shao, "Typical takagi-sugeno pi and pd fuzzy controllers: Analytical structures and stability analysis," *Information Sciences*, vol. 151, pp. 245–262, 2003.

[33] L. K. Wong, F. H. F. Leung, and P. K. S. Tam, "Design of fuzzy logic controllers for t-s fuzzy model based system with guaranteed performance," *International Journal of Approximate Reasoning*, vol. 30, pp. 41–55, 2002.

[34] W. Tang, G. Chen, and R. Lu, "A modified fuzzy pi controller for a flexible-joint robot arm with uncertainties," *Fuzzy Sets and Systems*, vol. 118, pp. 109–119, 2001.

[35] E. Carrera and M. A. Serna, "Inverse dynamics of flexible robots," *Mathematics and Computers in Simulation*, vol. 41, pp. 485–508, 1996.

[36] J. H. Yang, F. L. Lian, and L. C. Fu, "Nonlinear adaptive control for flexible-link manipulators," *IEEE Transactions on Robotics and Automation*, vol. 13, pp. 140–148, February 1997.

[37] S. Nicosia and P. Tomei, "A global output feedback controller for flexible joint robots," *Automatica*, vol. 31, no. 10, pp. 1465–1469, 1995.

[38] S. S. Ge, "Adaptive controller design for flexible joint manipulators," *Automatica*, vol. 32, no. 2, pp. 273–278, 1996.

[39] M. W. Spong, "Adaptive control of flexible joint manipulators: Comments on two papers," *Automatica*, vol. 31, no. 4, pp. 585–590, 1995.

[40] A. Ailon, "Output controllers based on iterative schemes for set-point regulation of uncertain flexible-joint robot models," *Automatica*, vol. 32, no. 10, pp. 1455–1461, 1996.

[41] D. Wang, "A simple iterative learning controller for manipulators with flexible joints," *Automatica*, vol. 31, no. 9, pp. 1341–1344, 1995.

[42] E. S. Choi and B. H. Ahn, "Technical communique: More on the applications of the contraction mapping method in robotics," *Automatica*, vol. 36, pp. 1755–1760, 2000.

[43] G. Klir and M. J. Wierman, *Uncertainty-Based Information: Elements of Generalized Information Theory.* Physica-Verlag, 1999.

[44] H. J. Zimmermann, "An application-oriented view of modeling uncertainty," *European Journal of Operational Research*, vol. 122, pp. 190–198, 2000.

[45] G. Shafer, *A Mathematical Theory of Evidence.* Princeton University Press, 1976.

[46] E. Gürkan, A. M. Erkmen, and I. Erkmen, "Reducing inconsistencies in intuitionistic 2-way adaptive fuzzy control systems," in *Proc. of the 4th International FLINS Conference*, pp. 289–296, 2000.

[47] Y. Jin, W. Seelen, and B. Sendhoff, "On generating fc^3 fuzzy rule systems from data using evolution strategies," *IEEE Transactions on Systems, Man and Cybernetics - Part B*, vol. 29, no. 6, pp. 829–845, 1999.

[48] J. Pearl, *Probabilistic Reasoning in Intelligent Systems: Networks of Plausible Inference.* Morgan Kaufmann Publishers Inc., 1988.

[49] G. Shafer, "Probability judgement in artificial intelligence," *Uncertainty in Artificial Intelligence, L. N. Kanal and J. F. Lemmer Eds., Elsevier Science Publishers*, pp. –, 1986.

[50] A. M. Erkmen, *Information Fractals for Approximate Reasoning in Sensor based Robot Grasp Control.* PhD thesis, George Mason University, 1989.

[51] H. E. Stephanou and S. Y. Lu, "Measuring consensus effectiveness by a generalized entropy criterion," *IEEE Transactions on Pattern Analysis and Machine Intelligence*, vol. 10, no. 4, pp. 544–554.

[52] W. Pedrycz, *Fuzzy Control and Fuzzy Systems.* Research Studies Press Ltd., 1993.

[53] K. T. Atanassov, "Intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol. 20, pp. 87–96, 1986.

[54] Z. Pawlak, "Rough set approach to knowledge-based decision," *European Journal of Operational Research*, vol. 99, pp. 48–57, 1997.

[55] H. B. Kaygısız, A. M. Erkmen, and I. Erkmen, "Smoothing stability roughness of a robot arm under dynamic load using reinforcement learning," in *Proc. of the 2001 IEEE/RJS Intl. Conference on Intelligent Robots and Systems*, vol. 4, pp. 2392–2397, 2001.

[56] J. Aczel and Z. Daroczy, *On Measure of Information and Their Characterizations.* Academic Press, 1975.

[57] S. Kullback, *Information Theory and Statistics.* Dover, 1959.

[58] G. Katona and G. Tusnady, "The principle of conservation of entropy in a noiseless channel," *Studia Scientiarum Mathemticarum Hungarica, Tomus II*, vol. 1-2, pp. 29–35, 1967.

[59] I. Csiszar, G. Katona, and G. Tusnady, "Information sources with different cost scales and the principle of conservation of entropy," *Z. Wahrscheinlichkeitstheorie Verw. Geb.*, vol. 12, pp. 185–222, 1969.

[60] I. J. Good, "Information, weightof evidence, the singularity between probability measures and signal detection," in *Lecture Notes in Mathematics, David Bridston Osteyee*, pp. 39–53, 1974.

[61] P. Diaconis and S. L. Zabell, "Updating subjective probability," *J. American Statistical Association*, vol. 77, no. 380, pp. 822–830, 1982.

[62] J. E. Shore and R. W. Johnson, "Axiomatic derivation of the principle of maximum entropy and the principle of minimum cross-entropy," *IEEE Transactions on Information Theory*, vol. 26, no. 1, pp. 26–36, 1980.

[63] J. E. Shore, "Relative entropy, probabilistic inference and ai," *Uncertainty in Artificial Intelligence, L. N. Kanal and J. F. Lemmer Eds., Elsevier Science Publishers*, pp. –, 1986.

[64] J. E. Shore and R. M. Gray, "Minimum cross-entropy pattern classification and cluster analysis," *IEEE Transactions on Pattern Analysis and Machine Intelligence*, vol. 4, no. 1, pp. 11–17, 1982.

[65] C. R. Rao, "Diversity and dissimilarity coefficients: a unified approach," *Theoretical Population Biology*, vol. 21, pp. 24–43, 1982.

[66] C. R. Rao and T. K. Kayak, "Cross-entropy, dissimilarity measures and characterizations of quadratic entropy," *IEEE Transactions on Information Theory*, vol. 31, no. 5, pp. 589–593, 1985.

[67] R. R. Yager, "Entropy and specificity in a mathematical theory of evidence," *Int. J. General Systems*, vol. 9, pp. 249–260, 1983.

[68] A. Tzouvaras, "Modeling vagueness by nonstandardness," *Fuzzy Sets and Systems*, vol. 94, pp. 385–396, 1998.

[69] S. Raha and K. S. Ray, "Reasoning with vague default," *Fuzzy Sets and Systems*, vol. 91, pp. 327–338, 1997.

[70] T. Denœux, "Modeling vague beliefs using fuzzy-valued belief structures," *Fuzzy Sets and Systems*, vol. 116, pp. 167–199, 2000.

[71] W. Pedrycz, "Shadowed sets: Representing and processing fuzzy sets," *IEEE Transactions on Systems, Man and Cybernetics - Part B: Cybernetics*, vol. 28, pp. 103–109, February 1998.

[72] K. Sridharan and H. E. Stephanou, "Fuzzy distances for proximity characterization under uncertainty," *Fuzzy Sets and Systems*, vol. 103, pp. 427–434, 1999.

[73] T. Chávez, "Modeling and measuring the effects of vagueness in decision models," *IEEE Transactions on Systems, Man and Cybernetics - Part A: Systems and Humans*, vol. 26, no. 3, pp. 311–323, 1996.

[74] A. Mosleh and V. M. Bier, "Uncertainty about probability: A reconciliation with the subjectivist viewpoint," *IEEE Transactions on Systems, Man and Cybernetics - Part A: Systems and Humans*, vol. 26, no. 3, pp. 303–310, 1996.

[75] G. D. Kleiter, "Propagating imprecise probabilities in bayesian networks," *Artificial Intelligence*, vol. 88, pp. 143–161, 1996.

[76] M. Pradhan, M. Henrion, G. Provan, B. D. Favero, and K. Huang, "The sensitivity of belief networks to imprecise probabilities: An experimental investigation," *Artificial Intelligence*, vol. 85, pp. 363–397, 1996.

[77] J. Lee, K. F. R. Liu, and W. Chiang, "A possibilistic-logic-based approach to integrating imprecise and uncertain information," *Fuzzy Sets and Systems*, vol. 113, pp. 309–322, 2000.

[78] R. R. Yager and D. P. Filev, "Including probabilistic uncertainty in fuzzy logic controller modeling using dempster-shafer theory," *IEEE Transactions on Fuzzy Systems, Man and Cybernetics*, vol. 25, no. 8, pp. 1221–1230, 1995.

[79] Z. X. Wan, *Lie Algebras.* Pergamon Press, 1975.

[80] S. P. Banks and K. Dinesh, "Approximate optimal control and stability of nonlinear finite- and infinite-dimensional systems," *Annals of Operations Research*, vol. 98, pp. 19–44, 2000.

[81] S. Haykin, *Neural Networks: A Comprehensive Foundation.* Prentice Hall, 1999.

[82] S. P. Bhattacharyya, H. Chapellat, and L. H. Keel, *Robust Control: The Parametric Approach.* Prentice Hall, 1995.

[83] J. J. E. Slotine and W. Li, *Applied Nonlinear Control.* Prentice Hall, 1991.

VITA

Evren Gürkan was born in Ankara, Turkey in 1974. She received her B.Sc. degree from Middle East Technical University (METU), Department of Electrical and Electronics Engineering in 1996 and M.Sc degree from The University of Sheffield Department of Automatic Control and Systems Engineering in 1997 respectively. She is working towards Ph.D. degree in Middle East Technical University (METU), Department of Electrical and Electronics Engineering since 1998.

She worked as a software engineer at ASELSAN Inc. during 1997-1999. She has joined METU, Department of Electrical and Electronic Engineering in 1999 where she is currently employed as a research assistant. Her areas of interest are fuzzy adaptive systems, nonlinear stability analysis and uncertainty modelling.

PUBLICATIONS DURING Ph.D. STUDY

International Peer-Reviewed Journal Papers

- **Gürkan E**, Erkmen İ and Erkmen A M, "Two-Way Fuzzy Adaptive Identification and Control of a Flexible-joint Robot Arm", Special Issue on Intelligent Learning and Control of Robotics and Intelligent Machines in Unstructured Environments, Information Sciences 145/1-2, 2002, pp. 13-43.
- **Banks S P**, Gürkan E and Erkmen İ, "Stability Analysis of T-S Fuzzy Systems based on Lie Algebras", *prepared for submission*
- **Gürkan E**, Erkmen A M and Erkmen İ, "Stability Analysis of 2-Way Fuzzy Systems Using Describing Function Technique", *prepared for submission*

International Conference Papers

- **Gürkan E**, Erkmen A M and Erkmen İ, "Intuitionistic, 2-Way Adaptive Fuzzy Control," Proceedings of IEEE International Conference on Robotics and Automation, 1999, pp. 2470-2475.

- **Gürkan E**, Erkmen A M and Erkmen İ, "Reducing Inconsistencies in Intuitionistic 2-Way Adaptive Fuzzy Control Systems," Proceedings of the 4th International FLINS Conference, 2000, pp. 289-296.
- **Gürkan E**, Erkmen A M and Erkmen İ, "Modelling and Minimizing Uncertainty", Proceedings of Brain Machine Workshop, 2000, pp. 157-170.
- **Gürkan E**, Erkmen A M and Erkmen İ, "Evaluation of Inconsistency in a 2-Way Fuzzy Adaptive System Using Shadowed Sets," Proceedings of 31st IEEE International Symposium on Multiple-Valued Logic (ISMVL 2001), 2001, pp. 109-115.
- **Gürkan E**, Banks S P, Erkmen A M and Erkmen İ, "Equivalent Linearization of 2-Way Fuzzy Adaptive System under Nonparametric Uncertainty and Inconsistency," Proceedings of the 5th International FLINS Conference, 2002, pp. 453-460.
- **Gürkan E**, Banks S P and Erkmen İ, "Stable Controller Design of the T-S Fuzzy Model of a Flexible-Joint Robot Arm based on Lie Algebra," accepted for CDC 2003.