FINSLER GEOMETRY AND ITS APPLICATIONS TO ELECTROMAGNETISM

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
THE MIDDLE EAST TECHNICAL UNIVERSITY

BY

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IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

IN

THE DEPARTMENT OF PHYSICS

SEPTEMBER 2003

Approval of the Graduate School of Natural and Applied Sciences.

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ABSTRACT

FINSLER GEOMETRY AND ITS APPLICATIONS TO ELECTROMAGNETISM

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SEPTEMBER 2003, 42 pages.

In this thesis Finsler geometry is extensively reviewed. The geometrization of fields by a Finslerian approach is considered. Also unification of electrodynamics and gravitation with suitable Finslerian metrics is examined.

Keywords: Geometrization of electrodynamics, Finsler geometry, Finsler spaces.

ÖZ

FİNSLER GEOMETRİSİ VE ELEKTROMANYETİĞE UYGULAMALARI

ÇAĞIL, AYŞE
Yüksek Lisans , Fizik Bölümü
Tez Yöneticisi: Doç. Dr. Yusuf İPEKOĞLU

EYLÜL 2003, 42 sayfa.

Bu tezde, Finsler Geometrisi geniş olarak ele alındı. Fiziksel alanların Finsler geometrisi kullanılarak geometrize edilmesi araştırıldı. Ayrıca elektrodinamik ve temel çekim kuramlarının uygun Finsler metrikleri kullanılarak birleştirilmeleri incelendi.

Anahtar Kelimeler: Elektrodinamik kuramının geometrize edilmesi, Finsler geometrisi, Finsler uzayları.

ACKNOWLEDGMENTS

I would like to express my thanks to my supervisor, Assoc. Prof. Dr. Yusuf İpekoğlu for introducing this interesting topic; for his patience and guidance. Also I would like to thank to Prof. Dr. Atalay Karasu and Özgün Süzer for enabling me to access some important references. Finally, I am glad to thank to my friends, Mustafa Çimşit and Hüseyin Dağ for their friendship and helpful discussions.

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CHAPTER 1

INTRODUCTION

One of the main aims of theoretical physics is to express all known forces of nature in one unified theory. Practically, all unification efforts nowadays proceed from the assumption that quantum field theory is fundamental and gravitation must be squeezed into a quantum context.

On the other hand, there exist other approaches to unification which uses some geometrical theories. They assumed that a geometrical theory, Einstein's general relativity is a fundamental theory, thus electromagnetism and other fields can be unified by means of a geometrical theory [1-7].

The most known geometrical approaches to unification are the theories of

Weyl and Kaluza-Klein, which aim to geometrize electromagnetism like gravitation. These theories faced with series problems such as, in Weyl's theory, the norms of vectors are not invariant under parallel transport, and in the approach of Kaluza-Klein theories, electrodynamics is geometrized in a five dimensional space-time. Also quantization of space-time is another existing problem when the electromagnetic field is quantized.

Finsler geometry is an alternative approach to geometrization of fields, and its fundamental idea can be traced back to a lecture of Riemann, in 1854 [1]. In this lecture Riemann suggested that the positive fourth root of a fourth order differential form might serve as a metric function. This function has three properties that it is convex and common with the Riemannian quadratic form it is positive and homogeneous of degree one in the differentials. Therefore, it is a natural generalization of the notion of distance between two neighboring points $x^i, x^i + dx^i$ to consider as given by some function $F(x^i, dx^i)$, where $i = 0, ..., n$, satisfying these three properties.

A systematic study of these kind of manifolds was first considered by Finsler in 1918 [2], and in 1925, the method of tensor calculus were applied to the theory [4]. It was found that the second derivatives of $\frac{1}{2}F^2(x^i, dx^i)$ with respect to differentials serves as components of a metric tensor in analogy with Riemann geometry. By this process, parallel displacements and connection coefficients in Finsler spaces are defined, but with these connections Ricci lemma was no longer valid. In 1934, Cartan [3] showed that it was indeed possible to define connections and a covariant derivative so that Ricci lemma is preserved. This development is closely related to the present application of Finsler geometry in physics, namely, to geometrize both electromagnetism and gravity simultaneously [10].

Finsler geometry was first applied in gravitational theory, and this application lead to corrections to observational results predicted by general relativity [11-19].

As mentioned before, the main application of Finsler geometry is the geometrization of electromagnetism and gravitation. A Finslerian approach to this geometrization was first introduced by Randers [5], but in his work Finsler geometry was not mentioned, although it was used. Randers metric produces a geodesic equation identical with Lorentz equation for a charged particle. But the metric depends on $\frac{q}{m}$ and defines a different space for each type of particle [6].

In the approach given in this study, unified theory of gravitation and electrodynamics is developed from a Finslerian tangent space gauge transformation [22, 25]. The transformation physically interpreted as containing physical fields and the resulting metric is similar to the metric introduced by Kaluza-Klein, but has different physical interpretations in the scheme of Finsler geometry [10, 23, 24, 26, 27].

In this study, our main aim is to review Finsler geometry and geometrization of electrodynamics by a Finslerian approach. This thesis is organized as follows:

In chapter two, a general view of Finsler geometry is given in detail. Finsler metric function, Finsler metric tensor and Cartan torsion tensor are defined. Then geodesics in Finslerian space-time and covariant differentiation methods in Finsler spaces are given. Finally curvature tensors in Finsler geometry are defined.

In chapter three, application of Finsler geometry to geometrization of electrodynamics is given. First Finsler gauge transformations are considered. Then by a specific transformation, a Finslerian metric function is calculated and properties of this metric function are studied. Finally, general forms of Finsler metric functions resulting from this transformation are considered.

CHAPTER 2

FINSLER GEOMETRY

2.1 Finsler Metric Function

In Riemannian geometry, length of a vector $|x|$ in a manifold M endowed by a metric tensor $g_{ij}(x)$ is given by the quadratic form

$$|x|^2 = g_{ij}(x)x^i x^j, \tag{2.1}$$

where $i, j, ... = 0, 1, 2, 3$ are the indices referring to space components.

Finsler geometry on the other hand offers a more general method to determine the norms of the vectors. From this point of view, Finsler geometry is a generalization of Riemann geometry to the effect that length of the vectors are determined by a general method which is not restricted by Riemann definition of length in terms of square root of the quadratic form [7-9,20,21].

Definition 2.1 *Consider an N dimensional manifold M which is endowed with a positive scalar function $F(x, y)$ such that*

$$F(x, y) : TM \rightarrow [0, \infty) \tag{2.2}$$

where $x = x^i = (x^0, ..., x^n) \in M$, $y = y^i = (y^0, ..., y^n) \in T_x M$ (tangent space) and TM is the tangent bundle.

The value of $F(x, y)$ corresponds to the length of the vector $y^i \in T_x M$ attached to the point $x^i \in M$. And the function $F(x, y)$ is called Finsler metric function.

Finsler metric function satisfies three basic properties;

i. Regularity;

   $F(x, y)$ is differentiable on the slit tangent bundle $TM \backslash 0$.

ii. Positive Homogeneity;

   $F(x, y)$ is homogenous function of degree one in y. Thus

$$F(x, ky) = kF(x, y) \tag{2.3}$$

for any number $k > 0$.

iii. Strong convexity;

The quadratic form

$$\frac{1}{2}\frac{\partial^2 F(x,y)}{\partial y^i \partial y^j} y^i y^j \tag{2.4}$$

is assumed to be positive definite for all variables $y^i \in T_x M$.

Under these conditions the doublet $(M, F(x,y))$ forms an N dimensional Finsler space.

2.2 Finslerian Metric Tensor and Cartan Torsion Tensor

Theorem 2.1 (Euler's theorem) *Consider any function $Z(x,y)$ which is differentiable and positively homogeneous of degree r with respect to y^i, that is $Z(x,ky) = k^r Z(x,y)$ for any $k > 0$.*

Euler's theorem states that $Z(x,ky) = k^r Z(x,y)$ implies

$$y_i \frac{\partial Z(x,y)}{\partial y^i} = rZ(x,y). \tag{2.5}$$

Definition 2.2 (Finslerian metric tensor) *The Finslerian metric tensor is defined as;*

$$g_{ij} = \frac{1}{2}\frac{\partial^2}{\partial y^i \partial y^j}(F^2(x,y)). \tag{2.6}$$

Since $F(x,y)$ is homogeneous of degree one in y^i, by Euler's theorem, the Finslerian metric tensor $g_{ij}(x,y)$ is homogeneous of degree zero in y.

Another property of Finslerian metric tensor is that it is symmetric in its indices, such that

$$g_{ij}(x,y) = g_{ji}(x,y) \quad . \tag{2.7}$$

Definition 2.3 (Cartan Torsion Tensor) *The Cartan torsion tensor is defined as;*

$$C_{ijk} = \frac{1}{2}\frac{\partial g_{ij}(x,y)}{\partial y^k}. \tag{2.8}$$

Cartan torsion tensor is symmetric in its all indices, and by Euler's theorem it is homogeneous of degree -1 in y^i.

One other important property of Cartan torsion tensor is that it satisfies the relation

$$y^i C_{ijk} = y^j C_{ijk} = y^k C_{ijk} = 0. \tag{2.9}$$

A Finsler geometry will reduce to Riemann geometry, if g_{ij} is assumed to be independent of y^i, that is $C_{ijk} = 0$. Thus all Finslerian relation generalize their Riemannian analogue as a result of presence of Cartan torsion tensor.

2.3 Geodesics in Finsler Spaces

The geodesics of a Finsler space can be defined in a way similar to that of Riemannian geometry.

In Finsler geometry

$$ds = F(x, dx) \quad , \qquad \delta \int F(x, dx) = 0 \tag{2.10}$$

gives the geodesic equations in Finslerian space time.

Consider the functional

$$I(C) = \int_{P_1}^{P_2} F(x, dx), \tag{2.11}$$

where the integration is carried along a curve C, joining two fixed points P_1 and P_2 of the manifold M. The stationary curves of the variational problem $\delta I(C) = 0$ are called Finslerian geodesics.

Since Finslerian metric function is homogeneous of degree 1 in dx, integral can be parameterized as

$$\int_{P_1}^{P_2} F(x, dx) = \int_{P_1}^{P_2} F(x, \frac{dx}{dt})dt \tag{2.12}$$

for any $t = t(s)$ subject to condition $\frac{dt}{ds} \neq 0$, where $y = \dot{x} = \frac{dx}{dt}$. So variational problem takes the form

$$\delta \int_{P_1}^{P_2} F(x, \dot{x})dt = 0, \tag{2.13}$$

where $\dot{x} = (\dot{x}^0, ..., \dot{x}^n) = (\frac{dx^0}{dt}, ..., \frac{dx^n}{dt})$.

Evaluating this integral gives rise to Euler-Lagrange equation for Finslerian geodesics, as

$$\frac{d}{dt}\left(\frac{\partial F(x, \dot{x})}{\partial \dot{x}^i}\right) - \frac{\partial F(x, \dot{x})}{\partial x^i} = 0. \tag{2.14}$$

Since $F^2(x, \dot{x}) = g_{ij}(x, \dot{x})\dot{x}^i\dot{x}^j$, and in the case when the parameter t is chosen to be Finslerian arclength s, the equation (2.14), can be rewritten as

$$\frac{d^2x^i}{ds^2} + \gamma^i_{mn}(x, x')x'^m x'^n = 0, \tag{2.15}$$

where $x' = \frac{dx}{ds}$ and

$$\gamma^i_{mn}(x, x') = \frac{g^{ik}(x, x')}{2}\left\{\frac{\partial g_{mk}(x, x')}{\partial x^n} + \frac{\partial g_{nk}(x, x')}{\partial x^m} - \frac{\partial g_{mn}(x, x')}{\partial x^k}\right\} \tag{2.16}$$

are the Finslerian Christoffel symbols.

Although the Finslerian Christoffel symbols are defined by the same rule as in the Riemannian case, their transformation properties under coordinate transformation $x^i = x^i(x^j)$ differs from the transformation properties of Riemannian Christoffel symbols due to their dependence on tangent vectors y^i.

2.4 Covariant Differentiation

2.4.1 δ -Derivative

If a tensor depends on coordinates alone, its covariant differentiation can be constructed by comparing the transformation of the metric from one tangent space to other tangent space, and the transformation of the tensor.

Consider a vector field $X^i(t)$ along a curve $C : X^i = X^i(t)$. In a new coordinate system $X^{i'}$ is given by the coordinate transformation

$$X^{i'} = X^{i'}(x^j), \tag{2.17}$$

and the vector field transforms as

$$X^i = A^i_{i'} X^{i'}, \tag{2.18}$$

where

$$A^i_{i'} = \frac{\partial x^i}{\partial x^{i'}} \quad . \tag{2.19}$$

Thus

$$\frac{dX^i}{dt} = A^i_{i'} \frac{dX^{i'}}{dt} + (\partial_{j'} A^i_{i'}) X^{i'} \frac{dx^{j'}}{dt} \quad , \tag{2.20}$$

where the term $(\partial_{j'} A^i_{i'}) X^{i'} \frac{dx^{j'}}{dt}$ does not yield a tensor.

To find an appreciate term, the transformation of the metric tensor $g_{ij}(x, \dot{x})$ is considered.

Under the same coordinate transformation (2.17), metric tensor transforms as

$$g_{i'j'}(x^{k'}, \dot{x}^{k'}) = g_{ij}(x^k, \dot{x}^k) A^i_{i'} A^j_{j'}. \tag{2.21}$$

Differentiating with respect to $x^{k'}$, we get

$$\begin{aligned} \frac{\partial g_{i'j'}}{\partial x^{k'}} &= \frac{\partial g_{ij}}{\partial x^k} A^i_{i'} A^j_{j'} A^k_{k'} + g_{ij}(A^i_{i'} \partial_{k'} A^j_{j'} + A^j_{j'} A^i_{i'} \partial_{k'} A^j_{j'} A^i_{i'}) \\ &\quad + \frac{\partial g_{ij}}{\partial \dot{x}^h} \partial_{k'} A^h_{h'} \frac{dx^{h'}}{dt} (A^i_{i'} A^j_{j'}), \end{aligned} \tag{2.22}$$

where $\partial_{k'} = \frac{\partial}{\partial x^{k'}}$, and $\dot{x}^{i'} = A^{i'}_i \dot{x}^i$.

By cyclic interchange of indices of equation (2.22), two similar equations can be written, then summing the first two and subtracting the third one, Christoffel symbols can be expressed as

$$\begin{aligned} \gamma_{i'j'k'} &= A^i_{i'} A^j_{j'} A^k_{k'} \gamma_{ijk} + g_{ik} A^k_{k'} (\partial_{j'} A^i_{i'}) \\ &\quad + C_{ijh} \{ A^i_{j'} A^j_{k'} \partial_{i'} A^h_{h'} + A^i_{k'} A^j_{i'} \partial_{j'} A^h_{h'} - A^i_{i'} A^j_{j'} \partial_{k'} A^h_{h'} \} \dot{x}^{h'}. \end{aligned} \tag{2.23}$$

Then solving for $(\partial_{j'} A^i_{i'}) \dot{x}^{j'}$, we get

$$\begin{aligned} (\partial_{j'} A^i_{i'}) \dot{x}^{j'} &= A^i_{r'} \{ \gamma^{r'}_{i'j'} - g^{r'h'} C_{h'i'l'} \gamma^{l'}_{p'j'} \dot{x}^{p'} \} \dot{x}^{j'} \\ &\quad - A^k_{i'} \{ \gamma^i_{kj} - g^{ih} C_{hkl} \gamma^l_{pj} \dot{x}^p \} \dot{x}^j. \end{aligned} \tag{2.24}$$

By defining

$$P^i_{kj}(x, \dot{x}) = \gamma^i_{kj}(x, \dot{x}) - C^i_{kl}(x, \dot{x}) \gamma^l_{pj}(x, \dot{x}) \dot{x}^p, \tag{2.25}$$

equation (2.24) reduces to

$$(\partial_{j'} A^i_{i'})\dot{x}^{j'} = A^i_{r'} P^{r'}_{i'j'} \dot{x}^{j'} - A^k_{i'} P^i_{kj} \dot{x}^j. \tag{2.26}$$

Substituting this expression into (2.20), and rearranging the terms we get

$$\left(\frac{dX^i}{dt} + P^i_{kj} X^k \dot{x}^j \right) = A^i_{i'} \left(\frac{dX^{i'}}{dt} + P^{i'}_{k'j'} X^{k'} \dot{x}^{j'} \right). \tag{2.27}$$

Definition 2.4 (δ Differentiation of a vector field along a curve) *Now the expression given as*

$$\frac{\delta X^i}{\delta t} = \frac{dX^i}{dt} + P^i_{kj}(x, \dot{x}) X^k \dot{x}^j \tag{2.28}$$

forms the components of a contravariant tensor, and $\frac{\delta X^i}{\delta t}$ is called the δ differentiation (of vector field along a curve).

If a vector field $X^i(x^k)$ is given, then

$$\frac{\partial X^i}{\partial x^j} = A^i_{i'} A^{j'}_j \frac{\partial X^{i'}}{\partial x^{j'}} + A^{j'}_j (\partial_{j'} A^i_{i'}) X^{i'}. \tag{2.29}$$

For $A^{j'}_j (\partial_{j'} A^i_{i'}) X^{i'}$ term, $P^i_{kj}(x, \dot{x})\dot{x}^j$ gives the correct transformation property where $P^i_{kj}(x, \dot{x})$ does not alone. Thus new coefficients $P^*_{ikj}(x, \dot{x})$, are defined as

$$\begin{aligned} P^*_{ikj}(x, \dot{x}) &= \gamma_{ikj}(x, \dot{x}) - \{C_{jkh} P^h_{il}(x, \dot{x}) \\ &\quad + C_{kih} P^h_{jl}(x, \dot{x}) - C_{ijh} P^h_{kl}(x, \dot{x})\} \dot{x}^l. \end{aligned} \tag{2.30}$$

Definition 2.5 (δ differentiation of a vector field) *For a vector field $X^i(x^k)$,*

$$X^i_{;j}(x,\dot{x}) = \frac{\partial X^i(x,\dot{x})}{\partial x^j} + P^{*i}_{hj}(x,\dot{x})X^h(x,\dot{x}) \tag{2.31}$$

is defined as the δ differentiation, and the relation between $\frac{\delta X^i}{\delta t}$ and $X^i_{;j}(x,\dot{x})$ is given as

$$\frac{\delta X^i(x,\dot{x})}{\delta t} = X^i_{;j}(x,\dot{x})\dot{x}^j. \tag{2.32}$$

Thus

$$P^{*h}_{ij}(x,\dot{x})\dot{x}^j = P^h_{ij}(x,\dot{x})\dot{x}^j. \tag{2.33}$$

Due to the symmetries of Christoffel symbols, connection terms P^{*h}_{ij} are also symmetric in their lower indices.

The δ differentiation process can be extended to the differentiation of tensors not only on the position points x^i, but also on a contravariant vector field $\xi^i(x)$. This new form of δ differentiation is defined as;

Definition 2.6 (δ Differentiation of tensors depending on $\xi^i(x)$) *When considering the tensors of the form $X(x^i, \xi^i(x))$, the term $\frac{\partial X^i}{\partial x^j}$ in the equation (2.31) is replaced by $\left\{\frac{\partial X^i}{\partial x^j} + \frac{\partial X^i}{\partial \dot{x}^k}.\frac{\partial \xi^k(x)}{\partial x^j}\right\}$.And δ differentiation of $X(x^i, \xi^i(x))$ is defined as;*

$$\begin{aligned} X^i_{;j}(x,\xi(x)) &= \left\{\frac{\partial X^i}{\partial x^j} + \frac{\partial X^i}{\partial \dot{x}^k}.\frac{\partial \xi^k(x)}{\partial x^j}\right\} \\ &\quad + P^{*i}_{hj}(x,\dot{x})X^h(x,\xi(x)). \end{aligned} \tag{2.34}$$

This equation can be extended to define δ differentiation of an arbitrary tensor $T^{i_1 \ldots i_r}_{j_1 \ldots j_s}$, with respect to x^k in the direction of $\dot{x}$, such as

$$\begin{aligned} T^{i_1 \ldots i_r}_{j_1 \ldots j_s \ ; \, k} &= \frac{\partial}{\partial x^k} T^{i_1 \ldots i_r}_{j_1 \ldots j_s} \\ &+ \frac{\partial}{\partial \dot{x}^k} (T^{i_1 \ldots i_r}_{j_1 \ldots j_s}) \frac{\partial x^{i_h}}{\partial x^k} \\ &+ \sum_{\mu=1}^{r} T^{i_1 \ldots i_{\mu-1} h i_{\mu+1} \ldots i_r}_{j_1 \ldots j_s} P^{* i_\mu}_{hk}(x, \dot{x}) \\ &- \sum_{\nu=0}^{s} T^{i_1 \ldots i_r}_{j_1 \ldots j_{\nu-1} h j_{\nu+1} \ldots j_s} P^{*h}_{j_\nu k}(x, \dot{x}). \end{aligned} \tag{2.35}$$

δ differentiation obeys the following properties;

i. The δ derivative of the sums of two tensors is equal to the sum of δ derivative of the tensors.

ii. The δ derivative obeys the same product rules as the ordinary derivative.

iii. The δ derivative of a scalar is its ordinary derivative.

Consider the δ derivative of the metric tensor $g_{ij}(x, \xi)$, in the direction $\dot{x}^i$ corresponding to same line element (x, ξ),

$$\begin{aligned} g_{ij \ ;k}(x, \xi) &= \frac{\partial g_{ij}(x, \xi)}{\partial x^k} + 2C_{ijh}(x, \xi) \frac{\partial \xi^h}{\partial x^k} \\ & \quad - g_{hj}(x, \xi) P^{*h}_{ik}(x, \dot{x}) - g_{hi}(x, \xi) P^{*h}_{jk}(x, \dot{x}). \end{aligned} \tag{2.36}$$

In this equation, $\frac{\partial \xi^h}{\partial x^k}$ has to be specified. By choosing $\dot{x}^i = \xi^i$, equation (2.36) can be simplified as

$$\begin{aligned} g_{ij \ ;k}(x, \xi) &= 2C_{ijh}(x, \xi) \left\{ \frac{\partial \xi^h}{\partial x^k} + P^{*h}_{lk}(x, \dot{x}) \xi^l \right\} \\ &= 2C_{ijh}(x, \xi) \xi^h_{;k}. \end{aligned} \tag{2.37}$$

This result represents the generalization of Ricci lemma of Riemannian geometry. An immediate consequence of the fact that the δ covariant derivative of the metric tensor with arbitrary argument does not vanish is the fact that under parallel displacement of a vector X^i, the length does not remain invariant.

The change in length of a vector X^i from a point $P(x^i)$ to another point $Q(x^i + dx^i)$, can be written as

$$\frac{\delta}{\delta s}(g_{ij}(x,X)X^iX^j)ds = \frac{\delta g_{ij}(x,X)}{\delta s}X^iX^j. \tag{2.38}$$

However if Γ is a unique geodesic of Finslerian manifold F^n joining the points P and Q, the scalar product of X^i with the tangent vector $\dot{x}^i$ to the geodesic at P, $g_{ij}(x,\dot{x})\dot{x}^iX^j$, remains constant, due to property (2.9) of Cartan torsion tensor.

Thus the length of the vector remains invariant under displacement if the displacement is taken in the direction of the vector.

2.4.2 Cartan Covariant Derivative

When considering tensors which are functions of independent variables of positions (x^i) and directions $(y^i = \dot{x}^i)$, δ differentiation is not sufficient. Also in δ differentiation covariant derivative of the metric function does not generally vanish. But Ricci lemma states that, covariant derivative of the metric tensor should vanish so that the space can be regarded to be locally Minkowskian. In

Finsler spaces, Cartan covariant derivative is constructed so that an analogue of Ricci lemma is valid. This construction is achieved by Euclidian connection of Cartan.

To endow the Finsler space F^n with an Euclidian connection, Cartan considers the manifold X_{2n-1} of the line elements (element of support), $(x^i, \dot{x}^i)$, which is $(2n-1)$ dimensional since only ratios of the $\dot{x}^i$ are necessary to define the direction in the tangent space $T_n(X)$.

In the space F^n a metric is defined by means of Finsler metric function $F(x^i, \dot{x}^i)$, but the manifold X_{2n-1} is said to be endowed with an Euclidian connection, if the following construction is imposed on X_{2n-1}. These construction is also called the fundamental postulates of Cartan such that

1. A metric tensor $g_{ij}(x, \dot{x})$ is given such that the square of distance between two neighboring points $(x^i, \dot{x}^i)$ and $(x^i + dx^i, \dot{x}^i + d\dot{x}^i)$ is given by;

$$ds^2 = g_{ij}(x, \dot{x}) dx^i dx^j. \tag{2.39}$$

2. Variation of vector X^i, when $(x^i, \dot{x}^i)$ goes to an infinitesimal change, becoming $(x^i + dx^i, \dot{x}^i + d\dot{x}^i)$ is represented by covariant differentiation

$$DX^i = dX^i + C^i_{kh} X^k d\dot{x}^h + \Gamma^i_{kh} X^k dx^h, \tag{2.40}$$

where C^i_{kh} and Γ^i_{kh} are functions of element of support $(x, \dot{x})$.

If a vector is parallel displaced, length of X^i should remain invariant. Here

$$dX^i = -C^i_{kh} X^k d\dot{x}^h - \Gamma^i_{kh} X^k dx^h, \tag{2.41}$$

and introducing the notation

$$\omega^i_k = C^i_{kh} d\dot{x}^h + \Gamma^i_{kh} dx^h, \tag{2.42}$$

and differentiating $g_{ij}(x, \dot{x}) X^i X^j$, invariance of X^i under parallel displacement implies

$$dg_{ij} = \omega_{ij} + \omega_{ji}. \tag{2.43}$$

Hence C^i_{jh} and Γ^i_{jh} should satisfy the relations

$$\frac{\partial g_{ij}}{\partial x^h} = \Gamma_{ijh} + \Gamma_{jih}, \tag{2.44}$$

$$\frac{\partial g_{ij}}{\partial \dot{x}^h} = C_{ijh} + C_{jih}. \tag{2.45}$$

3. a. If the direction of a vector X^i coincides with that of its element of support $(x, \dot{x})$, then its arclength is to be equal to $F(x, \dot{x})$, where

$$F^2(x, \dot{x}) = g_{ij}(x, \dot{x}) \dot{x}^i \dot{x}^j. \tag{2.46}$$

   b. Let X^i and Y^i represent two vectors with a common element of support $(x^k, \dot{x}^k)$. When the latter performs an infinitesimal rotation about its own center x^k, thus becoming $(x^k, \dot{x}^k + d\dot{x}^k)$, the following symmetry condition is required.

$$g_{ij}(x,\dot{x})X^i DY^j = g_{ij}(x,\dot{x})Y^i DX^j \tag{2.47}$$

This equation can be simplified as

$$C^k_{ih} Y^k X^i d\dot{x}^h = C^k_{ih} X^k Y^i d\dot{x}^h. \tag{2.48}$$

This equation is satisfied only if the symmetry relation $C_{kih} = C_{ikh}$ holds, and from equation (2.45), it can be deduced that

$$C_{ijh} = \frac{1}{2}\frac{\partial g_{ij}}{\partial \dot{x}^h}. \tag{2.49}$$

c. If the direction of a vector with fixed components x^i, coincides with that of its element of support, then its covariant differential corresponding to an infinitesimal rotation of its element of support about its own center vanishes identically. This implies

$$C^i_{kh}\dot{x}^k d\dot{x}^h = 0. \tag{2.50}$$

Since this is to hold for all possible values of dx^i then $C_{kih}\dot{x}^k = 0$.

d. The coefficients which appear in the covariant differential when the displacements are such that element of support is transported parallel to itself, are symmetric in their lower indices.

Consider a unit vector l^i in the direction of element of support $(x^i, \dot{x}^i)$, then

$$l^i = \frac{\dot{x}^i}{F(x, \dot{x})}. \tag{2.51}$$

Its covariant differential can be calculated as

$$Dl^i = dl^i + \Gamma^i_{kh} l^k dx^h, \tag{2.52}$$

and when l^i is displaced parallel to itself, then $Dl^i = 0$. Thus

$$d\left(\frac{\dot{x}^i}{F(x, \dot{x})}\right) = -\Gamma^i_{kh} l^k dx^h, \tag{2.53}$$

or

$$d\dot{x}^i = \dot{x}^i \frac{dF}{F} - \Gamma^i_{kh} \dot{x}^k dx^h. \tag{2.54}$$

Substituting this result into equation (2.40), we get

$$DX^i = dX^i + \Gamma^{*i}_{kj} X^k dx^j, \tag{2.55}$$

where $\Gamma^{*i}_{kj} = \Gamma^i_{kj} - C^{ikh}\Gamma^h_{jr}\dot{x}^r$. And from postulate (3.d), $\Gamma^{*i}_{kj} = \Gamma^{*i}_{jk}$, so

$$\Gamma_{kij} - \Gamma_{jik} = (C_{kih}\Gamma^h_{ij} - C_{ijh}\Gamma^h_{ik})\dot{x}^k. \tag{2.56}$$

By combining this equation and equation (2.44), we get a unique relation for Γ^i_{jk}, such that

$$\Gamma_{kij} = \gamma_{kij} - C_{jhi}\frac{\partial G^h}{\partial \dot{x}^k} + C_{kjh}\frac{\partial G^h}{\partial \dot{x}^i}, \tag{2.57}$$

where G^i is defined as

$$2G^i(x, \dot{x}) = \gamma^i_{hk}(x, \dot{x})\dot{x}^h\dot{x}^k. \tag{2.58}$$

and $\frac{\partial G^i(x,\dot{x})}{\partial \dot{x}^l}$ can be calculated as

$$\begin{aligned}\frac{\partial G^i(x,\dot{x})}{\partial \dot{x}^l} &= \frac{1}{2}\frac{\partial}{\partial \dot{x}^l}(\gamma^i_{hk}\dot{x}^h\dot{x}^k)\\ &= P^i_{lh}\dot{x}^h\\ &= \Gamma^i_{lh}\dot{x}^h, \qquad (2.59)\end{aligned}$$

where property (2.9) of Cartan torsion tensor is used in calculations of this equality.

Definition 2.7 (Cartan Covariant Differentiation) *Covariant differential of a vector field $X^i = X^i(x,\dot{x})$ in equation (2.40) can also be written in a different form if $d\dot{x}^i$ term is replaced by the covariant differential of the unit vector l^i,*

$$d\dot{x}^h = FDl^h + \dot{x}^h\frac{dF}{F} - \Gamma^h_{rs}. \qquad (2.60)$$

So that the equation (2.40) can be written as

$$DX^i = X^i\mid_h Dl^h + X^i_{|h}dx^h, \qquad (2.61)$$

where DX^i is the differential, and Cartan covariant derivative is defined as

$$X^i\mid_h = F\frac{\partial X^i}{\partial \dot{x}^h} + A^i_{kh}X^k \quad , \qquad (A^i_{kj} = FC^i_{kj}). \qquad (2.62)$$

Also

$$X^i_{|h} = \frac{\partial X^i}{\partial x^h} - \frac{\partial X^i}{\partial \dot{x}^k}\frac{\partial G^k}{\partial \dot{x}^h} + \Gamma^{*i}_{kh}X^k, \qquad (2.63)$$

is defined as the covariant derivative with respect to x^k. *The expression* $X^i_{|h}dx^h$ *would represent the variation of* X^i, *if the element of support were transformed by parallel displacement from point* (x^i) *to* $(x^i + dx^i)$.

Clearly above construction of a covariant differential of tensor in terms of covariant differential of the unit vector in the direction of element of support is applicable to tensors of any rank.

2.5 Curvature

In previous section it is mentioned that there exists two types of covariant differentiation in Finsler spaces. So, this two different differentiations result two cases for curvature, one of which is resulting from δ differentiation, and the other one arises from Cartan covariant derivative.

2.5.1 Curvature Tensors Resulting from δ Differentiation

Consider a vector field $X^i(x^k, \xi^k)$ such that the vector field ξ^k depends on position x^i. Then the δ derivative of this vector field $X^i(x^k, \xi^k)$ at the point x^k in the direction of ξ^k is given by

$$X^i_{;h} = \frac{\partial X^i}{\partial x^h} + \frac{\partial X^i}{\partial \dot{x}^l} . \frac{\partial \xi^l}{\partial x^h} + \Gamma^{*i}_{rh} X^r. \tag{2.64}$$

The curvature tensor can be found by straightforward calculation of the commutator of the δ derivative, such that

$$X^i_{;h;k} - X^i_{;k;h} = \tilde{K}^i_{jhk}(x,\xi)X^j, \tag{2.65}$$

where

$$\begin{aligned}\tilde{K}^i_{jhk} &= \left(\frac{\partial \Gamma^{*i}_{jh}}{\partial x^k} + \frac{\partial \Gamma^{*i}_{jh}}{\partial \dot{x}^l}\frac{\partial \xi^l}{\partial x^k}\right) \\ &\quad - \left(\frac{\partial \Gamma^{*i}_{jk}}{\partial x^h} + \frac{\partial \Gamma^{*i}_{jk}}{\partial \dot{x}^l}\frac{\partial \xi^l}{\partial x^h}\right) \\ &\quad + \Gamma^{*i}_{mk}\Gamma^{*m}_{jh} - \Gamma^{*i}_{mh}\Gamma^{*m}_{jk}. \end{aligned} \tag{2.66}$$

The tensor $\tilde{K}^i_{jhk}$ is called the relative curvature tensor since it depends on the derivatives $\frac{\partial \xi^l}{\partial x^k}$ of the vector field ξ^l.

Suppose that the vector field ξ^l is stationary at the point point under consideration, then it satisfies the relation $\xi^l_{;k}(x,\xi) = 0$. Then at this particular point

$$\frac{\partial \xi^l}{\partial x^h} = -\frac{\partial G^l(x,\xi)}{\partial \dot{x}^h} \tag{2.67}$$

is satisfied. So a new tensor can be defined as

$$\begin{aligned}K^i_{jhk} = &= \left(\frac{\partial \Gamma^{*i}_{jh}}{\partial x^k} - \frac{\partial \Gamma^{*i}_{jh}}{\partial \dot{x}^l}\frac{\partial G^l}{\partial x^k}\right) \\ &\quad - \left(\frac{\partial \Gamma^{*i}_{jk}}{\partial x^h} - \frac{\partial \Gamma^{*i}_{jk}}{\partial \dot{x}^l}\frac{\partial G^l}{\partial x^h}\right) \\ &\quad + \Gamma^{*i}_{mk}\Gamma^{*m}_{jh} - \Gamma^{*i}_{mh}\Gamma^{*m}_{jk}, \end{aligned} \tag{2.68}$$

which was called the K tensor of curvature.

2.5.2 Curvature Tensors of Cartan

Covariant derivation of Cartan was given by (2.61), and curvature tensors can be derived by evaluating the commutator relations of Cartan derivatives. In the previous section of covariant derivative, two distinct processes of partial differentiation was involved, namely $_{|h}$ and $|_h$. In order to obtain a complete set of commutations, mixed derivatives involving one or both of the processes has to be considered.

First, consider the commutation relation of $|_h$ derivative.

$$\begin{aligned} X^i|_h|_k - X^i|_k|_h &= F\left\{\frac{\partial F}{\partial \dot{x}^k}\frac{\partial x^i}{\partial \dot{x}^h} - \frac{\partial F}{\partial \dot{x}^h}\frac{\partial x^i}{\partial \dot{x}^k}\right\} \\ &\quad +X^r\{F\left(\frac{\partial A^i_{rh}}{\partial \dot{x}^k} - \frac{\partial A^i_{rk}}{\partial \dot{x}^h}\right) \\ &\quad +A^i_{km}A^m_{rh} - A^i_{mh}A^m_{rk}\}. \end{aligned} \tag{2.69}$$

Since $F\left(\frac{\partial A^i_{rh}}{\partial \dot{x}^k} - \frac{\partial A^i_{rk}}{\partial \dot{x}^h}\right) = \frac{\partial F}{\partial \dot{x}^k}A^i_{rk} - \frac{\partial F}{\partial \dot{x}^h}A^i_{rk}$, then equation (2.69) can be written as

$$\begin{aligned} X^i|_h|_k - X^i|_k|_h &= \left\{\frac{\partial F}{\partial \dot{x}^k}X^i|_h - \frac{\partial F}{\partial \dot{x}^h}X^i|_k\right\} \\ &\quad +S^i_{jkh}X^j, \end{aligned} \tag{2.70}$$

where

$$S^i_{jkh} = A^i_{kr}A^r_{jh} - A^i_{hr}A^r_{jk} \tag{2.71}$$

is the first of Cartan's curvature tensors. It is related to the curvature of the

Minkowski spaces at a point x^i.

At a fixed point x^i, the Finslerian metric function will play the role of the metric tensor of the tangent Minkowski space so that

$$ds^2_{minkowskian} = g_{ij}(x, y) dy^i dy^j. \tag{2.72}$$

By usual Riemannian geometry, the Christoffel symbols of Minkowski space are

$$\gamma^k_{ij} = \frac{1}{2} g^{kn} \left(\frac{\partial g_{in}}{\partial y^j} + \frac{\partial g_{jn}}{\partial y^i} - \frac{\partial g_{ij}}{\partial y^n} \right), \tag{2.73}$$

and they reduce to Cartan torsion tensor. The curvature tensor of the tangent Minkowski space can be calculated by ordinary Riemannian geometry rules. It is found that the curvature is given by $(F^{-2} S^i_{jkn})$.

Secondly, considering the commutation relation involving both $_{|h}$ and $|_h$, second curvature tensor of Cartan can be found, such that

$$\begin{aligned} X^i|_{h|k} - X^i_{|k}|_h &= F \frac{\partial X^i}{\partial \dot{x}^l} A^l_{hk|r} l^r \\ &\quad - \left\{ F \frac{\partial \Gamma^{*i}_{rk}}{\partial \dot{x}^h} - A^i_{hr|k} \right\} X^r \\ &\quad + A^r_{hk} X^i_{|r}. \end{aligned} \tag{2.74}$$

Eliminating the term $\frac{\partial X^i}{\partial \dot{x}^l}$ equation (2.74) will be written

$$\begin{aligned} X^i|_{h|k} - X^i_{|k}|_h &= -P^i_{jhk} X^j + X^i|_j A^i_{hk|lr} l^r \\ &\quad + X^i_{|j} A^j_{hk}, \end{aligned} \tag{2.75}$$

where

$$P^i_{jhk} = F\frac{\partial \Gamma^{*i}_{jk}}{\partial \dot{x}^h} + A^i_{jm}A^m_{hk|r} - A^i_{jh|k}, \tag{2.76}$$

is the second of Cartan's curvature tensor.

The curvature tensors S_{ijmn} and P_{ijmn} vanish in the Riemann case.

In order to find Cartan's third curvature tensor, commutator of $_{|h}$ type covariant derivatives are calculated, such that

$$X^i_{|h|k} - X^i_{|k|h} = R^i_{jhk}X^j - K^i_{rhk}l^r X^i_{|j}, \tag{2.77}$$

where $R^i_{jhk} = K^i_{jhk} + C^i_{jm}K^m_{rhk}\dot{x}^r$.

This tensor is the third curvature tensor of Cartan, and can be written in explicit form

$$\begin{aligned} R^i_{jhk} \ = \ & \left(\frac{\partial \Gamma^{*i}_{jh}}{\partial x^k} - \frac{\partial \Gamma^{*i}_{jh}}{\partial \dot{x}^l}\frac{\partial G^l}{\partial x^k}\right) \\ & - \left(\frac{\partial \Gamma^{*i}_{jk}}{\partial x^h} - \frac{\partial \Gamma^{*i}_{jk}}{\partial \dot{x}^l}\frac{\partial G^l}{\partial x^h}\right) \\ & + C^i_{jm}\left(\frac{\partial^2 G^m}{\partial \dot{x}^h \partial x^k} - \frac{\partial^2 G^m}{\partial \dot{x}^k \partial x^h} - G^m_{hl}\frac{\partial G^l}{\partial \dot{x}^k} + G^m_{kl}\frac{\partial G^l}{\partial \dot{x}^h}\right) \\ & + \Gamma^{*i}_{mk}\Gamma^{*m}_{jh} - \Gamma^{*i}_{mh}\Gamma^{*m}_{kj}. \end{aligned} \tag{2.78}$$

This tensor is Finslerian generalization of the Riemann curvature tensor.

CHAPTER 3

GEOMETRIZATION OF ELECTROMAGNETISM IN FINSLERIAN SPACES

3.1 Finsler Gauge Transformations

If a particle in a space-time moves along a curved, non-geodesic path, then it is said that the particle is under the influence of some external force. In such a case, an external force term is added to the equation of motions to explain the path of motion. Alternative point of view is that motion can be explained by a new metric, which would result from a gauge transformation. In this way, physical force fields can be geometrized, and general relativistic idea of space-time curvature determining the path of the particle will also include fields other than gravitation. For this purpose a class of gauge transformations which act on tangent space is considered [22].

Under these kind of transformations, the tangent vector y^μ transforms as

$$\overline{y}^\mu = Y^{*\mu}_\nu y^\nu, \tag{3.1}$$

where $\mu, \nu, ... = 0, 1, 2, 3$ are indices corresponding the space components, and

$$Y^{*\mu}_\nu = \frac{\partial \overline{y}^\mu}{\partial y^\nu}, \tag{3.2}$$

$$Y^{*\mu}_\alpha Y^\alpha_\nu = \delta^\mu_\nu, \tag{3.3}$$

where, $Y^\mu_\nu = \frac{\partial y^\mu}{\partial \overline{y}^\nu}$ is the inverse transformation, and these transformations (Y^μ_ν) are called Y transformations.

Even though the transformation does not act on the base space coordinates, it will seen to produce changes in the base space. Thus, these transformations also depend on the base coordinates, such as

$$Y^{*\mu}_\nu = Y^{*\mu}_\nu(x, y). \tag{3.4}$$

The Y transformation of the metric tensor is given as

$$\overline{g}_{\mu\nu}(x, y) = Y^\alpha_\mu(x, y) Y^\beta_\nu g_{\alpha\beta}(x, y). \tag{3.5}$$

Under this transformation, Finsler metric function is invariant, such as

$$\overline{F}^2(x, \overline{y}) = \overline{g}_{\mu\nu}\overline{y}^\mu \overline{y}^\nu$$

$$= g_{\alpha\beta}(x,y) Y^{\alpha}_{\mu} Y^{\beta}_{\nu} Y^{*\mu}_{\gamma} Y^{*\nu}_{\sigma} y^{\gamma} y^{\sigma}$$

$$= F^2(x,y). \tag{3.6}$$

Here y^{ν} is the contravariant vector and the covariant vector associated with it is y_{μ}, where $y_{\mu} = g_{\mu\nu} y^{\nu}$. Covariant vector y_{μ} transforms as

$$\overline{y}_{\mu} = Y^{\alpha}_{\mu} y_{\alpha}. \tag{3.7}$$

Since

$$\begin{aligned} \frac{\partial \overline{y}_{\mu}}{\partial \overline{y}^{\nu}} &= \overline{g}_{\mu\nu} \\ &= Y^{\alpha}_{\mu} Y^{\beta}_{\nu} g_{\alpha\beta} + Y^{\beta}_{\nu} \frac{\partial Y^{\alpha}_{\mu}}{\partial y^{\beta}} y_{\alpha}. \end{aligned} \tag{3.8}$$

The Y transformation of the Finslerian metric tensor does not yield a tensor unless

$$\frac{\partial Y^{\alpha}_{\mu}}{\partial y^{\beta}} y_{\alpha} = 0. \tag{3.9}$$

The condition (3.9) is called as the metric condition [7].

It is obvious that Y transformations, when Y^{μ}_{ν} is a function of x only, that is

$$Y^{\mu}_{\nu} = Y^{\mu}_{\nu}(x), \tag{3.10}$$

satisfy the metric condition. These type of transformations are called K-group or linear transformations [7].

Y transformations can be interpreted as the transformations from an original space where there exists no external field, to a space that also contains external fields which are turned on by some physical potentials contained in Y^μ_ν [23].

A specific example to Y transformations was given as [22],

$$Y^\mu_\nu = \delta^\mu_\nu - B^{-2}\left\{1-(1+kB^2)^{\frac{1}{2}}\right\} B^\mu_\nu, \tag{3.11}$$

where B_ν is a vector which can be associated to a physical potential, and $B^2 = g_{\mu\nu}B^\mu B^\nu$. Here k is a constant depending on the physical space that will be geometrized. The inverse transformation is given by the inverse of the matrix (3.11), such as

$$Y^{*\nu}_\mu = \delta^\nu_\mu - B^{-2}\left\{1-(1+kB^2)^{-\frac{1}{2}}\right\} B^\nu B_\mu. \tag{3.12}$$

3.2 Charged Classical Particle in Finsler Space-time

In this section, an original metric tensor is used to produce the Finsler metric function by a specific Y transformation. The original metric is assumed to be Minkowskian for simplicity. In this case gravitational field effects are neglected, but even in the presence of electromagnetic fields alone, the physical space-time can be described as curved Finsler space-time. The results calculated are same as usual classical electrodynamics which is based on the flat Minkowski space-time, with an additional electromagnetic field [22, 25].

3.2.1 Geodesic Equation

The original metric is chosen as ordinary Minkowskian metric $\eta_{\mu\nu}$ in the form

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{3.13}$$

After applying Y transformation (3.11) to this metric, the resulting metric will be

$$\overline{g}_{\mu\nu} = \eta_{\mu\nu} + kB_{\mu}B_{\nu}. \tag{3.14}$$

In this case, vector B_{μ} is related to electromagnetic vector potential A_{μ}. The contravariant form of the metric tensor (3.14) can be written as

$$\overline{g}^{\mu\nu} = \eta^{\mu\nu} - k(1 + kB^2)^{-1}B^{\mu}B^{\nu}, \tag{3.15}$$

where $B^2 = \eta_{\alpha\beta}B^{\alpha}B^{\beta}$, so that

$$\overline{g}^{\mu\gamma}\overline{g}_{\gamma\nu} = \delta^{\mu}_{\nu}. \tag{3.16}$$

If we calculate the geodesic equation resulting from the new metric (3.14), we get

$$\frac{dy^{\mu}}{d\tau} + kB_{\lambda}y^{\lambda}\left(\frac{\partial B_{\mu}}{\partial x^{\alpha}} - \frac{\partial B_{\alpha}}{\partial x^{\mu}}\right)y^{\alpha} = 0, \tag{3.17}$$

where $y^\alpha = \frac{dx^\alpha}{d\tau}$, and τ is the proper time.

Since we deal with the geometrization of electrodynamics, with conditions

$$B_\mu y^\mu = \frac{e}{mck} \tag{3.18}$$

and

$$B_\mu = A_\mu, \tag{3.19}$$

where e is the charge of the electron, m is the mass of the electron and c is the velocity of light and k is a constant and will be determined by the field equations. The geodesic equation (3.17) will take the form

$$\frac{dy^\mu}{d\tau} + \frac{e}{mc} F_{\mu\nu} y^\nu = 0, \tag{3.20}$$

where

$$F_{\mu\nu} = \left(\frac{\partial A_\mu}{\partial x^\nu} - \frac{\partial A_\nu}{\partial x^\mu} \right) \tag{3.21}$$

is the electromagnetic field tensor.

The geodesic equation (3.20) is identical with the Lorentz equation in Minkowskian space-time, with corresponding velocity y^μ.

An important point is that the laws of physics must be invariant under arbitrary gauge transformations. If we consider an electromagnetic gauge transformation

$$\overline{A}_\mu = A_\mu + \frac{\partial\Lambda(x)}{\partial x^\mu} \quad , \tag{3.22}$$

where $\Lambda(x)$ is any arbitrary function, the metric tensor (3.14) is invariant and the geodesic equation (3.17) remains unchanged.

3.2.2 Field Equations

By introducing the condition (3.18), the velocity dependent metric (3.14) reduces to a Riemannian metric. So field equations are calculated by Riemann geometry.

The Ricci tensor for the metric (3.14) is calculated as

$$\begin{aligned} R_{\eta\gamma} \;=\; & -\frac{1}{4}k^2\overline{g}^{\alpha\gamma}\overline{g}^{\tau\mu}F_{\tau\lambda}F_{\alpha\mu}B_\eta B_\gamma - \frac{1}{2}k\overline{g}^{\mu\lambda}F_{\gamma\lambda}F_{\eta\mu} \\ & -\frac{1}{2}k^2(1+kB^2)^{-1}\eta^{\mu\lambda}F_{\alpha\mu}B^\alpha(B_\eta B_{\lambda,\gamma} + B_\gamma B_{\lambda,\eta}) \\ & -\frac{1}{2}k^2(1+kB^2)^{-1}\overline{g}^{\mu\lambda}B_{\mu,\lambda}B^\alpha(B_\eta F_{\gamma\alpha} + B_\gamma F_{\eta\alpha}) \\ & +\frac{1}{2}k\overline{g}^{\mu\lambda}(F_{\eta\lambda,\mu}B_\gamma + F_{\gamma\lambda,\mu}B_\eta \\ & -\frac{1}{2}k(1+kB^2)^{-1}\eta^{\mu\lambda}B_{\gamma,\lambda}B_{\eta,\mu} \\ & -k\left[\frac{1}{2}(1+kB^2)^{-1}\eta^{\mu\lambda} - k(1+kB^2)^{-2}B^\mu B^\lambda\right]B_{\lambda,\gamma}B_{\mu,\eta} \\ & +\frac{1}{2}(1+kB^2)^{-1}\overline{g}^{\mu\lambda}B_{\mu,\lambda}(B_{\gamma,\eta} + B_{\eta,\gamma}) \\ & +\frac{1}{2}k(1+kB^2)^{-1}B^\alpha(F_{\alpha\gamma,\eta} + F_{\alpha\eta,\gamma}), \end{aligned} \tag{3.23}$$

where $_{,\mu}$ denotes $\frac{\partial}{\partial x^\mu}$.

The curvature scalar, $R = \overline{g}^{\eta\gamma} R_{\eta\gamma}$ is found to be

$$\begin{aligned} R \;\; = \;\; & -\frac{1}{4}k^2 B^2 (1+kB^2)^{-1} \overline{g}^{\alpha\lambda} \overline{g}^{\tau\mu} F_{\tau\lambda} F_{\alpha\mu} \\ & -\frac{1}{2}k \overline{g}^{\eta\gamma} \overline{g}^{\mu\lambda} F_{\gamma\lambda} F_{\eta\mu} \\ & +2k(1+kB^2)^{-1} \overline{g}^{\mu\lambda} B^\eta F_{\eta\lambda,\mu} \\ & -k(1+kB^2)^{-1} \overline{g}^{\mu\lambda} \overline{g}^{\eta\gamma} (B_{\gamma,\lambda} B_{\eta,\mu} - B_{\mu,\lambda} B_{\eta,\gamma}) \\ & -\frac{1}{2}k^2 (1+kB^2)^{-2} \eta^{\mu\lambda} B^\eta B^\gamma F_{\eta\mu} F_{\gamma\lambda}. \end{aligned} \tag{3.24}$$

If the two highest order terms of these equations are considered, then Einstein tensor can be written as

$$\begin{aligned} G_{\eta\gamma} \;\; = \;\; & R_{\eta\gamma} - \frac{1}{2} \overline{g}_{\eta\gamma} R \\ = \;\; & -\frac{1}{2}k^2 \overline{g}^{\alpha\lambda} \overline{g}^{\tau\mu} F_{\tau\lambda} F_{\alpha\mu} B_\eta B_\gamma \\ & +\frac{1}{8}k \overline{g}_{\eta\gamma} \overline{g}^{\alpha\lambda} \overline{g}^{\tau\mu} F_{\alpha\mu} F_{\lambda\tau} \\ & -\frac{1}{2}k \overline{g}^{\mu\lambda} F_{\gamma\lambda} F_{\eta\mu} \\ & -\frac{1}{8}k B^{-2} \overline{g}^{\alpha\lambda} \overline{g}^{\tau\mu} F_{\alpha\mu} F_{\lambda\tau} B_\eta B_\gamma \\ & -\frac{1}{2}k B3{-}2\eta^{\mu\lambda} F_{\alpha\mu} B3\alpha (B_\eta B_{\lambda,\gamma} + B_\gamma B_{\lambda,\eta}) \\ & +\frac{1}{2}k \overline{g}^{\mu\lambda} (F_{\eta\lambda,\mu} B_\gamma + F_{\gamma\lambda,\mu} B_\eta) \\ & -k B^{-2} \overline{g}^{\mu\lambda} B^\alpha F_{\alpha\gamma,\mu} B_\eta B_\gamma \\ & +\frac{1}{2}k B^{-2} \overline{g}^{\mu\lambda} \overline{g}^{\alpha\tau} (B_{\alpha,\lambda} B_{\tau,\mu} - B_{\mu,\lambda} B_{\tau,\alpha}) B_\eta B_\gamma \end{aligned}$$

$$+\frac{1}{4}kB^{-4}\eta^{\mu\lambda}B^{\alpha}B^{\tau}F_{\alpha\mu}F_{\tau\lambda}B_{\eta}B_{\gamma}. \tag{3.25}$$

Again by taking the highest order terms and by same simplifications, equation (3.25) reduces to

$$\begin{aligned} G_{\eta\gamma} &= \frac{1}{2}k^2\overline{g}^{\kappa\lambda}\overline{g}^{\sigma\rho}F_{\sigma\lambda}F_{\rho\kappa}B_{\mu}B_{\nu} \\ &+\frac{1}{2}k\left(g^{\kappa\lambda}F_{\mu\kappa}F_{\lambda\nu}+\frac{1}{4}g_{\mu\nu}g^{\kappa\lambda}g^{\sigma\rho}F_{\sigma\lambda}F_{\rho\kappa}\right) \end{aligned} \tag{3.26}$$

It is accepted that the field equations of a particle under the influence of an electromagnetic field will be

$$G_{\eta\gamma} = 8\pi\kappa c^{-4}(\rho_0 v_{\eta}v_{\gamma} + T_{\eta\gamma}), \tag{3.27}$$

where κ is the gravitational constant, ρ_0 is the proper matter density and $T_{\eta\gamma}$ is the electromagnetic energy tensor. From classical Riemannian geometry, the electromagnetic energy tensor is

$$\overline{T}_{\eta\gamma} = \frac{1}{4\pi}\left(\overline{g}^{\alpha\lambda}F_{\eta\lambda}F_{\gamma\alpha} - \frac{1}{4}\overline{g}_{\eta\gamma}\overline{g}^{\mu\alpha}\overline{g}^{\nu\beta}F_{\mu\nu}F_{\alpha\beta}\right). \tag{3.28}$$

If we compare the electromagnetic energy tensor (3.28) with Einstein tensor (3.25) calculated from metric (3.14), a value for the constant k can be determined as $k = \frac{4\kappa}{c^{-4}}$.

By this relation, electromagnetic energy tensor, has appeared as part of Einstein tensor. And also the matter density has appeared as part of curvature.

Since everything is expressed in terms of curvature tensor, electromagnetic field is completely geometrized. An important consequence of comparison of equations (3.25) and (3.28) is that the particle mass can be derived from electromagnetic field [28-30].

3.3 General Finsler Spaces

For more general purposes, a general class of Finsler spaces is given [27] which is determined by the metric function as

$$F(x,y) = \left[(\eta_{\alpha\beta} + kB_\alpha B_\beta)y^\alpha y^\beta\right]^{\frac{1}{2}} \tag{3.29}$$

For this metric y^μ can be identified as the velocity $v^\alpha = \frac{dx^\alpha}{d\tau}$. The possible choices of $B^\mu = B^\mu(x,y)$ are determined by the physical system that is being modelled.

In previous section, condition (3.19) was determined to geometrize the electrodynamics, and the resulting space contains electromagnetic field as an intrinsic property.

In general, there exists no restrictions on $B^\mu = B^\mu(x,y)$, except homogeneity, which comes from the homogeneity condition (2.3) of Finslerian metric function, such as

$$B_\mu(x,ky) = kB_\mu(x,y). \tag{3.30}$$

This is the only condition restricted on $B_\mu(x, y)$.

The general form of the metric tensor is then

$$\begin{aligned} f_{\alpha\beta} &= \frac{1}{2}\frac{\partial^2 F^2(x,y)}{\partial y^\alpha \partial y^\beta} \\ &= \eta_{\alpha\beta} + k\tilde{B}_\alpha \tilde{B}_\beta + kB_\nu y^\nu \frac{\partial \tilde{B}_\beta}{\partial y^\alpha}, \end{aligned} \tag{3.31}$$

where

$$\tilde{B}_\mu = \frac{\partial}{\partial y^\mu}(B_\nu y^\nu). \tag{3.32}$$

In previous section, by choosing $B_\mu = A_\mu$ we get the metric (3.14). And for simplicity the original metric is chosen as the ordinary Minkowskian metric (3.13). In fact it is also possible to use a metric $g_{\mu\nu}$ of curved space-time instead of the flat metric (3.13). By such choice, it will be possible to geometrize both gravitation and other physical fields by a Finslerian approach.

CHAPTER 4

CONCLUSION

The main purpose of this study was to review Finsler geometry and its applications to the geometrization of electromagnetism and gravitation.

In chapter two, Finsler geometry is reviewed in detail. The definition of Finsler metric function and three properties that it must satisfy are given. Then Finsler metric tensor and Cartan torsion tensor are defined. By a variational method, geodesics in Finsler spaces are calculated. Finally covariant differentiations, namely δ and Cartan derivatives, and curvature tensors of Finsler spaces are given. The δ differentiation was presented similar as covariant differentiation in Riemann geometry, but unless the differentiation is taken along a geodesic, norm of the vectors are not invariant under parallel displacement, and also Ricci lemma is not preserved. This problem is overcome with Cartan covariant differentiation. By Cartan's definition of element of support, Ricci lemma is preserved so that Finsler geometry can be applied to physics. Since there exist two covariant differentiations, the curvature tensors resulting from these δ and Cartan covariant differentiations are also given.

In chapter three, electrodynamics is dealt in the Finslerian space-time. First, Finsler gauge transformations and Y transformations are reviewed. Then a specific Y transformation is given (3.11). If this transformation is applied to a Minkowskian space-time, the resulting metric tensor is given in equation (3.31). In this equation, the physical potentials, $B^{\mu}(x, y)$, are determined by the physical system being modelled. If the conditions (3.18) and (3.19) are imposed on this metric tensor as explained in section 3.2, then the resulting Finslerian metric tensor contains electromagnetic effects as a property of space-time. The geodesic equation resulting from this metric tensor is equivalent to Lorentz equation for a charged particle. Moreover, from the field equations, it is deduced that electromagnetic energy-momentum tensor is a consequence of the curvature, arising from the metric, and also electromagnetic mass is contained in the curvature term.

In the presence of gravitational fields, Minkowskian metric tensor in equation (3.29) is replaced by a Riemannian metric tensor $g_{\mu\nu}$ for a curved space-time.

Then both gravitation and electromagnetism is geometrized simultaneously. This geometrization is done on the bases of the fact that, physical laws are invariant under arbitrary gauge and coordinate transformations. This implies charge and energy-momentum conservations.

One of the important property of the Finslerian approach is that it allows quantization of fields which are geometrized. In quantum electrodynamics, the metric tensor is not quantized, rather the electromagnetic fields contained in the metric tensor within the framework of the Minkowski space-time is quantized. This will be considered for future work.

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