A CAPACITATED INVENTORY MODEL WITH A FIXED ORDERING COST UNDER STOCHASTIC DEMAND

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ABSTRACT

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In this study, we investigate a single item, periodic review inventory problem where the amount that can be ordered is limited. The demand for the item is a random variable. Linear holding and backorder cost are charged per unit at the end of a period. Other than variable cost charged per unit ordered, a positive fixed ordering cost is incurred with each order given. The optimization criterion is minimization of discounted cost over a planning horizon. We examine a special case with a finite planning horizon, where optimality conditions can be determined. In this special case, demand values are assumed to be integer multiples of the capacity. As a result, we show that an all-or-nothing policy is optimal. Then, we investigate the infinite horizon problem of the same special case under average cost criterion by defining the problem as a Discrete Time Markov Chain.

Also in the light of these results for the special case, we develop a heuristic to the original problem. We complement these results with a computational study.

Keywords: Capacity Constraint, Stochastic Demand, Fixed Ordering Cost, All-or-Nothing Policy

ÖZ

RASSAL TALEP ALTINDA SABİT ISMARLAMA MALİYETLİ KAPASİTELİ ENVANTER MODELİ

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Bu çalışmada, üretim miktarının sınırlı olduğu, tek ürünlü dönemsel gözden geçirmeli, envanter problemi incelenmiştir. Ürün için talep rassal bir değişkendir. Doğrusal stok taşıma ve yok-satma maliyetleri dönem sonunda envanter durumuna göre hesaplanmaktadır. Birim başına ödenen üretim maliyeti dışında, her ısmarlama yapıldığında ödenen pozitif bir sabit ısmarlama maliyeti de bulunmaktadır. Eniyileme kriteri, bir planlama dönemi için indirimli maliyetin enazlanmasıdır. Sınırlı planlama dönemi olan özel bir problem incelenmiş ve bu problem için en iyi üretim politikası belirlenmiştir. Bu özel problemde, talep değerlerinin kapasitenin katları şeklinde geldiği kabul edilmiştir. Daha sonra, sonsuz dönemlik özel problem Kesikli Zamanlı Markov Zinciri olarak tanımlanıp, dönemsel ortalama maliyet kriteri altinda incelenmiştir. Bu sonuçların ışığı altında, genel problem için sezgisel bir çözüm geliştirilmiştir. Bu sonuçlar sayısal çalışmalarla desteklenmiştir.

Anahtar Kelimeler: Kapasite Kısıtı, Rassal Talep, Sabit Üretim Maliyeti, Hep yada Hiç Politikası

To My family

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CHAPTER 1

INTRODUCTION

To reduce the costs and gain a competitive advantage, firms should improve the efficiency of their operations. Inventory management has a key role in increasing efficiency. Matching supply and demand is a critical challenge and to become successful in this challenge, firms must lower the inventory related costs while satisfying the demand of their customers. Although identifying the most effective inventory control policy may be challenging, the benefits can be worthwhile.

Inventory control policies depend on the setting of the problem. Some properties of the problem are nature of demand, review policies, cost evaluation criteria, presence of fixed ordering cost, presence of capacity constraint of the production and finally length of the planning horizon. There are several analytical models developed by researchers changing due to setting of the problem addressed. Uncertainty of production systems is the main reason for holding inventories. Since the nature of customer demand is the major factor creating the uncertainty in the system, many inventory related problems have stochastic demand. For example, the basic newsvendor model balances the cost of holding inventory and cost of shortage of a single period problem under stochastic demand. Extensions of basic newsvendor model to multi period settings, capacitated production case, and fixed ordering cost case are available in the literature. However, the simultaneous inclusion of fixed ordering costs and capacitated production case has not been investigated in detail.

With this in mind, we analyze the single item periodic review, capacitated inventory model with fixed ordering cost in this study. Under periodic review policy, inventory position is reviewed in every period and production/order decision is made with considering inventory position. Production/order decisions are made before observing the actual demand in a period. Demand in each period is a discrete random variable independent of demands in other periods. Because of the production capacity, it may not be possible to reach some inventory levels by ordering. Production/order leadtime is assumed to be zero. Performance measures of inventory systems are calculated in terms of costs. Although customer satisfaction (service level) may be another performance measure, it is embedded to system cost by shortage (backorder) cost term. The costs of the system consist of holding and backorder costs which are charged per unit at the end of the period, fixed and variable costs of ordering which are charged at the beginning of the period. The variable production cost, inventory holding cost and backorder cost are assumed to be linear. The optimization criterion is either minimizing average cost or discounted cost of the system over a planning horizon. Although the optimal solutions have been developed for the cases in which either fixed ordering cost or capacitated production individually exists, their simultaneous consideration has not been studied extensively. Although there are several studies available in the literature which partially characterize the optimal policy structure of this problem setting, optimal policy has not been fully revealed yet.

This study attempts to solve a specific problem structure where demand is defined as multiples of the capacity of production and develops a heuristic solution for the general problem structure. This specific problem structure resembles batch production and batch ordering type problems. In these types of problems, demand comes in batches and production is made in batches. Process type industries can be a good example of batch production. In such inventory/production systems, all-or-nothing policy is optimal due to high fixed ordering cost. In the specific problem structure defined above, capacity of production is one batch in each period and demand also comes in batches. Hence, intuitively, optimal ordering policy may be an all-or-nothing policy even fixed ordering cost is not too high.

The study is organized as follows: In Chapter 2 motivation of this study is stated and the related work in the literature is summarized. In Chapter 3, we give the formal statement and basic notation of the problem, state optimal policy of the single period problem, and provide some computations and findings of general case. In Chapter 4, we define the structure of a special case and we investigate and analyze the characterization of this special case and optimal policy of this special case. In Chapter 5, we investigate the infinite horizon problem of the special case under average cost criterion by defining the problem as a Discrete Time Markov Chain Model. In Chapter 6, in the light of the results of Chapter 5, we develop a heuristic approach to the infinite horizon problem of the general case and evaluate the performance of the heuristic by comparing the results with the dynamic programming solution.

CHAPTER 2

MOTIVATION AND LITERATURE

2.1 Motivation of the Study

Inventory management in efficient and cost effective manner is a major challenge for companies. Many companies try to apply an effective inventory control policy, in order to reduce the costs and increase service level. Companies have many objectives regarding the inventory control strategies. Handling demand uncertainty and variability, best utilization of available production capacity, minimization of the production setup costs, minimization of holding cost of inventory, and satisfying maximum number of customer (or achieving desired service level) are some examples of these conflicting objectives.

Many studies have been carried out, showing how inventory decisions should be made and many analytical models, depending on the type of inventory system, have been developed. An inventory control system has many attributes such as, inventory review policy, demand type, cost functions, production capacity, planning horizon, and cost criterion. Each possible combination of these factors creates a problem setting. There are many studies available in the literature, addressing several types of problem settings and providing optimal solution for these settings. However, optimal solution to the single item periodic review problem with fixed ordering/production cost and capacity constraint under stochastic demand has not been fully characterized yet.

When the fixed ordering/prodcution cost is positive and there is a finite upper limit on the order amount in each period, problem becomes too complex and balance between holding, backorder and fixed ordering cost becomes too hard to identify. Relative benefit of ordering cannot be clearly assessed when capacity restriction is imposed. The goal of this research is to understand system dynamics and interactions between cost items and capacity restriction and provide efficient solutions for this problem setting.

2.2 Classification of Inventory Problems

The single item inventory problem can have many different settings depending on properties of the problem. Many of these problems have been studied extensively and optimal solutions to these problems have been developed. However there are some directions remaining relatively unexplored.

The problem we address is a periodic review problem. Unlike continuous review policies where inventory position is monitored constantly, inventory position is monitored at specific time intervals (periods) under periodic review policy.

Nature of the demand is another basis in classification of inventory problems. In deterministic demand case, optimization criteria is deterministic and also costs are deterministic.

In deterministic case, demand in each period is known at the beginning of the planning horizon. However, demand may have high variability over periods and presence of a production capacity may complicate the problem. These problems are known as lot-sizing problems since the optimal policy determines the optimal ordering quantity. The objective of lot sizing model is to minimize fixed ordering costs and inventory holding costs over the planning horizon. Although most of the studies in the literature on lot sizing problems do not allow backordering due to deterministic demand, under production capacity restriction shortages may occur and backorder costs may be applicable. The optimal solution to the basic lot sizing problem (without capacity constraint and backordering) is Wagner-Within algorithm which is a dynamic programming algorithm. Wagner-Within algorithm states that either amount of inventory carried to a period from the previous period or amount of production in that period is positive. In other words, demand in a period is satisfied by either inventory from previous periods, or production of this period but not both. Hence, inventory carried from one period to others should be exactly equal to the sum of these future periods' demands (Wagner-Within 1958). For example, in a problem with a finite planning horizon of T periods, the optimal production amount in period t should be equal to total demand of periods $t, t+1, \ldots, i$ where $i \leq T$.

If a stationary capacity restriction is imposed, the optimal policy can be found by using a shortest path algorithm (Florian-Klein 1971). Florian-Klein show that if production occurs in any period, it should be equal to the capacity except one period in which excess amount is produced. If production capacity is non-stationary, optimal solution has a property which is a modification of Wagner-Within. This property tells us, either amount inventory carried to a period from the previous period is zero, or production in that period is either zero or equal to capacity.

The capacitated single-item discrete lot sizing problem with backorders can be considered as the deterministic version of our problem. In these types of problems, determining true optimal policies is difficult and time consuming. Hence many heuristic solutions are used, which provide solutions that are either optimal or very close to optimal. The major difference of our problem and the deterministic capacitated lot sizing problem with backorders resulted from uncertainty. In deterministic problem, backorder cost is incurred only when it is more economical to backorder some amount than paying the fixed ordering cost per item plus holding cost. In our problem setting, backorders are also faced because of the stochastic demand. This uncertainty causes a need for safety stock consideration which is not included in deterministic problem.

In stochastic demand case, costs are defined as expected costs. These costs consist of four cost components, holding cost, backorder cost, variable production cost and fixed ordering cost. Holding cost, backorder cost and variable production cost are charged per unit whereas the fixed ordering cost is charged per order. In most of the problem settings, variable production costs are taken zero without loss of generality. Problems with stochastic demand are reviewed in the following section.

Another classification basis is cost optimization criteria. In average cost criterion, total cost incurred in multiple periods is divided to the number of periods and in discounted cost criterion, cost in previous periods contribute to current period's cost through a discount factor.

Planning horizon of the problem is the last factor in classification. Planning horizon in inventory problems can be either a finite horizon or infinite horizon. In finite horizon problems, parameters that identify the optimal policies, such as base stock level, are usually non-stationary whereas in infinite horizon problem these parameters are stationary.

2.3 Literature Review

2.3.1 General Review of Periodic Inventory Problems

The literature on inventory control systems is extensive therefore we only review the studies which are most relevant to our work.

The basic single period model with stochastic demand is the newsvendor model. In the newsvendor model, holding cost and shortage cost are balanced and optimal inventory level is the point which satisfies a critical ratio. This critical ratio is expressed by ratio of unit backorder cost to unit backorder cost plus unit holding cost. Cumulative probability of demand at optimal level should be equal to the critical ratio. Since the cumulative distribution of the demand $F(x)$ is the probability that demand is below x, the probability of satisfying all the demand at the optimal inventory level is equal to the critical ratio.

When newsvendor model is extended to multiple periods with linear production costs (no fixed ordering cost), then optimal ordering policy becomes a base stock policy. In base stock policy, if the initial inventory position in any period is below a critical level, enough should be ordered to bring the inventory position up to critical parameter called S, otherwise nothing should be ordered. (Scarf

1960).

When a positive fixed ordering cost K is introduced to the multi period problem, then optimal policy is defined by two parameters (s, S). Scarf (1960) proves the optimality of such policy. In (s, S) policy, if the inventory position falls below the critical level s, we order enough to bring the inventory position up to S, otherwise we do not order.

When there is no fixed ordering cost, with a finite stationary upper bound of ordering amount in a period, then modified base stock policy, which is proved by Federgruen and Zipkin (1986) is optimal. Federgruen and Zipkin show that under discounted cost optimization criterion, modified base stock policy is optimal. In modified base stock policy, if the inventory position falls below a critical level S, we order enough to bring the inventory position up to S, if not possible then we order capacity.

In the finite horizon problems, the critical parameters of the problems, such as base stock level, order up to level, may vary from period to period. However, in the infinite horizon cases, these parameters are stationary.

2.3.2 Problems with Fixed Ordering Cost and Capacity Constraint

When fixed ordering cost and finite upper bound on ordering amount are both present in single item periodic inventory problem, optimal ordering policy has not been fully identified. Wijngaard (1972) have presented a 3-period problem with deterministic demand where modified (s, S) type policies are not optimal. In modified (s, S) policy, if the inventory level falls below a critical level s, we order enough to bring the inventory position up to S or as close as possible to S. Wijngaard (1972) also come up with following result in the same paper: If the holding and stockout costs are linear and the demand is negative exponentially distributed, and if among the (s, S) strategies there is a best one with capacity is greater than $S - s$ and $s \geq 0$, then this strategy is optimal overall.

Shaoxiang and Lambrecht (1996) attempt to characterize the optimal solution and suggest that optimal policy in capacitated production, non-linear cost case shows a pattern of X-Y band structure. This X-Y band structure can be explained as follows: It is optimal to order full capacity when inventory drops below X, and it is optimal to order nothing when inventory is above Y. Between X and Y the ordering pattern depends on the problem. By this observation, computational effort for optimal policies is reduced. Although the X and Y bounds may vary from period to period, they prove the existence of global bounds that can be applied for all periods. They also provide a counter example for why a modified (s, S) is not optimal. In modified (s, S), order quantity should be non-increasing functions which is not the case in numerical example.

Gallego and Wolf (2000) extend the work of Shaoxiang and Lambrecht (1996), and partially characterize the optimal order policy. Following the X-Y band structure of Shaoxiang and Lambrecht (1996), they suggest to divide the space into four regions by using parameters s and s', and investigate the optimal policies in these regions, by defining CK convexity which is a generalization of K convexity of Scarf. They show that optimal capacitated policy has an (s, S)-like structure depending on the regions. In two of these regions optimal policy is completely specified, while in the other two, it is partially specified. Depending upon the relationship between s, s' and capacity, optimal policy will take one of the two pre-specified forms. These two optimal policy forms and regions in these forms are presented in Figure 2.1. However, this policy structure cannot fully reveal the optimal order policy in the region between s and s' and there may exist some intervals in this region where it is optimal to start and stop ordering.

Chan and Song (2003) provide an efficient algorithm to compute the optimal ordering policy parameters and show that it is enough to compute optimal ordering quantities for only a subset of inventory positions falling between X and Y bounds. As a result of this, computational effort of dynamic programming is reduced. They introduce (alpha,beta) convexity which is similar to CK convexity of Gallego and Wolf (2000). If the expected optimal cost for the n-period planning horizon problem is (K, CP) (fixed ordering cost, capacity) convex, then an efficient algorithm is possible for computing the optimal ordering quantities for inventory positions between X and Y. They illustrate the algorithm on a nu-

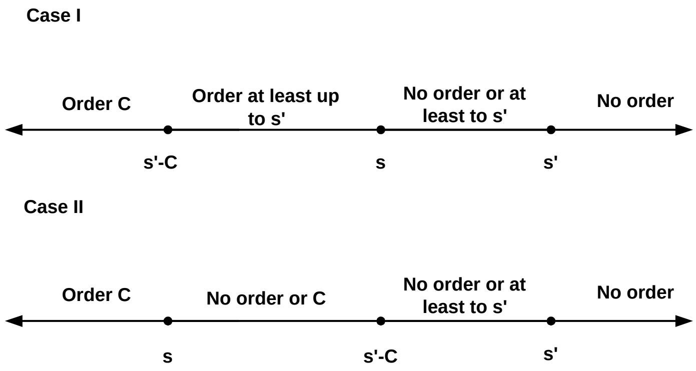

Figure 2.1: Optimal policy forms

merical example. Steps of the algorithm can be summarized as follows: After determining X-Y band, ordering quantities are assigned for inventory position values which are not between X and Y. Then, find s_0 value and find ordering quantities at s_0. Then, assign optimal order quantities increasing by one as inventory position is decreasing by one. When ordering quantity reaches capacity, restart by computing the ordering quantity for the following inventory position by dynamic programming and continue the algorithm.

2.3.3 Stochastic Lot Sizing Problem

The deterministic version of the lot sizing problem and the solution procedures are reviewed in Subsection 2.3.1. In this subsection, we present a brief review of literature on the stochastic lot scheduling problem and discuss similarities of this problem to our problem. Stochastic lot scheduling problem deals with scheduling production of single/multi products with stochastic demand structure with limited/unlimited production capacity (Sox et all 1999). Many analytical methods are applied to this problem differ based on the structure of the problem. Capacitated single item discrete time models, which are called capacitated lot sizing problem, is most similar one to our problem.

Sox et all (1999), survey the literature on the stochastic lot scheduling problem. They provide a framework for comparing several approaches in the existing literature. They discuss the complexity of the problem by comparing with the deterministic problem. They state some differences with deterministic problem such as safety stock consideration. The finite production capacity, which should be shared among the products, increases the amount of safety stock which is needed to maintain a specified service level. Inventory reduces the setup cost of production and serves as a buffer against stockouts because of the variation in the demand. They classify the studies in the literature into two categories: dynamic sequencing and cyclic sequencing. However this classification belongs to multi item problem setting, and is not related to our problem.

Sox and Muckstadt (1997) formulate the stochastic lot scheduling problem with setup times and costs as a stochastic mathematical model with a finite planning horizon. The objective function is minimization of total cost of planned production and setups and expected cost of inventory holding cost and backorder cost. They use Lagrangian decomposition and subgradient optimization to solve the problem. The subproblems are single item stochastic lot sizing problems that are solved by branch and bound algorithm.

Dellaert and Melo (1996) provide a heuristic solution to a single item capacitated stochastic lot sizing problem in a make to order environment. The objective is to determine the optimal size of the production in each period in order to minimize expected costs, which include setup costs, holding costs, and penalty costs, over a planning horizon. They suggest a (x, T, δ)-rule which means, production only takes place during a period for which the required deliveries are at least x units. In that case, the known orders for the next T periods are manufactured if the available capacity is not exceeded. Otherwise, the parameter δ controls the production amount.

The major differences of stochastic scheduling problems from our problem, are multi item setting and setup time consideration in scheduling problems. Multi item setting requires scheduling approaches which are not related to our problem setting. When problem is reduced to single item setting without setup time, it becomes stochastic lot sizing problem which is similar to our problem.

CHAPTER 3

DESCRIPTION OF THE GENERAL MODEL

In this chapter, we define our problem setting and present some properties about problem. In Section 3.1, description of the problem and in Section 3.2 basic notation of the problem is given. In Section 3.3, optimal policy of the single period problem is stated and some computations and findings for the general case are provided. A computational analysis on a numerical example is also presented in Section 3.3.

3.1 Description of the Problem

As we mentioned before, we analyze the single item periodic review production/inventory problem in this study. We assume that demand in any period is a discrete random variable and this random variable is independent but not necessarily identically distributed from period to period. Linear holding and backorder costs are charged per unit of inventory at the end of each period. A fixed ordering cost is associated with each order decision. Furthermore, amount of order in any period is limited with a positive capacity value. We assume that leadtime is zero. The objective is to minimize the expected discounted cost of system over a finite or infinite horizon.

The sequence of events are as follows: The state of the system at the beginning of each period is defined by the inventory position. At the beginning of each period considering the inventory position value, decision of how much to produce is given. Order amount corresponds to the decision variable in dynamic programming formulation. If production occurs, other than variable production cost per unit ordered, a fixed ordering cost is incurred for that period. Ordered amount arrives instantaneously, then demand is realized and satisfied with on-hand inventory, unsatisfied demand is fully backordered, and holding and backorder costs are assessed at the end of the period.

The difficulty about this problem setting can be explained as follows. In base stock policy, cost of holding inventory and cost of shortage are balanced. In multi period setting of same problem, with positive fixed ordering cost, there is a critical point s, where it is preferable to pay the fixed ordering cost to achieve the optimal inventory level S, where holding and backorder costs are balanced. Therefore at initial inventory levels below s, it is economical to order, otherwise it is not. However, if a finite ordering capacity is introduced to the problem, it may not be possible to reach the optimal inventory level S, at some initial inventory positions. Therefore under capacity limitations, tradeoff between holding, backorder and fixed ordering cost in a multi period setting is hard to determine. Many local minimum points in expected cost functions will be formed depending on the relationship between demand distribution, capacity and cost parameters. Therefore, ordering amount versus initial inventory position curve will be an unpredictable curve due to the nature of the problem. For example, unlike other problem settings, ordering amount may increase in some regions as initial inventory value increases. As a result of this, optimal ordering policy cannot be determined like other cases.

3.2 Notation of the Problem

The following notation is used throughout this study.

n = period index

D_n = non-negative random demand of period n (when there are n periods left in planning horizon). Demand in any period is independent and identically distributed from period to period with a probability distribution P_r, ie., $P_r = Pr(D = r)$, $r = 0, 1, \ldots$. If demand distribution is not identical in each period, probability distribution should have a period index also like P_{rn}

C = capacity of production, a positive integer

b = backorder cost per unit per period

h = holding cost per unit per period

K = fixed cost of ordering

v = unit variable cost

x_n = inventory level prior to placing any order in period n

y_n = inventory level after placing an order, before demand is realized in period n,

For any parameters given above, a period index can be used (such as h_n, C_n) if that parameter is non-stationary. If a parameter is stationary then it is used without an index.

γ = discount factor $(0 \leq \gamma \leq 1)$

The following assumptions are standard to pose as otherwise the analysis becomes non conventional.

1. Unit backorder cost should be greater than the unit variable cost $(b > v)$. Otherwise, ordering never takes place, and it would be optimal to backorder all demand instead of purchasing.

2. Unit backorder cost times production capacity should be greater than the fixed cost of ordering $(b * C > K)$. Otherwise ordering never takes place, and it would be optimal to backorder all demand instead of paying the fixed cost for single period problem. As number of periods to go increases, this assumption can be relaxed.

3. Expected value of the demand should be less than capacity $(E[D] < C)$. Otherwise system will not be stable in the long run.

3.3 General Case of the Problem

In this section, we present some findings about the general case of the problem. First, recursive cost functions of the dynamic problem are explained and some properties of these functions are presented. In the second subsection, optimal policy of the single period problem is discussed. In the third subsection, multi period extension of the problem is analyzed; a numerical example is given to show that modified (s, S) policy is not optimal. Finally, in the fourth subsection, a computational analysis for the multi period problem is presented.

3.3.1 Properties of Cost Functions

$L(y)$ is the one period expected holding and backorder cost function with given inventory position y. $L(y)$ can be expressed as;

$$L(y) = hE[(y - D)^+] + bE[(D - y)^+] \tag{3.1}$$

Some properties about $L(y)$ are as follows;

1. $L(y)$ is a non-negative convex function. Since holding and backorder costs are linear, $L(y)$ is a discrete convex function. It can be shown by taking the second difference of $L(y)$ since the demand is a discrete random variable.
2. If it is assumed that either unit holding cost h or unit backorder cost b is positive then $lim_{y \to +\infty}[L(y)] = \infty$.

These two properties are required for $L(y)$ to have a minimum point at a finite inventory position. Moreover, if unit holding or unit backorder cost is non-stationary, then this function should have a period index $L_n(y)$.

$J_n(x)$ is the expected cost for an n-period horizon problem if the beginning inventory level is x.

$G_n(y)$ is the expected cost for an n-period horizon problem if the inventory level after production decision is y.

$\delta(x)$ is an indicator function whether a fixed ordering cost is incurred or not. $\delta(x) = 1$, when $x > 0$.

$G_n(y)$ and $J_n(x)$ are as follows;

$$G_n(y) = vy + L(y) + \gamma E[J_{n-1}(y - D_n)] \tag{3.2}$$

$$J_n(x) = -vx + \min_{x \leq y \leq x+C} \{G_n(y) + K\delta(y - x)\} \tag{3.3}$$

Here, we should state one important assumption about these functions. At the end of the planning horizon, all leftover inventory is salvaged with zero cost/profit. Therefore, at the end of the planning horizon when number of periods to go is equal to zero, all cost are zero. In other words, $J_0(.) = G_0(.) = 0$ for any inventory position value.

$O_n(x)$ denote the optimal order amount when the initial inventory is equal to x in period n. It is obvious that $O_n(x)$ can have a maximum value equal to capacity in our problem setting.

Let $H_n(x)$ be the cost difference between ordering decision and order nothing decision when the initial inventory is equal to x in period n.

$$H_n(x) = \min_{x < y \leq x+C} \{G_n(y) + K - G_n(x)\} \tag{3.4}$$

Therefore, it is optimal to order when $H_n(x)$ is negative and not to order when $H_n(x)$ is positive. If $H_n(x)$ is equal to zero, then one is indifferent to order or not.

Before advancing to the next step, we should define two policy parameters;

S_n = inventory level where the minimum of $G_n(y)$ is achieved in any period n. Global minimum of $G_n(y)$ may not be unique, in that case minimum of these inventory levels is taken as S_n.

s_n = greatest inventory level such that $G_n(s_n) \geq G_n(S_n) + K$ in any period n.

Some properties about $J_n(x)$ and $G_n(y)$ are as follows;

1. $G_1(y)$ is equal to $L(y)$ when variable cost of ordering v is equal to zero.

2. $G_1(y)$ is a convex function as the sum of two convex functions. Therefore $G_1(y)$ has a global minimum point at a finite inventory position S_1.

3. $G_n(y)$ is a decreasing function for $y \leq S_1$. (Shaoxiang and Lambrecht 1996)

4. $J_n(x)$ is a non increasing function for $x \leq S_1$. (Shaoxiang and Lambrecht 1996)

5. $G_n(y)$ has a finite minimizing argument S_n. (Gallego and Wolf 2000)

6. $J_n(x) \geq J_{n-1}(x)$ for $n \geq 1$. (Chan and Song 2003)

These properties of functions $L(y)$, $J_n(x)$ and $G_n(y)$ will help us understanding the dynamics of the problem. For example, X-Y band structure of Shaoxiang and Lambrecht (1996), and optimal policy of the single period problem can be explained by these properties of functions. In the following subsection, optimal policy of the single period problem is discussed by using the properties shown above.

3.3.2 Single Period Problem

The single period problem resembles the last period of an N-period problem. This is the last period to go, and all costs are zero when n is equal to zero so, $G_1(y) = vy + L(y)$ and $J_1(x) = -vx + \min_{x \leq y \leq x+C} \{G_1(y) + K\delta(y-x)\}$. We have already stated that at the previous subsection that $G_1(y)$ is convex and has a global minimum point at the inventory level S_1. If there were not a capacity restriction on the ordering amount, the optimal policy would be an (s_1, S_1) policy. However capacity restriction prevents the optimal policy to have (s, S) structure. Fortunately, introduction of capacity constraint to the single period problem does not complicate things too much since $G_1(y)$ is convex. However, due to relationship between capacity value and parameters (s_1, S_1) two cases have to be considered to define the optimal policy. These are,

Case 1: If $S_1 - s_1 \leq C$: In this case, capacity restriction is not that binding and order up to level S_1 can be reached from reorder level s_1 with this capacity value. In this case, function $O_1(x)$ is defined in three regions and optimal ordering amounts in these regions are;

$$O_1(x) = \begin{cases} C & x \leq S_1 - C \\ S_1 - x & S_1 - C \leq x \leq s_1 \\ 0 & s_1 \leq x \end{cases} \tag{3.5}$$

In the first region where initial inventory position, x, is less than order up to level minus capacity $(S_1 - C)$, $G_1(x) > G_1(x + C) + K$ due to convexity of $G_1(x)$ and definition of s_1. Ordering amount, $O_1(x)$, is equal to capacity since we want to be as close as possible to S_1 due to convexity of $G_1(x)$. In other two regions, capacity is no longer a constraint for the order amount since we never want to exceed the global minimum of function $G_1(y)$, S_1, and between $S_1 - C$ and s_1, it is optimal to order up to S_1. After s_1 it is optimal not to order, since benefit of ordering is less than the fixed ordering cost in this region.

Case 2: $S_1 - s_1 > C$: In this case, capacity constraint is more restrictive and order up to level S_1 cannot be reached from reorder level s_1 with this capacity value. Hence, the initial inventory positions where it is optimal to order is less than s_1. Let z be the greatest point such that it is economical to order full capacity; $z = \max\{x : G_1(x) \geq G_1(x + C) + K\}$. By definition z is less than s_1. In this case, function $O_1(x)$ is defined in two regions and optimal ordering amounts in these regions are;

$$O_1(x) = \begin{cases} C & x \leq z \\ 0 & z \leq x \end{cases} \tag{3.6}$$

In this case, the capacity restriction does not allow us to reach the global minimum S_1 from the region where it is economical to order. By definition of z , the region where it is economical to order $(x \leq z)$ is defined. The order quantity in this region is equal to capacity since we want to be as close as possible to S_1 due to convexity of $G_1(y)$. Order amount is equal to zero when $x > z$, benefit of ordering cannot cover the fixed ordering cost in this region.

Here we present an example for the cases above. Assume that $h = 1.0$, $b = 12.0$, $K = 55.0$, $v = 1.0$, and $C^{(1)} = 15$ in the first case and $C^{(2)} = 6$ in the second case. Demand is discrete uniform between 0 and 9. Function $G_1(y)$ is presented in Figure 3.1.

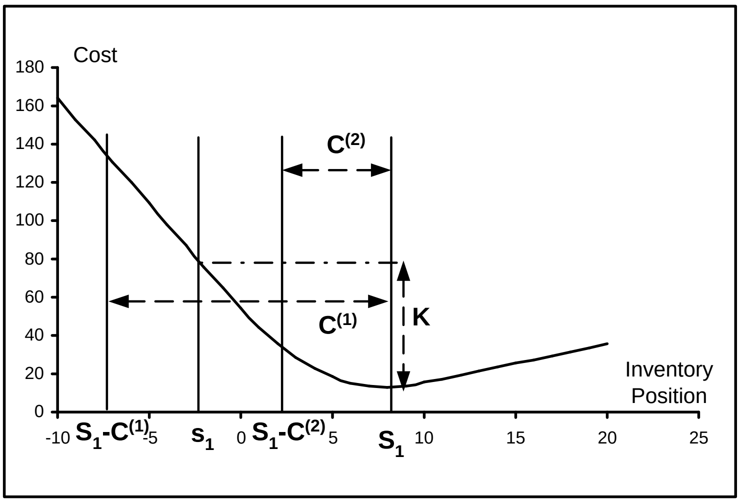

Figure 3.1: Curve of $G_1(y)$

It can be seen from Figure 3.1 that $S_1 = 8$ and $s_1 = -2$, so in the first case $S_1 - C^{(1)}$ is equal to -7 which is smaller than s_1, so optimal order policy is a modified (s, S) type policy. In Figure 3.2 below, optimal order quantities $O_n(x)$, when capacity is equal to 15, are presented.

When capacity is equal 6, then $S_1 - C^{(2)}$ is equal to 2 which is greater than s_1, so optimal order policy is a modified (s, S) type policy but ordering amounts are equal to capacity since $z < s_1$. The parameter z is equal to -3 and $z = -3 < s_1 = -2$ as it is stated before. In Figure 3.3 below, optimal order quantities $O_n(x)$, when capacity is equal to 6, are presented.

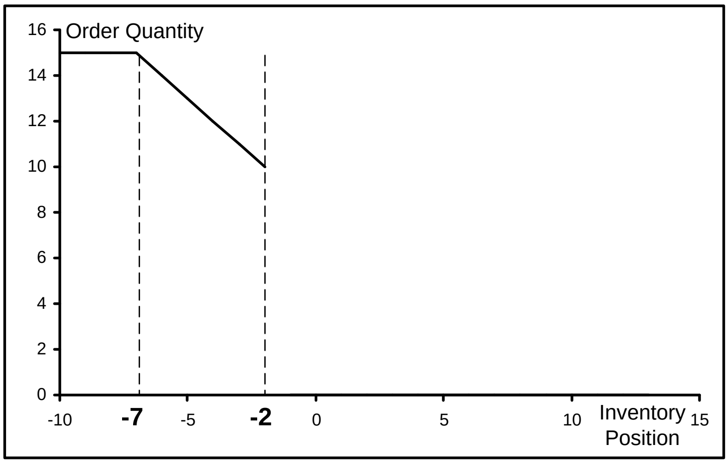

Figure 3.2: Curve of $O_1(x)$ when capacity is equal to 15

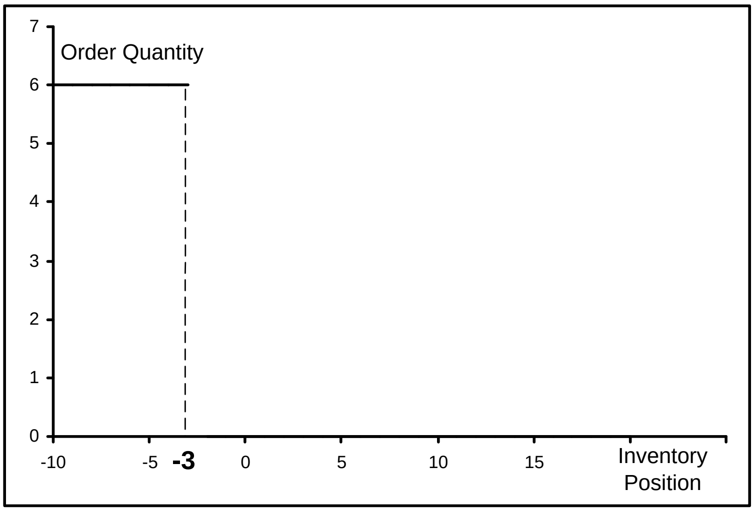

Figure 3.3: Curve of $O_1(x)$ when capacity is equal to 6

The optimal order policy for the single period problem is a modified (s, S) policy depending on relationship between s_1, S_1 and capacity. Either $S_1 - C$ or z, depending on which one is active, serves as the $X - bound$ of Shaoxiang and Lambrecht (1996).

3.3.3 Multi Period Problem

The single period optimal policy is proved to be a modified (s, S) policy. However, this policy cannot be extended to multi period problem. Under a modified (s, S) policy, order quantity should be non increasing function of the inventory level at the beginning of a period (Shaoxiang and Lambrecht 1996) which is not the case for this problem. Consider the example below,

Assume that $h = 1.0$, $b = 15.0$, $K = 55.0$, $v = 1.0$, and $C = 20$. Demand is equal to 8 with a probability of 0.95 and 9 with a probability of 0.05. Function $G_2(y)$ is presented in Figure 3.4.

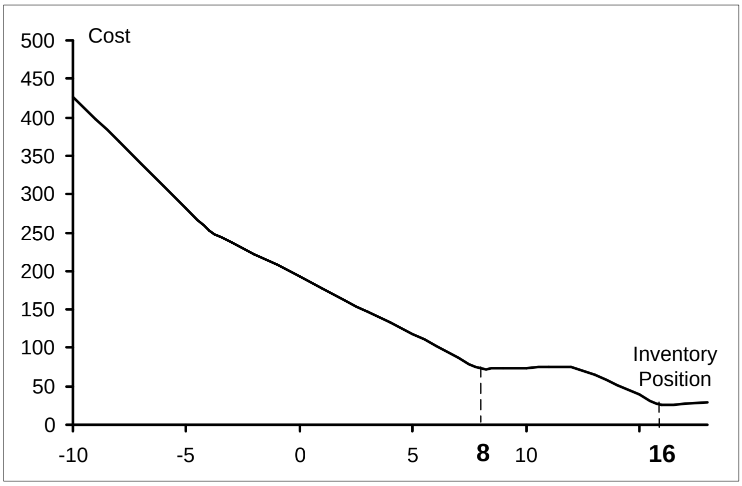

Figure 3.4: Curve of $G_2(y)$

From the graph of function $G_2(y)$, we can identify one local minimum point which is at 8 and one global minimum point which is at 16. As the number of periods to go, n, increases, number of local minimum points increases. For example, in Figure 3.5, the graph of function $G_7(y)$ is presented. The function $G_7(y)$ has 7 local minimum points at inventory values; 8, 17, 20, 24, 40, 48, and 56, and a global minimum point at 36.

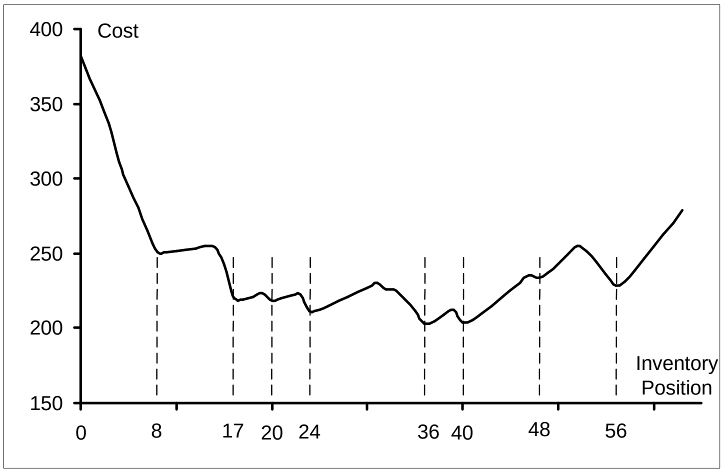

Figure 3.5: Curve of $G_7(y)$

These local minimum points of $G_n(y)$ are the points which are the combinations of possible demand values and capacity. These local minimums cause the order amount functions to have an unusual pattern.

It can be clearly seen from the order quantity versus inventory position graph (Figure 3.6) that order quantity is not a non increasing function of the inventory level. In fact, it looks like a combination of several modified (s, S) order quantity functions. The order amount is equal to capacity when inventory level is below -12, then it starts to decrease with a step size of one until -5. After -5 ordering amount is equal to capacity again and after -3 it starts to decrease again and goes on like this. It can be seen from Figure 3.6 that there are many 'kinks' in the curve of $O_n(y)$. These 'kinks' in the order amount function are resulted from capacity constraint and those local minimum points. Function $G_n(y)$ is

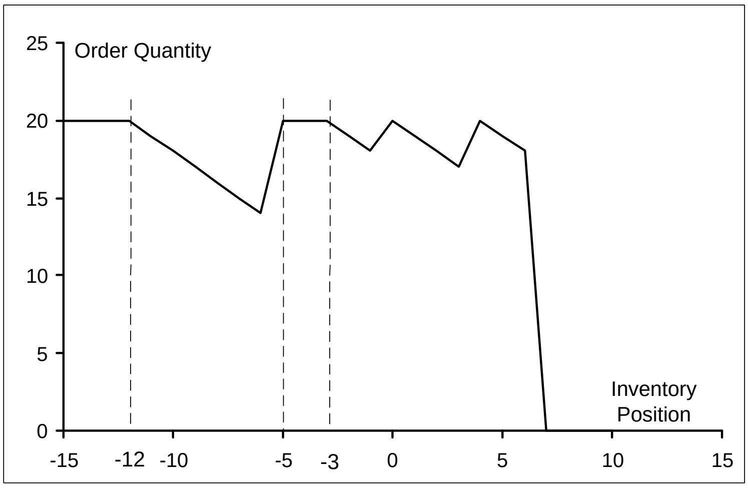

Figure 3.6: Curve of $O_7(y)$

neither convex nor K-convex. Gallego and Wolf (2000) prove that $G_n(y)$ is not a K-convex function but a CK-convex function. Therefore, at a specific inventory position y, it may be optimal to order less than capacity to reach a local minimum point rather than full capacity, even if it is optimal to order full capacity at an inventory position greater than y. This property of order quantity function is the reason for difficulty of defining the optimal policy of multi period problem.

Shaoxiang and Lambrecht (1996) attempt to characterize the optimal solution to multi period problem and proposed X-Y band structure which implies that there is a point X, until where the ordering amount is equal to capacity and a point Y, from where the ordering capacity is zero. Gallego and Wolf (2000) extend the work of Shaoxiang and Lambrecht (1996), partially characterize the optimal order policy and investigated the optimal policies between X-Y band. However, this policy structure cannot fully reveal the optimal order policy and cannot explain the 'kinked' curve of order quantity function.

These kinks can be identified by determining the local minimums of the function $G_n(y)$. Unfortunately, as the number of periods to go increases, number of local minimum points also increases since local minimum points are combinations of several demand values and capacity. Hence, it is difficult to identify the general behavior of the order amount curve and so the optimal policy structure of the multi period problem.

3.3.4 Computational Analysis

In this subsection, we present some findings from a computational study on the multi period problem with respect to various parameters. Here, we need to redefine parameter z_n. Let z_n be the highest inventory level such that order amount is positive. That is $z_n = \max\{x : O_n(x) \geq 0\}$ for any period n. Computational analysis on a numerical example will be presented in order to show how functions $J_n(x)$, $G_n(y)$, and parameters z_n, S_n are affected by cost parameters, order capacity, demand distribution, and number of periods to go.

Consider the example; $h = 2.0$, $b = 20.0$, $K = 80.0$, $v = 2.0$, and $C = 20$. Demand is equal to 8 with a probability of 0.70 and 10 with a probability of 0.30. Now, we itemize the factors, which affect the cost functions and parameters, and illustrate their effects on this example.

- Number of periods to go: As the number of periods to go increases, values of cost functions at the same inventory level increase. That is, $J_n(x) > J_{n-1}(x)$, $G_n(y) > G_{n-1}(y)$ (Chan and Song 2003). However the policy parameter S_n does not increase as in the modified base stock policy (Federgruen and Zipkin 1986), and also z_n values do not show a steady pattern (initially increase then decrease and again increase). The reason for this behavior can be explained as follows: In base stock policy, in each period a critical point S is tried to be achieved. When capacity constraint is present, then S_n value may increase as n increases. Because fixed ordering cost is zero, number of orders in a planning horizon is not important in this case. However in our case, the number of orders in a planning horizon becomes a major concern because of the fixed ordering cost. For example, when capacity is equal to three times of expected demand in a period, it may be optimal to order only once in a three period interval which results in a cyclic pattern in the values of S_n.

In Table 3.1 below, S_n and z_n values for different values of n are presented.

Table 3.1: Values of S_n and z_n with respect to number of periods to go

Period	1	2	3	4	5	6	7	8	9	10
S_n	10	18	26	34	26	34	26	34	26	34
z_n	3	7	6	7	6	7	6	6	6	6

- Unit holding cost: As unit holding cost increases, it is observed that value of cost function at the same inventory level at the same period increases. Because either inventory holding cost increases or amount of inventory on hand decreases which increases the expected backorder cost. On the other hand, the policy parameters S_n and z_n are observed to decrease since inventory holding becomes less beneficial.

- Unit backorder cost: As unit backorder cost increases, it is observed that value of cost function at the same inventory level at the same period increases. Because either total backorder cost increases or inventory on hand value increases which increases the inventory holding cost. Moreover, the policy parameters S_n and z_n are observed to increase since an increase in penalty cost of stockout increases the incentive to hold inventory.

- Fixed ordering cost: As fixed cost of ordering increases, it is observed that value of cost function at the same inventory level increases. Since for each order, higher costs are incurred and any attempt to decrease fixed ordering cost incurred in a planning horizon will increase inventory holding and backorder cost. Moreover, the policy parameter S_n is observed to increase since inventory on hand becomes more valuable due to increased allocated fixed cost to each ordered item. On the other hand, policy parameter z_n

is observed to decrease since the relative benefit of ordering full capacity decreases.

In Table 3.2 below, $G_n(y)$, S_n, and z_n values are presented. Holding cost, backorder costs, and fixed ordering costs are decreased by 50 percent and increased by 50, 100, and 400 percent, and corresponding $G_n(y)$, S_n and z_n values for $n = 10$ are given in Table 3.2. It can be inferred from the table that cost function and parameters values are most sensitive to the percentage change in fixed ordering cost. Moreover, percentage change in unit holding cost has greater effect on the cost function and parameters than percentage change in unit backorder cost.

Table 3.2: Values of $G_n(y)$, S_n and z_n with respect to cost parameters

	Holding Cost			Backorder Cost			Fixed Ordering Cost		
% Increase	S_{10}	z_{10}	G_{10}	S_{10}	z_{10}	G_{10}	S_{10}	z_{10}	G_{10}
-50%	48	7	496	26	5	585	18	7	455
0%	34	6	602	34	6	602	34	6	602
50%	18	6	667	34	7	610	44	6	718
100%	18	6	719	34	7	628	44	6	799
400%	16	4	979	36	8	628	82	6	948

- Capacity: An increase in capacity result in a decrease in cost, since the problem's feasible region becomes larger. In capacitated problems, there is motivation for holding inventories for future periods because future demand may not be satisfied with the capacity in future periods. As capacity value increases, this motivation decreases because capacity constraint becomes less binding. Also, inventory on hand becomes less valuable since order cost per item decreases. Therefore, policy parameter S_n is observed to decrease. However, policy parameter z_n may increase at the beginning since it may become possible to reach some desired inventory levels with a higher capacity value. But from a certain point, z_n may decrease since the relative benefit of order full capacity decreases.

In Table 3.3 below, $G_n(y)$, S_n, and z_n values are presented. Production capacity is decreased by 50 percent and increased by 50, 100, and 400 percent, and corresponding $G_n(y)$, S_n, and z_n values for $n = 10$ are given in Table 3.3. It can be inferred from the table that increasing capacity value causes considerable decrease in the total cost of the system at the beginning. However, it is interesting that 50 percent increase in capacity and 100 percent increase in capacity have approximately same minimum cost value although they are achieved at different inventory levels.

Table 3.3: Values of $G_n(y)$, S_n, and z_n with respect to Capacity Value

	Capacity		
% Increase	S_{10}	z_{10}	G_{10}
-50%	44	7	754
0%	34	6	602
50%	34	5	577
100%	26	5	577
400%	26	5	277

- Expected demand: Increase in expected demand causes an increase in costs in our example. Moreover, the parameter z_n is observed to increase. However, parameter S_n does not show any pattern. In Table 3.4 below, $G_n(y)$, S_n, and z_n values for $n = 10$ are presented when expected value of demand is equal to 7.6, 8.6, 9.6, 10.6 and 11.6. S_n value peaks (=40) when expected demand is equal to 9.6. This situation may be explained as follows: When expected demand is equal to $9.6 \cong 10$ and capacity is equal to 20, it is optimal to order once in every two periods and local and global minimum points of cost function $G_n(y)$ occurs at multiples of capacity.
- Maximum demand: Increase in maximum demand (while maintaining the same expected demand value) causes an increase in costs in our example.

Table 3.4: Values of $G_n(y)$, S_n, and z_n with respect to Expected Demand

Expected demand	S_{10}	z_{10}	G_{10}
7.6	44	5	555
8.6	34	6	602
9.6	34	8	648
10.6	26	9	708
11.6	26	10	764

Moreover, the parameter z_n is observed to increase. However policy parameter S_n does not show any pattern. In Table 3.5 below, $G_n(y)$, S_n, and z_n values for $n = 10$ are presented when maximum value of demand is equal to 10, 15, 20 and 30. S_n value peaks (=32) when maximum demand is equal to 15. This situation may be explained as follows: When maximum demand is equal to $15 \cong 16$ and other possible demand value is equal to 8 (which is equal to half of the maximum demand), minimum points of cost function $G_n(y)$ occur at multiples of 8. Hence S_{10} value is equal to 32 rather than 34 or 36.

Table 3.5: Values of $G_n(y)$, S_n and z_n with respect to Maximum Demand

Maximum demand	S_{10}	z_{10}	G_{10}
10	34	6	602
15	32	7	682
20	36	7	742
30	44	15	872

We conduct this computational analysis to gain some insight about the problem structure. Due to the results presented in Subsections 3.3.3 and 3.3.4, we believe that any monotone ordering policy cannot capture the essence of the system behavior. Therefore, a simple policy is not expected to be optimal for this problem structure. With this in mind, we define a special case of this problem where optimality conditions can be achieved with simple monotone policies. We expect to utilize the results of this special case in developing effective heuristic solution for the general case of the problem.

CHAPTER 4

A SPECIAL CASE WITH RESTRICTED DEMAND DISTRIBUTION AND NON-STATIONARY PARAMETERS

In Chapter 3, we showed that the optimal policy for single period problem is a modified (s, S) policy. Also, we showed that this policy cannot be extended to multi period problem and we presented some properties of the multi period problem. Due to these properties, we believe that optimal policy for general case of the problem is difficult to identify. In this chapter, we analyze a special case of the problem where we prove the optimality of an all-or-nothing policy for a specific structure of demand and capacity relation. In this specific problem structure, demand is defined as multiples of the capacity of production. In other words, capacity of production is one batch in each period and demand also comes in batches each period. Process type industries can be a good example of batch production. In these types of systems, production usually has high fixed cost and setup time. As a result, all-or-nothing policy may be optimal. However, due to relation between demand structure and capacity in our special case, all-or-nothing is optimal even for the problems with low fixed ordering cost. This simple monotone policy may be utilized in developing a heuristic solution for the general case of the problem. In Section 4.1, definition of the special case and differences between the special case and the general case are given. In Section 4.2, we set out characteristics for the finite horizon problem of the special case, and prove that optimal policy for this problem is an all-or-nothing policy with a threshold level. Finally, in Section 4.3, a computational analysis for the special case with a numerical example is given.

4.1 Definition of the Special Case

In general case, it is assumed that demand in any period is a discrete random variable and this random variable is independent and identically distributed from one period to another. In this special case, demand in any period is equal to any integer multiple of a base demand d with a known probability. These demand probabilities change over periods. That is; demand in a period when n-periods left in the planning horizon, (D_n), is equal to rd $(r = 0, 1 \ldots, R)$ with a probability of p_{rn}. Therefore, we may have a non-stationary demand distribution in the special case. Moreover, the capacity in a period is equal to base demand value (d). Linear holding and backorder costs are charged per unit of inventory at the end of each period and a fixed ordering cost is associated with each order decision. However, the cost parameters can also be period dependent and we denote unit backorder cost, unit holding cost, fixed (ordering) cost and unit variable cost in period n as b_n, h_n, K_n, and v_n, respectively. We assume that the leadtime is zero. The objective is to minimize the expected discounted cost of system over a finite horizon.

Assumptions of the general case still holds for this special case. The system is stable: That is, expected value of demand in a period is smaller than the capacity ($\sum_{r=0}^{\infty} p_{rn} r < 1$ for all n). Fixed ordering cost should be less than or equal to unit backorder cost times capacity for the single period problem. This assumption is necessary in the special case as well, otherwise production will not occur. Finally, unit backorder cost should be greater than unit variable cost $(b_n > v_n)$.

With new problem setting, some of the difficulties aroused in the general case are eliminated. Difficulties about the general case were explained in the previous chapter as follows: Under capacity limitations, tradeoff between holding, backorder and fixed ordering costs in a multi period setting is hard to determine. Function $G_n(y)$ is neither convex nor K-convex and have many local minimum points which are combinations of several demand values and capacity. Moreover, order quantity is not a non increasing function of the inventory level, and order quantity curve has kinks which are resulted from capacity constraint and these local minimum points. Also, number of local minimum points increases as number of periods to go increases. An approximation for this number in any period n can be possible values of convolution of demand distribution minus a multiple of capacity. This multiple of capacity should be smaller than n times capacity since it is possible to order at most n times in an n period horizon. It can be easily concluded that if demand points do not have a relation with capacity as in the special case, there will be many local minimum points of the function $G(y)$.

In the special case, all possible demand values are multiples of the base demand (d) and also order capacity is equal to d. This greatly reduces the total number of different combinations of demand values and capacity. Therefore, number of local minimum points of function $G_n(y)$ becomes smaller and it becomes easier to determine these local minimum points. Since each possible combination is a multiple of d, intuitively, it is expected that local minimum points occur at points which are multiples of base demand. Although function $G_n(y)$ is still neither convex nor K-convex in the special case, $G_n(y)$, restricted to certain points may exhibit convexity property. If so, the kinks in the order quantity versus initial inventory curve may be eliminated. Due to these simplifications, it may be possible to characterize the optimal order policy of the special case completely.

4.2 Characteristics of the Finite Horizon Problem

4.2.1 Properties of Local Minimum Points of Expected Cost Function

Before discussing the properties of local minimum points, first we have to define the term *local minimum point* as it is used in this chapter. A point is called a local minimum point, if it is the minimum point of an interval of length equal to capacity (d). Therefore for any interval with length d, there can be only one local minimum point. This definition is illustrated in Figure 4.1. Although point a is a local minimum point in general definition, it is not by our definition. Local minimum point of that interval is point b. If the local minimum point of any interval is not unique, smallest point of these local minimum points is taken as the local minimum point of that interval.

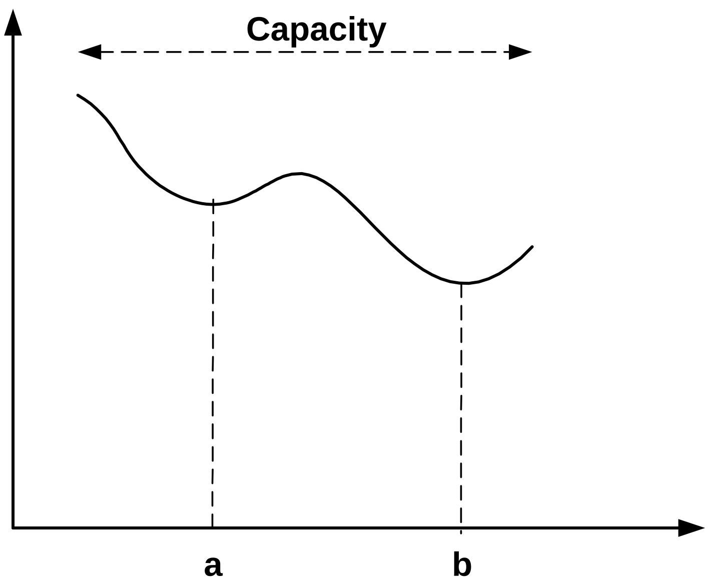

Figure 4.1: Local minimum point of an interval with length C

As stated earlier, it is expected that local minimum points of expected cost function $G_n(y)$ occurs at points which are multiples of base demand (d). In Figure 4.2, local minimum points of function $G(y)$ are highlighted. All of these points are multiples of d ($d = 10$ in the example) as seen in Figure 4.2. If this property is proved, then at any initial inventory point x, it would be optimal to order either nothing or up to a multiple of base demand. Capacity is equal to d, hence in an interval $(x, x+C]$, there can be only one point which is a multiple of d. Therefore optimal ordering decision is restricted to two alternatives; order nothing or order up to nearest multiple of d. Now, assume that initial inventory position at the beginning of a planning horizon is a multiple of d. Then, optimal ordering policy becomes an 'all-or-nothing policy' which means either order nothing or order full capacity. If proved, order amount can take values only zero or d when we limit our state space to multiples of d. In that case, kinks will not exist and behavior of order amount function may be identified clearly.

Figure 4.2: Local minimum points of function $G_n(y)$

We intend to prove that any point that is not a multiple of d cannot be a local minimum of an interval with length equal to capacity. To prove that, we shall show that, for any interval with length equal to d, minimum point of $G_n(y)$ is achieved at a multiple of d. Therefore, for any integer m, $\min_{md \leq y \leq (m+1)d} \{G_n(y)\} = \min \{G_n(md), G_n((m+1)d)\}$. Following theorem shows the properties of local minimum points.

Theorem 1 *For any integer m and $i \epsilon \{0, 1, \ldots, d\}$ the following statements are true for all n:*

a. $G_n(md + i) \geq \alpha G_n(md) + (1 - \alpha) G_n((m + 1)d)$ *where* $\alpha = (d - i)/d$

b. $G_n(md + i) \geq min \{G_n(md), G_n((m + 1)d)\}$

c. Minimum point of $G_n(y)$ occurs at an integer multiple of base demand d.

d. $J_n(md) = -v_n md + \min\{G_n(md), G_n((m+1)d) + K_n\}$

Proof: Proof of Theorem 1 is available at Appendix A.1. Here, we present a sketch for this proof. We prove Theorem 1 by induction.

1. For $n = 1$, we proved the part (a) of the induction and all other parts follow the part (a) of the induction.

2. For $n = k$, we assumed that all parts of the induction are true.

3. For $n = k + 1$, we proved the part (a) of the induction by using the part (d) of the induction assumption in period k. And all other parts follow the part (a) of the induction.

Part (a) of Theorem 1 states that for any period n and integer m, $G_n(md+i)$ is greater than or equal to the convex combination of $G_n(md)$ and $G_n((m+1)d)$.

Part (b) of Theorem 1 follows part (a). By part (b), the point $G_n(md+i)$ lies above the line connecting two other points; $G_n(md)$ and $G_n((m+1)d)$. In other words either $G_n(md)$ or $G_n((m+1)d)$ is the local minimum point of the interval $[md, (m+1)d]$.

Parts (c) immediately follows part (b) and states that S_n is an integer multiple of d due to fact that all of local minimum points are multiples of d.

Part (d) of Theorem 1 also follows part (b). Due to part (b) of Theorem 1, it is not reasonable to order any amount less than capacity if initial inventory is an integer multiple of d. Therefore, if the inventory position is equal to md, then it is optimal to order full capacity (C) or not to order in any period for any integer m which is part (d) of Theorem 1.

Corollary 1 *Optimal policy is an 'all-or-nothing policy' if we restricted the initial inventory levels to the multiples of d.*

Proof: Due to part (d) of Theorem 1, we can conclude that optimal ordering policy is an 'all-or-nothing' policy when state space is limited to the multiples of d. Since ordering decision is restricted to two alternatives at a multiple of d; either order nothing or order full capacity.

By Theorem 1 and Corollary 1, we show that optimal ordering policy is an 'all-or-nothing' policy. In other words, if we start with an inventory position value which is a multiple of base demand (capacity), we will surely end up with an inventory position value which is also a multiple of base demand (capacity) and the order amounts are always either zero or capacity, therefore there will be no kinks in the order quantity function. However, we do not yet show that order quantity is a non increasing function of the inventory level. There could exist a number of intervals where it is optimal to start and stop ordering and this will prevent us from identifying the optimal order policy clearly.

4.2.2 Convexity of the Function $G_n(y)$

In the previous section, we have proved that local minimum points of the function $G_n(y)$ occurs at the multiples of d and optimal ordering policy is an all-or-nothing policy at these points. This leads to; if the initial inventory position is a multiple of base demand (capacity), we will surely end up with an inventory position which is also a multiple of base demand (capacity) at the end of the planning horizon. Therefore, now we only care about the points that are multiples of d and we limit our state space to the multiples of d.

Our goal in this section is to show that order quantity is a non increasing function of the inventory level so in optimal policy, there can be no intervals where it is optimal to start and stop ordering. In fact, we aim to prove that a threshold policy, which is characterized by a period dependent *threshold* level, is optimal. This threshold policy can be explained as follows; at each period if initial inventory position is below a certain threshold level of that period, order full capacity, otherwise order nothing. So function $O_n(x)$ is as follows;

$$O_n(x) = \begin{cases} C & x < s_n \\ 0 & x \geq s_n \end{cases} \tag{4.1}$$

At the previous subsection, we have shown that ordering decision at any initial inventory position is limited to two options. Therefore,

$$J_k(md) = -v_k md + min\{G_k(md), G_k((m+1)d) + K_k\} \tag{4.2}$$

So, when making the ordering decision, one has to compare benefit of having a capacity more inventory on hand and paying the fixed and variable cost of ordering. We redefine the function $H_n(md)$ since only two options are available at an initial inventory value md. $H_n(md)$ determines the tradeoff between ordering and do nothing;

$$H_n(md) = G_k((m+1)d) + K_k - G_k(md) \tag{4.3}$$

and function $O_n(md)$ can be expressed as follows;

$$O_n(md) = \begin{cases} 0 & H_n(md) \geq 0 \\ C & 0 \geq H_n(md) \end{cases} \tag{4.4}$$

If function $H_n(md)$ is a non increasing function, which means relative benefit of ordering full capacity cannot increase as inventory position value increases, then function $O_n(md)$ will be a non increasing function also. Behavior of function $H_n(md)$ depends on the properties of function $G_n(md)$. If function $G_n(md)$ is convex, then both $H_n(md)$ and $O_n(md)$ will be non increasing functions. As a result, when it is optimal not to order at an inventory position, it can never be optimal to order at a greater inventory position. Hence, we can conclude that a period dependent threshold policy is optimal for our problem structure. On the other hand, if $G_n(md)$ is not convex, there will be more than a single region where it is optimal to order full capacity and also more than a single region, where it is optimal to order nothing. This complicates identifying the optimal ordering policy. In Figure 4.3 graph of function $G_n(y)$ is presented. The bold line passing through the points which are multiples of d (=10 in the example) form a piecewise linear function. We attempt to show that this piecewise linear function is a convex function and if so $G_n(md)$ is convex.

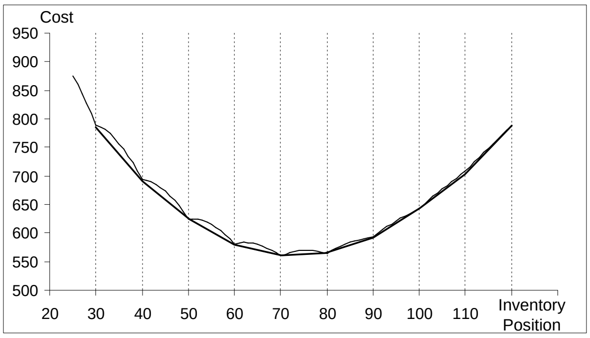

Figure 4.3: Graph of function $G_n(y)$

Before going through the theorem about convexity of $G_n(md)$, a lemma should be stated which will be required in Theorem 2.

Lemma 1 *Let G(x) be a function only defined at integer points. G(x) is a convex function if G(x) satisfies that* $G(m+1) \leq \frac{1}{2}G(m) + \frac{1}{2}G(m+2)$ *for any integer* m.

Proof: G(x) is a function only defined at integer points. Inequality $\frac{1}{2}G(m)+\frac{1}{2}G(m+2) \geq G(m+1)$ holds for any integers m. If we rearrange the terms of the inequality, we get;

$$G(m+2) \geq 2G(m+1) - G(m)$$
$$G(m+2) - G(m+1) \geq G(m+1) - G(m) \tag{4.5}$$

which is the standard convexity definition for discrete functions. □

The following theorem shows the convexity of $G_n(md)$ and establishes the optimality of threshold policy.

Theorem 2 *For any integer* m*, the following statements are true for all* n*:*

a. $G_n((m+1)d) \leq \frac{1}{2}G_n(md) + \frac{1}{2}G_n((m+2)d)$

b. Function $G_n(md)$ *is convex*

Proof: Proof of Theorem 2 is presented in Appendix A.2.

Corollary 2 *Optimal policy is an 'all-or-nothing policy with a threshold level'. If initial inventory is below a threshold level then order full capacity, otherwise order nothing.*

Proof: By using Lemma 1 and Theorem 1, we prove Theorem 2 which states that $G_n(md)$ is convex. So if the inventory position is less than a threshold value, then it is optimal to order full capacity (C) otherwise it is optimal not to order in any period. By Theorem 2 and Corollary 2, we show that optimal ordering policy is a period dependent threshold policy. As a result, order quantity is a non increasing function of the inventory level so in optimal policy there is only one region where it is optimal to order full capacity, and only one region where it is optimal to not order. Let s_n be the threshold level in period n, then it is optimal to order full capacity in region $(-\infty, s_n)$ and it is optimal to order nothing in region $[s_n, \infty)$.

We call this policy as a period dependent threshold policy since threshold level s_n is non-stationary because cost parameters and demand distributions are period dependent. Also number of periods left to go is another factor for non-stationarity even if cost parameters and demand distributions are stationary.

4.2.3 Behavior of the Decision Variables with respect to the Period Index under Stationary Parameters

For our problem instance, we prove that the function $G_n(md)$ is convex and the optimal order policy is an 'all-or-nothing policy'. This threshold policy can be characterized by a period dependent decision variable; s_n. The decision variable s_n is defined as the smallest integer multiple of capacity (d) that satisfies $G_n(s_n) \leq G_n(s_n + d) + K$. In other words $s_n = \min\{x : H_n(x) \geq 0\}$ which means smallest inventory level that it is optimal not to order. Therefore, by definition at all inventory levels below s_n, it is optimal to order which is exactly the definition of threshold point. Decision variable s_n is non-stationary because of many factors such as; period dependent cost parameters and demand distribution, and number of periods left in the planning horizon.

Recall that, decision variable S_n is the point where G_n is minimized. From the properties of G_n which are discussed is Subsection 4.2.1, S_n is also an integer multiple of capacity (d).

Next we intend to show the relationship between these two variables and periods. However, in order to define such a relationship, we restrict ourselves to the case where parameters are stationary so we drop the period index from unit backorder, unit holding, unit variable and fixed ordering costs and also from demand probabilities.

As the number of periods to go decreases, (as we get closer to the end of the planning horizon), threshold level and inventory level where global minimum is reached are expected to decrease. If proved, s_n and S_n are non decreasing in n. This is not the case in the general case. Recall that in general case, S_n shows a cyclic pattern in some specific problem settings. This is due to capacity and demand distribution relationship when fixed ordering cost is present. In order to avoid fixed ordering cost, it may be optimal to order i times in a j period interval ($i < j$), which results in a cyclic pattern in S_n values. However this cyclic pattern is not expected in this special case since there exists a specific relation between demand values and capacity. So we may attempt to define a behavior of variables (s_n, S_n) with respect to number of period left to go n.

Before going the next step, we should state the following lemmas:

Lemma 2 *For the stationary problem, for any period n, $s_n \leq S_n$.*

Proof: It is obvious that the smallest point that satisfies $G_n(s_n) \leq G_n(s_n + d) + K$ is less than or equal to S_n due to convexity of function G_n. Since function $H_n(x)$ is always negative when inventory level is greater than or equal to S_n. Assumption about fixed ordering cost (K is less than unit backorder cost multiplied by capacity $b.C$) is needed to guarantee the existence of such point for single period problem.

Lemma 3 *For the stationary problem, for any period n , j being an integer multiple of d, and for $j \leq S_n$, we have $G_n(j-d)-G_n(j) \geq 0$ and $J_n(j-d)-J_n(j) \geq 0$.*

Proof: By definition of S_n and convexity of G_n, proof is obvious.

Theorem 3 *For any j being an integer multiple of d, the following statements are true for any period n, $n = 1, 2, \ldots$ for the stationary problem:*

a. $J_n(j-d) - J_n(j) \geq J_{n-1}(j-d) - J_{n-1}(j)$ *where* $j \leq S_n$

b. $G_{n+1}(j-d) - G_{n+1}(j) \geq G_n(j-d) - G_n(j)$ *where* $j \leq S_n$

c. $S_{n+1} \geq S_n$

d. $s_{n+1} \geq s_n$

Proof: Proof of Theorem 3 is presented in Appendix A.3. Here, we present a sketch for this proof. We prove Theorem 3 by induction.

1. For $n = 1$, we proved the part (a) of the induction and parts (b) and (c) follow part (a). After proving part (c), statement in part (b) is extended from '$j \leq S_1$' to '$j \leq S_2$'. Then, part (d) of the induction is proved.

2. For $n = k$, we assumed that all parts of the induction are true.

3. For $n = k + 1$, we proved the part (a) of the induction by using a property of cost difference and parts (b) and (c) follow part (a). After proving part (c), statement in part (b) is extended from '$j \leq S_n$' to '$j \leq S_{n+1}$'. Then, part (d) of the induction is proved.

We prove Theorem 3 and by this theorem we show that, order up to level S_n and threshold point s_n are nondecreasing as the number of periods to go increases under stationary parameters. As the number of periods to go decreases, less inventory is carried since threshold level is reduced when cost parameters and demand distribution are stationary.

4.3 Computational Analysis for the Special Case

In this section, we present some computational findings about problem investigated in this chapter. Although this section is very similar to the computational analysis carried out in Chapter 3, we expect to verify our analytical findings here. Like computational analysis of the general case, examples are presented in order to show how functions $J_n(x)$ and $G_n(y)$, and variables s_n and S_n are affected by cost parameters, demand distribution, and number of periods to go.

Consider the example; $h = 2.0$, $b = 20.0$, $K = 80.0$, $v = 2.0$, and $C = 10$. Demand is equal to 0 with a probability of 0.50, 10 with a probability of 0.30, 20 with a probability of 0.10 and 30 with a probability of 0.10. Expected value of the demand is equal to 8 which is smaller than capacity value and finally discount factor is equal to 0.95.

- Number of periods to go: As the number of periods to go increases, values of cost functions at the same inventory level increase. That is, $J_n(x) > J_{n-1}(x)$, $G_n(y) > G_{n-1}(y)$. In Subsection 4.2.3, it is proved that the variables S_n and s_n are non decreasing as number of periods to go increases. In Table 4.1 below, S_n and s_n values for different values of n are presented.

Table 4.1: Values of S_n and s_n with respect to number of periods to go

Period	1	2	3	4	5	6	7	8	9	10
S_n	20	30	30	40	40	40	50	50	50	50
s_n	0	10	20	30	30	30	30	30	30	30

- Unit holding cost: As unit holding cost increases, it is observed that value of cost function at the same inventory level increases. Because either inventory holding cost increases or amount of inventory on hand decreases which increases the expected backorder cost. On the other hand, the variables S_n and s_n are expected to decrease since inventory holding becomes less beneficial.

- Unit backorder cost: As unit backorder cost increases, it is observed that value of cost function at the same inventory level increases. Because either total backorder cost increases or inventory on hand value increases which increases the inventory holding cost. Moreover, the variables S_n and s_n are expected to increase since an increase in penalty cost of stockout increases the incentive to hold inventory.

- Fixed ordering cost: As fixed cost of ordering increases, it is observed that value of cost function at the same inventory level increases since each order costs more. Any attempt to decrease total fixed ordering cost incurred in a planning horizon increases total inventory holding cost and backorder cost. Moreover, the variable S_n is observed to increase since inventory on hand becomes more valuable due to increased allocated fixed cost to each ordered item. However, variable s_n is observed to decrease since the relative benefit of ordering full capacity decreases.

Table 4.2: Values of $G_n(y)$, S_n, and s_n with respect to cost parameters

	Holding Cost			Backorder Cost			Fixed Ordering Cost		
% Increase	S_{10}	s_{10}	G_{10}	S_{10}	s_{10}	G_{10}	S_{10}	s_{10}	G_{10}
-50%	70	40	665	50	30	795	50	40	810
0%	50	30	913	50	30	913	50	30	913
50%	40	30	1097	60	40	980	60	30	983
100%	40	30	1243	60	40	1026	70	30	1044
400%	20	20	1826	70	60	1172	90	30	1478

In Table 4.2, $G_n(y)$, S_n, and s_n values are presented. Holding cost, backorder costs, and fixed ordering costs are decreased by 50 percent and increased by 50, 100, and 400 percent, and corresponding $G_n(y)$, S_n, and s_n values for $n = 10$ are given in Table 4.2. Unlike analysis of the general case, cost function and variables values are most sensitive to the percentage change in unit holding cost rather than fixed ordering cost. Moreover percentage change in fixed ordering cost has greater effect on the cost function and variables than percentage change in unit backorder cost. This difference can be explained as follows; Capacity is restricted in the special case and expected value of the demand is very close to capacity value in this example and also unit backorder cost is relatively high. Therefore, it is not possible to benefit from economies of scale in ordering and number of orders in a 10 period problem is somewhat stable. Hence, system behavior is less sensitive to changes in fixed ordering cost.

- Expected demand: Increase in expected demand causes an increase in cost in our example. Moreover, the variables S_n and s_n also increase. In Table 4.3 below, $G_n(y)$, S_n, and s_n values for different expected values of demand such as 7, 8, 9, and 10 when $n = 10$ are presented. Recall that in Subsection 3.3.4, there are some conditions where variable behavior does not have any pattern. Those types of situations do not occur in the special case analysis.

Table 4.3: Values of $G_n(y)$, S_n, and s_n with respect to Expected Demand

Expected demand	S_{10}	s_{10}	G_{10}
7	50	30	867
8	50	30	913
9	60	40	955
10	70	40	1000

- Maximum demand: Increase in maximum demand (while maintaining the same expected demand value) causes an increase in cost in our example. Moreover, the variables S_n and s_n increase. In Table 4.4, $G_n(y)$, S_n, and s_n values for maximum demand values 20, 30, 40 and 50 when $n = 10$ are presented. Recall that in general case there are some conditions where variable behavior does not show any pattern, which may be resulted from capacity demand relationship. Those types of situations do not occur in the special case analysis.

Table 4.4: Values of $G_n(y)$, S_n, and z_n with respect to Maximum Demand

Maximum demand	S_{10}	s_{10}	G_{10}
20	50	30	760
30	50	30	913
40	60	40	1018
50	60	50	1091

CHAPTER 5

CHARACTERISTICS OF THE INFINITE HORIZON PROBLEM WITH RESTRICTED DEMAND DISTRIBUTION AND AVERAGE COST CRITERION

In Chapter 4, we defined a special case for the general problem and established the optimal policy for that special case under discounted cost criterion. The optimal policy is an all-or-nothing policy with a threshold level. In this chapter, we analyze infinite horizon problem of the same special case under average cost criterion and under stationary parameters. In Section 5.1, Discrete Time Markov Chain model of the infinite horizon problem is presented. Each state of Markov Chain is the shortfall level from the threshold/order up to level. Determination of the threshold level using steady state distribution of shortfall level is also discussed in Section 5.1. In Section 5.2, we present an analysis of the optimal threshold level.

5.1 Infinite Horizon Model

In the previous chapter, we define a special case of the general problem. In this special case, all possible demand values are multiples of a unit capacity. The objective is to minimize the expected discounted cost of system over a finite horizon. We show that if the initial inventory is a multiple of capacity, then optimal order policy is an all-or-nothing policy specified by a period-dependent threshold level. So inventory levels other than multiples of capacity can never be reached with this policy. As a result of this, at the beginning of each period, we are at an inventory position which is a multiple of d we make our order decision depending on our inventory position relative to threshold level; either order full capacity or nothing, then order arrives and demand is faced and we carry positive or negative inventory which is also a multiple of d to the following period. Therefore, in this special case, we restrict our state space of inventory level to the multiples of capacity.

In this chapter, we are interested with the infinite horizon problem of the special case under average cost criterion. We apply an all-or-nothing policy characterized by a threshold level as the ordering policy for this problem. We should state a conjecture here: All-or-nothing policy is optimal for the infinite horizon problem with special demand structure under average cost criterion.

An all-or-nothing policy with a threshold level in our setting implies that the inventory position will always be less than or equal to the threshold level. Since capacity is equal to d in our special case, inventory level can never be above the threshold level once it goes below the threshold level. In fact, problem becomes a base stock policy, whenever inventory level is below base stock / threshold level, s, order full capacity, otherwise order nothing. Hence, parameter s can be viewed as the order up to level. Inventory level can take a maximum value of threshold level in the long run. We suggest that the inventory positions be represented as shortfalls from the threshold level. The shortfall w is defined as the amount of inventory that is less than threshold level: $s - x = w$. Shortfalls are not same as backorders; backorder means we have negative inventory values (amount of unsatisfied customer demand), on the other hand, shortfall means amount that cannot be produced because of capacity constraint. Steady state distribution of these shortfall levels represent the system condition in the long run and can be used in determination of threshold level and calculation of average cost in the infinite horizon problem.

5.1.1 Discrete Time Markov Chain Model of the Special Case

As stated before, the inventory position can be represented as shortfall from the threshold level. These shortfall levels represent the states of the Markov chain. In this subsection, we present a Markov model of the infinite horizon problem of the special case. By using the state transition probabilities, we determine the steady state distribution of shortfalls from the threshold level. Then, using this steady state distribution, average cost per period of the inventory system is calculated and the optimal threshold level where the average cost is minimum is determined. In order to do these, we first describe our Markov model. The length of the epoch of our Markov chain is the period of the inventory model. The state of the system at the beginning of each period is defined by the shortfall from the threshold level. The sequence of events for this Discrete Time Markov chain model is presented in Figure 5.1;

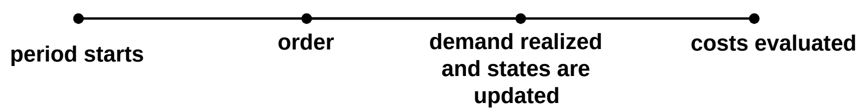

Figure 5.1: Sequence of events in an epoch

At the beginning of each period depending on the shortfall level, order decision is given. That is, if shortfall is equal to zero, nothing is ordered. If shortfall is greater than zero production occurs. Other than variable production cost per unit ordered (total variable cost paid is equal to unit variable cost multiplied by capacity in each order since order amount is equal to capacity), a fixed ordering cost is incurred for that period. Ordered amount arrives instantaneously raising inventory position value by one (thus reducing shortfall level by one). Then, demand is realized and satisfied with on-hand inventory, unsatisfied demand is fully backordered, and holding and backorder costs are assessed at the end of the period. Shortfall level of the system is updated. Shortfall level follows a discrete time Markov chain with an infinite state space. That is, shortfall level can take values between zero and infinity $\{w = 0, 1, \ldots\}$. Demand in each period is discrete and stationary. Let p_j be the probability that demand is equal to jd for all $j \epsilon \{0, 1, \ldots\}$. These demand probabilities form the state transition probability matrix of the Markov chain. Let a_{ij} be the transition probabilities from state i to state j. Then a_{ij}'s are;

$$\begin{aligned} a_{0j} &= p_j \\ a_{ij} &= \begin{cases} 0 & j+1-i < 0 \\ p_{j+1-i} & j+1-i \geq 0 \end{cases} \end{aligned} \tag{5.1}$$

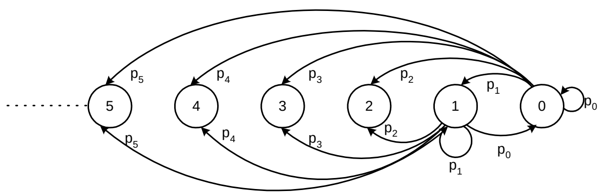

Figure 5.2: State Transition Diagram

If the shortfall level is equal to zero, than the shortfall level in the next period is equal to demand level in this period because when the shortfall is zero nothing is ordered. If the shortfall level is positive, than shortfall level in the next period is equal to shortfall level plus demand level minus one because a unit capacity is ordered when shortfall is positive. Therefore, from any positive shortfall level i, we cannot reach a shortfall level j if $j > 1 - i$. In Figure 5.2, a partial state transition diagram of Markov chain is presented where only transitions from state 0 and state 1 are presented as examples.

The state transition probability matrix is illustrated in Figure 5.3;

	0	1	2	3	4	5	6	.	.	.
0	p_0	p_1	p_2	p_3	p_4	p_5	p_6	.	.	.
1	p_0	p_1	p_2	p_3	p_4	p_5	p_6	.	.	.
2	0	p_0	p_1	p_2	p_3	p_4	p_5	.	.	.
3	.	0	p_0	p_1	p_2	p_3	p_4	.	.	.
4	.	.	0	p_0	p_1	p_2	p_3	.	.	.
5	.	.	.	0	p_0	p_1	p_2	.	.	.
6	.	.	.	.	0	p_0	p_1	.	.	.
.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.

Figure 5.3: State Transition Probability Matrix

It can be inferred from Figure 5.3 that transition probabilities depend only on demand probabilities of the problem. This Markov chain is recurrent if $\sum_{r=0}^{\infty} p_r r \leq 1$. This makes sense intuitively: $\sum_{r=0}^{\infty} p_r r$ is the ratio of expected value of demand in a period to capacity. If this ratio is greater than 1, shortfall will grow without bound and the Markov chain will be transient. Since Markov chain is assumed to be irreducible, two assumption for demand distribution should be made also; $p_0 > 0$ and $p_0 + p_1 < 1$. First assumption is identical to our assumption about expected value of demand (expected value of demand should be smaller than or equal to capacity). Second assumption is needed to have shortfalls greater than 1. Otherwise, shortfall level can take only two values; zero or one. For a detailed proof of existence of steady state distribution see Kulkarni (1995).

Let π_i be the steady state probability of state i. Since our Markov chain is irreducible, recurrent and aperiodic, we solve $\pi = \pi P$. We have already stated that transition probabilities of Markov chain depend only on demand probabilities. Hence, steady state distribution also depend only on demand probabilities. As a result; any problem having the same demand probabilities will have the same shortfall distribution in the longrun whatever the cost parameters or threshold levels are. In other words, changing cost parameters or setting different threshold levels do not affect the steady state distribution of shortfall. So, by $\pi = \pi P$, we get following equations;

$$\pi_j = \sum_{i=0}^{\infty} \pi_i a_{ij} \tag{5.2}$$

this yields

$$\pi_j = \pi_0 p_j + \sum_{i=1}^{j+1} \pi_i p_{j-i+1} \tag{5.3}$$

Kulkarni (1995) determines the π_i values by using generating function of limiting distributions and comes up with following result.

$$\pi_0 = 1 - \sum_{r=0}^{\infty} p_r r \tag{5.4}$$

This result turns out to be what we expected. In each state other than 0, we order full capacity, therefore average number of order in a period can be denoted by $1 - \pi_0$ which should be equal to the ratio of expected value of the demand to capacity. Moreover, π_j 's are computed as follows;

$$\begin{aligned}
\pi_1 &= \frac{1 - p_0}{p_0} \pi_0 \\
\pi_2 &= \frac{1 - p_0 - p_1}{p_0} (\pi_0 + \pi_1) \\
\pi_3 &= \frac{1 - p_0 - p_1 - p_2}{p_0} (\pi_0 + \pi_1 + \pi_2) + \frac{p_2}{p_0} \pi_2
\end{aligned} \tag{5.5}$$

and in general;

$$\pi_{j+1} = \frac{1 - \sum_{i=0}^{j} p_i}{p_0} (\sum_{i=0}^{j} \pi_i) + \sum_{i=2}^{j} \pi_i \sum_{k=j-i+2}^{j} (p_k / p_0) \tag{5.6}$$

By using $\pi_0 = 1 - \sum_{r=0}^{\infty} p_r r = \frac{E[D]}{C}$ all the steady state probabilities can be computed using a simple computer code. After finding the steady state distribution, long run expected cost per period can be written as follows;

$$
\begin{aligned}
AC(s) &= \sum_{i=1}^{\infty}\pi_i[\sum_{r=0}^{s-i+1} hp_r(s-i+1-r) + \sum_{r=s-i+1}^{\infty} bp_r(r-s+i-1)] \\
&+ \pi_0[\sum_{r=0}^{s} hp_r(s-r) + \sum_{r=s}^{\infty} bp_r(r-s)] + (1-\pi_0)K + (1-\pi_0)vC(5.7)
\end{aligned}
$$

Average cost per period has four components; holding cost, backorder cost, fixed ordering cost and variable cost. Holding and backorder cost calculation is based on shortage level and threshold level. For each shortage level other than zero, inventory carried to the following period is positive if demand is less than threshold level minus shortage plus one (amount that is ordered in that period), and negative if demand is greater than that value. If shortage is zero, inventory carried to the following period is positive if demand is less than threshold level minus shortage since nothing is ordered in that period, and negative if demand is greater than that value. Moreover, for each shortage level other than zero, a fixed ordering cost and capacity times unit variable cost are incurred. Shortfall distribution only depends on the demand probabilities therefore average cost per period is a function of threshold level so we may define it as $AC(s)$. Average cost function is a convex function (goes to infinity as s goes to minus and plus infinity) and has a finite minimizing point s^*. Optimal threshold level is equal to the one when minimum average cost per period is achieved so equals to s^*. Because demand is a discrete random variable, function $AC(s)$ is not continuous. So we cannot take derivative to find the global minimum point. Instead of derivative, we will evaluate the differences between $AC(s+1)$ and $AC(s)$. Since $AC(s)$ is convex, this statement $AC(s+1) - AC(s)$ goes from negative to positive as s increases and s^* is the point where this difference becomes positive for the first time. Although it corresponds to different inventory positions, steady state distribution of shortfall is the same for all threshold levels, so determining the differences are quite easy;

$$
AC(s+1) - AC(s) = \sum_{i=1}^{\infty}\pi_i[\sum_{r=0}^{s-i+2} hp_r(s-i+2-r) - \sum_{r=0}^{s-i+1} hp_r(s-i+1-r)
$$

$$
\begin{aligned}
&+ \sum_{r=s-i+2}^{\infty} bp_r(r-s+i-2) - \sum_{r=s-i+1}^{\infty} bp_r(r-s+i-1)] \\
&+ \pi_0[\sum_{r=0}^{s+1} hp_r(s+1-r) - \sum_{r=0}^{s} hp_r(s-r) \\
&+ \sum_{r=s+1}^{\infty} bp_r(r-s-1) - \sum_{r=s}^{\infty} bp_r(r-s)] \qquad (5.8)
\end{aligned}
$$

Note that, fixed ordering cost and capacity times variable cost terms cancel each other, since these costs are not related with the threshold level. By increasing threshold level, total inventory holding cost increases and total inventory backorder cost decreases and we aim to find the threshold level where total inventory related cost is minimum. Since, when determining the optimal threshold level, fixed ordering cost is not our concern in this problem structure, we expected a result somewhat similar to the optimal solution to base stock policy. If necessary eliminations are done in the Equation 5.8, it reduces to;

$$
\begin{aligned}
AC(s+1) - AC(s) &= \sum_{i=1}^{\infty} \pi_i [\sum_{r=0}^{s-i+1} hp_r - \sum_{r=s-i+2}^{\infty} bp_r] \\
&+ \pi_0[\sum_{r=0}^{s+1} hp_r - \sum_{r=s+1}^{\infty} bp_r]
\end{aligned}
$$

However in this statement there is a problem of the lower limit of the sum; $\sum_{r=s-i+2}^{\infty} bp_r$. When i is greater than $s+2$ lower limit becomes negative. To avoid this situation, we can replace $\sum_{r=s-i+2}^{\infty} p_r$ with $(1 - \sum_{r=0}^{s-i+1} p_r)$ and similarly $\sum_{r=s+1}^{\infty} p_r$ is replaced with $(1 - \sum_{r=0}^{s} p_r)$. Moreover we can update the upper limit of π_i as $s+1$ since upper limits of the sums inside are $s-i+1$.

$$
AC(s+1) - AC(s) = (h+b)[\sum_{i=1}^{s+1} \pi_i \sum_{r=0}^{s-i+1} p_r] + (h+b)\pi_0 \sum_{r=0}^{s} p_r - b \qquad (5.9)
$$

s^* is the smallest point that satisfies;

$$
AC(s+1) - AC(s) = \sum_{i=1}^{s+1} \pi_i \sum_{r=0}^{s-i+1} p_r + \pi_0 \sum_{r=0}^{s} p_r \geq \frac{b}{b+h} \qquad (5.10)
$$

This statement is the probability of shortfall plus demand minus order amount is less than threshold level, in other words, probability of not being stockout.

Optimal threshold level s^* is the smallest point where this probability is greater than or equal to the critical ratio $\frac{b}{b+h}$. This result is exactly the same as base stock policy result as expected due to insensitivity to fixed ordering cost. If we return back to our Markov model, in which each state represents shortage from threshold level. So stockout occurs when shortage level is greater than threshold level. Therefore, probability of not being stockout in the long run can be expressed as sum of the steady state probabilities of states {0,1,...,s} which is $\sum_{i=0}^{s} \pi_i$ intuitively.

Proposition 1 *Optimal threshold level is the smallest s that satisfies $\sum_{i=0}^{s} \pi_i \geq \frac{b}{b+h}$.*

Proof:

We know that $\pi_j = \sum_{i=1}^{j+1} \pi_i * p_{j+1-i} + \pi_0 p_j$ from properties of steady state probabilities. When we simplify the Equality 5.10;

$$\begin{aligned} AC(s+1) - AC(s) &= \sum_{i=1}^{s+1} \pi_i \sum_{r=0}^{s-i+1} p_r + \pi_0 \sum_{r=0}^{s} p_r \\ &= \pi_0 p_0 + \pi_1 p_0 + \pi_0 p_1 + \pi_1 p_1 + \pi_2 p_0 + \ldots \\ &+ \pi_0 p_s + \pi_1 p_s + \ldots + \pi_{s+1} p_0 \\ &= \pi_0 + \pi_1 \ldots + \pi_s \\ &= \sum_{i=0}^{s} \pi_i \end{aligned}$$

Optimal threshold point is the smallest point that satisfies;

$$\sum_{i=0}^{s} \pi_i \geq \frac{b}{b+h} \tag{5.11}$$

So algebraically, we show that $AC(s+1) - AC(s)$ is minimized when sum of the steady state probabilities of states {0,1,...,s} reaches the critical ratio $\frac{b}{b+h}$. Optimal threshold level is the smallest point that probability of not being stockout in any period is greater than or equal to critical ratio $\frac{b}{b+h}$. This newsvendor type result is expected since threshold level is only affected by holding cost, backorder cost, and demand distribution. As a result, for any given problem instance satisfying the properties of the special case, steady state distribution of shortfall levels can be computed by a simple computer code and optimal threshold level s^* can be calculated by using steady state distribution and unit backorder/holding cost. Again, average cost per period for the infinite horizon problem can be computed using the optimal threshold level s^*.

5.1.2 Analysis of the Optimal Threshold Level

In this subsection, we discuss how threshold level of the infinite horizon problem s^* is affected by cost parameters and demand distribution.

- Unit holding cost and unit backorder cost: Since the optimal threshold level is determined by the critical ratio $\frac{b}{b+h}$, as long as ratio of unit backorder cost to unit holding cost is constant, optimal threshold level is the same. If relative value of unit backorder cost to unit holding cost increases optimal threshold level also increases.

- Fixed ordering cost and unit variable cost: As stated before, optimal threshold level is not affected by values of fixed ordering cost and unit variable cost as long as the assumptions about fixed ordering cost and unit variable cost are valid ($K < bC$ for single period problem and $v < b$). Since every time the shortage level is positive, fixed ordering cost and capacity times unit variable cost are incurred regardless of threshold level.

- Expected demand: Increase in expected demand results in a decrease in π_0 value since it is equal to $(1 - \frac{E[D]}{c})$. Moreover, all other steady state probability values change not only for their dependency to π_0 value, but also for the change in state transition probabilities (demand probabilities). Nevertheless, we expect that optimal threshold level value is nondecreasing as expected value of demand increases.

- Maximum demand: Increase in maximum demand (while maintaining the same expected demand value) has no effect on π_0 value. Also, it is obvious that optimal threshold point is less than or equal to the maximum possible value of the demand. Since if inventory on hand is equal to maximum possible demand value, probability of stockout is zero. However, it is hard to determine how the threshold level is affected as maximum demand value increases.

CHAPTER 6

APPLICATION AND TESTING THE ALL-OR-NOTHING POLICY TO THE INFINITE HORIZON PROBLEM WITH A GENERAL DEMAND DISTRIBUTION AND AVERAGE COST CRITERION

In Chapter 5, we analyzed the infinite horizon problem of the special case under average cost criterion and presented a Discrete Time Markov Chain model of the special case and determination of the threshold level using steady state distribution of this Markov chain. In this chapter, we construct a similar Markov model for the general demand case where the ordering policy is restricted to an all-or-nothing threshold policy. In Section 6.1, a discrete time Markov chain model of the general problem is presented when policy is an all-or-nothing type. Determination of threshold level using the steady state distribution is discussed in Section 6.1 as well. Section 6.2 presents a comparison of this model with the optimal solution found by the dynamic programming. Performance of all-or-nothing heuristic in different problem settings is illustrated with examples.

6.1 Infinite Horizon Model

In the previous chapter, we present a Discrete Time Markov Chain model for the infinite horizon problem of the special case under average cost criterion. The optimal policy for the special case is well defined as an all-or-nothing policy characterized by a threshold level. It turned out to be that this threshold level is a base stock level at the same time in the infinite horizon problem because of the relationship between possible demand values and capacity. Therefore, we suggest that the inventory positions can be represented as shortfalls from the threshold level. Steady state distribution of shortfall levels represent the long run condition of the system, and this distribution are computed by using demand probabilities. Average cost per period is defined using these steady state probabilities. Finally, optimal threshold level for the infinite horizon problem is determined which turns out to be similar to the base stock level policy for the special case.

Now, we attempt to model the infinite horizon problem for the general discrete demand case as a discrete time Markov chain, by restricting the ordering policy to an all-or-nothing policy with a threshold level. We determine the optimal threshold level and by using this threshold level, average cost per period is calculated and compared with the average cost calculated by the dynamic programming formulation to test the performance of the all-or-nothing policy as a heuristic for the general case. Since, capacity does not have any relationship with the possible demand values, problem cannot turn out to be a base stock policy like the special case in Chapter 5. Application of the all-or-nothing policy to the general case may force certain state to take values higher than the threshold level, as we are only allowed to produce at capacity. Hence, states of the Markov chain are defined as shortfall from threshold level, so we may have negative shortfalls which means our states of Markov chain may start with a negative number.

6.1.1 Discrete Time Markov Chain Model of the General Case

In this subsection, a Discrete Time Markov Chain model of the infinite horizon problem with general demand case is presented. As in the model for the special case, the inventory positions are represented as shortfalls from the threshold level. These shortfall levels represent the states of the Markov chain. Unlike previous case, shortfall levels can take negative values since capacity may be greater than many possible demand values. In the long run, independent of the inventory position at the beginning of the planning horizon, inventory level eventually drops below the threshold level. The smallest shortfall level that ordering is possible is 1 and ordering brings the shortfall level to $1 - C$ (where C is the capacity). Therefore the smallest value that shortfall can take is $1 - C$. We first determine the state transition probabilities of Markov chain. Then, by using these state transition probabilities, we determine the steady state distribution of shortfall from threshold level. By using this steady state distribution, average cost per period of the inventory system is calculated and the optimal threshold level where the average cost is minimum can be determined. Like the model presented in Chapter 5, the length of the epoch of our Markov chain is the period of the inventory model. The state of the system at the beginning of each period is defined by the shortfall from threshold level. The sequence of events for this Markov chain is very similar to the one defined in the previous chapter and presented in Figure 6.1;

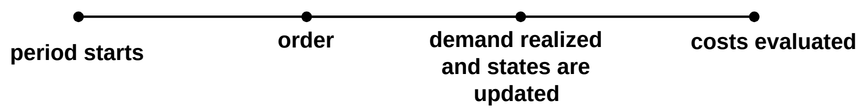

Figure 6.1: Sequence of events in an epoch

At the beginning of each period depending on the shortfall level, order decision is given. In other words, if shortfall is not positive, nothing is ordered. If shortfall is positive production occurs, other than variable production cost per unit ordered (total variable cost paid is equal to unit variable cost multiplied by capacity in each order), a fixed ordering cost is incurred for that period. Ordered amount arrives instantaneously raising inventory position value by capacity value (thus reducing shortfall level by capacity value). Then, demand is realized and satisfied with on-hand inventory, unsatisfied demand is fully backordered, and holding and backorder costs are assessed at the end of the period. Shortfall level of the system is updated. Shortfall level follows a Markov process with infinite state space, that is, shortfall level can take values between $-C+1$ and infinity $\{t = -C+1, -C+2, \ldots\}$. Demand in each period is discrete and stationary and no longer multiples of d. Let p_j be the probability that demand is equal to j for all $j \epsilon \{0, 1, \ldots\}$. These demand probabilities form the state transition probability matrix of the Markov chain. Let a_{ij} be the transition probabilities from state i to state j. Then a_{ij}'s are;

$$a_{ij} = \begin{cases} 0 & j - i < 0 \quad i \leq 0 \\ p_{j-i} & j - i \geq 0 \quad i \leq 0 \\ 0 & j + C - i < 0 \quad i > 0 \\ p_{j+C-i} & j + C - i \geq 0 \quad i > 0 \end{cases} \tag{6.1}$$

If the shortfall level is not positive, than the shortfall level in the next period is equal to shortfall level plus demand level in this period because when the shortfall is not positive, nothing is ordered. If the shortfall level is positive, than shortfall level in the next period is equal to shortfall level plus demand level minus capacity because a unit capacity is ordered when shortfall is positive. Therefore from any positive shortfall level i, we cannot reach a shortfall level j if $j > C - i$. In Figure 6.2, a partial state transition diagram of Markov chain is presented where only transitions from state -C+1 and 1 are presented as examples.

The state transition probability matrix is presented in Figure 6.3;

It can be inferred by the Figure 6.3, transition probabilities depend only

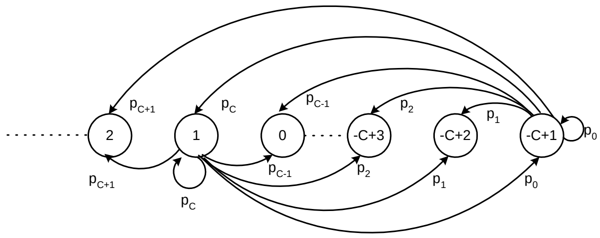

Figure 6.2: State Transition Diagram for General Case

	-C+1	-C+2	-C+3	.	.	0	1	2	.	.
-C+1	p_0	p_1	p_2	.	.	p_{C-1}	p_C	p_{C+1}	.	.
-C+2	0	p_0	p_1	.	.	p_{C-2}	p_{C-1}	p_C	.	.
-C+3	0	0	p_0	.	.	p_{C-3}	p_{C-2}	p_{C-1}	.	.
.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.
0	0	0	0	.	.	p_0	p_1	.	.	.
1	p_0	p_1	p_2	.	.	p_{C-1}	p_C	p_{C+1}	.	.
2	0	p_0	p_1	.	.	p_{C-2}	p_{C-1}	p_C	.	.
.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.

Figure 6.3: State Transition Probability Matrix for General Case

on demand probabilities of the problem like in the special case. This Markov chain is recurrent if $\sum_{r=0}^{\infty} p_r r \leq C$. This makes sense intuitively: $\sum_{r=0}^{\infty} p_r r$ is the expected value of demand in a period. If expected demand in a period is greater than capacity, shortfall will grow without bound and the Markov chain will be transient. Let π_i be the steady state probability of state i. Since our Markov chain is irreducible, recurrent and aperiodic, if we solve $\pi = \pi P$.

We have already stated that transition probabilities of Markov chain depend only on the demand probabilities like in the special case. Hence, steady state distribution also depend only on demand probabilities. As a result; any problem having the same demand probabilities have the same shortfall distribution in the longrun whatever the cost parameters or threshold levels are. In other words, changing cost parameters or setting different threshold levels do not affect the steady state distribution of shortfall. On the other hand, capacity value affects the steady state distribution of shortfall because changing capacity value also changes the state space of the Markov chain. So, by $\pi = \pi P$, we get the following equations ;

$$\pi_j = \sum_{i=-C+1}^{\infty} \pi_i a_{ij} \tag{6.2}$$

Also $\pi_i \geq 0$ for all i. Unfortunately determining the π_i values is not as easy as the previous Markov chain. Therefore a state reduction algorithm described in Heyman et all (1991) is used for determining the steady state values. In this state reduction algorithm, r_i which gives the expected number of visits to state i between two visits to state 0 ($-C+1$ in our case), are computed. Then, normalized r_i values are taken as steady state probabilities of states of the Markov chain. All the steady state probabilities are computed by a simple computer code using state reduction algorithm. After finding the steady state distribution, long run expected cost per period can be written as follows;

$$AC(s) \;=\; \sum_{i=1}^{\infty} \pi_i [\sum_{r=0}^{s-i+C} hp_r(s-i+C-r) + \sum_{r=s-i+C}^{\infty} bp_r(r-s+i-C)]$$

$$
\begin{aligned}
&+ \sum_{i=-C+1}^{0} \pi_i[\sum_{r=0}^{s-i} hp_r(s-r-i) + \sum_{r=s-i}^{\infty} bp_r(r-s+i)] \\
&+ K\sum_{i=1}^{\infty}\pi_i + vC\sum_{i=1}^{\infty}\pi_i \qquad (6.3)
\end{aligned}
$$

Average cost per period has four components; holding cost, backorder cost, fixed ordering cost and variable cost. Holding and backorder cost calculation is based on shortage level and threshold level. For each positive shortage level, inventory carried to the following period is positive if demand is less than threshold level minus shortage plus capacity (amount that is ordered in that period), and negative if demand is greater than that value. If shortage is zero or negative, inventory carried to the following period is positive if demand is less than threshold level minus shortage since nothing is ordered in that period, and negative if demand is greater than that value. Moreover for each positive shortage level, a fixed ordering cost and capacity times unit variable cost are incurred. Given the capacity value, shortfall distribution only depends on the demand probabilities therefore average cost per period is a function of threshold level so we may define it as $AC(s)$ as in the special case. Average cost function is a convex function (goes to infinity as s goes to minus and plus infinity) and has a finite minimizing point s^*. Optimal threshold level is equal to the one when minimum average cost per period is achieved so is equal to s^*. Because demand is a discrete random variable, function $AC(s)$ is not continuous. So we cannot take derivative to find the global minimum point. Instead of derivative, we will evaluate the differences between $AC(s+1)$ and $AC(s)$. Since $AC(s)$ is convex, this statement $AC(s+1)-AC(s)$ goes from negative to positive as s increases and s^* is the point where this difference becomes positive for the first time. Although it corresponds to different inventory position value, steady state distribution of shortfall is the same for all threshold levels, so determining the differences are quite easy like in the special case;

$$
AC(s+1) - AC(s) = \sum_{i=1}^{\infty}\pi_i[\sum_{r=0}^{s-i+C+1} hp_r(s-i+C+1-r) - \sum_{r=0}^{s-i+C} hp_r(s-i+C-r)
$$

$$
\begin{aligned}
&+ \sum_{r=s-i+C+1}^{\infty} bp_r(r-s+i-C-1) - \sum_{r=s-i+C}^{\infty} bp_r(r-s+i-C)] \\
&+ \sum_{i=-C+1}^{0} \pi_i [\sum_{r=0}^{s-i+1} hp_r(s+1-i-r) - \sum_{r=0}^{s-i} hp_r(s-i-r) \\
&+ \sum_{r=s-i+1}^{\infty} bp_r(r-s+i-1) - \sum_{r=s-i}^{\infty} bp_r(r-s+i)] \qquad (6.4)
\end{aligned}
$$

Note that, fixed ordering cost and capacity times variable cost terms cancel each other, since these costs are not related with the threshold level. By increasing threshold level, total inventory holding cost increases and total inventory backorder cost decreases and we aim to find the threshold level where total inventory related cost is minimum. Since, when determining the optimal threshold level, fixed ordering cost is not our concern in this problem structure, we expected a result somewhat similar to the optimal solution to base stock policy like in Chapter 5. If necessary eliminations are done in the Equation 6.4, it reduces to;

$$
\begin{aligned}
AC(s+1) - AC(s) &= \sum_{i=1}^{\infty} \pi_i [\sum_{r=0}^{s-i+C} hp_r - \sum_{r=s-i+C+1}^{\infty} bp_r] \\
&+ \sum_{i=-C+1}^{0} \pi_i [\sum_{r=0}^{s-i+1} hp_r - \sum_{r=s-i+1}^{\infty} bp_r]
\end{aligned}
$$

However, in this statement there is a problem of the lower limit of the sum; $\sum_{r=s-i+C+1}^{\infty} bp_r$. When i is greater than $s+C+1$ lower limit becomes negative. To avoid this situation, we can replace $\sum_{r=s-i+C+1}^{\infty} p_r$ with $(1 - \sum_{r=0}^{s-i+C} p_r)$ and similarly $\sum_{r=s-i+1}^{\infty} p_r$ is replaced with $(1 - \sum_{r=0}^{s-i} p_r)$. Moreover, we can update the upper limit of π_i as $s+C$ in the first part since upper limits of the sums inside are $s-i+C$.

$$
AC(s+1) - AC(s) = (h+b)[\sum_{i=1}^{s+C} \pi_i \sum_{r=0}^{s-i+C} p_r] + (h+b)[\sum_{i=-C+1}^{0} \pi_i \sum_{r=0}^{s-i} p_r] - b \qquad (6.5)
$$

s^* is the smallest point that satisfies;

$$
AC(s+1) - AC(s) = \sum_{i=1}^{s+C} \pi_i \sum_{r=0}^{s-i+C} p_r + \sum_{i=-C+1}^{0} \pi_i \sum_{r=0}^{s-i} p_r \geq \frac{b}{b+h} \qquad (6.6)
$$

This statement is the probability of shortfall plus demand minus order amount is less than threshold level, in other words probability of not being stockout. Optimal threshold level s^* is the smallest point where this probability is greater than or equal to the critical ratio $\frac{b}{b+h}$. This result is exactly the same as the result in the previous chapter. Therefore, base stock policy type solution is also applicable for our model of general case as expected due to insensitivity to fixed ordering cost. In the special case model, this result is simplified by using the properties of steady state probabilities of Markov chain. Therefore, we may apply the same method here. Stockout occurs when shortage level is greater than threshold level. Therefore, probability of not being stockout in the long run can be expressed as the sum of the steady state probabilities of states {-C+1,-C+2,...,s} which is $\sum_{i=-C+1}^{s} \pi_i$ intuitively.

Proposition 2 *Optimal threshold level is the smallest s that satisfies* $\sum_{i=-C+1}^{s} \pi_i \geq \frac{b}{b+h}$.

Proof:

We know that $\pi_j = \sum_{i=-C+1}^{\infty} \pi_i * a_{ij}$ from properties of steady state probabilities. When we simplify the Equation 6.6;

$$\begin{aligned} AC(s+1) - AC(s) &= \sum_{i=1}^{s+C} \pi_i \sum_{r=0}^{s-i+C} p_r + \sum_{i=-C+1}^{0} \pi_i \sum_{r=0}^{s-i} p_r \\ &= \pi_1 p_0 + \pi_{-C+1} p_0 + \pi_1 p_1 + \pi_2 p_0 + \pi_{-C+1} p_1 + \pi_{-C+2} p_0 + \ldots \\ &+ \pi_1 p_{s+C-1} + \pi_2 p_{s+C-2} + \ldots + \pi_{-C+1} p_{s+C-1} + \pi_{-C+2} p_{s+C-2} \cdots \\ &= \pi_{-C+1} + \pi_{-C+2} \cdots + \pi_s \\ &= \sum_{i=-C+1}^{s} \pi_i \end{aligned}$$

Optimal threshold point is the smallest point that satisfies;

$$\sum_{i=-C+1}^{s} \pi_i \geq \frac{b}{b+h} \tag{6.7}$$

So algebraically, we show that $AC(s+1) - AC(s)$ is minimized when sum of the steady state probabilities of states {-C+1,-C+2,...,s} reaches to the critical ratio $\frac{b}{b+h}$. Optimal threshold level is the smallest point that probability of not being stockout in any period is greater than or equal to the critical ratio $\frac{b}{b+h}$. This newsvendor type result is expected since threshold level is only affected by holding cost, backorder cost, and demand distribution. As a result, if the policy is restricted to all-or-nothing policy with a threshold level, steady state distribution of shortfall levels can be computed by a simple computer code and optimal threshold level for all-or-nothing policy, s^*, can be calculated by using steady state distribution and unit backorder/holding cost. Average cost per period can be determined using the optimal threshold level s^*.

6.2 Testing Performance of All-or-nothing Heuristic

In the previous section, a Discrete Time Markov Chain model for the general case is presented when policy is restricted to all-or-nothing policy for the infinite horizon problem. In this section, we test the performance of this heuristic on a set of example problem settings. Different combinations of problem parameters such as, demand distribution, cost parameters and capacity values are used in testing procedure. The steps of the testing procedure is as follows;

- We find the optimal cost per period through dynamic programming model.
- For the same problem setting, we compute the average cost per period by the discrete time Markov chain formulation.
- Then, we compare the results of the heuristic with the results of the dynamic programming model.

Unfortunately, a problem arises in computing infinite horizon average cost with dynamic programming. The main handicap of the dynamic programming, as the number of periods-to-go increases, the number of initial inventory values increases rapidly. Therefore, length of planning horizon can be taken relatively short. As a result, the planning horizon is not long enough for the average cost per period to converge. If we could solve the dynamic program for sufficiently large planning horizons, average cost per period would converge and would be independent of the initial inventory value at the beginning of the planning horizon. However, this is not the case and average cost values cannot recover from the dependence to the initial inventory value in our planning horizon and is a function of initial inventory value. As a result, some parameters like capacity and maximum demand are taken relatively smaller to have a longer planning horizon ($N = 70$ in our computations) and to obtain a good approximation for the infinite horizon average cost of the problem.

One problem still remains. Although we tried to have a longer planning horizon in dynamic programming, average cost value did not converge. Therefore, an approximate value for average cost should be used in comparison with heuristic result. One alternative can be using the average cost for a set of beginning inventory level. For example, using the average cost for beginning inventory levels -5, 0, 5, 10, 15. However, these beginning inventory levels will not be robust to the changes in the problem setting. Consider the situation, demand is zero or one and capacity is one. In this problem setting, an inventory value of 15 or -15 will have a great impact on the average cost. Therefore, including or excluding the cost at these inventory levels in calculating average cost leads to over or underestimation of the average cost per period.

So, we suggest another approximation (actually lower bound) for the average cost in the long run which is taking the minimum average cost for all initial inventory values. In other words, taking the average cost at the initial inventory level S_n. Since this value is the smallest possible value of the all possible average cost per period, this will be a lower bound for the actual average cost per period of the system. Comparing the result of heuristic with this approximation will underestimate the performance of heuristic since we assume that average cost per period is equal to minimum average cost.

We can itemize the parameters used to generate different problem settings as follows;

- Demand Distribution: Two sets of demand distribution are used in the problems. In the first set, possible demand values are $\{5, 6, 7, 8, 9, 10\}$ with their respective probabilities $\{0.06, 0.05, 0.35, 0.35, 0.15, 0.04\}$. This distribution resembles normal distribution with a mean of 7.6. The reason for choosing small values for possible demands is explained above as the limitation of the computation of the dynamic programming formulation. The other set consists of two possible demand values $\{9, 10\}$ with probabilities of $\{0.95, 0.05\}$ respectively and the mean value of demand is equal to 9.05.

- Unit Holding Cost: It is obvious that, the system's dynamics is affected by the relative values of the cost parameters not necessarily by their absolute values. Therefore, unit holding cost is taken as 1 and all other cost parameters are selected relative to the unit holding cost

- Unit Backorder Cost: Unit backorder cost is taken as 3, 5, and 10 in our problems. These unit backorder values, lead the ratio, $\frac{b}{b+h}$ to take values of 0.75, 0.83 and 0.91.

- Unit Variable Cost: Important point in determining the unit variable cost value, it should be always less than unit backorder cost by the assumptions of the problem. In fact, unit variable cost taken as zero in all our problem settings, as the total demand will always be satisfied.

- Capacity: To test the effects of capacity restriction on the problem, different capacity values are used in the test examples. Ratio of the expected demand in a period to the capacity value gives us the utilization of the capacity in the long run. The capacity values are selected so as to have capacity utilization values 95 percent, 84 percent, 76 percent, 69 percent for the first demand distribution and 90, 82, 75, 69 percent for second the distribution.

- Fixed Ordering Cost: Fixed ordering cost is allowed to take values of 15, 40 and 100 in our examples.

Table 6.1 summarizes the computational result of performance of our heuristic when possible demand values are $\{5, 6, 7, 8, 9, 10\}$ with respective probabilities of

Table 6.1: Performance test of Heuristic for demand set 1

No	h	b	K	C	Low. Bou.	Heu. Res.	Dev
1	1	3	15	8	16.91	17.96	6.2
2	1	3	15	9	15.92	16.28	2.2
3	1	3	15	10	15.08	15.38	2
4	1	3	15	11	14.42	14.69	1.9
5	1	3	40	8	39.47	41.71	5.7
6	1	3	40	9	35.64	37.39	4.9
7	1	3	40	10	32.91	34.38	4.5
8	1	3	40	11	30.7	31.96	4.1
9	1	3	100	8	87.45	98.71	12.9
10	1	3	100	9	78.8	88.05	11.7
11	1	3	100	10	72.16	79.98	10.8
12	1	3	100	11	66.74	73.42	10
13	1	5	15	8	17.45	18.62	6.7
14	1	5	15	9	16.27	16.77	3
15	1	5	15	10	15.63	15.92	1.9
16	1	5	15	11	14.97	15.23	1.7
17	1	5	40	8	40.28	42.37	5.2
18	1	5	40	9	36.28	37.88	4.4
19	1	5	40	10	33.56	34.92	4
20	1	5	40	11	31.35	32.51	3.7
21	1	5	100	8	89.2	99.37	11.4
22	1	5	100	9	80.19	88.54	10.4
23	1	5	100	10	73.43	80.52	9.7
24	1	5	100	11	67.89	73.96	9
25	1	10	15	8	18.28	19.54	6.9
26	1	10	15	9	16.6	17.39	4.8
27	1	10	15	10	16.15	16.44	1.9
28	1	10	15	11	15.53	15.8	1.7
29	1	10	40	8	41.22	43.29	5
30	1	10	40	9	36.99	38.5	4.1
31	1	10	40	10	34.18	35.44	3.7
32	1	10	40	11	32	33.08	3.4
33	1	10	100	8	90.87	100.3	10.4
34	1	10	100	9	81.48	89.17	9.4
35	1	10	100	10	74.51	81.04	8.8
36	1	10	100	11	68.92	74.53	8.1
						Average Dev.	6
						Max. Dev	12.9
						Min Dev.	1.7

$\{0.06, 0.05, 0.35, 0.35, 0.15, 0.04\}$. Column *Low.Bou.* is the average cost approximation calculated by minimum average cost. Column *Heu.Res* is the average cost calculated by heuristic. Finally, column *Dev* is the percent deviation of heuristic result from *Low.Bou.*, which is equal to $\frac{Heu.Res-Low.Bou.}{Low.Bou.} * 100$.

We evaluate the performance of heuristic by looking at the deviation from *Low.Bou.*. Although using deviation from *Low.Bou.* as a performance criteria underestimates the performance of the heuristic, it is an upper bound for the deviation from the actual average cost. Average deviation from *Low.Bou.* is 6 percent and maximum deviation is 13 percent as shown in Table 6.1. So, we may conclude that our heuristic performs well for this demand distribution.

For the second demand set, there are only two possible demand values. This type of demand distribution causes more trouble such as kinked curve of order quantity function. Therefore, for most of the problem sets, we do not expect to have an optimal policy which is an all-or-nothing policy in the long run with this demand structure. So, we expect to get worse performance of heuristic compared to the performance on first demand set. Table 6.2 summarizes the computational result of our performance test of heuristic when possible demand values are $\{9, 10\}$ with probabilities $\{0.95, 0.05\}$.

As expected, average deviation, maximum deviation and minimum deviation from *Low.Bou.* are greater than corresponding values of previous demand set. Average deviation from *Low.Bou.* is 8 percent and maximum deviation is 19 percent as shown in Table 6.2. So, we may conclude that our heuristic performs well for this demand distribution also.

As a conclusion, for the infinite horizon problem under average cost criterion, all-or-nothing policy with a threshold level is a well performing heuristic, even in the problematic demand structures.

Table 6.2: Performance test of Heuristic for demand set 2

No	h	b	K	C	Low. Bou.	Heu. Res.	Dev
1	1	3	15	10	14.85	17.29	16.4
2	1	3	15	11	14.85	16.44	10.7
3	1	3	15	12	14.85	15.81	6.5
4	1	3	15	13	14.78	15.3	3.5
5	1	3	40	10	38.15	39.91	4.6
6	1	3	40	11	35.52	37	4.2
7	1	3	40	12	33.37	34.67	3.9
8	1	3	40	13	31.57	32.7	3.6
9	1	3	100	10	85.49	94.21	10.2
10	1	3	100	11	78.84	86.37	9.5
11	1	3	100	12	73.36	79.92	8.9
12	1	3	100	13	68.72	74.47	8.4
13	1	5	15	10	14.95	17.69	18.3
14	1	5	15	11	14.95	16.89	13
15	1	5	15	12	14.95	16.31	9.1
16	1	5	15	13	14.85	15.85	6.7
17	1	5	40	10	38.59	40.31	4.5
18	1	5	40	11	36.09	37.46	3.8
19	1	5	40	12	33.74	35.17	4.2
20	1	5	40	13	32.07	33.25	3.7
21	1	5	100	10	86.78	94.61	9
22	1	5	100	11	80.03	86.82	8.5
23	1	5	100	12	74.44	80.42	8
24	1	5	100	13	69.8	75.02	7.5
25	1	10	15	10	15.2	18.08	19
26	1	10	15	11	15.2	17.34	14.1
27	1	10	15	12	15.2	16.77	10.3
28	1	10	15	13	14.98	16.32	8.9
29	1	10	40	10	38.92	40.71	4.6
30	1	10	40	11	36.46	37.91	4
31	1	10	40	12	33.98	35.63	4.9
32	1	10	40	13	32.43	33.73	4
33	1	10	100	10	87.86	95	8.1
34	1	10	100	11	81.02	87.27	7.7
35	1	10	100	12	75.2	80.88	7.5
36	1	10	100	13	70.64	75.5	6.9
						Average Dev.	8
						Max. Dev	19
						Min Dev.	3.5

CHAPTER 7

CONCLUSION

In this study, we analyzed the single item periodic review, capacitated inventory model with fixed ordering cost. Demand in each period was assumed to be a discrete random variable independent of demands in other periods. The optimization criterion was either minimizing average cost or discounted cost of the system over a planning horizon.

For this problem setting, we first showed that the optimal policy for a single period problem is a modified (s, S) type policy. Then, we showed that this modified (s, S) type policy cannot be optimal for the multi period problem and we discussed the reasons why an optimal policy for the multi period problem is hard to identify. Afterwards, we performed a computational analysis on a numerical example and investigated the behavior of the inventory system respect to changes in some problem parameters. By this computational analysis, we identified some points where system behavior contradicts with our expectations. As a result, we concluded that any simple monotone policy cannot be optimal for this problem structure. This result gave us motivation to define a special case of this problem where optimal order policy can be achieved with simple monotone policies.

Next, we defined a specific problem structure where demand is defined as multiples of the capacity of production. We investigated the characteristics of this special case problem with finite planning horizon. We came up with the following result for that special case: If the initial inventory position at the beginning of a planning horizon is a multiple of capacity value, then partial ordering never occurs and optimal policy is an all-or-nothing policy. Then, we proved the convexity of the expected cost function defined at the points which are multiples of capacity. These results, led us to define the optimal policy as an all-or-nothing policy with a threshold level. Then, we showed that threshold level is non decreasing as the number of periods to go increases, and we performed a computational analysis for the special case.

Furthermore, we investigated the infinite horizon problem of the special case under average cost criterion by defining the problem as a Discrete Time Markov Chain Model. The states of the Markov chain were defined as the shortfall levels from the threshold value. We showed that by using the steady state distribution, optimal threshold level can be computed. It turned out to be that optimal threshold level is insensitive to fixed ordering cost and unit variable cost. Optimal threshold point is the point that satisfies a critical ratio of $\frac{b}{b+h}$ as in base stock policy. We also showed that average cost per period can be computed by using the optimal threshold level.

We extended our results of Discrete Time Markov Chain Model to infinite horizon problem of the general case and developed a heuristic which is again all-or-nothing policy. In this heuristic, optimal threshold level and average cost per period is computed by using the steady state distribution of shortfall level and critical ratio of $\frac{b}{b+h}$. To test the performance of heuristic, we created a set of problems with different demand distributions, cost parameters and capacity value. We compared the result of the heuristic with the average cost value obtained from the dynamic programming model. Heuristic performed well for all of the cases and deviation from optimal solution remain under ten percent in most of the cases. On average, deviation from optimal solution (lower bound) was around seven percent.

A possible further research point can be determining the conditions where all-or-nothing policy performs better (optimal or very close to optimal) and developing hybrid policies. In these hybrid policies, total capacity value is divided into small values and total fixed ordering cost is allocated to each small capacity. As an example, more than one supplier (capacity) are available with different capacity values and fixed ordering costs. The ordering policy is restricted to an all-or-nothing policy with each supplier (capacity). Therefore, order policy will be a combination of all-or-nothing policies with different suppliers. For some initial inventory levels, it will be optimal to give order to all suppliers, and for some inventory levels it will be optimal to give order to a subset of suppliers. This multi supplier model can be reduced to single supplier model by dividing the capacity into smaller capacity values. This is expected to be a better approximation for our problem especially when the capacity is much larger than the expected demand of a period. The main problem that arises here is to allocate the fixed ordering cost to smaller capacity values.

Another possible path for further research is to utilize the result for special case in developing heuristics for the problems with general demand structure. By small modifications, this general demand structure may be expressed by a special demand structure or by a combination of special demand structures like in Chapter 4. As a result, optimal order policy for the problem with general demand structure can be approximated by all-or-nothing type policies.

Finally, a possible path for further research is to define a partial characterization of optimal policy for problems with shorter planning horizons. Then, resulting policy may be used as a myopic approximation to problems with longer planning horizons. This partial characterization of optimal policy should capture the unsystematic behavior of the system such as not monotone behavior of order amount curve. We use the term partial characterization since it is hard to determine optimal policy even for two period planning horizon problem.

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APPENDIX A

PROOFS OF LEMMAS AND THEOREMS

A.1 Proof of Theorem 1

We prove Theorem 1 by induction. Before induction, we should show a proposition that will be used in proof of Theorem 1.

Proposition 3 *When unit variable cost is zero, than function $J_n(x)$ is less than equal to $G_n(y) + K\delta(y-x)$ where $y\epsilon[x, x+C]$.*

Proof: If the unit variable cost is zero, then $J_n(x)$ is equal to $\min\{G_n(y) + K\delta(y-x)\}$ where $y\epsilon[x, x+C]$. Hence all possible values $G_n(y) + K\delta(y-x)$ is greater than equal to minimum of these values which is $J_n(x)$. □

Now, we can return back to our proof of Theorem 1. Part (a) of induction states that any point between two consecutive multiples of d (ϵ [md,(m+1)d]), has an expected cost value $G_n(md+i)$ which is greater than or equal to convex combination of expected costs of extreme points $G_n(md)$ and $G_n((m+1)d)$. This also means that, point $G_n(md+i)$ lies on or above the line connecting two points at two consecutive multiples of d, $G_n(md)$ and $G_n((m+1)d)$ as shown in Figure A.1.

For $n = 1$,

$$G_1(md) = v_1 md + \sum_{r=0}^{m} p_{r1}(md - rd)h_1 + \sum_{r=m+1}^{M} p_{r1}(rd - md)b_1$$

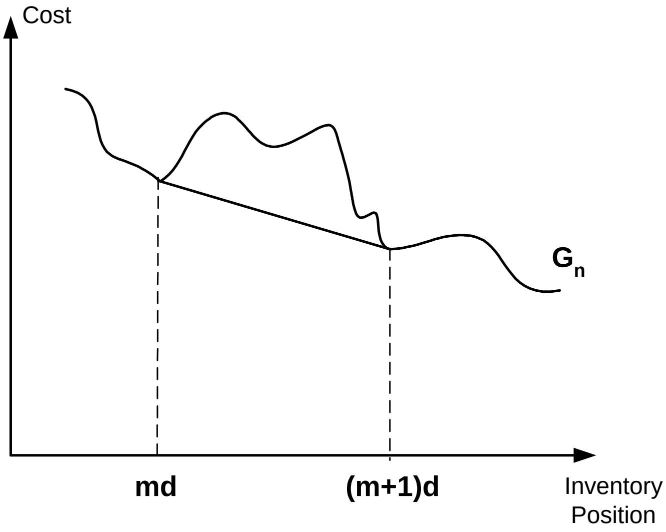

Figure A.1: Convex combination of two consecutive multiple of d

$$
\begin{aligned}
G_1((m+1)d) &= v_1(m+1)d + \sum_{r=0}^{m} p_{r1}[(m+1)d - rd]h_1 \\
&+ \sum_{r=m+1}^{M} p_{r1}[rd - (m+1)d]b_1 \\
G_1(md+i) &= v_1(md+i) + \sum_{r=0}^{m} p_{r1}(md+i-rd)h_1 \\
&+ \sum_{r=m+1}^{M} p_{r1}(rd - md - i)b_1
\end{aligned}
$$

$G_1(y)$ has three parts; variable cost part, holding cost part and backorder cost part. First note that, variable cost part of $md + i$ is exactly equal to the convex combination of the others': $v_1(md+i) = \alpha v_1 md + (1-\alpha)v_1(m+1)d$.

Next, each component of holding cost part $p_{r1}(md+i-rd)h_1$ is exactly equal to convex combination of others' so total expected holding cost when initial inventory $md+i$ is equal to the convex combination of the total expected holding cost of extreme points.

$$
p_{r1}(md+i-rd)h_1 = \alpha p_{r1}(md-rd)h_1 + (1-\alpha)p_{r1}[(m+1)d - rd]h_1
$$

$$
\begin{aligned}
\sum_{r=0}^{m} p_{r1}(md+i-rd)h_1 &= \alpha \sum_{r=0}^{m} p_{r1}(md-rd)h_1 \\
&+ (1-\alpha)\sum_{r=0}^{m} p_{r1}[(m+1)d-rd]h_1
\end{aligned}
$$

Similarly for backorder costs;

$$
\begin{aligned}
p_{r1}(rd-md-i)b_1 &= \alpha p_{r1}(rd-md)b_1 + (1-\alpha)p_{r1}[rd-(m+1)d]b_1 \\
\sum_{r=m+1}^{M} p_{r1}(rd-md-i)b_1 &= \alpha \sum_{r=m+1}^{M} p_{r1}(rd-md)b_1 \\
&+ (1-\alpha)\sum_{r=m+1}^{M} p_{r1}[rd-(m+1)d]b_1
\end{aligned}
$$

which means total expected backorder cost when initial inventory $md+i$ is equal to convex combination of total expected backorder cost of extreme points. So,

$$
G_1(md+i) = \alpha G_1(md) + (1-\alpha)G_1[(m+1)d].
$$

Therefore we proved (a) which means $G_1(md+i)$ is greater than or equal to the convex combination of $G_1(md)$ and $G_1((m+1)d)$.

Part (b) of induction states that any point between two consecutive multiples of d, that is in [md,(m+1)d], has an expected cost value $G_n(md+i)$ greater than minimum of expected costs of extreme points $G_n(md)$ and $G_n((m+1)d)$. If a point lies above a line connecting two extreme points of the interval, minimum point of this interval cannot be this point and also it is greater than at least one of the extreme points of the interval. Since this statement is true for all $i \; \epsilon$ [md,(m+1)d], either $G_1(md)$ or $G_1((m+1)d)$ is minimum in the interval $[md,(m+1)d]$.

Part (c) immediately follows part (b) and states that S_1 is an integer multiple of d. Because all of local minimum points are multiples of d, the global minimum point should also be a multiple of d.

Part (d) of induction also follows part (b). Due to part (b) of induction, it is not reasonable to order any amount less than capacity if initial inventory is an integer multiple of d. So if the inventory position is equal to md, then it is optimal to order full capacity (C) or not to order in the first period which is part (d) of the induction.

For $n = 1$, we show that all parts of the induction argument are true. For $n = k$ we assume all statements of induction to be true.

For $n = k + 1$,

Part (a) of induction;

$$\begin{aligned} G_{k+1}(md) &= G_1(md) + \gamma \sum_{r=0}^{R} p_{rk+1}[J_k(md - rd)] \\ G_{k+1}((m+1)d) &= G_1[(m+1)d] + \gamma \sum_{r=0}^{R} p_{rk+1}[J_k(md + d - rd)] \\ G_{k+1}(md + i) &= G_1(md + i) + \gamma \sum_{r=0}^{R} p_{rk+1}[J_k(md - rd + i)] \end{aligned}$$

We have shown that $G_1(md + i) = \alpha G_1(md) + (1 - \alpha)G_1((m + 1)d)$ for any set of cost parameters in the first part of induction. If we can prove that expected cost before ordering $J_k(md + i)$ is greater than or equal to convex combination of expected costs before ordering at extreme points of the interval; $\alpha J_k(md) + (1 - \alpha)J_k((m + 1)d)$. Then, we can prove that $G_n(md + i) \geq \alpha G_n(md) + (1 - \alpha)G_n((m + 1)d)$ for all m and for any $i\, \epsilon \{1, 2 \ldots d - 1\}$.

By part (d) of induction assumption for $n = k$ ordering decision at a point multiple of d is limited to two options; order nothing or order full capacity so that we can write the following equations;

$$J_k(md) = -v_k md + min\,\{G_k(md), G_k((m+1)d) + K_k\} \quad \text{(A.1)}$$

$$\begin{aligned} J_k((m+1)d) &= -v_k(m+1)d \\ &+ \min\{G_k((m+1)d), G_k((m+2)d) + K_k\} \end{aligned} \quad \text{(A.2)}$$

However, if the initial inventory value is not a multiple of d, there are more than two options for ordering decision. (But not as much as the general case since local minimum points are at multiples of d when $n = k$ and this limits the number of options for ordering decision). So;

$$J_k(md+i) = -v_k(md+i) + min \begin{cases} G_k(md+i) & \\ G_k(md+j)+K_k & j\epsilon\{i+1,\ldots,d-1\} \\ G_k((m+1)d)+K_k & \\ G_k((m+1)d+j)+K_k & j\epsilon\{1,2,\ldots,i\} \end{cases} \tag{A.3}$$

We want to prove that $J_k(md+i) \geq \alpha J_k(md) + (1-\alpha)J_k((m+1)d)$ when $\alpha = (d-i)/d$. For variable cost parts, $v_k(md+i) = \alpha v_k md + (1-\alpha)v_k(m+1)d$, so we can take unit variable cost v_k is equal to zero without loss of generality in this proof. When unit variable cost is zero, Proposition 3 is valid. So by Proposition 3 and Equation A.1, we can state that, both possible options' costs; $G_k(md)$ and $G_k((m+1)d)+K_k$ are either greater than or equal to $J_k(md)$ when unit variable cost v_k is equal to zero by. Similarly, from equation A.2, both $G_k((m+1)d)$ and $G_k((m+2)d)+K_k$ are also either greater than or equal to $J_k((m+1)d)$ when $v_k = 0$. There are four possible values $J_k(md+i)$ as can be seen in equation A.3 which means we have four cases to consider.

Case 1: If $G_k(md+i)$ is minimum: In this case, it is optimal to stay at the initial inventory position and order nothing. Either fixed ordering cost is to high, or relative benefit of being at a greater inventory position does not cover the the fixed ordering cost.

$$\begin{aligned} J_k(md+i) = G_k(md+i) &\geq \alpha G_k(md) + (1-\alpha)G_k((m+1)d) \\ &\geq \alpha J_k(md) + (1-\alpha)J_k((m+1)d) \end{aligned}$$

First inequality follows from the part (a) of the induction assumption in period k. Second inequality is justified by by Proposition 3 which states that $G_k(md) \geq J_k(md)$ and $G_k((m+1)d) \geq J_k((m+1)d)$ when $v_k = 0$. Therefore, $\alpha G_k(md) \geq \alpha J_k(md)$ and $(1-\alpha)G_k((m+1)d) \geq (1-\alpha)J_k((m+1)d)$.

Case 2: If $G_k(md+j)+K_k$ is minimum where $j\epsilon\{i+1,\ldots,d-1\}$: In this case, it is optimal to order but not up to the nearest multiple of d, which is $(m+1)d$. Therefore we can conclude that $G_k((m+1)d)$ cannot be the local minimum of interval [md,(m+1)d] which means that local minimum point is $G_k(md)$ ($\min\{G_k(md), G_k((m+1)d)\} = G_k(md)$). Hence the line connecting two extreme points of the interval has a positive slope as shown in Figure A.2. Otherwise it would be optimal to order at least up to (m+1)d with an expected cost of $G_k((m+1)d) + K_k$. Let β be the coefficient used in evaluating the convex combination cost functions at point $md+j$. Since $i < j$, $\alpha = (d-i)/d > \beta = (d-j)/d$. So;

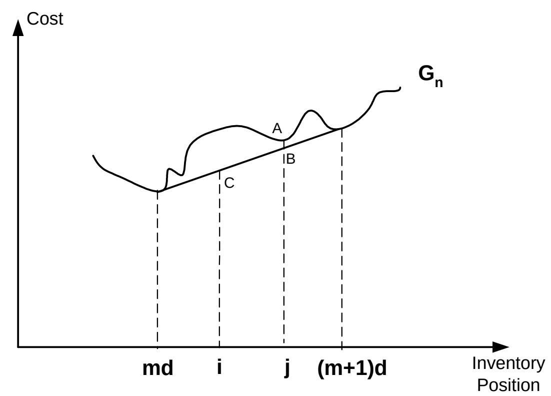

Figure A.2: Graph of situation in Case 2

$$
\begin{aligned}
J_k(md+i) = G_k(md+j) + K_k &\geq G_k(md+j) \\
&\geq \beta G_k(md) + (1-\beta) G_k((m+1)d) \\
&\geq \alpha G_k(md) + (1-\alpha) G_k((m+1)d) \\
&\geq \alpha J_k(md) + (1-\alpha) J_k((m+1)d)
\end{aligned}
$$

First inequality holds because K_k is non-negative. Second inequality follows from the part (a) of the induction assumption in period k. Third one is due to two properties given above; $\alpha > \beta$ and $G_k(md) \leq G_k((m+1)d)$. These two inequalities are illustrated in Figure A.2. Point A ($G_k(md+j)$) is greater than or equal to point B (convex combination at point $md + j$), therefore than point C (convex combination at point $md+i$) as shown on Figure A.2. Fourth inequality is justified by Proposition 3 which states that $G_k(md) \geq J_k(md)$ and similarly $G_k((m+1)d) \geq J_k((m+1)d)$ when $v_k = 0$. Therefore, $\alpha G_k(md) \geq \alpha J_k(md)$ and $(1-\alpha)G_k((m+1)d) \geq (1-\alpha)J_k((m+1)d)$.

Case 3: If $G_k((m+1)d) + K_k$ is minimum: In this case, it is optimal to order up to the nearest multiple of d, which is $(m+1)d$.

$$\begin{aligned} J_k(md+i) &= G_k((m+1)d) + K_k \\ &\geq \alpha \{G_k((m+1)d) + K_k\} + (1-\alpha)G_k((m+1)d) \\ &\geq \alpha J_k(md) + (1-\alpha)J_k((m+1)d) \end{aligned}$$

First inequality holds as $K_k \geq \alpha K_k$ remains when cancellations are done and $\alpha \leq 1$, and second inequality is justified by Proposition 3 which states that, $G_k((m+1)d)+K_k \geq J_k(md)$ and similarly $G_k((m+1)d) \geq J_k((m+1)d)$ when $v_k = 0$. Therefore, $\alpha G_k((m+1)d)+K_k \geq \alpha J_k(md)$ and $(1-\alpha)G_k((m+1)d) \geq (1-\alpha)J_k((m+1)d)$.

Case 4: If $G_k((m+1)d + j) + K_k$ is minimum where $j \epsilon \{1, 2, \ldots, i\}$: In this case, it is optimal to order not up to the nearest multiple of d, which is $(m+1)d$ but to a point greater than $(m+1)d$ which means, there exist at least one point between [(m+1)d,(m+1)d+i] where the expected cost is smaller than $G_k((m+1)d$. Therefore, we can conclude that $G_k((m+1)d$ cannot be the local minimum point of the interval [(m+1)d,(m+2)d] due to part (b) of the induction argument for $n = k$. So local minimum point is $\min\{G_k((m+1)d), G_k((m+2)d)\} = G_k((m+2)d)$ and the line connecting points $G_k((m+1)d)$ and $G_k((m+2)d$ has a negative slope as shown in Figure A.3. Otherwise, it would be optimal to order up to (m+1)d with a cost of $G_k((m+1)d) + K_k$. Let β be the coefficient used in evaluating the convex combination cost functions at point $md + j$. Since $i \geq j$, $\alpha = (d-i)/d \leq \beta(d-j)/d$. So;

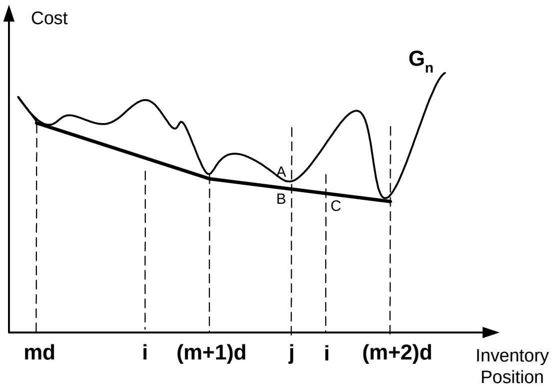

Figure A.3: Graph of situation in Case 4

$$\begin{aligned}
J_k(md+i) &= G_k((m+1)d+j)+K_k \\
&\geq \beta\left\{G_k((m+1)d)+K_k\right\}+(1-\beta)\left\{G_k((m+2)d)+K_k\right\} \\
&\geq \alpha\left\{G_k((m+1)d)+K_k\right\}+(1-\alpha)\left\{G_k((m+2)d)+K_k\right\} \\
&\geq \alpha J_k(md)+(1-\alpha)J_k((m+1)d)
\end{aligned}$$

First inequality holds because of $K_k = \beta K_k + (1-\beta)K_k$ and part (a) of the induction assumption in period k. Second inequality is true due to two properties given above; $\alpha \leq \beta$ and $G_k((m+2)d) \leq G_k((m+1)d)$. These two inequalities are illustrated in Figure A.3. Point A ($G_k(md+j)$) is equal to or greater than point B (convex combination at point $md+j$), therefore than point C (convex combination at point $md+i$) as shown on Figure A.2. Third inequality is justified by Proposition 3 which states that, $G_k((m+1)d)+K_k \geq J_k(md)$ and similarly $G_k((m+2)d)+K_k \geq J_k((m+1)d)$ when $v_k = 0$. Therefore, $\alpha G_k((m+1)d)+K_k \geq \alpha J_k(md)$ and $(1-\alpha)G_k((m+2)d)+K_k \geq (1-\alpha)J_k((m+1)d)$.

For all cases we proved that $J_k(md+i) \geq \alpha J_k(md) + (1-\alpha)J_k((m+1)d)$. The coefficient p_{rk+1} is non-negative so,

$$\begin{aligned}
J_k(md+i) &\geq \alpha J_k(md) + (1-\alpha)J_k((m+1)d) \\
J_k(md+i-rd) &\geq \alpha J_k(md-rd) + (1-\alpha)J_k((m+1)d-rd) \\
\sum_{r=0}^{R} p_{rk+1} J_k(md+i-rd) &\geq \alpha \sum_{r=0}^{R} p_{rk+1} J_k(md-rd) \\
&+ (1-\alpha) \sum_{r=0}^{R} p_{rk+1} J_k((m+1)d-rd) \\
\gamma \sum_{r=0}^{R} p_{rk+1} J_k(md+i-rd) &\geq \gamma\alpha \sum_{r=0}^{R} p_{rk+1} J_k(md-rd) \\
&+ \gamma(1-\alpha) \sum_{r=0}^{R} p_{rk+1} J_k((m+1)d-rd) \\
G_1(md) + \gamma \sum_{r=0}^{R} p_{rk+1} J_k(md+i-rd) &\geq G_1(md) + \gamma\alpha \sum_{r=0}^{R} p_{rk+1} J_k(md-rd) \\
&+ \gamma(1-\alpha) \sum_{r=0}^{R} p_{rk+1} J_k((m+1)d-rd) \\
G_{k+1}(md+i) &\geq \alpha G_{k+1}(md) + (1-\alpha)G_{k+1}((m+1)d)
\end{aligned}$$

So we prove that $G_{k+1}(md+i)$ is greater than or equal to $\alpha G_{k+1}(md) + (1-\alpha)G_{k+1}((m+1)d)$.

Part (a) of Theorem 1 states that for any period n and integer m, $G_n(md+i)$ is greater than or equal to the convex combination of $G_n(md)$ and $G_n((m+1)d)$.

Part (b) of Theorem 1 follows part (a). By part (b), the point $G_n(md+i)$ lies above the line connecting two other points; $G_n(md)$ and $G_n((m+1)d)$. In other words either $G_n(md)$ or $G_n((m+1)d)$ is the local minimum point of the interval $[md, (m+1)d]$.

Parts (c) immediately follows part (b) and states that S_n is an integer multiple of d due to fact that all of local minimum points are multiples of d.

Part (d) of Theorem 1 also follows part (b). Due to part (b) of Theorem 1, it is not reasonable to order any amount less than capacity if initial inventory is an integer multiple of d. Therefore, if the inventory position is equal to md, then it is optimal to order full capacity (C) or not to order in any period for any integer m which is part (d) of Theorem 1. □

A.2 Proof of Theorem 2

We prove Theorem 2 by induction.

Part (a) of induction states that any point ((m+1)d) which is multiple of d, has an expected cost value $G_n((m+1)d)$ less than or equal to convex combination of (in fact, average of) expected costs of two neighbour points $G_n(md)$ and $G_n((m+2)d)$.

For $n = 1$,

$$\begin{aligned} G_1(md) &= v_1 md + \sum_{r=0}^{m} p_{r1}(md - rd)h_1 + p_{(m+1)1} db_1 \\ &+ \sum_{r=m+2}^{M} p_{r1}(rd - md)b_1 \\ G_1((m+1)d) &= v_1(m+1)d + \sum_{r=0}^{m} p_{r1}[(m+1)d - rd]h_1 + 0 \\ &+ \sum_{r=m+2}^{M} p_{r1}[rd - (m+1)d]b_1 \\ G_1((m+2)d) &= v_1(m+2)d + \sum_{r=0}^{m} p_{r1}(md + i - rd)h_1 + p_{(m+1)1} dh_1 \\ &+ \sum_{r=m+2}^{M} p_{r1}(rd - md - i)b_1 \end{aligned}$$

$G_1(y)$ has three parts; variable cost part, holding cost part and backorder cost part. First, note that, variable cost part of $(m+1)d$ is exactly equal to convex combination of others': $v_1((m+1)d) = \frac{1}{2}v_1 md + \frac{1}{2}v_1(m+2)d$. Next each component of holding cost part $p_{r1}((m+1)d - rd)h_1$ is exactly equal to the convex combination of others', so total expected holding cost when initial inventory $(m+1)d$ is equal to convex combination of total expected holding cost of extreme points.

$$\begin{aligned} p_{r1}((m+1)d - rd)h_1 &= \frac{1}{2}p_{r1}(md - rd)h_1 + \frac{1}{2}p_{r1}[(m+2)d - rd]h_1 \\ \sum_{r=0}^{m} p_{r1}((m+1)d - rd)h_1 &= \frac{1}{2}\sum_{r=0}^{m} p_{r1}(md - rd)h_1 \\ &+ \frac{1}{2}\sum_{r=0}^{m} p_{r1}[(m+2)d - rd]h_1 \end{aligned}$$

Similarly for backorder costs;

$$\begin{aligned} p_{r1}(rd-(m+1)d)b_1 &= \frac{1}{2}p_{r1}(rd-md)b_1+\frac{1}{2}p_{r1}[rd-(m+2)d]b_1 \\ \sum_{r=m+2}^{M} p_{r1}(rd-(m+1)d)b_1 &= \frac{1}{2}\sum_{r=m+2}^{M} p_{r1}(rd-md)b_1 \\ &+ \frac{1}{2}\sum_{r=m+2}^{M} p_{r1}[rd-(m+2)d]b_1 \end{aligned}$$

which means total expected backorder cost when initial inventory $(m+1)d$ is equal to convex combination of total expected backorder cost of extreme points. Finally $0 \leq \frac{1}{2}p_{(m+1)1}db_1 + \frac{1}{2}p_{(m+1)1}dh_1$ since righthand side is positive. So we have;

$$G_1((m+1)d) \leq \frac{1}{2}G_1(md) + \frac{1}{2}G_1((m+2)d) \tag{A.4}$$

We proved part (a) of induction for $n=1$ which means $G_1((m+1)d)$ is less than or equal to the convex combination of $G_1(md)$ and $G_1((m+2)d)$. In the light of Lemma 1 and Theorem 1, this yields (b) immediately, which tells us that $G_1(md)$ is convex (which we already know from optimality analysis of single period problem). So if the inventory position less than a threshold value, then it is optimal to order full capacity (C) otherwise it is optimal not to order in the first period. For $n=k$ we assume all statements to be true.

For $n=k+1$,

$$\begin{aligned} G_{k+1}(md) &= G_1(md) + \gamma\sum_{r=0}^{R} p_{rk+1}[J_k(md-rd)] \\ G_{k+1}((m+1)d) &= G_1((m+1)d) + \gamma\sum_{r=0}^{R} p_{rk+1}[J_k(md+d-rd)] \\ G_{k+1}((m+2)d) &= G_1((m+2)d) + \gamma\sum_{r=0}^{R} p_{rk+1}[J_k(md+2d-rd)] \end{aligned}$$

We have shown that $G_1((m+1)d) \leq \frac{1}{2}G_1(md)+\frac{1}{2}G_1((m+2)d)$ in the first part of induction. If we can prove that expected cost before ordering $J_k((m+1)d)$ is less than or equal to convex combination of expected costs before ordering at md and $(m+2)d$; $\frac{1}{2}J_k(md)+\frac{1}{2}J_k((m+2)d)$ then we can prove that $G_n((m+1)d)) \leq \frac{1}{2}G_n(md) + \frac{1}{2}G_n((m+2)d)$ for all m (integer). Function J_k has been proved to be as follows at points md, $(m+1)d$ and $(m+2)d$;

$$J_k(md) = -v_k md + \min \begin{cases} G_k(md) \\ G_k((m+1)d) + K_k \end{cases} \tag{A.5}$$

$$J_k((m+1)d) = -v_k(m+1)d + min \begin{cases} G_k((m+1)d) \\ G_k((m+2)d) + K_k \end{cases} \tag{A.6}$$

$$J_k((m+2)d) = -v_k(m+2)d + min \begin{cases} G_k((m+2)d) \\ G_k((m+3)d) + K_k \end{cases} \tag{A.7}$$

We want to prove that $J_k((m+1)d) \leq \frac{1}{2}J_k(md) + \frac{1}{2}J_k((m+2)d)$. For variable cost components, $v_k(m+1)d = \frac{1}{2}v_k md + \frac{1}{2}v_k(m+2)d$, so we can take unit variable cost v_k is equal to zero without loss of generality in this proof. There are two alternatives for ordering decision at each inventory level, so two possible values for each term $J_k(md)$, $J_k((m+1)d)$, $J_k((m+2)d)$ which means we have eight cases to consider. However due to convexity assumption in period k, some combinations are not possible. As an example, if $J_k(md) = G_k(md)$, it is optimal to order nothing when initial inventory is equal to md. Since $G_k(md)$ convex, it is not optimal to order anything if initial inventory value is greater than md. Therefore if $J_k(md) = G_k(md)$, $J_k((m+1)d)$ cannot be equal to $G_k((m+2)d) + K_k$. For this reason, four of the eight cases are eliminated, so there are four possible cases left to be considered.

Case 1: If $J_k(md) = G_k(md)$, $J_k((m+1)d) = G_k((m+1)d)$, $J_k((m+2)d) = G_k((m+2)d)$: In this case, it is optimal to stay at the initial inventory position and order nothing for all three inventory levels. Either fixed ordering cost is too high, or relative benefit of being at a greater inventory position does not cover the fixed ordering cost. Being at an inventory level, which is greater than global minimum point for all three inventory levels may be another reason for this situation.

$$G_k((m+1)d) \quad \leq \quad \frac{1}{2}G_k(md) + \frac{1}{2}G_k((m+2)d)$$

$$J_k((m+1)d) \;\leq\; \frac{1}{2}J_k(md) + \frac{1}{2}J_k((m+2)d)$$

First inequality is true due to part (a) of the induction assumption for $n = k$ and second one follows the first inequality.

Case 2: If $J_k(md) = G_k((m+1)d) + K_k$, $J_k((m+1)d) = G_k((m+1)d)$, $J_k((m+2)d) = G_k((m+2)d)$: In this case, it is optimal to order capacity when initial inventory position is md, and order nothing for other two inventory levels. So threshold level is equal to $(m+1)d$ at period k.

$$\begin{aligned}
G_k((m+1)d) &\leq [G_k((m+2)d) + K_k] \\
&\leq \frac{1}{2}G_k((m+1)d) + \frac{1}{2}[G_k((m+2)d) + K_k] \\
&= \frac{1}{2}[G_k((m+1)d) + K_k] + \frac{1}{2}G_k((m+2)d) \\
J_k((m+1)d) &\leq \frac{1}{2}J_k(md) + \frac{1}{2}J_k((m+2)d)
\end{aligned}$$

First inequality holds since $J_k((m+1)d)$ is equal to $G_k((m+1)d)$, therefore $G_k((m+2)d) + K_k$ is greater than or equal to $G_k((m+1)d)$. Otherwise, $J_k((m+1)d)$ would be equal to $G_k((m+2)d)+K_k$. Second one immediately follows first one and in third inequality terms are rearranged.

Case 3: If $J_k(md) = G_k((m+1)d) + K_k$, $J_k((m+1)d) = G_k((m+2)d) + K_k$, $J_k((m+2)d) = G_k((m+2)d)$: In this case, it is optimal to order capacity when initial inventory position is md or $(m+1)d$, and order nothing for the other two inventory level $(m+2)d$. So threshold level is equal to $(m+2)d$ at period k.

$$\begin{aligned}
G_k((m+2)d) + K_k &\leq G_k((m+1)d) \\
&\leq \frac{1}{2}G_k((m+1)d) + \frac{1}{2}[G_k((m+2)d) + K_k] \\
&= \frac{1}{2}[G_k((m+1)d) + K_k] + \frac{1}{2}G_k((m+2)d) \\
J_k((m+1)d) &\leq \frac{1}{2}J_k(md) + \frac{1}{2}J_k((m+2)d)
\end{aligned}$$

First inequality holds since $J_k((m+1)d)$ is equal to $G_k((m+2)d) + K_k$, therefore $G_k((m+1)d)$ is greater than or equal to $G_k((m+2)d) + K_k$. Otherwise, $J_k((m+1)d)$ would be equal to $G_k((m+1)d)$. Second one immediately follows first one and in third inequality terms are rearranged.

Case 4: If $J_k(md) = G_k((m+1)d) + K_k$, $J_k((m+1)d) = G_k((m+2)d) + K_k$, $J_k((m+2)d) = G_k((m+3)d) + K_k$: In this case, it is optimal to order capacity at all three inventory levels. So threshold level is greater than $(m+2)d$ at period k.

$$\begin{aligned}
G_k((m+2)d) &\leq \frac{1}{2}G_k((m+1)d) + \frac{1}{2}G_k((m+3)d) \\
G_k((m+2)d) + K_k &\leq \frac{1}{2}[G_k((m+1)d) + K_k] + \frac{1}{2}[G_k((m+3)d) + K_k] \\
J_k((m+1)d) &\leq \frac{1}{2}J_k(md) + \frac{1}{2}J_k((m+2)d)
\end{aligned}$$

First inequality is true due to part (a) of the induction assumption for $n = k$ and in second one same term K_k is added to both side of the inequality.

For all cases, we proved that $J_k((m+1)d) \leq \frac{1}{2}J_k(md) + \frac{1}{2}J_k((m+2)d)$ and these conditions still hold when each component is multiplied by a non negative term p_{rk+1}. Therefore sums of these terms also have the same property even multiplied by a positive discount factor γ. So we prove that $G_{k+1}((m+1)d)$ is less than or equal to $\frac{1}{2}G_{k+1}(md) + \frac{1}{2}G_{k+1}((m+2)d)$. We conclude the induction argument for part (a) which is $G_n((m+1)d)$ is less than or equal to the convex combination of $G_n(md)$ and $G_n((m+2)d)$ for any integer m and period n. By using Lemma 1 and Theorem 1, parts (b) follows immediately which states that $G_n(md)$ is convex.□

A.3 Proof of Theorem 3

We prove Theorem 3 by induction. Before induction, we should show a property that is used in proof of Theorem 3.

Let j be an integer multiple of d; Recall that $J_k(j)$ can take two values; $G_k(j)$ or $G_k(j+d) + K$ (Theorem 1 part (d)). Because of this property and convexity;

$J_k((j-d)) - J_k(j)$ can take three different values. For any period k and j being an integer multiple of d;

$$J_k((j-d)) - J_k(j) = vd + min \begin{cases} G_k((j-d)) - G_k(j) & j-d \geq s_k \\ G_k(j) + K - G_k(j) & j-d \leq s_k \leq j \\ G_k(j) - G_k(j+d) & j \leq s_k \end{cases} \quad \text{(A.8)}$$

Since $J_k((j-d)) - J_k(j)$ can take three different values, we have three cases to consider;

Case 1: If $J_k((j-d)) - J_k(j) = vd + G_k((j-d)) - G_k(j)$: In this case, it is optimal not to order at both inventory levels; ($j-d$ and j) which means threshold level in period k is less than or equal to $j-d$. At point $(j-d)$, it is optimal not to order which means that $K + G_k(j) \geq G_k((j-d)) = J_k((j-d)) + v(j+d)$. If we subtract $J_k(j) + vj = G_k(j)$ from both side of the inequality, $K + vd \geq vd + G_k((j-d)) - G_k(j) = J_k((j-d)) - J_k(j)$

Case 2: If $J_k((j-d)) - J_k(j) = vd + G_k(j) + K - G_k(j)$: In this case, it is optimal to order at $(j-d)$ and not to order at j, which means threshold level in period k is j and $J_k((j-d)) - J_k(j) = vd + K$ since $G_k(j)$ terms cancel each other.

Case 3: If $J_k((j-d)) - J_k(j) = vd + G_k(j) - G_k(j+d)$: In this case, it is optimal to order at both inventory levels; ($j-d$ and j) which means threshold level in period k is greater than j. At point j, it is optimal to order which means that $G_k(j) \geq K + G_k((j+d))$. If we rearrange the terms and vd to both side of the equation, we get $J_k((j-d)) - J_k(j) = vd + G_k(j) - G_k(j+d) \geq K + vd$.

Now, we can return to our induction argument. This induction states that cost difference between two consecutive points (multiples of d) increases as number of period to go increases if these points lies at the left of the S_n. As a result, if it is optimal to order at an inventory level in current period, it is certain that it is optimal to order at the same inventory level in previous periods. Moreover, it states that policy parameters s_n and S_n increase as number of periods to go n increases.

We shall prove all parts of theorem by induction.

For $n = 1$ and j is an integer multiple of d,

a. $J_1(j-d) - J_1(j) \geq J_0(j-d) - J_0(j) = 0$ where $j \leq S_1$. By assumption, $J_0(x) = 0$ for all x. By Lemma 3, left hand side of inequality is always non-negative when $j \leq S_1$. Therefore, part (a) is proved for $n = 1$.

b. $G_2(j-d) - G_2(j) \geq G_1(j-d) - G_1(j)$ where $j \leq S_1$. $G_2(j-d) - G_2(j)$ is equal to $G_1(j-d) - G_1(j) + \sum_{r=0}^{R} p_r[J_1(j-d-r) - J_1(j-r)]$. Terms $G_1(j-d) - G_1(j)$ on both side of inequality cancel each other. Moreover, $J_1(j-d-r) - J_1(j-r)$ is non-negative for any positive integer r where $j \leq S_1$ by part (a) of induction. Therefore, part (b) is proved for $n = 1$.

c. $S_2 \geq S_1$. Due part (b) of induction, term $G_2(j-d) - G_2(j)$ is always positive for all $j \leq S_1$. So it is clear that minimum of this period is greater than or equal to the minimum of the previous period. Due to convexity of G_2, left hand side of the equation $(G_2(j-d) - G_1(j))$ is greater than or equal to zero when $S_1 \leq j \leq S_2$ and right hand side is always zero in this region. Therefore statement in part (b) can be extended from $j \leq S_1$ to $j \leq S_2$.

d. $s_2 \geq s_1$. By definition s_2 is the smallest point that satisfies $G_2(s_2) - G_2(s_2 + d) \leq K$. Similarly, s_1 is the smallest point that satisfies $G_1(s_1) - G_1(s_1 + d) \leq K$. Due to Lemma 2 above, $s_2 \leq S_2$ and we will investigate condition in two cases;

   Case 1: If $s_2 < S_2$: In this case, by part (b) of induction argument is valid. Hence, $G_1(s_2) - G_1(s_2 + d)$ is less than or equal to $G_2(s_2) - G_2(s_2 + d)$. Second statement is less than or equal to K by definition of s_2, so first statement is also less than or equal to K. Since, s_1 is the smallest point that satisfies $G_1(s_1) - G_1(s_1 + d) \leq K$ so all other values that satisfy the same inequality is greater than s_1 by definition of s_1 so that $s_2 \geq s_1$.

Case 2: If $s_2 = S_2$: In this case, part (b) of induction argument is not valid since $s_2 + d = S_2 + d$ which is greater that S_2 so we cannot compare $G_1(s_2) - G_1(s_2 + d)$ with $G_1(s_1) - G_1(s_1 + d)$. However, we already know that $S_2 \geq S_1$ again by part (c) of the induction and by Lemma 2, we can conclude that $s_2 = S_2 \geq S_1 \geq s_1$ so $s_2 \geq s_1$.

So we proved all parts of induction for $n = 1$ and we continue to the next step. For $n = k$ we assume all statements to be true.

For $n = k + 1$,

a. $J_{k+1}(j - d) - J_{k+1}(j) \geq J_k(j - d) - J_k(j)$ where $j \leq S_{k+1}$. In proving this statement, the main difficulty faced that because of ordering decision in both inventory points $(j - d)$ and j in both periods k and $k + 1$, this statement can take different values. We have shown that $J_{k+1}(j - d) - J_{k+1}(j)$ can take three different values;

$$J_{k+1}(j-d)-J_{k+1}(j) = vd+min \begin{cases} G_{k+1}(j-d) - G_{k+1}(j) & j-1 \geq s_{k+1} \\ G_{k+1}(j) + K - G_{k+1}(j) & j-1 < s_{k+1} \leq j \\ G_{k+1}(j) - G_{k+1}(j+d) & j < s_{k+1} \end{cases} \tag{A.9}$$

By induction assumption when $n = k$, we know that $G_{k+1}(j - d) - G_{k+1}(j)$ is greater than or equal to $G_k(j - d) - G_k(j)$ where $j \leq S_{k+1}$ and $s_{k+1} \geq s_k$.

We have three cases to consider for $J_{k+1}(j - d) - J_{k+1}(j)$:

Case 1: If $J_{k+1}(j-d) - J_{k+1}(j) = vd + G_{k+1}(j-d) - G_{k+1}(j)$: In this case, it is optimal not to order at both initial inventory position $(j - d)$ and j in period $k + 1$. Therefore threshold level in period $k + 1$ is smaller than or equal to $j - d$. From part (d) of induction , we know that $s_{k+1} \geq s_k$. As a result, threshold level in period k is also smaller than or equal to $j - d$. Therefore statement $J_k(j - d) - J_k(j)$ is equal to $vd + G_k(j - d) - G_k(j)$ because both initial inventory position $(j - d)$ and j is greater than or equal to threshold level of period k and it is optimal not to order at both inventory position. When two statements $vd + G_{k+1}(j-d) - G_{k+1}(j)$ and $vd + G_k(j-d) - G_k(j)$ are compared vd terms cancel out and we conclude that $G_{k+1}(j-d) - G_{k+1}(j) \geq G_k(j-d) - G_k(j)$.

Case 2: If $J_{k+1}(j-d) - J_{k+1}(j) = vd + G_{k+1}(j) + K - G_{k+1}(j) = vd + K$: In this case, it is optimal to order at $(j-d)$ and not to order j in period $k+1$. Therefore, threshold level in period $k+1$ is equal to j. From part (d) of induction , we know that $s_{k+1} \geq s_k$. As a result, threshold level in period k is smaller than or equal to j. Therefore statement $J_k(j-d) - J_k(j)$ can be equal to either $vd + G_k(j-d) - G_k(j)$ or $vd + G_k(j) + K - G_k(j) = vd + K$ depending on the order decision at $(j-d)$ in period k. From the property of $J_k(j-d) - J_k(j)$ given above, we know that if $J_k(j-d) - J_k(j) = vd + G_k(j-d) - G_k(j)$ then $J_k(j-d) - J_k(j) \leq vd + K$. So in both cases $J_{k+1}(j-d) - J_{k+1}(j) = vd + K \geq J_k(j-d) - J_k(j)$

Case 3: If $J_{k+1}(j-d) - J_{k+1}(j) = vd + G_{k+1}(j) - G_{k+1}(j+d)$: In this case, it is optimal to order at both inventory position $(j-d)$ and j in period $k+1$. Therefore, threshold level in period $k+1$ is greater than to j. From the property of $J_k(j-d) - J_k(j)$ given above, we know that $J_{k+1}(j-d) - J_{k+1}(j) = vd + G_{k+1}(j) - G_{k+1}(j+d) \geq K + vd$. Moreover from part (d) of induction , we know that $s_{k+1} \geq s_k$. Unlike the previous cases, this property does not mean much since there is no relation between s_k and j, no bound for s_k can be obtained from this property. Therefore statement $J_k(j-d) - J_k(j)$ can be equal to $vd + G_k(j-d) - G_k(j)$ or $vd + G_k(j) + K - G_k(j) = vd + K$ or $vd + G_k(j) - G_k(j+d)$ depending on the order decision at $(j-d)$ and j in period k. In the first two conditions we know that $J_k(j-d) - J_k(j) \leq K + vd$, so $J_{k+1}(j-1) - J_{k+1}(j) \geq K + vd \geq J_k(j-d) - J_k(j)$ due to the property given above.

If $J_k(j-d) - J_k(j) = vd + G_k(j) - G_k(j+d)$, we have to show that

$vd+G_{k+1}(j)-G_{k+1}(j+d) \geq vd+G_k(j)-G_k(j+d)$ for all $j \leq S_{k+1}$. By parts (b) and (c) of induction, statement above holds for all $j < S_{k+1}$. The problem arise when $j = S_{k+1}$, since $j+d$ is out of the region that we consider at the other parts of the induction. However, we know that it is optimal to order when initial inventory position is equal to j in this case ($J_{k+1}(j) = G_{k+1}(j+d)+K$) so j is smaller than threshold level s_{k+1} in this case, which means j cannot be equal to S_{k+1} by Lemma 2. Therefore problem does not actually exist. so in all three case $J_{k+1}(j-d) - J_{k+1}(j) \geq J_k(j-d) - J_k(j)$.

For all cases, we proved that $J_{k+1}(j-d) - J_{k+1}(j) \geq J_k(j-d) - J_k(j)$ where $j \leq S_{k+1}$.

b. $G_{k+2}(j-d)-G_{k+2}(j) \geq G_{k+1}(j-d)-G_{k+1}(j)$ where $j \leq S_{k+1}$. $G_{k+2}(j-d)-G_{k+2}(j)$ is equal to $G_1(j-d)-G_1(j)+\sum_{r=0}^{R} p_r[J_{k+1}(j-d-r)-J_{k+1}(j-r)]$. Similarly, $G_{k+1}(j-d)-G_{k+1}(j)$ is equal to $G_1(j-d)-G_1(j)+\sum_{r=0}^{R} p_r[J_k(j-d-r)-J_k(j-r)]$. Terms $G_1(j-d)-G_1(j)$ cancel each other. Moreover, $J_{k+1}(j-d-r)-J_{k+1}(j-r)$ is greater than or equal to $J_k(j-d-r)-J_k(j-r)$ for integer r where $j \leq S_{k+1}$ by part (a) of induction. Therefore, part (b) is proved for $n = k+1$.

c. $S_{k+2} \geq S_{k+1}$. Due part (b) of induction, term $G_{k+2}(j-d) - G_{k+2}(j)$ is always positive for all $j \leq S_{k+1}$. So, it is clear that minimum of this period is greater than or equal to the minimum of the previous period. Due to convexity of G_{k+1}, left hand side of the equation ($G_{k+2}(j-d) - G_{k+2}(j)$) is greater than or equal to zero when $S_{k+1} \leq j \leq S_{k+2}$ and right hand side is always non-positive in this region. Therefore statement in part (b) can be extended from $j \leq S_{k+1}$ to $j \leq S_{k+2}$.

d. $s_{k+2} \geq s_{k+1}$. By definition s_{k+2} is the smallest point that satisfies $G_{k+2}(s_{k+2})-G_{k+2}(s_{k+2}+d) \leq K$. Similarly, s_{k+1} is the smallest point that satisfies $G_{k+1}(s_{k+1}) - G_{k+1}(s_{k+1}+d) \leq K$. Due to Lemma 2 above, $s_{k+2} \leq S_{k+2}$ and we will investigate condition in two cases;

Case 1: If $s_{k+2} < S_{k+2}$: In this case, by part (b) of induction argument is valid. Hence, $G_{k+1}(s_{k+2}) - G_{k+1}(s_{k+2} + d)$ is less than or equal to $G_{k+2}(s_{k+2}) - G_{k+2}(s_{k+2} + d)$. Second statement is less than or equal to K by definition of s_{k+2}, so first statement is also less than or equal to K. Since, s_{k+1} is the smallest point that satisfies $G_{k+1}(s_{k+1}) - G_{k+1}(s_{k+1} + d) \leq K$ so all other values that satisfy the same inequality is greater than s_{k+1} by definition of s_{k+1} so that $s_{k+2} \geq s_{k+1}$.

Case 2: If $s_{k+2} = S_{k+2}$: In this case, part (b) of induction argument is not valid since $s_{k+2} + d = S_{k+2} + d$ which is greater that S_{k+2} so we cannot compare $G_{k+1}(s_{k+2}) - G_{k+1}(s_{k+2} + d)$ with $G_{k+1}(s_{k+1}) - G_{k+1}(s_{k+1} + d)$. However, we already know that $S_{k+2} \geq S_{k+1}$ again by part (c) of the induction and by Lemma 2, we can conclude that $s_{k+2} = S_{k+2} \geq S_{k+1} \geq s_{k+1}$ so $s_{k+2} \geq s_{k+1}$.

So, we proved that $s_{k+2} \geq s_{k+1}$ and conclude the induction argument. □