A CLASS OF SUPER INTEGRABLE KORTEWEG-DE VRIES SYSTEMS

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ABSTRACT

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In this thesis, we investigate the integrability of a class of multicomponent super integrable Korteweg-de Vries (KdV) systems in $(1+1)$ dimensions in the context of recursion operator formalism. Integrability conditions are obtained for the system with arbitrary number of components. In particular, from these conditions we construct two new subclasses of multicomponent super integrable KdV systems.

Keywords: integrability, KdV equation, recursion operator, super KdV equations.

ÖZ

BİR SINIF SÜPER ENTEGRE EDİLEBİLİR KORTEWEG-DE VRIES DENKLEM SİSTEMLERİ

DAĞ, HÜSEYİN

Yüksek Lisans , Fizik Bölümü

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Bu çalışmada, $(1+1)$ boyutta, bir tür çok bileşenli süper Korteweg-de Vries (KdV) sistemlerinin entegre edilebilirliği simetri adım operatörü formalizmi ile araştırıldı. Entegre edilebilme koşulları keyfi sayılardaki bileşenli sistemler için elde edildi. Entegre edilebilme koşullarından, iki alt tür çok bileşenli süper entegre edilebilir KdV sistemleri elde edildi.

Anahtar Kelimeler: entegre edilebilirlik, KdV denklemleri, süper KdV denklemleri, simetri adım operatörü.

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CHAPTER 1

INTRODUCTION

It is known that nonlinear evolution equations which are solved exactly possesses various surprising features such as infinitely many symmetries and conserved covariants, and they are integrable [1,2,4-7] . Such models arise in many branches of physics such as classical and quantum field theories, particle physics, relativity, statistical physics and quantum gravity. Also for integrable nonlinear systems, there exists a remarkable property that they have soliton like solutions to the equations of motions [8].

The theory of the nonlinear integrable equations or soliton equations started in 1967 with the paper by Gardner, Greene, Kruskal and Miura [9] on exact solution of the Korteweg-de Vries (KdV) equation. Historically, KdV equation was introduced as a mathematical description by Korteveg and de Vries in 1895 [11], to explain the well known observation of S. Russell [10] that travelling water waves maintains their shapes for long distances. Being nonlinear and difficult to solve, this equation generates lots of interest [3]. As mentioned, Kruskal, Zabusky, Gardner and Greene studied the KdV equation and reach the exiting result that the solutions of the equation does maintain for a long time even they go through a scattering, and Kruskal named those solutions as solitons. Besides being nonlinear, having the soliton like solutions, increased studies on KdV equation, and it became the first nonlinear equation solved exactly.

After the exact solution of the KdV equation, the properties of nonlinear equations are studied with great interest and integrability became an important property to nonlinear evolution systems [1, 2, 4, 14]. So, studies on integrability increased, and various attempts to find a universal definition to integrability had started. On this direction various integrability tests are developed [2, 26, 27]. Some of those are: the method of Lax pair, the method of bi-Hamiltonian formulation, Painlavé analysis, the method of prolongation structure and the method of recursion operator. In this work, we will focus on the construction of recursion operators and their related nonlinear systems.

All known integrable equations posses infinite number of symmetries. Usually, symmetries of an integrable equation are related to one another by a certain distinguished operator which is called recursion operator. It was first presented in its general form by Olver [27] in 1977.

The theories of super integrable systems have drawn a lot of attention in the last two decades [6,8,12-20,22]. Super systems contain anticommuting fields (fermions) of Grasmann algebra and commuting fields (bosons). The KdV equation is a completely integrable nonlinear evolution equation for a bosonic field. The first super integrable KdV system was discovered by Kupershmidt [12, 13] in 1984. A different system was later obteined by Manin and Radul [15] from their super Kadomptsev-Petviashvili (KP) hierarchy. A supersymmetric fermionic extension of KdV has been given by Mathieu [19].

There are several extensions of the classical KdV equation, and their integrability have been investigated in [20, 23, 24, 25]. More recently, Oğuz et al [26] have found a new class of multicomponent super KdV equations in the context of Hamiltonian formalism.

In this work, motivated by the above works on super integrable KdV systems, we consider a class of autonomous multicomponent super KdV system for integrability classification. For this purpose we proposed a recursion operator. From integrability conditions we construct two new multicomponent super KdV systems.

In chapter II, we briefly review the topics on integrability and super KdV systems.

In chapter III we consider a class of super multicomponent KdV systems and propose a recursion operator of degree 2. From integrability (compatibility) conditions, we construct two new multicomponent super KdV systems.

In chapter IV, we present our conclusion and discuss some technical aspects of our partial classification.

CHAPTER 2

BASIC DEFINITIONS AND INTEGRABILITY

2.1 Basic Definitions

2.1.1 Evolution Equation

In this study we consider evolution (system of) equations of the form

$$u_t = F[u], \tag{2.1}$$

where F is a suitable C^∞ vector field on some manifold M. It is assumed that the space of smooth vector fields on M is some space S of C^∞ functions on the real line vanishing rapidly at $\pm\infty$. Equation (2.1) gives the time evolution of some variable u and called the evolution equation. Here $F = F(u, u_x, u_{xx}, ...)$ depends on u in a non-linear way.

The Korteweg-de Vries equation which is also a subject to this study is an illustrative example for an evolution equation. KdV equation first introduced by Korteweg and de Vries in 1895 [11] to explain the solitary behavior of a plane water wave. The famous form of this equation which we shall use throughout this work is

$$u_t = u_{xxx} + 6uu_x, \tag{2.2}$$

where the dynamical variable u can be thought as the height of the wave from the surface, and the subscripts represent differentiation with respect to the corresponding variables.

2.1.2 Fréchet Derivative

Definition 2.1 *The Fréchet derivative of F at the point u in the direction of v is defined as*

$$F^*(u)[v] = F^*[v] \doteq \frac{\partial}{\partial \xi} F(u + \xi v)\bigg|_{\xi=0}. \tag{2.3}$$

Than the Fréchet derivative operator which is a linear differentiable operator denoted by F^*.

Example 2.1 *For the KdV equation (2.2), the Fréchet derivative is*

$$F^*[v] = v_{3x} + 6vu_x + 6uv_x, \tag{2.4}$$

and the operator form is

$$F^* = D^3 + 6uD + 6u_x, \tag{2.5}$$

where $D = \frac{d}{dx}$ is the total derivative with respect to x.

2.1.3 Symmetry

Definition 2.2 *For an evolution equation, a function $\sigma \in S$ is a symmetry if it satisfies*

$$\sigma^*[F] - F^*[\sigma] = 0. \tag{2.6}$$

Example 2.2 *For the KdV equation equation (2.2), u_x is a symmetry and satisfies equation (2.6), such that*

$$\begin{aligned} \sigma &= u_x \\ \sigma^* &= D \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} D(u_{3x} + 6uu_x) - (D^3 + 6uD + 6u_x)u_x &= (u_{4x} + 6u_x^2 + 6uu_{xx} - (u_{4x} + 6u_x^2 + 6uu_{xx}) \\ &= 0. \end{aligned}$$

2.1.4 Conserved Covariant

Definition 2.3 *For an evolution equation, a function $\gamma \in S^*$ is a conserved covariant (conserved gradient, i.e. gradient of a conserved functional) if it satisfies*

$$\gamma^*[F] + (F^*)^\dagger[\gamma] = 0. \tag{2.8}$$

Here $(\gamma^*)^\dagger = \gamma^*$ and γ is a given functional $I : S \to R$, defined as

$$\gamma \doteq I^*(u)[v] = \langle gradI, v\rangle \tag{2.9}$$

to satisfy equation (2.8), I must be conserved such as;

$$I^*[F] = I^*[u_t] = \langle gradI, F\rangle = 0. \tag{2.10}$$

Example 2.3 *For the KdV equation in the form equation (2.2) , the first conserved covariant and the conserved functional are given as*

$$\gamma^{(1)} = u \tag{2.11}$$

and

$$I^{(1)} = \int_{-\infty}^{\infty} \frac{u_{2x}}{2} dx. \tag{2.12}$$

Here $\gamma^* = id$ and $\gamma = gradI$ obviously.

Than equation (2.8) is satisfied such as

$$\gamma^*[F] + (F^*)^\dagger[\gamma] \quad = \quad 1(u_{3x} + 6uu_x) + (-D^3 - 6Du + u_x)u$$

$$= u_{3x} + 6uu_x - u_{3x} - 6uu_x - 6u_x u + 6uu_x \qquad (2.13)$$
$$= 0.$$

2.1.5 Recursion Operators

A recursion operator is a linear integro-differential operator which maps symmetries onto symmetries. The recursion operators were first presented in their general form by Olver [27]. If we know a recursion operator for a system of differential equations, we can generate infinitely many symmetries by applying recursion operator successively, starting with some symmetry $\sigma^{(0)}$. The resulting hierarchy of symmetries is

$$\sigma^{(n+1)} = R\sigma^{(n)} \qquad , \quad n = 0, 1, 2, ... \quad . \qquad (2.14)$$

Therefore, the existence of a recursion operator is strongly related to the integrability properties of the equation, since an equation which admits a recursion operator admits symmetries.

In order to be a recursion operator for a system, an operator need to satisfy the following criteria, together with the Fréchet derivative of the system.

Theorem 2.1 *Suppose $u(t) - F[u] = 0$ is a system of q differential equations. If*

$R : S \to S$ is a linear operator such that

$$F^*.R - D_t.R = \overline{R}.F^* - \overline{R}.D_t \tag{2.15}$$

on the solution manifold, where $\overline{R} : S \to S$ is a linear differential operator, then R is a recursion operator for the system.

The proof of the theorem is given in [27, 28]. In general, a recursion operator of a system depends on independent variables (t, x), and the dependent variables u and its derivatives.

For evolutionary system of equations given as $u_t - F(u)$, where $F(u) \in S$, since we have

$$D_t.R = R_t + R.D_t \tag{2.16}$$

criterion(2.15) implies that $R = \overline{R}$ and we get the following commutator relation for recursion operators;

$$R_t = [F^*, R]. \tag{2.17}$$

The recursion operator for the equation $u_t = F[u]$ is also called the strong symmetry for the given equation.

Example 2.4 *The KdV equation (2.2) possesses a recursion operator*

$$R = D^2 + 4u + 2u_x D^{-1} \tag{2.18}$$

and

$$R_t = 4u_t + 2u_{tx}D^{-1} \qquad F^* = D^3 + 6uD + 6u_x \ ,$$

then

$$\begin{aligned}
R_t &= 4u_{3x} + 2u_{4x}D^{-1} + 24uu_x + 12u_x^2 D^{-1} + 12uu_{2x}D^{-1} \\
F^*R &= (D^3 + 6uD + 6u_x)(D^2 + 4u + 2u_x D^{-1}) \\
&= D^5 + 10u_{3x} + 18u_{2x}D + 20u_x D^2 + 10uD^3 + 60uu_x \\
&\quad +24u^2 D + 12uu_{2x}D^{-1} + 12u_x^2 D^{-1} + 2u_{4x}D^{-1} \\
RF^* &= (D^2 + 4u + 2u_x D^{-1}))(D^3 + 6uD + 6u_x) \\
&= D^5 + 10uD^3 + 20u_x D^2 + 6u_{3x} + 18u_{2x}D \\
&\quad +24u^2 D + 36uu_x
\end{aligned}$$

such that $R_t - [F^*, R] = 0$*. Here*

$$D^{-1} = \int^{\infty} .d\xi, \tag{2.19}$$

with the property

$$D.D^{-1} = D^{-1}.D = id \tag{2.20}$$

so that it is inverse of D *.*

Example 2.5 *Since* $R = D^2 + 4u + 2u_x D^{-1}$ *is a recursion operator for the KdV equation (2.2), it must create new symmetries from a given symmetry. In previous* *sections we had shown that $\sigma^{(1)} = u_x$ is a symmetry of the KdV system. If R is operated on u_x, we get*

$$\begin{aligned}
\sigma^{(2)} &= R\sigma^{(1)} \\
&= (D^2 + 4u + 2u_x D^{-1})(u_x) \\
&= u_{3x} + 6uu_x \\
&= u_t.
\end{aligned}$$

Here, it is obvious that $\sigma^{(2)} = u_t$ is a symmetry of the KdV equation.

2.1.6 Hereditary Operators

In previous Section, we have mentioned that by applying the recursion operator onto a starting symmetry of a given evolution equation, we can create infinite symmetry hierarchy. But it is not necessary to reach all the existing symmetries of the hierarchy (commuting family of symmetries).

Fuchssteiner and Fokas [1] have shown that, a recursion operator, which can create the infinite hierarchy of a nonlinear evolution equation with all existing members of the hierarchy is possible. Such recursion operators are called hereditary operators and a recursion operator is hereditary when,

$$R^*[Rv]w + R[R^*v]w \tag{2.21}$$

is symmetric in v and w.

Example 2.6 *For the KdV equation (2.2), the recursion operator $R = D^2+4u+2u_xD^{-1}$ proves to be hereditary since*

$$\begin{aligned} R^*[v] &= 4v + 2v_xD^{-1} \\ R^*R[v] &= 4Rv + 2(Rv)_xD^{-1} \end{aligned}$$

and $(4Rv + 2(Rv)_xD^{-1})w - R(4vw + 2v_xD^{-1}w)$ is symmetric in v and w.

2.1.7 Hamiltonian Formalism

The Hamiltonian formalism is very important in the theory of integrability. In particular, bi-Hamiltonian systems, that is systems that admit two Hamiltonian representations on the same set of coordinates, are of great interest.

Furthermore, the existence of a commuting family of symmetries, is related to the Hamiltonian formalism.

Definition 2.4 *An operator valued function θ is called a Hamiltonian operator, if and only if it is skew-symmetric and it satisfies the Jacobi identity*

$$\{a,b,c\} + \{b,c,a\} + \{c,a,b\} = 0, \tag{2.22}$$

where $\{\}$ *defined as* $\{a,b,c\} = \{a,b,c\}(u) = \langle b, \theta'(u)[\theta(u)a]c\rangle$, *and then the Hamiltonian system will be*

$$u_t = \theta f, \tag{2.23}$$

where θ *is a Hamiltonian operator and* f *is a suitable gradient function (i.e.* $f^* = (f^*)^\dagger$*).*

It is possible for an evolution equation to be a Hamiltonian system with two hamiltonian operators θ_1 and θ_2, operating on two suitable gradient functions f_1 and f_2. Then the system can be written as

$$u_t = \theta_1 f_1 = \theta_2 f_2 \tag{2.24}$$

such systems are called bi-Hamiltonian systems.

For a bi-Hamiltonian system, if linear combination of the operators θ_1, θ_2

$$\theta_3 = \theta_1 + a\theta_2 \tag{2.25}$$

is still a Hamiltonian operator, then θ_1 and θ_2 are said to be compatible, and the system is called a compatible bi-Hamiltonian system.

If an evolution equation is a compatible bi-Hamiltonian system and θ_1 is invertible, then

$$R = \theta_2 \theta_1^{-1} \tag{2.26}$$

is a hereditary operator for the system and creates all members of the infinite hierarchy.

Example 2.7 *The KdV equation (2.2) ($u_t = u_{3x} + 6uu_x$) can be written in the forms*

$$
\begin{aligned}
u_t &= D(u_{2x} + 3u^2) && (2.27) \\
u_t &= (D^3 + 4uD + 2u_x)u && (2.28)
\end{aligned}
$$

where

$$
\begin{aligned}
\theta_1 &= D \quad , \quad \theta_2 = D^3 + 4uD + 2u_x && (2.29) \\
f_1 &= u_{2x} + 3u^2 \quad , \quad f_2 = u. && (2.30)
\end{aligned}
$$

are the compatible bi-Hamiltonian operators with ($\theta_1 + \theta_2$) being Hamiltonian, and the corresponding gradient functions.

Example 2.8 *Since the given equation is a compatible bi-Hamiltonian system, and the the operator*

$$
\begin{aligned}
R &= \theta_2 \theta_1^{-1} \\
R &= (D^3 + 4uD + 2u_x)(D^{-1}) \\
&= D^2 + 4u + 2u_x D^{-1}.
\end{aligned}
$$

is a hereditary operator. Then all of the members of the KdV hierarchy can be found by beginning with a suitable starting symmetry of the hierarchy.

2.1.8 The Lax Pair

For an evolution equation, one important property is to have Lax pair. Let us consider a linear, hermitian operator $L(t)$, satisfying an eigenvalue equation with a suitable wave function $\psi(t)$, such as

$$L(t)\psi = \lambda\psi, \tag{2.31}$$

where $\lambda_t = 0$.

Time evolution of the wave function is given by an operator $P(t)$ which is anti-symmetric and not necessarily hermitian

$$\frac{\partial\psi}{\partial t} = P\psi. \tag{2.32}$$

By taking the time derivative of the eigenvalue equation (2.31), and substituting equation (2.32), one can reach the commutator relation

$$\frac{\partial L}{\partial t} = [P, L]. \tag{2.33}$$

If the above relation is satisfied with suitable operators L and P, than those operators are called as a Lax pair. And if the commutator relation (2.33) yields a given nonlinear evolution equation, then, the system will have a strong slue of being solvable. Advanced discussions on this topic can be found in [2].

Example 2.9 *For the KdV equation (2.2), the Lax pair that provides the equation is given by*

$$L = D^2 + u \tag{2.34}$$

$$P = 4D^3 + 3uD + 3Du. \tag{2.35}$$

2.2 Integrability

In this section, we will give a brief summary of integrability which is an important topic when dealing with nonlinear systems of equations. It is a remarkable property that non linear systems which are solved exactly possess infinitely many symmetries and conserved covariants (if the model is conserved), or they are integrable [1, 2, 5, 8, 27]. It is not certain that is it possible for all integrable systems to be solved exactly, but one can easily say that integrability is a strong clue for the solvability of the nonlinear equation.

There does not exist a unique definition for an integrability of a nonlinear evolution equation. However, since the integrable systems possess infinitely many symmetries, the following criterion introduced by Olver [27] is the most acceptable approach. In the rest of this study, we will use the following definition when considering integrability.

Definition 2.5 *A system of equations is said to be integrable if it admits a non-trivial recursion operator.*

This means that for an evolution equation, if there exists a recursion operator R satisfying the condition (2.17) such that $R_t - [F^*, R] = 0$ the system is integrable.

In previous sections it was shown that

$$R = D^2 + 4u + 2u_x D^{-1}$$

is a recursion operator for the KdV equation satisfying (2.17). So the KdV equation is an integrable system.

There also exists alternative criterions for integrability. Since we will mention integrability in the point of view of the Lax pairs and bi-hamiltonian systems, it will be better to give their approaches briefly.

For an evolution equation, if there exists a Lax pair L and P, such that the commutator relation $L_t - [P, L] = 0$ results the given evolutionary system $u_t - F(u) = 0$, then the system is integrable.

It is given that there exists a Lax pair satisfying the commutator relation (2.17), which provides the KdV equation (2.2). So KdV equation is also integrable when considering the Lax representation.

Finally, if an evolutionary system is a compatible bi-hamiltonian system, it is obviously integrable. In previous sections it was shown that KdV equation (2.2) is a compatible bi-hamiltonian system.

2.3 Super Integrable KdV Type Equations

After the rise of the quantum theories, studies to write super partners to classical evolution equations had increased [12,15,16,19-25,29]. Since, u is the classical parameter in evolution equations, it can be regarded as a commuting (bosonic) field in super theories. And a fermionic (anti commuting) partner can be written such that the evolution equation (2.1) will take the form

$$\begin{pmatrix} u_t \\ \\ \xi_t \end{pmatrix} = F \begin{pmatrix} u \\ \\ \xi \end{pmatrix}. \tag{2.36}$$

Here the bosonic parameter commutes while the fermionic parameter anti-commutes, such that

$$\begin{aligned} u_\alpha u_\beta &= u_\alpha u_\beta, \\ \xi_a \xi_b &= -\xi_b \xi_a, \end{aligned} \tag{2.37}$$

and the evolution of the system is given by the operator F

$$\mathbf{F} = \begin{pmatrix} F^{(0)}_{\alpha\beta} & F^{(1)}_{\alpha b} \\ \\ F^{(2)}_{a\beta} & F^{(3)}_{ab} \end{pmatrix} \tag{2.38}$$

with $\alpha, \beta, ... = 1, 2, ..., m$ are the bosonic labels and $a, b, ... = 1, 2, ..., n$ are fermionic labels.

The main way to get a super integrable system is the supersymmetrization of the bosonic field. And starting from the classical evolution equation, one can end up with a supersymmetric system [19-22,30]. But there also exist successful attempts which result a super system just by introducing a super partner to the bosonic component, which is not necessary to be supersymmetric [12-14]. Such systems are called super systems.

In the case of the KdV equations; the first supersymmetric integrable KdV system was introduced by Manin- Radul [15], and the first super extension of the KdV equation was introduced by Kupershmidt [12].

In the following sections, we will give the famous super KdV equation and supersymmetrization of the classical KdV equation.

2.3.1 Super Integrable KdV System

Kupershmidt, in 1984, introduced a super integrable KdV system [12]. This system is not supersymmetric but the integrability of the equation is provided by a Lax pair. Since the system is not supersymmetric, some authors call it as the Kuper-KdV equation. The super KdV system introduced by Kupershmidt is

$$\begin{aligned} u_t &= u_{3x} + 6uu_x - 12\xi\xi_{2x} \ , \\ \xi_t &= 4\xi_{3x} + 6u\xi_x + 3\xi u_x \ , \end{aligned} \tag{2.39}$$

where u is the bosonic field and ξ is the fermionic field.

This system admits a Lax pair

$$L = D^2 + u + \xi D^{-1}\xi$$

$$P = 4D^3 + 3uD + 3Du \tag{2.40}$$

which satisfy the commutation relation (2.33).

2.3.2 Supersymmetric Integrable KdV Systems

When studying the super spaces, supersymmetry is a powerful tool. Supersymmetrization of the bosonic field could be used when constructing a supersymmetric system. In the case of the KdV equation, the construction of the supersymmetric KdV system is as follows.

We will formulate the fermionic extension of the KdV equation in a superspace formalism. The variable x will be extended to a doublet (x, θ), where θ is an anti-commuting (i.e. fermionic, Grasmanian) variable: $\theta^2 = 0$. So, the bosonic (commuting) field $u(x)$, will be replaced by a superfield, such that

$$u(x) \rightarrow \Psi(x, \theta). \tag{2.41}$$

Then the Taylor expansion of the superfield Ψ in terms of component fields $u(x)$ and $\xi(x)$ is given as

$$\Psi(x,\theta) = \xi(x) + \theta u(x). \tag{2.42}$$

Here ξ is the super partner of the commuting variable u, and $\Psi(x,\theta)$ is a fermionic superfield. It has the same character of the θ independent term in the expansion, which is ξ here.

It is also possible to have a bosonic expansion of a superfield, or to have a bosonic superfield such that

$$\Omega(x,\theta) = u(x) + \theta\xi(x). \tag{2.43}$$

The fermionic and bosonic extensions of the KdV equation is given in the following sections.

2.3.2.2 Fermionic Supersymmetrization of the KdV Equation

As mentioned before the superspace can be expanded as a fermionic superfield such that; $\Psi(x,\theta) = \xi(x) + \theta u(x)$. In superspace formalism, the fundamental differential operator is the super derivative defined as

$$d_s = \theta\partial x + \partial_\theta \ , \tag{2.44}$$

where $\partial_x, \partial_\theta$ denote the partial derivatives with respect to x, θ, and $d_s^2 = D$ obviously.

The space supersymmetric invariance refers to invariance with respect to transformations $x \to \eta\theta$ and $\theta \to \theta + \eta$ where η is the anticommuting constant parameter. The generator of the transformation is given by

$$Q = \partial_\theta - \theta\partial x, \tag{2.45}$$

and for the fermionic superfield, the supersymmetric transformation is

$$\delta\Psi = \eta Q\theta, \tag{2.46}$$

which transforms in components

$$\begin{aligned} \delta u(x) &= \eta\xi_x(x), \\ \delta\xi(x) &= \eta u(x). \end{aligned} \tag{2.47}$$

If we consider the KdV equation given in (2.2), its superfield expansion with $\xi = 0$ is given as

$$\begin{aligned} \Psi(x,\theta) &= \theta u(x) \\ \Psi_t &= \theta u_t \\ &= \theta u_{3x} + \theta(6uu_x) \end{aligned} \tag{2.48}$$

$$
\begin{aligned}
&= d_s^6\Psi + 3d_s^2(\Psi d_s\Psi) \\
&= d_s^6\Psi + 6d_s\Psi d_s^2\Psi.
\end{aligned}
$$

In general

$$\Psi_t = d_s^6\Psi + ad_s^2(\Psi d_s\Psi) + (6-2a)d_s\Psi d_s^2\Psi, \tag{2.49}$$

where a is any constant.

After finding the general form of Ψ_t, again doing the expansion $\Psi_t = \xi + u\theta$ with $\xi \neq 0$, we get

$$
\begin{aligned}
\Psi_t &= \xi_t + u_t\theta \qquad (2.50) \\
&= d_s^6 + ad_s^2(\Psi d_s\Psi) + (6-2a)d_s\Psi d_s^2\Psi.
\end{aligned}
$$

After calculating the terms d_s^6, $ad_s^2(\Psi d_s\Psi)$ and $(6-2a)d_s\Psi d_s^2\Psi$, one can reach the equation

$$
\begin{aligned}
\xi_t + u_t\theta &= a\xi u_x + a\xi_x u \\
&\quad +(6-2a)u\xi_x\xi_{3x} \qquad (2.51) \\
&\quad +[u_{3x} + a\xi\xi_{2x} + 2auu_x + (6-2a)uu_x]\theta.
\end{aligned}
$$

This will result in a system (with $a = 3$, for the integrability) given by Mathieu [19, 21, 22] such that

$$
\begin{aligned}
u_t &= u_{3x} + 6uu_x + 3\xi\xi_{2x} \\
\xi_t &= \xi_{3x} + 3u\xi_x + 3u_x\xi. \qquad (2.52)
\end{aligned}
$$

The integrability of the above system was given by the Lax pair

$$
\begin{aligned}
L &= D^2 + u + \theta\xi_x \qquad (2.53) \\
P &= 4D^3 + 6uD + 3u_x + 3\theta(2\xi_x D + \xi_{2x}).
\end{aligned}
$$

There also exists bosonic expansion of the superfield, which will be dealt in the next Section.

2.3.2.2 Bosonic Supersymmetrization of the KdV Equation

One other way to extend $u(x)$ into a superfield is to expand it in terms bosonic field such that

$$u(x) \to \Omega(x,\theta) = u(x) + \theta\xi(x),$$

and the KdV equation (2.2) can be expanded as

$$
\begin{aligned}
\Omega_t(x,\theta) &= \Omega_{3x} + 6\Omega\Omega_x \\
&= d_s^6\Omega + 6\Omega d_s^2\Omega \qquad (2.54) \\
u_t + \theta\xi_t &= u_{3x} + 6uu_x + \theta(-\xi_{3x} + 6u\xi_x + 6\xi u_x)\ ,
\end{aligned}
$$

which yields the system

$$
\begin{aligned}
u_t &= u_{3x} + 6uu_x \\
\xi_t &= \xi_{3x} + 6u\xi_x + 6\xi u_x.
\end{aligned}
\tag{2.55}
$$

This equation is called the trivial supersymmetric KdV equation, because the commuting part of the system is the KdV equation itself and there is no coupling from the anticommuting parameter.

CHAPTER 3

MULTICOMPONENT SUPER INTEGRABLE KdV TYPE EQUATIONS AND THEIR RECURSION OPERATORS

In this chapter, we are concerned with the integrability of a class of multicomponent super KdV systems admitting a recursion operator of order 2.

3.1 Integrability of Multicomponent Systems

In this thesis we aimed to study the integrability of a class of multicomponent super KdV systems by introducing a suitable recursion operator. Systems admitting a recursion operator are integrable in the existence of infinitely many symmetries sense because by definition, a recursion operator maps symmetries of a system to other symmetries.

The integrability criterion for evolutionary systems was given by equation (2.17). In the case of multicomponent systems, this criterion will take the form

$$R_{AB} = F^*_{AC} R_{CA} - R_{AC} F^*_{CA}, \tag{3.1}$$

where the Fréchet derivative of the system takes the matrix form

$$F^*_{AB} = \begin{pmatrix} F^{*(0)}_{\alpha\beta} & F^{*(1)}_{\alpha b} \\ F^{*(2)}_{a\beta} & F^{*(3)}_{ab} \end{pmatrix}, \tag{3.2}$$

and the recursion operator will be

$$R_{AB} = \begin{pmatrix} R^{(0)}_{\alpha\beta} & R^{(1)}_{\alpha b} \\ R^{(2)}_{a\beta} & R^{(3)}_{ab} \end{pmatrix}, \tag{3.3}$$

then the component form of the criterion (3.1) will be

$$\begin{aligned} R_{\alpha\beta} &= F^*_{\alpha\gamma} R_{\gamma\beta} + F^*_{\alpha d} R_{d\beta} - R_{\alpha\gamma} F^*_{\gamma\beta} - R_{\alpha d} F^*_{d\beta} \\ R_{\alpha b} &= F^*_{\alpha\gamma} R_{\gamma b} + F^*_{\alpha d} R_{db} - R_{\alpha\gamma} F^*_{\gamma b} - R_{\alpha d} F^*_{db} \\ R_{a\beta} &= F^*_{a\gamma} R_{\gamma\beta} + F^*_{ad} R_{d\beta} - R_{a\gamma} F^*_{\gamma\beta} - R_{ad} F^*_{d\beta} \\ R_{ab} &= F^*_{a\gamma} R_{\gamma b} + F^*_{ad} R_{db} - R_{a\gamma} F^*_{\gamma b} - R_{ad} F^*_{db}. \end{aligned} \tag{3.4}$$

Motivated by the form of two-components super and supersymmetric KdV equations, we consider the following multicomponent super KdV systems.

$$\begin{aligned} u_{\alpha,t} &= b_{\alpha\beta}u_{\beta,3x} + 3C_{\alpha\beta\gamma}u_\beta u_{\gamma,x} + K_{\alpha ab}\xi_a\xi_{b,2x}, \\ \xi_{a,t} &= \Lambda_{ab}\xi_{b,3x} + L_{a\alpha b}\xi_b u_{\alpha,x} + N_{a\alpha b}\xi_{b,x}u_\alpha, \end{aligned} \tag{3.5}$$

where bosonic indices $\alpha, \beta, \gamma, ... = 0, 1, ..., m$ and fermionic indices $a, b, c, ... = 0, 1, ..., n$. All the coefficients are constant parameters, which will be determined by the integrability conditions of the system.

We associate the integrability of these systems with the existence of a recursion operator (3.3) with the components

$$\begin{aligned} R^{(0)}_{\alpha\beta} &= b_{\alpha\beta}D^2 + C_{\alpha\beta\gamma}(2u\gamma + u_{\gamma,x}D^{-1}) \\ &\quad + F_{\alpha\beta ab}\xi_{a,x}D^{-1}\xi_b D^{-1}, \\ R^{(1)}_{\alpha b} &= L^1_{\alpha bc}\xi_c D + L^2_{\alpha bc}\xi_{c,x} + L^3_{\alpha bc}\xi_{c,2x}D^{-1} \\ &\quad + P_{\alpha b\beta c}u_\beta\xi_c D^{-1} + S_{\alpha bc\beta}\xi_{c,x}D^{-1}u_\beta D^{-1} \\ &\quad + R_{\alpha b\beta c}u_{\beta,x}D^{-1}\xi_c D^{-1}, \\ R^{(2)}_{a\beta} &= Q^1_{a\beta c}\xi_c + Q^2_{a\beta c}\xi_{c,x}D^{-1} \\ &\quad + T_{a\beta\alpha c}u_\alpha D^{-1}\xi_c D^{-1}, \end{aligned} \tag{3.6}$$

$$
\begin{aligned}
R^{(3)}_{ab} &= \Lambda_{ab}D^2 + Z^1_{ab\alpha}u_\alpha + Z^2_{ab\alpha}u_{\alpha x}D^{-1} \\
&\quad + M_{abcd}\xi_{c,x}D^{-1}\xi_d D^{-1} + Q_{ab\alpha\beta}u_\alpha D^{-1}u_\beta D^{-1},
\end{aligned}
$$

where all coefficients are constants. The components of the Fréchet derivative of the system (3.5) is calculated as

$$
\begin{aligned}
F^{*(0)}_{\alpha\beta} &= b_{\alpha\beta}D^3 + 3C_{\alpha\gamma\beta}u_\gamma D + 3C_{\alpha\beta\gamma}u_{\gamma,x}, \\
F^{*(1)}_{\alpha b} &= -K_{\alpha ba}\xi_{a,2x} + K_{\alpha ab}\xi_a D^2, \\
F^{*(2)}_{a\beta} &= L_{a\beta d}\xi_d D + N_{a\beta d}\xi_{d,x}, \\
F^{*(3)}_{ab} &= \Lambda_{ab}D^3 + L_{a\gamma b}u_{\gamma,x} + N_{a\gamma b}u_\gamma D.
\end{aligned} \tag{3.7}
$$

The integrability criterion (3.1) gives the relation among the coefficient terms of system (3.5) and the recursion operator (3.6). We present computational details of the integrability criterion (3.1) in Appendix A.

Having obtained the necessary conditions of integrability and specified the numbers m and n, our basic aim is to identify integrable cases and to give a complete description and classification of integrable systems. This procedure is based on the coefficients of the higher order terms, which are $b_{\alpha\beta}$ and Λ_{ab} (main matrices) in our system [31]. In other words, system (3.5) can be reduced to a system with main matrices in Jordan canonical form by a linear invertible change of dependent variables.

In this work we shall not give a complete classification. Motivated by the two components super and supersymmetric KdV systems. we construct two new subclasses multicomponent super KdV systems. They correspond to $N_{a\beta c} = 2L_{a\beta c}$ and $N_{a\beta c} = L_{a\beta c}$ (from integrability conditions).

3.2 Multicomponent Super Integrable KdV System I

The multicomponent super KdV system (3.5) admits a recursion operator for the case $N_{a\beta c} = 2L_{a\beta c}$ and it is integrable. All of the equations in the appendix A are satisfied with the conditions

$$\begin{aligned}
b_{\alpha\beta} = \delta_{\alpha\beta} \quad &, \quad \Lambda_{ab} = 4\delta_{ab}, \\
C_{\alpha\mu\nu} = C_{\alpha\nu\mu} \quad &, \quad K_{\alpha mn} = K_{\alpha nm}, \\
L^1_{\alpha mn} = K_{\alpha mn} \quad &, \quad L^2_{\alpha mn} = \frac{1}{3}K_{\alpha mn}, \\
Q^1_{a\beta c} = L_{a\beta c} \quad &, \quad Q^2_{a\beta c} = \frac{2}{3}L_{a\beta c}, \\
N_{a\beta c} = 2L_{a\beta c} \quad &, \quad Z^1_{ac\beta} = \frac{4}{3}L_{a\beta c}, \\
F_{\alpha\beta ab} = M_{abcd} = P_{\alpha a\beta b} \quad &= \quad R_{\alpha a\beta b} = S_{\alpha ab\beta} = 0, \\
T_{a\alpha\beta b} = Q_{ab\alpha\beta} \quad &= \quad L^3_{\alpha ab} = Z^2_{ab\alpha} = 0,
\end{aligned} \tag{3.8}$$

and the relations

$$C_{\alpha\beta\gamma}C_{\gamma\mu\nu} = C_{\alpha\mu\gamma}C_{\gamma\beta\nu},$$

$$
\begin{aligned}
K_{\alpha n d}L_{d\beta m} &= K_{\alpha m d}L_{d\beta n},\\
C_{\alpha\beta\gamma}K_{\gamma m n} &= \frac{2}{3}K_{\alpha d n}L_{d\beta m},\\
L_{a\gamma n}C_{\delta\beta\mu} &= \frac{2}{3}L_{a\beta d}L_{d\mu n},\\
L_{a\beta d}L_{d\mu n} &= L_{a\mu d}L_{d\beta n},\\
L_{a\gamma b}K_{\gamma m n} &= L_{a\gamma n}K_{\gamma b n}.
\end{aligned}
\tag{3.9}
$$

Then we get the first new subclass of integrable multicomponent super KdV system as

$$
\begin{aligned}
u_{\alpha,t} &= \delta_{\alpha\beta}u_{\beta,3x}+3C_{\alpha\beta\gamma}u_{\beta}u_{\gamma,x}+K_{\alpha ab}\xi_a\xi_{b,2x},\\
\xi_{a,t} &= 4\delta_{ab}\xi_{b,3x}+L_{a\alpha b}u_{\alpha,x}\xi_b+2L_{a\alpha b}\xi_{b,x}u_{\alpha},
\end{aligned}
\tag{3.10}
$$

and the recursion operator for the above system is

$$
\begin{aligned}
R^{(0)}_{\alpha\beta} &= \delta_{\alpha\beta}D^2+2C_{\alpha\beta\gamma}u_{\gamma}+C_{\alpha\beta\gamma}u_{\gamma,x}D^{-1},\\
R^{(1)}_{\alpha b} &= K_{\alpha bc}\xi_c D+\frac{1}{3}K_{\alpha bc}\xi_{c,x},\\
R^{(2)}_{a\beta} &= L_{a\beta c}\xi_c+\frac{2}{3}L_{a\beta c}\xi_{c,x}D^{-1},\\
R^{(3)}_{ab} &= 4\delta_{ab}D^2+\frac{4}{3}L_{a\gamma b}u_{\gamma}.
\end{aligned}
\tag{3.11}
$$

This system can be considered as the multicomponent form of the super KdV equation given by (2.39). Because the above system reduces to one of the known two component super KdV systems (2.39) which contains of one bosonic and one fermionic variables ($m=0,n=0$). The corresponding recursion operator for this system with suitable choice of coefficients ($C_{000} = 2$, $K_{000} = -12$, and $L_{000} = 3$) is

$$
\begin{aligned}
R^{(0)} &= D^2 + 4u + 2u_x D^{-1}, \\
R^{(1)} &= -12\xi D - 4\xi_x, \\
R^{(2)} &= 3\xi + 2\xi_x D^{-1}, \\
R^{(3)} &= 4D^2 + 4u,
\end{aligned}
\tag{3.12}
$$

where $u_0 = u$ and $\xi_0 = \xi$.

Furthermore our system (3.10) reduces to the system of Oğuz et al [26], when $K_{\alpha cd} = L_{c\alpha d}$. In this case the constructed recursion operator (3.12) can be written as a product of two Hamiltonian operators. These Hamiltonian operators constitute a compatible pair and the recursion operator becomes hereditary.

3.3 Multicomponent Super Integrable KdV System II

The integrability conditions (Appendix A) yields the other possible subclass $N_{a\beta c} = L_{a\beta c}$ to system (3.5). All of the equations given in appendix A are satisfied with a recursion operator resulting from (3.6), when conditions

$$
b_{\alpha\beta} = \delta_{\alpha\beta} \qquad , \qquad \Lambda_{ab} = \delta_{ab},
$$

$$
\begin{aligned}
L_{a\beta c} = N_{a\beta c} \quad &, \quad C_{\alpha\beta\gamma} = C_{\alpha\gamma\beta}, \\
K_{\alpha ab} = K_{\alpha ba} \quad &, \quad M_{abcd} = M_{adcb}, \\
R_{\alpha b\beta c} = R_{\alpha c\beta b} \quad &, \quad P_{\alpha b\beta c} = P_{\alpha c\beta b}, \\
F_{\alpha\beta cd} = F_{\alpha\beta dc} \quad &, \quad S_{\alpha dc\beta} = -F_{\alpha\beta cd}, \\
T_{a\beta\alpha c} = -Q_{ac\beta\alpha} \quad &, \quad L^1_{\alpha mn} = \frac{2}{3}K_{\alpha mn}, \\
L^2_{\alpha mn} = L^3_{\alpha mn} = -\frac{1}{3}K_{\alpha mn} \quad &, \quad Q^1_{a\beta d} = Z^1_{ad\beta} = \frac{2}{3}L_{a\beta d}, \\
Q^2_{a\beta d} = Z^2_{ad\beta} = \frac{1}{3}L_{a\beta d} \quad &, \quad R_{\alpha b\mu d} = -\frac{2}{9}K_{\alpha bc}L_{c\mu d}, \\
M_{abcd} = -\frac{2}{9}L_{a\gamma b}K_{\gamma cd} \quad &, \quad F_{\alpha\beta cd} = -\frac{1}{9}K_{\alpha cm}L_{m\beta d} = -S_{\alpha cd\beta} = P_{\alpha b\beta d}, \\
T_{a\mu\nu b} = -Q_{ab\mu\nu} \quad &= \quad -\frac{1}{3}L_{a\gamma b}C_{\gamma\mu\nu} + \frac{1}{9}L_{a\mu c}L_{c\nu b},
\end{aligned}
\tag{3.13}
$$

and also the relations

$$
\begin{aligned}
L_{a\beta c}L_{c\gamma d} &= L_{a\gamma c}L_{c\beta d}, \\
L_{a\beta c}L_{b\gamma d} &= L_{a\beta d}L_{b\gamma c}, \\
K_{\alpha cm}L_{m\gamma d} &= K_{\alpha dm}L_{m\gamma c}, \\
L_{a\gamma d}K_{\gamma mn} &= L_{a\gamma n}K_{\gamma md}, \\
C_{\alpha\beta\gamma}C_{\gamma\mu\nu} &= C_{\alpha\nu\gamma}C_{\gamma\mu\beta}, \\
C_{\alpha\beta\gamma}K_{\gamma cd} &= \frac{2}{3}K_{\alpha cm}L_{m\beta d}, \\
L_{a\gamma d}C_{\gamma\beta\mu} &= \frac{2}{3}L_{a\beta c}L_{c\mu d}.
\end{aligned}
\tag{3.14}
$$

are satisfied. Then the multicomponent KdV system will take the form

$$
\begin{aligned}
u_{\alpha,t} &= \delta_{\alpha\beta} u_{\beta,3x} + 3C_{\alpha\beta\gamma} u_\beta u_{\gamma,x} + K_{\alpha ab} \xi_a \xi_{b,2x}, \\
\xi_{a,t} &= \delta_{ab} \xi_{b,3x} + L_{a\gamma c}(\xi_{c,x} u_\gamma + \xi_c u_{\gamma,x}). \qquad (3.15)
\end{aligned}
$$

For the above multicomponent system, integrability is guaranteed by the existence of a recursion operator which has components

$$
\begin{aligned}
R^{(0)}_{\alpha\beta} &= \delta_{\alpha\beta} D^2 + C_{\alpha\beta\gamma}(2u_\gamma + u_{\gamma,x} D^{-1}) - \frac{1}{9} K_{\alpha cm} L_{m\beta d} \xi_{c,x} D^{-1} \xi_d D^{-1}, \\
R^{(1)}_{\alpha b} &= \frac{2}{3} K_{\alpha bc} \xi_c D - \frac{1}{3} K_{\alpha bc}(\xi_{c,x} + \xi_{c,2x} D^{-1}) \\
&\quad + K_{\alpha bm} L_{m\beta c}\left(\frac{1}{9}\xi_{c,x} D^{-1} u_\beta D^{-1} - \frac{1}{9} u_\beta \xi_c D^{-1} - \frac{2}{9} u_{\beta,x} D^{-1} \xi_c D^{-1}\right), \\
R^{(2)}_{a\beta} &= L_{a\beta c}\left(\frac{2}{3}\xi_c + \frac{1}{3}\xi_{c,x} D^{-1}\right) - \frac{1}{9} L_{a\beta m} L_{m\gamma c} u_\gamma D^{-1} \xi_c D^{-1}, \\
R^{(3)}_{ab} &= \delta_{ab} D^2 + L_{a\gamma b}\left(\frac{2}{3} u_\gamma + \frac{1}{3} u_{\gamma,x} D^{-1}\right) - \frac{2}{9} L_{a\gamma b} K_{\gamma cd} \xi_{c,x} D^{-1} \xi_d D^{-1} \\
&\quad + \frac{1}{9} L_{a\mu c} L_{c\nu b} u_\mu D^{-1} u_\nu D^{-1}. \qquad (3.16)
\end{aligned}
$$

The above system contains the two component supersymmetric KdV systems (2.52). If we choose the coefficients $C_{000} = 2$, $K_{000} = 3$ and $L_{000} = 3$ where $m = n = 0$, we get a recursion operator to supersymmetric integrable KdV system (2.52) with components

$$
\begin{aligned}
R^{(0)} &= D^2 + 4u + 2u_x D^{-1} - \xi_x D^{-1} u D^{-1}, \\
R^{(1)} &= -\xi_x + 2\xi D - \xi_{2x} D^{-1} - u\xi D^{-1} \\
&\quad - \xi_x D^{-1} u D^{-1} - 2u_x D^{-1} \xi D^{-1}, \\
R^{(2)} &= \xi_x D^{-1} + 2\xi - u D^{-1} \xi D^{-1}, \qquad (3.17)
\end{aligned}
$$

$$R^{(3)} = +D^2 + 2u + u_x D^{-1} + uD^{-1}uD^{-1} - 2\xi_x D^{-1}\xi D^{-1}.$$

The multicomponent system given in (3.15) also yields the trivial supersymmetric system (2.55), with suitable coefficients ($C_{000} = 2$, $K_{000} = 0$ and $L_{000} = 3$) satisfying the conditions (3.13) and (3.14). The components of the recursion operator which provides the integrability of the trivial supersysmmetric integrable KdV system (2.55) are

$$R^{(0)} = D^2 + 4u + 2u_x D^{-1}, \tag{3.18}$$

$$R^{(1)} = 0, \tag{3.19}$$

$$R^{(2)} = 2\xi_x D^{-1} + 4\xi, \tag{3.20}$$

$$R^{(3)} = D^2 + 4u + 2u_x D^{-1}. \tag{3.21}$$

CHAPTER 4

CONCLUSION

In this work, we have considered a class of multicomponent super KdV systems. We have investigated the integrability of these systems in terms of the existence of a certain recursion operator.

We have given the necessary integrability conditions for arbitrary numbers m and n . From these integrability conditions we have found two new subclasses of multicomponent super KdV systems. One of them contains the two component super KdV system (2.39) while the other contains two component supersymmetric KdV systems (2.52) and (2.55).

As we mentioned in the previous chapter, the complete classification of integrable systems is based on the canonical form of the coefficients of the higher order terms of the systems. Moreover, the canonical form of the integrable systems is unique and well-defined. One of the interesting problems is to give a complete classification of our multicomponent super KdV systems. The other is to investigate the Hamiltonian (and bi-Hamiltonian) structure of these systems. These will be considered for future work.

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APPENDIX A

INTEGRABILITY CONDITIONS

The equations in this appendix are the integrability conditions of multicomponent KdV system (3.4), with the recursion operator (3.6) which is introduced, resulting from the integrability condition (3.1). For practical purposes, symmetric part of an equation is denoted as $\{,\}$, where antisymmetrization is denoted as $[,]$.

$$b_{\alpha\mu}C_{\mu\beta\nu} - b_{\gamma\nu}C_{\mu\beta\gamma} = 0 \tag{A.1}$$

$$C_{\alpha\gamma\beta}C_{\gamma\mu\nu} - C_{\alpha\nu\beta}C_{\gamma\beta\mu} = 0 \tag{A.2}$$

$$\begin{aligned}&2C_{\alpha\beta\gamma}K_{\gamma cd} + 3b_{\alpha\mu}F_{\mu\beta dc} - K_{\alpha md}Q^1_{m\beta c}\\&-K_{\alpha cm}Q^1_{m\beta d} - 2K_{\alpha cm}Q^2_{m\beta d} + L^1_{\alpha mc}N_{m\beta d}\\&-L^3_{\alpha md}N_{m\beta c} = 0\end{aligned} \tag{A.3}$$

$$C_{\alpha\beta\gamma}K_{\gamma cd} + 3b_{\alpha\mu}F_{\mu\beta dc} - K_{\alpha cm}Q^2_{m\beta d} = 0 \tag{A.4}$$

$$2b_{\alpha\mu}F_{\mu\beta cd} - L^2_{\alpha mc}N_{m\beta d} + F_{\alpha\beta cm}\Lambda_{md} + \{c,d\} = 0 \tag{A.5}$$

$$\begin{aligned}
&-b_{\alpha\mu}F_{\mu\beta dc} + 2K_{\alpha cm}Q^1_{m\beta d} + K_{\alpha cm}Q^2_{m\beta d} \\
&-L^1_{\alpha mc}L_{m\beta d} - L^1_{\alpha mc}N_{m\beta d} + F_{\alpha\beta db}\Lambda_{bc} \\
&-L^2_{\alpha md}L_{m\beta c} = 0
\end{aligned} \tag{A.6}$$

$$K_{\alpha cm}Q^1_{m\beta d} - L^1_{\alpha mc}L_{m\beta d} + [c,d] = 0 \tag{A.7}$$

$$K_{\alpha cm}T_{m\beta\mu d} - P_{\alpha m\mu c}N_{m\beta d} + [c,d] = 0 \tag{A.8}$$

$$2K_{\alpha cm}T_{m\beta\mu d} + [c,d] = 0 \tag{A.9}$$

$$3C_{\alpha\mu\gamma}F_{\gamma\beta dc} - K_{\alpha cm}T_{m\beta\mu d} - F_{\alpha\beta cm}L_{m\mu d} = 0 \tag{A.10}$$

$$3C_{\alpha\mu\gamma}F_{\gamma\beta dc} - K_{\alpha cm}T_{m\beta\mu d} - F_{\alpha\beta\ cm}L_{m\mu d} = 0 \tag{A.11}$$

$$3C_{\alpha\mu\gamma}F_{\gamma\beta cd} - K_{\alpha mc}T_{m\beta\mu d} - F_{\alpha\beta\ md}N_{m\mu c} = 0 \tag{A.12}$$

$$b_{\alpha\gamma}F_{\gamma\beta cd} - F_{\alpha\beta md}\Lambda_{mc} = 0 \tag{A.13}$$

$$F_{\alpha\beta md}L_{m\mu c} + F_{\alpha\beta md}N_{m\mu c} - 3C_{\alpha\mu\gamma}F_{\gamma\beta cd} = 0 \tag{A.14}$$

$$R_{\alpha m\mu c}N_{m\beta d} + [c,d] = 0 \tag{A.15}$$

$$K_{\alpha cm}T_{m\beta\mu d} - F_{\alpha\beta md}L_{m\mu c} = 0 \tag{A.16}$$

$$F_{\alpha\mu cd}b_{\mu\beta} - F_{\alpha\beta cm}\Lambda_{md} = 0 \tag{A.17}$$

$$3F_{\alpha\gamma cd}C_{\gamma\mu\beta} + S_{\alpha mc\beta}N_{m\mu d} - F_{\alpha\beta cm}L_{m\mu d} = 0 \tag{A.18}$$

$$F_{\alpha\beta cm}(N_{c\mu d} - L_{c\mu d}) = 0 \tag{A.19}$$

$$L^3_{\alpha mc}(N_{m\mu d} - L_{m\mu d}) = 0 \tag{A.20}$$

$$P_{\alpha m\mu c}(N_{m\mu d} - L_{m\mu d}) = 0 \tag{A.21}$$

$$R_{\alpha m\mu c}(N_{m\mu d} - L_{m\mu d}) = 0 \tag{A.22}$$

$$S_{\alpha mc\mu}(N_{m\mu d} - L_{m\mu d}) = 0 \tag{A.23}$$

$$\begin{aligned} &L^1_{\alpha bc}\Lambda_{cd} - b_{\alpha\mu}L^1_{\mu bd} - 3b_{\alpha\mu}L^2_{\mu bd} - 3b_{\alpha\mu}L^3_{\mu bd} \\ &-2b_{\alpha\mu}K_{\mu bd} = 0 \end{aligned} \tag{A.24}$$

$$\begin{aligned} &L^1_{\alpha bc}L_{c\mu d} + R_{\alpha b\mu c}\Lambda_{cd} - 3b_{\alpha\gamma}P_{\gamma b\mu d} - b_{\alpha\gamma}R_{\gamma b\mu d} \\ &-3C_{\alpha\gamma\mu}L^1_{\gamma bd} - 2K_{\alpha dm}Z^1_{mb\mu} - K_{\alpha dm}Z^2_{mb\mu} \\ &+L^1_{\alpha md}L_{m\mu b} + L^1_{\alpha md}N_{m\mu b} + C_{\alpha\gamma\mu}K_{\gamma bd} = 0 \end{aligned} \tag{A.25}$$

$$\begin{aligned} &L^1_{\alpha bc}N_{c\mu d} + S_{\alpha bd\beta}b_{\beta\mu} - b_{\alpha\gamma}S_{\gamma bd\mu} - 3C_{\alpha\mu\gamma}L^1_{\gamma bd} \\ &-3C_{\alpha\mu\gamma}L^2_{\gamma bd} - 3b_{\alpha\gamma}P_{\gamma m\mu d} + L^2_{\alpha md}N_{m\mu b} = 0 \end{aligned} \tag{A.26}$$

$$L^2_{\alpha bc}\Lambda_{cd} - b_{\alpha\mu}L^2_{\mu bd} - 3b_{\alpha\mu}L^3_{\mu bd} - b_{\alpha\mu}K_{\mu bd} = 0 \tag{A.27}$$

$$
\begin{aligned}
&L^2_{\alpha bc}L_{c\mu d}+L^2_{\alpha bc}N_{c\mu d}-S_{\alpha bd\beta}b_{\beta\mu}-R_{\alpha b\mu c}\Lambda_{cd}\\
&-6b_{\alpha\gamma}P_{\gamma b\mu d}-2b_{\alpha\gamma}S_{\gamma bd\mu}-2b_{\alpha\gamma}R_{\gamma b\mu d}-3C_{\alpha\gamma\mu}L^2_{\gamma bd}\\
&+L^2_{\alpha md}L_{m\mu b}-C_{\alpha\gamma\mu}K_{\gamma bd}=0
\end{aligned}
\tag{A.28}
$$

$$
\begin{aligned}
&L^2_{\alpha bc}L_{c\mu d}-3b_{\alpha\gamma}P_{\gamma b\mu d}-3b_{\alpha\gamma}R_{\gamma b\mu d}-K^{\alpha dm}Z^1_{mb\mu}\\
&-2K^{\alpha dm}Z^2_{mb\mu}+L^1_{\alpha cd}L_{c\mu b}=0
\end{aligned}
\tag{A.29}
$$

$$
\begin{aligned}
&L^2_{\alpha bc}N_{c\mu d}-3b_{\alpha\gamma}P_{\gamma b\mu d}-3b_{\alpha\gamma}S_{\gamma bd\mu}-3C_{\alpha\gamma\mu}L^2_{\gamma bd}\\
&-3C_{\alpha\gamma\mu}L^3_{\gamma bd}+K^{\alpha md}Z^1_{mb\mu}-2C_{\alpha\gamma\mu}K_{\gamma bd}\\
&+L^3_{\alpha cd}L_{c\mu b}=0
\end{aligned}
\tag{A.30}
$$

$$
\begin{aligned}
&3b_{\alpha\mu}L^1_{\mu bc}+3b_{\alpha\mu}L^2_{\mu bc}+b_{\alpha\mu}L^3_{\mu bc}-K_{\alpha mc}\Lambda_{mb}\\
&-L^3_{\alpha mc}\Lambda_{mb}=0
\end{aligned}
\tag{A.31}
$$

$$
3b_{\alpha\mu}L^1_{\mu bc}+b_{\alpha\mu}L^2_{\mu bc}-2b_{\alpha\mu}K_{\mu bc}-L^2_{\alpha mc}\Lambda_{mb}=0
\tag{A.32}
$$

$$
b_{\alpha\mu}L^1_{\mu bc}+K_{\alpha cm}\Lambda_{mb}-b_{\alpha\mu}K_{\mu bc}-L^1_{\alpha cm}\Lambda_{mb}=0
\tag{A.33}
$$

$$
\begin{aligned}
&b_{\alpha\mu}P_{\mu b\beta c}+3C_{\alpha\beta\mu}L^1_{\mu bc}+k_{\alpha cm}Z^1_{mb\beta}-2C_{\alpha\beta\mu}K_{\mu cb}\\
&-L^1_{\alpha mc}N_{m\beta b}-P_{\alpha m\beta c}\Lambda_{mb}=0
\end{aligned}
\tag{A.34}
$$

$$
L^3_{\alpha bc}\Lambda_{cd}-b_{\alpha\mu}L^3_{\mu bd}=0
\tag{A.35}
$$

$$L^3_{\alpha bc}L_{c\mu d}+2L^3_{\alpha bc}N_{c\mu d}+R_{\alpha b\mu c}\Lambda_{cd}-3b_{\alpha\gamma}P_{\gamma b\mu d}$$
$$-3b_{\alpha\gamma}S_{\gamma bd\mu}-b_{\alpha\gamma}R_{\gamma b\mu d}-3C_{\alpha\gamma\mu}L^3_{\gamma bd}$$
$$+K_{\alpha md}Z^2_{mb\mu}=0 \tag{A.36}$$

$$2L^3_{\alpha bc}L_{c\mu d}+L^3_{\alpha bc}N_{c\mu d}+S_{\alpha bd\beta}b_{\beta\mu}-3b_{\alpha\gamma}P_{\gamma b\mu d}$$
$$-b_{\alpha\gamma}S_{\gamma bd\mu}-3b_{\alpha\gamma}R_{\gamma b\mu d}=0 \tag{A.37}$$

$$L^3_{\alpha bc}L_{c\mu d}+P_{\alpha b\beta d}b_{\beta\mu}-b_{\alpha\gamma}P_{\gamma b\mu d}$$
$$-3b_{\alpha\gamma}R_{\gamma b\mu d}-K_{\alpha dm}Z^2_{mb\mu}=0 \tag{A.38}$$

$$L^3_{\alpha bc}N_{c\mu d}+P_{\alpha b\mu c}\Lambda_{cd}-b_{\alpha\gamma}S_{\gamma bd\mu}$$
$$-b_{\alpha\gamma}P_{\gamma b\mu d}-3C_{\alpha\gamma\mu}L^3_{\gamma bd}=0 \tag{A.39}$$

$$3P_{\alpha b\beta d}C_{\beta\mu\nu}+P_{\alpha b\mu c}L_{c\nu d}+R_{\alpha b\mu c}L_{c\nu d}$$
$$-3C_{\alpha\mu\gamma}P_{\gamma b\nu d}-3C_{\alpha\mu\gamma}R_{\gamma b\nu d}-3C_{\alpha\gamma\nu}P_{\gamma b\mu d}$$
$$-2K_{\alpha dm}Q_{mb\nu\mu}-K_{\alpha dm}Q_{mb\mu\nu}=0 \tag{A.40}$$

$$P_{\alpha b\beta c}K_{\beta mn}-2K_{\alpha md}M_{dbnc}+[m,c]=0 \tag{A.41}$$

$$P_{\alpha b\beta c}N_{c\mu d}+\frac{3}{2}S_{\alpha bd\gamma}C_{\gamma\beta\mu}-3C_{\alpha\mu\gamma}P_{\gamma b\beta d}$$
$$-3C_{\alpha\mu\gamma}S_{\gamma bd\beta}+\{\beta,\mu\}=0 \tag{A.42}$$

$$S_{\alpha bc\beta}\Lambda_{cd} - b_{\alpha\mu}S_{\mu bd\beta} = 0 \tag{A.43}$$

$$R_{\alpha b\beta d}b_{\beta\mu} - b_{\alpha\beta}R_{\beta b\mu d} = 0 \tag{A.44}$$

$$S_{\alpha bc\beta}L_{c\mu d} + S_{\alpha bc\beta}N_{c\mu d} - 3C_{\alpha\gamma\mu}S_{\gamma bd\beta} = 0 \tag{A.45}$$

$$S_{\alpha bc\beta}L_{c\mu d} - K_{\alpha dm}Q_{mb\mu\beta} = 0 \tag{A.46}$$

$$S_{\alpha bc\beta}N_{c\mu d} - 3C_{\alpha\gamma\mu}S_{\gamma bd\beta} + K_{\alpha md}Q_{mb\mu\beta} = 0 \tag{A.47}$$

$$S_{\alpha bd\beta}b_{\beta\mu} - S_{\alpha md\mu}\Lambda_{mb} = 0 \tag{A.48}$$

$$S_{\alpha bm\beta}K_{\beta cn} - K_{\alpha cd}M_{dbmn} + [m,n] = 0 \tag{A.49}$$

$$S_{\alpha bm\beta}K_{\beta cn} + [m,n] = 0 \tag{A.50}$$

$$F_{\alpha\beta cm}K_{\beta bn} + F_{\alpha\beta cn}K_{\beta bm} + S_{\alpha bc\beta}(K_{\beta mn} + K_{\beta nm}) = 0 \tag{A.51}$$

$$R_{\alpha b\beta d}K_{\beta mn} - K_{\alpha cn}M_{cbmd} = 0 \tag{A.52}$$

$$R_{\alpha b\beta d}K_{\beta mn} - K_{\alpha mc}M_{cbnd} = 0 \tag{A.53}$$

$$3R_{\alpha b\beta d}C_{\beta\mu\nu} - 3C_{\alpha\mu\gamma}R_{\gamma b\nu d} = 0 \tag{A.54}$$

$$R_{\alpha b\mu c}\Lambda_{cd} - R_{\alpha m\mu d}\Lambda_{mb} + C_{\alpha\gamma\mu}(K_{\gamma bd} - K_{\gamma db}) = 0 \tag{A.55}$$

$$R_{\alpha b\beta c}L_{c\mu d} - R_{\alpha c\beta d}L_{c\mu b} = 0 \tag{A.56}$$

$$R_{\alpha b\beta c}(L_{c\mu d} - N_{c\mu d}) = 0 \tag{A.57}$$

$$3C_{\alpha\nu\mu}P_{\mu b\beta c} + K_{\alpha cm}Q_{mb\beta\nu} - P_{\alpha m\beta c}L_{m\nu b} + \{\beta, \nu\} = 0 \tag{A.58}$$

$$\begin{aligned} &F_{\alpha\beta cm}K_{\beta bn} + F_{\alpha\beta cn}K_{\beta bm} + S_{\alpha bc\beta}(K_{\beta mn} + K_{\beta nm}) \\ &-K_{\alpha md}M_{dbcn} = 0 \end{aligned} \tag{A.59}$$

$$F_{\alpha\mu cd}(K_{\mu mb} - K_{\mu bm}) = 0 \tag{A.60}$$

$$\begin{aligned} &S_{\alpha bc\beta}(K_{\beta mn} - K_{\beta nm}) + F_{\alpha\beta cm}K_{\beta bn} - F_{\alpha\beta cn}K_{\beta bm} \\ &2F_{\alpha\beta cm}K_{\beta bn} - F_{\alpha\beta cn}K_{\beta bm} + [m, d] = 0 \end{aligned} \tag{A.61}$$

$$L^3_{\alpha mn}(L_{m\nu b} - N_{m\nu b}) = 0 \tag{A.62}$$

$$P_{\alpha m\beta n}(L_{m\nu b} - N_{m\nu b}) = 0 \tag{A.63}$$

$$S_{\alpha mn\beta}(L_{m\nu b} - N_{m\nu b}) = 0 \tag{A.64}$$

$$R_{\alpha m\beta n}(L_{m\nu b} - N_{m\nu b}) = 0 \tag{A.65}$$

$$\frac{3}{2}S_{\alpha bn\gamma}C_{\gamma\beta\mu} - S_{\alpha mn\beta}L_{m\mu b} + \{\mu,\beta\} = 0 \tag{A.66}$$

$$Q^1_{a\beta c}\Lambda_{cd} - \Lambda_{ac}Q^1_{c\beta d} - 3\Lambda_{ac}Q^2_{c\beta d} + \Lambda_{ac}N_{c\beta d} = 0 \tag{A.67}$$

$$\begin{aligned} &Q^1_{a\beta c}L_{c\mu d} - 3L_{a\gamma d}C_{\gamma\beta\mu} - 3\Lambda_{ac}T_{c\beta\mu d} \\ &-L_{a\mu c}Q^1_{c\beta d} + 3Q^1_{a\gamma d}C_{\gamma\beta\mu} + Z^2_{ac\mu}N_{c\beta d} = 0 \end{aligned} \tag{A.68}$$

$$\begin{aligned} &Q^1_{a\beta c}N_{c\mu d} - T_{a\beta\mu c}\Lambda_{cd} - 2N_{a\gamma d}C_{\gamma\beta\mu} - 2\Lambda_{ac}T_{c\beta\mu d} \\ &-N_{a\mu c}Q^1_{c\beta d} - N_{a\mu c}Q^2_{c\beta d} + 3Q^2_{a\gamma d}C_{\gamma\beta\mu} \\ &+Z^1_{ac\mu}N_{c\beta d} = 0 \end{aligned} \tag{A.69}$$

$$Q^2_{a\beta c}\Lambda_{cd} - \Lambda_{ac}Q^2_{c\beta d} = 0 \tag{A.70}$$

$$\begin{aligned} &Q^2_{a\beta c}N_{c\mu d} + Q^2_{a\beta c}L_{c\mu d} - N_{a\gamma d}C_{\gamma\beta\mu} - 3\Lambda_{ac}T_{c\beta\mu d} \\ &-L_{a\mu c}Q^2_{c\beta d} = 0 \end{aligned} \tag{A.71}$$

$$Q^2_{a\beta c}L_{c\mu d} - L_{a\gamma d}C_{\gamma\beta\mu} - 3\Lambda_{ac}T_{c\beta\mu d} = 0 \tag{A.72}$$

$$Q^2_{a\beta c}N_{c\mu d} + T_{a\beta\mu c}\Lambda_{cd} - \Lambda_{ac}T_{c\beta\mu d} - N_{a\mu c}Q^2_{c\beta d} = 0 \tag{A.73}$$

$$T_{a\beta\gamma d}b_{\gamma\mu} - \Lambda_{ac}T_{c\beta\mu d} = 0 \tag{A.74}$$

$$3T_{a\beta\gamma d}C_{\gamma\mu\nu} - L_{a\nu c}T_{c\beta\mu d} - N_{a\nu c}T_{c\beta\mu d} = 0 \tag{A.75}$$

$$Z^2_{ad\mu}(L_{d\beta c} - N_{d\beta c}) = 0 \tag{A.76}$$

$$T_{a\beta\gamma d}K_{\gamma mn} - L_{a\gamma m}F_{\gamma\beta nd} = 0 \tag{A.77}$$

$$T_{a\beta\mu m}\Lambda_{md} - T_{a\gamma\mu d}b_{\gamma\beta} = 0 \tag{A.78}$$

$$T_{a\beta\gamma d}L_{d\mu m} - N_{a\gamma d}T_{d\beta\mu m} + \{\gamma.\mu\} = 0 \tag{A.79}$$

$$T_{a\beta\gamma d}L_{d\mu m} - 3C_{a\nu\gamma m}C_{\nu\beta\mu} - Q_{ad\gamma\mu}N_{d\beta m} = 0 \tag{A.80}$$

$$T_{a\beta\gamma d}(L_{d\mu m} - N_{d\mu m}) = 0 \tag{A.81}$$

$$L_{a\mu d}b_{\mu\beta} - Q^1_{a\mu d}b_{\mu\beta} + \Lambda_{ac}Q^1_{c\beta d} - \Lambda_{ac}L_{c\beta d} = 0 \tag{A.82}$$

$$\begin{aligned}&2L_{a\gamma d}C_{\gamma\beta\mu} + \Lambda_{ac}T_{c\beta\mu d} + N_{a\mu c}Q^1_{c\beta d}\\ &-3Q^1_{c\gamma d}C_{\gamma\mu\beta} - T_{a\beta\mu m}\Lambda_{md} - Z^1_{ac\mu}L_{c\beta d} = 0\end{aligned} \tag{A.83}$$

$$L_{a\gamma c}F_{\gamma\beta md} + [c,d] = 0 \tag{A.84}$$

$$\begin{aligned}&N_{a\gamma d}b_{\gamma\beta} - Q^2_{a\gamma d}b_{\gamma\beta} + 3\Lambda_{ac}Q^1_{c\beta d} + \Lambda_{ac}Q^2_{c\beta d}\\ &-2\Lambda_{ac}L_{c\beta d} - \Lambda_{ac}N_{c\beta d} = 0\end{aligned} \tag{A.85}$$

$$N_{a\gamma m}F_{\gamma\beta nd} + [m,n] = 0 \tag{A.86}$$

$$M_{admn}(L_{d\beta c} - N_{d\beta c}) = 0 \tag{A.87}$$

$$3\Lambda_{ac}Q^1_{c\beta d} + 3\Lambda_{ac}Q^2_{c\beta d} - \Lambda_{ac}L_{c\beta d} - 2\Lambda_{ac}L_{c\beta d} = 0 \tag{A.88}$$

$$Q_{ad\mu\nu}(L_{d\beta c} - N_{d\beta c}) = 0 \tag{A.89}$$

$$M_{anmc}N_{n\beta d} - M_{anmd}N_{n\beta c} = 0 \tag{A.90}$$

$$Z^1_{ab\mu}b_{\mu\nu} - \Lambda_{ac}Z^1_{cb\nu} - 3\Lambda_{ac}Z^2_{cb\nu} + \Lambda_{ac}L_{cb\nu} = 0 \tag{A.91}$$

$$\begin{aligned}&3Z^1_{ab\gamma}C_{\gamma\mu\nu} - Q_{ab\mu\gamma}b_{\gamma\nu} - N_{a\mu c}Z^1_{cb\nu} - N_{a\mu c}Z^2_{cb\nu}\\&+Z^1_{ac\mu}L_{c\nu b} + Z^2_{ac\mu}L_{c\nu b} - 2\Lambda_{ac}Q_{cb\mu\nu} - 3\Lambda_{ac}Q_{cb\nu\mu}\\&-L_{a\nu c}Z^1_{cb\mu} = 0\end{aligned} \tag{A.92}$$

$$\begin{aligned}&Z^1_{ab\mu}K_{\mu cd} - L_{a\gamma c}L^2_{\gamma bd} - L_{a\gamma c}L^3_{\gamma bd} + 3\Lambda_{am}M_{mbdc}\\&-Q^1_{a\gamma c}K_{\gamma bd} = 0\end{aligned} \tag{A.93}$$

$$Z^2_{ab\mu}b_{\mu\nu} - \Lambda_{ac}Z^2_{cb\nu} = 0 \tag{A.94}$$

$$3Z^2_{ab\gamma}C_{\gamma\mu\nu} - 3\Lambda_{ac}Q_{cb\mu\nu} - L_{a\mu c}Z^2_{cb\nu} + \{\mu,\nu\} = 0 \tag{A.95}$$

$$\begin{aligned}&3Z^2_{ab\gamma}C_{\gamma\mu\nu} + Q_{ab\mu\gamma}b_{\gamma\nu} - N_{a\mu c}Z^2_{cb\nu} - 3\Lambda_{ac}Q_{cb\mu\nu}\\&-\Lambda_{ac}Q_{cb\mu\nu} = 0\end{aligned} \tag{A.96}$$

$$Z^2_{ab\mu}K_{\mu cd} + M_{abcm}\Lambda_{md} - N_{a\gamma c}L^3_{\gamma bd}$$

$$+3\Lambda_{am}M_{mbdc} - \Lambda_{am}M_{mbcd} = 0 \tag{A.97}$$

$$Z^2_{ab\mu}K_{\mu cd} - L_{a\gamma c}L^3_{\gamma bd} + 3\Lambda_{am}M_{mbdc} = 0 \tag{A.98}$$

$$\Lambda_{am}M_{mbcd} - M_{abmd}\Lambda_{mc} = 0 \tag{A.99}$$

$$M_{abmc}(L_{m\mu d} + N_{m\mu d}) - N_{a\gamma c}R_{\gamma b\mu d} - L_{a\gamma c}M_{mbcd} = 0 \tag{A.100}$$

$$M_{abmd}L_{m\mu c} - L_{a\gamma c}R_{\gamma b\mu d} = 0 \tag{A.101}$$

$$M_{abcm}(L_{m\mu d} - N_{m\mu d}) = 0 \tag{A.102}$$

$$M_{abcm}N_{m\mu d} - Q_{ab\beta\gamma}K_{\gamma dc} + L_{a\gamma d}(P_{\gamma b\beta c} + S_{\gamma bc\beta})$$

$$-N_{a\gamma c}P_{\gamma b\beta d} - N_{a\beta m}M_{mbcd} = 0 \tag{A.103}$$

$$M_{abcm}\Lambda_{md} + 2\Lambda_{am}M_{mbcd} + N_{a\gamma c}L^2_{\gamma bd}$$

$$+Q^2_{a\gamma c}K_{\gamma bd} + [c,d] = 0 \tag{A.104}$$

$$M_{abcm}\Lambda_{md} + Q^2_{a\gamma c}K_{\gamma bd} - Q^2_{a\gamma c}K_{\gamma db}$$

$$-M_{amcd}\Lambda_{mb} = 0 \tag{A.105}$$

$$Q_{ab\gamma\nu}b_{\gamma\mu} - \Lambda_{ac}Q_{cb\mu\nu} = 0 \tag{A.106}$$

$$3Q_{ab\mu\nu}C_{\mu\beta\gamma} - N_{a\beta c}Q_{cb\gamma\nu} - L_{a\beta c}Q_{cb\gamma\nu} = 0 \tag{A.107}$$

$$Q_{ab\mu\nu}K_{\mu cd} - L_{a\gamma c}S_{\gamma bd\nu} = 0 \tag{A.108}$$

$$Q_{ab\mu\nu}b_{\nu\beta} - Q_{ac\mu\beta}\Lambda_{cb} = 0 \tag{A.109}$$

$$Q_{ab\mu\nu}K_{\mu cd} - Q_{ab\mu\nu}K_{\mu dc} + [c,d] = 0 \tag{A.110}$$

$$Q_{ab\mu\nu}(K_{\mu cd} - K_{\mu dc}) - T_{a\gamma\mu c}K_{\gamma bd} + T_{a\gamma\mu d}K_{\gamma bc} + [c,d] = 0 \tag{A.111}$$

$$3Q_{ab\mu\nu}C_{\nu\beta\gamma} - 2Q_{ac\mu\beta}L_{c\gamma b} + \{\beta,\gamma\} = 0 \tag{A.112}$$

$$\begin{aligned} &L_{a\gamma c}L^1_{\gamma bd} + L_{a\gamma c}L^2_{\gamma bd} - N_{a\gamma d}L^1_{\gamma bc} \\ &-\Lambda_{am}M_{mbcd} + Q^2_{a\gamma d}K_{\gamma bc} + M_{abdm}\Lambda_{mc} = 0 \end{aligned} \tag{A.113}$$

$$L_{a\gamma c}L^1_{\gamma bd} - Q^1_{a\gamma c}K_{\gamma db} + [c,d] = 0 \tag{A.114}$$

$$L_{a\gamma c}P_{\gamma b\beta d} + L_{a\gamma c}R_{\gamma b\beta d} + [c,d] = 0 \tag{A.115}$$

$$N_{a\gamma c}S_{\gamma bd\beta} + [c,d] = 0 \tag{A.116}$$

$$\frac{3}{2}Q_{ab\mu v}C_{v\beta\gamma} - N_{a\mu c}Q_{cb\beta\gamma} + \{\mu,\beta,\gamma\} = 0 \tag{A.117}$$

$$3\Lambda_{ac}Z^1_{cb\beta} + 3\Lambda_{ac}Z^2_{cb\beta} - 2\Lambda_{ac}L_{c\beta b} - \Lambda_{ac}N_{c\beta b} = 0 \tag{A.118}$$

$$
\begin{aligned}
&3\Lambda_{ac}Z^1_{cb\beta} + \Lambda_{ac}Z^2_{cb\beta} - 2\Lambda_{ac}N_{c\beta b} - \Lambda_{ac}L_{c\beta b} \\
&+L_{a\beta c}\Lambda_{cb} - Z^2_{ac\beta}\Lambda_{cb} = 0
\end{aligned} \tag{A.119}
$$

$$
\Lambda_{ac}Z^1_{cb\beta} - \Lambda_{ac}N_{c\beta b} - Z^1_{ac\beta}\Lambda_{cb} + N_{a\beta c}\Lambda_{cb} = 0 \tag{A.120}
$$

$$
\begin{aligned}
&\Lambda_{ac}Q_{cb\mu\nu} + N_{a\mu c}Z^1_{cb\nu} - Z^1_{ac\nu}N_{c\mu b} \\
&-Q_{ab\mu\beta}b_{\beta\nu} + \{\mu,\nu\} = 0
\end{aligned} \tag{A.121}
$$

$$
T_{a\gamma\beta c}K_{\gamma bd} - T_{a\gamma\beta c}K_{\gamma db} = 0 \tag{A.122}
$$

$$
\begin{aligned}
&2(L_{a\gamma c}P_{\gamma b\beta d} - L_{a\gamma d}P_{\gamma b\beta c}) + Q_{ab\mu\nu}(K_{\nu cd} - K_{\nu dc}) \\
&-(T_{a\gamma\mu c}K_{\gamma bd} - T_{a\gamma\mu d}K_{\gamma bc}) + [c,d] = 0
\end{aligned} \tag{A.123}
$$

$$
Z^2_{ac\beta}(L_{c\mu d} - N_{c\mu d}) = 0 \tag{A.124}
$$

$$
M_{acmn}(L_{c\mu d} - N_{c\mu d}) = 0 \tag{A.125}
$$

$$
Q_{ac\gamma\beta}(L_{c\mu d} - N_{c\mu d}) = 0 \tag{A.126}
$$

$$
M_{abmd}N_{m\mu c} - N_{a\mu m}M_{mbcd} = 0 \tag{A.127}
$$

$$
T_{a\gamma\mu c}K_{\gamma bd} + T_{a\gamma\mu d}K_{\gamma bc} + Q_{ab\mu\nu}(K_{\nu cd} + K_{\nu dc}) = 0 \tag{A.128}
$$