BAYESIAN INFERENCE IN ANOVA MODELS

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ABSTRACT

BAYESIAN INFERENCE IN ANOVA MODELS

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Estimation of location and scale parameters from a random sample of size n is of paramount importance in Statistics. An estimator is called fully efficient if it attains the Cramer-Rao minimum variance bound besides being unbiased. The method that yields such estimators, at any rate for large n, is the method of modified maximum likelihood estimation. Apparently, such estimators cannot be made more efficient by using sample based classical methods. That makes room for Bayesian method of estimation which engages prior distributions and likelihood functions. A formal combination of the prior knowledge and the sample information is called posterior distribution. The posterior distribution is maximized with respect to the unknown parameter(s). That gives HPD (highest probability density) estimator(s). Locating the maximum of the posterior distribution is, however, enormously difficult (computationally and analytically) in most situations. To alleviate these difficulties, we use modified likelihood function in the posterior distribution instead of the likelihood function. We derived the HPD estimators of location and scale parameters of distributions in the family of Generalized Logistic. We have extended the work to experimental design, one way ANOVA. We have obtained the HPD estimators of the block effects and the scale parameter (in the distribution of errors); they have beautiful algebraic forms. We have shown that they are highly efficient. We have given real life examples to illustrate the usefulness of our results. Thus, the enormous computational and analytical difficulties with the traditional Bayesian method of estimation are circumvented at any rate in the context of experimental design.

Key Words: Modified maximum likelihood, Bayesian estimation, Prior distribution, Posterior distribution, Experimental design.

ÖZ

ANOVA MODELLERİNE BAYESIAN YAKLAŞIM

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Boyutu n olan rassal örneklemden gelen konum ve ölçek parametrelerinin tahmini İstatistikte büyük öneme sahiptir. Bir tahmin edici, yansız olmakla birlikte, Cramer_Rao en küçük varyans sınırına erişirse bütünüyle etkin olarak adlandırılır. Bu özelliklere sahip tahmin edicileri veren yöntem uyarlanmış en çok olabilirlik yöntemidir. Görüldüğü gibi, bu tahmin ediciler, örneklem bazlı en çok olabilirlik yöntemleri gibi klasik analizler ile daha etkin hale getirilemezler. Bu durum öncül olasılık dağılımları ile olabilirlik fonksiyonunu birleştiren Bayesci yöntemlere olanak vermektedir. Öncül dağılımın ve örneklemden elde edilen bilginin formal kombinasyonu soncul dağılım olarak adlandırılır. Soncul dağılım bilinmeyen parametrelere göre maksimize edilir. Bu işlem sonucunda en yüksek soncul olasılık yoğunluk (HPD) tahmin edicileri elde edilir. Fakat çoğu durumda soncul dağılımın maksimum noktasını bulmak analitik ve hesapsal açılardan oldukça zorlayıcı olabilir. Bu sorunları aşabilmek için, soncul dağılımda olabilirlik fonksiyonu yerine uyarlanmış olabilirlik fonksiyonunu kullandık. Genelleştirilmiş Lojistik dağılımlarında konum ve ölçek parametleri için HPD tahmin edicileri türettik. Bu analizleri tek yönlü deney tasarımı çalışmaları ile genişlettik. Blok etkileri ve (hata terimlerinin dağılımındaki) ölçek parametresi için cebirsel olarak uygun formda olan HPD tahmin edicileri elde ettik. Bu tahmin edicilerin yüksek derecede etkin olduklarını gösterdik. Sonuçlarımızın yararlılığını göstermek adına gerçek hayattan elde edilen veriler ile örnekler verdik. Böylece deney tasarımında geleneksel Bayesci tahmin yöntemlerinde karşılaşılan analitik ve hesaplama zorluklarını aşmış olduk.

Anahtar Kelimeler: Uyarlanmış En Çok Olabilirlik, Bayesci Tahmin, Öncül Dağılım, Soncul Dağılım, Deney Tasarımı.

To my family for their unconditional love....

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TABLE OF CONTENTS

LIST OF TABLES

TABLES

LIST OF FIGURES

FIGURES

CHAPTER 1

INTRODUCTION and LITERATURE SURVEY

The main purpose of statistical theory is to derive inferences for unknown parameters under a specified model. Basically, there are two different approaches to statistics. Classical statistics, also called the frequentist approach, deals with parametric statistical models in which random variables are assumed to be distributed according to a function $f(y,\theta)$ where the parameter θ is unknown and fixed. On the other hand, Bayesian statisticians argue that data is not the only source of information about the underlying population. Since true values of parameters are unknown, they may be considered as random variables. Therefore, Bayesian methods include external information to the analysis by considering a model for the random variable y with pdf (probability density function) $f(y,\theta)$ where θ is unknown and not fixed.

In Bayesian framework the parameter θ is treated as a random variable and has a distribution itself, which is called prior distribution. Prior distribution has a major role to play because it represents the information about the unknown parameter before the data are observed. By using the prior distribution and the data itself, it is possible to obtain the posterior distribution.

1.1. Basic Definitions:

Theorem: Prior distributions are combined with the information obtained from sample data and updated to posterior distributions. The theorem for this process is called Bayes theorem stated by Thomas Bayes in 1764. Given the data y and the prior information about θ, the posterior distribution $p(\theta \mid y)$ can be written as the product of the prior distribution $p(\theta)$ and the likelihood function $L(y \mid \theta)$. Formally,

$$p(\theta \mid y) \propto L(y \mid \theta) p(\theta). \tag{1.1}$$

Posterior distribution contains all the information about an unknown parameter. It combines the sampling information and information from the prior distribution (which depends on previous experiences). It is possible to make inferences about the unknown parameter(s) by using the posterior densities.

In order to obtain posterior densities, one can choose informative or non informative priors, proper or improper.

1.1.1. Noninformative Prior Distributions:

Non informative prior distributions give equal, or nearly equal, weights to all θ values. Such distributions may also be called as flat or diffuse priors. Uniform distribution is an obvious choice for a noninformative prior. Generally, the most favored non-informative prior used by Bayesians is Jeffreys' prior. Jeffreys' principle gives the non informative prior density as $p(\theta) \propto |J(\theta)|^{1/2}$ where $J(\theta)$ is the Fisher information for θ (Marin and Robert, 2007). There is considerable work on Jeffreys' prior. For example, Ibrahim and Laud (1991) use the Jeffreys' prior with generalized linear models and show that proper posterior are produced. Hartigan (1983) studies different exponential family distributions and gives their associated Jeffreys' priors. Poirer (1994) works with logit model using Jeffrey's prior and Kass (1989) produces the full properties of these priors with geometric interpretation.

Non informative prior distributions have been investigated by many statisticians. Jeffreys (1961) and Hartigan (1964) state invariance principles for noninformative prior distributions. Box and Tiao (1973), Berger (1985), Bernardo (1979) present some definitions and discussions about them. Barnard (1985) examines the relationship between pivotal quantities and noninformative Bayesian inference. Kass and Wasserman (1996) discuss the ways of obtaining noninformative priors based on Jeffres' rule. Dawid, Stone and Zidek (1973) point out the difficulties by using these priors.

1.1.2. Proper and Improper Prior Distributions:

A prior density $p(\theta)$ is called improper if it is non-negative for all θ values but $\int p(\theta)d\theta = \infty$. Otherwise, it is called proper prior (Gelman et al., 2004). The main distinction between proper and improper priors is that **i**mproper priors may lead to improper posterior distributions. For large sample sizes this problem disappears but in some cases it remains the same, therefore the resulting posterior distributions should be investigated more carefully when an improper prior is used.

1.1.3. Informative Prior Distributions:

Informative or informed priors incorporate all the information about the parametes. The information may come from experience.

1.1.3.1.Conjugate Prior Distributions:

The main difficulty of Bayesian anlaysis is that the posterior distribution may not be in analytically convenient form. In order to alleviate this difficulty, a conjugate prior may be used. Let P be a class of prior distributions for θ. The class P is said to be conjugate to a class of sampling distributions $p(y \mid \theta)$ if the resulting posterior distribution $p(\theta \mid y)$ is in the same family as $p(\theta)$. Conjugate families are practical to use and mathematically convenient since posterior distributions have known parametric forms (Gill, 2008). Although, they are good starting points, in some cases it is not possible to use conjugate distributions because of their complicated forms.

Remark: If there exists a sufficient statistics $T(y)$ then the posterior distribution can be written as, $p(\theta \mid y) \propto f(y \mid \theta) p(\theta) \propto g(T(y) \mid \theta) p(\theta)$ which implies that posterior distribution depends on the sampling distribution through sufficient statistics.

Remark: Probability distributions that belong to exponential family have conjugate prior distributions. (Gelman et al., 2004).

Bayesian way of thinking has been applied to almost all statistical problems. In this thesis, we are fundemantally interested in the estimation of unknown parameters coming from different symmetric and skewed families. Gill (2008) considers Bayesian linear regression models by assuming different prior distributions. Lindley and Smith (1972) deal with Bayesian linear regression while Geweke (1993) investigate the regression model by using error terms having t distribution and shows that when the error terms are not normal, complex solutions will result. While investigating Bayesian regression, most of the researchers assume homoscedasticity. However, Leonard (1975), Mouchart and Simar (1984), Le Cam (1986) deal with heteroscedasticity. Inference problems like hypothesis testing is also possible with Bayesian approaches. A good discussion of hypothesis testing is given by Marden (2000). Lindley (1961) develops a procedure that provides two sided hypothesis testing which coincidences with classical methods. Moreover, Berger et al. (1994,1997) and Lee (2004) deal with Bayesian hypothesis. Bayesian inference is discussed more generally by Jeffreys (1961), Zellner (1971), Box and Tiao (1973). Bayesian techniques are also used in nonparametric analyses. Bernardo and Smith (1994), Dey et al. (1998), Walker et al. (1999) deal with nonparametric Bayesian, a brief review can be found in Sinha and Dey (1997).

The purpose of this thesis is to show that most of the analytical and computational difficulties with Bayesian methodology can be alleviated by using Tiku's method of parameter estimation. The results one gets are simple and amazingly highly efficient.

1.2. Bayesian Estimation with Single Parameter Models: Normal Distribution with Unknown Mean but Known Variance

Consider a sample $y_1, y_2, \ldots, y_n$ assumed to come from a normal distribution with unknown mean μ and known variance σ^2:

$$p(y_i \mid \mu) \; \alpha \; \exp(-\frac{1}{2\sigma^2}(y_i - \mu)^2), -\infty < y, \mu < \infty. \tag{1.2}$$

In order to find Bayesian estimator of unknown mean μ, assume a conjugate prior density $p(\mu)$,

$$p(\mu) = \frac{1}{2\pi\sigma_0^2}\exp(-\frac{1}{2\sigma_0^2}(\mu - \mu_0)^2), \qquad -\infty < \mu < \infty. \tag{1.3}$$

The hyperparameters of the prior distribution μ_0 and σ_0^2 are assumed to be known. The posterior density is,

$$p(\mu \mid y) \propto p(\mu) p(y \mid \mu) = p(\mu) \prod_{i=1}^{n} p(y_i \mid \mu)$$

$$\propto \exp\left(-\frac{1}{2} \left[\frac{1}{\sigma_0^2} \left(\mu - \mu_0^2 \right) + \frac{1}{\sigma^2} \sum_{i=1}^{n} \left(y_i - \mu \right)^2 \right] \right)$$

(1.4)

which can be written as

$$p(\mu \mid y) \propto \exp\left(-\frac{1}{2\sigma^2 \sigma_0^2} \left[\mu^2 \left(n\sigma_0^2 + \sigma^2 \right) - 2\mu \left[n\bar{y}\sigma_0^2 + \sigma^2 \mu_0 \right] \right] \right). \qquad (1.5)$$

Making equation (1.5) a complete square by adding and subtracting $\left(\frac{n\bar{y}\sigma_0^2 + \sigma^2 \mu_0}{\sigma_0^2 + \sigma^2} \right)^2$ we get

$$p(\mu \mid y) \propto \exp\left(-\frac{1}{2\sigma_0^2} \left(n\sigma_0^2 + \sigma^2 \right) \left[\mu - \frac{n\bar{y}\sigma_0^2 + \sigma^2 \mu_0}{\sigma_0^2 + \sigma^2} \right]^2 \right). \qquad (1.6)$$

The conjugate prior distribution implies that the posterior distribution of μ is normal density also. After some algebraic simplification, the posterior density $p(\mu \mid y)$ can be written as

$$p(\mu \mid y) \propto \exp\left(-\frac{1}{2\sigma_1^2} \left(\mu - \mu_1 \right)^2 \right) \qquad (1.7)$$

where

$$\mu_1 = \frac{\frac{1}{\sigma_0^2}\mu_0 + \frac{n}{\sigma^2}\bar{y}}{\frac{1}{\sigma_0^2} + \frac{n}{\sigma^2}} = w\mu_0 + (1-w)\bar{y} \ ,$$

and

$$\frac{1}{\sigma_1^2} = \frac{1}{\sigma_0^2} + \frac{n}{\sigma^2}. \qquad (1.8)$$

Remark: The posterior mean is a weighted average of sample mean and prior mean with weights proportional to inverse of the variances. The posterior mean μ_1 may also be expressed as adjusted prior mean and sample mean as

$$\mu_1 = \mu_0 + n(\bar{y} - \mu_0)\frac{\sigma_0^2}{\sigma^2 + n\sigma_0^2}. \qquad (1.9)$$

Alternatively,

$$\mu_1 = \bar{y} - (\bar{y} - \mu_0)\frac{\sigma^2}{\sigma^2 + n\sigma_0^2}. \qquad (1.10)$$

Moreover, as the number of observations increases, posterior mean approaches $\bar{y}$. Therefore, the asymptotic distribution of μ_1 is normal with mean $\bar{y}$ and variance σ^2 / n. In addition to this, Gill (2008) states that large values of prior variance will yield the same results as the frequentists approach.

The variances of posterior density and $\bar{y}$ can be compared by computing the relative efficiency $RE = \frac{Var(\mu_1)}{Var(\bar{y})}$. The numerical results are given in Table 1.1 below for $\tau^2 = \sigma^2{}_0 / \sigma^2$.

Table 1.1 RE values for normal distribution with unknown mean but known variance

τ^2	2	4	7	20
n=3	0.8571	0.9230	0.9545	0.9836
n=5	0.9090	0.9523	0.9722	0.9900
n=10	0.9523	0.9756	0.9859	0.9950
n=50	0.9900	0.9950	0.9971	0.9990

Remark: From Table 1.1, it is seen that as τ^2 increases, relative efficiency increases. In other words, Bayesian estimator of μ looses efficiency as prior distribution variance becomes large. On the other hand, if sample variance increases, the Bayesian estimator of mean μ becomes more efficient than the sample mean.

Remark: It is seen from Table 1.1 that as sample size increases, likelihood dominates over the prior information. Therefore, Bayes estimator is not advantageous as compared to sample mean when number of observations is large.

1.3. Bayesian Estimation with Multiparameter Models:

1.3.1 Normal Distribution with Unknown Mean and Unknown Variance: A Non-Informative Prior:

Consider normal distribution with unknown mean μ and unknown variance σ^2 and assume a non informative prior density as,

$$p(\mu,\sigma^2) \propto \left(\sigma^2\right)^{-1} \tag{1.11}$$

Combining the likelihood with (1.11) the joint posterior density $p(\mu,\sigma^2 \mid y)$ is obtained as,

$$p(\mu,\sigma^2 \mid y) \propto \sigma^{-(n+2)} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\mu)^2\right)$$

$$= \sigma^{-(n+2)} \exp\left(-\frac{1}{2\sigma^2}\left[(n-1)s^2 + n(\bar{y}-\mu)^2\right]\right) . \qquad (1.12)$$

Note that, we can determine the posterior density of μ given σ^2 with uniform prior density as normal $(\bar{y},\sigma^2/n)$. Moreover, the marginal posterior density of σ^2 can be found by integrating (1.12) as

$$p(\sigma^2 \mid y) \propto \int \sigma^{-(n+2)} \exp\left(-\frac{1}{2\sigma^2}\left[(n-1)s^2 + n(\bar{y}-\mu)^2\right]\right) d\mu$$

$$\propto \left(\sigma^2\right)^{-(n+1)/2} \exp\left(-\frac{(n-1)s^2}{2\sigma^2}\right) \qquad (1.13)$$

which is a scaled inverse chi square distribution with $n-1$ degrees of freedom. As a result, the joint posterior density (1.12), is factorized as $p(\mu,\sigma^2 \mid y) = p(\mu \mid \sigma^2, y)\,p(\sigma^2 \mid y)$. In order to draw samples from joint posterior density (1.12) first samples should be selected from (1.13) and then from conditional posterior density of μ which is normal $(\bar{y},\sigma^2/n)$. Moreover, if unconditional posterior density of μ is to be obtained,

$$p(\mu \mid y) = \int p(\mu,\sigma^2 \mid y)\, d\sigma^2 \qquad (1.14)$$

which is obtained as

$$p(\mu \mid y) \propto \left[1+\frac{n(\mu-\bar{y})^2}{(n-1)s^2}\right]^{-n/2}. \qquad (1.15)$$

1.3.2. Normal Distribution with Unknown Mean and Unknown Variance: Dependent Prior Distributions of μ and σ^2

Consider normal distribution with unknown mean μ and unknown variance σ^2 with prior distributions specified by (1.16) and (1.17) below:

$$\mu \mid \sigma^2 \sim Normal(\mu_0, \sigma^2 / \kappa_0) \qquad (1.16)$$

and

$$\sigma^2 \sim Inv-\chi^2(v_0, \sigma_0^2). \qquad (1.17)$$

The dependent prior distribution of μ on σ^2 implies that they will have joint conjugate prior density as

$$p(\mu, \sigma^2) \propto \sigma^{-1}(\sigma^2)^{-(v_0/2+1)} \exp\left(-\frac{1}{2\sigma^2}\left[v_0\sigma_0^2 + \kappa_0(\mu_0-\mu)^2\right]\right). \qquad (1.18)$$

Gelman et al. (2004) calls density (1.18) as $N-Inv-\chi^2(\mu_0, \sigma_0^2/\kappa_0; v_0, \sigma_0^2)$. Multiplying (1.18) with the likelihood function yields the joint posterior distribution:

$$p(\mu, \sigma^2 \mid y) \propto \sigma^{-1}(\sigma^2)^{-(v_0/2+1)} \exp\left(-\frac{1}{2\sigma^2}\left[v_0\sigma_0^2 + \kappa_0(\mu-\mu_0)^2\right]\right) \times$$
$$(\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2}\left[(n-1)s^2 + n(\bar{y}-\mu)^2\right]\right) \qquad (1.19)$$

which is $N-Inv-\chi^2(\mu_n), \sigma_n^2/\kappa_n; v_n, \sigma_n^2)$ where $\mu_n = \frac{\kappa_0}{\kappa_0+n}\mu_0 + \frac{n}{\kappa_0+n}\bar{y}$, $\kappa_n = \kappa_0 + n$, $v_n = v_0 + n$, $v_n\sigma_n^2 = v_0\sigma_0^2 + (n-1)s^2 + \frac{\kappa_0 n}{\kappa_0+n}(\bar{y}-\mu_0)^2$.

Remark: It is seen that posterior distribution uses information from prior distribution and sample distribution with appropriate weights. It can easily be shown that μ_n is the weighted average of prior mean and sample mean.

One can show that, the conditional density of $\mu \mid \sigma^2, y$ is $Normal\left(\frac{\frac{\kappa_0}{\sigma^2}\mu_0 + \frac{n}{\sigma^2}\bar{y}}{\frac{\kappa_0}{\sigma^2}+\frac{n}{\sigma^2}}, \frac{1}{\frac{\kappa_0}{\sigma^2}+\frac{n}{\sigma^2}}\right)$ and marginal posterior density of $\sigma^2 \mid y$ is $Inv-\chi^2(v_n, \sigma_n^2)$. Moreover, the marginal posterior of μ obtained by integrating (1.19) over σ^2, is obtained as $p(\mu \mid y) \propto \left[1+\frac{\kappa_n(\mu-\mu_n)^2}{v_n\sigma_n^2}\right]^{-(v_n+1)/2}$.

1.3.3. Normal Distribution with Unknown Mean and Unknown Variance: Independent Prior Distributions of μ and σ^2

Consider normal distribution with unknown mean μ and unknown variance σ^2 with independent conjugate prior distributions as $p(\mu) \sim Normal(\mu_0, \sigma_0^2)$ and $p(\sigma^2) \sim Inv-Gamma\left(\frac{\delta_0}{2}, \frac{\delta_0 s_0^2}{2}\right)$. The joint posterior distribution of μ and σ^2 can be written as

$$p(\mu,\sigma^2 \mid y) = L(y \mid \mu,\sigma^2)p(\mu,\sigma^2)$$

$$\propto (\sigma^{-2})^{\frac{n+\delta_0}{2}+1} \exp\left(-\frac{1}{2\sigma^2}(\delta_0 s_0^2 + n\hat{\sigma}^2)\right)\times$$

$$\exp\left(-\frac{n}{2\hat{\sigma}^2}(\hat{\mu}-\mu)^2 - \frac{1}{2\sigma_0^2}(\mu-\mu_0)^2\right). \qquad (1.20)$$

From joint posterior density (1.20) it is easy to show that,

$$p(\sigma^2 \mid y) \sim scaled - Inv - \chi^2\left(n+\delta_0, \frac{\delta_0 s_0^2 + n\hat{\sigma}^2}{n+\delta_0}\right) \qquad (1.21)$$

and

$$p(\mu \mid y) \sim Normal\left(\frac{\frac{\mu_0}{\sigma_0^2}+\frac{n\hat{\mu}}{\hat{\sigma}^2}}{\frac{1}{\sigma_0^2}+\frac{n}{\hat{\sigma}^2}}, \frac{\hat{\sigma}^2\sigma_0^2}{\hat{\sigma}^2+n\sigma_0^2}\right) \qquad (1.22)$$

Remark. It can be seen from the marginal posterior density (1.22) that the precision of μ is equal to the sum of the precision of prior distribution and the precision of the sample.

Remark: It can be stated from (1.22) that posterior mean μ is a weighted average of sample mean and prior mean with weights proportional to inverse of the variances.

Relative efficiency values of Bayes estimator of μ, denoted as μ_1, with respect to sample estimator $\hat{\mu}$ with respect to different sample sizes is given by Table 1.2 below.

Table 1.2 RE values for normal distribution with unknown mean and unknown variance: independent prior distributions

		σ =1.5	σ =2.5
n=5	σ_0=2 σ_0=3	0.8183 0.9102	0.6213 0.7855
n=10	σ_0=2 σ_0=3	0.8974 0.9530	0.7556 0.8777
n=15	σ_0=2 σ_0=3	0.9291 0.9675	0.8248 0.9135
n=20	σ_0=2 σ_0=3	0.9466 0.9757	0.8624 0.9345
n=30	σ_0=2 σ_0=3	0.9635 0.9835	0.9049 0.9550

Remark: Table 1.2 indicates that, Bayes estimator of μ will be less efficient if the variance of prior distribution becomes large, and it gains efficiency when sample dispersion increases.

Remark: As sample size increases, likelihood dominates and Bayes estimator of mean is not advantageous. This result is also the same as the results shown by Table 1.1 given in section 1.2.

CHAPTER 2

BAYESIAN ANALYSES OF SYMMETRIC and SKEWED FAMILIES ONE SAMPLE CASE

In this chapter, location parameters of some symmetric and skewed distributions are estimated by using both classical and Bayesian techniques. First, we apply Modified Maximum Likelihood (MML) estimation method and find MML estimators (MMLE) of unknown parameters. After that, posterior densities are provided by considering parameters as random variables and assuming appropriate robust priors for them. Highest posterior density (HPD) estimators are derived from resulting posterior distributions. Relative efficiencies of MMLEs and Bayes estimators are compared and presented in the following pages. We start with symmetric distributions and extend our analyzes to skewed distributions. It may be noted that maximum likelihood estimators are intractable in most situations and that makes Bayesian methodology enormously problematic.

2.1. Symmetric Distributions:

2.1.1.Type II Censored Samples Coming from Normal Distribution

2.1.1.1. MML Estimators:

Consider a symmetric type II censored sample from normal distribution with mean μ and variance σ^2. After censoring smallest and largest r

observations, remaining observations can be displayed in terms of order statistics as $y_{(r+1)},\ldots,y_{(n-r)}$. In order to find Maximum Likelihood (ML) estimators of unknown parameters, the likelihood function L is

$$L \propto \sigma^{(n-2r)} F(z_{(r+1)})^r \left[1 - F(z_{(n-r)})\right]^r \prod_{i=r+1}^{n-r} f\left(z_{(i)}\right) \tag{2.1}$$

where $z_{(i)} = \dfrac{y_{(i)} - \mu}{\sigma}$, $r+1 \le i \le n-r$, $F(z) = \int_{-\infty}^{z} f(z)dz$ and $f(z) \propto e^{-\frac{z^2}{2}}$.

The derivatives of lnL which give ML estimators are

$$\frac{d\ln L}{d\mu} = \frac{1}{\sigma}\left[\sum_{i=r+1}^{n-r} z_{(i)} - rg_1\left(z_{(r+1)}\right) + rg_2\left(z_{(n-r)}\right)\right] = 0 \tag{2.2}$$

and

$$\frac{d\ln L}{d\sigma} = \frac{1}{\sigma}\left[-\left(n-2r\right) + \sum_{i=r+1}^{n-r} z_i^2 - rz_{(r+1)} g_1\left(z_{(r+1)}\right) + rz_{(n-r)} g_2\left(z_{(n-r)}\right)\right] = 0 \tag{2.3}$$

where $g_1(z) = f(z)/F(z)$ and $g_2(z) = f(z)/(1-F(z))$ (Tiku, 1967).

These equations are very difficult to solve since they involve non-linear functions $g_i(z)$. They have no explicit solutions and iterative methods have to be used to obtain ML estimators. Newton-Raphson method can be applied to solve them (Schneider,1986) but the resulting estimators will be implicit and difficult to use (Tiku and Stewart,1977). Moreover, iterative methods may yield multiple roots, iterations may not converge or they may converge to wrong values, see Barnett (1966), Lee et al.(1980), and Vaughan(1992). Puthenpura and Sinha (1986) also indicate that iterative solutions might not converge if the data contains outliers; see also Qumsiyeh (2007, pp.8-14). In order to alleviate these difficulties, method of modified maximum likelihood estimation (Tiku (1967, 1968a,b,c, 1970, 1973), Tiku and Suresh,1992) is used which gives explicit solutions. The intractable terms in likelihood equations are linearized and resulting equations yield MML estimators. They are highly efficient and asymptotically equivalent to the ML estimators (Bhattacharyya, 1985; Vaughan and Tiku, 2000) and they maintain high efficiencies for small samples also (Tiku and Suresh, 1992; Vaughan, 1992, 2002). Tan (1985) and Tan and Balakrishnan (1986) study MML method in Bayesian point of view in case of censored normal samples.

In order to formulate MML equations, the functions $g_1\left(z_{(r+1)}\right)$ and $g_2\left(z_{(n-r)}\right)$ can be linearized in the vicinity of z (Tiku,1967) such that, $g_1\left(z_{(r+1)}\right) \cong \alpha - \beta z_{(r+1)}$ and $g_2\left(z_{(n-r)}\right) \cong \alpha + \beta z_{(n-r)}$. From Taylor series expansion, the coefficients α and β are found as

$$\beta = \frac{f(t)}{q}\left(t + \frac{f(t)}{q}\right), \quad \alpha = \frac{f(t)}{q} + t\beta \tag{2.4}$$

where $q = r/n$, $\int_{-\infty}^{t} f(z)dz = 1 - q$. Note that, α and β are both positive and between 0 and 1. (Tiku and Akkaya, 2004).

MML equations are obtained by incorporating (2.4) in (2.2) and (2.3) as

$$\frac{d \ln L^*}{d\mu} = \frac{1}{\sigma}\left[\sum_{i=r+1}^{n-r} z_{(i)} - r\left(\alpha - \beta z_{(r+1)}\right) + r\left(\alpha - \beta z_{(n-r)}\right)\right]$$

$$= \frac{m}{\sigma^2}\left(K - \mu\right) = 0 \tag{2.5}$$

and

$$\frac{d\ln L^*}{d\sigma} = \frac{1}{\sigma}\left[-(n-2r) + \sum_{i=r+1}^{n-r} z_i^2 - rz_{(r+1)}\left(\alpha - \beta z_{(r+1)}\right) + rz_{(n-r.)}\left(\alpha - \beta z_{(n-r)}\right)\right]$$

$$= -\frac{1}{\sigma^3}\left[\left(A\sigma^2 - B\sigma - C\right) - m(K-\mu)^2\right] = 0 \qquad (2.6)$$

where

$$A = n - 2r\,,\ m = n - 2r + 2r\beta\,,\ K = \frac{\sum_{i=r+1}^{n-r} y_{(i)} + r\beta\left(y_{(r+1)} + y_{(n-r)}\right)}{m}\,,$$

$$B = r\alpha\left(y_{(n-r)} - y_{(r+1)}\right)$$

and

$$C = \sum_{i=r+1}^{n-r} y_{(i)}^2 + r\beta\left[y_{(r+1)}^2 - y_{(n-r)}^2\right] - mK^2. \qquad (2.7)$$

The solutions of MML equations give MML estimators as

$$\hat{\mu}_{MML} = \frac{\sum_{i=r+1}^{n-r} y_{(i)} + r\beta\left(y_{(r+1)} + y_{(n-r)}\right)}{m} \qquad (2.8)$$

and,

$$\hat{\sigma}_{MML} = \left[B + \sqrt{B^2 + 4AC}\right] / 2\sqrt{A(A-1)} \qquad (2.9)$$

Note that, MML estimators $\hat{\mu}_{MML}$ and $\hat{\sigma}_{MML}$ are asymtotically fully efficient since likelihood and modified likelihood equations are asymptotically equivalent.

Remark: Asymptotically, MMLE $\hat{\sigma}_{MML}$ is unbiased and $Var(\hat{\sigma}_{MML}) \cong (\sigma^2 / 2A)[1 - r\alpha t / A]^{-1}$.

Remark: $\hat{\mu}_{MML}$ is unbiased for all n and independent of $\hat{\sigma}_{MML}$. It is easy to show that $\frac{d \ln L}{d\mu} \cong \frac{d \ln L^*}{d\mu} = \frac{m}{\sigma^2}(\hat{\mu}_{MMl} - \mu)$ and $E(\partial^{r+s} \ln L / \partial\mu^r \partial\sigma^s) = 0$ for all $r, s \geq 1$. Moreover, the minumum variance bound for estimating μ is $\sigma^2 / (n - 2r)$ for large n (Tiku and Akkaya, 2004).

Comment: The complete sample results can readily be obtained by taking $r = 0$, of course, certain regularity conditions have to be satisfied. If the variances of estimators are compared between complete sample and censored sample, it is seen that censoring observations can improve efficiencies if miscreant observations (e.g.,outliers) occur in the data.

2.1.1.2. Posterior Distribution:

In order to find posterior distribution of μ and σ^2 given sample observations, first modified likelihood equation L^* is obtained (by solving the two differential equations (2.5) and (2.6)):

$$L^* \propto \sigma^{-A} \exp\left[-\frac{1}{2\sigma^2}\left(A\hat{\sigma}^2_{MML}\right) + m\left(\hat{\mu}_{MML} - \mu\right)^2 \right] H(y) \qquad (2.10)$$

where $H(y)$ is a function free of μ and σ. Then, the joint distribution of $\hat{\mu}_{MML}$ and $\hat{\sigma}_{MML}$ can be written as

$$f(\hat{\mu},\hat{\sigma}^2) \propto \left(\sigma^{-2}\right)^{\frac{A-1}{2}} \left(\hat{\sigma}^2_{MML}\right)^{\frac{A}{2}-1} \exp\left(-\frac{A}{2\sigma^2}\hat{\sigma}^2_{MML}\right)\times$$

$$\exp\left(-\frac{m}{2\hat{\sigma}^2_{MML}}\left(\hat{\mu}_{MML}-\mu\right)^2\right) \tag{2.11}$$

since $\hat{\mu}_{MML}$ and $\hat{\sigma}_{MML}$ are independently distributed.

The priors for μ and σ^2 are taken as independent normal and inverse chi-square distribution as

$$p(\mu) \propto \left(\sigma_0^2\right)^{-\frac{1}{2}} \exp\left(-\frac{1}{2\sigma_0^2}\left(\mu-\mu_0\right)^2\right) \tag{2.12}$$

and

$$p(\sigma^2) \propto \left(\sigma^{-2}\right)^{\frac{\delta_0}{2}+1} \exp\left(-\frac{1}{2\sigma^2}\delta_0 s_0^2\right) \tag{2.13}$$

where μ_0, σ_0^2 and δ_0 are hyperparamaters of prior distributions. Then,

$$p\left(\mu,\sigma^2\right) = p(\mu)p\left(\sigma^2\right) \tag{2.14}$$

and

$$p\left(\mu,\sigma^2 \middle| y\right) \propto p\left(\mu,\sigma^2\right)L^*\left(y \middle| \mu,\sigma^2\right). \tag{2.15}$$

From (2.15), the joint posterior density can be expressed as

$$p\left(\mu,\sigma^2 \middle| y\right) \propto \left(\sigma^{-2}\right)^{\frac{A-1+\delta_0}{2}+1} \exp\left[-\frac{1}{2\sigma^2}\left(\delta_0 s_0^2 + A\hat{\sigma}^2\right)\right]\times$$

$$\exp\left[-\frac{m}{2\hat{\sigma}^2}(\hat{\mu}-\mu)^2-\frac{1}{2\sigma_0^2}(\mu-\mu_0)^2\right] \quad (2.16)$$

where $\hat{\mu}$ and $\hat{\sigma}^2$ are MMLE of μ and σ^2.

2.1.1.3. HPD Estimators:

The estimators given by mode of posterior densities are called HPD (Highest Posterior Density) estimators. From joint posterior density (2.16), it is easy to show that

$$p(\mu|y)\sim Normal\left(\frac{m\hat{\mu}\sigma_0^2+\mu_0\hat{\sigma}^2}{m\sigma_0^2+\hat{\sigma}^2},\frac{\hat{\sigma}^2\sigma_0^2}{m\sigma_0^2+\hat{\sigma}^2}\right) \quad (2.17)$$

and,

$$p(\sigma^2|y)\sim InverseGamma\left(\frac{A-1+\delta_0}{2},\frac{\delta_0 s_0^2+A\hat{\sigma}^2}{2}\right). \quad (2.18)$$

2.1.1.4. Comparing Efficiencies of MML and Bayes Estimators: Simulation Results

In order to evaluate the performence of Bayesian estimator and MML estimator, simulated relative efficiencies, $RE=Var(\hat{\mu}_{Bayes})/Var(\hat{\mu}_{MML})$,are calculated and given in Table 2.1 below. Symmetric type II censoring is considered with fixed q=r/n. The observations are assumed to have normal distribution with $\mu=0$ having likelihood function (2.1). HPD estimator of μ given by equation (2.17) is calculated by assuming (2.12) as prior density with hyperparameters $\mu_0=0$ and $\sigma_0=3$. Using IMSL algorithms 10,000 simulations are performed for different sample sizes n.

Table 2.1 Simulated means, variances and RE values for censored normal distribution with $\sigma = 1.5$

$q = 0.2$		$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
$n = 5$	$\sigma_0 = 2.0$	-0.0037	0.5174	-0.0031	0.4285	0.8281
	$\sigma_0 = 2.5$	0.0040	0.4934	0.0033	0.4331	0.8778
$n = 10$	$\sigma_0 = 2.0$	-0.0044	0.2428	-0.0043	0.2169	0.8934
	$\sigma_0 = 2.5$	-0.0013	0.2482	-0.0011	0.2308	0.9301
$n = 15$	$\sigma_0 = 2.0$	0.0064	0.1668	0.0062	0.1548	0.9280
	$\sigma_0 = 2.5$	0.0029	0.1676	0.0028	0.1595	0.9513
$n = 20$	$\sigma_0 = 2.0$	-0.0040	0.1234	-0.0039	0.1163	0.9427
	$\sigma_0 = 2.5$	-0.0021	0.1230	0.0021	0.1183	0.9623
$n = 30$	$\sigma_0 = 2.0$	-0.0041	0.0850	-0.0040	0.0817	0.9612
	$\sigma_0 = 2.5$	-0.0007	0.0817	-0.0007	0.0796	0.9746

Table 2.2 Simulated means, variances and RE values for censored normal distribution with $\sigma_0 = 3$

$q = 0.2$		$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
$n = 5$	$\sigma = 1.5$	0.0017	0.5234	0.0015	0.4773	0.9119
	$\sigma = 2.5$	0.0055	1.4100	0.0050	1.1309	0.8020
$n = 10$	$\sigma = 1.5$	-0.0026	0.2517	-0.0024	0.2392	0.9503
	$\sigma = 2.5$	-0.0116	0.6995	-0.0107	0.6123	0.8754
$n = 15$	$\sigma = 1.5$	-0.0037	0.1688	-0.0036	0.1630	0.9657
	$\sigma = 2.5$	-0.0013	0.4671	-0.0015	0.4248	0.9095
$n = 20$	$\sigma = 1.5$	-0.0046	0.1249	-0.0045	0.1216	0.9736
	$\sigma = 2.5$	-0.0073	0.3421	-0.0071	0.3184	0.9308
$n = 30$	$\sigma = 1.5$	-0.0003	0.0839	0.0003	0.0824	0.9821
	$\sigma = 2.5$	0.0030	0.2277	0.0029	0.2167	0.9518

Table 2.1 shows the values of means, variances and relative efficiencies while sample observations are coming from normal distribution with $\sigma = 1.5$. In order to see the impact of prior dispersion on the effectiveness of Bayesian estimator, two different σ_0 values are considered. It can be concluded that when prior dispersion increases, relative effiency value increases, that means MML estimator gains efficiency. Table 2.2 is constructed to show the changes in relative efficiencies with respect to different sample dispersions. Therefore, means and variances of estimators are calculated by assuming prior density (2.12) with $\mu_0 = 0$ and $\sigma_0 = 3$ and sample observations are assumed to have $\mu = 0$ and $\sigma = 1.5$ and $\sigma = 2.5$. According to the results in Table 2.2 we can say that as sample dispersion increases relative efficiency decreases, which indicates that Bayes estimator gains efficiency relative to MMLE. Moreover, both Table 2.1 and Table 2.2 show that increasing sample size causes increase in relative efficiencies. That is to say, MML estimator gets close to HPD estimators as number of observation increases which is in agreement with our earlier statements given in Chapter I.

Comment: One of the most important aspects of Bayesian techiques is that one can obtain estimators whose variances are smaller than the minum variance bound (MVB). By combining prior information with likelihood function, one can cross the MVB barrier which is not possible at all in classicals statistical analyses.

2.1.2.Type II Censored Samples Coming from LTS Distributions

2.1.2.1. MML Estimators:

It has been a tradition to assume a normal distribution but as indicated by Geary (1947) and Scheffé (1959), normality assumption may not be very realistic. In practice, it may be more reasonable to assume that underlying distribution belongs to a family which includes a wide class of symmetric distributions. Therefore, in this part of the study, we consider type II symmetric censored sample $y_{(r+1)},\dots,y_{(n-r)}$ coming from the distribution

$$f(y)=\frac{1}{\sigma\sqrt{k}\beta(1/2,p-1/2)}\left\{1+\frac{(y-\mu)^2}{k\sigma^2}\right\}^{-p}, \qquad -\infty<y<\infty, \qquad (2.19)$$

where $k=2p-3$ and $p\geq 2$. We assume p is known. It is easy to show that the ratio $t=\dfrac{\sqrt{v}(y-\mu)}{\sigma\sqrt{k}}$ has a Student's t distribution with degree of freedom $v=2p-1$. Moreover, (2.18) is a normal distribution for $p=\infty$.

In order to find ML estimators, the likelihood equation can be expressed as,

$$L\propto\sigma^{-A}\left[\prod_{i=r+1}^{n-r}\left(1+\frac{z_{(i)}^2}{k}\right)^{-p}\right]F\left(z_{(r+1)}\right)^r\left(1-F\left(z_{(n-r)}\right)\right)^r \qquad (2.20)$$

where $A=n-2r$, $z_{(i)}=\dfrac{y_{(i)}-\mu}{\sigma}$ and $F(z)=\int_{-\infty}^{z}f(z)dz$.

The derivatives of log of L are

$$\frac{d\ln L}{d\mu}=\frac{2p}{k\sigma}\sum_{i=r+1}^{n-r}g\left(z_{(i)}\right)-\frac{r}{\sigma}h_1\left(z_{(r+1)}\right)+\frac{r}{\sigma}h_2\left(z_{(n-r)}\right)=0 \qquad (2.21)$$

and

$$\frac{d\ln L}{d\sigma}=-\frac{A}{\sigma}+\frac{2p}{k\sigma}\sum_{i=r+1}^{n-r}z_{(i)}g\left(z_{(i)}\right)-\frac{r}{\sigma}z_{(r+1)}h_1\left(z_{(r+1)}\right)+\frac{r}{\sigma}z_{(n-r)}h_2\left(z_{(n-r)}\right)=0$$

(2.22)

where $g(z)=\dfrac{z}{1+z^2/k}$, $h_1(z)=\dfrac{f(z)}{F(z)}$ and $h_2(z)=\dfrac{f(z)}{1-F(z)}$.

Since these equations do not have explicit solutions, solving them iteratively may have convergence and multiple root problems especially for small values of p. Therefore, MML estimators are used instead of MLEs. MML equations are obtained by linearizing the intractable functions as $g(z_{(i)})\cong\alpha_{(i)}+\beta_{(i)}z_{(i)}$, $h_1(z_{(r+1)})\cong a-bz_{(r+1)}$ and $h_2(z_{(n-r)})\cong a+bz_{(n-r)}$. The coefficients of these functions can be obtained from first two terms of Taylor series expansion around $t_{(i)}=E(z_{(i)})$. The estimators are found by solving modified likelihood equations:

$$\hat{\mu}_{MML}=\frac{1}{M}\left(\frac{2p}{k}\sum_{i=r+1}^{n-r}\beta_i y_{(i)}+rb\left[y_{(r+1)}+y_{(n-r)}\right]\right)$$

and

$$\hat{\sigma}_{MML}=\frac{1}{2A}\left[B+\sqrt{B^2+4AC}\right] \qquad (2.23)$$

where

$$A=n-2r\,,\; M=\frac{2p}{k}\sum_{i=r+1}^{n-r}\beta_i+2rb\,,\; B=\frac{2p}{k}\sum_{i=r+1}^{n-r}\alpha_i y_{(i)}+ra\left[y_{(n-r)}-y_{(r+1)}\right]$$

and

$$C=\frac{2p}{k}\sum_{i=r+1}^{n-r}\beta_i y_{(i)}^2+rb\left[y_{(r+1)}^2+y_{(n-r)}^2\right]-M\hat{\mu}^2. \qquad (2.24)$$

Moreover, $\alpha_i = \dfrac{(2/k)t_{(i)}^3}{\left[1+(1/k)t_{(i)}^2\right]^2}$, $\beta_i = \dfrac{1-(1/k)t_{(i)}^2}{\left[1+(1/k)t_{(i)}^2\right]^2}$, $a = \left[\dfrac{f(t_{(r+1)})}{F(t_{(r+1)})}\right] + bt_{(r+1)}$ and $b = \left[-f'(t_{(r+1)})F(t_{(r+1)}) + f^2(t_{(r+1)})\right]F^{-2}(t_{(r+1)})$. We can extend these results to complete samples by taking $r = 0$.

Note: It may be noted that if $r = 0$, β_1 (and few other β_i coefficients) can be negative if n is large and p is small. Consequently, C can be negative resulting in an irrational estimator of σ. To rectify this situation we replace α_i and β_i by $\alpha_i^* = \dfrac{(1/k)t_{(i)}^3}{\left[1+(1/k)t_{(i)}^2\right]^2}$ and $\beta_i^* = \dfrac{1}{\left[1+(1/k)t_{(i)}^2\right]^2}$, respectively, if $\beta_1 < 0$. This does not affect the asymptotic results since $g(z_{(i)}) \cong \alpha_i^* + \beta_i^* z_{(i)}$.

Tiku and Suresh (1992) and Vaughan (1992) indicate that for fixed $q = r/n$, $\hat{\mu}_{MML}$ and $\hat{\sigma}_{MML}$ are asypmtotically minumum variance bound estimators. For small n also they are highly efficient. Note also that $\hat{\mu}_{MML}$ is unbiased for all $n-2r$ and $\mathrm{cov}(\hat{\mu},\hat{\sigma}) = 0$. This result follows from symmetry: see Tiku et al. (1986).

Remark: As stated by Tiku and Akkaya (2004), for large A (with fixed r/n), the asymptotic properties of MML estimators are as follows:

i) $\hat{\mu}_{MML}$ is minumum variance bound estimator of μ and $Var(\hat{\mu}_{MML}) \cong \sigma^2 / M$,

ii) $A\hat{\sigma}^2/\sigma^2$ is distributed as chi-square with A-1 degrees of freedom,

iii) $\hat{\mu}_{MML}$ and $\hat{\sigma}_{MML}$ are distributed independently.

Also, $\frac{\sqrt{M}\left(\hat{\mu}_{MML}-\mu\right)}{\sigma}$ is a standard normal variate and, therefore, $T=\frac{\sqrt{M}\left(\hat{\mu}-\mu\right)}{\hat{\sigma}}$ is distributed as Student's t with $A-1$ degrees of freedom (asymptotically).

2.1.2.2. Prior Distributions:

In order to develop Bayesian estimators of of μ and σ^2, prior distribution for them should be specified. Traditionally, a normal-inverted gamma density is assumed such that

$$p(\mu|\sigma)\propto\sigma^{-1}\exp\left[-\frac{n}{2\sigma^2}\left(\mu-\mu_0^2\right)\right] \tag{2.25}$$

and

$$p\left(\sigma^2\right)\propto\left(\sigma^{-2}\right)^{\delta_0/2+1}\exp\left(-\frac{1}{2\sigma^2}\delta_0 s_0^2\right). \tag{2.26}$$

The joint prior distribution is

$$p\left(\mu,\sigma^2\right)\propto p(\mu|\sigma)p\left(\sigma^2\right). \tag{2.27}$$

However, conjugate prior for a normal distribution may not be robust to outliers. As stated by Stone (1964), Tiao and Zellner (1964), Dickey (1968, 1974), Berger (1984), Bian (1989), O'Hagan (1990), Angers and Berger (1991), Bian and Dickey (1996) assuming prior distributions which give higher probabilities to extreme values of μ will be more reliable. Such distributions are called robust priors. Therefore, instead of normal-inverted gamma density an independent t and inverted gamma distribution is assumed as

$$p\left(\mu,\sigma^2\right) \propto p(\mu)p\left(\sigma^2\right) \tag{2.28}$$

where the prior of σ^2 is given by (2.26) and the prior of μ is given by

$$p(\mu) \propto \left[1+\frac{(\mu-\mu_0)^2}{v_0\sigma_0^2}\right]^{-(v_0+1)/2}. \tag{2.29}$$

The prior distribution of μ reduces to a non informative prior if σ_0 is infinite and it becomes a normal distribution if v_0 is infinte (Bian and Tiku, 1997).

2.1.2.3. Posterior Distribution:

HPD estimators are derived from posterior densities of unknown parameters which are obtained by combining information from prior distributions and sample itself. In order to find joint posterior density, first joint distribution of $\hat{\mu}$ and $\hat{\sigma}^2$ is expressed as

$$f\left(\hat{\mu},\hat{\sigma}^2\right) \propto \left(\sigma^{-2}\right)^{\frac{A-1}{2}}\left(\hat{\sigma}^2\right)^{\frac{A}{2}-1}\exp\left(-\frac{A}{2\sigma^2}\hat{\sigma}^2\right)\exp\left(-\frac{M}{2\hat{\sigma}^2}(\hat{\mu}-\mu)^2\right). \tag{2.30}$$

Thus, the posterior density is obtained:

$$p\left(\mu,\sigma^2 \middle| y\right) \propto p(\mu)p\left(\sigma^2\right)f\left(\hat{\mu},\hat{\sigma}^2\right) \tag{2.31}$$

which is

$$p\left(\mu,\sigma^2 \middle| y\right) \propto \left(\sigma^{-2}\right)^{(\delta_0+A+1)/2-1}\exp\left(-\frac{\sigma^{-2}}{2}\left[\delta_0 s_0^2 + A\hat{\sigma}^2\right]\right)\times$$

$$\exp\left(-\frac{M}{2\hat{\sigma}^2}(\hat{\mu}-\mu)^2\right)\left[1+\frac{(\mu-\mu_0)^2}{v_0\sigma_0^2}\right]^{-(v_0+1)/2}. \tag{2.32}$$

It is seen from (2.32) that μ and σ^2 are posteriorly independent where the marginal posterior densities are

$$p(\mu|y) \propto \left[1 + \frac{(\mu - \mu_0)^2}{v_0 \sigma_0^2}\right]^{-\frac{(v_0+1)}{2}} \exp\left(-\frac{M}{2\hat{\sigma}^2}(\mu - \hat{\mu})^2\right) \qquad (2.33)$$

and

$$p(\sigma^2|y) \propto \left(\sigma^{-2}\right)^{\frac{\delta_0+A+1}{2}-1} \exp\left(-\frac{\sigma^{-2}}{2}\left[\delta_0 s_0^2 + A\hat{\sigma}^2\right]\right). \qquad (2.34)$$

2.1.2.4. HPD Estimators:

The HPD estimator of σ^2 is the mode of the scale inverse chi-square density given by (2.34),

$$\hat{\sigma}^2_{Bayes} \cong \frac{\left(\delta_0 s_0^2 + A\hat{\sigma}^2\right)}{\delta_0 + A} \qquad (2.35)$$

Remark: From equation (2.35) it is seen that Bayesian estimator of σ^2 is the weighted average of MML estimators and prior information.

In order to find HPD estimator of μ, the mode of the marginal posterior density of μ given by equation (2.33) should be found. However, it is a poly t density which includes a t and a normal factor. Poly t densities are usually bimodal and asymmetric, so the mode is different than their mean. Therefore, two different cases are considered in finding Bayesian estimator of μ. In the first case, the degree of freedom of prior distribution of μ is taken as infinite, and in the second case it is considered as finite.

Case1: When $v_0 = \infty$, marginal posterior density of μ become a product of two normal densities, since the prior distribution of μ expressed by (2.29) reduces to a normal distribution. Under this assumption the marginal posterior density of μ is

$$p(\mu|y) \propto \exp\left(-\frac{1}{2\sigma_0^2}(\mu - \mu_0)^2 - \frac{M}{2\hat{\sigma}^2}(\mu - \hat{\mu})^2 \right) \tag{2.36}$$

which yields a normal distribution with mean $\dfrac{\sigma_0^{-2}\mu_0 + M\hat{\sigma}^{-2}\hat{\mu}}{\sigma_0^{-2} + M\hat{\sigma}^{-2}}$ and variance $\dfrac{1}{\sigma_0^{-2} + M\hat{\sigma}^{-2}}$. Therefore we can write,

$$\hat{\mu}_{Bayes} = \frac{\sigma_0^{-2}\mu_0 + M\hat{\sigma}^{-2}\hat{\mu}}{\sigma_0^{-2} + M\hat{\sigma}^{-2}} \tag{2.37}$$

Comment: We may also express Bayesian estimator of μ as $\hat{\mu}_{Bayes} = w\mu_0 + (1-w)\hat{\mu}$ where $w = \sigma_0^{-2}\left(\sigma_0^{-2} + M\hat{\sigma}^{-2}\right)^{-1}$. From this expression it is clear that Bayes estimator of μ is a weighted (or convex) combination of prior mean and sample information. If M (i.e.,n) goes to infinity then $w = 0$. In which case Bayesian estimator of μ will reduce to MML estimator. On the other hand, Bayes estimator will be equal to the prior mean μ_0 when $w = 1$ and prior variance is zero which occurs in case $\sigma_0 = 0$.

Case2: When v_0 is finite, the posterior density of μ is

$$p(\mu|y) \propto \left[1 + (\mu - \mu_0)^2 / v_0\sigma_0^2\right]^{-\frac{(v_0+1)}{2}} \left[1 + \frac{n(\bar{y} - \mu)^2}{(n-1)s^2}\right]^{-\frac{n}{2}} . \tag{2.38}$$

In order to obtain the mode of the density of (2.38) the modal equation is obtained by taking derivative of $\ln p(\mu|y)$. The equation simplifies to

$$\ln p(\mu|y) \propto -\frac{(v_0+1)}{2}\ln\left[1+\frac{(\mu-\mu_0)^2}{v_0\sigma_0^2}\right]-\frac{M}{2\hat{\sigma}^2}(\mu-\hat{\mu})^2$$

$$=(v_0+1)\hat{\sigma}^2(\mu-\mu_0)+M(\mu-\hat{\mu})\left[v_0\sigma_0^2+(\mu-\mu_0)^2\right]. \qquad (2.39)$$

HPD estimator of μ will be found by equating the modal equation to zero and find the root for μ .

In order to solve (2.39), it is re-formulated as

$$x^3-(\Delta\mu)x^2+\left[\frac{(v_0+1)}{gM}+v_0\right]x-v_0(\Delta\mu)=0 \qquad (2.40)$$

where $x=\frac{\mu-\mu_0}{\sigma_0}$, $\Delta\mu=\frac{\hat{\mu}-\mu_0}{\sigma_0}$ and $g=\frac{\sigma_0^2}{\hat{\sigma}^2}$. Equation (2.40) is a cubic equation which will have either one or three real roots because conjugate roots occur in pairs. If the discriminant of (2.40) is greater than or equal to zero, we can say that it has only one real root, otherwise there will be three distinct real roots.

Let $a=-\Delta\mu$, $b=\left(\frac{(v_0+1)}{gM}+v_0\right)$ and $c=-v_0\Delta\mu$. The discriminant of equation (2.40) is

$$D=\frac{1}{27}\left[\frac{(v_0+1)}{gM}+v_0-\frac{1}{3}(\Delta\mu)^2\right]^3+\frac{1}{4}\left[-\frac{2}{27}(\Delta\mu)^3+\frac{1}{3}\Delta\mu\left(\frac{(v_0+1)}{gM}+v_0\right)-v_0\Delta\mu\right]^2. \qquad (2.41)$$

Re organization of (2.41) yields

$$27D = v_0(\Delta\mu)^4 + \left[2v_0^2 - \frac{5v_0(v_0+1)}{gM} - \left(\frac{(v_0+1)}{2gM}\right)^2\right](\Delta\mu)^2 + \left[\frac{(v_0+1)}{gM} + v_0\right]^3; \quad (2.42)$$

(2.42) is a quadratic function of $(\Delta\mu)^2$ and the discriminant of (2.42),

$$D_1 = \left[2v_0^2 - \frac{5v_0(v_0+1)}{gM} - \left(\frac{(v_0+1)}{2gM}\right)^2\right]^2 - 4v_0\left(\frac{(v_0+1)}{gM} + v_0\right)^3 \quad (2.43)$$

which can be simplified as

$$D_1 = \frac{(v_0+1)}{16gM}\left[\frac{(v_0+1)}{gM} - 8v_0\right]^3. \quad (2.44)$$

We can say that $D \geq 0$ and so posterior of μ is unimodal if $D_1 < 0$. From (2.44) it is seen that $D_1 < 0$ if $\frac{(v_0+1)}{gM} - 8v_0 < 0$ which implies $gM > \frac{1}{8} + \frac{1}{8v_0}$. In the extreme case of $v_0 = 1$, $D_1 > 0$ if and only if $\frac{\sigma_0^2}{\hat{\sigma}^2} M < 0.25$. However, since $\text{var}(\hat{\mu}/\sigma)$ is proportional to $1/M$. $M\frac{\sigma_0^2}{\hat{\sigma}^2}$ will hardly ever be smaller than 0.25. Therefore, it can be stated that modal equation will have one real root,almost always.

In order to find posterior estimator of μ, consider two cases:

Case1: $\hat{\mu}_{Bayes}$ is close to μ_0 :

The modal equation (2.39) is re-written as

$$(v_0+1)\hat{\sigma}^2 + Mv_0\sigma_0^2 = (\hat{\mu}-\mu_0)^2 \times$$

$$\left[\frac{M(\hat{\mu}_{Bayes}-\mu_0)}{(\hat{\mu}-\mu_0)} + \frac{Mv_0\sigma_0^2}{(\hat{\mu}_{Bayes}-\mu_0)(\hat{\mu}-\mu_0)} - \frac{M(\hat{\mu}_{Bayes}-\mu_0)^2}{(\hat{\mu}-\mu_0)^2}\right]. \quad (2.45)$$

Since $\hat{\mu}_{Bayes}$ is close to μ_0 we can ignore $\frac{(\hat{\mu}_{Bayes}-\mu_0)}{(\hat{\mu}-\mu_0)}$ and $\frac{(\hat{\mu}_{Bayes}-\mu_0)^2}{(\hat{\mu}-\mu_0)^2}$ and the resulting HPD estimator $\hat{\mu}_{Bayes}$ will be

$$\hat{\mu}_{Bayes} = \frac{\sigma_0^{-2}\left(1+\frac{1}{v_0}\right)\mu_0 + M\hat{\sigma}^{-2}\hat{\mu}}{\sigma_0^{-2}\left(1+\frac{1}{v_0}\right) + M\hat{\sigma}^{-2}} \quad (2.46)$$

which is a convex combination of μ_0 and $\hat{\mu}$.

Case2: $\hat{\mu}_{Bayes}$ is close to $\hat{\mu}$:

The modal equation (2.39) is re-written as

$$Mv_0\sigma_0^2 = (\mu_0-\hat{\mu})^2 \times$$

$$\left[\frac{2M(\hat{\mu}_{Bayes}-\hat{\mu})}{(\mu_0-\hat{\mu})} - \frac{M(\hat{\mu}_{Bayes}-\hat{\mu})^2}{(\mu_0-\hat{\mu})^2} - \frac{(v_0+1)\hat{\sigma}^2(\hat{\mu}_{Bayes}-\mu_0)}{(\hat{\mu}_{Bayes}-\mu_0)(\mu_0-\hat{\mu})^2} - M\right]. \quad (2.47)$$

Since $\hat{\mu}_{Bayes}$ is close to $\hat{\mu}$ we can ignore $\frac{(\hat{\mu}_{Bayes}-\hat{\mu})}{(\hat{\mu}-\mu_0)}$ and $\frac{(\hat{\mu}_{Bayes}-\hat{\mu})^2}{(\hat{\mu}-\mu_0)^2}$ and after some algebra (2.47) becomes

$$\hat{\mu}_{Bayes} \times \left[M v_0 \sigma_0^2 + M(\Delta\mu)^2 \sigma_0^2 + (v_0+1)\hat{\sigma}^2 \right] = M v_0 \sigma_0^2 \hat{\mu} + M(\Delta\mu)^2 \sigma_0^2 \hat{\mu} \times (v_0+1)\hat{\sigma}^2 \mu_0. \tag{2.48}$$

The resulting HPD estimator $\hat{\mu}_{Bayes}$ is

$$\hat{\mu}_{Bayes} = \frac{\sigma_0^{-2}\mu_0 + \left\{ \left[v_0 + (\Delta\mu)^2 \right] / (v_0+1) \right\} M \hat{\sigma}^{-2} \hat{\mu}}{\sigma_0^{-2} + \left\{ \left[v_0 + (\Delta\mu)^2 \right] / (v_0+1) \right\} M \hat{\sigma}^{-2}} \tag{2.49}$$

which is a convex combination of μ_0 and $\hat{\mu}$, a beautiful result indeed.

Comments:

i) In the case of normal distribution with a conjugate prior we have seen that Bayesian estimator of μ is the weighted average of prior mean μ_0 and the sample mean $\bar{y}$, namely $\hat{\mu}_{Bayes} = w\mu_0 + (1-w)\bar{y}$. In this case we have also a similar form with weights which are reciprocals of variances. This beautiful result is an outcome of applying the method of modified likelihood estimation. It can also be seen from (2.46) and (2.49) that the weight of prior mean is higher when the posterior distribution is governed by prior information, and weight for $\hat{\mu}$ is higher when the case is vice versa. This weighted form of HPD estimators makes them robust to outliers.

ii) When we have non-informative prior distribution of μ, i.e., when σ_0 goes to infinity, the Bayesian estimator of μ converges to MMLE. On the other hand, as prior variances become smaller, $\hat{\mu}_{Bayes}$ tends to prior mean μ_0; see also Bian and Tiku (1997).

iii) As variance of sample observations increases, Bayesian estimator of μ converges to μ_0 since it includes weights which depend on $\hat{\sigma}$.

iv) As sample size increases, the information coming from likelihood function dominates over the prior distribution and Bayesian estimator converges to MML estimator.

2.1.2.5 Comparing Efficiencies of MML and Bayes Estimators: Simulation Results

The performence of MML and Bayes estimators are compared by simulations. For illustration, we consider observations coming from (2.19) with p=3.5, i.e., Student's t distribution with 6 degree of freedom. The mean of the prior distribution (2.29) is taken as 0 and prior degrees of freedom is taken as 6, to be compatible with the sampling distribution. IMSL subroutine is used to generate indepent random variables of size n. First we obtain simulated values with type II censoring with fixed q=r/n, while $\hat{\mu}_{Bayes}$ is close to μ_0. Then, simulations are done with type II censoring while $\hat{\mu}_{Bayes}$ is close to $\hat{\mu}_{MML}$. The results are given by Table 2.3, Table 2.4 and Table 2.5, Table 2.6, respectively. Moreover, full sample results are also reported for both of the cases in Table 2.7, Table 2.8, Table 2.9 and Table 2.10.

Table 2.3 Simulated means, variances and RE values for censored Student t distribution with $v_0 = 6, \mu = 0, \sigma = 1.5, \mu_0 = 0$ when $\hat{\mu}_{Bayes}$ is close to μ_0

$q = 0.2$		$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
$n = 5$	$\sigma_0 = 2.0$	-0.0046	0.6457	-0.0028	0.4543	0.7037
	$\sigma_0 = 2.5$	0.0165	0.6382	0.0148	0.4952	0.7760
$n = 10$	$\sigma_0 = 2.0$	0.0057	0.3163	0.0052	0.2606	0.8237
	$\sigma_0 = 2.5$	-0.0051	0.3080	-0.0046	0.2707	0.8788
$n = 15$	$\sigma_0 = 2.0$	0.0002	0.2033	0.0002	0.1785	0.8783
	$\sigma_0 = 2.5$	-0.0077	0.1996	-0.0074	0.1834	0.9188
$n = 20$	$\sigma_0 = 2.0$	-0.0058	0.1540	-0.0056	0.1397	0.9071
	$\sigma_0 = 2.5$	-0.0010	0.1585	-0.0009	0.1441	0.9387
$n = 30$	$\sigma_0 = 2.0$	-0.0081	0.1006	-0.0079	0.0944	0.9384
	$\sigma_0 = 2.5$	-0.0048	0.1030	-0.0047	0.0989	0.9593

Table 2.4 Simulated means, variances and RE values for censored Student t distribution with $v_0 = 6, \mu = 0, \sigma_0 = 3, \mu_0 = 0$ when $\hat{\mu}_{Bayes}$ is close to μ_0

$q = 0.2$		$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
$n = 5$	$\sigma = 1.5$	-0.0008	0.6528	-0.0007	0.5403	0.8277
	$\sigma = 2.5$	-0.0176	1.7614	-0.0148	1.1464	0.6509
$n = 10$	$\sigma = 1.5$	0.0007	0.3080	0.0007	0.2805	0.9106
	$\sigma = 2.5$	-0.0036	0.8622	-0.0032	0.6802	0.7889
$n = 15$	$\sigma = 1.5$	-0.0020	0.2055	-0.0019	0.1936	0.9418
	$\sigma = 2.5$	-0.0115	0.5592	-0.0103	0.4768	0.8526
$n = 20$	$\sigma = 1.5$	-0.0018	0.1527	-0.0017	0.1460	0.9563
	$\sigma = 2.5$	0.0027	0.4200	0.0023	0.3731	0.8883
$n = 30$	$\sigma = 1.5$	0.0047	0.1027	0.0046	0.0998	0.9716
	$\sigma = 2.5$	-0.0002	0.2822	-0.0002	0.2606	0.9235

Table 2.5 Simulated means, variances and RE values for censored Student t distribution with $v_0 = 6, \mu = 0, \sigma = 1.5, \mu_0 = 0$ when $\hat{\mu}_{Bayes}$ is close to $\hat{\mu}_{MML}$

$q = 0.2$		$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
$n = 5$	$\sigma_0 = 2.0$	-0.0070	0.6356	-0.0058	0.4499	0.7079
	$\sigma_0 = 2.5$	-0.0026	0.6345	-0.0015	0.4943	0.7791
$n = 10$	$\sigma_0 = 2.0$	0.0072	0.3057	0.0068	0.2528	0.8269
	$\sigma_0 = 2.5$	-0.0082	0.3116	-0.0076	0.2747	0.8816
$n = 15$	$\sigma_0 = 2.0$	-0.0036	0.2036	-0.0033	0.1793	0.8804
	$\sigma_0 = 2.5$	-0.0062	0.2056	-0.0060	0.1890	0.9194
$n = 20$	$\sigma_0 = 2.0$	0.0007	0.1519	0.0007	0.1382	0.9097
	$\sigma_0 = 2.5$	0.0007	0.1520	0.0007	0.1429	0.9401
$n = 30$	$\sigma_0 = 2.0$	-0.0012	0.1004	-0.0011	0.0943	0.9389
	$\sigma_0 = 2.5$	-0.0015	0.0998	-0.0015	0.0957	0.9594

Table 2.6 Simulated means, variances and RE values for censored Student t distribution with $v_0 = 6, \mu = 0, \sigma_0 = 3, \mu_0 = 0$ when $\hat{\mu}_{Bayes}$ is close to $\hat{\mu}_{MML}$

$q = 0.2$		$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
$n = 5$	$\sigma = 1.5$	-0.0012	0.6368	-0.0015	0.5306	0.8332
	$\sigma = 2.5$	0.0091	1.7537	0.0073	1.1890	0.6780
$n = 10$	$\sigma = 1.5$	0.0052	0.3100	0.0051	0.2829	0.9125
	$\sigma = 2.5$	-0.0142	0.8551	-0.0124	0.6821	0.7977
$n = 15$	$\sigma = 1.5$	0.0018	0.2031	0.0016	0.1913	0.9420
	$\sigma = 2.5$	-0.0051	0.5684	-0.0047	0.4866	0.8561
$n = 20$	$\sigma = 1.5$	-0.0034	0.1483	-0.0033	0.1418	0.9565
	$\sigma = 2.5$	-0.0119	0.4292	-0.0113	0.3819	0.8899
$n = 30$	$\sigma = 1.5$	0.0019	0.1025	0.0018	0.0996	0.9717
	$\sigma = 2.5$	0.0026	0.2803	-0.0025	0.2592	0.9247

Table 2.7 Simulated means, variances and RE values for Student t distribution with $v_0 = 6, \mu = 0, \sigma = 1.5, \mu_0 = 0$ when $\hat{\mu}_{Bayes}$ is close to μ_0

		$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
$n=5$	$\sigma_0 = 2.0$	0.0014	0.6270	0.0006	0.4249	0.6777
	$\sigma_0 = 2.5$	-0.0072	0.6271	-0.0057	0.4826	0.7695
$n=10$	$\sigma_0 = 2.0$	0.0031	0.3042	0.0026	0.2498	0.8211
	$\sigma_0 = 2.5$	0.0043	0.2935	0.0043	0.2574	0.8772
$n=15$	$\sigma_0 = 2.0$	0.0035	0.1957	0.0035	0.1718	0.8778
	$\sigma_0 = 2.5$	0.0094	0.1983	0.0090	0.1818	0.9169
$n=20$	$\sigma_0 = 2.0$	-0.0024	0.1460	-0.0022	0.1323	0.9065
	$\sigma_0 = 2.5$	-0.0030	0.1495	-0.0029	0.1402	0.9381
$n=30$	$\sigma_0 = 2.0$	-0.0040	0.0997	-0.0039	0.0943	0.9455
	$\sigma_0 = 2.5$	0.0013	0.0986	0.0013	0.0951	0.9646

Table 2.8 Simulated means, variances and RE values for Student t distribution with $v_0 = 6, \mu = 0, \sigma_0 = 3, \mu_0 = 0$ when $\hat{\mu}_{Bayes}$ is close to μ_0

		$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
$n=5$	$\sigma = 1.5$	0.0073	0.6430	0.0064	0.5248	0.8162
	$\sigma = 2.5$	0.0066	1.7081	0.0075	1.0882	0.6371
$n=10$	$\sigma = 1.5$	0.0031	0.2987	0.0030	0.2719	0.9104
	$\sigma = 2.5$	0.0123	0.8301	0.0113	0.6503	0.7834
$n=15$	$\sigma = 1.5$	-0.0002	0.1990	-0.0002	0.1873	0.9409
	$\sigma = 2.5$	0.0007	0.5527	-00001	0.4711	0.8525
$n=20$	$\sigma = 1.5$	-0.0026	0.1453	-0.0026	0.1390	0.9565
	$\sigma = 2.5$	-0.0059	0.4034	-0.0058	0.3575	0.8863
$n=30$	$\sigma = 1.5$	0.0055	0.1017	0.0054	0.0991	0.9749
	$\sigma = 2.5$	-0.0050	0.2787	-0.0047	0.2601	0.9331

Table 2.9 Simulated means, variances and RE values for Student t distribution with $v_0 = 6, \mu = 0, \sigma = 1.5, \mu_0 = 0$ when $\hat{\mu}_{Bayes}$ is close to $\hat{\mu}_{MML}$

		$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
$n=5$	$\sigma_0=2.0$	-0.0030	0.6261	-0.0031	0.4386	0.7005
	$\sigma_0=2.5$	-0.0050	0.6268	-0.0032	0.4831	0.7707
$n=10$	$\sigma_0=2.0$	-0.0046	0.3036	-0.0040	0.2509	0.8265
	$\sigma_0=2.5$	0.0025	0.3060	0.0024	0.2690	0.8791
$n=15$	$\sigma_0=2.0$	-0.0024	0.1965	-0.0023	0.1727	0.8788
	$\sigma_0=2.5$	-0.0044	0.2004	-0.0043	0.1843	0.9194
$n=20$	$\sigma_0=2.0$	-0.0086	0.1476	-0.0083	0.1340	0.9076
	$\sigma_0=2.5$	-0.0024	0.1468	-0.0023	0.1377	0.9384
$n=30$	$\sigma_0=2.0$	0.0023	0.0989	0.0022	0.0936	0.9469
	$\sigma_0=2.5$	-0.0005	0.0975	-0.0004	0.0941	0.9650

Table 2.10 Simulated means, variances and RE values for Student t distribution with $v_0 = 6, \mu = 0, \sigma_0 = 3, \mu_0 = 0$ when $\hat{\mu}_{Bayes}$ is close to $\hat{\mu}_{MML}$

		$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
$n=5$	$\sigma=1.5$	-0.0046	0.6106	-0.0043	0.5064	0.8293
	$\sigma=2.5$	-0.0206	1.7110	-0.0175	1.1312	0.6533
$n=10$	$\sigma=1.5$	-0.0150	0.2983	-0.0145	0.2730	0.9151
	$\sigma=2.5$	-0.0049	0.8460	-0.0042	0.6705	0.7926
$n=15$	$\sigma=1.5$	0.0002	0.1997	0.0001	0.1880	0.9414
	$\sigma=2.5$	-0.0124	0.5550	-0.0111	0.4747	0.8554
$n=20$	$\sigma=1.5$	0.0021	0.1466	0.0021	0.1403	0.9572
	$\sigma=2.5$	-0.0087	0.4060	-0.0080	0.3602	0.8872
$n=30$	$\sigma=1.5$	0.0017	0.0995	0.0017	0.0970	0.9754
	$\sigma=2.5$	-0.0041	0.2773	-0.0040	0.2591	0.9346

Table 2.3 and Table 2.4 deal with the censoring case while $\hat{\mu}_{Bayes}$ is close to prior mean μ_0. Table 2.3 indicates the effect of increasing prior dispersion on both of the HPD and MML estimators, while Table 2.4 shows the changes in simulated values with respect to different σ values. We know that Bayes estimator looses efficiency as prior distribution variance becomes larger. On the other hand, increasing the sample variance makes HPD estimator better than MML estimator. These results are shown in Table 2.3 and Table 2.4 above. After that, simulations are carried out for the case when $\hat{\mu}_{Bayes}$ is close to $\hat{\mu}_{MML}$. All of the inferences are the same for this case also. The relative efficiencies are almost the same as in the case when $\hat{\mu}_{Bayes}$ is close to μ_0 but a little larger as expected.

All of the statements made above are valid also for the full sample cases which are given in Table 2.7 -Table 2.10.

Moreover, all of the tables above show that both MML and HPD estimators have almost no bias. Also, as sample size increases $\hat{\mu}_{Bayes}$ converges to $\hat{\mu}_{MML}$, which is expected.

2.2. HPD Estimators under Skewed Sample Distributions

We extend the results of the previous section to skewed distributions. Primarily, gamma distribution is considered since it represents an important class of skewed distributions.

2.2.1. Gamma Distribution:

2.2.1.1. MML Estimators:

Consider a random sample $y_1, y_2, \ldots, y_n$ coming from gamma distribution with unknown origin and scale as

$$f(y)=\frac{1}{\Gamma(k)\sigma}(\frac{y-\mu}{\sigma})^{k-1}\exp(-\frac{y-\mu}{\sigma}), \qquad y\geq\mu . \tag{2.50}$$

Assume $k>1$ and known. We have $E(Y)=\mu+k\sigma$, $Var(Y)=k\sigma^2$ and $Mode=\mu+(k-1)\sigma$. It is known that for $k>1$, (2.50) is unimodal and positively skewed.

In order to find the ML estimators of μ and σ, the likelihood equations are

$$\frac{d\ln L}{d\mu}\propto\frac{n}{\sigma}-(k-1)\sum_{i=1}^{n}(y_i-\mu)^{-1}=0$$

and

$$\frac{d\ln L}{d\sigma}\propto-\frac{nk}{\sigma}+\frac{\sum_{i=1}^{n}(y_i-\mu)}{\sigma^2}=0 \tag{2.51}$$

We re-write (2.51) in terms of order statistics, since complere sums are invariant to ordering. Thus,

$$\frac{d\ln L}{d\mu}\propto\frac{n}{\sigma}-\frac{(k-1)}{\sigma}\sum_{i=1}^{n}z_{(i)}^{-1}=0$$

and

$$\frac{d\ln L}{d\sigma}\propto-\frac{nk}{\sigma}+\frac{\sum_{i=1}^{n}(y_{(i)}-\mu)}{\sigma^2}=0 \tag{2.52}$$

where $z_{(i)}=\frac{y_{(i)}-\mu}{\sigma}$. These equations have no explicit solutions. Therefore, instead of maximum likelihood estimation method, Tiku's modified maximum likelihood estimation method is used. Let $t_i = E(z_{(i)})$. Taylor series expansion yields the equation $z_{(i)}^{-1} = 2t_i^{-1} - t_i^{-2} z_{(i)}$. The likelihood functions are

$$\frac{d\ln L}{d\mu} \cong \frac{d\ln L^*}{d\mu} = \frac{n}{\sigma} - \frac{(k-1)}{\sigma}\sum_{i=1}^{n}(2t_i^{-1} - t_i^{-2} z_{(i)}) = 0$$

and

$$\frac{d\ln L}{d\sigma} = \frac{d\ln L^*}{d\sigma} = -\frac{nk}{\sigma} + \frac{1}{\sigma}\sum_{i=1}^{n} z_{(i)} = 0 \ . \qquad (2.53)$$

The MML equations are solutions of $\frac{d\ln L^*}{d\mu} = 0$ and $\frac{d\ln L^*}{d\sigma} = 0$ and they are found as

$$\hat{\mu} = K - D\hat{\sigma} \quad \text{and} \quad \hat{\sigma} = \sum_{i=1}^{n} \beta_i y_{(i)}$$

where,

$$K = \frac{\sum_{i=1}^{n} \delta_i y_{(i)}}{m}, \quad D = \frac{1}{m}\left(2\sum_{i=1}^{n} t_i^{-1} - \frac{n}{k-1}\right), \quad \beta_i = \frac{1-(n/m)\delta_i}{n\left[k - (1/m)\sum t_i^{-1}\right]},$$

$$m = \sum_{i=1}^{n} \delta_i \ , \ \delta_i = t_i^{-2}. \text{ Note that, } \sum_{i=1}^{n} \beta_i = 0 .$$

As stated by Bian and Tiku (1997), MML estimators are linear functions of order statistics. Therefore, their variance-covariance matrix can be obtained from the expected values of order statistics which are available in Gupta (1960), Pearson and Hartley(1972) and Prescott (1974).

Since $(1/n)\{(d\ln L/d\mu)-(d\ln L^{*}/d\mu)\}$ converges to zero as n goes to infinity because $\lim_{n\to\infty}\left(\frac{1}{n}\right)\sum_{i=1}^{n}t_i^{-1}=\frac{1}{\Gamma(k)}\int_0^{\infty}e^{-z}z^{k-2}dz=\frac{1}{k-1}$, the following results are true (Bian and Tiku ,1997):

Lemma1: $\frac{d\ln L^{*}}{d\mu}=\frac{m(k-1)}{\sigma^2}(K-D\sigma-\mu)$ which shows that the MML estimator of μ is MVB estimator (asymptotically) when σ is known.

Lemma2. $\frac{d\ln L^{*}}{d\sigma}=\frac{nk}{\sigma^2}\left[\frac{1}{nk}\sum_{i=1}^{n}(y_{(i)}-\mu)-\sigma\right]$ which shows that MML estimator of σ is MVB estimator asymptotically when μ is known.

Moreover, The asymptotic variance-covariance matrix of $\hat{\mu}$ and $\hat{\sigma}$ is given by

$$V^{*}=\left[I^{*}(\mu,\sigma)\right]^{-1},$$

where

$$I^{*}(\mu,\sigma)=\begin{bmatrix} m(k-1) & n \\ n & nk \end{bmatrix}\sigma^{-2}. \qquad (2.54)$$

Remark: Since the joint density of $\left[\frac{d\ln L^{*}}{d\mu},\frac{d\ln L^{*}}{d\sigma}\right]$ is bivariate normal with mean vector (μ,σ) and variance covariance matrix (2.54) , asymptotically, we can write

$$f_1(\hat{\mu},\hat{\sigma})\propto\sigma^{-2}\exp\left(-\frac{1}{2\sigma^2}\left[m(k-1)(\hat{\mu}-\mu)^2+2n(\hat{\mu}-\mu)(\hat{\sigma}-\sigma)+nk(\hat{\sigma}-\sigma)^2\right]\right). \qquad (2.55)$$

Remark: Let $\xi = \mu + k\sigma$. The asymptotic distribution of $\hat{\mu}$ and $\hat{\xi} = \hat{\mu} + k\hat{\sigma}$ is bivariate normal with mean vector (μ, ξ) and variance-covariance matrix $\begin{bmatrix} \Delta / m(k-1) & 0 \\ 0 & k/n \end{bmatrix}\sigma^2$ where $\Delta = [1 - n/mk(k-1)]^{-1}$. Therefore, we can write the density function of $(\hat{\mu}, \hat{\xi})$ as

$$f_2(\hat{\mu}, \hat{\xi}) \propto \sigma^{-2} \exp\left(-\frac{1}{2\sigma^2} \left[\{m(k-1) - n/k\}(\hat{\mu} - \mu)^2 + (n/k)(\hat{\xi} - \xi)^2 \right] \right). \tag{2.56}$$

Note that the joint density of $(\hat{\mu}, \hat{\sigma})$ and $(\hat{\mu}, \hat{\xi})$ can be written respectively as

$$f_1(\hat{\mu}, \hat{\sigma}) = g_1(\hat{\mu} \mid \hat{\sigma}) h_1(\hat{\sigma})$$

and

$$f_2(\hat{\mu}, \hat{\xi}) = g_2(\hat{\mu}) h_2(\hat{\xi}) \tag{2.57}$$

where

$$g_1(\hat{\mu} \mid \hat{\sigma}) \propto \exp\left\{ \frac{-h_1}{2\hat{\sigma}^2} \left[(\hat{\mu} - \mu) + k\rho^2 (\hat{\sigma} - \sigma)^2 \right] \right\}$$

and

$$g_2(\hat{\mu}) \propto \exp\left\{ \frac{-h_2}{2\hat{\sigma}^2} (\hat{\mu} - \mu)^2 \right\}; \tag{2.58}$$

$h_1(\hat{\sigma})$ and $h_2(\hat{\xi})$ are approximated by a chi-square distributions by using three-moment chi-square approximation as (Tiku 1996a,b)

$$h_1(\hat{\sigma}) \propto \sigma^{-nk(1-\rho^2)} \exp\left[-nk(1-\rho^2)\hat{\sigma}/\sigma\right]\hat{\sigma}^{nk(1-\rho^2)-1}$$

and

$$h_2(\hat{\xi}) \propto \xi^{-n} \exp\left(-n\hat{\xi}/\xi\right)\hat{\xi}^{n-1} \qquad (2.59)$$

where,

$\rho = -\left[n/\{mk(k-1)\}\right]^{1/2}$ is the correlation coefficient between $\hat{\mu}$ and $\hat{\sigma}$. h_1 and h_2 are provided by Vaughan(1992) as variance adjusting factors that approximate sampling distributions much closer to normal. They can be expressed as

$$h_1 = m(k-1)\left[1-1/nk(1-\rho^2)\right]^2$$

and

$$h_2 = \left[m(k-1)-n/k\right]\left[1-1/nk(1-\rho^2)\right]^2. \qquad (2.60)$$

Considering the functional forms of $f_1(\hat{\mu},\hat{\sigma})$ and $f_2(\hat{\mu},\hat{\xi})$ given by (2.57) the likelihood functions $L_1(\mu,\sigma)$ and $L_2(\mu,\xi)$ can be approximated as

$$L_1(\mu,\sigma) \propto \sigma^{-(nk(1-\rho^2)+1)} \exp\left[\frac{-nk(1-\rho^2)\hat{\sigma}}{\sigma}\right] \times$$
$$\exp\left\{\frac{-h_1}{2\hat{\sigma}^2}\left[(\hat{\mu}-\mu)+k\rho^2(\hat{\sigma}-\sigma)^2\right]\right\}$$

and

$$L_2(\mu,\xi) \propto \xi^{-n} \exp\left(-n\hat{\xi}/\xi\right)\hat{\xi}^{n-1} \exp\left\{-\frac{h_2}{2\hat{\sigma}^2}(\hat{\mu}-\mu)^2\right\} \quad (2.61)$$

It has been shown by Tan(1985) that (2.61) are close approximations to the corresponding likelihood functions so that HPD estimators can be obtained by using them.

2.2.1.2. Prior Distributions:

In order to find the HPD estimators, robust priors for μ and ξ are assumed as an independent t and inverse gamma distribution:

$$p(\mu) \propto p(\mu)p(\xi) \quad (2.62)$$

where,

$$p(\mu) \propto \left[1+(\mu-\mu_0)^2/\delta_0 s_0^2\right]^{-(\delta_0+1)/2}$$

and

$$p(\xi) \propto \xi^{-n_0} \exp(-n_0\xi_0/\xi) \quad (2.63)$$

2.2.1.3. Posterior Distributions:

The posterior distribution of μ and ξ can be found by combining the prior distribution with sample information as

$$p(\mu,\xi \mid y) \propto p(\mu)p(\xi)f_2(\hat{\mu},\hat{\xi}) \quad (2.64)$$

and

$$f(\mu, \xi \mid y) \propto \xi^{-(n_0+n)} \exp\left[-\left(n_0\xi_0 + n\xi\right)/\xi\right]\left[1 + (\mu - \mu_0)^2 / \delta_0 s_0^2\right]^{-(\delta_0+1)/2} \times$$
$$\exp\left[-h_2(\hat{\mu} - \mu)^2 / 2\hat{\sigma}^2\right] \qquad (2.65)$$

2.2.1.4. HPD Estimators:

It can be seen from (2.65) that μ and ξ are posteriorly independent. The marginal posterior density of ξ^{-1} is the scaled gamma given by

$$f(\xi \mid y) \propto \xi^{-(n_0+n)} \exp\left[-\left(n_0\xi_0 + n\hat{\xi}\right)/\xi\right] \qquad (2.66)$$

which is $inversegamma\left(n_0 + n - 1, n_0\xi_0 + n\hat{\xi}\right)$ .The HPD estimator of ξ is obtained from (2.66) as

$$\hat{\xi}_{Bayes} = E(\xi \mid y) = \frac{n_0\xi_0 + n\xi}{n_0 + n} \qquad (2.67)$$

Comment: As is seen from (2.67), Bayesian estimator of ξ is a convex combination of the prior location ξ_0 and MML estimator $\hat{\xi}$. Bayes estimator will be close to its prior value or sample estimate depending on the weights of n_0 and n (Bian and Tiku, 1997).

The marginal posterior density of μ is poly t density with t factor and normal factor that represents prior information and sampling information jointly:

$$f(\mu \mid y) \propto \left[1 + \frac{(\mu - \mu_0)^2}{\delta_0 s_0^2}\right]^{-(\delta_0+1)/2} \exp\left[\frac{-h_2}{2\hat{\sigma}^2}(\mu - \hat{\mu})^2\right]. \qquad (2.68)$$

In order to find the HPD estimator of μ, two cases need to be considered.

Case1: If δ_0 is infinite, then the prior of μ reduces to a normal density and the posterior will be normal density as

$$p(\mu \mid y) \propto \exp(-\frac{1}{2s_0^2}(\mu-\mu_0)^2 \exp(-\frac{h_2}{2\hat{\sigma}^2}(\mu-\hat{\mu})^2). \qquad (2.69)$$

After some algebra, the posterior density (2.69) reduces to a normal density with $(\mu_{Bayes}, \sigma^2_{Bayes})$ as

$$E(\mu \mid y) = \hat{\mu}_{Bayes} = \frac{s_0^{-2}\mu_0 + h_2\hat{\sigma}^{-2}\hat{\mu}}{s_0^{-2} + h_2\hat{\sigma}^{-2}} \ ,$$

$$Var(\mu \mid y) = \hat{\sigma}^2_{Bayes} = \left(s_0^{-2} + h_2\hat{\sigma}^{-2}\right)^{-1}. \qquad (2.70)$$

Comment: Similar to the symmetric family case, we can express the Bayes estimator of μ as the weighted average of prior and sample mean as $\hat{\mu}_{Bayes} = w\mu_0 + (1-w)\hat{\mu}$ where $w = s_0^{-2} / \left(s_0^{-2} + h_2\hat{\sigma}^{-2}\right)$. From this weighted form it is seen that if $s_0 = 0$, Bayes estimator of μ will be equal to the prior value, as expected. If h_2 tends to infinity then the weight w becomes zero and Bayes estimator of μ converges to $\hat{\mu}$. Moreover, HPD estimator of μ become robust to outliers because of the weights depend on the variances.

Case2*:* When δ_0 is finite, the posterior density of μ will be a poly t density which can be expressed as

$$p(\mu \mid y) \propto \left[1 + \frac{(\mu-\mu_0)^2}{\delta_0 s_0^2}\right]^{-(\delta_0+1)/2} \exp(-\frac{h_2}{2\hat{\sigma}^2}(\mu-\hat{\mu})^2). \qquad (2.71)$$

In order to solve this poly t density we obtain the modal equation by taking derivative of log posterior density of μ and find

$$(\delta_0+1)(\mu-\mu_0)\hat{\sigma}^2+h_2(\mu-\hat{\mu})\left[\delta_0 s_0^2+(\mu-\mu_0)^2\right]=0; \qquad (2.72)$$

$\hat{\mu}_{Bayes}$ is the solution of the modal equation. Repeating the same mathematical procedures, as in the symmetric family case, we consider two cases for the HPD estimator of μ.

Case1: $\hat{\mu}_{Bayes}$ is close to μ_0, we can write HPD estimator of μ as

$$\hat{\mu}_b=\frac{s_0^{-2}\left(1+\dfrac{1}{\delta_0}\right)\mu_0+h_2\hat{\sigma}^{-2}\hat{\mu}}{s_0^{-2}\left(1+\dfrac{1}{\delta_0}\right)+h_2\hat{\sigma}^{-2}}. \qquad (2.73)$$

Case2: $\hat{\mu}_{Bayes}$ is close to $\hat{\mu}$, then HPD estimator of μ is,

$$\hat{\mu}_b=\frac{s_0^{-2}\mu_0+\left\{\left[\delta_0+(\Delta\mu)^2\right]/(\delta_0+1)\right\}h_2\hat{\sigma}^{-2}\hat{\mu}}{s_0^{-2}+\left\{\left[\delta_0+(\Delta\mu)^2\right]/(\delta_0+1)\right\}h_2\hat{\sigma}^{-2}} \qquad (2.74)$$

where $\Delta\mu=\dfrac{(\hat{\mu}-\mu_0)}{s_0}$.

2.2.1.5. Comparing Efficiencies of MML and Bayes Estimators: Simulation Results

Simulated means, variances and relative efficiencies of HPD and MML estimators are given by Table 2.11- Table 2.14 below while the underlying distribution is (2.50). Results are obtained from 10,000 simulations in which we assume $k=3$, $\mu=0$ and hyperparameters $\mu_0=0$ and $\delta_0=6$ with prior distribution given by (2.63).

Note that Table 2.11 and Table 2.12 show the results while $\hat{\mu}_{Bayes}$ is close to μ_0. Table 2.13 and Table 2.14 are obtained under the assumption that $\hat{\mu}_{Bayes}$ is close to $\hat{\mu}_{MML}$. The inference obtained from these tables are not different than that of symmetric distributions. In case of skewed distributions, Bayesian estimator is again a weighted combination of prior mean μ_0 and MML estimator $\hat{\mu}_{MML}$ where the weights are adjusted by variances. Like in the symmetric case, it is expected here also that as σ_0 increases, relative efficiencies should increase, which is shown in Table 2.11. Moreover, if σ increases, relative efficiencies should decrease and this result is shown in Table 2.12. Similar arguments can also be stated for Table 2.13 and Table 2.14 in which relative efficiencies are slightly larger as we expect.

Comment: Note that $\hat{\mu}_{Bayes}$ has a little bias especially for small n values. In that cases, mean squared error values would be compared instead of variances. However, since $\hat{\mu}_{Bayes}$ is a convex combination of μ_0 and $\hat{\mu}_{MML}$ with weights w and $(1-w)$, respectively, mean square error of $\hat{\mu}_{Bayes}$ will be smaller than that of $\hat{\mu}_{MML}$ since $(1-w)^2<1$. Therefore, any bias correction will yield relative efficiencies which are favorable to $\hat{\mu}_{Bayes}$.

Table 2.11 Simulated means, variances and RE values for Gamma distribution with $\delta_0 = 6, \mu = 0, \mu_0 = 0, \sigma = 1$, when $\hat{\mu}_{Bayes}$ is close to μ_0

		$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
$n=5$	$\sigma_0 = 3$	0.0686	0.8257	0.1159	0.6056	0.7333
	$\sigma_0 = 4$	0.0650	0.8455	0.0982	0.6935	0.8203
$n=7$	$\sigma_0 = 3$	0.0156	0.5248	0.0452	0.4094	0.7800
	$\sigma_0 = 4$	0.0171	0.5255	0.0353	0.4546	0.8650
$n=10$	$\sigma_0 = 3$	-0.0161	0.3309	-0.0165	0.2833	0.8561
	$\sigma_0 = 4$	-0.0120	0.3384	-0.0033	0.3095	0.9144
$n=15$	$\sigma_0 = 3$	-0.0426	0.1990	-0.0360	0.1821	0.9153
	$\sigma_0 = 4$	-0.0374	0.1996	-0.0337	0.1896	0.9501
$n=20$	$\sigma_0 = 3$	-0.0401	0.1390	-0.0366	0.1311	0.9435
	$\sigma_0 = 4$	-0.0473	0.1347	-0.0452	0.1303	0.9672

Table 2.12 Simulated means, variances and RE values for Gamma distribution with $\delta_0 = 6, \mu = 0, \mu_0 = 0, \sigma_0 = 3$, when $\hat{\mu}_{Bayes}$ is close to μ_0

		$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
$n=5$	$\sigma = 1.0$	0.0706	0.8358	0.1186	0.6165	0.7375
	$\sigma = 1.5$	0.1023	1.9027	0.2183	1.0832	0.5693
$n=7$	$\sigma = 1.0$	0.0551	0.5254	0.0357	0.4104	0.7811
	$\sigma = 1.5$	0.0125	1.2056	0.0904	0.7502	0.6222
$n=10$	$\sigma = 1.0$	-0.0194	0.3397	-0.0042	0.2906	0.8554
	$\sigma = 1.5$	-0.0326	0.7556	0.0099	0.5512	0.7294
$n=15$	$\sigma = 1.0$	-0.0389	0.1921	-0.0325	0.1760	0.9159
	$\sigma = 1.5$	-0.0620	0.4444	-0.0412	0.3667	0.8253
$n=20$	$\sigma = 1.0$	-0.0408	0.1376	-0.0373	0.1298	0.9433
	$\sigma = 1.5$	-0.0555	0.3065	-0.0449	0.2699	0.8804

Table 2.13 Simulated means, variances and RE values for Gamma distribution with $\delta_0 = 6, \mu = 0, \mu_0 = 0, \sigma = 1$, when $\hat{\mu}_{Bayes}$ is close to $\hat{\mu}_{MML}$

		$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
$n=5$	$\sigma_0=3$	0.0846	0.8452	0.1295	0.6307	0.7462
	$\sigma_0=4$	0.0537	0.8503	0.0872	0.6987	0.8217
$n=7$	$\sigma_0=3$	0.0099	0.5412	0.0401	0.4268	0.7886
	$\sigma_0=4$	0.0170	0.5432	0.0357	0.4704	0.8660
$n=10$	$\sigma_0=3$	-0.0214	0.3325	-0.0066	0.2848	0.8563
	$\sigma_0=4$	-0.0195	0.3367	-0.0105	0.3081	0.9151
$n=15$	$\sigma_0=3$	-0.0369	0.1980	-0.0306	0.1816	0.9169
	$\sigma_0=4$	-0.0379	0.2007	-0.0341	0.1909	0.9513
$n=20$	$\sigma_0=3$	-0.0384	0.1336	-0.0351	0.1261	0.9440
	$\sigma_0=4$	-0.0360	0.1379	-0.0340	0.1335	0.9676

Table 2.14 Simulated means, variances and RE values for Gamma distribution with $\delta_0 = 6, \mu = 0, \mu_0 = 0, \sigma_0 = 3$, when $\hat{\mu}_{Bayes}$ is close to $\hat{\mu}_{MML}$

		$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
$n=5$	$\sigma=1.0$	0.0713	0.8187	0.1163	0.6090	0.7438
	$\sigma=1.5$	0.1230	1.8887	0.2282	1.1200	0.5930
$n=7$	$\sigma=1.0$	0.0048	0.5411	0.0358	0.4235	0.7827
	$\sigma=1.5$	-0.0051	1.1972	0.0738	0.7584	0.6335
$n=10$	$\sigma=1.0$	-0.0180	0.3290	-0.0036	0.2828	0.8597
	$\sigma=1.5$	-0.0373	0.7497	0.0046	0.5494	0.7329
$n=15$	$\sigma=1.0$	-0.0356	0.1947	-0.0293	0.1785	0.9164
	$\sigma=1.5$	-0.0559	0.4413	-0.0365	0.3670	0.8315
$n=20$	$\sigma=1.0$	-0.0448	0.1406	-0.0411	0.1327	0.9439
	$\sigma=1.5$	-0.0702	0.3116	-0.0587	0.2749	0.8824

2.2.2. Generalized Logistic Distribution:

2.2.2.1. MML Estimators:

Consider the family of generalized logistic distributions as

$$f(y)=\frac{b}{\sigma}\frac{\exp\left(-\frac{y-\mu}{\sigma}\right)}{\left[1+\exp\left(-\frac{y-\mu}{\sigma}\right)\right]^{b+1}},\ -\infty<y<\infty. \quad (2.75)$$

If $b<1$, $f(y)$ is negatively skewed; if $b>1$, $f(y)$ is positively skewed; if $b=1$, $f(y)$ is symmetric. We will consider the cases where $b\neq 1$.

For a random sample $y_1, y_2, \ldots, y_n$ from (2.75), the derivatives of the likelihood function *L* are,

$$\frac{d\ln L}{d\mu}=\frac{n}{\sigma}-\frac{(b+1)}{\sigma}\sum_{i=1}^{n}g(z_i)=0$$

and

$$\frac{d\ln L}{d\sigma}=-\frac{n}{\sigma}+\frac{1}{\sigma}\sum_{i=1}^{n}z_i-\frac{(b+1)}{\sigma}\sum_{i=1}^{n}z_i g(z_i)=0 \quad (2.76)$$

where $z=(y-\mu)/\sigma$ and $g(z)=e^{-z}/(1+e^{-z})=1/(1+e^{z})$. These equations do not have explicit solutions, therefore MML estimators will be obtained by first expressing likelihood functions in terms of order statistics $z_{(i)}$ and then linearizing $g(z_{(i)})$ as $g(z_{(i)})=\alpha_i-\beta_i z_{(i)}$. This yields,

$$\alpha_i = \frac{1+e^{t_{(i)}} + t_{(i)}e^{t_{(i)}}}{\left(1+e^{t_{(i)}}\right)^2}$$

and

$$\beta_i = \frac{e^{t_{(i)}}}{\left(1+e^{t_{(i)}}\right)^2} \tag{2.77}$$

where $t_{(i)} = E\left(z_{(i)}\right)$. Balakrishnan and Leung (1988) give values of $t_{(i)}$ for $n \le 15$. For other cases, the values of $t_{(i)}$ can be obtained as $t_{(i)} = -\ln\left(q_i^{-1/b} - 1\right)$ and $q_i = i/\left(n+1\right)$ (Tiku and Akkaya, 2004).

The modified likelihood equations are expressed as

$$\frac{d\ln L}{d\mu} \cong \frac{d\ln L^*}{d\mu} = \frac{n}{\sigma} - \frac{(b+1)}{\sigma}\sum_{i=1}^{n}(\alpha - \beta_i z_{(i)}) = 0$$

$$= \frac{M}{\sigma^2}(K + D\sigma - \mu) = 0 \tag{2.78}$$

and

$$\frac{d\ln L}{d\sigma} \cong \frac{d\ln L^*}{d\sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma}\sum_{i=1}^{n} z_{(i)} - \frac{(b+1)}{\sigma}\sum_{i=1}^{n} z_{(i)}(\alpha_i - \beta_i z_{(i)}) = 0$$

$$= \frac{1}{\sigma^3}\left[\left(n\sigma^2 - B\sigma - C\right) - M(K-\mu)(K + D\sigma - \mu)\right] = 0$$

(2.79)

where $M = (b+1)m$, $K = \left(\sum_{i=1}^{n}\beta_i y_{(i)}\right)/m$, $m = \sum_{i=1}^{n}\beta_i$, $D = \sum_{i=1}^{n}\Delta_i / m$

$$\Delta_i = (b+1)^{-1} - \alpha_i ,\quad B = (b+1)\sum_{i=1}^{n} \Delta_i (y_i - K) \text{ and } C = (b+1)\left(\sum_{i=1}^{n} \beta_i y_{(i)}^2 - mK^2 \right).$$

MML estimators given below are the solutions of (2.78) and (2.79),

$$\hat{\mu} = K + D\hat{\sigma} \quad \text{and} \quad \hat{\sigma} = \left\{ B + \sqrt{\left(B^2 + 4nC\right)} \right\} / 2\sqrt{\{n(n-1)\}} . \tag{2.80}$$

Tiku and Akkaya (2004) gives the asymptotic variance- covariance matrix of $\hat{\mu}$ and $\hat{\sigma}$ as

$$I(\mu, \sigma) = \frac{n}{\sigma^2} \times \begin{bmatrix} \frac{b}{(b+2)} & \frac{b\{\psi(b+1) - \psi(2)\}}{(b+2)} \\ \frac{b\{\psi(b+1) - \psi(2)\}}{(b+2)} & 1 + \frac{b\left\{ \left[\psi'(b+1) + \psi'(2)\right] + \left[\psi(b+1) - \psi(2)\right]^2 \right\}}{(b+2)} \end{bmatrix} \tag{2.81}$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the psi function and $\psi'(x)$ is the derivative of $\psi(x)$ with respect to x. The variances are calculated from (2.81) as

$$Var(\hat{\mu}) = \frac{\sigma^2}{n} \frac{b+2}{b} \left[1 + \frac{b\left[\psi(b+1) - \psi(2)\right]^2}{b + 2 + b\left[\psi'(b+1) + \psi'(2)\right]} \right] \tag{2.82}$$

$$Var(\hat{\sigma}) = \frac{\sigma^2}{n} \left[\frac{b+2}{b + 2 + b\left[\psi'(b+1) + \psi'(2)\right]} \right] \tag{2.83}$$

and

$$\text{cov}(\hat{\mu},\hat{\sigma}) = -\frac{\sigma^2}{n}\frac{(b+2)[\psi(b+1)-\psi(2)]}{b+2+b[\psi'(b+1)+\psi'(2)]}. \tag{2.84}$$

2.2.2.2. Prior and Posterior Distributions:

We will find the posterior density of μ and ξ where $\hat{\xi} = \hat{\mu} + k\hat{\sigma}$ and k is the constant that makes $\hat{\mu}$ and $\hat{\xi}$ uncorraleted. Therefore we can specify k as $k = -\text{var}(\hat{\mu})/\text{cov}(\hat{\mu},\hat{\sigma})$. Thus,

$$k = \frac{b+2+b[\psi'(b+1)+\psi'(2)]+b[\psi(b+1)-\psi(2)]^2}{b[\psi(b+1)-\psi(2)]} \tag{2.85}$$

Variance-covariance matrix of $(\hat{\mu},\hat{\xi})$ is,

$$\text{var}(\hat{\mu},\hat{\xi}) = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}\frac{\sigma^2}{n} \tag{2.86}$$

where $\text{var}(\hat{\xi}) = \text{var}(\hat{\mu}) + Var(\hat{\sigma})k^2 + 2k\,\text{cov}(\hat{\mu},\hat{\sigma})$ which simplifies to

$$\text{var}(\hat{\xi}) = \frac{\sigma^2}{n}\frac{b+2}{b}\left[1+\frac{[b+2+b[\psi'(b+1)+\psi'(2)]]^2}{b(b+2)+b^2[\psi'(b+1)+\psi'(2)][\psi(b+1)-\psi(2)]^2}\right]. \tag{2.87}$$

Remark: Since $\left(\frac{d\ln L}{d\mu},\frac{d\ln L}{d\sigma}\right)$ is distributed as bivariate normal, we can say that $\hat{\mu}$ and $\hat{\xi}$ are distributed as bivariate normal with mean (μ,ξ) and variance-covariance matrix (2.86) Therefore, the joint distribution of $(\hat{\mu},\hat{\xi})$ is written as

$$f(\hat{\mu},\hat{\xi}) \propto \sigma^{-2n} \exp\left[-\frac{1}{2\sigma^2}\left\{\frac{(\hat{\mu}-\mu)^2}{k_1}+\frac{\left(\hat{\xi}-\xi\right)^2}{k_2}\right\}\right]. \tag{2.88}$$

We can obtain marginal distributions of $\hat{\mu}$ and $\hat{\xi}$ from (2.88) as

$$f(\hat{\mu}) \propto \sigma^{-n} \exp\left\{-\frac{1}{2k_1\sigma^2}(\hat{\mu}-\mu)^2\right\} \tag{2.89}$$

and

$$h(\hat{\xi}) \propto \sigma^{-n} \exp(-\frac{1}{2k_2\sigma^2}\left(\hat{\xi}-\xi\right)^2)^2 \tag{2.90}$$

since $\hat{\mu}$ and $\hat{\xi}$ are independent.

Remark: In order to have more accurate results, chi-square approximation for $\hat{\xi}$ is applied. As a result, $2n\hat{\xi}/\xi$ is distributed as chi-square with degree of freedom $2n$. The approximation of $\hat{\mu}$ yields an infinite degree of freedom so the distribution of $\hat{\mu}$ can be considered as normal which is given above.

Therefore, we can write the joint distribution of $\hat{\mu}$ and $\hat{\xi}$ as

$$f\left(\hat{\mu},\hat{\xi}\right) \propto \xi^{-n} \exp\left(-n\hat{\xi}/\xi\right)\hat{\xi}^{n-1} \exp\left\{-\frac{k_1}{2\hat{\sigma}^2}(\hat{\mu}-\mu)^2\right\}. \tag{2.91}$$

Bayes theorem is used to obtain posterior densities and to do that the priors for μ and ξ are assumed as

$$p(\mu) \propto \left[1+(\mu-\mu_0)^2/\delta_0 s_0^2\right]^{-(\delta_0+1)/2}$$

and

$$p(\xi) \propto \xi^{-n_0} \exp(-n_0\xi_0 / \xi) \qquad (2.92)$$

where $p(\mu,\xi\ ) \propto p(\mu)p(\xi)$. Combining the likelihood (2.91) and priors (2.92), the posterior density of μ and ξ is obtained:

$$f(\mu,\xi \mid y) \propto \xi^{-(n_0+n)} \exp\left[-\left(n_0\xi_0 + n\hat{\xi}\right)/\xi\right]\left[1+(\mu-\mu_0)^2 / \delta_0 s_0^2\right]^{-(\delta_0+1)/2} \times \exp\left[-k_1(\hat{\mu}-\mu)^2 / 2\hat{\sigma}^2\right]. \qquad (2.93)$$

Comment: It is seen from $f(\mu,\xi \mid y)$ that μ and ξ are posteriorly independent. The marginal posterior density of ξ^{-1} is the scaled gamma

$$f(\xi \mid y) \propto \xi^{-(n_0+n)} \exp\left[-\left(n_0\xi_0 + n\hat{\xi}\right)/\xi\right] \qquad (2.94)$$

and the marginal posterior density of μ is poly t density with t factor and normal factor that represents prior information and sampling information together,

$$f(\mu \mid y) \propto \left[1+\frac{(\mu-\mu_0)^2}{\delta_0 s_0^2}\right]^{-(\delta_0+1)/2} \exp\left[\frac{-k_1}{2\hat{\sigma}^2}(\mu-\hat{\mu})^2\right]. \qquad (2.95)$$

2.2.2.3. HPD Estimators:

The HPD estimator of ξ is $\hat{\xi}_{Bayes} = E(\xi \mid y) = \left(n_0\xi_0 + n\hat{\xi}\right)/(n_0+n)$. As in the gamma distribution, $\hat{\xi}_{Bayes}$ is a combination of the prior location ξ_0 and MML estimator $\hat{\xi}$. In this situation we may emphasize that if n_0 is large, the posterior estimate is close to the prior location ξ_0 but if n is large it will be close to MML estimator.

In order to find the HPD estimator of μ, two cases need to be considered.

Case1: If δ_0 is infinite, than the prior of μ reduces to a normal density and the posterior is

$$p(\mu \mid y) \propto \exp(-\frac{1}{2s_0^2}(\mu-\mu_0)^2 \exp(-\frac{k_1}{2\hat{\sigma}^2}(\mu-\hat{\mu})^2)\,. \tag{2.96}$$

It is seen from (2.96) that $p(\mu \mid y)$ is normal (μ_b, σ_b^2) where $\mu_b = \dfrac{s_0^{-2}\mu_0 + k_1\hat{\sigma}^{-2}\hat{\mu}}{s_0^{-2} + k_1\hat{\sigma}^{-2}}$ and $\sigma_b^2 = \left(s_0^{-2} + k_1\hat{\sigma}^{-2}\right)^{-1}$. Therefore,

$$E(\mu \mid y) = \hat{\mu}_b = \frac{s_0^{-2}\mu_0 + k_1\hat{\sigma}^{-2}\hat{\mu}}{s_0^{-2} + k_1\hat{\sigma}^{-2}} \tag{2.97}$$

Case2: When δ_0 is finite, then the posterior density reduces to a a poly t density as

$$p(\mu \mid y) \propto \left[1 + \frac{(\mu-\mu_0)^2}{\delta_0 s_0^2}\right]^{-(\delta_0+1)/2} \exp(-\frac{k_1}{2\hat{\sigma}^2}(\mu-\hat{\mu})^2)\,; \tag{2.98}$$

$\hat{\mu}_b$ is the solution of the modal equation,

$$(\delta_0+1)(\mu-\mu_0)\hat{\sigma}^2 + k_1(\mu-\hat{\mu})\left[\delta_0 s_0^2 + (\mu-\mu_0)^2\right] = 0\,. \tag{2.99}$$

When the posterior density of μ is governed by the prior density, then the HPD estimator of μ is

$$\hat{\mu}_b = \frac{s_0^{-2}\left(1+\dfrac{1}{\delta_0}\right)\mu_0 + k_1\hat{\sigma}^{-2}\hat{\mu}}{s_0^{-2}\left(1+\dfrac{1}{\delta_0}\right) + k_1\hat{\sigma}^{-2}} \tag{2.100}$$

When the posterior density of μ is governed by the sampling density, then the HPD estimator of μ is,

$$\hat{\mu}_b = \frac{s_0^{-2}\mu_0 + \left\{\left[\delta_0 + (\Delta\mu)^2\right]/(\delta_0+1)\right\}k_1\hat{\sigma}^{-2}\hat{\mu}}{s_0^{-2} + \left\{\left[\delta_0 + (\Delta\mu)^2\right]/(\delta_0+1)\right\}k_1\hat{\sigma}^{-2}} \tag{2.101}$$

where $\Delta\mu = \dfrac{(\hat{\mu}-\mu_0)}{\sigma_0}$.

Comment: Note that the posterior estimates of μ is a weighted combination of μ_0 and $\hat{\mu}$ with weights proportional to variances. The form of Bayesian estimator of μ is similar to (2.17), (2.46), (2.49), (2.70), (2.73) and (2.74). Consequently, similar arguments about the efficiencies can be made under the assumption of the generalized logistic distribution also. This is made possible by an application of modified maximum likelihood estimation. Maximum likelihood estimation would get us no where because computations are too involved.

2.2.2.4. Comparing Efficiencies of MML and Bayes Estimators:Simulations Results

In case of generalized logistic family we consider relative efficiencies of HPD and MML estimators for $b = 0.5$, 1, 4, 6 and 8 with different sample sizes. The parameters of prior distribution of μ specified by (2.92) is taken as $\mu_0 = 0, s_0 = 2$ and $s_0 = 2.5$ while observations are assumed to have the distribution (2.75) with $\mu = 0$, $\sigma = 1$ and, alternatively, $\sigma = 1.5$. The results of

10,000 simulations with respect to different b values are given in Table 2.15 - Table 2.34 below. According to these results we can say that Bayesian estimator is generally better when b=0.5. Both MML and HPD estimators are unbiased for large n but they have a little bias especially for n=5. In that cases any bias correction would make HPD estimators more efficient as they are now and does not make any difference in the interpretation. Therefore, we leave these values as they are.

Like previous sections, the situations of HPD estimator being close to prior mean or HPD estimator being close to MML estimator are considered separately. In both cases, very similar results are obtained which are in favor of Bayesian estimator. Note that, HPD is a little less efficient when $\hat{\mu}_{Bayes}$ is close to $\hat{\mu}_{MML}$, as ecpected. One can see the results of simulations of case1 from Table 2.15 to Table 2.24 and case 2 from Table 2.25 to Table 2.34 below.

Generally, all of the simulated values tell us that HPD estimator is negatively affected with the increasing prior dispersion. As prior dispersion increases, prior distribution converges to non-informative prior and therefore HPD estimator converges to MML estimator. On the other hand, if the value of prior dispersion decreases, Bayesian estimator will converge to prior mean. From simulated efficiencies it can also be inferred that HPD estimator looses efficiency with increasing σ_0. However, if we look at relative efficiencies, with different σ values in order to see the impact of the change in σ, we see that as it increases, weight of prior mean increases and MML estimator looses efficiency. These results are also the same as the statements we made in previous sections.

Moreover, we can state that HPD estimators are preferable for small sample sizes. When number of observations increases, prior distribution is dominated by likelihood function in which case the MML estimators are more advantegeous to use when sample size is large.

Table 2.15 Simulated means, variances and RE values for Generalized Logistic distribution with $\mu_0 = 0, \delta_0 = 6, s_0 = 3, \mu = 0, b = 0.5$ when $\hat{\mu}_{Bayes}$ is close to μ_0

$b = 0.5$		$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
$n = 5$	$\sigma = 1.0$ $\sigma = 1.5$	-0.0940 -0.1645	1.1234 2.5673	-0.0960 -0.1586	0.8068 1.4063	0.7182 0.5478
$n = 10$	$\sigma = 1.0$ $\sigma = 1.5$	-0.0433 -0.0416	0.5383 1.2146	-0.0448 -0.0480	0.4603 0.8750	0.8551 0.7204
$n = 15$	$\sigma = 1.0$ $\sigma = 1.5$	-0.0169 -0.0294	0.3429 0.8165	-0.0182 -0.0321	0.3105 0.6595	0.9055 0.8076
$n = 20$	$\sigma = 1.0$ $\sigma = 1.5$	-0.0114 -0.0271	0.2595 0.5862	-0.0121 -0.0281	0.2413 0.5000	0.9299 0.8530

Table 2.16 Simulated means, variances and RE values for Generalized Logistic distribution with $\mu_0 = 0, \delta_0 = 6, s_0 = 3, \mu = 0, b = 1$, when $\hat{\mu}_{Bayes}$ is close to μ_0

$b = 1$		$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
$n = 5$	$\sigma = 1.0$ $\sigma = 1.5$	-0.0087 0.0246	0.6363 1.4195	-0.0085 0.0297	0.5225 0.9483	0.8212 0.6680
$n = 10$	$\sigma = 1.0$ $\sigma = 1.5$	0.0026 -0.0002	0.3050 0.6944	0.0027 0.0001	0.2780 0.5703	0.9115 0.8214
$n = 15$	$\sigma = 1.0$ $\sigma = 1.5$	0.0051 -0.0062	0.2042 0.4534	0.0048 -0.0057	0.1927 0.3988	0.9437 0.8796
$n = 20$	$\sigma = 1.0$ $\sigma = 1.5$	0.0042 -0.0050	0.1594 0.3379	0.0041 -0.0046	0.1532 0.3071	0.9611 0.9090

Table 2.17 Simulated means, variances and RE values for Generalized Logistic distribution with $\mu_0 = 0, \delta_0 = 6, s_0 = 3, \mu = 0, b = 4$, when $\hat{\mu}_{Bayes}$ is close to μ_0

$b = 4$		$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
$n = 5$	$\sigma = 1.0$	0.1519	0.4965	0.1604	0.4342	0.8745
	$\sigma = 1.5$	0.2155	1.1162	0.2392	0.8480	0.7597
$n = 10$	$\sigma = 1.0$	0.0691	0.2351	0.0714	0.2202	0.9366
	$\sigma = 1.5$	0.1028	0.5281	0.1095	0.4587	0.8686
$n = 15$	$\sigma = 1.0$	0.0389	0.1504	0.0399	0.1442	0.9588
	$\sigma = 1.5$	0.0584	0.3595	0.0621	0.3273	0.9105
$n = 20$	$\sigma = 1.0$	0.0342	0.1136	0.0347	0.1101	0.9692
	$\sigma = 1.5$	0.0405	0.2584	0.0424	0.2414	0.9341

Table 2.18 Simulated means, variances and RE values for Generalized Logistic distribution with $\mu_0 = 0, \delta_0 = 6, s_0 = 3, \mu = 0, b = 6$, when $\hat{\mu}_{Bayes}$ is close to μ_0

$b = 6$		$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
$n = 5$	$\sigma = 1.0$	0.1905	0.5994	0.2046	0.5109	0.8524
	$\sigma = 1.5$	0.2829	1.3741	0.3217	0.9987	0.7268
$n = 10$	$\sigma = 1.0$	0.0843	0.2878	0.0882	0.2664	0.9256
	$\sigma = 1.5$	0.1331	10.6188	0.1428	0.5265	0.8508
$n = 15$	$\sigma = 1.0$	0.0564	0.1846	0.0580	0.1756	0.9512
	$\sigma = 1.5$	0.0835	0.4109	0.0886	0.3685	0.8967
$n = 20$	$\sigma = 1.0$	0.0420	0.1356	0.0429	0.1307	0.9639
	$\sigma = 1.5$	0.0626	0.3106	0.0656	0.2865	0.9222

Table 2.19 Simulated means, variances and RE values for Generalized Logistic distribution with $\mu_0 = 0, \delta_0 = 6, s_0 = 3, \mu = 0, b = 8$, when $\hat{\mu}_{Bayes}$ is close to μ_0

$b = 8$		$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
$n = 5$	$\sigma = 1.0$	0.2261	0.7304	0.2458	0.6005	0.8222
	$\sigma = 1.5$	0.3209	1.6426	0.3693	1.1477	0.6987
$n = 10$	$\sigma = 1.0$	0.0927	0.3372	0.0981	0.3081	0.9137
	$\sigma = 1.5$	0.1436	0.7526	0.1584	0.6230	0.8278
$n = 15$	$\sigma = 1.0$	0.0671	0.2163	0.0693	0.2043	0.9445
	$\sigma = 1.5$	0.0890	0.5032	0.0967	0.4430	0.8804
$n = 20$	$\sigma = 1.0$	0.0534	0.1611	0.0546	0.1544	0.9584
	$\sigma = 1.5$	0.0737	0.3567	0.0774	0.3251	0.9116

Table 2.20 Simulated means, variances and RE values for Generalized Logistic distribution with $\mu_0 = 0, \delta_0 = 6, \mu = 0, \sigma = 1, b = 0.5$, when $\hat{\mu}_{Bayes}$ is close to μ_0

$b = 0.5$		$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
$n = 5$	$s_0 = 2.0$	-0.0882	1.1225	-0.0909	0.6084	0.5420
	$s_0 = 2.5$	-0.0873	1.1127	-0.0897	0.7160	0.6435
$n = 10$	$s_0 = 2.0$	-0.0047	0.5383	-0.0460	0.3871	0.7191
	$s_0 = 2.5$	-0.0344	0.5340	-0.0363	0.4281	0.8018
$n = 15$	$s_0 = 2.0$	-0.0309	0.3572	-0.0313	0.2893	0.8100
	$s_0 = 2.5$	-0.0194	0.3525	-0.0207	0.3060	0.8682
$n = 20$	$s_0 = 2.0$	-0.0100	0.2593	-0.0113	0.2212	0.8531
	$s_0 = 2.5$	-0.0164	0.2678	-0.0169	0.2413	0.9010

Table 2.21 Simulated means, variances and RE values for Generalized Logistic distribution with $\mu_0 = 0, \delta_0 = 6, \mu = 0, \sigma = 1, b = 1$, when $\hat{\mu}_{Bayes}$ is close to μ_0

$b = 1$		$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
$n = 5$	$s_0 = 2.0$	-0.0034	0.6365	0.0017	0.4276	0.6719
	$s_0 = 2.5$	0.0162	0.6404	0.0131	0.4878	0.7616
$n = 10$	$s_0 = 2.0$	0.0017	0.2981	0.0016	0.2437	0.8173
	$s_0 = 2.5$	-0.0036	0.3023	-0.0031	0.2650	0.8768
$n = 15$	$s_0 = 2.0$	0.0080	0.2059	0.0075	0.1812	0.8798
	$s_0 = 2.5$	0.0016	0.2018	0.0018	0.1857	0.9203
$n = 20$	$s_0 = 2.0$	0.0029	0.1533	0.0028	0.1394	0.9094
	$s_0 = 2.5$	0.0068	0.1554	0.0065	0.1462	0.9407

Table 2.22 Simulated means, variances and RE values for Generalized Logistic distribution with $\mu_0 = 0, \delta_0 = 6, \mu = 0, \sigma = 1, b = 4$, when $\hat{\mu}_{Bayes}$ is close to μ_0

$b = 4$		$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
$n = 5$	$s_0 = 2.0$	0.1346	0.5054	0.1535	0.3798	0.7516
	$s_0 = 2.5$	0.1482	0.5050	0.1606	0.4196	0.8309
$n = 10$	$s_0 = 2.0$	0.0680	0.2342	0.0726	0.2032	0.8677
	$s_0 = 2.5$	0.0694	0.2362	0.0726	0.2154	0.9118
$n = 15$	$s_0 = 2.0$	0.0382	0.1569	0.0407	0.1430	0.9115
	$s_0 = 2.5$	0.0452	0.1522	0.0465	0.1433	0.9416
$n = 20$	$s_0 = 2.0$	0.0368	0.1139	0.0378	0.1064	0.9342
	$s_0 = 2.5$	0.0244	0.1133	0.0253	0.1084	0.9564

Table 2.23 Simulated means, variances and RE values for Generalized Logistic distribution with $\mu_0 = 0, \delta_0 = 6, \mu = 0, \sigma = 1, b = 6$, when $\hat{\mu}_{Bayes}$ is close to μ_0

$b = 6$		$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
$n = 5$	$s_0 = 2.0$	0.1932	0.5986	0.2162	0.4362	0.7288
	$s_0 = 2.5$	0.1815	0.6249	0.2019	0.4989	0.7984
$n = 10$	$s_0 = 2.0$	0.0928	0.2859	0.0995	0.2432	0.8507
	$s_0 = 2.5$	0.0929	0.2802	0.0977	0.2514	0.8974
$n = 15$	$s_0 = 2.0$	0.0585	0.1835	0.0616	0.1648	0.8979
	$s_0 = 2.5$	0.0561	0.1864	0.0584	0.1736	0.9317
$n = 20$	$s_0 = 2.0$	0.0409	0.1405	0.0430	0.1295	0.9212
	$s_0 = 2.5$	0.0470	0.1383	0.0481	0.1314	0.9495

Table 2.24 Simulated means, variances and RE values for Generalized Logistic distribution with $\mu_0 = 0, \delta_0 = 6, \mu = 0, \sigma = 1, b = 8$, when $\hat{\mu}_{Bayes}$ is close to μ_0

$b = 8$		$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
$n = 5$	$s_0 = 2.0$	0.1959	0.7224	0.2302	0.5045	0.6984
	$s_0 = 2.5$	0.2065	0.7397	0.2326	0.5712	0.7721
$n = 10$	$s_0 = 2.0$	0.1046	0.3412	0.1143	0.2816	0.8252
	$s_0 = 2.5$	0.1018	0.3317	0.1084	0.2925	0.8819
$n = 15$	$s_0 = 2.0$	0.0640	0.2116	0.0684	0.1867	0.8825
	$s_0 = 2.5$	0.0645	0.2174	0.0676	0.2002	0.9205
$n = 20$	$s_0 = 2.0$	0.0488	0.1611	0.0514	0.1468	0.9109
	$s_0 = 2.5$	0.0463	0.1592	0.0481	0.1499	0.9414

Table 2.25 Simulated means, variances and RE values for Generalized Logistic distribution with $\mu_0 = 0, \delta_0 = 6, s_0 = 3, \mu = 0, b = 0.5$, when $\hat{\mu}_{Bayes}$ is close to $\hat{\mu}_{MML}$

$b = 0.5$		$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
$n = 5$	$\sigma = 1.0$	-0.0932	1.1132	-0.0957	0.8213	0.7378
	$\sigma = 1.5$	-0.1332	2.5805	-0.1404	1.4813	0.5756
$n = 10$	$\sigma = 1.0$	-0.0230	0.5361	-0.0257	0.4618	0.8614
	$\sigma = 1.5$	-0.545	1.2093	-0.0585	0.8914	0.7371
$n = 15$	$\sigma = 1.0$	-0.0270	0.3567	-0.0278	0.3232	0.9060
	$\sigma = 1.5$	-0.0153	0.7792	-0.0194	0.6356	0.8157
$n = 20$	$\sigma = 1.0$	-0.0138	0.2616	-0.0143	0.2435	0.9308
	$\sigma = 1.5$	-0.0288	0.5973	-0.0299	0.5123	0.8577

Table 2.26 Simulated means, variances and RE values for Generalized Logistic distribution with $\mu_0 = 0, \delta_0 = 6, s_0 = 3, \mu = 0, b = 1$, when $\hat{\mu}_{Bayes}$ is close to $\hat{\mu}_{MML}$

$b = 1$		$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
$n = 5$	$\sigma = 1.0$	0.0037	0.6302	0.0031	0.5210	0.8267
	$\sigma = 1.5$	-0.0070	1.4016	-0.0055	0.9593	0.6844
$n = 10$	$\sigma = 1.0$	-0.0112	0.3096	-0.0109	0.2828	0.9135
	$\sigma = 1.5$	-0.0049	0.6943	-0.0041	0.5735	0.8259
$n = 15$	$\sigma = 1.0$	-0.0064	0.2040	-0.0062	0.1927	0.9446
	$\sigma = 1.5$	-0.0015	0.4557	-0.0013	0.4015	0.8811
$n = 20$	$\sigma = 1.0$	0.0047	0.1506	0.0046	0.1443	0.9588
	$\sigma = 1.5$	0.0098	0.3431	0.0094	0.3128	0.9117

Table 2.27 Simulated means, variances and RE values for Generalized Logistic distribution with $\mu_0 = 0, \delta_0 = 6, s_0 = 3, \mu = 0, b = 4$, when $\hat{\mu}_{Bayes}$ is close to $\hat{\mu}_{MML}$

$b = 4$		$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
$n = 5$	$\sigma = 1.0$	0.1447	0.4944	0.1539	0.4338	0.8773
	$\sigma = 1.5$	0.2112	1.1408	0.2355	0.8720	0.7644
$n = 10$	$\sigma = 1.0$	0.0723	0.2345	0.0744	0.2198	0.9374
	$\sigma = 1.5$	0.0708	0.5265	0.0795	0.4582	0.8703
$n = 15$	$\sigma = 1.0$	0.0367	0.1534	0.0378	0.1473	0.9596
	$\sigma = 1.5$	0.0686	0.3459	0.0716	0.3158	0.9131
$n = 20$	$\sigma = 1.0$	0.0310	0.1133	0.0316	0.1099	0.9698
	$\sigma = 1.5$	0.0501	0.2614	0.0520	0.2443	0.9345

Table 2.28 Simulated means, variances and RE values for Generalized Logistic distribution with $\mu_0 = 0, \delta_0 = 6, s_0 = 3, \mu = 0, b = 6$, when $\hat{\mu}_{Bayes}$ is close to $\hat{\mu}_{MML}$

$b = 6$		$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
$n = 5$	$\sigma = 1.0$	0.1812	0.5999	0.1958	0.5120	0.8535
	$\sigma = 1.5$	0.2748	1.4024	0.3119	1.0363	0.7389
$n = 10$	$\sigma = 1.0$	0.0824	0.2848	0.0863	0.2637	0.9258
	$\sigma = 1.5$	0.1426	0.6390	0.1519	0.5464	0.8551
$n = 15$	$\sigma = 1.0$	0.0516	0.1817	0.0533	0.1729	0.9518
	$\sigma = 1.5$	0.0841	0.4122	0.0890	0.3705	0.8988
$n = 20$	$\sigma = 1.0$	0.0454	0.1351	0.0462	0.1303	0.9647
	$\sigma = 1.5$	0.0547	0.3098	0.0579	0.2861	0.9233

Table 2.29 Simulated means, variances and RE values for Generalized Logistic distribution with $\mu_0 = 0, \delta_0 = 6, s_0 = 3, \mu = 0, b = 8$, when $\hat{\mu}_{Bayes}$ is close to $\hat{\mu}_{MML}$

$b = 8$		$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
$n = 5$	$\sigma = 1.0$	0.2238	0.7209	0.2418	0.6004	0.8329
	$\sigma = 1.5$	0.2981	1.7155	0.3516	1.2106	0.7057
$n = 10$	$\sigma = 1.0$	0.0908	0.3357	0.0962	0.3074	0.9157
	$\sigma = 1.5$	0.1618	0.7576	0.1752	0.6328	0.8352
$n = 15$	$\sigma = 1.0$	0.0064	0.2154	0.0685	0.2035	0.9448
	$\sigma = 1.5$	0.1012	0.4821	0.1073	0.4269	0.8855
$n = 20$	$\sigma = 1.0$	0.0467	0.1587	0.0479	0.1522	0.9588
	$\sigma = 1.5$	0.0789	0.3558	0.0822	0.3252	0.9139

Table 2.30 Simulated means, variances and RE values for Generalized Logistic distribution with $\mu_0 = 0, \delta_0 = 6, \mu = 0, \sigma = 1, b = 0.5$, when $\hat{\mu}_{Bayes}$ is close to $\hat{\mu}_{MML}$

$b = 0.5$		$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
$n = 5$	$s_0 = 2.0$	-0.0710	1.1025	-0.0799	0.6306	0.5719
	$s_0 = 2.5$	-0.0879	1.1390	-0.0931	0.7539	0.6619
$n = 10$	$s_0 = 2.0$	-0.0195	0.5526	-0.0250	0.4053	0.7335
	$s_0 = 2.5$	-0.0285	0.5402	-0.0322	0.4383	0.8113
$n = 15$	$s_0 = 2.0$	-0.0246	0.3532	-0.0258	0.2877	0.8145
	$s_0 = 2.5$	-0.0127	0.3550	-0.0145	0.3094	0.8717
$n = 20$	$s_0 = 2.0$	-0.0154	0.2596	-0.0164	0.2225	0.8572
	$s_0 = 2.5$	-0.0180	0.2645	-0.0186	0.2388	0.9029

Table 2.31 Simulated means, variances and RE values for Generalized Logistic distribution with $\mu_0 = 0, \delta_0 = 6, \mu = 0, \sigma = 1, b = 1$, when $\hat{\mu}_{Bayes}$ is close to $\hat{\mu}_{MML}$

$b = 1$		$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
$n = 5$	$s_0 = 2.0$	-0.0052	0.6273	-0.0047	0.4311	0.6873
	$s_0 = 2.5$	-0.0035	0.6285	-0.0031	0.4080	0.7651
$n = 10$	$s_0 = 2.0$	-0.0077	0.3119	-0.072	0.2577	0.8263
	$s_0 = 2.5$	0.0054	0.3177	0.0051	0.2797	0.8806
$n = 15$	$s_0 = 2.0$	-0.0005	0.1976	-0.0006	0.1745	0.8829
	$s_0 = 2.5$	0.0045	0.2024	0.0044	0.1865	0.9216
$n = 20$	$s_0 = 2.0$	-0.0022	0.1513	-0.0019	0.1381	0.9125
	$s_0 = 2.5$	-0.0010	0.1517	-0.0011	0.1429	0.9421

Table 2.32 Simulated means, variances and RE values for Generalized Logistic distribution with $\mu_0 = 0, \delta_0 = 6, \mu = 0, \sigma = 1, b = 4$, when $\hat{\mu}_{Bayes}$ is close to $\hat{\mu}_{MML}$

$b = 4$		$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
$n = 5$	$s_0 = 2.0$	0.1422	0.5018	0.1583	0.3873	0.7718
	$s_0 = 2.5$	0.1441	0.4937	0.1555	0.4111	0.8327
$n = 10$	$s_0 = 2.0$	0.0642	0.2350	0.0687	0.2045	0.8704
	$s_0 = 2.5$	0.0686	0.2312	0.0716	0.2111	0.9129
$n = 15$	$s_0 = 2.0$	0.0368	0.1540	0.0391	0.1407	0.9132
	$s_0 = 2.5$	0.0395	0.1542	0.0410	0.1453	0.9424
$n = 20$	$s_0 = 2.0$	0.0317	0.1140	0.0329	0.1065	0.9346
	$s_0 = 2.5$	0.0297	0.1153	0.0305	0.1104	0.9570

Table 2.33 Simulated means, variances and RE values for Generalized Logistic distribution with $\mu_0 = 0, \delta_0 = 6, \mu = 0, \sigma = 1, b = 6$, when $\hat{\mu}_{Bayes}$ is close to $\hat{\mu}_{MML}$

$b = 6$		$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
$n = 5$	$s_0 = 2.0$	0.1876	0.6158	0.2116	0.4581	0.7439
	$s_0 = 2.5$	0.1754	0.5950	0.1936	0.4818	0.8097
$n = 10$	$s_0 = 2.0$	0.0928	0.2887	0.0995	0.2467	0.8545
	$s_0 = 2.5$	0.0821	0.2865	0.0873	0.2581	0.9008
$n = 15$	$s_0 = 2.0$	0.0571	0.1823	0.0603	0.1641	0.9002
	$s_0 = 2.5$	0.0614	0.1817	0.0634	0.1698	0.9343
$n = 20$	$s_0 = 2.0$	0.0430	0.1352	0.0447	0.1249	0.9240
	$s_0 = 2.5$	0.0438	0.1375	0.0450	0.1307	0.9503

Table 2.34 Simulated means, variances and RE values for Generalized Logistic distribution with $\mu_0 = 0, \delta_0 = 6, \mu = 0, \sigma = 1, b = 8$, when $\hat{\mu}_{Bayes}$ is close to $\hat{\mu}_{MML}$

$b = 8$		$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
$n = 5$	$s_0 = 2.0$	0.2130	0.7156	0.2440	0.5053	0.7061
	$s_0 = 2.5$	0.2156	0.7184	0.2399	0.5613	0.7813
$n = 10$	$s_0 = 2.0$	0.1045	0.3416	0.1142	0.2840	0.8315
	$s_0 = 2.5$	0.1045	0.3256	0.1107	0.2882	0.8849
$n = 15$	$s_0 = 2.0$	0.0661	0.2177	0.0705	0.1927	0.8849
	$s_0 = 2.5$	0.0688	0.2124	0.0715	0.1960	0.9225
$n = 20$	$s_0 = 2.0$	0.0500	0.1588	0.0523	0.1451	0.9138
	$s_0 = 2.5$	0.0460	0.1612	0.0478	0.1519	0.9417

CHAPTER 3

BAYESIAN ANALYSES OF SYMMETRIC and SKEWED FAMILIES ONE WAY EXPERIMENTAL DESIGN

In this chapter, we are interested in developing Bayesian estimators for one way Anova model by considering Student's t and generalized logistic families of distributions. We develop MML and Bayesian estimators of the main effects in one-factor experimental design. As in the one sample case, robust priors are asummed for unknown parameters and marginal posterior densities are derived by combining them with the likelihood function. The resulting HPD estimators are a convex combination of MML estimators and prior hyperparameters which are demonstrated in the following sections. We reiterate, this was made possible by applying the method of modified maximum likelihood estimation.

We start with Student's t family which consists of long-tailed symmetric distributions.

3.1. Student t family:

Consider a one way ANOVA model,

$$y_{ij} = \mu + \gamma_i + \varepsilon_{ij} \; i = 1,\ldots,a \;\; j = 1,\ldots,n \qquad (3.1)$$

where random errors are assumed to have a scaled Student's t distribution. Thus, we can write the density function of y as

$$f(y) = \frac{1}{\sigma\sqrt{k}\beta\left(\frac{1}{2}, p-\frac{1}{2}\right)}\left[1+\frac{(y_{ij}-\mu_i)^2}{k\sigma^2}\right]^{-p}, \quad -\infty < y < \infty, \tag{3.2}$$

where $\mu_i = \mu + \gamma_i$ $(1 \le \text{i} \le \text{a})$, $k = 2p-3$, $\beta(.,.)$ is the beta function and $p \ge 2$. $E(y_{ij}) = \mu_i$ $(1 \le \text{j} \le \text{n})$ and $Var(y_{ij}) = \sigma^2$.

3.1.1. MML Estimators:

A type II symmetric sample $y_{i,(r)} \le y_{i,(r+1)} \le \ldots \le y_{i,(n-r)}$ from (3.2) is given which yields the likelihood function

$$L \propto \sigma^{-a(n-2r)} \prod_{i=1}^{a}\prod_{j=r+1}^{n-r}\left(1+\frac{z_{i(j)}^2}{k}\right)^{-p}\left[F(z_{i,(r+1)})\right]^r\left[1-F(z_{i,(n-r)})\right]^r \tag{3.3}$$

where $z_{i,(j)} = \dfrac{y_{i,(j)}-\mu_i}{\sigma}$.

In order to find the estmators of μ_i and σ, the derivatives of log likelihood function are obtained as

$$\frac{d\ln L}{d\mu_i} \propto \frac{2p}{k\sigma}\sum_{i=r+1}^{n-r} g(z_{i,(j)}) - \frac{r}{\sigma}h_1(z_{i,(r+1)}) + \frac{r}{\sigma}h_2(z_{i,(n-r)})$$

and

$$\frac{d\ln L}{d\sigma} \propto -\frac{A}{\sigma} + \frac{2p}{k\sigma}\sum_{i=r+1}^{n-r} z_{i,(j)} g\left(z_{i,(j)}\right) - \frac{r}{\sigma} z_{i,(r+1)} h_1(z_{i,(r+1)}) + \frac{r}{\sigma} z_{i,(n-r)} h_2(z_{i,(n-r)})$$

(3.4)

where $A = a(n-2r)$, $z_{i,(j)} = \dfrac{y_{i(j)} - \mu_i}{\sigma}$, $g\left(z_{i,(j)}\right) = \dfrac{z_{i,(j)}}{1 + \dfrac{z_{i,(j)}^2}{k}}$, $h_1(z_{i,(j)}) = \dfrac{f(z_{i,(j)})}{F(z_{i,(j)})}$

and $h_2(z_{i,(j)}) = \dfrac{f(z_{i,(j)})}{1 - F(z_{i,(j)})}$.

These equations do not have explicit solutions. Therefore, MML method is used in estimating the parameters instead of ML estimation which is enormously problematic. In order to do this, $g(z_{i,(j)})$, $h_1(z_{i,(j)})$ and $h_2(z_{i,(j)})$ are linerazied as $g(z_{i,(j)}) \cong \alpha_{i,(j)} + \beta_{i,(j)} z_{i,(j)}$, $h_1(z_{i,(r+1)}) \cong a_i - b_i z_{i,(r+1)}$ and $h_2(z_{i,(n-r)}) \cong a_i + b_i z_{i,(n-r)}$. Incorporating these linear approximations in the first derivatives of lnL, the MML estimators are obtained as (Tiku and Suresh, 1992, Section 6):

$$\hat{\mu}_i = \frac{1}{m}\left\{\frac{2p}{k}\sum_{j=r+1}^{n-r} \beta_{(j)} y_{i,(j)} + rb\left[y_{i,(r+1)} + y_{i,(n-r)}\right]\right\}, \quad i = 1,\ldots,a$$

and,

$$\hat{\sigma} = \frac{1}{2A}\left(B + \sqrt{B^2 + 4AC}\right) \tag{3.5}$$

where,

$$m = \frac{2p}{k}\sum_{j=r+1}^{n-r} \beta_j + 2rb,\ B_i = \frac{2p}{k}\sum_{j=r+1}^{n-r} \alpha_j y_{i,(j)} + ra\left[y_{i,(n-r)} + y_{i,(r+1)}\right]$$

$$C_i = \frac{2p}{k}\sum_{j=r+1}^{n-r}\beta_j y_{i,(j)}{}^2 + rb\left[y_{i,(r+1)}^2 + y_{i,(n-r)}^2\right] - m\hat{\mu}_i^2 ,$$

$$B = \sum_{i=1}^{a} B_i \;,\; C = \sum_{i=1}^{a} C_i \;,$$

$$\alpha_j = \frac{(2/k)t_{(j)}^3}{\left[1+(1/k)t_{(j)}^2\right]^2} \;, \beta_j = \frac{1-(1/k)t_{(j)}^2}{\left[1+(1/k)t_{(j)}^2\right]^2} \;,$$

$$b = \left[-f'\left(t_{(r+1)}F(t_{(r+1)}\right) + f^2(t_{(r+1)}\right]F^{-2}(t_{(r+1)}) \;, a = \left\lfloor f(t_{(r+1)})/F(t_{(r+1)})\right\rfloor + b\, t_{(r+1)}$$

Incidentally, the MML estimator of μ is $\hat{\mu} = (1/a)\sum_{i=1}^{a}\hat{\mu}_i$ and $\hat{\gamma}_i = \hat{\mu}_i - \hat{\mu}$.

Note that complete sample results can easily be obtained by taking $r = 0$. Of course, certain regularity conditions have to be satisfied as said earlier.

Remark: Tiku and Suresh (1992) and Vaughan(1992) indicate in their works that for fixed $q = r/n$, these MML estimators are MVB estimators asymptotically. Also, they are almost fully efficient for small sample sizes.

The asymptotic properties of MML estimators are the following:

i) $\frac{\sqrt{m}\,(\hat{\mu}_i - \hat{\mu})}{\hat{\sigma}}$ is distributed as normal (0,1),

ii) $\frac{A\hat{\sigma}^2}{\sigma^2}$ is distributed as $\chi^2_{(A-a)}$,

iii) $\hat{\mu}_i$ and $\hat{\sigma}$ are independently distributed.

According to these results, the joint density of $\hat{\mu}_i$ $(1 \le \text{i} \le \text{a})$ and $\hat{\sigma}^2$ is

$$f(\hat{\mu}_1,\ldots,\hat{\mu}_a,\hat{\sigma}^2) \propto \left(\sigma^{-2}\right)^{(A-a)/2}(\hat{\sigma}^2)^{A-a/2-1} \times$$
$$\exp(-\frac{A}{2\sigma^2})\prod \exp(-\frac{m}{2\hat{\sigma}^2}(\hat{\mu}_i - \mu_i)^2) \qquad (3.6)$$

3.1.2. Prior Distribution:

Robust prior distributions with known hyperparameters are assumed as independent Student's t and inverted gamma distribution which yields

$$p(\sigma^2, \mu_1, \ldots \mu_a) \propto \left(\sigma^{-2}\right)^{\delta_0/2+1} \exp(-\frac{1}{2\sigma^2}\delta_0 s_0^2) \times$$

$$\prod \left(1 + \frac{(\mu_i - \mu_{i0})^2}{v_{i0}\sigma_{i0}^2}\right)^{-\frac{(v_{i0}+1)}{2}} . \qquad (3.7)$$

Bian (1996) considers the situation when the joint prior density of μ_i $(1 \le \text{i} \le \text{a})$ is a product of normal densities. His results immediately follow from ours as a particular case by taking υ_{i0} equal to infinity for all $i = 1, \ldots a$. Bian also takes $r = 0$ in (3.3). Our results are much more general.

3.1.3. Posterior Distribution:

Combining the prior distribution (3.7) with the likelihood function (3.6), the posterior distribution are obtained and given by (3.8) below:

$$f\left(\mu_1, \ldots, \mu_a, \sigma^2 \mid y\right) \propto (\sigma^{-2})^{\frac{A-a+\delta_0}{2}+1} \exp\left(-\frac{\sigma^{-2}}{2}\left(\delta_0 s_0^2 + A\hat{\sigma}^2\right)\right) \times$$

$$\prod_{i=1}^{a} \exp\left(-\frac{m}{2\hat{\sigma}^2}(\mu_i - \hat{\mu}_i)^2\right)\left(1 + \frac{\left(\mu_i - \mu_{i0}\right)^2}{v_{i0}\sigma_{i0}^2}\right)^{-\frac{(v_{i0}+1)}{2}} . \qquad (3.8)$$

It is seen from (3.8) that the posterior density of $\mu_1, \ldots, \mu_a$ and σ^2 are independent. Marginal posterior density of σ^2 is inverse gamma with

$inverse-gamma(\frac{\delta_0 + A - a}{2}, \frac{\delta_0 s_0^2 + A\hat{\sigma}^2}{2})$ while marginal posterior of μ is a polyt density $f(\mu_i \mid y) \propto \exp\left(-\frac{m}{2\hat{\sigma}^2}(\mu_i - \hat{\mu}_i)^2\right)\left(1 + \frac{(\mu_i - \mu_{i0})^2}{v_{i0}\sigma_{i0}^2}\right)^{-\frac{(v_{i0}+1)}{2}}$.

3.1.4 HPD Estimators:

The HPD estimator of σ^2 is the mode of inverse gamma density, which can be expressed as

$$\hat{\sigma}^2{}_b = \left(\delta_0 s^2{}_0 + A\hat{\sigma}^2\right)/\left(\delta_0 + A\right) \tag{3.9}$$

As in previous cases, it is a weighted average of prior information and MML estimator of σ^2.

Since the marginal posterior density of μ is a poly t density two different cases are considered to find HPD estimator of μ.

Case1: If v_{i0} is infinite, than the prior of μ_i reduces to a normal density and the posterior is

$$p(\mu_i \mid y) \propto \exp(-\frac{1}{2\sigma_{i0}^2}(\mu_i - \mu_{i0})^2 \exp(-\frac{m}{2\hat{\sigma}^2}(\mu_i - \hat{\mu}_i)^2). \tag{3.10}$$

It has been shown in Chapter 2 that the posterior distribution is normal, and the Bayesian estimator of μ is the mode, (same as the mean) of the posterior distribution which can be expressed as

$$\hat{\mu}_{b,i} = \frac{\sigma_{i0}^{-2}\mu_{i0} + m\ \hat{\sigma}^{-2}\hat{\mu}_i}{\sigma_{i0}^{-2} + m\ \hat{\sigma}^{-2}} \tag{3.11}$$

with variance $Var(\mu_i \mid y) = \left(\sigma_{i0}^{-2} + m\hat{\sigma}^{-2}\right)^{-1}$.

The overall esitmator of μ is obtained as

$$\hat{\mu}_b = \sum_{i=1}^{a} c_i \hat{\mu}_{b,i} \ , \ \ c_i = \left(\sigma_{i0}^{-2} + m\hat{\sigma}^{2}\right)\left[\sum_{i=1}^{a}\left(\sigma_{i0}^{-2} + m\ \hat{\sigma}^{-2}\right)\right]^{-1} \tag{3.12}$$

and the Bayesian estimator of factor effects is,

$$\hat{\gamma}_{i,b} = \hat{\mu}_{i,b} - \hat{\mu}_b, i = 1,\ldots,a\ . \tag{3.13}$$

Alternatively, the weighted form of $\hat{\mu}_{b,i}$ can be written as $\hat{\mu}_{b,i} = w_i \mu_{i0} + (1 - w_i)\hat{\mu}_i$ where $w_i = \sigma_{i0}^{-2}\left(\sigma_{i0}^{-2} + m\ \hat{\sigma}^{-2}\right)^{-1}$**.** These are beautiful results indeed.

Comment: As in the one sample case, $\hat{\mu}_{b,i}$ is a weighted combination of MML estimator and prior hyperparameter. From the weighted form of $\hat{\mu}_{b,i}$ we can see that as m goes to infinity Bayesian estimator converges to MML estimator since weight w_i becomes zero. On the other hand, if $\sigma_{i0} = 0$ and so $w_i = 1$ then the Bayes estimator will be equal to μ_{i0}**,** which is expected.

Case2: When v_{i0} is finite, then $\hat{\mu}_{b,i}$ is the solution of the modal equation

$$(v_{i0} + 1)(\mu_i - \mu_{i0})\hat{\sigma}^2 + m\ (\mu_i - \hat{\mu}_i)\left[v_{i0}\sigma_{i0}^2 + (\mu_i - \mu_{i0})^2\right] = 0. \tag{3.14}$$

Applying exactly the same procedures as in Chapter 2 and re-organizing the resulting equations, we obtain Bayesian estimator $\hat{\mu}_{b,i}$ in two different forms as follows:

When the posterior density of μ_i is governed by the prior density, the HPD estimator of μ_i is

$$\hat{\mu}_{b,i} = \frac{\sigma_{i0}^{-2}\left(1+\frac{1}{v_{i0}}\right)\mu_{i0} + m\,\hat{\sigma}^{-2}\hat{\mu}_i}{\sigma_{i0}^{-2}\left(1+\frac{1}{v_{i0}}\right) + m\,\hat{\sigma}^{-2}}. \tag{3.15}$$

On the other hand, when the posterior density of μ_i is governed by the sampling density, the HPD estimator of μ_i is

$$\hat{\mu}_{i,b} = \frac{\sigma_{i0}^{-2}\mu_{i0} + \left\{\left[v_{i0} + (\Delta\mu_i)^2\right]/(v_{i0}+1)\right\} m\,\hat{\sigma}^{-2}\hat{\mu}_i}{\sigma_{i0}^{-2} + \left\{\left[v_{i0} + (\Delta\mu_i)^2\right]/(v_{i0}+1)\right\} m_i\hat{\sigma}^{-2}} \tag{3.16}$$

where $\Delta\mu_i = \frac{(\hat{\mu}_i - \mu_{i0})}{\sigma_{i0}}$.

Comments:

The HPD estimators of μ_i given by (3.11), (3.15) and (3.16) contain information from robust priors and MML estimators. They have the weighted form which makes them robust to outliers. If we look at some special cases we see that,

i) The prior density of μ_i converges to a non-informative prior if dispersion of it goes to infinity. Therefore, for large σ_{i0} values HPD estimator of μ_i converges to MML esitmator.

ii) If prior dispersion σ_{i0} goes to zero then HPD estimator of μ_i tends to prior hyperparameter μ_{i0}.

iii) When sample observation have a large dispersion, HPD estimator of μ_i converges to μ_{i0} since prior information takes much importance as sample variance increases.

iv) If sample size increses, the information coming from likelihood function dominates over prior density. As a result, Bayes estimator converges to MML estimators.

3.1.5. Comparing Efficiencies of MML and Bayes Estimators: Simulation Results

In order to compare relative efficiencies of HPD estimators and MMLEs we simulate random variables from (3.1) with p=3.5. Simulated mean and variances of MML estimators and two Bayesian estimators given by (3.5), (3.15) and (3.16) are obtained by 10,000 simulations. The degree of freedom of prior distribution (3.7) is taken as 6, while $\mu_{i0} = 0$ and μ_i =0. We assign different values to σ_{i0} and σ to see the effect on relative efficiencies. We generate independent random variables for two treatments containing fixed number of observations. According to the simulation results given in Table 3.1 to Table 3.8 below we can say the followings:

Comparing the results for both censoring, with q=r/n being fixed, and full sample cases it is clear that Bayes estimator have lower variances. However, they loose their advantage when prior dispersion increases. On the other hand, they gain efficiency when sample dispersion increases. Moreover, as the number of observations in each treatment increase the relative efficiencies increases since prior information is dominated by the likelihood funtion for large sample sizes. That is to say, all of the statments made in the one sample case are valid for the one way classification model that is the beauty in the convex combination of MML estimators and prior information.

Table 3.1 Simulated means, variances and RE values for censored Student t distribution with $\mu_{i0}=0, s_{i0}=3, \delta_{i0}=6, \mu_i=0$, when $\hat{\mu}_{Bayes}$ is close to μ_0

q=0.2			$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
ntr=2 nbl=5	σ=1.5	i=1	-0.0101	0.6332	-0.0092	0.5454	0.8614
		i=2	-0.0108	0.6320	-0.0107	0.5450	0.8624
	σ=2.5	i=1	-0.0109	1.7824	-0.0062	1.2612	0.7076
		i=2	-0.0050	1.7715	-0.0047	1.2509	0.7062
ntr=2 nbl=10	σ=1.5	i=1	-0.0064	0.3049	-0.0062	0.2814	0.9229
		i=2	0.0023	0.3094	0.0022	0.2856	0.9230
	σ=2.5	i=1	0.0030	0.8438	0.0023	0.6852	0.8120
		i=2	0.0194	0.8565	0.0176	0.6928	0.8089
ntr=2 nbl=15	σ=1.5	i=1	0.0030	0.2078	0.0030	0.1969	0.9475
		i=2	-0.0037	0.2019	-0.0037	0.1914	0.9478
	σ=2.5	i=1	-0.0046	0.5695	-0.0042	0.4922	0.8643
		i=2	-0.0041	0.5613	-0.0039	0.4845	0.8633

Table 3.2 Simulated means, variances and RE values for censored Student t distribution with $\mu_{i0}=0, \sigma=1.5, \delta_{i0}=6, \mu_i=0$, when $\hat{\mu}_{Bayes}$ is close to μ_0

q=0.2			$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
ntr=2 nbl=5	s_0=2.0	i=1	-0.0011	0.6357	0.0005	0.4701	0.7395
		i=2	-0.0015	0.6463	-0.0003	0.4767	0.7376
	s_0=2.5	i=1	-0.0179	0.6475	-0.0172	0.5269	0.8137
		i=2	0.0029	0.6555	0.0033	0.5317	0.8112
ntr=2 nbl=10	s_0=2.0	i=1	-0.0081	0.3118	-0.0075	0.2617	0.8394
		i=2	-0.0021	0.3042	-0.0018	0.2562	0.8422
	s_0=2.5	i=1	0.0085	0.3075	0.0082	0.2746	0.8929
		i=2	0.0022	0.3061	0.0021	0.2732	0.8927
ntr=2 nbl=15	s_0=2.0	i=1	0.0006	0.2059	0.0006	0.1828	0.8880
		i=2	-0.0104	0.2035	-0.0098	0.1809	0.8887
	s_0=2.5	i=1	-0.0002	0.2043	-0.0002	0.1891	0.9258
		i=2	0.0017	0.1988	0.0016	0.1841	0.9256

Table 3.3 Simulated means, variances and RE values for Student t distribution with $\mu_{i0} = 0, s_{i0} = 3, \delta_{i0} = 6, \mu_i = 0$, when $\hat{\mu}_{Bayes}$ is close to μ_0

			$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
ntr=2 nbl=5	σ=1.5	i=1	-0.0074	0.6315	-0.0067	0.5336	0.8451
		i=2	-0.0017	0.6307	-0.0016	0.5326	0.8446
	σ=2.5	i=1	0.0150	1.7509	0.0139	1.1768	0.6721
		i=2	0.0315	1.7010	0.0257	1.1342	0.6668
ntr=2 nbl=10	σ=1.5	i=1	0.0014	0.3044	0.0015	0.2793	0.9176
		i=2	-0.0084	0.3048	-0.0080	0.2800	0.9187
	σ=2.5	i=1	-0.0029	0.8322	-0.0025	0.6642	0.7981
		i=2	-0.0025	0.8324	-0.0027	0.6634	0.7969
ntr=2 nbl=15	σ=1.5	i=1	-0.0072	0.2009	-0.0070	0.1899	0.9454
		i=2	0.0026	0.2005	0.0025	0.1894	0.9446
	σ=2.5	i=1	-0.0052	0.5506	-0.0047	0.4773	0.8595
		i=2	-0.0099	0.5558	-0.0092	0.4776	0.8594

Table 3.4 Simulated means, variances and RE values for Student t distribution with $\mu_{i0} = 0, \sigma = 1.5, \delta_{i0} = 6, \mu_i = 0$, when $\hat{\mu}_{Bayes}$ is close to μ_0

			$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
ntr=2 nbl=5	s_0=2.0	i=1	0.0064	0.6222	0.0064	0.4425	0.7112
		i=2	-0.0040	0.6203	-0.0036	0.4415	0.7118
	s_0=2.5	i=1	-0.008	0.6282	-0.0012	0.5003	0.7964
		i=2	0.0042	0.6230	0.0048	0.4958	0.7959
ntr=2 nbl=10	s_0=2.0	i=1	-0.0019	0.3008	-0.0018	0.2502	0.8316
		i=2	0.0030	0.2993	0.0028	0.2494	0.8334
	s_0=2.5	i=1	0.0032	0.3026	0.0032	0.2682	0.8863
		i=2	0.0013	0.3033	0.0013	0.2689	0.8866
ntr=2 nbl=15	s_0=2.0	i=1	0.0018	0.1953	0.0018	0.1725	0.8834
		i=2	0.0061	0.1976	0.0057	0.1742	0.8815
	s_0=2.5	i=1	0.0013	0.2021	0.0011	0.1863	0.9220
		i=2	0.0001	0.1956	-0.0001	0.1804	0.9224

Table 3.5 Simulated means, variances and RE values for censored Student t distribution with $\mu_{i0} = 0, s_{i0} = 3, \delta_{i0} = 6, \mu_i = 0$, when $\hat{\mu}_{Bayes}$ is close to $\hat{\mu}_{MML}$

q=0.2			$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
ntr=2 nbl=5	σ=1.5	i=1 i=2	-0.0187 -0.0125	0.6451 0.6230	-0.0174 -0.0116	0.5603 0.5409	0.8686 0.8682
	σ=2.5	i=1 i=2	-0.0105 0.0166	1.7576 1.7678	-0.0094 0.0148	1.2622 1.2826	0.7181 0.7256
ntr=2 nbl=10	σ=1.5	i=1 i=2	-0.0013 0.0041	0.3122 0.3007	-0.0014 0.0037	0.2884 0.2780	0.9236 0.9245
	σ=2.5	i=1 i=2	0.0043 0.0080	0.8433 0.8741	0.0040 0.0079	0.6888 0.7124	0.8168 0.8150
ntr=2 nbl=15	σ=1.5	i=1 i=2	-0.0064 0.0086	0.2028 0.2037	-0.0062 0.0084	0.1922 0.1931	0.9480 0.9479
	σ=2.5	i=1 i=2	-0.0114 -0.0003	0.5553 0.5470	-0.0106 0.0001	0.4817 0.4839	0.8674 0.8688

Table 3.6 Simulated means, variances and RE values for censored Student t distribution with $\mu_{i0} = 0, \sigma = 1.5, \delta_{i0} = 6, \mu_i = 0$, when $\hat{\mu}_{Bayes}$ is close to $\hat{\mu}_{MML}$

q=0.2			$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
ntr=2 nbl=5	s_0=2.0	i=1 i=2	0.0094 0.0118	0.6367 0.6297	0.0083 0.0101	0.4804 0.4765	0.7545 0.7567
	s_0=2.5	i=1 i=2	0.0069 -0.0155	0.6349 0.6354	0.0069 -0.0145	0.526 0.5251	0.8231 0.8263
ntr=2 nbl=10	s_0=2.0	i=1 i=2	0.0029 0.0058	0.3072 0.3103	0.0026 0.0053	0.2594 0.2620	0.8445 0.8444
	s_0=2.5	i=1 i=2	-0.0012 0.0013	0.3052 0.3120	-0.0013 0.0014	0.2732 0.2790	0.8954 0.8941
ntr=2 nbl=15	s_0=2.0	i=1 i=2	-0.0049 -0.0019	0.2036 0.1997	-0.0047 -0.0019	0.1811 0.1779	0.8896 0.8908
	s_0=2.5	i=1 i=2	-0.0055 -0.0048	0.2056 0.2012	-0.0053 -0.0046	0.1904 0.1865	0.9261 0.9268

Table 3.7 Simulated means, variances and RE values for Student t distribution with $\mu_{i0}=0, s_{i0}=3, \delta_{i0}=6, \mu_i=0$, when $\hat{\mu}_{Bayes}$ is close to $\hat{\mu}_{MML}$

			$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
ntr=2 nbl=5	σ=1.5	i=1	0.0016	0.6373	0.0020	0.5414	0.8495
		i=2	-0.0036	0.6172	-0.0023	0.5239	0.8488
	σ=2.5	i=1	-0.0251	1.6943	-0.0212	1.1721	0.6918
		i=2	-0.0252	1.7447	-0.0202	1.2036	0.6899
ntr=2 nbl=10	σ=1.5	i=1	0.0011	0.3042	0.0011	0.2796	0.9189
		i=2	-0.0031	0.3003	-0.0031	0.2762	0.9197
	σ=2.5	i=1	-0.0177	0.8159	-0.0160	0.6598	0.8087
		i=2	0.0056	0.8290	0.0052	0.6694	0.8075
ntr=2 nbl=15	σ=1.5	i=1	0.0014	0.1957	0.0013	0.1850	0.9456
		i=2	-0.0061	0.1994	-0.0060	0.1886	0.9459
	σ=2.5	i=1	-0.0060	0.5545	-0.0054	0.4786	0.8631
		i=2	-0.0045	0.5455	-0.0043	0.4710	0.8635

Table 3.8 Simulated means, variances and RE values for Student t distribution with $\mu_{i0}=0, \sigma=1.5, \delta_{i0}=6, \mu_i=0$, when $\hat{\mu}_{Bayes}$ is close to $\hat{\mu}_{MML}$

			$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
ntr=2 nbl=5	s_0=2.0	i=1	-0.0052	0.6291	-0.0034	0.4579	0.7279
		i=2	0.0067	0.6260	0.0061	0.4542	0.7256
	s_0=2.5	i=1	-0.0019	0.6272	-0.0027	0.5022	0.8007
		i=2	0.0040	0.6105	0.0039	0.4915	0.8052
ntr=2 nbl=10	s_0=2.0	i=1	-0.0036	0.3057	-0.0032	0.2562	0.8380
		i=2	-0.0055	0.2957	-0.0049	0.2479	0.8381
	s_0=2.5	i=1	0.0036	0.3020	0.0033	0.2688	0.8901
		i=2	0.0024	0.2964	0.0025	0.2637	0.8896
ntr=2 nbl=15	s_0=2.0	i=1	-0.0008	0.1965	-0.0008	0.1742	0.8867
		i=2	0.0024	0.1995	0.0023	0.1766	0.8853
	s_0=2.5	i=1	-0.0036	0.1999	-0.0035	0.1846	0.9236
		i=2	-0.0002	0.1992	-0.0002	0.1841	0.9240

3.2. Generalized Logistic Distribution:

Consider the model (3.1) with error terms having generalized logistic distribution. The density function of y_{ij} is

$$f(y_{ij}) = \frac{b}{\sigma} \frac{\exp\left(-\frac{y_{ij}-\mu_i}{\sigma}\right)}{\left[1+\exp\left(-\frac{y_{ij}-\mu_i}{\sigma}\right)\right]^{b+1}}, -\infty < y < \infty \tag{3.17}$$

3.2.1. MML Estimators:

In order to find MML estimators, the derivatives of likelihood functions are,

$$\frac{d\ln L}{d\mu_i} \propto \frac{n}{\sigma} - \frac{b+1}{\sigma}\sum_{j=1}^{n} g(z_{i(j)})$$

and

$$\frac{d\ln L}{d\sigma} \propto -\frac{N}{\sigma} + \frac{1}{\sigma}\sum_{i=1}^{a}\sum_{j=1}^{n} z_{i(j)} - \frac{b+1}{\sigma}\sum_{i=1}^{a}\sum_{j=1}^{n} z_{i(j)} g(z_{i(j)}) \tag{3.18}$$

where $g(z_{i(j)}) = \frac{e^{-z_{i(j)}}}{1+e^{-z_{i(j)}}}$ and $z_{i(j)} = \frac{y_{i(j)}-\mu_i}{\sigma}$.

In order to find MML estimators, we linearize $g(z_{i(j)})$ as $g(z_{i(j)}) = \alpha_{i(j)} - \beta_{i(j)} z_{i(j)}$ where $\alpha_{i(j)}$ and $\beta_{i(j)}$ are obtained from first two terms of Taylor series expansion as before:

$$\alpha_{i(j)} = \frac{1+e^{t_{(j)}} + t_{(j)}e^{t_{(j)}}}{\left(1+e^{t_{(j)}}\right)^2} \text{ and } \beta_{i(j)} = \frac{e^{t_{(j)}}}{\left(1+e^{t_{(j)}}\right)^2}; \tag{3.19}$$

where $t_{i(j)} = t_{(j)} = -\ln\left(q_j^{-\frac{1}{b}} - 1\right)$ and $q_j = \frac{j}{n+1}$.

Note that $\alpha_{1(j)} = \alpha_{2(j)} = \ldots = \alpha_{a(j)} = \alpha_j$ and $\beta_{1(j)} = \beta_{2(j)} = \ldots \beta_{a(j)} = \beta_j$ for all $j = 1, \ldots, n$. Therefore, equation (3.18) is written as

$$\frac{d \ln L}{d\mu_i} \propto \frac{n}{\sigma} - \frac{b+1}{\sigma}\sum_{j=1}^{n}\left(\alpha_j - \beta_j z_{i(j)}\right)$$

and

$$\frac{d \ln L}{d\sigma} \propto -\frac{N}{\sigma} + \frac{1}{\sigma}\sum_{i=1}^{a}\sum_{j=1}^{n} z_{i(j)} - \frac{b+1}{\sigma}\sum_{i=1}^{a}\sum_{j=1}^{n} z_{i(j)}\left(\alpha_j - \beta_j z_{i(j)}\right). \tag{3.20}$$

MML estimators are obtained by solving (3.20) and are given by Tiku and Akkaya (2004) as follows:

$$\hat{\mu}_i = \frac{\sum_{j=1}^{n}\beta_j y_{i(j)}}{\sum_{j=1}^{n}\beta_j} - \hat{\sigma}\frac{D}{m}$$

and

$$\hat{\sigma} = \frac{-B + \sqrt{B^2 + 4NC}}{2N} \tag{3.21}$$

where

$$\Delta_j = \alpha_j - \frac{1}{b+1}, \; D = \sum_{j=1}^{n} \Delta_j \; , \; m = \sum_{j=1}^{n} \beta_j \; , \; K_i = \frac{\sum_{j=1}^{n} \beta_j y_{i(j)}}{\sum_{j=1}^{n} \beta_j} ,$$

$$B_i = (b+1)\sum_{j=1}^{n} \Delta_j \left(y_{i(j)} - K_i\right), \; B = \sum_{i=1}^{a} B_i \; ,$$

$$C_i = (b+1)\sum_{j=1}^{n} \beta_j \left(y_{i(j)} - K_i\right)^2 , \; C = \sum_{i=1}^{a} C_i \; .$$

Then, we can write, $\hat{\mu} = (1/a)\sum_{i=1}^{a} \hat{\mu}_i$ and $\hat{\gamma}_i = \hat{\mu}_i - \hat{\mu}$.

Variances of estimators are obtained from the inverse of the Fisher information matrix given below:

$$I = \frac{n}{\sigma^2} \times$$

$$\begin{bmatrix} \frac{b}{b+2} & \frac{b}{b+2}\left[\psi(b+1) - \psi(2)\right] \\ \frac{b}{b+2}\left[\psi(b+1) - \psi(2)\right] & a + \frac{ab}{b+2}\left[\psi'(b+1) + \psi'(2) + \left(\psi(b+1) - \psi(2)\right)^2\right] \end{bmatrix} \quad (3.22)$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the psi-function and $\psi'(x)$ is the derivative of $\psi(x)$ with respect to x.

3.2.2. Prior and Posterior Distributions:

We will find marginal posterior densities of μ_i and ξ_i where $\hat{\xi}_i = \hat{\mu}_i + k\hat{\sigma}$. The value of k is determined as to make $\hat{\mu}_i$ and $\hat{\xi}_i$ uncorrelated and it is simply $k = -Var(\hat{\mu}_i) / \text{cov}(\hat{\mu}_i, \hat{\sigma})$ where

$$k = \frac{a(b+2) + ab[\psi'(b+1) + \psi'(2)] + [\psi(b+1) - \psi(2)]^2}{b[\psi(b+1) - \psi(2)]}. \tag{3.23}$$

In order to write the joint density function of $\hat{\mu}_i$ and $\hat{\xi}_i$ we use the bivariate normality of $d \ln L / d\mu_i$ and $d \ln L / d\xi_i$ and write

$$f\left(\hat{\mu}_i, \hat{\xi}_i\right) \propto \sigma^{-2an} \exp\left(-\frac{1}{2\sigma^2} \left(\frac{(\hat{\mu}_i - \mu)^2}{k_1} + \frac{\left(\hat{\xi}_i - \xi\right)^2}{k_2} \right)\right) \tag{3.24}$$

where by k_1 and k_2 are obtained from

$$Var(\hat{\mu}_i, \hat{\xi}_i) = \sigma^2 \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}. \tag{3.25}$$

From (3.24) it is also seen that $\hat{\mu}_i$ and $\hat{\xi}_i$ are uncorrelated and the marginal densities of them can be written as

$$f(\hat{\mu}_i) \propto \sigma^1 \exp\left[-\frac{1}{2\sigma^2} \frac{(\hat{\mu}_i - \mu_i)^2}{k_1} \right]$$

and

$$f\left(\hat{\xi}_i\right) \propto \sigma^{-1} \exp\left[-\frac{1}{2\sigma^2}\frac{\left(\hat{\xi}_i - \xi_i\right)^2}{k_2}\right]. \tag{3.26}$$

In order to obtain more accurate results the distribution of $\hat{\xi}_i$ can be approximated by chi-square distribution with $2n$ degrees of freedom.

In order to obtain posterior densities of parameters, independent robust priors for μ_i and ξ_i are assumed:

$$p\left(\mu_i, \xi_i\right) \propto p(\mu_i) p(\xi_i) \tag{3.27}$$

where

$$p(\mu_i) \propto \left[1 + \frac{(\mu_i - \mu_{i0})^2}{\delta_{i0} s_{i0}^2}\right]^{-\frac{(\delta_{i0}+1)}{2}}$$

and,

$$p\left(\xi_i\right) \propto \xi_i^{-n_{i0}} \exp\left(-\frac{n_{i0}\xi_{i0}}{\xi_i}\right); \tag{3.28}$$

$\mu_{i0}, \delta_{i0}, s_{i0}^2, n_{i0}, \xi_{i0}$ are hyperparameters considered as known and fixed.

Combining sample information with prior densities, the joint posterior distribution of parameters is given by,

$$f\left(\mu_i, \xi_i \middle| y\right) \propto \xi_i^{-(n+n_{i0})} \exp\left[-\frac{\left(n_{i0}\xi_{i0} + n\hat{\xi}_i\right)}{\xi_i}\right] \times$$

$$\exp\left[-\frac{k_1}{2\hat{\sigma}^2}(\hat{\mu}_i - \hat{\mu})^2\right]\left[1 + \frac{(\mu_i - \mu_{i0})^2}{\delta_{i0} s_{i0}^2}\right]^{-\frac{(\delta_{i0}+1)}{2}} \tag{3.29}$$

The form (3.30) indicates that μ_i and ξ_i^{-1} are posteriorly independent and marginal posterior densities are

$$f(\xi|y) \propto \xi_i^{-(n+n_{i0})} \exp\left(-\frac{n_{i0}\xi_{i0} + n\hat{\xi}_i}{\xi}\right)$$

and

$$f(\mu_i|y) \propto \exp\left[-\frac{h_2}{2\hat{\sigma}^2}(\hat{\mu}_i - \hat{\mu})^2\right]\left[1 + \frac{(\mu_i - \mu_{i0})^2}{\delta_{i0}s_{i0}^2}\right]^{-\frac{(\delta_{i0}+1)}{2}} \tag{3.30}$$

where $h_2 = 1/k_1$.

3.2.3. HPD Estimators:

$\hat{\xi}_{Bayes,i}$ is the mode of the inverse gamma density given by (3.30) and it can be expressed as

$$\hat{\xi}_{Bayes,i} = \frac{n_{i0}\xi_{i0} + n\hat{\xi}_i}{n_{i0} + n}. \tag{3.31}$$

In order to find the Bayesian estimator of μ_i, the poly-t density given by (3.30) is solved by following exactly the same procedures as in the previous sections and two cases for the estimator of μ_i are obtained.

***Case1*:** If δ_{i0} is infinite, then the marginal density of μ_i is,

$$f(\mu_i|y) \propto \exp\left[-\frac{1}{2s_{i0}^2}(\mu_i - \mu_{i0})^2\right]\exp\left[-\frac{h_2}{2\hat{\sigma}^2}(\mu_i - \hat{\mu}_i)^2\right] \tag{3.32}$$

which yields the Bayes estimator as

$$\hat{\mu}_{Bayes,i} = \frac{s_{i0}^{-2}\mu_{i0} + h_2\hat{\sigma}^{-2}\hat{\mu}_i}{s_{i0}^{-2} + h_2\hat{\sigma}^{-2}}. \tag{3.33}$$

Case2: If δ_{i0} is finite, then the solution of modal equation yields the Bayes estimators as follows:

i) When Bayesian estimator $\hat{\mu}_{Bayes,i}$ is closes to prior mean μ_{i0}, then

$$\hat{\mu}_{Bayesi} = \frac{s_{i0}^{-2}\left[1+\frac{1}{\delta_{i0}}\right]\mu_{i0} + h_2\hat{\sigma}^{-2}\hat{\mu}_i}{s_{i0}^{-2}\left[1+\frac{1}{\delta_{i0}}\right] + h_2\hat{\sigma}^{-2}}. \tag{3.34}$$

ii) When Bayesian estimator $\hat{\mu}_{Bayes,i}$ is closes to sample mean $\hat{\mu}_i$, then

$$\hat{\mu}_{Bayes,i} = \frac{s_{i0}^{-2}\mu_{i0} + \left[\left(\delta_{i0} + (\Delta\mu_i)^2\right)/(\delta_{i0}+1)\right] h_2\hat{\sigma}^{-2}\hat{\mu}_i}{s_{i0}^{-2} + \left[\left(\delta_{i0} + (\Delta\mu_i)^2\right)/(\delta_{i0}+1)\right] h_2\hat{\sigma}^{-2}} \tag{3.35}$$

where $\Delta\mu_i = (\hat{\mu}_i - \mu_{i0})/s_{i0}$.

As is seen from (3.33), (3.34) and (3.35), the form of Bayes estimator is exactly the same as in the case of Student's t distribution. So we can write

$$\hat{\mu}_{Bayes,i} = w_i\mu_{i0} + (1-w_i)\hat{\mu}_i. \tag{3.36}$$

The weighted form of Bayesian estimator makes it possible to produce exactly the same arguments as in previous section which assumes Student's t distribution for error terms.

3.2.4. Comparing Efficiencies of MML and Bayes Estimators: Simulation Results

Considering generalized logistic family we carry out simulations for two treatments containing 3,5,10 and 15 observations by assuming different b values. HPD estimators (3.34) and (3.35) are obtained by assuming prior density as (3.28) with μ_0=0 and $s_0 = 2$ and $s_0 = 2.5$. The random observations are assumed to have the form (3.17) with $\mu = 0$, $\sigma = 1$ and $\sigma = 1.5$. The simulated means, variances and relative efficiencies shown below indicate that for b=0.5 HPD estimators are generally better, like in the one sample case. Since the HPD estimator is a convex combination, the relationship with prior and sample dispersion and relative efficiencies are the same as before. Increasing prior variance yields less favorable HPD estimator while increasing σ values gives more efficient Bayesian estimators. Similar statements can be made for Bayesian estimators (3.34) and (3.35) except that the estimator given by (3.34) has a little higher efficiencies. In addition to these, we can also say that, HPDs are better for small sample sizes although for all sample sizes they break the MVB barrier which is not possible in the classical statistical analyses.

Remark: All of the simulation results are obtained when we have two treatments. However, they can be generalized to more than two treatments by using exactly similar arguments.

Table3.9 Simulated values for Generalized Logistic distribution with $\mu_{i0}=0, s_{i0}=3, \delta_{i0}=6, \mu_i=0, \sigma=0.75, b=0.5$ $\hat{\mu}_{Bayes}$ is close to μ_0

$b=0.5$		$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
ntr=2 nbl=3	**i=1** **i=2**	-0.1348 -0.1475	1.0497 1.0621	-0.1232 -0.1378	0.7543 0.7637	0.7187 0.7190
ntr=2 nbl=5	**i=1** **i=2**	-0.0876 -0.0598	0.6315 0.6354	-0.0839 -0.0592	0.5240 0.5237	0.8299 0.8243
ntr=2 nbl=10	**i=1** **i=2**	-0.0225 -0.0244	0.2965 0.2975	-0.0228 -0.0243	0.2721 0.2728	0.9178 0.9167
ntr=2 nbl=15	**i=1** **i=2**	-0.0102 -0.0108	0.1990 0.2000	-0.0104 -0.0109	0.1884 0.1893	0.9469 0.9466

Table 3.10 Simulated values for Generalized Logistic distribution with $\mu_{i0}=0, s_{i0}=3, \delta_{i0}=6, \mu_i=0, \sigma=0.75, b=1$, when $\hat{\mu}_{Bayes}$ is close to μ_0

$b=1$		$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
ntr=2 nbl=3	**i=1** **i=2**	-0.0080 0.0176	0.6038 0.5975	-0.0080 0.0160	0.4897 0.4844	0.8111 0.8107
ntr=2 nbl=5	**i=1** **i=2**	0.0010 0.0003	0.3535 0.3476	0.0011 0.0004	0.3153 0.3103	0.8918 0.8925
ntr=2 nbl=10	**i=1** **i=2**	0.0065 0.0017	0.1747 0.1697	0.0064 0.0017	0.1662 0.1614	0.9510 0.9509
ntr=2 nbl=15	**i=1** **i=2**	0.0010 -0.0032	0.1151 0.1161	0.0010 -0.0031	0.1114 0.1123	0.9682 0.9681

Table 3.11 Simulated values for Generalized Logistic distribution with $\mu_{i0} = 0, s_{i0} = 3, \delta_{i0} = 6, \mu_i = 0, \sigma = 0.75, b = 4$, when $\hat{\mu}_{Bayes}$ is close to μ_0

$b = 4$		$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
ntr=2 nbl=3	**i=1** **i=2**	0.1498 0.1400	0.4232 0.4156	0.1546 0.1450	0.3785 0.3729	0.8945 0.8974
ntr=2 nbl=5	**i=1** **i=2**	0.0816 0.0795	0.2341 0.2342	0.0828 0.0810	0.2199 0.2198	0.9397 0.9383
ntr=2 nbl=10	**i=1** **i=2**	0.0365 0.0425	0.1086 0.1076	0.0368 0.0426	0.1056 0.1046	0.9720 0.9724
ntr=2 nbl=15	**i=1** **i=2**	0.0251 0.0254	0.0737 0.0734	0.0252 0.0255	0.0723 0.0720	0.9816 0.9816

Table 3.12 Simulated values for Generalized Logistic distribution with $\mu_{i0} = 0, s_{i0} = 3, \delta_{i0} = 6, \mu_i = 0, \sigma = 0.75, b = 6$, when $\hat{\mu}_{Bayes}$ is close to μ_0

$b = 6$		$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
ntr=2 nbl=3	**i=1** **i=2**	0.1759 0.1865	0.4699 0.4881	0.1829 0.1931	0.4211 0.4363	0.8962 0.8939
ntr=2 nbl=5	**i=1** **i=2**	0.1000 0.1052	0.2626 0.2555	0.1022 0.1071	0.2470 0.2402	0.9399 0.9401
ntr=2 nbl=10	**i=1** **i=2**	0.0513 0.0438	0.1165 0.1193	0.0516 0.0442	0.1133 0.1160	0.9727 0.9721
ntr=2 nbl=15	**i=1** **i=2**	0.0342 0.0346	0.0778 0.0797	0.0343 0.0347	0.0764 0.0783	0.9821 0.9821

Table 3.13 Simulated values for Generalized Logistic distribution with $\mu_{i0}=0, s_{i0}=3, \delta_{i0}=6, \mu_i=0, \sigma=0.75, b=8$, when $\hat{\mu}_{Bayes}$ is close to μ_0

$b=8$		$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
ntr=2	**i=1**	0.1986	0.5453	0.2082	0.4849	0.8893
nbl=3	**i=2**	0.1909	0.5575	0.2008	0.4958	0.8894
ntr=2	**i=1**	0.1194	0.2846	0.1216	0.2679	0.9414
nbl=5	**i=2**	0.1145	0.2857	0.1169	0.2687	0.9405
ntr=2	**i=1**	0.0568	0.1333	0.0573	0.1297	0.9724
nbl=10	**i=2**	0.0525	0.1323	0.0530	0.1286	0.9723
ntr=2	**i=1**	0.0367	0.0844	0.0369	0.0829	0.9822
nbl=15	**i=2**	0.0405	0.0861	0.0407	0.0845	0.9821

Table 3.14 Simulated values for Generalized Logistic distribution with $\mu_{i0}=0, s_{i0}=3, \delta_{i0}=6, \mu_i=0, b=0.5$, when $\hat{\mu}_{Bayes}$ is close to μ_0

$b=0.5$			$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
ntr=2 nbl=3	σ=1	**i=1**	-0.1807	1.9211	-0.1633	1.1425	0.5947
		i=2	-0.1767	1.9012	-0.1617	1.1321	0.5955
	σ=1.5	**i=1**	-0.2609	4.2632	-0.2119	1.7321	0.4063
		i=2	-0.2573	4.2912	-0.2029	1.7347	0.4042
ntr=2 nbl=5	σ=1	**i=1**	-0.0901	1.1176	-0.0853	0.8134	0.7278
		i=2	-0.0926	1.0956	-0.0876	0.8005	0.7306
	σ=1.5	**i=1**	-0.1463	2.4436	-0.1304	1.3194	0.5399
		i=2	-0.1449	2.5662	-0.1312	1.3921	0.5425
ntr=2 nbl=10	σ=1	**i=1**	-0.0286	0.5273	-0.0291	0.4545	0.8619
		i=2	-0.0215	0.5218	-0.0220	0.4499	0.8624
	σ=1.5	**i=1**	-0.0532	1.1811	-0.0517	0.8615	0.7294
		i=2	-0.0299	1.2033	-0.0327	0.8773	0.7291
ntr=2 nbl=15	σ=1	**i=1**	-0.0205	0.3528	-0.0205	0.3206	0.9086
		i=2	-0.0192	0.3496	-0.0192	0.3177	0.9090
	σ=1.5	**i=1**	-0.0227	0.8015	-0.0276	0.6506	0.8117
		i=2	-0.0137	0.7821	-0.0150	0.6350	0.8119

Table 3.15 Simulated values for Generalized Logistic distribution with $\mu_{i0} = 0, s_{i0} = 3, \delta_{i0} = 6, \mu_i = 0, b = 1$, when $\hat{\mu}_{Bayes}$ is close to μ_0

$b = 1$			$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
ntr=2 nbl=3	σ=1	i=1	0.0061	1.0580	0.0067	0.7546	0.7132
		i=2	0.0054	1.0664	0.0056	0.7655	0.7178
	σ=1.5	i=1	-0.0196	2.4235	-0.0124	1.2811	0.5286
		i=2	-0.0114	2.4468	-0.0047	1.2883	0.5265
ntr=2 nbl=5	σ=1	i=1	-0.0108	0.6526	-0.0108	0.5363	0.8217
		i=2	-0.0047	0.6550	-0.0039	0.5390	0.8229
	σ=1.5	i=1	-0.0013	1.4344	-0.0002	0.9610	0.6700
		i=2	-0.0089	1.4430	-0.0066	0.9627	0.6671
ntr=2 nbl=10	σ=1	i=1	0.0057	0.3058	0.0053	0.2794	0.9140
		i=2	0.0043	0.3159	0.0041	0.2890	0.9151
	σ=1.5	i=1	0.0061	0.7007	0.0055	0.5779	0.8247
		i=2	0.0112	0.6891	0.0100	0.5677	0.8238
ntr=2 nbl=15	σ=1	i=1	0.0065	0.2056	0.0062	0.1941	0.9440
		i=2	-0.0027	0.2045	-0.0026	0.1931	0.9443
	σ=1.5	i=1	0.0028	0.4668	0.0026	0.4117	0.8818
		i=2	-0.0035	0.4511	-0.0034	0.3978	0.8818

Table 3.16 Simulated values for Generalized Logistic distribution with $\mu_{i0} = 0, s_{i0} = 3, \delta_{i0} = 6, \mu_i = 0, b = 4$, when $\hat{\mu}_{Bayes}$ is close to μ_0

$b = 4$			$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
ntr=2 nbl=3	σ=1	i=1	0.1962	0.7419	0.2049	0.6233	0.8402
		i=2	0.1881	0.7433	0.1983	0.6178	0.8312
	σ=1.5	i=1	0.3042	1.6408	0.3222	0.1543	0.7035
		i=2	0.2827	1.6718	0.3042	0.1691	0.6993
ntr=2 nbl=5	σ=1	i=1	0.1103	0.4137	0.1129	0.3724	0.9001
		i=2	0.0968	0.4054	0.0998	0.3646	0.8995
	σ=1.5	i=1	0.1412	0.9159	0.1509	0.7296	0.7966
		i=2	0.1542	0.9216	0.1616	0.7357	0.7983
ntr=2 nbl=10	σ=1	i=1	0.0484	0.1987	0.0490	0.1890	0.9512
		i=2	0.0633	0.1940	0.0635	0.1846	0.9512
	σ=1.5	i=1	0.0756	0.4349	0.0772	0.3893	0.8950
		i=2	0.0770	0.4271	0.0785	0.3826	0.8959
ntr=2 nbl=15	σ=1	i=1	0.0350	0.1294	0.0352	0.1252	0.9677
		i=2	0.0354	0.1233	0.0356	0.1194	0.9683
	σ=1.5	i=1	0.0492	0.2905	0.0500	0.2700	0.9293
		i=2	0.0442	0.2929	0.0452	0.2724	0.9300

Table 3.17 Simulated values for Generalized Logistic distribution with $\mu_{i0}=0, \delta_{i0}=6, \mu_i=0, \sigma=1, b=0.5$, when $\hat{\mu}_{Bayes}$ is close to μ_0

$b=0.5$			$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
ntr=2 nbl=3	S=2.0	i=1	-0.1798	1.8935	-0.1427	0.7742	0.4088
		i=2	-0.1699	1.9061	-0.1380	0.7671	0.4024
	S=2.5	i=1	-0.1819	1.9303	-0.1537	0.9793	0.5073
		i=2	-0.1890	1.9358	-0.1613	0.9913	0.5074
ntr=2 nbl=5	S=2.0	i=1	-0.0922	1.1379	-0.0809	0.6135	0.5392
		i=2	-0.0797	1.0935	-0.0709	0.5900	0.5396
	S=2.5	i=1	-0.0983	1.1269	-0.0912	0.7314	0.6490
		i=2	-0.0954	1.1158	-0.0884	0.7265	0.6511
ntr=2 nbl=10	S=2.0	i=1	-0.0488	0.5269	-0.0459	0.3854	0.7314
		i=2	-0.0311	0.5416	-0.0302	0.3940	0.7275
	S=2.5	i=1	-0.0301	0.5268	-0.0300	0.4276	0.8116
		i=2	-0.0390	0.5224	-0.0381	0.4242	0.8121
ntr=2 nbl=15	S=2.0	i=1	-0.0194	0.3396	-0.0193	0.2763	0.8135
		i=2	-0.0160	0.3561	-0.0165	0.2891	0.8119
	S=2.5	i=1	-0.0117	0.3509	-0.0120	0.3063	0.8729
		i=2	-0.0208	0.3608	-0.0208	0.3148	0.8725

Table 3.18 Simulated values for Generalized Logistic distribution with $\mu_{i0}=0, \delta_{i0}=6, \mu_i=0, \sigma=1, b=1$, when $\hat{\mu}_{Bayes}$ is close to μ_0

$b=1$			$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
ntr=2 nbl=3	S=2.0	i=1	-0.0090	1.0898	-0.0047	0.5800	0.5323
		i=2	0.0117	1.0990	0.0091	0.5853	0.5326
	S=2.5	i=1	0.0022	1.0650	0.0028	0.6774	0.6332
		i=2	-0.0006	1.1070	-0.0016	0.6797	0.6352
ntr=2 nbl=5	S=2.0	i=1	0.0046	0.6411	0.0034	0.4289	0.6691
		i=2	-0.0075	0.6414	-0.0061	0.4307	0.6715
	S=2.5	i=1	-0.0016	0.6250	-0.0013	0.4754	0.7607
		i=2	0.0082	0.6313	0.0074	0.4786	0.7582
ntr=2 nbl=10	S=2.0	i=1	-0.0010	0.3120	-0.0018	0.2569	0.8233
		i=2	-0.0068	0.3096	-0.0062	0.2550	0.8238
	S=2.5	i=1	-0.0077	0.3114	-0.0071	0.2739	0.8793
		i=2	-0.0048	0.3124	-0.0044	0.2749	0.8801
ntr=2 nbl=15	S=2.0	i=1	0.0025	0.2052	0.0024	0.1810	0.8817
		i=2	-0.0007	0.2046	-0.0006	0.1804	0.8816
	S=2.5	i=1	0.0095	0.2052	0.0091	0.1891	0.9215
		i=2	-0.0033	0.2036	-0.0031	0.1875	0.9210

Table 3.19 Simulated values for Generalized Logistic distribution with $\mu_{i0}=0, \delta_{i0}=6, \mu_i=0, \sigma=1, b=4$, when $\hat{\mu}_{Bayes}$ is close to μ_0

$b=4$			$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
ntr=2 nbl=3	S =2.0	**i=1**	0.1958	0.7427	0.2095	0.5264	0.7087
		i=2	0.1944	0.7322	0.2072	0.5155	0.7040
	S =2.5	**i=1**	0.1977	0.7426	0.2080	0.5836	0.7859
		i=2	0.1944	0.7236	0.2056	0.5652	0.7811
ntr=2 nbl=5	S =2.0	**i=1**	0.1117	0.4107	0.1158	0.3294	0.8021
		i=2	0.1124	0.4098	0.1167	0.3289	0.8025
	S =2.5	**i=1**	0.1172	0.4055	0.1196	0.3512	0.8660
		i=2	0.1108	0.4029	0.1137	0.3487	0.8654
ntr=2 nbl=10	S =2.0	**i=1**	0.0430	0.1975	0.0445	0.1765	0.8938
		i=2	0.0494	0.1975	0.0507	0.1766	0.8941
	S =2.5	**i=1**	0.0534	0.1935	0.0541	0.1802	0.9314
		i=2	0.0501	0.1938	0.0509	0.1804	0.9310
ntr=2 nbl=15	S =2.0	**i=1**	0.0326	0.1272	0.0332	0.1182	0.9295
		i=2	0.0309	0.1284	0.0316	0.1194	0.9299
	S =2.5	**i=1**	0.0034	0.1266	0.0347	0.1208	0.9543
		i=2	0.0319	0.1278	0.0323	0.1219	0.9540

Table 3.20 Simulated values for Generalized Logistic distribution with $\mu_{i0}=0, s_{i0}=3, \delta_{i0}=6, \mu_i=0, \sigma=0.75, b=0.5$, when $\hat{\mu}_{Bayes}$ is close to $\hat{\mu}_{MML}$

$b=0.5$		$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
ntr=2 nbl=3	**i=1**	-0.1560	1.0894	-0.1447	0.7993	0.7337
	i=2	-0.1413	1.0641	-0.1323	0.7814	0.7343
ntr=2 nbl=5	**i=1**	-0.0660	0.6249	-0.0640	0.5204	0.8328
	i=2	-0.0543	0.6333	-0.0546	0.5284	0.8344
ntr=2 nbl=10	**i=1**	-0.0263	0.2929	-0.0262	0.2691	0.9187
	i=2	-0.0254	0.3016	-0.0254	0.2769	0.9181
ntr=2 nbl=15	**i=1**	-0.0203	0.1943	-0.0202	0.1841	0.9473
	i=2	-0.0103	0.1952	-0.0105	0.1848	0.9470

Table 3.21 Simulated values for Generalized Logistic distribution with $\mu_{i0}=0, s_{i0}=3, \delta_{i0}=6, \mu_i=0, \sigma=0.75, b=1$, when $\hat{\mu}_{Bayes}$ is close to $\hat{\mu}_{MML}$

$b=1$		$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
ntr=2 nbl=3	**i=1** **i=2**	-0.0012 0.0139	0.6028 0.6152	-0.0007 0.0129	0.4942 0.5035	0.8198 0.8184
ntr=2 nbl=5	**i=1** **i=2**	0.0025 0.0008	0.3576 0.3642	0.0026 0.0006	0.3197 0.3256	0.8940 0.8941
ntr=2 nbl=10	**i=1** **i=2**	-0.0009 -0.0022	0.1724 0.1740	-0.0008 -0.0021	0.1640 0.1654	0.9511 0.9509
ntr=2 nbl=15	**i=1** **i=2**	-0.0055 0.0058	0.1175 0.1137	-0.0054 0.0057	0.1138 0.1102	0.9685 0.9686

Table 3.22 Simulated values for Generalized Logistic distribution with $\mu_{i0}=0, s_{i0}=3, \delta_{i0}=6, \mu_i=0, \sigma=0.75, b=4$, when $\hat{\mu}_{Bayes}$ is close to $\hat{\mu}_{MML}$

$b=4$		$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
ntr=2 nbl=3	**i=1** **i=2**	0.1508 0.1496	0.4119 0.4156	0.1551 0.1539	0.3713 0.3736	0.9015 0.8989
ntr=2 nbl=5	**i=1** **i=2**	0.0833 0.0872	0.2294 0.2375	0.0844 0.0884	0.2161 0.2237	0.9420 0.9418
ntr=2 nbl=10	**i=1** **i=2**	0.0411 0.0420	0.1072 0.1078	0.0413 0.0422	0.1042 0.1048	0.9722 0.9724
ntr=2 nbl=15	**i=1** **i=2**	0.0245 0.0311	0.0714 0.0728	0.0246 0.0312	0.0701 0.0715	0.9818 0.9820

Table 3.23 Simulated values for Generalized Logistic distribution with $\mu_{i0}=0, s_{i0}=3, \delta_{i0}=6, \mu_i=0, \sigma=0.75, b=6$, when $\hat{\mu}_{Bayes}$ is close to $\hat{\mu}_{MML}$

$b=6$		$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
ntr=2 nbl=3	**i=1** **i=2**	0.1889 0.1728	0.4766 0.4884	0.1955 0.1803	0.4285 0.4382	0.8991 0.8972
ntr=2 nbl=5	**i=1** **i=2**	0.1077 0.1084	0.2532 0.2629	0.1093 0.1101	0.2387 0.2478	0.9427 0.9428
ntr=2 nbl=10	**i=1** **i=2**	0.0604 0.0529	0.1200 0.1178	0.0606 0.0532	0.1168 0.1146	0.9732 0.9729
ntr=2 nbl=15	**i=1** **i=2**	0.0348 0.0285	0.0775 0.0790	0.0350 0.0287	0.0762 0.0776	0.9823 0.9823

Table 3.24 Simulated values for Generalized Logistic distribution with $\mu_{i0}=0, s_{i0}=3, \delta_{i0}=6, \mu_i=0, \sigma=0.75, b=8$, when $\hat{\mu}_{Bayes}$ is close to $\hat{\mu}_{MML}$

$b=8$		$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
ntr=2 nbl=3	**i=1** **i=2**	0.1969 0.2070	0.5449 0.5483	0.2061 0.2157	0.4869 0.4902	0.8936 0.8940
ntr=2 nbl=5	**i=1** **i=2**	0.1082 0.1120	0.2894 0.2892	0.1107 0.1144	0.2728 0.2723	0.9424 0.9415
ntr=2 nbl=10	**i=1** **i=2**	0.0579 0.0577	0.1338 0.1298	0.0584 0.0581	0.1301 0.1263	0.9729 0.9727
ntr=2 nbl=15	**i=1** **i=2**	0.0409 0.0392	0.0841 0.0864	0.0411 0.0394	0.0826 0.0848	0.9824 0.9823

Table 3.25 Simulated values for Generalized Logistic distribution with $\mu_{i0}=0, s_{i0}=3, \delta_{i0}=6, \mu_i=0, b=0.5$, when $\hat{\mu}_{Bayes}$ is close to $\hat{\mu}_{MML}$

$b=0.5$			$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
ntr=2 nbl=3	σ=1	i=1	-0.1770	1.8892	-0.1587	1.1710	0.6199
		i=2	-0.1828	1.8972	-0.1654	1.1828	0.6235
	σ=1.5	i=1	-0.2675	4.2832	-0.2262	2.0147	0.4704
		i=2	-0.3067	4.3164	-0.2478	2.0142	0.4666
ntr=2 nbl=5	σ=1	i=1	-0.1089	1.1295	-0.1025	0.8412	0.7447
		i=2	-0.0972	1.1105	-0.0931	0.8243	0.7423
	σ=1.5	i=1	-0.1263	2.4737	-0.1158	1.4199	0.5740
		i=2	-0.1223	2.4461	-0.1136	1.4058	0.5747
ntr=2 nbl=10	σ=1	i=1	-0.0358	0.5357	-0.0358	0.4643	0.8667
		i=2	-0.0383	0.5458	-0.0379	0.4638	0.8656
	σ=1.5	i=1	-0.0431	1.1859	-0.0428	0.8846	0.7460
		i=2	-0.0338	1.2096	-0.0360	0.9030	0.7465
ntr=2 nbl=15	σ=1	i=1	-0.0214	0.3507	-0.0215	0.3194	0.9107
		i=2	0.0113	0.3462	-0.0118	0.3153	0.9110
	σ=1.5	i=1	-0.0271	0.7936	-0.0273	0.6505	0.8197
		i=2	-0.0274	0.7874	-0.0282	0.6450	0.8191

Table 3.26 Simulated values for Generalized Logistic distribution with $\mu_{i0}=0, s_{i0}=3, \delta_{i0}=6, \mu_i=0, b=1$, when $\hat{\mu}_{Bayes}$ is close to $\hat{\mu}_{MML}$

$b=1$			$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
ntr=2 nbl=3	σ=1	i=1	-0.0011	1.0933	-0.0019	0.7941	0.7264
		i=2	0.0002	1.0951	0.0003	0.7920	0.7233
	σ=1.5	i=1	00013	2.3836	0.0024	1.3554	0.5686
		i=2	-0.0081	2.4094	-0.0062	1.3737	0.5701
ntr=2 nbl=5	σ=1	i=1	-0.0015	0.6476	-0.0013	0.5358	0.8274
		i=2	0.0011	0.6245	-0.0004	0.5166	0.8272
	σ=1.5	i=1	-0.0068	1.4099	-0.0051	0.9743	0.6910
		i=2	-0.0191	1.4461	-0.0164	0.9967	0.6892
ntr=2 nbl=10	σ=1	i=1	0.0067	0.3138	0.0065	0.2873	0.9155
		i=2	-0.0079	0.3156	-0.0074	0.2890	0.9156
	σ=1.5	i=1	-0.0054	0.7024	-0.0051	0.5831	0.8301
		i=2	0.0004	0.6913	0.0006	0.5741	0.8304
ntr=2 nbl=15	σ=1	i=1	-0.0102	0.1983	-0.0099	0.1875	0.9452
		i=2	0.0068	0.2056	0.0068	0.1943	0.9450
	σ=1.5	i=1	0.0022	0.4607	0.0021	0.4073	0.8840
		i=2	-0.0032	0.4587	-0.0030	0.4054	0.8837

Table 3.27 Simulated values for Generalized Logistic distribution with $\mu_{i0}=0, s_{i0}=3, \delta_{i0}=6, \mu_i=0, b=4$, when $\hat{\mu}_{Bayes}$ is close to $\hat{\mu}_{MML}$

$b=4$			$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
ntr=2 nbl=3	σ=1	i=1	0.1869	0.7180	0.1957	0.6054	0.8432
		i=2	0.1838	0.7550	0.1948	0.6345	0.8405
	σ=1.5	i=1	0.3165	1.6325	0.3321	1.1782	0.7217
		i=2	0.3016	1.6760	0.3214	1.2165	0.7259
ntr=2 nbl=5	σ=1	i=1	0.1113	0.4057	0.1140	0.3662	0.9026
		i=2	0.1055	0.4062	0.1082	0.3670	0.9033
	σ=1.5	i=1	0.1674	0.9160	0.1732	0.7626	0.78107
		i=2	0.1772	0.9186	0.1824	0.7454	0.8114
ntr=2 nbl=10	σ=1	i=1	0.0452	0.1953	0.0459	0.1858	0.9513
		i=2	0.0533	0.1949	0.0539	0.1855	0.9516
	σ=1.5	i=1	0.0794	0.4348	0.0809	0.3905	0.8980
		i=2	0.0673	0.4377	0.0694	0.3930	0.8979
ntr=2 nbl=15	σ=1	i=1	0.0326	0.1306	0.0329	0.1264	0.9680
		i=2	0.0344	0.1303	0.0346	0.1261	0.9683
	σ=1.5	i=1	0.0465	0.2914	0.0475	0.2712	0.9308
		i=2	0.0503	0.2935	0.0509	0.2736	0.9321

Table 3.28 Simulated values for Generalized Logistic distribution with $\mu_{i0}=0, \delta_{i0}=6, \mu_i=0, \sigma=1, b=0.5$, when $\hat{\mu}_{Bayes}$ is close to $\hat{\mu}_{MML}$

$b=0.5$			$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
ntr=2 nbl=3	S=2.0	i=1	-0.1857	1.9184	-0.1482	0.8797	0.4585
		i=2	0.1933	1.9151	-0.1580	0.8685	0.4535
	S=2.5	i=1	-0.1729	1.8877	-0.1515	1.0421	0.5521
		i=2	-0.2043	1.9264	-0.1746	1.0757	0.5584
ntr=2 nbl=5	S=2.0	i=1	-0.0624	1.1064	-0.0616	0.6321	0.5713
		i=2	-0.0783	1.0993	-0.0739	0.6298	0.5729
	S=2.5	i=1	-0.0904	1.1204	-0.0863	0.7575	0.6761
		i=2	-0.0883	1.1041	-0.0833	0.7451	0.6749
ntr=2 nbl=10	S=2.0	i=1	-0.0266	0.5325	-0.0280	0.3965	0.7445
		i=2	-0.0352	0.5411	-0.0341	0.4039	0.7463
	S=2.5	i=1	-0.0289	0.5311	-0.0292	0.4342	0.8176
		i=2	-0.0279	0.5256	-0.0278	0.4304	0.8188
ntr=2 nbl=15	S=2.0	i=1	-0.0348	0.3394	-0.0332	0.2784	0.8201
		i=2	-0.0160	0.3472	-0.0165	0.2851	0.8211
	S=2.5	i=1	-0.0237	0.3470	-0.0235	0.3038	0.8754
		i=2	-0.0163	0.3527	-0.0169	0.3090	0.8762

Table 3.29 Simulated values for Generalized Logistic distribution with $\mu_{i0} = 0, \delta_{i0} = 6, \mu_i = 0, \sigma = 1, b = 1$, when $\hat{\mu}_{Bayes}$ is close to $\hat{\mu}_{MML}$

$b=1$			$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
ntr=2 nbl=3	S=2.0	i=1	0.0063	1.0682	0.0072	0.6031	0.5646
		i=2	0.0012	1.0946	0.0008	0.6163	0.5630
	S=2.5	i=1	-0.0180	1.0781	-0.0149	0.7034	0.6524
		i=2	0.0175	1.0917	0.0146	0.7229	0.6622
ntr=2 nbl=5	S=2.0	i=1	0.0137	0.6363	0.0129	0.4423	0.6919
		i=2	0.0114	0.6297	0.0098	0.4331	0.6878
	S=2.5	i=1	-0.0002	0.6277	-0.0003	0.4845	0.7718
		i=2	-0.0163	0.6397	-0.0140	0.4936	0.7717
ntr=2 nbl=10	S=2.0	i=1	-0.0035	0.3126	-0.0030	0.2595	0.8302
		i=2	0.0007	0.3049	0.0007	0.2529	0.8295
	S=2.5	i=1	-0.0028	0.3134	-0.0028	0.2765	0.8822
		i=2	-0.0036	0.3075	-0.0034	0.2716	0.8833
ntr=2 nbl=15	S=2.0	i=1	0.0044	0.2011	0.0042	0.1781	0.8860
		i=2	0.0016	0.2029	0.0015	0.1796	0.8851
	S=2.5	i=1	0.0074	0.2106	0.0072	0.1943	0.9226
		i=2	0.0009	0.2064	0.0009	0.1905	0.9233

Table 3.30 Simulated values for Generalized Logistic distribution with $\mu_{i0} = 0, \delta_{i0} = 6, \mu_i = 0, \sigma = 1, b = 4$, when $\hat{\mu}_{Bayes}$ is close to $\hat{\mu}_{MML}$

$b=4$			$\hat{\mu}_{MML}$	$V(\hat{\mu}_{MML})$	$\hat{\mu}_{HPD}$	$V(\hat{\mu}_{HPD})$	RE
ntr=2 nbl=3	S=2.0	i=1	0.1963	0.7287	0.2082	0.5257	0.7214
		i=2	0.2040	0.7238	0.2133	0.5226	0.7220
	S=2.5	i=1	0.1958	0.7395	0.2068	0.5846	0.7906
		i=2	0.2034	0.7328	0.2127	0.5841	0.7971
ntr=2 nbl=5	S=2.0	i=1	0.1056	0.4103	0.1104	0.3323	0.8098
		i=2	0.1120	0.4188	0.1161	0.3392	0.8100
	S=2.5	i=1	0.1137	0.4087	0.1168	0.3555	0.8699
		i=2	0.1169	0.4072	0.1197	0.3530	0.8670
ntr=2 nbl=10	S=2.0	i=1	0.0572	0.1994	0.0580	0.1792	0.8986
		i=2	0.0541	0.1976	0.0551	0.1776	0.8985
	S=2.5	i=1	0.0473	0.1925	0.0481	0.1794	0.9319
		i=2	0.0530	0.1952	0.0537	0.1820	0.9325
ntr=2 nbl=15	S=2.0	i=1	0.0388	0.1256	0.0342	0.1169	0.9308
		i=2	0.0319	0.1276	0.0324	0.1188	0.9313
	S=2.5	i=1	0.0319	0.1269	0.0323	0.1211	0.9546
		i=2	0.0325	0.1276	0.0329	0.1217	0.9544

CHAPTER 4

APPLICATION

4.1.Application with Real Data

We start with the observations that represents the differences (in heights) between cross and self fertilizied plants of the same pair grown in one pot. The data, which is known as Darwin's data, is given below:

49, -67, 8, 16, 6, 23, 28, 41, 14, 29, 56, 24, 75, 60, -48

The Q-Q plot based on a normal distribution is represented by Figure 4.1 below. We see that two smallest and one largest observations are different than the bulk of the observations, that means they are possibly outliers. In order to obtain more reliable results, formal outlier tests are applied to the data and it has been found that these observations are in fact outliers. (Tiku and Akkaya, 2004). Since existence of extreme values adversely affects the efficiency of estimators, these observations are given zero weights. That is to say, they are censored from the data. Since we deal with symmetric censoring in Chapter 2, we censor two smallest and two largest observations from the data and calculate MML and HPD estimators with the remaining observations.

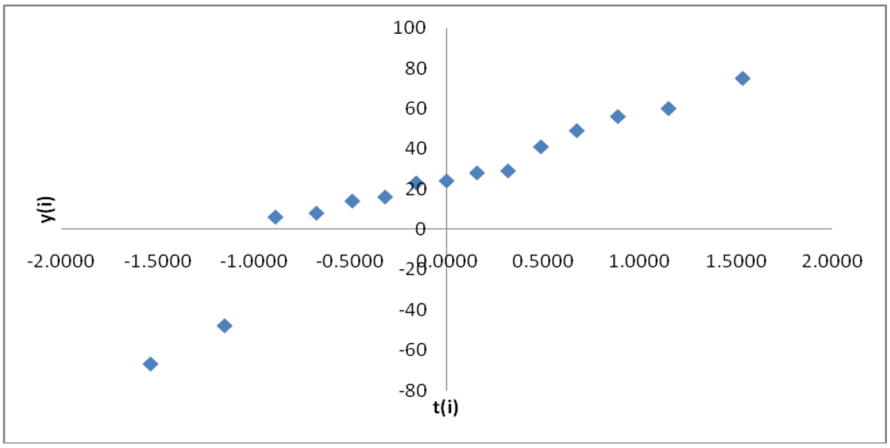

Figure 4.1. Q-Q Plot for example1 with normal distribution

In order to see the affect of outliers, we start with full data and obtain Least Square (LS) estimators with fifteen observations as,

$$\hat{\mu}_{LS} = 20.933 \text{ and } Var(\hat{\mu}_{LS}) = 94.975 \quad (4.1)$$

We aim to calculate Bayesian estimator of μ also, therefore we assume a normal prior with hyperparameters μ_0, σ_0^2. If we assume μ_0 = 20 and σ_0^2 = 7 and use (1.22) we get

$$\hat{\mu}_{Bayes} = 20.317 \text{ and } Var(\hat{\mu}_{Bayes}) = 32.323 \quad (4.2)$$

From (4.1) and (4.2) we see that LS and Bayes estimators of μ are very close to each other, however, the variance of Bayes estimator is much smaller.

In order to eliminate the affect of extreme values, we censor two smallest and largest observatios from the data and calculate MML estimator (2.8) as

$$\hat{\mu}_{MML} = 27.703 \text{ and } Var(\hat{\mu}_{MML}) = 45.197 \quad (4.3)$$

Moreover, HPD estimator of μ given by (2.17) is obtained by assuming $p(\mu) \sim N(\mu_0, \sigma_0^2)$. If we take μ_0 and σ_0^2 as 20 and 7, respectively, we get

$$\hat{\mu}_{Bayes} = 24.007 \text{ and } Var(\hat{\mu}_{Bayes}) = 23.510 \tag{4.4}$$

Like in the full sample case, we see that MMLE and Bayesian estimator of μ are similar to each other but HPD estimator has smaller variance. Moreover, if we compare full sample and censored sample results for both MML and Bayes estimators, we see that extreme values affect the efficiencies adversely. Therefore, they need to be removed from the data in order to get more reliable estimates.

Example2: Following data represents the average annual erosion rates of thirteen states in US (Tiku and Akkaya, 2004):

-0.4 -0.5 -0.9 -0.5 0.1 -1.0 0.1 -1.5 -4.2 -0.6 -2.0 0.7 -0.1

The general pattern of the data indicates that a negatively skewed distribution may be appropriate. According to the Q-Q plot shown by Figure 4.2 we can say that GL distribution may be a good choice. In order to determine the value of b, estimates of $dLnL/n$ are calculated. We find that b=0.5 is the maximizing point and should be used.

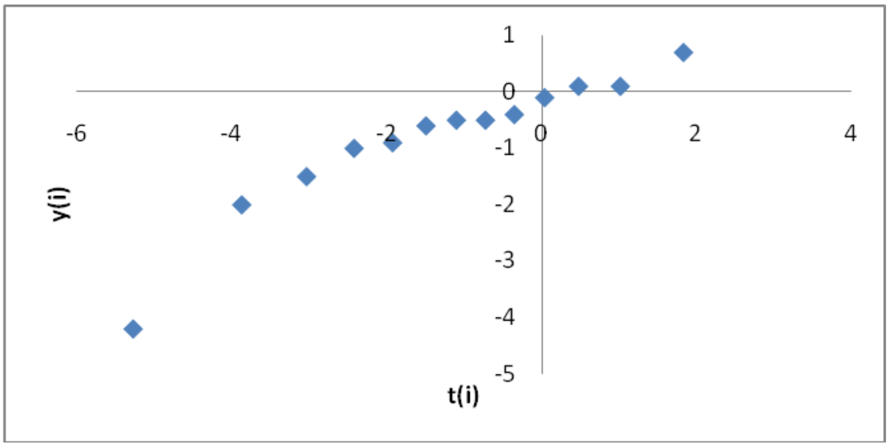

Figure 4.2. Q-Q plot for Example2 with GL distribution (b=0.5).

Assuming GL distribution with b=0.5, MML estimators are calculated from (2.80) as

$$\hat{\mu}_{MML} = \text{-}0.1812 \text{ and } Var(\hat{\mu}_{MML}) = 0.0685 \tag{4.5}$$

In order to calculate HPD estimators, we assume a robust prior (2.92) with $\mu_0 = 0$ and $\sigma_0^2 = 1$ and obtain Bayes estimator of μ from (2.100) as

$$\hat{\mu}_{Bayes} = \text{-}0.1796 \text{ and } Var(\hat{\mu}_{Bayes}) = 0.0587 \tag{4.6}$$

If we compare (4.5) and (4.6) we see that the estimates of μ are very close to each other but variance of HPD estimator is lower than the variance of MMLE. Relative efficiency of HPD estimator is calculated as

$$RE = \frac{Var(\hat{\mu}_{Bayes})}{Var(\hat{\mu}_{MML})} = 0.8574 \tag{4.7}$$

which indicates the HPD estimator is more efficient than MML estimator. However, if we change the prior variance from $\sigma_0^2 = 1$ to $\sigma_0^2 = 3$ we see that,

$$RE = \frac{Var(\hat{\mu}_{Baye})}{Var(\hat{\mu}_{MML})} = 0.9824 \tag{4.8}$$

That means, if prior dispersion increases, Bayes estimator looses efficiency since the weight of MML estimator increases. However, as σ_0^2 becomes larger and larger, RE converges to 1 which indicates HPD estimator converges to MML estimator. That is to say, for reasonable prior variance, HPD is more efficient than MML estimators but for large σ_0^2 it becomes at least as efficient as MMLE.

Example3. The following data represents the gain (in pounds) of 20 pigs with respect to two different feeds, A and B (Tiku and Akkaya, 2004).

A: 0.09 1.43 2.79 1.60 1.71 3.37 2.06 2.67 8.42 3.67
B: 1.96 1.79 2.60 1.40 2.22 3.45 1.16 5.71 2.93 1.40

In order to fit a reasonable distribution we look at different Q-Q plots and see that GL distribution with b>1 may be an appropriate choice for the error terms. The value of b is chosen as 8 since it maximezes $d \ln \hat{L} / n$.

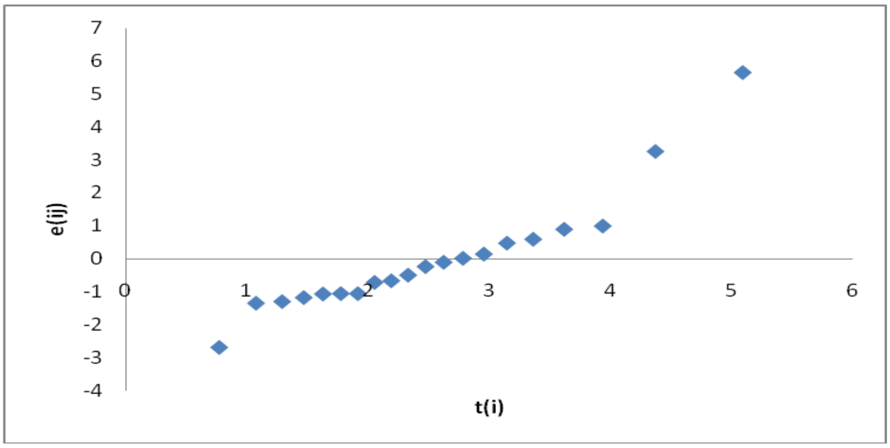

Figure 4.3. Q-Q plot for Example 3 withGL distribution(b=8).

We may fit a one way experimental design model to the data with two treatments each having ten observations. MML estimators (3.21) are calculated as

$$\hat{\mu}_{A,MML} = \text{-}0.4893,\ \hat{\mu}_{B,MML} = \text{-}0.3945 \text{ and } Var(\hat{\mu}_{i,MML}) = 0.2600 \quad (4.9)$$

We find HPD estimators of treatments by assuming prior distribution (3.28) with $\mu_{i,0} = 0$, $\sigma^2_{i,0} = 1$ and $v_{i,0} = 6$ as

$$\hat{\mu}_{A,HPD} = \text{-}0.4312,\ \hat{\mu}_{B,HPD} = \text{-}0.3476 \text{ and } Var(\hat{\mu}_{i,HPD}) = 0.2072 \quad (4.10)$$

The estimates of location parameters are close to each other but their variances become smaller if Bayesian techniques are used in estimation. Assuming prior distribution with parameters mentioned above, we get the relative efficiency of HPD estimators as

$$RE = \frac{Var(\hat{\mu}_{Bayes})}{Var(\hat{\mu}_{MML})} = 0.79 \tag{4.11}$$

However, if we change prior dispersion and assume $\sigma_{i,0}^2 = 1.5$, then relative efficiency become

$$RE = \frac{Var(\hat{\mu}_{Bayes})}{Var(\hat{\mu}_{MML})} = 0.89 \tag{4.12}$$

Relative efficiency increases with prior dispersion and converges to 1 for large values of $\sigma_0^2 = 1.5$, as expected.

Example4. The following data comes from the study of Columbian molasess. Brix Degrees, which is a measure of the quantity of solids in a molasses, is one of the qualities of importance. The sources of the molasses are three different areas in the country. In order to see whether these three locations provide the same Brix Degrees of molasses, eight observations are obtainded from each location (Johnson and Leone, 1964).

LocationI :	81.6	81.3	82.0	79.6	78.4	81.8	80.2	80.7
Location II:	81.8	84.7	82.0	85.6	79.9	83.2	84.1	85.0
Location III:	82.1	79.6	83.1	80.7	81.8	79.9	82.6	81.9

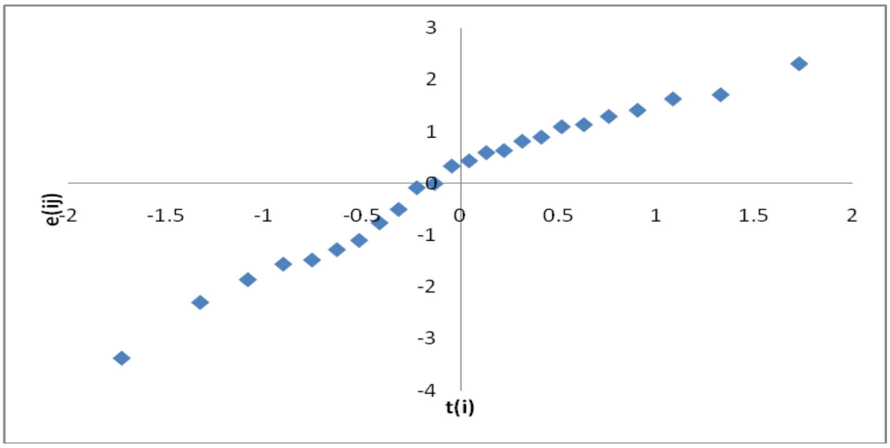

Figure 4.4. Q-Q plot for Example 4 with LTS distribution(p=4).

One way ANOVA model is considered with three treatments each having eight observations. Q-Q plot, shown by Figure 4.4, indicates that LTS family may be appropriate for the data. The suitable value of p is found as 4 since it maximizes $d \ln \hat{L}/n$.

According to one way classification model with LTS distribution p=4, MML estimators (3.5) are calculated as

$$\hat{\mu}_{1,MML} = 80.778,\ \hat{\mu}_{2,MML} = 83.367,\ \hat{\mu}_{3,MML} = 81.498 \text{ and } Var(\hat{\mu}_{i,MML}) = 0.265 \quad (4.13)$$

Assuming robust prior (3.7) with $\mu_{i,0} = 80$, $\sigma_{i,0} = 1$ and $v_{i,0} = 6$, HPD estimators (3.15) are calculated as

$$\hat{\mu}_{1,Bayes} = 80.594,\ \hat{\mu}_{2Bayes} = 82.572,\ \hat{\mu}_{3,Bayes} = 81.145 \text{and } Var(\hat{\mu}_{i,Bayes}) = 0.210 \quad (4.14)$$

The results agree with the simulation results of Chapter 3. The MML and HPD estimates of μ are close to each other but the latter have smaller variance. However, as in the previous cases, variance of HPD estimators converges to variance of MML estimators as σ_0 increases. For example, If we take $\sigma_{i,0} = 1$, the relative efficiency is %79.4 while it becomes %89.7 for $\sigma_{i,0} = 1.5$.

Comment: The method we have developed in this chapter can be extended to more complex data structures, e.g., two-way-classification ANOVA with interaction, BIB design, linear regression, etc. That will be the subject matter of our future research.

CHAPTER 5

CONCLUSION

In the classical statistical framework, any estimator of an unknown population parameter is a function of sample observations. The variance of the estimator, however, cannot be less than the Cramer – Rao minumum variance bound. The only way to obtain an estimator having its variance smaller than the minumum variance bound is to engage Bayesian methodology: An unknown population parameter θ is assumed to have a probability density function $p(\theta)$ called prior distribution. The posterior distribution $f(\theta \mid y)$ of θ is defined to be the product of the prior distribution $p(\theta)$ and the likelihood function $L(y_1, y_2, \ldots, y_n \mid \theta)$, the likelihood function representing the sample information. Thus, the posterior distribution of θ is given by

$$f(\theta \mid y) \propto p(\theta) L(y_1, y_2, y_3, \ldots, y_n \mid \theta) \tag{5.1}$$

θ is a single parameter or vector of parameters. The Bayesian estimator of θ, popularly called the HPD (highest posterior density) estimator, is defined to be the mode of $f(\theta \mid y)$.

If $f(\theta \mid y)$ is a symmetric distribution, its mode, median and mean are the same. Otherwise, they are different and called Bayesian mode, Bayesian median and Bayesian mean estimators, respectively. HPD estimator Bayesian mode is most popular. To locate the mode of $f(\theta \mid y)$ one differentiates its logarithm with respect to θ and equates it to zero. The solution is the HPD estimator provided $\partial^2 \ln L(\theta \mid y)/d\theta^2 < 0$. In most situations, however, $\partial \ln L(\theta \mid y)/d\theta = 0$ has no explicit solution and finding its zero(es) becomes a very difficult task analytically and computationally. Certain probing techniques, e.g., Gibbs sampling, are available to locate the mode of $f(\theta \mid y)$. Such solutions, however, are not conducive to algebraic treatment of the subject matter. To alleviate these difficulties, Bain and Tiku (1997, a,b) defined posterior distribution as the product of the prior distribution and Tiku's modified maximum likelihood function $L^*(y_1, y_2, \ldots, y_n \mid \theta)$; L^* is obtained by solving differential equations which yield modified maximum likelihood estimators (Tiku and Akkaya, 2004, p.53). An intresting feature of L^* is that it resembles a normal – theory likelihood function irrespective of the underlying distribution $f(y \mid \theta)$. Consequently, HPD estimators take the form of convex combinations of the prior perceived value of a population parameter and its modified maximum likelihood estimator. The latter are known to be asymptotically equivalent to maximum likelihood estimators; for finite sample sizes, they are essentially as efficient as maximum likelihood estimators and numerically very close to them.

Bian and Tiku (1997,a,b) used this new posterior to find the HPD estimators of location and scale parameters of distributions in two families: a) long-tailed symmetric distributions, and b) Gamma distributions. We have extended this work to the prominent family of Generalized Logistic distributions. Further, we have extended the work to one – way – classification ANOVA models. Our estimators are convex combinations and have beautiful algebraic forms. We have shown that they have variances smaller than the minimum variance bounds. In one – way – classification our estimator of ith block effect, for example, is

$$\hat{\mu}_{b,i} = \frac{\sigma_{i0}^{-2}\mu_{i0} + m\ \hat{\sigma}^{-2}\hat{\mu}_i}{\sigma_{i0}^{-2} + m\ \hat{\sigma}^{-2}} \tag{5.2}$$

and its variance is

$$Var(\mu_i \mid y) = \left(\sigma_{i0}^{-2} + m\hat{\sigma}^{-2}\right)^{-1} \tag{5.3}$$

The underlying distribution being (3.2). The estimated variance of the corresponding MMLE is $\hat{\sigma}^2 / m$ (almost) which is larger than (5.3). Note that, $\hat{\sigma}^2 / m$ is only marginially bigger than the minimum variance bound $\{(p-3/2)(p+1)/np(p-1/2)\}\hat{\sigma}^2$ (Tiku and Suresh, 1992); the MVB estimator of μ does not exist for the LTS family.

We have given in Chapter 4 of the thesis real life examples to illustrate the usefulness of our method and the HPD estimators.

Our method can be extended to more complex data structures, e.g., two-way- classification ANOVA with interaction, linear regression, etc. That will be the subject matter of future research.

REFERENCES

Angers, J.F. and Berger, J.O. (1991). Robust hierarchical Bayesian estimation of exchangeable means. *Canadian Journal of Statistics*, 19, 39-56.

Balakrishnan, N. and Leung, W.Y. (1988). Means, variances and covariances of order statistics, BLUE for the Type I generalized logistic distribution and some applications. *Communications in Statistics-Simulation and Computation*, 17, 51-84.

Barnard,G. A. (1985). *Pivotal Inference*, In Encyclopedia of Statistical Sciences, Vol.6, ed. S. Kotz, N.L. Johnson, and C. B.. Read, 743-747. New York, Wiley.

Barnett, V.D. (1966). Evaluation of the maximum likelihood estimator where the likelihood equation has multiple roots. *Biometrika*, 53,151-165.

Berger, J.O. (1984). The robust Bayesian viewpoint (with discussion). In "*Robustness of Bayesian Analysis*",edited by J. Kadane, North-Holland-Amsterdam.

Berger, J. O. (1985). *Statistical Decision Theory and Bayesian Analysis*, second edition, New-York: Springer-Verlag.

Berger, J.O., Brown, L.D. and Wolbert, R.L. (1994). A unified conditional frequentist and Bayesian test for fixed and sequential simple hypothesis testing, *Annals of Statistics, 22, 1787-1807.*

Berger, J.O., Boukai, B. And Wang, Y. (1997). Unified frequentist and Bayesian testing of a precise hypothesis, *Statistical Science, 12, 133-160.*

Bernardo, J. M. (1979). Reference posterior distributions for Bayesian inference (with discussion). *Journal of the Royal Statistical Society*, B 41, 113-147.
Bernardo, J. M., and Smith, A.F. M. (1994). *Bayesian Theory,* Wiley, New York.

Bhattacharyya, G.K. (1985). The asymptotics of maximum likelihod and related estimators based on type II censored data. *Journal of the American Statistical Association***,** 80,398-404.

Bian, G. (1989). Robust Bayesian estimators in one way ANOVA model. *Test*, 4, 115-135.

Bian, G. (1997). Bayesian inference in location-scale distribution with independent bivariate priors. *Test*, Vol6, 137-157.

Bian, G. and Dickey, J.M. (1996). Properties of multivariate Cauchy and poly-Cauchy distributions with Bayesian g-prior applications. In " *Bayesian Analysis in Statistics and Econometrics: Essays in Honor of Arnold Zellner*" edited by D.A. Berry, K.M. Chaloner and J.K. Geweke. John Wiley & Sons, New York.

Bian, G. and Tiku, M.L. (1997,a), Bayesian inference based on robust priors and MML estimators: Part I, symmetric location-scale distributions. *Statistics*, 29, 317-345.

Bian, G. and Tiku, M.L. (1997,b), Bayesian inference based on robust priors and MML estimators: Part II, skew location-scale distributions. *Statistics*, 29, 81-89

Box, G. E. P, and Tiao, G. C. (1973). *Bayesian Inference in Statistical Analysis*, New York: Wiley Classics.

Carlin, B. P., Chaloner, K., Church, T., Louis, T. A. and Matts, J. P. (1993). Bayesian Approaches for Monitoring Clinical Trials with an application to toxoplasmic encephalitis prophylaxis. *The Statistician, 42, 355-367.*

Dawid, A. P., Stone, M., and Zidek, J. V. (1973). Marginaliztion paradoxes in Bayesian and structural inferences (with discussion). *Journal of the Royal Statistical Society*, B, 35, 189-233.

Dey, D., Müller, P. And Sinha, D. (eds) (1998). *Practical Nonparametric and Semiparametric Bayesian Statistics*, Sipringer-Verlag.

Dickey, J. M. (1968). Three multidimensional-integral identities with Bayesian applications. *Annals of Mathematical Statistics*, 39, 1615-1627.

Dickey, J. M. (1974). Bayesian alternatives to the F test and least square estimate in the normal linear model. In "*Studies in Bayesian Econometrics and Statistics*" edited by S.E. Fienberg and A. Zellner, North-Holland, Amsterdam, 515-554.

Geary, R.C. (1947). Testing for Normality. *Biometrika* 34, 209-220.

Gelman, A., Carlin, B.J., Stern,S.H. and Rubin, D. (2004). *Bayesian Data Analysis,* Cahpman&HALL/CRC.

Geweke, J. (1993). Bayesian treatment of the independent student t linear model, *Journal of Applied Econometrics, 8, 19-40.*

Ghosh, K.J., Delampady, M., Samanta, T. (2006). *An Introduction to Bayesian Analysis (Theory and Methods),* New-York: Springer.

Gill, J. (2008). *Bayesian Methods: A social and Behavioral Sciences Approach*, Chapman and Hall, New York.

Gupta, S.S. (1960). Order statistics from the gamma distribution. *Technometrics*, 2, 243-262.

Hartigan , J. (1964). Invariant prior distributions, *Annals of Mathematical Statistics*, 35, 836-845.

Hartigan, J.A. (1983). *Bayes Theory*, Springer-Verlag, New York.

Ibrahim, J.G. and Chen, M.H. (2000). Prior elicitation and variable selection for generalized linear mixed models. *In Generalized Linear Models: A Bayesian*

Perspective, Dipak K. Dey, Sujit K.Ghosh and Bani K. Mallick (eds.). Marcel Dekker, New York.

Ibrahim, J. G. And Laud, P.W. (1991). On Bayesian analysis of generalized linear models using Jeffreys Priors, *Journal of the American Statistical Association, 86, 981-986.*

Jeffreys, H. (1961*). Theory of Probability*, third edition. Oxford University Press.

Johnson, L.N. and Leone, C. F. (1964). *Statistics and Experimental Design: In Engineering and the Physical Sciences, Volume II*, John Wiley and Sons, Inc.

Kass, R. E. (1989). The geometry of asymptotic inference, *Statistical Science, 4, 188-219*.

Kass, R. E., and Wasserman, L. (1996). The selection of prior distributions by formal rules. *Journal of the American Statistical Association*, 91, 1343--1370.

Le Cam, L. (1986). *Asymptotic Methods in Statistical Decision Theory*, Springer-Verlag, New York.

Lee, P.M. (2004*). Bayesian Statistics: An Introduction*. Second Edition, Hodder Arnold, New York.

Lee, K.R., Kapadia, C.H., and Dwight, B.B. (1980). On estimating the scale parameter of Rayleigh distribution from censored samples. *Statistische Hefte,* 21,14-29.

Leonard, T. (1975). A Bayesian approach to the linear model with unequal variances, *Technometrics, 17, 95-102.*

Lindley, D. V. (1961). The use of prior probability distributions in statistical inference and decision. *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, Berkeley, University of California Press, 453-468.*

Lindley, D.V. and Smith, A. F. M. (1972). Bayes estimates for the linear model, *Journal of the Royal Statistics Society, Series B, 34, 1-41.*

Marden, J. I. (2000). Hypothesis testing, from p values to Bayes Factors, *Journal of the American Statistical Association, 95, 1316-1320.*

Marin, J.M. and Robert, J.P. (2007). *Bayesian Core: A Practical Approach to Computational Bayesian Statistics*, Springer, New York.

Mouchart, M. and Simar,L. (1984). A note on least squares approximation in the Bayesian analysis of regression models, *Journal of the Royal Statistical Society, B, 46, 124-133.*

O'Hagan, A. (1990). Outliers and credence for location parameter inference. *Journal of the American Statistical Association*, 85, 172-176.

Pearson, E.S. and Hartley, H.O. (1972). *Biometrika Tables for Statisticians*. Vol. 2, Cambridge University Press.

Poirer, D. J. (1994). Jeffreys prior for logit models, *Journal of Econometrics, 63, 327-339.*

Prescott, P. (1974). Variances and covariances of order statistics from the gamma distribution. *Biometrika*, 61, 607-613.

Puthenpura, S. and Sinha, N.K. (1986). Modified maximum likelihood method for the robust estimation of system parameters from very noisy data. *Automatica*, 22, 231-235.

Qumsiyeh, S.B. (2007). Non-normal bivariate distributions: Estimation and hypothesis testing. *Ph. D. Thesis, Faculty of Art and Science, Department of Statistics, Middle East Technical University, Turkey*

Senoglu, B. and Tiku, M.L. (2001). Analysis of variance in experimental design with nonnomal error distribution. *Commun. Statist. Theory and Methods*, 30(7),1335-1352.

Schneider, H. (1986). *Truncated and Censored Samples from Normal Populations*. Marcel Dekker:New York.

Scheffé, H. (1959). *The Analysis of Variance*. John Wiley: New York

Sinha, D. And Dey, D. K. (1997). Semiparametric Bayesian analysis of survival data, *Journal of the American Statistical Association*, *92, 1195-1212.*

Stone, M. (1964). Comments on a posterior distribution of Geisser and Cornfield. *Journal of Royal Statistical Society, Series B,* 26, 274-276.

Tan, W.Y. (1985). On Tiku's robust procedure- a bayesian insight. *Journal of Statistical Planning and Inference,*11,329-340.

Tan, W.Y. and Balakrishnan, N. (1986). Bayesian insight into Tiku's robust procedures based on asymmetric censored samples. *Journal of Statistical Computation and Simul*ation, 24, 17-31.

Tiao, G.C. and Zellner, A. (1964). On the Bayesian estimation of multivariate regression. *Journal of Royal Statistical Society,* Series B, 26, 277-285.

Tiku, M.L. (1967). Estimating the mean and standard deviation from a censored normal sample. *Biometrika,* 54, 155-165.

Tiku, M.L. (1968a). Estimating the parameters of log-normal distribution from censored samples. *Journal of the American Statistical Association,* 63, 134-140.

Tiku, M.L. (1968b). Estimating the parameters of normal and logistic distributions from censored samples. *Australian Journal of Statistics,* 10, 64-74.

Tiku, M.L. (1968c). Estimating the mean and standard deviation from progressively censored normal samples. *Journal of Indian Society Agricultural Statistics*, 20, 20-25.

Tiku, M.L. (1970). Monte Carlo study of some simple estimators in censored normal samples. *Biometrika*, 57, 207-210.

Tiku, M.L. (1973). Testing group effects from Type II censored normal samples in experimental design. *Biometrics*, 29, 25-33.

Tiku, M.L. and Akkaya, A. D. (2004). *Robust Estimation and Hypothesis Testing.* New Age International (P) Ltd: Delhi.

Tiku, M.L., Tan, W.Y. and Balakrihnan, N. (1986). *Robust Inference.* Marcel Dekker, New York.

Tiku, M.L. and Suresh, R.P. (1992). A new method of estimation for location and scale parameters, *Journal of Statistical Planning and Inference***,** 30, 281-292.

Vaughan, D.C. (1992). On the Tiku-Suresh method of estimation. *Communications in Statistics-Theory and Methods***,** 21,451-469.

Vaughan, D.C. and Tiku, M.L. (2000). Estimation and hypothesis testing for a non-normal bivariate distrtibution and applications. *Journal of Mathematical and Computer Modelling*, 32, 53-67.

Vaughan, D.C. (2002). The generalized hyperbolic distribution and its properties. *Communications in Statistics Theory and Methods***,** 31, 219-238.

Walker, S., Damien, P., Laud, P. and Smith, A. (1999). Bayesian nonparametric inference for distributions and related functions (with discussion), *Journal of the Royal Statistical Society, B, 61.*

APPENDIX A

VISUAL FORTRAN PROGRAM FOR CENSORED STUDENT T DISTRIBUTION ONE SAMPLE CASE

```
C   *** Written by Pelin Özbozkurt, 2009, Ankara***

      use numerical_libraries
      integer simnumber,n
      real r,q,k,p,DF,t(1000),u,h(1000)
      real mu,sigma, order,ordery, alpha(1000)
      real alphapay(1000),alphapayda(1000)
      real betapay(1000),betapayda(1000),beta1(1000)
      real t1,g1,fpay1,fpay,fpayda,f1,b1,b2,b,a
      real sumbeta,M,sumbetay,muhat1,muhat
      real muhatMML(10000), Abig,sumalphay
      real Bbig,Cbig,sum1, z(10000),y(10000)
      real sigmahat1,sigmahat,sigmahatMML(10000)
      real sum2,muhatMMLsimmean,sum3,muhatMMLsimvar
      real muzero,sigmazero,vzero,muhatbayespay1,muhatbayespay
      real muhatbayespayda1,muhatbayespayda,sigmahatpayda
      real muhatbayesson,muhatbayes(10000)
      real sum5,muhatBayessimmean,sum6,muhatBayessimvar
      real RE,sigmazeroinv,vzeroinv,sigmahatinv
      real muhatbayespay2, muhatbayespayda2
      real deltamu,mubcase2pay1,mubcase2pay2,mubcase2pay3
      real mubcase2pay,mubcase2payda1,mubcase2payda2
      real mubcase2payda3,mubcase2payda,mubcase
      real muhatbayesCase2(10000),sum5case2,
      real muhatBayesCase2simmean,sum6case2
      real muhatBayesCase2simvar,RECase2
      real d1
```

```
      open(unit=1,file='C:\Documents and Settings\peli\Desktop\
      &CENSOREDstudentT.txt')

      print*, 'enter n and r'

      read*, n,r

      simnumber=10000
      mu=0
      sigma=1.5
      q=r/n
      p=3.5
      k=(2*p)-3
      DF=(2.0*p)-1

C *** Specify prior probabilities as:

      muzero=0
      sigmazero=3
      vzero=6

C *** Calculate t(i) as:

      do i=1,n
      u=i/((1.0*n)+1)
      t(i)=TIN(u,DF)
      enddo

C *** Calculate alpha(i) as:

      do i=1,n
      alphapay(i)=(2/k)*t(i)*t(i)*t(i)
      alphapayda(i)=(1+((1/k)*t(i)*t(i)))**2
      alpha(i)=alphapay(i)/alphapayda(i)
      enddo
```

```
C *** Calculate beta(i) as:

      do i=1,n
      betapay(i)=1-((1/k)*t(i)*t(i))
      betapayda(i)=(1+((1/k)*t(i)*t(i)))**2
      beta1(i)=betapay(i)/betapayda(i)
      enddo

C *** IF BETA(r+1) ISNEGATIVE, SO USE BETASTAR AND ALPHASTAR as:

      if (beta1(r+1)<0) then
      do i=1,n
      beta1(i)=1/((1+(1/k)*t(i)*t(i))**2)
      alphapay(i)=(1/k)*t(i)*t(i)*t(i)
      alphapayda(i)=(1+((1/k)*t(i)*t(i)))**2
      alpha(i)=alphapay(i)/alphapayda(i)
      enddo
      endif

C *** Calculate a and b as:

      d1=1.0*(1-q)
      t1=TIN(d1,DF)
      g1=t1/(1+(1/k)*(t1)*(t1))
      fpay1=1/(1+(1/k)*(t1)*(t1))
      fpay=fpay1**p
      fpayda=sqrt(k)*BETA(0.5,p-0.5)
      f1=fpay/fpayda
      b1=((-1.0)*f1)/q
      b2=(2.0*p/k)*g1-(1.0)*(f1/q)
      b=b1*b2
      a=f1/q-b*t1

C *** Start simulation:

      DO 212 s=1,simnumber
      call RNSTT(n,DF,h)
```

```
      do i=1,n
      y(i)=sigma*h(i)+mu
      enddo

C *** Find ordered y(i) values as:

      order=1
5     if (order.eq.1) then
      order=0
      do 8 i=1,n-1
      if (y(i).gt.y(i+1)) then
      ordery=y(i)
      y(i)=y(i+1)
      y(i+1)=ordery
      order=1
      endif
8     continue
      go to 5
      endif

C *** Start to calculate MML estimators:

C *** Calculate MML of muhat

      sumbeta=0
      do i=r+1,n-r
      sumbeta=sumbeta+beta1(i)
      enddo

      M=((2.0*p*sumbeta)/k)+(2.0*r*b)

      sumbetay=0
      do i=r+1,n-r
      sumbetay=sumbetay+beta1(i)*y(i)
      enddo
      muhat1=((2.0*p*sumbetay)/k)+r*b*(y(r+1)+y(n-r))
      muhat=muhat1/M
      muhatMML(s)=muhat
```

```
C *** Calculate MML of sigmahat

      Abig=(1.0*n)-(2.0*r)

      sumalphay=0
      do i=r+1,n-r
      sumalphay=sumalphay+alpha(i)*y(i)
      enddo

      Bbig=((2*p*sumalphay)/k)+r*a*(y(n-r)-y(r+1))

      sum1=0
      do i=r+1,n-r
      sum1=sum1+beta1(i)*((y(i)-muhat)**2)
      enddo

      Cbig=((2.0*p*sum1)/k)+r*b*(y(r+1)-muhat)**2
     &+r*b*(y(n-r)-muhat)**2

      sigmahat1=sqrt((Bbig**2)+(4.0*Abig*Cbig))
      sigmahat=(Bbig+sigmahat1)/(2*sqrt((Abig)*(Abig-1)))
      sigmahatMML(s)=sigmahat

C *** Calculate HPD Estimators:

C *** CASE1: When mubayes is close to muzero:

      sigmazeroinv=(1.0/sigmazero)
      vzeroinv=(1.0/vzero)
      sigmahatinv=(1.0/sigmahat)
      muhatbayespay1=(sigmazeroinv**2)*(1+vzeroinv)*muzero
      muhatbayespay2=M*(sigmahatinv**2)*muhat
      muhatbayespay=muhatbayespay1+muhatbayespay2
      muhatbayespayda1=(sigmazeroinv**2)*(1+vzeroinv)
      muhatbayespayda2=M*(sigmahatinv**2)
      muhatbayespayda=muhatbayespayda1+muhatbayespayda2
      muhatbayesson=muhatbayespay/muhatbayespayda
      muhatbayes(s)=muhatbayesson
```

```
C *** Case2: When mubayes close to muMML:

      deltamu=(muhat-muzero)/sigmazero
      mubcase2pay1=(sigmazeroinv*sigmazeroinv*muzero)
      mubcase2pay2=(1.0*vzero+(deltamu*deltamu))/(vzero+1.0)
      mubcase2pay3=mubcase2pay2*M*(sigmahatinv**2)*muhat
      mubcase2pay=(mubcase2pay1+mubcase2pay3)
      mubcase2payda1=(sigmazeroinv*sigmazeroinv)
      mubcase2payda2=(vzero+(deltamu*deltamu))/(vzero+1)
      mubcase2payda3=mubcase2payda2*M*(sigmahatinv**2)
      mubcase2payda=mubcase2payda1+mubcase2payda3
      mubcase2=mubcase2pay/mubcase2payda
      muhatbayesCase2(s)=mubcase2
212   continue

C *** Finding simulated MMLmean:

      sum2=0
      do i=1,simnumber
      sum2=sum2+muhatMML(i)
      enddo
      muhatMMLsimmean=sum2/simnumber

C *** Finding simulated MMLvariance:

      sum3=0
      do i=1,simnumber
      sum3=sum3+((muhatMML(i)-muhatMMLsimmean)**2)
      enddo
      muhatMMLsimvar=sum3/(simnumber-1)

C *** Finding simulated Bayes mean: CASE1:

      sum5=0
      do i=1,simnumber
      sum5=sum5+muhatbayes(i)
      enddo

      muhatBayessimmean=sum5/simnumber
```

```
C *** Finding simulated Bayes mean: CASE2:

      sum5case2=0
      do i=1,simnumber
      sum5case2=sum5case2+muhatbayesCase2(i)
      enddo

      muhatBayesCase2simmean=sum5case2/simnumber

C *** Finding Simulated Bayes variance:CASE1:

      sum6=0
      do i=1,simnumber
      sum6=sum6+((muhatbayes(i)-muhatBayessimmean)**2)
      enddo

      muhatBayessimvar=sum6/(simnumber-1)

C *** Finding simulated Bayes variance:CASE2:

      sum6case2=0
      do i=1,simnumber
      sum6case2=sum6case2+((muhatbayesCase2(i)-
                &muhatBayesCase2simmean)**2)
      enddo

      muhatBayesCase2simvar=sum6case2/(simnumber-1)

C ** Relative Efficiency, Case1:

      RE=muhatBayessimvar/muhatMMLsimvar

C ** Relative Efficiency, Case2:

      RECase2=muhatBayesCase2simvar/muhatMMLsimvar

202   format(a8,6x,a12,6x,a10,6x,a14,6x,a13)
      write(1,202)'MML_Mean','MML_Variance','Bayes_Mean',
      &'Bayes_Variance','RE(Bayes/MML)'
```

```
C *** Case1: When mubayes close to muzero:

203   format(f7.4,8x,f7.4,12x,f7.4,9x,f7.4,13x,f7.4)
      write(1,203) muhatMMLsimmean,muhatMMLsimvar,
      &muhatBayessimmean,muhatBayessimvar,RE

C *** Case2: When mubayes close to muMML:

C 203 format(f7.4,8x,f7.4,12x,f7.4,9x,f7.4,13x,f7.4)
C write(1,203) muhatMMLsimmean,muhatMMLsimvar,
C &muhatBayesCase2simmean,muhatBayesCase2simvar
C &RECase2

stop
end
```

APPENDIX B

VISUAL FORTRAN PROGRAM FOR GENERALIZED LOGISTIC DISTRIBUTION ONE WAY CLASSIFICATION

```
C   *** Written by Pelin Özbozkurt, 2009, Ankara***

      use numerical_libraries
      integer N,simnumber,p,k,order,ntr,nbl,f
      real b,mu(1000),sigma,ordery,pu
      real q(1000),tara1(1000),tara2(1000),t(1000)
      real bbetapay(1000),bbetapayda(1000),bbeta(1000)
      real alphapay(1000),alphapayda(1000),alpha(1000)
      real u(1000),z(1000),sumbbeta, u2(100,100)
      real ya1(100,100),ya2(100,100),ya3(100,100),y(100,100)
      real delta(1000),sumdelta,D2,K2(1000)
      real B2ara1(1000),B2(1000),B2sum
      real C2ara(1000),C2(1000),C2sum,sigmahatpay1
      real sigmahatpayda1,sigmahatpayda2,sigmahatpay2
      real sMML,mMML(100),sigmahatMML(100000)
      real summuhat(1000),sim_Meanmuhat_MML(1000)
      real sumsigmahat,sim_Meansigmahat_MML muhatMML(100,100000)
      real summuvar(1000),sim_Variancemuhat_MML(1000)
      real sumsigmavar,sim_Variancesigmahat_MML
      real muzero(1000),szero(1000),deltazero(1000)
      real sifprime2,sif2,sibarti1,siprimebarti1
      real muhatpriorpay1(1000),muhatpriorpay2(1000)
      real muhatpriorpayda(1000),muhatBAYESprior(1000)
      real muhat_prior_BAYES(100,100000),muhatpriorpay(1000)
      real h2inversepay,h2inversepayda1,h2inversepayda2
      real h2inversepayda3,h2inverse,h2
      real summubayeshat(1000),sim_Meanmuhat_Bayes(1000)
```

```
 real summubayesvar(1000),sim_Variancemuhat_Bayes(1000)
 real RE_Bayes_MML(1000)
 real fmupay1,fmupay2,fmupay,fmupayda1,fmupayda2
 real fmupayda3,fmupayda,MVBmuKONTROL
 real fsigmapay,fsigmapayda1,fsigmapayda2
 real fsigmapayda3,fsigmapayda
 real fcovpay,fcovpayda1,fcovpayda2,fcovpayda3,fcovpayda
 real fmupayda1ara, fsigmapayda1ara,fcovpayda1ara
 real K2ara(1000), szeroinv(10000)
 real deltamu(10000),mubcase2pay1(10000)
 real mubcase2pay2(10000), mubcase2pay3(10000)
 real mubcase2pay(10000),mubcase2payda1(10000)
 real mubcase2payda2(10000), mubcase2(10000)
 real mubcase2payda3(10000),mubcase2payda(10000)
 real muhatbayesCase2(100,10000)
 real sum5case2(10000), sum6case2(10000)
 real muhatBayesCase2simmean(10000)
 real muhatBayesCase2simvar(10000)
 real RECase2(10000), deltazeroinv(10000),sMMLinv
 real muhatpriorpayda1(1000),muhatpriorpayda2(1000)

 open(unit=1,file='C:\Documents and Settings\peli\Desktop\
&GL_ANOVA.txt')

 print*, 'enter ntr, nbl,b'

 read*,ntr
 read*,nbl
 read*,b

 simnumber=10000
 do i=1,ntr
 mu(i)=0
 enddo

 N=ntr*nbl
 sigma=1
 sifprime2=0.6449
 sif2=0.4228
```

```
      if (b.eq.0.5) sibarti1=0.0365
      if (b.eq.1)  sibarti1=0.4228
      if (b.eq.2)  sibarti1=0.9228
      if (b.eq.3)  sibarti1=1.2561
      if (b.eq.4)  sibarti1=1.5061
      if (b.eq.5)  sibarti1=1.7061
      if (b.eq.6)  sibarti1=1.8728
      if (b.eq.7)  sibarti1=2.0156
      if (b.eq.8)  sibarti1=2.1406
      if (b.eq.9)  sibarti1=2.2518
      if (b.eq.10) sibarti1=2.3518
      if (b.eq.0.5) siprimebarti1=0.9348
      if (b.eq.1)  siprimebarti1=0.6449
      if (b.eq.2)  siprimebarti1=0.3949
      if (b.eq.3)  siprimebarti1=0.2838
      if (b.eq.4)  siprimebarti1=0.2213
      if (b.eq.5)  siprimebarti1=0.1813
      if (b.eq.6)  siprimebarti1=0.1536
      if (b.eq.7)  siprimebarti1=0.1331
      if (b.eq.8)  siprimebarti1=0.1175
      if (b.eq.9)  siprimebarti1=0.1051
      if (b.eq.10) siprimebarti1=0.0951

C *** Specify prior parameters as:

      do i=1,ntr
      muzero(i)=0
      szero(i)=2.5
      deltazero(i)=6
      end

C ** Find h2 as:

      fmupay1=((sibarti1-sif2)*(sibarti1-sif2))+
     &(siprimebarti1+sifprime2)
      fmupay2=(1.0*ntr*b*fmupay1)+(1.0*ntr*(b+2.0))
      fmupay=fmupay2
      fmupaydalara=1.0*(ntr-1)*b*b*((sibarti1-sif2)*
     &(sibarti1-sif2))
```

```
      fmupayda1=fmupayda1ara/(b+2.0)
      fmupayda2=(1.0*ntr*b*b*(siprimebarti1+sifprime2))/(b+2.0)
      fmupayda3=(1.0*ntr*b)+fmupayda1+fmupayda2
      fmupayda=nbl*fmupayda3
      h2inverse=fmupay/fmupayda
      h2=1.0/h2inverse

C *** Start to calculate MML estimators:

      do j=1,nbl
      p=j
      q(j)=p/(nbl+1.0)
      enddo

      do j=1,nbl
      tara1(j)=(1.0/q(j))**(1.0/b)
      tara2(j)=tara1(j)-1
      t(j)=(-1.0)*alog(tara2(j))
      enddo

      do j=1,nbl
      bbetapay(j)=exp(t(j))
      bbetapayda(j)=(1+exp(t(j)))*(1+exp(t(j)))
      bbeta(j)=bbetapay(j)/bbetapayda(j)
      enddo

      sumbbeta=0
      do j=1,nbl
      sumbbeta=sumbbeta+bbeta(j)
      enddo

      do j=1,nbl
      alphapay(j)=1.0+exp(t(j))+t(j)*exp(t(j))
      alphapayda(j)=(1+exp(t(j)))*(1+exp(t(j)))
      alpha(j)=alphapay(j)/alphapayda(j)
      enddo
```

```
C *** Start simulation as:

      DO 212 s=1,simnumber

      call rnun(N,z)

      do i=1,N
      u(i)=z(i)
      enddo

      f=1
            do i=1,ntr
                  do j=1,nbl
                        u2(i,j)=u(f)
      f=f+1
      enddo
      enddo

C *** Obtain random variables from GL as:

      do i=1,ntr
            do j=1,nbl
                  ya1(i,j)=(1.0/u2(i,j))**((1.0/b))
                  ya2(i,j)=ya1(i,j)-1
                  ya3(i,j)=(-1.0)*alog(ya2(i,j))
                  y(i,j)=mu(i)+sigma*ya3(i,j)
            enddo
      enddo

C *** Find ordered y(i,(j)) values as:

      order=1
5     if (order.eq.1) then
      order=0
      do 8 i=1,ntr
      do 10 j=1,nbl-1
      if (y(i,j).gt.y(i,j+1)) then
      ordery=y(i,j)
      y(i,j)=y(i,j+1)
```

```
      y(i,j+1)=ordery
      order=1
      endif
10    continue
8     continue
      go to 5
      endif

C *** Find muMML and sigmaMML as:

      do j=1,nbl
      delta(j)=alpha(j)-(1.0/(b+1.0))
      enddo

      sumdelta=0
      do j=1,nbl
      sumdelta=sumdelta+delta(j)
      enddo
      D2=sumdelta/sumbbeta

      do i=1,ntr
      K2ara(i)=0
      K2(i)=0
            do j=1,nbl
            K2ara(i)=K2ara(i)+bbeta(j)*y(i,j)
            enddo
      K2(i)=K2ara(i)/sumbbeta
      enddo

      do i=1,ntr
      B2ara1(i)=0
            do j=1,nbl
            B2ara1(i)=B2ara1(i)+1.0*delta(j)*(y(i,j)-K2(i))
            enddo
      enddo


      do i=1,ntr
      B2(i)=(b+1.0)*B2ara1(i)
```

```
enddo
B2sum=0
do i=1,ntr
B2sum=B2sum+B2(i)
enddo

do i=1,ntr
C2ara(i)=0
      do j=1,nbl
      C2ara(i)=C2ara(i)+bbeta(j)*(y(i,j)-K2(i))*
     &(y(i,j)-K2(i))
      enddo
enddo

do i=1,ntr
C2(i)=C2ara(i)*(b+1.0)
enddo

C2sum=0
do i=1,ntr
C2sum=C2sum+C2(i)
enddo

sigmahatpay1=(B2sum*B2sum)+(4.0*ntr*nbl*C2sum)
sigmahatpay2=(-1.0)*B2sum+sqrt(1.0*sigmahatpay1)
sigmahatpayda1=1.0*N*(N-ntr)
sigmahatpayda2=2.0*sqrt(1.0*sigmahatpayda1)
sMML=sigmahatpay2/sigmahatpayda2

do i=1,ntr
mMML(i)=K2(i)-(1.0*sMML*D2)
enddo
sigmahatMML(s)=sMML

do i=1,ntr
muhatMML(i,s)=mMML(i)
enddo
```

```
C *** Find muBayes as:

C *** Case1: When mubayes close to muzero:

      sMMLinv=1/sMML

      do i=1,ntr
      deltazeroinv(i)=1/deltazero(i)
      muhatpriorpay1(i)=(1.0/szero(i))*(1.0/szero(i))*
     &(1+deltazeroinv(i))*muzero(i)
      muhatpriorpay2(i)=1.0*h2*mMML(i)*sMMLinv*sMMLinv
      muhatpriorpay(i)=muhatpriorpay1(i)+muhatpriorpay2(i)
      muhatpriorpayda1(i)=(1.0/szero(i))*(1.0/szero(i))*
     &(1+deltazeroinv(i))
      muhatpriorpayda2(i)=1.0*h2*sMMLinv*sMMLinv
      muhatpriorpayda(i)=muhatpriorpayda1(i)+muhatpriorpayda2(i)
      muhatBAYESprior(i)=muhatpriorpay(i)/muhatpriorpayda(i)
      enddo

      do i=1,ntr
      muhat_prior_BAYES(i,s)=muhatBAYESprior(i)
      enddo

C *** Case2: When mubayes close to muMML:

      do i=1,ntr
      szeroinv(i)=1/szero(i)
      deltamu(i)=(mMML(i)-muzero(i))/szero(i)
      mubcase2pay1(i)=(szeroinv(i)*szeroinv(i)*muzero(i))
      mubcase2pay2(i)=(1.0*deltazero(i)+(deltamu(i)*
     &deltamu(i)))/(deltazero(i)+1.0)
      mubcase2pay3(i)=h2*(mubcase2pay2(i)*
     &(sMMLinv**2)*mMML(i))
      mubcase2pay(i)=(mubcase2pay1(i)+mubcase2pay3(i))
      mubcase2payda1(i)=(szeroinv(i)*szeroinv(i))
      mubcase2payda2(i)=(deltazero(i)+(deltamu(i)*
     &deltamu(i)))/(deltazero(i)+1)
      mubcase2payda3(i)=mubcase2payda2(i)*h2*(sMMLinv**2)
      mubcase2payda(i)=mubcase2payda1(i)+mubcase2payda3(i)
```

```
      mubcase2(i)=mubcase2pay(i)/mubcase2payda(i)
      muhatbayesCase2(i,s)=mubcase2(i)
      enddo
212   continue

C *** Finding simulated MMLmean:

      do i=1,ntr
      summuhat(i)=0
            do s=1,simnumber
            summuhat(i)=summuhat(i)+muhatMML(i,s)
            enddo
      enddo

      do i=1,ntr
      sim_Meanmuhat_MML(i)=summuhat(i)/simnumber
      enddo

C *** Finding simulated Bayes mean: CASE1:

      do i=1,ntr
      summubayeshat(i)=0
            do s=1,simnumber
            summubayeshat(i)=summubayeshat(i)+
                             &muhat_prior_BAYES(i,s)
            enddo
      enddo
      do i=1,ntr
      sim_Meanmuhat_Bayes(i)=summubayeshat(i)/simnumber
      enddo

C *** Finding simulated Bayes mean: CASE2:

      do i=1,ntr
      sum5case2(i)=0
            do s=1,simnumber
            sum5case2(i)=sum5case2(i)+muhatbayesCase2(i,s)
            enddo
      enddo
```

```
      do i=1,ntr
      muhatBayesCase2simmean(i)=sum5case2(i)/simnumber
      enddo

C *** Finding simulated MMLvariance:

      do i=1,ntr
      summuvar(i)=0
            do s=1,simnumber
            summuvar(i)=summuvar(i)+
            &(muhatMML(i,s)-sim_Meanmuhat_MML(i))*
            &(muhatMML(i,s)-sim_Meanmuhat_MML(i))
            enddo
      enddo

      do i=1,ntr
      sim_Variancemuhat_MML(i)=summuvar(i)/(simnumber-1.0)
      enddo

C *** Finding Simulated Bayes variance:CASE1:

      do i=1,ntr
      summubayesvar(i)=0
            do s=1,simnumber
            summubayesvar(i)=summubayesvar(i)+
            &(muhat_prior_BAYES(i,s)-sim_Meanmuhat_Bayes(i))*
      &(muhat_prior_BAYES(i,s)-sim_Meanmuhat_Bayes(i))
            enddo
      enddo

      do i=1,ntr
      sim_Variancemuhat_Bayes(i)=summubayesvar(i)/
      &(simnumber-1.0)
      enddo

C *** Finding Simulated Bayes variance:CASE2:

      do i=1,ntr
      sum6case2(i)=0
```

```
            do s=1,simnumber
            sum6case2(i)=sum6case2(i)+
                        &((muhatbayesCase2(i,s)-
                         &muhatBayesCase2simmean(i))**2)
            enddo
      enddo

      do i=1,ntr
      muhatBayesCase2simvar(i)=sum6case2(i)/(simnumber-1)
      enddo

C *** Relative efficency, Case1:

      do i=1,ntr
      RE_Bayes_MML(i)=sim_Variancemuhat_Bayes(i)/
                     &sim_Variancemuhat_MML(i)
      enddo

C *** Relative efficency, Case2:

      do i=1,ntr
      RECase2(i)=muhatBayesCase2simvar(i)/
                &sim_Variancemuhat_MML(i)
      enddo
202   format(a9,11x,a10,6x,a12,6x,a14,10x,a13)
      write(1,202) 'MML_Mean','MML_Variance','Bayes_Mean',
     &'Bayes_Variance','RE(Bayes/MML)'

C *** Case1: When mubayes close to muzero:

203   format(f7.4,11x,f7.4,13x,f7.4,11x,f7.4,15x,f7.4,11x,f7.4,
    &11x,f7.4)

      do i=1,ntr
      write(1,203) sim_Meanmuhat_MML(i),
     &sim_Variancemuhat_MML(i),sim_Meanmuhat_Bayes(i),
     &sim_Variancemuhat_Bayes(i),RE_Bayes_MML(i)
      Enddo
```

```
C *** Case2: When mubayes close to muMML:

C 203 format(f7.4,11x,f7.4,13x,f7.4,11x,f7.4,15x,f7.4,11x,f7.4,
C      &11x,f7.4)

C     do i=1,ntr
C     write(1,203) sim_Meanmuhat_MML(i),
C     &sim_Variancemuhat_MML(i),muhatBayesCase2simmean(i),
C     &muhatBayesCase2simvar(i),RECase2(i)
C     enddo

      stop
      end
```

VITA

Pelin (Özkan) Özbozkurt was born in Ankara on January 3, 1980. She received her B.S. degree in Statistics from Middle East Technical University in June, 2001. After that, she became research assistant in the department of Statistics. She received her M.S. degree of Statistics in September 2004 and M.S. degree of Economics on September 2005. In April 2008, she started to work in Turk Telekom as Data Mining Specialist and she is still working there. Her main interests are Bayesian analysis and experimental designs.