NEW CLASSES OF DIFFERENTIAL EQUATIONS AND BIFURCATION OF DISCONTINUOUS CYCLES

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

MEHMET TURAN

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY
IN
MATHEMATICS

JULY 2009

Approval of the thesis:

NEW CLASSES OF DIFFERENTIAL EQUATIONS AND BIFURCATION OF DISCONTINUOUS CYCLES

submitted by **MEHMET TURAN** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy in Mathematics Department, Middle East Technical University** by,

Prof. Dr. Canan Özgen
Dean, Graduate School of **Natural and Applied Sciences** ________________

Prof. Dr. Zafer Nurlu
Head of Department, **Mathematics** ________________

Prof. Dr. Marat Akhmet
Supervisor, **Mathematics Department, METU** ________________

Examining Committee Members:

Prof. Dr. Hasan Taşeli
Mathematics Department, METU ________________

Prof. Dr. Marat Akhmet
Mathematics Department, METU ________________

Prof. Dr. Tanıl Ergenç
Mathematics Department, Atılım University ________________

Prof. Dr. Gerhard Wilhelm Weber
Institute of Applied Mathematics, METU ________________

Assist. Prof. Dr. Ömür Uğur
Institute of Applied Mathematics, METU ________________

Date: **July 14, 2009**

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Last Name: MEHMET TURAN

Signature :

ABSTRACT

NEW CLASSES OF DIFFERENTIAL EQUATIONS AND BIFURCATION OF DISCONTINUOUS CYCLES

Turan, Mehmet
Ph.D., Department of Mathematics
Supervisor : Prof. Dr. Marat Akhmet

July 2009, 91 pages

In this thesis, we introduce two new classes of differential equations, which essentially extend, in several directions, impulsive differential equations and equations on time scales. Basics of the theory for quasilinear systems are discussed, and particular results are obtained so that further investigations of the theory are guaranteed.

Applications of the newly-introduced systems are shown through a center manifold theorem, and further, Hopf bifurcation Theorem is proved for a three-dimensional discontinuous dynamical system.

Keywords: Periodic solution, stability, center manifold, Hopf bifurcation

ÖZ

YENİ TÜR DİFERANSİYEL DENKLEMLER SINIFLARI VE SÜREKSİZ DÖNGÜLERİN ÇATALLANMASI

Turan, Mehmet
Doktora, Matematik Bölümü
Tez Yöneticisi : Prof. Dr. Marat Akhmet

Temmuz 2009, 91 sayfa

Bu tezde iki tür yeni diferansiyel denklem sınıfı tanıttık. Aslında bu denklem sınıfları impalsif diferansiyel denklemlerini ve zaman skalalarında diferansiyel denklemleri çeşitli açılardan genişletirler. Yarı doğrusal denklemlerin temel teorisi tartışılmış ve teorinin daha ileri düzeyde incelenebilmesini garantilemek için belirli sonuçlar elde edilmiştir.

Yeni tanıtılan sistemlerin uygulamaları merkez çok katlı teoremi aracılığıyla gösterilmiş ve bir üç boyutlu süreksiz dinamik sistem için Hopf Çatallanma Teoremi kanıtlanmıştır.

Anahtar Kelimeler: Periyodik çözüm, kararlılık, merkez çok katlısı, Hopf çatallanması

To my mother, Meliha,
and
to the memory of my late father, Kıtmir

ACKNOWLEDGMENTS

I would like to express my sincere gratitude to my supervisor, Prof. Dr. Marat Akhmet, for his invaluable guidance and encouragement throughout this research. He has been more than a supervisor to me. During our extensive conversations, I have learned a lot about not only my academic life, but also about real life. I appreciate his motivation, helpful discussions, encouragement, patience and constant guidance. It has been a great privilege for me working with him.

I shall also thank to the examining committee members, Prof. Dr. Tanıl Ergenç, Prof. Dr. Hasan Taşeli, Prof. Dr. Gerhard Wilhelm Weber, and Assist. Prof. Dr. Ömür Uğur for their helpful comments and suggestions. It has been an enjoyable experience discussing various issues with Dr. Ergenç and Dr. Taşeli during the lectures I have attended with them and during our dialogs outside the classroom.

The members of the Applied Dynamics Group at METU also deserve to be mentioned here. It has been a great pleasure for me to work and discuss with my friends. Furthermore, I thank the members of the Departments of Mathematics at METU and at Atılım University for providing me with a friendly atmosphere; especially, my roommates Abdullah Özbekler at METU and Ahmet Yantır at Atılım University.

The financial support of The Scientific and Technical Research Council of Turkey (TÜBİTAK) is also acknowledged.

This thesis is dedicated to the memory of my late father, Kıtmir, and mother, Meliha. I would like to extend my deepest thanks to them because of everything they have done for me throughout my whole life.

Last, but not least, my special thanks go to my beloved sisters Sabiha and Gurbet along with their husbands for unconditional love, Enes and Rumeysa for being true manifestations of a more colorful side of life.

TABLE OF CONTENTS

LIST OF FIGURES

FIGURES

CHAPTER 1

INTRODUCTION

Most of the real world processes are studied by means of differential equations. The invention of the theory of ordinary differential equations dates back to the second half of the Seventeenth Century. Newton (1642-1727) was the first person to consider the differential equations. He regarded this observation so important that he used the phrase "...the laws of nature are expressed by differential equations..." to emphasize the importance of his discovery.

A new era in the development of the theory of differential equations starts with Poincaré (1854-1912). Instead of traditional methods, he considered new topological ideas. The *Qualitative Theory of Differential Equations* - or, as it is known nowadays, *the theory of dynamical systems* - is the starting point to discuss the nonlinear differential equations. Birkhoff (1884-1944) understood the idea of Poincaré and developed it at the beginning of the Twentieth Century. Russian mathematicians have taken an important role in the development of this subject, beginning with the works of Lyapunov (1857-1918) on the stability of motion, Andronov (1901-1952) on bifurcation theory, Krylov (1879-1955) and Bogolyubov (1909-1992) on the theory of averaging, Kolmogorov (1903-1987) on the theory of perturbations of conditionally periodic motions.

In the last quarter of the last century, there has been an explosion of interest in the study of nonlinear dynamical systems; geometric and qualitative techniques developed during this period makes it possible to better investigate nonlinear dynamical systems. The theory of nonlinear dynamical systems is one of the most developing subjects of the theory of differential equations since it is mostly applied in physics, chemistry, biology, ecology, economics, mechanics, electrics, and electronics, all of which have yielded valuable results. In fact, the systems which seemed to be hard to grasp from the analytical point of view are now easily understandable from the geometric or qualitative points of view.

The history of discontinuous dynamical systems is relatively short. In [58], the first investigation into the discontinuous dynamical system can be seen. There, the authors considered the model of a clock; a pendulum which experiences a strike when the angle between the current position and the equilibrium position reaches a specific level was taken into account. In that work, it was shown that the approximation method used in nonlinear mechanics can be applied for a study of differential equations with impulse action. This has attracted the attention of scientists from other disciplines since it made it possible to investigate the processes in nonlinear oscillations.

With this accomplished, scientists became interested in Impulsive Differential Equations (IDE's). IDE's characterize many real life evolutionary processes whose state experiences a change called the 'impulse'. Impulses are short-term perturbations of the process. When the changes occur at the specified times, we talk about the IDE's with fixed moments of time. Generally, this is not the case and, at times, the impulse actions take place depending on the state. These kinds of systems are called IDE's with variable moments of impulse actions. For long, scientists have considered merely those IDE's with fixed moments of impulse actions and stayed away from those with variable moments of impulse actions; this is due to the fact that they did not have enough material to handle these problems with. The truth is that, these problems were not so easy to overcome. Many results concerning the IDE's with fixed moments of impulses have been provided in different references [60, 85]. These also contain some results about the IDE's with variable moments of impulses. Once B-topology was introduced by Akhmetov and Perestyuk [5, 6, 13], handling IDE's with variable moments of impulses became easier compared to the past. The method which is based on the B-topology enables us to deal with these kinds of systems. This method is our main tool in the investigation of the system considered in this study.

The center manifold theory is another main tool used in this thesis. In the literature and among the first efforts regarding the subject of center manifold theory, one can see the paper by Pliss [82]. Also, the book by Carr [30] provides us with useful information about the applications of center manifold. However, in neither one of these works can one find the center manifold and its applications related to discontinuous dynamical systems.

The main point that we are going to utilize from regarding the center manifold theory is to apply it to discontinuous dynamical systems to prove the existence of a periodic solution in multi-dimensional discontinuous dynamical systems. In fact, this will be a discontinuous limit cycle; that is, we shall prove the Hopf Bifurcation Theorem in three-dimensional discontinuous dynamical systems.

While dealing with the Hopf Bifurcation Theorem, naturally a new class of differential equations arises (which we abbreviate as the *differential equations on time scales with transition condition (DETC)*). The concept of the time scale was first introduced by Hilger [51]. In his work, the author tends to unify and extend the differential and discrete equations. The DETC introduced here and the one proposed by Hilger, have both similarities and differences.

1.1 Elements of Impulsive Differential Equations

Many evolutionary processes are subject to short-term perturbation whose duration is negligible when compared to that of the whole process. This perturbation results in a change in the state of the process. For example, when a bouncing ball strikes against a fixed surface, then a change in the velocity of the ball occurs. Another example is the pendulum of a clock showing a change in momentum when passing through its equilibrium position. Models like these have played a significant role in the development of impulsive differential equations. In [24, 25, 85, 86], many theoretical results are given for impulsive differential equations such as the existence and uniqueness of solutions, stability, periodic solutions.

In principle, there are two different kinds of impulsive differential equations: the ones with fixed moments of impulse actions and those with variable moments of impulse actions. The former is a system of the form

$$\frac{dx}{dt} = f(t,x), \quad t \neq \theta_i, \\ \Delta x|_{t=\theta_i} = I_i(x), \tag{1.1}$$

which is called an 'IDE with fixed moments of pulse actions'. In (1.1), $x \in \mathbb{R}^n$ is the state (phase) variable. The sequence $\{\theta_i\}$, where i is an index belonging to a finite or infinite index set as a subset of $\mathbb{Z}$, denotes the fixed moments at which the impulse actions take place. The right-hand side function $f(t,x)$ is the continuous rate of change of the phase variable, and $I_i(x)$ is the discrete (sudden) change of the phase variable. Moreover, $\Delta x|_{t=\theta_i} = x(\theta_i^+) - x(\theta_i)$ denotes the jump in the phase point at the time $t = \theta_i$. That is, a phase point of (1.1) moves along one of the trajectories of $x' = f(t,x)$ until the time $t = \theta_i$. At the moment $t = \theta_i$, the phase point jumps to the point $x(\theta_i^+) = x(\theta_i) + I_i(x(\theta_i))$, and continues along a trajectory of $x' = f(t,x)$ until the next moment of impulse action, and so on. Therefore, a solution, $x(t)$, of (1.1) is a piecewise continuous function with discontinuities of the first kind at $t = \theta_i$.

In the latter one, however, impulse action takes place when the phase point meets one of the prescribed surfaces in the phase space. These kinds of systems are more challenging to investigate when compared to the first category since different solutions possess different moments of impulses. Nevertheless, they arise more naturally than the first kind. An impulsive differential equation with variable (or non-fixed) moments of impulses is a system of the following form

$$\frac{dx}{dt} = f(t,x), \quad t \neq \tau_i(x), \\ \Delta x|_{t=\tau_i(x)} = I_i(x), \tag{1.2}$$

where x, $f(t,x)$ and $I_i(x)$ have been described before, and for each i, $\tau_i(x)$ stands for the surface of discontinuity. As it can be seen easily in (1.2), the moments when the impulse actions take place depend on the phase point, $x(t)$ and, hence, each solution will perform the jumps at different times. For this reason, system (1.2) is more difficult than system (1.1) to investigate.

The systems in (1.1) and (1.2) are both non-autonomous. There exists another important class of differential equations which is autonomous, also known as *Discontinuous*

Dynamical Systems (DDS's). A discontinuous dynamical system can be expressed as

$$\begin{aligned} &\frac{dx}{dt} = f(x), \quad x \notin \Gamma, \\ &\Delta x|_{x\in\Gamma} = I(x), \end{aligned} \tag{1.3}$$

where Γ denotes the set of discontinuity. A phase point of (1.3) moves along one of the trajectories of the autonomous differential equation $x' = f(x)$ until the time when this solution, say $x(t)$, meets the set Γ. After this meeting, the phase point is mapped to the point $x + I(x)$, if x is the phase point just before the meeting, and continues its motion along the trajectory of $x' = f(x)$ with the initial point at $x + I(x)$, and so on. It is clear that the discontinuities of a solution of (1.3) also depend on the solution, like in (1.2). This is one of the reasons why the theory of systems (1.2) and (1.3) have not been addressed adequately until now. However, they have started to be noticed by many scientists since they have a wide range of applications. In [6, 13], a method has been introduced and developed to handle these systems. In the present study, we intend to use these methods as well.

Here, our main system will be of the form (1.3). Needless to mention that while studying this system, a new type of differential equation came up, which we call as *differential equations on variable time scales* (DETCV). In the next section, we shall provide the conventional differential equations on time scales as well as the DETC (the ones that we introduced to deal with the DETCV).

1.2 An Overview of the Differential Equations on Time Scales

Some dynamic processes have been modeled by difference equations or differential equations. As far as the modeling is concerned, the idea to involve both continuous and discrete times to model a process is more realistic. For this reason, except for impulsive differential equations, there exist another class of systems called dynamic systems on time scales or measure chains [64]. The notion of time scales was introduced by Aulbach and Hilger back in the 80's [22, 51, 52]: the idea there was to unify the discrete and continuous dynamics. Recently, many results in the theory of discrete dynamics have been obtained as discrete analogs of the corresponding results of continuous dynamics. However, in the discrete case, there are some topological deficiencies, including lack of connectedness. Some assumptions have been made to overcome these topological deficiencies. For the same purpose, we will be using a special kind of time scale.

Any nonempty closed subset of $\mathbb{R}$ is called a 'time scale', generally denoted by $\mathbb{T}$. A differential equation of the form

$$x^{\Delta}(t) = f(t, x), \quad t \in \mathbb{T}, \tag{1.4}$$

where $x^{\Delta}(t)$ denotes the Δ-derivative of x at the point $t \in \mathbb{T}$, is called a 'dynamic equation on the time scale $\mathbb{T}$'.

The theory of dynamic equations on time scales (DETS) has been developed in the last couple of decades [2, 29, 64]. After a literature survey about DETS, one can conclude that there are not as many theoretical results on the existence of periodic solutions and almost periodic solutions. To this date, the investigations concerning linear DETS, integral manifolds, and the stability of equations have not been developed in full. It goes without saying that, such results need to be obtained so as to able us to benefit from the applications of the theory. We also propose a method to obtain such theoretical results, and to investigate differential equations on certain time scales with transition conditions (DETC) which are, in a way, more general than DETS.

Here, effort is made to expand our knowledge of these aspects of the theory, and to introduce a new class of differential equations on time scales. In fact, this class of equations arises naturally when we solve the problem of Hopf bifurcation, which is our main goal in this study.

The time scale that we consider in this thesis is of the form

$$\mathbb{T}_c = \bigcup_{i \in \mathbb{Z}} [t_{2i-1}, t_{2i}], \tag{1.5}$$

where $t_n, n \in \mathbb{Z}$, is a strictly increasing sequence such that $t_n \to \pm\infty$ as $n \to \pm\infty$. On a time scale as in (1.5), the differential equation with transition conditions (DETC) is defined as a system of the form

$$\begin{aligned} y' &= f(t, y), \quad t \in \mathbb{T}_c, \\ y(t_{2i+1}) &= y(t_{2i}) + J_i(y(t_{2i})), \end{aligned} \tag{1.6}$$

where $f : \mathbb{T}_c \times \mathbb{R}^n \to \mathbb{R}^n$, and $J_i : \mathbb{R}^n \to \mathbb{R}^n$ are continuous functions in their domains.

At the same time, we should recognize that significant theoretical results have been achieved concerning oscillations, boundary value problems, positive solutions, hybrid systems, etc. [1, 2, 20, 28, 29, 34, 37, 38, 41, 64, 65, 91]. We assume that our proposals may initiate new ideas by which the theory can also be developed, thus adding to the previous significant achievements in that direction.

The DETC will be discussed in the next chapter. The main idea in the investigation of DETC is to apply the results of the theory of impulsive differential equations (IDE's), the investigation of which started in the late sixties of the last century [43, 60, 85]. We note that certain classes of DETC, concerned with time scales, can be reduced to IDE if we apply a special transformation [6] of the independent argument - the time variable. This transformation allows the reduced IDE to inherit all similar properties of the corresponding DETC. Then, the investigation of the IDE can proceed using the existing results. Finally, by taking into account the properties of the independent argument transformation, we can have an interpretation of the obtained results for the DETC. The approach we are using to connect the DETC with another type of differential equations is close to that in paper [65], where hybrid systems on time scales have been considered. Besides the DETC, in this study, we introduce the non-linearity on time scales and consider, as we call *the variable time scales*.

1.3 Basics of Center Manifold and Hopf Bifurcation

Roughly speaking, a bifurcation is a qualitative change in an attractor's structure as a control parameter is varied smoothly. For example, a simple equilibrium or fixed point attractor might give way to a periodic oscillation as the stress on a system increases. Similarly, a periodic attractor might become unstable and be replaced by a chaotic attractor.

The bifurcation theory is the mathematical study of changes in the qualitative or topological structure of a given family. Examples of such families are the integral curves of a family of vector fields or the solutions of a family of differential equations. Most commonly applied to the mathematical study of dynamical systems, a bifurcation occurs when a small smooth change made to the parameter values (the bifurcation parameters) of a system causes a sudden "qualitative" or topological change in its behavior. Bifurcations occur in both continuous systems (described by ODE's, DDE's or PDE's) and discrete systems (described by maps).

At times, bifurcations are divided into two principle classes. The first one is local bifurcations, which can be analyzed entirely through changes in the local stability properties of equilibria, periodic orbits or other invariant sets as parameters cross through critical thresholds. The second one is global bifurcations, which often occur when larger invariant sets of the system "collide" with each other, or with the equilibria of the system; these cannot be detected purely by a local stability analysis of the equilibria (fixed points).

A local bifurcation occurs when a parameter change causes the stability of an equilibrium (or fixed point) to change. In continuous systems, this corresponds to the real part of an eigenvalue of an equilibrium passing through zero. In discrete systems (those described by maps rather than ODE's), this corresponds to a fixed point having a Floquet multiplier with modulus equal to one. In both cases, the equilibrium is non-hyperbolic at the bifurcation point. The topological changes in the phase portrait of the system can be confined to arbitrarily small neighborhoods of the bifurcating fixed points by moving the bifurcation parameter close to the bifurcation point (hence, 'local').

Global bifurcations occur when 'larger' invariant sets, such as periodic orbits, collide with the equilibria. This causes changes in the topology of the trajectories in the phase space which cannot be confined to a small neighborhood, as is the case with local bifurcations. In fact, the changes in topology extend out to an arbitrarily large distance (hence, 'global').

Examples of global bifurcations include the following:

- Homoclinic bifurcation, in which a limit cycle collides with a saddle point;
- Heteroclinic bifurcation, in which a limit cycle collides with two or more saddle points;
- Infinite-period bifurcation, in which a stable node and saddle point simultane-

ously occur on a limit cycle; and

- Blue sky catastrophe, in which a limit cycle collides with a non-hyperbolic cycle.

It deserves mentioning that global bifurcations can also involve more complicated sets such as chaotic attractors.

Named after Eberhard Hopf and Aleksandr Andronov, a Hopf or Andronov-Hopf bifurcation, is a local bifurcation. Here, a fixed point of a dynamical system loses stability as a pair of complex conjugate eigenvalues of the linearization around the fixed point cross the imaginary axis of the complex plane. Under reasonably generic assumptions about the dynamical system, we can expect to see a small amplitude limit cycle branching from the fixed point. This bifurcation was studied by Poincaré who, in his work on the gravitational three-body problem, obtained certain periodic solutions. Later, though, Andronov and Hopf provided a more explicit discussion on that issue.

One of the main methods of simplifying dynamical systems is to reduce the dimension of the system. The center manifold theory is a rigorous mathematical technique that makes this reduction possible, at least near the equilibria. Due to the power of this theory in investigating systems, it became very popular, and attracted many scientists.

The history of center manifolds is very short, going back to 1960's. The ideas for center manifolds in finite dimensions have been developed by Kelly (1967), Carr (1981), Guckenheimer and Holmes (1983), Vanderbauwhede (1989), and others. For recent developments in the approximation of center manifolds, see Jolly and Rosa (2005). Pages 1-5 of the book by Li and Wiggins (1997) provide an extensive list of the applications of center manifold theory to infinite dimensional problems. Mielke (1996) developed center manifold theory for elliptic partial differential equations, and applied the theory to elasticity and hydrodynamical problems. Haken (2004), in turn, investigated the applications to phase transitions in biological, chemical and physical systems.

When a system loses stability, the number of eigenvalues and eigenvectors associated with this change is typically small. Hence, bifurcation problems usually involve systems where the linearization has a very large - and possibly infinite - dimensional stable part and a small number of “critical” modes which change from stable to unstable as the bifurcation parameter exceeds a threshold. The central idea of the bifurcation theory is that the dynamics of the system near the onset of instability is governed by the evolution of these critical modes, while the stable modes follow in a passive fashion and become ‘enslaved’. The center manifold theorem is the rigorous formulation of this idea; it allows us to reduce a large problem to a small and manageable one. Therefore, after the reduction on the center manifold, it becomes easier to investigate the system since - in a local neighborhood of the fixed point - the quantitative behavior of the reduced system is the same as that of the whole system.

In this study, we shall also utilize the center manifold theory to investigate the Hopf bifurcation in a three-dimensional discontinuous dynamical system.

1.4 Description of B-equivalence Method

A challenge in investigating systems with discontinuities on nonlinear surfaces is that each solution has different moments of impulse effects, or discontinuities. In the literature surrounding the object, many results can be found related to linear surfaces of impulse actions [60, 63, 71, 84, 85, 86]. However, rarely can one see the works containing nonlinearities on the surfaces. Although they are more realistic for real world applications, many authors tend to avoid these systems due to this difficulty. In [6, 11], the authors have introduced a new method to handle this difficulty. There, the so-called B-equivalence and B-topology have been proposed. This method is a powerful tool to deal with the variable moments of impulse actions. Here, we shall apply the method proposed for impulsive differential equations with variable moments of impulses and, in Chapter 3, we will adopt this method to the differential equations on variable time scales. Subsequently in Chapter 4, this method will be applied to a system in three dimension and the Hopf bifurcation theorem will be proved.

Consider the systems

$$\begin{aligned} x' &= f(t,x), \quad t \neq \tau_i(x), \\ \Delta x|_{t=\tau_i(x)} &= I_i(x), \end{aligned} \tag{1.7}$$

and

$$\begin{aligned} y' &= f(t,y), \quad t \neq \theta_i, \\ \Delta y|_{t=\theta_i} &= J_i(y), \end{aligned} \tag{1.8}$$

where the hyper-surfaces $t = \tau_i(x)$ are small perturbations of the hyper-planes $t = \theta_i$, and the functions J_i to be supplied in the thesis. Akhmet and Perestyuk [6, 11] have shown that, corresponding to each solution $x(t, t_0, x_0)$ of (1.7) satisfying $x(t_0, t_0, x_0) = x_0$, there exists a solution $y(t, t_0, x_0)$ of (1.8) satisfying $y(t_0, t_0, x_0) = x_0$ such that these two solutions are the same for all t in their common domains except for the ϵ-neighborhoods of the discontinuity points, and vice versa. In fact, a discontinuity point of the solution of one system lies in an ϵ-neighborhood of the corresponding discontinuity point of the solution of the other system. In this study, we have adopted this important technique to our system and, by means of this compelling material, we have successfully obtained the required results.

1.5 A Transformation of the Independent Variable: ψ-substitution

It is common to simplify a given equation by a proper transformation in the theory of differential equations. Likewise in this study, we use a transformation introduced by Akhmet in [6] and developed in [19]. This is a transformation of the independent variable and serves as a bridge in the passage from DETC, as in (1.6), to an IDE.

For a time scale $\mathbb{T}_c$ as in (1.5), on the set $\mathbb{T}_c' = \mathbb{T}_c \setminus \bigcup_{i=-\infty}^{\infty}\{t_{2i-1}\}$, the ψ-substitution is defined as

$$\psi(t) = \begin{cases} t - \sum\limits_{0<t_{2k}<t} \delta_k, & t \geq 0 \\ t + \sum\limits_{t \leq t_{2k}<0} \delta_k, & t < 0 \end{cases}, \tag{1.9}$$

where $\delta_k = t_{2k+1} - t_{2k}$. Notice that the ψ-substitution is a one-to-one map, $\psi(0) = 0$, and the structure of the sequence $\{t_n\}$ implies that ψ maps $\mathbb{T}_c'$ onto $\mathbb{R}$. The inverse transformation is

$$\psi^{-1}(s) = \begin{cases} s + \sum\limits_{0<s_k<s} \delta_k, & s \geq 0 \\ s - \sum\limits_{s\leq s_k<0} \delta_k, & s < 0 \end{cases}. \tag{1.10}$$

Note that the inverse transformation is a piecewise continuous function with discontinuity of the first kind at the points $s = s_i = \psi(t_{2i})$, $i \in \mathbb{Z}$ and $\psi^{-1}(s_i+) - \psi^{-1}(s_i) = \delta_i$.

The aim of the ψ-substitution is to make the domain of the system (1.6) a connected domain. Besides, it carries the significant properties of the function it is applied to. For example, if $\phi(s)$ is a periodic function on $\mathbb{R}$, then $\phi(\psi(t))$ is a periodic function on $\mathbb{T}'_c$, and vice versa. A number of properties of the ψ-substitution will be given throughout the thesis when necessary.

1.6 Motivation for the Main Study

For a motivation, let us consider how the idea of variable time scales emerged before we begin with the main part of the thesis. The following planar system was considered in [6]

$$\begin{aligned} &\frac{dx}{dt} = Ax + f(x), \quad x \notin \Gamma, \\ &\Delta x|_{x\in\Gamma} = B(x)x, \end{aligned} \tag{1.11}$$

where $\Gamma = \cup_{i=1}^{p} \ell_i$ is a set of curves starting at the origin. Using polar coordinates, the system is written in the form:

$$\begin{aligned} &\frac{dr}{d\phi} = \lambda r + P(r,\phi), \quad (r,\phi) \notin \Gamma, \\ &\phi^+ \mid_{(r,\phi)\in\ell_i} = \phi + \theta_i + \gamma(r,\phi), \\ &r^+ \mid_{(r,\phi)\in\ell_i} = (1+k_i)r + \omega(r,\phi). \end{aligned} \tag{1.12}$$

Denote by ℓ'_i the image of ℓ_i under the transition operator $\Pi_i(\phi, r)$ where $\Pi_i^1(\phi, r) = \phi + \theta_i + \gamma(r,\phi)$, and $\Pi_i^2(\phi, r) = (1 + k_i)r + \omega(r,\phi)$. Let $\mathcal{D}_i$ be the set bounded by ℓ'_i and ℓ_{i+1}. In [6], it is shown that this set is non-empty, and ℓ'_i is between ℓ_i and ℓ_{i+1} if the equation is considered in a small neighborhood of the origin.

Denoting $\mathbb{T}(r) = \bigcup_{i=1}^{p} \mathcal{D}_i$, we have the following DETCV:

$$\begin{aligned} \frac{dr}{d\phi} &= \lambda r + P(r,\phi), \qquad (\phi, r) \in \mathbb{T}(r), \\ \phi^{+} &= \Pi_i^1(\phi, r), \quad r^{+} = \Pi_i^2(\phi, r), \quad (\phi, r) \in \ell_i. \end{aligned} \tag{1.13}$$

This equation is an example of a differential equation on a variable time scale. In this study, we shall consider a generalization of this equation and prove the Hopf Bifurcation Theorem for our system.

1.7 Organization of the Thesis

This dissertation has been arranged in the following way:

In Chapter 2, we introduce the differential equation with transition conditions on time scales (DETC) and investigate it on the basis of reduction to the impulsive differential equations. We give the basic definitions on time scales and consider the basic properties of linear systems, the existence and stability of periodic solutions.

Chapter 3 is devoted to differential equations on variable time scales (DETCV), and contains the definition of a variable time scale, existence and uniqueness theorem for DETCV, the method used to investigate the DETCV, existence of periodic solutions, stability of solutions and finally bounded solutions. The results given in that chapter will be used in our main study.

In Chapter 4, we consider the Hopf Bifurcation Theorem where we illustrate the bifurcation of three-dimensional discontinuous cycles. Also proved in this chapter is the existence of a center manifold. To demonstrate the work throughout the thesis, each chapter contains a number of examples.

Finally, the last chapter is devoted to a conclusion.

CHAPTER 2

DIFFERENTIAL EQUATIONS WITH TRANSITION CONDITION ON TIME SCALES

In this chapter we investigate differential equations on certain time scales with transition conditions (DETC) on the basis of a reduction to the impulsive differential equations (IDE). DETC are in some sense more general than dynamic equations on time scales [29, 64]. Basic properties of linear systems, existence and stability of periodic solutions are considered. Appropriate examples are given to illustrate the theory.

2.1 Introduction

The theory of dynamic equations on time scales (DETS) has been developed in the last several decades [2, 29, 64]. After a literature survey about DETS, one can conclude that there are not so many results of the theory on the existence of periodic solutions. Up to this moment, the investigations concerning linear DETS, integral manifolds and the stability of equations have not been fully developed. Certainly, these results should be obtained in order to benefit from the applications of the theory. In this chapter, we make an attempt to expand our knowledge of these aspects of the theory. We also propose a way to obtain these theoretical results. Moreover, we investigate differential equations on certain time scales with transition conditions (DETC), which are in some sense more general than DETS. At the same time, we should recognize that significant theoretical results concerning oscillations, boundary value problems, positive solutions, hybrid systems etc., have been achieved

[1, 2, 20, 28, 29, 34, 37, 38, 41, 64, 65, 91]. We assume that our proposals may stimulate new ideas by which the theory can also be developed adding to the previous significant achievements in that direction. The main idea of the chapter is to apply the results of the theory of impulsive differential equations (IDE) the investigation of which started in the last century in the late 1960s [6, 11, 13, 43, 60, 85]. We note that certain classes of DETC, particular with their time scales, can be reduced to IDE, if we apply a special transformation [6] of the independent argument (the time variable). This transformation allows the reduced IDE to inherit all similar properties of the corresponding DETC. Then the investigation of the IDE can proceed using the known results. Finally, by taking the properties of the independent argument transformation into account, we can make an interpretation of the obtained results for DETC. The approach we are using to connect the DETC with another type of differential equations is close to that in the paper [65], where hybrid systems on time scales were considered.

This chapter is organized as follows. In the next section the time scale with its specific properties is considered. Moreover, the general form of DETC is described. The special transformation is given in Section 2.3. Reduction of DETC to IDE is done in Section 2.4. In Section 2.5, periodic solutions of linear equations and elements of Floquet's theory are considered also Massera theorem is proved. The last section of this chapter is devoted to the problem of existence and stability of almost periodic solutions.

2.2 Description of the Differential Equations with Transition Condition on Time Scales

Throughout this chapter we consider a specific time scale of the following type. Fix a sequence $\{t_i\} \in \mathbb{R}$ such that $t_i < t_{i+1}$ for all $i \in \mathbb{Z}$, and $|t_i| \to \infty$ as $|i| \to \infty$. Denote $\delta_i = t_{2i+1} - t_{2i}, \kappa_i = t_{2i} - t_{2i-1}$ and assume that:

(C0) $\sum_{i=-\infty}^{n} \kappa_i = \infty, \sum_{i=m}^{\infty} \kappa_i = \infty$, for any $n, m \in \mathbb{Z}$.

The time scale $\mathbb{T}_c = \bigcup_{i=-\infty}^{\infty} [t_{2i-1}, t_{2i}]$, is going to be considered throughout this study.

Consider the following system of differential equations

$$\begin{aligned} \frac{dy}{dt} &= f(t, y), \quad t \in \mathbb{T}_c, \\ y(t_{2i+1}) &= J_i(y(t_{2i})) + y(t_{2i}), \end{aligned} \tag{2.1}$$

where the derivative is one sided at the boundary points of $\mathbb{T}_c$, $f : \mathbb{T}_c \times \mathbb{R}^n \to \mathbb{R}^n$, $J_i : \mathbb{R}^n \to \mathbb{R}^n$, for all $i \in \mathbb{Z}$. We assume that functions f and J_i are continuous on their respective domains. Let us introduce the following *transition operator*, Π_i : $\{t_{2i}\}\times\mathbb{R}^n \to \{t_{2i+1}\}\times\mathbb{R}^n$, $i \in \mathbb{Z}$, such that $\Pi_i(t_{2i}, y) = (t_{2i+1}, J_i(y)+y)$. Thus the evolution of the process is described by:

1. the system of differential equations
$$\frac{dy}{dt} = f(t, y), \quad t \in \mathbb{T}_c; \tag{2.2}$$
2. the transition operator $\Pi_i, i \in \mathbb{Z}$;
3. the set $\mathbb{T}_c \times \mathbb{R}^n$.

We shall call equation (2.1) the *differential equation on time scales with transition condition* (DETC). Let us show how to construct a solution of (2.1). Denote, by $\phi(t, \kappa, z)$, a solution of system (2.2) with an initial condition $y(\kappa) = z$, $\kappa \in \mathbb{T}_c$, $z \in \mathbb{R}^n$, and, by $y(t)$, a solution of system (2.1) with an initial condition $y(t^0) = y_0$. Fix $t^0 \in \mathbb{T}_c$ such that $t_{2k-1} < t^0 < t_{2k}$ for some $k \in \mathbb{Z}$. If $t^0 \le t < t_{2k}$ the solution is equal to $\phi(t, t^0, y_0)$, and $y(t_{2k}) = \phi(t_{2k}-, t^0, y_0)$, where the left limit is assumed to exist. Now, applying the transition operator, we obtain that $y(t_{2k+1}) = J_k(y(t_{2k})) + y(t_{2k})$. Note that the solution is not defined in the interval (t_{2k}, t_{2k+1}). Next, on the interval $[t_{2k+1}, t_{2(k+1)})$ the solution is equal to $\phi(t, t_{2k+1}, y(t_{2k+1}))$, and $y(t_{2(k+1)}) = \phi(t_{2(k+1)}-, t_{2k+1}, y(t_{2k+1}))$, and so on. If solution $y(t)$ is defined on a set $I \subset \mathbb{T}_c$, then the set $\{(t, y) : y = y(t), t \in I\}$ is called an *integral curve* of the solution.

Let us start with the general information about differential equations on time scales. We provide only those facts of the theory which directly concern our needs in this chapter. More detailed description on the subject can be found in [2, 29, 64].

Any nonempty closed subset, $\mathbb{T}$, of $\mathbb{R}$ is called a time scale. For instance, $\mathbb{R}$ (real numbers), $\mathbb{Z}$ (integers), $\mathbb{N}$ (natural numbers) and $\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ are examples of time scales while $\mathbb{Q}$ (rational numbers), $\mathbb{R} \setminus \mathbb{Q}$ (irrational numbers) and $(0, 1)$ are not time scales [29, 64].

On a time scale $\mathbb{T}$, the functions $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ and $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$ are called the forward and backward jump operators, respectively. In case when any of these sets is empty, that is, if $\mathbb{T}$ is bounded above (below), this definition is supplemented by $\sigma(\max \mathbb{T}) = \max \mathbb{T}$ ($\rho(\min \mathbb{T}) = \min \mathbb{T}$). The point $t \in \mathbb{T}$ is called right-scattered if $\sigma(t) > t$, and right-dense if $\sigma(t) = t$. Similarly, it is called left-scattered if $\rho(t) < t$, and left-dense if $\rho(t) = t$. Note that on time scale $\mathbb{T}_c$, the points t_{2i-1}, $i \in \mathbb{Z}$, are left-scattered and right-dense, and the points t_{2i}, $i \in \mathbb{Z}$, are right-scattered and left-dense. Moreover, it is worth mentioning that $\sigma(t_{2i}) = t_{2i+1}$, $\rho(t_{2i+1}) = t_{2i}$, $i \in \mathbb{Z}$, and $\sigma(t) = \rho(t) = t$ for any other $t \in \mathbb{T}_c$.

The Δ-derivative of a continuous function f, at a right-scattered point is defined as

$$f^{\Delta}(t) := \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t},$$

and at a right-dense point it is defined as

$$f^{\Delta}(t) := \lim_{s \to t} \frac{f(t) - f(s)}{t - s},$$

if the limit exists.

Let $\mathbb{T}$ be an arbitrary time scale. A function $\varphi : \mathbb{T} \to \mathbb{R}$ is called rd-continuous if :

(i) it is continuous at each right-dense or maximal $t \in \mathbb{T}$;

(ii) the left sided limit $\varphi(t-) = \lim_{\xi \to t^-} \varphi(\xi)$ exists at each left-dense t.

Similarly, a function $\varphi : \mathbb{T} \to \mathbb{R}$ is called ld-continuous if :

(i) it is continuous at each left-dense or minimal $t \in \mathbb{T}$;

(ii) the right sided limit $\varphi(t+) = \lim_{\xi \to t^+} \varphi(\xi)$ exists at each right-dense t.

An equation of the form

$$y^{\Delta}(t) = f(t, y), \quad t \in \mathbb{T}, \tag{2.3}$$

is said to be a differential equation on time scale [64], where function $f : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be rd-continuous on $\mathbb{T} \times \mathbb{R}^n$.

In our specific case we denote, by $\mathcal{T}_0$, the set of all functions which are rd-continuous on $\mathbb{T}_c$. Moreover, we define a set of functions $\mathcal{T}_0^1 \subset \mathcal{T}_0$ which are continuously differentiable on $\mathbb{T}_c$, assuming that the functions have a one-sided derivative at the boundary points of $\mathbb{T}_c$, that is if $\phi \in \mathcal{T}_0^1$, then $\phi' \in \mathcal{T}_0$.

2.3 The ψ-substitution

It is common to simplify a given equation by a proper transformation in every theory of differential equations. Likewise, in this section, we introduce a transformation which plays the role of a bridge in the passage from DETC, as in (2.1), to an IDE.

Without loss of generality, we assume that $t_{-1} < 0 < t_0$. The ψ-substitution, on the set $\mathbb{T}'_c = \mathbb{T}_c \setminus \bigcup_{i=-\infty}^{\infty} \{t_{2i-1}\}$, is defined as

$$\psi(t) = \begin{cases} t - \sum\limits_{0<t_{2k}<t} \delta_k, & t \geq 0 \\ t + \sum\limits_{t \leq t_{2k}<0} \delta_k, & t < 0 \end{cases} \tag{2.4}$$

where $\delta_k = t_{2k+1} - t_{2k}$. Notice that the ψ-substitution is a one-to-one map, $\psi(0) = 0$, and the condition $(C0)$ implies that $\psi(\mathbb{T}'_c) = \mathbb{R}$. The inverse transformation is

$$\psi^{-1}(s) = \begin{cases} s + \sum\limits_{0<s_k<s} \delta_k, & s \geq 0 \\ s - \sum\limits_{s \leq s_k<0} \delta_k, & s < 0 \end{cases}. \tag{2.5}$$

Note that the inverse transformation is a piecewise continuous function with discontinuity of the first kind at the points $s = s_i$, $i \in \mathbb{Z}$, and $\psi^{-1}(s_i+) - \psi^{-1}(s_i) = \delta_i$.

Lemma 2.3.1 $\psi'(t) = 1$ *if* $t \in \mathbb{T}'_c$.

Proof. Assume that $t \geq 0$. Then,

$$\begin{aligned} \psi'(t) &= \lim_{h\to 0} \frac{\psi(t+h) - \psi(t)}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left[\left(t + h - \sum_{0<t_{2k}<t+h} \delta_k\right) - \left(t - \sum_{0<t_{2k}<t} \delta_k\right)\right] \\ &= 1. \end{aligned}$$

The assertion for $t < 0$ can be proved similarly. ■

Denote $s_i = \psi(t_{2i}), i \in \mathbb{Z}$. To make the reduction of DETC to IDE, we also need the following sets of functions. A function $\varphi : \mathbb{R} \to \mathbb{R}^n$ is said to be in $\mathcal{PC}_0$ if :

(i) $\varphi(s)$ is left continuous on $\mathbb{R}$ and continuous on $\mathbb{R} \setminus \bigcup_{i=-\infty}^{\infty}\{s_i\}$;

(ii) $\varphi(s)$ has discontinuities of the first kind at the points s_i.

Similarly, a function φ is said to be in $\mathcal{PC}_0^1$ if $\varphi \in \mathcal{PC}_0$ and φ' is in $\mathcal{PC}_0$ where

$$\varphi'(s_i) = \lim_{s \to s_i^-} \frac{\varphi(s) - \varphi(s_i)}{s - s_i}.$$

One can easily check that $\psi^{-1} \in \mathcal{PC}_0^1$, and $\frac{d}{ds}(\psi^{-1}(s)) = 1$ if $s \neq s_i, i \in \mathbb{Z}$.

In the next lemma we show that the spaces of functions $\mathcal{T}_0$ and $\mathcal{PC}_0$ are closely related. This relation is set up by ψ-substitution. In the same manner the relations between $\mathcal{T}_0^1$ and $\mathcal{PC}_0^1$ are going to be constructed. In what follows assume that $s = \psi(t)$.

Lemma 2.3.2 *If $\varphi \in \mathcal{T}_0$ then $\varphi \circ \psi^{-1} \in \mathcal{PC}_0$, and $\varphi \circ \psi \in \mathcal{T}_0$ if $\varphi \in \mathcal{PC}_0$.*

Proof. Since ψ is a one-to-one transformation we see that if t is not one of the points t_k, then $\psi(t)$ is not one of the points s_i. Now, the continuity of ψ-substitution concludes the proof. ■

Corollary 2.3.3 *If $\varphi \in \mathcal{T}_0^1$ then $\varphi \circ \psi^{-1} \in \mathcal{PC}_0^1$, and $\varphi \circ \psi \in \mathcal{T}_0^1$ if $\varphi \in \mathcal{PC}_0^1$.*

2.4 The Reduction to an Impulsive Differential Equation

From the definition of Δ-derivative at a right-scattered point [64], we have

$$y^{\Delta}(t_{2i}) = \frac{y(t_{2i+1}) - y(t_{2i})}{t_{2i+1} - t_{2i}}, \; i \in \mathbb{Z},$$

and hence equation (2.3) can be written as

$$\begin{aligned} y'(t) &= f(t, y), \quad t \in \mathbb{T}_c, \\ y(t_{2i+1}) &= f(t_{2i}, y(t_{2i}))\delta_i + y(t_{2i}), \end{aligned} \tag{2.6}$$

where $\delta_i = t_{2i+1} - t_{2i}$.

We generalize the last equation if the specific term $f(t_{2i}, y(t_{2i}))\delta_i$ in (2.6) is replaced by an expression $J_i(y(t_{2i}))$, where J_i can be an arbitrary function.

Thus the following equation is considered

$$\begin{aligned} y'(t) &= f(t, y), \quad t \in \mathbb{T}_c, \\ y(t_{2i+1}) &= J_i(y(t_{2i})) + y(t_{2i}). \end{aligned} \tag{2.7}$$

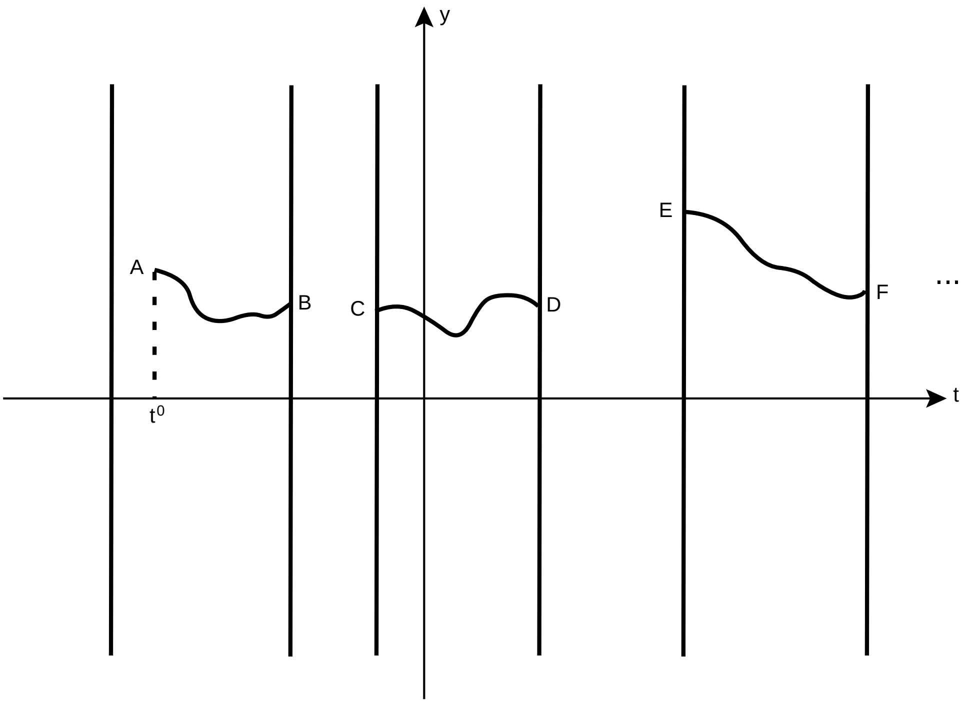

Figure 2.1: A trajectory of (2.7)

We name (2.7) as a *differential equation on time scale with transition condition* and we abbreviate its name as DETC. In Figure 2.1, a trajectory of the system (2.7) is shown. There, a solution starting at the initial point A at the time $t = t^0$ is sketched. The solution moves along one of the trajectories of $y' = f(t, y)$ until the time $t = t_{2i}$ when it touches the next hyperplane at the point, say B. At this moment a transition is performed and the solution jumps to the point C on the hyperplane $t = t_{2i+1}$. This transition is performed by means of the function J_i. In classical DETS, the transition from the hyperplane $t = t_{2i}$ to the hyperplane $t = t_{2i+1}$ is performed as follows: First, the tangent line to the graph of the solution at the point B is drawn, and then the intersection point of this tangent line with the hyperplane $t = t_{2i+1}$ is found. This intersection is the point where the phase point will be after the transition. However, in practice, this is not the case and the transition is done by a more general function, as J_i that we use in this study. Clearly, (2.6) is a specification of (2.7) with $J_i(y) = f(t,y)\delta_i$.

A function $\varphi \in \mathcal{T}_0^1$ is a solution of (2.7) if $\varphi'(t) = f(t,\varphi(t))$ for $t \in \mathbb{T}_c$, and $\varphi(t_{2i+1}) = J_i(\varphi(t_{2i})) + \varphi(t_{2i})$ for $t = t_{2i+1}, i \in \mathbb{Z}$.

Let us now apply the transformation of the independent argument to (2.7). If y is a solution of (2.7), then $x = y \circ \psi^{-1}$ is a solution of the equation $x' = f(\psi^{-1}(s), x)$ for $s \neq s_i$. Moreover, if $t = t_{2i+1}$, then $s = \psi(t) = s_i^+$, and hence, the second equation in (2.7) leads to

$$x(s_i^+) = J_i(x(s_i)) + x(s_i),$$

which can be written as

$$\Delta x|_{s=s_i} = J_i(x(s_i)),$$

where $\Delta x|_{s=s_i} = x(s_i^+) - x(s_i)$. Thus, x is a solution of the following IDE

$$\begin{aligned} x' &= f(\psi^{-1}(s), x), \quad s \neq s_i, \\ \Delta x|_{s=s_i} &= J_i(x(s_i)). \end{aligned} \tag{2.8}$$

The connection between DETC (2.7) and IDE (2.8) is established. The solution $x(s), x(s^0) = x_0, (s^0, x_0) \in \mathbb{R} \times \mathbb{R}^n$, of (2.8) satisfies the following integral equation

$$x(s) = x_0 + \int_{s^0}^{s} f(\psi^{-1}(\xi), x(\xi))d\xi + \sum_{s^0 \leq s_i < s} J_i(x(s_i^+)), \tag{2.9}$$

if $s \geq s^0$, and

$$x(s) = x_0 + \int_{s^0}^{s} f(\psi^{-1}(\xi), x(\xi))d\xi - \sum_{s \leq s_i < s^0} J_i(x(s_i^+)), \tag{2.10}$$

if $s < s^0$.

Let a, b be in $\mathbb{T}_c$ such that $a \leq b$. We denote

$$\mathbb{T}_c(a,b) = [a, t_{2m}] \cup \sum_{k=m+1}^{p-1} [t_{2k-1}, t_{2k}] \cup [t_{2p-1}, b],$$

where m and p are integers which satisfy $t_{2m-1} \leq a \leq t_{2m} < \cdots < t_{2p-1} \leq t \leq t_{2p}$, and for $f \in \mathcal{T}_0$ we set

$$\int_{\mathbb{T}_c(a,b)} f(\tau)d\tau := \int_a^{t_{2m}} f(\tau)d\tau + \int_{t_{2m+1}}^{t_{2m+2}} f(\tau)d\tau + \cdots + \int_{t_{2p-1}}^{b} f(\tau)d\tau.$$

Now, the solution, $y(t), y(t^0) = y_0$, of (2.7), where $t^0 = \psi^{-1}(s^0)$, satisfies

$$y(t) = y_0 + \int_{\mathbb{T}_c(t^0,t)} f(\tau, y(\tau))d\tau + \sum_{t^0 \le t_{2i} < t} J_i(y(t_{2i+1})), \tag{2.11}$$

if $t \geq t^0$, and

$$y(t) = y_0 - \int_{\mathbb{T}_c(t,t^0)} f(\tau, y(\tau))d\tau - \sum_{t \le t_{2i} < t^0} J_i(y(t_{2i+1})). \tag{2.12}$$

if $t < t^0$.

2.5 Linear Systems

In this section, we shall consider the linear differential equations with transition conditions on time scales. The results of this section will be needed in the next section where we investigate the existence of periodic solutions.

2.5.1 A Homogeneous Linear System

Let $f(t, y) = A(t)y$ and $J_i(y) = B_i y$ in (2.1), where $A(t) \in C(\mathbb{R}, \mathbb{R}^{n\times n})$ and $B_i \in \mathbb{R}^{n\times n}$. Consider the linear time scale differential equation

$$\begin{aligned} y'(t) &= A(t)y, \quad t \in \mathbb{T}_c, \\ y(t_{2i+1}) &= B_i y(t_{2i}) + y(t_{2i}). \end{aligned} \tag{2.13}$$

By means of ψ-substitution, system (2.13) turns out to be the IDE

$$\begin{aligned} x' &= \tilde{A}(s)x, \quad s \neq s_i, \\ \Delta x|_{s=s_i} &= B_i x, \end{aligned} \tag{2.14}$$

where $\tilde{A}(s) = A(\psi^{-1}(s))$. Since the solutions of system (2.14) form a linear space of dimension n [60, 85], and ψ-substitution transforms only the time variable, the solutions of (2.13) also form a linear space of the same dimension, n.

Let $e_j = (0, \cdots, 0, 1, 0, \cdots, 0)^T$ be the n-tuple whose $j-th$ component is 1 and all others are 0 and assume that $x_j(s),\ x_j(0) = e_j$, is a solution of (2.14) for $j = 1, \cdots, n$. Then [85] for any other solution $x(s),\ x(0) = x_0$, of (2.14) we have

$$x(s) = \sum_{j=1}^{n} c_j x_j(s), \tag{2.15}$$

where the coefficients c_j are uniquely determined from $x_0 = \sum_{j=1}^{n} c_j e_j$.

Now, forming the matriciant $X(s) = [x_1(s) \;\; x_2(s) \;\; \cdots \;\; x_n(s)]$ of system (2.14), equality (2.15) can be written as

$$x(s) = X(s)x_0.$$

If $\mathcal{X}(s,r) = \mathcal{X}(s)\mathcal{X}^{-1}(r)$ is a transition matrix of $x' = \tilde{A}(s)x$ then

$$X(s) = \begin{cases} I, & s = 0 \\ \mathcal{X}(s, s_p)(I + B_p)\prod_{k=p}^{1}\mathcal{X}(s_k, s_{k-1})(I + B_{k-1})\mathcal{X}(s_0, 0), & s > 0 \\ \mathcal{X}(s, s_l)(I + B_l)^{-1}\prod_{k=l+1}^{-1}\mathcal{X}(s_{k-1}, s_k)(I + B_k)^{-1}\mathcal{X}(s_{-1}, 0), & s < 0 \end{cases}$$

where for $s > 0$ we have assumed that $0 < s_0 < \cdots < s_p < s < s_{p+1}$ and for $s < 0$ that $s_{l-1} < s < s_l < \cdots < s_{-1} < 0$.

On the other hand, ψ-substitution yields that a solution $y_j(t)$, $y_j(0) = e_j$, is determined by

$$y_j(t) = x_j(\psi(t)).$$

Hence, any solution $y(t)$, $y(0) = y_0$, of (2.13) is given by $y(t) = Y(t)y_0$ where the matriciant $Y(t)$ is defined and determined by

$$Y(t) = \begin{cases} I, & t = 0 \\ \mathcal{Y}(t, t_{2p+1})(I + B_p)\prod_{k=p}^{1}\mathcal{Y}(t_{2k}, t_{2k-1})(I + B_{k-1})\mathcal{Y}(t_1, 0), & t > 0 \\ \mathcal{Y}(t, t_{2l})(I + B_l)^{-1}\prod_{k=l+1}^{-1}\mathcal{Y}(t_{2k-1}, t_{2k})(I + B_k)^{-1}\mathcal{Y}(t_{-1}, 0), & t < 0 \end{cases}$$

in which $\mathcal{Y}(t,\tau) = \mathcal{Y}(t)\mathcal{Y}^{-1}(\tau)$ is a transition matrix of $y' = A(t)y$ and for $t > 0$ we have assumed that $0 \leq t_{2p+1} < t < t_{2(p+1)}$ and for $t < 0$ that $t_{2l-1} < t < t_{2l} \leq 0$.

2.5.2 A Non-homogeneous Linear System

Consider the system

$$\begin{aligned} y'(t) &= A(t)y + g(t), \quad t \in \mathbb{T}_c, \\ y(t_{2i+1}) &= B_i y(t_{2i}) + W_i + y(t_{2i}), \end{aligned} \tag{2.16}$$

where $y \in \mathbb{R}^n$, $A(t)$, B_i are as described for system (2.13), $g(t) \in \mathcal{T}_0$ and $\{W_i\}$, $i \in \mathbb{Z}$, is a sequence of n-vectors.

Applying the transformations $y(t) = Y(t)u(t)$ and $s = \psi(t)$ one can obtain

$$\begin{aligned} z' &= X^{-1}(s)\tilde{g}(s), \quad s \neq s_i, \\ \Delta z|_{s=s_i} &= X^{-1}(s_i^+)W_i \end{aligned} \tag{2.17}$$

where $z(s) = u(\psi^{-1}(s))$, $\tilde{g}(s) = g(\psi^{-1}(s))$. The solution of (2.17) satisfying $z(s^0) = z_0$ is

$$z(s) = z_0 + \int_{s^0}^{s} X^{-1}(\xi)\tilde{g}(\xi)d\xi + \sum_{s^0 \leq s_i < s} X^{-1}(s_i^+)W_i, \tag{2.18}$$

if $s \geq s^0$, and

$$z(s) = z_0 + \int_{s^0}^{s} X^{-1}(\xi)\tilde{g}(\xi)d\xi - \sum_{s \leq s_i < s^0} X^{-1}(s_i^+)W_i, \tag{2.19}$$

if $s < s^0$. Consequently, the general solution of (2.16) is

$$y(t) = Y(t,t^0)y_0 + \int_{\mathbb{T}_c(t^0,t)} Y(t,\tau)g(\tau)d\tau + \sum_{t^0 \leq t_{2i} < t} Y(t,t_{2i+1})W_i, \tag{2.20}$$

if $t \geq t^0$, and

$$y(t) = Y(t,t^0)y_0 - \int_{\mathbb{T}_c(t,t^0)} Y(t,\tau)g(\tau)d\tau - \sum_{t < t_{2i} \leq t^0} Y(t,t_{2i+1})W_i, \tag{2.21}$$

if $t < t^0$.

2.5.3 Linear Systems with Constant Coefficients

Let $A(t) \equiv A$ and $B_i \equiv B$ be constant matrices in (2.13) and consider the linear system with constant coefficients

$$\begin{aligned} y' &= Ay, \quad t \in \mathbb{T}_c, \\ y(t_{2i+1}) &= By(t_{2i}) + y(t_{2i}), \end{aligned} \tag{2.22}$$

where $A, B \in \mathbb{R}^{n\times n}$. The following assumptions, for system (2.22), are needed:

(C1) the matrices A and B commute, $AB = BA$;

(C2) $\det(I + B) \neq 0$;

(C3) the limits

$$\lim_{t\to\infty}\frac{\psi(t)-\psi(t^0)}{t-t^0}=\ell,\quad \lim_{t\to\infty}\frac{i(t^0,t)}{t-t^0}=p$$

exist, where $i(t^0,t)$ is the number of gaps, (t_{2k},t_{2k+1}), in $\mathbb{T}_c$ between t^0 and t.

Denote $\Lambda_0=\ell A+p\ln(I+B)$.

Theorem 2.5.1 *Let conditions* $(C0)-(C3)$ *hold. Then the zero solution of (2.22) is*

(a) asymptotically stable if the real parts of all eigenvalues of the matrix Λ_0 *are negative;*

(b) unstable if the real part of at least one eigenvalue of the matrix Λ_0 *is positive.*

Proof. It is easily seen that $\mathcal{Y}(t,\tau)=e^{A(t-\tau)}$ and hence, if $t_{2m-1}\le t^0\le t_{2m}<\cdots<t_{2n-1}\le t\le t_{2n}$, we get

$$Y(t,t^0)=e^{A(t-t_{2n-1})}(I+B)\prod_{k=n-1}^{m+1}\left[e^{A(t_{2k}-t_{2k-1})}(I+B)\right]e^{A(t_{2m}-t^0)}.$$

Condition $(C1)$ implies $Y(t,t^0)=e^{A[\psi(t)-\psi(t^0)]}(I+B)^{i(t^0,t)}$. Due to condition $(C3)$ we can write

$$\psi(t)-\psi(t^0)=[\ell+\epsilon_1(t)](t-t^0),\ \text{and,}\ i(t^0,t)=[p+\epsilon_2(t)](t-t^0)$$

where $\epsilon_j(t)\to 0$ as $t\to\infty$, $j=1,2$. In general the functions $\epsilon_j(t)$, $j=1,2$, are piecewise continuous functions.

Now, the solution $y(t)$, $y(t^0)=y_0$, of (2.22) is written as $y(t)=e^{\Lambda(t)(t-t^0)}y_0$, where $\Lambda(t)=\Lambda_0+\epsilon_1(t)A+\epsilon_2(t)\ln(I+B)$ for $t\ge t^0$.

Assume that $\max_j \mathrm{Re}\lambda_j(\Lambda_0)=\gamma<0$. The properties of functions ϵ_j, $j=1,2$, imply that for a fixed positive ϵ there exists a sufficiently large $T>0$ such that if $t\ge T$ then $|\epsilon_j(t)|<\epsilon$, $j=1,2$.

Therefore,

$$\|y(t)\|\le K(\bar{\epsilon})e^{\varkappa(\epsilon)(t-t^0)}e^{(\gamma+\bar{\epsilon})(t-t^0)},$$

where $\varkappa(\epsilon)=\|\epsilon_1(t)A+\epsilon_2(t)\ln(I+B)\|$. Since $\gamma<0$ and $\epsilon,\bar{\epsilon}$ can be chosen so small that $\gamma+\bar{\epsilon}+\varkappa(\epsilon)<0$, part (a) of the theorem is proved.

Let λ_0 be the eigenvalue of Λ_0, whose real part is positive, and y_0 be a corresponding eigenvector in a small neighborhood of the origin. We can obtain that

$$\|y(t)\| \geq \mathrm{e}^{-\varkappa(\epsilon)(t-t^0)}\mathrm{e}^{\mathrm{Re}\lambda_0(t-t^0)}\|y_0\|.$$

Since, $\mathrm{Re}\lambda_0 > 0$ we can choose $\epsilon > 0$ so small that $-\varkappa(\epsilon) + \mathrm{Re}\lambda_0 > 0$, and the last inequality completes the proof. ■

Example 2.5.2 *Let $t_i = i + (-1)^i\kappa,\ 0 < \kappa \leq \frac{1}{3}$, and consider the system*

$$\begin{aligned} y_1' &= \alpha y_1 - \beta y_2, \\ y_2' &= \beta y_1 + \alpha y_2, \quad t \in \mathbb{T}_c, \\ y_1(t_{2i+1}) &= (1+k)y_1(t_{2i}), \\ y_2(t_{2i+1}) &= (1+k)y_2(t_{2i}), \end{aligned} \tag{2.23}$$

where β is a positive real number and $k > -1$ is a constant. One can easily see that the matrices $A = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$ and $B = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ commute with each other and $\ell = \frac{1}{2} + \kappa$, $p = \frac{1}{2}$. Therefore, we have

$$\Lambda_0 = \begin{bmatrix} (\frac{1}{2}+\kappa)\alpha + \frac{1}{2}\ln(1+k) & -(\frac{1}{2}+\kappa)\beta \\ (\frac{1}{2}+\kappa)\beta & (\frac{1}{2}+\kappa)\alpha + \frac{1}{2}\ln(1+k) \end{bmatrix}$$

which has eigenvalues $\lambda_{1,2} = (\frac{1}{2}+\kappa)\alpha + \frac{1}{2}\ln(1+k) \pm (\frac{1}{2}+\kappa)\beta i$. Hence, the zero solution of (2.23) is asymptotically stable if $(\frac{1}{2}+\kappa)\alpha + \frac{1}{2}\ln(1+k) < 0$, unstable if $(\frac{1}{2}+\kappa)\alpha + \frac{1}{2}\ln(1+k) > 0$.

2.6 Periodic Solutions

2.6.1 Description of Periodic Time Scales

Definition 2.6.1 *The time scale $\mathbb{T}_c$ is said to have an ω-property if there exists a number $\omega \in \mathbb{R}^+$ such that $t + \omega \in \mathbb{T}_c$ whenever $t \in \mathbb{T}_c$.*

From this definition, by simply using mathematical induction, we prove the following lemma.

Lemma 2.6.2 *If* $\mathbb{T}_c$ *has an* ω*-property then* $t + n\omega \in \mathbb{T}_c$ *for all* $t \in \mathbb{T}_c$, $n \in \mathbb{Z}$.

Definition 2.6.3 *A sequence* $\{a_i\} \subset \mathbb{R}$ *is said to satisfy an* (ω, p)*-property if there exist numbers* $\omega \in \mathbb{R}^+$ *and* $p \in \mathbb{N}$ *such that* $a_{i+p} = a_i + \omega$ *for all* $i \in \mathbb{Z}$.

Lemma 2.6.4 *If* t *is a right-dense (respectively, left-dense) point of* $\mathbb{T}_c$ *which has an* ω*-property, then* $t + n\omega$ *is also a right-dense (respectively, left-dense) point of* $\mathbb{T}_c$ *for all* $n \in \mathbb{Z}$.

Proof. We will prove the statement just for $n = 1$, since the remaining part is an obvious application of mathematical induction. Let t be a right-dense point. Then

$$\begin{aligned}\sigma(t+\omega) &= \inf\{s > t+\omega : s \in \mathbb{T}_c\} = \inf\{s > t : s \in \mathbb{T}_c\} + \omega \\ &= \sigma(t) + \omega = t + \omega,\end{aligned}$$

that is, $t + \omega$ is a right-dense point. Similarly, one can prove the lemma for left-dense points. ∎

Corollary 2.6.5 *If* $\mathbb{T}_c$ *has an* ω*-property, then there exists* $p \in \mathbb{N}$, *such that the sequences* $\{t_{2i}\}$ *and* $\{t_{2i+1}\}$ *satisfy* (ω, p)*-property.*

Corollary 2.6.6 *If* $\mathbb{T}_c$ *has an* ω*-property, the sequence* $\{\delta_k\}$, *is* p*-periodic, that is,* $\delta_{k+p} = \delta_k$ *for all* $k \in \mathbb{Z}$.

The next lemma assumes that p_0 is the minimal of these numbers $p \in \mathbb{N}$ in Corollary 2.6.6.

Lemma 2.6.7 *If* $\mathbb{T}_c$ *has an* ω*-property then the sequence* $\{s_i\}$, $s_i = \psi(t_{2i})$, *is* $(\tilde{\omega}, p_0)$*-periodic with*

$$\tilde{\omega} = \omega - \sum_{0<t_{2k}<\omega} \delta_k = \psi(\omega).$$

That is, $s_{i+p_0} = s_i + \tilde{\omega}$ *for all* i.

Proof. Assume that $i \geq 0$, $i = np_0 + j$ for some $n \in \mathbb{Z}$, $0 \leq j < p_0$ and $0 < t_0 < \cdots < t_{2(p_0-1)} < \omega$. Then

$$\begin{aligned} s_{i+p_0} &= \psi(t_{2(i+p_0)}) = t_{2(i+p_0)} - \sum_{0<t_{2k}<t_{2(i+p_0)}} \delta_k \\ &= t_{2i} + \omega - \sum_{0<t_{2k}<t_{2i}} \delta_k - \sum_{t_{2i}\leq t_{2k}<t_{2(i+p_0)}} \delta_k = \psi(t_{2i}) + \omega - \sum_{k=i}^{i+p_0-1} \delta_k \\ &= s_i + \omega - \sum_{k=j}^{j+p_0-1} \delta_{k+np_0} = s_i + \omega - \sum_{k=j}^{j+p_0-1} \delta_k = s_i + \omega - \sum_{k=0}^{p_0-1} \delta_k \\ &= s_i + \omega - \sum_{0<t_{2k}<\omega} \delta_k = s_i + \tilde{\omega}, \end{aligned}$$

where we have used the fact that

$$\begin{aligned} \sum_{k=j}^{j+p_0-1} \delta_k &= \sum_{k=j}^{p_0-1} \delta_k + \sum_{k=p_0}^{j+p_0-1} \delta_k = \sum_{k=j}^{p_0-1} \delta_k + \sum_{k=0}^{j-1} \delta_{k+p_0} \\ &= \sum_{k=j}^{p_0-1} \delta_k + \sum_{k=0}^{j-1} \delta_k = \sum_{k=0}^{p_0-1} \delta_k. \end{aligned}$$

All other cases can be proved similarly. ■

Corollary 2.6.8 *If $\mathbb{T}_c$ has an ω-property, then $\psi(t + \omega) = \psi(t) + \psi(\omega)$.*

Denote the set of all T-periodic functions, defined on the set $\mathcal{A} \subset \mathbb{R}$, by $\mathcal{P}_T(\mathcal{A})$.

Lemma 2.6.9 *If $\phi \in \mathcal{P}_\omega(\mathbb{T}_c)$ and $\mathbb{T}_c$ has an ω-property, then $\phi \circ \psi^{-1} \in \mathcal{P}_{\tilde{\omega}}(\mathbb{R})$ with $\tilde{\omega} = \psi(\omega)$.*

Proof. By Corollary 2.6.8, $s + \tilde{\omega} = \psi(t + \omega)$. Then the equality

$$\phi(\psi^{-1}(s + \tilde{\omega})) = \phi(t + \omega) = \phi(t) = \phi(\psi^{-1}(s))$$

completes the proof. ■

Similar to the proof of the last lemma the following assertion can easily be proved.

Lemma 2.6.10 *If $\phi \in \mathcal{P}_{\tilde{\omega}}(\mathbb{R})$, then $\phi \circ \psi \in \mathcal{P}_\omega(\mathbb{T}_c)$.*

2.6.2 The Floquet Theory

Consider

$$\begin{aligned} y'(t) &= A(t)y + f(t), \quad t \in \mathbb{T}_c, \\ y(t_{2i+1}) &= B_i y(t_{2i}) + J_i + y(t_{2i}), \end{aligned} \tag{2.24}$$

where $A, f \in \mathcal{P}_\omega(\mathbb{T}_c)$, sequences B_i and J_i are p-periodic, $\mathbb{T}_c$ has an ω-property, and let $Y(t),\ Y(0) = I$, be the fundamental matrix solution of the corresponding homogeneous system

$$\begin{aligned} y'(t) &= A(t)y, \quad t \in \mathbb{T}_c, \\ y(t_{2i+1}) &= B_i y(t_{2i}) + y(t_{2i}). \end{aligned} \tag{2.25}$$

Recall that a solution $y(t)$, $y(t^0) = y_0$, of (2.24) is given by

$$y(t) = Y(t)y_0 + \int_{\mathbb{T}_c(0,t)} Y(t,\tau)f(\tau)d\tau + \sum_{0<t_{2i}<t} Y(t, t_{2i+1})J_i.$$

Now, for this solution to be ω-periodic, we need $y(\omega) = y(0) = y_0$, that is,

$$[I - Y(\omega)]y_0 = b \tag{2.26}$$

where

$$b = \int_{\mathbb{T}_c(0,\omega)} Y(\omega,\tau)f(\tau)d\tau + \sum_{0<t_{2i}<\omega} Y(\omega, t_{2i+1})J_i. \tag{2.27}$$

Definition 2.6.11 *The eigenvalues, ρ_j, of the matrix of monodromy, $Y(\omega)$, are called Floquet multipliers (or simply multipliers) of system (2.24).*

The following Theorems 16, 17, 18 can be proved as similar assertions for ordinary differential equations.

Theorem 2.6.12 *If ρ is a multiplier then there exists a nontrivial solution, $y(t)$, of (2.25) such that $y(t + \omega) = \rho y(t)$. Conversely, if there exists a nontrivial solution, $y(t)$, of (2.25) such that $y(t + \omega) = \rho y(t)$ then ρ is a multiplier.*

Theorem 2.6.13 *System (2.25) has a $k\omega$-periodic solution if and only if there exists a multiplier, ρ, such that $\rho^k = 1$.*

Now, if we have $\rho \neq 1$ for all multipliers, then the system in (2.26) has a unique solution: this may be stated as a theorem.

Theorem 2.6.14 *If unity is not one of the multipliers, then (2.24) has a unique ω-periodic solution, $y(t)$, such that $y(0) = y_0 = [I - Y(\omega)]^{-1}b$.*

Now, we can write the matriciant, $Y(t)$, in the Floquet form

$$Y(t) = \Phi(t)\mathrm{e}^{P\psi(t)}$$

where $\Phi(t) = Y(t)\mathrm{e}^{-P\psi(t)}$, $P = \frac{1}{\tilde{\omega}} \ln Y(\omega)$, $\tilde{\omega} = \psi(\omega)$. Then

$$\begin{aligned} \Phi(t+\omega) &= Y(t+\omega)\mathrm{e}^{-P\psi(t+\omega)} = Y(t)Y(\omega)\mathrm{e}^{-P\psi(\omega)}\mathrm{e}^{-P\psi(t)} \\ &= Y(t)\mathrm{e}^{-P\psi(t)} = \Phi(t) \end{aligned}$$

and hence, $\Phi(t)$ is ω-periodic. From the definition of $\Phi(t)$ we see that it is continuously differentiable, bounded (because of its periodicity), and is nonsingular for all $t \in \mathbb{T}_c$. One can easily verify that the transformation $y = \Phi(t)u$, transforms system (2.25) into a system with constant coefficients

$$\begin{aligned} &u' = Pu, \quad t \in \mathbb{T}_c \\ &u(t_{2i+1}) = u(t_{2i}), \end{aligned} \tag{2.28}$$

where we have used

$$\lim_{t \to t_{2i+1}^+} \psi(t) = \psi(t_{2i}).$$

Definition 2.6.15 *The eigenvalues, λ_j, of the matrix, $P = \frac{1}{\tilde{\omega}} \ln Y(\omega)$, are called the Floquet exponents (or simply exponents).*

Similar to ODE, and applying the Floquet theory for IDE, [85], one can prove that the following theorems are valid.

Theorem 2.6.16 *Let $\{\lambda_j\}$ be the exponents. Then the solutions of (2.25) are*

(a) asymptotically stable if and only if $\mathrm{Re}(\lambda_j) < 0$ *for all j;*

(b) stable if $\mathrm{Re}(\lambda_j) \leq 0$ *for all j and λ_j is simple when* $\mathrm{Re}\lambda_j = 0$*;*

(c) unstable if there exists an exponent λ_j *such that* $\mathrm{Re}(\lambda_j) > 0$.

Theorem 2.6.17 *Let* $\{\rho_j\}$ *be the multipliers. Then the solutions of (2.25) are*

(a) asymptotically stable if and only if all multipliers lie inside the unit circle;

(b) stable if $|\rho_j| \le 1$ *for all* j *and* ρ_j *is simple when* $|\rho_j| = 1$*;*

(c) unstable if there exists a multiplier ρ_j *which lies outside the unit circle.*

Example 2.6.18 *Let* $t_i = i\pi + (-1)^i \frac{\pi}{4}$ *and consider the system*

$$\begin{aligned} y_1' &= -y_2 + f_1(t), \\ y_2' &= y_1 + f_2(t), \quad t \in \mathbb{T}_c, \\ y_1(t_{2i+1}) &= (1+k)y_1(t_{2i}), \\ y_2(t_{2i+1}) &= (1+k)y_2(t_{2i}), \end{aligned} \tag{2.29}$$

where $f_1(t) = \mathrm{e}^{t-t_{2i-1}}$, $f_2(t) = \sin(t - t_{2i-1})$ *for* $t_{2i-1} < t \le t_{2i}$ *and* $k \in \mathbb{R}$ *is a constant. It is easy to see that this system is* 2π*-periodic and the matriciant of the corresponding homogeneous system is*

$$\mathcal{Y}(t,\tau) = \begin{bmatrix} \cos(t-\tau) & -\sin(t-\tau) \\ \sin(t-\tau) & \cos(t-\tau) \end{bmatrix}$$

and hence the matrix of monodromy is

$$Y(2\pi) = \mathcal{Y}(2\pi, \frac{3\pi}{4})(I+B)\mathcal{Y}(\frac{\pi}{4}, 0) = \begin{bmatrix} 1+k & 0 \\ 0 & 1+k \end{bmatrix}.$$

Therefore, the multipliers are $\rho_{1,2} = 1 + k$*. Now, if* $k \neq 0$ *then, by Theorem (2.6.14), the system in (2.29) has a unique* 2π*-periodic solution and, by Theorem (2.6.17), this periodic solution is asymptotically stable for* $-2 < k < 0$, *unstable for* $k < -2$ *or* $k > 0$, *stable for* $k = -2$.

2.6.3 The Massera Theorem

Let us consider the following analogue of the famous Massera theorem [68].

Theorem 2.6.19 *If system (2.24) has a bounded solution $y^*(t)$ on the set $\{t \in \mathbb{T}_c : t \geq 0\}$ then there exists a periodic solution of system (2.24).*

Proof. Assume on the contrary that there exists no periodic solution. Let $y^*(t), y^*(0) = y_0$, be a bounded solution of (2.24), then

$$y^*(t) = Y(t)y_0 + \int_{\mathbb{T}_c(0,t)} Y(t)Y^{-1}(\tau)f(\tau)d\tau + \sum_{0<t_{2i}<t} Y(t)Y^{-1}(t_{2i+1})J_i$$

and $y^*(\omega) = Y(\omega)y_0 + b$ where b is as in (2.27). Now, $x^*(s) = y^*(\psi^{-1}(s))$ is a solution of

$$\begin{aligned} x' &= A(\psi^{-1}(s))x + f(\psi^{-1}(s)), \quad s \neq s_i \\ \Delta x|_{s=s_i} &= B_i x + J_i. \end{aligned} \tag{2.30}$$

Since $x^*(s+\tilde{\omega}) = y^*(\psi^{-1}(s+\tilde{\omega})), \tilde{\omega} = \psi(\omega)$, is also a solution of (2.30), it implies that $y^*(t+\omega)$ is also a solution of (2.24).

Thus, we have

$$\begin{aligned} y^*(t+\omega) &= Y(t+\omega)y_0 + \int_{\mathbb{T}_c(0,t+\omega)} Y(t+\omega)Y^{-1}(\tau)f(\tau)d\tau \\ &\quad + \sum_{0<t_{2i}<t+\omega} Y(t+\omega)Y^{-1}(t_{2i+1})J_i \\ &= Y(t)y^*(\omega) + \int_{\mathbb{T}_c(0,t)} Y(t)Y^{-1}(\tau)f(\tau)d\tau + \sum_{0<t_{2i}<t} Y(t)Y^{-1}(t_{2i+1})J_i \end{aligned}$$

and

$$y^*(2\omega) = Y(\omega)y^*(\omega) + b = Y^2(\omega)y_0 + Y(\omega)b + b.$$

Continuing in this way, by mathematical induction, we see that

$$y^*(n\omega) = Y^n(\omega)y_0 + \sum_{k=0}^{n-1} Y^k(\omega)b.$$

If there is no ω-periodic solution, then the system $[I - Y(\omega)]y_0 = b$ has no solution. However, this means that there is a solution, c, of the system $[I - Y(\omega)]^T y = 0$ such that $\langle b, c\rangle \neq 0$. Thus,

$$\langle y^*(n\omega), c\rangle = \langle Y^n(\omega)y_0 + \sum_{k=0}^{n-1} Y^k(\omega)b, c\rangle =$$

$$\langle y_0, [Y^n(\omega)]^T c\rangle + \sum_{k=0}^{n-1}\langle b, [Y^k(\omega)]^T c\rangle = \langle y_0, c\rangle + \sum_{k=0}^{n-1}\langle b, c\rangle = \langle y_0, c\rangle + n\langle b, c\rangle$$

which becomes unbounded as $n \to \infty$. On the other hand, since $y^*(t)$ is bounded, we have

$$|\langle y^*(n\omega), c\rangle| \leq |y^*(n\omega)||c| \leq M|c|$$

which contradicts with the previous equality. Hence, the proof is completed. ■

Corollary 2.6.20 *If system (2.24) does not have an ω-periodic solution, then all solutions of system (2.24) are unbounded on both $\{t \in \mathbb{T}_c : t \geq 0\}$ and $\{t \in \mathbb{T}_c : t < 0\}$.*

2.7 Deduction

In this chapter, the connection between a specific type of differential equations on time scales (DETC) and the impulsive differential equations is established. Some benefits of this established connection include knowledge about properties of linear DETC, the investigation of existence of periodic and almost periodic solutions and their stability. We suppose that the problems of stability, oscillations, smoothness of solutions, integral manifolds, theory of functional differential equations can be investigated applying our results. Another interesting opportunity is to analyze equations with more sophisticated time scales.

CHAPTER 3

DIFFERENTIAL EQUATIONS ON VARIABLE TIME SCALES

In this chapter, we introduce a class of differential equations on *variable* time scales with a transition condition between two consecutive parts of the scale. Conditions for existence and uniqueness of solutions are obtained. Periodicity, boundedness, stability of solutions are considered. The method of investigation is by means of two successive reductions: B-equivalence of the system [4, 6, 11] on a variable time scale to a system on a time scale, a reduction to an impulsive differential equation [6, 19]. Appropriate examples are constructed to illustrate the theory.

3.1 Introduction

In the last several decades, the theory of dynamic equations on time scales (DETS) has been developed very intensively. For a full description of the equations we refer to the nicely written books [29, 64] and papers [65, 88]. The equations have a very special transition condition for adjoint elements of time scales. To enlarge the field of applications of the DETS, and to have more theoretical opportunities we, in [19], proposed to generalize the transition operator, correspondingly to investigate *differential equations on time scales with the transition condition* (DETC).

In our recent investigations [6], it was found that the idea of the equations can be extended, if one: 1) involves in the discussion of certain union of separated sets in the (t, x) space such that intersection of each line x =constant with the union is a time scale in the sense of Hilger (we call these separated sets altogether as the variable time scale); 2) introduces the differential equations, the domain of which are variable time scales. We call the systems as differential equations on variable time scales with transition condition (DETCV). The present chapter is devoted to the development of methods to study these systems, and some theoretical results are obtained. To give an outline of the way of the study, we can shortly say that two consequent reductions are in the base: (a) reduction of DETCV to DETC, using B-equivalence method [4, 6, 11]; (b) the method of ψ-substitution [8, 19] to reduce DETC to impulsive differential equations.

This chapter is organized as follows. The next section has detailed description of variable time scales with examples. Section 3.3 describes the differential equations on variable time scales. The existence and uniqueness of solutions and B-equivalence and B-stability are considered in Sections 3.4 and 3.5. The description of the reduction process is given in Section 3.6. In the last two sections, we apply the procedure to investigate periodic solutions and stability of an equilibrium position.

3.2 Description of a Variable Time Scale

In this section, we give, first, a general definition of a variable time scale, and next, we describe a specific variable time scale, which will be used to introduce DETCV.

Definition 3.2.1 *A nonempty closed set $\mathbb{T}(x)$ in $\mathbb{R} \times \mathbb{R}^n$ is said to be a variable time scale if for any $x_0 \in \mathbb{R}^n$ the projection of $\mathbb{T}(x_0)$ on time axis, that is the set $\{t \in \mathbb{R} : (t, x_0) \in \mathbb{T}(x_0)\}$, is a time scale in Hilger's sense.*

To illustrate this definition let us consider the following example.

Example 3.2.2 *Let $\{r_i\}_{i=1}^{\infty}$ be an increasing sequence of positive real numbers such that $r_i \to \infty$ as $i \to \infty$, and*

$$\mathcal{D}_i = \{(t, x) \in \mathbb{R} \times \mathbb{R} : r_{2i-1}^2 \leq t^2 + x^2 \leq r_{2i}^2\}.$$

Then, we define the variable time scale as $\mathbb{T}(x) = \bigcup_{i=1}^{\infty} \mathcal{D}_i$ (See Figure 3.1).

For a fixed $x_0 \in \mathbb{R}$, there exists a smallest k such that $r_{2k} \geq |x_0|$. Thus, we have

$$\mathbb{T}(x_0) = \bigcup_{i=k}^{\infty} \{(t, x_0) : t \in \mathbb{R}, r_{2i-1}^2 \leq t^2 + x_0^2 \leq r_{2i}^2\}.$$

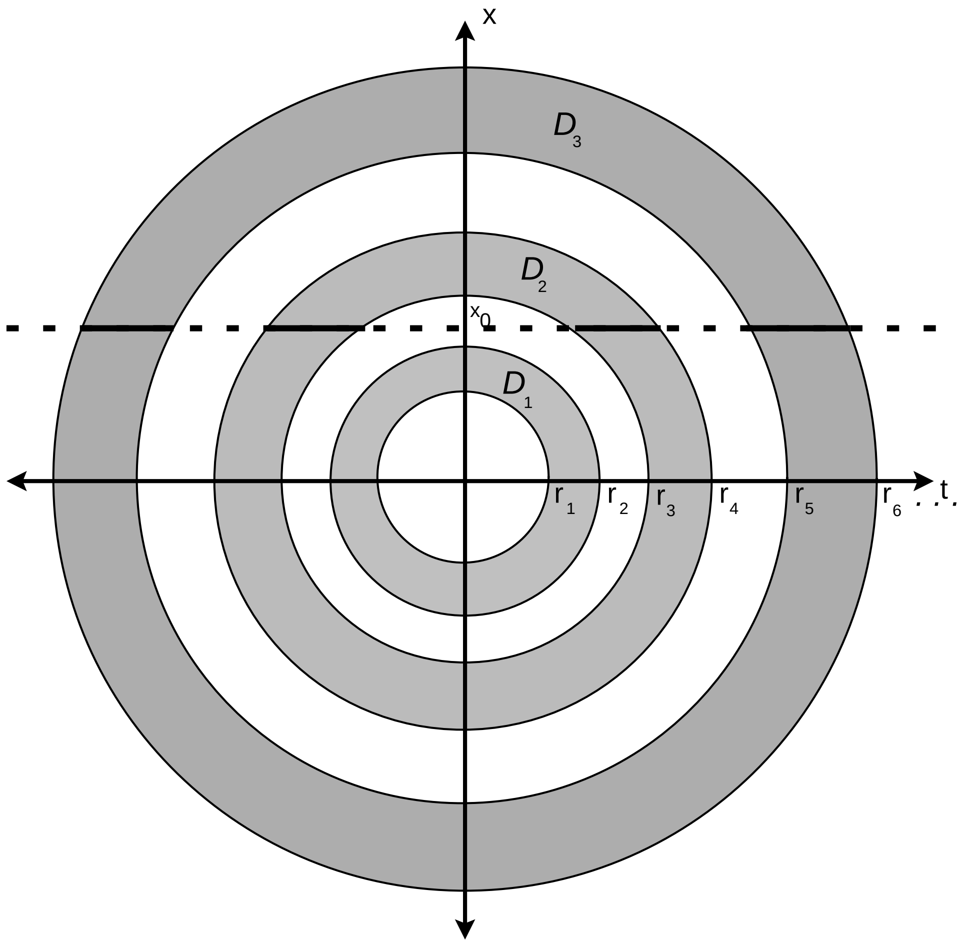

Figure 3.1: An example of a variable time scale

The projection of $\mathbb{T}(x_0)$ *on time axis is*

$$\mathbb{T}_c = \bigcup_{i=k}^{\infty}\left(\left[-\sqrt{r_{2i}^2 - x_0^2}, -\sqrt{r_{2i-1}^2 - x_0^2}\right] \cup \left[\sqrt{r_{2i-1}^2 - x_0^2}, \sqrt{r_{2i}^2 - x_0^2}\right]\right),$$

which is a time scale in Hilger's sense.

The following variable time scale may be considered as another example. However, it is an essential element in the definition of differential equations with transition conditions on a variable time scale, discussed in this chapter. Fix a sequence $\{t_i\} \subset \mathbb{R}$ such that $t_i < t_{i+1}$ for all $i \in \mathbb{Z}$, and $t_i \to \pm\infty$ as $i \to \pm\infty$. Denote $\delta_i = t_{2i+1} - t_{2i}, \kappa_i = t_{2i} - t_{2i-1}$ and take a sequence of continuous functions $\tau_i : \mathbb{R}^n \to \mathbb{R}$. Assume that:

(C4) for some positive numbers $\theta', \theta \in \mathbb{R}$, we have $\theta' \leq t_{i+1} - t_i \leq \theta$ for all $i \in \mathbb{Z}$;

(C5) there exists $\ell_0, 0 < 2\ell_0 < \theta'$, such that $\|\tau_i(x)\| \leq \ell_0$ for all $x \in \mathbb{R}^n, i \in \mathbb{Z}$.

Denote

$$l_i := \inf_{x \in \mathbb{R}^n} \{t_i + \tau_i(x)\}, \qquad r_i := \sup_{x \in \mathbb{R}^n} \{t_i + \tau_i(x)\}. \tag{3.1}$$

From (C4) and (C5) it follows that there exist positive numbers θ_l and θ_r such that

(C4′) $\theta_l \leq l_{i+1} - r_i \leq \theta_r, i \in \mathbb{Z}$.

We set

$$\begin{aligned} \mathcal{E}_i &= \{(t, x) \in \mathbb{R} \times \mathbb{R}^n : t_{2i} + \tau_{2i}(x) < t < t_{2i+1} + \tau_{2i+1}(x)\}, \\ \mathcal{S}_i &= \{(t, x) \in \mathbb{R} \times \mathbb{R}^n : t = t_i + \tau_i(x)\}, \\ \mathcal{D}_i &= \{(t, x) \in \mathbb{R} \times \mathbb{R}^n : t_{2i-1} + \tau_{2i-1}(x) \leq t \leq t_{2i} + \tau_{2i}(x)\}. \end{aligned} \tag{3.2}$$

Due to (C4′), none of $\mathcal{D}_i$ is empty and we introduce the set

$$\mathbb{T}_0(x) := \bigcup_{i=-\infty}^{\infty} \mathcal{D}_i. \tag{3.3}$$

In the previous chapter, we considered a special time scale $\mathbb{T}_c = \bigcup_{i=-\infty}^{\infty} [t_{2i-1}, t_{2i}]$; however, now, we have the set $\mathbb{T}_0(x)$. It seems reasonable to call the latter as *the variable time scale*, and in our study we are going to use, for sets of type $\mathbb{T}_c$, the term *non-variable time scales* to emphasize the difference.

For the convenience of the reader let us consider the following example.

Example 3.2.3 *Let* $t_i = \pi i, \tau_i(x) = \frac{\sin(\|x\|)}{\|x\|^2 + |i| + 1}$ *where* $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$ *is the Euclidean norm of* $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$. *Then, we have*

$$l_i = \pi i - \frac{1}{\sqrt{(c_i^2 + |i| + 1)^2 + 4c_i^2}}, \quad r_i = \pi i + \frac{1}{\sqrt{(c_i^2 + |i| + 1)^2 + 4c_i^2}}$$

where the number $c_i > 0$ *is the smallest real number which satisfies the equation* $\tan(c_i) = (c_i^2 + |i| + 1)/(2c_i)$. *Thus, for* $\theta_l = \frac{\pi}{2}$ *and* $\theta_r = \pi$, *we see that (C4) is satisfied and*

$$\mathcal{D}_i = \{(t, x) \in \mathbb{R} \times \mathbb{R}^n : t_{2i-1} + \tau_{2i-1}(x) \leq t \leq t_{2i} + \tau_{2i}(x)\}.$$

Then, the variable time scale could be established as in (3.3).

3.3 Differential Equations on Variable Time Scales

In what follows, we introduce a special operator which plays an important role in describing the differential equations on variable time scales as well as methods for investigation of these equations through the reduction to impulsive differential equations.

Let us consider a transition operator $\Pi_i : \mathcal{S}_{2i} \to \mathcal{S}_{2i+1}$, for all $i \in \mathbb{Z}$, such that $\Pi_i(t, y) = (\Pi_i^1(t, y), \Pi_i^2(t, y))$ where $\Pi_i^1 : \mathcal{S}_{2i} \to \mathbb{R}$ and $\Pi_i^2 : \mathcal{S}_{2i} \to \mathbb{R}^n$, and

$$\Pi_i^1(t, y) = t_{2i+1} + \tau_{2i+1}(\Pi_i^2(t, y)) \quad \text{and} \quad \Pi_i^2(t, y) = I_i(y) + y, \tag{3.4}$$

where $I_i : \mathbb{R}^n \to \mathbb{R}^n$ is a function. One can easily see that $\Pi_i^1(t, y)$ is the time coordinate of $(t^+, y^+) := \Pi_i(t, y)$, the image of $(t, y) \in \mathcal{S}_{2i}$ under the operator Π_i, and $\Pi_i^2(t, y)$ is the space coordinate of the image.

The differential equation which we are going to deal with is:

$$\begin{aligned} &y' = F(t, y), \quad (t, y) \in \mathbb{T}_0(y), \\ &t^+ = \Pi_i^1(t, y), \quad y^+ = \Pi_i^2(t, y), \quad (t, y) \in \mathcal{S}_{2i}, \end{aligned} \tag{3.5}$$

where the derivative at the boundary points of the variable time scale in (3.5) is one-sided derivative and $F : \mathbb{T}_0(y) \to \mathbb{R}^n$ is assumed to be continuous on its domain.

We call (3.5) *a differential equation on a variable time scale with transition condition* and abbreviate it as DETCV.

To describe the solutions of differential equations with transition conditions on a variable time scale carefully, we begin the definition with *the graph* of a solution of (3.5). Accordingly, we start with the following construction. Consider a piece-wise curve C such that:

1. C lies in $\mathbb{T}_0(y)$;
2. the part of C in each $\mathcal{D}_i, i \in \mathbb{Z}$, is a continuous arc;
3. if C has points in $\mathcal{D}_j$ and $\mathcal{D}_{j+1}$ for some fixed $j \in \mathbb{Z}$, then C intersects each of the surfaces $\mathcal{S}_{2j}$ and $\mathcal{S}_{2j+1}$ exactly once;
4. C intersects each hyperplane $t = \theta, \theta \in \mathbb{R}$, at most at one point.

The curve can be viewed as the graph of a piece-wise function $y = \varphi(t)$. Let $t = \alpha_i$ and $t = \beta_i$ be the moments that the graph of $y = \varphi(t)$ intersects the surfaces $\mathcal{S}_{2i-1}$ and $\mathcal{S}_{2i}$, respectively, where the surfaces are defined previously. From (C4) and (C5) or (C4') it is easily seen that $\alpha_i < \beta_i$ for all $i \in \mathbb{Z}$. Then, we set the non-variable time scale

$$\mathbb{T}_c^{\varphi} := \bigcup_{i=-\infty}^{\infty} [\alpha_i, \beta_i],$$

which is the domain of φ, and define the Δ-derivative as given in the previous chapter. That is, for $t = \beta_i$, we have

$$\varphi^{\Delta}(\beta_i) = \frac{\varphi(\alpha_{i+1}) - \varphi(\beta_i)}{\alpha_{i+1} - \beta_i},$$

and

$$\varphi^{\Delta}(t) = \lim_{s \to t} \frac{\varphi(s) - \varphi(t)}{s - t},$$

for any other $t \in \mathbb{T}_c^{\varphi}$, whenever the limit exists.

Thus, to define a DETCV, we need:

1. the variable time scale $\mathbb{T}_0(y) = \bigcup_{i=-\infty}^{\infty} \mathcal{D}_i$;
2. the system of differential equations
$$\frac{dy}{dt} = F(t, y), \quad (t, y) \in \mathbb{T}_0(y); \tag{3.6}$$
3. the transition operator $\Pi_i : \mathcal{S}_{2i} \to \mathcal{S}_{2i+1}, i \in \mathbb{Z}$.

Setting $\Delta t := t^+ - t$ and $\Delta y := y^+ - y$, we can rewrite (3.5) as

$$\begin{aligned} y' &= F(t, y), \quad (t, y) \in \mathbb{T}_0(y), \\ \Delta t|_{(t,y) \in \mathcal{S}_{2i}} &= \Pi_i^1(t, y) - t, \\ \Delta y|_{(t,y) \in \mathcal{S}_{2i}} &= \Pi_i^2(t, y) - y. \end{aligned} \tag{3.7}$$

The class of equations is important as it can be reduced from the discontinuous dynamics [6]. Particularly, they are needed to develop the center manifold theory of these equations, and, consequently, the Hopf bifurcation theory which will be covered in the next chapter.

Let us show how to construct a solution of (3.5), or equivalently of (3.7). Denote by $\phi(t,\kappa,\eta)$ a solution of the initial value problem $y(\kappa)=\eta$ for system

$$\frac{dy}{dt} = F(t,y), \tag{3.8}$$

and $y = y(t,t^0,y_0)$ a solution of the initial value problem $y(t^0) = y_0$ for the system (3.5). Assume that (t^0,y_0) is an interior point of $\mathcal{D}_k$ for some $k \in \mathbb{Z}$. We construct the solution for increasing t. The process of definition of $y(t)$ goes as follows: starting from (t^0,y_0), the solution is equal to $y(t) = \phi(t,t^0,y_0)$ up to a point $(\beta_k, y(\beta_k))$, where β_k is the first from the left solution of the equation $\beta = t_{2k}+\tau_{2k}(y(\beta))$, that is the first meeting point of the solution $\phi(t,t^0,y_0)$ with the surface $\mathcal{S}_{2k}$, and $y_k := y(\beta_k) = \phi(\beta_k,t^0,y_0)$. Then, applying the transition operator Π_k, we obtain $(\beta_k^+, y_k^+) = \left(\Pi_k^1(\beta_k,y_k), \Pi_k^2(\beta_k,y_k)\right)$. Denote $\alpha_{k+1} = \Pi_k^1(\beta_k,y_k)$. After α_{k+1}, there is no meeting of the solution with $\mathcal{S}_{2k+1}$. (A sufficient condition which ensures this fact will be given later.) The solution is not defined on the time interval (β_k,α_{k+1}). Next, on $\mathcal{D}_{k+1}$ the solution is equal to $y(t) = \phi(t,\alpha_{k+1},y_k^+)$ and so on (See Figure 3.2).

The way of investigation of DETCV has not been considered yet, except for the short episode in [6]. So, in what follows, we consider a quasilinear system as it is convenient to develop the methods of reductions proposed in [6, 11, 19]. That is, we shall assume $F(t,y)$ and $I_i(y)$ in a special form: $F(t,y) = A(t)y + f(t,y)$, $I_i(y) = B_i y + J_i(y)$ where $A(t) : \mathbb{R} \to \mathbb{R}^{n\times n}$ is an $n\times n$ continuous real valued matrix-function, B_i is an $n\times n$ matrix, functions $f(t,y) : \mathbb{T}_0(y) \to \mathbb{R}^n$ and $J_i(y) : \mathbb{R}^n \to \mathbb{R}^n$ are continuous. Thus, the system which we will consider is:

$$\begin{aligned} y' &= A(t)y + f(t,y), \quad (t,y) \in \mathbb{T}_0(y),\\ \Delta t|_{(t,y)\in\mathcal{S}_{2i}} &= \Pi_i^1(t,y) - t,\\ \Delta y|_{(t,y)\in\mathcal{S}_{2i}} &= \Pi_i^2(t,y) - y, \end{aligned} \tag{3.9}$$

where $\Pi_i^1(t,y) = t_{2i+1} + \tau_{2i+1}(\Pi_i^2(t,y))$ and $\Pi_i^2(t,y) = B_i y + J_i(y) + y$.

Example 3.3.1 *The following planar system was considered in* [6]*:*

$$\begin{aligned} \frac{dx}{dt} &= Ax + f(x), \quad x \notin \Gamma,\\ \Delta x|_{x\in\Gamma} &= B(x)x, \end{aligned} \tag{3.10}$$

where $\Gamma = \bigcup_{i=1}^{p} \ell_i$ *is a set of curves starting at the origin and which are defined by the*

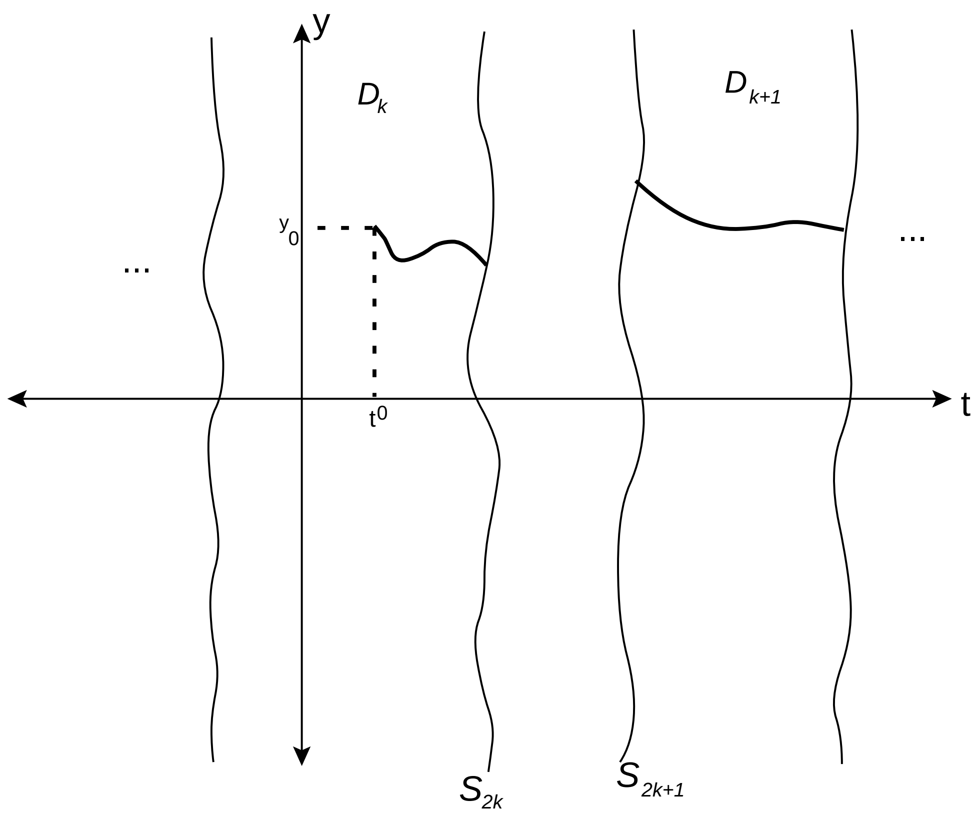

Figure 3.2: A solution of a differential equation on a variable time scale.

equations $\langle a^i, x\rangle + \tau_i(x) = 0,\ i = \overline{1, p},$

$$B(x) = (k + \kappa(x))Q \begin{pmatrix} \cos(\theta + \upsilon(x)) & -\sin(\theta + \upsilon(x)) \\ \sin(\theta + \upsilon(x)) & \cos(\theta + \upsilon(x)) \end{pmatrix} Q^{-1} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where the functions f, k, v *are smooth,* $f(x) = o(\|x\|)$, $\kappa(x) = o(\|x\|)$, $\upsilon(x) = o(\|x\|)$, $\tau_i(x) = o(\|x\|^2)$, $i = \overline{1, p}$ *and* Q *is some nonsingular matrix. By using polar coordinates, the system is written in the form:*

$$\begin{aligned} &\frac{dr}{d\phi} = \lambda r + P(r, \phi),\ (r, \phi) \notin \Gamma, \\ &\phi^+ \mid_{(r,\phi)\in l_i} = \phi + \theta_i + \gamma(r, \phi), \\ &r^+ \mid_{(r,\phi)\in l_i} = (1 + k_i)r + \omega(r, \phi), \end{aligned} \tag{3.11}$$

where Γ *is presented as* $\ell_i : \phi = \gamma_i + r\psi_i(r, \phi),\ i = \overline{1, p}.$

Denote by ℓ_i' *the image of* ℓ_i *under the transition operator* $\Pi_i(\phi, r)$ *where* $\Pi_i^1(\phi, r) = \phi + \theta_i + \gamma(r, \phi)$, *and* $\Pi_i^2(\phi, r) = (1 + k_i)r + \omega(r, \phi)$, *Let* $\mathcal{D}_i$ *be the set bounded by* ℓ_i' *and*

ℓ_{i+1}. In [6], *it is shown that this set is non-empty and ℓ_i' is between ℓ_i and ℓ_{i+1} if the equation is considered in a small neighborhood of the origin.*

Denoting $\mathbb{T}(r) = \bigcup_{i=1}^{p} \mathcal{D}_i$, we have that one deals with the following DETCV:

$$\begin{aligned} &\frac{dr}{d\phi} = \lambda r + P(r,\phi), \qquad (\phi, r) \in \mathbb{T}(r), \\ &\phi^+ = \Pi_i^1(\phi, r), \quad r^+ = \Pi_i^2(\phi, r), \quad (\phi, r) \in \ell_i. \end{aligned} \tag{3.12}$$

Equations of the form (3.10) could be effectively applied as a model for the various mechanical processes with impacts [31, 54, 56, 74, 94]. That is why, the last example could be considered as a good **motivation** to investigate DETCV.

We are going to develop the theory starting with the present chapter and discuss such problems as center manifold theorem, multidimensional Hopf bifurcation, in the next chapter. We intend to investigate the problems using our approach to discontinuous dynamical systems [6].

Summarizing all the materials discussed above, we could say that there is a demand to develop the Hilger's differential equation on non-variable time scales to the differential equations on variable time scales of general type, as a particular case of DETCV. For this reason, let us specify the transition operator in the previous part, assuming $\Pi_k^2(\beta_k, y(\beta_k)) = F(\beta_k, y(\beta_k))(\alpha_{k+1} - \beta_k) + y(\beta_k)$, then (3.9) has a specified form

$$y^{\Delta} = F(t, y), \quad (t, y) \in \mathbb{T}_0(y). \tag{3.13}$$

The last system could be considered as *the differential equation on variable time scale* (DEVTS). We suppose that the theory of DEVTS should be developed as well as the theory of DETS has been [29]. One can expect that many interesting problems connected with topology of the variable time scale may appear. Some of these problems are going to be discussed in this chapter.

3.4 Existence and Uniqueness of Solutions

Among the properties of a differential equation, the problem of existence and uniqueness of solutions has great importance. In this section, we are going to investigate the problem for (3.9) for increasing t.

Remark 3.4.1 *The continuation of the solution to the left can not be considered yet, since the invertibility of the transition operator* Π_i *is not assumed.*

Consider the following ordinary differential equation

$$\frac{dy}{dt} = A(t)y + f(t, y), \tag{3.14}$$

where the matrix $A(t)$ and the function $f(t, y)$ are the same as in (3.9). We will assume that the following Lipschitz condition holds uniformly with respect to $t \in \mathbb{R}$ and $i \in \mathbb{Z}$, for arbitrary $x, y \in \mathbb{R}^n$:

(C6) $\|\tau_i(x) - \tau_i(y)\| + \|J_i(x) - J_i(y)\| + \|f(t, x) - f(t, y)\| \leq \ell\|x - y\|.$

Moreover, we assume that

(C7) $\sup_{t\in\mathbb{R}} \|f(t, 0)\| + \sup_{i\in\mathbb{Z}} \|J_i(0)\| = M < \infty;$

(C8) $\sup_{t\in\mathbb{R}} \|A(t)\| + \sup_{i\in\mathbb{Z}} \|B_i\| = N < \infty;$

(C9) $\tilde{M}\ell < 1$, where $\tilde{M} = \sup_{(t,y)\in\mathbb{T}_0(y)} \|A(t)y + f(t, y)\|.$

Then, we have the following theorem.

Theorem 3.4.2 *Assume that* $(C4) - (C9)$ *hold and the function* f *is continuous. Then for any* $(t^0, y_0) \in \mathbb{T}_0(y)$ *the system*

$$\begin{aligned} y' &= A(t)y + f(t, y), \quad (t, y) \in \mathbb{T}_0(y), \\ t^+ &= \Pi_i^1(t, y), \quad y^+ = \Pi_i^2(t, y), \quad (t, y) \in \mathcal{S}_{2i}, \end{aligned} \tag{3.15}$$

with the initial condition $y(t^0) = y_0$, *has a unique solution,* $y(t, t^0, y_0)$, *which can be continued to the right of* t^0, *to* ∞.

Proof. For the following discussion, it is important that if $(\gamma, y_\gamma) \in \mathbb{T}_0(y)$, then there exists an index $i \in \mathbb{Z}$ such that $(\gamma, y_\gamma) \in \mathcal{D}_i$, $i \in \mathbb{Z}$. And hence, because of (C6) and (C8), there exists a unique solution of the ordinary differential equation,

$$\begin{aligned} y' &= A(t)y + f(t, y), \\ y(\gamma) &= y_\gamma, \end{aligned} \tag{3.16}$$

which is continuable to $\mathcal{S}_{2i}$, the right boundary surface of $\mathcal{D}_i$ [35, 49].

Assume that $(t^0, y_0) \in \mathcal{D}_k$ for some $k \in \mathbb{Z}$. On $\mathcal{D}_k$, we will consider (3.16) for $\gamma = t^0$, $y_\gamma = y_0$, which has the unique solution $y(t) = \phi(t, t^0, y_0)$ defined throughout $\mathcal{D}_k$. Let β_k be the first from left solution of $\beta = t_{2k} + \tau_{2k}(y(\beta))$. Then, by means of jump operators we obtain $\alpha_{k+1} := \Pi_k^1(\beta_k, y(\beta_k))$ and $y_k^+ := \Pi_k^2(\beta_k, y(\beta_k))$.

Next, on $\mathcal{D}_{k+1}$, we consider the ordinary differential equation (3.16) with the initial condition $y(\alpha_{k+1}) = y_k^+$, which has the unique solution $\phi(t, \alpha_{k+1}, y_k^+)$. Thus, the solution is not defined on the time interval (β_k, α_{k+1}).

Assume that the solution intersects the surface $\mathcal{S}_{2k+1}$ at any other point, say α_{k+1}^*, which is going to be a solution of the equation

$$\alpha^* = t_{2k+1} + \tau_{2k+1}(\phi(\alpha^*, \alpha_{k+1}, y_k^+)).$$

Clearly, we have $\alpha_{k+1}^* > \alpha_{k+1}$ and (C6) implies that

$$(\alpha_{k+1}^* - \alpha_{k+1})(1 - \ell \sup_{t \in [\alpha_{k+1}, \alpha_{k+1}^*]} \|A(t)\phi(t, \alpha_{k+1}, y_k^+) + f(t, \phi(t, \alpha_{k+1}, y_k^+))\|) \leq 0$$

which yields a contradiction since $\tilde{M}\ell < 1$. Therefore, the solution does not have any other meeting point with the surface $\mathcal{S}_{2k+1}$. Hence, on $\mathcal{D}_{k+1}$, the unique solution is obtained as $\phi(t, \alpha_{k+1}, y_k^+)$. In this way, we can continue this solution to ∞. ■

3.5 *B*-Equivalence, *B*-Stability

A difficulty in investigating the system (3.9) is that the discontinuity moments of distinct solutions are not, in general, the same. To investigate the asymptotic properties of solutions of (3.9), we introduce the following concepts.

In what follows, we are going to adopt, for DETCV, the techniques of B- topology and B-equivalence which were introduced and developed in [6, 11, 44, 57, 89, 93] for equations with impulses at variable moments of time.

Let $u(t) = y(t, t^0, y_0)$ be a solution of (3.9) and h be a sufficiently small positive real number such that the open neighborhood, $B((t^0, y_0), h)$, centered at (t^0, y_0) with radius h belongs to $\mathcal{D}_k$ for some $k \in \mathbb{Z}$. Let β_i^u be the moment when the solution $u(t)$ meets the surface $\mathcal{S}_{2i}$, and $\alpha_{i+1}^u = \Pi_i^1(\beta_i^u, y(\beta_i^u))$, for $i = k, k+1, \cdots$. We set the non-variable time scale

$$\mathbb{T}_{t^0}^u := [t^0, \beta_k^u] \cup \bigcup_{i=k+1}^{\infty} [\alpha_i^u, \beta_i^u].$$

Let $v(t) = y(t, t^1, y_1)$ be another solution of (3.9) with $(t^1, y_1) \in B((t^0, y_0), h)$ and let β_i^v be the moment when the solution $v(t)$ meets the surface $\mathcal{S}_{2i}$, and $\alpha_{i+1}^v = \Pi_i^1(\beta_i^v, y(\beta_i^v))$, for $i = k, k+1, \cdots$. We, similarly, define the non-variable time scale

$$\mathbb{T}_{t^1}^v := [t^1, \beta_k^v] \cup \bigcup_{i=k+1}^{\infty} [\alpha_i^v, \beta_i^v].$$

Define the distance between two non-variable time scales, $\mathbb{T}_{t^0}^u$ and $\mathbb{T}_{t^1}^v$, by

$$d(\mathbb{T}_{t^0}^u, \mathbb{T}_{t^1}^v) = \max\left\{ \sup_{i \ge k+1} |\alpha_i^u - \alpha_i^v|, \sup_{i \ge k} |\beta_i^u - \beta_i^v| \right\}.$$

We say that two solutions u and v are in an ϵ-neighborhood of each other on $\mathbb{T}_{t^0}^u$ and $\mathbb{T}_{t^1}^v$ if:

(i) $d(\mathbb{T}_{t^0}^u, \mathbb{T}_{t^1}^v) < \epsilon$;

(ii) $|u(t) - v(t)| < \epsilon$ for all $t \in \mathbb{T}_{t^0}^u \cap \mathbb{T}_{t^1}^v$.

The topology defined by ϵ-neighborhoods of rd-continuous solutions will be called B-topology. It is easily seen that it is a Hausdorff topology. Topologies and metrics for spaces of discontinuous functions were introduced and developed in [6, 11, 57].

For any $\alpha, \beta \in \mathbb{R}$ we define the oriented interval $[\alpha \hat{,} \beta]$ as

$$[\alpha \hat{,} \beta] = \begin{cases} [\alpha, \beta], & \text{if } \alpha \le \beta \\ [\beta, \alpha], & \text{otherwise} \end{cases}. \tag{3.17}$$

Consider the non-variable time scale

$$\mathbb{T}_c^0 = \bigcup_{i=-\infty}^{\infty} [l_{2i-1}, r_{2i}], \tag{3.18}$$

where $l_i, r_i, i \in \mathbb{Z}$, are as defined by (3.1) for the variable time scale $\mathbb{T}_0(y)$, and take a continuation $\tilde{f} : \mathbb{T}_c^0 \times \mathbb{R}^n \to \mathbb{R}^n$ of $f : \mathbb{T}_0(y) \to \mathbb{R}^n$ which is Lipschitzian with the same Lipschitz constant ℓ.

Set $\mathbb{T}_c := \bigcup_{i=-\infty}^{\infty} [t_{2i-1}, t_{2i}]$. We start with proving the following lemma.

Lemma 3.5.1 *There are mappings $W_i(z) : \mathbb{R}^n \to \mathbb{R}^n, i \in \mathbb{Z}$, such that, corresponding to each solution $y(t)$ of (3.9), there is a solution $z(t)$ of the system*

$$\begin{aligned} z' &= A(t)z + \tilde{f}(t,z), \quad t \neq t_{2i}, \\ z(t_{2i+1}) &= B_i z(t_{2i}) + W_i(z(t_{2i})) + z(t_{2i}), \end{aligned} \tag{3.19}$$

such that $y(t) = z(t)$ for all $t \in \mathbb{T}_c$ except possibly on $[t_{2i-1}\widehat{,} \alpha_i]$ and $[\beta_i\widehat{,} t_{2i}]$ where α_i and β_i are the moments that $y(t)$ meets the surfaces $\mathcal{S}_{2i-1}$ and $\mathcal{S}_{2i}$, respectively.

The functions W_i satisfy the inequality

$$\|W_i(z) - W_i(y)\| \leq k(\ell)\ell\|z - y\|, \tag{3.20}$$

uniformly with respect to $i \in \mathbb{Z}$, for all $z, y \in \mathbb{R}^n$ such that $\|z\| \leq h$ and $\|y\| \leq h$; here $k(\ell) = k(\ell, h)$ is a bounded function.

Remark 3.5.2 *We say that systems (3.9) and (3.19) are B-equivalent.*

Proof. Fix $i \in \mathbb{Z}$. Let $z(t)$ be the solution of (3.9) such that $z(t_{2i}) = z$, and assume that α_i and β_i are solutions of $\alpha = t_{2i-1} + \tau_{2i-1}(z(\alpha))$, and $\beta = t_{2i} + \tau_{2i}(z(\beta))$, respectively. Let $z_1(t)$ be the solution of the system

$$\frac{dz}{dt} = A(t)z + \tilde{f}(t,z) \tag{3.21}$$

with the initial condition $z_1(\alpha_{i+1}) = \Pi_i^2(\beta_i, z(\beta_i))$.

We first note that $z_1(\alpha_{i+1}) = (I + B_i)z(\beta_i) + J_i(z(\beta_i))$. Moreover, for $t \in [t_{2i}\widehat{,} \beta_i]$,

$$z(t) = z(t_{2i}) + \int_{t_{2i}}^{t} \Big[A(s)z(s) + \tilde{f}(s, z(s))\Big] ds, \tag{3.22}$$

and for $t \in [\alpha_{i+1}\widehat{,} t_{2i+1}]$,

$$\begin{aligned} z_1(t) &= z_1(\alpha_{i+1}) + \int_{\alpha_{i+1}}^{t} \Big[A(s)z_1(s) + \tilde{f}(s, z_1(s))\Big] ds \\ &= (I + B_i)z(\beta_i) + J_i(z(\beta_i)) + \int_{\alpha_{i+1}}^{t} \Big[A(s)z_1(s) + \tilde{f}(s, z_1(s))\Big] ds \\ &= (I + B_i)\left\{z(t_{2i}) + \int_{t_{2i}}^{\beta_i} \Big[A(s)z(s) + \tilde{f}(s, z(s))\Big] ds\right\} \\ &\quad + J_i(z(\beta_i)) + \int_{\alpha_{i+1}}^{t} \Big[A(s)z_1(s) + \tilde{f}(s, z_1(s))\Big] ds. \end{aligned} \tag{3.23}$$

Thus, we set

$$W_i(z) = (I + B_i)\int_{t_{2i}}^{\beta_i}\Big[A(s)z(s) + \tilde{f}(s, z(s))\Big]ds + J_i(z(\beta_i))$$
$$+ \int_{\alpha_{i+1}}^{t_{2i+1}}\Big[A(s)z_1(s) + \tilde{f}(s, z_1(s))\Big]ds. \tag{3.24}$$

Substituting (3.24) in (3.19), we see that $W_i(z)$ satisfies the first conclusion of the lemma. Figure 3.3 illustrates the procedure of the construction of $W_i(z)$.

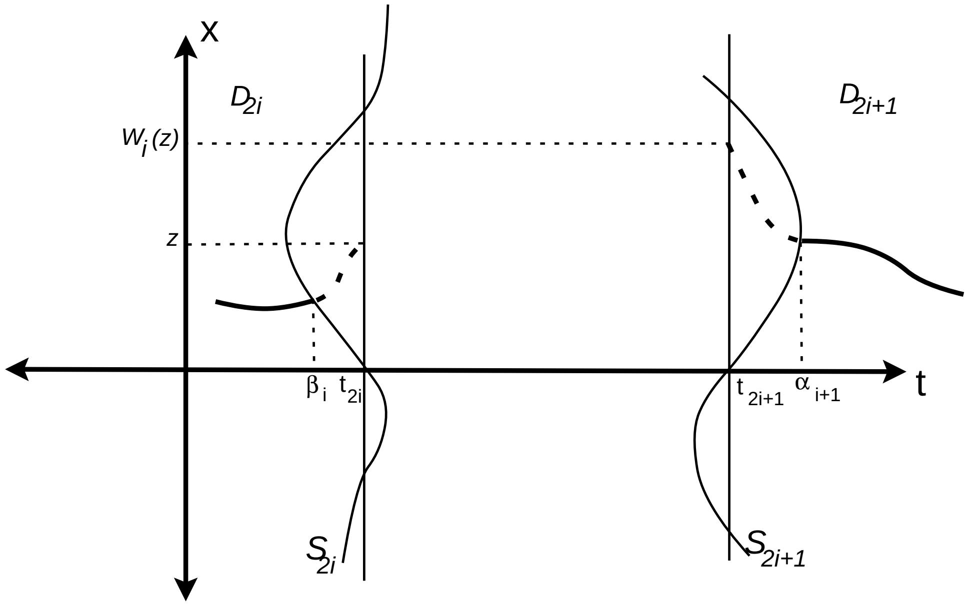

Figure 3.3: The construction of W_i.

We next prove (3.20). Let $\|z(t_{2i})\| \leq h$. By employing integrals (3.22) and (3.23), we find that the solutions $z(t)$ and $z_1(t)$ determined above satisfy the inequalities $\|z(t)\| \leq H$ and $\|z_1(t)\| \leq H$ on $[\beta_i, \hat{t}_{2i}]$ and $[\alpha_{i+1}, \hat{t}_{2i+1}]$, where

$$H = \Big[M(1+\ell) + (1+N+\ell)(h+M\ell)e^{N\ell+\ell^2}\Big]e^{N\ell+\ell^2}.$$

Let $y(t)$ be the solution of (3.9) such that $y(t_{2i}) = y$, and assume that $\overline{\alpha}_i$ and $\overline{\beta}_i$ are solutions of $\overline{\alpha} = t_{2i-1} + \tau_{2i-1}(y(\overline{\alpha}))$, and $\overline{\beta} = t_{2i} + \tau_{2i}(y(\overline{\beta}))$, respectively. Let $y_1(t)$ be the solution of (3.21) with the initial condition $y_1(\overline{\alpha}_{i+1}) = \Pi_i^2(\overline{\beta}_i, y(\overline{\beta}_i))$. Without loss of any generality, we assume that $\overline{\beta}_i \geq \beta_i$ and $\overline{\alpha}_{i+1} \leq \alpha_{i+1}$. Application of the Gronwall-Bellman lemma shows that, for $t \in [\beta_i, \hat{t}_{2i}]$,

$$\|z(t) - y(t)\| \leq e^{(N+\ell)\ell}\|z - y\|. \tag{3.25}$$

The equation,

$$y(\overline{\beta}_i) = y(\beta_i) + \int_{\beta_i}^{\overline{\beta}_i} \left[A(s)y(s) + \tilde{f}(s, y(s))\right] ds \tag{3.26}$$

gives us

$$\|y(\overline{\beta}_i) - y(\beta_i)\| \leq (NH + \ell H + M)(\overline{\beta}_i - \beta_i). \tag{3.27}$$

Thus, we obtain

$$\|z(\beta_i) - y(\overline{\beta}_i)\| \leq e^{(N+\ell)\ell}\|z - y\| + (NH + \ell H + M)(\overline{\beta}_i - \beta_i). \tag{3.28}$$

Now, condition (C6) together with (3.28) leads to

$$\overline{\beta}_i - \beta_i \leq \frac{\ell e^{(N+\ell)\ell}}{1 - \ell(NH + \ell H + M)}\|z - y\|. \tag{3.29}$$

Hence (3.28) becomes

$$\|z(\beta_i) - y(\overline{\beta}_i)\| \leq \frac{e^{(N+\ell)\ell}}{1 - \ell(NH + \ell H + M)}\|z - y\|. \tag{3.30}$$

On the other hand,

$$y_1(\alpha_{i+1}) = y_1(\overline{\alpha}_{i+1}) + \int_{\overline{\alpha}_{i+1}}^{\alpha_{i+1}} \left[A(s)y_1(s) + \tilde{f}(s, y_1(s))\right] ds \tag{3.31}$$

gives us

$$\|y_1(\alpha_{i+1}) - y_1(\overline{\alpha}_{i+1})\| \leq (NH + \ell H + M)(\alpha_{i+1} - \overline{\alpha}_{i+1}). \tag{3.32}$$

Using the transition operators and (3.30) we get,

$$\|z_1(\alpha_{i+1}) - y_1(\overline{\alpha}_{i+1})\| \leq \frac{(1 + N + \ell)e^{(N+\ell)\ell}}{1 - \ell(NH + \ell H + M)}\|z - y\|. \tag{3.33}$$

Condition (C6) and (3.33) imply that

$$\alpha_{i+1} - \overline{\alpha}_{i+1} \leq \frac{\ell(1 + N + \ell)e^{(N+\ell)\ell}}{1 - \ell(NH + \ell H + M)}\|z - y\|. \tag{3.34}$$

From (3.32), (3.33) and (3.34) we obtain

$$\|z_1(\alpha_{i+1}) - y_1(\alpha_{i+1})\| \leq H_1 e^{(N+\ell)\ell}\|z - y\|. \tag{3.35}$$

where $H_1 = (1 + N + \ell)[1 + \ell(NH + \ell H + M)]/[1 - \ell(NH + \ell H + M)]$. Solutions $z_1(t)$ and $y_1(t)$ on $[\alpha_{i+1}, \hat{t}_{2i+1}]$ satisfy the inequality

$$\|z_1(t) - y_1(t)\| \leq H_1 e^{2(N+\ell)\ell}\|z - y\|. \tag{3.36}$$

Now, subtracting the expression

$$W_i(y) = (I + B_i)\int_{t_{2i}}^{\bar{\beta}_i} \left[A(s)y(s) + \tilde{f}(s, y(s))\right] ds + J_i(y(\bar{\beta}_i))$$
$$+ \int_{\bar{\alpha}_{i+1}}^{t_{2i+1}} \left[A(s)y_1(s) + \tilde{f}(s, y_1(s))\right] ds. \qquad (3.37)$$

from equation (3.24), and using equations (3.25), (3.29), (3.34) and (3.36), we conclude that equation (3.20) holds. This proves the lemma. ∎

Definition 3.5.3 *A solution $y(t)$ is said to be B-stable, if for arbitrary $\epsilon > 0$, there is $\delta > 0$ such that a solution $\varphi(t)$ for which $\|\varphi(t^0) - y(t^0)\| < \delta$ is in the ϵ-neighborhood of $y(t)$ on $\mathbb{T}^y_{t^0}$ and $\mathbb{T}^\varphi_{t^0}$.*

Definition 3.5.4 *A B-stable solution $y(t)$ is called B-asymptotically stable, if there is $\delta > 0$ such that for arbitrary $\epsilon > 0$, there is $\theta > t^0$ such that a solution $\varphi(t)$ for which $\|\varphi(t^0) - y(t^0)\| < \delta$ is in the ϵ-neighborhood of $y(t)$ on $\mathbb{T}^y_\theta$ and $\mathbb{T}^\varphi_\theta$.*

3.6 Reduction to an Impulsive Differential Equation

Previously we have shown that a differential equation on a variable time scale is B-equivalent to a corresponding differential equation on a non-variable time scale. Now, we are going to reduce (3.19), which is B-equivalent to (3.9), into a system of impulsive differential equations.

Now, using the substitution of the independent variable (that is, the ψ-substitution) in (3.19) and letting $x(s) = z(\psi^{-1}(s))$, we obtain, for $t \neq t_{2i}$,

$$x' = A(\psi^{-1}(s))x + \tilde{f}(\psi^{-1}(s), x(s)),$$

and, for $t = t_{2i}$, we get

$$\begin{aligned} x(s_i^+) &= z(t_{2i+1}) \\ &= (I + B_i)z(t_{2i}) + W_i(z(t_{2i})) \\ &= (I + B_i)x(s_i) + W_i(x(s_i)). \end{aligned}$$

Thus, the second equation in (3.19) leads to,

$$\Delta x\,|_{s=s_i} = B_i x(s_i) + W_i(x(s_i)),$$

where $\Delta x \mid_{s=s_i} = x(s_i^+) - x(s_i)$. Hence, $x(s)$ is a solution of the impulsive differential equation:

$$\begin{aligned} x' &= A(\psi^{-1}(s))x + \tilde{f}(\psi^{-1}(s), x(s)), \quad s \neq s_i, \\ \Delta x \mid_{s=s_i} &= B_i x(s_i) + W_i(x(s_i)). \end{aligned} \tag{3.38}$$

It is known that, a solution of (3.38) satisfying $x(s^0) = x_0$, for $s \geq 0$ is given by

$$\begin{aligned} x(s) &= X(s, s^0)x_0 + \int_{s^0}^{s} X(s,\xi)\tilde{f}(\psi^{-1}(\xi), x(\xi)d\xi \\ &\quad + \sum_{s^0 \leq s_i < s} X(s, s_i^+)W_i(x(s_i^+)), \end{aligned} \tag{3.39}$$

where $X(s, s^0) = X(s)X^{-1}(s^0)$ and $X(s)$ is defined by

$$X(s) = \begin{cases} I, & s = 0, \\ \mathcal{X}(s, s_p)(I + B_p)\prod_{k=p}^{1}\left[\mathcal{X}(s_k, s_{k-1})(I + B_{k-1})\right]\mathcal{X}(s, 0), & s > 0, \end{cases}$$

in which $\mathcal{X}(s, r) = \mathcal{X}(s)\mathcal{X}^{-1}(r)$ is a transition matrix of $x' = A(\psi^{-1}(s))x$ and it is assumed that $0 < s^0 < \cdots < s_p < s < s_{p+1}$.

Now, using back substitution, we see that a solution $y(t), y(t^0) = y_0$, of (3.19), for $t \geq t^0$, is given by,

$$\begin{aligned} y(t) &= Y(t, t^0)t_0 + \int_{\mathbb{T}(t^0,t)} Y(t,\tau)f(\tau, y(\tau))d\tau \\ &\quad + \sum_{t^0 \leq t_{2i} < t} Y(t, t_{2i+1})W_i(t_{2i+1}), \end{aligned} \tag{3.40}$$

where $Y(t, t^0) = Y(t)Y^{-1}(t^0)$ and $Y(t)$, for $0 < t^0 < \cdots < t_{2p+1} < t < t_{2p+2}$, is defined by

$$Y(t) = \begin{cases} I, & t = 0, \\ \mathcal{Y}(t, t_{2p+1})(I + B_p)\prod_{k=p}^{1}\left[\mathcal{Y}(t_{2k}, t_{2k-1})(I + B_{2k-1})\right]\mathcal{Y}(t, 0), & t > 0, \end{cases}$$

in which $\mathcal{Y}(t,\tau) = \mathcal{Y}(t)\mathcal{Y}^{-1}(\tau)$ is a transition matrix of $y' = A(t)y$. The notation $\int_{\mathbb{T}(a,b)} f(\tau)d\tau$ was introduced in the previous chapter, [19].

Thus, instead of investigating system (3.9), we are going to deal with (3.19) which turns out to an IDE, as in (3.38), after ψ-substitution.

On the bases of the discussion in Sections 3.5 and 3.6, one may conclude that the method of investigation of DETCV may be realized as consecutive reductions: a) using a B-equivalence method to get a DETC; and b) applying the ψ-substitution to DETC to obtain an IDE. We finalize the reductions with the interpretation of results for the issue DETCV. Figure 3.4 illustrates the method.

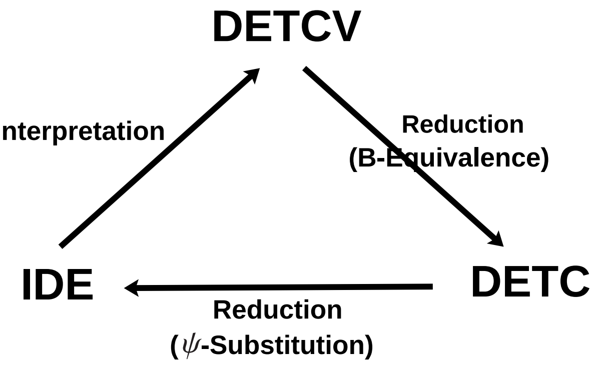

Figure 3.4: The investigation method of DETCV

3.7 Periodic Systems

The variable time scale $\mathbb{T}_0(y)$ is said to satisfy an (ω, p)-property if $(t \pm \omega, y)$ is in $\mathbb{T}_0(y)$ whenever (t, y) is. In this case, one can easily see that, there exists $p \in \mathbb{N}$ such that the sequences $\{t_{2i-1}\}$ and $\{t_{2i}\}$ satisfy the (ω, p)-property, [19], and $\tau_{i+p}(y) = \tau_i(y)$ for all $i \in \mathbb{Z}$.

Suppose now that (3.9) is ω-periodic, i.e. $\mathbb{T}_0(y)$ satisfies the (ω, p)-property, $A(t)$ and $f(t, y)$ are ω-periodic functions of t, and $B_{i+p} = B_i, J_{i+p}(y) = J_i(y)$ uniformly with respect to $i \in \mathbb{Z}$.

Since (3.9) satisfies the conditions for the uniqueness of a solution, and is periodic it can be shown that the following result holds.

Lemma 3.7.1 *If (3.9) is periodic, then the sequence $W_i(z)$ is p-periodic uniformly with respect to $z \in \mathbb{R}^n$.*

Hence (3.19) is also periodic.

Lemma 3.7.2 *[19] If* $\mathbb{T}_0(y)$ *satisfies an* (ω, p)*-property, then* $\psi(t+\omega) = \psi(t) + \psi(\omega)$.

Consider the ω-periodic system

$$\begin{aligned} \frac{dz}{dt} &= A(t)z + f(t), \quad t \neq t_{2i}, \\ z(t_{2i+1}) &= B_i z(t_{2i}) + W_i + z(t_{2i}), \end{aligned} \tag{3.41}$$

and let $Z(t)$ be the fundamental matrix of the corresponding homogeneous system,

$$\begin{aligned} \frac{dz}{dt} &= A(t)z, \quad t \neq t_{2i}, \\ z(t_{2i+1}) &= B_i z(t_{2i}) + z(t_{2i}). \end{aligned} \tag{3.42}$$

Using ψ-substitution, systems (3.41) and (3.42) reduce to

$$\begin{aligned} \frac{du}{ds} &= A(\psi^{-1}(s))u + f(\psi^{-1}(s)), \quad s \neq s_i, \\ \Delta u \mid_{s=s_i} &= B_i u + W_i, \end{aligned} \tag{3.43}$$

and

$$\begin{aligned} \frac{du}{ds} &= A(\psi^{-1}(s))u, \quad s \neq s_i, \\ \Delta u \mid_{s=s_i} &= B_i u, \end{aligned} \tag{3.44}$$

respectively, where $u(s) = z(\psi^{-1}(s))$.

According to [14], there is a piece-wise continuous Floquet-Lyapunov transformation $u = \Phi(s)v$ reducing (3.43) to a system with a constant matrix. There is, therefore, a constant nonsingular matrix S such that the transformation, $u = \Phi(s)Sw$, reduces (3.43) to

$$\begin{aligned} \frac{dw}{ds} &= \Lambda w + g(s), \quad s \neq s_i, \\ \Delta w \mid_{s=s_i} &= I_i, \end{aligned} \tag{3.45}$$

where $\Lambda = diag(\Lambda_+, \Lambda_-)$ is a constant matrix with $\mathrm{Re}\lambda_j(\Lambda_+) > 0$ for $j = 1, 2, \cdots, m$, and $\mathrm{Re}\lambda_j(\Lambda_-) < 0$ for $j = m, m+1, \cdots, n$,

$$\begin{aligned} \Lambda &= S^{-1}\Phi^{-1}(s)\left[A(\psi^{-1}(s)) - \frac{d\Phi(s)}{ds}\Phi^{-1}(s)\right]\Phi(s)S, \\ g(s) &= S^{-1}\Phi^{-1}(s)f(\psi^{-1}(s)), \\ I_i &= S^{-1}\Phi^{-1}(s_i^+)W_i. \end{aligned}$$

It is natural to call $\lambda_j = \lambda_j(\Lambda)$ the characteristic indices and $\rho_j = e^{\lambda_j}$ the characteristic multipliers of (3.45), respectively [19, 85]. Similarly, we call the numbers λ_j and ρ_j the characteristic indices and characteristic multipliers of (3.42).

Lemma 3.7.3 *If the real parts of the characteristic indices of (3.42) do not vanish, then (3.41) has a unique ω-periodic solution, which will be B-asymptotically stable when all characteristic indices of (3.42) have negative real parts.*

Proof. Let

$$G(s) = \begin{cases} diag(\exp(\Lambda_+ s), 0), & \text{for} \quad s < 0, \\ diag(0, -\exp(\Lambda_- s)), & \text{for} \quad s > 0, \end{cases}$$

and let $\alpha = \min_{1 \le j \le n} |\mathrm{Re}\lambda_j(\Lambda)| + \epsilon$ where ϵ is an arbitrary positive number. In this case, it is known that there exists a number $K = K(\epsilon) > 1$, such that

$$\|G(s-r)\| \le K \exp(-\alpha|s-r|), \quad s, r \in \mathbb{R}.$$

By using this inequality, it was shown in [11] that

$$w_0(s) = \int_{-\infty}^{\infty} G(s-r)g(r)dr + \sum_{-\infty}^{\infty} G(s-s_i)I_i$$

is an $\tilde{\omega}$-periodic solution of (3.45), for which

$$\|w_0(s)\| \le 2Km(\alpha) \max\left\{\max_s \|g(s)\|, \max_i \|I_i\|\right\},$$
$$m(\alpha) = \frac{1}{\alpha} + \frac{\exp(\alpha\theta)}{1 - \exp(-\alpha\theta')}.$$

Hence, $u_0(s) = \Phi(s)S\,w_0(s)$ is a periodic solution of (3.43) and, for $s \in \mathbb{R}$, we have

$$\|u_0(s)\| \le 2Km_1 m(\alpha) \max\left\{\max_s \|f(\psi^{-1}(s))\|, \max_i \|I_i\|\right\},$$

where $m_1 = \max_s \|\Phi(s)S\|$. Therefore, $z_0(t) = u_0(\psi(t))$ is a periodic solution of (3.41) which satisfies the inequality,

$$\|z_0(t)\| \le 2Km_1 m(\alpha) \max\left\{\max_{t \in \mathbb{T}_c} \|f(t)\|, \max_i \|I_i\|\right\},$$

for $t \in \mathbb{T}_c$. This proves the lemma. ■

Now, let $C = 2Km_1 m(\alpha)$ and fix $\gamma > 1$. Let $k(\ell) = k(\ell, h)$ be the function defined in Lemma 3.5.1, for $h = \gamma CM$. By applying Lemmas 3.7.1 and 3.7.3 and the successive-approximation method, exactly as it was done in [11], we can prove the following lemma.

Lemma 3.7.4 *Suppose that $\tilde{f}(t,z), W_i(z)$ and $\tau_i(z)$ in (3.19) satisfy conditions (C6), (C7) and (3.20). If all characteristic indices of system (3.42) have non-vanishing real parts, then, when $\ell C \max\{1, k(\ell)\} < (\gamma - 1)/\gamma$, system (3.19) has a unique ω-periodic solution $z_0(t)$ such that $\|z_0(t)\| \leq h$ for $t \in \mathbb{T}_c$.*

The solution $z_0(t)$ is B-asymptotically stable, if the real parts of all characteristic indices of system (3.42) are negative.

On the bases of B-equivalence, Lemmas 3.5.1, 3.7.4 and continuous dependence of solution on initial data for ordinary differential equation, one can prove the following theorem.

Theorem 3.7.5 *Suppose that system (3.9) satisfies conditions (C4)-(C8) and is ω-periodic. If the characteristic indices of system (3.42) have non-vanishing real parts, then for a sufficiently small lipschitz constant ℓ, system (3.9) has a unique ω-periodic solution, which is B-asymptotically stable when all characteristic indices of system (3.42) have negative real parts.*

Example 3.7.6 *Let us consider the variable time scale $\mathbb{T}_0(y)$ constructed by $t_i = i$, $\tau_i(y) = (-1)^i \ell \sin(y)$, where $y \in \mathbb{R}$, $0 < \ell < \frac{1}{2}$, and consider the 2-periodic system*

$$
\begin{aligned}
y' &= ky + \cos(\pi t), \quad (t,y) \in \mathbb{T}_0(y), \\
y^+ &= (p+1)y + I, \\
t^+ &= 2i + 1 - \ell \sin(y),
\end{aligned}
\tag{3.46}
$$

with $k, p, I \in \mathbb{R}, I > 0$. The system which is B-equivalent to (3.46) is

$$
\begin{aligned}
z' &= kz + \cos(\pi t), \quad t \neq 2i, \\
z(2i+1) &= (p+1)z(2i) + W_i(z),
\end{aligned}
\tag{3.47}
$$

where

$$
\begin{aligned}
W_i(z) &= (1+p)\int_{2i}^{\beta_i} [kz(s) + \cos(\pi s)]ds + I + \int_{\alpha_{i+1}}^{2i+1} [kz(s) + \cos(\pi s)]ds \\
&= (1+p)\int_{2i}^{\beta_i} kz(s)ds + \int_{\alpha_{i+1}}^{2i+1} kz(s)ds + I + \frac{(1+p)\sin(\pi\beta_i) - \sin(\pi\alpha_{i+1})}{\pi}
\end{aligned}
$$

where $z(t)$ is a solution of (3.46) satisfying $z(2i) = z$ and α_i and β_i are solutions of $\alpha = 2i - 1 - \ell\sin(z(\alpha))$ and $\beta = 2i + \ell\sin(z(\beta))$, respectively.

The homogeneous system corresponding to (3.47) is

$$\begin{aligned} z' &= kz, \quad t \neq 2i, \\ z(2i+1) &= (p+1)z(2i). \end{aligned} \tag{3.48}$$

It is easily seen that for (3.46), the conditions (C4)-(C8) are satisfied and

$$Z(2) = [(p+1)e^k]$$

is the matrix of monodromy and $\lambda = \ln(p+1) + k$ *is the characteristic index of (3.48). By Theorem 3.7.5, if* $\ln(p+1) + k \neq 0$, *then system (3.46) has a unique 2-periodic solution which is B-asymptotically stable when* $\ln(p+1) + k < 0$.

3.8 Stability of an Equilibrium

In this part, we are again going to consider the quasilinear system

$$\begin{aligned} &y' = A(t)y + f(t,y), \quad (t,y) \in \mathbb{T}_0(y), \\ &t^+ = \Pi_i^1(t,y), \quad y^+ = \Pi_i^2(t,y), \quad (t,y) \in \mathcal{S}_{2i}, \end{aligned} \tag{3.49}$$

on the variable time scale $\mathbb{T}_0(y)$. However, this time, the condition for existence of a Green's function is replaced by a more general one, namely, exponential dichotomy, [39]. Let

$$\begin{aligned} y' &= A(t)y, \quad t \neq t_{2i}, \\ y(t_{2i+1}) &= B_i y(t_{2i}) + y(t_{2i}), \end{aligned} \tag{3.50}$$

be the homogeneous system corresponding to (3.49). Moreover, suppose that the system which is B-equivalent to (3.49) is:

$$\begin{aligned} z' &= A(t)z + \tilde{f}(t,z), \quad t \neq t_{2i}, \\ z(t_{2i+1}) &= B_i z(t_{2i}) + W_i(z(t_{2i})) + z(t_{2i}). \end{aligned} \tag{3.51}$$

Suppose that there are m- and $(n-m)$-dimensional hyperplanes $Y_+(t)$ and $Y_-(t)$ in $\mathbb{T}_c \times \mathbb{R}^n$ such that if $y(t)$ is a solution of (3.50) and $y(t) \in Y_+(t)$, then $\|y(t)\| \leq a_1\|y(\tau)\| \exp(-\gamma_1(t-\tau))$, $-\infty < \tau \leq t < +\infty$ and, if $y(t) \in Y_-(t)$ then $\|y(t)\| \geq a_2\|y(\tau)\| \exp(\gamma_2(t-\tau))$, $-\infty < \tau \leq t < +\infty$. Here, a_j, γ_j, $j = 1,2$ are positive constants. If (3.50) satisfies these conditions, then we say that (3.50) is *exponentially dichotomous*.

In this case, using the inequality $\psi(t) - \psi(\tau) \leq t - \tau$ when $\tau \leq t$, one can show that, for the reduced impulsive linear system

$$\begin{aligned} x' &= A(\psi^{-1}(s))x, \quad s \neq s_i, \\ \Delta x\,|_{s=s_i} &= B_i x(s_i), \end{aligned} \tag{3.52}$$

where $x(s) = y(\psi^{-1}(s))$, $s_i = \psi(t_{2i})$, there are m- and $(n-m)$-dimensional hyperplanes $X_+(s)$ and $X_-(s)$ in $\mathbb{R} \times \mathbb{R}^n$ such that if $x(s)$ is a solution of (3.52) and $x(s) \in X_+(s)$, then $\|x(s)\| \leq a_1\|x(r)\| \exp(-\gamma_1(s-r))$, $-\infty < r \leq s < +\infty$ and, if $x(s) \in X_-(s)$, then $\|x(s)\| \geq a_2\|x(r)\| \exp(\gamma_2(s-r))$, $-\infty < r \leq s < +\infty$. Then, the linear system (3.52) with impulse action is said to be *exponentially dichotomous* (e.d.) [39].

If (3.52) is e.d., then by applying the orthogonalization method to a given set of linearly independent solutions $x_1(s), x_2(s), \cdots, x_n(s)$, we can construct a piecewise-continuous Lyapunov-Schmidt transformation $x = L(s)w$ reducing (3.52) to a block-diagonal system [11], i.e., a system splitting into two systems:

$$\frac{d\xi}{ds} = P_1(s)\xi, \quad s \neq s_i, \quad \Delta\xi\,|_{s=s_i} = Q_i^1\xi, \tag{3.53}$$

and

$$\frac{d\eta}{ds} = P_2(s)\eta, \quad s \neq s_i, \quad \Delta\eta\,|_{s=s_i} = Q_i^2\eta, \tag{3.54}$$

where $w = (\xi, \eta)$, with ξ an m-vector and η an $(n-m)$-vector. Corresponding to fundamental matrices $X_1(s,r)$ and $X_2(s,r)$ of (3.53) and (3.54), there are positive constants a and γ such that

$$\|X_1(s,r)\| \leq a\exp(-\gamma(s-r)), \quad s \geq r,$$

and

$$\|X_2(s,r)\| \leq a\exp(\gamma(s-r)), \quad s \leq r.$$

Similarly, (3.49) can be reduced to the system

$$\begin{aligned} \frac{d\xi}{ds} &= P_1(s)\xi + \tilde{f}_1(s,w), \quad s \neq s_i, \\ \frac{d\eta}{ds} &= P_2(s)\eta + \tilde{f}_2(s,w), \quad s \neq s_i, \\ \Delta\xi\,|_{s=s_i} &= Q_i^1\xi(s_i) + W_i^1(w(s_i)), \\ \Delta\eta\,|_{s=s_i} &= Q_i^2\eta(s_i) + W_i^2(w(s_i)), \end{aligned} \tag{3.55}$$

after applying ψ-substitution and Lyapunov-Schmidt transformation, successively.

Besides the conditions imposed before, suppose that (3.49) satisfies

$$f(t,0) = J_i(0) = 0 \tag{3.56}$$

uniformly with respect to $t \in \mathbb{T}_c$ and $i \in \mathbb{Z}$.

We investigate the stability of an equilibrium position of (3.49), first noting that (3.56) implies $W_i(0) = 0$ for $i \in \mathbb{Z}$.

It follows from Lemma 3.5.1, B-equivalence, and the continuous dependence of solutions of (3.49) on initial data, that the following analog of the Lyapunov-Perron theorem holds.

Theorem 3.8.1 *Suppose that system (3.49) satisfies conditions (C4)-(C7) and (3.56), and system (3.50) is e.d. Then, for a sufficiently small Lipschitz constant ℓ, the equilibrium position of (3.49) is conditionally asymptotically stable with respect to an m-dimensional manifold of initial values containing the origin. If $m = n$, then the zero solution of (3.49) is asymptotically stable.*

Proof. By virtue of the reasoning given above, we consider the system (3.51) which can be reduced to the form (3.55). We assume that the functions on the right side of (3.55) satisfy conditions analogous to (C6), (C7) and (3.20) with the same constants. Then, the integral-equation system

$$\begin{aligned} \xi &= X_1(s,s^0)c + \int_{s^0}^{s} X_1(s,r)\tilde{f}_1(r,w)dr + \sum_{s_i<s} X_1(s,s_i)W_i^1(w), \\ \eta &= -\int_{s}^{\infty} X_2(s,r)\tilde{f}_2(r,w)dr - \sum_{s_i>s} X_2(s,s_i)W_i^2(w), \end{aligned} \tag{3.57}$$

under the conditions

$$a\ell\|c\|(a+\epsilon)\left[\frac{2}{\gamma^2-\sigma^2} + \frac{2}{1-\exp(-\theta'(\gamma-\sigma))}\right] < \epsilon$$

and

$$2a\left[\frac{1}{\gamma-\sigma} + \frac{1}{1-\exp(-\theta'(\gamma-\sigma))}\right] < 1,$$

where ϵ and σ are arbitrary fixed constants such that $\epsilon > 0$ and $0 < \sigma < \gamma$, has a solution $w(s) = w(s, s^0, c)$ for which

$$\|w(s)\| \leq (a + \epsilon)\|c\| \exp\left(-\sigma\left(s - s^0\right)\right). \tag{3.58}$$

If $s = s^0$ in (3.57), then

$$\begin{aligned} &\xi(s^0, s^0, c) = c \\ &\eta(s^0, s^0, c) = -\int_{s^0}^{\infty} X_2(s^0, r)\tilde{f}_2(r, w)dr - \sum_{s_i > s^0} X_2(s^0, s_i)W_i^2(w(s_i)). \end{aligned} \tag{3.59}$$

By using the customary method [85], we can easily show that $w(s)$ is also a solution of (3.55). Hence, by virtue of (3.57) and (3.58), we conclude that (3.59) determines a set of initial values of solutions of (3.55) tending to an equilibrium state when $s \to \infty$. Since, B-equivalence and ψ-substitution do not change the dimension of the manifold, the theorem is proved. ■

3.9 Bounded Solutions

Theorem 3.9.1 *If conditions (C4)-(C8) are satisfied for system (3.49) and (3.50) is e.d., then for a sufficiently small Lipschitz constant ℓ, system (3.49) has a unique solution, continuable to $+\infty$ and $-\infty$, uniformly bounded for all $t, (t, y(t)) \in \mathbb{T}_0(y)$.*

Proof. System (3.51) which is B-equivalent to (3.49) can be reduced to

$$\begin{aligned} &x' = A(\psi^{-1}(s))x + \tilde{f}(\psi^{-1}(s), x), \quad s \neq s_i, \\ &\Delta x\,|_{s=s_i} = B_i x(s_i) + W_i(x(s_i)), \end{aligned} \tag{3.60}$$

by means of ψ-substitution, where $x(s) = z(\psi^{-1}(s))$. In [11], it was shown that for $h = \nu a M\{1/\gamma + [\exp(\gamma\theta)/(1 - \exp(-\gamma\theta')]\}$, where $\nu > 1$ is fixed, under the condition

$$\ell a\left(\frac{1}{\gamma} + \frac{k(\ell)\exp(\gamma\theta)}{1 - \exp(-\gamma\theta')}\right) < \frac{\nu - 1}{\nu},$$

the system (3.60) has a unique bounded solution $x_0(s)$. Using the inverse substitution, we see that $y_0(t) = x_0(\psi(t))$ is a bounded solution of (3.51), and B-equivalence between (3.51) and (3.49) proves the theorem. ■

3.10 Deduction

In this chapter, we have introduced a new class of differential equations, differential equations on variable time scales with transition conditions. These systems naturally appear when we investigate discontinuous dynamics with non-fixed moments of impulses. Consequently, our results will be needed to develop methods of investigation of mechanical models with impacts. Particularly, interesting problems are related to bifurcations [31, 54, 56, 94], chaos [54], etc. We are going to develop the theory of introduced equations according to these demands.

CHAPTER 4

BIFURCATION OF THREE-DIMENSIONAL DISCONTINUOUS CYCLES

In this chapter, we consider three-dimensional discontinuous dynamical systems with non-fixed moments of impacts. Existence of the center manifold is proved for the system. The result is applied for the extension of the planar Hopf bifurcation theorem [6]. Illustrative examples are constructed for the theory.

4.1 Introduction

Dynamical systems are used to describe real world motions using differential (continuous time) or difference (discrete time) equations. In the last several decades, the need for discontinuous dynamical systems has been increased because they, often, describe the model better when the discontinuous and continuous motions are mingled. This need has made scientists to improve and develop the theory of these systems. Many new results have arised. One must mention that namely systems with not prescribed time of discontinuities were apparently first introduced for investigation of the real world [58, 78], and this fact emphasizes very much the practical sense of the theory. The problem is one of the most difficult and interesting subjects of investigations [36, 40, 61, 62, 63, 72, 84]. It was emphasized in early stage of theory's development, [71].

In [6], the Hopf bifurcation for the planar discontinuous dynamical system has been studied. Here, we extend this result to three-dimensional space based on the center manifold. The advantage is that we use the method of $B-$equivalence [5, 6, 19] as well as the results of time scales which are developed in [6, 19].

This chapter is organized as follows. In the next section, we start to analyze the non-perturbed system. Section 4.3 describes the perturbed system. The center manifold is given in Section 4.4. In Section 4.5, the bifurcation of periodic solutions is studied. Seciton 4.6 is devoted to examples in order to illustrate the theory. In the last section a brief conclusion is given.

4.2 The Non-perturbed System

We shall consider in $\mathbb{R}^3$ the following dynamical system:

$$\begin{aligned} \frac{dx}{dt} &= Ax, \\ \frac{dz}{dt} &= \hat{b}z, \quad (x,z) \notin \Gamma_0, \\ \Delta x\mid_{(x,z)\in\Gamma_0} &= B_0 x, \\ \Delta z\mid_{(x,z)\in\Gamma_0} &= c_0 z, \end{aligned} \tag{4.1}$$

where $A, B_0 \in \mathbb{R}^{2\times 2}, \hat{b}, c_0 \in \mathbb{R}$, Γ_0 is a subset of $\mathbb{R}^3$ and will be described below. The phase point of (4.1) moves between two consecutive intersections with the set Γ_0 along one of the trajectories of the system $x' = Ax, z' = \hat{b}z$. When the solution meets the set Γ_0 at the moment τ, the point $x(t)$ has a jump $\Delta x\mid_\tau := x(\tau+) - x(\tau)$ and the point $z(t)$ has a jump $\Delta z\mid_\tau := z(\tau+) - z(\tau)$. Thus, we suppose that the solutions are left continuous functions.

From now on, G denotes a neighborhood of the origin.

The following assumptions will be needed throughout this chapter:

(C10) $\Gamma_0 = \bigcup_{i=1}^{p} \mathcal{P}_i, p \in \mathbb{N}$, where $\mathcal{P}_i = \ell_i \times \mathbb{R}, \ell_i$ are half-lines starting at the origin defined by $\langle a^i, x\rangle = 0$ for $i = 1, \cdots, p$, $a^i = (a_1^i, a_2^i) \in \mathbb{R}^2$ are constant vectors;

(C11) $A = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$, where $\beta \neq 0$;

(C12) there exists a regular matrix $Q \in \mathbb{R}^{2\times 2}$ and nonnegative real numbers k and θ

such that

$$B_0 = kQ\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} Q^{-1} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

For the sake of brevity, in what follows, every angle for a point or a line is considered with respect to the half-line of the first coordinate axis in $x-$plane. Denote $\ell_i' = (I + B_0)\ell_i, i = 1, \cdots, p$. Let γ_i and ζ_i be the angles of ℓ_i and ℓ_i' for $i = 1, \cdots, p$, respectively, and

$$B_0 = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix};$$

(C13) $0 < \gamma_1 < \zeta_1 < \gamma_2 < \cdots < \gamma_p < \zeta_p < 2\pi$, and $(b_{11} + 1)\cos\gamma_i + b_{12}\sin\gamma_i \neq 0$ for $i = 1, \ldots, p$.

In Figure 4.1, the discontinuity set and a trajectory of the system (4.1) are shown. The planes $\mathcal{P}_i$ form the set Γ_0 and each $\mathcal{P}_i'$ is the image of $\mathcal{P}_i$ under the transformation $(I + B)x$.

The system (4.1) is said to be a $D_0 -$ *system* if conditions (C10)-(C13) hold. It is easy to see that the origin is a unique singular point of $D_0 -$ *system* and (4.1) is not linear.

Let us subject (4.1) to the transformation $x_1 = r\cos\phi, x_2 = r\sin\phi, z = z$ and exclude the time variable t. The solution $(r(\phi, r_0, z_0), z(\phi, r_0, z_0))$ which starts at the point $(0, r_0, z_0)$ satisfies the following system in cylindrical coordinates:

$$\begin{aligned} &\frac{dr}{d\phi} = \lambda r, \\ &\frac{dz}{d\phi} = bz, \quad \phi \neq \gamma_i \ (\mathrm{mod}\ 2\pi), \\ &\Delta r\ |_{\phi=\gamma_i\ (\mathrm{mod}\ 2\pi)} = k_i r, \\ &\Delta z\ |_{\phi=\gamma_i\ (\mathrm{mod}\ 2\pi)} = c_0 z, \end{aligned} \tag{4.2}$$

where $\lambda = \alpha/\beta, b = \hat{b}/\beta$, the variable ϕ is ranged over the time scale

$$\mathbb{R}_\phi = \mathbb{R} \setminus \bigcup_{i=-\infty}^{\infty} \bigcup_{j=1}^{p} (2\pi i + \gamma_j, 2\pi i + \zeta_j]$$

and

$$k_i = \left[((b_{11} + 1)\cos\gamma_i + b_{12}\sin\gamma_i)^2 + (b_{21}\cos\gamma_i + (b_{22} + 1)\sin\gamma_i)^2 \right]^{1/2} - 1.$$

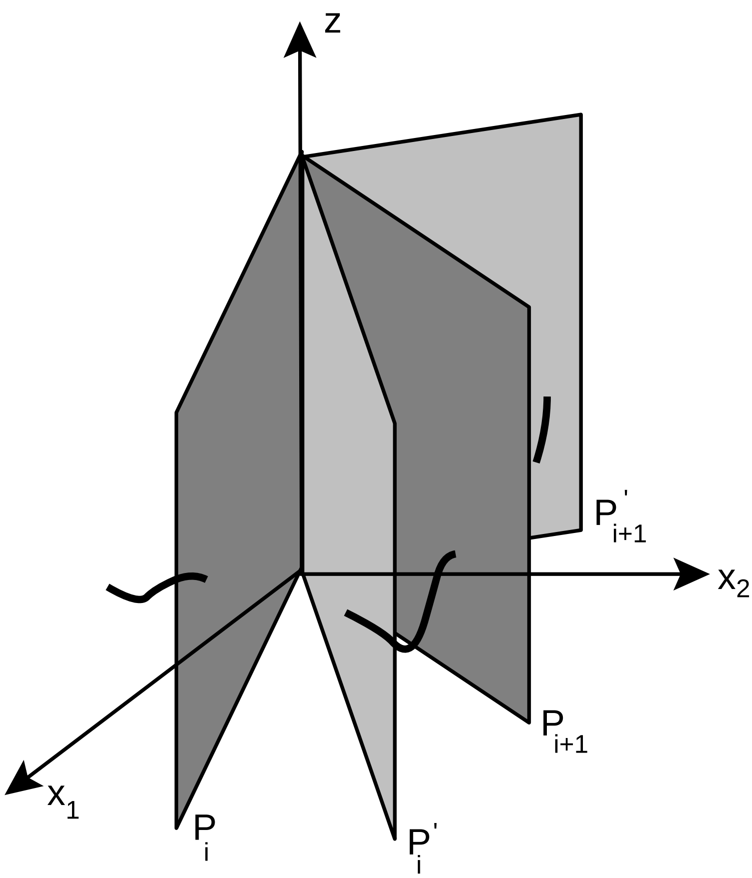

Figure 4.1: The discontinuity set and a trajectory of (4.1)

Equation (4.2) is $2\pi-$periodic, so, in what follows we shall consider just the section $[0, 2\pi]$. That is, the system

$$
\begin{aligned}
&\frac{dr}{d\phi} = \lambda r,\\
&\frac{dz}{d\phi} = bz, \quad \phi \neq \gamma_i,\\
&\Delta r\,|_{\phi=\gamma_i} = k_i r,\\
&\Delta z\,|_{\phi=\gamma_i} = c_0 z,
\end{aligned}
\tag{4.3}
$$

is provided for discussion, where $\phi \in [0, 2\pi]_\phi = [0, 2\pi] \setminus \cup_{i=1}^{p}(\gamma_i, \zeta_i]$. System (4.3) is a sample of time-scale differential equation. Let us use the *ψ − substitution*, $\varphi = \psi(\phi) = \phi - \sum_{0<\gamma_j<\phi} \theta_j, \theta_j = \zeta_j - \gamma_j$, which was introduced and developed in [6, 19]. The range of this new variable is $[0, 2\pi - \sum_{i=1}^{p} \theta_i]$.

It is easy to check that upon $\psi-$substitution (4.3) reduces to the following impulsive equations:

$$
\begin{aligned}
&\frac{dr}{d\varphi} = \lambda r,\\
&\frac{dz}{d\varphi} = bz, \quad \varphi \neq \varphi_i,\\
&\Delta r \mid_{\varphi=\varphi_i} = k_i r,\\
&\Delta z \mid_{\varphi=\varphi_i} = c_0 z,
\end{aligned}
\tag{4.4}
$$

where $\varphi_i = \psi(\gamma_i)$. Solving (4.4) as an impulsive system [60, 86] and using $\psi-$ substitution one can obtain that a solution of (4.3) is of the form

$$r(\phi) = \exp\left(\lambda\left(\phi - \sum_{0<\gamma_i<\phi} \theta_i\right)\right)\left[\prod_{0<\gamma_i<\phi}(1+k_i)\right] r_0, \tag{4.5}$$

$$z(\phi) = \exp\left(b\left(\phi - \sum_{0<\gamma_i<\phi} \theta_i\right)\right)\left[\prod_{0<\gamma_i<\phi}(1+c_0)\right] z_0, \tag{4.6}$$

for $\phi \in [0, 2\pi]_\phi$. Denote

$$q_1 = \exp\left(\lambda\left(2\pi - \sum_{i=1}^{p} \theta_i\right)\right)\prod_{i=1}^{p}(1+k_i), \tag{4.7}$$

$$q_2 = \exp\left(b\left(2\pi - \sum_{i=1}^{p} \theta_i\right)\right)\prod_{i=1}^{p}(1+c_0). \tag{4.8}$$

Depending on q_1 and q_2 we may see that the following lemmas are valid.

Lemma 4.2.1 *Assume that $q_1 = 1$. Then, if*

(i) $q_2 = 1$ *then all solutions are periodic with period* $\omega = \left(2\pi - \sum_{i=1}^{p} \theta_i\right)\beta^{-1}$;

(ii) $q_2 = -1$ *then a solution that starts to its motion on x_1x_2-plane is ω-periodic and all other solutions are 2ω-periodic;*

(iii) $|q_2| > 1$ *then a solution that starts to its motion on x_1x_2-plane is ω-periodic and all other solutions lie on the surface of a cylinder and they move away the origin (i.e. zero solution is unstable);*

(iv) $|q_2| < 1$ *then a solution that starts to its motion on x_1x_2-plane is ω-periodic and all other solutions lie on the surface of a cylinder and they move toward the x_1x_2-plane (i.e. zero solution is stable).*

Lemma 4.2.2 *Assume that* $q_1 < 1$*. Then, if*

(i) $|q_2| < 1$ *all solutions will spiral toward the origin, i.e., origin is an asymptotically stable fixed point;*

(ii) $|q_2| > 1$ *a solution that starts to its motion on* x*-plane spirals toward the origin and a solution that starts to its motion on z-axis will move away from the origin. In this case the origin is half stable (or conditionally stable);*

(iii) $q_2 = 1(q_2 = -1)$ *then a solution that starts to its motion on z-axis is periodic with period* $\omega(2\omega)$ *and all other solutions will approach to z-axis.*

Lemma 4.2.3 *Assume that* $q_1 > 1$*. Then, if*

(i) $|q_2| < 1$ *then origin is a stable focus;*

(ii) $|q_2| > 1$ *then origin is an unstable focus;*

(iii) $q_2 = 1(q_2 = -1)$ *then a solution that starts to its motion on z-axis is periodic with period* $\omega(2\omega)$ *and all other solutions will approach to z-axis.*

We note that when $q_2 = -1$, (this means z may be negative, too) the solutions starting their motion out of x_1x_2-plane, will move above and below the x_1x_2-plane. More explicitly, if a solution starts to its motion above the x-plane, then after the time corresponding to ω, it will be below the x-plane, and in the next duration corresponding to ω, it will try to move above x-plane and at the end of that duration it will be above the x-plane, and so on.

From now on, we assume that $q_1 = 1$ and $|q_2| < 1$.

4.3 The Perturbed System

Let G denote a sufficiently small neighborhood of the origin and consider the system

$$\begin{aligned}
&\frac{dx}{dt} = Ax + f(x,z),\\
&\frac{dz}{dt} = \hat{b}z + g(x,z), \quad (x,z) \notin \Gamma,\\
&\Delta x|_{(x,z)\in\Gamma} = B(x)x,\\
&\Delta z|_{(x,z)\in\Gamma} = c(z)z,
\end{aligned} \tag{4.9}$$

where the following assumptions are assumed to be true:

(C14) $\Gamma = \bigcup_{i=1}^{p} \mathcal{S}_i$, where $\mathcal{S}_i = s_i \times \mathbb{R}$ and the equation of s_i is given by $s_i : \langle a^i, x\rangle + \tau_i(x) = 0$, for $i = 1, \cdots, p$;

(C15)

$$B(x) = (k + \kappa(x))Q \begin{bmatrix} \cos(\theta + \Theta(x)) & -\sin(\theta + \Theta(x)) \\ \sin(\theta + \Theta(x)) & \cos(\theta + \Theta(x)) \end{bmatrix} Q^{-1} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and $c(z) = c_0 + \tilde{c}(z)$;

(C16) functions $f, g, \kappa, \tilde{c}$ and Θ are in C^1 and τ_i is in C^2;

(C17) $f(x,z) = \mathcal{O}(\|(x,z)\|^2)$, $g(x,z) = \mathcal{O}(\|(x,z)\|^2)$, $\kappa(x) = \mathcal{O}(\|x\|)$, $\Theta(x) = \mathcal{O}(\|x\|)$, $\tilde{c}(z) = \mathcal{O}(z)$, $\tau_i(x) = \mathcal{O}(\|x\|^2), i = 1, \cdots, p$, and $f(0,z) = 0$, $g(0,z) = 0$ for all $z \in \mathbb{R}$.

Moreover, it is supposed that the matrices A, Q, the vectors $a^i, i = 1, \cdots, p$, constants k, θ are the same as for (4.1), i.e.,

(C18) the associated with (4.9) is D_0 system.

Remark 4.3.1 *Conditions (C14) and (C15) imply that surfaces $\mathcal{S}_i$ do not intersect each other except on $z-$axis and neither of them intersects itself.*

The system (4.9) is said to be a $D-$ *system* if the conditions (C10)-(C17) hold.

In what follows we assume without loss of generality that $\gamma_i \neq \frac{\pi}{2} j$, $j = 1, 2, 3$. Then one can transform the equation in (C14) to the polar coordinates so that $s_i : a_i^1 r\cos\phi + a_i^2 r\sin\phi + \tau_i(r\cos\phi, r\sin\phi) = 0$ and, hence

$$\phi = \tan^{-1}\left(\tan\gamma_i - \frac{\tau_i(r\cos\phi, r\sin\phi)}{a_i^2 r\cos\phi}\right).$$

Using Taylor expansion gives that the previous equation can be written, for sufficiently small r, as

$$s_i : \phi = \gamma_i + \Psi_i(r, \phi), \quad i = 1, \cdots, p$$

where functions Ψ_i are $2\pi-$periodic in ϕ, continuously differentiable and $\Psi_i = O(r)$.

If the phase point $(x_1(t), x_2(t), z(t))$ meets the discontinuity surface $\mathcal{S}_i$ at the angle θ, then after the jump, the point $(x_1(\theta+), x_2(\theta+), z(\theta+))$ will be on the surface $\mathcal{S}'_i = \{(u, v) \in \mathbb{R}^3 : u = (I + B(x))x, v = (1 + c_0)z + c(z), (x, z) \in \mathcal{S}_i\}$. For the remaining part of the paper the following assertion is very important and the proof can be found in [6].

Lemma 4.3.2 *If the conditions (C16) and (C17) are valid then the surface $\mathcal{S}'_i$ is placed between the surfaces $\mathcal{S}_i$ and $\mathcal{S}_{i+1}$ for every i if G is sufficiently small.*

Using the cylindrical coordinates $x_1 = r\cos\phi, x_2 = r\sin\phi, z = z$, one can find that the differential part of (4.9) has the following form:

$$\begin{aligned} \frac{dr}{d\phi} &= \lambda r + P(r, \phi, z), \\ \frac{dz}{d\phi} &= bz + Q(r, \phi, z), \end{aligned} \tag{4.10}$$

where, as is known [87], the functions $P(r, \phi, z)$ and $Q(r, \phi, z)$ are $2\pi-$periodic in ϕ, continuously differentiable in all variables and $P = O(r, z)$, $Q = O(r, z)$, with $P(0, \phi, z) = 0$, $Q(0, \phi, z) = 0$, for all $\phi, z \in \mathbb{R}$. Denote $x^+ = (x_1^+, x_2^+) = (I + B(x))x$, $x^+ = r^+(\cos\phi^+, \sin\phi^+)$, $\tilde{x}^+ = (\tilde{x}_1^+, \tilde{x}_2^+) = (I + B(0))x$, where $x = (x_1, x_2) \in s_i, i = 1, \cdots, p$. The inequality $\|x^+ - \tilde{x}^+\| \leq \|B(x) - B(0)\| \cdot \|x\|$ implies that $r^+ = (1 + k_i)r + \omega(r, \phi)$. Moreover, using the relation $\frac{x_2^+}{x_1^+}$ and $\frac{\tilde{x}_2^+}{\tilde{x}_1^+}$ and condition (C14) one can conclude that $\phi^+ = \phi + \theta_i + \gamma(r, \phi)$. Functions ω and γ are $2\pi-$periodic in ϕ and $\omega = O(r^2)$, $\gamma = O(r)$.

Finally, the transformed system is of the following form:

$$\begin{aligned}
\frac{dr}{d\phi} &= \lambda r + P(r,\phi,z),\\
\frac{dz}{d\phi} &= bz + Q(r,\phi,z), \quad (r,\phi,z) \notin \Gamma,\\
\Delta r|_{(r,\phi)\in s_i} &= k_i r + \omega(r,\phi),\\
\Delta\phi|_{(r,\phi)\in s_i} &= \theta_i + \gamma(r,\phi),\\
\Delta z|_{(r,\phi)\in s_i} &= c_0 z + \tilde{c}(z).
\end{aligned} \tag{4.11}$$

Let us introduce the following system besides (4.11):

$$\begin{aligned}
\frac{d\rho}{d\phi} &= \lambda\rho + P(\rho,\phi,z),\\
\frac{dz}{d\phi} &= bz + Q(\rho,\phi,z), \quad \phi \neq \gamma_i,\\
\Delta\rho|_{\phi=\gamma_i} &= k_i\rho + W_i^1(\rho,z),\\
\Delta\phi|_{\phi=\gamma_i} &= \theta_i,\\
\Delta z|_{\phi=\gamma_i} &= c_0 z + W_i^2(\rho,z),
\end{aligned} \tag{4.12}$$

where all elements, except for $W_i = (W_i^1, W_i^2), i = 1, \cdots, p$, are the same as in (4.11) and the domain of (4.12) is $[0, 2\pi]_\phi$. We shall define the functions W_i below.

Let $(r(\phi, r_0, z_0), z(\phi, r_0, z_0))$ be a solution of (4.11) ϕ_i be the angle where the phase point intersects $\mathcal{S}_i$. Denote also by $\chi_i = \phi_i + \theta_i + \gamma(r(\phi_i, r_0, z_0), \phi_i)$ the angle where the phase point has to be after the jump.

Further $(\widehat{\alpha,\beta}]$, $\{\alpha,\beta\} \subset \mathbb{R}$ denotes the oriented inetrval, that is

$$(\widehat{\alpha,\beta}] = \begin{cases} (\alpha,\beta] & \text{if } \alpha \le \beta,\\ (\beta,\alpha] & \text{otherwise.} \end{cases}$$

Definition 4.3.3 *We shall say that systems (4.11) and (4.12) are B−equivalent in G if for every solution $(r(\phi, r_0, z_0), z(\phi, r_0, z_0))$ of (4.11) whose trajectory is in G for all $\phi \in [0, 2\pi]_\phi$ there exists a solution $(\rho(\phi, r_0, z_0), z(\phi, r_0, z_0))$ of (4.12) which satisfies the relation*

$$r(\phi, r_0, z_0) = \rho(\phi, r_0, z_0), \quad \phi \in [0, 2\pi]_\phi \setminus \bigcup_{i=1}^{p} \{(\widehat{\phi_i, \gamma_i}] \cup (\widehat{\zeta_i, \chi_i}]\}, \tag{4.13}$$

and, conversely, for every solution $(\rho(\phi, r_0, z_0), z(\phi, r_0, z_0))$ of (4.12) whose trajectory is in G, there exists a solution $(r(\phi, r_0, z_0), z(\phi, r_0, z_0))$ of (4.11) which satisfies (4.13).

Fix $i = 1, \cdots, p$. Let $(r_1(\phi), z_1(\phi)), (r_1(\gamma_i), z_1(\gamma_i)) = (\rho, z)$, be a solution of

$$\begin{aligned} \frac{dr}{d\phi} &= \lambda r + P(r, \phi, z), \\ \frac{dz}{d\phi} &= bz + Q(r, \phi, z), \end{aligned} \tag{4.14}$$

and let $\phi = \eta_i$ be the meeting angle of the solution with $\mathcal{P}_i$. Then

$$\begin{aligned} r_1(\eta_i) &= e^{\lambda(\eta_i - \gamma_i)}\rho + \int_{\gamma_i}^{\eta_i} e^{\lambda(\eta_i - s)} P(r_1(s), s, z_1(s))ds, \\ z_1(\eta_i) &= e^{b(\eta_i - \gamma_i)}z + \int_{\gamma_i}^{\eta_i} e^{b(\eta_i - s)} Q(r_1(s), s, z_1(s))ds. \end{aligned}$$

Set $\eta_i' = \eta_i + \theta_i + \gamma(r_1(\eta_i), \eta_i)$ and $(\rho', z') = ((1+k_i)r_1(\eta_i) + \omega(r_1(\eta_i), \eta_i), (1+c_0)z_1(\eta_i) + c(z_1(\eta_i)))$. Let $(r_2(\phi), z_2(\phi)), (r_2(\eta_i'), z_2(\eta_i')) = (\rho', z')$, be a solution of (4.14). Then,

$$\begin{aligned} r_2(\zeta_i) &= e^{\lambda(\zeta_i - \eta_i')}\rho' + \int_{\eta_i'}^{\zeta_i} e^{\lambda(\zeta_i - s)} P(r_2(s), s, z_2(s))ds, \\ z_2(\zeta_i) &= e^{b(\zeta_i - \eta_i')}z' + \int_{\eta_i'}^{\zeta_i} e^{b(\zeta_i - s)} Q(r_2(s), s, z_2(s))ds. \end{aligned}$$

We define that

$$\begin{aligned} W_i^1(\rho, z) &= r_2(\zeta_i) - (1+k_i)\rho \\ &= e^{\lambda(\zeta_i - \eta_i')}\Big[(1+k_i)\Big(e^{\lambda(\eta_i - \gamma_i)}\rho + \int_{\gamma_i}^{\eta_i} e^{\lambda(\eta_i - s)} P(r_1(s), s, z_1(s))ds\Big) \\ &\quad +\omega(r_1(\eta_i), \eta_i)\Big] + \int_{\eta_i'}^{\zeta_i} e^{\lambda(\zeta_i - s)} P(r_1(s), s, z_1(s))ds - (1+k_i)\rho, \end{aligned}$$

or, if simplified

$$\begin{aligned} W_i^1(\rho, z) &= (1+k_i)(e^{-\lambda\gamma(r_1(\eta_i), \eta_i)} - 1)\rho \\ &\quad +(1+k_i)\int_{\gamma_i}^{\eta_i} e^{\lambda(\zeta_i - \theta_i - s - \gamma(r_1(\eta_i), \eta_i))} P(r_1(s), s, z_1(s))ds \\ &\quad + \int_{\eta_i'}^{\zeta_i} e^{\lambda(\zeta_i - s)} P(r_2(s), s, z_2(s))ds + e^{\lambda(\zeta_i - \eta_i')}\omega(r_1(\eta_i), \eta_i). \end{aligned} \tag{4.15}$$

We, similarly, define

$$\begin{aligned} W_i^2(\rho, z) &= z_2(\zeta_i) - (1+c_0)z \\ &= e^{b(\zeta_i - \eta_i')}\Big[(1+c_0)\Big(e^{b(\eta_i - \gamma_i)}z + \int_{\gamma_i}^{\eta_i} e^{b(\eta_i - s)} Q(r_1(s), s, z_1(s))ds\Big) \\ &\quad +\tilde{c}(z_1(\eta_i))\Big] + \int_{\eta_i'}^{\zeta_i} e^{b(\zeta_i - s)} Q(r_1(s), s, z_1(s))ds - (1+c_0)z, \end{aligned}$$

or,

$$\begin{aligned} W_i^2(\rho, z) &= (1+k_i)(e^{-b\gamma(r_1(\eta_i),\eta_i)} - 1)z \\ &\quad +(1+c_0)\int_{\gamma_i}^{\eta_i} e^{(\zeta_i-\theta_i-s-\gamma(r_1(\eta_i),\eta_i))} Q(r_1(s), s, z_1(s))ds \\ &\quad + \int_{\eta_i'}^{\zeta_i} e^{b(\zeta_i-s)} Q(r_2(s), s, z_2(s))ds + e^{b(\zeta_i-\eta_i')}\tilde{c}(z_1(\eta_i)). \end{aligned} \tag{4.16}$$

We note that there exists a Lipschitz constant ℓ and a bounded function $m(\ell)$ such that

$$||W_i^j(\rho_1, z_1) - W_i^j(\rho_2, z_2)|| \leq m(\ell)\ell(||\rho_1 - \rho_2|| + ||z_1 - z_2||), \tag{4.17}$$

for all $\rho_1, \rho_2, z_1, z_2 \in \mathbb{R}$, $j = 1, 2$. For detailed proof and explanation about (4.17) we refer to [6, 19].

4.4 Center Manifold

Now, using $\psi-$substitution (4.12) becomes:

$$\begin{aligned} \frac{d\rho}{d\varphi} &= \lambda\rho + F(\rho, \varphi, z), \\ \frac{dz}{d\varphi} &= bz + G(\rho, \varphi, z), \quad \varphi \neq \varphi_i, \\ \Delta\rho|_{\varphi=\varphi_i} &= k_i\rho + W_i^1(\rho, z), \\ \Delta z|_{\varphi=\varphi_i} &= c_0 z + W_i^2(\rho, z), \end{aligned} \tag{4.18}$$

where $\varphi = \psi(\phi)$, $\varphi_i = \psi(\gamma_i)$, $F(\rho, \varphi, z) = P(\rho, \psi^{-1}(\varphi), z)$, $G(\rho, \varphi, z) = Q(\rho, \psi^{-1}(\varphi), z)$. Functions F and G are $T-$ periodic in φ, with $T = \psi(2\pi)$, and satisfy

$$||F(\rho, \varphi, z) - F(\rho', \varphi, z')|| \leq L(||\rho - \rho'|| + ||z - z'||), \tag{4.19}$$

$$||G(\rho, \varphi, z) - G(\rho', \varphi, z')|| \leq L(||\rho - \rho'|| + ||z - z'||), \tag{4.20}$$

for some Lipschitz constant L.

Following the methods given in [5], one can see that system (4.18) has two integral manifolds whose equations are given by:

$$\begin{aligned} \Phi_0(\varphi, \rho) &= \int_{-\infty}^{\varphi} \pi_0(\varphi, s) G(\rho(s, \varphi, \rho), s, z(s, \varphi, \rho))ds \\ &\quad + \sum_{\varphi_i < \varphi} \pi_0(\varphi, \varphi_i^+) W_i^2(\rho(\varphi_i^+, \varphi, \rho), z(\varphi_i^+, \varphi, \rho)), \end{aligned} \tag{4.21}$$

and

$$\Phi_-(\varphi, z) \;=\; -\int_{\varphi}^{\infty} \pi_-(\varphi, s)F(\rho(s,\varphi,z), s, z(s,\varphi,z))ds + \sum_{\varphi_i<\varphi} \pi_-(\varphi,\varphi_i^+)W_i^1(\rho(\varphi_i^+,\varphi,z), z(\varphi_i^+,\varphi,z)), \tag{4.22}$$

where

$$\pi_0(\varphi, s) = e^{b(\varphi-s)} \prod_{s\le\varphi_j<\varphi} (1 + c_0)$$

and

$$\pi_-(\varphi, s) = e^{\lambda(\varphi-s)} \prod_{s\le\varphi_j<\varphi} (1 + k_j).$$

In (4.21), the pair $(\rho(s,\varphi,\rho), z(s,\varphi,\rho))$ denotes a solution of (4.18) which satisfies $\rho(\varphi,\varphi,\rho) = \rho$. Similarly, $(\rho(s,\varphi,z), z(s,\varphi,z))$, in (4.22), is solution of (4.18) with $z(\varphi,\varphi,z) = z$.

In [5], it was shown that there exist constants K_0, M_0, σ_0 such that Φ_0 satisfies:

$$\Phi_0(\varphi, 0) = 0, \tag{4.23}$$

$$||\Phi_0(\varphi,\rho_1) - \Phi_0(\varphi,\rho_2)|| \le K_0\ell||\rho_1 - \rho_2||, \tag{4.24}$$

for all ρ_1, ρ_2 such that a solution $w(\varphi) = (\rho(\varphi), z(\varphi))$ of (4.18) with the initial condition $w(\varphi_0) = (\rho_0, \Phi_0(\varphi_0,\rho_0)), \rho_0 \ge 0$, is defined on $\mathbb{R}$ and satisfies

$$||w(\varphi)|| \le M_0\rho_0 e^{-\sigma_0(\varphi-\varphi_0)}, \quad \varphi \ge \varphi_0. \tag{4.25}$$

Similarly, it was shown that there exist constants K_-, M_-, σ_- such that Φ_- satisfies:

$$\Phi_-(\varphi, 0) = 0, \tag{4.26}$$

$$||\Phi_-(\varphi, z_1) - \Phi_-(\varphi, z_2)|| \le K_-\ell||z_1 - z_2||, \tag{4.27}$$

for all z_1, z_2 such that a solution $w(\varphi) = (\rho(\varphi), z(\varphi))$ of (4.18) with the initial condition $w(\varphi_0) = (\Phi_-(\varphi_0, z_0), z_0), z_0 \in \mathbb{R}$, is defined on $\mathbb{R}$ and satisfies

$$||w(\varphi)|| \le M_-||z_0||e^{-\sigma_-(\varphi-\varphi_0)}, \quad \varphi \le \varphi_0. \tag{4.28}$$

Set $S_0 = \{(\rho,\varphi,z) : z = \Phi_0(\varphi,\rho)\}$ and $S_- = \{(\rho,\varphi,z) : \rho = \Phi_-(\varphi,z)\}$. Here, S_0 is called the *center manifold* and S_- is called the *stable manifold*. A sketch of an arbitrary center manifold is shown in Figure 4.2.

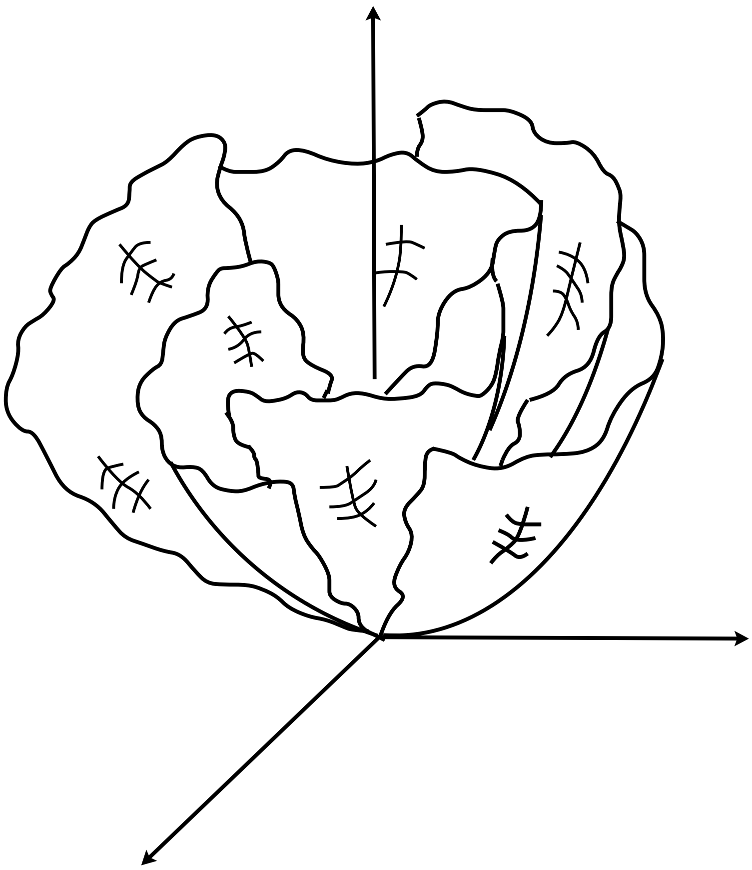

Figure 4.2: A discontinuous center manifold

The analogues of the following two Lemma's together with their proofs can be found in [5].

Lemma 4.4.1 *If the Lipschitz constant ℓ is sufficiently small, then for every solution $w(\varphi) = (\rho(\varphi), z(\varphi))$ of (4.18) there exists a solution $\mu(\varphi) = (u(\varphi), v(\varphi))$ on the center manifold, S_0, such that*

$$\begin{aligned} \|\rho(\varphi) - u(\varphi)\| &\leq 2M_0\|\rho(\varphi_0) - u(\varphi_0)\|e^{-\sigma_0(\varphi-\varphi_0)}, \\ \|z(\varphi) - v(\varphi)\| &\leq M_0\|z(\varphi_0) - v(\varphi_0)\|e^{-\sigma_0(\varphi-\varphi_0)}, \quad \varphi \geq \varphi_0, \end{aligned} \tag{4.29}$$

where M_0 and σ_0 are the constants used in (4.25).

Lemma 4.4.2 *For sufficiently small Lipschitz constant ℓ the surface S_0 is stable in large.*

On the local center manifold S_0, the first coordinate of the solutions of (4.18) satisfies the following system:

$$\begin{aligned} \frac{d\rho}{d\varphi} &= \lambda\rho + F(\rho, \varphi, \Phi_0(\varphi, \rho)), \quad \varphi \neq \varphi_i, \\ \Delta\rho|_{\varphi=\varphi_i} &= k_i\rho + W_i^1(\rho, \Phi_0(\varphi, \rho)). \end{aligned} \tag{4.30}$$

Now, it is time to consider the reduction principle:

Theorem 4.4.3 *Assume that conditions (C10)-(C19) are fulfilled. Then the trivial solution of (4.18) is stable, asymptotically stable or unstable if the trivial solution of (4.30) is stable, asymptotically stable or unstable, respectively.*

Using inverse of $\psi-$substitution and $B-$equivalence, one can see that the following theorem holds:

Theorem 4.4.4 *Assume that conditions (C10)-(C19) are fulfilled. Then the trivial solution of (4.9) is stable, asymptotically stable or unstable if the trivial solution of (4.30) is stable, asymptotically stable or unstable, respectively.*

4.5 Bifurcation of Periodic Solutions

This section is devoted to the bifurcation of a periodic solution for the discontinuous dynamical system. Let us consider the system,

$$\begin{aligned} \frac{dx}{dt} &= Ax + f(x, z) + \mu\tilde{f}(x, z, \mu), \\ \frac{dz}{dt} &= \hat{b}z + g(x, z) + \mu\tilde{g}(x, z, \mu), \quad (x, z) \notin \Gamma(\mu), \\ \Delta x|_{(x,z)\in\Gamma(\mu)} &= B(x, \mu)x, \\ \Delta z|_{(x,z)\in\Gamma(\mu)} &= c(z, \mu)z. \end{aligned} \tag{4.31}$$

Assume that the following conditions are satisfied:

(C18) the set $\Gamma(\mu) = \bigcup_{i=1}^{p} \mathcal{S}_i(\mu)$, where $\mathcal{S}_i(\mu) = s_i(\mu) \times \mathbb{R}$ and the equation of $s_i(\mu)$ is given by $s_i(\mu) : \langle a^i, x\rangle + \tau_i(x) + \mu\nu(x, \mu) = 0$, for $i = 1, \cdots, p$;

(C19) there exists a matrix $Q(\mu) \in \mathbb{R}^{2\times 2}, Q(0) = Q$, analytic in $(-\mu_0, \mu_0)$, and real numbers γ, χ such that $Q^{-1}(\mu)B(x,\mu)Q(\mu) =$

$$(k + \mu\gamma + \kappa(x))\begin{bmatrix} \cos(\theta + \mu\chi + \Theta(x)) & -\sin(\theta + \mu\chi + \Theta(x)) \\ \sin(\theta + \mu\chi + \Theta(x)) & \cos(\theta + \mu\chi + \Theta(x)) \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and $c(z,\mu) = c_0 + \tilde{c}(z) + \mu d(z,\mu)$;

(C20) associated with (4.31) systems

$$\begin{aligned} \frac{dx}{dt} &= Ax, \\ \frac{dz}{dt} &= \hat{b}z, \quad (x,z) \notin \Gamma_0, \\ \Delta x|_{(x,z)\in\Gamma_0} &= B_0 x, \\ \Delta z|_{(x,z)\in\Gamma_0} &= c_0 z. \end{aligned} \tag{4.32}$$

and

$$\begin{aligned} \frac{dx}{dt} &= Ax + f(x,z), \\ \frac{dz}{dt} &= \hat{b}z + g(x,z), \quad (x,z) \notin \Gamma(0), \\ \Delta x|_{(x,z)\in\Gamma(0)} &= B(x,0)x, \\ \Delta z|_{(x,z)\in\Gamma(0)} &= c(z,0)z. \end{aligned} \tag{4.33}$$

are $D_0 - system$ and $D - system$ respectively;

(C21) functions $\tilde{f}$ and ν are analytic in their all arguments;

(C22) $\tilde{f}(0,0,\mu) = 0, \nu(0,\mu) = 0$, uniformly for $\mu \in (-\mu_0, \mu_0)$.

We, first of all, linearize system (4.31) around origin. Note that the eigenvalues of the linearized system are continuously depend on μ, and hence for sufficiently small values of μ, the eigenvalues of the coefficient matrix in the linearized system will be in a similar form with the eigenvalues of the coefficient matrix in (4.1). Thus, by means of a regular transformation, one can show that the right hand side of (4.31) is like the right hand side of (4.9) with the only difference that all coefficients depend on μ. This is why, without loss of any generality, we assume that (4.31) is in linearized form.

Using polar coordinates one can write system (4.31) in the following form:

$$\begin{aligned}\frac{dr}{d\phi} &= \lambda(\mu)r + P(r,\phi,z,\mu),\\ \frac{dz}{d\phi} &= b(\mu)z + Q(r,\phi,z,\mu), \quad (r,\phi,z)\notin\Gamma(\mu),\\ \Delta r|_{(r,\phi)\in\ell_i(\mu)} &= k_i(\mu)r + \omega(r,\phi,\mu),\\ \Delta\phi|_{(r,\phi)\in\ell_i(\mu)} &= \theta_i(\mu) + \gamma(r,\phi,\mu),\\ \Delta z|_{(r,\phi)\in\ell_i(\mu)} &= c_0(\mu)z + \tilde{c}(z,\mu).\end{aligned} \tag{4.34}$$

Let the system

$$\begin{aligned}\frac{d\rho}{d\phi} &= \lambda(\mu)\rho + P(\rho,\phi,z,\mu),\\ \frac{dz}{d\phi} &= b(\mu)z + Q(\rho,\phi,z,\mu), \quad \phi\neq\gamma_i(\mu),\\ \Delta\rho|_{\phi=\gamma_i(\mu)} &= k_i(\mu)\rho + W_i^1(\rho,z,\mu),\\ \Delta\phi|_{\phi=\gamma_i(\mu)} &= \theta_i(\mu),\\ \Delta z|_{\phi=\gamma_i(\mu)} &= c_0(\mu)z + W_i^2(\rho,z,\mu),\end{aligned} \tag{4.35}$$

where $\gamma_i(\mu), i = 1,\cdots,p$, are angles of $m_i(\mu)$, be $B-$equivalent to (4.34). Here, for each i, the line $m_i(\mu)$ is obtained by linearizing $s_i(\mu)$ around the origin. That is, we have $m_i(\mu) : \langle a^i, x\rangle + \mu\frac{\partial v(0,\mu)}{\partial x} = 0$. The functions $W_i^1(\rho,z,\mu)$ and $W_i^2(\rho,z,\mu)$ can be defined in the same manner as in (4.15) and (4.16), respectively. Appyling $\psi-$substitution to (4.35) we get,

$$\begin{aligned}\frac{d\rho}{d\varphi} &= \lambda(\mu)\rho + F(\rho,\varphi,z,\mu),\\ \frac{dz}{d\varphi} &= b(\mu)z + G(\rho,\varphi,z,\mu), \quad \varphi\neq\varphi_i(\mu),\\ \Delta\rho|_{\varphi=\varphi_i(\mu)} &= k_i(\mu)\rho + W_i^1(\rho,z,\mu),\\ \Delta z|_{\varphi=\varphi_i(\mu)} &= c_0(\mu)z + W_i^2(\rho,z,\mu).\end{aligned} \tag{4.36}$$

Following the methods, as we did to obtain (4.21) and (4.22) one can see that system (4.36) has two integral manifolds whose equations are given by:

$$\begin{aligned}\Phi_0(\varphi,\rho,\mu) \;=\; &\int_{-\infty}^{\varphi}\pi_0(\varphi,s,\mu)G(\rho(s,\varphi,\rho,\mu),s,z(s,\varphi,\rho,\mu),\mu)ds\\ &+\sum_{\varphi_i(\mu)<\varphi}\pi_0(\varphi,\varphi_i^+,\mu)W_i^2(\rho(\varphi_i^+,\varphi,\rho,\mu),z(\varphi_i^+,\varphi,\rho,\mu)),\end{aligned} \tag{4.37}$$

and

$$\begin{aligned}\Phi_-(\varphi,z,\mu) \;=\; &-\int_{\varphi}^{\infty}\pi_-(\varphi,s,\mu)F(\rho(s,\varphi,z,\mu),s,z(s,\varphi,z,\mu),\mu)ds\\ &+\sum_{\varphi_i(\mu)<\varphi}\pi_-(\varphi,\varphi_i^+,\mu)W_i^1(\rho(\varphi_i^+,\varphi,z,\mu),z(\varphi_i^+,\varphi,z,\mu)),\end{aligned} \tag{4.38}$$

where

$$\pi_0(\varphi, s, \mu) = e^{b(\varphi - s)} \prod_{s \le \varphi_j(\mu) < \varphi} (1 + c_0(\mu)),$$

and

$$\pi_-(\varphi, s, \mu) = e^{\lambda(\varphi - s)} \prod_{s \le \varphi_j(\mu) < \varphi} (1 + k_j(\mu)).$$

In (4.37), the pair $(\rho(s,\varphi,\rho,\mu), z(s,\varphi,\rho,\mu))$ denotes a solution of (4.36) satisfying $\rho(\varphi,\varphi,\rho,\mu) = \rho$. Similarly, $(\rho(s,\varphi,z,\mu), z(s,\varphi,z,\mu))$, in (4.38), is a solution of (4.36) with $z(\varphi,\varphi,z,\mu) = z$.

Set $S_0(\mu) = \{(\rho,\varphi,z) : z = \Phi_0(\varphi,\rho,\mu)\}$ and $S_-(\mu) = \{(\rho,\varphi,z) : \rho = \Phi_-(\varphi,z,\mu)\}$.

On the local center manifold, $S_0(\mu)$, the first coordinate of the solutions of (4.36) satisfies the following system:

$$\begin{aligned} \frac{d\rho}{d\varphi} &= \lambda(\mu)\rho + F(\rho,\varphi,\Phi_0(\varphi,\rho,\mu)), \quad \varphi \ne \varphi_i(\mu), \\ \Delta\rho|_{\varphi=\varphi_i(\mu)} &= k_i(\mu)\rho + W_i^1(\rho,\Phi_0(\varphi,\rho,\mu)). \end{aligned} \tag{4.39}$$

Similar to (4.7) and (4.8) one can define the functions

$$q_1(\mu) = \exp\left(\lambda(\mu)\left(2\pi - \sum_{i=1}^{p}\theta_i(\mu)\right)\right)\prod_{i=1}^{p}(1 + k_i(\mu)), \tag{4.40}$$

and

$$q_2(\mu) = \exp\left(b(\mu)\left(2\pi - \sum_{i=1}^{p}\theta_i(\mu)\right)\right)\prod_{i=1}^{p}(1 + c_0(\mu)). \tag{4.41}$$

System (4.39) is the system studied in [6] and there it was shown that this system, for sufficiently small μ, has a periodic solution with period $T = \psi(2\pi)$. Here we will show that if the first coordinate of a solution of (4.36) is $T-$ periodic, then so is the second coordinate.

Now, since

$$\begin{aligned} \pi_0(\varphi + T, s + T, \mu) &= \pi_0(\varphi, s, \mu), \\ \rho(s + T, \varphi + T, \rho, \mu) &= \rho(s, \varphi, \rho, \mu), \\ z(s + T, \varphi + T, \rho, \mu) &= z(s, \varphi, \rho, \mu), \end{aligned}$$

and G is $T-$periodic in φ, we have,

$$
\begin{aligned}
& \Phi_0(\varphi + T, \rho, \mu) \\
= & \int_{-\infty}^{\varphi+T} \pi_0(\varphi + T, s, \mu) G(\rho(s, \varphi + T, \rho, \mu), s, z(s, \varphi + T, \rho, \mu), \mu) ds \\
& + \sum_{\varphi_i(\mu) < \varphi + T} \pi_0(\varphi + T, \varphi_i^+, \mu) W_i^2(\rho(\varphi_i^+, \varphi + T, \rho, \mu), z(\varphi_i^+, \varphi + T, \rho, \mu)) \\
= & \int_{-\infty}^{\varphi} \pi_+(\varphi, t, \mu) G(\rho(t, \varphi, \rho, \mu), t, z(t, \varphi, \rho, \mu), \mu) dt \\
& + \sum_{\bar{\varphi}_i(\mu) < \varphi} \pi_+(\varphi, \bar{\varphi}_i^+, \mu) W_i^2(\rho(\bar{\varphi}_i^+, \varphi, \rho, \mu), z(\bar{\varphi}_i^+, \varphi, \rho, \mu)) \\
= & \Phi_0(\varphi, \rho, \mu),
\end{aligned}
$$

where in the second equation we have used the substitutions $s = t + T$ and $\varphi_i(\mu) = \bar{\varphi}_i(\mu) + T$. Now, we have the following theorem which, in case of two dimension, can be found in [6].

Theorem 4.5.1 *Assume that* $q_1(0) = 1, q_1'(0) \neq 0, |q_2(0)| < 1$, *and the origin is a focus for (4.33). Then, for sufficiently small* r_0 *and* z_0, *there exists a function* $\mu = \delta(r_0, z_0)$ *such that the solution* $(r(\phi, \delta(r_0, z_0)), z(\phi, \delta(r_0, z_0)))$ *of (4.34), with the initial condition* $r(0, \delta(r_0, z_0)) = r_0, z(0, \delta(r_0, z_0)) = z_0$, *is periodic with a period,* $T' = \left(2\pi - \sum_{i=1}^{p} \theta_i\right)\beta^{-1} + o(|\mu|)$.

4.6 Examples

Example 4.6.1 *Consider the following dynamical system:*

$$
\begin{aligned}
& x_1' = (0.1 - \mu)x_1 - 20x_2 + 2x_1x_2, \\
& x_2' = 20x_1 + (0.1 - \mu)x_2 + 3x_1^2 z, \\
& z' = (-0.3 + \mu)z + \mu^2 x_1 z, \quad (x_1, x_2, z) \notin \mathcal{S}, \\
& \Delta x_1|_{(x_1,x_2,z)\in\mathcal{S}} = \left((\kappa_1 + \mu^3)\cos(\tfrac{\pi}{3}) - 1\right)x_1 - (\kappa_1 + \mu^3)\sin(\tfrac{\pi}{3})x_2, \\
& \Delta x_2|_{(x_1,x_2,z)\in\mathcal{S}} = (\kappa_1 + \mu^3)\sin(\tfrac{\pi}{3})x_1 + \left((\kappa_1 + \mu^3)\cos(\tfrac{\pi}{3}) - 1\right)x_2, \\
& \Delta z|_{(x_1,x_2,z)\in\mathcal{S}} = (\kappa_2 + \mu - 1)z,
\end{aligned}
\tag{4.42}
$$

where $\kappa_1 = \exp(-\frac{\pi}{120})$, $\kappa_2 = \exp(-\frac{\pi}{400})$, $\mathcal{S} = s \times \mathbb{R}$, *the curve s is given by the equation* $x_2 = x_1^2 + \mu x_1^3$, $x_1 > 0$. *Using (4.40) and (4.41), one can define*

$$q_1(\mu) = (\kappa_1 + \mu^3)\exp((0.1 - \mu)\frac{5\pi}{60}),$$

and

$$q_2(\mu) = (\kappa_2 + \mu)\exp((-0.3 + \mu)\frac{5\pi}{60}).$$

It is easily seen that $q_1(0) = \kappa_1 \exp(\frac{\pi}{120}) = 1$, $q_1'(0) = -\frac{\pi}{12} \neq 0$, *and* $q_2(0) = \exp(-\frac{11\pi}{200}) < 1$. *Therefore, by Theorem 4.5.1, system (4.42) has a periodic solution with period* $\approx \frac{5\pi}{60}$ *if* $|\mu|$ *is sufficiently small.*

Figure 4.3 shows the trajectory of (4.42) with the parameter $\mu = 0.05$ *and the initial value* $(x_{10}, x_{20}, z_0) = (0.02, 0, 0.05)$. *Since there is an asymptotically stable center manifold, no matter which initial condition is taken, the trajectory will get closer and closer to the center manifold as time increases.*

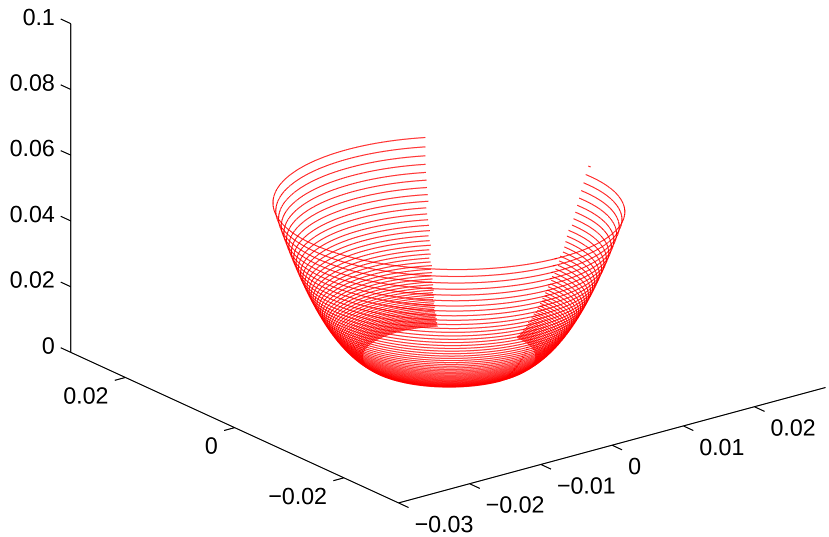

Figure 4.3: A trajectory of (4.42)

In Figure 4.4, the existence of a discontinuous limit cycle is illustrated. There an outer and an inner solution are shown which spiral to a trajectory lying between these two.

Since the exact value of the initial point for the periodic solution is not known we have shown two trajectories of (4.42).

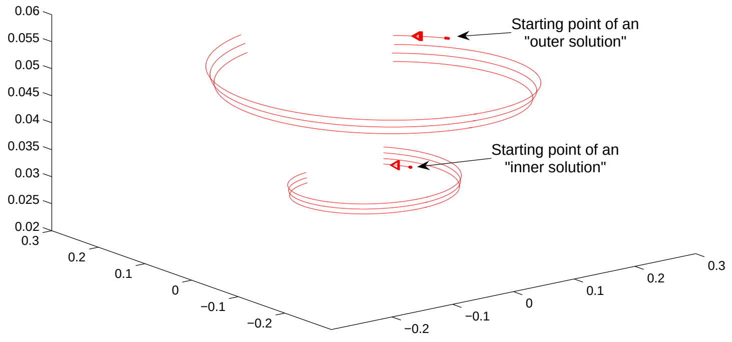

Figure 4.4: There must exist a discontinuous limit cycle of (4.42)

Example 4.6.2 *Consider the following dynamical system:*

$$
\begin{aligned}
x_1' &= (-2+\mu)x_1 - x_2 + \mu z^2,\\
x_2' &= x_1 + (-2+\mu)x_2,\\
z' &= (-1+\mu)z + \mu^2 x_1 z, \quad (x_1,x_2,z) \notin \mathcal{S},\\
\Delta x_1|_{(x_1,x_2,z)\in\mathcal{S}} &= \Big((\kappa_1 - x_1^2 - x_2^2)\cos(\tfrac{\pi}{3}) - 1\Big)x_1 - (\kappa_1 - x_1^2 - x_2^2)\sin(\tfrac{\pi}{3})x_2,\\
\Delta x_2|_{(x_1,x_2,z)\in\mathcal{S}} &= (\kappa_1 - x_1^2 - x_2^2)\sin(\tfrac{\pi}{3})x_1 + \Big((\kappa_1 - x_1^2 - x_2^2)\cos(\tfrac{\pi}{3}) - 1\Big)x_2,\\
\Delta z|_{(x_1,x_2,z)\in\mathcal{S}} &= (\kappa_2 - 1 - z^2)z,
\end{aligned}
\tag{4.43}
$$

where $\kappa_1 = \exp(\frac{10\pi}{3}), \kappa_2 = \exp(\frac{5\pi}{6}), \mathcal{S} = s \times \mathbb{R}, s$ *is a curve given by the equation* $x_2 = x_1 + \mu^2 x_1^3, x_1 > 0$. *Using (4.40) and (4.41), one can define*

$$q_1(\mu) = \kappa_1 \exp((-2+\mu)\frac{5\pi}{3}),$$

and

$$q_2(\mu) = \kappa_2 \exp((-1+\mu)\frac{5\pi}{3}).$$

Now, $q_1(0) = \kappa_1 \exp(-\frac{10\pi}{3}) = 1,\ q_1'(0) = \frac{5\pi}{3} \neq 0,\ q_2(0) = \kappa_2 \exp(\frac{5\pi}{3}) = \exp(-\frac{5\pi}{6})$.

Moreover, associated $D-$system is:

$$\begin{aligned}
&x_1' = -2x_1 - x_2,\\
&x_2' = x_1 - 2x_2,\\
&z' = -z, \quad (x_1, x_2, z) \notin \mathcal{P},\\
&\Delta x_1|_{(x_1,x_2,z)\in\mathcal{P}} = \Big((\kappa_1 - x_1^2 - x_2^2)\cos(\tfrac{\pi}{3}) - 1\Big)\, x_1 - (\kappa_1 - x_1^2 - x_2^2)\sin(\tfrac{\pi}{3})x_2,\\
&\Delta x_2|_{(x_1,x_2,z)\in\mathcal{P}} = (\kappa_1 - x_1^2 - x_2^2)\sin(\tfrac{\pi}{3})x_1 + \Big((\kappa_1 - x_1^2 - x_2^2)\cos(\tfrac{\pi}{3}) - 1\Big)\, x_2,\\
&\Delta z|_{(x_1,x_2,z)\in\mathcal{P}} = (\kappa_2 - 1 - z^2)z,
\end{aligned} \tag{4.44}$$

where $\mathcal{P} = \ell \times \mathbb{R}$, ℓ is given by the equation $x_2 = x_1, x_1 > 0$, and the origin is stable focus. Indeed, using cylindrical coordinates, denote the solution of (4.44) starting at the angle $\phi = \frac{\pi}{4}$ by $(r(\phi, r_0, z_0), z(\phi, r_0, z_0))$.

We obtain

$$r_n = (\kappa_1 - r_{n-1}^2)r_{n-1}\exp(-\frac{10\pi}{3}),$$

and

$$z_n = (\kappa_2 - z_{n-1}^2)z_{n-1}\exp(-\frac{5\pi}{3}),$$

where $r_n = r(\frac{\pi}{4} + 2\pi n, r_0, z_0)$ and $z_n = z(\frac{\pi}{4} + 2\pi n, r_0, z_0)$. It is easily seen that the sequences r_n and z_n are monotonically decreasing for sufficiently small (r_0, z_0), and there exists a limit of (r_n, z_n). Assume that this limit is $(\xi, \eta) \neq (0, 0)$. Then it implies that there exists a periodic solution of (4.44) and $\xi = (\kappa_1 - \xi^2)\xi\exp(-\frac{10\pi}{3})$ and $\eta = (\kappa_2 - \eta^2)\eta\exp(-\frac{5\pi}{3})$ which give us a contradiction. Thus, $(\xi, \eta) = (0, 0)$. Consequently, the origin is a stable focus of (4.44) and by Theorem 4.5.1 the system (4.43) has a limit cycle with period $\approx \frac{5\pi}{3}$ if $|\mu|$ is sufficiently small.

4.7 Deduction

In this chapter, we have studied the existence of a center manifold and the Hopf bifurcation for a certain three dimensional discontinuous dynamical system. The bifurcation of discontinuous cycle is observed by means of the $B-$equivalence method and its consequences. These results will be extended to arbitrary dimension for a more general type of equations.

CHAPTER 5

CONCLUSION

In this thesis, we have introduced two new classes of differential equations: differential equations with transition conditions on time scales and differential equations on variable time scales. It is necessary to introduce these classes of equations for the investigation of the main results: Hopf-Bifurcation in three-dimensional discontinuous dynamical systems.

The first class of the systems that we introduce in this thesis is the DETC. We make a connection between this class of equations and impulsive differential equations. This connection is given by means of a specific transformation of the independent variable called the ψ-substitution [6]. Some benefits of the established connection include knowledge about properties of linear DETC, the investigation of existence of periodic and almost periodic solutions and their stability. We suppose that the problems of stability, oscillations, smoothness of solutions, integral manifolds, theory of functional differential equations can be investigated applying our results. Another interesting opportunity is to analyze equations with more sophisticated time scales.

The second class of the introduced systems is the differential equations on variable time scales. These systems naturally appear when we investigate discontinuous dynamics with non-fixed moments of impulses. Consequently, these results will be needed to develop methods of investigation of mechanical models with impacts. Particularly, interesting problems are related to bifurcations [31, 54, 56, 94], chaos [54]. In this thesis, the theory of this class of equations that we introduce are developed according to these demands.

After introducing these new classes of differential equations, we study the existence of a center manifold and the Hopf bifurcation for a certain three-dimensional discontinuous dynamical system. The bifurcation of discontinuous cycles is observed by means of the B-equivalence method and its consequences [5, 6]. These results will be extended to arbitrary dimension for a more general type of equations.

We expect that these results will be helpful for further investigation of multi dimensional discontinuous dynamical systems. In fact, a study related to a three-dimensional hybrid system has been submitted as an invited paper.

REFERENCES

[1] R. P. Agarwal, M. Benchohra, D. O'Regan and A. Ouahab, *Second order impulsive dynamic equations on time scales*, Funct. Differ. Equ., **11** (2004) 223-234.

[2] R. P. Agarwal, M. Bohner, D. O'Regan and A. Peterson, *Dynamic equations on time scales: a survey. Dynamic equations on time scales*, J. Comput. Appl. Math. 141 (2002) 1-26.

[3] R. P. Agarwal and M. Bohner, *Basic calculus on time scales and some of its applications*, Results Math. **35** (1999) 3-22.

[4] E. Akalın, M. U. Akhmet, *The principles of B-smooth discontinuous flows*, Computers and Mathematics with applications **49** (2005) 981-995.

[5] M. U. Akhmet, *On the reduction principle for differential equations with piecewise constant argument of generalized type*, Journal of Mathematical Analysis and Applications, **336** (2007) 646-663.

[6] M. U. Akhmet, *Perturbations and Hopf bifurcation of the planar discontinuous dynamical system*, Nonlinear Analysis, **60** (2005) 163-178.

[7] M. Akhmet, *On the smoothness of solutions of impulsive autonomous systems*, Nonlinear Analysis **60** (2005) 311-324.

[8] M. U. Akhmet, *On the general problem of stability for impulsive differential equations*, J. Math. Anal. Appl., **288** (2003) 182-196.

[9] M. U. Akhmetov, *On the expansion of solutions to differential equations with discontinuous right-hand side in a series in initial data and parameters*, Ukrainian Math. J., **45** (1993) 786-789.

[10] M. U. Akhmetov, *On the smoothness of solutions of differential equations with a discontinuous right-hand side*, Ukrainian Math. J., **45** (1993) 1785-17892.

[11] M. U. Akhmet and N. A. Perestyuk, *The comparison method for differential equations with impulse action*, Differential Equations, **26** (1990), no. 9, 1079-1086.

[12] M. U. Akhmetov, *On motion with impulse actions on a surface*, Izv.-Acad. Nauk Kaz. SSR, Ser. Fiz.-Mat., **1** (1988) 11-14.

[13] M. U. Akhmetov and N. A. Perestyuk, *Differential properties of solutions and integral surfaces for nonlinear pulse systems*, Differential Equations, **28** (1992) 445-453.

[14] M. U. Akhmetov, N. A. Perestyuk and A. M. Samoilenko, *Almost-periodic solutions of differential equations with impulse action,* Akad. Nauk Ukrain. SSR Inst. **26** (1983) 49. Math. Preprint (in Russian).

[15] M. U. Akhmet and M. Turan, *Bifurcation in a 3D Hybrid system*, Communications in Applied Analysis (in press).

[16] M. U. Akhmet and M. Turan, *Bifurcation of three-dimensional discontinuous cycles*, Nonlinear Analysis: Theory, Methods and Applications (in press).

[17] M. U. Akhmet and M. Turan, *Differential equations on variable time scales*, Nonlinear Analysis: Theory, Methods and Applications **70** (2009) 1175-1192.

[18] M. U. Akhmet and M. Turan, *A new class of differential equations: systems on variable time scales with transition conditions*, Proceedings of the Fifth International Conference on Dynamic Systems and Applications held at Morehouse College, Atlanta, USA, May 30-June 2, 2007. Proceedings of Dynamic Systems and Applications, 5 (2008) 21-27.

[19] M. U. Akhmet and M. Turan, *The differential equations on time scales through impulsive differential equations*, Nonlinear Analysis, **65** (2006) 2043-2060.

[20] D. Anderson, J. Bullock, L. Erbe, A. Peterson and H. Tran, *Nabla dynamic equations on time scales*, Panamer. Math. J. **13** (2003) 1-47.

[21] A. A. Andronov and C. E. Chaikin, *Theory of oscillations*, Princeton Uni. Press, Princeton, 1949.

[22] B. Aulbach and S. Hilger, *A unified approach to continuous and discrete dynamics*, **53**, Qualitative theory of Differential Equations, Szeged, Hungary, 1988.

[23] J. Awrejcewicz and C. H. Lamarque, *Bifurcation and Chaos in Nonsmooth Mechanical Systems*, World Scientific, Singapore, 2003.

[24] D. D. Bainov and P. S. Simeonov, *Impulsive Differential Equations: Asymptotic Properties of the Solutions*, World Scientific, 1995.

[25] D. D. Bainov and P. S. Simeonov, *Impulsive Differential Equations: Periodic Solutions and Applications*, Longman Scientific and Technical, 1993.

[26] G. D. Birkhoff, *Dynamical Systems*, American Mathematical Society, 1927.

[27] C. J. Budd, *Non-smooth dynamical systems and the grazing bifurcation*, Nonlinear Mathematics and its Applications, Cambridge Univ. Press, Cambridge, (1996) 219-235.

[28] M. Bohner and V. Lakshmikantham, *Formulas of Bendixson and Alekseev for difference equations*, Bull. London Math. Soc. **36** (2004) 65-71.

[29] M. Bohner and A. Peterson, *Dynamic equations on time scales. An introduction with applications*, Birkhäuser Boston, Inc., Boston, MA, 2001.

[30] J. Carr, *Applications of Centre Manifold Theory*, Springer-Verlag, New York, 1981.

[31] W. Chin, E. Ott, H. E. Nusse, C. Grebogi, *Grazing bifurcations in impact oscillators*, Physical Review E(3), **50** (1994) no. 6, 4427-4444.

[32] S. N. Chow, J. K. Hale, *Methods of Bifurcation Theory*, Springer, New York, Heidelberg, Berlin, 1982.

[33] B. Coll, A. Gasull and R. Prohens, *Degenerate Hopf bifurcations in discontinuous planar systems*, J. Math. Anal. Appl., **253** (2001) 671-690.

[34] J. DaCunha, *Stability for time varying linear dynamic systems on time scales*, J. Comput. Appl. Math. **176** (2005) 381-410.

[35] S. G. Deo, V. Lakshmikantham, V. Raghavendra, *Textbook of ordinary differential equations*, Tata McGraw-Hill, New Delhi, 1997.

[36] A. Domoshnitsky, M. Drakhlin, E. Litsyn, *Nonoscillation and positivity of solutions to first order state-dependent differential equations with impulses in variable moments*, J. Differential Equations **228** (2006) 39 - 48.

[37] W. P. Eloe, *Sign properties of Green's functions for disconjugate dynamic equations on time scales*, J. Math. Anal. Appl. **287** (2003) 444-454.

[38] L. Erbe and A. Peterson, *Some recent results in linear and nonlinear oscillation*, Dynam. Systems Appl. **13** (2004) 381-395.

[39] A. M. Fink, *Almost periodic differential equations*, Springer-Verlag, Berlin, New York, 1974.

[40] M. Frigon and D. O'Regan, *Impulsive differential equations with variable times*, Nonlinear Analysis **26** (1996) 1913-1922.

[41] A. Gray and A. Lovit, *Oscillation theorems for a self-adjoint dynamic equation on time scales*, Panamer. Math. J. **15** (2005) 49-65.

[42] J. Guckenheimer and P. Holmes, *Nonlinear oscillations, Dynamical systems, and Bifurcations of Vector Fields*, Springer-Verlag, New York, 1983.

[43] A. Halanay and D. Wexler, *Qualitative theory of systems with impulses (Romanian)*, Editura Acad. Rep. Soc. Romania, Bucuresti, 1968.

[44] A. Halanay, D. Wexler, *Qualitative theory of impulsive systems (Romanian)*, Republici Socialiste Romania, Bucuresti (1982).

[45] J. K. Hale, *Ordinary Differential Equations*, Robert E.Krieger Publishing Company, 1980.

[46] J. K. Hale, *Ordinary Differential Equations*, Wiley-Interscience, New York, 1969.

[47] J. K. Hale and H. Koçak, *Dynamics and bifurcations*, Springer-Verlag Press, New York, 1991.

[48] J. K. Hale and J. P. LaSalle, *Differential Equation and Dynamical systems*, Acedemic Press, 1967.

[49] P. Hartman, *Ordinary differential equations*, Wiley, New York, 1964.

[50] B. D. Hassard, N. D. Kazarinoff and Y. H. Wan, *Theory and Applications of Hopf Bifurcations*, Cambridge University Press, Cambridge, 1981.

[51] S. Hilger, *Ein Maßkettenkalkül mit Anvendung auf Zentrums* , PhD thesis, Universität Würzburg, 1988.

[52] S. Hilger, *Analysis on measure chains - a unified approach to continuous and discrete calculus*, Results Math. **18** (1990) 18-56.

[53] M. W. Hirsch and S. Smale, *Differential equations, Dynamical systems, and Linear algebra*, Academic Press, New York and London, 1974.

[54] P. J. Holmes, *The dynamics of repeated impacts with a sinusoidally vibrating table*, Journal of Sound and Vibratiom, **84**(2) (1982) 173-189.

[55] G. Iooss and D. D. Joseph, *Elementary stability and bifurcation theory*, Springer-Verlag, New York, 1990.

[56] A. P. Ivanov, *Impact Oscillations: Linear Theory of Stability and Bifurcations*, Journal of Sound and Vibratiom, **178**(3) (1994) 361-378.

[57] A. N. Kolmogorov, *On the Skorohod convergence (Russian. English summary)*, Teor. Veroyatnost. i Primenen., **1** (1956) 239-247.

[58] N. M. Krylov and N. N. Bogolyubov, *Introduction to nonlinear mechanics*, Acad. Nauk Ukrainy, Kiev, 1937.

[59] Y. A. Kuznetsov, *Elements of Applied Bifurcation Theory*, Applied Mathematical Sciences, Springer-Verlag, New York, 1995.

[60] V. Lakshmikantham, D. D. Bainov, and P. S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific, Singapore, 1989.

[61] V. Lakshmikantham, S. Leela and S. Kaul, *Comparison principle for impulsive differential equations with variable times and Stability theory*, Nonlinear Anal.: TMA, **22** (1994) 499-503.

[62] V. Lakshmikantham and X. Liu, *On quasistability for impulsive differential equations*, Nonlinear analysis **13** (1989) 819-828.

[63] X. Liu and R. Pirapakaran, *Global stability results for impulsive differential equations*, Appl. Anal. **33** (1989) 87-102.

[64] V. Laksmikantham, S. Sivasundaram and B. Kaymakcalan, *Dynamical systems on measure chains*, Math. and its Appl. 370, Kluwer Academic, Dordrecht, (1996).

[65] V. Lakshmikantham and A. S. Vatsala, *Hybrid systems on time scales. Dynamic equations on time scales*, J. Comput. Appl. Math. **141** (2002) 227-235.

[66] R. I. Leine, D. H. Van Campen and B. L. Van de Vrande, *Bifurcations in nonlinear discontinuous systems*, Nonlinear Dynam., **23** (2000) 105-164.

[67] J. E. Marsden and M. McCracken, *The Hopf bifurcation and its applications*, Springer-Verlag, New York, 1976.

[68] J. L. Massera, *The existence of periodic solutions of differential equations*, Duke Math. J., **17** (1950) 457-475.

[69] R. K. Miller and A. Michel, *Ordinary Differential Equations*, Academic Press, New York, 1982.

[70] N. Minorsky, *Nonlinear Oscillations*, Van Nostrand, Princeton, 1962.

[71] A. D. Myshkis and A. M. Samoilenko, *Systems with shocks at given moments*, Matematicheskii Zbornik, **74**(2) (1967) 202-208.

[72] R. F. Nagaev, *Periodic solutions of piecewise continuous systems with a small parameter (in Russian)*, Prikl. Mat. Mech., **36** (1972) 1059-1069.

[73] A. B. Nordmark, *Existence of periodic orbits in grazing bifurcaitons of impacting mechanical oscillators*, Nonlinearity, **14**(6) (2001) 1517-1542.

[74] A. B. Nordmark, *Non-periodic motion caused by grazing incidence in an impact oscillator*, Journal of Sound and Vibration, **145**(2) (1991) 279-297.

[75] H. E. Nusse and J. A. Yorke, *Border-collision bifurcations for piecewise smooth one-dimensional maps*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., **5** (1995) 189-207.

[76] H. E. Nusse, E. Ott and J. A. Yorke, *Border-collision bifurcations: and explanation for observed bifurcation phenomena*, Phys. Rev. E, **49**(2) (1994) 1073-1076.

[77] H. E. Nusse and J. A. Yorke, *Border-collision bifurcations including "period two to period three" for piecewise smooth systems*, Phys. D, **57** (1992) 39-57.

[78] T. Pavlidis, *A new model for simple neural nets and its application in the design of neural oscillator*, Bulletin of Mathematical Biology **27**(2) (1965) 215-229.

[79] T. Pavlidis, *Stability of a class of discontinuous dynamical systems*, Information and Control, **9** (1966) 298-322.

[80] L. Perko, *Differential Equation and Dynamical systems*, Springer-Verlag Press, 1991.

[81] V. A. Pliss, *Integral Sets of Periodic Systems of Differential Equations*, Izdat. Nauka, Moscow, 1977.

[82] V. A. Pliss, *A reduction principle in the theory of stability of motion*, Izv. Akad. Nauk SSSR Ser. Mat., **28** (1964) 1297-1324.

[83] H. Poincaré, *Les Méthodes Nouvelles de la Mécanique Céleste*, Gauthier-Villars, Paris, 1899.

[84] V. F. Rozhko, *Lyapunov stability in discontinuous dynamic systems (in Russian)*, Differential Equations **11** (1975) 761-766.

[85] A. M. Samoilenko and N. A. Perestyuk, *Impulsive Differential Equations*, World Scientifc, Singapore, 1995.

[86] A. M. Samoilenko and N. A. Perestyuk, *Impulsive Differential Equations*, (In Russian) Vischa Shkola, Kiev, 1987.

[87] G. Sansone and R. Conti, *Nonlinear differential equations*, McMillan, New York, (1964).

[88] S. Sivasundaram, *Stability of dynamic systems on the time scales*, Nonlinear Dyn. Syst. Theory **2** (2002) no. 2, 185-202.

[89] A. V. Skorokhod, *Limit theorems for random processes*, Theory Probab. Appl., **39** (1994) 289-319.

[90] S. H. Strogatz, *Nonlinear dynamics and Chaos : with applications to physics, biology, chemistry, and engineering*, Addison-Wesley Pub., Reading, Mass., 1994.

[91] H. R. Sun and W. T. Li, *Positive solutions of second-order half-linear dynamic equations on time scales*, Appl. Math. Comput. **158** (2004) 331-344.

[92] P. Thota and H. Dankowicz, *Continuous and discontinuous grazing bifurcations in impact oscillators*, Phys. D, **214** (2006) 187-197.

[93] D. Wexler, *Solutions périodiques et presque-périodiques des systémes d'équations difféletielles linéaires en distributions*, J. Differential Equations, **2** (1966) 12-32.

[94] G. S. Whiston, *The vibro-impact response of a harmonically excited and preloaded one-dimensional linear oscillator*, Journal of Sound and Vibration, **115**(2) (1987) 303-319.

[95] S. Wiggins, *Introduction to Applied Nonlinear Dynamical Systems and Chaos*, Springer, New York, 2003.

VITA

PERSONAL INFORMATION

Surname, Name:	Turan, Mehmet
Nationality:	Turkish (TC)
Date and Place of Birth:	April 20, 1980, Diyarbakır
Marital Status:	Single
email:	mehmetturan21@gmail.com

EDUCATION

Degree	**Institution**	**Year of Graduation**
Ph.D.	METU, Mathematics	2009
B.S.	METU, Mathematics	2002
High School	Diyarbakır Birlik High School	1997

WORK EXPERIENCE

Year	**Place**	**Enrollment**
2008-present	Atılım University, Mathematics	Instructor
2002-2008	METU, Mathematics Department	Research Assistant

FOREIGN LANGUAGES

English (fluent), German (basic)

AWARDS and SCHOLARSHIPS

- Graduate study award, The Scientific and Technological Research Council of Turkey (TÜBİTAK), 2005-2006.

PUBLICATIONS

- M. U. Akhmet and M. Turan, *Bifurcation in a 3D Hybrid system*, Communications in Applied Analysis (in press).

- M. U. Akhmet, D. Aruğaslan and M. Turan, *Hopf bifurcation for the 3D Filippov system*, Dynamics of Continuous, Discrete and Impulsive Systems, Series A (in press).

- M. U. Akhmet and M. Turan, *Bifurcation of three-dimensional discontinuous cycles*, Nonlinear Analysis: Theory, Methods and Applications (in press).

- M. U. Akhmet and M. Turan, *Differential equations on variable time scales*, Nonlinear Analysis: Theory, Methods and Applications **70** (2009) 1175-1192.

- M. U. Akhmet and M. Turan, *A new class of differential equations: systems on variable time scales with transition conditions*, Proceedings of the Fifth International Conference on Dynamic Systems and Applications held at Morehouse College, Atlanta, USA, May 30-June 2, 2007. Proceedings of Dynamic Systems and Applications, 5 (2008) 21-27.

- M. U. Akhmet and M. Turan, *The differential equations on time scales through impulsive differential equations*, Nonlinear Analysis **65** (2006) 2043-2060.

CONFERENCE PRESENTATIONS

- *Bifurcation of Discontinuous Cycles*, World Congress of Nonlinear Analysts, WCNA - 2008, July 2 - July 9, 2008, Hyatt Grand Cypress Resort in Orlando, Florida, USA (with Marat Akhmet).

- *Differential equations on variable time scales*, III. Ankara Matematik Günleri, Ankara Üniversitesi, Mayıs 25-28, 2008, Ankara (with Marat Akhmet).

- *Differential equations on variable time scales*, Fifth International Conference on Dynamic Systems and Applications, May 30 - June 2, 2007, Morehouse College, Atlanta, Georgia, USA (with Marat Akhmet).

SEMINARS IN APPLIED DYNAMICS GROUP
(Department of Mathematics and Institute of Applied Mathematics, METU)

- November 2008. Title: *Bifurcation of Discontinuous Cycles.*
- May 2006. Title: *A new model of Oscillators with impacts.*
- October 2005. Title: *The differential equations on time scales through impulsive differential equations.*
- March 2005. Title: *Continuous dependence of solutions of Discontinuous Dynamical Systems on initial data.*
- December 2004. Title: *On smoothness of Discontinuous Dynamical Systems.*

PROJECTS

- Development of Theory of Discontinuous Dynamical Systems and its Applications to Mechanics and Electronics, TÜBİTAK Project No: 106T418, November 2006 - November 2008.