NEW CLASSES OF DIFFERENTIAL EQUATIONS AND BIFURCATION OF DISCONTINUOUS CYCLES

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ABSTRACT

NEW CLASSES OF DIFFERENTIAL EQUATIONS AND BIFURCATION OF
DISCONTINUOUS CYCLES

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In this thesis, we introduce two new classes of differential equations, which essentially extend, in several directions, impulsive differential equations and equations on time scales. Basics of the theory for quasilinear systems are discussed, and particular results are obtained so that further investigations of the theory are guaranteed.

Applications of the newly-introduced systems are shown through a center manifold theorem, and further, Hopf bifurcation Theorem is proved for a three-dimensional discontinuous dynamical system.

Keywords: Periodic solution, stability, center manifold, Hopf bifurcation
ÖZ

YENİ TÜR DİFERANSİYEL DENKLEMLER SINİFLARI VE SÜREKSİZ DÖNGÜLERİN ÇATALLANMASI

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Bu tezde iki tür yeni diferansiyel denklem sınıfı tanıttık. Aslında bu denklem sınıfları impulsif diferansiyel denklemlerini ve zaman skalalarında diferansiyel denklemleri çeşitli açılarından genişletirler. Yarı doğrusal denklemlerin temel teorisi tartışılmış ve teorinin daha ileri düzeyde incelenebilmesini garantilemek için belirli sonuçlar elde edilmiştir.

Yeni tanıtılan sistemlerin uygulamaları merkez çok katlı teoremi aracılığıyla gösterilmiş ve bir üç boyutlu süreksiz dinamik sistem için Hopf Çatallama Teoremi kanıtlanmıştır.

Anahtar Kelimeler: Periyodik çözüm, kararlılık, merkez çok katlı, Hopf çatallanması
To my mother, Meliha, and

to the memory of my late father, Kitmir
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CHAPTER 1

INTRODUCTION

Most of the real world processes are studied by means of differential equations. The invention of the theory of ordinary differential equations dates back to the second half of the Seventeenth Century. Newton (1642-1727) was the first person to consider the differential equations. He regarded this observation so important that he used the phrase “...the laws of nature are expressed by differential equations...” to emphasize the importance of his discovery.

A new era in the development of the theory of differential equations starts with Poincaré (1854-1912). Instead of traditional methods, he considered new topological ideas. The Qualitative Theory of Differential Equations - or, as it is known nowadays, the theory of dynamical systems - is the starting point to discuss the nonlinear differential equations. Birkhoff (1884-1944) understood the idea of Poincaré and developed it at the beginning of the Twentieth Century. Russian mathematicians have taken an important role in the development of this subject, beginning with the works of Lyapunov (1857-1918) on the stability of motion, Andronov (1901-1952) on bifurcation theory, Krylov (1879-1955) and Bogolyubov (1909-1992) on the theory of averaging, Kolmogorov (1903-1987) on the theory of perturbations of conditionally periodic motions.

In the last quarter of the last century, there has been an explosion of interest in the study of nonlinear dynamical systems; geometric and qualitative techniques developed during this period makes it possible to better investigate nonlinear dynamical systems. The theory of nonlinear dynamical systems is one of the most developing subjects of the theory of differential equations since it is mostly applied in physics,
chemistry, biology, ecology, economics, mechanics, electrics, and electronics, all of which have yielded valuable results. In fact, the systems which seemed to be hard to grasp from the analytical point of view are now easily understandable from the geometric or qualitative points of view.

The history of discontinuous dynamical systems is relatively short. In [58], the first investigation into the discontinuous dynamical system can be seen. There, the authors considered the model of a clock; a pendulum which experiences a strike when the angle between the current position and the equilibrium position reaches a specific level was taken into account. In that work, it was shown that the approximation method used in nonlinear mechanics can be applied for a study of differential equations with impulse action. This has attracted the attention of scientists from other disciplines since it made it possible to investigate the processes in nonlinear oscillations.

With this accomplished, scientists became interested in Impulsive Differential Equations (IDE’s). IDE’s characterize many real life evolutionary processes whose state experiences a change called the ‘impulse’. Impulses are short-term perturbations of the process. When the changes occur at the specified times, we talk about the IDE’s with fixed moments of time. Generally, this is not the case and, at times, the impulse actions take place depending on the state. These kinds of systems are called IDE’s with variable moments of impulse actions. For long, scientists have considered merely those IDE’s with fixed moments of impulse actions and stayed away from those with variable moments of impulse actions; this is due to the fact that they did not have enough material to handle these problems with. The truth is that, these problems were not so easy to overcome. Many results concerning the IDE’s with fixed moments of impulses have been provided in different references [60, 85]. These also contain some results about the IDE’s with variable moments of impulses. Once B-topology was introduced by Akhmetov and Perestyuk [5, 6, 13], handling IDE’s with variable moments of impulses became easier compared to the past. The method which is based on the B-topology enables us to deal with these kinds of systems. This method is our main tool in the investigation of the system considered in this study.

The center manifold theory is another main tool used in this thesis. In the literature and among the first efforts regarding the subject of center manifold theory, one can
see the paper by Pliss [82]. Also, the book by Carr [30] provides us with useful information about the applications of center manifold. However, in neither one of these works can one find the center manifold and its applications related to discontinuous dynamical systems.

The main point that we are going to utilize from regarding the center manifold theory is to apply it to discontinuous dynamical systems to prove the existence of a periodic solution in multi-dimensional discontinuous dynamical systems. In fact, this will be a discontinuous limit cycle; that is, we shall prove the Hopf Bifurcation Theorem in three-dimensional discontinuous dynamical systems.

While dealing with the Hopf Bifurcation Theorem, naturally a new class of differential equations arises (which we abbreviate as the differential equations on time scales with transition condition (DETC)). The concept of the time scale was first introduced by Hilger [51]. In his work, the author tends to unify and extend the differential and discrete equations. The DETC introduced here and the one proposed by Hilger, have both similarities and differences.

1.1 Elements of Impulsive Differential Equations

Many evolutionary processes are subject to short-term perturbation whose duration is negligible when compared to that of the whole process. This perturbation results in a change in the state of the process. For example, when a bouncing ball strikes against a fixed surface, then a change in the velocity of the ball occurs. Another example is the pendulum of a clock showing a change in momentum when passing through its equilibrium position. Models like these have played a significant role in the development of impulsive differential equations. In [24, 25, 85, 86], many theoretical results are given for impulsive differential equations such as the existence and uniqueness of solutions, stability, periodic solutions.

In principle, there are two different kinds of impulsive differential equations: the ones with fixed moments of impulse actions and those with variable moments of impulse
actions. The former is a system of the form

\[
\frac{dx}{dt} = f(t, x), \quad t \neq \theta_i, \\
\Delta x|_{t=\theta_i} = I_i(x),
\]

(1.1)

which is called an ‘IDE with fixed moments of pulse actions’. In (1.1), \(x \in \mathbb{R}^n\) is the state (phase) variable. The sequence \(\{\theta_i\}\), where \(i\) is an index belonging to a finite or infinite index set as a subset of \(\mathbb{Z}\), denotes the fixed moments at which the impulse actions take place. The right-hand side function \(f(t, x)\) is the continuous rate of change of the phase variable, and \(I_i(x)\) is the discrete (sudden) change of the phase variable. Moreover, \(\Delta x|_{t=\theta_i} = x(\theta_i^+) - x(\theta_i)\) denotes the jump in the phase point at the time \(t = \theta_i\). That is, a phase point of (1.1) moves along one of the trajectories of \(x' = f(t, x)\) until the time \(t = \theta_i\). At the moment \(t = \theta_i\), the phase point jumps to the point \(x(\theta_i^+) = x(\theta_i) + I_i(x(\theta_i))\), and continues along a trajectory of \(x' = f(t, x)\) until the next moment of impulse action, and so on. Therefore, a solution, \(x(t)\), of (1.1) is a piecewise continuous function with discontinuities of the first kind at \(t = \theta_i\).

In the latter one, however, impulse action takes place when the phase point meets one of the prescribed surfaces in the phase space. These kinds of systems are more challenging to investigate when compared to the first category since different solutions possess different moments of impulses. Nevertheless, they arise more naturally than the first kind. An impulsive differential equation with variable (or non-fixed) moments of impulses is a system of the following form

\[
\frac{dx}{dt} = f(t, x), \quad t \neq \tau_i(x), \\
\Delta x|_{t=\tau_i(x)} = I_i(x),
\]

(1.2)

where \(x, f(t, x)\) and \(I_i(x)\) have been described before, and for each \(i\), \(\tau_i(x)\) stands for the surface of discontinuity. As it can be seen easily in (1.2), the moments when the impulse actions take place depend on the phase point, \(x(t)\) and, hence, each solution will perform the jumps at different times. For this reason, system (1.2) is more difficult than system (1.1) to investigate.

The systems in (1.1) and (1.2) are both non-autonomous. There exists another important class of differential equations which is autonomous, also known as \textit{Discontinuous}
Dynamical Systems (DDS’s). A discontinuous dynamical system can be expressed as

\[
\frac{dx}{dt} = f(x), \quad x \not\in \Gamma, \\
\Delta x|_{x \in \Gamma} = I(x),
\]

where \(\Gamma\) denotes the set of discontinuity. A phase point of (1.3) moves along one of the trajectories of the autonomous differential equation \(x' = f(x)\) until the time when this solution, say \(x(t)\), meets the set \(\Gamma\). After this meeting, the phase point is mapped to the point \(x + I(x)\), if \(x\) is the phase point just before the meeting, and continues its motion along the trajectory of \(x' = f(x)\) with the initial point at \(x + I(x)\), and so on.

It is clear that the discontinuities of a solution of (1.3) also depend on the solution, like in (1.2). This is one of the reasons why the theory of systems (1.2) and (1.3) have not been addressed adequately until now. However, they have started to be noticed by many scientists since they have a wide range of applications. In [6, 13], a method has been introduced and developed to handle these systems. In the present study, we intend to use these methods as well.

Here, our main system will be of the form (1.3). Needless to mention that while studying this system, a new type of differential equation came up, which we call as differential equations on variable time scales (DETCV). In the next section, we shall provide the conventional differential equations on time scales as well as the DETC (the ones that we introduced to deal with the DETCV).

### 1.2 An Overview of the Differential Equations on Time Scales

Some dynamic processes have been modeled by difference equations or differential equations. As far as the modeling is concerned, the idea to involve both continuous and discrete times to model a process is more realistic. For this reason, except for impulsive differential equations, there exist another class of systems called dynamic systems on time scales or measure chains [64]. The notion of time scales was introduced by Aulbach and Hilger back in the 80’s [22, 51, 52]: the idea there was to unify the discrete and continuous dynamics. Recently, many results in the theory of discrete dynamics have been obtained as discrete analogs of the corresponding results of continuous dynamics. However, in the discrete case, there are some topo-
logical deficiencies, including lack of connectedness. Some assumptions have been made to overcome these topological deficiencies. For the same purpose, we will be using a special kind of time scale.

Any nonempty closed subset of $\mathbb{R}$ is called a ‘time scale’, generally denoted by $\mathbb{T}$. A differential equation of the form

$$x^\Delta(t) = f(t,x), \quad t \in \mathbb{T},$$

where $x^\Delta(t)$ denotes the $\Delta$-derivative of $x$ at the point $t \in \mathbb{T}$, is called a ‘dynamic equation on the time scale $\mathbb{T}$’.

The theory of dynamic equations on time scales (DETS) has been developed in the last couple of decades [2, 29, 64]. After a literature survey about DETS, one can conclude that there are not as many theoretical results on the existence of periodic solutions and almost periodic solutions. To this date, the investigations concerning linear DETS, integral manifolds, and the stability of equations have not been developed in full. It goes without saying that, such results need to be obtained so as to able us to benefit from the applications of the theory. We also propose a method to obtain such theoretical results, and to investigate differential equations on certain time scales with transition conditions (DETC) which are, in a way, more general than DETS.

Here, effort is made to expand our knowledge of these aspects of the theory, and to introduce a new class of differential equations on time scales. In fact, this class of equations arises naturally when we solve the problem of Hopf bifurcation, which is our main goal in this study.

The time scale that we consider in this thesis is of the form

$$\mathbb{T}_c = \bigcup_{i \in \mathbb{Z}} [t_{2i-1}, t_{2i}],$$

where $t_n, n \in \mathbb{Z}$, is a strictly increasing sequence such that $t_n \to \pm \infty$ as $n \to \pm \infty$. On a time scale as in (1.5), the differential equation with transition conditions (DETC) is defined as a system of the form

$$y' = f(t, y), \quad t \in \mathbb{T}_c,$$

$$y(t_{2i+1}) = y(t_{2i}) + J_i(y(t_{2i})),$$

where $f : \mathbb{T}_c \times \mathbb{R}^n \to \mathbb{R}^n$, and $J_i : \mathbb{R}^n \to \mathbb{R}^n$ are continuous functions in their domains.
At the same time, we should recognize that significant theoretical results have been achieved concerning oscillations, boundary value problems, positive solutions, hybrid systems, etc. \cite{1, 2, 20, 28, 29, 34, 37, 38, 41, 64, 65, 91}. We assume that our proposals may initiate new ideas by which the theory can also be developed, thus adding to the previous significant achievements in that direction.

The DETC will be discussed in the next chapter. The main idea in the investigation of DETC is to apply the results of the theory of impulsive differential equations (IDE’s), the investigation of which started in the late sixties of the last century \cite{43, 60, 85}. We note that certain classes of DETC, concerned with time scales, can be reduced to IDE if we apply a special transformation \cite{6} of the independent argument - the time variable. This transformation allows the reduced IDE to inherit all similar properties of the corresponding DETC. Then, the investigation of the IDE can proceed using the existing results. Finally, by taking into account the properties of the independent argument transformation, we can have an interpretation of the obtained results for the DETC. The approach we are using to connect the DETC with another type of differential equations is close to that in paper \cite{65}, where hybrid systems on time scales have been considered. Besides the DETC, in this study, we introduce the non-linearity on time scales and consider, as we call the variable time scales.

1.3 Basics of Center Manifold and Hopf Bifurcation

Roughly speaking, a bifurcation is a qualitative change in an attractor’s structure as a control parameter is varied smoothly. For example, a simple equilibrium or fixed point attractor might give way to a periodic oscillation as the stress on a system increases. Similarly, a periodic attractor might become unstable and be replaced by a chaotic attractor.

The bifurcation theory is the mathematical study of changes in the qualitative or topological structure of a given family. Examples of such families are the integral curves of a family of vector fields or the solutions of a family of differential equations. Most commonly applied to the mathematical study of dynamical systems, a bifurcation occurs when a small smooth change made to the parameter values (the bifurcation
parameters) of a system causes a sudden “qualitative” or topological change in its behavior. Bifurcations occur in both continuous systems (described by ODE’s, DDE’s or PDE’s) and discrete systems (described by maps).

At times, bifurcations are divided into two principle classes. The first one is local bifurcations, which can be analyzed entirely through changes in the local stability properties of equilibria, periodic orbits or other invariant sets as parameters cross through critical thresholds. The second one is global bifurcations, which often occur when larger invariant sets of the system “collide” with each other, or with the equilibria of the system; these cannot be detected purely by a local stability analysis of the equilibria (fixed points).

A local bifurcation occurs when a parameter change causes the stability of an equilibrium (or fixed point) to change. In continuous systems, this corresponds to the real part of an eigenvalue of an equilibrium passing through zero. In discrete systems (those described by maps rather than ODE’s), this corresponds to a fixed point having a Floquet multiplier with modulus equal to one. In both cases, the equilibrium is non-hyperbolic at the bifurcation point. The topological changes in the phase portrait of the system can be confined to arbitrarily small neighborhoods of the bifurcating fixed points by moving the bifurcation parameter close to the bifurcation point (hence, ‘local’).

Global bifurcations occur when ‘larger’ invariant sets, such as periodic orbits, collide with the equilibria. This causes changes in the topology of the trajectories in the phase space which cannot be confined to a small neighborhood, as is the case with local bifurcations. In fact, the changes in topology extend out to an arbitrarily large distance (hence, ‘global’).

Examples of global bifurcations include the following:

- Homoclinic bifurcation, in which a limit cycle collides with a saddle point;
- Heteroclinic bifurcation, in which a limit cycle collides with two or more saddle points;
- Infinite-period bifurcation, in which a stable node and saddle point simultane-
ously occur on a limit cycle; and

- Blue sky catastrophe, in which a limit cycle collides with a non-hyperbolic cycle.

It deserves mentioning that global bifurcations can also involve more complicated sets such as chaotic attractors.

Named after Eberhard Hopf and Aleksandr Andronov, a Hopf or Andronov-Hopf bifurcation, is a local bifurcation. Here, a fixed point of a dynamical system loses stability as a pair of complex conjugate eigenvalues of the linearization around the fixed point cross the imaginary axis of the complex plane. Under reasonably generic assumptions about the dynamical system, we can expect to see a small amplitude limit cycle branching from the fixed point. This bifurcation was studied by Poincaré who, in his work on the gravitational three-body problem, obtained certain periodic solutions. Later, though, Andronov and Hopf provided a more explicit discussion on that issue.

One of the main methods of simplifying dynamical systems is to reduce the dimension of the system. The center manifold theory is a rigorous mathematical technique that makes this reduction possible, at least near the equilibria. Due to the power of this theory in investigating systems, it became very popular, and attracted many scientists.

The history of center manifolds is very short, going back to 1960’s. The ideas for center manifolds in finite dimensions have been developed by Kelly (1967), Carr (1981), Guckenheimer and Holmes (1983), Vanderbauwhede (1989), and others. For recent developments in the approximation of center manifolds, see Jolly and Rosa (2005). Pages 1-5 of the book by Li and Wiggins (1997) provide an extensive list of the applications of center manifold theory to infinite dimensional problems. Mielke (1996) developed center manifold theory for elliptic partial differential equations, and applied the theory to elasticity and hydrodynamical problems. Haken (2004), in turn, investigated the applications to phase transitions in biological, chemical and physical systems.

When a system loses stability, the number of eigenvalues and eigenvectors associated with this change is typically small. Hence, bifurcation problems usually involve sys-
tems where the linearization has a very large - and possibly infinite - dimensional stable part and a small number of “critical” modes which change from stable to unstable as the bifurcation parameter exceeds a threshold. The central idea of the bifurcation theory is that the dynamics of the system near the onset of instability is governed by the evolution of these critical modes, while the stable modes follow in a passive fashion and become ‘enslaved’. The center manifold theorem is the rigorous formulation of this idea; it allows us to reduce a large problem to a small and manageable one. Therefore, after the reduction on the center manifold, it becomes easier to investigate the system since - in a local neighborhood of the fixed point - the quantitative behavior of the reduced system is the same as that of the whole system.

In this study, we shall also utilize the center manifold theory to investigate the Hopf bifurcation in a three-dimensional discontinuous dynamical system.

1.4 Description of B-equivalence Method

A challenge in investigating systems with discontinuities on nonlinear surfaces is that each solution has different moments of impulse effects, or discontinuities. In the literature surrounding the object, many results can be found related to linear surfaces of impulse actions [60, 63, 71, 84, 85, 86]. However, rarely can one see the works containing nonlinearities on the surfaces. Although they are more realistic for real world applications, many authors tend to avoid these systems due to this difficulty. In [6, 11], the authors have introduced a new method to handle this difficulty. There, the so-called B-equivalence and B-topology have been proposed. This method is a powerful tool to deal with the variable moments of impulse actions. Here, we shall apply the method proposed for impulsive differential equations with variable moments of impulses and, in Chapter 3, we will adopt this method to the differential equations on variable time scales. Subsequently in Chapter 4, this method will be applied to a system in three dimension and the Hopf bifurcation theorem will be proved.
Consider the systems
\[
x' = f(t, x), \quad t \neq \tau_i(x),
\]
\[
\Delta x|_{t=\tau_i(x)} = I_i(x),
\]
and
\[
y' = f(t, y), \quad t \neq \theta_i,
\]
\[
\Delta y|_{t=\theta_i} = J_i(y),
\]
where the hyper-surfaces \( t = \tau_i(x) \) are small perturbations of the hyper-planes \( t = \theta_i \), and the functions \( J_i \) to be supplied in the thesis. Akhmet and Perestyuk [6, 11] have shown that, corresponding to each solution \( x(t, t_0, x_0) \) of (1.7) satisfying \( x(t_0, t_0, x_0) = x_0 \), there exists a solution \( y(t, t_0, x_0) \) of (1.8) satisfying \( y(t_0, t_0, x_0) = x_0 \) such that these two solutions are the same for all \( t \) in their common domains except for the \( \epsilon \)-neighborhoods of the discontinuity points, and vice versa. In fact, a discontinuity point of the solution of one system lies in an \( \epsilon \)-neighborhood of the corresponding discontinuity point of the solution of the other system. In this study, we have adopted this important technique to our system and, by means of this compelling material, we have successfully obtained the required results.

1.5 A Transformation of the Independent Variable: \( \psi \)-substitution

It is common to simplify a given equation by a proper transformation in the theory of differential equations. Likewise in this study, we use a transformation introduced by Akhmet in [6] and developed in [19]. This is a transformation of the independent variable and serves as a bridge in the passage from DETC, as in (1.6), to an IDE.

For a time scale \( \mathbb{T}_c \) as in (1.5), on the set \( \mathbb{T}_c' = \mathbb{T}_c \setminus \bigcup_{i=-\infty}^{\infty} \{ t_{2i-1} \} \), the \( \psi \)-substitution is defined as

\[
\psi(t) = \begin{cases} 
  t - \sum_{0 \leq \delta_k < t} \delta_k, & t \geq 0 \\
  t + \sum_{t \leq \delta_k < 0} \delta_k, & t < 0
\end{cases}
\]

(1.9)

where \( \delta_k = t_{2k+1} - t_{2k} \). Notice that the \( \psi \)-substitution is a one-to-one map, \( \psi(0) = 0 \), and the structure of the sequence \( \{ t_n \} \) implies that \( \psi \) maps \( \mathbb{T}_c' \) onto \( \mathbb{R} \). The inverse
transformation is

$$\psi^{-1}(s) = \begin{cases} 
  s + \sum_{0 \leq s_k < s} \delta_k, & s \geq 0 \\
  s - \sum_{s_k \leq s < 0} \delta_k, & s < 0
\end{cases} \quad (1.10)$$

Note that the inverse transformation is a piecewise continuous function with discontinuity of the first kind at the points $s = s_i = \psi(t_{2i}), i \in \mathbb{Z}$ and $\psi^{-1}(s_i+) - \psi^{-1}(s_i) = \delta_i$.

The aim of the $\psi$-substitution is to make the domain of the system (1.6) a connected domain. Besides, it carries the significant properties of the function it is applied to. For example, if $\phi(s)$ is a periodic function on $\mathbb{R}$, then $\phi(\psi(t))$ is a periodic function on $\mathbb{T}_c'$, and vice versa. A number of properties of the $\psi$-substitution will be given throughout the thesis when necessary.

### 1.6 Motivation for the Main Study

For a motivation, let us consider how the idea of variable time scales emerged before we begin with the main part of the thesis. The following planar system was considered in [6]

$$\frac{dx}{dt} = Ax + f(x), \quad x \notin \Gamma,$$

$$\Delta x|_{x \in \Gamma} = B(x)x, \quad (1.11)$$

where $\Gamma = \cup_{i=1}^\ell \ell_i$ is a set of curves starting at the origin. Using polar coordinates, the system is written in the form:

$$\frac{dr}{d\phi} = \lambda r + P(r, \phi), \quad (r, \phi) \notin \Gamma,$$

$$\phi^+|_{(r,\phi) \in \ell_i} = \phi + \theta_i + \gamma(r, \phi),$$

$$r^+|_{(r,\phi) \in \ell_i} = (1 + k_i)r + \omega(r, \phi). \quad (1.12)$$

Denote by $\ell_i'$ the image of $\ell_i$ under the transition operator $\Pi_i(\phi, r)$ where $\Pi_i^1(\phi, r) = \phi + \theta_i + \gamma(r, \phi)$, and $\Pi_i^2(\phi, r) = (1 + k_i)r + \omega(r, \phi)$. Let $\mathcal{D}_i$ be the set bounded by $\ell_i'$ and $\ell_{i+1}$. In [6], it is shown that this set is non-empty, and $\ell_i'$ is between $\ell_i$ and $\ell_{i+1}$ if the equation is considered in a small neighborhood of the origin.
Denoting $\mathbb{T}(r) = \bigcup_{i=1}^{P} \mathcal{D}_i$, we have the following DETCV:

$$
\frac{dr}{d\phi} = \lambda r + P(r, \phi), \quad (\phi, r) \in \mathbb{T}(r),
$$
$$
\phi^+ = \Pi_1^i(\phi, r), \quad r^+ = \Pi_2^i(\phi, r), \quad (\phi, r) \in \ell_i.
$$

(1.13)

This equation is an example of a differential equation on a variable time scale. In this study, we shall consider a generalization of this equation and prove the Hopf Bifurcation Theorem for our system.

1.7 Organization of the Thesis

This dissertation has been arranged in the following way:

In Chapter 2, we introduce the differential equation with transition conditions on time scales (DETC) and investigate it on the basis of reduction to the impulsive differential equations. We give the basic definitions on time scales and consider the basic properties of linear systems, the existence and stability of periodic solutions.

Chapter 3 is devoted to differential equations on variable time scales (DETCV), and contains the definition of a variable time scale, existence and uniqueness theorem for DETCV, the method used to investigate the DETCV, existence of periodic solutions, stability of solutions and finally bounded solutions. The results given in that chapter will be used in our main study.

In Chapter 4, we consider the Hopf Bifurcation Theorem where we illustrate the bifurcation of three-dimensional discontinuous cycles. Also proved in this chapter is the existence of a center manifold. To demonstrate the work throughout the thesis, each chapter contains a number of examples.

Finally, the last chapter is devoted to a conclusion.
CHAPTER 2

DIFFERENTIAL EQUATIONS WITH TRANSITION CONDITION ON TIME SCALES

In this chapter we investigate differential equations on certain time scales with transition conditions (DETC) on the basis of a reduction to the impulsive differential equations (IDE). DETC are in some sense more general than dynamic equations on time scales [29, 64]. Basic properties of linear systems, existence and stability of periodic solutions are considered. Appropriate examples are given to illustrate the theory.

2.1 Introduction

The theory of dynamic equations on time scales (DETS) has been developed in the last several decades [2, 29, 64]. After a literature survey about DETS, one can conclude that there are not so many results of the theory on the existence of periodic solutions. Up to this moment, the investigations concerning linear DETS, integral manifolds and the stability of equations have not been fully developed. Certainly, these results should be obtained in order to benefit from the applications of the theory. In this chapter, we make an attempt to expand our knowledge of these aspects of the theory. We also propose a way to obtain these theoretical results. Moreover, we investigate differential equations on certain time scales with transition conditions (DETC), which are in some sense more general than DETS. At the same time, we should recognize that significant theoretical results concerning oscillations, boundary value problems, positive solutions, hybrid systems etc., have been achieved.
We assume that our proposals may stimulate new ideas by which the theory can also be developed adding to the previous significant achievements in that direction. The main idea of the chapter is to apply the results of the theory of impulsive differential equations (IDE) the investigation of which started in the last century in the late 1960s [6, 11, 13, 43, 60, 85]. We note that certain classes of DETC, particular with their time scales, can be reduced to IDE, if we apply a special transformation [6] of the independent argument (the time variable). This transformation allows the reduced IDE to inherit all similar properties of the corresponding DETC. Then the investigation of the IDE can proceed using the known results. Finally, by taking the properties of the independent argument transformation into account, we can make an interpretation of the obtained results for DETC. The approach we are using to connect the DETC with another type of differential equations is close to that in the paper [65], where hybrid systems on time scales were considered.

This chapter is organized as follows. In the next section the time scale with its specific properties is considered. Moreover, the general form of DETC is described. The special transformation is given in Section 2.3. Reduction of DETC to IDE is done in Section 2.4. In Section 2.5, periodic solutions of linear equations and elements of Floquet’s theory are considered also Massera theorem is proved. The last section of this chapter is devoted to the problem of existence and stability of almost periodic solutions.

### 2.2 Description of the Differential Equations with Transition Condition on Time Scales

Throughout this chapter we consider a specific time scale of the following type. Fix a sequence \( \{t_i\} \in \mathbb{R} \) such that \( t_i < t_{i+1} \) for all \( i \in \mathbb{Z} \), and \( |t_i| \to \infty \) as \( |i| \to \infty \). Denote \( \delta_i = t_{2i+1} - t_{2i}, \kappa_i = t_{2i} - t_{2i-1} \) and assume that:

\[(C0) \quad \sum_{i=-\infty}^{n} \kappa_i = \infty, \quad \sum_{i=m}^{\infty} \kappa_i = \infty, \quad \text{for any } n, m \in \mathbb{Z}.\]

The time scale \( \mathbb{T}_c = \bigcup_{i=-\infty}^{\infty} [t_{2i-1}, t_{2i}] \), is going to be considered throughout this study.
Consider the following system of differential equations

\[
\frac{dy}{dt} = f(t, y), \quad t \in \mathbb{T}_c, \quad (2.1)
\]

\[
y(t_{2i+1}) = J_i(y(t_{2i}))+ y(t_{2i}),
\]

where the derivative is one sided at the boundary points of \(\mathbb{T}_c\), \(f : \mathbb{T}_c \times \mathbb{R}^n \rightarrow \mathbb{R}^n\), \(J_i : \mathbb{R}^n \rightarrow \mathbb{R}^n\), for all \(i \in \mathbb{Z}\). We assume that functions \(f\) and \(J_i\) are continuous on their respective domains. Let us introduce the following transition operator, \(\Pi_i : \{t_{2i}\} \times \mathbb{R}^n \rightarrow \{t_{2i+1}\} \times \mathbb{R}^n\), \(i \in \mathbb{Z}\), such that \(\Pi_i(t_{2i}, y) = (t_{2i+1}, J_i(y)+y)\). Thus the evolution of the process is described by:

1. the system of differential equations
\[
\frac{dy}{dt} = f(t, y), \quad t \in \mathbb{T}_c; \quad (2.2)
\]

2. the transition operator \(\Pi_i, i \in \mathbb{Z}\);

3. the set \(\mathbb{T}_c \times \mathbb{R}^n\).

We shall call equation (2.1) the differential equation on time scales with transition condition (DETC). Let us show how to construct a solution of (2.1). Denote, by \(\phi(t, \kappa, z)\), a solution of system (2.2) with an initial condition \(y(\kappa) = z, \kappa \in \mathbb{T}_c, z \in \mathbb{R}^n\), and, by \(y(t)\), a solution of system (2.1) with an initial condition \(y(t^0) = y_0\). Fix \(t^0 \in \mathbb{T}_c\) such that \(t_{2k-1} < t^0 < t_{2k}\) for some \(k \in \mathbb{Z}\). If \(t^0 \leq t < t_{2k}\) the solution is equal to \(\phi(t, t^0, y_0)\), and \(y(t_{2k}) = \phi(t_{2k-1}, t^0, y_0)\), where the left limit is assumed to exist. Now, applying the transition operator, we obtain that \(y(t_{2k+1}) = J_k(y(t_{2k}))+ y(t_{2k})\). Note that the solution is not defined in the interval \((t_{2k}, t_{2k+1})\). Next, on the interval \([t_{2k+1}, t_{2k+1})\) the solution is equal to \(\phi(t, t_{2k+1}, y(t_{2k+1}))\), and \(y(t_{2k+1}) = \phi(t_{2k+1}, t^0, y(t_{2k+1}))\), and so on. If solution \(y(t)\) is defined on a set \(I \subset \mathbb{T}_c\), then the set \(\{(t, y) : y = y(t), t \in I\}\) is called an integral curve of the solution.

Let us start with the general information about differential equations on time scales. We provide only those facts of the theory which directly concern our needs in this chapter. More detailed description on the subject can be found in [2, 29, 64].

Any nonempty closed subset, \(\mathbb{T}\), of \(\mathbb{R}\) is called a time scale. For instance, \(\mathbb{R}\) (real numbers), \(\mathbb{Z}\) (integers), \(\mathbb{N}\) (natural numbers) and \(\{1/n : n \in \mathbb{N}\} \cup \{0\}\) are examples of

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time scales while \( \mathbb{Q} \) (rational numbers), \( \mathbb{R} \setminus \mathbb{Q} \) (irrational numbers) and \((0, 1)\) are not time scales [29, 64].

On a time scale \( T \), the functions \( \sigma(t) := \inf\{s \in T : s > t\} \) and \( \rho(t) := \sup\{s \in T : s < t\} \) are called the forward and backward jump operators, respectively. In case when any of these sets is empty, that is, if \( T \) is bounded above (below), this definition is supplemented by \( \sigma(\max T) = \max T \) (\( \rho(\min T) = \min T \)). The point \( t \in T \) is called right-scattered if \( \sigma(t) > t \), and right-dense if \( \sigma(t) = t \). Similarly, it is called left-scattered if \( \rho(t) < t \), and left-dense if \( \rho(t) = t \). Note that on time scale \( T_c \), the points \( t_{2i-1}, i \in \mathbb{Z} \), are left-scattered and right-dense, and the points \( t_{2i}, i \in \mathbb{Z} \), are right-scattered and left-dense. Moreover, it is worth mentioning that \( \sigma(t_2) = t_{2i+1}, \rho(t_{2i+1}) = t_{2i}, i \in \mathbb{Z} \), and \( \sigma(t) = \rho(t) = t \) for any other \( t \in T_c \).

The \( \Delta \)-derivative of a continuous function \( f \), at a right-scattered point is defined as

\[
f^\Delta(t) := \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t},
\]

and at a right-dense point it is defined as

\[
f^\Delta(t) := \lim_{s \to t} \frac{f(t) - f(s)}{t - s},
\]

if the limit exists.

Let \( T \) be an arbitrary time scale. A function \( \varphi : T \to \mathbb{R} \) is called rd-continuous if:

(i) it is continuous at each right-dense or maximal \( t \in T \);

(ii) the left sided limit \( \varphi(t-) = \lim_{\xi \to t-} \varphi(\xi) \) exists at each left-dense \( t \).

Similarly, a function \( \varphi : T \to \mathbb{R} \) is called ld-continuous if:

(i) it is continuous at each left-dense or minimal \( t \in T \);

(ii) the right sided limit \( \varphi(t+) = \lim_{\xi \to t+} \varphi(\xi) \) exists at each right-dense \( t \).

An equation of the form

\[
y^\Delta(t) = f(t, y), \quad t \in T, \tag{2.3}
\]

is said to be a differential equation on time scale [64], where function \( f : T \times \mathbb{R}^n \to \mathbb{R}^n \) is assumed to be rd-continuous on \( T \times \mathbb{R}^n \).
In our specific case we denote, by $T_0$, the set of all functions which are rd-continuous on $\mathbb{T}_c$. Moreover, we define a set of functions $T_0^1 \subset T_0$ which are continuously differentiable on $\mathbb{T}_c$, assuming that the functions have a one-sided derivative at the boundary points of $\mathbb{T}_c$, that is if $\phi \in T_0^1$, then $\phi' \in T_0$.

2.3 The $\psi$-substitution

It is common to simplify a given equation by a proper transformation in every theory of differential equations. Likewise, in this section, we introduce a transformation which plays the role of a bridge in the passage from DETC, as in (2.1), to an IDE.

Without loss of generality, we assume that $t_{-1} < 0 < t_0$. The $\psi$-substitution, on the set $T_c' = \mathbb{T}_c \setminus \bigcup_{i=-\infty}^{\infty} \{t_{2i-1}\}$, is defined as

$$
\psi(t) = \begin{cases} 
  t - \sum_{0 < t_{2k} < t} \delta_k, & t \geq 0 \\
  t + \sum_{t_{2k} < 0} \delta_k, & t < 0 
\end{cases}
$$

(2.4)

where $\delta_k = t_{2k+1} - t_{2k}$. Notice that the $\psi$-substitution is a one-to-one map, $\psi(0) = 0$, and the condition $(C0)$ implies that $\psi(T_c') = \mathbb{R}$. The inverse transformation is

$$
\psi^{-1}(s) = \begin{cases} 
  s + \sum_{0 < t_{2k} < s} \delta_k, & s \geq 0 \\
  s - \sum_{t_{2k} < s < 0} \delta_k, & s < 0 
\end{cases}
$$

(2.5)

Note that the inverse transformation is a piecewise continuous function with discontinuity of the first kind at the points $s = s_i$, $i \in \mathbb{Z}$, and $\psi^{-1}(s_i+) - \psi^{-1}(s_i) = \delta_i$.

Lemma 2.3.1 $\psi'(t) = 1$ if $t \in \mathbb{T}_c'$.

Proof. Assume that $t \geq 0$. Then,

$$
\psi'(t) = \lim_{h \to 0} \frac{\psi(t + h) - \psi(t)}{h} = \lim_{h \to 0} \frac{1}{h} \left[ \left( t + h - \sum_{0 < t_{2k} < t + h} \delta_k \right) - \left( t - \sum_{0 < t_{2k} < t} \delta_k \right) \right] = 1.
$$
The assertion for \( t < 0 \) can be proved similarly.

Denote \( s_i = \psi(t_{2i}), \) \( i \in \mathbb{Z}. \) To make the reduction of DETC to IDE, we also need the following sets of functions. A function \( \varphi : \mathbb{R} \to \mathbb{R}^n \) is said to be in \( \mathcal{PC}_0 \) if:

(i) \( \varphi(s) \) is left continuous on \( \mathbb{R} \) and continuous on \( \mathbb{R} \setminus \bigcup_{i=-\infty}^{\infty} \{s_i\}; \)

(ii) \( \varphi(s) \) has discontinuities of the first kind at the points \( s_i. \)

Similarly, a function \( \varphi \) is said to be in \( \mathcal{PC}_1^0 \) if \( \varphi \in \mathcal{PC}_0 \) and \( \varphi' \) is in \( \mathcal{PC}_0 \) where

\[
\varphi'(s_i) = \lim_{s \to s_i^-} \frac{\varphi(s) - \varphi(s_i)}{s - s_i}.
\]

One can easily check that \( \psi^{-1} \in \mathcal{PC}_0^1, \) and \( \frac{d}{ds}(\psi^{-1}(s)) = 1 \) if \( s \neq s_i, i \in \mathbb{Z}. \)

In the next lemma we show that the spaces of functions \( \mathcal{T}_0 \) and \( \mathcal{PC}_0 \) are closely related. This relation is set up by \( \psi \)-substitution. In the same manner the relations between \( \mathcal{T}_0^1 \) and \( \mathcal{PC}_0^1 \) are going to be constructed. In what follows assume that \( s = \psi(t). \)

**Lemma 2.3.2** If \( \varphi \in \mathcal{T}_0 \) then \( \varphi \circ \psi^{-1} \in \mathcal{PC}_0, \) and \( \varphi \circ \psi \in \mathcal{T}_0 \) if \( \varphi \in \mathcal{PC}_0. \)

**Proof.** Since \( \psi \) is a one-to-one transformation we see that if \( t \) is not one of the points \( t_k, \) then \( \psi(t) \) is not one of the points \( s_i. \) Now, the continuity of \( \psi \)-substitution concludes the proof. ■

**Corollary 2.3.3** If \( \varphi \in \mathcal{T}_0^1 \) then \( \varphi \circ \psi^{-1} \in \mathcal{PC}_0^1, \) and \( \varphi \circ \psi \in \mathcal{T}_0^1 \) if \( \varphi \in \mathcal{PC}_0^1. \)

### 2.4 The Reduction to an Impulsive Differential Equation

From the definition of \( \Delta \)-derivative at a right-scattered point [64], we have

\[
y^\Delta(t_{2i}) = \frac{y(t_{2i+1}) - y(t_{2i})}{t_{2i+1} - t_{2i}}, \quad i \in \mathbb{Z},
\]

and hence equation (2.3) can be written as

\[
y'(t) = f(t, y), \quad t \in T_c,
\]

\[
y(t_{2i+1}) = f(t_{2i}, y(t_{2i}))\delta_i + y(t_{2i}), \quad (2.6)
\]
where $\delta_i = t_{2i+1} - t_{2i}$.

We generalize the last equation if the specific term $f(t_{2i}, y(t_{2i}))\delta_i$ in (2.6) is replaced by an expression $J_i(y(t_{2i}))$, where $J_i$ can be an arbitrary function.

Thus the following equation is considered

$\begin{align*}
y'(t) &= f(t, y), \quad t \in T_c, \\
y(t_{2i+1}) &= J_i(y(t_{2i})) + y(t_{2i}).
\end{align*}$

(2.7)

![Figure 2.1: A trajectory of (2.7)](image)

We name (2.7) as a differential equation on time scale with transition condition and we abbreviate its name as DETC. In Figure 2.1, a trajectory of the system (2.7) is shown. There, a solution starting at the initial point $A$ at the time $t = t^0$ is sketched. The solution moves along one of the trajectories of $y' = f(t, y)$ until the time $t = t_{2i}$ when it touches the next hyperplane at the point, say $B$. At this moment a transition is performed and the solution jumps to the point $C$ on the hyperplane $t = t_{2i+1}$. This transition is performed by means of the function $J_i$. In classical DETS, the transition from the hyperplane $t = t_{2i}$ to the hyperplane $t = t_{2i+1}$ is performed as follows: First, the tangent line to the graph of the solution at the point $B$ is drawn, and then the
intersection point of this tangent line with the hyperplane \( t = t_{2j+1} \) is found. This intersection is the point where the phase point will be after the transition. However, in practice, this is not the case and the transition is done by a more general function, as \( J_i \) that we use in this study. Clearly, (2.6) is a specification of (2.7) with \( J_i(y) = f(t, y)\delta_i \).

A function \( \varphi \in \mathcal{T}_0^{-1} \) is a solution of (2.7) if \( \varphi'(t) = f(t, \varphi(t)) \) for \( t \in \mathbb{T}_c \), and \( \varphi(t_{2j+1}) = J_i(\varphi(t_{2i})) + \varphi(t_{2i}) \) for \( t = t_{2j+1}, i \in \mathbb{Z} \).

Let us now apply the transformation of the independent argument to (2.7). If \( y \) is a solution of (2.7), then \( x = y \circ \psi^{-1} \) is a solution of the equation \( x' = f(\psi^{-1}(s), x) \) for \( s \neq s_i \). Moreover, if \( t = t_{2j+1} \), then \( s = \psi(t) = s^+_i \), and hence, the second equation in (2.7) leads to

\[ x(s^+_i) = J_i(x(s_i)) + x(s_i), \]

which can be written as

\[ \Delta x|_{s=s_i} = J_i(x(s_i)), \]

where \( \Delta x|_{s=s_i} = x(s^+_i) - x(s_i) \). Thus, \( x \) is a solution of the following IDE

\[ x' = f(\psi^{-1}(s), x), \quad s \neq s_i, \]

\[ \Delta x|_{s=s_i} = J_i(x(s_i)). \quad (2.8) \]

The connection between DETC (2.7) and IDE (2.8) is established. The solution \( x(s), x(s^0) = x_0, (s^0, x_0) \in \mathbb{R} \times \mathbb{R}^n \), of (2.8) satisfies the following integral equation

\[ x(s) = x_0 + \int_{s^0}^s f(\psi^{-1}(\xi), x(\xi))d\xi + \sum_{s^0 \leq s_i < s} J_i(x(s^+_i)), \quad (2.9) \]

if \( s \geq s^0 \), and

\[ x(s) = x_0 + \int_{s^0}^s f(\psi^{-1}(\xi), x(\xi))d\xi - \sum_{s \leq s_i < s^0} J_i(x(s^+_i)), \quad (2.10) \]

if \( s < s^0 \).

Let \( a, b \) be in \( \mathbb{T}_c \) such that \( a \leq b \). We denote

\[ \mathbb{T}_c(a, b) = [a, t_{2m}] \cup \sum_{k=m+1}^{p-1} [t_{2k-1}, t_{2k}] \cup [t_{2p-1}, b], \]

where \( m \) and \( p \) are integers which satisfy \( t_{2m-1} \leq a \leq t_{2m} < \cdots < t_{2p-1} \leq t \leq t_{2p} \), and for \( f \in \mathcal{T}_0 \) we set

\[ \int_{\mathbb{T}_c(a, b)} f(\tau)d\tau := \int_a^{t_{2m}} f(\tau)d\tau + \int_{t_{2m+1}}^{t_{2m+2}} f(\tau)d\tau + \cdots + \int_{t_{2p-1}}^{b} f(\tau)d\tau. \]
Now, the solution, \( y(t) \), of (2.7), where \( t^0 = \psi^{-1}(s^0) \), satisfies

\[ y(t) = y(0) + \int_{t^0}^{t} f(t, y(t))dt + \sum_{t^0 < t < t^1} J_i(y(t^0)), \quad t \geq t^0, \tag{2.11} \]

if \( t \geq t^0 \), and

\[ y(t) = y(0) - \int_{t^0}^{t} f(t, y(t))dt - \sum_{t < t^0 < t^1} J_i(y(t^0)), \quad t < t^0. \tag{2.12} \]

2.5 \textbf{Linear Systems}

In this section, we shall consider the linear differential equations with transition conditions on time scales. The results of this section will be needed in the next section where we investigate the existence of periodic solutions.

2.5.1 \textbf{A Homogeneous Linear System}

Let \( f(t, y) = A(t)y \) and \( J_i(y) = B_iy \) in (2.1), where \( A(t) \in C(\mathbb{R}^n) \) and \( B_i \in \mathbb{R}^{n \times n} \). Consider the linear time scale differential equation

\[ y'(t) = A(t)y, \quad t \in T_c, \tag{2.13} \]

\[ y(t_{2i+1}) = B_iy(t_{2i}) + y(t_{2i}). \]

By means of \( \psi \)-substitution, system (2.13) turns out to be the IDE

\[ x' = \tilde{A}(s)x, \quad s \neq s_i, \tag{2.14} \]

\[ \Delta x|_{s=s_i} = B_i x, \]

where \( \tilde{A}(s) = A(\psi^{-1}(s)) \). Since the solutions of system (2.14) form a linear space of dimension \( n \) \cite{60, 85}, and \( \psi \)-substitution transforms only the time variable, the solutions of (2.13) also form a linear space of the same dimension, \( n \).

Let \( e_j = (0, \cdots, 0, 1, 0, \cdots, 0)^T \) be the \( n \)-tuple whose \( j \)-th component is 1 and all others are 0 and assume that \( x_j(s), x_j(0) = e_j \), is a solution of (2.14) for \( j = 1, \cdots, n \). Then \cite{85} for any other solution \( x(s), x(0) = x_0 \), of (2.14) we have

\[ x(s) = \sum_{j=1}^{n} c_j x_j(s), \tag{2.15} \]
where the coefficients \( c_j \) are uniquely determined from \( x_0 = \sum_{j=1}^n c_j e_j \).

Now, forming the matriciant \( X(s) = [x_1(s) \ x_2(s) \ \cdots \ x_n(s)] \) of system (2.14), equality (2.15) can be written as

\[
x(s) = X(s)x_0.
\]

If \( X(s, r) = X(s)X^{-1}(r) \) is a transition matrix of \( x' = \tilde{A}(s)x \) then

\[
X(s) = \begin{cases} 
I, & s = 0 \\
X(s, s_p)(I + B_p) \prod_{k=p}^{1} X(s_k, s_{k-1})(I + B_{k-1})X(s_0, 0), & s > 0 \\
X(s, s_l)(I + B_l)^{-1} \prod_{k=l+1}^{1} X(s_{k-1}, s_k)(I + B_k)^{-1}X(s_{-1}, 0), & s < 0
\end{cases}
\]

where for \( s > 0 \) we have assumed that \( 0 < s_0 < \cdots < s_p < s < s_{p+1} \) and for \( s < 0 \) that \( s_{l-1} < s < s_l < \cdots < s_{-1} < 0 \).

On the other hand, \( \psi \)-substitution yields that a solution \( y_j(t) \), \( y_j(0) = e_j \), is determined by

\[
y_j(t) = x_j(\psi(t)).
\]

Hence, any solution \( y(t) \), \( y(0) = y_0 \), of (2.13) is given by \( y(t) = Y(t)y_0 \) where the matriciant \( Y(t) \) is defined and determined by

\[
Y(t) = \begin{cases} 
I, & t = 0 \\
Y(t, t_{2p+1})(I + B_{p}) \prod_{k=p}^{1} Y(t_{2k}, t_{2k-1})(I + B_{k-1})Y(t_1, 0), & t > 0 \\
Y(t, t_{2l})(I + B_{l})^{-1} \prod_{k=l+1}^{1} Y(t_{2k-1}, t_{2k})(I + B_k)^{-1}Y(t_{-1}, 0), & t < 0
\end{cases}
\]

in which \( Y(t, \tau) = Y(t)Y^{-1}(\tau) \) is a transition matrix of \( y' = A(t)y \) and for \( t > 0 \) we have assumed that \( 0 \leq t_{2p+1} < t < t_{2(p+1)} \) and for \( t < 0 \) that \( t_{2l-1} < t < t_{2l} \leq 0 \).

### 2.5.2 A Non-homogeneous Linear System

Consider the system

\[
y'(t) = A(t)y + g(t), \quad t \in \mathbb{T}_c, \\
y(t_{2i+1}) = B_i y(t_{2i}) + W_i + y(t_{2i}),
\]

(2.16)
where \( y \in \mathbb{R}^n \), \( A(t) \), \( B_i \) are as described for system (2.13), \( g(t) \in T_0 \) and \( \{W_i\}, i \in \mathbb{Z} \), is a sequence of \( n \)-vectors.

Applying the transformations \( y(t) = Y(t)u(t) \) and \( s = \psi(t) \) one can obtain

\[
\begin{align*}
    z' &= X^{-1}(s) \tilde{g}(s), \quad s \neq s_i, \\
    \Delta z|_{s=s_i} &= X^{-1}(s_i^+) W_i
\end{align*}
\]

where \( z(s) = u(\psi^{-1}(s)), \tilde{g}(s) = g(\psi^{-1}(s)) \). The solution of (2.17) satisfying \( z(s^0) = z_0 \) is

\[
z(s) = z_0 + \int_{s^0}^{s} X^{-1}(\xi) \tilde{g}(\xi) d\xi + \sum_{s_i \leq s < s_i^+} X^{-1}(s_i^+) W_i,
\]

if \( s \geq s^0 \), and

\[
z(s) = z_0 + \int_{s^0}^{s} X^{-1}(\xi) \tilde{g}(\xi) d\xi - \sum_{s_i < s \leq s^0} X^{-1}(s_i^+) W_i,
\]

if \( s < s^0 \). Consequently, the general solution of (2.16) is

\[
y(t) = Y(t, t^0) y_0 + \int_{\mathbb{T}_c, (t^0, t)} Y(t, \tau) g(\tau) d\tau + \sum_{t^0 \leq t < t^1} Y(t, t_{2i+1}) W_i,
\]

if \( t \geq t^0 \), and

\[
y(t) = Y(t, t^0) y_0 - \int_{\mathbb{T}_c, (t^0, t)} Y(t, \tau) g(\tau) d\tau - \sum_{t < t_{2i} \leq t^0} Y(t, t_{2i+1}) W_i,
\]

if \( t < t^0 \).

### 2.5.3 Linear Systems with Constant Coefficients

Let \( A(t) \equiv A \) and \( B_i \equiv B \) be constant matrices in (2.13) and consider the linear system with constant coefficients

\[
\begin{align*}
    y' &= Ay, \quad t \in \mathbb{T}_c, \\
    y(t_{2i+1}) &= By(t_{2i}) + y(t_{2i}),
\end{align*}
\]

where \( A, B \in \mathbb{R}^{n \times n} \). The following assumptions, for system (2.22), are needed:

(C1) the matrices \( A \) and \( B \) commute, \( AB = BA \);

(C2) \( \det(I + B) \neq 0 \);
Theorem 2.5.1. Let conditions (C0) – (C3) hold. Then the zero solution of (2.22) is

(a) asymptotically stable if the real parts of all eigenvalues of the matrix \( \Lambda_0 \) are negative;

(b) unstable if the real part of at least one eigenvalue of the matrix \( \Lambda_0 \) is positive.

Proof. It is easily seen that \( \mathcal{Y}(t, \tau) = e^{A(t-\tau)} \) and hence, if \( t_{2m-1} \leq t^0 \leq t_2 < \cdots < t_{2n-1} \leq t \leq t_{2n} \), we get

\[
Y(t, t^0) = e^{A(t-t_{2n-1})}(I + B) \prod_{k=n-1}^{m+1} [e^{A(t_{2k-1}-t_{2k-2})}(I + B)] e^{A(t_{2n-1}-t^0)}.
\]

Condition (C1) implies \( Y(t, t^0) = e^{A(\phi(t)-\psi(t^0))(I + B)i(t, t^0)} \). Due to condition (C3) we can write

\[
\psi(t) - \psi(t^0) = [\ell + \epsilon_1(t)](t - t^0), \quad \text{and,} \quad i(t^0, t) = [p + \epsilon_2(t)](t - t^0)
\]

where \( \epsilon_j(t) \to 0 \) as \( t \to \infty \), \( j = 1, 2 \). In general the functions \( \epsilon_j(t), j = 1, 2 \), are piecewise continuous functions.

Now, the solution \( y(t), y(t^0) = y_0 \), of (2.22) is written as \( y(t) = e^{A(t-t^0)}y_0 \), where

\[
\Lambda(t) = \Lambda_0 + \epsilon_1(t)A + \epsilon_2(t)\ln(I + B) \quad \text{for} \quad t \geq t^0.
\]

Assume that \( \max_j \Re \lambda_j(\Lambda_0) = \gamma < 0 \). The properties of functions \( \epsilon_j, j = 1, 2 \), imply that for a fixed positive \( \epsilon \) there exists a sufficiently large \( T > 0 \) such that if \( t \geq T \) then \( |\epsilon_j(t)| < \epsilon, \quad j = 1, 2 \).

Therefore,

\[
\|y(t)\| \leq K(\bar{\epsilon})e^{\gamma(t-t^0)}e^{(\gamma + \bar{\epsilon})(t-t^0)},
\]

where \( \gamma(\epsilon) = \|\epsilon_1(t)A + \epsilon_2(t)\ln(I + B)\| \). Since \( \gamma < 0 \) and \( \epsilon, \bar{\epsilon} \) can be chosen so small that \( \gamma + \bar{\epsilon} + \gamma(\epsilon) < 0 \), part (a) of the theorem is proved.
Let $\lambda_0$ be the eigenvalue of $\Lambda_0$, whose real part is positive, and $y_0$ be a corresponding eigenvector in a small neighborhood of the origin. We can obtain that

$$||y(t)|| \geq e^{-\varepsilon(t-t_0)} e^{\Re(\lambda_0)(t-t_0)} ||y_0||.$$ 

Since, $\Re(\lambda_0) > 0$ we can choose $\varepsilon > 0$ so small that $-\varepsilon\varepsilon + \Re(\lambda_0) > 0$, and the last inequality completes the proof. 

Example 2.5.2 Let $t_i = i + (-1)^i \kappa$, $0 < \kappa \leq \frac{1}{3}$, and consider the system

$$\begin{align*}
y_1' &= \alpha y_1 - \beta y_2, \\
y_2' &= \beta y_1 + \alpha y_2, \quad t \in \mathbb{T}_c, \\
y_1(t_{2i+1}) &= (1+k)y_1(t_{2i}), \\
y_2(t_{2i+1}) &= (1+k)y_2(t_{2i}),
\end{align*}$$

(2.23)

where $\beta$ is a positive real number and $k > -1$ is a constant. One can easily see that the matrices $A = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$ and $B = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ commute with each other and $\ell = \frac{1}{2} + \kappa, p = \frac{1}{2}$. Therefore, we have

$$\Lambda_0 = \begin{bmatrix} (\frac{1}{2} + \kappa)\alpha + \frac{1}{2} \ln(1+k) & -(\frac{1}{2} + \kappa)\beta \\ (\frac{1}{2} + \kappa)\beta & (\frac{1}{2} + \kappa)\alpha + \frac{1}{2} \ln(1+k) \end{bmatrix}$$

which has eigenvalues $\lambda_{1,2} = (\frac{1}{2} + \kappa)\alpha + \frac{1}{2} \ln(1+k) \pm (\frac{1}{2} + \kappa)\beta i$. Hence, the zero solution of (2.23) is asymptotically stable if $(\frac{1}{2} + \kappa)\alpha + \frac{1}{2} \ln(1+k) < 0$, unstable if $(\frac{1}{2} + \kappa)\alpha + \frac{1}{2} \ln(1+k) > 0$.

2.6 Periodic Solutions

2.6.1 Description of Periodic Time Scales

Definition 2.6.1 The time scale $\mathbb{T}_c$ is said to have an $\omega$-property if there exists a number $\omega \in \mathbb{R}^+$ such that $t + \omega \in \mathbb{T}_c$ whenever $t \in \mathbb{T}_c$.

From this definition, by simply using mathematical induction, we prove the following lemma.
Lemma 2.6.2 If $T_c$ has an $\omega$-property then $t + n\omega \in T_c$ for all $t \in T_c$, $n \in \mathbb{Z}$.

Definition 2.6.3 A sequence $\{a_i\} \subset \mathbb{R}$ is said to satisfy an $(\omega, p)$-property if there exist numbers $\omega \in \mathbb{R}^+$ and $p \in \mathbb{N}$ such that $a_{i+p} = a_i + \omega$ for all $i \in \mathbb{Z}$.

Lemma 2.6.4 If $t$ is a right-dense (respectively, left-dense) point of $T_c$ which has an $\omega$-property, then $t + n\omega$ is also a right-dense (respectively, left-dense) point of $T_c$ for all $n \in \mathbb{Z}$.

Proof. We will prove the statement just for $n = 1$, since the remaining part is an obvious application of mathematical induction. Let $t$ be a right-dense point. Then

$$
\sigma(t + \omega) = \inf\{s > t + \omega : s \in T_c\} = \inf\{s > t : s \in T_c\} + \omega = \sigma(t) + \omega = t + \omega,
$$

that is, $t + \omega$ is a right-dense point. Similarly, one can prove the lemma for left-dense points.

Corollary 2.6.5 If $T_c$ has an $\omega$-property, then there exists $p \in \mathbb{N}$, such that the sequences $\{t_{2i}\}$ and $\{t_{2i+1}\}$ satisfy $(\omega, p)$-property.

Corollary 2.6.6 If $T_c$ has an $\omega$-property, the sequence $\{\delta_k\}$, is $p$-periodic, that is, $\delta_{k+p} = \delta_k$ for all $k \in \mathbb{Z}$.

The next lemma assumes that $p_0$ is the minimal of these numbers $p \in \mathbb{N}$ in Corollary 2.6.6.

Lemma 2.6.7 If $T_c$ has an $\omega$-property then the sequence $\{s_i\}$, $s_i = \psi(t_{2i})$, is $(\tilde{\omega}, p_0)$-periodic with

$$
\tilde{\omega} = \omega - \sum_{0 < t_{2k} < \omega} \delta_k = \psi(\omega).
$$

That is, $s_{i+p_0} = s_i + \tilde{\omega}$ for all $i$. 

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Proof. Assume that \( i \geq 0 \), \( i = np_0 + j \) for some \( n \in \mathbb{Z} \), \( 0 \leq j < p_0 \) and \( 0 < t_0 < \cdots < t_{2(p_0-1)} < \omega \). Then

\[
\psi(t_{2(i+p_0)}) = t_{2(i+p_0)} - \sum_{0 < t_{2k} < t_{2(i+p_0)}} \delta_k
\]

\[
= t_{2i} + \omega - \sum_{0 < t_{2k} < t_{2i}} \delta_k - \sum_{t_{2i} < t_{2k} < t_{2(i+p_0)}} \delta_k = \psi(t_{2i}) + \omega - \sum_{k=i}^{j+p_0-1} \delta_k
\]

\[
= s_i + \omega - \sum_{k=j}^{j+p_0-1} \delta_k = s_i + \omega - \sum_{k=0}^{p_0-1} \delta_k = s_i + \omega - \sum_{k=j}^{j+p_0-1} \delta_k
\]

All other cases can be proved similarly. ■

Corollary 2.6.8 If \( \mathbb{T}_c \) has an \( \omega \)-property, then \( \psi(t + \omega) = \psi(t) + \psi(\omega) \).

Denote the set of all \( T \)-periodic functions, defined on the set \( \mathcal{A} \subset \mathbb{R} \), by \( \mathcal{P}_T(\mathcal{A}) \).

Lemma 2.6.9 If \( \phi \in \mathcal{P}_\omega(\mathbb{T}_c) \) and \( \mathbb{T}_c \) has an \( \omega \)-property, then \( \phi \circ \psi^{-1} \in \mathcal{P}_\tilde{\omega}(\mathbb{R}) \) with \( \tilde{\omega} = \psi(\omega) \).

Proof. By Corollary 2.6.8, \( s + \tilde{\omega} = \psi(t + \omega) \). Then the equality

\[
\phi(\psi^{-1}(s + \tilde{\omega})) = \phi(t + \omega) = \phi(t) = \phi(\psi^{-1}(s))
\]

completes the proof. ■

Similar to the proof of the last lemma the following assertion can easily be proved.

Lemma 2.6.10 If \( \phi \in \mathcal{P}_\tilde{\omega}(\mathbb{R}) \), then \( \phi \circ \psi \in \mathcal{P}_\omega(\mathbb{T}_c) \).
2.6.2 The Floquet Theory

Consider

\[ y'(t) = A(t)y + f(t), \quad t \in \mathbb{T}_c, \]
\[ y(t_{2i+1}) = B_i y(t_{2i}) + J_i + y(t_{2i}), \quad i \in \mathbb{Z}, \]

where \( A, f \in \mathcal{P}_\omega(\mathbb{T}_c) \), sequences \( B_i \) and \( J_i \) are \( p \)-periodic, \( \mathbb{T}_c \) has an \( \omega \)-property, and let \( Y(t), Y(0) = I \), be the fundamental matrix solution of the corresponding homogeneous system

\[ y'(t) = A(t)y, \quad t \in \mathbb{T}_c, \]
\[ y(t_{2i+1}) = B_i y(t_{2i}) + y(t_{2i}). \]

Recall that a solution \( y(t), y(t^0) = y_0 \), of (2.24) is given by

\[ y(t) = Y(t)y_0 + \int_{\mathbb{T}_c(0,t)} Y(t,\tau)f(\tau)d\tau + \sum_{0<t_2<\omega} Y(t,t_{2i+1})J_i. \]

Now, for this solution to be \( \omega \)-periodic, we need \( y(\omega) = y(0) = y_0 \), that is,

\[ [I - Y(\omega)]y_0 = b \]

where

\[ b = \int_{\mathbb{T}_c(0,\omega)} Y(\omega,\tau)f(\tau)d\tau + \sum_{0<t_2<\omega} Y(\omega,t_{2i+1})J_i. \]

**Definition 2.6.11** The eigenvalues, \( \rho_j \), of the matrix of monodromy, \( Y(\omega) \), are called Floquet multipliers (or simply multipliers) of system (2.24).

The following Theorems 16, 17, 18 can be proved as similar assertions for ordinary differential equations.

**Theorem 2.6.12** If \( \rho \) is a multiplier then there exists a nontrivial solution, \( y(t) \), of (2.25) such that \( y(t + \omega) = \rho y(t) \). Conversely, if there exists a nontrivial solution, \( y(t) \), of (2.25) such that \( y(t + \omega) = \rho y(t) \) then \( \rho \) is a multiplier.

**Theorem 2.6.13** System (2.25) has a \( k\omega \)-periodic solution if and only if there exists a multiplier, \( \rho \), such that \( \rho^k = 1 \).
Now, if we have \( \rho \neq 1 \) for all multipliers, then the system in (2.26) has a unique solution: this may be stated as a theorem.

**Theorem 2.6.14** If unity is not one of the multipliers, then (2.24) has a unique \( \omega \)-periodic solution, \( y(t) \), such that \( y(0) = y_0 = [I - Y(\omega)]^{-1}b \).

Now, we can write the matriciant, \( Y(t) \), in the Floquet form

\[ Y(t) = \Phi(t)e^{P\phi(t)} \]

where \( \Phi(t) = Y(t)e^{-P\phi(t)} \), \( P = \frac{1}{\omega} \ln Y(\omega) \), \( \omega = \psi(\omega) \). Then

\[
\Phi(t + \omega) = Y(t + \omega)e^{-P\phi(t+\omega)} = Y(t)Y(\omega)e^{-P\phi(\omega)}e^{-P\phi(t)} = Y(t)e^{-P\phi(t)} = \Phi(t)
\]

and hence, \( \Phi(t) \) is \( \omega \)-periodic. From the definition of \( \Phi(t) \) we see that it is continuously differentiable, bounded (because of its periodicity), and is nonsingular for all \( t \in T_c \). One can easily verify that the transformation \( y = \Phi(t)u \), transforms system (2.25) into a system with constant coefficients

\[
u' = Pu, \quad t \in T_c
\]

\[
u(t_{2i+1}) = u(t_{2i}), \quad (2.28)
\]

where we have used

\[
\lim_{t \to t_{2i+1}} \psi(t) = \psi(t_{2i}).
\]

**Definition 2.6.15** The eigenvalues, \( \lambda_j \), of the matrix, \( P = \frac{1}{\omega} \ln Y(\omega) \), are called the Floquet exponents (or simply exponents).

Similar to ODE, and applying the Floquet theory for IDE, [85], one can prove that the following theorems are valid.

**Theorem 2.6.16** Let \( \{\lambda_j\} \) be the exponents. Then the solutions of (2.25) are

(a) asymptotically stable if and only if \( \text{Re}(\lambda_j) < 0 \) for all \( j \);

(b) stable if \( \text{Re}(\lambda_j) \leq 0 \) for all \( j \) and \( \lambda_j \) is simple when \( \text{Re}\lambda_j = 0 \);
(c) unstable if there exists an exponent $\lambda_j$ such that $\text{Re}(\lambda_j) > 0$.

**Theorem 2.6.17** Let $\{\rho_j\}$ be the multipliers. Then the solutions of (2.25) are

(a) asymptotically stable if and only if all multipliers lie inside the unit circle;
(b) stable if $|\rho_j| \leq 1$ for all $j$ and $\rho_j$ is simple when $|\rho_j| = 1$;
(c) unstable if there exists a multiplier $\rho_j$ which lies outside the unit circle.

**Example 2.6.18** Let $t_i = i\pi + (-1)^{\frac{i}{4}}$ and consider the system

\[
\begin{align*}
   y'_1 &= -y_2 + f_1(t), \\
   y'_2 &= y_1 + f_2(t), \quad t \in \mathbb{T}, \\
   y_1(t_{2i-1}) &= (1 + k)y_1(t_{2i}), \\
   y_2(t_{2i+1}) &= (1 + k)y_2(t_{2i}),
\end{align*}
\]

(2.29)

where $f_1(t) = e^{t_{2i-1}}$, $f_2(t) = \sin(t - t_{2i-1})$ for $t_{2i-1} < t \leq t_{2i}$ and $k \in \mathbb{R}$ is a constant. It is easy to see that this system is $2\pi$-periodic and the matriciant of the corresponding homogeneous system is

\[
\mathcal{Y}(t, \tau) = \begin{bmatrix}
   \cos(t - \tau) & -\sin(t - \tau) \\
   \sin(t - \tau) & \cos(t - \tau)
\end{bmatrix}
\]

and hence the matrix of monodromy is

\[
Y(2\pi) = \mathcal{Y}(2\pi, \frac{3\pi}{4})(I + B)\mathcal{Y}(\frac{\pi}{4}, 0) = \begin{bmatrix}
   1 + k & 0 \\
   0 & 1 + k
\end{bmatrix}.
\]

Therefore, the multipliers are $\rho_{1,2} = 1 + k$. Now, if $k \neq 0$ then, by Theorem (2.6.14), the system in (2.29) has a unique $2\pi$-periodic solution and, by Theorem (2.6.17), this periodic solution is asymptotically stable for $-2 < k < 0$, unstable for $k < -2$ or $k > 0$, stable for $k = -2$.

### 2.6.3 The Massera Theorem

Let us consider the following analogue of the famous Massera theorem [68].
Theorem 2.6.19 If system (2.24) has a bounded solution \( y^*(t) \) on the set \( \{ t \in \mathbb{T}_c : t \geq 0 \} \) then there exists a periodic solution of system (2.24).

Proof. Assume on the contrary that there exists no periodic solution. Let \( y^*(t), y^*(0) = y_0 \), be a bounded solution of (2.24), then

\[
y^*(t) = Y(t)y_0 + \int_{T_c(0,t)} Y(t)Y^{-1}(\tau)f(\tau)d\tau + \sum_{0 < t_2 < t} Y(t)Y^{-1}(t_{2i+1})J_i
\]

and \( y^*(\omega) = Y(\omega)y_0 + b \) where \( b \) is as in (2.27). Now, \( x^*(s) = y^*(\psi^{-1}(s)) \) is a solution of

\[
x' = A(\psi^{-1}(s))x + f(\psi^{-1}(s)), \quad s \neq s_i
\]

\[
\Delta x|_{s=s_i} = B_i x + J_i.
\]

Since \( x^*(s + \omega) = y^*(\psi^{-1}(s + \omega)), \omega = \psi(\omega) \), is also a solution of (2.30), it implies that \( y^*(t + \omega) \) is also a solution of (2.24).

Thus, we have

\[
y^*(t + \omega) = Y(t + \omega)y_0 + \int_{T_c(0,t+\omega)} Y(t + \omega)Y^{-1}(\tau)f(\tau)d\tau + \sum_{0 < t_2 < t+\omega} Y(t + \omega)Y^{-1}(t_{2i+1})J_i
\]

\[
= Y(t)y^*(\omega) + \int_{T_c(0,t)} Y(t)Y^{-1}(\tau)f(\tau)d\tau + \sum_{0 < t_2 < t} Y(t)Y^{-1}(t_{2i+1})J_i
\]

and

\[
y^*(2\omega) = Y(\omega)y^*(\omega) + b = Y^2(\omega)y_0 + Y(\omega)b + b.
\]

Continuing in this way, by mathematical induction, we see that

\[
y^*(n\omega) = Y^n(\omega)y_0 + \sum_{k=0}^{n-1} Y^k(\omega)b.
\]

If there is no \( \omega \)-periodic solution, then the system \([I - Y(\omega)]y_0 = b\) has no solution. However, this means that there is a solution, \( c \), of the system \([I - Y(\omega)]^Ty = 0 \) such that \( \langle b, c \rangle \neq 0 \). Thus,

\[
\langle y^*(n\omega), c \rangle = \langle Y^n(\omega)y_0 + \sum_{k=0}^{n-1} Y^k(\omega)b, c \rangle = \langle y_0, [Y^n(\omega)]^Tc \rangle + \sum_{k=0}^{n-1} \langle b, [Y^k(\omega)]^Tc \rangle = \langle y_0, c \rangle + \sum_{k=0}^{n-1} \langle b, c \rangle = \langle y_0, c \rangle + n\langle b, c \rangle
\]

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which becomes unbounded as $n \to \infty$. On the other hand, since $y^*(t)$ is bounded, we have

$$|\langle y^*(n\omega), c \rangle| \leq |y^*(n\omega)||c| \leq M|c|$$

which contradicts with the previous equality. Hence, the proof is completed. ■

**Corollary 2.6.20** If system (2.24) does not have an $\omega$-periodic solution, then all solutions of system (2.24) are unbounded on both $\{t \in \mathbb{T}_c : t \geq 0\}$ and $\{t \in \mathbb{T}_c : t < 0\}$.

### 2.7 Deduction

In this chapter, the connection between a specific type of differential equations on time scales (DETC) and the impulsive differential equations is established. Some benefits of this established connection include knowledge about properties of linear DETC, the investigation of existence of periodic and almost periodic solutions and their stability. We suppose that the problems of stability, oscillations, smoothness of solutions, integral manifolds, theory of functional differential equations can be investigated applying our results. Another interesting opportunity is to analyze equations with more sophisticated time scales.
CHAPTER 3

DIFFERENTIAL EQUATIONS ON VARIABLE TIME SCALES

In this chapter, we introduce a class of differential equations on variable time scales with a transition condition between two consecutive parts of the scale. Conditions for existence and uniqueness of solutions are obtained. Periodicity, boundedness, stability of solutions are considered. The method of investigation is by means of two successive reductions: $B$-equivalence of the system [4, 6, 11] on a variable time scale to a system on a time scale, a reduction to an impulsive differential equation [6, 19]. Appropriate examples are constructed to illustrate the theory.

3.1 Introduction

In the last several decades, the theory of dynamic equations on time scales (DETS) has been developed very intensively. For a full description of the equations we refer to the nicely written books [29, 64] and papers [65, 88]. The equations have a very special transition condition for adjoint elements of time scales. To enlarge the field of applications of the DETS, and to have more theoretical opportunities we, in [19], proposed to generalize the transition operator, correspondingly to investigate differential equations on time scales with the transition condition (DETC).

In our recent investigations [6], it was found that the idea of the equations can be extended, if one: 1) involves in the discussion of certain union of separated sets in the $(t, x)$ space such that intersection of each line $x =$constant with the union is a time scale in the sense of Hilger (we call these separated sets altogether as the variable time scale); 2) introduces the differential equations, the domain of which are variable
time scales. We call the systems as differential equations on variable time scales with transition condition (DETCV). The present chapter is devoted to the development of methods to study these systems, and some theoretical results are obtained. To give an outline of the way of the study, we can shortly say that two consequent reductions are in the base: (a) reduction of DETCV to DETC, using $B$-equivalence method [4, 6, 11]; (b) the method of $\psi$-substitution [8, 19] to reduce DETC to impulsive differential equations.

This chapter is organized as follows. The next section has detailed description of variable time scales with examples. Section 3.3 describes the differential equations on variable time scales. The existence and uniqueness of solutions and $B$-equivalence and $B$-stability are considered in Sections 3.4 and 3.5. The description of the reduction process is given in Section 3.6. In the last two sections, we apply the procedure to investigate periodic solutions and stability of an equilibrium position.

3.2 Description of a Variable Time Scale

In this section, we give, first, a general definition of a variable time scale, and next, we describe a specific variable time scale, which will be used to introduce DETCV.

**Definition 3.2.1** A nonempty closed set $\mathbb{T}(x)$ in $\mathbb{R} \times \mathbb{R}^n$ is said to be a variable time scale if for any $x_0 \in \mathbb{R}^n$ the projection of $\mathbb{T}(x_0)$ on time axis, that is the set $\{t \in \mathbb{R} : (t, x_0) \in \mathbb{T}(x_0)\}$, is a time scale in Hilger’s sense.

To illustrate this definition let us consider the following example.

**Example 3.2.2** Let $\{r_i\}_{i=1}^\infty$ be an increasing sequence of positive real numbers such that $r_i \to \infty$ as $i \to \infty$, and

$$\mathcal{D}_i = \{(t, x) \in \mathbb{R} \times \mathbb{R} : r_{2i-1}^2 \leq t^2 + x^2 \leq r_{2i}^2\}.$$

Then, we define the variable time scale as $\mathbb{T}(x) = \bigcup_{i=1}^\infty \mathcal{D}_i$ (See Figure 3.1).

For a fixed $x_0 \in \mathbb{R}$, there exists a smallest $k$ such that $r_{2k} \geq |x_0|$. Thus, we have

$$\mathbb{T}(x_0) = \bigcup_{i=k}^\infty \{(t, x_0) : t \in \mathbb{R}, r_{2i-1}^2 \leq t^2 + x_0^2 \leq r_{2i}^2\}.$$
The projection of $\mathbb{T}(x_0)$ on time axis is

$$\mathbb{T}_c = \bigcup_{i=k}^{\infty} \left( \left[ -\sqrt{r_{2i}^2 - x_0^2}, -\sqrt{r_{2i-1}^2 - x_0^2} \right] \cup \left[ \sqrt{r_{2i-1}^2 - x_0^2}, \sqrt{r_{2i}^2 - x_0^2} \right] \right),$$

which is a time scale in Hilger’s sense.

The following variable time scale may be considered as another example. However, it is an essential element in the definition of differential equations with transition conditions on a variable time scale, discussed in this chapter. Fix a sequence $\{t_i\} \subset \mathbb{R}$ such that $t_i < t_{i+1}$ for all $i \in \mathbb{Z}$, and $t_i \to \pm \infty$ as $i \to \pm \infty$. Denote $\delta_i = t_{2i+1} - t_{2i}, \kappa_i = t_{2i} - t_{2i-1}$ and take a sequence of continuous functions $\tau_i : \mathbb{R}^n \to \mathbb{R}$. Assume that:

**(C4)** for some positive numbers $\theta', \theta \in \mathbb{R}$, we have $\theta' \leq t_{i+1} - t_i \leq \theta$ for all $i \in \mathbb{Z}$.
(C5) there exists $0 < 2 \ell_0 < \theta'$, such that $\|\tau_i(x)\| \leq \ell_0$ for all $x \in \mathbb{R}^n$, $i \in \mathbb{Z}$.

Denote
\[
l_i := \inf_{x \in \mathbb{R}^n} \{t_i + \tau_i(x)\}, \quad r_i := \sup_{x \in \mathbb{R}^n} \{t_i + \tau_i(x)\}.
\]

(3.1)

From (C4) and (C5) it follows that there exist positive numbers $\theta_l$ and $\theta_r$ such that

(C4') $\theta_l \leq l_{i+1} - r_i \leq \theta_r$, $i \in \mathbb{Z}$.

We set
\[
E_i := \{(t, x) \in \mathbb{R} \times \mathbb{R}^n : t_2i + \tau_2i(x) < t < t_2i+1 + \tau_2i+1(x)\},
\]
\[
S_i := \{(t, x) \in \mathbb{R} \times \mathbb{R}^n : t = t_i + \tau_i(x)\},
\]
\[
D_i := \{(t, x) \in \mathbb{R} \times \mathbb{R}^n : t_2i-1 + \tau_2i-1(x) \leq t \leq t_2i + \tau_2i(x)\}.
\]

(3.2)

Due to (C4'), none of $D_i$ is empty and we introduce the set
\[
T_0(x) := \bigcup_{i=-\infty}^{\infty} D_i.
\]

(3.3)

In the previous chapter, we considered a special time scale $T_c = \bigcup_{i=-\infty}^{\infty} [t_{2i-1}, t_{2i}]$; however, now, we have the set $T_0(x)$. It seems reasonable to call the latter as the variable time scale, and in our study we are going to use, for sets of type $T_c$, the term non-variable time scales to emphasize the difference.

For the convenience of the reader let us consider the following example.

**Example 3.2.3** Let $t_i = \pi i$, $\tau_i(x) = \frac{\sin(||x||)}{||x||}$ where $||x|| = \sqrt{x_1^2 + \cdots + x_n^2}$ is the Euclidean norm of $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$. Then, we have
\[
l_i = \pi i - \frac{1}{\sqrt{(c_i^2 + |i| + 1)^2 + 4c_i^2}}, \quad r_i = \pi i + \frac{1}{\sqrt{(c_i^2 + |i| + 1)^2 + 4c_i^2}}
\]
where the number $c_i > 0$ is the smallest real number which satisfies the equation
\[
\tan(c_i) = (c_i^2 + |i| + 1)/(2c_i).
\]
Thus, for $\theta_l = \frac{\pi}{2}$ and $\theta_r = \pi$, we see that (C4) is satisfied and
\[
D_i = \{(t, x) \in \mathbb{R} \times \mathbb{R}^n : t_2i-1 + \tau_2i-1(x) \leq t \leq t_2i + \tau_2i(x)\}.
\]

Then, the variable time scale could be established as in (3.3).
3.3 Differential Equations on Variable Time Scales

In what follows, we introduce a special operator which plays an important role in describing the differential equations on variable time scales as well as methods for investigation of these equations through the reduction to impulsive differential equations.

Let us consider a transition operator $\Pi_i : S_{2i} \to S_{2i+1}$, for all $i \in \mathbb{Z}$, such that $\Pi_i(t, y) = (\Pi^1_i(t, y), \Pi^2_i(t, y))$ where $\Pi^1_i : S_{2i} \to \mathbb{R}$ and $\Pi^2_i : S_{2i} \to \mathbb{R}^n$, and

$$\Pi^1_i(t, y) = t_{2i+1} + \tau_{2i+1}(\Pi^2_i(t, y)) \quad \text{and} \quad \Pi^2_i(t, y) = I_i(y) + y, \quad (3.4)$$

where $I_i : \mathbb{R}^n \to \mathbb{R}^n$ is a function. One can easily see that $\Pi^1_i(t, y)$ is the time coordinate of $(t^+, y^+) := \Pi_i(t, y)$, the image of $(t, y) \in S_{2i}$ under the operator $\Pi_i$, and $\Pi^2_i(t, y)$ is the space coordinate of the image.

The differential equation which we are going to deal with is:

$$y' = F(t, y), \quad (t, y) \in T_0(y),$$

$$t^+ = \Pi^1_i(t, y), \quad y^+ = \Pi^2_i(t, y), \quad (t, y) \in S_{2i}, \quad (3.5)$$

where the derivative at the boundary points of the variable time scale in (3.5) is one-sided derivative and $F : T_0(y) \to \mathbb{R}^n$ is assumed to be continuous on its domain.

We call (3.5) a differential equation on a variable time scale with transition condition and abbreviate it as DETCV.

To describe the solutions of differential equations with transition conditions on a variable time scale carefully, we begin the definition with the graph of a solution of (3.5). Accordingly, we start with the following construction. Consider a piece-wise curve $C$ such that:

1. $C$ lies in $T_0(y)$;
2. the part of $C$ in each $D_i, i \in \mathbb{Z}$, is a continuous arc;
3. if $C$ has points in $D_j$ and $D_{j+1}$ for some fixed $j \in \mathbb{Z}$, then $C$ intersects each of the surfaces $S_{2j}$ and $S_{2j+1}$ exactly once;
4. $C$ intersects each hyperplane $t = \theta, \theta \in \mathbb{R}$, at most at one point.
The curve can be viewed as the graph of a piece-wise function \( y = \varphi(t) \). Let \( t = \alpha_i \) and \( t = \beta_i \) be the moments that the graph of \( y = \varphi(t) \) intersects the surfaces \( S_{2i-1} \) and \( S_{2i} \), respectively, where the surfaces are defined previously. From (C4) and (C5) or (C4’) it is easily seen that \( \alpha_i < \beta_i \) for all \( i \in \mathbb{Z} \). Then, we set the non-variable time scale

\[
T^c_\varphi := \bigcup_{i=-\infty}^{\infty} [\alpha_i, \beta_i],
\]

which is the domain of \( \varphi \), and define the \( \Delta \)-derivative as given in the previous chapter. That is, for \( t = \beta_i \), we have

\[
\varphi^\Delta(\beta_i) = \frac{\varphi(\alpha_{i+1}) - \varphi(\beta_i)}{\alpha_{i+1} - \beta_i},
\]

and

\[
\varphi^\Delta(t) = \lim_{s \to t} \frac{\varphi(s) - \varphi(t)}{s - t},
\]

for any other \( t \in T^c_\varphi \), whenever the limit exists.

Thus, to define a DETCV, we need:

1. the variable time scale \( T_0(y) = \bigcup_{i=-\infty}^{\infty} D_i \);
2. the system of differential equations

\[
\frac{dy}{dt} = F(t, y), \quad (t, y) \in T_0(y);
\]

(3.6)

3. the transition operator \( \Pi_i : S_{2i} \to S_{2i+1}, i \in \mathbb{Z} \).

Setting \( \Delta t := t^+ - t \) and \( \Delta y := y^+ - y \), we can rewrite (3.5) as

\[
y' = F(t, y), \quad (t, y) \in T_0(y),
\]

\[
\Delta t_{|(t, y) \in S_{2i}} = \Pi_1^i(t, y) - t,
\]

\[
\Delta y_{|(t, y) \in S_{2i}} = \Pi_2^i(t, y) - y.
\]

(3.7)

The class of equations is important as it can be reduced from the discontinuous dynamics [6]. Particularly, they are needed to develop the center manifold theory of these equations, and, consequently, the Hopf bifurcation theory which will be covered in the next chapter.
Let us show how to construct a solution of (3.5), or equivalently of (3.7). Denote by \( \phi(t, \kappa, \eta) \) a solution of the initial value problem \( y(\kappa) = \eta \) for system

\[
\frac{dy}{dt} = F(t, y),
\]

and \( y = y(t, t^0, y_0) \) a solution of the initial value problem \( y(t^0) = y_0 \) for the system (3.5). Assume that \((t^0, y_0)\) is an interior point of \( \mathcal{D}_k \) for some \( k \in \mathbb{Z} \). We construct the solution for increasing \( t \). The process of definition of \( y(t) \) goes as follows: starting from \((t^0, y_0)\), the solution is equal to \( y(t) = \phi(t, t^0, y_0) \) up to a point \((\beta_k, y(\beta_k))\), where \( \beta_k \) is the first from the left solution of the equation \( \beta = t_{2k} + \tau_{2k}(y(\beta)) \), that is the first meeting point of the solution \( y_0 = \phi(\beta_k, t^0, y_0) \). Then, applying the transition operator \( \Pi_k \), we obtain \((\beta_k^+, y_k^+) = \left( \Pi_k^0(\beta_k, y_k), \Pi_k^1(\beta_k, y_k) \right) \). Denote \( \alpha_{k+1} = \Pi_k^1(\beta_k, y_k) \). After \( \alpha_{k+1} \), there is no meeting of the solution with \( S_{2k+1} \). (A sufficient condition which ensures this fact will be given later.) The solution is not defined on the time interval \((\beta_k, \alpha_{k+1})\). Next, on \( \mathcal{D}_{k+1} \) the solution is equal to \( y(t) = \phi(t, \alpha_{k+1}, y_k^+) \) and so on (See Figure 3.2).

The way of investigation of DETCV has not been considered yet, except for the short episode in [6]. So, in what follows, we consider a quasilinear system as it is convenient to develop the methods of reductions proposed in [6, 11, 19]. That is, we shall assume \( F(t, y) \) and \( I_i(y) \) in a special form: \( F(t, y) = A(t)y + f(t, y), \ I_i(y) = B_iy + J_i(y) \) where \( A(t) : \mathbb{R} \to \mathbb{R}^{n \times n} \) is an \( n \times n \) continuous real valued matrix-function, \( B_i \) is an \( n \times n \) matrix, functions \( f(t, y) : \mathbb{T}_0(y) \to \mathbb{R}^n \) and \( J_i(y) : \mathbb{R}^n \to \mathbb{R}^n \) are continuous. Thus, the system which we will consider is:

\[
\begin{align*}
    y' &= A(t)y + f(t, y), \quad (t, y) \in \mathbb{T}_0(y), \\
    \Delta t_{(t,y) \in S_{2i}} &= \Pi_i^1(t, y) - t, \\
    \Delta y_{(t,y) \in S_{2i}} &= \Pi_i^2(t, y) - y,
\end{align*}
\]

(3.9)

where \( \Pi_i^1(t, y) = t_{2i+1} + \tau_{2i+1}(\Pi_i^2(t, y)) \) and \( \Pi_i^2(t, y) = B_iy + J_i(y) + y \).

**Example 3.3.1** The following planar system was considered in [6]:

\[
\begin{align*}
    \frac{dx}{dt} &= Ax + f(x), \quad x \notin \Gamma, \\
    \Delta x|_{x \in \Gamma} &= B(x)x,
\end{align*}
\]

(3.10)

where \( \Gamma = \bigcup_{i=1}^p \ell_i \) is a set of curves starting at the origin and which are defined by the
Figure 3.2: A solution of a differential equation on a variable time scale.

\[ \begin{align*}
\text{equations} \langle a_i, x \rangle &+ \tau_i(x) = 0, \quad i = 1, p, \\
B(x) &= (k + \kappa(x))Q \begin{pmatrix} \cos(\theta + \upsilon(x)) & -\sin(\theta + \upsilon(x)) \\ \sin(\theta + \upsilon(x)) & \cos(\theta + \upsilon(x)) \end{pmatrix} Q^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\end{align*} \]

where the functions \( f, k, \upsilon \) are smooth, \( f(x) = o(||x||), \kappa(x) = o(||x||), \upsilon(x) = o(||x||), \)
\( \tau_i(x) = o(||x||^2), \quad i = 1, p \) and \( Q \) is some nonsingular matrix. By using polar coordinates, the system is written in the form:

\[ \begin{align*}
\frac{dr}{d\phi} &= \lambda r + P(r, \phi), \quad (r, \phi) \notin \Gamma, \\
\phi^+ \mid_{(r, \phi) \in \ell_i} &= \phi + \theta_i + \gamma(r, \phi), \\
r^+ \mid_{(r, \phi) \in \ell_i} &= (1 + k_i)r + \omega(r, \phi),
\end{align*} \]  

where \( \Gamma \) is presented as \( \ell_i : \phi = \gamma_i + r \psi_i(r, \phi), \quad i = 1, p. \)

Denote by \( \ell'_i \) the image of \( \ell_i \) under the transition operator \( \Pi_i(\phi, r) \) where \( \Pi_i^1(\phi, r) = \phi + \theta_i + \gamma(r, \phi), \) and \( \Pi_i^2(\phi, r) = (1 + k_i)r + \omega(r, \phi). \) Let \( D \) be the set bounded by \( \ell'_i \) and
In [6], it is shown that this set is non-empty and \( \ell_i \) is between \( \ell_i \) and \( \ell_{i+1} \) if the equation is considered in a small neighborhood of the origin.

Denoting \( \mathbb{T}(r) = \bigcup_{i=1}^p D_i \), we have that one deals with the following DETCV:

\[
\frac{dr}{d\phi} = \lambda r + P(r, \phi), \quad (\phi, r) \in \mathbb{T}(r), \\
\phi^+ = \Pi_1^i(\phi, r), \quad r^+ = \Pi_2^i(\phi, r), \quad (\phi, r) \in \ell_i.
\]  

Equations of the form (3.10) could be effectively applied as a model for the various mechanical processes with impacts [31, 54, 56, 74, 94]. That is why, the last example could be considered as a good motivation to investigate DETCV.

We are going to develop the theory starting with the present chapter and discuss such problems as center manifold theorem, multidimensional Hopf bifurcation, in the next chapter. We intend to investigate the problems using our approach to discontinuous dynamical systems [6].

Summarizing all the materials discussed above, we could say that there is a demand to develop the Hilger’s differential equation on non-variable time scales to the differential equations on variable time scales of general type, as a particular case of DETCV. For this reason, let us specify the transition operator in the previous part, assuming \( \Pi_2^i(\beta_k, y(\beta_k)) = F(\beta_k, y(\beta_k))(\alpha_{k+1} - \beta_k) + y(\beta_k) \), then (3.9) has a specified form

\[
y^\Delta = F(t, y), \quad (t, y) \in \mathbb{T}_0(y).
\]  

The last system could be considered as the differential equation on variable time scale (DEVTS). We suppose that the theory of DEVTS should be developed as well as the theory of DETS has been [29]. One can expect that many interesting problems connected with topology of the variable time scale may appear. Some of these problems are going to be discussed in this chapter.

### 3.4 Existence and Uniqueness of Solutions

Among the properties of a differential equation, the problem of existence and uniqueness of solutions has great importance. In this section, we are going to investigate the problem for (3.9) for increasing \( t \).
Remark 3.4.1 The continuation of the solution to the left can not be considered yet, since the invertibility of the transition operator $\Pi_i$ is not assumed.

Consider the following ordinary differential equation
\[
\frac{dy}{dt} = A(t)y + f(t, y),
\] (3.14)
where the matrix $A(t)$ and the function $f(t, y)$ are the same as in (3.9). We will assume that the following Lipschitz condition holds uniformly with respect to $t \in \mathbb{R}$ and $i \in \mathbb{Z}$, for arbitrary $x, y \in \mathbb{R}^n$:

(C6) $\|\tau_i(x) - \tau_i(y)\| + \|J_i(x) - J_i(y)\| + \|f(t, x) - f(t, y)\| \leq \ell \|x - y\|.$

Moreover, we assume that

(C7) $\sup_{t \in \mathbb{R}} \|f(t, 0)\| + \sup_{i \in \mathbb{Z}} \|J_i(0)\| = M < \infty$;

(C8) $\sup_{t \in \mathbb{R}} \|A(t)\| + \sup_{i \in \mathbb{Z}} \|B_i\| = N < \infty$;

(C9) $\bar{M} \ell < 1$, where $\bar{M} = \sup_{(t, y) \in \mathbb{T}_0(y)} \|A(t)y + f(t, y)\|$.

Then, we have the following theorem.

**Theorem 3.4.2** Assume that (C4) – (C9) hold and the function $f$ is continuous. Then for any $(t^0, y_0) \in \mathbb{T}_0(y)$ the system
\[
y' = A(t)y + f(t, y), \quad (t, y) \in \mathbb{T}_0(y),
\]
\[
t^* = \Pi^1_i(t, y), \quad y^* = \Pi^2_i(t, y), \quad (t, y) \in \mathbb{S}_2,
\] (3.15)

with the initial condition $y(t^0) = y_0$, has a unique solution, $y(t, t^0, y_0)$, which can be continued to the right of $t^0$, to $\infty$.

**Proof.** For the following discussion, it is important that if $(\gamma, y_\gamma) \in \mathbb{T}_0(y)$, then there exists an index $i \in \mathbb{Z}$ such that $(\gamma, y_\gamma) \in \mathbb{D}_i$, $i \in \mathbb{Z}$. And hence, because of (C6) and (C8), there exists a unique solution of the ordinary differential equation,
\[
y' = A(t)y + f(t, y),
\]
\[
y(\gamma) = y_\gamma,
\] (3.16)
which is continuably to $S_{2i}$, the right boundary surface of $D_i$ [35, 49].

Assume that $(i^0, y_0) \in D_k$ for some $k \in \mathbb{Z}$. On $D_k$, we will consider (3.16) for $y = i^0$, $y_i = y_0$, which has the unique solution $y(t) = \phi(t, i^0, y_0)$ defined throughout $D_k$. Let $\beta_k$ be the first from left solution of $\beta = t_{2k} + \tau_{2k}(y(\beta))$. Then, by means of jump operators we obtain $\alpha_{k+1} := \Pi_1^k(\beta_k, y(\beta_k))$ and $y_+^k := \Pi_2^k(\beta_k, y(\beta_k))$.

Next, on $D_{k+1}$, we consider the ordinary differential equation (3.16) with the initial condition $y(\alpha_{k+1}) = y_+^k$, which has the unique solution $\phi(t, \alpha_{k+1}, y_+^k)$. Thus, the solution is not defined on the time interval $(\beta_k, \alpha_{k+1})$.

Assume that the solution intersects the surface $S_{2k+1}$ at any other point, say $\alpha_{k+1}^*$, which is going to be a solution of the equation

$$\alpha^* = t_{2k+1} + \tau_{2k+1}(\phi(\alpha^*, \alpha_{k+1}, y_+^k)).$$

Clearly, we have $\alpha_{k+1}^* > \alpha_{k+1}$ and (C6) implies that

$$(\alpha_{k+1}^* - \alpha_{k+1})(1 - \ell \sup_{\alpha_{k+1} \leq \alpha \leq \alpha_{k+1}^*} \|A(t)\phi(t, \alpha_{k+1}, y_+^k) + f(t, \phi(t, \alpha_{k+1}, y_+^k))\|) \leq 0$$

which yields a contradiction since $\tilde{M} \ell < 1$. Therefore, the solution does not have any other meeting point with the surface $S_{2k+1}$. Hence, on $D_{k+1}$, the unique solution is obtained as $\phi(t, \alpha_{k+1}, y_+^k)$. In this way, we can continue this solution to $\infty$. $\blacksquare$

### 3.5 $B$-Equivalence, $B$-Stability

A difficulty in investigating the system (3.9) is that the discontinuity moments of distinct solutions are not, in general, the same. To investigate the asymptotic properties of solutions of (3.9), we introduce the following concepts.

In what follows, we are going to adopt, for DETCV, the techniques of $B$-topology and $B$-equivalence which were introduced and developed in [6, 11, 44, 57, 89, 93] for equations with impulses at variable moments of time.

Let $u(t) = y(t, i^0, y_0)$ be a solution of (3.9) and $h$ be a sufficiently small positive real number such that the open neighborhood, $B((i^0, y_0), h)$, centered at $(i^0, y_0)$ with radius $h$ belongs to $D_k$ for some $k \in \mathbb{Z}$. Let $\beta_i^h$ be the moment when the solution $u(t)$ meets
the surface \( S_{2i} \), and \( \alpha_{i+1} = \Pi^i_t (\beta^v_i, y(\beta^v_i)) \), for \( i = k, k + 1, \cdots \). We set the non-variable time scale
\[
T^u_{\bar{\psi}} := [t^0, \beta^v_k] \cup \bigcup_{i=k+1}^{\infty} [\alpha^u_i, \beta^u_i].
\]
Let \( v(t) = y(t, t^1, y_1) \) be another solution of (3.9) with \((t^1, y_1) \in B((t^0, y_0), h)\) and let \( \beta^v_i \) be the moment when the solution \( v(t) \) meets the surface \( S_{2i} \), and \( \alpha_{i+1}^u = \Pi^i_t (\beta^v_i, y(\beta^v_i)) \), for \( i = k, k + 1, \cdots \). We, similarly, define the non-variable time scale
\[
T^u_i := [t^1, \beta^u_k] \cup \bigcup_{i=k+1}^{\infty} [\alpha^u_i, \beta^u_i].
\]
Define the distance between two non-variable time scales, \( T^u_{\bar{\psi}} \) and \( T^u_i \), by
\[
d(T^u_{\bar{\psi}}, T^u_i) = \max \left\{ \sup_{i=\bar{\psi}+1} |\alpha^u_i - \alpha^u_{i_1}|, \sup_{i=\bar{\psi}} |\beta^u_i - \beta^u_{i_1}| \right\}.
\]
We say that two solutions \( u \) and \( v \) are in an \( \epsilon \)-neighborhood of each other on \( T^u_{\bar{\psi}} \) and \( T^u_i \) if:

(i) \( d(T^u_{\bar{\psi}}, T^u_i) < \epsilon \);

(ii) \( |u(t) - v(t)| < \epsilon \) for all \( t \in T^u_{\bar{\psi}} \cap T^u_i \).

The topology defined by \( \epsilon \)-neighborhoods of rd-continuous solutions will be called \( B \)-topology. It is easily seen that it is a Hausdorff topology. Topologies and metrics for spaces of discontinuous functions were introduced and developed in [6, 11, 57].

For any \( \alpha, \beta \in \mathbb{R} \) we define the oriented interval \([\hat{\alpha}, \beta] \) as
\[
[\hat{\alpha}, \beta] = \begin{cases} 
[\alpha, \beta], & \text{if } \alpha \leq \beta \\
[\beta, \alpha], & \text{otherwise}
\end{cases}
\]
(3.17)
Consider the non-variable time scale
\[
T^0_c := \bigcup_{l=-\infty}^{\infty} [l_{2i-1}, l_{2i}],
\]
(3.18)
where \( l_i, r_i \in \mathbb{Z} \), are as defined by (3.1) for the variable time scale \( T_0(y) \), and take a continuation \( \tilde{f} : T^0_c \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) of \( f : T_0(y) \rightarrow \mathbb{R}^n \) which is Lipschitzian with the same Lipschitz constant \( \ell \).

Set \( T_c := \bigcup_{l=-\infty}^{\infty} [l_{2i-1}, l_{2i}] \). We start with proving the following lemma.
Lemma 3.5.1 There are mappings \( W_i(z) : \mathbb{R}^n \rightarrow \mathbb{R}^n, i \in \mathbb{Z}, \) such that, corresponding to each solution \( y(t) \) of (3.9), there is a solution \( z(t) \) of the system

\[
\begin{align*}
z' &= A(t)z + \tilde{f}(t, z), \quad t \neq t_{2i}, \\
z(t_{2i+1}) &= B_i(z(t_{2i})) + W_i(z(t_{2i})) + z(t_{2i}),
\end{align*}
\]

(3.19)
such that \( y(t) = z(t) \) for all \( t \in \mathbb{T} \) except possibly on \( [t_{2i-1}, \alpha_i] \) and \( [\beta_i, t_{2i}] \) where \( \alpha_i \) and \( \beta_i \) are the moments that \( y(t) \) meets the surfaces \( S_{2i-1} \) and \( S_{2i} \), respectively.

The functions \( W_i \) satisfy the inequality

\[
\|W_i(z) - W_i(y)\| \leq k(\ell)\|z - y\|,
\]

(3.20)
uniformly with respect to \( i \in \mathbb{Z} \), for all \( z, y \in \mathbb{R}^n \) such that \( \|z\| \leq h \) and \( \|y\| \leq h \); here \( k(\ell) = k(\ell, h) \) is a bounded function.

Remark 3.5.2 We say that systems (3.9) and (3.19) are B-equivalent.

Proof. Fix \( i \in \mathbb{Z} \). Let \( z(t) \) be the solution of (3.9) such that \( z(t_{2i}) = z \), and assume that \( \alpha_i \) and \( \beta_i \) are solutions of \( \alpha = t_{2i-1} + \tau_{2i-1}(z(\alpha)) \), and \( \beta = t_{2i} + \tau_{2i}(z(\beta)) \), respectively. Let \( z_1(t) \) be the solution of the system

\[
\frac{dz}{dt} = A(t)z + \tilde{f}(t, z)
\]

(3.21)
with the initial condition \( z_1(\alpha_i + 1) = \Pi_i^2(\beta_i, z(\beta_i)) \).

We first note that \( z_1(\alpha_i + 1) = (I + B_i)z(\beta_i) + J_i(z(\beta_i)) \). Moreover, for \( t \in [t_{2i}, \beta_i] \),

\[
z(t) = z(t_{2i}) + \int_{t_{2i}}^t \left[ A(s)z(s) + \tilde{f}(s, z(s)) \right] ds,
\]

(3.22)
and for \( t \in [\alpha_i + 1, t_{2i+1}] \),

\[
\begin{align*}
z_1(t) &= z_1(\alpha_i + 1) + \int_{\alpha_i + 1}^t \left[ A(s)z_1(s) + \tilde{f}(s, z_1(s)) \right] ds \\
&= (I + B_i)z(\beta_i) + J_i(z(\beta_i)) + \int_{t_{2i}}^{\beta_i} \left[ A(s)z_1(s) + \tilde{f}(s, z_1(s)) \right] ds \\
&= (I + B_i) \left\{ z(t_{2i}) + \int_{t_{2i}}^{\beta_i} \left[ A(s)z(s) + \tilde{f}(s, z(s)) \right] ds \right\} \\
&\quad + J_i(z(\beta_i)) + \int_{\alpha_i + 1}^\beta \left[ A(s)z_1(s) + \tilde{f}(s, z_1(s)) \right] ds.
\end{align*}
\]

(3.23)
Thus, we set

\[
W_i(z) = (I + B_i) \int_{t_2}^{\beta_i} \left[ A(s)z(s) + \tilde{f}(s, z(s)) \right] ds + J_i(z(\beta_i)) \\
+ \int_{\alpha_{i+1}}^{t_{2i+1}} \left[ A(s)z_1(s) + \tilde{f}(s, z_1(s)) \right] ds.
\]  

(3.24)

Substituting (3.24) in (3.19), we see that \(W_i(z)\) satisfies the first conclusion of the lemma. Figure 3.3 illustrates the procedure of the construction of \(W_i(z)\).

![Figure 3.3: The construction of \(W_i(z)\).](image)

We next prove (3.20). Let \(\|z(t_2)\| \leq h\). By employing integrals (3.22) and (3.23), we find that the solutions \(z(t)\) and \(z_1(t)\) determined above satisfy the inequalities \(\|z(t)\| \leq H\) and \(\|z_1(t)\| \leq H\) on \([\beta_i, t_{2i}]\) and \([\alpha_{i+1}, t_{2i+1}]\), where

\[
H = \left[ M(1 + \ell) + (1 + N + \ell)(h + M\ell)e^{N\ell+\ell^2} \right] e^{N\ell+\ell^2}.
\]

Let \(y(t)\) be the solution of (3.9) such that \(y(t_2) = y\), and assume that \(\tilde{\alpha}_i\) and \(\tilde{\beta}_i\) are solutions of \(\tilde{\alpha} = t_{2i-1} + \tau_{2i-1}(y(\tilde{\alpha}))\), and \(\tilde{\beta} = t_{2i} + \tau_{2i}(y(\tilde{\beta}))\), respectively. Let \(y_i(t)\) be the solution of (3.21) with the initial condition \(y_1(\tilde{\alpha}_{i+1}) = \Pi_i^j(\tilde{\beta}_i, y(\tilde{\beta}_i))\). Without loss of any generality, we assume that \(\tilde{\beta}_i \geq \beta_i\) and \(\tilde{\alpha}_{i+1} \leq \alpha_{i+1}\). Application of the Gronwall-Bellman lemma shows that, for \(t \in [\beta_i, t_{2i}]\),

\[
\|z(t) - y(t)\| \leq e^{(N+\ell^2)}\|z - y\|.
\]  

(3.25)
The equation,
\[ y(\bar{\beta}_i) = y(\beta_i) + \int_{\bar{\beta}_i}^{\beta_i} \left[ A(s)y(s) + \bar{f}(s, y(s)) \right] ds \]  \hspace{1cm} (3.26)
gives us
\[ \|y(\bar{\beta}_i) - y(\beta_i)\| \leq (NH + \ell H + M)(\bar{\beta}_i - \beta_i). \]  \hspace{1cm} (3.27)
Thus, we obtain
\[ \|z(\beta_i) - y(\bar{\beta}_i)\| \leq e^{(N + \ell)\ell} \|z - y\| + (NH + \ell H + M)(\bar{\beta}_i - \beta_i). \]  \hspace{1cm} (3.28)
Now, condition (C6) together with (3.28) leads to
\[ \bar{\beta}_i - \beta_i \leq \frac{\ell e^{(N + \ell)\ell}}{1 - \ell (NH + \ell H + M)} \|z - y\|. \]  \hspace{1cm} (3.29)
Hence (3.28) becomes
\[ \|z(\beta_i) - y(\bar{\beta}_i)\| \leq \frac{e^{(N + \ell)\ell}}{1 - \ell (NH + \ell H + M)} \|z - y\|. \]  \hspace{1cm} (3.30)
On the other hand,
\[ y_1(\alpha_{i+1}) = y_1(\bar{\alpha}_{i+1}) + \int_{\bar{\alpha}_{i+1}}^{\alpha_{i+1}} \left[ A(s)y_1(s) + \bar{f}(s, y_1(s)) \right] ds \]  \hspace{1cm} (3.31)
gives us
\[ \|y_1(\alpha_{i+1}) - y_1(\bar{\alpha}_{i+1})\| \leq (NH + \ell H + M)(\alpha_{i+1} - \bar{\alpha}_{i+1}). \]  \hspace{1cm} (3.32)
Using the transition operators and (3.30) we get,
\[ \|z_1(\alpha_{i+1}) - y_1(\bar{\alpha}_{i+1})\| \leq \frac{(1 + N + \ell)e^{(N + \ell)\ell}}{1 - \ell (NH + \ell H + M)} \|z - y\|. \]  \hspace{1cm} (3.33)
Condition (C6) and (3.33) imply that
\[ \alpha_{i+1} - \bar{\alpha}_{i+1} \leq \frac{\ell (1 + N + \ell)e^{(N + \ell)\ell}}{1 - \ell (NH + \ell H + M)} \|z - y\|. \]  \hspace{1cm} (3.34)
From (3.32), (3.33) and (3.34) we obtain
\[ \|z_1(\alpha_{i+1}) - y_1(\alpha_{i+1})\| \leq H_1 e^{(N + \ell)\ell} \|z - y\|. \]  \hspace{1cm} (3.35)
where \( H_1 = (1 + N + \ell)[1 + \ell(NH + \ell H + M)]/[1 - \ell(NH + \ell H + M)]. \) Solutions \( z_1(t) \) and \( y_1(t) \) on \([\alpha_{i+1}, \tau_{i+1}]\) satisfy the inequality
\[ \|z_1(t) - y_1(t)\| \leq H_1 e^{2(N + \ell)\ell} \|z - y\|. \]  \hspace{1cm} (3.36)
Now, subtracting the expression
\[ W_i(y) = (I + B_i) \int_{\tilde{\beta}_i}^{\tilde{\beta}_i} \left[ A(s)y(s) + \tilde{f}(s, y(s)) \right] ds + J_i(y(\tilde{\beta}_i)) \]
\[ + \int_{\tilde{\alpha}_i}^{\beta_i+1} \left[ A(s)y_1(s) + \tilde{f}(s, y_1(s)) \right] ds. \] (3.37)
from equation (3.24), and using equations (3.25), (3.29), (3.34) and (3.36), we conclude that equation (3.20) holds. This proves the lemma. ■

**Definition 3.5.3** A solution \( y(t) \) is said to be \( B \)-stable, if for arbitrary \( \epsilon > 0 \), there is \( \delta > 0 \) such that a solution \( \varphi(t) \) for which \( \|\varphi(t^0) - y(t^0)\| < \delta \) is in the \( \epsilon \)-neighborhood of \( y(t) \) on \( \mathbb{T}_\mu^\nu \) and \( \mathbb{T}_\nu^\rho \).

**Definition 3.5.4** A \( B \)-stable solution \( y(t) \) is called \( B \)-asymptotically stable, if there is \( \delta > 0 \) such that for arbitrary \( \epsilon > 0 \), there is \( \theta > t^0 \) such that a solution \( \varphi(t) \) for which \( \|\varphi(t^0) - y(t^0)\| < \delta \) is in the \( \epsilon \)-neighborhood of \( y(t) \) on \( \mathbb{T}_\mu^\nu \) and \( \mathbb{T}_\nu^\rho \).

### 3.6 Reduction to an Impulsive Differential Equation

Previously we have shown that a differential equation on a variable time scale is \( B \)-equivalent to a corresponding differential equation on a non-variable time scale. Now, we are going to reduce (3.19), which is \( B \)-equivalent to (3.9), into a system of impulsive differential equations.

Now, using the substitution of the independent variable (that is, the \( \psi \)-substitution) in (3.19) and letting \( x(s) = z(\psi^{-1}(s)) \), we obtain, for \( t \neq t_{2i} \),
\[ x' = A(\psi^{-1}(s))x + \tilde{f}(\psi^{-1}(s), x(s)), \]
and, for \( t = t_{2i} \), we get
\[ x(s_{i+1}) = z(t_{2i+1}) \]
\[ = (I + B_i)z(t_{2i}) + W_i(z(t_{2i})) \]
\[ = (I + B_i)x(s_i) + W_i(x(s_i)). \]

Thus, the second equation in (3.19) leads to,
\[ \Delta x \big|_{s=s_i} = B_i x(s_i) + W_i(x(s_i)), \]
where $\Delta x \mid_{x=s_i} = x(s_i^+) - x(s_i)$. Hence, $x(s)$ is a solution of the impulsive differential equation:

\[
x' = A(\psi^{-1}(s))x + f(\psi^{-1}(s), x(s)), \quad s \neq s_i,
\]

\[
\Delta x \mid_{x=s_i} = B_i(x(s_i)) + W_i(x(s_i)).
\]

It is known that, a solution of (3.38) satisfying $x(s^0) = x_0$, for $s \geq 0$ is given by

\[
x(s) = X(s, s^0)x_0 + \int_{s^0}^{s} X(s, \xi)f(\psi^{-1}(\xi), x(\xi))d\xi
\]

\[
+ \sum_{s^0 \leq \xi < s} X(s, s_i^+)W_i(x(s_i^+)),
\]

(3.39)

where $X(s, s^0) = X(s)X^{-1}(s^0)$ and $X(s)$ is defined by

\[
X(s) = \begin{cases} 
I, & s = 0, \\
X(s, s_p)(I + B_p) \prod_{k=p}^{1} [X(s_k, s_{k-1})(I + B_{k-1})]X(s, 0), & s > 0,
\end{cases}
\]

in which $X(s, r) = X(s)X^{-1}(r)$ is a transition matrix of $x' = A(\psi^{-1}(s))x$ and it is assumed that $0 < s^0 < \cdots < s_p < s < s_{p+1}$.

Now, using back substitution, we see that a solution $y(t), y(t^0) = y_0$, of (3.19), for $t \geq t^0$, is given by

\[
y(t) = Y(t, t^0)y_0 + \int_{T_{(t)}^{(t^0)}} Y(t, \tau)f(\tau, y(\tau))d\tau
\]

\[
+ \sum_{t^0 \leq \tau < t} Y(t, \tau^{t+1})W_i(\tau^{t+1}),
\]

(3.40)

where $Y(t, t^0) = Y(t)Y^{-1}(t^0)$ and $Y(t)$, for $0 < t^0 < \cdots < t_{2p+1} < t < t_{2p+2}$, is defined by

\[
Y(t) = \begin{cases} 
I, & t = 0, \\
Y(t, t_{2p+1})(I + B_p) \prod_{k=p}^{1} [Y(t_{2k}, t_{2k-1})(I + B_{2k-1})]Y(t, 0), & t > 0,
\end{cases}
\]

in which $Y(t, \tau) = Y(t)Y^{-1}(\tau)$ is a transition matrix of $y' = A(t)y$. The notation $\int_{T_{(a,b)}} f(\tau)d\tau$ was introduced in the previous chapter, [19].

Thus, instead of investigating system (3.9), we are going to deal with (3.19) which turns out to an IDE, as in (3.38), after $\psi$-substitution.
On the bases of the discussion in Sections 3.5 and 3.6, one may conclude that the method of investigation of DETCV may be realized as consecutive reductions: a) using a $B$-equivalence method to get a DETC; and b) applying the $\psi$-substitution to DETC to obtain an IDE. We finalize the reductions with the interpretation of results for the issue DETCV. Figure 3.4 illustrates the method.

![Figure 3.4: The investigation method of DETCV](image)

### 3.7 Periodic Systems

The variable time scale $T_0(y)$ is said to satisfy an $(\omega, p)$-property if $(t \pm \omega, y)$ is in $T_0(y)$ whenever $(t, y)$ is. In this case, one can easily see that, there exists $p \in \mathbb{N}$ such that the sequences $\{t_{2i-1}\}$ and $\{t_2\}$ satisfy the $(\omega, p)$-property, [19], and $\tau_{i+p}(y) = \tau_i(y)$ for all $i \in \mathbb{Z}$.

Suppose now that (3.9) is $\omega$-periodic, i.e. $T_0(y)$ satisfies the $(\omega, p)$-property, $A(t)$ and $f(t, y)$ are $\omega$-periodic functions of $t$, and $B_{i+p} = B_i$, $J_{i+p}(y) = J_i(y)$ uniformly with respect to $i \in \mathbb{Z}$.

Since (3.9) satisfies the conditions for the uniqueness of a solution, and is periodic it can be shown that the following result holds.

**Lemma 3.7.1** If (3.9) is periodic, then the sequence $W_i(z)$ is $p$-periodic uniformly with respect to $z \in \mathbb{R}^n$. 

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Hence (3.19) is also periodic.

Lemma 3.7.2 [19] If $T_0(y)$ satisfies an $(\omega, p)$-property, then $\psi(t + \omega) = \psi(t) + \psi(\omega)$.

Consider the $\omega$-periodic system

$$
\begin{align*}
\frac{dz}{dt} &= A(t)z + f(t), \quad t \neq t_{2i}, \\
z(t_{2i+1}) &= B_i z(t_{2i}) + W_i + z(t_{2i}),
\end{align*}
$$

and let $Z(t)$ be the fundamental matrix of the corresponding homogeneous system,

$$
\begin{align*}
\frac{dz}{dt} &= A(t)z, \quad t \neq t_{2i}, \\
z(t_{2i+1}) &= B_i z(t_{2i}) + z(t_{2i}).
\end{align*}
$$

Using $\psi$-substitution, systems (3.41) and (3.42) reduce to

$$
\begin{align*}
\frac{du}{ds} &= A(\psi^{-1}(s))u + f(\psi^{-1}(s)), \quad s \neq s_i, \\
\Delta u \big|_{s=s_i} &= B_i u + W_i,
\end{align*}
$$

and

$$
\begin{align*}
\frac{du}{ds} &= A(\psi^{-1}(s))u, \quad s \neq s_i, \\
\Delta u \big|_{s=s_i} &= B_i u,
\end{align*}
$$

respectively, where $u(s) = z(\psi^{-1}(s))$.

According to [14], there is a piece-wise continuous Floquet-Lyapunov transformation $u = \Phi(s)v$ reducing (3.43) to a system with a constant matrix. There is, therefore, a constant nonsingular matrix $S$ such that the transformation, $u = \Phi(s)S w$, reduces (3.43) to

$$
\begin{align*}
\frac{dw}{ds} &= \Lambda w + g(s), \quad s \neq s_i, \\
\Delta w \big|_{s=s_i} &= I_i,
\end{align*}
$$

where $\Lambda = \text{diag}(\Lambda_+, \Lambda_-)$ is a constant matrix with $\text{Re}\lambda_j(\Lambda_+) > 0$ for $j = 1, 2, \cdots, m$, and $\text{Re}\lambda_j(\Lambda_-) < 0$ for $j = m, m + 1, \cdots, n$,

$$
\begin{align*}
\Lambda &= S^{-1} \Phi^{-1}(s) \left[ A(\psi^{-1}(s)) - \frac{d\Phi(s)}{ds} \Phi^{-1}(s) \right] \Phi(s)S, \\
g(s) &= S^{-1} \Phi^{-1}(s) f(\psi^{-1}(s)), \\
I_i &= S^{-1} \Phi^{-1}(s_i^+) W_i.
\end{align*}
$$
It is natural to call \( \lambda_j = \lambda_j(\Lambda) \) the characteristic indices and \( \rho_j = e^{\lambda_j} \) the characteristic multipliers of (3.45), respectively [19, 85]. Similarly, we call the numbers \( \lambda_j \) and \( \rho_j \) the characteristic indices and characteristic multipliers of (3.42).

**Lemma 3.7.3** If the real parts of the characteristic indices of (3.42) do not vanish, then (3.41) has a unique \( \omega \)-periodic solution, which will be B-asymptotically stable when all characteristic indices of (3.42) have negative real parts.

**Proof.** Let

\[
G(s) = \begin{cases} 
    \text{diag}(\exp(\Lambda_s), 0), & \text{for } s < 0, \\
    \text{diag}(0, -\exp(\Lambda_s)), & \text{for } s > 0,
\end{cases}
\]

and let \( \alpha = \min_{1 \leq j \leq n} |\text{Re}\lambda_j(\Lambda) + \epsilon | \) where \( \epsilon \) is an arbitrary positive number. In this case, it is known that there exists a number \( K = K(\epsilon) > 1 \), such that

\[
\|G(s - r)\| \leq K \exp(-\alpha|s - r|), \quad s, r \in \mathbb{R}.
\]

By using this inequality, it was shown in [11] that

\[
w_0(s) = \int_{-\infty}^{\infty} G(s - r)g(r)dr + \sum_{s_i} G(s - s_i)I_i
\]

is an \( \tilde{\omega} \)-periodic solution of (3.45), for which

\[
\|w_0(s)\| \leq 2Km(\alpha) \max \left\{ \max_s \|g(s)\|, \max_i \|I_i\| \right\},
\]

\[
m(\alpha) = \frac{1}{\alpha} + \frac{\exp(\alpha\theta)}{1 - \exp(-\alpha\theta)}.
\]

Hence, \( u_0(s) = \Phi(s)S w_0(s) \) is a periodic solution of (3.43) and, for \( s \in \mathbb{R} \), we have

\[
\|u_0(s)\| \leq 2Km_1m(\alpha) \max \left\{ \max_s \|f(\psi^{-1}(s))\|, \max_i \|I_i\| \right\},
\]

where \( m_1 = \max_s \|\Phi(s)S\| \). Therefore, \( z_0(t) = u_0(\psi(t)) \) is a periodic solution of (3.41) which satisfies the inequality,

\[
\|z_0(t)\| \leq 2Km_1m(\alpha) \max \left\{ \max_{t \in \mathbb{T}_c} \|f(t)\|, \max_i \|I_i\| \right\},
\]

for \( t \in \mathbb{T}_c \). This proves the lemma.

Now, let \( C = 2Km_1m(\alpha) \) and fix \( \gamma > 1 \). Let \( k(\ell) = k(\ell, h) \) be the function defined in Lemma 3.5.1, for \( h = \gamma CM \). By applying Lemmas 3.7.1 and 3.7.3 and the successive-approximation method, exactly as it was done in [11], we can prove the following lemma.
Lemma 3.7.4 Suppose that \( \tilde{f}(t, z), W_i(z) \) and \( \tau_i(z) \) in (3.19) satisfy conditions (C6), (C7) and (3.20). If all characteristic indices of system (3.42) have non-vanishing real parts, then, when \( \ell C \max[1, k(\ell)] < (\gamma - 1)/\gamma \), system (3.19) has a unique \( \omega \)-periodic solution \( z_0(t) \) such that \( \|z_0(t)\| \leq h \) for \( t \in \mathbb{T}_c \).

The solution \( z_0(t) \) is \( B \)-asymptotically stable, if the real parts of all characteristic indices of system (3.42) are negative.

On the bases of \( B \)-equivalence, Lemmas 3.5.1, 3.7.4 and continuous dependence of solution on initial data for ordinary differential equation, one can prove the following theorem.

Theorem 3.7.5 Suppose that system (3.9) satisfies conditions (C4)-(C8) and is \( \omega \)-periodic. If the characteristic indices of system (3.42) have non-vanishing real parts, then for a sufficiently small lipschitz constant \( \ell \), system (3.9) has a unique \( \omega \)-periodic solution, which is \( B \)-asymptotically stable when all characteristic indices of system (3.42) have negative real parts.

Example 3.7.6 Let us consider the variable time scale \( \mathbb{T}_0(y) \) constructed by \( t_i = i \), \( \tau_i(y) = (-1)^i \ell \sin(y) \), where \( y \in \mathbb{R}, 0 < \ell < \frac{1}{2} \), and consider the 2-periodic system

\[
\begin{align*}
y' &= ky + \cos(\pi t), \quad (t, y) \in \mathbb{T}_0(y), \\
y^+ &= (p + 1)y + I, \\
t^+ &= 2i + 1 - \ell \sin(y),
\end{align*}
\]

(3.46)

with \( k, p, I \in \mathbb{R}, I > 0 \). The system which is \( B \)-equivalent to (3.46) is

\[
\begin{align*}
z' &= kz + \cos(\pi t), \quad t \neq 2i, \\
z(2i + 1) &= (p + 1)z(2i) + W_i(z),
\end{align*}
\]

(3.47)

where

\[
W_i(z) = (1 + p) \int_{2i}^{2i+1} [kz(s) + \cos(\pi s)]ds + I + \int_{\alpha_i}^{\beta_i} [kz(s) + \cos(\pi s)]ds
\]

\[
= (1 + p) \int_{2i}^{2i+1} kz(s)ds + \int_{\alpha_i}^{\beta_i} kz(s)ds + I + \frac{(1 + p) \sin(\pi \beta_i) - \sin(\pi \alpha_{i+1})}{\pi}
\]

where \( z(t) \) is a solution of (3.46) satisfying \( z(2i) = z \) and \( \alpha_i \) and \( \beta_i \) are solutions of \( \alpha = 2i - 1 - \ell \sin(z(\alpha)) \) and \( \beta = 2i + \ell \sin(z(\beta)) \), respectively.
The homogeneous system corresponding to (3.47) is
\[ z' = kz, \quad t \neq 2i, \]
\[ z(2i + 1) = (p + 1)z(2i). \]
(3.48)

It is easily seen that for (3.46), the conditions (C4)-(C8) are satisfied and
\[ Z(2) = [(p + 1)e^k] \]
is the matrix of monodromy and \( \lambda = \ln(p + 1) + k \) is the characteristic index of (3.48).

By Theorem 3.7.5, if \( \ln(p + 1) + k \neq 0 \), then system (3.46) has a unique 2-periodic solution which is \( B \)-asymptotically stable when \( \ln(p + 1) + k < 0 \).

### 3.8 Stability of an Equilibrium

In this part, we are again going to consider the quasilinear system
\[ y' = A(t)y + f(t, y), \quad (t, y) \in T_0(y), \]
\[ t^+ = \Pi^1(t, y), \quad y^+ = \Pi^2(t, y), \quad (t, y) \in S_2, \]
(3.49)
on the variable time scale \( T_0(y) \). However, this time, the condition for existence of a Green's function is replaced by a more general one, namely, exponential dichotomy, [39]. Let
\[ y' = A(t)y, \quad t \neq 2i, \]
\[ y(t_{2i+1}) = B_i y(t_{2i}) + y(t_{2i}), \]
(3.50)
be the homogeneous system corresponding to (3.49). Moreover, suppose that the system which is \( B \)-equivalent to (3.49) is:
\[ z' = A(t)z + \tilde{f}(t, z), \quad t \neq 2i, \]
\[ z(t_{2i+1}) = B_i z(t_{2i}) + W_i(z(t_{2i})) + z(t_{2i}). \]
(3.51)

Suppose that there are \( m \)- and \( n - m \)-dimensional hyperplanes \( Y_+(t) \) and \( Y_-(t) \) in \( T_c \times \mathbb{R}^n \) such that if \( y(t) \) is a solution of (3.50) and \( y(t) \in Y_+(t) \), then \( ||y(t)|| \leq a_1||y(\tau)|| \exp(-\gamma_1(t - \tau)), -\infty < \tau \leq t < +\infty \) and, if \( y(t) \in Y_-(t) \) then \( ||y(t)|| \geq a_2||y(\tau)|| \exp(\gamma_2(t - \tau)), -\infty < \tau \leq t < +\infty \). Here, \( a_j, \gamma_j, j = 1, 2 \) are positive constants. If (3.50) satisfies these conditions, then we say that (3.50) is exponentially dichotomous.
In this case, using the inequality \( \psi(t) - \psi(\tau) \leq t - \tau \) when \( \tau \leq t \), one can show that, for the reduced impulsive linear system

\[
x' = A(\psi^{-1}(s))x, \quad s \neq s_i,
\]
\[
\Delta x |_{s=s_i} = B_i x(s_i),
\]
(3.52)

where \( x(s) = y(\psi^{-1}(s)), s_i = \psi(t_{2i}) \), there are \( m \)- and \( (n-m) \)-dimensional hyperplanes \( X_+(s) \) and \( X_-(s) \) in \( \mathbb{R} \times \mathbb{R}^n \) such that if \( x(s) \) is a solution of (3.52) and \( x(s) \in X_+(s) \), then \( \|x(s)\| \leq a_1 \|x(r)\| \exp(-\gamma_1(s-r)), -\infty < r \leq s < +\infty \) and, if \( x(s) \in X_-(s) \), then \( \|x(s)\| \geq a_2 \|x(r)\| \exp(\gamma_2(s-r)), -\infty < r \leq s < +\infty \). Then, the linear system (3.52) with impulse action is said to be exponentially dichotomous (e.d.) [39].

If (3.52) is e.d., then by applying the orthogonalization method to a given set of linearly independent solutions \( x_1(s), x_2(s), \ldots, x_n(s) \), we can construct a piecewise-continuous Lyapunov-Schmidt transformation \( x = L(s)w \) reducing (3.52) to a block-diagonal system [11], i.e., a system splitting into two systems:

\[
\frac{d\xi}{ds} = P_1(s)\xi, \quad s \neq s_i, \quad \Delta \xi |_{s=s_i} = Q^1_i \xi,
\]
(3.53)

and

\[
\frac{d\eta}{ds} = P_2(s)\eta, \quad s \neq s_i, \quad \Delta \eta |_{s=s_i} = Q^2_i \eta,
\]
(3.54)

where \( w = (\xi, \eta) \), with \( \xi \) an \( m \)-vector and \( \eta \) an \((n-m)\)-vector. Corresponding to fundamental matrices \( X_1(s, r) \) and \( X_2(s, r) \) of (3.53) and (3.54), there are positive constants \( a \) and \( \gamma \) such that

\[
\|X_1(s, r)\| \leq a \exp(-\gamma(s-r)), \quad s \geq r,
\]

and

\[
\|X_2(s, r)\| \leq a \exp(\gamma(s-r)), \quad s \leq r.
\]

Similarly, (3.49) can be reduced to the system

\[
\frac{d\xi}{ds} = P_1(s)\xi + \tilde{f}_1(s,w), \quad s \neq s_i,
\]
\[
\frac{d\eta}{ds} = P_2(s)\eta + \tilde{f}_2(s,w), \quad s \neq s_i,
\]
\[
\Delta \xi |_{s=s_i} = Q^1_i \xi(s_i) + W^1_i(w(s_i)),
\]
\[
\Delta \eta |_{s=s_i} = Q^2_i \eta(s_i) + W^2_i(w(s_i))
\]
(3.55)
after applying \( \psi \)-substitution and Lyapunov-Schmidt transformation, successively.

Besides the conditions imposed before, suppose that (3.49) satisfies

\[
f(t, 0) = J_i(0) = 0	ag{3.56}
\]

uniformly with respect to \( t \in T_c \) and \( i \in \mathbb{Z} \).

We investigate the stability of an equilibrium position of (3.49), first noting that (3.56) implies \( W_i(0) = 0 \) for \( i \in \mathbb{Z} \).

It follows from Lemma 3.5.1, \( B \)-equivalence, and the continuous dependence of solutions of (3.49) on initial data, that the following analog of the Lyapunov-Perron theorem holds.

**Theorem 3.8.1** Suppose that system (3.49) satisfies conditions (C4)-(C7) and (3.56), and system (3.50) is e.d. Then, for a sufficiently small Lipschitz constant \( \ell \), the equilibrium position of (3.49) is conditionally asymptotically stable with respect to an \( m \)-dimensional manifold of initial values containing the origin. If \( m = n \), then the zero solution of (3.49) is asymptotically stable.

**Proof.** By virtue of the reasoning given above, we consider the system (3.51) which can be reduced to the form (3.55). We assume that the functions on the right side of (3.55) satisfy conditions analogous to (C6), (C7) and (3.20) with the same constants. Then, the integral-equation system

\[
\begin{align*}
\xi &= X_1(s, s^0)c + \int_{s^0}^{s} X_1(s, r)f_1(r, w)dr + \sum_{s_i < s} X_1(s, s_i)W^1_i(w), \\
\eta &= -\int_{s}^{\infty} X_2(s, r)f_2(r, w)dr - \sum_{s_i > s} X_2(s, s_i)W^2_i(w),
\end{align*}
\tag{3.57}
\]

under the conditions

\[
al\|c\|(a + \epsilon)
\left[ \frac{2}{\gamma^2 - \sigma^2} + \frac{2}{1 - \exp(-\theta'(\gamma - \sigma))} \right] < \epsilon
\]

and

\[
2a
\left[ \frac{1}{\gamma - \sigma} + \frac{1}{1 - \exp(-\theta'(\gamma - \sigma))} \right] < 1,
\]
where \( \epsilon \) and \( \sigma \) are arbitrary fixed constants such that \( \epsilon > 0 \) and \( 0 < \sigma < \gamma \), has a solution \( w(s) = w(s, s^0, c) \) for which

\[
\|w(s)\| \leq (a + \epsilon)\|c\| \exp\left(-\sigma\left(s - s^0\right)\right).
\]  
\[(3.58)\]

If \( s = s^0 \) in (3.57), then

\[
\xi(s^0, s^0, c) = c, \\
\eta(s^0, s^0, c) = -\int_{s^0}^{\infty} X_2(s^0, r)\tilde{f}_2(r, w)\,dr - \sum_{s_i > s^0} X_2(s^0, s_i)W_i^2(w(s_i)).
\]  
\[(3.59)\]

By using the customary method [85], we can easily show that \( w(s) \) is also a solution of (3.55). Hence, by virtue of (3.57) and (3.58), we conclude that (3.59) determines a set of initial values of solutions of (3.55) tending to an equilibrium state when \( s \to \infty \).

Since, \( B \)-equivalence and \( \psi \)-substitution do not change the dimension of the manifold, the theorem is proved.

\section{3.9 Bounded Solutions}

\textbf{Theorem 3.9.1} If conditions (C4)-(C8) are satisfied for system (3.49) and (3.50) is e.d., then for a sufficiently small Lipschitz constant \( \ell \), system (3.49) has a unique solution, continuable to \( +\infty \) and \( -\infty \), uniformly bounded for all \( t, (t, y(t)) \in \mathbb{T}_0(y) \).

\textbf{Proof.} System (3.51) which is \( B \)-equivalent to (3.49) can be reduced to

\[
x' = A(\psi^{-1}(s))x + \tilde{f}(\psi^{-1}(s), x), \quad s \neq s_i, \\
\Delta x \big|_{s=s_i} = B_i x(s_i) + W_i(x(s_i)),
\]  
\[(3.60)\]

by means of \( \psi \)-substitution, where \( x(s) = z(\psi^{-1}(s)) \). In [11], it was shown that for

\[
h = v a M \left\{1/\gamma + [\exp(\gamma \theta)/(1 - \exp(-\gamma \theta))]\right\}, \quad \text{where } v > 1 \text{ is fixed, under the condition}
\]

\[
\ell a \left(\frac{1}{\gamma} + \frac{k(\ell) \exp(\gamma \theta)}{1 - \exp(-\gamma \theta')}\right) < \frac{v - 1}{v},
\]

the system (3.60) has a unique bounded solution \( x_0(s) \). Using the inverse substitution, we see that \( y_0(t) = x_0(\psi(t)) \) is a bounded solution of (3.51), and \( B \)-equivalence between (3.51) and (3.49) proves the theorem.

\section{58}
3.10 Deduction

In this chapter, we have introduced a new class of differential equations, differential equations on variable time scales with transition conditions. These systems naturally appear when we investigate discontinuous dynamics with non-fixed moments of impulses. Consequently, our results will be needed to develop methods of investigation of mechanical models with impacts. Particularly, interesting problems are related to bifurcations [31, 54, 56, 94], chaos [54], etc. We are going to develop the theory of introduced equations according to these demands.
In this chapter, we consider three-dimensional discontinuous dynamical systems with non-fixed moments of impacts. Existence of the center manifold is proved for the system. The result is applied for the extension of the planar Hopf bifurcation theorem [6]. Illustrative examples are constructed for the theory.

4.1 Introduction

Dynamical systems are used to describe real world motions using differential (continuous time) or difference (discrete time) equations. In the last several decades, the need for discontinuous dynamical systems has been increased because they, often, describe the model better when the discontinuous and continuous motions are mingled. This need has made scientists to improve and develop the theory of these systems. Many new results have arised. One must mention that namely systems with not prescribed time of discontinuities were apparently first introduced for investigation of the real world [58, 78], and this fact emphasizes very much the practical sense of the theory. The problem is one of the most difficult and interesting subjects of investigations [36, 40, 61, 62, 63, 72, 84]. It was emphasized in early stage of theory’s development, [71].

In [6], the Hopf bifurcation for the planar discontinuous dynamical system has been studied. Here, we extend this result to three-dimensional space based on the center manifold. The advantage is that we use the method of $B$–equivalence [5, 6, 19] as
well as the results of time scales which are developed in [6, 19].

This chapter is organized as follows. In the next section, we start to analyze the non-perturbed system. Section 4.3 describes the perturbed system. The center manifold is given in Section 4.4. In Section 4.5, the bifurcation of periodic solutions is studied. Section 4.6 is devoted to examples in order to illustrate the theory. In the last section a brief conclusion is given.

4.2 The Non-perturbed System

We shall consider in $\mathbb{R}^3$ the following dynamical system:

$$
\begin{align*}
\frac{dx}{dt} &= Ax, \\
\frac{dz}{dt} &= \hat{b}z, \quad (x, z) \notin \Gamma_0, \\
\Delta x \big|_{(x,z)\in\Gamma_0} &= B_0x, \\
\Delta z \big|_{(x,z)\in\Gamma_0} &= c_0z,
\end{align*}
$$

(4.1)

where $A, B_0 \in \mathbb{R}^{2 \times 2}, \hat{b}, c_0 \in \mathbb{R}$, $\Gamma_0$ is a subset of $\mathbb{R}^3$ and will be described below. The phase point of (4.1) moves between two consecutive intersections with the set $\Gamma_0$ along one of the trajectories of the system $x' = Ax, z' = \hat{b}z$. When the solution meets the set $\Gamma_0$ at the moment $\tau$, the point $x(t)$ has a jump $\Delta x \big|_{\tau} := x(\tau^+) - x(\tau)$ and the point $z(t)$ has a jump $\Delta z \big|_{\tau} := z(\tau^+) - z(\tau)$. Thus, we suppose that the solutions are left continuous functions.

From now on, $G$ denotes a neighborhood of the origin.

The following assumptions will be needed throughout this chapter:

(C10) $\Gamma_0 = \bigcup_{i=1}^p \mathcal{P}_i$, $p \in \mathbb{N}$, where $\mathcal{P}_i = \ell_i \times \mathbb{R}, \ell_i$ are half-lines starting at the origin defined by $\langle a^i, x \rangle = 0$ for $i = 1, \cdots, p$, $a^i = (a^i_1, a^i_2) \in \mathbb{R}^2$ are constant vectors;

(C11) $A = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$, where $\beta \neq 0$;

(C12) there exists a regular matrix $Q \in \mathbb{R}^{2 \times 2}$ and nonnegative real numbers $k$ and $\theta$
such that
\[ B_0 = kQ \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} Q^{-1}, \]

For the sake of brevity, in what follows, every angle for a point or a line is considered with respect to the half-line of the first coordinate axis in \(x\)-plane. Denote \( \ell'_i = (I + B_0)\ell_i, i = 1, \ldots, p. \) Let \( \gamma_i \) and \( \zeta_i \) be the angles of \( \ell_i \) and \( \ell'_i \) for \( i = 1, \ldots, p, \) respectively, and

\[ B_0 = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}; \]

(C13) \( 0 < \gamma_1 < \zeta_1 < \gamma_2 < \cdots < \gamma_p < \zeta_p < 2\pi, \) and \( (b_{11} + 1) \cos \gamma_i + b_{12} \sin \gamma_i \neq 0 \) for \( i = 1, \ldots, p. \)

In Figure 4.1, the discontinuity set and a trajectory of the system (4.1) are shown. The planes \( P_i \) form the set \( \Gamma_0 \) and each \( P'_i \) is the image of \( P_i \) under the transformation \((I + B)x.\)

The system (4.1) is said to be a \( D_0 - system \) if conditions (C10)-(C13) hold. It is easy to see that the origin is a unique singular point of \( D_0 - system \) and (4.1) is not linear.

Let us subject (4.1) to the transformation \( x_1 = r \cos \phi, x_2 = r \sin \phi, z = z_0 \) and exclude the time variable \( t \). The solution \((r(\phi, r_0, z_0), z(\phi, r_0, z_0))\) which starts at the point \((0, r_0, z_0)\) satisfies the following system in cylindrical coordinates:

\[ \begin{align*}
\frac{dr}{d\phi} &= \lambda r, \\
\frac{dz}{d\phi} &= b z, \quad \phi \neq \gamma_i \pmod{2\pi}, \\
\Delta r \big|_{\phi=\gamma_i \pmod{2\pi}} &= k_i r, \\
\Delta z \big|_{\phi=\gamma_i \pmod{2\pi}} &= c_0 z,
\end{align*} \]

where \( \lambda = \frac{\alpha}{\beta}, b = \frac{\dot{b}}{\beta}, \) the variable \( \phi \) is ranged over the time scale

\[ \mathbb{R}_\phi = \mathbb{R} \setminus \bigcup_{i=-\infty}^{\infty} \bigcup_{j=1}^{p} (2\pi i + \gamma_j, 2\pi j + \zeta_j) \]

and

\[ k_i = \left[ ((b_{11} + 1) \cos \gamma_i + b_{12} \sin \gamma_i)^2 + (b_{21} \cos \gamma_i + (b_{22} + 1) \sin \gamma_i)^2 \right]^{1/2} - 1. \]
Figure 4.1: The discontinuity set and a trajectory of (4.1)

Equation (4.2) is $2\pi$-periodic, so, in what follows we shall consider just the section $[0, 2\pi]$. That is, the system

\[
\begin{align*}
\frac{dr}{d\phi} &= \lambda r, \\
\frac{dz}{d\phi} &= bz, \quad \phi \neq \gamma_i, \\
\Delta r \mid_{\phi=\gamma_i} &= k_i r, \\
\Delta z \mid_{\phi=\gamma_i} &= c_0 z,
\end{align*}
\]

(4.3)

is provided for discussion, where $\phi \in [0, 2\pi]_\phi = [0, 2\pi] \setminus \bigcup_{i=1}^{p} (\gamma_i, \zeta_i]$. System (4.3) is a sample of time-scale differential equation. Let us use the $\psi$-substitution, $\varphi = \psi(\phi) = \phi - \sum_{0<\gamma_j<\phi} \theta_j, \theta_j = \zeta_j - \gamma_j$, which was introduced and developed in [6, 19]. The range of this new variable is $[0, 2\pi - \sum_{i=1}^{p} \theta_j]$.

It is easy to check that upon $\psi$-substitution (4.3) reduces to the following impulsive
where $\varphi_i = \psi(\gamma_i)$. Solving (4.4) as an impulsive system [60, 86] and using $\psi-$ substitution one can obtain that a solution of (4.3) is of the form

$$
\frac{dr}{d\varphi} = \lambda r, \\
\frac{dz}{d\varphi} = b z, \quad \varphi \neq \varphi_i, \\
\Delta r \big|_{\varphi = \varphi_i} = k_i r, \\
\Delta z \big|_{\varphi = \varphi_i} = c_0 z,
$$

(4.4)

for $\varphi \in [0, 2\pi]_{\phi}$. Denote

$$
q_1 = \exp \left( a \left( 2\pi - \sum_{i=1}^{p} \theta_i \right) \right) \prod_{i=1}^{p} (1 + k_i), \\
q_2 = \exp \left( b \left( 2\pi - \sum_{i=1}^{p} \theta_i \right) \right) \prod_{i=1}^{p} (1 + c_0).
$$

(4.5) (4.6)

Depending on $q_1$ and $q_2$ we may see that the following lemmas are valid.

**Lemma 4.2.1** Assume that $q_1 = 1$. Then, if

(i) $q_2 = 1$ then all solutions are periodic with period $\omega = \left( 2\pi - \sum_{i=1}^{p} \theta_i \right) \beta^{-1}$;

(ii) $q_2 = -1$ then a solution that starts to its motion on $x_1x_2$-plane is $\omega$-periodic and all other solutions are $2\omega$-periodic;

(iii) $|q_2| > 1$ then a solution that starts to its motion on $x_1x_2$-plane is $\omega$-periodic and all other solutions lie on the surface of a cylinder and they move away the origin (i.e. zero solution is unstable);

(iv) $|q_2| < 1$ then a solution that starts to its motion on $x_1x_2$-plane is $\omega$-periodic and all other solutions lie on the surface of a cylinder and they move toward the $x_1x_2$-plane (i.e. zero solution is stable).
Lemma 4.2.2 Assume that $q_1 < 1$. Then, if

(i) $|q_2| < 1$ all solutions will spiral toward the origin, i.e., origin is an asymptotically stable fixed point;

(ii) $|q_2| > 1$ a solution that starts to its motion on $x$-plane spirals toward the origin and a solution that starts to its motion on $z$-axis will move away from the origin. In this case the origin is half stable (or conditionally stable);

(iii) $q_2 = 1(q_2 = -1)$ then a solution that starts to its motion on $z$-axis is periodic with period $\omega(2\omega)$ and all other solutions will approach to $z$-axis.

Lemma 4.2.3 Assume that $q_1 > 1$. Then, if

(i) $|q_2| < 1$ then origin is a stable focus;

(ii) $|q_2| > 1$ then origin is an unstable focus;

(iii) $q_2 = 1(q_2 = -1)$ then a solution that starts to its motion on $z$-axis is periodic with period $\omega(2\omega)$ and all other solutions will approach to $z$-axis.

We note that when $q_2 = -1$, (this means $z$ may be negative, too) the solutions starting their motion out of $x_1x_2$-plane, will move above and below the $x_1x_2$-plane. More explicitly, if a solution starts to its motion above the $x$-plane, then after the time corresponding to $\omega$, it will be below the $x$-plane, and in the next duration corresponding to $\omega$, it will try to move above $x$-plane and at the end of that duration it will be above the $x$-plane, and so on.

From now on, we assume that $q_1 = 1$ and $|q_2| < 1$. 
4.3 The Perturbed System

Let $G$ denote a sufficiently small neighborhood of the origin and consider the system

$$\frac{dx}{dt} = Ax + f(x, z),$$
$$\frac{dz}{dt} = \hat{b}z + g(x, z), \quad (x, z) \notin \Gamma,$$
$$\Delta x|_{(x, z) \in \Gamma} = B(x)x,$$
$$\Delta z|_{(x, z) \in \Gamma} = c(z)z,$$

where the following assumptions are assumed to be true:

(C14) $\Gamma = \bigcup_{i=1}^{p} S_i$, where $S_i = s_i \times \mathbb{R}$ and the equation of $s_i$ is given by $s_i : \langle a^i, x \rangle + \tau_i(x) = 0$, for $i = 1, \cdots, p$;

(C15) $B(x) = (k + \kappa(x))Q \begin{bmatrix} \cos(\theta + \Theta(x)) & -\sin(\theta + \Theta(x)) \\ \sin(\theta + \Theta(x)) & \cos(\theta + \Theta(x)) \end{bmatrix} Q^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

and $c(z) = c_0 + \tilde{c}(z)$;

(C16) functions $f, g, \kappa, \tilde{c}$ and $\Theta$ are in $C^1$ and $\tau_i$ is in $C^2$;

(C17) $f(x, z) = O(||(x, z)||^2)$, $g(x, z) = O(||(x, z)||^2)$, $\kappa(x) = O(||x||)$, $\Theta(x) = O(||x||)$, $\tilde{c}(z) = O(z)$, $\tau_i(x) = O(||x||^2)$, $i = 1, \cdots, p$, and $f(0, z) = 0$, $g(0, z) = 0$ for all $z \in \mathbb{R}$.

Moreover, it is supposed that the matrices $A, Q$, the vectors $a^i, i = 1, \cdots, p$, constants $k, \theta$ are the same as for (4.1), i.e.,

(C18) the associated with (4.9) is $D_0$ system.

**Remark 4.3.1** Conditions (C14) and (C15) imply that surfaces $S_i$ do not intersect each other except on $z$–axis and neither of them intersects itself.

The system (4.9) is said to be a $D$–system if the conditions (C10)-(C17) hold.
In what follows we assume without loss of generality that $\gamma_i \neq \frac{\pi}{2} j, j = 1, 2, 3$. Then one can transform the equation in (C14) to the polar coordinates so that $s_i : a_i^1 r \cos \phi + a_i^2 r \sin \phi + \tau_i (r \cos \phi, r \sin \phi) = 0$ and, hence

$$\phi = \tan^{-1} \left( \tan \gamma_i - \frac{\tau_i (r \cos \phi, r \sin \phi)}{a_i^2 r \cos \phi} \right).$$

Using Taylor expansion gives that the previous equation can be written, for sufficiently small $r$, as

$$s_i : \phi = \gamma_i + \Psi_i(r, \phi), \quad i = 1, \ldots, p$$

where functions $\Psi_i$ are $2\pi$–periodic in $\phi$, continuously differentiable and $\Psi_i = O(r)$. If the phase point $(x_1(t), x_2(t), z(t))$ meets the discontinuity surface $S_i$ at the angle $\theta$, then after the jump, the point $(x_1(\theta^+), x_2(\theta^+), z(\theta^+))$ will be on the surface $S_i' = \{(u, v) \in \mathbb{R}^3 : u = (I + B(x))x, v = (1 + c_0)z + c(z), (x, z) \in S_i\}$. For the remaining part of the paper the following assertion is very important and the proof can be found in [6].

**Lemma 4.3.2** If the conditions (C16) and (C17) are valid then the surface $S_i'$ is placed between the surfaces $S_i$ and $S_{i+1}$ for every $i$ if $G$ is sufficiently small.

Using the cylindrical coordinates $x_1 = r \cos \phi, x_2 = r \sin \phi, z = z$, one can find that the differential part of (4.9) has the following form:

$$\begin{align*}
\frac{dr}{d\phi} &= \lambda r + P(r, \phi, z), \\
\frac{dz}{d\phi} &= b z + Q(r, \phi, z),
\end{align*}$$

(4.10)

where, as is known [87], the functions $P(r, \phi, z)$ and $Q(r, \phi, z)$ are $2\pi$–periodic in $\phi$, continuously differentiable in all variables and $P = O(r, z), Q = O(r, z)$, with $P(0, \phi, z) = 0, Q(0, \phi, z) = 0$, for all $\phi, z \in \mathbb{R}$. Denote $x^+ = (x_1^+, x_2^+) = (I + B(x))x, x^+ = r^+(\cos \phi^+, \sin \phi^+), \tilde{x}^+ = (\tilde{x}_1^+, \tilde{x}_2^+) = (I + B(0))x$, where $x = (x_1, x_2) \in s_i, i = 1, \ldots, p$. The inequality $\|x^+ - \tilde{x}^+\| \leq \|B(x) - B(0)\| \cdot \|x\|$ implies that $r^+ = (1 + k_i)r + \omega(r, \phi)$. Moreover, using the relation $\frac{\dot{x}_1}{x_1}$ and $\frac{\dot{x}_2}{x_2}$ and condition (C14) one can conclude that $\dot{\phi}^+ = \dot{\phi} + \theta_i + \gamma(r, \phi)$. Functions $\omega$ and $\gamma$ are $2\pi$–periodic in $\phi$ and $\omega = O(r^2), \gamma = O(r)$. 67
Finally, the transformed system is of the following form:

\[
\begin{align*}
\frac{dr}{d\phi} &= \lambda r + P(r, \phi, z), \\
\frac{dz}{d\phi} &= b z + Q(r, \phi, z), \quad (r, \phi, z) \notin \Gamma, \\
\Delta r|_{(r, \phi) \in S} &= k_i r + \omega(r, \phi), \\
\Delta \phi|_{(r, \phi) \in S} &= \theta_i + \gamma(r, \phi), \\
\Delta z|_{(r, \phi) \in S} &= c_0 z + \bar{c}(z).
\end{align*}
\] (4.11)

Let us introduce the following system besides (4.11):

\[
\begin{align*}
\frac{d\rho}{d\phi} &= \lambda \rho + P(\rho, \phi, z), \\
\frac{dz}{d\phi} &= b z + Q(\rho, \phi, z), \quad \phi \neq \gamma_i, \\
\Delta \rho|_{\rho = \gamma_i} &= k_i \rho + W^1_i(\rho, z), \\
\Delta \phi|_{\rho = \gamma_i} &= \theta_i, \\
\Delta z|_{\rho = \gamma_i} &= c_0 z + W^2_i(\rho, z),
\end{align*}
\] (4.12)

where all elements, except for \( W_i = (W^1_i, W^2_i), i = 1, \cdots, p, \) are the same as in (4.11) and the domain of (4.12) is \([0, 2\pi]_\phi\). We shall define the functions \( W_i \) below.

Let \((r(\phi, r_0, z_0), z(\phi, r_0, z_0))\) be a solution of (4.11) \( \phi_i \) be the angle where the phase point intersects \( S_i \). Denote also by \( \chi_i = \phi_i + \theta_i + \gamma(r(\phi_i, r_0, z_0), \phi_i) \) the angle where the phase point has to be after the jump.

Further \((\alpha, \beta), (\alpha, \beta) \in \mathbb{R}\) denotes the oriented interval, that is

\[
(\alpha, \beta) = \begin{cases} 
(\alpha, \beta) & \text{if } \alpha \leq \beta, \\
(\beta, \alpha) & \text{otherwise.}
\end{cases}
\]

**Definition 4.3.3** We shall say that systems (4.11) and (4.12) are \( B \)-equivalent in \( G \) if for every solution \((r(\phi, r_0, z_0), z(\phi, r_0, z_0))\) of (4.11) whose trajectory is in \( G \) for all \( \phi \in [0, 2\pi]_\phi \) there exists a solution \((\rho(\phi, r_0, z_0), z(\phi, r_0, z_0))\) of (4.12) which satisfies the relation

\[
r(\phi, r_0, z_0) = \rho(\phi, r_0, z_0), \quad \phi \in [0, 2\pi]_\phi \setminus \bigcup_{i=1}^{p} ((\phi_i, \gamma_i) \cup (\zeta_i, \chi_i)), \] (4.13)

and, conversely, for every solution \((\rho(\phi, r_0, z_0), z(\phi, r_0, z_0))\) of (4.12) whose trajectory is in \( G \), there exists a solution \((r(\phi, r_0, z_0), z(\phi, r_0, z_0))\) of (4.11) which satisfies (4.13).
Fix $i = 1, \cdots, p$. Let $(r_1(\phi), z_1(\phi)), (r_1(\gamma_i), z_1(\gamma_i)) = (\rho, z)$, be a solution of

\begin{align*}
\frac{dr}{d\phi} &= \lambda r + P(r, \phi, z), \\
\frac{dz}{d\phi} &= b z + Q(r, \phi, z),
\end{align*}

(4.14)

and let $\phi = \eta_i$ be the meeting angle of the solution with $\mathcal{P}_i$. Then

\begin{align*}
 r_1(\eta_i) &= e^{\lambda(\eta_i - \gamma_i)} \rho + \int_{\gamma_i}^{\eta_i} e^{\lambda(\eta_i - s)} P(r_1(s), s, z_1(s)) ds, \\
 z_1(\eta_i) &= e^{b(\eta_i - \gamma_i)} z + \int_{\gamma_i}^{\eta_i} e^{b(\eta_i - s)} Q(r_1(s), s, z_1(s)) ds.
\end{align*}

Set $\eta_i' = \eta_i + \gamma(r_1(\eta_i), \eta_i)$ and $(\rho', z') = ((1 + k_i) r_1(\eta_i) + \omega(r_1(\eta_i), \eta_i), (1 + c_0) z_1(\eta_i) + c(z_1(\eta_i)))$. Let $(r_2(\phi), z_2(\phi)), (r_2(\eta_i'), z_2(\eta_i')) = (\rho', z')$, be a solution of (4.14). Then,

\begin{align*}
 r_2(\xi_i) &= e^{\lambda(\xi_i - \eta_i')} \rho' + \int_{\eta_i'}^{\xi_i} e^{\lambda(\xi_i - s)} P(r_2(s), s, z_2(s)) ds, \\
 z_2(\xi_i) &= e^{b(\xi_i - \eta_i')} z' + \int_{\eta_i'}^{\xi_i} e^{b(\xi_i - s)} Q(r_2(s), s, z_2(s)) ds.
\end{align*}

We define that

\begin{align*}
 W_i^1(\rho, z) &= r_2(\xi_i) - (1 + k_i) \rho \\
 &= e^{\lambda(\xi_i - \eta_i')} \left[ (1 + k_i) e^{\lambda(\eta_i - \gamma_i)} \rho + \int_{\gamma_i}^{\eta_i} e^{\lambda(\eta_i - s)} P(r_1(s), s, z_1(s)) ds \right] \\
 &\quad + \omega(r_1(\eta_i), \eta_i) + \int_{\eta_i}^{\eta_i'} e^{\lambda(\xi_i - s)} P(r_1(s), s, z_1(s)) ds - (1 + k_i) \rho,
\end{align*}

or, if simplified

\begin{align*}
 W_i^1(\rho, z) &= (1 + k_i)(e^{-\lambda r(1(r_1(\eta_i), \eta_i))} - 1) \rho \\
 &\quad + (1 + k_i) \int_{\gamma_i}^{\eta_i} e^{\lambda(\xi_i - s - \gamma(r_1(\eta_i), \eta_i))} P(r_1(s), s, z_1(s)) ds \\
 &\quad + \int_{\eta_i}^{\eta_i'} e^{\lambda(\xi_i - s)} P(r_2(s), s, z_2(s)) ds + e^{b(\xi_i - \eta_i')} \omega(r_1(\eta_i), \eta_i). \quad (4.15)
\end{align*}

We, similarly, define

\begin{align*}
 W_i^2(\rho, z) &= z_2(\xi_i) - (1 + c_0) z \\
 &= e^{b(\xi_i - \eta_i')} \left[ (1 + c_0) e^{b(\eta_i - \gamma_i)} z + \int_{\gamma_i}^{\eta_i} e^{b(\eta_i - s)} Q(r_1(s), s, z_1(s)) ds \right] \\
 &\quad + \omega(z_1(\eta_i)) + \int_{\eta_i}^{\eta_i'} e^{b(\xi_i - s)} Q(r_1(s), s, z_1(s)) ds - (1 + c_0) z,
\end{align*}
We note that there exists a Lipschitz constant $\ell$ and a bounded function $m(\ell)$ such that

$$
\|W_j^i(\rho_1, z_1) - W_j^i(\rho_2, z_2)\| \leq m(\ell)\ell (\|\rho_1 - \rho_2\| + \|z_1 - z_2\|),
$$

(4.17)

for all $\rho_1, \rho_2, z_1, z_2 \in \mathbb{R}, j = 1, 2$. For detailed proof and explanation about (4.17) we refer to [6, 19].

4.4 Center Manifold

Now, using $\psi-$substitution (4.12) becomes:

$$
\begin{align*}
\frac{d\rho}{d\varphi} &= \lambda \rho + F(\rho, \varphi, z), \\
\frac{dz}{d\varphi} &= b z + G(\rho, \varphi, z), \quad \varphi \neq \varphi_i, \\
\Delta \rho |_{\varphi = \varphi_i} &= k_i \rho + W^1_i(\rho, z), \\
\Delta z |_{\varphi = \varphi_i} &= c_0 z + W^2_i(\rho, z),
\end{align*}
$$

(4.18)

where $\varphi = \psi(\phi), \varphi_i = \psi(\gamma_i), F(\rho, \varphi, z) = P(\rho, \psi^{-1}(\varphi), z), G(\rho, \varphi, z) = Q(\rho, \psi^{-1}(\varphi), z)$. Functions $F$ and $G$ are $T-$ periodic in $\varphi$, with $T = \psi(2\pi)$, and satisfy

$$
\begin{align*}
\|F(\rho, \varphi, z) - F(\rho', \varphi, z')\| &\leq L(\|\rho - \rho'\| + \|z - z'\|), \\
\|G(\rho, \varphi, z) - G(\rho', \varphi, z')\| &\leq L(\|\rho - \rho'\| + \|z - z'\|),
\end{align*}
$$

(4.19, 4.20)

for some Lipschitz constant $L$.

Following the methods given in [5], one can see that system (4.18) has two integral manifolds whose equations are given by:

$$
\begin{align*}
\Phi_0(\varphi, \rho) &= \int_{-\infty}^{\varphi} \pi_0(\varphi, s) G(\rho(s, \varphi, \rho), s, z(s, \varphi, \rho)) ds \\
&\quad + \sum_{\varphi_i < \varphi} \pi_0(\varphi_i, \varphi_i^+) W^2_i(\rho(\varphi_i^+, \varphi, \rho), z(\varphi_i^+, \varphi, \rho)),
\end{align*}
$$

(4.21)
and
\[
\Phi_-(\varphi, z) = -\int_\varphi^{\infty} \pi_-(\varphi, s) F(\rho(s, \varphi, z), s, z(s, \varphi, z)) \, ds \\
+ \sum_{\varphi_i < \varphi} \pi_-(\varphi, \varphi_i^+) W_i(\rho(\varphi_i^+, \varphi, z), z(\varphi_i^+, \varphi, z)),
\]
(4.22)
where
\[
\pi_0(\varphi, s) = e^{b(\varphi - s)} \prod_{s \leq \varphi} (1 + c_i)
\]
and
\[
\pi_-(\varphi, s) = e^{b(\varphi - s)} \prod_{s \leq \varphi} (1 + k_j).
\]

In (4.21), the pair \((\rho(s, \varphi, \rho), z(s, \varphi, \rho))\) denotes a solution of (4.18) which satisfies \(\rho(\varphi, \varphi, \rho) = \rho\). Similarly, \((\rho(s, \varphi, z), z(s, \varphi, z))\), in (4.22), is solution of (4.18) with \(z(\varphi, \varphi, z) = z\).

In [5], it was shown that there exist constants \(K_0, M_0, \sigma_0\) such that \(\Phi_0\) satisfies:
\[
\Phi_0(\varphi, 0) = 0,
\]
(4.23)
\[
\|\Phi_0(\varphi, \rho_1) - \Phi_0(\varphi, \rho_2)\| \leq K_0 \ell \|\rho_1 - \rho_2\|,
\]
(4.24)
for all \(\rho_1, \rho_2\) such that a solution \(w(\varphi) = (\rho(\varphi), z(\varphi))\) of (4.18) with the initial condition \(w(\varphi_0) = (\rho_0, \Phi_0(\varphi_0, \rho_0)), \rho_0 \geq 0\), is defined on \(\mathbb{R}\) and satisfies
\[
\|w(\varphi)\| \leq M_0 \rho_0 e^{-\sigma_0(\varphi - \varphi_0)}, \quad \varphi \geq \varphi_0.
\]
(4.25)

Similarly, it was shown that there exist constants \(K_-, M_-, \sigma_-\) such that \(\Phi_-\) satisfies:
\[
\Phi_-(\varphi, 0) = 0,
\]
(4.26)
\[
\|\Phi_-(\varphi, z_1) - \Phi_-(\varphi, z_2)\| \leq K_- \ell \|z_1 - z_2\|,
\]
(4.27)
for all \(z_1, z_2\) such that a solution \(w(\varphi) = (\rho(\varphi), z(\varphi))\) of (4.18) with the initial condition \(w(\varphi_0) = (\Phi_-(\varphi_0, z_0), z_0), z_0 \in \mathbb{R}\), is defined on \(\mathbb{R}\) and satisfies
\[
\|w(\varphi)\| \leq M_- \|z_0\| e^{-\sigma_-(\varphi - \varphi_0)}, \quad \varphi \leq \varphi_0.
\]
(4.28)

Set \(S_0 = \{(\rho, \varphi, z) : z = \Phi_0(\varphi, \rho)\}\) and \(S_- = \{(\rho, \varphi, z) : \rho = \Phi_-(\varphi, z)\}\). Here, \(S_0\) is called the center manifold and \(S_-\) is called the stable manifold. A sketch of an arbitrary center manifold is shown in Figure 4.2.
Figure 4.2: A discontinuous center manifold

The analogues of the following two Lemma’s together with their proofs can be found in [5].

**Lemma 4.4.1** If the Lipschitz constant $\ell$ is sufficiently small, then for every solution $w(\varphi) = (\rho(\varphi), z(\varphi))$ of (4.18) there exists a solution $\mu(\varphi) = (u(\varphi), v(\varphi))$ on the center manifold, $S_0$, such that

$$
\|\rho(\varphi) - u(\varphi)\| \leq 2M_0\|\rho(\varphi_0) - u(\varphi_0)\|e^{-\sigma_0(\varphi - \varphi_0)},
$$

$$
\|z(\varphi) - v(\varphi)\| \leq M_0\|z(\varphi_0) - v(\varphi_0)\|e^{-\sigma_0(\varphi - \varphi_0)}, \quad \varphi \geq \varphi_0,
$$

(4.29)

where $M_0$ and $\sigma_0$ are the constants used in (4.25).

**Lemma 4.4.2** For sufficiently small Lipschitz constant $\ell$ the surface $S_0$ is stable in large.
On the local center manifold $S_0$, the first coordinate of the solutions of (4.18) satisfies the following system:

\[
\begin{aligned}
\frac{d\rho}{d\varphi} &= \lambda \rho + F(\rho, \varphi, \Phi_0(\varphi, \rho)), \quad \varphi \neq \varphi_i, \\
\Delta \rho|_{\varphi=\varphi_i} &= k_i \rho + W^1_i(\rho, \Phi_0(\varphi, \rho)).
\end{aligned}
\]  

(4.30)

Now, it is time to consider the reduction principle:

**Theorem 4.4.3** Assume that conditions (C10)-(C19) are fulfilled. Then the trivial solution of (4.18) is stable, asymptotically stable or unstable if the trivial solution of (4.30) is stable, asymptotically stable or unstable, respectively.

Using inverse of $\psi$–substitution and $B$–equivalence, one can see that the following theorem holds:

**Theorem 4.4.4** Assume that conditions (C10)-(C19) are fulfilled. Then the trivial solution of (4.30) is stable, asymptotically stable or unstable if the trivial solution of (4.31) is stable, asymptotically stable or unstable, respectively.

\[
\begin{aligned}
\frac{dx}{dt} &= Ax + f(x, z) + \mu \tilde{f}(x, z, \mu), \\
\frac{dz}{dt} &= \tilde{b}z + g(x, z) + \mu \tilde{g}(x, z, \mu), \quad (x, z) \notin \Gamma(\mu), \\
\Delta x|_{(x, z) \in \Gamma(\mu)} &= B(x, \mu)x, \\
\Delta z|_{(x, z) \in \Gamma(\mu)} &= c(z, \mu)z.
\end{aligned}
\]  

(4.31)

4.5 **Bifurcation of Periodic Solutions**

This section is devoted to the bifurcation of a periodic solution for the discontinuous dynamical system. Let us consider the system,

\[
\begin{aligned}
\frac{dx}{dt} &= Ax + f(x, z) + \mu \tilde{f}(x, z, \mu), \\
\frac{dz}{dt} &= \tilde{b}z + g(x, z) + \mu \tilde{g}(x, z, \mu), \quad (x, z) \notin \Gamma(\mu), \\
\Delta x|_{(x, z) \in \Gamma(\mu)} &= B(x, \mu)x, \\
\Delta z|_{(x, z) \in \Gamma(\mu)} &= c(z, \mu)z.
\end{aligned}
\]  

Assume that the following conditions are satisfied:

(C18) the set $\Gamma(\mu) = \bigcup_{i=1}^{p} S_i(\mu)$, where $S_i(\mu) = s_i(\mu) \times \mathbb{R}$ and the equation of $s_i(\mu)$ is given by $s_i(\mu) : \langle d', x \rangle + \tau_i(x) + \mu v(x, \mu) = 0$, for $i = 1, \ldots, p$;
(C19) there exists a matrix \( Q(\mu) \in \mathbb{R}^{2 \times 2}, Q(0) = Q \), analytic in \((-\mu_0, \mu_0)\), and real numbers \( \gamma, \chi \) such that \( Q^{-1}(\mu)B(x, \mu)Q(\mu) = (k + \mu \gamma + \kappa(x)) \begin{bmatrix} \cos(\theta + \mu \chi + \Theta(x)) & -\sin(\theta + \mu \chi + \Theta(x)) \\ \sin(\theta + \mu \chi + \Theta(x)) & \cos(\theta + \mu \chi + \Theta(x)) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \)

and \( c(z, \mu) = c_0 + \tilde{c}(z) + \mu d(z, \mu); \)

(C20) associated with (4.31) systems

\[
\begin{align*}
\frac{dx}{dt} &= Ax, \\
\frac{dz}{dt} &= \hat{b}z, \quad (x, z) \notin \Gamma_0, \\
\Delta x|_{(x, z) \in \Gamma_0} &= B_0x, \\
\Delta z|_{(x, z) \in \Gamma_0} &= c_0z.
\end{align*}
\]

and

\[
\begin{align*}
\frac{dx}{dt} &= Ax + f(x, z), \\
\frac{dz}{dt} &= \hat{b}z + g(x, z), \quad (x, z) \notin \Gamma(0), \\
\Delta x|_{(x, z) \in \Gamma(0)} &= B(x, 0)x, \\
\Delta z|_{(x, z) \in \Gamma(0)} &= c(z, 0)z.
\end{align*}
\]

are \( D_0 - system \) and \( D - system \) respectively;

(C21) functions \( \tilde{f} \) and \( \nu \) are analytic in their all arguments;

(C22) \( \tilde{f}(0, 0, \mu) = 0, \nu(0, \mu) = 0, \) uniformly for \( \mu \in (-\mu_0, \mu_0)\).

We, first of all, linearize system (4.31) around origin. Note that the eigenvalues of the linearized system are continuously depend on \( \mu \), and hence for sufficiently small values of \( \mu \), the eigenvalues of the coefficient matrix in the linearized system will be in a similar form with the eigenvalues of the coefficient matrix in (4.1). Thus, by means of a regular transformation, one can show that the right hand side of (4.31) is like the right hand side of (4.9) with the only difference that all coefficients depend on \( \mu \). This is why, without loss of any generality, we assume that (4.31) is in linearized form.
Using polar coordinates one can write system (4.31) in the following form:

\[
\begin{align*}
\frac{dr}{d\phi} &= \lambda(\mu)r + P(r, \phi, z, \mu), \\
\frac{dz}{d\phi} &= b(\mu)z + Q(r, \phi, z, \mu), \quad (r, \phi, z) \notin \Gamma(\mu), \\
\Delta r_{1(\rho, \phi)\in\Gamma(\mu)} &= k_1(\mu)r + \omega(r, \phi, \mu), \\
\Delta \phi_{1(\rho, \phi)\in\Gamma(\mu)} &= \theta_1(\mu) + \gamma(r, \phi, \mu), \\
\Delta z_{1(\rho, \phi)\in\Gamma(\mu)} &= c_0(\mu)z + \tilde{c}(z, \mu).
\end{align*}
\] (4.34)

Let the system

\[
\begin{align*}
\frac{d\rho}{d\phi} &= \lambda(\mu)\rho + P(\rho, \varphi, z, \mu), \\
\frac{dz}{d\varphi} &= b(\mu)z + Q(\rho, \varphi, z, \mu), \quad \varphi \neq \gamma_i(\mu), \\
\Delta \rho|_{\varphi=\gamma_i(\mu)} &= k_i(\mu)\rho + W^1_i(\rho, z, \mu), \\
\Delta \phi|_{\varphi=\gamma_i(\mu)} &= \theta_i(\mu), \\
\Delta z|_{\varphi=\gamma_i(\mu)} &= c_0(\mu)z + W^2_i(\rho, z, \mu),
\end{align*}
\] (4.35)

where \(\gamma_i(\mu), i = 1, \cdots, p\), are angles of \(m_i(\mu)\), be \(B\)-equivalent to (4.34). Here, for each \(i\), the line \(m_i(\mu)\) is obtained by linearizing \(s_i(\mu)\) around the origin. That is, we have \(m_i(\mu) : (a', x) + \mu \frac{\partial s_i(\mu)}{\partial x} = 0\). The functions \(W^1_i(\rho, z, \mu)\) and \(W^2_i(\rho, z, \mu)\) can be defined in the same manner as in (4.15) and (4.16), respectively. Applying \(\psi\)-substitution to (4.35) we get,

\[
\begin{align*}
\frac{d\rho}{d\varphi} &= \lambda(\mu)\rho + F(\rho, \varphi, z, \mu), \\
\frac{dz}{d\varphi} &= b(\mu)z + G(\rho, \varphi, z, \mu), \quad \varphi \neq \varphi_i(\mu), \\
\Delta \rho|_{\varphi=\varphi_i(\mu)} &= k_i(\mu)\rho + W^1_i(\rho, z, \mu), \\
\Delta z|_{\varphi=\varphi_i(\mu)} &= c_0(\mu)z + W^2_i(\rho, z, \mu).
\end{align*}
\] (4.36)

Following the methods, as we did to obtain (4.21) and (4.22) one can see that system (4.36) has two integral manifolds whose equations are given by:

\[
\Phi_0(\varphi, \rho, \mu) = \int_{-\infty}^{\varphi} \pi_0(\varphi, s, \mu)G(\rho(s, \varphi, \rho, \mu), s, z(s, \varphi, \rho, \mu), \mu)ds + \sum_{\varphi_i(\mu) < \varphi} \pi_0(\varphi, \varphi_i^+, \mu)W^2_i(\rho(\varphi_i^+, \varphi, \rho, \mu), z(\varphi_i^+, \varphi, \rho, \mu)),
\] (4.37)

and

\[
\Phi_-(\varphi, z, \mu) = -\int_{\varphi}^{\infty} \pi_-(\varphi, s, \mu)F(\rho(s, \varphi, z, \mu), s, z(s, \varphi, z, \mu), \mu)ds + \sum_{\varphi_i(\mu) < \varphi} \pi_-(\varphi, \varphi_i^+, \mu)W^1_i(\rho(\varphi_i^+, \varphi, z, \mu), z(\varphi_i^+, \varphi, z, \mu)),
\] (4.38)
where
\[ \pi_0(\varphi, s, \mu) = e^{b(\varphi - s)} \prod_{s \leq \varphi / \mu < \varphi} (1 + c_0(\mu)), \]
and
\[ \pi_-(\varphi, s, \mu) = e^{a(\varphi - s)} \prod_{s \leq \varphi / \mu < \varphi} (1 + k_j(\mu)). \]

In (4.37), the pair \((\rho(s, \varphi, \rho, \mu), z(s, \varphi, \rho, \mu))\) denotes a solution of (4.36) satisfying \(\rho(\varphi, \varphi, \rho, \mu) = \rho\). Similarly, \((\rho(s, \varphi, z, \mu), z(s, \varphi, z, \mu))\), in (4.38), is a solution of (4.36) with \(z(\varphi, \varphi, z, \mu) = z\).

Set \(S_0(\mu) = \{ (\rho, \varphi, z) : z = \Phi_0(\varphi, \rho, \mu) \}\) and \(S_-(\mu) = \{ (\rho, \varphi, z) : \rho = \Phi_-(\varphi, z, \mu) \}\).

On the local center manifold, \(S_0(\mu)\), the first coordinate of the solutions of (4.36) satisfies the following system:
\[ \frac{dp}{d\varphi} = \lambda(\mu)p + F(\rho, \varphi, \Phi_0(\varphi, \rho, \mu)), \quad \varphi \neq \varphi_*(\mu), \tag{4.39} \]
\[ \Delta \rho|_{\varphi = \varphi_*(\mu)} = k_i(\mu)p + W_i(\rho, \Phi_0(\varphi, \rho, \mu)). \]

Similar to (4.7) and (4.8) one can define the functions
\[ q_1(\mu) = \exp \left( \lambda(\mu) \left( 2\pi - \sum_{i=1}^{p} \theta_i(\mu) \right) \right) \prod_{i=1}^{p} (1 + k_i(\mu)), \tag{4.40} \]
and
\[ q_2(\mu) = \exp \left( b(\mu) \left( 2\pi - \sum_{i=1}^{p} \theta_i(\mu) \right) \right) \prod_{i=1}^{p} (1 + c_0(\mu)). \tag{4.41} \]

System (4.39) is the system studied in [6] and there it was shown that this system, for sufficiently small \(\mu\), has a periodic solution with period \(T = \psi(2\pi)\). Here we will show that if the first coordinate of a solution of (4.36) is \(T-\) periodic, then so is the second coordinate.

Now, since
\[ \pi_0(\varphi + T, s + T, \mu) = \pi_0(\varphi, s, \mu), \]
\[ \rho(s + T, \varphi + T, \rho, \mu) = \rho(s, \varphi + T, \mu), \]
\[ z(s + T, \varphi + T, \rho, \mu) = z(s, \varphi + T, \rho, \mu), \]

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and \( G \) is \( T \)-periodic in \( \varphi \), we have,

\[
\Phi_0(\varphi + T, \rho, \mu) = \int_{-\infty}^{\infty} \pi_0(\varphi + T, s, \mu)G(\rho(s, \varphi + T, \rho, \mu), s, z(s, \varphi + T, \rho, \mu), \mu)ds \\
+ \sum_{\varphi, (\mu) < \varphi + T} \pi_0(\varphi + T, \varphi_i^+, \mu)W_i^2(\rho(\varphi_i^+, \varphi + T, \rho, \mu), z(\varphi_i^+, \varphi + T, \rho, \mu)) \\
= \int_{-\infty}^{\infty} \pi_0(\varphi, t, \mu)G(\rho(t, \varphi, \rho, \mu), t, z(t, \varphi, \rho, \mu), \mu)dt \\
+ \sum_{\varphi, (\mu) < \varphi} \pi_+(\varphi, \varphi_i^+, \mu)W_i^2(\rho(\varphi_i^+, \varphi, \rho, \mu), z(\varphi_i^+, \varphi, \rho, \mu)) \\
= \Phi_0(\varphi, \rho, \mu),
\]

where in the second equation we have used the substitutions \( s = t + T \) and \( \varphi_i(\mu) = \tilde{\varphi}_i(\mu) + T \). Now, we have the following theorem which, in case of two dimension, can be found in [6].

**Theorem 4.5.1** Assume that \( q_1(0) = 1, q_1'(0) \neq 0, |q_2(0)| < 1 \), and the origin is a focus for (4.33). Then, for sufficiently small \( r_0 \) and \( z_0 \), there exists a function \( \mu = \delta(r_0, z_0) \) such that the solution \((r(\phi, \delta(r_0, z_0)), z(\phi, \delta(r_0, z_0)))\) of (4.34), with the initial condition \( r(0, \delta(r_0, z_0)) = r_0, z(0, \delta(r_0, z_0)) = z_0 \), is periodic with a period, \( T' = (2\pi - \sum_{i=1}^{n} \tilde{\theta}_i)\beta^{-1} + o(|\mu|) \).

### 4.6 Examples

**Example 4.6.1** Consider the following dynamical system:

\[
\begin{align*}
    x_1' &= (0.1 - \mu)x_1 - 20x_2 + 2x_1x_2, \\
    x_2' &= 20x_1 + (0.1 - \mu)x_2 + 3x_1^2z, \\
    z' &= (-0.3 + \mu)z + \mu^2x_1z, \\
    \Delta x_1\|_{(x_1, x_2, z) \in S} &= (\kappa_1 + \mu^3)\cos\left(\frac{\pi}{3}\right)x_1 - (\kappa_1 + \mu^3)\sin\left(\frac{\pi}{3}\right)x_2, \\
    \Delta x_2\|_{(x_1, x_2, z) \in S} &= (\kappa_1 + \mu^3)\sin\left(\frac{\pi}{3}\right)x_1 + (\kappa_1 + \mu^3)\cos\left(\frac{\pi}{3}\right) - 1)x_2, \\
    \Delta z\|_{(x_1, x_2, z) \in S} &= (\kappa_2 + \mu - 1)z,
\end{align*}
\]

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where $\kappa_1 = \exp\left(-\frac{\pi}{120}\right)$, $\kappa_2 = \exp\left(-\frac{\pi}{400}\right)$, $S = s \times \mathbb{R}$, the curve $s$ is given by the equation $x_2 = x_1^2 + \mu x_1^3$, $x_1 > 0$. Using (4.40) and (4.41), one can define

$$q_1(\mu) = (\kappa_1 + \mu^3) \exp((0.1 - \mu) \frac{5\pi}{60}),$$

and

$$q_2(\mu) = (\kappa_2 + \mu) \exp((-0.3 + \mu) \frac{5\pi}{60}).$$

It is easily seen that $q_1(0) = \kappa_1 \exp\left(-\frac{\pi}{120}\right) = 1$, $q_1'(0) = -\frac{\pi}{12} \neq 0$, and $q_2(0) = \exp\left(-\frac{11\pi}{200}\right) < 1$. Therefore, by Theorem 4.5.1, system (4.42) has a periodic solution with period $\approx \frac{5\pi}{60}$ if $|\mu|$ is sufficiently small.

Figure 4.3 shows the trajectory of (4.42) with the parameter $\mu = 0.05$ and the initial value $(x_{10}, x_{20}, z_0) = (0.02, 0, 0.05)$. Since there is an asymptotically stable center manifold, no matter which initial condition is taken, the trajectory will get closer and closer to the center manifold as time increases.

Figure 4.3: A trajectory of (4.42)

In Figure 4.4, the existence of a discontinuous limit cycle is illustrated. There an outer and an inner solution are shown which spiral to a trajectory lying between these two.
Since the exact value of the initial point for the periodic solution is not known we have shown two trajectories of (4.42).

Figure 4.4: There must exist a discontinuous limit cycle of (4.42)

**Example 4.6.2** Consider the following dynamical system:

\[
  \begin{align*}
    x_1' &= (-2 + \mu)x_1 - x_2 + \mu z^2, \\
    x_2' &= x_1 + (-2 + \mu)x_2, \\
    z' &= (-1 + \mu)z + \mu^2 x_1 z, \quad (x_1, x_2, z) \notin S, \\
    \Delta x_1|_{(x_1, x_2, z)\in S} &= (\kappa_1 - x_1^2 - x_2^2) \cos(\frac{\pi}{3}) - 1 \quad x_1 - (\kappa_1 - x_1^2 - x_2^2) \sin(\frac{\pi}{3}) x_2, \\
    \Delta x_2|_{(x_1, x_2, z)\in S} &= (\kappa_2 - 1 - z^2) z, \quad (4.43)
  \end{align*}
\]

where \( \kappa_1 = \exp(\frac{10\pi}{3}) \), \( \kappa_2 = \exp(\frac{5\pi}{6}) \), \( S = s \times \mathbb{R} \), \( s \) is a curve given by the equation \( x_2 = x_1 + \mu^2 x_1^3 \), \( x_1 > 0 \). Using (4.40) and (4.41), one can define

\[
  q_1(\mu) = \kappa_1 \exp((-2 + \mu)\frac{5\pi}{3}),
\]

and

\[
  q_2(\mu) = \kappa_2 \exp((-1 + \mu)\frac{5\pi}{3}).
\]

Now, \( q_1(0) = \kappa_1 \exp(-\frac{10\pi}{3}) = 1 \), \( q_1'(0) = \frac{5\pi}{3} \neq 0 \), \( q_2(0) = \kappa_2 \exp(\frac{5\pi}{3}) = \exp(-\frac{5\pi}{6}) \).
Moreover, associated $D-$system is:

\[
\begin{align*}
x_1' &= -2x_1 - x_2, \\
x_2' &= x_1 - 2x_2, \\
z' &= -z, \quad (x_1, x_2, z) \notin \mathcal{P}, \\
\Delta x_1\big|_{(x_1, x_2, z) \in \mathcal{P}} &= \left( (\kappa_1 - x_1^2 - x_2^2) \cos(\frac{\pi}{3}) - 1 \right) x_1 - (\kappa_1 - x_1^2 - x_2^2) \sin(\frac{\pi}{3}) x_2, \\
\Delta x_2\big|_{(x_1, x_2, z) \in \mathcal{P}} &= (\kappa_1 - x_1^2 - x_2^2) \sin(\frac{\pi}{3}) x_1 + \left( (\kappa_1 - x_1^2 - x_2^2) \cos(\frac{\pi}{3}) - 1 \right) x_2, \\
\Delta z\big|_{(x_1, x_2, z) \in \mathcal{P}} &= (\kappa_2 - 1 - z^2) z, \\
\end{align*}
\]

where $\mathcal{P} = \ell \times \mathbb{R}$, $\ell$ is given by the equation $x_2 = x_1$, $x_1 > 0$, and the origin is stable focus. Indeed, using cylindrical coordinates, denote the solution of (4.44) starting at the angle $\phi = \frac{\pi}{4}$ by $(r(\phi, r_0, z_0), z(\phi, r_0, z_0))$.

We obtain

\[
r_n = (\kappa_1 - r_{n-1}^2) r_{n-1} \exp(-\frac{10\pi}{3}),
\]

and

\[
z_n = (\kappa_2 - z_{n-1}^2) z_{n-1} \exp(-\frac{5\pi}{3}),
\]

where $r_n = r\left(\frac{\pi}{4} + 2\pi n, r_0, z_0\right)$ and $z_n = z\left(\frac{\pi}{4} + 2\pi n, r_0, z_0\right)$. It is easily seen that the sequences $r_n$ and $z_n$ are monotonically decreasing for sufficiently small $(r_0, z_0)$, and there exists a limit of $(r_n, z_n)$. Assume that this limit is $(\xi, \eta) \neq (0, 0)$. Then it implies that there exists a periodic solution of (4.44) and $\xi = (\kappa_1 - \xi^2) \xi \exp(-\frac{10\pi}{3})$ and $\eta = (\kappa_2 - \eta^2) \eta \exp(-\frac{5\pi}{3})$ which give us a contradiction. Thus, $(\xi, \eta) = (0, 0)$. Consequently, the origin is a stable focus of (4.44) and by Theorem 4.5.1 the system (4.43) has a limit cycle with period $\approx \frac{5\pi}{3}$ if $|\mu|$ is sufficiently small.

### 4.7 Deduction

In this chapter, we have studied the existence of a center manifold and the Hopf bifurcation for a certain three dimensional discontinuous dynamical system. The bifurcation of discontinuous cycle is observed by means of the $B-$equivalence method and its consequences. These results will be extended to arbitrary dimension for a more general type of equations.
In this thesis, we have introduced two new classes of differential equations: differential equations with transition conditions on time scales and differential equations on variable time scales. It is necessary to introduce these classes of equations for the investigation of the main results: Hopf-Bifurcation in three-dimensional discontinuous dynamical systems.

The first class of the systems that we introduce in this thesis is the DETC. We make a connection between this class of equations and impulsive differential equations. This connection is given by means of a specific transformation of the independent variable called the $\psi$-substitution [6]. Some benefits of the established connection include knowledge about properties of linear DETC, the investigation of existence of periodic and almost periodic solutions and their stability. We suppose that the problems of stability, oscillations, smoothness of solutions, integral manifolds, theory of functional differential equations can be investigated applying our results. Another interesting opportunity is to analyze equations with more sophisticated time scales.

The second class of the introduced systems is the differential equations on variable time scales. These systems naturally appear when we investigate discontinuous dynamics with non-fixed moments of impulses. Consequently, these results will be needed to develop methods of investigation of mechanical models with impacts. Particularly, interesting problems are related to bifurcations [31, 54, 56, 94], chaos [54]. In this thesis, the theory of this class of equations that we introduce are developed according to these demands.

After introducing these new classes of differential equations, we study the existence
of a center manifold and the Hopf bifurcation for a certain three-dimensional discontinuous dynamical system. The bifurcation of discontinuous cycles is observed by means of the $B$-equivalence method and its consequences [5, 6]. These results will be extended to arbitrary dimension for a more general type of equations.

We expect that these results will be helpful for further investigation of multidimensional discontinuous dynamical systems. In fact, a study related to a three-dimensional hybrid system has been submitted as an invited paper.
REFERENCES


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EDUCATION

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<td>Ph.D.</td>
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<td>2009</td>
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<tr>
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WORK EXPERIENCE

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AWARDS and SCHOLARSHIPS

• Graduate study award, The Scientific and Technological Research Council of Turkey (TÜBİTAK), 2005-2006.
PUBLICATIONS


CONFERENCE PRESENTATIONS


- *Differential equations on variable time scales*, Fifth International Conference on Dynamic Systems and Applications, May 30 - June 2, 2007, Morehouse College, Atlanta, Georgia, USA (with Marat Akhmet).
SEMINARS IN APPLIED DYNAMICS GROUP
(Department of Mathematics and Institute of Applied Mathematics, METU)

• November 2008. Title: *Bifurcation of Discontinuous Cycles*.

• May 2006. Title: *A new model of Oscillators with impacts*.

• October 2005. Title: *The differential equations on time scales through impulsive differential equations*.

• March 2005. Title: *Continuous dependence of solutions of Discontinuous Dynamical Systems on initial data*.

• December 2004. Title: *On smoothness of Discontinuous Dynamical Systems*.

PROJECTS