DIFFERENTIAL EQUATIONS WITH DISCONTINUITIES AND POPULATION DYNAMICS

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

DUYGU ARUĞASLAN ÇİNÇİN

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY
IN
MATHEMATICS

JUNE 2009

Approval of the thesis:

DIFFERENTIAL EQUATIONS WITH DISCONTINUITIES AND POPULATION DYNAMICS

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ABSTRACT

DIFFERENTIAL EQUATIONS WITH DISCONTINUITIES AND POPULATION DYNAMICS

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June 2009, 124 pages

In this thesis, both theoretical and application oriented results are obtained for differential equations with discontinuities of different types: impulsive differential equations, differential equations with piecewise constant argument of generalized type and differential equations with discontinuous right-hand sides. Several qualitative problems such as stability, Hopf bifurcation, center manifold reduction, permanence and persistence are addressed for these equations and also for Lotka-Volterra predator-prey models with variable time of impulses, ratio-dependent predator-prey systems and logistic equation with piecewise constant argument of generalized type.

For the first time, by means of Lyapunov functions coupled with the Razumikhin method, sufficient conditions are established for stability of the trivial solution of differential equations with piecewise constant argument of generalized type. Appropriate examples are worked out to illustrate the applicability of the method. Moreover, stability analysis is performed for the logistic equation, which is one of the most widely used population dynamics models.

The behaviour of solutions for a 2-dimensional system of differential equations with discontinuous right-hand side, also called a Filippov system, is studied. Discontinuity sets intersect at a vertex, and are of the quasilinear nature. Through the $B-$equivalence of that system to an impulsive differential equation, Hopf bifurcation is investigated. Finally, the obtained results are extended to a 3-dimensional discontinuous system of Filippov type. After the existence of a center manifold is proved for the 3-dimensional system, a theorem on the bifurcation of periodic solutions is provided in the critical case. Illustrative examples and numerical simulations are presented to verify the theoretical results.

Keywords: Differential equations with discontinuities, Hopf bifurcation, Lyapunov-Razumikhin method, Center manifold theory, Population dynamics

ÖZ

SÜREKSİZLİKLERİ OLAN DİFERENSİYEL DENKLEMLER VE POPÜLASYON DİNAMİĞİ

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Haziran 2009, 124 sayfa

Bu tezde, süreksizlikleri olan farklı tipteki diferensiyel denklemler: impalsif diferensiyel denklemler, genelleştirilmiş parçalı sabit argümanlı diferensiyel denklemler ve sağ tarafı süreksiz diferensiyel denklemler için hem teorik hem de uygulamaya yönelik sonuçlar elde edilmiştir. Bu denklemler ve değişken zamanlı impalsif etkili Lotka-Volterra avcı-av modelleri, genelleştirilmiş parçalı sabit argüman içeren oran-bağımlı avcı-av sistemleri ve lojistik denklem için kararlılık, Hopf bifürkasyonu, merkez manifolda indirgeme, devamlılık ve direngenlik gibi birçok kalitatif problem ele alınmıştır.

Razumikhin metodu ile birleştirilen Lyapunov fonksiyonları, genelleştirilmiş parçalı sabit argüman içeren diferensiyel denklemlerde ilk kez kullanılarak aşikar çözümün kararlılığı için yeter koşullar elde edilmiştir. Metodun uygulanabilirliğini göstermek amacıyla uygun örnekler sunulmuştur. Ayrıca, en çok kullanılan popülasyon dinamik modellerinden biri olan lojistik denklem için kararlılık analizi yapılmıştır.

Filippov sistemi diye de adlandırılan sağ tarafı süreksiz iki boyutlu bir diferensiyel denklemler sistemi için çözümlerin davranışları araştırılmıştır. Süreksizlik kümeleri yarı doğrusal karakterde olup bir köşede kesişmektedirler. Bu sistemin impalsif diferensiyel denkleme B-denkliğinden faydalanılarak, Hopf bifürkasyonu incelenmiştir. Bulunan sonuçlar son olarak Filippov tipindeki üç boyutlu süreksiz bir sistem için genelleştirilmiştir. Üç boyutlu sistemde merkez manifoldun varlığı gösterildikten sonra kritik durum için periyodik çözümlerin bifürkasyonu üzerine bir teorem elde edilmiştir. Teorik bulguları doğrulamak adına açıklayıcı örnekler ile birlikte sayısal simülasyonlar sunulmuştur.

Anahtar Kelimeler: Süreksizlikleri olan diferensiyel denklemler, Hopf bifürkasyonu, Lyapunov-Razumikhin metodu, Merkez manifold teorisi, Popülasyon dinamiği

To my parents, Hatice and Hadi,
my sister, Emine,
and my husband, Kemal

ACKNOWLEDGMENTS

I am very indebted to the following people who gave me the possibility to complete this thesis.

First of all, I would like to express my deepest gratitude to my supervisor Prof. Dr. Marat Akhmet for his constant support, continuous encouragement and kindly attitude not only relating to my thesis but in many aspects along the way of this study. He has been a great source of inspiration through his stimulating enthusiasm and broad knowledge in the subject matter. His scientific vision and excellent guidance are of considerable value to me.

I offer sincere thanks to my co-supervisor Prof. Dr. Meryem Beklioğlu for her lovely care and intensive efforts during my research. I enjoyed learning more on the population dynamics through the discussions with her.

I would like to extend my thanks to the members of the examining committee for their valuable comments and feedback on the manuscipt of my thesis. I deeply thank all members of the Department of Mathematics for the friendly atmosphere they provided. I am also grateful to my friends, especially to Rezan Sevinik, for their support, suggestions and good will. Furthermore, I would like to express the pleasure of working in a cohesive team with my friends Cemil Büyükadalı, Mehmet Turan and Enes Yılmaz, and thank them for the fruitful discussions during our meetings.

My admirable parents, Hatice and Hadi Aruğaslan, deserve the greatest appreciation and respect for the unconditional love, support and trust they have provided in every stage of my life. I owe my loving thanks to my beloved sister and dearest friend, Emine Aruğaslan, who is not tired of listening me in all my happy and difficult times. I also thank warmly to my family-in-law for being so considerate and supportive.

Last, but not the least, I would like to give my heartfelt thanks to my husband, Kemal Çinçin, for his presence, understanding and belief in me during this long process.

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LIST OF SYMBOLS

$\mathbb{N}$	:	the set of all positive integers
$\mathbb{N}_0$	:	the set of all non-negative integers
$\mathbb{Z}$	:	the set of all integers
$\mathbb{R}$	:	the set of all real numbers
$\mathbb{R}^+$	:	the set of all non-negative real numbers
$\mathbb{R}^n$	:	n-dimensional real space, $n \in \mathbb{N}$
$\langle x, y\rangle$	:	scalar product for all vectors $x, y \in \mathbb{R}^n$, $n \in \mathbb{N}$
$\|\cdot\|$	:	Euclidean norm, $\|x\| = \sqrt{\langle x, x\rangle}$, $x \in \mathbb{R}^n$ and $n \in \mathbb{N}$
$u' = u'(t) = \frac{du}{dt}$	:	derivative of u with respect to t
$u(x) = O(v(x))$	:	$u(x)$ is big-oh of $v(x)$ as $x \to 0$
$u(x) = o(v(x))$	:	$u(x)$ is small-oh of $v(x)$ as $x \to 0$
$(a \hat{,} b]$	:	an oriented interval, i.e., $(a \hat{,} b] = \begin{cases} (a, b], & \text{if} \quad a \leq b, \\ [b, a), & \text{if} \quad b < a. \end{cases}$

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FIGURES

CHAPTER 1

INTRODUCTION

Very few ordinary differential equations can be solved explicitly. Fortunately, in many situations exact solutions are not necessary and only qualitative aspects of the solutions are of interest. Even if an exact solution is obtainable, qualitative analysis can provide a more comprehensive understanding of the situation than the solution itself. That being the case, in the qualitative theory of differential equations, rather than finding exact solutions, it is essential to study their certain characteristics. This theory originated in the groundbreaking works of Lyapunov [122] and Poincaré [149], and has been developed during the last several decades, proving to be extremely effective in the investigation of various physical and biological phenomena.

The principal results of the qualitative approach include existence and uniqueness of solutions, stability of equilibrium points, existence and stability of periodic solutions, bifurcation of equilibrium points, bifurcation of periodic solutions and so on. A solution starting at a certain initial value may evolve towards an equilibrium point or a periodic solution. Equilibrium points and periodic solutions can be stable or unstable, thus attracting or repulsing neighbouring solutions, respectively. Number and stability of equilibrium points or periodic solutions can change as parameters are varied. This qualitative change in the structural behaviour of solutions is called bifurcation, an originally French word introduced by Poincaré [149].

The qualitative theory of ordinary differential equations is rather well developed [24, 48, 51, 53, 63, 64, 79, 82, 87, 88, 111, 136, 151, 172]. Recently, studies that address the qualitative behaviour of systems with discontinuous characteristics have received increasing attention as they naturally arise in real phenomena. In this thesis, we survey several results on differential equations with discontinuities and consider their applications in population dynamics. Before defining the objective of the thesis in detail, we shall describe different types of discontinuous systems and the related problems.

1.1 General Description of Differential Equations with Discontinuities

The theory of differential equations with discontinuities plays an increasingly important role in applications. Many real processes which appear in various problems of biology, chemistry, control theory, ecology, economics, electronics, mechanics, medicine and physics are studied by means of mathematical models with some kind of discontinuity [11, 12, 15, 31, 41, 55, 90, 91, 110, 119, 132]. This fact has increased the need to establish a comprehensive theory for differential equations with discontinuities [2, 8, 20, 22, 52, 69, 72, 83, 98, 105, 108, 113, 129, 134, 137, 140, 142, 152, 153, 157].

In what follows, it is sensible to distinguish between different types of discontinuities that will be treated in this thesis. The first one is the discontinuous external forces also called impulse effects [83, 113, 152]. Another type is the piecewise constant arguments [52, 156, 170] of generalized type. The last one is the case when the right-hand sides of the equations depend discontinuously on the state variables [22, 72, 108]. Containing impulsive differential equations, differential equations with piecewise constant argument of generalized type and differential equations with discontinuous right-hand sides, the range of differential equations with discontinuities is quite vast.

1.1.1 Impulsive Differential Equations

Evolution of a real process can be subject to short-term perturbations whose duration is negligible compared to the duration of the process itself. These perturbations are realized momentarily in the form of impulses causing an instantaneous change in the state of the process. For example, when an oscillating string is struck by a hammer, it experiences a sudden change of velocity; a pendulum of a clock undergoes a rapid change of momentum when it crosses its equilibrium position; harvesting and epidemics lead to a significant decrease in the population density of a species, etc. In order to explain such processes mathematically, it becomes necessary to study impulsive differential equations, also called differential equations with discontinuous trajectories.

Particular examples such as mathematical model of clock [25, 105, 129] played a leading role in the development of the mathematical theory of differential equations with impulsive actions. However, general notions of impulsive differential equations were introduced by Pavlidis [142]-[144]. The book of Samoilenko and Perestyuk [152] is also a fundamental work in the area as it covers many theoretical problems including the existence and uniqueness of solutions, stability, integral sets, periodic and almost periodic solutions, etc.

The interest in the theory of systems with discontinuous trajectories has recently grown due to the needs of modern science [32, 34, 79, 83, 113, 142, 152] and technology [25, 33, 35, 105, 129, 143, 144]. The theory is now being recognized to be not only richer than the corresponding theory of differential equations without impulses, but also represents a more natural framework for mathematical modeling of real processes [99] investigated in various fields of physics, mechanics, economics, population dynamics, ecology, biological systems and optimal control [34, 113].

There are two principally different types of impulsive differential equations: with impulses at fixed times; and with impulsive action at variable times. The mathematical model of a process with impulse effects at fixed times can be described by the following impulsive system [152]

$$\begin{aligned} \frac{dx}{dt} &= f(t,x), \qquad t \neq \tau_i, \\ \Delta x|_{t=\tau_i} &= I_i(x), \end{aligned} \tag{1.1}$$

where $x \in \mathbb{R}^n$, $n \in \mathbb{N}$, $t \in \mathbb{R}$, $\{\tau_i\}$ is a given sequence of times indexed by a finite or an infinite set J, f and I_i are $n-$dimensional vector-valued, continuous functions. A phase point of (1.1) moves along one of the trajectories of the differential equation $x' = f(t,x)$ for all $t \neq \tau_i$. When $t = \tau_i$, the point has a jump $\Delta x|_{t=\tau_i} = x(\tau_i^+) - x(\tau_i^-) = I_i(x(\tau_i^-))$. Hence, a solution $x(t)$ of (1.1) is a piecewise continuous function that has discontinuities of the first kind at $t = \tau_i$.

In the variable case, impulses occur when the phase point of a system intersects the prescribed surfaces in the phase space. It is well known that systems with impulses at variable times generate more difficult theoretical challenges compared to the systems which experience impulses at fixed times. They are mostly used to describe processes in mechanics and electronics. Generally, these systems take the form

$$\begin{aligned} \frac{dx}{dt} &= f(t,x), \quad t \neq \tau_i(x), \\ \Delta x|_{t\,=\,\tau_i(x)} &= I_i(x), \end{aligned} \tag{1.2}$$

where $\tau_i(x)$, $i \in J$, stand for the surfaces of discontinuities. In opposition to the system (1.1), points of discontinuity in (1.2) depend on the solution, which results in a more complicated situation.

Most of the mathematical problems encountered in the study of impulsive differential equations can not be treated by standard techniques developed for ordinary differential equations, especially when the impulses take place at variable moments [14, 20, 21, 145]. Effective methods for the investigation of systems with impulses at variable times can be found in [58, 76, 113, 152].

There also exists an important class of impulsive differential equations that are known as discontinuous dynamical systems. A discontinuous dynamical system [142, 152] can be written as

$$\begin{aligned} \frac{dx}{dt} &= f(x), \quad x \notin \Gamma, \\ \Delta x|_{x\,\in\,\Gamma} &= I(x). \end{aligned} \tag{1.3}$$

A phase point of system (1.3) moves between two consecutive intersections with the set $\Gamma \subset \mathbb{R}^n$ along one of the trajectories of the differential equation $x' = f(x)$, and when the point, say x, intersects with Γ it is mapped into the point $x + I(x)$. Clearly, for discontinuous dynamical systems moments of intersection with the set Γ depend on the solution and hence they are of variable nature. The properties of such systems have not been thoroughly discussed so far. Since a wide range of applications demonstrate the necessity of studying such systems, they attract the attention of many researchers nowadays, see, for example, [2, 14] and the references therein. Hence, the theory of discontinuous dynamical systems is a rapidly developing field at present.

1.1.2 Differential Equations with Piecewise Constant Argument of Generalized Type

Differential equations with delay provide very useful mathematical models for a variety of systems in which the rate of change of the system depends somehow on its past history. It is recognized that differential equations with piecewise constant arguments are closely related to delay differential equations [78, 80] as they contain arguments of delayed or advanced type [5]. These equations have come into existence in an attempt to extend the theory of functional differential equations with continuous arguments to differential equations with discontinuous arguments [170].

The theory of differential equations with piecewise constant argument of the form

$$\frac{dx(t)}{dt} = f(t, x(t), x([t])), \tag{1.4}$$

where $t \in \mathbb{R}$, $x \in \mathbb{R}^n$, and $[\cdot]$ denotes the greatest integer function, was initiated in [52, 156] and has been intensively developed by many authors in the last few decades [1, 80, 141, 169, 170]. Studies of such equations were motivated by the fact that they represent a hybrid of continuous and discrete systems and thus combine the properties of both differential and difference equations.

There exists an extensive literature dealing with differential equations with piecewise constant argument. Results concerning oscillatory behaviour of solutions are included in [1, 55, 158], [169]-[171] and the references cited therein. Existence and uniqueness of solutions, their backward continuation on $(-\infty, 0]$ and asymptotic stability of the trivial solution has been studied in [52, 170]. The problem of existence of periodic and almost periodic solutions for differential equations with piecewise constant argument has been considered in [1, 155, 165, 173] and the references therewith. Later, Cooke and Wiener gathered all previous results including stability, oscillation properties and existence of periodic solutions in their comprehensive survey paper [50]. A brief summary of the theory can also be found in [170]. Other than mathematicians, this class of differential equations has attracted the attention of many scientists due to their wide range of applications in the fields of biology, control theory, neural networks, biomedical models of disease, etc. [12, 41, 55, 77, 125, 132, 155, 158, 167, 169, 177].

Most of the results for differential equations with piecewise constant argument are obtained by reducing them into discrete equations and by applying numerical methods [12, 41, 52, 55, 78, 81, 132, 156, 170]. The method of reduction to discrete equations has been the main instrument of investigation. As a consequence of the existing method, initial value problems are considered only for the case when initial moments are integers or their multiples. In addition, one can not study stability in the complete form as only integers or their multiples are allowed to be discussed for initial moments.

In [5], [8]-[10], the concept of differential equations with piecewise constant argument has been generalized by considering arbitrary piecewise constant functions as arguments. It has been assumed that there is no restriction on the distance between the switching moments of the argument. There, it has been proposed to investigate differential equations of the form

$$\frac{dx(t)}{dt} = A(t)x(t) + f(t, x(t), x(\beta(t))), \tag{1.5}$$

where $x \in \mathbb{R}^n$, $n \in \mathbb{N}$, $t \in \mathbb{R}$, $A(t)$ is a continuous $n \times n$ matrix, $\beta(t) = \theta_i$ if $\theta_i \leq t < \theta_{i+1}$, $i \in \mathbb{Z}$, and θ_i is a strictly ordered sequence of real numbers with $|\theta_i| \to \infty$ as $|i| \to \infty$. Clearly, the greatest integer function $[t]$ is a particular case of the function $\beta(t)$. Indeed, if we choose $\theta_i = i$, $i \in \mathbb{Z}$, then $\beta(t) = [t]$. System (1.5) is called a differential equation with piecewise constant argument of generalized type. For the investigation of such systems [5]-[9], a new approach based on the construction of an equivalent integral equation has been used. By means of this approach, it was shown that the definition of the initial value problem for differential equations with piecewise constant argument of generalized type is similar to the one given for classical ordinary differential equations. Results on the existence and uniqueness of solutions, continuous dependence on the initial value imply that one can investigate stability by taking any real number as an initial moment. Hence, definitions of stability for differential equations with piecewise constant argument of generalized type coincide with the definitions used for ordinary differential equations [88].

1.1.3 Differential Equations with Discontinuous Right-Hand Sides

It is well known that systems modeled by ordinary differential equations can be written in the vector form $x' = f(t, x)$, where $t \in \mathbb{R}$, $x \in \mathbb{R}^n$, $n \in \mathbb{N}$, and f is an $n-$dimensional vector-valued, continuous function. However, there exist many practical situations in which the function on the right-hand sides is discontinuous with respect to the state variable x or to the time variable t, resulting in a differential equation with discontinuous right-hand sides.

The theory of differential equations with discontinuous right-hand sides has been to a great extent developed by the needs of physical problems requiring automatic controls such as relays and switches [72]. These equations are also specific for a wide range of applications arising from mechanical systems with dry friction, electrical circuits with small inductivities, systems with small inertia, dynamical systems with non-differentiable potential, optimization problems with non-smooth data, electrical networks with switches, oscillations in visco-elasticity, optimal control, etc. (see, for example, [25, 72, 73] and the references therein). Mathematical modeling of such applications leads to discontinuous systems which switch between different states and the vector field of each state is associated with a different set of differential equations [31, 114].

Systems described by differential equations with a discontinuous right-hand sides are also called Filippov systems. For these systems, depending on the vector field, either a transversal intersection or a sliding mode may appear. From the mathematical point of view, several ways exist to handle such systems. For example, one way is to use the theory of differential inclusions [72]. Systems with sliding mode are generally extended to a set valued vector field, that is, to differential inclusions for investigational purposes. Another way is a continuous approximation of discontinuities to get smooth differential equations [30]. Method of B-equivalence [18, 19, 22] can also be used effectively in the analysis of differential equations with discontinuous right-hand sides, especially when the sets of discontinuities are of quasilinear nature.

Stimulated by the problems of applied nature, qualitative theory of classical differential equations including the notions of existence, uniqueness, continuous dependence, stability and bifurcation has been adapted for equations with discontinuous right-hand sides. Hence, the amount of literature on the theory of differential equations with discontinuous right-hand sides is vast. Different aspects of the modified theory are elucidated in a variety of books and papers. The books [25, 35, 129] can be considered as an important basis for the development of such systems. A nice introduction can be found in [59, 72, 108]. The fundamental work of Filippov extends a discontinuous differential equation to a differential inclusion [72, 73]. In his book [72] many results from the classical theory of differential equations were shown to be valid for equations with discontinuous right-hand sides as well, and rather than applications, it presents the main trends of the theory of differential equations with discontinuous right-hand sides such as existence and uniqueness, dependence on the initial data, bounded and periodic solutions, stability and so on. Moreover, there exist many publications that consider dry friction problems, existence and bifurcation of periodic solutions for Filippov type systems by means of differential inclusions, see for example [31, 49, 69, 109, 112, 114, 179, 180]. The description of bifurcations for these systems can be found in [114].

In the literature, discontinuities on the right-hand sides are mostly assumed to appear on straight lines [49, 108, 112, 179, 180]. However, Akhmetov and Perestyuk [22] and Akhmetov [18, 19] obtained several theoretical results for such equations with nonlinear sets of discontinuities. The main tool of investigation in these papers was the *B*-equivalence method introduced by the authors. This method was firstly proposed to reduce impulsive systems with variable time of impulses to the systems with fixed moments of impulsive actions [20, 21]. Then it appeared that the method is also applicable to differential equations with discontinuous right-hand sides. That is, differential equations with discontinuous vector fields with nonlinear discontinuity sets can be reduced to impulsive differential equations with fixed moments of impulses. Method of *B*-equivalence will be thoroughly discussed in Chapter 4.

We provide general overviews of the Lyapunov-Razumikhin method, bifurcation phenomena and center manifold theory with extensive literature in the next two sections. These concepts will be treated in detail for differential equations with discontinuities in the following chapters.

1.2 A Brief History of the Lyapunov-Razumikhin Method

In his seminal thesis, Lyapunov [122] proposed two methods, named by himself the first and second method, for stability analysis of motions. It is well known that Lyapunov's second method has proved to be an indispensable tool in the qualitative theory of differential equations. It has been widely used in the investigation of stability of various systems in mathematics and those considered as models in ecology, biology, epidemiology, mechanics and economics [17, 29, 43, 47, 77, 82, 101, 103, 127, 128, 151, 174, 178]. The significance of the method stems from the facts that it enables one to discuss qualitative properties of solutions of a system without solving the differential equation explicitly and that it can be utilized effectively to deal with nonlinear systems.

Based on the Lyapunov's second method, many results from the stability theory of differential equations without delay have been successfully extended and adjusted to systems with time delay. This extension has been carried out in two directions by Krasovskii [104] and Razumikhin [150] individually. The first direction makes use of Lyapunov functionals and is known as Lyapunov-Krasovskii method. On the other hand, functions are much simpler to handle and more practical to determine sufficient conditions for stability. Thus, in the second direction, Lyapunov functions are combined with the Razumikhin technique, which is generally referred to as Lyapunov-Razumikhin method.

Geometrically, Lyapunov function method involves finding a system of closed surfaces that contain and approach the origin. The vector field of motion should be directed inside the areas enclosed by these surfaces which form the level surfaces of a Lyapunov function, say $V(t, x)$. If a solution enters such an area, then it will never leave it again. For systems without deviating argument, the speed vector on the level surfaces is determined only by the present moment of time, i.e., by the point lying on the given surface. However, the speed in systems with argument deviation depends, in some way, also on the previous history which is usually hard to find. In order to be able to estimate the full derivative of Lyapunov function along the solutions, Razumikhin [150] proposed to consider a previous history to lie inside the level surface $V(t, x) = c$, $c \geq 0$. That is to say, the idea was to evaluate the derivative not for all curves that correspond to solutions of the system, but only for those that leave areas enclosed by the level surfaces. The standard technique of proving Lyapunov theorems on stability made such assumption both natural and logical. This led to an additional Razumikhin condition for the Lyapunov theorems, which included the estimation of the derivative of the Lyapunov function on the curve that satisfies $V(s, x(s)) < V(t, x(t))$, $s < t$ [84, 101, 150].

There are many publications in which the Lyapunov function method together with Razumikhin type techniques presents itself as the main and general approach used for stability analysis of a variety of delay differential equations, e.g., see [44, 121, 159, 166] and the references therein.

1.3 An Overview of Bifurcation and Center Manifold Theories

Bifurcation theory is concerned with the topological changes in the qualitative nature of solutions of a family of differential equations as parameters are varied. Bifurcation appears when a small change made on the parameter values of a system causes a sudden qualitative change in its behaviour, e.g., number and type of equilibrium points and periodic solutions may change as parameters vary. Generally, equilibrium solutions are stable to small perturbations if the parameter is in a certain range, and become unstable when it passes through a critical value, called a bifurcation point. Moreover, periodic solutions around equilibrium points may exist in a small neighborhood of a bifurcation point. Bifurcations occur in many physical systems, examples of which can be found in morphodynamics (the forming of meanders in rivers), structural mechanics (the buckling of an elastic beam), utter oscillation of suspension bridges, biochemical reactions (reaction-diffusion models) and cardiac arrhythmias in malfunctioning hearts. More examples of bifurcation can be found in the mathematical studies of physics, chemistry, biology, engineering and population ecology.

Since many practical problems in nature are influenced by discontinuous characteristics of physical phenomena, it is desirable to know whether periodic solutions of a system exist for a certain parameter set and how these periodic solutions can change for a varying parameter of the system. The appearance of a new branch of periodic solutions from a branch of equilibrium points is known as Hopf bifurcation, named after Hopf [92]. The phenomenon of the Hopf bifurcation is local in the sense that all events happen in a small neighborhood of the equilibrium point and the description of the changes in amplitude and period of the oscillation is only correct nearby the bifurcation point of the parameter at which the number of periodic solutions changes. Geometrically, classical Hopf bifurcation for a smooth system means that an equilibrium solution changes its stability as a pair of complex conjugate eigenvalues of the linearization around the equilibrium point cross the imaginary axis of the complex plane. That is, stability changes from stable to unstable through a center type of equilibrium point, or vice versa. In this way, bifurcating periodic orbits are generated by nonlinear perturbation.

During the last decades many results about bifurcation theory have appeared and bifurcations of periodic solutions, i.e., Hopf bifurcation, in smooth vector fields are well understood [79, 89, 111, 124]. Recently, bifurcation features of a system under the influence of a discontinuity have received increasing attention as the variety of the discontinuities leads to rich bifurcation phenomena not observed in smooth systems [40, 57, 60, 62, 70, 114, 140].

The study and classification of various kinds of bifurcation phenomena for non-smooth systems can be summarized as follows. Feigin [70] and Di Bernardo et al. [60, 62] study non-conventional bifurcations, also called C-bifurcation, in Filippov systems. The C-bifurcation concept was first mentioned in [71] and later accepted as a collective name for bifurcations caused by discontinuity [62]. Since then many mathematicians, engineers, and physicists have paid attention to the study and classification of different types of C-bifurcation in piecewise smooth systems. Border-collision bifurcation of fixed points in maps explores the phenomena when a family of fixed points transversely crosses the line of discontinuity as the parameter varies. This bifurcation phenomenon has been studied in different applications [139, 140]. A special case is the corner-collision bifurcation in which some solutions graze corners of the discontinuity sets, and this implies a border-collision phenomenon [61]. Another type of non-conventional bifurcation is the grazing bifurcation which studies the corresponding properties when a periodic orbit intersects the line of discontinuity tangentially [40]. This kind of bifurcation usually occurs in impacting systems [56, 137, 162]. Sliding bifurcation appears when part of a periodic orbit coincides with the line of discontinuity, which has vast application backgrounds [60, 108]. Dankowicz and Nordmark [57] study bifurcations of stick-slip oscillations in a friction model, a non-smooth continuous system. Non-conventional bifurcations of non-smooth discrete mappings are addressed by Nusse and Yorke [138, 140]. Another type of C-bifurcation concerns the creation or disappearance of a periodic orbit that is related to Hopf bifurcation or generalized Hopf bifurcation [14, 49, 69, 108, 109, 112, 114, 116, 179, 180].

Several approaches have been proposed in the literature to analyze the nature of Hopf bifurcation including integral averaging [46], the Fredholm alternative [97], the implicit function theorem [85], the method of multiple scales [135], and center-manifold reduction [42, 89, 172]. The study of center manifolds forms one of the cornerstones of the qualitative theory of differential equations.

The center manifold theory emerged in the sixties of the last century [100, 148], and soon became a very powerful tool for the investigation of stability and bifurcation of various systems [42]. Due to the existence of such manifolds, the analysis of local bifurcations (bifurcations of equilibrium points and periodic orbits) can be reduced to the study of the systems on the center manifolds.

When the linearized system possesses a pair of purely imaginary eigenvalues as well as a finite or infinite number of eigenvalues with negative real parts, center manifold theory guarantees that there exists a two dimensional subspace, i.e., the center manifold, which is tangent to the subspace spanned by the eigenvectors corresponding to the eigenvalues with zero real part. This subspace is invariant under the flow generated by the nonlinear equations. Since the idea of center manifold analysis is to reduce a system, which is high or infinite dimensional, to a two dimensional system by projecting the original dynamics onto the eigenvectors corresponding to purely imaginary eigenvalues, it provides a low dimensional picture of a high or infinite dimensional flow. Accordingly, after a reduction to the center manifold, it becomes easier to determine the quantitative behaviour on it, and in turn the behaviour of the whole system locally. For instance, stability in the full nonlinear equations will be the same as its stability in the flow on the center manifold. Besides, any bifurcations which occur in the neighborhood of the equilibrium point on the center manifold are guaranteed to occur also in the full nonlinear system. In particular, if a limit cycle is born in a Hopf bifurcation in the center manifold, then it will also be born in the full high or infinite dimensional system.

In the last couple of decades many authors have contributed towards developing the general theories of bifurcation and center manifold reduction. For much more detail, we refer to the books [42, 45, 79, 172].

1.4 Models of Population Dynamics

Population dynamics is the branch of mathematical biology which uses mathematical models as a tool to solve biological problems. It studies short and long term changes in the size of populations, and in the meantime, describes the biological and environmental factors leading to those changes. During the last two decades, the growth of population dynamics and the diversity of applications has been astonishing.

When species interact the population dynamics of each species is affected. An interaction between species can occur in several ways that can be classified as one of the three:

(i) predator-prey situation (one benefits by eating the other): the growth rate of one population is decreased and the other increased;

(ii) competition (both are mutually derogative): the growth rate of each population is decreased;

(iii) mutualism or symbiosis (mutually beneficial): the growth rate of each population is enhanced.

The increasing use of mathematics in population dynamics is inevitable as it requires quantitative and qualitative measurements of several ecological activities. The theory of differential equations has been extensively used for decades to study fluctuations in the populations of species, interactions of species with the environment, and competition and mutualism between the species. Therefore, they play an important role for addressing many fundamental questions in population dynamics [126]. Various mathematical models have been proposed in the study of population dynamics in the literature [38, 133]. The dynamic relationship between predators and their prey has long been and will continue to be one of the dominant themes in both ecology and mathematical ecology due to its universal existence and importance [38]. Although these problems appear to be mathematically simple at first sight, they are very complicated and challenging.

To obtain a better understanding of ecological communities, it is necessary to clarify how density of species may change as members are included in or excluded from communities. Mathematical models of many processes in population dynamics are expressed by impulsive differential equations. These processes are characterized by a sudden change in their state. For example, we can consider a fish population in a pool and suppose that some fish are taken out to be sold every week. This action will affect not only the number of fish population in the pool, but it will also affect the rate of change of the population, depending on the number of male or female fish remained within the pool for reproduction. In a predator-prey environment, predators themselves can sometimes change instantaneously due to immigration. There are still some other perturbations in ecology such as epidemics, harvesting, fires, floods, etc. that are not suitable to be treated continually. These perturbations also bring sudden changes to the systems. Recently, some impulsive equations have been introduced in population dynamics in relation to population ecology [118, 119].

One of the common deficiences of population models, especially models of single species, is that the birth rate is considered to act instantaneously whereas there may be a time delay to take the time to reach maturity into account. In fact, time delays occur in almost every situation that to neglect them is to ignore reality. More realistic models thereby should include some of the past states of the systems, that is, a real system should be modeled by differential equations with time delays. Time delays in the dynamics of a single population or of a more interacting species can arise from a great variety of causes. One frequently considered mechanism which introduces delays into the dynamics of population growth is that of age structure. Other delay mechanisms which have been mentioned in the literature include the feeding time, hunger coefficients in predator-prey interactions, replenishment or regeneration time for resources. Time delay due to gestation is also a common example as the consumption of prey by the predator throughout its past history governs the present birth rate of the predator. On various time scales, further causes for delays in population dynamics are food storage of predators, gatherers, reaction times, threshold levels, etc. [54, 123]. Recently, delayed biological systems have received much attention from biologists and mathematicians [37, 54, 67, 68, 78, 107, 123, 132, 160, 168, 175, 176].

In this thesis, we will deal with predator-prey systems involving two species: Lotka-Volterra models with impulses and ratio-dependent models with piecewise constant argument of generalized type in Chapter 2 and a system of one species: the logistic equation again with piecewise constant argument of generalized type in Chapter 3. Below, we briefly describe these models in their most familiar forms. More information will be provided in the forthcoming chapters where we incorporate discontinuities such as impulses and piecewise constant arguments into the models.

1.4.1 A Single Species Model: Logistic Equation

It is well known that the logistic equation of population growth plays an important role in the development of ecological thinking. The classical logistic equation was introduced by Verhulst [163] to describe the population growth in a limited environment. This model is formalized by the differential equation

$$N'(t) = rN(t)(1 - \frac{N(t)}{K}),$$

where $N(t)$ represents the number of individuals at time t, $r > 0$ is the intrinsic growth rate and $K > 0$ is the carrying capacity or the maximum number of individuals that the environment can support. The logistic equation constitutes a significant part of models involving more than one interacting population as well, since it is often assumed that the growth rate of one or more of the populations satisfy the logistic equation in the absence of the other populations, see for example [133].

It is well recognized that delays occur in a variety of biological processes, especially in single population models as mentioned above. It was pointed out by Hutchinson [96] that the logistic equation would be inappropriate for the description of population growth when there is a delay in some of the processes involved. Since then

Hutchinson's equation, known as the delayed logistic equation has been investigated in many papers [54, 78, 107, 133] and the references therein. There also exist several results obtained for the logistic equation with piecewise constant arguments [23, 115, 125, 132, 167].

1.4.2 Predator-Prey Models: Lotka-Volterra Systems

The study of predator-prey systems began with the independent work of Lotka (a physical chemist) [120] and Volterra (a mathematician) [164] in 1920s. Volterra (1926) proposed a simple model to describe the interaction of two species. Since the same system of equations was also derived by Lotka (1920,1925) from a chemical reaction, it is known as the Lotka-Volterra system. It is still one of the most famous models of predator-prey interactions in an ecosystem. If we let $N(t)$ and $P(t)$ denote, respectively, the prey population and the predator population present at time t, then the Lotka-Volterra model is described by

$$\begin{aligned} N' &= aN - bNP, \\ P' &= -cP + dNP, \end{aligned} \tag{1.6}$$

where a, b, c and d are positive constants that stand for the natural growth rate of the prey in the absence of predators, the rate at which predators consume prey, the natural death rate of the predator in the absence of prey and the rate at which predators increase by consuming prey, respectively.

The classical models that study the interaction of two or more species are mostly variations of the Lotka-Volterra system. Owing to its theoretical and practical significances, it is commonly used for modeling predator-prey type of interactions [78, 99, 107, 118]. In recent years, Lotka-Volterra model has also been used in physics, chemistry, economics and other fields [126, 133]. The analysis of qualitative behaviours including stability, periodic oscillation, chaos and bifurcation plays a key role in the studies of this model.

In general, an equilibrium point is said to be a *center* if there exists a neighborhood of the equilibrium where all trajectories are cycles containing it. Besides, if we can find a neighborhood of the equilibrium such that all trajectories starting in it spiral to this equilibrium as $t \to \infty$ ($t \to -\infty$), we call such an equilibrium as a stable (unstable) *focus*. It is well known that system (1.6) possesses two equilibria: the origin $(0,0)$ as saddle, and the positive equilibrium $(c/d, a/b)$ as center, i.e., a unique closed trajectory passes through any point in the first quadrant containing $(c/d, a/b)$ in it (see Figure 1.1). Hence, except the positive equilibrium and the coordinate axes, all solutions of the classical Lotka-Volterra system are periodic.

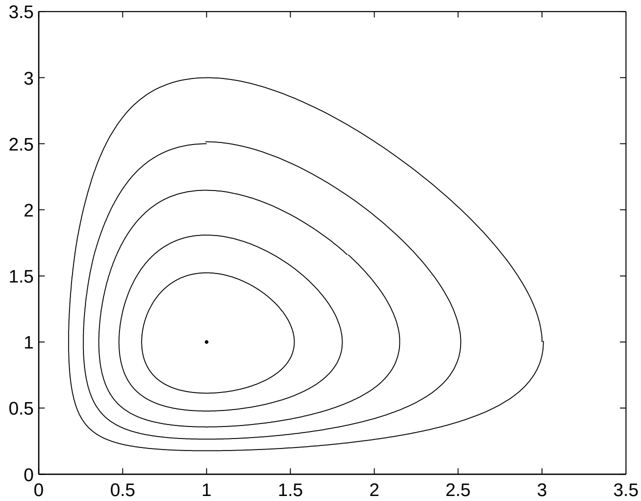

Figure 1.1: A family of closed orbits around the equilibrium $(c/d, a/b) = (1,1)$ for the Lotka-Volterra system (1.6) with $a = b = c = d = 1$.

A major inadequacy of the Lotka-Volterra model is that solutions are not structurally stable as a small perturbation can have a very marked effect. Although the Lotka-Volterra model is unrealistic, it suggests that predator-prey interactions can show a periodic behaviour. In fact, this is not an unexpected phenomena. Because if a prey population increases, it enables the growth of its predator. As the predator population increases, they consume more prey and reduce the prey population. With less food available for the predator, the predator population declines and when it is low enough, this allows the prey population to increase and the whole cycle starts over and over again.

Based on the construction of the Lotka-Volterra system (1.6), if a number of prey equal to c/d and of predator equal to a/b are put into an empty lake, there will be no biological fluctuations. If initial numbers are close to those values, there will be a small fluctuation represented by a small closed curve around the point $(c/d, a/b)$. To produce large fluctuations in the numbers of both species, it is sufficient to begin the experiment with a few members (see Figure 1.1). However, this is not in accordance with the observations and it is improbable that a few members placed in a lake would give rise to large fluctuations. On the contrary, it seems more probable that by putting a certain number of each species into an empty lake, a state of equilibrium should be reached after a certain time [129]. As a consequence, we understand that having the equilibrium point as center the Lotka-Volterra system is not realistic for biological applications. Later, this fact has been developed by modeling more general systems of differential equations [102] or by introducing impulses and delays into the system [99, 107, 118, 132, 160, 176], which give results different from those of (1.6). For example, under certain conditions, instead of a center, equilibrium point may be either a stable focus or a stable node. Moreover, this point may be an unstable focus surrounded by a stable limit cycle [168], which is a closed trajectory in the predator-prey space and not a member of a continuous family of closed trajectories. Limit cycles exhibit a persistent pattern of regular fluctuations. However, it is different from the fluctuations in the Lotka-Volterra system, where the amplitude of oscillation depends entirely on the initial conditions whereas the amplitude of a limit cycle is fixed by intrinsic parameters of the model such as birth rates, predation rates, etc.

One of the other unrealistic assumptions in the Lotka-Volterra model is that the prey growth is unbounded in the absence of predation. After the intensive study of predator-prey systems through the Lotka-Volterra model, various complications have been included to understand the dynamical behaviour of predator-prey systems better [106]. One complication is that the per capita growth rate of predators should be a function of the ratio of prey to predator abundance as suggested by the ratio-dependent theory.

1.4.3 Ratio-Dependent Predator-Prey Models

Standard Lotka-Volterra type models, on which a large body of existing predator-prey theory is built, assume that the per capita rate of predation depends only on the prey number. Recently, there are growing explicit biological and physiological evidences [3], [26]-[28] that in many situations, especially when predators have to search for food and therefore have to share or compete for food, a more suitable and general predator-prey theory should be based on the so-called ratio-dependent theory. Moreover, when the number of predators changes slowly relative to prey number, there is often competition among the predators, and the per capita rate of predation thereby depends on the numbers of both prey and predator, most likely and simply on their ratio. These hypotheses are strongly supported by numerous laboratory experiments and observations [26]-[28] and for mathematicians, ratio-dependent theory seems to be more realistic and capable of producing richer, more reasonable and acceptable dynamics [28, 37, 106] than the usual predator-prey models based on the prey-dependent theory.

Generally, a ratio-dependent predator-prey model takes the form

$$\begin{aligned} x' &= x(a - bx) - \frac{cxy}{my + x}, \\ y' &= -dy + \frac{fxy}{my + x}, \end{aligned} \tag{1.7}$$

where x and y denote, respectively, the densities of the prey and the predator; a, c, d, f and m are the prey intrinsic growth rate, capture rate, death rate of the predator, the conversion rate and the half saturation constant, respectively, a/b gives the carrying capacity of the prey in the absence of predation. Since the model (1.7) contains several parameters, it requires a more complex analysis. Although the idea of ratio-dependent functional response has been in the literature since 1937 [161], the number of publications that study ratio-dependent models is not so large. However, they have received increasing attention in the last couple of decades [3, 26, 27, 65, 66, 74, 94, 106].

We see that system (1.7) describes populations whose members can respond immediately to any change in the environment. However, in real populations both prey and predator require reaction time delays and they can be subject to short term perturbations. Being aware of these facts, several studies have appeared that deal with delayed as well as impulsive ratio-dependent predator-prey models [11, 37, 67, 68, 95, 160, 175, 176].

1.5 Objective of the Thesis

In this thesis, we deal with differential equations with discontinuities and obtain several results on the qualitative properties of these equations. Moreover, we attempt to establish a bridge between mathematics-oriented and application-oriented research in this field.

Models of population dynamics under certain conditions do not satisfy realities. Naturally, more realistic and interesting models of populations should take the impulsive effects, the seasonality of the changing environment and the effects of time delays into account. In this context, differential equations with discontinuities play an important role in the improvement of these models.

Ecological systems are often perturbed by human exploit activities such as planting and harvesting. Such processes are modeled by impulsive differential equations. From this point of view, we consider the classical Lotka-Volterra system (1.6), which has the positive equilibrium point as center and thus ecologically undesirable, with variable time of impulses. These impulses have an artificial character and they occur when the state of species satisfies prescribed conditions. Due to impulse effects, it is possible to obtain the positive equilibrium point as a stable or an unstable focus under the conditions formulated through the parameters of the model. Further, having the positive equilibrium as focus enables us to discuss the bifurcation of periodical processes. We assume that two different types of impulse effects, called 'vertical' jumps in this thesis, appear in the model, i.e., the number of prey remains unchanged whereas predator number decreases (vertical jump going down) or increases (vertical jump going up) abruptly.

Moreover, the ratio-dependent type predator-prey model (1.7) is extended by using generalized piecewise constant delays. Then the problems such as permanence and long term coexistence (or persistence) of species, which are among the most important and ubiquitous concepts in the predator-prey theory, are addressed.

The amount of publications which deal with the stability analysis of differential equations with piecewise constant argument is vast. However, they generally use the method of reduction to discrete equations. Consequently, the analysis of solutions starting at moments which are not integers or their multiples has been unattainable. Particularly, one can not investigate the problem of stability completely, as only integers or their multiples are allowed to be discussed for initial moments. One of the principal goals of this thesis is to meet these challenges by employing the Lyapunov-Razumikhin method for differential equations with piecewise constant argument of generalized type.

In the literature, there are numerous papers in which Lyapunov-Razumikhin method has been successfully utilized on the stability analysis of delay differential equations, functional differential equations, impulsive delay differential equations and impulsive functional differential equations [44, 84, 121, 159, 166]. However, this method has not been used on the stability investigation of differential equations with piecewise constant argument, although they are close to delay differential equations. In this thesis, Lyapunov's second method coupled with the Razumikhin technique is developed for differential equations with piecewise constant argument of generalized type. The application range of the results is illustrated by discussing a logistic equation with piecewise constant delay, and including a comparison with the earlier results obtained by Gopalsamy and Liu in [77].

Bifurcation theory is one of the most developing fields of modern mathematics. Bifurcations in ordinary differential equations are well understood [45, 79, 86, 89, 97, 111, 124, 172]. However, appearance of discontinuities in real processes motivates to improve the qualitative level of investigation and construct a similar theory for differential equations with discontinuities. Thus, bifurcations in non-smooth systems of Filippov type have recently attracted the attention of many mathematicians [31, 49, 69, 109, 112, 114, 116, 179, 180]. We address bifurcation of periodic solutions, i.e., Hopf bifurcation, for 2-dimensional and 3-dimensional systems with discontinuous right-hand sides and try to provide a theoretical basis which can be useful for practical investigations in other fields of the science. First, we consider a planar non-smooth system of differential equations with discontinuous right-hand sides and obtain sufficient conditions for the existence of focus, center and Hopf bifurcation. There are several papers in which Hopf bifurcation is considered for planar non-smooth systems. However, most of these papers consider the systems with discontinuities on a single straight line. We attempt to generalize the bifurcation problem by considering discontinuities on nonlinear sets which consist of arbitrarily finite number of curves intersecting at a vertex. We realize this idea by using the results of the papers [18, 19, 22] which concern different qualitative aspects of differential equations with discontinuous right-hand sides by means of the B-equivalence method [2, 14, 18, 21, 22]. These results, especially the ones on smoothness of solutions lead us to investigate bifurcation problems for non-smooth planar systems of differential equations with discontinuous right-hand sides. It is the advantage of the B-equivalence method that we can analyze systems with nonlinear sets of discontinuities. Second, we study the behaviour of solutions for a 3-dimensional non-smooth system with discontinuities on nonlinear cylindrical surfaces. We show that all solutions that remain sufficiently close to the origin can be captured on a two dimensional invariant center manifold. This reduction allows us to extend the Hopf bifurcation theorem obtained for the planar system to the 3-dimensional system. The approach used in the proof of existence of the center manifold could be considered classical, and consists of using the differential equation to express the invariance of the center manifold under the dynamics to conclude that it must be the graph of a function satisfying a certain fixed point problem.

1.6 Structure of the Thesis

This thesis contains an introductory part which provides elementary notions and a background for the theory of differential equations with discontinuities, their qualitative properties and applications, especially in population dynamics.

In Chapter 2, we investigate the dynamics of Lotka-Volterra predator-prey models influenced by variable time of impulse effects and nonautonomus ratio-dependent systems with piecewise constant argument of generalized type. For the impulsive Lotka-Volterra models, existence of focus and center is proved both in the noncritical and critical cases. Bifurcation of periodic solutions is considered in the critical case. As for the ratio-dependent systems, after constructing equivalent integral equations, problems such as positive invariance, permanence and non-persistence are addressed.

Chapter 3 presents the stability analysis for differential equations with piecewise constant argument of generalized type. Some preliminary definitions and basic problems are discussed for the issue system. Based on the Lyapunov's second method, Razumikhin-type theorems are presented on stability, uniform stability and uniform asymptotic stability. Appropriate examples, one of which contains the logistic equation, are worked out to illustrate the applicability of the results. The stability analysis performed for the logistic equation is compared with the previous ones.

Chapter 4 deals with bifurcations of periodic solutions for 2-dimensional and 3-dimensional non-smooth systems. The notion of B-equivalent impulsive systems is explained. For these systems, problems such as existence of focus and center in the noncritical case, distinguishing between the center and the focus in the critical case and Hopf bifurcation are solved. The center manifold theory is given for the 3-dimensional system. Appropriate examples together with numerical simulations are presented to illustrate the findings.

Finally, in Chapter 5 a short overview and the contributions of the thesis are presented. Some concluding remarks are also given in this chapter.

CHAPTER 2

ANALYSIS OF PREDATOR-PREY MODELS WITH DISCONTINUITIES

2.1 Dynamics of Lotka-Volterra Predator-Prey Models Effected by Impulses

The Lotka-Volterra system describes the interaction of two species in an ecosystem, a prey and a predator. Since there are two species, this system involves two equations

$$\begin{aligned} x' &= ax - bxy, \\ y' &= -cy + dxy, \end{aligned} \tag{2.1}$$

where x and y denote, respectively, the prey and predator population densities; a (the growth rate of prey), b (the rate at which predators consume prey), c (the death rate of predator) and d (the rate at which predators increase by consuming prey) are positive constants. The assumptions in the model (2.1) are as follows.

(i) The prey in the absence of any predation grows unboundedly, which is described by the term ax.

(ii) The effect of the predation is to reduce the prey's per capita growth rate by a term proportional to the prey and predator populations, this is the $-bxy$ term.

(iii) In the absence of any prey, for sustenance the death rate of predator results in exponential decay, this is given by the $-cy$ term.

(iv) Contribution of the prey to the growth rate of predators is proportional to the available prey as well as to the size of the predator population, this is the dxy term.

The xy term can be thought of as representing the conversion of energy from one source to another: bxy is taken from the prey and dxy is given to the predators. We know that system (2.1) has only one positive equilibrium $(c/d, a/b)$ as center. However, having the equilibrium as center, the system is considered to be ecologically undesirable. In other words, the hypotheses of the model (2.1) do not seem to be in accordance with the observations [129].

The Lotka-Volterra population growth model (2.1) does not assume human activities at all. We aim to introduce human intervention by impulsive perturbation. In general, the appearence of such discontinuities can be explained by the fact that development of a biological system may have sudden changes. It is natural that the obtained systems can be written in the form of impulsive differential equations [113, 152]. In this section, our idea is to perturb system (2.1) by impulses at variable moments of time. These impulses, in particular, may include man-made controls which are introduced when the state of species satisfies certain criteria. That is, we consider introducing or removing some members as impulsive control. The approach of impulsive control was also proposed by Liu in [117, 118] and in the paper [11]. However, the research on the Lotka-Volterra system with impulses is not too much yet.

We mainly use the results which were obtained in [2, 14]. One can verify that our sytems satisfy the properties of discontinuous dynamical systems described in [2], that is, existence and uniqueness, continuation of solutions on $\mathbb{R}$, the group property, continuous dependence of solutions on initial value and differentiability of solutions in initial value.

In Section 2.1.1, we formulate two problems: Problem D and Problem U. In the next section, we investigate these problems. Lastly, the Hopf bifurcation for two systems which are associated with Problems D and U is considered in Section 2.1.3.

2.1.1 Formulation of the Problems

In order to be more convenient, we first translate the equilibrium $(c/d, a/b)$ to the origin by the linear transformation

$$\begin{bmatrix} x - c/d \\ y - a/b \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & \dfrac{2d\sqrt{ac}}{bc} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

This transformation takes system (2.1) into the following form

$$\begin{aligned} x_1' &= -\sqrt{ac}\, x_2 - \frac{2d\sqrt{ac}}{c} x_1 x_2, \\ x_2' &= \sqrt{ac}\, x_1 + 2d x_1 x_2. \end{aligned} \tag{2.2}$$

We have new variables x_1 and x_2 possibly with negative values. But, the positiveness of the issue variables x and y are certainly saved in a neighborhood of the equilibrium $(c/d, a/b)$. Clearly, systems (2.1) and (2.2) are qualitatively equivalent. Since $(c/d, a/b)$ is a center of (2.1), the origin is a center of (2.2).

In what follows, we will consider how an impulsive perturbation may change the behaviour of the system (2.2) around the origin. We introduce impulses into the system (2.2) with a careful assumption that they are considered as impulsive control and we are sure that a more adequate explanation of discontinuous population dynamics is a deal of future as well as a closer collaboration of mathematicians and biologists. For that reason, we simply consider the impulsive control as the ability to instantly introduce or remove some members from the environment. It is acceptable and easily realizable as an ecological project. From this point of view, we formulate two problems to investigate: Problem D and Problem U.

2.1.1.1 Problem D: Downsizing the Predator Population as Impulsive Control

Our objective is to bioregulate the Lotka-Volterra system by impulsive perturbation. Ecologically, it seems reasonable to control only the predator density. On the basis of that idea, we consider the impulsive action by means of removing some members of predators from the system. In other words, we downsize the predator population as an impulsive control. For example, if we have fish as predator (and Daphnia as prey) in a lake, the decrease in its density can be expressed by harvesting for commercial fishery. This type of dynamics can be modeled as follows

$$
\begin{aligned}
x_1' &= -\sqrt{ac}\,x_2 - \frac{2d\sqrt{ac}}{c}x_1x_2,\\
x_2' &= \sqrt{ac}\,x_1 + 2dx_1x_2, \quad (x_1, x_2) \notin \Gamma_1,\\
\Delta x_1|_{(x_1,x_2)\in\Gamma_1} &= 0,\\
\Delta x_2|_{(x_1,x_2)\in\Gamma_1} &= \kappa x_2,
\end{aligned}
\tag{2.3}
$$

where $\kappa < 0$ and Γ_1 is a half-straight line in the second quadrant defined by the equation $x_2 = -\sqrt{3}x_1$ for $x_1 < 0$. When the solution meets the set Γ_1 at the time t_1, there exists a vertical jump, $\Delta x_2|_{t_1} = \kappa x_2(t_1) = x_2(t_1+) - x_2(t_1)$ going *down*. That is why, we propose to call determining the behaviour of solutions of system (2.3) around the origin as Problem D.

Additionally, in Section 2.1.3 we will introduce a system with a small parameter μ associated with (2.3) and the problem of Hopf bifurcation for that sytem will be considered as Problem DH.

Remark 2.1.1 Writing (2.3) in x, y coordinates, we obtain the following system

$$
\begin{aligned}
x' &= ax - bxy,\\
y' &= -cy + dxy, \quad (x, y) \notin \tilde{\Gamma}_1,\\
\Delta x|_{(x,y)\in\tilde{\Gamma}_1} &= 0,\\
\Delta y|_{(x,y)\in\tilde{\Gamma}_1} &= \kappa(y - a/b),
\end{aligned}
$$

where $\tilde{\Gamma}_1$ is a half-line defined by the equation $y - a/b = -\frac{d\sqrt{3ac}}{bc}(x - c/d)$ with $x < c/d$. We note that the corresponding impulsive control is only applied to the predator density in x, y coordinates as well.

2.1.1.2 Problem U: Upsizing the Predator Population as Impulsive Control

Similar to the Problem D, we can formulate Problem U for the system

$$
\begin{aligned}
x_1' &= -\sqrt{ac}\,x_2 - \frac{2d\sqrt{ac}}{c}x_1x_2,\\
x_2' &= \sqrt{ac}\,x_1 + 2dx_1x_2, \quad (x_1, x_2) \notin \Gamma_2,\\
\Delta x_1|_{(x_1,x_2)\in\Gamma_2} &= 0,\\
\Delta x_2|_{(x_1,x_2)\in\Gamma_2} &= \kappa x_2,
\end{aligned}
\tag{2.4}
$$

where $\kappa < 0$ and Γ_2 is a straight line which is placed in the fourth quadrant and described by $x_2 = -\sqrt{3}x_1$, $x_1 > 0$. In this sytem, we control the predator density by introducing new members into the environment and thus we have a vertical jump going *up*. In other words, we consider upsizing the predator population as an impulsive control. For the Hopf bifurcation, we shall define Problem UH in a manner similar to the Problem DH, which will be presented later in the subsequent sections.

Remark 2.1.2 Since we aim to construct a method for investigation of impulsive control in the Lotka-Volterra model, we choose particular sets Γ_1 and Γ_2 in the systems (2.3) and (2.4), respectively. Indeed, these systems can be generalized by taking Γ_1 and Γ_2 as unions of arbitrary finite curves emanating from the origin as well as considering impulsive parts in a larger class [14], and by all means they can be analyzed using a similar approach that will be constructed below.

2.1.2 Existence of Foci and Centers

2.1.2.1 Investigation of Problem D

System (2.3) experiences discontinuities when $(x_1, x_2) \in \Gamma_1$. Applying the polar transformation $x_1 = r\cos\phi$, $x_2 = r\sin\phi$, time variable t is excluded and impulse effects appear when the angle variable ϕ is equal to $\frac{2\pi}{3} + 2\pi n$, $n \in \mathbb{Z}$. In this thesis, every angle for a point is considered with respect to the positive half-line of the first coordinate axis.

We can rewrite (2.3) in polar coordinates (r, ϕ) in the following form

$$\begin{aligned} &\frac{dr}{d\phi} = P(r,\phi), \qquad \phi \neq \frac{2\pi}{3} (\text{mod } 2\pi), \\ &\Delta r|_{\phi=\frac{2\pi}{3}(\text{mod } 2\pi)} = \lambda r. \end{aligned} \tag{2.5}$$

Here the independent variable ϕ is ranged over the set

$$\bigcup_{i=-\infty}^{\infty} (2\pi i + \frac{2\pi}{3} + \theta, 2\pi(i+1) + \frac{2\pi}{3}],$$

where $\theta = \tan^{-1}(\frac{-\sqrt{3}\kappa}{4+3\kappa})$. The function $P(r,\phi)$ and the constant λ are given by

$$P(r,\phi) = \frac{r^2(-\frac{2d}{c}\cos\phi + \frac{2d}{\sqrt{ac}}\sin\phi)\cos\phi\sin\phi}{1 + r(\frac{2d}{\sqrt{ac}}\cos\phi + \frac{2d}{c}\sin\phi)\cos\phi\sin\phi}, \; \lambda = \frac{1}{2}\sqrt{1+3(1+\kappa)^2} - 1.$$

Clearly, P is a $2\pi-$periodic function in ϕ and $P = o(r)$.

Since (2.5) is a $2\pi-$periodic system, it is enough to consider it only for the interval $\phi \in [0, 2\pi]$. That is, the system

$$\begin{aligned} &\frac{dr}{d\phi} = P(r,\phi), \quad \phi \neq \frac{2\pi}{3}, \\ &\Delta r|_{\phi=\frac{2\pi}{3}} = \lambda r, \end{aligned} \tag{2.6}$$

where $\phi \in [0, 2\pi] \setminus (\frac{2\pi}{3}, \frac{2\pi}{3} + \theta]$, is provided for investigation.

Indeed, system (2.6) is a “time-scale” differential equation. In order to obtain an impulsive differential equation, we shall use the $\psi-$substitution method which was introduced in [14]. The $\psi-$substitution, on the set $\phi \in [0, 2\pi] \setminus (\frac{2\pi}{3}, \frac{2\pi}{3} + \theta]$, is defined by

$$\psi = \begin{cases} \phi, & \text{if} \quad 0 \le \phi \le \frac{2\pi}{3}, \\ \phi - \theta, & \text{if} \quad \frac{2\pi}{3} + \theta < \phi \le 2\pi. \end{cases}$$

After the substitution, (2.6) reduces to the following impulsive system

$$\begin{aligned} &\frac{dr}{d\psi} = P(r,\psi), \quad \psi \neq \frac{2\pi}{3}, \\ &\Delta r|_{\psi=\frac{2\pi}{3}} = \lambda r, \end{aligned} \tag{2.7}$$

where $\psi \in [0, 2\pi - \theta]$. If we solve (2.7) as an impulsive system [113, 152] and use the backward $\psi-$substitution, we can see that the solution $r(\phi, r_0)$ of (2.6) with $r(0, r_0) = r_0$ is of the form

$$r(\phi, r_0) = \begin{cases} r_0 + \int_0^{\phi} P ds, & \text{if} \quad 0 \le \phi \le \frac{2\pi}{3}, \\ (1+\lambda)\left(r_0 + \int_0^{\frac{2\pi}{3}} P ds\right) + \int_{\frac{2\pi}{3}+\theta}^{\phi} P ds, & \text{if} \quad \frac{2\pi}{3} + \theta < \phi \le 2\pi, \end{cases}$$

where $P = P(r(s, r_0), s)$. We can now construct the Poincaré return map $r(2\pi, r_0)$ on the positive half side of the x_1-axis as follows

$$r(2\pi, r_0) = (1+\lambda)r_0 + (1+\lambda)\int_0^{\frac{2\pi}{3}} P du + \int_{\frac{2\pi}{3}+\theta}^{2\pi} P du.$$

The last equation implies that the origin of (2.6) is a stable focus if $\lambda < 0$ and it is an unstable focus if $\lambda > 0$. As for (2.3), we reach the following conclusion regarding the noncritical case.

THEOREM 2.1.1 *If*

(i) $-2 < \kappa < 0$, *then the origin is a stable focus;*

(ii) $\kappa < -2$, *then the origin is an unstable focus of system (2.3).*

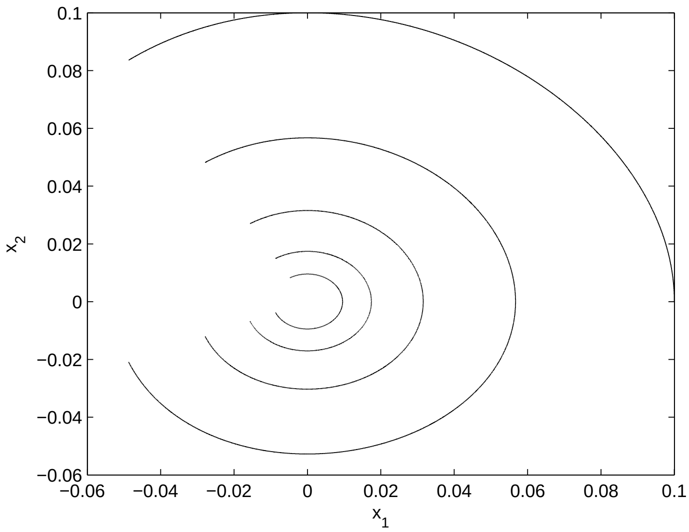

Figure 2.1: A solution of (2.3) with initial condition $(0.1, 0)$, $a = b = c = d = 0.5$ and $\kappa = -1.25$.

If we take, for example, $\kappa = -1.25$, we see from the Figure 2.1 that the origin is a stable focus of (2.3) as stated in the last theorem. For $\kappa = -2.25$, it becomes an unstable focus as shown in the Figure 2.2.

However, if $\lambda = 0$, equivalently if $\kappa = -2$, we have the critical case in which the origin is either a focus or a center. In what follows, we solve this problem of distinguishing between the focus and the center.

We can easily see that the angle θ is equal to $\dfrac{2\pi}{3}$ when $\kappa = -2$. Results of the paper [22] imply, for sufficiently small r_0, that solution $r(\phi, r_0)$ of (2.6), $r(0, r_0) = r_0$, has

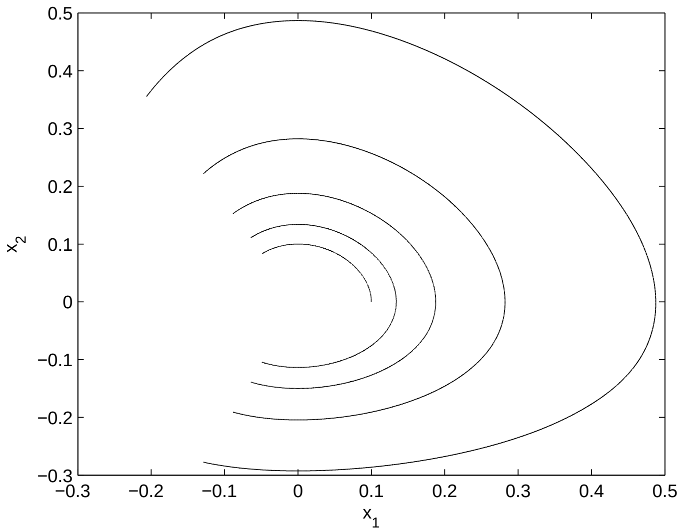

Figure 2.2: A solution of (2.3) with initial condition $(0.1, 0)$, $a = b = c = d = 0.5$ and $\kappa = -2.25$.

the expansion

$$r(\phi, r_0) = \sum_{j=0}^{\infty} r_j(\phi) r_0^j,$$

with $\phi \in [0, 2\pi] \backslash (\frac{2\pi}{3}, \frac{4\pi}{3}]$, $r_0(\phi) = 0$, and $r_1(\phi) = 1$. Then, we can define the Poincaré return map

$$r(2\pi, r_0) = \sum_{j=1}^{\infty} a_j r_0^j,$$

where $a_j = r_j(2\pi)$ and $a_1 = 1$. The function P can also be expanded in a series

$$P(r, \phi) = \sum_{j=2}^{\infty} P_j(\phi) r^j, \tag{2.8}$$

for sufficiently small r. The functions $P_j(\phi)$ in the expansion (2.8) can be found using the definition of the function P. For example, the first two of them are given by

$$P_2(\phi) = (-\frac{2d}{c}\cos\phi + \frac{2d}{\sqrt{ac}}\sin\phi)\cos\phi\sin\phi,$$
$$P_3(\phi) = (\frac{\cos^2\phi - \sin^2\phi}{c\sqrt{ac}} + \frac{\cos\phi\sin\phi}{c^2} - \frac{\cos\phi\sin\phi}{ac})4d^2\cos^2\phi\sin^2\phi,$$

and the functions $P_j(\phi)$, $j = 4, 5, \ldots$, can be evaluated in a similar manner.

From the differential part of (2.6) and the expansion (2.8), one can find that

$$\frac{dr_j(\phi)}{d\phi} = \tilde{P}_j(\phi), \ \ j \geq 2,$$

where $\tilde{P}_2(\phi) = P_2(\phi)$, $\tilde{P}_3(\phi) = 2P_2(\phi)r_2(\phi) + P_3(\phi)$ and we can define $\tilde{P}_j(\phi)$ for $j = 4, 5, \ldots$ similarly.

Since we consider the critical case $\kappa = -2$, which implies that $\lambda = 0$ in the second equation of (2.6), we have $r_j(\frac{4\pi}{3}) - r_j(\frac{2\pi}{3}) = 0$ for all $j = 2, 3, \ldots$. Hence, the coefficients $r_j(\phi)$, $j = 2, 3, \ldots$ with $\phi \in [0, 2\pi] \setminus (\dfrac{2\pi}{3}, \dfrac{4\pi}{3}]$, $r_j(0) = 0$, are solutions of the system

$$\begin{aligned} &\frac{dr}{d\phi} = \tilde{P}_j(\phi), \qquad \phi \neq \frac{2\pi}{3}, \\ &\Delta r|_{\phi=\frac{2\pi}{3}} = 0. \end{aligned}$$

As $a_j = r_j(2\pi)$, we can now evaluate a_j in the expansion of $r(2\pi, r_0)$:

$$a_j = \int_0^{\frac{2\pi}{3}} \tilde{P}_j(\phi)d\phi + \int_{\frac{4\pi}{3}}^{2\pi} \tilde{P}_j(\phi)d\phi$$

for $j = 2, 3, \ldots$.

For the critical case, the sign of the first nonzero element of the sequence a_j, $j = 2, 3, \ldots$, determines what type of an equilibrium point the origin is. The origin is a stable (unstable) focus if the first nonzero element is negative (positive). If all $a_j = 0$, $j = 2, 3, \ldots$, then the origin is a center [14]. That is why, in order to distinguish between the center and the focus we first need the value of a_2:

$$a_2 = \int_0^{\frac{2\pi}{3}} P_2(\phi)d\phi + \int_{\frac{4\pi}{3}}^{2\pi} P_2(\phi)d\phi = \frac{d\sqrt{3}}{2\sqrt{ac}}.$$

Since a_2 is positive, the following theorem holds.

Theorem 2.1.2 *If $\kappa = -2$ then the origin of system (2.6) is an unstable focus, which implies in turn that the origin is an unstable focus for (2.3).*

A simulation result for the critical case $\kappa = -2$ can be seen in the Figure 2.3, which shows that the origin is an unstable focus of (2.3).

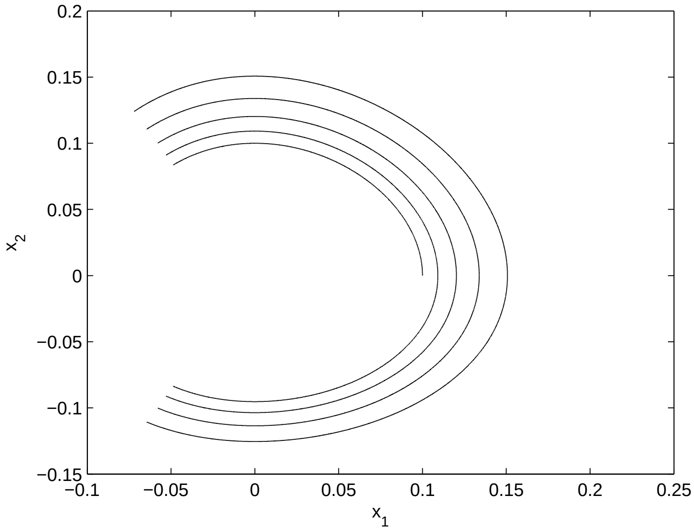

Figure 2.3: A solution of (2.3) with initial condition $(0.1, 0)$, $a = b = c = d = 0.5$ and $\kappa = -2$.

2.1.2.2 Investigation of Problem *U*

Introducing polar coordinates as well as taking the $2\pi-$periodicity into account, system (2.4) can be written in the following form

$$\begin{aligned} &\frac{dr}{d\phi} = P(r, \phi), \quad \phi \neq \frac{5\pi}{3}, \\ &\Delta r|_{\phi = \frac{5\pi}{3}} = \lambda r, \end{aligned} \tag{2.9}$$

for $\phi \in [\pi, 3\pi] \setminus (\frac{5\pi}{3}, \frac{5\pi}{3} + \theta]$, where $P(r, \phi)$, λ and θ are the same as described in the investigation of Problem D. For a solution $r(\phi, r_0)$, $r(\pi, r_0) = r_0$ of (2.9), the Poincaré return map defined on the negative half side of the x_1-axis is

$$r(3\pi, r_0) = (1 + \lambda)r_0 + (1 + \lambda)\int_{\pi}^{\frac{5\pi}{3}} Pdu + \int_{\frac{5\pi}{3}+\theta}^{3\pi} Pdu.$$

Clearly, the noncritical case, that is, $\lambda < 0$ or $\lambda > 0$, can be treated similarly as discussed for Problem D. We shall consider the critical case $\lambda = 0$ in the way described hereinabove. We know that the following expansion

$$r(\phi, r_0) = \sum_{j=0}^{\infty} r_j(\phi) r_0^j,$$

exists with $\phi \in [\pi, 3\pi] \setminus (\frac{5\pi}{3}, \frac{7\pi}{3}]$, $r_0(\phi) = 0$, and $r_1(\phi) = 1$. Then, we can define the Poincaré return map:

$$r(3\pi, r_0) = \sum_{j=1}^{\infty} k_j r_0^j,$$

where $k_j = r_j(3\pi)$ and $k_1 = 1$. Evaluating the element k_2, we obtain

$$k_2 = \int_{\pi}^{\frac{5\pi}{3}} P_2(\phi)d\phi + \int_{\frac{7\pi}{3}}^{3\pi} P_2(\phi)d\phi = -\frac{d\sqrt{3}}{2\sqrt{ac}} < 0.$$

It is seen that the critical case of Problem U leads to a different result with regard to the corresponding case of Problem D. Combining the results obtained both in the noncritical and critical cases, following assertion can be formulated.

Theorem 2.1.3 *If*

(i) $-2 \leq \kappa < 0$, *then the origin is a stable focus;*

(ii) $\kappa < -2$, *then the origin is an unstable focus of system (2.4).*

We can see from the figures below that the origin is a stable (unstable) focus of (2.4) for $\kappa = -1.25$ ($\kappa = -2.25$). In the critical case $\kappa = -2$, the origin is a stable focus (see Figure 2.6).

2.1.3 Bifurcation of Periodic Solutions

It is always the case with realistic biological models that they involve parameters, generally denoted by μ. In order to study existence of limit cycle solutions in such models, Hopf bifurcation theory plays a crucial role. To put the theory geometrically, we can say that if an equilibrium solution changes its stability as a pair of complex conjugate eigenvalues of the linearization around the equilibrium point cross the imaginary axis of the complex plane at a bifurcation point, then at least one small amplitude limit cycle exists about the equilibrium solution and in a small neighborhood of the bifurcation point.

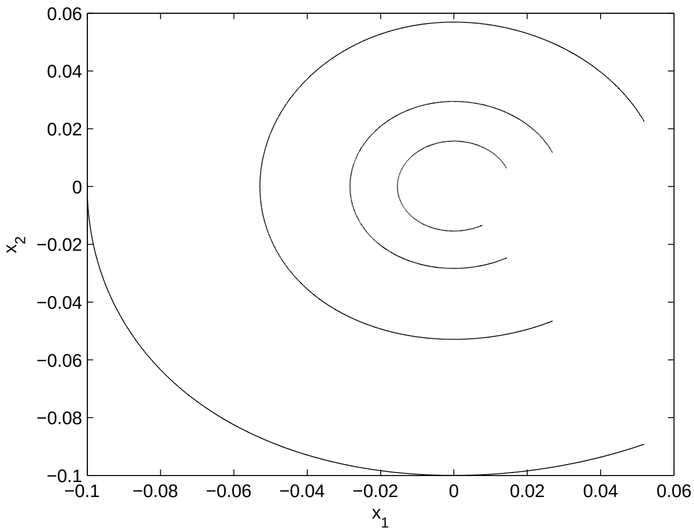

Figure 2.4: A solution of (2.4) with initial condition $(-0.1, 0)$, $a = b = c = d = 0.5$ and $\kappa = -1.25$.

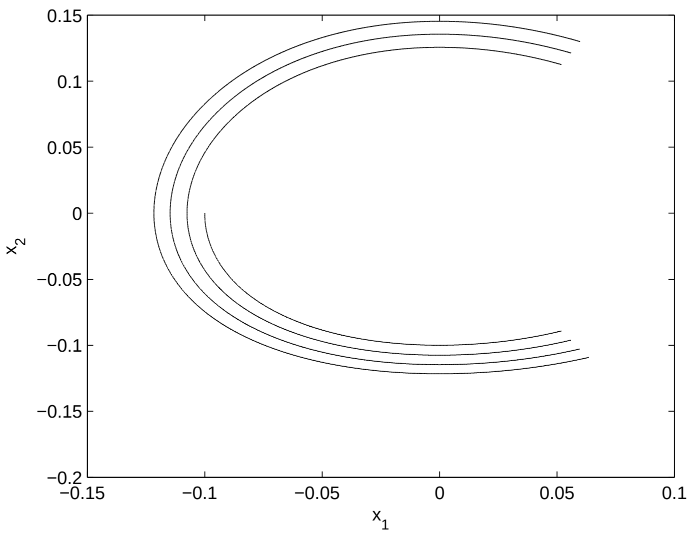

Figure 2.5: A solution of (2.4) with initial condition $(-0.1, 0)$, $a = b = c = d = 0.5$ and $\kappa = -2.25$.

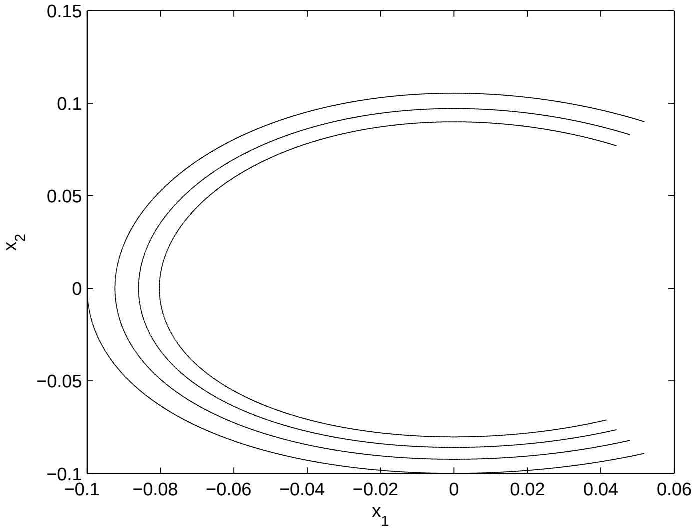

Figure 2.6: A solution of (2.4) with initial condition $(-0.1, 0)$, $a = b = c = d = 0.5$ and $\kappa = -2$.

Since the origin is a center, and not a focus, it is not possible to apply Hopf bifurcation theory for system (2.2) which is the transformed Lotka-Volterra population growth model in x_1, x_2 coordinates. Nevertheless, one can obtain the origin as a stable or an unstable focus through impulsive control so that it becomes possible to investigate the problem of Hopf bifurcation, i.e., bifurcation of periodic solutions, for the Lotka-Volterra system.

2.1.3.1 Problem DH: Hopf Bifurcation Related to Problem D

We introduce the following discontinuous dynamical system

$$\begin{aligned}
x_1' &= \mu x_1 - \sqrt{ac}x_2 - \frac{2d\sqrt{ac}}{c}x_1x_2, \\
x_2' &= \sqrt{ac}x_1 + \mu x_2 + 2dx_1x_2, \qquad (x_1, x_2) \notin \Gamma_1(\mu), \\
\Delta x_1|_{(x_1,x_2)\in\Gamma_1(\mu)} &= 0, \\
\Delta x_2|_{(x_1,x_2)\in\Gamma_1(\mu)} &= (\kappa+\mu)x_2,
\end{aligned} \tag{2.10}$$

where $\Gamma_1(\mu)$ is determined by the equation $x_2 = -\sqrt{3}x_1 + \mu x_1$ for $x_1 < 0$. Let $\gamma(\mu)$ denote the angle of the points lying on $\Gamma_1(\mu)$. In (2.10), μ appears to be an internal control parameter of the populations. When $\mu = 0$, (2.10) reduces to the system (2.3) described in Section 2.1.2. That is why, we say that system (2.10) is associated with (2.3).

Using polar coordinates and keeping the $2\pi-$periodicity in mind, (2.10) can be written as follows

$$\begin{aligned} \frac{dr}{d\phi} &= \frac{\mu}{\sqrt{ac}} r + P(r,\phi,\mu),\ \phi \neq \gamma(\mu), \\ \Delta r|_{\phi=\gamma(\mu)} &= \lambda(\mu) r, \end{aligned} \tag{2.11}$$

for $\phi \in [0, 2\pi] \setminus (\gamma(\mu), \gamma(\mu) + \theta(\mu)]$, where

$$\theta(\mu) = \tan^{-1}\left(\frac{(-\sqrt{3}+\mu)(\kappa+\mu)}{1+(1+\kappa+\mu)(-\sqrt{3}+\mu)^2}\right),$$

$$P(r,\phi,\mu) = \frac{r^2\left(-\frac{2d}{c}(1+\frac{\mu}{a})\cos\phi + \frac{2d}{\sqrt{ac}}(1-\frac{\mu}{c})\sin\phi\right)\cos\phi\sin\phi}{1+r(\frac{2d}{\sqrt{ac}}\cos\phi + \frac{2d}{c}\sin\phi)\cos\phi\sin\phi},$$

and

$$\lambda(\mu) = \sqrt{\frac{1+(1+\kappa+\mu)^2(-\sqrt{3}+\mu)^2}{1+(-\sqrt{3}+\mu)^2}} - 1.$$

Let $r(\phi, r_0, \mu)$, $r(0, r_0, \mu) = r_0$, be the solution of (2.11). On the interval $[0, \gamma(\mu)]$, we have

$$r(\phi, r_0, \mu) = \exp(\frac{\mu}{\sqrt{ac}}\phi) r_0 + \int_0^\phi \exp(\frac{\mu}{\sqrt{ac}}(\phi - s)) P ds.$$

Next, the solution $r(\phi, r_0, \mu)$ of (2.11) on $(\gamma(\mu) + \theta(\mu), 2\pi]$ is given by

$$\begin{aligned} r(\phi, r_0, \mu) &= (1+\lambda(\mu))\exp(\frac{\mu}{\sqrt{ac}}(\phi - \theta(\mu))) r_0 \\ &+ (1+\lambda(\mu))\int_0^{\gamma(\mu)} \exp(\frac{\mu}{\sqrt{ac}}(\phi - \theta(\mu) - s)) P ds \\ &+ \int_{\gamma(\mu)+\theta(\mu)}^{\phi} \exp(\frac{\mu}{\sqrt{ac}}(\phi - s)) P ds, \end{aligned}$$

where $P = P(r(s, r_0, \mu), s, \mu)$. We can evaluate the Poincaré map $r(2\pi, r_0, \mu)$ by means of the last equation. Let $q(\mu)$ denote the coefficient of r_0 in $r(2\pi, r_0, \mu)$. Then, we have

$$r(2\pi, r_0, \mu) = q(\mu) r_0 + o(r_0),$$

where

$$q(\mu) = (1 + \lambda(\mu)) \exp(\frac{\mu}{\sqrt{ac}}(2\pi - \theta(\mu))).$$

Results from [14] imply that conditions $q(0) = 1$ and $q'(0) \neq 0$ are sufficient for the existence of periodical processes in system (2.11). It can be evaluated easily that when $\kappa = -2$, $q(0) = 1$ and $q'(0) = -\frac{3}{4} + \frac{4\pi}{3\sqrt{ac}}$.

Applying the technique which is used in the paper [14], we can state the following theorem, which will be proven in Chapter 4 (see Theorem 4.1.4) for a more general case.

Theorem 2.1.4 *If $\kappa = -2$ and $\sqrt{ac} \neq \frac{16\pi}{9}$ then for sufficiently small r_0, there exists a function $\mu = \delta(r_0)$, $\delta(0) = 0$, such that the solution $r(\phi, r_0, \delta(r_0))$ of (2.11) is periodic with period 2π. Moreover, the closed trajectory is an unstable limit cycle. The period of the corresponding periodic solution of (2.10) is $T = \frac{4\pi}{3\sqrt{ac}} + o(|\mu|)$.*

Simulated for two different initial values, it can be seen from Figure 2.7 that system (2.10) admits an unstable periodic solution.

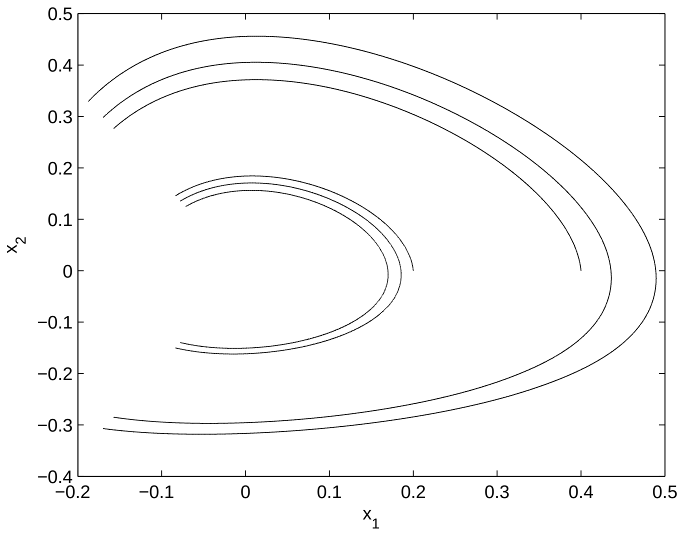

Figure 2.7: The simulation result showing the existence of an unstable closed trajectory of (2.10) with $a = b = c = d = 0.5$, $\mu = -0.03$ and $\kappa = -2$.

2.1.3.2 Problem UH: Hopf Bifurcation Related to Problem U

We consider the system

$$\begin{aligned}
&x_1' = \mu x_1 - \sqrt{ac}x_2 - \frac{2d\sqrt{ac}}{c}x_1x_2,\\
&x_2' = \sqrt{ac}x_1 + \mu x_2 + 2dx_1x_2, \quad (x_1,x_2) \notin \Gamma_2(\mu),\\
&\Delta x_1|_{(x_1,x_2)\in\Gamma_2(\mu)} = 0,\\
&\Delta x_2|_{(x_1,x_2)\in\Gamma_2(\mu)} = (\kappa+\mu)x_2,
\end{aligned} \tag{2.12}$$

where $\Gamma_2(\mu)$ is a curve given by $x_2 = -\sqrt{3}x_1 + \mu x_1$ with $x_1 > 0$. We denote the angle of the points on $\Gamma_2(\mu)$ by $\xi(\mu)$. Clearly, system (2.12) is associated with (2.4). In polar coordinates, this system can be written as

$$\begin{aligned}
&\frac{dr}{d\phi} = \frac{\mu}{\sqrt{ac}}r + P(r,\phi,\mu),\ \phi \neq \xi(\mu),\\
&\Delta r|_{\phi=\xi(\mu)} = \lambda(\mu)r,
\end{aligned} \tag{2.13}$$

for $\phi \in [\pi, 3\pi] \setminus (\xi(\mu), \xi(\mu) + \theta(\mu)]$, where $P(r,\phi,\mu)$, $\lambda(\mu)$ and $\theta(\mu)$ are the same as defined above. Using the similar discussions made in Problem DH, we can conclude the following result.

Theorem 2.1.5 *If $\kappa = -2$ and $\sqrt{ac} \neq \frac{16\pi}{9}$ then for sufficiently small r_0, there exists a function $\mu = \delta(r_0)$, $\delta(0) = 0$, such that the solution $r(\phi, r_0, \delta(r_0))$ of (2.13) is periodic with 2π. Moreover, the closed trajectory is a stable limit cycle. The period of the corresponding periodic solution of (2.12) is $T = \frac{4\pi}{3\sqrt{ac}} + o(|\mu|)$.*

We can see from Figure 2.8 that system (2.12) admits a stable limit cycle. That is, two different solutions of the system approach a periodic orbit from inside and outside.

2.1.4 Conclusion

Under the assumption that the coefficients a, b, c, d of the Lotka-Volterra system are positive, we may conclude that the complex behaviour of solutions depends on the values of the coefficient κ which appears in the impulsive part of systems (2.3), (2.4), (2.10) and (2.12). That is, the problem of controllability of the Lotka-Volterra system by the proposed impulsive control is constructive.

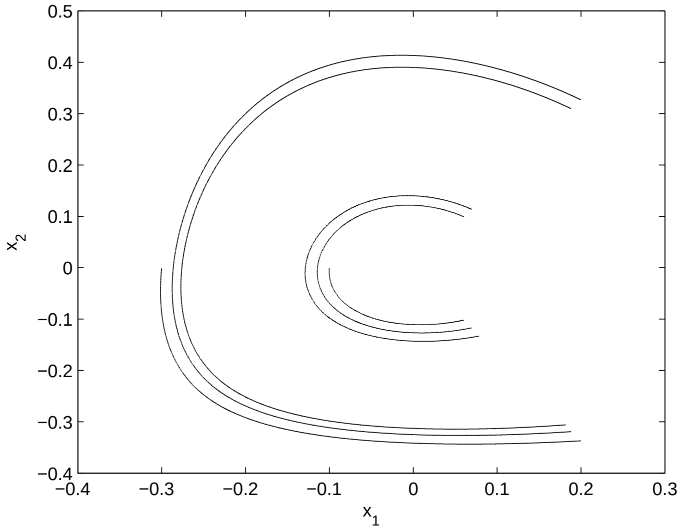

Figure 2.8: The simulation result showing the existence of a stable closed trajectory of (2.12) with $a = b = c = d = 0.5$, $\mu = 0.03$ and $\kappa = -2$.

2.2 Dynamics of Ratio-Dependent Predator-Prey Systems with Piecewise Constant Argument of Generalized Type

Predator-prey systems with functional response have received great attention in recent years. Problems which appear in the analysis of such systems are quite complicated and challenging due to their complex dynamics. Predator-prey models with prey-dependent functional response of the form $p(x) = \dfrac{x}{m+x}$, where $m > 0$ is the half saturation constant, have been well studied (see, e.g., Freedman [75] and the references cited therein). The traditional prey-dependent model is described by the system

$$\begin{aligned} x' &= x(a - bx) - cy\frac{x}{m+x}, \\ y' &= y(-d + f\frac{x}{m+x}), \end{aligned} \tag{2.14}$$

where a prey population x serves as food for a predator population y. The model parameters a, b, c, d, f and m are assumed to be positive and they denote the growth rate of prey, strength of competition among individuals of prey species, capturing rate, death rate of the predator, conversion rate and the half saturation constant, respectively. Here, a/b is the carrying capacity of the prey population which has a logistic growth rate in the absence of predation.

On the other hand, it was recently argued by many biologists that a more suitable functional response should depend on the ratio of prey to predator abundance, particularly when predators have to search for food and hence, have to share or compete for food. Empirical evidence from field and experimental studies also shows that most natural systems are closer to ratio dependence than to prey dependence [3], [26]-[28]. In this light, Arditi and Ginzburg [28], proposed a ratio-dependent response function of the form $p(x/y) = \dfrac{x/y}{m + x/y} = \dfrac{x}{my + x}$ and the following ratio-dependent predator-prey model

$$\begin{aligned} x' &= x(a - bx) - cy\frac{x}{my + x}, \\ y' &= y(-d + f\frac{x}{my + x}). \end{aligned} \tag{2.15}$$

Analyses of such ratio-dependent models show that they produce richer and more admissible dynamics [74, 94, 106]. Most of these analyses assume the model parameters as constant. Ratio-dependent models have not been well studied yet in the sense that most results are for models with constant environment. This means that the models have been assumed to be autonomous where all biological or environmental parameters are constant in time. However, this is rarely the case in real life as many biological and environmental parameters do vary in time. For example, these parameters can be variable due to seasonal fluctuations. When this is taken into account, a model must be nonautonomous, which is, of course, more difficult to analyze in general. Fan et al. [65] incorporate the varying property of the parameters into the model and carry out systematic studies on the global dynamics of the following ratio-dependent model, i.e., the nonautonomous version of (2.15)

$$\begin{aligned} x' &= x(a(t) - b(t)x) - \frac{c(t)xy}{m(t)y + x}, \\ y' &= y(-d(t) + \frac{f(t)x}{m(t)y + x}), \end{aligned} \tag{2.16}$$

where variable parameters $a(t), b(t), c(t), d(t), f(t)$ and $m(t)$ have the same biological significances as described for system (2.14). Additionally, Fan and Wang [66] proposed a discrete analogue of (2.16) by reducing the following system of differential equations with piecewise constant argument

$$
\begin{aligned}
\frac{1}{x_1(t)}\frac{dx_1(t)}{dt} &= a([t]) - b([t])x_1([t]) - \frac{c([t])x_2([t])}{m([t])x_2([t]) + x_1([t])},\\
\frac{1}{x_2(t)}\frac{dx_2(t)}{dt} &= -d([t]) + \frac{f([t])x_1([t])}{m([t])x_2([t]) + x_1([t])}, \qquad t \neq 0, 1, 2, \ldots,
\end{aligned}
\tag{2.17}
$$

to discrete equations. Here, $[t]$ denotes the maximal integer not greater than t.

The theory of differential equations with piecewise constant arguments was initiated by Cooke and Wiener [52] and Shah and Wiener [156]. It is well recognized that these equations are closely related to delay differential equations [78, 80] and that predator-prey systems with time delays are more realistic and more relevant in ecology. Regarding this approach, dynamics of populations modeled by differential equations with piecewise constant arguments have been studied quite extensively. Examples of the application of these equations to the problems of biology can be found in [12, 23, 41, 78, 115, 125, 131, 132].

2.2.1 Descripton of the Models

The principal aim of this section is to incorporate piecewise constant (delayed) argument of generalized type [5, 8] into model (2.16). The existing method of investigation of differential equations with piecewise constant arguments is based on the reduction to discrete equations. For example, in (2.17), the piecewise constant argument appears in all arguments on the right-hand side, allowing the reduction of this system to discrete equations. However, we discuss the case when not all arguments on the right-hand side are piecewise constant argument of generalized type and for the equations that we shall propose below, it is not possible to make the reduction to discrete equations. That is why, our approach is interesting and valuable. We replace different types of delayed arguments, which are introduced previously in some ratio-dependent predator-prey models [37, 67, 68, 95, 160, 175, 176], by piecewise constant argument of generalized type. First, following the logic of [67, 68, 95], we incorporate the piecewise constant argument of generalized type into the prey growth rate response to resources limitations as well as into the positive feedback in the average growth rate of the predator due, for example, to gestation or digestion. Second, the effect of introducing a piecewise constant delay into the predator's reaction to changes in the prey population will be considered. Hence, the piecewise constant argument of generalized type will appear only in the predator equation [37, 160, 175, 176], which requires more easily verifiable conditions. These ideas lead us, in turn, to consider two models of the form

$$\begin{aligned} x' &= x\left(a(t) - b(t)x(\beta(t))\right) - \frac{c(t)xy}{m(t)y + x}, \\ y' &= y\left(-d(t) + \frac{f(t)x(\beta(t))}{m(t)y(\beta(t)) + x(\beta(t))}\right), \end{aligned} \tag{2.18}$$

and

$$\begin{aligned} x' &= x\left(a(t) - b(t)x\right) - \frac{c(t)xy}{m(t)y + x}, \\ y' &= y\left(-d(t) + \frac{f(t)x(\beta(t))}{m(t)y(\beta(t)) + x(\beta(t))}\right), \end{aligned} \tag{2.19}$$

where $t \in \mathbb{R}$, $\beta(t) = \theta_i$ if $\theta_i \leq t < \theta_{i+1}$, $i \in \mathbb{Z}$, is an identification function, $\{\theta_i\}$, $i \in \mathbb{Z}$, is a strictly ordered sequence of real numbers, $|\theta_i| \to \infty$ as $|i| \to \infty$. In the models (2.18) and (2.19), both the fundamental information in memory and predecisions of the present time drive the state.

In the present section, we shall obtain analogue of the results such as positive invariance, permanence and other related properties discussed in [65]. The proofs are adapted to our case, taking the deviation of piecewise constant argument of generalized type into account.

Clearly, the greatest integer function $[t]$ is a particular case of the identification function $\beta(t)$. Indeed, if we take $\theta_i = i$, $i \in \mathbb{Z}$, then we obtain $\beta(t) = [t]$. Therefore, systems (2.18) and (2.19) belong to the class of differential equations with piecewise constant argument of generalized type [8].

In the rest of this section, following assumptions will be needed.

(B1) The model parameters $a(t)$, $b(t)$, $c(t)$, $d(t)$, $f(t)$ and $m(t)$ are continuous and bounded from below and above by positive constants.

(B2) There exist a positive number θ such that $\theta_{i+1} - \theta_i \leq \theta$, $i \in \mathbb{Z}$.

Meanwhile, for convenience, we adopt the notations below throughout this section.

(N1) $F^u = \sup_{t\in\mathbb{R}} F(t)$ and $F^l = \inf_{t\in\mathbb{R}} F(t)$ for a continuous bounded function $F(t)$ on $\mathbb{R}$;

(N2) $x_i = x(\theta_i),\ \ y_i = y(\theta_i),\ i \in \mathbb{Z}$;

(N3) $\phi(t,x,y,z) = a(t) - b(t)z - \dfrac{c(t)y}{m(t)y + x}$,

$\psi(t,x,y) = -d(t) + \dfrac{f(t)x}{m(t)y + x}$,

$\varphi(t,x,y) = a(t) - b(t)x - \dfrac{c(t)y}{m(t)y + x}$.

Using the functions ϕ, ψ and φ introduced in (N3), systems (2.18) and (2.19) can be represented simply as

$$\begin{aligned} x'(t) &= x(t)\phi(t, x(t), y(t), x(\beta(t))), \\ y'(t) &= y(t)\psi(t, x(\beta(t)), y(\beta(t))), \end{aligned}$$

and

$$\begin{aligned} x'(t) &= x(t)\varphi(t, x(t), y(t)), \\ y'(t) &= y(t)\psi(t, x(\beta(t)), y(\beta(t))), \end{aligned}$$

respectively.

In Section 2.2.2, equivalent integral equations are constructed for the issue systems. Section 2.2.3 addresses properties such as positive invariance, permanence and persistence for systems (2.18) and (2.19).

2.2.2 Construction of the Equivalent Integral Equations

We shall use the following definition, which is similar to the one in [141] and modified for our general case as in [5, 8]. For the sake of simplicity, we consider solutions starting at the moment θ_0, which is the element of the sequence $\{\theta_i\}$, $i \in \mathbb{Z}$. But, it does not reduce the generality of our results since they could be considered similarly for an arbitrary initial moment [5].

Definition 2.2.1 *A pair of functions* $(x(t), y(t))$ *is a solution of (2.18) on* $[\theta_0, \infty)$ *if it satisfies the following conditions:*

(i) the functions $x(t)$ *and* $y(t)$ *are continuous on* $[\theta_0, \infty)$;

(ii) the derivatives $x'(t)$ *and* $y'(t)$ *exist for* $t \in [\theta_0, \infty)$ *with the possible exception of the points* θ_i, $i \geq 0$, *where one-sided derivatives exist;*

(iii) $(x(t), y(t))$ *satisfies (2.18) on each interval* $[\theta_i, \theta_{i+1})$, $i \geq 0$.

Since the first equation of (2.19) is an ordinary differential equation, it is convenient to write the following definition.

Definition 2.2.2 *A pair of functions* $(x(t), y(t))$ *is a solution of (2.19) on* $[\theta_0, \infty)$ *if it satisfies the conditions:*

(i) *the functions* $x(t)$ *and* $y(t)$ *are continuous on* $[\theta_0, \infty)$;

(ii) *the derivative* $x'(t)$ *exists for all* $t \in [\theta_0, \infty)$ *whereas* $y'(t)$ *exists for* $t \in [\theta_0, \infty)$ *with the possible exception of the points* θ_i, $i \geq 0$, *where one-sided derivatives exist;*

(iii) $x(t)$ *satisfies the first equation in (2.19) for all* $t \in [\theta_0, \infty)$ *whereas* $y(t)$ *satisfies the second equation in (2.19) on each interval* $[\theta_i, \theta_{i+1})$, $i \geq 0$.

In what follows, dealing with predator-prey models (2.18) and (2.19), we shall just consider solutions $(x(t), y(t))$ with $x(\theta_0) = x_0 > 0$, $y(\theta_0) = y_0 > 0$. Moreover, it is supposed that for any given (x_0, y_0), both (2.18) and (2.19) have unique solutions in the sense of Definitions 2.2.1 and 2.2.2, respectively. We shall discuss the existence and uniqueness theorem for differential equations with piecewise constant argument of generalized type in Chapter 3 (see Lemma 3.1.2 and Theorem 3.1.1).

Lemma 2.2.1 *Suppose (B1) is satisfied. System (2.18) with* $x(\theta_0) = x_0$, $y(\theta_0) = y_0$ *is equivalent to the following system of integral equations*

$$\begin{aligned} x(t) &= x_0 \exp\left(\int_{\theta_0}^{t} \phi(s, x(s), y(s), x(\beta(s)))ds\right), \\ y(t) &= y_0 \exp\left(\int_{\theta_0}^{t} \psi(s, x(\beta(s)), y(\beta(s)))ds\right). \end{aligned} \tag{2.20}$$

Proof: *Necessity*. Let $(x(t), y(t))$ be the solution of (2.18) with $x(\theta_0) = x_0$, $y(\theta_0) = y_0$. From the condition (*iii*) of Definition 2.2.1, we know that this solution satisfies (2.18) on each interval $[\theta_i, \theta_{i+1})$, $i \geq 0$. Hence, for $t \in [\theta_0, \theta_1)$, we have

$$\begin{aligned} x(t) &= x_0 \exp\left(\int_{\theta_0}^{t} \phi(s, x(s), y(s), x(\beta(s)))ds\right), \\ y(t) &= y_0 \exp\left(\int_{\theta_0}^{t} \psi(s, x(\beta(s)), y(\beta(s)))ds\right). \end{aligned}$$

Letting $t \to \theta_1$, it follows from the condition (i) of Definition 2.2.1 that

$$x_1 = x_0 \exp\left(\int_{\theta_0}^{\theta_1} \phi(s, x(s), y(s), x(\beta(s)))ds\right),$$
$$y_1 = y_0 \exp\left(\int_{\theta_0}^{\theta_1} \psi(s, x(\beta(s)), y(\beta(s)))ds\right).$$

Hence, (2.20) holds on $[\theta_0, \theta_1]$. Suppose that (2.20) is valid on the interval $[\theta_0, \theta_k]$ for some $k \geq 1$. Then, for $t \in [\theta_k, \theta_{k+1})$

$$\begin{aligned} x(t) &= x_k \exp\left(\int_{\theta_k}^{t} \phi(s, x(s), y(s), x(\beta(s)))ds\right) \\ &= x_0 \exp\left(\int_{\theta_0}^{t} \phi(s, x(s), y(s), x(\beta(s)))ds\right), \end{aligned}$$

and

$$\begin{aligned} y(t) &= y_k \exp\left(\int_{\theta_k}^{t} \psi(s, x(\beta(s)), y(\beta(s)))ds\right) \\ &= y_0 \exp\left(\int_{\theta_0}^{t} \psi(s, x(\beta(s)), y(\beta(s)))ds\right). \end{aligned}$$

As $t \to \theta_{k+1}$, we can observe that

$$x_{k+1} = x_0 \exp\left(\int_{\theta_0}^{\theta_{k+1}} \phi(s, x(s), y(s), x(\beta(s)))ds\right),$$
$$y_{k+1} = y_0 \exp\left(\int_{\theta_0}^{\theta_{k+1}} \psi(s, x(\beta(s)), y(\beta(s)))ds\right).$$

Hence, (2.20) is satisfied on $[\theta_0, \theta_{k+1}]$. By induction, this proves that it is valid for all $t \geq \theta_0$.

$Sufficiency$. Let $(x(t), y(t))$ be a solution of (2.20). Fix $i \geq 0$ and consider the interval $[\theta_i, \theta_{i+1})$. Differentiating (2.20) on (θ_i, θ_{i+1}), we can see that $(x(t), y(t))$ satisfies (2.18). Furthermore, letting $t \to \theta_i+$ and taking into account that $(x(\beta(t)), y(\beta(t)))$ is a pair of right continuous functions, we obtain that $(x(t), y(t))$ satisfies (2.18) on $[\theta_i, \theta_{i+1})$. This completes the proof. □

Predator equation in (2.19) coincides with the one in (2.18) and the prey equation is nothing but an ordinary differential equation. Thus, system (2.19) is more easily analyzable compared to (2.18). Using a similar method as in the proof of Lemma 2.2.1, one can prove the following result for (2.19).

LEMMA 2.2.2 *Suppose (B1) is satisfied. System (2.19) with* $x(\theta_0) = x_0$*,* $y(\theta_0) = y_0$ *is equivalent to*

$$x(t) = x_0 \exp\left(\int_{\theta_0}^{t} \varphi(s, x(s), y(s))ds\right),$$
$$y(t) = y_0 \exp\left(\int_{\theta_0}^{t} \psi(s, x(\beta(s)), y(\beta(s)))ds\right).$$

Lemma 2.2.1 (Lemma 2.2.2) implies immediately that the next assertion is valid.

THEOREM 2.2.1 *The positive quadrant* $int(\mathbb{R}_+^2) = \{(x, y) \in \mathbb{R}^2 \mid x > 0,\ y > 0\}$ *is positively invariant for systems (2.18) and (2.19).*

In the subsequent section, it will be shown that the region of invariance can be significantly narrowed.

2.2.3 Positive Invariance, Permanence and Persistence

In this section, assuming that conditions (B1) and (B2) are fulfilled, the results concerning positive invariance, permanence and related properties will be addressed.

THEOREM 2.2.2 *If* $m^l a^l > c^u$*,* $f^l > d^u$*,* $a^u b^u \theta < b^l$ *and* $f^u \theta < 4$*, then the set*

$$\Gamma = \{(x, y) \in \mathbb{R}^2 \mid r_1 \le x \le R_1,\ r_2 \le y \le R_2\} \tag{2.21}$$

is positively invariant for system (2.18), where

$$r_1 = \frac{m^l a^l - c^u}{m^l b^u}, \qquad R_1 = \frac{a^u}{b^l},$$
$$r_2 = \frac{f^l - d^u}{m^u d^u} r_1, \qquad R_2 = \frac{f^u - d^l}{m^l d^l} R_1.$$

Proof: Let $(x(t), y(t))$ be the solution of (2.18) initiating at the point $(x(\theta_0), y(\theta_0)) = (x_0, y_0)$ with $r_1 \le x_0 \le R_1$ and $r_2 \le y_0 \le R_2$.

We first consider the prey equation in (2.18). It follows from the positivity of the solutions of (2.18) that

$$x'(t) \le x(t)\left(a^u - b^l x(\beta(t))\right), \quad t \ge \theta_0.$$

For $t \in [\theta_0, \theta_1)$, we have

$$x'(t) \le x(t)\left(a^u - b^l x_0\right) = b^l x(t)(R_1 - x_0),$$

which implies together with (B2) that

$$x(t) \le x_0 \exp(b^l(R_1 - x_0)(t - \theta_0)) \le x_0 \exp(b^l(R_1 - x_0)\theta) \equiv G(x_0).$$

We want to find the maximum value of the continuous function $G(x_0)$ on the closed interval $[r_1, R_1]$. The hypothesis $a^u b^u \theta < b^l$ shows that $G'(x_0) > 0$ on $[r_1, R_1]$. Hence, $G(x_0) \le G(R_1) = R_1$ on $[r_1, R_1]$. All these discussions result in

$$x(t) \le R_1 \;\text{ for }\; t \in [\theta_0, \theta_1) \;\text{ whenever }\; r_1 \le x_0 \le R_1. \tag{2.22}$$

From the prey equation in (2.18), we obtain for $t \in [\theta_0, \theta_1)$

$$x'(t) \ge x(t)\left(a^l - b^u x_0 - \frac{c^u}{m^l}\right) = b^u x(t)(r_1 - x_0),$$

which clearly implies that

$$x(t) \ge x_0 \exp(b^u(r_1 - x_0)(t - \theta_0)) \ge x_0 \exp(b^u(r_1 - x_0)\theta) \equiv g(x_0).$$

Following the same way that we have used for $G(x_0)$, we find that the function $g(x_0)$ attains its minimum value at r_1, i.e., $g(x_0) \ge g(r_1) = r_1$ on $[r_1, R_1]$. Thus,

$$x(t) \ge r_1 \;\text{ for }\; t \in [\theta_0, \theta_1) \;\text{ whenever }\; r_1 \le x_0 \le R_1. \tag{2.23}$$

Combining (2.22) and (2.23), we have

$$r_1 \le x_0 \le R_1 \;\Rightarrow\; r_1 \le x(t) \le R_1 \;\text{ for }\; t \in [\theta_0, \theta_1).$$

This, together with the continuity of $x(t)$, implies that $r_1 \le x(\theta_1) = x_1 \le R_1$. Hence, when the same technique used for the interval $[\theta_0, \theta_1)$ is repeated for $t \in [\theta_1, \theta_2)$, it can be easily seen that

$$r_1 \le x_1 \le R_1 \;\Rightarrow\; r_1 \le x(t) \le R_1 \;\text{ for }\; t \in [\theta_1, \theta_2),$$

which in turn implies that $r_1 \le x(\theta_2) = x_2 \le R_1$. Continuing the process on each interval $[\theta_i, \theta_{i+1})$, $i = 2, 3, 4, \ldots$, in a similar manner, we can conclude that

$$r_1 \le x_0 \le R_1 \;\Rightarrow\; r_1 \le x(t) \le R_1 \;\text{ for all }\; t \ge \theta_0.$$

From the predator equation in (2.18) and the positiveness of $y(t)$, we have

$$y'(t) \leq y(t)\left(-d^l + \frac{f^u x(\beta(t))}{m^l y(\beta(t)) + x(\beta(t))}\right), \quad t \geq \theta_0.$$

For $t \in [\theta_0, \theta_1)$ this inequality takes the form

$$\begin{aligned} y'(t) \leq y(t)\left(-d^l + \frac{f^u x_0}{m^l y_0 + x_0}\right) &\leq y(t)\left(-d^l + \frac{f^u R_1}{m^l y_0 + R_1}\right) \\ &= \frac{m^l d^l y(t)}{m^l y_0 + R_1}(R_2 - y_0) \end{aligned}$$

which produces

$$\begin{aligned} y(t) &\leq y_0 \exp\left(\left(\frac{m^l d^l}{m^l y_0 + R_1}\right)(R_2 - y_0)(t - \theta_0)\right) \\ &\leq y_0 \exp\left(\left(\frac{m^l d^l}{m^l y_0 + R_1}\right)(R_2 - y_0)\theta\right) \equiv H(y_0). \end{aligned}$$

Using the hypothesis $f^u\theta < 4$, we find that the derivative of the function $H(y_0)$ is always positive. That being the case, $H(y_0) \leq H(R_2) = R_2$ on the interval $[r_2, R_2]$. Then,

$$y(t) \leq R_2 \;\text{ for }\; t \in [\theta_0, \theta_1) \;\text{ whenever }\; r_2 \leq y_0 \leq R_2. \tag{2.24}$$

We now continue with the predator equation for the other direction on $[\theta_0, \theta_1)$,

$$\begin{aligned} y'(t) \geq y(t)\left(-d^u + \frac{f^l x_0}{m^u y_0 + x_0}\right) &\geq y(t)\left(-d^u + \frac{f^l r_1}{m^u y_0 + r_1}\right) \\ &= \frac{m^u d^u y(t)}{m^u y_0 + r_1}(r_2 - y_0) \end{aligned}$$

and these inequalities lead to

$$\begin{aligned} y(t) &\geq y_0 \exp\left(\left(\frac{m^u d^u}{m^u y_0 + r_1}\right)(r_2 - y_0)(t - \theta_0)\right) \\ &\geq y_0 \exp\left(\left(\frac{m^u d^u}{m^u y_0 + r_1}\right)(r_2 - y_0)\theta\right) \equiv h(y_0). \end{aligned}$$

By straightforward evaluation of $h'(y_0)$, we arrive at $h(y_0) \geq h(r_2) = r_2$ on $[r_2, R_2]$ for the reason that $f^l\theta \leq f^u\theta < 4$. Therefore, we have

$$y(t) \geq r_2 \;\text{ for }\; t \in [\theta_0, \theta_1) \;\text{ whenever }\; r_2 \leq y_0 \leq R_2. \tag{2.25}$$

From (2.24) and (2.25), it follows that

$$r_2 \leq y_0 \leq R_2 \;\Rightarrow\; r_2 \leq y(t) \leq R_2 \;\text{ for }\; t \in [\theta_0, \theta_1).$$

Since $y(t)$ is continuous, we can construct the desired result on each interval $[\theta_i, \theta_{i+1})$, $i = 1, 2, 3, \ldots$, following the same way discussed previously for $x(t)$. That is to say,

$$r_2 \le y_1 \le R_2 \;\Rightarrow\; r_2 \le y(t) \le R_2 \text{ for all } t \ge \theta_0,$$

proving the theorem. □

Theorem 2.2.3 *Let the conditions $m^l a^l > c^u$, $f^l > d^u$ and $f^u \theta < 4$ be fulfilled. Then the set Γ defined by (2.21) is positively invariant for system (2.19).*

Proof: Let $(x(t), y(t))$ be the solution of (2.19) passing through (x_0, y_0) where $r_1 \le x_0 \le R_1$ and $r_2 \le y_0 \le R_2$. In that case, the prey equation does not contain any piecewise constant argument. That is why, it follows for all $t \ge \theta_0$ that

$$b^u x(t)\,(r_1 - x(t)) \le x'(t) \le b^l x(t)\,(R_1 - x(t))\,, \quad t \ge \theta_0. \tag{2.26}$$

Then, $x(t)$ being differentiable for all $t \ge \theta_0$, a standard comparison argument shows that

$$r_1 \le x_0 \le R_1 \;\Rightarrow\; r_1 \le x(t) \le R_1 \text{ for all } t \ge \theta_0.$$

As the predator equation in (2.19) coincides with the one in (2.18), we apply exactly the same technique that is used for $y(t)$ in the proof of Theorem 2.2.2 to reach the desired conclusion. □

Lemma 2.2.3 *For system (2.18), $\limsup_{t \to +\infty} x(t) \le S_1$, where $S_1 = \dfrac{a^u}{b^l} \exp(a^u \theta)$.*

Proof: From the first equation of (2.18), we see that $x'(t) \le a^u x(t)$ for all $t \ge \theta_0$. This inequality leads to

$$x(t) \le x(\theta_i) \exp(a^u (t - \theta_i)) \le x(\beta(t)) \exp(a^u \theta)$$

on each interval $[\theta_i, \theta_{i+1})$, $i \ge 0$. In fact, using the continuity of $x(t)$, this result can be generalized as

$$x(t) \le x(\beta(t)) \exp(a^u \theta) \text{ for all } t \ge \theta_0,$$

which is equivalent to $x(\beta(t)) \geq x(t)\exp(-a^u\theta)$ for all $t \geq \theta_0$. Therefore, the prey equation satisfies

$$\begin{aligned}
x'(t) &\leq x(t)\left(a^u - b^l x(\beta(t))\right) \\
&\leq x(t)\left(a^u - b^l \exp(-a^u\theta)x(t)\right) \\
&= b^l \exp(-a^u\theta)x(t)\,(S_1 - x(t))\,, \;\; t \geq \theta_0.
\end{aligned}$$

Since $x'(t)$ exists for $t \in [\theta_0, \infty)$ with the possible exception of the points θ_i, $i \geq 0$, where one-sided derivatives exist, we should modify the standard comparison argument. Consider the solution $\tilde{x}(t)$ of the following ordinary differential equation

$$\begin{aligned}
&\tilde{x}\,'(t) = b^l \exp(-a^u\theta)\tilde{x}(t)\,(S_1 - \tilde{x}(t))\,, \\
&\tilde{x}(\theta_0) = \tilde{x}_0,
\end{aligned}$$

where $\tilde{x}_0 \geq x_0$. For $t \in [\theta_0, \theta_1)$, we obtain $x(t) \leq \tilde{x}(t)$ by a standard comparison argument. Since the solutions $x(t)$ and $\tilde{x}(t)$ are continuous, one can conclude that $x(t) \leq \tilde{x}(t)$ on each interval $[\theta_i, \theta_{i+1})$, $i = 1, 2, 3, \ldots$, and hence, $x(t) \leq \tilde{x}(t)$ for all $t \geq \theta_0$. This clearly shows that

$$\limsup_{t\to+\infty} x(t) \leq \limsup_{t\to+\infty} \tilde{x}(t) = \lim_{t\to+\infty} \tilde{x}(t) = S_1,$$

proving the lemma. □

Lemma 2.2.4 *For system (2.18), if* $m^l a^l > c^u$ *holds true, then* $\liminf_{t\to+\infty} x(t) \geq s_1$, *where* $s_1 = \dfrac{m^l a^l - c^u}{m^l b^u} \exp((a^l - b^u S_1 - \dfrac{c^u}{m^l})\theta)$.

Proof: Since $\limsup_{t\to+\infty} x(t) \leq S_1$, for any $\epsilon > 1$, there is some $T_\epsilon > \theta_0$ such that for $t \geq T_\epsilon$, $x(t) < \epsilon S_1$. Therefore, $x(\beta(t)) < \epsilon S_1$ for $t \geq \beta(T_\epsilon) + \theta$. For $t \geq \beta(T_\epsilon) + \theta$, we derive from the prey equation of (2.18)

$$x'(t) \geq x(t)\left(a^l - b^u x(\beta(t)) - \frac{c^u}{m^l}\right) \geq x(t)\left(a^l - b^u \epsilon S_1 - \frac{c^u}{m^l}\right),$$

which, together with the same arguments used in the proof of Lemma 2.2.3, leads us to

$$x(\beta(t)) \leq x(t)\exp(-(a^l - b^u\epsilon S_1 - \frac{c^u}{m^l})\theta), \tag{2.27}$$

for $t \geq \beta(T_\epsilon) + \theta$. From (2.27), it follows that

$$x'(t) \geq \left(a^l - b^u x(t) \exp(-(a^l - b^u \epsilon S_1 - \frac{c^u}{m^l})\theta) - \frac{c^u}{m^l}\right).$$

Using a comparison argument in a similar manner in the proof of Lemma 2.2.3 and letting $\epsilon \to 1$, we see that $\liminf_{t\to+\infty} x(t) \geq s_1$. □

LEMMA 2.2.5 *Let $f^l > d^u$ and $m^l a^l > c^u$ hold true. Then $\limsup_{t\to+\infty} y(t) \leq S_2$ and $\liminf_{t\to+\infty} y(t) \geq s_2$, where $S_2 = \dfrac{f^u - d^l}{m^l d^l} S_1 \exp((f^u - d^l)\theta)$ and $s_2 = \dfrac{f^l - d^u}{m^u d^u} s_1 \exp(-d^u\theta)$ for system (2.18).*

Proof: We observed above that for any $\epsilon > 1$, there exists a $T_\epsilon > \theta_0$ such that $x(\beta(t)) < \epsilon S_1$ for $t \geq \beta(T_\epsilon) + \theta$. From the predator equation of (2.18), we have

$$-d^u y(t) \leq y'(t) \leq (f^u - d^l)y(t).$$

Hence, by a similar argument used for the prey population, we derive that

$$y(\beta(t)) \exp(-d^u\theta) \leq y(t) \leq y(\beta(t)) \exp((f^u - d^l)\theta), \ \ t \geq \theta_0. \tag{2.28}$$

According to (2.28), we get for $t \geq \beta(T_\epsilon) + \theta$,

$$\begin{aligned} y'(t) &\leq y(t)\left(-d^l + \frac{f^u \epsilon S_1}{m^l y(\beta(t)) + \epsilon S_1}\right) \\ &\leq y(t)\left(-d^l + \frac{f^u \epsilon S_1}{m^l y(t)\exp(-(f^u - d^l)\theta) + \epsilon S_1}\right) \\ &= y(t)\left(\frac{(f^u - d^l)\epsilon S_1 - m^l d^l y(t)\exp(-(f^u - d^l)\theta)}{m^l y(t)\exp(-(f^u - d^l)\theta) + \epsilon S_1}\right). \end{aligned}$$

A standard comparison argument, as in the proof Lemma 2.2.3, shows that

$$\limsup_{t\to+\infty} y(t) \leq \frac{f^u - d^l}{m^l d^l} \epsilon S_1 \exp((f^u - d^l)\theta)$$

and the conclusion $\limsup_{t\to+\infty} y(t) \leq S_2$ holds by letting $\epsilon \to 1$.

On the other hand, it follows from Lemma 2.2.4 that for any $\eta > 1$, there exists a constant $T_\eta > \beta(T_\epsilon) + \theta$ such that for $t \geq T_\eta$, $x(t) > \dfrac{s_1}{\eta}$. Then, $x(\beta(t)) > \dfrac{s_1}{\eta}$ for $t \geq \beta(T_\eta) + \theta$. Now, using (2.28), it is easy to see that

$$\begin{aligned} y'(t) &\geq y(t)\left(-d^u + \frac{f^l(s_1/\eta)}{m^u y(\beta(t)) + (s_1/\eta)}\right) \\ y(t) &\geq \left(-d^u + \frac{f^l(s_1/\eta)}{m^u y(t)\exp(d^u\theta) + (s_1/\eta)}\right) \\ &= y(t)\left(\frac{(f^l - d^u)s_1/\eta - m^u d^u y(t)\exp(d^u\theta)}{m^u y(t)\exp(d^u\theta) + (s_1/\eta)}\right). \end{aligned}$$

Hence, by the comparison theorem and the arbitrariness of η, we have

$$\liminf_{t\to+\infty} y(t) \geq \frac{f^l - d^u}{m^u d^u} s_1 \exp(-d^u) = s_2. \quad \square$$

We can also establish similar results for system (2.19). The inequalities in (2.26) imply by a comparison argument that the following lemma is valid [65].

Lemma 2.2.6 *For system (2.19),* $\limsup_{t\to+\infty} x(t) \leq R_1$. *In particular, if* $m^l a^l > c^u$ *then* $\liminf_{t\to+\infty} x(t) \geq r_1$.

Moreover, similar to Lemma 2.2.5, following assertion can be easily verified.

Lemma 2.2.7 *For (2.19), if* $f^l > d^u$ *and* $m^l a^l > c^u$ *are satisfied, then* $\limsup_{t\to+\infty} y(t) \leq \tilde{S}_2$ *and* $\liminf_{t\to+\infty} y(t) \geq \tilde{s}_2$*, where* $\tilde{S}_2 = \dfrac{f^u - d^l}{m^l d^l} R_1 \exp((f^u - d^l)\theta)$, $\tilde{s}_2 = \dfrac{f^l - d^u}{m^u d^u} r_1 \exp(-d^u\theta)$.

For the rest of the section, we need the following definitions of the concepts; ultimate boundedness, permanence and non-persistence.

Definition 2.2.3 *The solution of (2.18) ((2.19)) is said to be ultimately bounded if there exists a* $B > 0$ *such that for every solution* $(x(t), y(t))$ *of (2.18) ((2.19)), there exists* $T > 0$ *such that* $\|(x(t), y(t))\| \leq B$*, for all* $t \geq t_0 + T$*, where* B *is independent of the particular solution while* T *may depend on the solution.*

Definition 2.2.4 *System (2.18) ((2.19)) is said to be permanent if there exist* δ *and* Δ *with* $0 < \delta < \Delta$ *such that*

$$\min\left\{\liminf_{t\to+\infty} x(t),\ \liminf_{t\to+\infty} y(t)\right\} \geq \delta,$$

and

$$\max\left\{\limsup_{t\to+\infty} x(t),\ \limsup_{t\to+\infty} y(t)\right\} \leq \Delta,$$

for all solutions of (2.18) ((2.19)) with positive initial values.

Definition 2.2.5 *System (2.18) ((2.19)) is said to be non-persistent if there exists a positive solution* $(x(t), y(t))$ *of (2.18) ((2.19)) satisfying*

$$\min\left\{\liminf_{t\to+\infty} x(t),\ \liminf_{t\to+\infty} y(t)\right\} = 0.$$

From the proofs of Lemma 2.2.3-2.2.5 for (2.18) (Lemma 2.2.6 and 2.2.7 for (2.19)), it is easy to conclude the following statement for ultimate boundedness.

THEOREM 2.2.4 *If $m^l a^l > c^u$ and $f^l > d^u$, then the set Ω defined by*

$$\Omega = \{(x, y) \in \mathbb{R}^2 \mid s_1 \le x \le S_1,\ \ s_2 \le y \le S_2\},$$

and the set Σ defined by

$$\Sigma = \{(x, y) \in \mathbb{R}^2 \mid r_1 \le x \le R_1,\ \ \tilde{s}_2 \le y \le \tilde{S}_2\}$$

are ultimately bounded regions for systems (2.18) and (2.19), respectively.

Meanwhile, from Lemma 2.2.3-2.2.5 (Lemma 2.2.6 and 2.2.7) we have already shown the following statement.

THEOREM 2.2.5 *If $m^l a^l > c^u$ and $f^l > d^u$, then system (2.18) ((2.19)) is permanent.*

THEOREM 2.2.6 *If $f^u < d^l$ or $\dfrac{c^l}{m^u} > a^u + d^u$ then system (2.18) ((2.19)) is not persistent.*

Proof: If $f^u < d^l$ is true, then the inequality $y'(t) \le (f^u - d^l)y(t)$ implies that $\lim_{t\to+\infty} y(t) = 0$. In this case, (2.18) ((2.19)) is not persistent by Definition 2.2.5.

If $\dfrac{c^l}{m^u} > a^u + d^u$, then there exists an $\alpha > 0$ such that $\dfrac{c^l}{m^u + \alpha} = a^u + d^u$. Let $(x(t), y(t))$ be the solution of (2.18) ((2.19)) with $\dfrac{x(\theta_0)}{y(\theta_0)} < \alpha$. We claim that $\dfrac{x(t)}{y(t)} < \alpha$ for all $t \ge \theta_0$ and $\lim_{t\to+\infty} x(t) = 0$. If not, there exists a first time $t_1 > \theta_0$ such that

$$\frac{x(t_1)}{y(t_1)} = \alpha \ \text{ and } \ \frac{x(t)}{y(t)} < \alpha \ \text{ for } \ t \in [\theta_0, t_1).$$

Then, for $t \in [\theta_0, t_1]$, we have

$$x'(t) \le x(t)\left(a^u - \frac{c^l}{m^u + \dfrac{x(t)}{y(t)}}\right) \le x(t)\left(a^u - \frac{c^l}{m^u + \alpha}\right) = -d^u x(t),$$

which implies that $x(t) \le x(\theta_0)\exp(-d^u(t - \theta_0))$. Moreover, for all $t \ge \theta_0$, we have $y'(t) \ge -d^u y(t)$ which leads to $y(t) \ge y(\theta_0)\exp(-d^u(t - \theta_0))$. All these discussions show that

$$\frac{x(t)}{y(t)} \le \frac{x(\theta_0)\exp(-d^u(t - \theta_0))}{y(\theta_0)\exp(-d^u(t - \theta_0))} = \frac{x(\theta_0)}{y(\theta_0)} < \alpha \ \text{ for } \ t \in [\theta_0, t_1],$$

which is a contradiction to the existence of t_1, justifying our claim. This in turn implies that $x(t) \le x(\theta_0)\exp(-d^u(t-\theta_0))$ for all $t \ge \theta_0$. Therefore, $\lim_{t\to+\infty} x(t) = 0$, which completes the proof. □

THEOREM 2.2.7 *If* $\frac{c^l}{m^u} > a^u + d^u$ *and* $f^u < d^l(1+\frac{m^l}{\alpha})$, *where* $\alpha = \frac{c^l}{a^u+d^u} - m^u$, *then there exist positive solutions* $(x(t), y(t))$ *of (2.18) ((2.19)) such that* $\lim_{t\to+\infty}(x(t), y(t)) = (0,0)$.

Proof: From the proof of Theorem 2.2.6, we have $\frac{x(t)}{y(t)} < \alpha$ for all $t \ge \theta_0$ and $\lim_{t\to+\infty} x(t) = 0$ provided that $\frac{x(\theta_0)}{y(\theta_0)} < \alpha$. These arguments imply directly that $\frac{x(\beta(t))}{y(\beta(t))} < \alpha$ for $t \ge \theta_0$. Then, for $t \ge \theta_0$,

$$y'(t) \le y(t)\left(-d^l + \frac{f^u \dfrac{x(\beta(t))}{y(\beta(t))}}{m^l + \dfrac{x(\beta(t))}{y(\beta(t))}}\right) \le y(t)\left(-d^l + \frac{f^u\alpha}{m^l+\alpha}\right) \equiv -\lambda y(t),$$

where $\lambda < 0$ by the hypothesis $f^u < d^l(1+\frac{m^l}{\alpha})$. This immediately shows that $\lim_{t\to+\infty} y(t) = 0$. The proof is completed. □

CHAPTER 3

LYAPUNOV-RAZUMIKHIN METHOD FOR DIFFERENTIAL EQUATIONS WITH PIECEWISE CONSTANT ARGUMENT OF GENERALIZED TYPE

In [5, 8, 9], the concept of differential equations with piecewise constant argument [52, 78, 80, 170] has been generalized by considering arbitrary piecewise constant functions as arguments. In this chapter, using stability definitions from [5], we develop the Lyapunov's second method for stability of differential equations with piecewise constant argument of generalized type by employing the Razumikhin technique [84, 150]. To the best of our knowledge, there have been no results on stability obtained by Lyapunov-Razumikhin method for differential equations with piecewise constant argument, despite the fact that they are delay differential equations.

Differential equations with piecewise constant arguments play an important role in numerous applications [12, 41, 55, 77, 78, 125, 130, 132, 167, 177] as well as they can be applied successfully to approximate solutions of delay differential equations [55, 81]. There are many interesting results of the theory of differential equations with piecewise constant argument [155, 165, 173], which include complex behaviour of solutions [78]. A great part of the theory has been summarized in [170]. The theoretical depth of investigation of these equations was determined by the papers [41, 52, 156], where the reduction to discrete equations had been chosen as the main instrument of study. Consequently, analysis of solutions, starting at moments which are not integers has been unattainable. Particularly, one can not investigate the problem of stability completely, as only integers or their multiples are allowed to be discussed for initial moments.

The approach developed in [5, 8, 9] has a goal to meet the challenges mentioned above. In fact, the detailed comparison of values of a solution at a point and at neighbor moments, where the argument function has discontinuitities, helps to extend the discussion. It embraces several results on the existence and uniqueness of solutions, dependence on initial data, and exceptionally stability, which we intend to consider in the present chapter. To give more sense to the last words, in Example 3.3.3 at the end of this chapter, we will present additional stability analysis for the results obtained by Gopalsamy and Liu [77] for the logistic type equation

$$N'(t) = rN(t)(1 - aN(t) - bN([t])), \quad t > 0, \tag{3.1}$$

where $[t]$ denotes the maximal integer not greater than t.

3.1 Preliminaries

We fix a real-valued sequence θ_i, $i \in \mathbb{N}_0$ such that $0 = \theta_0 < \theta_1 < \cdots < \theta_i < \cdots$ with $\theta_i \to \infty$ as $i \to \infty$.

In the present chapter, we shall consider the following equation [8]

$$x'(t) = f(t, x(t), x(\beta(t))), \tag{3.2}$$

where $x \in S(\rho)$, $S(\rho) = \{x \in \mathbb{R}^n : \|x\| < \rho\}$, $t \in \mathbb{R}^+$, $\beta(t) = \theta_i$ if $t \in [\theta_i, \theta_{i+1})$, $i \in \mathbb{N}_0$.

System (3.2) on $[\theta_i, \theta_{i+1})$, $i \in \mathbb{N}_0$, has the form of a special functional differential equation

$$x'(t) = f(t, x(t), x(\theta_i)). \tag{3.3}$$

Hence, we can see that (3.2) has the structure of a continuous dynamical system within the intervals $[\theta_i, \theta_{i+1})$, $i \in \mathbb{N}_0$. We assume that the solutions of the equation are continuous functions, but the deviating function $\beta(t)$ is discontinuous. In general, the right-hand side of (3.2) has discontinuities at the moments θ_i. In other words, we consider the solutions of the equation as functions, which are continuous and continuously differentiable within the intervals $[\theta_i, \theta_{i+1})$, $i \in \mathbb{N}_0$.

The following assumptions will be needed throughout the chapter:

(C1) $f(t, y, z) \in C(\mathbb{R}^+ \times S(\rho) \times S(\rho))$ is an $n \times 1$ real-valued function;

(C2) $f(t, 0, 0) = 0$ for all $t \geq 0$;

(C3) $f(t, y, z)$ satisfies the condition

$$\|f(t, y_1, z_1) - f(t, y_2, z_2)\| \leq \ell(\|y_1 - y_2\| + \|z_1 - z_2\|)$$

for all $t \in \mathbb{R}^+$ and $y_1,\ y_2,\ z_1,\ z_2 \in S(\rho)$, where $\ell > 0$ is a Lipschitz constant;

(C4) there exists a positive number θ such that $\theta_{i+1} - \theta_i \leq \theta$, $i \in \mathbb{N}_0$;

(C5) $\ell\theta[1 + (1 + \ell\theta)e^{\ell\theta}] < 1$;

(C6) $3\ell\theta e^{\ell\theta} < 1$.

Let us use the following sets of functions:

$\mathcal{K} = \{\, a \in C(\mathbb{R}^+, \mathbb{R}^+) : a$ is strictly increasing and $a(0) = 0\}$,

$\Omega = \{\, b \in C(\mathbb{R}^+, \mathbb{R}^+)\ :\ b(0) = 0,\ b(s) > 0$ for $s > 0\}$.

Definition 3.1.1 *[8] A function $x(t)$ is a solution of (3.2) on $\mathbb{R}^+$ if*

(i) $x(t)$ is continuous on $\mathbb{R}^+$;

(ii) the derivative $x'(t)$ exists for $t \in \mathbb{R}^+$ with the possible exception of the points θ_i, $i \in \mathbb{N}_0$, where one-sided derivatives exist;

(iii) equation (3.2) is satisfied by $x(t)$ on each interval (θ_i, θ_{i+1}), $i \in \mathbb{N}_0$, and it holds for the right derivative of $x(t)$ at the points θ_i, $i \in \mathbb{N}_0$.

Notation 3.1.1 $K(\ell) = \dfrac{1}{1 - \ell\theta[1 + (1 + \ell\theta)e^{\ell\theta}]}$.

The following lemma is an important auxiliary result of the present chapter as it will be used in the stability analysis.

LEMMA 3.1.1 *Let (C1)-(C5) be fulfilled. Then the following inequality*

$$\|x(\beta(t))\| \leq K(\ell)\,\|x(t)\| \tag{3.4}$$

holds for all $t \geq 0$.

Proof: Let us fix $t \in \mathbb{R}^+$. Then there exists $k \in \mathbb{N}_0$ such that $t \in [\theta_k, \theta_{k+1})$. We have

$$x(t) = x(\theta_k) + \int_{\theta_k}^{t} f(s, x(s), x(\theta_k))ds, \quad t \in [\theta_k, \theta_{k+1}).$$

Hence,

$$\begin{aligned} \|x(t)\| &\leq \|x(\theta_k)\| + \ell \int_{\theta_k}^{t} (\|x(s)\| + \|x(\theta_k)\|)\, ds \\ &\leq (1 + \ell\theta)\,\|x(\theta_k)\| + \ell \int_{\theta_k}^{t} \|x(s)\|\, ds. \end{aligned}$$

The Gronwall-Bellman Lemma yields that $\|x(t)\| \leq (1 + \ell\theta)e^{\ell\theta}\,\|x(\theta_k)\|$. Moreover, for $t \in [\theta_k, \theta_{k+1})$ we have

$$x(\theta_k) = x(t) - \int_{\theta_k}^{t} f(s, x(s), x(\theta_k))ds.$$

Thus,

$$\begin{aligned} \|x(\theta_k)\| &\leq \|x(t)\| + \ell \int_{\theta_k}^{t} (\|x(s)\| + \|x(\theta_k)\|)\, ds \\ &\leq \|x(t)\| + \ell \int_{\theta_k}^{t} \left[(1 + \ell\theta)e^{\ell\theta} + 1\right] \|x(\theta_k)\|\, ds \\ &\leq \|x(t)\| + \ell\theta \left[(1 + \ell\theta)e^{\ell\theta} + 1\right] \|x(\theta_k)\|\,. \end{aligned}$$

It follows from condition (C5) that $\|x(\theta_k)\| \leq K(\ell)\,\|x(t)\|$ for $t \in [\theta_k, \theta_{k+1})$. Hence, (3.4) holds for all $t \geq 0$. □

We give the following assertion which establishes the existence and uniqueness of solutions of (3.3).

LEMMA 3.1.2 *[9] Let (C1) and (C3)-(C6) be satisfied and $i \in \mathbb{N}_0$ be fixed. Then for every $(\xi, x_0) \in [\theta_i, \theta_{i+1}] \times S(\rho)$, there exists a unique solution $x(t) = x(t, \xi, x_0)$ of (3.3) on $[\theta_i, \theta_{i+1}]$.*

Proof: *Existence*. Fix $i \in \mathbb{N}_0$ and assume without loss of generality that $\theta_i \leq \xi \leq \theta_{i+1}$.

Define a norm $\|x(t)\|_0 = \max\limits_{[\theta_i,\xi]} \|x(t)\|$. Take $x_0(t) = x_0$ and a sequence

$$x_{m+1}(t) = x_0 + \int_{\xi}^{t} f(s, x_m(s), x_m(\theta_i))ds, \quad m \geq 0.$$

It can be easily checked that $\|x_{m+1}(t) - x_m(t)\|_0 \leq (2l\theta)^{m+1} \|x_0\|$. Then condition (C6) implies that the sequence $x_m(t)$ is convergent and its limit $x(t)$ satisfies

$$x(t) = x_0 + \int_{\xi}^{t} f(s, x(s), x(\theta_i))ds$$

on $[\theta_i, \xi]$. The existence is proved.

Uniqueness. Let $x_j(t) = x(t, \xi, x_0^j)$, $x_j(\xi) = x_0^j$, $j = 1, 2$, denote the solutions of (3.3) where $\theta_i \leq \xi \leq \theta_{i+1}$. It is sufficient to show that $x_0^1 \neq x_0^2$ implies $x_1(t) \neq x_2(t)$ for every $t \in [\theta_i, \theta_{i+1}]$.

The solutions $x_1(t)$ and $x_2(t)$ satisfy, respectively, the following integral equations

$$x_1(t) = x_0^1 + \int_{\xi}^{t} f(s, x_1(s), x_1(\theta_i))ds,$$

$$x_2(t) = x_0^2 + \int_{\xi}^{t} f(s, x_2(s), x_2(\theta_i))ds$$

for all $t \in [\theta_i, \theta_{i+1}]$. Subtracting we obtain that

$$\begin{aligned}\|x_1(t) - x_2(t)\| &\leq \left\|x_0^1 - x_0^2\right\| + l \int_{\xi}^{t} \left(\|x_1(s) - x_2(s)\| + \|x_1(\theta_i) - x_2(\theta_i)\|\right) ds \\ &\leq \left\|x_0^1 - x_0^2\right\| + l\theta \|x_1(\theta_i) - x_2(\theta_i)\| + l \left|\int_{\xi}^{t} \|x_1(s) - x_2(s)\| ds\right|.\end{aligned}$$

It follows from the Gronwall-Bellman inequality that

$$\|x_1(t) - x_2(t)\| \leq \left(\left\|x_0^1 - x_0^2\right\| + l\theta \|x_1(\theta_i) - x_2(\theta_i)\|\right) e^{l\theta}.$$

Particularly,

$$\|x_1(\theta_i) - x_2(\theta_i)\| \leq \left(\left\|x_0^1 - x_0^2\right\| + l\theta \|x_1(\theta_i) - x_2(\theta_i)\|\right) e^{l\theta}.$$

Then,

$$\|x_1(\theta_i) - x_2(\theta_i)\| \leq \frac{e^{l\theta}}{1 - l\theta e^{l\theta}} \left\|x_0^1 - x_0^2\right\|.$$

Hence,

$$\|x_1(t) - x_2(t)\| \le e^{l\theta}\left(1 + l\theta\frac{e^{l\theta}}{1 - l\theta e^{l\theta}}\right)\left\|x_0^1 - x_0^2\right\|. \tag{3.5}$$

If we assume on the contrary that there exists $t^* \in [\theta_i, \theta_{i+1}]$ such that $x_1(t^*) = x_2(t^*)$, then

$$x_0^1 - x_0^2 = \int_{\xi}^{t^*} \left(f(s, x_2(s), x_2(\theta_i)) - f(s, x_1(s), x_1(\theta_i))\right) ds.$$

The last expression, together with (3.5) and (C6), leads us to

$$\begin{aligned}
\left\|x_0^1 - x_0^2\right\| &\le l\left|\int_{\xi}^{t^*} \left(\|x_2(s) - x_1(s)\| + \|x_2(\theta_i) - x_1(\theta_i)\|\right) ds\right| \\
&\le \frac{2l\theta e^{l\theta}}{1 - l\theta e^{l\theta}}\left\|x_0^1 - x_0^2\right\| \\
&< \left\|x_0^1 - x_0^2\right\|,
\end{aligned}$$

which is a contradiction. The theorem is proved. □

Theorem 3.1.1 *[9] Assume that conditions (C1) and (C3)-(C6) hold true. Then for every* $(t_0, x_0) \in \mathbb{R}^+ \times S(\rho)$*, there exists a unique solution* $x(t) = x(t, t_0, x_0)$ *of (3.2) on* $\mathbb{R}^+$ *in the sense of Definition 3.1.1 such that* $x(t_0) = x_0$.

Proof: Without loss of generality, assume that $\theta_i \le t_0 \le \theta_{i+1}$ for some $i \in \mathbb{N}_0$. By Lemma 3.1.2 for $\xi = t_0$, there exists a unique solution $x(t) = x(t, t_0, x_0)$ of (3.2) on $[\theta_i, \theta_{i+1}]$ as a solution of (3.3). Using the lemma again, we can continue $x(t)$ from $t = \theta_i$ to $t = \theta_{i-1}$. Clearly, $x(t)$ can be continued to $t = 0$.

Similarly, for increasing t, one can easily see that the solution $x(t)$ can be continued from $t = \theta_{i+1}$ to $t = \theta_{i+2}$. Since $\theta_i \to \infty$ as $i \to \infty$, we can complete the proof by using induction. □

Definition 3.1.2 *Let* $V : \mathbb{R}^+ \times S(\rho) \to \mathbb{R}^+$. *Then, V is said to belong to the class* ϑ *if*

(i) *V is continuous on* $\mathbb{R}^+ \times S(\rho)$ *and* $V(t, 0) = 0$ *for all* $t \in \mathbb{R}^+$;

(ii) $V(t, x)$ *is continuously differentiable on* $(\theta_i, \theta_{i+1}) \times S(\rho)$ *and for each* $x \in S(\rho)$, *the right derivative exists at* $t = \theta_i$, $i \in \mathbb{N}_0$.

Definition 3.1.3 *Given a function $V \in \vartheta$, the derivative of V with respect to system (3.2) is defined by*

$$V'(t,x,y) = \frac{\partial V(t,x)}{\partial t} + \langle \nabla V(t,x), f(t,x,y) \rangle ,$$

for all $t \neq \theta_i$ in $\mathbb{R}^+$ and $x\,, y \in S(\rho)$, where ∇V denotes the gradient vector of V with respect to x.

3.2 Stability Analysis

In this section, we assume that conditions (C1)-(C6) are satisfied and we will obtain the stability of the zero solution of (3.2) based on the Lyapunov-Razumikhin method. We can formulate the definitions of Lyapunov stability in the same way as for ordinary differential equations.

Definition 3.2.1 *[5] The zero solution of (3.2) is said to be*

(i) stable if for any $\varepsilon > 0$ and $t_0 \in \mathbb{R}^+$, there exists a $\delta = \delta(t_0, \varepsilon) > 0$ such that $\|x_0\| < \delta$ implies $\|x(t, t_0, x_0)\| < \varepsilon$ for all $t \geq t_0$;

(ii) uniformly stable if δ is independent of t_0.

Definition 3.2.2 *[5] The zero solution of (3.2) is said to be uniformly asymptotically stable if it is uniformly stable and there is a $\delta_0 > 0$ such that for every $\varepsilon > 0$ and $t_0 \in \mathbb{R}^+$, there exists a $T = T(\varepsilon) > 0$ such that $\|x(t, t_0, x_0)\| < \varepsilon$ for all $t > t_0 + T$ whenever $\|x_0\| < \delta_0$.*

Theorem 3.2.1 *Assume that there exists a function $V \in \vartheta$ such that*

(i) $u(\|x\|) \leq V(t,x)$ on $\mathbb{R}^+ \times S(\rho)$, where $u \in \mathcal{K}$;

(ii) $V'(t,x,y) \leq 0$ for all $t \neq \theta_i$ in $\mathbb{R}^+$ and $x,\, y \in S(\rho)$ such that $V(\beta(t), y) \leq V(t,x)$.

Then the zero solution of (3.2) is stable.

Proof: At first, we show stability for $t_0 = \theta_j$ for some $j \in \mathbb{N}_0$. Then it will allow us to prove stability for an arbitrary $t_0 \in \mathbb{R}^+$ due to Lemma 3.1.1.

Let $\rho_1 \in (0, \rho)$. Given $\varepsilon \in (0, \rho_1)$ and $t_0 = \theta_j$, choose $\delta_1 > 0$ sufficiently small that $V(\theta_j, x(\theta_j)) < u(\varepsilon)$ if $\left\|x(\theta_j)\right\| < \delta_1$. Define $\delta = \delta_1/K(\ell)$. We note $\delta < \delta_1$ as $K(\ell) > 1$ and show that this δ is the needed one.

Let us fix $k \in \mathbb{N}_0$ and consider the interval $[\theta_k, \theta_{k+1})$. Using the condition (ii), we shall show that

$$V(t, x(t)) \leq V(\theta_k, x(\theta_k)) \ \text{ for } \ t \in [\theta_k, \theta_{k+1}). \tag{3.6}$$

Set $V(t) = V(t, x(t))$. If (3.6) is not true, then there exist points κ and τ, $\theta_k \leq \kappa < \tau < \theta_{k+1}$, such that

$$V(\kappa) = V(\theta_k) \ \text{ and } \ V(t) > V(\theta_k) \ \text{ for } \ t \in (\kappa, \tau].$$

By applying the Mean-Value Theorem to the function V, we get

$$\frac{V(\tau) - V(\kappa)}{\tau - \kappa} = V'(\zeta) > 0 \tag{3.7}$$

for some $\zeta \in (\kappa, \tau)$. Indeed, being $V(\zeta) > V(\theta_k)$, it follows from the condition (ii) that $V'(\zeta) \leq 0$, which contradicts (3.7). Hence, (3.6) is true. Using the continuity of V and $x(t)$, we can obtain by induction that

$$V(t, x(t)) \leq V(\theta_j, x(\theta_j)) \ \text{ for all } \ t \geq \theta_j. \tag{3.8}$$

If $\left\|x(\theta_j)\right\| < \delta$, we have $V(\theta_j, x(\theta_j)) < u(\varepsilon)$ since $\delta < \delta_1$. This together with (3.8) leads us to the inequality $V(t, x(t)) < u(\varepsilon)$ which implies immediately that $||x(t)|| < \varepsilon$ for all $t \geq \theta_j$. Hence, stability for the case $t_0 = \theta_j$, $i \in \mathbb{N}_0$ is proved.

Now let us consider the case $t_0 \in \mathbb{R}^+$, $t_0 \neq \theta_i$ for all $i \in \mathbb{N}_0$. Then there is $j \in \mathbb{N}_0$ such that $\theta_j < t_0 < \theta_{j+1}$. Given $\varepsilon > 0$ $(\varepsilon < \rho_1)$, we choose $\delta_1 > 0$ such that $V(\theta_j, x(\theta_j)) < u(\varepsilon)$ if $\left\|x(\theta_j)\right\| < \delta_1$. Take a solution $x(t)$ of (3.2) such that $||x(t_0)|| < \delta$, where $\delta = \delta_1/K(\ell)$. By Lemma 3.1.1, $||x(t_0)|| < \delta$ results in $\left\|x(\theta_j)\right\| < \delta_1$. Then by the discussion used for $t_0 = \theta_j$, we obtain that $||x(t)|| < \varepsilon$ for all $t \geq \theta_j$ and hence for all $t \geq t_0$, proving the stability of the zero solution. □

Theorem 3.2.2 *Assume that there exists a function $V \in \vartheta$ such that*

(i) $u(\|x\|) \le V(t,x) \le v(\|x\|)$ *on* $\mathbb{R}^+ \times S(\rho)$*, where* $u, v \in \mathcal{K}$*;*

(ii) $V'(t,x,y) \le 0$ *for all* $t \ne \theta_i$ *in* $\mathbb{R}^+$ *and* $x, y \in S(\rho)$ *such that* $V(\beta(t), y) \le V(t,x)$*.*

Then the zero solution of (3.2) is uniformly stable.

Proof: Let $\rho_1 \in (0,\rho)$. Fix $\varepsilon > 0$ in the range $0 < \varepsilon < \rho_1$ and choose $\delta_1 > 0$ such that $v(\delta_1) \le u(\varepsilon)$. Define $\delta = \delta_1/K(\ell)$. Similar to the previous discussion, we consider two cases when $t_0 = \theta_j$ for some $j \in \mathbb{N}_0$ and another one when $t_0 \ne \theta_i$ for all $i \in \mathbb{N}_0$, to show that this δ is the needed one. If $t_0 = \theta_j$, where j is a fixed non-negative integer and $\left\|x(\theta_j)\right\| < \delta$, then as a consequence of the condition (i) we have $V(\theta_j, x(\theta_j)) < v(\delta) < v(\delta_1) \le u(\varepsilon)$. Using the same argument used in the proof of Theorem 3.2.1, we get the inequality $V(t, x(t)) \le V(\theta_j, x(\theta_j))$ for all $t \ge \theta_j$ and see that $V(t, x(t)) < u(\varepsilon)$ for all $t \ge \theta_j$. Hence $\|x(t)\| < \varepsilon$ for all $t \ge \theta_j$. We note that evaluation of δ does not depend on the choice of $j \in \mathbb{N}_0$.

Now, take $t_0 \in \mathbb{R}^+$ with $t_0 \ne \theta_i$ for all $i \in \mathbb{N}_0$. Then there exists $j \in \mathbb{N}_0$ such that $\theta_j < t_0 < \theta_{j+1}$. Take a solution $x(t)$ of (3.2) such that $\|x(t_0)\| < \delta$. It follows by Lemma 3.1.1 that $\left\|x(\theta_j)\right\| < \delta_1$. From a similar idea used for the case $t_0 = \theta_j$, we conclude that $\|x(t)\| < \varepsilon$ for $t \ge \theta_j$ and indeed for all $t \ge t_0$. Finally, one can see that the evaluation is independent of $j \in \mathbb{N}_0$ and correspondingly of all $t_0 \in \mathbb{R}^+$. □

Theorem 3.2.3 *Assume that all of the conditions in Theorem 3.2.2 are valid and there exist a continuous nondecreasing function ψ such that $\psi(s) > s$ for $s > 0$ and a function $w \in \Omega$. If condition (ii) is replaced by*

(iii) $V'(t,x,y) \le -w(\|x\|)$ *for all* $t \ne \theta_i$ *in* $\mathbb{R}^+$ *and* $x, y \in S(\rho)$ *such that* $V(\beta(t), y) < \psi(V(t,x))$*,*

then the zero solution of (3.2) is uniformly asymptotically stable.

Proof: When $V(\beta(t), y) \leq V(t, x)$, we have $V(\beta(t), y) < \psi(V(t, x))$. Then by the condition (*iii*), we have $V'(t, x, y) \leq 0$. From Theorem 3.2.2, it follows that the zero solution of (3.2) is uniformly stable.

First, we show "uniform" asymptotic stability with respect to all elements of the sequence θ_i, $i \in \mathbb{N}_0$.

Fix $j \in \mathbb{N}_0$ and $\rho_1 \in (0, \rho)$. If $t_0 = \theta_j$ and $\delta > 0$ is such that $v(K(\ell)\delta) = u(\rho_1)$, $K(\ell) > 1$, arguments of Theorem 3.2.2 shows that $V(t, x(t)) < v(\delta) < v(K(\ell)\delta)$ for all $t \geq \theta_j$ and hence $\|x(t)\| < \rho_1$ if $\left\|x(\theta_j)\right\| < \delta$. In what follows, we shall present that this δ can be taken as δ_0 in the Definition 3.2.2 of uniform asymptotic stability. That is, for arbitrary ε, $0 < \varepsilon < \rho_1$, we need to show that there exists a $T = T(\varepsilon) > 0$ such that $\|x(t)\| < \varepsilon$ for $t > \theta_j + T$ if $\left\|x(\theta_j)\right\| < \delta$.

Set $\gamma = \inf\{w(s) : v^{-1}(u(\varepsilon)) \leq s \leq \rho_1\}$. We note that this set is not empty since $\varepsilon < \rho_1$ and $u, v \in \mathcal{K}$ implies that $u(\varepsilon) < v(\rho_1)$, which, in turn, leads us to the inequality $v^{-1}(u(\varepsilon)) < \rho_1$.

Denote $\delta_1 = K(\ell)\delta$. From the properties of the function $\psi(s)$, there is a number $a > 0$ such that $\psi(s) - s > a$ for $u(\varepsilon) \leq s \leq v(\delta_1)$.

Let N be the smallest positive integer such that $u(\varepsilon) + Na \geq v(\delta_1)$.

Choose $t_k = k(\dfrac{v(\delta_1)}{\gamma} + \theta) + \theta_j$, $k = 1, 2, \ldots, N$. We will prove that

$$V(t, x(t)) \leq u(\varepsilon) + (N - k)a \ \text{ for } \ t \geq t_k, \ \ k = 0, 1, 2, \ldots, N. \tag{3.9}$$

We have $V(t, x(t)) < v(\delta_1) \leq u(\varepsilon) + Na$ for $t \geq t_0 = \theta_j$. Hence, (3.9) holds for $k = 0$. Now, we suppose that (3.9) holds true for some $0 \leq k < N$. Let us show that

$$V(t, x(t)) \leq u(\varepsilon) + (N - k - 1)a \ \text{ for } \ t \geq t_{k+1}. \tag{3.10}$$

Let $I_k = [\beta(t_k) + \theta, t_{k+1}]$. To prove (3.10), we first claim that there exists a $t^* \in I_k$ such that

$$V(t^*, x(t^*)) \leq u(\varepsilon) + (N - k - 1)a. \tag{3.11}$$

Otherwise, $V(t, x(t)) > u(\varepsilon) + (N - k - 1)a$ for all $t \in I_k$.

On the other side, we have

$$V(t, x(t)) \leq u(\varepsilon) + (N - k)a \;\text{ for }\; t \geq t_k,$$

which implies that $V(\beta(t), x(\beta(t))) \leq u(\varepsilon) + (N - k)a$ for $t \geq \beta(t_k) + \theta$.

Hence, for $t \in I_k$

$$\psi(V(t, x(t))) > V(t, x(t)) + a > u(\varepsilon) + (N - k)a \geq V(\beta(t), x(\beta(t))).$$

Since $v^{-1}(u(\epsilon)) \leq \|x(t)\| \leq \rho_1$ for $t \in I_k$, it follows from the hypothesis (*iii*) that

$$V'(t, x(t), x(\beta(t))) \leq -w(\|x(t)\|) \leq -\gamma \text{ for all } t \neq \theta_m \text{ in } I_k,\ m \in \mathbb{N}_0.$$

Using the continuity of the function V and the solution $x(t)$, we get

$$\begin{aligned} V(t_{k+1}, x(t_{k+1})) &\leq V(\beta(t_k) + \theta, x(\beta(t_k) + \theta)) - \gamma(t_{k+1} - \beta(t_k) - \theta) \\ &< v(\delta_1) - \gamma(t_{k+1} - t_k - \theta) = 0, \end{aligned}$$

which is a contradiction. Thus (3.11) holds, that is, there exists a $t^* \in I_k$ such that $V(t^*, x(t^*)) \leq u(\varepsilon) + (N - k - 1)a$.

Next, we show that

$$V(t, x(t)) \leq u(\varepsilon) + (N - k - 1)a \;\text{ for all }\; t \in [t^*, \infty). \tag{3.12}$$

If (3.12) does not hold, then there exists a $\hat{t} \in (t^*, \infty)$ such that

$$V(\hat{t}, x(\hat{t})) > u(\varepsilon) + (N - k - 1)a \geq V(t^*, x(t^*)).$$

Thus, we can find a $\tilde{t} \in (t^*, \hat{t})$ such that $\tilde{t} \neq \theta_m$, $m \in \mathbb{N}_0$, $V'(\tilde{t}, x(\tilde{t}), x(\beta(\tilde{t}))) > 0$ and satisfying $V(\tilde{t}, x(\tilde{t})) > u(\varepsilon) + (N - k - 1)a$. If there is no such $\tilde{t}$, then for all $t \in (t^*, \hat{t})$, $t \neq \theta_m$, we have $V'(t, x(t), x(\beta(t))) \leq 0$ or $V(t, x(t)) \leq u(\varepsilon) + (N - k - 1)a$. But, $V'(t, x(t), x(\beta(t))) \leq 0$ leads to $V(\hat{t}, x(\hat{t})) \leq V(t^*, x(t^*))$, a contradiction. If $V(t, x(t)) \leq u(\varepsilon) + (N - k - 1)a$, then $V(t, x(t)) < V(\hat{t}, x(\hat{t}))$ for $t \in (t^*, \hat{t})$, $t \neq \theta_m$, also yields a contradiction. Hence, $\tilde{t}$ exists.

However,

$$\psi(V(\tilde{t}, x(\tilde{t}))) > V(\tilde{t}, x(\tilde{t})) + a > u(\varepsilon) + (N - k)a \geq V(\beta(\tilde{t}), x(\beta(\tilde{t})))$$

implies that $V'(\tilde{t}, x(\tilde{t}), x(\beta(\tilde{t}))) \leq -\gamma < 0$, a contradiction. Then, we conclude that $V(t, x(t)) \leq u(\varepsilon) + (N - k - 1)a$ for all $t \geq t^*$ and thus for all $t \geq t_{k+1}$. This completes the induction and shows that (3.9) is valid. For $k = N$, we have

$$V(t, x(t)) \leq u(\varepsilon)\,,\ t \geq t_N = N\left(\frac{v(\delta_1)}{\gamma} + \theta\right) + t_0.$$

Hence, $\|x(t)\| < \varepsilon$ for $t > \theta_j + T$ where $T = N\left(\frac{v(\delta_1)}{\gamma} + \theta\right)$, proving the uniform asymptotic stability for $t_0 = \theta_j$, $j \in \mathbb{N}_0$.

Consider the case $t_0 \neq \theta_i$ for all $i \in \mathbb{N}_0$. Then $\theta_j < t_0 < \theta_{j+1}$ for some $j \in \mathbb{N}_0$. $\|x(t_0)\| < \delta$ implies by Lemma 3.1.1 that $\left\|x(\theta_j)\right\| < \delta_1$. Hence, the argument used above for the case $t_0 = \theta_j$ yields that $\|x(t)\| < \varepsilon$ for $t > \theta_j + T$ and in turn for all $t > t_0 + T$. $\square$

3.3 Examples and New Lights for the Logistic Equation

In the following examples, we assume that the sequence θ_i, which is used for the definition of the function $\beta(t)$, satisfies the condition (C4). For the logistic equation with piecewise constant argument of generalized type, we present stability results for all possible initial moments on $\mathbb{R}^+$. Hence, these results are advantegous compared to the previous ones which take integers as initial moments.

Example 3.3.1 *Consider the following linear equation*

$$x'(t) = -a(t)x(t) - b(t)x(\beta(t)), \tag{3.13}$$

where a and b are bounded continuous functions on $\mathbb{R}^+$ such that $|b(t)| \leq a(t)$ for all $t \geq 0$. We can check that conditions (C1)-(C2) and (C3) with the Lipschitz constant $\ell = \sup_{t\in\mathbb{R}^+} a(t)$ are fulfilled. Moreover, we assume that the sequence θ_i and ℓ satisfy (C5) and (C6). Let $V(x) = \frac{x^2}{2}$, then for $t \neq \theta_i$, $i \in \mathbb{N}_0$,

$$\begin{aligned} V'(x(t)) &= -a(t)x^2(t) - b(t)x(t)x(\beta(t)) \\ &\leq -a(t)x^2(t) + |b(t)|\,|x(t)|\,|x(\beta(t))| \\ &\leq -[a(t) - |b(t)|]x^2(t) \leq 0 \end{aligned}$$

whenever $|x(\beta(t))| \leq |x(t)|$. *Since* $V = x^2/2$, $V(x(\beta(t))) \leq V(x(t))$ *implies that* $V'(x(t)) \leq 0$. *Thus by Theorem 3.2.2, the trivial solution of (3.13) is uniformly stable.*

Next, let us investigate uniform asymptotic stability. If there are constants $\lambda > 0$, $\omega \in [0, 1)$ *and* $q > 1$ *with* $\lambda \leq a(t)$, $|b(t)| \leq \omega\lambda$ *and* $1 - q\omega > 0$, *then for* $\psi(s) = q^2 s$, $w(s) = (1 - q\omega)\lambda s^2$ *and* $V(x) = \dfrac{x^2}{2}$, *we obtain that*

$$V'(x(t)) \leq -w(|x(t)|), \ \ t \neq \theta_i,$$

whenever $V(x(\beta(t))) < \psi(V(x(t)))$. *Theorem 3.2.3 implies that* $x = 0$ *is uniformly asymptotically stable.*

The following illustration is a development of an example from [150].

Example 3.3.2 *Let us now consider a nonlinear scalar equation*

$$x'(t) = f(x(t), \mu x(\beta(t))), \tag{3.14}$$

where $f(x, y)$ *is a continuous function with* $f(0, 0) = 0$, $\dfrac{f(x, 0)}{x} = -\sigma$ *for some* $\sigma > 0$ *satisfying* $\sigma \geq \ell|\mu|$ *and* $|f(x_1, y_1) - f(x_2, y_2)| \leq \ell(|x_1 - x_2| + |y_1 - y_2|)$. *Then conditions (C1)-(C3) are valid. We consider a sequence* θ_i *such that (C5)-(C6) hold true together with the Lipschitz constant* ℓ.

Choosing $V(x) = x^2$, *we get for* $t \neq \theta_i$

$$\begin{aligned} V'(x(t)) &= 2x(t)f(x(t), \mu x(\beta(t))) \\ &= 2\left[\frac{f(x(t), \mu x(\beta(t))) - f(x(t), 0)}{x(t)} + \frac{f(x(t), 0)}{x(t)}\right]x^2(t) \\ &\leq 2\left[\frac{\ell|\mu||x(\beta(t))|}{|x(t)|} - \sigma\right]x^2(t) \leq 2(\ell|\mu| - \sigma)x^2(t) \leq 0 \end{aligned}$$

whenever $V(x(\beta(t))) \leq V(x(t))$. *It follows from Theorem 3.2.2 that the solution* $x = 0$ *of (3.14) is uniformly stable.*

Example 3.3.3 *(a logistic equation with harvesting)*

In [77], stability of the positive equilibrium $N^* = \dfrac{1}{a + b}$ *of equation (3.1) has been studied. Equation (3.1) models the dynamics of a logistically growing population subjected to a density-dependent harvesting. There,* $N(t)$ *denotes the population density of a single species and the model parameters* r , a *and* b *are assumed to be positive.*

Gopalsamy and Liu showed that N^* is globally asymptotically stable if $\alpha \geq 1$ where $\alpha = a/b$. Particularly, it was shown that the equilibrium state is stable for integer-valued initial moments. The restriction is caused by the method of investigation: reduction to difference equations. Our results are for all initial moments from $\mathbb{R}^+$, not only integers. Moreover, we consider uniform stability for the general case $\beta(t)$. Consequently, we may say that our approach allows to study stability of such equations in the complete form.

We consider the biological sense of the insertion of piecewise constant delay [77, 78, 131, 132] into a population model as follows. The delay means that the rate of the population depends both on the present size as well as the memorized values of the population. To illustrate the dependence, one may think populations, which meet at the beginning of a season, e.g., in springtime, with their instinctive evaluations of the population state, environment and implicitly decide which living conditions to prefer and where to go [12] in line with group hierarchy, communications, dynamics and then adapt to those conditions.

Let us discuss the following equation

$$N'(t) = rN(t)(1 - aN(t) - bN(\beta(t))), \quad t > 0, \tag{3.15}$$

which is a generalization of (3.1). One can see that (3.1) is of type (3.15) when $\beta(t) = [t]$.

For our needs, we translate the equilibrium point N^* to the origin by the transformation $x = b(N - N^*)$, which takes (3.15) into the following form

$$x'(t) = -r[x(t) + \frac{1}{1+\alpha}][\alpha x(t) + x(\beta(t))]. \tag{3.16}$$

Note that $f(x,y) := -r(x + \frac{1}{1+\alpha})(\alpha x + y)$ is a continuous function and has continuous partial derivatives for $x, y \in S(\rho)$. If we evaluate the first partial derivatives of the function $f(x,y)$, we see that

$$|\partial f/\partial x| \leq r(2\alpha\rho + \rho + \frac{\alpha}{1+\alpha}),$$
$$|\partial f/\partial y| \leq r(\rho + \frac{1}{1+\alpha}),$$

for $x, y \in S(\rho)$.

If we choose $\ell = r(2\alpha\rho + 2\rho + 1)$ as a Lipschitz constant, one can see that the conditions (C1)-(C3) are fulfilled for sufficiently small r. In addition, we assume that ℓ is sufficiently small so that the conditions (C5) and (C6) are satisfied.

Suppose that $\alpha \geq 1$ and $\rho < 1/(1+\alpha)$. Then for $V(x) = x^2$, $x \in S(\rho)$ and $t \neq \theta_i$, we have

$$\begin{aligned} V'(x(t), x(\beta(t))) &= -2rx(t)(x(t) + \frac{1}{1+\alpha})(\alpha x(t) + x(\beta(t))) \\ &\leq -2r(x(t) + \frac{1}{1+\alpha})(\alpha x^2(t) - |x(t)||x(\beta(t))|) \\ &\leq -2r(x(t) + \frac{1}{1+\alpha})(\alpha - 1)x^2(t) \leq 0 \end{aligned}$$

whenever $V(x(\beta(t))) \leq V(x(t))$. Theorem 3.2.2 implies that the zero solution of (3.16) is uniformly stable. This in turn leads to uniform stability of the positive equilibrium N^* of (3.15).

To prove uniform asymptotic stability, we need to satisfy the condition (iii) in Theorem 3.2.3. In view of uniform stability, given $\rho_1 \in (0, \rho)$ we know that there exists a $\delta > 0$ such that $x(t) \in S(\rho_1)$ for all $t \geq t_0$ whenever $|x(t_0)| < \delta$. Let us take a constant q such that $1 < q < \alpha$, then for $\psi(s) = q^2 s$, $w(s) = 2r(\alpha - q)\eta s^2$, $\eta = 1/(1+\alpha) - \rho_1$ and $V(x) = x^2$, we have

$$V'(x(t), x(\beta(t))) \leq -2r(x(t) + \frac{1}{1+\alpha})(\alpha - q)x^2(t) \leq -w(|x(t)|), \; t \neq \theta_i,$$

whenever $V(x(\beta(t))) < \psi(V(x(t)))$. Hence the solution $x = 0$ ($N = N^*$) of (3.16) ((3.15)) is uniformly asymptotically stable by Theorem 3.2.3.

CHAPTER 4

BIFURCATION OF NON-SMOOTH LIMIT CYCLES

4.1 Bifurcation of a Non-Smooth Planar Limit Cycle from a Vertex

The theory of differential equations with discontinuous right-hand sides has been substantially developed through numerous applications. There are many problems from mechanics, engineering sciences [25, 108, 109, 129], control theory [72] and economics [90] that are modeled by dynamical systems with discontinuous vector fields. Besides, the books [25, 35, 129], which concern mechanical systems with dry friction, periodic solutions of discontinuos systems and discontinuous oscillations, form an important basis for the development of such discontinuous systems. Owing to the problems of applied nature, qualitative theory of classical ordinary differential equations including the notions of existence, uniqueness, continuous dependence, stability and bifurcation has been carefully adapted for equations with discontinuous right-hand sides. The main trends of the theory can be found in [72].

Bifurcations in smooth systems are well understood [39, 45, 79, 124], but little is known in discontinuous systems. Stimulated by non-smooth phenomena in the real world, subject of Hopf bifurcation in discontinuous systems has received great attention in recent years [49, 108, 109, 112, 116, 129, 138, 140, 179, 180]. Dankowicz and Nordmark [57] study bifurcations of stick-slip oscillations for the friction model which leads to a non-smooth dynamical system having discontinuity at the first derivative of the vector field. Feigin [70, 71] considers $C-$bifurcations, also known as border-collision bifurcations, in Filippov systems being a subclass of discontinuous systems described by differential equations with a discontinuous right-hand side

[72]. Border-collision bifurcations for non-smooth discrete maps are also addressed by Nusse and Yorke [138, 140].

Kunze [108] and Küpper et al. [112, 179] address bifurcation of periodic solutions for planar Filippov systems with discontinuities on a single straight line. In [179], generalized Hopf bifurcation for a piecewise smooth planar system of the following form

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{cases} f^{+}(x,y,\lambda), & x > 0, \\ f^{-}(x,y,\lambda), & x < 0, \end{cases}$$

where $f^{\pm}(x,y,\lambda) = A^{\pm}(\lambda)(x,y)^T + g^{\pm}(x,y,\lambda)$, λ a real parameter, has been investigated using differential inclusions. Eigenvalues of the matrix $A^{\pm}(\lambda)$ were assumed to be complex conjugate, i.e., $\alpha^{\pm}(\lambda) \pm i\omega^{\pm}(\lambda)$. This system has been stimulated by a brake system of the form

$$\begin{aligned} &mu'' + d_1u' + c_1u = \sigma^{+}(u,u',\lambda), && \text{if } u > 0, \\ &mu'' + (d_1 + d_2)u' + (c_1 + c_2)u = \sigma^{-}(u,u',\lambda), && \text{if } u < 0, \end{aligned}$$

where a mass m rests on a smooth surface and is connected to the walls by springs (c_1 and c_2) and dampers (d_1 and d_2). $\sigma^{\pm}$ denotes the external force and the parameter λ controls its magnitude (see [179] for details).

In papers [14] and [180], possibly for the first time, a special structure of the domain has been developed for planar differential equations with discontinuities. To say more clearly, [14] treats bifurcation of periodic solutions for planar discontinuous dynamical systems where discontinuities in the state variable appear on countably many curves intersecting at the origin, and [180] studies generalized Hopf bifurcation for piecewise smooth planar systems with discontinuities on the right-hand side at several straight lines emanating from the origin. We suppose that domains of this type can be very useful in various mechanical and electrical models with discontinuities under proper transformations.

Most of the papers in the literature assume that discontinuity sets of non-smooth systems consist of a single surface, especially a straight line [49, 108, 112, 179]. However, due to exterior effects, discontinuities may appear on curves or surfaces of nonlinear feature. Hence, it is reasonable to perturb the sets of discontinuities. Differential equations whose right-hand sides are discontinuous on nonlinear surfaces were investigated in [18, 19, 22] by the method of B-equivalence [2, 14, 18, 21, 22]. This method was first proposed to reduce impulsive systems with variable time of impulses to the systems with fixed moments of impulse effects. It then turned out that this method could be used for differential equations with discontinuous right-hand sides as well [18, 22]. That is, through the B-equivalence method, differential equations with discontinuous vector fields with nonlinear discontinuity sets can be reduced to impulsive differential equations with fixed moments of impulses.

Our present work is an attempt to generalize the problem of Hopf bifurcation for a planar non-smooth system by considering discontinuities on finitely many nonlinear curves emanating from a vertex. We consider the domain in a neighborhood of a vertex which unites several curves. That is, the phase space is divided into subdomains and the system is described by a different set of differential equations in each domain. We can say that the system considered in this section is more general than the one in [180], where discontinuities occur at straight lines. We aim to give some theoretical background rather than applications, which will be very useful in many problems in the future. Using B-equivalence of the issue systems to impulsive differential equations, we obtain corresponding qualitative properties. It is the inherent advantage of the B-equivalence method that we can study equations with nonlinear discontinuity sets.

The section is organized in the following way. In Section 4.1.1, we introduce the nonperturbed system and study existence of foci and centers for that system. Section 4.1.2 presents the perturbed system and the notion of B-equivalent impulsive systems. The problem of distinguishing between the center and the focus is solved in Section 4.1.3. We investigate bifurcation of periodic solutions in the next section. We use the geometrical characterization given by the change from an unstable to a stable focus through a center for the nonperturbed system. Afterwards, an appropriate example is worked out to illustrate our results. Finally, we discuss the possible generalization of the present results in Section 4.1.6.

4.1.1 The Nonperturbed System

We consider every angle for a point with respect to the positive half-line of the first coordinate axis. In the rest of the present section, following assumptions will be needed.

(A1) Let $\{l_i\}_{i=1}^p$, $p \geq 2$, $p \in \mathbb{N}$, be a set of half-lines starting at the origin and given by the equations $\Phi_i(x) = 0$, $\Phi_i(x) = \left\langle a^i, x \right\rangle$, $i = 1, 2, \ldots, p$, where $a^i = ({a_1}^i, {a_2}^i) \in \mathbb{R}^2$ are constant vectors (see Figure 4.1). Let γ_i, $i = 1, 2, \ldots, p$, denote the angles of the lines l_i such that

$$0 < \gamma_1 < \gamma_2 < \cdots < \gamma_p < 2\pi.$$

(A2) There exist real-valued constant 2×2 matrices A_i defined by $A_i = \begin{bmatrix} \alpha_i & -\beta_i \\ \beta_i & \alpha_i \end{bmatrix}$ with $\beta_i > 0$ for each $i = 1, 2, \ldots, p$.

Meanwhile, for convenience throughout this section, we adopt the notations below.

(N1) $\theta_1 = (2\pi + \gamma_1) - \gamma_p$, $\theta_i = \gamma_i - \gamma_{i-1}$, $i = 2, 3, \ldots, p$.

(N2) Let D_i denote the region situated between the straight lines l_{i-1} and l_i and defined in polar coordinates (r, ϕ), where $x_1 = r\cos\phi$, $x_2 = r\sin\phi$, as follows

$$D_1 = \{(r, \phi) \mid r \geq 0 \text{ and } \gamma_p < \phi \leq \gamma_1 + 2\pi\},$$

$$D_i = \{(r, \phi) \mid r \geq 0 \text{ and } \gamma_{i-1} < \phi \leq \gamma_i\}, \; i = 2, 3, \ldots, p.$$

Now we define a function f such that $f(x) = A_i x$ for $x \in D_i$, $i = 1, 2, \ldots, p$, and consider a differential equation of the form

$$\frac{dx}{dt} = f(x). \tag{4.1}$$

According to the definition of the regions D_i, one can see that the function f in system (4.1) has discontinuities on the straight lines l_i, $i = 1, 2, \ldots, p$.

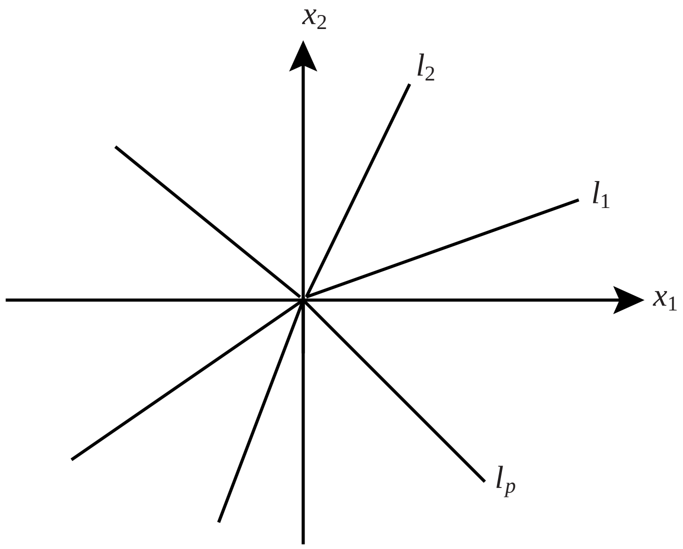

Figure 4.1: The domain of the nonperturbed system (4.1) with a vertex which unites the straight lines l_i, $i = 1, 2, \ldots, p$

Remark 4.1.1 It follows from the assumptions (A1) and (A2) that

$$\langle \frac{\partial \Phi_i(x)}{\partial x}, f(x) \rangle \neq 0 \quad \text{for } x \in l_i,\ i = 1, 2, \ldots, p.$$

That is, the vector field is transversal at every point on l_i for each i.

If we use the polar transformation, we can write the system (4.1) in the following form

$$\frac{dr}{d\phi} = g(r), \tag{4.2}$$

where

$$g(r) = \begin{cases} \lambda_1 r, & \text{if} \quad \phi \in (\gamma_p + 2k\pi, \gamma_1 + 2(k+1)\pi], \\ \lambda_i r, & \text{if} \quad \phi \in (\gamma_{i-1} + 2k\pi, \gamma_i + 2k\pi],\ i = 2, 3, \ldots, p, \end{cases}$$

with $\lambda_i = \dfrac{\alpha_i}{\beta_i}$, $i = 1, 2, \ldots, p$, and $k \in \mathbb{Z}$. Since equation (4.2) is $2\pi-$periodic, it will be enough to consider just the section $\phi \in [0, 2\pi]$. Thus, the function g in (4.2) can be defined shortly as $g(r) = \lambda_i r$ if $(r, \phi) \in D_i$. Clearly, this function has discontinuities when $\phi = \gamma_i$, $i = 1, 2, \ldots, p$.

The solution $r(\phi, r_0)$ of (4.2) starting at the point $(0, r_0)$ has the form

$$r(\phi, r_0) = \begin{cases} \exp(\lambda_1 \phi) r_0, & \text{if} \quad 0 \le \phi \le \gamma_1, \\ \exp\left(\lambda_1 \gamma_1 + \lambda_2 \theta_2 + \cdots + \lambda_i (\phi - \gamma_{i-1})\right) r_0, & \text{if} \quad \gamma_{i-1} < \phi \le \gamma_i, \\ \exp\left(\lambda_1 \left(\phi - (\gamma_p - \gamma_1)\right) + \sum_{i=2}^{p} \lambda_i \theta_i\right) r_0, & \text{if} \quad \gamma_p < \phi \le 2\pi, \end{cases}$$

where $i = 2, 3, \ldots, p$.

If we construct the Poincaré return map $r(2\pi, r_0)$ on the positive half-axis Ox_1, we can see that

$$r(2\pi, r_0) = \exp(\sum_{i=1}^{p} \lambda_i \theta_i) r_0.$$

Let us denote $q = \exp(\sum_{i=1}^{p} \lambda_i \theta_i)$. Since $r(2\pi, r_0) = q r_0$, we obtain the following theorem for the nonperturbed system.

Theorem 4.1.1 *If*
(i) $q = 1$, *then the origin is a center and all solutions are periodic with period* $T = \sum_{i=1}^{p} \frac{\theta_i}{\beta_i}$;
(ii) $q < 1$, *then the origin is a stable focus;*
(iii) $q > 1$, *then the origin is an unstable focus of (4.1).*

4.1.2 The Perturbed System

Let $\Omega \subset \mathbb{R}^2$ be a domain in the neighborhood of the origin. The following is the list of conditions assumed for this section.

(P1) Let $\{c_i\}_{i=1}^{p}$ be a set of curves in Ω which start at the origin and are determined by the equations $\tilde{\Phi}_i(y) = 0$, $\tilde{\Phi}_i(y) = \left\langle a^i, y \right\rangle + \tau_i(y)$, $i = 1, 2, \ldots, p$, where $\tau_i(y) = o(\|y\|)$ and for each i, the constant vectors a^i are the same as described in (A1).

We split the domain Ω into $p-$subdomains, which will be called $\tilde{D}_i$ and formulated soon, by means of the curves c_i, $i = 1, 2, \ldots, p$. We assume without loss of generality that $\gamma_i \neq \frac{\pi}{2} j,\ j = 1, 3.$ Then for sufficiently small r, equation of the curve c_i can be written in polar coordinates as follows [14]

$$c_i : \phi = \gamma_i + \psi_i(r, \phi),\ \ i = 1, 2, \ldots, p, \tag{4.3}$$

where ψ_i is a $2\pi-$periodic function in ϕ, continuously differentiable and moreover $\psi_i = O(r)$. Using this discussion which makes use of polar transformation, we get the following description for the regions $\tilde{D}_i$:

$$\tilde{D}_1 = \{(r, \phi) \mid r \geq 0 \text{ and } \gamma_p + \psi_p(r, \phi) < \phi \leq \gamma_1 + 2\pi + \psi_1(r, \phi)\},$$

$$\tilde{D}_i = \{(r, \phi) \mid r \geq 0 \text{ and } \gamma_{i-1} + \psi_{i-1}(r, \phi) < \phi \leq \gamma_i + \psi_i(r, \phi)\},\ \ i = 2, 3, \ldots, p.$$

Let ε be a positive number and $N_\varepsilon(\tilde{D}_i)$ denote the $\varepsilon-$neighborhoods of the regions $\tilde{D}_i$, $i = 1, 2, \ldots, p$.

(P2) Let f_i be a function defined on $N_\varepsilon(\tilde{D}_i)$ and $f_i \in C^{(2)}(N_\varepsilon(\tilde{D}_i))$ for each $i = 1, 2, \ldots, p$.

(P3) $\tau_i \in C^{(2)}(N_\varepsilon(\tilde{D}_i))$, $i = 1, 2, \ldots, p$.

(P4) $f_i(y) = o(\|y\|)$, $i = 1, 2, \ldots, p$.

We shall consider the function $\tilde{f}(y) = A_i y + f_i(y)$ for $y \in \tilde{D}_i$, where the matrix A_i is as described in the assumption (A2). On Ω, we now study the following differential equation associated with (4.1)

$$\frac{dy}{dt} = \tilde{f}(y), \tag{4.4}$$

where the function $\tilde{f}(y)$ has discontinuities on the curves c_i, $i = 1, 2, \ldots, p$. The domain of the system (4.4) can be seen in Figure 4.2.

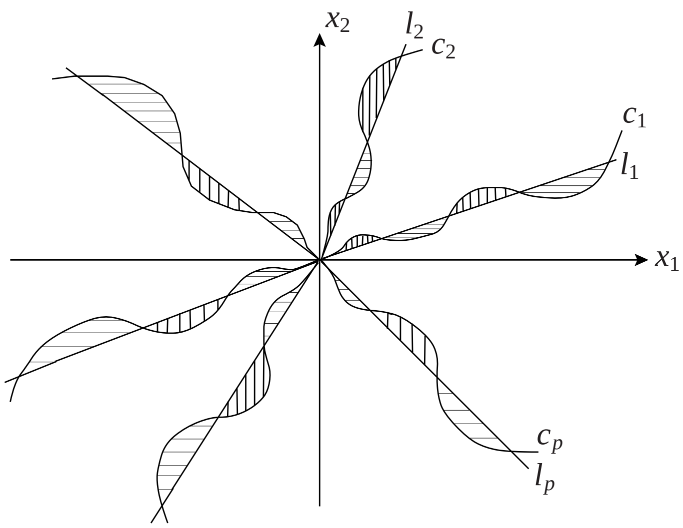

Figure 4.2: The domain of the perturbed system (4.4) near a vertex which unites the curves c_i associated with the straight lines l_i, $i = 1, 2, \ldots, p$

If Ω is sufficiently small, then conditions (A1) and (P1) imply that curves c_i intersect each other only at the origin, none of them can intersect itself and $\langle \frac{\partial \tilde{\Phi}_i(y)}{\partial y}, \tilde{f}(y) \rangle \neq 0$ for $y \in c_i$, $i = 1, 2, \ldots, p$.

Further, for system (4.4) if a solution which starts sufficiently close to the origin on a curve c_i with fixed i, then conditions mentioned above imply the continuation of the solution to the curve c_{i+1} or c_{i-1} depending on the direction of the time.

We can utilize polar coordinates and assume that system (4.4) transforms into an equivalent system of the form

$$\frac{dr}{d\phi} = \tilde{g}(r, \phi), \tag{4.5}$$

where $\tilde{g}(r, \phi) = \lambda_i r + P_i(r, \phi)$ for $(r, \phi) \in \tilde{D}_i$. The function P_i is $2\pi-$periodic in ϕ, continuously differentiable and $P_i = o(r)$, $i = 1, 2, \ldots, p$.

From the construction, we see that system (4.5) is a differential equation with discontinuous right-hand side and the discontinuities occur on the curves c_i, $i = 1, 2, \ldots, p$. In almost every area of differential equations, it is common to reduce a given equation into an equivalent form by proper methods. From this point of view, we shall use the B-equivalence method [18, 22] which plays the role of a bridge in the passage from differential equations with discontinuous right-hand sides to impulsive differential equations.

To reduce the system (4.5) with discontinuous vector fields into an impulsive differential equation, we redefine the function $\tilde{g}$ in the neighborhoods of the straight lines l_i, which contain the curve c_i. That is to say, we construct a new function g_N which is continuous everywhere except possibly at the points $(r, \phi) \in l_i$. The redefinition will be made at the points which lie between l_i and c_i and belong to the regions D_i or D_{i+1} for each i. Therefore, the construction is performed with minimal possible changes corresponding to the B-equivalence method, which is the main instrument of our investigation.

It is clear from the context that if $i = p$ then $D_{p+1} = D_1$. Using the argument above, we realize the following reconstruction of the domain. We consider the subregions of D_i and D_{i+1}, which are placed between the straight line l_i and the curve c_i. We refer to the subregions $D_i \cap \tilde{D}_{i+1}$ (horizontally shaded regions in Figure 4.2) and $D_{i+1} \cap \tilde{D}_i$ (vertically shaded regions in Figure 4.2) for all i. We extend the function $\tilde{g}$ from the region $D_i \cap \tilde{D}_{i+1}$ to D_i and from $D_{i+1} \cap \tilde{D}_i$ to D_{i+1} so that the new function g_N and its partial derivatives become continuous up to the angle $\phi = \gamma_i$, $i = 1, 2, \ldots, p$. According to all these discussions made for the definition of g_N, we conclude that $g_N(r, \phi) = \lambda_i r + P_i(r, \phi)$ for $(r, \phi) \in D_i$. Now we consider the following differential equation

$$\frac{dr}{d\phi} = g_N(r, \phi). \tag{4.6}$$

Fix $i \in \{1, 2, \ldots, p\}$ and consider a neighborhood of l_i based on the description above. We need to analyze the following three cases:

I. Assume that the point $(r, \gamma_i) \in \tilde{D}_{i+1}$. Let $r^0(\phi) = r(\phi, \gamma_i, \rho)$ be a solution of (4.5) satisfying $r^0(\gamma_i) = \rho$ and ξ_i be the angle where this solution crosses the curve c_i. We denote a solution of (4.6) by $r^1(\phi) = r(\phi, \xi_i, r^0(\xi_i))$, $r^1(\xi_i) = r^0(\xi_i)$, on the interval $[\xi_i, \gamma_i]$. By the variation of constant formula, these solutions have the form

$$r^0(\phi) = \exp(\lambda_{i+1}(\phi - \gamma_i))\rho + \int_{\gamma_i}^{\phi} \exp(\lambda_{i+1}(\phi - s))P_{i+1}(r^0(s), s)ds,$$

$$r^1(\phi) = \exp(\lambda_i(\phi - \xi_i))r^0(\xi_i) + \int_{\xi_i}^{\phi} \exp(\lambda_i(\phi - s))P_i(r^1(s), s)ds.$$

Now, we define a mapping I_i on the line $\phi = \gamma_i$ into itself as follows

$$\begin{aligned} I_i(\rho) &= r^1(\gamma_i) - \rho = (\exp((\lambda_i - \lambda_{i+1})(\gamma_i - \xi_i)) - 1)\,\rho \\ &+ \exp(\lambda_i(\gamma_i - \xi_i)) \int_{\gamma_i}^{\xi_i} \exp(\lambda_{i+1}(\xi_i - s))P_{i+1}ds \\ &+ \int_{\xi_i}^{\gamma_i} \exp(\lambda_i(\gamma_i - s))P_i ds. \end{aligned}$$

II. If the point $(r, \gamma_i) \in \tilde{D}_i$, one can find I_i in a similar manner:

$$\begin{aligned} I_i(\rho) &= (\exp((\lambda_i - \lambda_{i+1})(\xi_i - \gamma_i)) - 1)\,\rho \\ &+ \exp(\lambda_{i+1}(\gamma_i - \xi_i)) \int_{\gamma_i}^{\xi_i} \exp(\lambda_i(\xi_i - s))P_i ds \\ &+ \int_{\xi_i}^{\gamma_i} \exp(\lambda_{i+1}(\gamma_i - s))P_{i+1}ds. \end{aligned}$$

III. If $(r, \gamma_i) \in c_i$, then $I_i(\rho) = 0$.

Results from [14] imply that the functions I_i, $i = 1, 2, \ldots, p$, are continuously differentiable and the equation (4.3) leads us to $I_i = o(\rho)$.

Hereby we construct the following impulsive differential equation

$$\begin{aligned} &\frac{d\rho}{d\phi} = g_N(\rho, \phi), \quad \phi \neq \gamma_i, \\ &\Delta\rho|_{\phi = \gamma_i} = I_i(\rho). \end{aligned} \tag{4.7}$$

Let $r(\phi, r_0)$ be a solution of (4.5), $r(0, r_0) = r_0$, and ξ_i be the meeting angle of this solution with the curve c_i. Denote by $(\xi_i, \hat{\gamma}_i]$ the interval $(\xi_i, \gamma_i]$ whenever $\xi_i \leq \gamma_i$ and $[\gamma_i, \xi_i)$ if $\gamma_i < \xi_i$.

Definition 4.1.1 *We shall say that systems (4.5) and (4.7) are B-equivalent in Ω if for every solution $r(\phi, r_0)$ of (4.5) whose trajectory is in Ω for all $\phi \in [0, 2\pi]$ there exists a solution $\rho(\phi, r_0)$ of (4.7) which satisfies the relation*

$$r(\phi, r_0) = \rho(\phi, r_0), \quad \phi \in [0, 2\pi] \setminus \bigcup_{i=1}^{p} (\xi_i, \hat{\gamma}_i], \tag{4.8}$$

and, conversely, for every solution $\rho(\phi, r_0)$ of (4.7) whose trajectory is in Ω, there exists a solution $r(\phi, r_0)$ of (4.5) which satisfies (4.8).

From the discussion above and the construction of the impulsive system (4.7) with impulse actions at fixed angles, it follows that for sufficiently small Ω, solution $r(\phi, r_0)$ of (4.5) whose trajectory is in Ω for all $\phi \in [0, 2\pi]$ takes the same values with the exception of the oriented intervals $(\widehat{\xi_i, \gamma_i}]$ as the solution $\rho(\phi, r_0)$, $\rho(0, r_0) = r_0$, of (4.7). Hence, systems (4.5) and (4.7) are B-equivalent in the sense of the Definition 4.1.1. Moreover, solutions of (4.5) exist in the neighborhood Ω, they are continuous and have discontinuities in the derivative on the curves c_i. Correspondingly, a solution of system (4.4) for any initial value is continuous, continuously differentiable except possibly at the moments when the trajectories intersect the curves c_i, and it is unique.

Theorem 4.1.2 *Suppose (A1)-(A2), (P1)-(P4) are satisfied and $q < 1$ $(q > 1)$. Then the origin is a stable (unstable) focus of (4.4).*

Proof: Let $r(\phi, r_0)$ be the solution of (4.5) with $r(0, r_0) = 0$ and $\rho(\phi, r_0)$, $\rho(0, r_0) = r_0$, be the solution of (4.7). For the sake of simplicity, we shall use the notations $P_i = P_i(\rho(s, r_0), s)$ and $I_i = I_i(\rho(\gamma_i, r_0))$, $i = 1, 2, \ldots, p$.

On the interval $\phi \in [0, \gamma_1]$, we have

$$\rho(\phi, r_0) = \exp(\lambda_1\phi)r_0 + \int_0^{\phi} \exp(\lambda_1(\phi - s))P_1 ds.$$

For any i, $2 \le i \le p$, the solution $\rho(\phi, r_0)$ of (4.7) on $(\gamma_{i-1}, \gamma_i]$ is given by

$$\begin{aligned}
\rho(\phi, r_0) \;=\;& \exp\left(\lambda_i(\phi - \gamma_{i-1}) + \lambda_{i-1}\theta_{i-1} + \cdots + \lambda_2\theta_2 + \lambda_1\gamma_1\right) r_0 \\
+\;& \exp\left(\lambda_i(\phi - \gamma_{i-1}) + \cdots + \lambda_2\theta_2 + \lambda_1\gamma_1\right) \int_0^{\gamma_1} \exp(-\lambda_1 s)P_1 ds \\
+\;& \sum_{k=2}^{i-1} \exp\left(\lambda_i(\phi - \gamma_{i-1}) + \cdots + \lambda_{k+1}\theta_{k+1} + \lambda_k\gamma_k\right) \int_{\gamma_{k-1}}^{\gamma_k} \exp(-\lambda_k s)P_k ds \\
+\;& \int_{\gamma_{i-1}}^{\phi} \exp\left(\lambda_i(\phi - s)\right) P_i ds \\
+\;& \sum_{k=2}^{i} \exp\left(\lambda_i(\phi - \gamma_{i-1}) + \lambda_{i-1}\theta_{i-1} + \cdots + \lambda_k\theta_k\right) I_{k-1}.
\end{aligned}$$

For $\phi \in (\gamma_p, 2\pi]$, system (4.7) admits the solution

$$\rho(\phi, r_0) = \exp\left(\lambda_1(\phi - \gamma_p)\right)\left(\rho(\gamma_p, r_0) + I_p\right) + \int_{\gamma_p}^{\phi} \exp\left(\lambda_1(\phi - s)\right) P_1 ds.$$

Using the differentiable dependence of solutions of impulse systems on parameters [21] and the results from [14], we can conclude that the solution $\rho(\phi, r_0)$ is differentiable in r_0 and $\frac{\partial\rho(\phi, r_0)}{\partial r_0}|_{(\phi,r_0)=(2\pi,0)} = q$. Since systems (4.5) and (4.7), correspondingly (4.4) and (4.7), are B-equivalent, we derive

$$\frac{\partial r(\phi, r_0)}{\partial r_0}|_{(\phi,r_0)=(2\pi,0)} = q,$$

which completes the proof. □

4.1.3 The Problem of Distinguishing Between the Center and the Focus

If $q = 1$, then we have the critical case and the origin is either a focus or a center for system (4.4). In what follows, we solve this problem of distinguishing between the focus and the center.

We assume that f_i and τ_i, $i = 1, 2, \ldots, p$, are analytic functions in $N_\varepsilon(\tilde{D}_i)$. Then for sufficiently small ρ, the solution $\rho(\phi, r_0)$ of (4.7) satisfying $\rho(0, r_0) = r_0$ has the expansion [22]

$$\rho(\phi, r_0) = \sum_{j=0}^{\infty} \rho_j(\phi) r_0^j, \tag{4.9}$$

for all $\phi \in [0, 2\pi]$. From the expansion (4.9), it can be easily seen that $\rho_1(0) = 1$, $\rho_i(0) = 0$ for all $i = 0, 2, 3, 4, \ldots$, and $\rho_0(\phi) = 0$. The coefficient $\rho_1(\phi)$ with $\rho_1(0) = 1$ is the solution of the system

$$\frac{d\rho_1}{d\phi} = g(\rho_1),$$

where g is the function defined in system (4.2). It is clear that $\rho_1(2\pi) = q = 1$. We use the notation $k_j = \rho_j(2\pi)$, $j = 2, 3, \ldots$. For the solution $\rho(\phi, r_0)$ of (4.7), we construct the Poincaré return map

$$\rho(2\pi, r_0) = qr_0 + \sum_{j=2}^{\infty} k_j r_0^j.$$

In the critical case, the sign of the first nonzero element of the sequence k_j determines what type of a singular point the origin is. Moreover, for all $i = 1, 2, \ldots, p$, we have

$$P_i(\rho, \phi) = \sum_{j=2}^{\infty} P_{ij}(\phi) \rho^j, \tag{4.10}$$

and

$$I_i(\rho) = \sum_{j=2}^{\infty} I_{ij}\rho^j. \tag{4.11}$$

The existence of the expansions (4.10) and (4.11) has been proved in [22]. By means of (4.10) and (4.11), one can derive that the coefficients $\rho_j(\phi)$ with $\rho_j(0) = 0,\ j = 2, 3, \ldots$, are solutions of the following impulsive system

$$\begin{aligned} \frac{d\rho_j}{d\phi} &= h(\rho_j, \phi), \qquad \phi \neq \gamma_i, \\ \Delta\rho_j|_{\phi\,=\,\gamma_i} &= W_{ij}, \end{aligned} \tag{4.12}$$

where $h(\rho_j, \phi) = \lambda_i\rho_j + Q_{ij}(\phi)$ if $(\rho_j, \phi) \in D_i,\ i = 1, 2, \ldots, p$. From the differential part of (4.7) and the expansion (4.10), one can evaluate for any i, $1 \leq i \leq p$,

$$Q_{i2}(\phi) = P_{i2}(\phi)\rho_1^2(\phi), \quad Q_{i3}(\phi) = 2P_{i2}(\phi)\rho_1(\phi)\rho_2(\phi) + P_{i3}(\phi)\rho_1^3(\phi)$$

and $Q_{ij}(\phi)$ for $j = 4, 5, \ldots$, can be determined similarly. Further, the constants W_{ij} in (4.12) can be found from the impulsive part of (4.7) and the expansion (4.11). For instance,

$$W_{i2} = I_{i2}\rho_1^2(\gamma_i), \quad W_{i3} = 2I_{i2}\rho_1(\gamma_i)\rho_2(\gamma_i) + I_{i3}\rho_1^3(\gamma_i),$$

and W_{ij} can be evaluated, for $j = 4, 5, \ldots$, in the same manner.

As $k_j = \rho_j(2\pi)$, by solving the system (4.12) one can evaluate $k_j,\ j = 2, 3, \ldots$, which are the coefficients in the expansion of the Poincaré return map $\rho(2\pi, r_0)$:

$$\begin{aligned} k_j &= \int_0^{\gamma_1} \exp(-\lambda_1 s)Q_{1j}ds + \int_{\gamma_p}^{2\pi} \exp\left(\lambda_1(2\pi - s)\right) Q_{1j}ds + \\ &\sum_{i=2}^{p} \exp\Big(\lambda_1(2\pi - \gamma_p) + \cdots + \lambda_{i+1}\theta_{i+1} + \lambda_i\gamma_i\Big) \int_{\gamma_{i-1}}^{\gamma_i} \exp(-\lambda_i s)Q_{ij}ds + \\ &\sum_{i=2}^{p} \exp\Big(\lambda_1(2\pi - \gamma_p) + \lambda_p\theta_p + \cdots + \lambda_i\theta_i\Big) W_{i-1,j} + \exp\Big(\lambda_1(2\pi - \gamma_p)\Big) W_{pj}. \end{aligned} \tag{4.13}$$

From the expansion of $\rho(2\pi, r_0)$ and (4.13), it immediately follows that the following assertion is valid.

Lemma 4.1.1 *Let $q = 1$ and the first nonzero element of the sequence $k_j,\ j = 2, 3, \ldots$, be negative (positive). Then the origin is a stable (unstable) focus of (4.7). If $k_j = 0$ for all $j \geq 2$, then the origin is a center for system (4.7).*

Since systems (4.5) and (4.7), correspondingly (4.4) and (4.7), are B-equivalent, we have proved the following theorem.

THEOREM 4.1.3 *Let $q = 1$ and the first nonzero element of the sequence k_j, $j = 2, 3, \ldots$, be negative (positive). Then the origin is a stable (unstable) focus of (4.4). If $k_j = 0$ for all $j \geq 2$, then the origin is a center for system (4.4).*

4.1.4 Bifurcation of Periodic Solutions

In this section, we first introduce the system

$$\frac{dz}{dt} = \hat{f}(z, \mu), \tag{4.14}$$

where $\hat{f}(z, \mu) = A_i z + f_i(z) + \mu F_i(z, \mu)$ for $z \in \tilde{D}_i(\mu) \subset \mathbb{R}^2$ for analysis, and then we will describe it in detail with the help of the following assumptions.

(H1) Let $\{c_i(\mu)\}_{i=1}^p$ be a collection of curves in Ω which start at the origin and are given by the equations $\left\langle a^i, z \right\rangle + \tau_i(z) + \mu\kappa_i(z, \mu) = 0$, $i = 1, 2, \ldots, p$.

(H2) Let $\{l_i(\mu)\}_{i=1}^p$ be a union of half-lines which start at the origin and are defined by $\langle a^i + \mu \dfrac{\partial \kappa_i(0, \mu)}{\partial z}, z \rangle = 0$, $i = 1, 2, \ldots, p$. Denote by $\gamma_i(\mu)$ the angles of the lines $l_i(\mu)$, $i = 1, 2, \ldots, p$.

Similar to the construction of the regions D_i and $\tilde{D}_i$, we set for $\mu \in (-\mu_0, \mu_0)$ and $i = 2, 3, \ldots, p$:

$$\tilde{D}_1(\mu) = \{(r, \phi, \mu) \mid r \geq 0,\ \gamma_p(\mu) + \Psi_p < \phi \leq \gamma_1(\mu) + 2\pi + \Psi_1\},$$

$$\tilde{D}_i(\mu) = \{(r, \phi, \mu) \mid r \geq 0,\ \gamma_{i-1}(\mu) + \Psi_{i-1} < \phi \leq \gamma_i(\mu) + \Psi_i\},$$

$$D_1(\mu) = \{(r, \phi, \mu) \mid r \geq 0,\ \gamma_p(\mu) < \phi \leq \gamma_1(\mu) + 2\pi\},$$

$$D_i(\mu) = \{(r, \phi, \mu) \mid r \geq 0,\ \gamma_{i-1}(\mu) < \phi \leq \gamma_i(\mu)\},$$

where functions $\Psi_i = \Psi_i(r, \phi, \mu)$ are $2\pi-$periodic in ϕ, continuously differentiable, $\Psi_i = O(r)$, $i = 1, 2, \ldots, p$, and they can be defined applying a similar technique used in the construction of equation (4.3).

(H3) $F_i : N_\varepsilon(\tilde{D}_i(\mu)) \times (-\mu_0, \mu_0) \to \mathbb{R}^2$ and κ_i are analytical functions both in z and μ in the $\varepsilon-$neighborhood of their domains.

(H4) $F_i(0, \mu) = 0$ and $\kappa_i(0, \mu) = 0$ hold uniformly for each i and $\mu \in (-\mu_0, \mu_0)$.

(H5) The matrices A_i, the functions f_i, τ_i and the constant vectors a^i correspond to the ones described in systems (4.1) and (4.4).

Besides the system (4.14), we need the equation

$$\frac{dz}{dt} = \hat{f}_z(0, \mu)z, \tag{4.15}$$

where $\hat{f}_z(0, \mu) = A_i + \mu \dfrac{\partial F_i(0, \mu)}{\partial z}$ whenever $z \in D_i(\mu)$.

In polar coordinates, system (4.14) reduces to

$$\frac{dr}{d\phi} = \hat{g}(r, \phi, \mu), \tag{4.16}$$

where $\hat{g}(r, \phi, \mu) = \lambda_i(\mu)r + P_i(r, \phi, \mu)$ if $(r, \phi, \mu) \in \tilde{D}_i(\mu)$.

Let the following impulse system

$$\begin{aligned} \frac{d\rho}{d\phi} &= \hat{g}_N(\rho, \phi, \mu), \qquad \phi \neq \gamma_i(\mu), \\ \Delta\rho|_{\phi = \gamma_i(\mu)} &= I_i(\rho, \mu) \end{aligned} \tag{4.17}$$

be B-equivalent to (4.16), where $\hat{g}_N$ stands for the extension of $\hat{g}$ as we described in Section 4.1.2. That is, $\hat{g}_N(\rho, \phi, \mu) = \lambda_i(\mu)\rho + P_i(\rho, \phi, \mu)$ for $(\rho, \phi, \mu) \in D_i(\mu)$. We know that the function $\hat{g}_N$ and its partial derivatives become continuous up to the angle $\phi = \gamma_i(\mu)$ for $i = 1, 2, \ldots, p$. The function $I_i(\rho, \mu)$, for each $i = 1, 2, \ldots, p$ can be defined in the same way as done for $I_i(\rho)$.

Using a similar argument as in (4.1), we can obtain for system (4.15) that

$$q(\mu) = \exp(\sum_{i=1}^{p} \lambda_i(\mu)\theta_i(\mu)).$$

The last expression plays an important rule to establish the theorem on the bifurcation of periodic solutions as stated below.

Theorem 4.1.4 *Let $q(0) = 1$, $q'(0) \neq 0$ and the origin be a focus for (4.4). Then, for sufficiently small r_0, there exists a unique continuous function $\mu = \delta(r_0)$, $\delta(0) = 0$, such that the solution $r(\phi, r_0, \delta(r_0))$ of (4.16) is periodic with period 2π. Moreover, the closed trajectory is stable (unstable) if the origin of (4.4) is a stable (unstable) focus. The period of the corresponding periodic solution of (4.14) is $T = \sum_{i=1}^{p} \frac{\theta_i}{\beta_i} + o(|\mu|)$.*

Proof: Let $\rho(\phi, r_0, \mu)$ be the solution of (4.17) such that $\rho(0, r_0, \mu) = r_0$. To exclude the trivial solution, we consider $r_0 > 0$. The theorem of analyticity of solutions [22] imply that

$$\rho(2\pi, r_0, \mu) = \sum_{j=1}^{\infty} k_j(\mu) r_0^j,$$

where $k_j(\mu) = \sum_{i=0}^{\infty} k_{ji}\mu^i$. Since $k_1(\mu) = q(\mu)$, we have by the hypotheses of the theorem that $k_{10} = q(0) = 1$ and $k_{11} = q'(0) \neq 0$. For the existence of a periodic solution we require that $\rho(2\pi, r_0, \mu) = r_0$. Now we define $\mathcal{F}(r_0, \mu) = \rho(2\pi, r_0, \mu) - r_0$. Then, it can be derived that

$$\mathcal{F}(r_0, \mu) = q'(0)\mu r_0 + \sum_{j=2}^{\infty} k_{j0} r_0^j + \sum_{i+j \geq 3} k_{ji}\mu^i r_0^j,$$

where $i, j \in \mathbb{N}$ in the second summation. We call $\mathcal{F}(r_0, \mu) = 0$ as the bifurcation equation. If we cancel by r_0, we obtain the equation

$$\mathcal{H}(r_0, \mu) = 0, \tag{4.18}$$

where

$$\mathcal{H}(r_0, \mu) = q'(0)\mu + \sum_{j=2}^{\infty} k_{j0} r_0^{j-1} + \sum_{i+j \geq 2} k_{j+1,i}\mu^i r_0^j.$$

In the second summation of the last equation, we have $i \in \mathbb{N}$ and $j \in \mathbb{N}_0$. Since $\mathcal{H}(0,0) = 0$ and $\dfrac{\partial \mathcal{H}(0,0)}{\partial \mu} = q'(0) \neq 0$, one can say by the implicit function theorem that for sufficiently small r_0 there exists a function $\mu = \delta(r_0)$ such that $\rho(\phi, r_0, \delta(r_0))$ is a periodic solution.

We assume without loss of generality that $k_{j0} = 0$ for $j = 2, 3, \ldots, l-1$ and $k_{l0} \neq 0$. Then we can obtain from (4.18) that

$$\delta(r_0) = -\frac{k_{l0}}{q'(0)} r_0^{l-1} + \sum_{i=l}^{\infty} \delta_i r_0^i. \tag{4.19}$$

If we analyze the equation (4.19), we can conclude that the bifurcation of periodic solutions exists if a stable (unstable) focus for $\mu = 0$ becomes unstable (stable) for $\mu \neq 0$.

Let $\rho(\phi) = \rho(\phi, \bar{r_0}, \bar{\mu})$ be a periodic solution of (4.17). This periodic solution is a stable limit cycle if $\dfrac{\partial \mathcal{F}(\bar{r_0}, \bar{\mu})}{\partial r_0} < 0$. Assuming that the first nonzero element k_{l0} of the sequence k_{j0}, $j \geq 2$, is negative and using (4.19), we get

$$\frac{\partial \mathcal{F}(\bar{r_0}, \bar{\mu})}{\partial r_0} = (l-1)k_{l0}\bar{r_0}^{\,l-1} + \mathcal{G}(r_0),$$

where $\mathcal{G}$ starts with a member whose order is not less than l. Thus, $\dfrac{\partial \mathcal{F}(\bar{r_0}, \bar{\mu})}{\partial r_0} < 0$.

Since (4.16) and (4.17) are B-equivalent systems, the proof is completed. □

4.1.5 An Example

To be convenient, in the following example we use the corresponding notations that are adopted in Sections 4.1.1-4.1.4.

Example 4.1.1 *Let $c_1(\mu)$ and $c_2(\mu)$ be the curves defined by $z_2 = \dfrac{1}{\sqrt{3}}z_1 + (1+\mu)z_1^3$, $z_1 > 0$ and $z_2 = \sqrt{3}z_1 + z_1^5 + \mu z_1^2$, $z_1 < 0$, respectively. We take*

$$A_1 = \begin{bmatrix} -0.7 & -2 \\ 2 & -0.7 \end{bmatrix}, \; f_1(z) = \begin{bmatrix} z_1\sqrt{z_1^2+z_2^2} \\ z_2\sqrt{z_1^2+z_2^2} \end{bmatrix}, \; F_1(z,\mu) = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix},$$

and

$$A_2 = \begin{bmatrix} 0.5 & -2 \\ 2 & 0.5 \end{bmatrix}, \; f_2(z) = \begin{bmatrix} -2z_1\sqrt{z_1^2+z_2^2} \\ -2z_2\sqrt{z_1^2+z_2^2} \end{bmatrix}, \; F_2(z,\mu) = \begin{bmatrix} -z_1 \\ -z_2 \end{bmatrix}.$$

After these preparations, we consider the system

$$\frac{dz}{dt} = \hat{f}(z,\mu) \tag{4.20}$$

where $\hat{f}(z,\mu) = A_i z + f_i(z) + \mu F_i(z,\mu)$ if $z \in \tilde{D}_i(\mu)$, $i = 1,2$. Here $\tilde{D}_1(\mu)$ denotes the region situated between the curves $c_1(\mu)$ and $c_2(\mu)$, which contains the fourth quadrant. $\tilde{D}_2(\mu)$ is the region between $c_1(\mu)$ and $c_2(\mu)$ containing the second quadrant.

Since $q = 1$, by Theorem 4.1.1 the origin is a center for the nonperturbed system

$$\frac{dx}{dt} = f(x),$$

where $f(x) = A_i x$ *whenever* $x \in D_i$, $i = 1, 2$ *as shown in Figure 4.3. Here* D_1 *and* D_2 *are the regions between the half straight lines* $l_1 : z_2 = \frac{1}{\sqrt{3}} z_1$, $z_1 > 0$ *and* $l_2 : z_2 = \sqrt{3} z_1$, $z_1 < 0$, *which contain the fourth and second quadrants, respectively.*

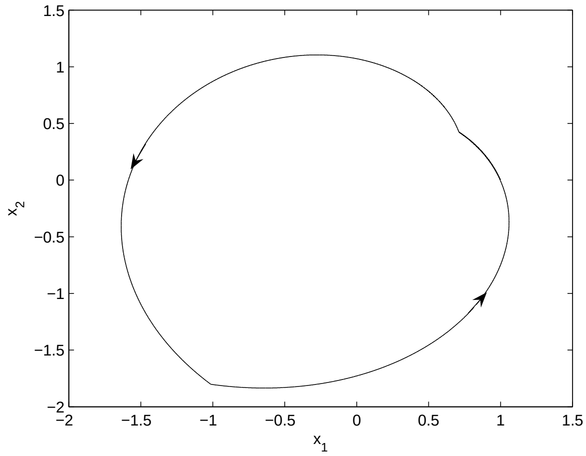

Figure 4.3: The simulation result showing the existence of a center for the nonperturbed system

One can see that $l_1(\mu)$ $(l_2(\mu))$ *coincides with* l_1 (l_2). *Hence,* $\gamma_1 = \gamma_1(\mu) = \frac{\pi}{6}$ *and* $\gamma_2 = \gamma_2(\mu) = \frac{4\pi}{3}$. *Using the given informaton, we obtain*

$$q(\mu) = \exp(-\frac{\pi}{6}\mu), \;\; q(0) = 1, \;\; q'(0) = -\frac{\pi}{6} \neq 0.$$

Moreover, for the associated system

$$\frac{dy}{dt} = \tilde{f}(y),$$

where $\tilde{f}(y) = A_i y + f_i(y)$ *whenever* $y \in \tilde{D}_i$, $i = 1, 2$, *it follows from Theorem 4.1.3 that the origin is a stable focus as* $k_2 < 0$ *for the perturbed system (see Figure 4.4). Here* $\tilde{D}_1$ *and* $\tilde{D}_2$ *are the regions between the curves* $c_1 : z_2 = \frac{1}{\sqrt{3}} z_1 + z_1^3$, $z_1 > 0$ *and* $c_2 : z_2 = \sqrt{3} z_1 + z_1^5$, $z_1 < 0$, *which contain the fourth and second quadrants, respectively.*

From Figure 4.5, we see that the trajectories approach a periodic solution from inside and outside. That is, system (4.20) has a stable limit cycle with period $\approx \pi$.

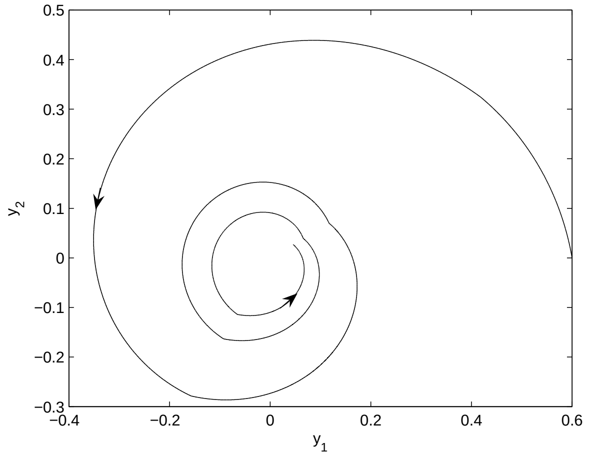

Figure 4.4: The simulation result showing the existence of a stable focus for the perturbed system ($\mu = 0$)

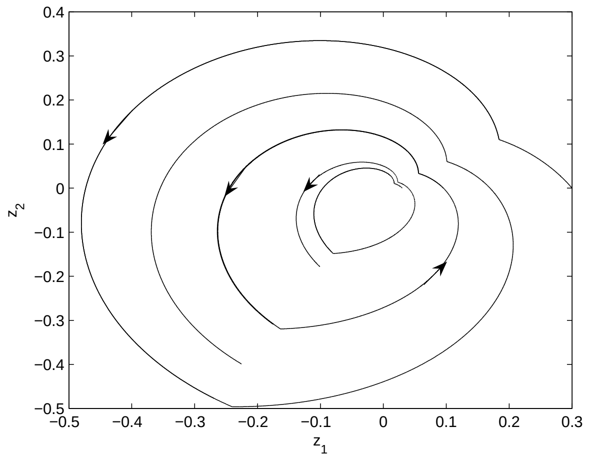

Figure 4.5: The simulation result with $\mu = -0.8$ showing the existence of a limit cycle for system (4.20)

4.1.6 Conclusion

Hopf bifurcation for smooth systems is characterized by a pair of complex conjugate eigenvalues of the linearized system. It is well known that it is not the case for systems of differential equations with discontinuities. Although the system specified in (4.14) together with the assumption (A2) reflects a special class of such systems, it is worthwhile to develop a technique for the investigation of bifurcation problem as it exhibits complicated bifurcation phenomena. Further, the problem can be generalized by taking the matrices A_i, $i = 1, 2, \ldots, p$, not only of focus type in all subregions but also of another types, e.g., they may be hyperbolic with real eigenvalues. Clearly, this problem can be analyzed in a similar way when it is required by concrete applications in mechanics, electronics, biology, etc.

4.2 Bifurcation of a Non-Smooth 3-Dimensional Limit Cycle

When we consider bifurcations of a given type in a neighborhood of the origin, the center manifold theory appears as one of the most effective tools in the investigation. The study of center manifolds can be traced back to the works of Pliss [147, 148] and Kelley [100]. When such manifolds exist, the investigation of local behaviours can be reduced to the study of the systems on the center manifolds. Any bifurcations which occur in the neighborhood of the origin on the center manifold are guaranteed to occur in the full nonlinear system as well. In particular, if a limit cycle exists on the center manifold, then it will also appear in the full system.

Physical phenomena are often modeled by discontinuous dynamical systems which switch between different vector fields in different modes. In the last several decades, existence of non-smooth dynamics in the real world has stimulated the study of bifurcation of periodic solutions in discontinuous systems as mentioned in Section 4.1. Furthermore, Bautin and Leontovich [35] and Küpper et al. [112, 179] have considered Hopf bifurcation for planar Filippov systems with discontinuities on a single straight line. However, to the best of our knowledge, there have been no results considering bifurcation in three and more dimensions for equations with discontinuous vector fields.

In [14], Hopf bifurcation has been investigated for planar discontinuous dynamical systems. Based on the method of B-equivalence [2, 14, 18, 21, 22] to impulsive differential equations and by using the projection on the center manifold, we extend the results of Section 4.1 to obtain qualitative properties for three dimensional systems with discontinuous right-hand sides. The present section deals with discontinuities on arbitrarily finite nonlinear surfaces.

The structure of this section is as follows. Section 4.2.1 describes the nonperturbed system and studies its qualitative properties. Section 4.2.2 is dedicated to the perturbed system and the notion of B-equivalent impulsive systems. The center manifold theory is given in Section 4.2.3. Our main results concerning the bifurcation of periodic solutions are formulated in Section 4.2.4. In the last section, we present an appropriate example to illustrate our findings.

Remark 4.2.1 To make the analysis more understandable, we shall use similar notations as given in Section 4.1. We note that though some notations below coincide with the ones used in Section 4.1, they all should be treated independently.

4.2.1 The Nonperturbed System

For the sake of brevity in the sequel, every angle for a point is considered with respect to the positive half-line of the first coordinate axis in x_1x_2-plane. Moreover, it is important to note that we shall consider angle values only in the interval $[0, 2\pi]$ because of the periodicity.

Before introducing the nonperturbed system, we give the following assumptions and notations which will be needed throughout the section.

($\mathcal{A}1$) Let $\{\mathcal{P}_i\}_{i=1}^p$, $p \geq 2$, $p \in \mathbb{N}$, be a set of half-planes starting at the $z-$axis, i.e., $\mathcal{P}_i = l_i \times \mathbb{R}$, where l_i are half-lines which start at the origin and are given by $\varphi_i(x) = 0$, $\varphi_i(x) = \left\langle a^i, x \right\rangle$, $x \in \mathbb{R}^2$ and $a^i = ({a_1}^i, {a_2}^i) \in \mathbb{R}^2$ are constant vectors (see Figure 4.6). Let γ_i denote the angle of the line l_i for each $i = 1, 2, \ldots, p$ such that

$$0 < \gamma_1 < \gamma_2 < \cdots < \gamma_p < 2\pi.$$

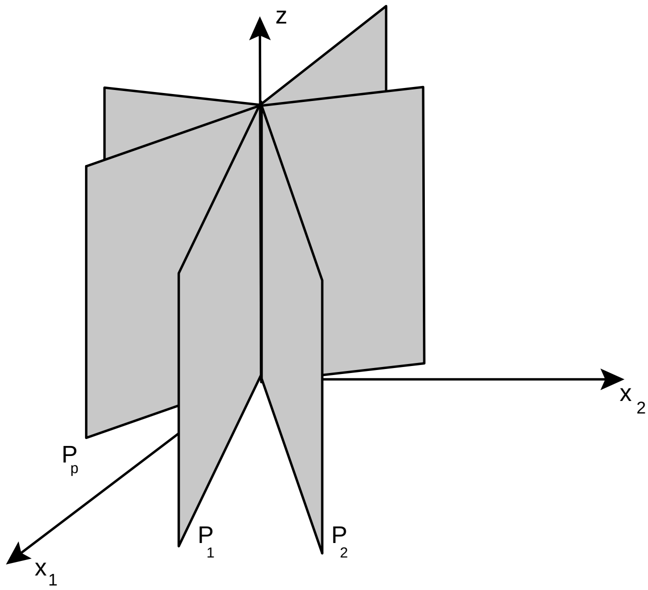

Figure 4.6: Half-planes $\mathcal{P}_i,\ i = 1, 2, \ldots, p,$ of discontinuities for the nonperturbed system (4.21)

($\mathcal{A}$2) There exist constant, real-valued 2×2 matrices A_i defined by $A_i = \begin{bmatrix} \alpha_i & -\beta_i \\ \beta_i & \alpha_i \end{bmatrix}$ where $\beta_i > 0$ and constants $b_i \in \mathbb{R}$, $i = 1, 2, \ldots, p$.

($\mathcal{N}$1) $\theta_1 = (2\pi + \gamma_1) - \gamma_p$ and $\theta_i = \gamma_i - \gamma_{i-1}$, $i = 2, 3, \ldots, p$.

($\mathcal{N}$2) Let $\mathcal{D}_i$ denote the region situated between the planes $\mathcal{P}_{i-1}$ and $\mathcal{P}_i$ and defined in cylindrical coordinates (r, ϕ, z), where $x_1 = r\cos\phi,\ x_2 = r\sin\phi$ and $z = z$, by

$$\mathcal{D}_1 = \{(r, \phi, z) \mid r \geq 0,\ \ \gamma_p < \phi \leq \gamma_1 + 2\pi,\ \ z \in \mathbb{R}\},$$

$$\mathcal{D}_i = \{(r, \phi, z) \mid r \geq 0,\ \ \gamma_{i-1} < \phi \leq \gamma_i,\ \ z \in \mathbb{R}\},\ \ i = 2, 3, \ldots, p.$$

Under the assumptions made above, we study in $\mathbb{R}^3$ the following nonperturbed system

$$\begin{aligned} \frac{dx}{dt} &= F(x), \\ \frac{dz}{dt} &= f(z), \end{aligned} \tag{4.21}$$

where $F(x) = A_i x$ and $f(z) = b_i z$ for $(x, z) \in \mathcal{D}_i$, $i = 1, 2, \ldots, p$.

We note that the functions F and f in system (4.21) are discontinuous on the planes $\mathcal{P}_i$, $i = 1, 2, \ldots, p$.

Remark 4.2.2 It follows from the assumptions $(\mathcal{A}1)$ *and* $(\mathcal{A}2)$ *that*

$$\langle \frac{\partial \varphi_i(x)}{\partial x}, F(x) \rangle \neq 0 \quad \text{for } x \in l_i,\ i = 1, 2, \ldots, p.$$

That is, the vector field is transversal at every point on $\mathcal{P}_i$ *for each* i*.*

Since the results can be most conveniently stated in terms of cylindrical coordinates, we use the transformation $x_1 = r\cos\phi$, $x_2 = r\sin\phi$, $z = z$ so that system (4.21) reduces to

$$\begin{aligned} \frac{dr}{d\phi} &= G(r), \\ \frac{dz}{d\phi} &= g(z), \end{aligned} \tag{4.22}$$

where $G(r) = \lambda_i r$ and $g(z) = k_i z$ if $(r, \phi, z) \in \mathcal{D}_i$, with $\lambda_i = \dfrac{\alpha_i}{\beta_i}$ and $k_i = \dfrac{b_i}{\beta_i}$, $i = 1, 2, \ldots, p$. We see that the functions G and g given in (4.22) have discontinuities when $\phi = \gamma_i$, $i = 1, 2, \ldots, p$.

The solution $(r(\phi, r_0), z(\phi, z_0))$ of (4.22) starting at the point $(0, r_0, z_0)$ is given by

$$r(\phi, r_0) = \begin{cases} \exp(\lambda_1 \phi)\, r_0, & \text{if } \ 0 \leq \phi \leq \gamma_1, \\ \exp\{\lambda_1\gamma_1 + \lambda_2\theta_2 + \cdots + \lambda_i(\phi - \gamma_{i-1})\} r_0, & \text{if } \ \gamma_{i-1} < \phi \leq \gamma_i, \\ \exp\{\lambda_1[\phi - (\gamma_p - \gamma_1)] + \sum_{i=2}^{p} \lambda_i\theta_i\} r_0, & \text{if } \ \gamma_p < \phi \leq 2\pi, \end{cases}$$

$$z(\phi, z_0) = \begin{cases} \exp(k_1 \phi) z_0, & \text{if } \ 0 \leq \phi \leq \gamma_1, \\ \exp\{k_1\gamma_1 + k_2\theta_2 + \cdots + k_i(\phi - \gamma_{i-1})\} z_0, & \text{if } \ \gamma_{i-1} < \phi \leq \gamma_i, \\ \exp\{k_1[\phi - (\gamma_p - \gamma_1)] + \sum_{i=2}^{p} k_i\theta_i\} z_0, & \text{if } \ \gamma_p < \phi \leq 2\pi, \end{cases}$$

for $i = 2, 3, \ldots, p$.

Now, we define a section $\mathsf{P} = \{(x_1, x_2, z) \mid x_2 = 0, x_1 > 0, z \in \mathbb{R}\}$. Constructing the Poincaré return map on P, we find that

$$(r(2\pi, r_0), z(2\pi, z_0)) = (\exp(\sum_{i=1}^{p} \lambda_i\theta_i) r_0, \exp(\sum_{i=1}^{p} k_i\theta_i) z_0).$$

Let us denote

$$q_1 = \exp(\sum_{i=1}^{p} \lambda_i \theta_i), \tag{4.23}$$

$$q_2 = \exp(\sum_{i=1}^{p} k_i \theta_i). \tag{4.24}$$

Since $r(2\pi, r_0) = q_1 r_0$, $z(2\pi, z_0) = q_2 z_0$, we can establish the following assertions.

LEMMA 4.2.1 *Assume that $q_1 = 1$. If*

(i) $q_2 = 1$, *then all solutions are periodic with period* $T = \sum_{i=1}^{p} \frac{\theta_i}{\beta_i}$, *i.e.,* $\mathbb{R}^3$ *is a center manifold;*

(ii) $q_2 < 1$, *then a solution that starts to its motion on* x_1x_2-*plane is* $T-$*periodic and all other solutions lie on the surface of a cylinder and they move toward the* x_1x_2-*plane, i.e.,* x_1x_2-*plane is a center manifold and* $z-$*axis is a stable manifold;*

(iii) $q_2 > 1$, *then a solution that starts to its motion on* x_1x_2-*plane is* $T-$*periodic and all other solutions lie on the surface of a cylinder and they move away from the origin, i.e.,* x_1x_2-*plane is a center manifold and* $z-$*axis is an unstable manifold.*

LEMMA 4.2.2 *Assume that $q_1 < 1$. If*

(i) $q_2 = 1$, *then a solution that starts to its motion on* $z-$*axis is* $T-$*periodic and all other solutions will approach the* $z-$*axis, i.e.,* x_1x_2-*plane is a stable manifold and* $z-$*axis is a center manifold;*

(ii) $q_2 < 1$, *all solutions will spiral toward the origin, i.e., the origin is asymptotically stable;*

(iii) $q_2 > 1$, *a solution that starts to its motion on* x_1x_2-*plane spirals toward the origin and a solution initiating on* $z-$*axis will move away from the origin, i.e.,* x_1x_2-*plane is a stable manifold and* $z-$*axis is a center manifold.*

LEMMA 4.2.3 *Assume that $q_1 > 1$. If*

(i) $q_2 = 1$, *then a solution that starts to its motion on $z-$axis is $T-$periodic and all other solutions move away from the $z-$axis, i.e., x_1x_2-plane is an unstable manifold and $z-$axis is a center manifold;*

(ii) $q_2 < 1$, *a solution that starts to its motion on x_1x_2-plane moves away from the origin and a solution initiating on $z-$axis spirals toward the origin, i.e., x_1x_2-plane is an unstable manifold and $z-$axis is a stable manifold;*

(iii) $q_2 > 1$, *all solutions move away from the origin, i.e., the origin is unstable.*

Remark 4.2.3 From now on, we assume that $q_1 = 1$ and $q_2 < 1$. In other words, x_1x_2-plane is a center manifold and $z-$axis is a stable manifold.

4.2.2 The Perturbed System

Let $\Upsilon \subset \mathbb{R}^3$ be a domain in the neighborhood of the origin. The following conditions are assumed to hold throughout the section.

($\mathcal{P}1$) Let $\{\mathcal{S}_i\}_{i=1}^p$, $p \geq 2$, be a set of cylindrical surfaces which start at the $z-$axis, i.e., $\mathcal{S}_i = c_i \times \mathbb{R}$, where c_i are curves starting at the origin and determined by the equations $\tilde{\varphi}_i(x) = 0$, $\tilde{\varphi} = \left\langle a^i, x \right\rangle + \tau_i(x)$, $x \in \mathbb{R}^2$, $\tau_i(x) = o(\|x\|)$ and the constant vectors a^i are the same as described in ($\mathcal{A}1$).

Without loss of generality, we may assume that $\gamma_i \neq \dfrac{\pi}{2} j$, $j = 1, 3$. Using the transformation $x_1 = r\cos\phi$, $x_2 = r\sin\phi$, equation of the curve c_i can be written, for sufficiently small r, as follows [14]

$$c_i : \phi = \gamma_i + \psi_i(r, \phi), \;\; i = 1, 2, \ldots, p, \tag{4.25}$$

where ψ_i is a $2\pi-$periodic function in ϕ, continuously differentiable and $\psi_i = O(r)$. Then, we can define the region situated between the surfaces $\mathcal{S}_{i-1}$ and $\mathcal{S}_i$ as follows:

$$\tilde{\mathcal{D}}_1 = \{(r,\phi,z) \mid r \geq 0,\ \gamma_p + \psi_p(r,\phi) < \phi \leq \gamma_1 + 2\pi + \psi_1(r,\phi),\ z \in \mathbb{R}\},$$

$$\tilde{\mathcal{D}}_i = \{(r,\phi,z) \mid r \geq 0,\ \gamma_{i-1} + \psi_{i-1}(r,\phi) < \phi \leq \gamma_i + \psi_i(r,\phi),\ z \in \mathbb{R}\},$$

where $i = 2, 3, \ldots, p$.

Let ε be a positive number and $N_\varepsilon(\tilde{\mathcal{D}}_i)$ denote the $\varepsilon-$neighborhoods of the regions $\tilde{\mathcal{D}}_i$, $i = 1, 2, \ldots, p$. In addition to $(\mathcal{P}1)$, we assume the following list of conditions.

$(\mathcal{P}2)$ Let the functions f_i, h_i, $i = 1, 2, \ldots, p$, be defined on the set $N_\varepsilon(\tilde{\mathcal{D}}_i)$ and satisfy $f_i,\ h_i \in C^{(2)}(N_\varepsilon(\tilde{\mathcal{D}}_i))$.

$(\mathcal{P}3)$ $\tau_i \in C^{(2)}(N_\varepsilon(\tilde{\mathcal{D}}_i))$, $i = 1, 2, \ldots, p$.

$(\mathcal{P}4)$ $f_i(x,z) = o(\|x,z\|)$, $h_i(x,z) = o(\|x,z\|)$, and $f_i(0,z) = 0$, $h_i(0,z) = 0$ for all $z \in \mathbb{R}$, $i = 1, 2, \ldots, p$.

We define for $(x,z) \in \tilde{\mathcal{D}}_i$, two functions by $\tilde{F}(x,z) = A_i x + f_i(x,z)$ and $\tilde{f}(x,z) = b_i z + h_i(x,z)$, where the matrix A_i and the constant b_i are as defined in $(\mathcal{A}2)$ above. In the neighborhood Υ, we consider the following system

$$\begin{aligned} \frac{dx}{dt} &= \tilde{F}(x,z), \\ \frac{dz}{dt} &= \tilde{f}(x,z). \end{aligned} \tag{4.26}$$

Here, it can be easily seen that the functions $\tilde{F}(x,z)$ and $\tilde{f}(x,z)$ have discontinuities on the surfaces $\mathcal{S}_i$, $i = 1, 2, \ldots, p$.

For sufficiently small neighborhood Υ, it follows from the conditions $(\mathcal{A}1)$ and $(\mathcal{P}1)$ that the surfaces $\mathcal{S}_i$ intersect each other only at $z-$axis, none of them can intersect itself and $\langle \frac{\partial \tilde{\varphi}_i(x)}{\partial x}, \tilde{F}(x,0)\rangle \neq 0$ for $x \in c_i$, $i = 1, 2, \ldots, p$. The surfaces of discontinuities, $\mathcal{S}_i$, associated with the planes $\mathcal{P}_i$ can be seen in Figure 4.7.

If a solution of system (4.26) starts at a point, which is sufficiently close to the origin and on the surface $\mathcal{S}_i$ with fixed i, then this solution can be continued either to the surface $\mathcal{S}_{i+1}$ or $\mathcal{S}_{i-1}$ depending on the direction of the time.

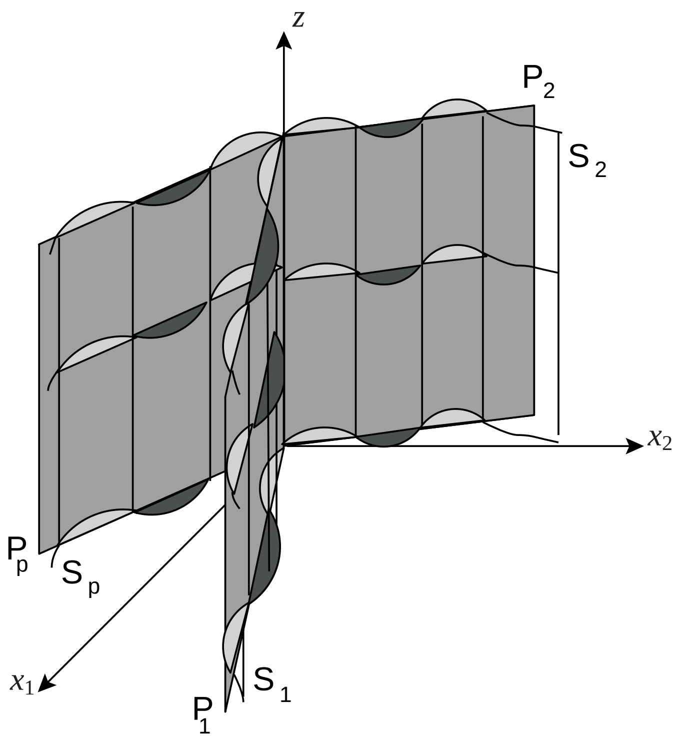

Figure 4.7: Surfaces $\mathcal{S}_i$, $i = 1, 2, \ldots, p$, of discontinuities for the perturbed system (4.26)

We make use of cylindrical coordinates and rewrite the system (4.26) in the following equivalent form

$$\begin{aligned}\frac{dr}{d\phi} &= \tilde{G}(r, \phi, z),\\ \frac{dz}{d\phi} &= \tilde{g}(r, \phi, z),\end{aligned} \tag{4.27}$$

where $\tilde{G}(r, \phi, z) = \lambda_i r + P_i(r, \phi, z)$ and $\tilde{g}(r, \phi, z) = k_i z + Q_i(r, \phi, z)$ whenever $(r, \phi, z) \in \tilde{\mathcal{D}}_i$. The functions P_i and Q_i are $2\pi-$periodic in ϕ, continuously differentiable and $P_i = o(\|(r, z)\|)$, $Q_i = o(\|(r, z)\|)$, $i = 1, 2, \ldots, p$.

From the construction, we see that system (4.27) is a differential equation with discontinuous right-hand side. For our needs, we redefine the functions $\tilde{G}$ and $\tilde{g}$ in the neighborhoods of the planes $\mathcal{P}_i$, which contain the surface $\mathcal{S}_i$. In other words, we construct new functions G_N and g_N which are continuous everywhere except possibly at the points $(r, \phi, z) \in \mathcal{P}_i$. The redefinition will be made exceptionally at the points which lie between $\mathcal{P}_i$ and $\mathcal{S}_i$ and belong to the regions $\mathcal{D}_i$ or $\mathcal{D}_{i+1}$ for each i. Therefore, this construction is performed with minimal possible changes corresponding to the B-equivalence method [2], which is the main instrument of our investigation.

It is clear from the context that if $i = p$ then $\mathcal{D}_{p+1} = \mathcal{D}_1$. Using the argument above, we realize the following reconstruction of the domain. We consider the subregions of $\mathcal{D}_i$ and $\mathcal{D}_{i+1}$, which are placed between the plane $\mathcal{P}_i$ and the surface $\mathcal{S}_i$. We refer to the subregions $\mathcal{D}_i \cap \tilde{\mathcal{D}}_{i+1}$ (light coloured closed regions in Figure 4.7) and $\mathcal{D}_{i+1} \cap \tilde{\mathcal{D}}_i$ (dark coloured closed regions in Figure 4.7) for all i. We extend the functions $\tilde{G}$ and $\tilde{g}$ from the region $\mathcal{D}_i \cap \tilde{\mathcal{D}}_{i+1}$ to $\mathcal{D}_i$ and from $\mathcal{D}_{i+1} \cap \tilde{\mathcal{D}}_i$ to $\mathcal{D}_{i+1}$ so that the new functions G_N and g_N and their partial derivatives become continuous up to the angle $\phi = \gamma_i$, $i = 1, 2, \ldots, p$. According to all these discussions for the definitions of G_N and g_N, we conclude that $G_N(r, \phi, z) = \lambda_i r + P_i(r, \phi, z)$ and $g_N(r, \phi, z) = k_i z + Q_i(r, \phi, z)$ for $(r, \phi, z) \in \mathcal{D}_i$. Now, we consider the following differential equation

$$\begin{aligned} \frac{dr}{d\phi} &= G_N(r, \phi, z), \\ \frac{dz}{d\phi} &= g_N(r, \phi, z). \end{aligned} \tag{4.28}$$

Let us fix $i \in \{1, 2, \ldots, p\}$ and consider a neighborhood of $\mathcal{P}_i$ based on the description above. We shall investigate the following three cases.

I. Assume that the point $(r, \gamma_i, z) \in \tilde{\mathcal{D}}_{i+1}$. Let $(r^0(\phi), (z^0(\phi))$ be a solution of (4.27) satisfying $(r^0(\gamma_i), (z^0(\gamma_i)) = (\rho, w)$ and ξ_i be the angle where this solution crosses the surface $\mathcal{S}_i$. We denote a solution of (4.28) on the interval $[\xi_i, \gamma_i]$ by $(r^1(\phi), z^1(\phi))$ with $(r^1(\xi_i), z^1(\xi_i)) = (r^0(\xi_i), z^0(\xi_i))$. Then

$$r^0(\phi) = \exp(\lambda_{i+1}(\phi - \gamma_i))\rho + \int_{\gamma_i}^{\phi} \exp(\lambda_{i+1}(\phi - s)) P_{i+1}(r^0(s), s, z^0(s)) ds,$$

$$z^0(\phi) = \exp(k_{i+1}(\phi - \gamma_i))w + \int_{\gamma_i}^{\phi} \exp(k_{i+1}(\phi - s)) Q_{i+1}(r^0(s), s, z^0(s)) ds,$$

and

$$r^1(\phi) = \exp(\lambda_i(\phi - \xi_i)) r^0(\xi_i) + \int_{\xi_i}^{\phi} \exp(\lambda_i(\phi - s)) P_i(r^1(s), s, z^1(s)) ds,$$

$$z^1(\phi) = \exp(k_i(\phi - \xi_i)) r^0(\xi_i) + \int_{\xi_i}^{\phi} \exp(k_i(\phi - s)) Q_i(r^1(s), s, z^1(s)) ds.$$

Define a mapping $W_i = (W_i^1, W_i^2)$ on the plane $\phi = \gamma_i$ into itself as follows

$$\begin{aligned}
W_i^1(\rho, w) &= r^1(\gamma_i) - \rho = [\exp((\lambda_i - \lambda_{i+1})(\gamma_i - \xi_i)) - 1]\rho \\
&+ \exp(\lambda_i(\gamma_i - \xi_i)) \int_{\gamma_i}^{\xi_i} \exp(\lambda_{i+1}(\xi_i - s)) P_{i+1} ds \\
&+ \int_{\xi_i}^{\gamma_i} \exp(\lambda_i(\gamma_i - s)) P_i ds,
\end{aligned}$$

$$\begin{aligned}
W_i^2(\rho, w) &= z^1(\gamma_i) - w = [\exp((k_i - k_{i+1})(\gamma_i - \xi_i)) - 1]w \\
&+ \exp(k_i(\gamma_i - \xi_i)) \int_{\gamma_i}^{\xi_i} \exp(k_{i+1}(\xi_i - s)) Q_{i+1} ds \\
&+ \int_{\xi_i}^{\gamma_i} \exp(k_i(\gamma_i - s)) Q_i ds.
\end{aligned}$$

II. If the point $(r, \gamma_i, z) \in \tilde{\mathcal{D}}_i$, we can evaluate W_i in the same way:

$$\begin{aligned}
W_i^1(\rho, w) &= [\exp((\lambda_i - \lambda_{i+1})(\xi_i - \gamma_i)) - 1]\rho \\
&+ \exp(\lambda_{i+1}(\gamma_i - \xi_i)) \int_{\gamma_i}^{\xi_i} \exp(\lambda_i(\xi_i - s)) P_i ds \\
&+ \int_{\xi_i}^{\gamma_i} \exp(\lambda_{i+1}(\gamma_i - s)) P_{i+1} ds,
\end{aligned}$$

$$\begin{aligned}
W_i^2(\rho, w) &= [\exp((k_i - k_{i+1})(\xi_i - \gamma_i)) - 1]w \\
&+ \exp(k_{i+1}(\gamma_i - \xi_i)) \int_{\gamma_i}^{\xi_i} \exp(k_i(\xi_i - s)) Q_i ds \\
&+ \int_{\xi_i}^{\gamma_i} \exp(\lambda_{i+1}(\gamma_i - s)) Q_{i+1} ds.
\end{aligned}$$

III. If $(r, \gamma_i, z) \in \mathcal{S}_i$, then $W_i(\rho, w) = 0$.

Results from [14] imply that the functions W_i^1 and W_i^2, $i = 1, 2, \ldots, p$, are continuously differentiable and we have $W_i^1 = o(\|(\rho, w)\|)$, $W_i^2 = o(\|(\rho, w)\|)$, which follows from the equation (4.25). In addition, we note that there exists a Lipschitz constant ℓ and a bounded function $m(\ell)$ [2, 14] such that

$$\|W_i^j(\rho_1, w_1) - W_i^j(\rho_2, w_2)\| \le \ell m(\ell)(\|\rho_1 - \rho_2\| + \|w_1 - w_2\|), \tag{4.29}$$

for all $\rho_1, \rho_2, w_1, w_2 \in \mathbb{R}$, $j = 1, 2$.

Consider the following impulsive differential equation

$$\begin{aligned}
&\frac{d\rho}{d\phi} = G_N(\rho, \phi, w),\\
&\frac{dw}{d\phi} = g_N(\rho, \phi, w), \qquad \phi \neq \gamma_i,\\
&\Delta\rho|_{\phi = \gamma_i} = W_i^1(\rho, w),\\
&\Delta w|_{\phi = \gamma_i} = W_i^2(\rho, w).
\end{aligned} \tag{4.30}$$

Let $(r(\phi, r_0), z(\phi, z_0))$ be a solution of (4.27) with $r(0, r_0) = r_0$, $z(0, z_0) = z_0$ and ξ_i be the meeting angle of this solution with the surface $\mathcal{S}_i$, $i = 1, 2, \ldots, p$.

Definition 4.2.1 *We shall say that systems (4.27) and (4.30) are B-equivalent in* Υ *if for every solution* $(r(\phi, r_0), z(\phi, z_0))$ *of (4.27) whose trajectory is in* Υ *for all* $\phi \in [0, 2\pi]$ *there exists a solution* $(\rho(\phi, r_0), w(\phi, z_0))$ *of (4.30) which satisfies the relation*

$$(r(\phi, r_0), z(\phi, z_0)) = (\rho(\phi, r_0), w(\phi, z_0)), \quad \phi \in [0, 2\pi] \setminus \bigcup_{i=1}^{p} (\xi_i, \hat{\gamma}_i], \tag{4.31}$$

and, conversely, for every solution $(\rho(\phi, r_0), w(\phi, z_0))$ *of (4.30) whose trajectory is in* Υ *there exists a solution* $(r(\phi, r_0), z(\phi, z_0))$ *of (4.27) which satisfies (4.31).*

For sufficiently small Υ, the solution $(r(\phi, r_0), z(\phi, z_0))$, whose trajectory is in Υ for all $\phi \in [0, 2\pi]$, takes the same values with the exception of the oriented intervals $(\xi_i, \hat{\gamma}_i]$ as the solution $(\rho(\phi, r_0), w(\phi, z_0))$ with $\rho(0, r_0) = r_0$, $w(0, z_0) = z_0$ of the impulsive differential equation (4.30). That is, systems (4.27) and (4.30) are said to be B-equivalent in the sense of the Definition 4.2.1. From the discussion and the construction above, it implies that solutions of (4.27) exist in the neighborhood Υ, they are continuous and have discontinuities in the derivative on the surface $\mathcal{S}_i$ for each i. Accordingly, a solution of system (4.26) starting at any initial point is continuous, continuously differentiable except possibly at the moments when the trajectories intersect the surface $\mathcal{S}_i$ and is unique.

4.2.3 Center Manifold Reduction

In this section, we establish a center manifold theorem for sufficiently small solutions to (4.30), that is, we show that these solutions can be captured on a 2-dimensional invariant manifold and we explicitly describe the dynamics on this manifold.

The functions G_N and g_N in (4.30) have been defined as $G_N(\rho,\phi,w) = \lambda_i\rho + P_i(\rho,\phi,w)$ and $g_N(\rho,\phi,w) = k_i w + Q_i(\rho,\phi,w)$, where $(\rho,\phi,w) \in \mathcal{D}_i$. Functions P_i and Q_i are $2\pi-$periodic in ϕ, and satisfy in a neighborhood of the origin

$$||P_i(\rho_1,\phi,w_1) - P_i(\rho_2,\phi,w_2)|| \leq L(||\rho_1 - \rho_2|| + ||w_1 - w_2||),$$
$$||Q_i(\rho_1,\phi,w_1) - Q_i(\rho_2,\phi,w_2)|| \leq L(||\rho_1 - \rho_2|| + ||w_1 - w_2||),$$

for sufficiently small positive constant L, $i = 1,2,\ldots,p$. Applying the methods of the paper [9], we can conclude that system (4.30) has two integral manifolds whose equations are given by

$$\begin{aligned} \Phi_0(\phi,\rho) &= \int_{-\infty}^{\phi} e^{k(\phi-s)} Q(\rho(s,\phi,\rho), s, w(s,\phi,\rho))ds \\ &\quad + \sum_{\gamma_i<\phi} e^{k_i(\phi-\gamma_i)} W_i^2(\rho(\gamma_i,\phi,\rho), w(\gamma_i,\phi,\rho)), \end{aligned} \tag{4.32}$$

and

$$\begin{aligned} \Phi_-(\phi,w) &= -\int_{\phi}^{\infty} e^{\lambda(\phi-s)} P(\rho(s,\phi,w), s, w(s,\phi,w))ds \\ &\quad + \sum_{\gamma_i<\phi} e^{\lambda_i(\phi-\gamma_i)} W_i^1(\rho(\gamma_i,\phi,w), w(\gamma_i,\phi,w)), \end{aligned} \tag{4.33}$$

where $k = k_i, \lambda = \lambda_i, P = P_i$ and $Q = Q_i$ whenever $(s,\cdot,\cdot) \in \mathcal{D}_i$. We note that the pair $(\rho(s,\phi,\rho), w(s,\phi,\rho))$ in (4.32) denotes a solution of (4.30) satisfying $\rho(\phi,\phi,\rho) = \rho$ and $(\rho(s,\phi,w), w(s,\phi,w))$ in (4.33) is a solution of (4.30) with $w(\phi,\phi,w) = w$.

It is also shown in [9] that there exist positive constants K_0, M_0, σ_0 such that

$$\Phi_0(\phi,0) = 0, \tag{4.34}$$
$$||\Phi_0(\phi,\rho_1) - \Phi_0(\phi,\rho_2)|| \leq K_0\ell||\rho_1 - \rho_2||, \tag{4.35}$$

for all ρ_1,ρ_2, where a solution $\eta(\phi) = (\rho(\phi), w(\phi))$ of impulsive system (4.30) with $\eta(\phi_0) = (\rho_0, \Phi_0(\phi_0,\rho_0))$, $\rho_0 \geq 0$, is defined on $\mathbb{R}$ and has the following property

$$||\eta(\phi)|| \leq M_0\rho_0 e^{-\sigma_0(\phi-\phi_0)}, \quad \phi \geq \phi_0. \tag{4.36}$$

Furthermore, it is shown that there exist positive constants K_-, M_-, σ_- such that Φ_- satisfies

$$\Phi_-(\phi,0) = 0, \tag{4.37}$$
$$||\Phi_-(\phi,w_1) - \Phi_-(\phi,w_2)|| \leq K_-\ell||w_1 - w_2||, \tag{4.38}$$

for all w_1, w_2, where a solution $\eta(\phi) = (\rho(\phi), w(\phi))$ of the system (4.30) with $\eta(\phi_0) = (\Phi_-(\phi_0, w_0), w_0)$, $w_0 \in \mathbb{R}$, is defined on $\mathbb{R}$ and satisfies

$$||\eta(\phi)|| \leq M_- ||w_0|| e^{-\sigma_-(\phi-\phi_0)}, \quad \phi \leq \phi_0. \tag{4.39}$$

Denote $S_0 = \{(\rho, \phi, w) : w = \Phi_0(\phi, \rho)\}$ and $S_- = \{(\rho, \phi, w) : \rho = \Phi_-(\phi, w)\}$. Here, S_0 is said to be the *center manifold* and S_- is said to be the *stable manifold*.

The following lemmas can be proven in a similar manner to the ones in [9] with slight changes.

Lemma 4.2.4 *If the Lipschitz constant ℓ is sufficiently small, then for every solution $\eta(\phi) = (\rho(\phi), w(\phi))$ of (4.30) there exists a solution $\mu(\phi) = (u(\phi), v(\phi))$ on the center manifold, S_0, such that*

$$\begin{aligned} &||\rho(\phi) - u(\phi)|| \leq 2M_0 ||\rho(\phi_0) - u(\phi_0)|| e^{-\sigma_0(\phi-\phi_0)}, \\ &||w(\phi) - v(\phi)|| \leq M_0 ||w(\phi_0) - v(\phi_0)|| e^{-\sigma_0(\phi-\phi_0)}, \quad \phi \geq \phi_0, \end{aligned}$$

where M_0 and σ_0 are the constants used in (4.36).

Lemma 4.2.5 *For sufficiently small Lipschitz constant ℓ, the surface S_0 is stable in large.*

The dynamics reduced to the local center manifold S_0 is governed by an impulsive differential equation that is satisfied by the first coordinate of the solutions of (4.30) and has the form

$$\begin{aligned} &\frac{d\rho}{d\phi} = G_N(\rho, \phi, \Phi_0(\phi, \rho)), \quad \phi \neq \gamma_i, \\ &\Delta\rho|_{\phi=\gamma_i} = W_i^1(\rho, \Phi_0(\phi, \rho)). \end{aligned} \tag{4.40}$$

The following theorem follows from the reduction principle.

Theorem 4.2.1 *The trivial solution of (4.30) is stable, asymptotically stable or unstable if the trivial solution of (4.40) is stable, asymptotically stable or unstable, respectively.*

Using B-equivalence, one can see that the following theorem holds.

Theorem 4.2.2 *Assume that the conditions given above are fulfilled. Then the trivial solution of (4.26) is stable, asymptotically stable or unstable if the trivial solution of (4.40) is stable, asymptotically stable or unstable, respectively.*

4.2.4 Bifurcation of Periodic Solutions

The center manifold reduction in the previous section allows us to establish a Hopf bifurcation theorem, yielding a very powerful tool to perform a bifurcation analysis on parameter dependent versions of the considered systems. During the last two decades, many authors have contributed towards developing the general theory.

In order to state the Hopf bifurcation theorem, we include parameter dependence into our framework. In particular, the bifurcation of periodic solutions under the influence of a single parameter μ, $\mu \in (-\mu_0, \mu_0)$, μ_0 a positive constant, is considered for the system

$$\begin{aligned} \frac{dx}{dt} &= \hat{F}(x,z,\mu), \\ \frac{dz}{dt} &= \hat{f}(x,z,\mu), \end{aligned} \tag{4.41}$$

where $\hat{F}(x,z,\mu) = A_i x + f_i(x,z) + \mu F_i(x,z,\mu)$ and $\hat{f}(x,z,\mu) = b_i z + h_i(x,z) + \mu H_i(x,z,\mu)$ whenever $(x,z) \in \tilde{\mathcal{D}}_i(\mu) \subset \mathbb{R}^3$, which will be defined below. We will need the following assumptions on the system (4.41).

($\mathcal{H}1$) Let $\{\mathcal{S}_i(\mu)\}_{i=1}^{p}$ be a collection of surfaces in Υ which start at the z-axis, i.e., $\mathcal{S}_i(\mu) = c_i(\mu) \times \mathbb{R}$, where $c_i(\mu)$ are curves given by $\left\langle a^i, x \right\rangle + \tau_i(x) + \mu\kappa_i(x,\mu) = 0$, $x \in \mathbb{R}^2$, $i = 1, 2, \ldots, p$.

($\mathcal{H}2$) Let $\{\mathcal{P}_i(\mu)\}_{i=1}^{p}$ be a union of half-planes which start at the $z-$axis, i.e., $\mathcal{P}_i(\mu) = l_i(\mu) \times \mathbb{R}$, where $l_i(\mu)$ is defined by $\langle a^i + \mu \dfrac{\partial \kappa_i(0,\mu)}{\partial x}, x \rangle = 0$, $i = 1, 2, \ldots, p$. Denote by $\gamma_i(\mu)$ the angle of the line $l_i(\mu)$, $i = 1, 2, \ldots, p$.

Like the construction of the regions $\mathcal{D}_i$ and $\tilde{\mathcal{D}}_i$, we define for $\mu \in (-\mu_0, \mu_0)$, $i = 2, 3, \ldots, p$, the ones associated to the system (4.41):

$$\tilde{\mathcal{D}}_1(\mu) = \{(r, \phi, z, \mu) \mid r \geq 0,\ \gamma_p(\mu) + \Psi_p < \phi \leq \gamma_1(\mu) + 2\pi + \Psi_1,\ z \in \mathbb{R}\},$$

$$\tilde{\mathcal{D}}_i(\mu) = \{(r, \phi, z, \mu) \mid r \geq 0,\ \gamma_{i-1}(\mu) + \Psi_{i-1} < \phi \leq \gamma_i(\mu) + \Psi_i,\ z \in \mathbb{R}\},$$

$$\mathcal{D}_1(\mu) = \{(r, \phi, z, \mu) \mid r \geq 0,\ \gamma_p(\mu) < \phi \leq \gamma_1(\mu) + 2\pi,\ z \in \mathbb{R}\},$$

$$\mathcal{D}_i(\mu) = \{(r, \phi, z, \mu) \mid r \geq 0,\ \gamma_{i-1}(\mu) < \phi \leq \gamma_i(\mu),\ z \in \mathbb{R}\}.$$

Here the functions $\Psi_i = \Psi_i(r, \phi, \mu)$ are $2\pi-$periodic in ϕ, continuously differentiable, $\Psi_i = O(r)$, $i = 1, 2, \ldots, p$ and can defined in a similar manner to ψ_i in (4.25).

To establish the Hopf bifurcation theorem, we also need the following assumptions:

- ($\mathcal{H}3$) The functions $F_i : N_\varepsilon(\tilde{\mathcal{D}}_i(\mu)) \to \mathbb{R}^2$ and κ_i are analytical functions in x, z and μ in the $\varepsilon-$neighbourhood of their domains;
- ($\mathcal{H}4$) $F_i(0, 0, \mu) = 0$ and $\kappa_i(0, \mu) = 0$ hold uniformly for $\mu \in (-\mu_0, \mu_0)$;
- ($\mathcal{H}5$) The matrices A_i, the constants b_i, the functions f_i g_i, τ_i and the constant vectors a^i correspond to the ones described in systems (4.21) and (4.26).

In cylindrical coordinates, system (4.41) reduces to

$$\begin{aligned} \frac{dr}{d\phi} &= \hat{G}(r, \phi, z, \mu), \\ \frac{dz}{d\phi} &= \hat{g}(r, \phi, z, \mu), \end{aligned} \tag{4.42}$$

$\hat{G}(r, \phi, z, \mu) = \lambda_i(\mu)r + P_i(r, \phi, z, \mu)$ and $\hat{g}(r, \phi, z, \mu) = k_i(\mu)z + Q_i(r, \phi, z, \mu)$ if $(r, \phi, z, \mu) \in \tilde{\mathcal{D}}_i(\mu)$.

Let the following impulsive system

$$\begin{aligned} \frac{d\rho}{d\phi} &= \hat{G}_N(\rho, \phi, w, \mu), \\ \frac{dw}{d\phi} &= \hat{g}_N(\rho, \phi, w, \mu), \quad \phi \neq \gamma_i(\mu), \\ \Delta\rho|_{\phi = \gamma_i(\mu)} &= W_i^1(\rho, w, \mu) \\ \Delta w|_{\phi = \gamma_i(\mu)} &= W_i^2(\rho, w, \mu) \end{aligned} \tag{4.43}$$

be B-equivalent to (4.42), where $\hat{G}_N$ and $\hat{g}_N$ stand, respectively, for the extensions of $\hat{G}$ and $\hat{g}$. That is, $\hat{G}_N(\rho, \phi, w, \mu) = \lambda_i(\mu)\rho + P_i(\rho, \phi, w, \mu)$ and $\hat{g}_N(\rho, \phi, w, \mu) =$

$k_i(\mu)w + Q_i(\rho,\phi,w,\mu)$ for $(\rho,\phi,w,\mu) \in \mathcal{D}_i(\mu)$. Then the functions $\hat{G}_N$ and $\hat{g}_N$ and their partial derivatives become continuous up to the angle $\phi = \gamma_i(\mu)$ for $i = 1,2,\ldots,p$. The functions $W_i^1(\rho,w,\mu)$ and $W_i^2(\rho,w,\mu)$ can be defined in the same manner as in Section 4.2.2.

Following the same methods which are used to obtain (4.32) and (4.33), we can say that system (4.43) has two integral manifolds whose equations are given by

$$\begin{aligned}\Phi_0(\phi,\rho,\mu) &= \int_{-\infty}^{\phi} e^{k(\mu)(\phi-s)} Q(\rho(s,\phi,\rho,\mu), s, w(s,\phi,\rho,\mu),\mu)ds \\ &+ \sum_{\gamma_i(\mu)<\phi} e^{k_i(\mu)(\phi-\gamma_i(\mu))} W_i^2(\rho(\gamma_i(\mu),\phi,\rho,\mu), w(\gamma_i(\mu),\phi,\rho,\mu),\mu), \end{aligned} \tag{4.44}$$

and

$$\begin{aligned}\Phi_-(\phi,w,\mu) &= -\int_{\phi}^{\infty} e^{\lambda(\mu)(\phi-s)} P(\rho(s,\phi,w,\mu), s, w(s,\phi,w,\mu),\mu)ds \\ &+ \sum_{\gamma_i(\mu)<\phi} e^{\lambda_i(\mu)(\phi-\gamma_i(\mu))} W_i^1(\rho(\gamma_i(\mu),\phi,w,\mu), w(\gamma_i(\mu),\phi,w,\mu),\mu), \end{aligned} \tag{4.45}$$

where $k(\mu) = k_i(\mu), \lambda(\mu) = \lambda_i(\mu), P = P_i$ and $Q = Q_i$ whenever $(s,\cdot,\cdot,\cdot) \in \mathcal{D}_i(\mu)$. In (4.44), the pair $(\rho(s,\phi,\rho,\mu), w(s,\phi,\rho,\mu))$ denotes a solution of (4.43) satisfying $\rho(\phi,\phi,\rho,\mu) = \rho$. Similarly, $(\rho(s,\phi,w,\mu), w(s,\phi,w,\mu))$, in (4.45), is a solution of (4.43) with $w(\phi,\phi,w,\mu) = w$.

Now, we set $S_0(\mu) = \{(\rho,\phi,w,\mu) : w = \Phi_0(\phi,\rho,\mu)\}$ and $S_-(\mu) = \{(\rho,\phi,w,\mu) : \rho = \Phi_-(\phi,w,\mu)\}$. The reduced system on the center manifold $S_0(\mu)$ is given by

$$\begin{aligned} &\frac{d\rho}{d\phi} = \hat{G}_N(\rho,\phi,\Phi_0(\phi,\rho,\mu),\mu), \quad \phi \neq \gamma_i(\mu), \\ &\Delta\rho\,|_{\phi=\phi_i(\mu)} = W_i^1(\rho,\Phi_0(\phi,\rho,\mu),\mu). \end{aligned} \tag{4.46}$$

Similar to (4.23) and (4.24) we can define the functions

$$q_1(\mu) = \exp(\sum_{i=1}^{p} \lambda_i(\mu)\theta_i(\mu)), \tag{4.47}$$

$$q_2(\mu) = \exp(\sum_{i=1}^{p} k_i(\mu)\theta_i(\mu)). \tag{4.48}$$

System (4.46) is a system of the type studied in [14] and there it is shown that this system, for sufficiently small μ, has a periodic solution with period 2π. For our needs, we shall show that if the first coordinate of a solution of (4.43) is $2\pi-$periodic, then so is the second one.

Now, since

$$\rho(s+2\pi,\phi+2\pi,\rho,\mu) = \rho(s,\phi,\rho,\mu),$$

$$w(s+2\pi,\phi+2\pi,\rho,\mu) = w(s,\phi,\rho,\mu),$$

and each Q_i is $2\pi-$periodic in ϕ, we have

$$\begin{aligned}
&\Phi_0(\phi+2\pi,\rho,\mu)\\
&= \int_{-\infty}^{\phi+2\pi} e^{k(\mu)(\phi+2\pi-s)} Q(\rho(s,\phi+2\pi,\rho,\mu),s,w(s,\phi+2\pi,\rho,\mu),\mu)ds\\
&\quad + \sum_{\gamma_i(\mu)<\phi+2\pi} e^{k_i(\mu)(\phi+2\pi-\gamma_i(\mu))} \times\\
&\qquad \times\ W_i^2(\rho(\gamma_i(\mu),\phi+2\pi,\rho,\mu),w(\gamma_i(\mu),\phi+2\pi,\rho,\mu),\mu)\\
&= \int_{-\infty}^{\phi} e^{k(\mu)(\phi-t)} Q(\rho(t,\phi,\rho,\mu),t,w(t,\phi,\rho,\mu),\mu)dt\\
&\quad + \sum_{\overline{\gamma_i}(\mu)<\phi} e^{k_i(\mu)(\phi-\overline{\gamma_i}(\mu))} W_i^2(\rho(\overline{\gamma_i}(\mu),\phi,\rho,\mu),w(\overline{\gamma_i}(\mu),\phi,\rho,\mu),\mu)\\
&= \Phi_0(\phi,\rho,\mu),
\end{aligned}$$

where the substitutions $s = t+2\pi$ and $\gamma_i(\mu) = \overline{\gamma_i}(\mu)+2\pi$ are used for the integral and summation in the second equality.

Then, we obtain the following theorem whose proof can easily be adapted from the two dimensional case given in Theorem 4.1.4 of Section 4.1.

Theorem 4.2.3 *Assume that* $q_1(0) = 1, q_1'(0) \neq 0, q_2(0) < 1$, *and the origin is a focus for (4.26). Then, for sufficiently small* r_0 *and* z_0, *there exists a unique continuous function* $\mu = \delta(r_0,z_0)$, $\delta(0,0) = 0$ *such that the solution* $(r(\phi,\delta(r_0,z_0)),z(\phi,\delta(r_0,z_0)))$ *of (4.42), with the initial condition* $(r(0,\delta(r_0,z_0),z(0,\delta(r_0,z_0)) = (r_0,z_0)$, *is periodic with period* 2π. *The period of the corresponding periodic solution of (4.41) is* $\sum_{i=1}^{p}\frac{\theta_i}{\beta_i} + o(|\mu|)$.

4.2.5 An Example

For convenience in this section, we shall use the corresponding notations that are adopted through Sections 4.2.1-4.2.4.

EXAMPLE 4.2.1 *Let $c_1(\mu)$ and $c_2(\mu)$ denote the curves determined by $x_2 = \frac{1}{\sqrt{3}}x_1 + (1+\mu)x_1^3$, $x_1 > 0$ and $x_2 = \sqrt{3}x_1 + x_1^5 + \mu x_1^2$, $x_1 < 0$, respectively. We choose*

$$A_1 = \begin{bmatrix} -0.7 & -2 \\ 2 & -0.7 \end{bmatrix}, \ f_1(x,z) = \begin{bmatrix} x_1 z\sqrt{x_1^2 + x_2^2} \\ x_2 z^2\sqrt{x_1^2 + x_2^2} \end{bmatrix}, \ F_1(x,z,\mu) = \begin{bmatrix} x_1(1+z) \\ x_2 \end{bmatrix},$$

$$b_1 = 2, \ h_1(x,z) = x_1^2 z, \ H_1(x,z,\mu) = z,$$

$$A_2 = \begin{bmatrix} 0.5 & -2 \\ 2 & 0.5 \end{bmatrix}, \ f_2(x,z) = \begin{bmatrix} -2x_1 z^2\sqrt{x_1^2 + x_2^2} \\ -2x_2\sqrt{x_1^2 + x_2^2} \end{bmatrix}, \ F_2(x,z,\mu) = \begin{bmatrix} x_1 \\ x_2(1 + x_1 z) \end{bmatrix},$$

$$b_2 = -1.5, \ h_2(x,z) = x_1 z, \ H_2(x,z,\mu) = [1 - (x_1^2 + x_2^2)]z.$$

Now, we consider the system

$$\begin{aligned} \frac{dx}{dt} &= \hat{F}(x,z,\mu), \\ \frac{dz}{dt} &= \hat{f}(x,z,\mu), \end{aligned} \tag{4.49}$$

where $\hat{F}(x,z,\mu) = A_i x + f_i(x,z) + \mu F_i(x,z,\mu)$ and $\hat{f}(x,z,\mu) = b_i z + h_i(x,z) + \mu H_i(x,z,\mu)$ whenever $(x,z) \in \tilde{\mathcal{D}}_i(\mu)$.

Since $l_1(\mu)$ ($l_2(\mu)$) coincides with l_1 (l_2), $\gamma_1 = \gamma_1(\mu) = \frac{\pi}{6}$ and $\gamma_2 = \gamma_2(\mu) = \frac{4\pi}{3}$. Now, we can evaluate $q_1(\mu)$ and $q_2(\mu)$ as follows

$$q_1(\mu) = \exp(\pi\mu), \tag{4.50}$$

$$q_2(\mu) = \exp(\pi(\mu - \frac{1}{24})). \tag{4.51}$$

From (4.50) and (4.51), we can see that $q_1(0) = 1$, $q_1'(0) > 0$ and $q_2(0) < 1$. Therefore, by Theorem 4.2.3, system (4.49) has a periodic solution with period $\approx \pi$. One can see from the Figures 4.8 and 4.9 below, which are obtained for the same initial conditions and $\mu = 0.1$, that the trajectories approach a periodic solution from above and below. In other words, system (4.49) admits a stable limit cycle.

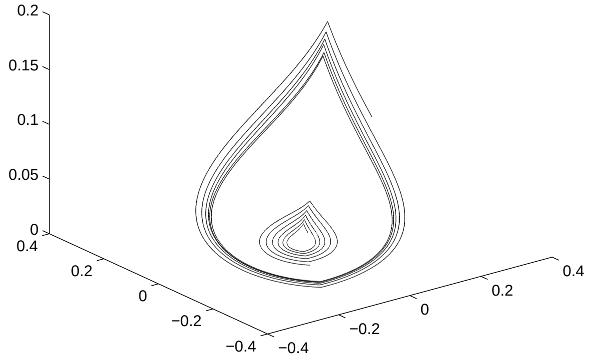

Figure 4.8: The simulation result showing the existence of a periodic solution for (4.49)

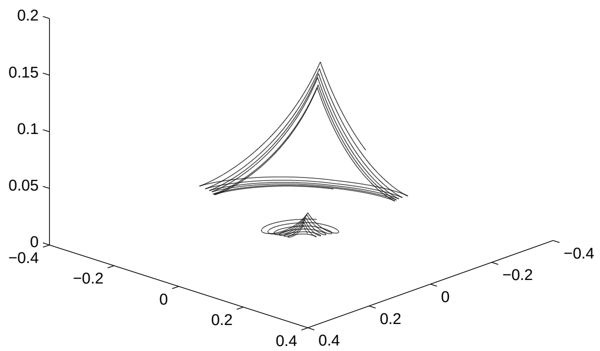

Figure 4.9: A different viewpoint of the Figure 4.8

CHAPTER 5

CONCLUSION

This thesis is devoted to the differential equations with discontinuities of different types: impulsive differential equations, differential equations with piecewise constant argument, differential equations with discontinuous right-hand sides [72, 113, 152, 52] and also to their applications in population dynamics.

In the last four decades, there has been a boom in the theory of differential equations with discontinuities. The importance of these equations is caused by the needs of modern science and technology as discontinuous characteristics are very often observed in the evolution of real processes in biology, chemistry, control theory, ecology, economics, electronics, mechanics, medicine and physics. The theory is not only richer than the corresponding theory of classical differential equations, but also represents a more natural framework for mathematical modeling of real world problems. Hence, we find it worthwhile to discuss several qualitative problems related to differential equations with some kind of discontinuity in the thesis.

It is well recognized that models of population dynamics are not suitable to be considered continually and thus not realistic when the seasonality of the changing environment, impulse and delay effects are not taken into account. In order to obtain more accurate results, it is desirable to study population dynamics models under these effects. In this context, we have improved the Lotka-Volterra and ratio-dependent predator-prey models with the help of differential equations with discontinuities in Chapter 2.

The subject of Chapter 3 is very new. We should emphasize that differential equations with piecewise constant arguments of generalized type have been very recently introduced by Akhmet [5], [8]-[10] and the novelty of those equations has been recognized in [146]. Through the Lyapunov-Razumikhin method, we have developed the previous results obtained on the stability of differential equations with piecewise constant arguments by considering the argument function in the general form and by taking any non-negative real number as an initial moment in Chapter 3. We improve the deficiencies arising from the classical method of reduction to discrete equations, which has been used as a main tool of investigation in the earlier works.

The last two problems considered in Chapter 4 have been investigated by using the B-equivalence method, which has been developed in papers of Akhmet [2, 14, 18, 21, 22]. The significance of the method stems from the fact that it enables us to consider discontinuity sets of nonlinear feature. The power and the effectiveness of this method for the analysis of problems of nonlinear feature have been proved once again. We can see that the method presents itself in the most complicated places of nonlinear problems such as bifurcation and center manifold reduction.

In the formulation of our problems, we have been motivated by the practical significances and challenges in population dynamics and mechanisms with dry friction. We are sure that the theoretical basis established in this thesis will be useful for practical investigations in other fields of the science and will lead to survey application problems including collision bifurcation theory, oscillation in mechanisms with vibration, neural networks, etc. more deeply compared to the previous ones. Moreover, we believe that the concept of nonlinearities can be significantly increased using the results of the thesis. The results of Chapter 3 can be used in the stability analysis of many real systems with piecewise constant arguments. We know that differential equations with discontinuous right-hand sides are also specific for a wide range of applications arising from mechanical systems with dry friction, electrical circuits with small inductivities, systems with small inertia, dynamical systems with non-differentiable potential, optimization problems with non-smooth data, electrical networks with switches, oscillations in visco-elasticity and optimal control. Thus, further investigations could be concentrated on the creation or disappearance of a periodic orbit in real world problems through the results of Chapter 4.

REFERENCES

[1] Aftabizadeh, A. R., Wiener, J. and Xu, J.-M. *Oscillatory and periodic solutions of delay differential equations with piecewise constant argument*, Proc. Amer. Math. Soc., **99**, 673–679, (1987).

[2] Akalın, E. and Akhmet, M. U. *The principles of B-smooth discontinuous flows*, Comput. Math. Appl., **49**, 981–995, (2005).

[3] Akçakaya, H. R., Arditi, R. and Ginzburg, L. R. *Ratio-dependent predation: an abstraction that works*, Ecology, **76**, 995–1004, (1995).

[4] Akhmet, M. U. and Bekmukhambetova, G. A. *A prototype compartmental model of blood pressure distribution*, Nonlinear Anal. Real World Appl., (in press).

[5] Akhmet, M. U. *Stability of differential equations with piecewise constant arguments of generalized type*, Nonlinear Anal., **68**, 794–803, (2008).

[6] Akhmet, M. U. *Asymptotic behavior of solutions of differential equations with piecewise constant arguments*, Appl. Math. Lett., **21**, 951–956, (2008).

[7] Akhmet, M. U. *Almost periodic solutions of differential equations with piecewise constant argument of generalized type*, Nonlinear Anal. Hybrid Syst., **2**, 456–467, (2008).

[8] Akhmet, M. U. *Integral manifolds of differential equations with piecewise constant argument of generalized type*, Nonlinear Anal., **66**, 367–383, (2007).

[9] Akhmet, M. U. *On the reduction principle for differential equations with piecewise constant argument of generalized type*, J. Math. Anal. Appl., **336**, 646–663, (2007).

[10] Akhmet, M. U. *On the integral manifolds of the differential equations with piecewise constant argument of generalized type*, Proceedings of the Conference on Differential and Difference Equations at the Florida Institute of Technology, August 1-5, 2005, Melbourne, Florida, Hindawi Publishing Corporation, 11–20 (2006).

[11] Akhmet, M. U., Beklioğlu, M., Ergenç, T. and Tkachenko, V. I. *An impulsive ratio-dependent predator-prey system with diffusion*, Nonlinear Anal. Real World Appl., **7**, 1255–1267, (2006).

[12] Akhmet, M. U., Öktem, H., Pickl, S. W. and Weber, G.-W. *An anticipatory extension of Malthusian model*, CASYS'05-Seventh International Conference, AIP Conference Proceedings, **839**, 260–264, (2006).

[13] Akhmet, M. U. and Turan, M. *The differential equations on time scales through impulsive differential equations*, Nonlinear Anal., **65**, 2043–2060, (2006).

[14] Akhmet, M. U. *Perturbations and Hopf bifurcation of the planar discontinuous dynamical system*, Nonlinear Anal., **60**, 163–178, (2005).

[15] Akhmet M. U., Bekmukhambetova G. A. and Serinağaoğlu Y. *Regular and irregular discontinous dynamics of the systemic arterial pressure,* Proceedings of the International Conference on Numerical Analysis and Applied Mathematics, Rodos, 44–46, (2005).

[16] Akhmet M. U. and Yılmaz, O. *Positive solutions of linear impulsive differential equations,* Nonlinear Oscil., **8(3)**, 289–295, (2005).

[17] Akhmetov, M. U. and Zafer, A. *Stability of the zero solution of impulsive differential equations by the Lyapunov second method*, J. Math. Anal. Appl., **248**, 69–82, (2000).

[18] Akhmetov, M. U. *On the expansion of solutions to differential equations with discontinuous right-hand side in a series in initial data and parameters*, Ukrainian Math. J., **45**, 786–789, (1993).

[19] Akhmetov, M. U. *On the smoothness of solutions of differential equations with a discontinuous right-hand side*, Ukrainian Math. J., **45**, 1785–1792, (1993).

[20] Akhmet, M. U. and Perestyuk, N. A. *Periodic and almost periodic solutions of strongly nonlinear impulsive systems*, J. Appl. Math. Mech., **56(6)**, 829–837, (1992).

[21] Akhmetov, M. U. and Perestyuk, N. A. *Differential properties of solutions and integral surfaces of nonlinear impulse systems,* Differ. Equ., **28**, 445–453, (1992).

[22] Akhmetov, M. U. and Perestyuk, N. A. *Asymptotic representation of solutions of regularly perturbed systems of differential equations with a nonclassical right-hand side,* Ukrainian Math. J., **43**, 1209–1214, (1991).

[23] Altıntan, D. *Extension of the logistic equation with piecewise constant arguments and population dynamics*, MSc. Thesis, Middle East Technical University, (2006).

[24] Andronov, A. A., Leontovich, E. A., Gordon, I. I. and Maier, A. G. *Theory of Bifurcations of Dynamic Systems on a Plane,* Halsted Press, New York, Wiley, (1973).

[25] Andronov, A. A. and Chaikin, C. E. *Theory of Oscillations*, Princeton Univ. Press, Princeton, (1949).

[26] Arditi, R. and Saiah, H. *Empirical evidence of the role of heterogeneity in ratio-dependent consumption*, Ecology, **73**, 1544–1551, (1992).

[27] Arditi, R., Ginzburg, L. R. and Akçakaya, H. R. *Variation in plankton densities among lakes: a case for ratio-dependent models*, American Naturalist, **138**, 1287–1296, (1991).

[28] Arditi, R. and Ginzburg, L. R. *Coupling in predator-prey dynamics: ratio-dependence,* J. Theoret. Biol., **139**, 311–326, (1989).

[29] Arnold, L. and Schmalfuss, B. *Lyapunov's second method for random dynamical systems*, J. Differential Equations, **177**, 235–265, (2001).

[30] Awrejcewicz, J., Fečkan, M. and Olejnik, P. *On continuous approximation of discontinuous systems*, Nonlinear Anal., **62**, 1317–1331, (2005).

[31] Awrejcewicz, J. and Lamarque, C.-H. *Bifurcation and Chaos in Nonsmooth Mechanical Systems,* World Scientific, Singapore, (2003).

[32] Bainov, D. D. and Simeonov, P. S. *Impulsive Differential Equations: Asymptotic Properties of the Solutions,* World Scientific, (1995).

[33] Bainov, D. D. and Simeonov, P. S. *Impulsive Differential Equations: Periodic Solutions and Applications*, Longman Scientific and Technical, (1993).

[34] Bainov, D. D. and Simeonov, P. S. *Systems with Impulse Effect, Theory and Applications*, Ellis Horwood Series in Mathematics and its Applications, Ellis Horwood, Chichister, U.K., (1989).

[35] Bautin, N. N. and Leontovich, E. A. *Methods and Rules for the Qualitative Study of Dynamical Systems on the Plane*, Nauka, Moscow, (1990).

[36] Beklioğlu, M. and Zorlu, P. *Role of food and predator predation avoidance and morphology of Daphnia pulex in Laboratory bioassays of diel vertical migration*, Cladocera 2005, VIIth International Symposium on Cladocera, September 3-9, Herzberg, Switzerland, (2005).

[37] Beretta, E. and Kuang, Y. *Global analyses in some delayed ratio-dependent predator-prey systems*, Nonlinear Anal., **32**, 381–408, (1998).

[38] Berryman, A. A. *The origins and evolution of predator-prey theory*, Ecology, **73**, 1530–1535, (1992).

[39] Blows, T. R. and Lloyd, N. G. *The number of limit cycles of certain polynomial differential equations*, Proc. Roy. Soc. Edinburgh Sect. A, **98**, 215–239, (1984).

[40] Budd, C. J. *Non-smooth dynamical systems and the grazing bifurcation*, Nonlinear Mathematics and its Applications, Cambridge Univ. Press, Cambridge, 219–235, (1996).

[41] Busenberg, S. and Cooke, K. L. *Models of vertically transmitted diseases with sequential-continuous dynamics*, Nonlinear Phenomena in Mathematical Sciences, Academic Press, New York, 179–187, (1982).

[42] Carr, J. *Applications of Center Manifold Theory*, Springer-Verlag, New York, (1981).

[43] Challamel, N. and Gilles, G. *Stability and dynamics of a harmonically excited elastic-perfectly plastic oscillator*, J. Sound Vibration, **301**, 608–634, (2007).

[44] Chen, Z. and Fu, X. *New Razumikhin-type theorems on the stability for impulsive functional differential systems*, Nonlinear Anal., **66**, 2040–2052, (2007).

[45] Chow, S.-N. and Hale, J. K. *Methods of Bifurcation Theory*, Springer, New York, Heidelberg, Berlin, (1982).

[46] Chow, S.-N. and Mallet-Paret, J. *Integral averaging and bifurcation*, J. Differential Equations, **26**, 112–159, (1977).

[47] Clarke, F. H., Ledyaev, Yu. S. and Stern, R. J. *Asymptotic stability and smooth Lyapunov functions*, J. Differential Equations, **149**, 69–114, (1998).

[48] Coddington, E. A. and Levinson, N. *Theory of Ordinary Differential Equations*, New York, McGraw-Hill, (1955).

[49] Coll, B., Gasull, A. and Prohens, R. *Degenerate Hopf bifurcations in discontinuous planar systems*, J. Math. Anal. Appl., **253**, 671–690, (2001).

[50] Cooke, K. L. and Wiener, J. *A survey of differential equation with piecewise continuous argument*, Lecture Notes in Math., **1475**, Springer, Berlin, 1–15, (1991).

[51] Corduneanu, C. *Principles of Differential and Integral Equations*, Boston, Allyn and Bacon, (1971).

[52] Cooke, K. L. and Wiener, J. *Retarded differential equations with piecewise constant delays*, J. Math. Anal. Appl., **99**, 265–297, (1984).

[53] Cronin, J. *Ordinary Differential Equations, Introduction and Qualitative Theory*, Boca Raton, CRC Press, (2007).

[54] Cushing, J. M. *Integrodifferential Equations and Delay Models in Population Dynamics*, Lecture Notes in Biomath., **20**, Springer-Verlag, Berlin-New York, (1977).

[55] Dai, L. and Singh, M. C. *On oscillatory motion of spring-mass systems subjected to piecewise constant forces*, J. Sound Vibration, **173**, 217–232, (1994).

[56] Dankowicz, H. and Zhao, X. *Local analysis of co-dimension-one and co-dimension-two grazing bifurcations in impact microactuators*, Phys. D, **202**, 238–257, (2005).

[57] Dankowicz, H. and Nordmark, A. B. *On the origin and bifurcations of stick-slip oscillations*, Phys. D, **136**, 280–302, (2000).

[58] Devi, J. V. and Vatsala, A. S. *Generalized quasilinearization for an impulsive differential equation with variable moments of impulse*, Dynam. Systems Appl., **12**, 369–382, (2003).

[59] Di Bernardo, M., Budd, C. J., Champneys, A. R. and Kowalczyk, P. *Piecewise Smooth Dynamical Systems, Theory and Applications*, London, Springer, (2008).

[60] Di Bernardo, M., Kowalczyk, P. and Nordmark, A. B. *Sliding bifurcations: a novel mechanism for the sudden onset of chaos in dry friction oscillators*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., **13**, 2935–2948, (2003).

[61] Di Bernardo, M., Budd, C. J. and Champneys, A. R. *Corner collision implies border-collision bifurcation*, Phys. D, **154(3-4)**, 171–194, (2001).

[62] Di Bernardo, M., Feigin, M. I., Hogan, S. J. and Homer, M. E. *Local analysis of C-bifurcations in $n-$dimensional piecewise-smooth dynamical systems*, Chaos Solitons Fractals, **10(11)**, 1881–1908, (1999).

[63] Ding, T.-R. *Approaches to the Qualitative Theory of Ordinary Differential Equations, Dynamical Systems and Nonlinear Oscillations*, New Jersey, World Scientific, (2007).

[64] Dumortier, F., Llibre, J. and Artés, J. C. *Qualitative Theory of Planar Differential Systems*, Berlin, Heidelberg, Springer, (2006).

[65] Fan, M., Wang, Q. and Zou, X. *Dynamics of a non-autonomous ratio-dependent predator-prey system*, Proc. Roy. Soc. Edinburgh Sect. A, **133**, 97–118, (2003).

[66] Fan, M. and Wang, K. *Periodic solutions of a discrete time nonautonomous ratio-dependent predator-prey system*, Math. Comput. Modelling, **35**, 951–961, (2002).

[67] Fan, M. and Wang, K. *Periodicity in a delayed ratio-dependent predator-prey system*, J. Math. Anal. Appl., **262**, 179–190, (2001).

[68] Fan, Y.-H. and Li, W.-T. *Permanence for a delayed discrete ratio-dependent predator-prey system with Holling type functional response*, J. Math. Anal. Appl., **299**, 357–374, (2004).

[69] Fečkan, M. *Bifurcations of periodic and chaotic solutions in discontinuous systems*, Arch. Math., 73–82, (1998).

[70] Feigin, M. I. *On the structure of C-bifurcation boundaries of piecewise-continuous systems*, J. Appl. Math. Mech., **42**, 820–829, (1978).

[71] Feigin, M. I. *Doubling of the oscillation period with C-bifurcations in piecewise continuous systems*, J. Appl. Math. Mech., **34**, 822–830, (1970).

[72] Filippov, A. F. *Differential Equations with Discontinuous Right-Hand Sides*, Kluwer Academic Publishers, Dordrecht, (1988).

[73] Filippov A. F. *Differential equations with discontinuous right-hand side*, Amer. Math. Soc. Transl. Ser. 2, **42**, 199–231, (1964).

[74] Freedman, H. I. and Mathsen, R. M. *Persistence in predator-prey systems with ratio-dependent predator influence*, Bull. Math. Biol., **55**, 817–827, (1993).

[75] Freedman, H. I. *Deterministic Mathematical Models in Population Ecology*, Marcel Dekker, New York, (1980).

[76] Frigon, M. and O'Regan, D. *Impulsive differential equations with variable times*, Nonlinear Anal., **26**, 1913–1922, (1996).

[77] Gopalsamy, K. and Liu, P. *Persistence and global stability in a population model*, J. Math. Anal. Appl., **224**, 59–80, (1998).

[78] Gopalsamy, K. *Stability and Oscillation in Delay Differential Equations of Population Dynamics*, Kluwer Academic Publishers, Dordrecht, (1992).

[79] Guckenheimer, J. and Holmes, P. *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Applied Mathematical Sciences, Springer-Verlag, (1983).

[80] Györi, I. and Ladas, G. *Oscillation Theory of Delay Differential Equations with Applications*, Oxford University Press, New York, (1991).

[81] Györi, I. *On approximation of the solutions of delay differential equations by using piecewise constant argument*, Int. J. Math. Math. Sci., **14**, 111–126, (1991).

[82] Hahn, W. *Stability of Motion*, Berlin, New York, Springer-Verlag, (1967).

[83] Halanay, A. and Wexler, D. *Qualitative Theory of Impulsive Systems*, Republici Socialiste Romania, Bucuresti, (1968).

[84] Hale, J. K. *Theory of Functional Differential Equations*, Springer-Verlag, New York, Heidelberg, Berlin, (1997).

[85] Hale, J. K. and Lunel, S. M. V. *Introduction to Functional Differential Equations*, Springer-Verlag, New York, (1993).

[86] Hale, J. K. and Koçak, H. *Dynamics and Bifurcations*, Springer-Verlag, New York, (1991).

[87] Hale, J. K. *Ordinary Differential Equations*, New York, Wiley-Interscience, (1969).

[88] Hartman, P. *Ordinary Differential Equations*, Philadelphia, Society for Industrial and Applied Mathematics, (2002).

[89] Hassard, B. D., Kazarinoff, N. D. and Wan, Y. H. *Theory and Applications of Hopf Bifurcations*, Cambridge University Press, Cambridge, (1981).

[90] Henry, C. *Differential equations with discontinuous right-hand side for planning procedure*, J. Econom. Theory, **4**, 541–551, (1972).

[91] Holmes, P. J. *The dynamics of repeated impacts with a sinusoidally vibrating table*, J. Sound Vibration, **84(2)**, 173–189, (1982).

[92] Hopf, E. *Abzweigung einer periodischen Lösung von einer stationären Löosung eines Differentialsystems*, Ber. Math. Phys. Sächsische Acad. Wiss. Leipzig, **94**, 1–22, (1942).

[93] Hoppensteadt, F. C. *Mathematical Methods of Population Biology*, Cambridge-New York, (1982).

[94] Hsu, S.-B., Hwang, T.-W. and Kuang, Y. *Global analysis of Michaelis-Menten type ratio-dependent predator-prey system*, J. Math. Biol., **42**, 489–506, (2001).

[95] Huo, H.-F. and Li, W.-T. *Existence and global stability of periodic solutions of a discrete ratio-dependent food chain model with delay*, Appl. Math. Comput., **162**, 1333–1349, (2005).

[96] Hutchinson, G. E. *Circular causal systems in ecology*, Ann. NY Acad. Sci., **50**, 221–246, (1948).

[97] Ioos, G. and Joseph, D. D. *Elementary Stability and Bifurcation Theory*, Springer-Verlag, New York, (1980).

[98] Ivanov, A. P. *Impact oscillations: linear theory of stability and bifurcations*, J. Sound Vibration, **178(3)**, 361–378, (1994).

[99] Jin, Z., Maoan, H. and Guihua, L. *The persistence in a Lotka-Volterra competition systems with impulsive*, Chaos Solitons Fractals, **24**, 1105–1117, (2005).

[100] Kelley, A. *The stable, center-stable, center, center-unstable, unstable manifolds*, J. Differential Equations, **3**, 546–570, (1967).

[101] Khusainov, D. *Application of the second Lyapunov method to stability investigation of differential equations with deviations*, Arch. Math., **34**, 127–142, (1998).

[102] Kolmogoroff, A. *Sulla teoria di Volterra della lotta per l'esistenza*, Giorn. Ist. Ital. Attuari, **7**, 74–80, (1936).

[103] Korobeinikov, A. *Lyapunov functions and global stability for SIR and SIRS epidemiological models with non-linear transmission*, Bull. Math. Biol., **68**, 615–626, (2006).

[104] Krasovskii, N. N. *Stability of Motion: Application of Lyapunov's Second Method to Differential Systems and Equations with Time-Delay*, SU Press, Stanford, (1963).

[105] Krylov, N. M. and Bogolyubov, N. N. *Introduction to Nonlinear Mechanics*, Academiya Nauk Ukrin. SSR, Kiev, (1937).

[106] Kuang, Y. and Beretta, E. *Global qualitative analysis of a ratio-dependent predator-prey system*, J. Math. Biol., **36**, 389–406, (1998).

[107] Kuang, Y. *Delay Differential Equations with Applications in Population Dynamics*, New York, Academic Press, (1993).

[108] Kunze, M. *Non-Smooth Dynamical Systems*, Lecture Notes in Math., Springer, Berlin, Heidelberg, New York, (2000).

[109] Kunze, M. and Küpper, T. *Qualitative bifurcation analysis of a non-smooth friction-oscillator model*, Z. Angew. Math. Phys., **48**, 87–101, (1997).

[110] Kuznetsov, Y. A., Rinaldi, S. and Gragnani, A. *One-parametric bifurcations in planar Filippov systems*, Int. J. Bif. Chaos, **13**, 2157–2188, (2003).

[111] Kuznetsov, Y. A. *Elements of Applied Bifurcation Theory*, Applied Mathematical Sciences, Springer-Verlag, New York, (1995).

[112] Küpper, T. and Moritz, S. *Generalized Hopf bifurcation for non-smooth planar systems*, Phil. Trans. R. Soc. Lond., **359**, 2483–2496, (2001).

[113] Lakshmikantham, V., Bainov, D. D. and Simeonov, P. S. *Theory of Impulsive Differential Equations*, World Scientific, Singapore, (1989).

[114] Leine, R. I., Van Campen, D. H. and Van de Vrande, B. L. *Bifurcations in nonlinear discontinuous systems*, Nonlinear Dynam., **23**, 105–164, (2000).

[115] Li, X. and Wang, Z. *Global attractivity for a logistic equation with piecewise constant arguments. Differences and differential equations*, Fields Inst. Commun., **42**, 215–222, (2004).

[116] Llibre, J. and Ponce, E. *Bifurcation of a periodic orbit from infinity in planar piecewise linear vector fields*, Nonlinear Anal., **36**, 623–653, (1999).

[117] Liu, X. *Impulsive stabilization and control of chaotic systems*, Nonlinear Anal., **47**, 1081–1092, (2001).

[118] Liu, X. and Rohlf, K. *Impulsive control of a Lotka-Volterra system*, IMA J. Math. Control Inform., **15**, 269–284, (1998).

[119] Liu, X. *Stability results for impulsive differential systems with applications to population growth models*, Dynam. Stability Systems, **9**, 163–174, (1994).

[120] Lotka, A. J. *Elements of Physical Biology*, William and Wilkins, Baltimore, (1925).

[121] Luo, Z. and Shen, J. *New Razumikhin type theorems for impulsive functional differential equations*, Appl. Math. Comput., **125**, 375–386, (2002).

[122] Lyapunov, A. M. *Probléme Général de la Stabilité du Mouvement*, Ann. Math. Studies, 17, Princeton Univ. Press, (1949) (Translation of the original paper published in Comm. Soc. Math. Kharkow, 1892).

[123] MacDonald, N. *Time lags in biological models*, Lecture Notes in Biomath., **27**, (1978).

[124] Marsden, J. E. and McCracken, M. *The Hopf Bifurcation and its Applications*, Applied Mathematical Sciences, Springer, New York, Berlin, (1976).

[125] Matsunaga, H., Hara, T. and Sakata, S. *Global attractivity for a logistic equation with piecewise constant argument*, NoDEA, **8**, 45–52, (2001).

[126] May, R. M. *Stability and Complexity in Model Ecosystems*, Princeton University, (1973).

[127] Michel, A. N. and Hu, B. *Stability analysis of discontinuous dynamical systems using vector Lyapunov functions*, Circuits Systems Signal Process, **19**, 221–243, (2000).

[128] Miller, R. K. and Michel, A. *Ordinary Differential Equations*, Academic Press, New York, (1982).

[129] Minorsky, N. *Nonlinear Oscillations*, Van Nostrand, Princeton, (1962).

[130] Murad, N. M. and Celeste, A. *Linear and nonlinear characterisation of loading systems under piecewise discontinuous disturbances voltage: analytical and numerical approaches*, Power Electronics Systems and Applications, 291–297, (2004).

[131] Muroya, Y. and Kato, Y. *On Gopalsamy and Liu's conjecture for global stability in a population model*, J. Comput. Appl. Math., **181**, 70–82, (2005).

[132] Muroya, Y. *Persistence, contractivity and global stability in logistic equations with piecewise constant delays*, J. Math. Anal. Appl., **270**, 602–635, (2002).

[133] Murray, J. D. *Mathematical Biology*, Springer, New York, (2001).

[134] Myshkis, A. D. and Samoilenko, A. M. *Systems with shocks at given moments*, Matematicheskii Zbornik, **74(2)**, 202–208, (1967).

[135] Nayfeh, A. H. *Order reduction of retarded nonlinear systems–the method of multiple scales versus center-manifold reduction*, Nonlinear Dynam., **51**, 483–500, (2008).

[136] Nemytskii, V. V. and Stepanov, V. V. *Qualitative Theory of Differential Equations*, Princeton, N.J., Princeton University Press, (1960).

[137] Nordmark, A. B. *Existence of periodic orbits in grazing bifurcations of impacting mechanical oscillators*, Nonlinearity, **14(6)**, 1517–1542, (2001).

[138] Nusse, H. E. and Yorke, J. A. *Border-collision bifurcations for piecewise smooth one-dimensional maps*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., **5**, 189–207, (1995).

[139] Nusse, H. E., Ott, E. and Yorke, J. A. *Border-collision bifurcations: an explanation for observed bifurcation phenomena*, Phys. Rev. E, **49(2)**, 1073–1076, (1994).

[140] Nusse, H. E. and Yorke, J. A. *Border-collision bifurcations including "period two to period three" for piecewise smooth systems*, Phys. D, **57**, 39–57, (1992).

[141] Papaschinopoulos, G. *On the integral manifold for a system of differential equations with piecewise constant argument*, J. Math. Anal. Appl., **201**, 75–90, (1996).

[142] Pavlidis, T. *Stability of a class of discontinuous dynamical systems*, Information and Control, **9**, 298–322, (1966).

[143] Pavlidis, T. and Jury, E. I. *Analysis of a new class of pulse-frequency modulated feedback systems*, IEEE Trans. Automat. Control, **10(1)**, 35–43, (1965).

[144] Pavlidis, T. *A new model for simple neural nets and its application in the design of a neural oscillator*, Bull. Math. Biophys., **27**, 215–229, (1965).

[145] Perestyuk, N. A., Samoilenko, A. M. and Stanzhitskii, A. N. *On the existence of periodic solutions for certain classes of systems of differential equations with random pulse influence*, Ukrainian Math. J., **53(8)**, 1266–1289, (2001).

[146] Pinto, M. *Asymptotic equivalence of nonlinear and quasi linear differential equations with piecewise constant arguments*, Math. Comput. Modelling, **49**, 1750–1758, (2009).

[147] Pliss, V. A. *Integral Sets of Periodic Systems of Differential Equations*, Izdat. Nauka, Moscow, (1977).

[148] Pliss, V. A. *A reduction principle in the theory of stability of motion*, Izv. Akad. Nauk SSSR Ser. Mat., **28**, 1297–1324, (1964).

[149] Poincaré, H. *Les Méthodes Nouvelles de la Mécanique Céleste*, Gauthier-Villars, Paris, (1899).

[150] Razumikhin, B. S. *Stability of delay systems*, Prikl. Mat. Mekh., **20**, 500–512, (1956).

[151] Rouche, N., Habets, P. and Laloy, M. *Stability Theory by Lyapunov's Direct Method*, New York, Springer-Verlag, (1977).

[152] Samoilenko, A. M. and Perestyuk, N. A. *Impulsive Differential Equations*, World Scientific, (1995).

[153] Samoilenko, A. M. and Perestyuk, N. A. *Impulsive Differential Equations*, (in Russian) Vischa Shkola, Kiev, (1987).

[154] Scheffer, M. *Ecology of Shallow Lakes*, Chapman and Hall, (1998).

[155] Seifert, G. *Almost periodic solutions of certain differential equations with piecewise constant delays and almost periodic time dependence,* J. Differential Equations, **164**, 451–458, (2000).

[156] Shah, S. M. and Wiener, J. *Advanced differential equations with piecewise constant argument deviations*, Int. J. Math. Math. Sci., **6**, 671–703, (1983).

[157] Shaw, S. W. and Holmes, P. J. *A periodically forced piecewise linear oscillator*, J. Sound Vibration, **90(1)**, 129–155, (1983).

[158] Shen, J. H. and Stavroulakis, I. P. *Oscillatory and nonoscillatory delay equation with piecewise constant argument*, J. Math. Anal. Appl., **248**, 385–401, (2000).

[159] Stamova, I. M. and Stamov, G. T. *Lyapunov-Razumikhin method for impulsive functional differential equations and applications to the population dynamics*, J. Comput. Appl. Math., **130**, 163–171, (2001).

[160] Tang, S. and Chen, L. *Global qualitative analysis for a ratio-dependent predator-prey model with delay*, J. Math. Anal. Appl., **266**, 401–419, (2002).

[161] Thompson, W. R. *Biological control and the theories of the interaction of populations*, Parasitology, **31**, 299–388, (1939).

[162] Thota, P. and Dankowicz, H. *Continuous and discontinuous grazing bifurcations in impact oscillators*, Phys. D, **214**, 187-197, (2006).

[163] Verhulst, P. F. *Notice sur la loi que la population pursuit dans son accroissement*, Corresp. Math. Phys., **10**, 113-121, (1838).

[164] Volterra, V. *Variations and fluctuations of the number of individuals in animal species living together*, Animal Ecology, McGraw-Hill, New York, p. 409, (1931).

[165] Wang, G. *Periodic solutions of a neutral differential equation with piecewise constant arguments*, J. Math. Anal. Appl., **326**, 736–747, (2007).

[166] Wang, Q. and Liu, X. *Impulsive stabilization of delay differential systems via the Lyapunov-Razumikhin method*, Appl. Math. Lett., **20**, 839–845, (2007).

[167] Wang, Z. and Wu, J. *The stability in a logistic equation with piecewise constant arguments*, Differential Equations Dynam. Systems, **14**, 179–193, (2006).

[168] Wangersky, P. J. and Cunningham, W. J. *Time lag in prey-predator population models*, Ecology, **38**, 136–139, (1957).

[169] Wiener, J. and Lakshmikantham, V. *A damped oscillator with piecewise constant time delay*, Nonlinear Stud., **7**, 78–84, (2000).

[170] Wiener, J. *Generalized Solutions of Functional Differential Equations*, World Scientific, Singapore, (1993).

[171] Wiener, J. and Cooke, K. L. *Oscillations in systems of differential equations with piecewise constant argument*, J. Math. Anal. Appl., **137**, 221–239, (1989).

[172] Wiggins, S. *Introduction to Applied Nonlinear Dynamical Systems and Chaos*, New York, Springer, (2003).

[173] Xia, Y., Huang, Z. and Han, M. *Existence of almost periodic solutions for forced perturbed systems with piecewise constant argument*, J. Math. Anal. Appl., **333**, 798–816, (2007).

[174] Xu, B. *Stability of retarded dynamical systems: a Lyapunov function approach*, J. Math. Anal. Appl., **253**, 590–615, (2001).

[175] Xu, R. and Chen, L. *Persistence and global stability for $n-$species ratio-dependent predator-prey system with time delays*, J. Math. Anal. Appl., **275**, 27–43, (2002).

[176] Xu, R., Davidson, F. A. and Chaplain, M. A. J. *Persistence and stability for a 2-species ratio-dependent predator-prey system with distributed time delay*, J. Math. Anal. Appl., **269**, 256–277, (2002).

[177] Yang, X. *Existence and exponential stability of almost periodic solution for cellular neural networks with piecewise constant argument*, Acta Math. Appl. Sin., **29**, 789–800, (2006).

[178] Yoshizawa, T. *Stability Theory by Lyapunov's Second Method*, The Mathematical Society of Japan, Tokyo, (1966).

[179] Zou, Y., Küpper, T. and Beyn, W.-J. *Generalized Hopf bifurcation for planar Filippov systems continuous at the origin*, J. Nonlinear Sci., **16**, 159–177, (2006).

[180] Zou, Y. and Küpper, T. *Generalized Hopf bifurcation emanated from a corner for piecewise smooth planar systems*, Nonlinear Anal., **62**, 1–17, (2005).

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FOREIGN LANGUAGES

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PUBLICATIONS

1. Akhmet, M. U., Aruğaslan, D. and Beklioğlu, M. *Impulsive control of the population dynamics*, Proceedings of the Conference on Differential and Difference

Equations and Applications at the Florida Institute of Technology, August 1-5, 2005, Melbourne, Florida, Hindawi Publ. Corp., New York, 21-29, (2006).

2. Akhmet, M. U., Aruğaslan, D. and Liu, X. *Permanence of nonautonomous ratio-dependent predator-prey systems with piecewise constant argument of generalized type*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal., **15**, 37-51, (2008).

3. Akhmet, M. U. and Aruğaslan, D. *Lyapunov-Razumikhin method for differential equations with piecewise constant argument*, Discrete Contin. Dyn. Syst., (accepted).

4. Akhmet, M. U. and Aruğaslan, D. *Bifurcation of a non-smooth planar limit cycle from a vertex*, Nonlinear Anal., (accepted).

5. Akhmet, M. U., Aruğaslan, D. and Turan, M. *Hopf bifurcation for a 3D Filippov system*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal., (accepted).

PARTICIPATION IN SCIENTIFIC MEETINGS

1. Fifth World Congress of Nonlinear Analysts, Orlando, USA, 2-9 July, 2008.

2. Second Ankara Mathematics Days Symposium, Ankara University, Ankara, Turkey, 22-23 May, 2008.

3. Conference on Differential and Difference Equations and Applications, Florida Institute of Technology, Melbourne, USA, 1-5 August, 2005.

PRESENTATIONS IN SCIENTIFIC MEETINGS

1. *Hopf bifurcation of a planar Filippov system*, WCNA-08, Orlando, 2008.

2. *Lyapunov-Razumikhin method for differential equations with piecewise constant argument*, Second Ankara Mathematics Days Symposium, Ankara, 2008.

3. *Differential Equations with Discontinuous Right-Hand Side: Sliding Motion*, Applied Dynamics Group Seminars, Department of Mathematics, IAM and METU, Ankara, 2007.

4. *Lyapunov-Razumikhin method for differential equations with piecewise constant argument of generalized type*, Applied Dynamics Group Seminars, Department of Mathematics, IAM and METU, Ankara, 2007.

5. *Control of discontinuous processes of population dynamics*, Department of Applied Mathematics, University of Waterloo, Canada, 2006.

6. *Permanence of nonautonomous ratio-dependent predator-prey systems with piecewise constant argument of generalized type*, Applied Dynamics Group Seminars, Department of Mathematics, IAM and METU, Ankara, 2006.

7. *Impulsive control of the population dynamics*, Differential and Difference Equations and Applications, Joint Seminar by Institute of Applied Mathematics and Biology Department, METU, 2005.

8. *Impulsive bioregulation*, Institute of Applied Mathematics, METU, 2004.

ORGANIZATION OF SCIENTIFIC EVENTS

1. Special session organization entitled "Bifurcation and Chaos in Differential Equations with Discontinuities and Applications" in the World Congress of Nonlinear Analysts together with M. U. Akhmet, Orlando, 2-9 July, 2008.

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PROJECTS

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