ENVIRONMENTAL EFFECTS ON QUANTUM GEOMETRIC PHASE AND QUANTUM ENTANGLEMENT

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ABSTRACT

ENVIRONMENTAL EFFECTS ON QUANTUM GEOMETRIC PHASE AND QUANTUM ENTANGLEMENT

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We investigate the geometric phase (GP) acquired by the states of a spin-1/2 nucleus which is subject to a static magnetic field. This nucleus as the carrier system of GP, is taken as coupled to a dissipative environment, so that it evolves non-unitarily. We study the effects of different characteristics of different environments on GP as nucleus evolves in time. We showed that magnetic field strength is the primary physical parameter that determines the stability of GP; its stability decreases as the magnetic field strength increases. (By decrease in stability what we mean is the increase in the time rate of change of GP.) We showed that this decrease can be very rapid, and so it could be impossible to make use of it as a quantum logic gate in quantum information theory (QIT). To see if these behaviors differ in different environments, we analyze the same system for a fixed temperature environment which is under the influence of an electromagnetic field in a squeezed state. We find that the general dependence of GP on magnetic field does not change, but this time the effects are smoother. Namely, increase in magnetic field decreases the stability of GP also for in this environment; but this decrease is slower in comparison with the former case, and furthermore it occurs gradually.

As a second problem we examine the entanglement of two atoms, which can be used as a two-qubit system in QIT. The entanglement is induced by an external quantum system. Both two-level atoms are coupled to a third two-level system by dipole-dipole interaction. The two atoms are assumed to be in ordinary vacuum and the third system is taken as influenced by a certain environment. We examined different types of environments. We show that the steady-state bipartite entanglement can be achieved in case the environment is a strongly fluctuating, that is a squeezed-vacuum, while it is not possible for a thermalized environment.

Keywords: Quantum information theory, quantum computation, entanglement, concurrence, geometric phase, Berry phase, quantum logic gate.

ÖZ

KUANTUM GEOMETRİK FAZ VE KUANTUM DOLANIKLIĞA ÇEVRESEL ETKİLER

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Statik bir manyetik alana maruz kalan bir çekirdeğin kuantum durumlarının kazanacağı geometrik faz (GF) incelendi. GF'nin taşıyıcısı olan bu çekirdek dağıtıcı bir sistemle birleşik alındığı için üniter olmayan bir şekilde evrilmektedir. Çekirdeğin zamanda evrimi esnasında farklı ortamların farklı özelliklerinin GF'ye etkileri çalışıldı. Uygulanan manyetik alan şiddetinin GF'nin kararlılığını belirleyen birincil fiziksel parametre olduğu gösterildi. Manyetik alan şiddeti arttırıldıkça GF'nin kararlılığının azaldığı gözlendi. (Burada, kararlılıktaki azalma GF'nin değişim hızındaki artşı tarif etmektedir.) Kararlılıktaki bu azalmanın çok hızlı olabileceği gösterildi. Öyle ki, bu durumda kuantum bilgi kuramında (KBK) bir kuantum kapı olarak kullanılamaz hale gelmektedir. Böyle davranışlarının farklı ortamlarda farklı olup olmadığının anlaşılması için aynı sistem, sıcaklığı sabit tutulmuş ve sıkıştırılmış bir durumda bulunan bir elektromanyetik dalgaya maruz olan bir çevrede çözümlendi. GFnin manyetik alan bağımlılığının genelde değişmediği, ancak etkilerin bu durumda daha yumuşak olduğu bulundu. Yani manyetik alandaki artış GF'nin kararlılığında bu ortam için de azalmaya sebep olumakta ama bu sefer bu azalış önceki ortamdakine göre daha yavaş ve kademe kademe gerçekleşmektedir.

İkinci bir problem olarak, KBK'nda bir iki-kuantum-bit olarak kullanılabilecek iki tane iki seviyeli atomun dolanıklığı incelendi. Bu sistemdeki dolanıklığa bir dış sistem neden olmaktadır. İki-seviyeli atomların her ikisi de üçüncü bir iki-seviyeli atomla dipol-dipol etkileşim halinde ve aynı zamanda sıradan vakum içinde bulunmaktadırlar. Üçüncü atom ise başka bir ortam içerisindedir. Bu sistem, üçüncü atomun içinde bulunduğu ortam farklılaştırılarak incelendi. İki parçacıklı durağan-durum dolanıklığının, üçüncü atom sıkıştırılmış-vakumda iken elde edilebilmesine rağmen sadece sıcaklığın olduğu bir ortamda bulunduğunda bunun mümkün olmadığı gösterildi.

Anahtar Kelimeler: Kuantum bilgi kuramı, kuantum hesaplama, dolanıklık, mutabıklık, geometrik faz, Berry fazı, kuantum lojik kapısı.

Tanrıçam, Işığım,

Habibim, Yavrum,

Kızım

ARİNNA'ya

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I would be incomplete without my sisters Meryem GÜNHAN's eyes, Nazan GÜNHAN's sensations, Emine GÜNHAN's thoughts, I would not feel to be cared for without my sister Işıl GÜNHAN's care.

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CHAPTER 1

INTRODUCTION

During the history of civilization, humankind invented and improved various methods to encode, to process data, and to send it to a receiver in such a way that it is secure and fast enough. These data have been for example, his love, commercial information, military secrets. For this purpose he educated the pigeons, wrote the messages on a piece of cloth that is wrapped around a rod, encoded the natural languages with symbols different than the known alphabets, created non-natural languages, etc.

Beginning from 1950s another method has been used. In this technique, which is now conventional, information is processed classically via logic gates, that performs logical operations on one or more logic inputs, and produces a single logic output. These inputs and outputs are register of classical bits, which are discrete Boolean states 0 and 1. Classical logic gates are primarily implemented electronically and electromagnetically. Though it is not reached yet, today's technology show us the limits of this technique. What we can see more is the existence of another method which takes these limits far beyond. It depends on the quantum theory. This theory's basic principles, which are constructed in the first quarter of 20th century, offer alternative entities for information processing, such as the states of any two-state quantum system. These states can be used as Boolean variables. Unlike those classical ones, such a system, which is called a quantum bit, can actually be in an arbitrary superposition of its bases states, $|0\rangle$ and $|1\rangle$. This makes it a much more powerful computational resource as a continuous variable. The devices that performs necessary operations on these quantum bits are also offered by quantum theory. These are called quantum logic gates. These gates can be implemented by an entity called, quantum geometric phase, $[1, 2]$. It involves intrinsic properties of a quantum system, so is independent from the parameters that determines the dynamics of evolution of the system.

One other entity offered by quantum theory is entanglement, [3–5]. It depends basically on the inseparability of degrees of freedom of a quantum system. This means that these degrees of freedom are instantly sending information of their state to each other. This information is called quantum information and does not necessitate man in nature.

On account of mentioned properties, if man makes it possible to use and to manipulate these quantum entities, in all the ways he wants, he would process and send the information in an amount [6], with a speed [7–9], and with a security [10, 11] never attained before. In chapter II, we introduce basic components of quantum information theory (QIT). First the definitions of necessary tools for processing information quantum mechanically will be given. Then, how computation can be performed with quantum logic gates operating on input states giving needed outputs will be explained. In this manner definition of quantum geometric phase is introduced. It is shown that it is a physical property with which one is capable to perform those computational tasks for implementing the necessary quantum logic gates. In computation, the input states can, and sometimes have to be entangled. Together with its structure, defining quantum information, a measure for entanglement is given. Because one needs to know whether a quantum state is entangled, and if so, the amount of entanglement it possesses.

Even though it is known where to find and how to take advantage of these quantum structures, there is one thing to battle against, that is, the environment in which the systems carrying these entities are placed. (The terms bath, environment and reservoir will be used interchangeably throughout this thesis.) Since we can not truly isolate a system, it constantly interacts with an environment. This interaction has a great potential to kill those fragile quantum characteristics of a system before it is made possible to utilize these characteristics. Thus, we have to know what can happen to our candidate carrier systems during their evolutions in a reservoir.

In chapter III, we give the basic properties of environments in which quantum systems are expected to be affected differently. These environments are assumed to be baths of harmonic oscillators. A general equation of motion for a collection of spin$-1/2$

quantum systems in a bath is given. This system is taken to be in interaction with a thermal environment driven by an electromagnetic field in a squeezed coherent state. But due to its infinitely many degrees of freedom this bath is assumed not to be affected significantly, so that it remains in its initial state. One other approximation is considered in simplifying the equation of motion for the system. It is the so called Markov approximation which assumes that the system does not have memory. In other words, it instantly forgets the past due to the damping caused by environmental degrees of freedom.

We assumed our carrier system to be in environments having different physical properties. One of these should be temperature. It is because first, there would always be a finite temperature in a real physical environment, and second, with increasing temperature a quantum system can easily leave the quantum regime. Another physical parameter is the magnetic field in and around the system. Even the background radiation, which exists everywhere in the universe, may cause significant changes, especially in geometric phase, so that it cannot be used in QIT. In taking into account these possibilities we analyzed, in chapter IV, the effects of temperature and magnetic field on carrier systems. We also studied the case when the environment is under the influence of a specific type of radiation field. We chose it to be in a squeezed state. The reason is that, squeezed state electromagnetic fields are in the farthest position to classical regime. Their this property makes one to think that they have a potential to make the systems, to which they affect, stay in the quantum regime, and demonstrate their quantum characteristics. Since we need quantum bits, all the carriers are assumed to be two-level quantum systems.

CHAPTER 2

INFORMATION PROCESSING AND COMPUTING QUANTUM MECHANICALLY

In this chapter we will introduce some basic concepts and definitions on quantum information processing and quantum computing together with some examples. The aim is to underline the nature of quantum states, so called *entangled states*, which are needed to be used as resources and the characteristics of the operations, *quantum gates*, manipulate those resources to get required outputs as results of the computational processes.

In classical information processing, binary digits 0 and 1 are used as logical states. A classical bit can be in either one of the boolean states 0 and 1. The information is classically encoded and decoded with these classical bits, and processed with classical logic gates.

In quantum information processing, the basic resource is two-level quantum systems. Two-level atoms, polarization states of polarized photons, spin states of spin$-1/2$ nuclei are examples of those resources, called as *quantum binary digits*, or *qubits*, in short. The quantum states, $|0\rangle, |1\rangle$, representing the two levels of a qubit, also represents the *quantum* Boolean states. But in this case, unlike its classical counterpart, a qubit can be in any arbitrary superposition state,

$$\alpha_0|0\rangle + \alpha_1|1\rangle, \tag{2.1}$$

with $\alpha_j \in \mathbb{C}$, making it a much more powerful resource. Here $|0\rangle$ and $|1\rangle$ states are prescribed as normalized and mutually orthogonal, $\langle j|k\rangle = \delta_{jk}$, where δ_{jk} is the Kronecker-delta symbol, being equal to 1 for $j = k$, and to 0 for $j \neq k$. For the superposition state (2.1) we have described normalization by $|\alpha_0|^2 + |\alpha_1|^2 = 1$.

A quantum computation is performed as follows: An input state is prepared on a register of qubits. The state of qubits is evolved unitarily by a building-block of operations. At the end of this unitary evolution the final state is taken as the output of the block. A '*device*' which performs fixed unitary operations on selected qubits of the input state of the block in which it stays is called a *quantum logic gate*, [12, 13].

At this point some examples of quantum logic gates will be given, using the following notation;

A matrix representation of the computational bases $|0\rangle$ and $|1\rangle$ can be as follows;

$$|0\rangle \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle \doteq \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{2.2}$$

where $\doteq$ will be read as '*is represented by*', following Sakurai, [14].

Schematic representation of a gate operated on input states resulting output states is

$$|input\ state\rangle \quad \longrightarrow \boxed{GATE} \longrightarrow \quad |output\ state\rangle.$$

One-Qubit Gates

A *NOT gate*, X, defined as a flip operation between states $|0\rangle$ and $|1\rangle$, is given in the computational basis $B_1 = \{|0\rangle, |1\rangle\}$ as

$$X \doteq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.3}$$

Throughout this thesis computational basis, which is defined for $n-$qubits as

$$\begin{aligned} B_n &\equiv \bigotimes_{j=1}^{n} \{|0\rangle, |1\rangle\}_j \\ &= \{|00\cdots 0\rangle, |0\cdots 01\rangle, \cdots, |11\cdots 1\rangle\}. \end{aligned} \tag{2.4}$$

will be used, unless otherwise stated.

A Y*-gate* is defined as a flip operation between states $|0\rangle$ and $|1\rangle$ followed by multiplication of them with $-i$ and i, respectively,

$$Y \doteq \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{2.5}$$

A *Z-gate* leaves $|0\rangle$ unchanged and flips the sign of $|1\rangle$,

$$Z \doteq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.6}$$

A *phase ϕ-gate* changes the relative phase between the states $|0\rangle$ and $|1\rangle$ by ϕ,

$$\phi \doteq \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix}. \tag{2.7}$$

One other important single-qubit gate is the *Hadamard gate*, H, sometimes described as 'square-root of NOT' gate, in that it turns $|0\rangle$ and $|1\rangle$ into $(|0\rangle + |1\rangle)/\sqrt{2}$ and $(|0\rangle - |1\rangle)/\sqrt{2}$, respectively, which are 'halfways' between $(|0\rangle$ and $|1\rangle$. It is defined as

$$H \doteq \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{2.8}$$

The schematic representations of these five single-qubit gates are

$$|x\rangle \quad \longrightarrow \boxed{X} \longrightarrow \quad |1-x\rangle, \tag{2.9}$$

$$|x\rangle, \quad \longrightarrow \boxed{Y} \longrightarrow \quad (-1)^{1-x}\, i\, |1-x\rangle, \tag{2.10}$$

$$|x\rangle \quad \longrightarrow \boxed{Z} \longrightarrow \quad (-1)^{x}\, |x\rangle, \tag{2.11}$$

$$|x\rangle \quad \longrightarrow \boxed{\phi} \longrightarrow \quad e^{ix\phi}|x\rangle, \tag{2.12}$$

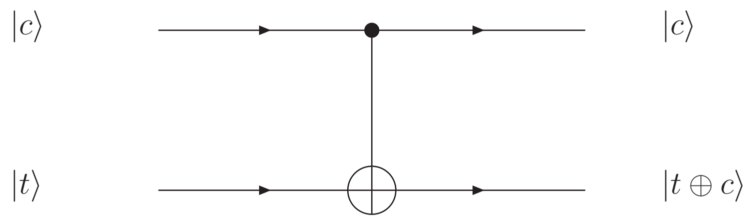
$$|x\rangle \quad \longrightarrow \boxed{H} \longrightarrow \quad \tfrac{1}{\sqrt{2}}(|1-x\rangle + (-1)^{x}|x\rangle), \tag{2.13}$$

where x is either 0 or 1.

Two-Qubit Gates

Quantum computation and quantum information processing requires to execute conditional dynamics between two qubits, where the state of one qubit influences the evolution of another qubit during a quantum computation, ([13, 15, 16]). So, in addition to those single-qubit ones we also need two-qubit gates. We will give two important ones.

Controlled-NOT or CNOT *gate* has two input qubits, known as the *control* qubit, $|c\rangle$, and the *target* qubit, $|t\rangle$, respectively. It is defined schematically as



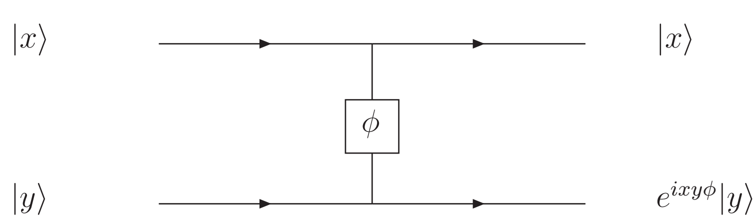
(2.14)

and represented by the matrix

$$U_{CN} \doteq \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{2.15}$$

for the second qubit being the target. Here the input state $|c, t\rangle$ is CNOTted by the gate to get the output state $|c, t \oplus c\rangle$, where $\oplus$ is addition modulo two. It is possible to think that control qubit does not change its state. But depending on the state of control qubit, a NOT operation may or may not be applied to target. One candidate implementation for the CNOT gate was proposed by Cory *et al.* ([17, 18]). In this technique it is showed that by applying a sequence of radio frequency pulses ([19–21]) to a nuclear magnetic resonance (NMR) liquid consisting of identical molecules each containing exactly two spin-1/2 nuclei of the same isotope, it is possible to obtain a *CNOT-gate operated* two qubit system. Here the two qubits are the two nuclei of the identical molecules, and the sequence of radio frequency (RF) pulses play the CNOT-gate role.

Controlled phase-shift gate is defined schematically as



(2.16)

and represented by the matrix

$$U_\phi \doteq \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\phi} \end{pmatrix}. \tag{2.17}$$

It performs the transformation $|11\rangle \to e^{i\phi}|11\rangle$, and leaves the other basis states unchanged. For implementation of the controlled phase-shift gate a similar technique used for implementation of CNOT gate is suggested by Ekert *et al.* ([13,22]). In this technique again the NMR spectroscopic methods are used. But this time an appropriate sequence of RF pulses is used to eliminate the dynamical phases gained by the states of the system in hand during an evolution. What remains at end is a conditionally phase shifted states of a system of two spin$-1/2$ particles. This remaining phase is the so-called *Berry phase*, or more generally ***geometric phase***, ([1,2]). By *conditionally* it is meant that the phase shift of the state of particle 1 depends on the state of particle 2, and vice versa, i.e. the amount of phase shifts of the states of particles, when their spins are aligned, is different than the amount of shift when they are anti-aligned, for example. By the same method it is also possible to obtain a single qubit phase shift gate defined by (2.7). In fact, all the single qubit gates can be implemented by NMR operations.

In order to be able to perform an arbitrary quantum computation on any number of qubits, one needs *"a set of gates that is sufficient for approximating any unitary operation to arbitrary accuracy"*. A set is called ***a universal set of quantum gates*** if it is possible to satisfy this approximation only with the gates in it [23]. A known result in the literature states that the Hadamard gate, H, and *all* controlled phase gates, U_ϕ, form a universal set, E, for quantum computations, ([13,16]),

$$E = \{H, U_\phi; \forall\phi \in \mathbb{R}\}. \tag{2.18}$$

Having a universal set of quantum gates, one is able to manipulate a register of n-qubit state. These states can, and for some quantum computations *have to be* ***entangled***.

Now we will give definitions of quantum geometric phase and quantum entanglement together with some properties that will be needed and used throughout this work.

2.1 Geometric Phase

During its evolution, the wave function of a quantum system gathers information on the geometric structure of the Hilbert space, $\mathcal{H}$, in which it lies. This information is retained in the form of a phase factor distinct from the familiar dynamical one, which is caused by and depends on the dynamical variables of the system. Unlike its dynamical counterpart, this phase is independent of all the external dependencies of the system, and depends only on the path traversed by the system's state in $\mathcal{H}$. For this reason it is named as *the geometric phase*, [24]. Its this property that makes it a possibly useful tool for fault tolerant quantum computation [25], since it is potentially robust against certain sources of noise, ([26] *and the references therein*).

Pancharatnam is the one who first introduced the concept of geometric phase in the classical context in 1956 [1]. He defined a phase characterizing the interference of classical light in distinct states of polarization. The quantum counterpart is defined 30 years later in 1984 by Berry, [2], for the case of cyclic evolution of a pure state. In this case the system is assumed to be closed, and the speed of change of the state is small enough in comparison with the change in its energy, so that it is in an adiabatic unitary evolution. In 1987 and 1988 Aharanov and Anandan relaxed the adiabaticity condition, ([27, 28]). They showed first that a quantum system which is initially in state $|\psi\rangle$ and completes a closed loop in its Hilbert space, $\mathcal{H}$, at time τ in state $|\tilde{\psi}\rangle \equiv e^{-i\,f}|\psi\rangle$, gains a geometric phase

$$\Phi_G \equiv i\int_0^\tau \langle\tilde{\psi}|\frac{d}{dt}|\tilde{\psi}\rangle dt, \tag{2.19}$$

where $f = f(t)$ is a real valued function of time so that $f(\tau) - f(0) = \Phi_G$ and the observable characteristics of $|\psi\rangle$ and $|\tilde{\psi}\rangle$ are the same. What they then showed is that this expression is *adiabaticity independent*, where a system is said to be adiabatically evolving if its speed of change between the states of different energies is too slow in comparison with the energy differences in convenient units. The following year Samuel and Bhandari were able to extend the concept to non-cyclic evolutions, viz they express the geometric phase between any two non-orthogonal states in a Hilbert space, [29].

The concept of mixed state geometric phase was first introduced as a purely mathematical problem by Uhlmann in 1991, [30]. Then in 2000 Sjöqvist *et al.* provided a physical prescription based on interferometry, and introduced the total phase of a mixed state [31]. They showed that it is possible to express the geometric part of this total phase by defining the parallel transport condition [32] for mixed states.

Finally in 2004 Tong *et al.* relaxed the unitarity condition and obtained the generalization of the mixed quantal state geometric phase to non-unitary evolutions, [33]. What they did is summarized below. The problem is to express the geometric phase of a state, $\hat{\rho}(t)$, of a system, s, gained during the course of its evolution in time $t \in [0,\tau]$. The mixed state $\hat{\rho}(t)$ can be expressed in its spectral decomposition as

$$\hat{\rho}(t) = \sum_{k=1}^{S} \rho_k(t)|\rho_k(t)\rangle\langle\rho_k(t)|, \tag{2.20}$$

where $\rho_k(t) \in [0,1]$, satisfying the normalization condition, $\sum_{k=1}^{S} \rho_k = 1$, are the eigenvalues, and $\{|\rho_k(t)\rangle\}_{k=1}^{S}$, spanning the Hilbert space $\mathcal{H}_s$ of dimension S of the system, is the set of eigenvectors of $\hat{\rho}(t)$. This state is *purified* by taking the system s as a subsystem of a larger system $s+a$, where the added system a is called *ancilla*, of dimension $dim(a) \geq S$. So, the pure state $|\psi(t)\rangle \in \mathcal{H}_s \otimes \mathcal{H}_a$, $\mathcal{H}_a$ being the Hilbert space in which the states of ancilla lies, can be expressed as

$$|\psi(t)\rangle = \sum_{k=1}^{S} \sqrt{\rho_k(t)}|\rho_k(t)\rangle \otimes |a_k\rangle. \tag{2.21}$$

Here $\mathcal{H}_a$ is assumed to be spanned by $\{|a_k\rangle\}_{k=1}^{S}$, and its dimension, without loss of generality, is taken to be S. The relative phase between $|\psi(\tau)\rangle$ and $|\psi(0)\rangle$ reads

$$\begin{aligned}\Phi_G(\tau) &\equiv arg(\langle\psi(0)|\psi(\tau)\rangle) \\ &= arg(\sum_{k=1}^{S} \sqrt{\rho_k(0)\rho_k(\tau)}\langle\rho_k(0)|\rho_k(\tau)\rangle) \\ &= arg(\sum_{k=1}^{S} \sqrt{\rho_k(0)\rho_k(\tau)}\langle\rho_k(0)|V(\tau)|\rho_k(0)\rangle),\end{aligned} \tag{2.22}$$

where in the last step the transformation between orthonormal basis $\{|\rho_k(t)\rangle\}_{k=1}^{S}$ and $\{|\rho_k(0)\rangle\}_{k=1}^{S}$ is performed by a unitary operator

$$V(\tau) \equiv \sum_{k=1}^{S} |\rho_k(\tau)\rangle\langle\rho_k(0)|. \tag{2.23}$$

But this operator is not the only one that realizes the path between $|\psi(0)\rangle$ and $|\psi(\tau)\rangle$. Instead, there is an equivalence set of unitary operators $\tilde{V}(t)$ of the form

$$\tilde{V}(t) = V(t) \sum_{k=1}^{S} e^{i\theta_k(t)} |\rho_k(0)\rangle\langle\rho_k(0)|, \tag{2.24}$$

where θ_k are time-dependent real parameters such that $\theta_k(0) = 0$. We know that $arg\langle\psi(0)|\psi(\tau)\rangle$ gives the pure geometric phase of $|\psi\rangle$ in case its evolution satisfies the parallel transport condition $\langle\psi(t)|\dot{\psi}(t)\rangle = 0$, and if so $|\rho_k\rangle$ should satisfy $\langle\rho_k(t)|\dot{\rho}_k(t)\rangle = 0$. Taking this condition into account with the operators $\tilde{V}(t)$, they removed the dependence of $\Phi_G(t)$ upon the purification type displayed by equation (2.21), and reached to the expression

$$\Phi_G(\tau) = arg\left(\sum_{k=1}^{S} \sqrt{\rho_k(0)\rho_k(\tau)}\langle\rho_k(0)|\rho_k(\tau)\rangle e^{-\int_0^\tau \langle\rho_k(t)|\dot{\rho}_k(t)\rangle dt}\right). \tag{2.25}$$

The equation (2.25) is the most general expression for the geometric phase of a quantum system evolving non-unitarily, i.e. in interaction with an environment.

2.2 Entanglement

The concept of entanglement was introduced as a tool for the test of completeness and, in some sense consistency of the quantum theory, [3–5, 34]. Today it is a vital concept for quantum information theory, [7,35,36]. In spite of this much significance, there is no general agreement on a single definition or a measure of entanglement, valid for all types of quantum states.

Mainly there are three difficulties against a general definition. These are *non-locality, violation of classical realism* and *separability.* First, the strong overlap of wavefunctions of individual atoms in a Bose-Einstein condensate, [37], and the concept of single particle entanglement with respect to its intrinsic degrees of freedom, [38,39], makes the non-locality requirement meaningless. Second, violation of classical realism which is described by Bell-type inequalities, [40–42], and was assumed to be a signal for the existence of entanglement, indicates only the quantum nature of states. This violation can be observed in truly unentangled states as well as entangled ones. And third, non-separability condition, which is undoubtedly necessary for a state to be entangled, is *not* sufficient for at least one definition. *Three-tangle* is defined for three-partite entanglement measure, [43], and gives 0 for the so-called inseparable W-states,

$$|W\rangle = \frac{1}{3}(|100\rangle + |010\rangle + |001\rangle). \tag{2.26}$$

Though it is still in need of an accurate definition, we have at least one necessary condition for a quantum state to be entangled. And without further details we will be content with this condition only, which will be mentioned below. It is because first, the very definition of this enigmatic phenomenon is beyond the scope of this work. Second, and fortunately, those quantum computation and quantum information tasks requires the ability of only one- and two-qubit manipulations for which the following necessity condition is also sufficient.

For an N-particle system's state to possess N-partite entanglement it is necessary that their total density matrix, $\hat{\rho}^{(N)}$, can *not* be written in the form

$$\hat{\rho}^{(N)} = \sum_{ij} p_{ij}^{(k)} \hat{\rho}_i^{(k)} \otimes \hat{\rho}_j^{(N-k)}, \tag{2.27}$$

for any $k = 1 \cdots N-1$, where $p_{ij}^{(k)} \in [0,1]$ satisfy $\sum_{ij} p_{ij}^{(k)} = 1$.

Since it is a 'kind of' information between the corresponding parties of a total system, and is purely quantum in nature, in order to quantify the entanglement in a two qubit system, one has to answer the question "*how much quantum information does the state of the pair involve?*". Before this, the quantification of classical information will be stated, with which the quantum counterpart is closely related.
By knowing a random variable X it is meant that one has the complete information about X. Or, with the complementary perspective, before knowing X there was some uncertainty about X, and to know it one should remove this uncertainty. The amount of uncertainty, before having the complete knowledge about X, and equivalently the information gained, on the average, by knowing X, is given by the Shannon entropy, [44], defined for a probability distribution $\{p_x\}$ as

$$H(X) \equiv -\sum_x p_x \log_2 p_x, \tag{2.28}$$

where p_x is the probability for X to get the value x, x is a possible symbol for X, and '$\log_a$' is the logarithm function in base a.

Now, let $|\psi^{(AB)}\rangle$ be the pure state of a pair of quantum systems, A and B. The appearing state from the viewpoint of either observer can be obtained by tracing over the degrees of freedom of the other observer,

$$\hat{\rho}^A = Tr_B\,\hat{\rho}^{(AB)}, \qquad\qquad \hat{\rho}^B = Tr_A\,\hat{\rho}^{(AB)}, \tag{2.29}$$

where $\hat{\rho}^{(AB)} = |\psi^{(AB)}\rangle\langle\psi^{(AB)}|$, and Tr_K is the partial trace operation with respect to part K.

What this operation does is more than tracing over one party, it takes the quantum information, shared by the pair, out of the picture. That is, it creates an uncertainty (in some sense) which does not appear when the pair is considered as a total system. So it destroys the quantum information between A and B. The meaning of the general physical quantity *entropy*, "*available space per available states*", makes it again the best candidate for the quantification of this uncertainty. Because, ignoring one subsystem of the whole creates some amount of space for the information between the parties, which means that the parties share *this* information with some probability, or share *that* information with some other probability, etc. Such probabilities did not appear before. There were no room for more than one possibility, which is for the state of the whole at which the subsystems can share *only this* information.

This time von Neumann entropy is the proper choice [45],

$$\begin{aligned} S(\hat{\rho}^K) &\equiv -Tr\,\hat{\rho}^K \log_2 \hat{\rho}^K \\ &\equiv E(|\psi^{(AB)}\rangle). \end{aligned} \tag{2.30}$$

Here K can be either one of A and B, the result does not change. The amount of entanglement, E, is the same from point of view of either observer, as it should be;

$$S(\hat{\rho}^A) = S(\hat{\rho}^B). \tag{2.31}$$

In the spectral decomposition of its argument von Neumann entropy reduces to the Shannon entropy,

$$S(\hat{\rho}^K) = -\sum_j \rho_j^K \log_2 \rho_j^K, \tag{2.32}$$

with ρ_j^K is the j^{th} eigenvalue of the density matrix of the system K.

Bennett *et al.* [45] verified the validity of this quantification for entanglement as the following. Take n copies of the bi-partite pure state $|\psi^{(AB)}\rangle$. Let this n qubit pairs be reversibly convertible to maximum m copies of one of the completely entangled Bell states, i.e. to the singlets

$$|\beta_{xy}\rangle \equiv \frac{1}{\sqrt{2}}(|0,y\rangle + (-1)^x|1,1-y\rangle), \qquad x,y=0,1, \tag{2.33}$$

by purely local operations and classical communication (LOCC). Then, the entanglement of concentration for $|\psi^{(AB)}\rangle$ is given as

$$E_C(|\psi^{(AB)}\rangle) = \frac{m(n)}{n}. \tag{2.34}$$

They showed that for large n, $E_C(|\psi^{(AB)}\rangle)$ approaches $E(|\psi^{(AB)}\rangle)$,

$$\lim_{n\to\infty} \frac{m(n)}{n} = E(|\psi^{(AB)}\rangle). \tag{2.35}$$

Entanglement of concentration gives the number of singlets that can be extracted from a state. The complementary perspective also gives a measure for bi-partite entanglement; the minimum number of singlets required to create a state. This measure is called *entanglement of formation* (EoF) and given by Bennet *et al.* [46]. It is defined as the following. Let m' copies of singlets (2.33) be reversibly convertible to n copies of $|\psi^{(AB)}\rangle$ by LOCC. Then, EoF for $|\psi^{(AB)}\rangle$ is given as

$$E_F(|\psi^{(AB)}\rangle) = \frac{m'(n)}{n}. \tag{2.36}$$

In case the state in question is pure, distillation (E_C) and dilution (E_F) gives same amount of entanglement $S(\hat{\rho}^K)$. But for mixed states they need not be equal, [45–49]. Moreover, for mixed states, $S(\hat{\rho}^K)$ can not be a *measure* of entanglement, since it gives different values for different decompositions

$$\hat{\rho}^{(AB)} = \sum_j p_j |\psi_j^{(AB)}\rangle\langle\psi_j^{(AB)}|, \tag{2.37}$$

of the same state $\hat{\rho}^{(AB)}$. Here $|\psi_j^{(AB)}\rangle$ are the pure states of the pair (AB), and p_j are the corresponding weights for the decomposition in hand, satisfying $\sum_j p_j = 1$.

Depending on this argument Wootters *et al.* [50,51] introduced a well defined definition for the quantification of entanglement of a bi-partite state $\hat{\rho}^{(AB)}$ as
"*the average entanglement of the pure states of the decomposition of the state $\hat{\rho}^{(AB)}$, minimized over all decompositions* ",

$$E(\hat{\rho}^{(AB)}) \equiv \min \sum_j p_j E(|\psi_j^{(AB)}\rangle), \tag{2.38}$$

where

$$E(|\psi_j^{(AB)}\rangle) \equiv S(|\psi_j^{(AB)}\rangle\langle\psi_j^{(AB)}|). \tag{2.39}$$

In case the parties A and B are *two-level systems*, it is possible to find a closed form for the amount of entanglement (2.38) as a function of their density matrix. First we define *concurrence* as

$$C(\hat{\rho}^{(AB)}) \equiv \max(0, 2\lambda_{max} - Tr\, R), \tag{2.40}$$

where λ_{max} is the maximum of the eigenvalues of the matrix

$$R = |\hat{\rho}^{(AB)} Y \otimes Y \hat{\rho^*}^{(AB)}|^{1/2}, \tag{2.41}$$

with Y given by (2.5) in the basis B_1 (2.4) is the Pauli matrix, σ_y, and $\hat{\rho^*}^{(AB)}$ is the complex conjugate of $\hat{\rho}^{(AB)}$ with respect to the basis B_2 (2.4). They proved that $E(\hat{\rho}^{(AB)})$, (2.38), can be given in the closed form

$$E(\hat{\rho}^{(AB)}) = h\left(\frac{1+\sqrt{1-C^2}}{2}\right), \tag{2.42}$$

with

$$h(x) \equiv -[x \log_2 x + (1-x)\log_2(1-x)]. \tag{2.43}$$

CHAPTER 3

EVOLUTION OF SYSTEMS IN INTERACTION WITH THE ENVIRONMENT

The quantum superposition of a system interacting with an environment decays, *in general*, into statistical mixtures. This phenomenon is called *decoherence*. For quantum computational and quantum informational tasks decoherence is one of the most important limiting factors. So, one has to deal with it. The reason for the word *in general* is that, the state of a system *may* evolve in a subspace of its total Hilbert space without decohering. Such a subspace is called *decoherence free* [52, 53]. Such subspaces may be obtained even with adiabatically [54] manipulating the environment [55]. Hodges *et al.* reported in [56] an experimental implementation of conditional gate, CNOT gate, between two qubits, each of which lies in their decoherence free (DF) subspaces. The gate is capable of creating entanglement between those qubits. Since they are in DF subspaces it is possible to control these qubits coherently so that the entanglement between them can live long enough.

But for more realistic cases we need to know the effects of environments on the tools that are needed to be used in computational and informational processes. For this purpose we have studied the effects of an environment at different temperatures and the effects of electromagnetic field in squeezed state that drives the environment in which our candidate systems live.

In this chapter some important properties of different kinds of reservoirs will be given. These reservoirs constitute possible environments for the quantum systems to live in and to interact with. Most of the notations and definitions used in the sections which follow are taken from [57] and [58].

3.1 Thermal States

In thermal equilibrium at temperature T the state of a system with Hamiltonian $\hat{H}$ is represented by the density matrix

$$\hat{\rho}_{thermal} = \frac{e^{-\frac{\hat{H}}{k_B T}}}{Tr\, e^{-\frac{\hat{H}}{k_B T}}} \tag{3.1}$$

where k_B is the Boltzmann constant. The only non-zero moments are those containing equal number of photon creation, $\hat{b}^\dagger$, and annihilation, $\hat{b}$, operators. First two of them are

$$\langle \hat{b}^\dagger(\omega^1)\hat{b}(\omega_1)\rangle_{\hat{\rho}_{thermal}} = \bar{n}(\omega_1)\delta(\omega^1 - \omega_1), \tag{3.2}$$

$$\begin{aligned}\langle \hat{b}^\dagger(\omega^1)\hat{b}^\dagger(\omega^2)\hat{b}(\omega_2)\hat{b}(\omega_1)\rangle_{\hat{\rho}_{thermal}} &= \bar{n}(\omega_1)\bar{n}(\omega_2)[\delta(\omega^1 - \omega_1)\delta(\omega^2 - \omega_2) - \\ &\qquad \delta(\omega^1 - \omega_2)\delta(\omega^2 - \omega_1)],\end{aligned} \tag{3.3}$$

where $\hbar$ is the Planck's constant divided 2π,

$$\bar{n}(\omega) = (e^{\frac{\hbar\omega}{k_B T}} - 1)^{-1} \tag{3.4}$$

is the average number of thermal photons with frequency ω, and $\hat{b}(\omega)$, $\hat{b}^\dagger(\omega)$ are the annihilation and the creation operators for the corresponding photons, satisfying

$$[\hat{b}(\omega), \hat{b}^\dagger(\omega')] = \delta(\omega - \omega'). \tag{3.5}$$

3.2 Squeezed States

Squeezed states are one of the most important types of states of radiation field. They are characterized by the property that the variance of a quadrature operator $\hat{x}_\lambda$, $\Delta\hat{x}_\lambda$ is less than 1/2. Here the operator $\hat{x}_\lambda$ is defined in terms of operators $\hat{b}^\dagger$ and $\hat{b}$ as

$$\hat{x}_\lambda \equiv \frac{1}{\sqrt{2}}(\hat{b}e^{-i\lambda} + \hat{b}^\dagger e^{i\lambda}), \tag{3.6}$$

with $[\hat{b}, \hat{b}^\dagger] = 1$, and $\lambda \in \mathbb{R}$ being a phase. And it is said that the quadrature $\hat{x}_\lambda$ is squeezed. Following the Heisenberg uncertainty relation,

$$\Delta\hat{x}_\lambda^2 \Delta\hat{x}_{\lambda+\pi/2}^2 \geq \frac{1}{4}|\langle[\hat{x}_\lambda, \hat{x}_{\lambda+\pi/2}]\rangle|^2 = \frac{1}{4}, \tag{3.7}$$

the variance in $\hat{x}_{\lambda+\pi/2}$ is greater than 1/2. Here $[\hat{x}_\lambda, \hat{x}_{\lambda+\pi/2}] = i$ is the commutator between quadrature operators. As an illustration with a physical quantity, consider a quantized single-mode electric field of frequency ν, amplitude E, and polarized in the direction $\hat{\epsilon}$,

$$\vec{E}(t) = \hat{\epsilon}E(\hat{b}e^{-i\nu t} + \hat{b}^\dagger e^{i\nu t}). \tag{3.8}$$

In terms of quadrature operators given above, $\vec{E}(t)$ is given as

$$\vec{E}(t) = \sqrt{2}\hat{\epsilon}E\left[\hat{x}_\lambda \cos(\lambda - \nu t) - \hat{x}_{\lambda+\pi/2} \sin(\lambda - \nu t)\right]. \tag{3.9}$$

Single mode squeezed states are generated by the action of the unitary squeezing operator $\hat{S}(\xi)$ on ordinary vacuum,

$$\begin{aligned} |\xi\rangle &\equiv \hat{S}(\xi)|0\rangle \\ &\equiv \exp\left(-\frac{\xi}{2}\hat{b}^{\dagger 2} + \frac{\xi^*}{2}\hat{b}^2\right)|0\rangle, \end{aligned} \tag{3.10}$$

with $\xi = r_\xi e^{i\phi_\xi} \in \mathbb{C}$, ξ^* the complex conjugate of ξ, and $r_\xi, \phi_\xi \in \mathbb{R}$ characterizing the squeezing.

The annihilation and creation operators $\hat{b}$ and $\hat{b}^\dagger$ transform by $\hat{S}(\xi)$ as

$$\hat{S}^\dagger(\xi)\hat{b}\hat{S}(\xi) = \hat{b}\cosh(r_\xi) - \hat{b}^\dagger e^{i\phi_\xi}\sinh(r_\xi), \tag{3.11}$$

$$\hat{S}^\dagger(\xi)\hat{b}^\dagger\hat{S}(\xi) = \hat{b}^\dagger\cosh(r_\xi) - \hat{b}\,e^{-i\phi_\xi}\sinh(r_\xi). \tag{3.12}$$

For two modes with annihilation operators $\hat{b}_1$ and $\hat{b}_2$, the two-mode squeezed vacuum state $|\xi_{12}\rangle$ is generated by the action of the two-mode squeezing operator

$$\hat{S}_{12}(\xi) \equiv \exp(-\xi\hat{b}_1^\dagger\hat{b}_2^\dagger + \xi^*\hat{b}_2\hat{b}_1), \tag{3.13}$$

on the vacuum $|0_1 0_2\rangle$. The operator $\hat{S}_{12}(\xi)$ is not simply the product of the single-mode squeezing operators for the modes 1 and 2. The generalization of the squeezed vacuum states to a continuum of modes is achieved by acting on the vacuum state $|0\rangle$ of the continuum field with the squeezing operator

$$\hat{S}[\xi(\omega)] = \exp\left(-\frac{1}{2}\int_\omega^{2\Omega} d\omega[\xi(\omega)\hat{b}^\dagger(\omega)\hat{b}^\dagger(2\Omega-\omega) - \xi^*(\omega)\hat{b}(2\Omega-\omega)\hat{b}(\omega)]\right), \tag{3.14}$$

where $\xi(\omega) = r_\xi(\omega)\exp[i\phi_\xi(\omega)]$, and 2Ω is the squeezing carrier frequency, which is the natural generalization of the two-mode squeezing operator (3.13). Since they multiply the same pairs in the integrand, we can set, without loss of generality, $\xi(\omega)$ and $\xi(2\Omega-\omega)$ as equal, i.e. $r(\omega) = r(2\Omega-\omega)$ and $\phi(\omega) = \phi(2\Omega-\omega)$. It is then said that the field is squeezed at frequency Ω. The statistical properties of the continuum squeezed vacuum state

$$|\{\xi(\omega)\}\rangle = \hat{S}[\xi(\omega)]|0\rangle \tag{3.15}$$

can be found with the aid of the transformations

$$\begin{aligned}\hat{S}^\dagger[\xi(\omega)]\,\hat{b}(\omega')\,\hat{S}[\xi(\omega)] = \hat{b}(\omega')\,\cosh(r_\xi(\omega'))-\\ \hat{b}^\dagger(2\Omega-\omega')\,\exp[i\phi_\xi(\omega')]\,\sinh(r_\xi(\omega')),\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}\hat{S}^\dagger[\xi(\omega)]\,\hat{b}^\dagger(\omega')\,\hat{S}[\xi(\omega)] = \hat{b}^\dagger(\omega')\,\cosh(r_\xi(\omega'))-\\ \hat{b}(2\Omega-\omega')\,\exp[-i\phi_\xi(\omega')]\,\sinh(r_\xi(\omega')).\end{aligned} \tag{3.17}$$

The lowest order moments in the continuum squeezed vacuum are

$$\langle\hat{b}(\omega)\rangle_{\xi(\omega)} = 0, \tag{3.18}$$

$$\langle\hat{b}^\dagger(\omega)\rangle_{\xi(\omega)} = 0, \tag{3.19}$$

$$\langle\hat{b}^\dagger(\omega)\hat{b}(\omega')\rangle_{\xi(\omega)} = \sinh^2 r_\xi(\omega)\;\delta(\omega-\omega'), \tag{3.20}$$

$$\langle\hat{b}(\omega)\hat{b}^\dagger(\omega')\rangle_{\xi(\omega)} = \cosh^2 r_\xi(\omega)\;\delta(\omega-\omega'), \tag{3.21}$$

$$\langle\hat{b}(\omega)\hat{b}(\omega')\rangle_{\xi(\omega)} = -e^{[i\phi_\xi(\omega)]}\;\sinh\,r_\xi(\omega)\;\cosh\,r_\xi(\omega)\;\delta(\omega+\omega'-2\Omega), \tag{3.22}$$

$$= \langle\hat{b}^\dagger(\omega)\hat{b}^\dagger(\omega')\rangle^*_{\xi(\omega)}, \tag{3.23}$$

where (3.20) defines the photon number function $N(\omega) = N(2\Omega-\omega)$, and (3.22) defines the two-photon correlation function $M(\omega) = M(2\Omega-\omega)$.

3.3 Coherent States

Coherent states are another important type of radiation field states. Single mode coherent state is defined as

$$|\alpha\rangle \equiv \hat{D}(\alpha)|0\rangle \tag{3.24}$$

$$\equiv \exp(\alpha\hat{b}^\dagger - \alpha^*\hat{b})|0\rangle \tag{3.25}$$

$$= \exp\left(-|\alpha|^2/2\right)\exp(\alpha\hat{b}^\dagger)\exp(-\alpha^*\hat{b})|0\rangle, \tag{3.26}$$

where α is any complex number. Here, $|\alpha\rangle$ is an eigenstate of the annihilation operator with eigenvalue α,

$$\hat{b}|\alpha\rangle = \alpha|\alpha\rangle, \tag{3.27}$$

and it follows from (3.26) that

$$|\alpha\rangle = \exp\left(-|\alpha|^2/2\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{(n!)}}|n\rangle, \tag{3.28}$$

where the single-mode number states $|n\rangle$ are the eigenatates of the number operator $\hat{n}$ with eigenvalues n. So, for a radiation field in a coherent state $|\alpha\rangle$ the mean number of photons is

$$\begin{aligned} \langle\hat{n}\rangle_\alpha &\equiv \langle\alpha|\hat{b}^\dagger\hat{b}|\alpha\rangle \\ &= |\alpha|^2, \end{aligned} \tag{3.29}$$

and the variance of $\hat{n}$ is easily calculated to be equal to it

$$\begin{aligned} \Delta n^2 &\equiv \langle\hat{n}^2\rangle_\alpha - \langle\hat{n}\rangle_\alpha^2 \\ &= |\alpha|^2. \end{aligned} \tag{3.30}$$

Their equivalence is a characteristic of the *Poissonian statistics*, namely the photon number probability distribution $P(n)$ for the coherent state $|\alpha\rangle$ is

$$\begin{aligned} P(n) &\equiv |\langle n|\alpha\rangle|^2 \\ &= \exp\left(-|\alpha|^2\right)\frac{|\alpha|^{2n}}{n!}. \end{aligned} \tag{3.31}$$

Another important property of coherent states to be mentioned is that, for all coherent states the variances of the quadrature operator, $\hat{x}_\lambda$, defined by (3.6), and its conjugate component, $\hat{x}_{\lambda+\pi/2}$, are same and equal to 1/2,

$$\Delta\hat{x}_\lambda^2 = \frac{1}{2} = \Delta\hat{x}_{\lambda+\pi/2}^2. \tag{3.32}$$

It follows that *coherent states are minimum uncertainty states.*

Under the action of the unitary operator $\hat{D}(\alpha)$ defined in (3.24) the annihilation and the creation operators $\hat{b}$ and $\hat{b}^\dagger$ transform as follows

$$\begin{aligned} \hat{D}^\dagger(\alpha)\hat{b}\hat{D}(\alpha) &= \hat{b}+\alpha && (3.33)\\ \hat{D}^\dagger(\alpha)\hat{b}^\dagger D(\alpha) &= \hat{b}^\dagger+\alpha^*. && (3.34)\end{aligned}$$

With this property the operator $\hat{D}(\alpha)$ takes its name as *the displacement operator.* Accordingly, continuum displacement operator $\hat{D}[\alpha(\omega)]$ acts on the continuum annihilation and creation operators as

$$\begin{aligned} \hat{D}^\dagger[\alpha(\omega)]\hat{b}(\omega^{'})\hat{D}[\alpha(\omega)] &= \hat{b}(\omega^{'})+\alpha(\omega^{'}) && (3.35)\\ \hat{D}^\dagger[\alpha(\omega)]\hat{b}^\dagger(\omega^{'})\hat{D}[\alpha(\omega)] &= \hat{b}^\dagger(\omega^{'})+\alpha^*(\omega^{'}). && (3.36)\end{aligned}$$

It defines the continuum coherent state

$$\begin{aligned} |\{\alpha(\omega)\}\rangle &\equiv \hat{D}[\alpha(\omega)]|0\rangle \\ &\equiv \exp\left(\int d\omega[\alpha(\omega)\hat{b}^\dagger(\omega)-\alpha^*(\omega)\hat{b}(\omega)]\right)|0\rangle, && (3.37)\end{aligned}$$

which is an eigenstate of the continuum annihilation operator $\hat{b}(\omega)$, so is an eigenstate of the boson annihilation operator

$$\hat{b}_f \equiv \int d\omega\; f^*(\omega)\,\hat{b}(\omega) \tag{3.38}$$

with eigenvalue

$$\alpha_f = \int d\omega\; f^*(\omega)\,\alpha(\omega), \tag{3.39}$$

where $f(\omega)$ is any complex function satisfying the normalization condition

$$\int d\omega|f(\omega)|^2 = 1. \tag{3.40}$$

The expectation value of the normal ordered product $(\hat{b}_f^\dagger)^l(\hat{b}_f)^m$ in the continuum coherent state is

$$\langle\{\alpha(\omega)\}|(\hat{b}_f^\dagger)^l(\hat{b}_f)^m|\{\alpha(\omega)\}\rangle = (\alpha_f^*)^l(\alpha_f)^m. \tag{3.41}$$

For $l=m$

$$\langle(\hat{b}_f^\dagger)^l(\hat{b}_f)^l\rangle_{\alpha(\omega)} = \langle\hat{b}_f^\dagger\hat{b}_f\rangle^l_{\alpha(\omega)},$$

showing that *statistics associated with quanta annihilated by* $\hat{b}_f$ *are Poissonian* i.e. *the variance is equal to the mean.*

3.4 System, Environment, and Their Interaction

The Hamiltonians for the system, environment and their interaction are given as $\hat{H}_S$, $\hat{H}_R$ and $\hat{H}_{RS}$, respectively,

$$\begin{aligned}\hat{H} &= \hat{H}_S + \hat{H}_R + \hat{H}_{RS} \\ &\equiv \hat{H}_0 + \hat{H}_{RS}.\end{aligned} \tag{3.42}$$

Environment will be assumed to be a bath of harmonic oscillators with creation operator $\hat{b}(\omega)^\dagger$ and annihilation operator $\hat{b}(\omega)$ for the ω-mode,

$$\hat{H}_R = \hbar \int \omega \hat{b}^\dagger(\omega)\hat{b}(\omega)d\omega. \tag{3.43}$$

The system is a collection of $2-$level systems each of which will be characterized by the dipole operators

$$\begin{aligned}\hat{s}_j^+ &\equiv |e_j\rangle\langle g_j|, \\ \hat{s}_j^- &\equiv |g_j\rangle\langle e_j|, \\ \hat{s}_j^z &\equiv \frac{1}{2}(|e_j\rangle\langle e_j| - |g_j\rangle\langle g_j|),\end{aligned} \tag{3.44}$$

with the transition frequency ω_j between the excited state, $|e_j\rangle$, and the ground state, $|g_j\rangle$, of the j^{th} $2-$level system. The dipole operators (3.44) satisfy the well-known commutation

$$\begin{aligned}&[\hat{s}_j^+, \hat{s}_k^-] = 2\hat{s}_j^z \delta_{jk}, \\ &[\hat{s}_j^z, \hat{s}_k^\pm] = \pm \hat{s}_j^\pm \delta_{jk},\end{aligned} \tag{3.45}$$

and anti-commutation

$$\{\hat{s}_j^+, \hat{s}_k^-\} = \delta_{jk} \tag{3.46}$$

relations, with $(\hat{s}_j^\pm)^2 = 0$.

The Hamiltonian of the J $2-$level systems is

$$\hat{H}_S = \hbar \sum_{j=1}^{J} \omega_j \hat{s}_j^z. \tag{3.47}$$

The interaction Hamiltonian $\hat{H}_{RS}$ in Schrödinger picture is assumed to be as

$$\hat{H}_{RS} = \hbar \sum_{j=1}^{J} \int_{-\infty}^{\infty} d\omega g_j(\omega) \hat{s}_j^+ \hat{b}(\omega) \; + h.c. \tag{3.48}$$

where $g_j(\omega)$, depending on ω_j and ω, is the coupling coefficient between the j^{th} 2-level system and the ω-mode of the reservoir, and $h.c.$ indicates the hermitian conjugate of the previous term.

The total (system + environment) is represented by the density matrix $\hat{\rho}_{RS}$ for which the equation of motion in the interaction picture reads

$$\frac{d}{dt}\hat{\rho}_{RS}(t) \equiv \dot{\hat{\rho}}_{RS}(t) = -\frac{i}{\hbar}[\hat{V}(t), \hat{\rho}_{RS}(t)], \tag{3.49}$$

where

$$\hat{V}(t) = e^{\frac{i}{\hbar}\hat{H}_0 t}\hat{H}_{RS}\, e^{-\frac{i}{\hbar}\hat{H}_0 t} \tag{3.50}$$

is the interaction Hamiltonian in the interaction picture.

Integrating (3.49) and inserting the result again in (3.49) gives

$$\dot{\hat{\rho}}_{RS}(t) = -\frac{i}{\hbar}[\hat{V}(t), \hat{\rho}_{RS}(0)] - \frac{1}{\hbar^2}\int_0^t [\hat{V}(t),\ [\hat{V}(t')\ , \hat{\rho}_{RS}(t')]\ ]\ dt'. \tag{3.51}$$

Since it has infinitely many degrees of freedom, evolution resulting from coupling with the finite system does not affect the environment significantly. This enables us to assume that the reservoir, represented by the density matrix $\hat{\rho}_R$, remains in its initial state during this evolution, i.e.

$$\hat{\rho}_R(t) = \hat{\rho}_R(0).$$

Ignoring the terms not affecting the dynamics, total density matrix can be written as

$$\hat{\rho}_{RS}(t) \simeq \hat{\rho}_S(t) \otimes \hat{\rho}_R(0).$$

Here $\hat{\rho}_S(t)$ is the system's density operator, and is obtained from $\hat{\rho}_{RS}(t)$ by tracing out the reservoir's degrees of freedom,

$$\hat{\rho}_S(t) = Tr_R\, \hat{\rho}_{RS}(t).$$

So, the equation of motion for $\hat{\rho}_S(t)$ is

$$\dot{\hat{\rho}}_S(t) = -\frac{i}{\hbar}Tr_R[\hat{V}(t), \hat{\rho}_S(0) \otimes \hat{\rho}_R(0)] - \frac{1}{\hbar^2}Tr_R\int_0^t [\hat{V}(t),\ [\hat{V}(t')\ , \hat{\rho}_S(t') \otimes \hat{\rho}_R(0)]\ ]\ dt'. \tag{3.52}$$

Substituting the explicit form of the interaction picture Hamiltonian $\hat{V}(t)$,

$$\hat{V}(t) = \hbar\sum_{j=1}^{J}\int_{-\infty}^{\infty} d\omega\, \left(g_j(\omega)\hat{s}_j^{+}\hat{b}(\omega)e^{i(\omega_j-\omega)t}\right)\ + h.c., \tag{3.53}$$

into (3.52) one gets

$$\dot{\hat{\rho}}_S(t) = \{-i\sum_{j=1}^{J}[\hat{s}_j^+, \hat{\rho}_S(0)]\int_{-\infty}^{\infty} d\omega\left(g_j(\omega)e^{i(\omega_j-\omega)t}\langle\hat{b}(\omega)\rangle_R\right)$$

$$+\int_0^t dt'\int_{-\infty}^{\infty} d\omega'\int_{-\infty}^{\infty} d\omega\sum_{j=1}^{J}\sum_{j'=1}^{J}$$

$$(g_j(\omega)g_{j'}(\omega')e^{i(\omega_j-\omega)t}e^{i(\omega_{j'}-\omega')t'}$$

$$\times[\,(\hat{s}_{j'}^+\hat{\rho}_S(t')\hat{s}_j^+ - \hat{s}_j^+\hat{s}_{j'}^+\hat{\rho}_S(t'))\langle\hat{b}(\omega)\hat{b}(\omega')\rangle_R$$

$$+(\hat{s}_j^+\hat{\rho}_S(t')\hat{s}_{j'}^+ - \hat{\rho}_S(t')\hat{s}_{j'}^+\hat{s}_j^+)\langle\hat{b}(\omega')\hat{b}(\omega)\rangle_R\,]$$

$$+g_j(\omega)g_{j'}^*(\omega')e^{i(\omega_j-\omega)t}e^{-i(\omega_{j'}-\omega')t'}$$

$$\times[\,(\hat{s}_{j'}^-\hat{\rho}_S(t')\hat{s}_j^+ - \hat{s}_j^+\hat{s}_{j'}^-\hat{\rho}_S(t'))\langle\hat{b}(\omega)\hat{b}^+(\omega')\rangle_R$$

$$+(\hat{s}_j^+\hat{\rho}_S(t')\hat{s}_{j'}^- - \hat{\rho}_S(t')\hat{s}_{j'}^-\hat{s}_j^+)\langle\hat{b}^+(\omega)\hat{b}(\omega')\rangle_R\,])\}$$

$$+\,h.c., \tag{3.54}$$

where $\langle operator\rangle_R$ means the expectation value of the reservoir *operator*.

In the next step, one more assumption will be taken into account. In order to illustrate this assumption explicitly, we write the second term in (3.54) that contains time integral as follows,

$$\hat{\rho}_{2.}(t) = \int_{-\infty}^{\infty} d\beta\hat{f}(\beta)\int_0^t dt'\hat{\rho}(t')e^{i(\beta-\beta_0)(t-t')}. \tag{3.55}$$

Here we confine all the terms, which are not necessary for this illustration, into the definitions of new variables. If $\hat{f}(\beta)$ is a slowly varying function function of β, then the β-integration would yield a strongly localized function in time at $t' = t$. This function approximates a Dirac-delta function the width of which depends on the β-rate-of-change-of-$\hat{f}$. In this case $\hat{\rho}(t')$ in (3.55) can be taken outside the integral as $\hat{\rho}(t)$,

$$\hat{\rho}_{2.}(t) \simeq \hat{\rho}(t)\int_{-\infty}^{\infty} d\beta\hat{f}(\beta)\int_0^t dt'e^{i(\beta-\beta_0)(t-t')}. \tag{3.56}$$

Since in the t' integration the contribution will be prominently at $t' \cong t$, the upper limit can be extended to infinity,

$$\hat{\rho}_{2.}(t) \simeq \hat{\rho}(t)\int_{-\infty}^{\infty} d\beta\hat{f}(\beta)\int_0^{\infty} dt'e^{i(\beta-\beta_0)(t-t')}. \tag{3.57}$$

Performing these integrals would yield as follows,

$$\hat{\rho}_{2.}(t) \simeq \hat{\rho}(t) \int_{-\infty}^{\infty} d\beta \hat{f}(\beta) \left(\pi\delta(\beta - \beta_0) + i\mathcal{P}\frac{1}{(\beta - \beta_0)} \right) \quad (3.58)$$

$$= \hat{\rho}(t) \left(\pi \hat{f}(\beta_0) + i\mathcal{P} \int_{-\infty}^{\infty} d\beta \frac{\hat{f}(\beta)}{(\beta - \beta_0)} \right), \quad (3.59)$$

where $\mathcal{P}$ denotes the principal part of the integral,

$$\mathcal{P} \int_{-\infty}^{\infty} d\beta \frac{\hat{f}(\beta)}{(\beta - \beta_0)} \equiv \lim_{\delta \to 0} \left(\int_{-\infty}^{\beta_0 - \delta} d\beta \frac{\hat{f}(\beta)}{(\beta - \beta_0)} + \int_{\beta_0 + \delta}^{\infty} d\beta \frac{\hat{f}(\beta)}{(\beta - \beta_0)} \right). \quad (3.60)$$

This principal part integral yields a shift in the natural frequency, β_0, of the system. Practically by redefining the natural frequency, these shifts can be incorporated into the free Hamiltonian. Thus, since it does not effect the dynamics of the system, we will ignore these shifts.

The physical motivation behind this approximation is that, the large number of reservoir degrees of freedom causes a damping on the system. And this damping destroys the memory of the past. It is the so called Markov approximation. Now we define the reservoir in which the Markov approximation will be assumed. The environment is taken to be a continuum in a thermal bath that is driven by a squeezed coherent field, i.e.

$$\hat{\rho}_R = \hat{S}[\xi(\omega)]\, \hat{D}[\alpha(\omega)]\, \hat{\rho}_{thermal}\, \hat{S}^{\dagger}[\xi(\omega)]\, \hat{D}^{\dagger}[\alpha(\omega)] \quad (3.61)$$

The operator moments in (3.54) can easily be calculated for this reservoir with the aid of the displacement and the squeezing operators given in (3.16, 3.35). They are listed below:

$$\langle \hat{b}(\omega) \rangle_R = \alpha(\omega) \cosh r(\omega) - \alpha^*(2\Omega - \omega) e^{i\phi(\omega)} \sinh r(\omega), \quad (3.62)$$

$$\begin{aligned}
\langle \hat{b}(\omega)\hat{b}(\omega') \rangle_R = {} & (\alpha(\omega) \cosh r(\omega) - \alpha^*(2\Omega - \omega) \sinh r(\omega) e^{i\phi(\omega)}) \\
& \times (\alpha(\omega') \cosh r(\omega') - \alpha^*(2\Omega - \omega') \sinh r(\omega') e^{i\phi(\omega')}) \\
& - [(n(\omega) + 1) \cosh r(\omega) \sinh r(\omega') e^{i\phi(\omega')} \\
& \quad + n(\omega') \sinh r(\omega) \cosh r(\omega') e^{i\phi(\omega)}] \delta(\omega + \omega' - 2\Omega) \\
= {} & \langle b(\omega) \rangle \langle b(\omega') \rangle \\
& - [(n(\omega) + 1) \cosh r(\omega) \sinh r(\omega') e^{i\phi(\omega')}
\end{aligned}$$

$$+\,n(\omega')\sinh r(\omega)\cosh r(\omega')e^{i\phi(\omega)}]\delta(\omega+\omega'-2\Omega), \quad (3.63)$$

$$\begin{aligned}
\langle \hat{b}(\omega)\hat{b}^{\dagger}(\omega')\rangle_R &= (\alpha(\omega)\cosh r(\omega) - \alpha^*(2\Omega-\omega)\sinh r(\omega)e^{i\phi(\omega)}) \\
&\quad \times(\alpha^*(\omega')\cosh r(\omega') - \alpha(2\Omega-\omega')\sinh r(\omega')e^{-i\phi(\omega')}) \\
&\quad +[(n(\omega)+1)\cosh r(\omega)\cosh r(\omega') \\
&\quad +n(2\Omega-\omega)\sinh r(\omega)\sinh r(\omega')e^{i(\phi(\omega)-\phi(\omega'))}]\delta(\omega-\omega') \\
&= \langle b(\omega)\rangle\langle b^{\dagger}(\omega')\rangle \\
&\quad +[(n(\omega)+1)\cosh r(\omega)\cosh r(\omega') \\
&\quad +n(2\Omega-\omega)\sinh r(\omega)\sinh r(\omega')e^{i(\phi(\omega)-\phi(\omega'))}]\delta(\omega-\omega'),
\end{aligned} \quad (3.64)$$

$$\begin{aligned}
\langle \hat{b}^{\dagger}(\omega)\hat{b}(\omega')\rangle_R &= (\alpha(\omega')\cosh r(\omega') - \alpha^*(2\Omega-\omega')\sinh r(\omega')e^{i\phi(\omega')}) \\
&\quad \times(\alpha^*(\omega)\cosh r(\omega) - \alpha(2\Omega-\omega)\sinh r(\omega)e^{-i\phi(\omega)}) \\
&\quad +[(n(2\Omega-\omega)+1)\sinh r(\omega)\sinh r(\omega')e^{-i(\phi(\omega)-\phi(\omega'))} \\
&\quad +n(\omega)\cosh r(\omega)\cosh r(\omega')]\delta(\omega-\omega') \\
&= \langle b^{\dagger}(\omega)\rangle\langle b(\omega')\rangle \\
&\quad +[(n(2\Omega-\omega)+1)\sinh r(\omega)\sinh r(\omega')e^{-i(\phi(\omega)-\phi(\omega'))} \\
&\quad +n(\omega)\cosh r(\omega)\cosh r(\omega')]\delta(\omega-\omega').
\end{aligned} \quad (3.65)$$

In the next chapter we will consider different systems in specified environments. For these systems we will find the solution for the equation (3.54) by taking into account the Markov approximation.

CHAPTER 4

EFFECTS ON GEOMETRIC PHASE AND ENTANGLEMENT DUE TO THE INTERACTION BETWEEN CARRIER SYSTEMS AND ENVIRONMENT

4.1 Geometric Phase

As it was mentioned in chapter two a full understanding of the nature of quantum geometric phase in different environments is vital for quantum computations. Because it is proposed to implement the controlled gates, [13,56], and these gates are of crucial importance for a universal set of quantum logic gates. In this section the analysis of the environmental characteristics affecting the geometric phases will be given.

The effects of such environments on the geometric phase that will be gained by the state of an open quantum system was studied before.

Wang *et al.* [59] analyzed the effects of a squeezed vacuum reservoir on geometric phase of a two-level atom in an electromagnetic field by a formulation entirely in terms of geometric structures. Carollo *et al.* [55] showed that geometric phase can be induced by cyclic evolution in an adiabatically manipulated the environment, which is again a squeezed vacuum reservoir.

Banerjee and Srikanth studied the effects of a squeezed-thermal environment on the geometric phase of a two-level atom, [60]. In this study two types of interaction between the two-level atom and the environment was considered; one is quantum non-demolitional and in the other weak Born-Markov approximation is considered.

Rezakhani and Zanardi analyzed the temperature effects on mixed-state geometric phase for one and two coupled spin-1/2 particles, [26].

In this work the geometric phase gained by a two-level nucleus in a mixed state, which is evolving non-unitarily, viz. in interaction with the environment, is studied. The effects of temperature, magnetic field and squeezing is analyzed in the time evolution of the system. The relationships between those effects is also examined.

Our specific system is **a single spin** $1/2$ **nucleus in a magnetic field** $\vec{B}$. We assume that the static magnetic is in the direction of the quantization axis which is assumed to define the $z-$axis, i.e. $\vec{B} = B\hat{z}$. In this case the Hamiltonian of the system is

$$\begin{aligned} \hat{H}_S &= -\vec{\hat{\mu}} \cdot \vec{B} \\ &= -\hbar\gamma_n B\hat{I}^z, \end{aligned} \tag{4.1}$$

where γ_n is the gyromagnetic ratio of the nucleus, and $\hat{I}^z$ is the $z-$component of the spin operator $\hat{I}$ of the nucleus,

$$\hat{I}^z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.2}$$

Defining $\omega_{(n)} \equiv \gamma_n B_z$, we have the interaction picture Hamiltonian as

$$V(t) = \hbar\int_{-\infty}^{\infty} d\omega[g_n(\omega)e^{-i(\omega_{(n)}+\omega)t}\hat{I}^+\hat{b}(\omega) + h.c.], \tag{4.3}$$

where $\hat{I}^\pm$ defined in terms of the $x-$ and $y-$components of spin operator $\hat{I}$ as $\hat{I}^\pm \equiv \hat{I}^x \pm i\hat{I}^y$, and satisfy the commutation relations

$$[\hat{I}^z, \hat{I}^\pm] = \pm\hat{I}^\pm, \tag{4.4}$$

$$[\hat{I}^+, \hat{I}^-] = 2\hat{I}^z. \tag{4.5}$$

4.1.1 Temperature and Applied Magnetic Field Dependencies of Geometric Phase

In order to analyze the effects of only temperature and magnetic field in time we set $\alpha = r = \phi = 0$ in equations (3.62-3.65). The operator moments in this case are

$$\begin{aligned}
\langle \hat{b}(\omega) \rangle_R &= 0 \\
\langle \hat{b}(\omega)\hat{b}(\omega') \rangle_R &= 0 \\
\langle \hat{b}(\omega)\hat{b}^\dagger(\omega') \rangle_R &= (n(\omega)+1)\delta(\omega-\omega') \\
\langle \hat{b}(\omega)\hat{b}^\dagger(\omega') \rangle_R &= n(\omega)\delta(\omega-\omega')
\end{aligned} \tag{4.6}$$

Inserting equations (4.3) and (4.6) into equation (3.54) and performing the integrals by taking into account the Markov approximation, we get the equation of motion for the density matrix of the nucleus as

$$\begin{aligned}
\dot{\hat{\rho}}_S(t) = \pi g^2 \{ \, (n(\omega_{(n)})+1)[\,(\hat{I}^-\hat{\rho}_S(t)\hat{I}^+ - \hat{I}^+\hat{I}^-\hat{\rho}_S(t)) \\
+ n(\omega_{(n)})(\hat{I}^+\hat{\rho}_S(t)\hat{I}^- - \hat{\rho}_S(t)\hat{I}^-\hat{I}^+)\,]\} \; + h.c. \; ,
\end{aligned} \tag{4.7}$$

where

$$n(\omega_{(n)}) \equiv n(B,T) = \left(\frac{\hbar\gamma_n}{k_B T} B - 1 \right)^{-1} , \tag{4.8}$$

T being the temperature of the environment. Here the coupling coefficient, $g_n(\omega)$, between the nucleus and the $\omega-$mode of the reservoir is taken to be constant, g.

We obtained the solution of the equation (4.7) as

$$\rho_{S11}(t) = \frac{n(\omega_n)}{2n(\omega_n)+1} + \left(\rho_{S11}(0) - \frac{n(\omega_n)}{2n(\omega_n)+1} \right) e^{-(4n(\omega_n)+2)\pi g^2} \tag{4.9}$$

$$\rho_{S12}(t) = \rho_{S12}(0) e^{-(2n(\omega_n)+1)\pi g^2} , \tag{4.10}$$

where $\rho_{Sij}(t)$ is the ij-th entity of the matrix $\hat{\rho}_S(t)$. In terms of these, the eigenvalues and the normalized eigenvectors of $\hat{\rho}_S(t)$ are

$$\lambda_{S\pm}(t) = \frac{1}{2}\left(1 \pm \sqrt{1 - 4(\rho_{S11}(t)(1-\rho_{S11}(t)) - |\rho_{S11}(t)|^2)} \right) , \tag{4.11}$$

$$|\rho_{S\pm}(t)\rangle \doteq N_\pm(t) \begin{pmatrix} \rho_{S12}(t) e^{-i\omega_{(n)}t} \\ (\lambda_{S\pm}(t) - \rho_{S12}(t)) e^{i\omega_{(n)}t} \end{pmatrix} , \tag{4.12}$$

with the normalization coefficient

$$N_{\pm}(t) = (|\rho_{S12}(t)|^2 + (\lambda_{S\pm}(t) - \rho_{S12}(t))^2)^{-1/2}. \quad (4.13)$$

Since $\hat{\rho}_S(t)$ is the density matrix in an interaction picture, and since geometric phase expression (2.25) assumes Schrödinger picture, we have transformed the eigenvectors of $\hat{\rho}_S(t)$ into Schödinger picture. The last term in the geometric phase (GP) expression (2.25), which is the exponent, can be expressed in terms of (4.9) and (4.10) as follows,

$$\begin{aligned} &-\int_0^t \langle \rho_{S\pm}(t')|\dot{\rho}_{S\pm}(t')\rangle dt' = \\ &\quad -i\int_0^t \frac{\mathrm{Im}[\dot{\rho}_{S12}(t')\rho^*_{S12}(t')] - \omega_n \left[|\rho_{S12}(t')|^2 - (\lambda_{S\pm}(t') - \rho_{S11}(t'))^2\right]}{|\rho_{S12}(t')|^2 - (\lambda_{S\pm}(t') - \rho_{S11}(t'))^2} dt'. \end{aligned} \quad (4.14)$$

We calculated (4.14) for the system in hand, and obtained explicitly

$$\begin{aligned} -\int_0^t \langle \rho_{S\pm}(t')|\frac{d}{dt'}|\rho_{S\pm}(t')\rangle dt' &= i\omega_{(n)}\left(\frac{|\rho_{S12}(0)|^2}{2\alpha\gamma+\beta} - 1\right)t \\ \pm i\omega_{(n)}\frac{|\rho_{S12}(0)|^2}{(4n(\omega_{(n)})+2)\pi g^2(2\alpha\gamma+\beta)}\Bigg[&\ \mathrm{Arcsinh}\left(\frac{2\alpha^2 e^{(4n(\omega_{(n)})+2)\pi g^2 t} + \beta}{\sqrt{4\alpha^2\gamma^2 - \beta^2}}\right) \\ &- \mathrm{Arcsinh}\left(\frac{2\alpha^2 + \beta}{\sqrt{4\alpha^2\gamma^2 - \beta^2}}\right) \\ &- \mathrm{Arcsinh}\left(\frac{2\gamma^2 e^{-(4n(\omega_{(n)})+2)\pi g^2 t} + \beta}{\sqrt{4\alpha^2\gamma^2 - \beta^2}}\right) \\ &+ \mathrm{Arcsinh}\left(\frac{2\gamma^2 + \beta}{\sqrt{4\alpha^2\gamma^2 - \beta^2}}\right)\Bigg], \end{aligned} \quad (4.15)$$

where

$$\alpha \equiv \frac{1}{4n(\omega_n)+2}, \quad (4.16)$$

$$\gamma \equiv \rho_{S11}(0) - \frac{n(\omega_n)}{2n(\omega_n)+1}, \quad (4.17)$$

$$\beta \equiv |\rho_{S12}(0)|^2 - \gamma. \quad (4.18)$$

Now, with equations (4.11), (4.12) and (4.15) we are able to calculate the GP, Φ_G, of the system. We will analyze the temperature and the magnetic field dependence of Φ_G as it evolves in time. For this purpose we first define new dimensionless parameters for time, magnetic field, and temperature as $g^2\,t$, $\frac{\gamma_n}{g^2}B$, $\frac{k_B}{g^2\hbar}T$, respectively. We take an arbitrary initial state, defined by $\rho_{S11}(0) = 0.6$ and $\rho_{S12}(0) = 0.3$, and evaluate the time evolution of GP.

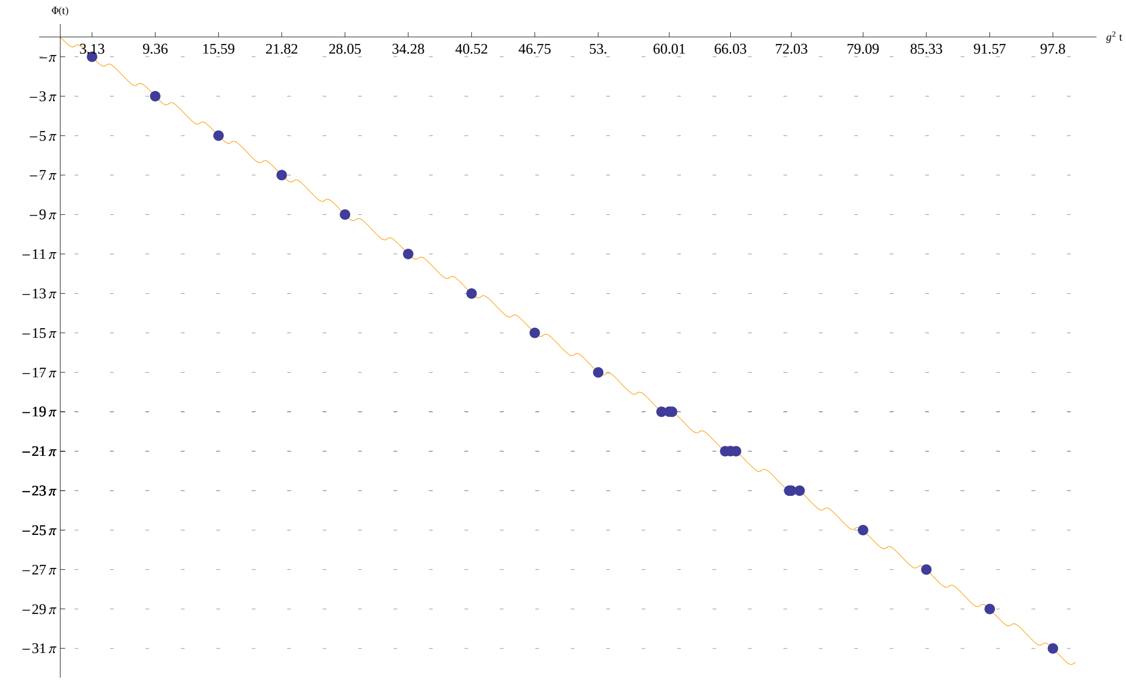


Figure 4.1: GP (Φ_G, in radians), versus time. Time, t, is in unit of g^{-2}. Here, we have taken $\frac{\gamma_n}{g^2}B = 1$, i.e. magnetic field, B, is in unit of $\frac{g^2}{\gamma_n}$, and $\frac{k_B}{g^2\hbar}T = 1$, i.e. temperature, T, is in unit of $\frac{g^2\hbar}{k_B}$.

The figure (4.1) shows the behavior of Φ_G for $g^2t \in [0, 100]$. Each blue dot shows the first point on the next 2π segment, which Φ_G sweeps. So, $(2 \times nd - 1)\pi$ per unit time gives the speed of Φ_G. Here, nd stands for the number of the dots, and we will use it as the measure of speed of GP. The increase in the negative direction shows that the angle that defines GP rotates clockwise. At some instants $60.01\, g^2t$, $66.03\, g^2t$ and $72.03\, g^2t$ it oscillates back-and-forth around -19π, -21π and -23π, respectively. But then it continues to rotate in the same direction.

By fixing temperature at $\frac{g^2\hbar}{k_B}T = 1$ we analyzed the behavior of Φ_G for different values of magnetic field $\frac{\gamma_n}{g^2}B$.

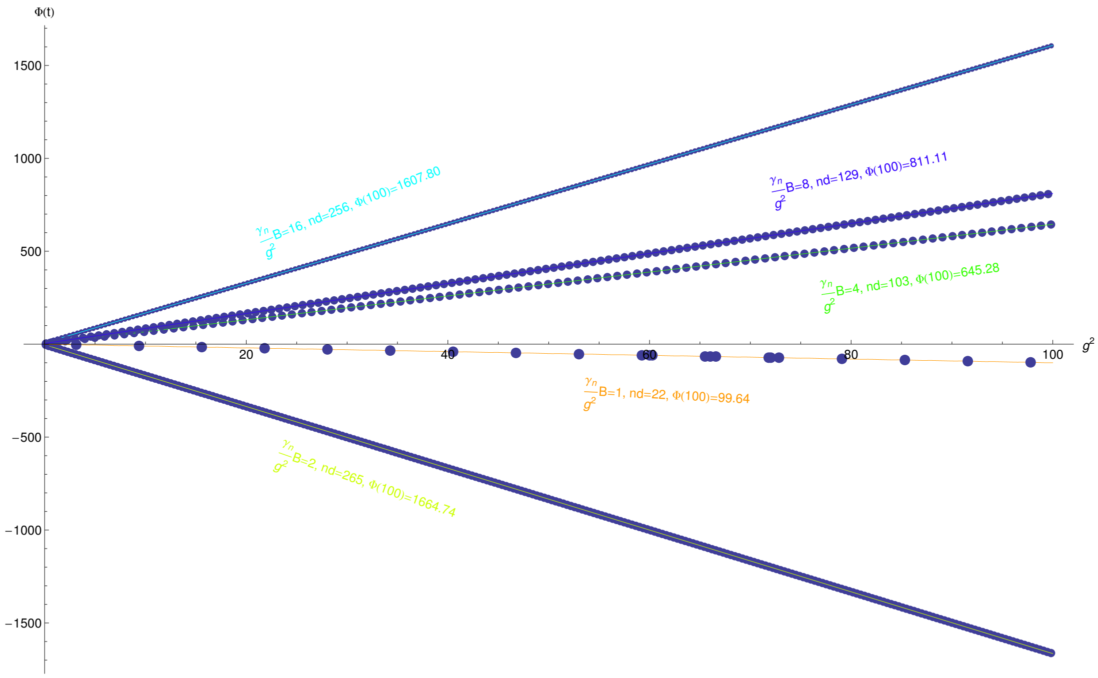


Figure 4.2: GP (Φ_G, in radians), versus time, for various values of $\frac{\gamma_n}{g^2}B$. Time, t, is in unit of g^{-2}. Here, we have taken $\frac{k_B}{g^2\hbar}T = 1$, i.e. temperature, T, is in unit of $\frac{g^2\hbar}{k_B}$.

It is understood from figure (4.2) that when we increase magnetic field twice of its first value, Φ_G becomes very unstable. Because it completes 265 rotations in a $100\, g^{-2}$ time interval by sweeping a $1664.74\, radians$ angle, while it completed 22 rotations by sweeping $99.64\, radians$ angle before the magnetic field is doubled. For a stronger magnetic field, B, $\frac{\gamma_n}{g^2}B = 4$, GP changes the direction of rotations, this time counter-clockwise. It becomes more stable, because number of full rotations is 103 and the swept angle is $645.28\, radians$ in this case. Further increases in B to the values $\frac{\gamma_n}{g^2}B = 8, 16$ decrease the stability of Φ_G that continues rotating counter-clockwise.

Next, we increase temperature twice, and look how does Φ_G evolves, first for a single value of magnetic field, B.

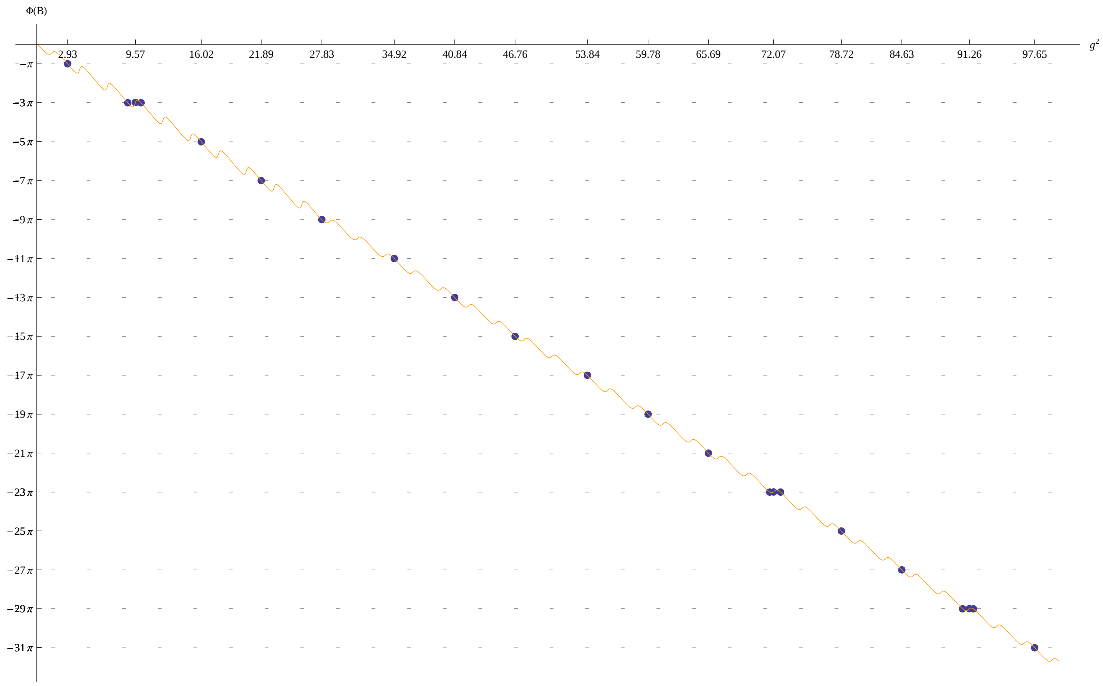


Figure 4.3: GP (Φ_G, in radians), versus time. Time, t, is in unit of g^{-2}. Here, we have taken $\frac{\gamma_n}{g^2}B = 1$, i.e. magnetic field, B, is in unit of $\frac{g^2}{\gamma_n}$, and $\frac{k_B}{g^2\hbar}T = 2$, i.e. temperature, T, is in unit of $\frac{g^2\hbar}{k_B}$.

Data of figure (4.3) differs from figure (4.1)'s only in temperature. Comparing these two figures we see two little differences when the other physical parameters are held fixed. First, the back-and-forth oscillations again occur but this time at different times $9.57\,g^2t$, $72.07\,g^2t$ and $91.26\,g^2t$ around -3π, -23π and -29π, respectively. Second, the total angle swept is $99.51\,radians$ while it was $99.64\,radians$ both with 22 surface-changes.

We will analyze again magnetic field dependence of Φ_G, but this time fixing temperature at a higher value, $\frac{k_B}{g^2\hbar}T = 2$.

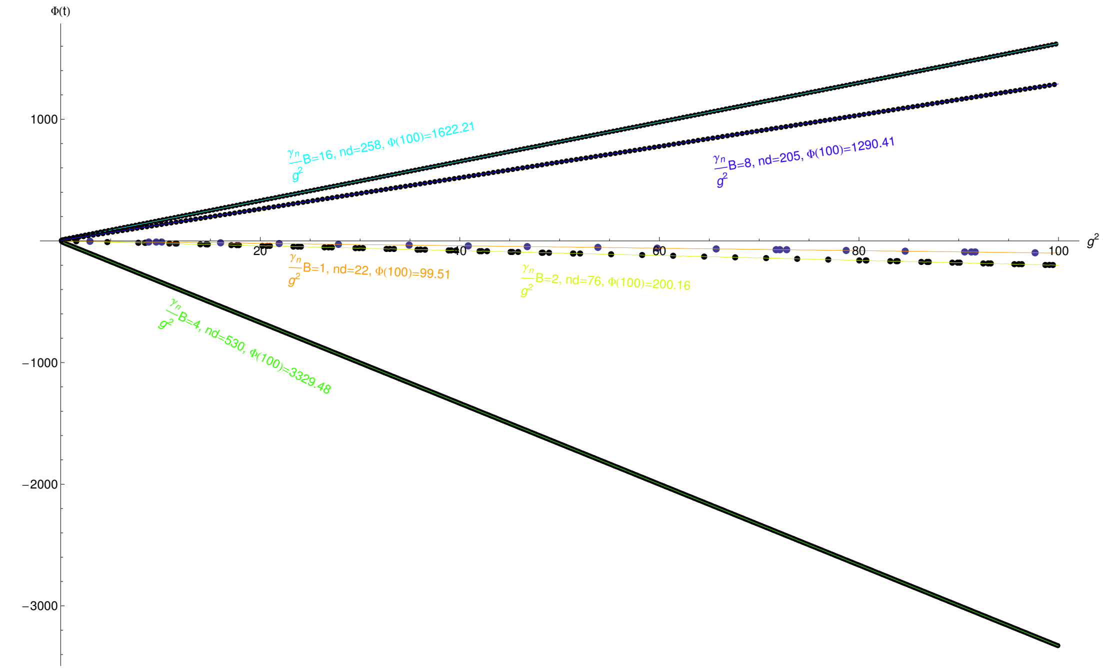


Figure 4.4: GP (Φ_G, in radians), versus time, for various values of $\frac{\gamma_n}{g^2}B$. Time, t, is in unit of g^{-2}. Here, we have taken $\frac{k_B}{g^2\hbar}T = 2$, i.e. temperature, T, is in unit of $\frac{g^2\hbar}{k_B}$.

How does the change in temperature affects the GP, that is understood from the figure (4.4), can be listed as follows; change in the stability occurs at a stronger magnetic field. The crazy increase in the speed takes place at a magnetic field value in $\frac{\gamma_n}{g^2}B \in (2, 4]$, while this value was $\frac{\gamma_n}{g^2}B \in (1, 2]$, for a lower temperature, (figure 4.2). The change in the direction of rotation occurs at a magnetic field strength between $4\frac{g^2}{\gamma_n}$ and $8\frac{g^2}{\gamma_n}$. While it was between $2\frac{g^2}{\gamma_n}$ and $4\frac{g^2}{\gamma_n}$ for $T = \frac{g^2\hbar}{k_B}$. When this change occurs, the stability increases, at least relatively.

In order to be able to make some generalizations we increase the temperature further, again in the same ratio. What we obtained is summarized in the next figure.

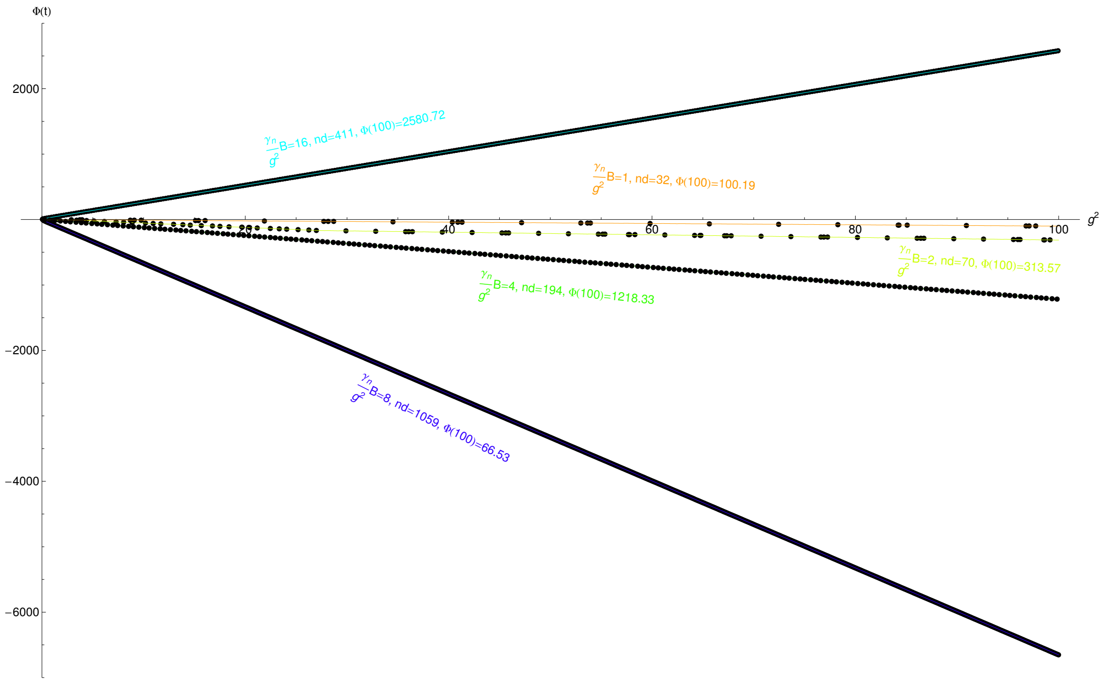


Figure 4.5: GP (Φ_G, in radians), versus time, for various values of $\frac{\gamma_n}{g^2}B$. Time, t, is in unit of g^{-2}. Here, we have taken $\frac{k_B}{g^2\hbar}T = 4$, i.e. temperature, T, is in unit of $\frac{g^2\hbar}{k_B}$.

The general characteristics does not change. Before it changes its direction with the changing magnetic field, there is a wild increase in the speed of GP; number of dots increases from 194 for $B = 4\frac{g^2}{\gamma_n}$ to 1059 for $B = 8\frac{g^2}{\gamma_n}$, whereas the increase is from 70 to 194 for the increase in magnetic field from $B = 2\frac{g^2}{\gamma_n}$ to $B = 4\frac{g^2}{\gamma_n}$. And a change in the direction of rotations takes place at a magnetic field stronger than $B = 4\frac{g^2}{\gamma_n}$. When we compare with the previous figures we saw that this change occurs always for the ratio $\frac{\hbar\gamma_n}{k_B}\frac{B}{T} > 2$.

These are reasonable results. Because, increase in magnetic field increases the differences between energy levels of a system, which means the system is more quantized, and increase in temperature decreases that difference, so that this less quantized system goes to a classical system for which the energy spectrum is a continuum. And we know that the more quantized a system is, the more those quantum characteristics come into prominence.

In order to reach more generalizations we analyze GP with decreasing magnetic field, again by holding temperature fixed at different values. First results are for a very low temperature. The figure (4.6) shows that for temperature $T = \frac{g^2\hbar}{k_B}$, decrease in magnetic field increases the stability of GP; number of dots decreases. It does not change its direction of rotation.

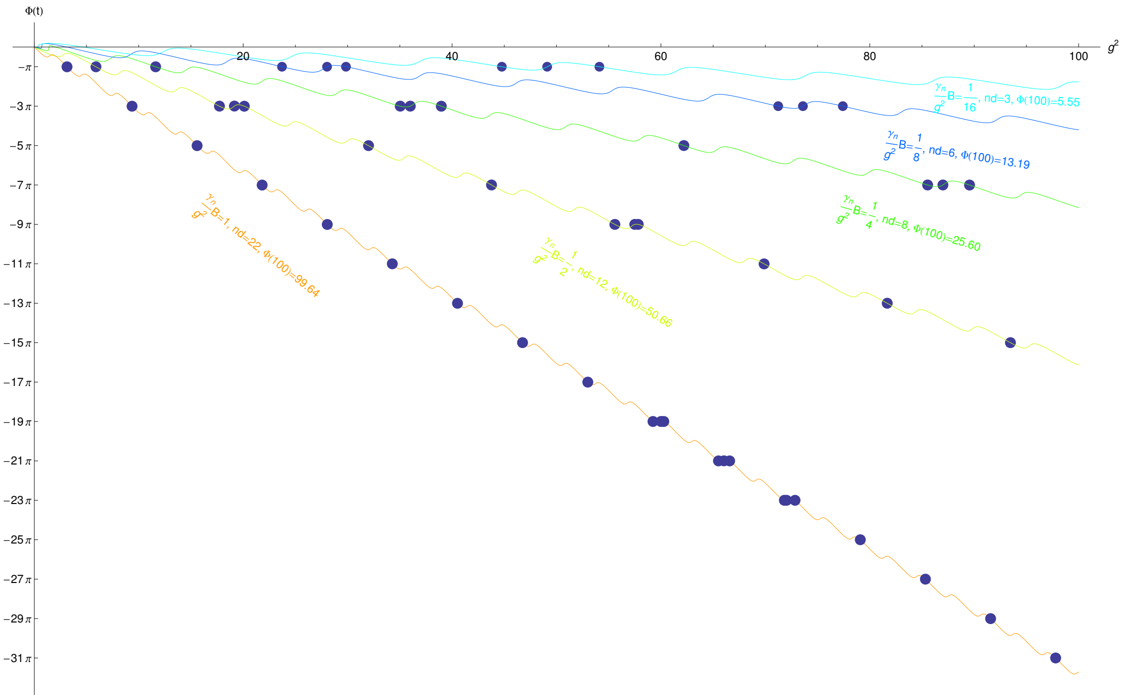


Figure 4.6: GP (Φ_G, in radians), versus time, for various values of $\frac{\gamma_n}{g^2}B$. Time, t, is in unit of g^{-2}. Here, we have taken $\frac{k_B}{g^2\hbar}T = 1$, i.e. temperature, T, is in unit of $\frac{g^2\hbar}{k_B}$.

In the next two figures we increase temperature to values $\frac{k_B}{g^2\hbar}T = 2, 16$, decrease the strength of magnetic field gradually, and see that the general behavior of GP does not change.

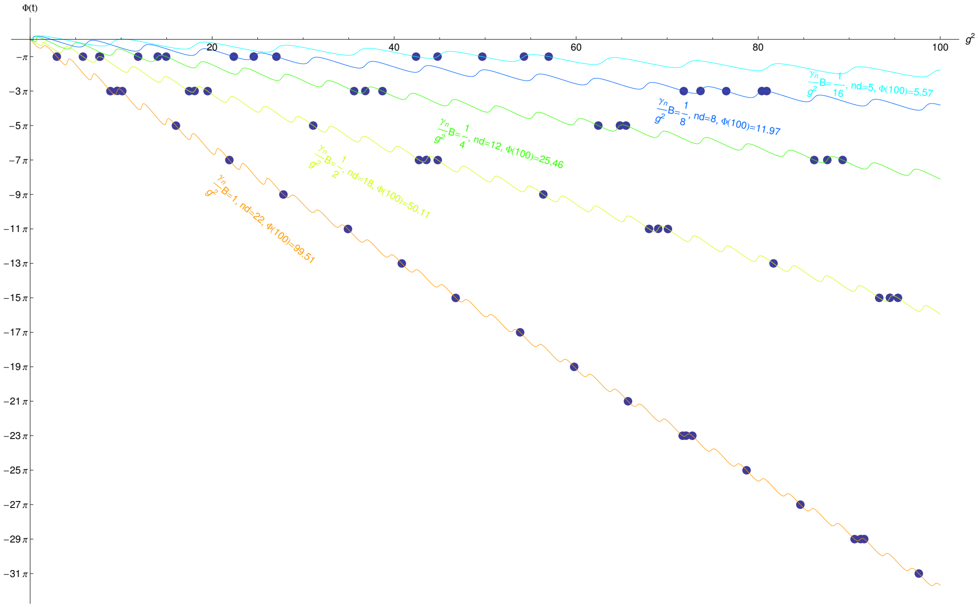


Figure 4.7: GP (Φ_G, in radians), versus time, for various values of $\frac{\gamma_n}{g^2}B$. Time, t, is in unit of g^{-2}. Here, we have taken $\frac{k_B}{g^2\hbar}T = 2$, i.e. temperature, T, is in unit of $\frac{g^2\hbar}{k_B}$.

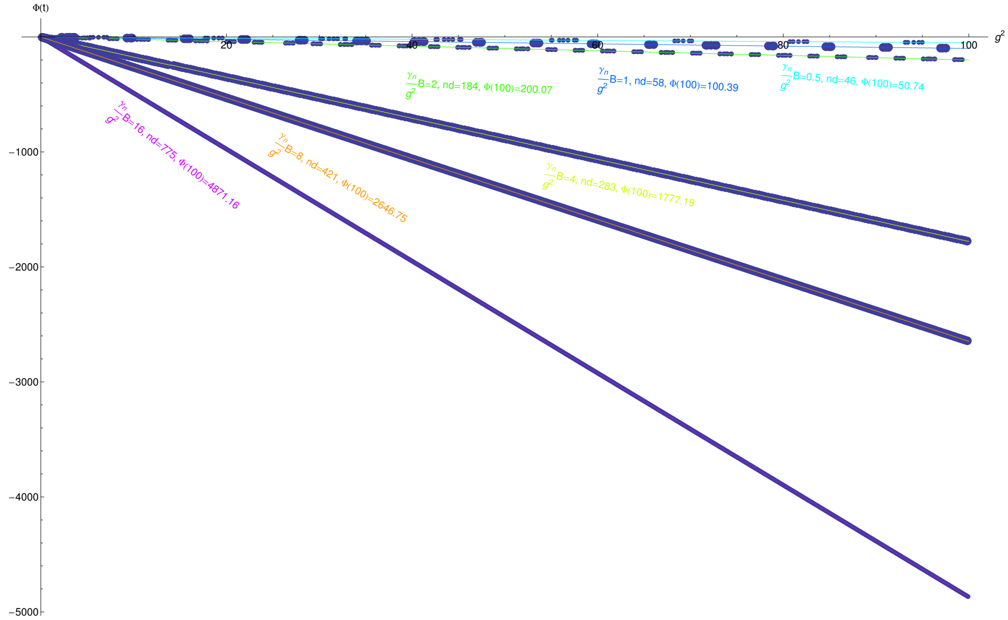


Figure 4.8: GP (Φ_G, in radians), versus time, for various values of $\frac{\gamma_n}{g^2}B$. Time, t, is in unit of g^{-2}. Here, we have taken $\frac{k_B}{g^2\hbar}T = 16$, i.e. temperature, T, is in unit of $\frac{g^2\hbar}{k_B}$.

Another point on the effects of magnetic field and temperature on GP is the magnetic field at which it changes its direction of rotation. In figure (4.9) its seen that GP begins to rotate really very fast before it changes its direction, number of dots are of the order 4000. For temperature $T = \frac{g^2\hbar}{k_B}$, this change occurs at a magnetic field strength $\frac{\gamma_n}{g^2}B$ between 2.3595 and 2.36.

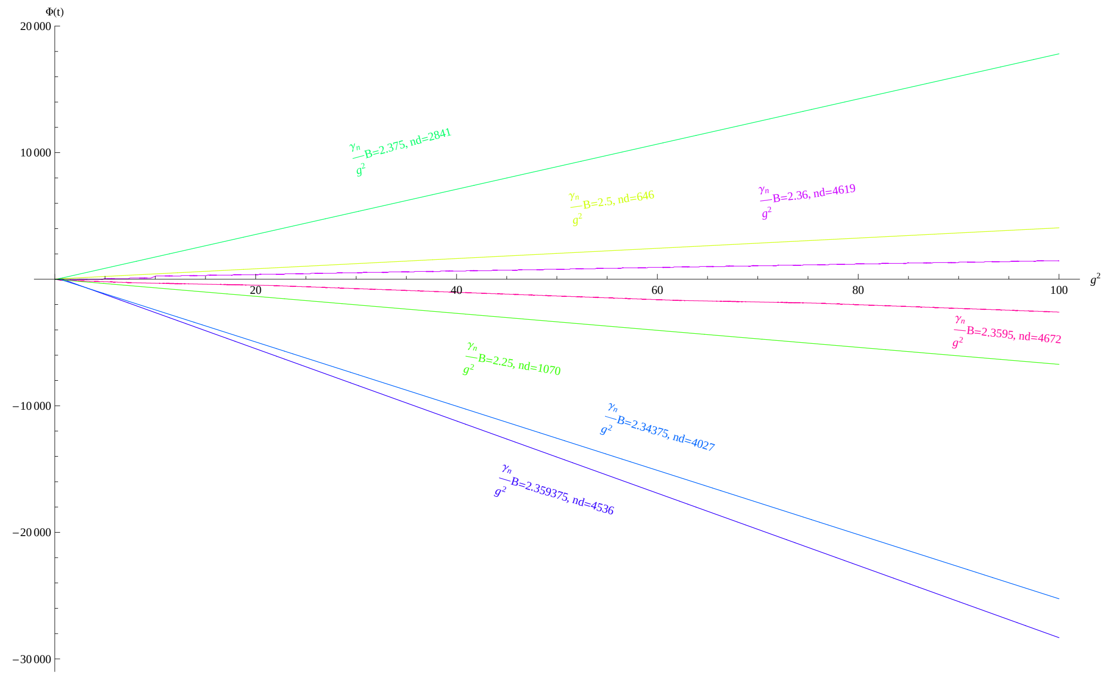


Figure 4.9: GP (Φ_G, in radians), versus time, for various values of $\frac{\gamma_n}{g^2}B$ around which GP changes its direction of rotation. Time, t, is in unit of g^{-2}. Here, we have taken $\frac{k_B}{g^2\hbar}T = 1$, i.e. temperature, T, is in unit of $\frac{g^2\hbar}{k_B}$.

Now, by taking into consideration all data above in the figures, the most general result that one can reach can be stated as *geometric phase depends on the* ***ratio***, $\frac{\hbar\gamma_n}{k_BT}B$, *more than other physical parameters*. This ratio determines the average number of excitations in the environment. The dependencies of GP can be listed as;

1. Stability of GP is determined by the magnetic field which influences the system.
2. GP is more stable when *ratio* is less than 1.
3. GP changes its direction of rotation when the *ratio* is between 2 and 3, and around these *ratios* it has a speed so that it would be interrogated whether GP can be used in QIT.

In the next section, we will analyze how GP is affected in another type of environment.

4.1.2 Applied Magnetic Field and Squeezed-State-Driven-Environment Dependencies of Geometric Phase

The environment we will consider is assumed to be held at a fixed temperature and driven by an electromagnetic field in a squeezed state. For such an environment the operator moments (3.62-3.65) are given as,

$$
\begin{aligned}
\langle \hat{b}(\omega) \rangle &= 0 \\
\langle \hat{b}(\omega)\hat{b}(\omega') \rangle &= -([n(\omega)+1] \cosh r(\omega) \sinh r(\omega') e^{i\phi(\omega')} \\
&\quad + n(\omega') \cosh r(\omega') \sinh r(\omega) e^{i\phi(\omega)}) \delta(\omega+\omega'-2\Omega) \\
\langle \hat{b}(\omega)\hat{b}^{\dagger}(\omega') \rangle &= ([n(\omega)+1] \cosh r(\omega) \cosh r(\omega') \\
&\quad + n(2\Omega-\omega) \sinh r(\omega') \sinh r(\omega) e^{i(\phi(\omega)-\phi(\omega'))}) \\
\langle \hat{b}^{\dagger}(\omega)\hat{b}(\omega') \rangle &= (n(\omega) \sinh r(\omega') \sinh r(\omega) \\
&\quad + [n(2\Omega-\omega)+1] \sinh r(\omega) \sinh r(\omega') e^{-i(\phi(\omega)-\phi(\omega'))}).
\end{aligned} \tag{4.19}
$$

Here the parameters 2Ω is the squeezing carrier frequency, and r and ϕ characterizes the squeezing, as defined in chapter II.

With these moments, the equation of motion (3.54) for the single spin-1/2 nucleus becomes,

$$
\begin{aligned}
\dot{\hat{\rho}}_S(t) &= \pi g^2 \left([n(\omega_n)+1] \cosh^2 r(\omega_n) + n(2\Omega-\omega_n) \sinh^2 r(\omega_n) \right) \\
&\quad \times \left(2\hat{I}^- \hat{\rho}_S(t) \hat{I}^+ - \hat{I}^+ \hat{I}^- \hat{\rho}_S(t) - \hat{\rho}_S(t) \hat{I}^+ \hat{I}^- \right) \\
&+ \pi g^2 \left(n(\omega_n) \cosh^2 r(\omega_n) + [n(2\Omega-\omega_n)+1] \sinh^2 r(\omega_n) \right) \\
&\quad \times \left(2\hat{I}^+ \hat{\rho}_S(t) \hat{I}^- - \hat{I}^- \hat{I}^+ \hat{\rho}_S(t) - \hat{\rho}_S(t) \hat{I}^- \hat{I}^+ \right) \\
&- \pi g^2 \left(n(2\Omega-\omega_n) + n(\omega_n) + 1 \right) \sinh 2r(\omega_n) \\
&\quad \times \left(e^{i[2(\omega_n-\Omega)t+\phi(\omega_n)]} \hat{I}^+ \hat{\rho}_S(t) \hat{I}^+ + e^{-i[2(\omega_n-\Omega)t+\phi(\omega_n)]} \hat{I}^- \hat{\rho}_S(t) \hat{I}^- \right).
\end{aligned} \tag{4.20}
$$

We obtained the solution of this equation in terms of matrix elements as,

$$
\rho_{S11}(t) = \rho_{S11}(0) e^{-2(A+B)t} + \frac{A}{A+B} \left(1 - e^{-2(A+B)t} \right), \tag{4.21}
$$

$$\rho_{S12}(t) = \left(C e^{\sqrt{k^2-(\omega_n-\Omega)^2}t} + (\rho_{S12}(0) - C) e^{-\sqrt{k^2-(\omega_n-\Omega)^2}t} \right) e^{i(\omega_n-\Omega)t-(B+C)t}, \tag{4.22}$$

with

$$A \equiv \pi g^2 \left(n(\omega_n)\cosh^2 r(\omega_n) + [n(2\Omega-\omega_n)+1]\sinh^2 r(\omega_n) \right),$$

$$B \equiv \pi g^2 \left([n(\omega_n)+1]\cosh^2 r(\omega_n) + n(2\Omega-\omega_n)\sinh^2 r(\omega_n) \right),$$

$$k \equiv -\pi g^2 [n(2\Omega-\omega_n) + n(\omega_n) + 1]\sinh 2r(\omega_n),$$

$$C \equiv \rho_{S12}(0) \left(\frac{1}{2} - i\frac{\omega_n-\Omega}{\sqrt{(\omega_n-\Omega)^2+k^2}} \right) + \rho^*_{S12}(0)\frac{k}{\sqrt{(\omega_n-\Omega)^2+k^2}} e^{i\phi(\omega_n)}.$$

For calculations of the first two terms in the GP expression (2.25) this solution is sufficient. For the last term we need the time derivative of $\rho_{S12}(t)$, which is given as

$$\begin{aligned} \dot{\rho}_{S12}(t) = & \left(C e^{\sqrt{k^2-(\omega_n-\Omega)^2}t} - (\rho_{S12}(0) - C) e^{-\sqrt{k^2-(\omega_n-\Omega)^2}t} \right) \\ & \times e^{i(\omega_n-\Omega)t-(B+C)t} \sqrt{k^2-(\omega_n-\Omega)^2} \\ & + \left(i(\omega_n-\Omega) - (B+C) \right) \rho_{S12}(t). \end{aligned} \tag{4.23}$$

But we could not evaluate the integral in (4.14) analytically. Thus, in the following we computed it numerically.

We will analyze the squeezing and the magnetic field dependence of GP, Φ_G, as it evolves in time. Temperature will assumed to be constant at $T = \frac{g^2\hbar}{k_B}$. We will consider dimensionless parameters, as was defined in the previous section, for time and magnetic field, $g^2\, t$, $\frac{\gamma_n}{g^2}B$, respectively. Initial state of the system will also assumed to be the same with the previous analysis, $\rho_{S11}(0) = 0.6$ and $\rho_{S12}(0) = 0.3$. Since they can always be adjusted, or can be incorporated into other physical parameters by redefining them without affecting the dynamics of the system, two of the three squeezing parameters will not be taken as variables. These are the squeezing carrier frequency, Ω, and one of the squeezing characteristic parameters, ϕ, which are assumed to be equal to g^2 and π, respectively.

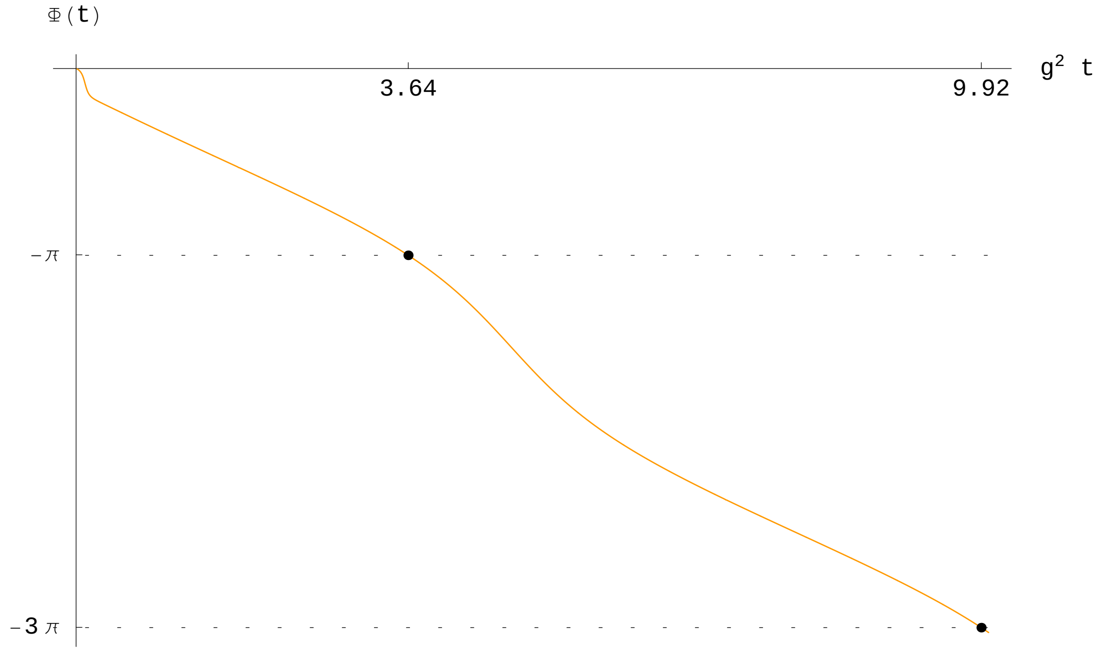


Figure 4.10: GP (Φ_G, in radians), versus time. Time, t, is in unit of g^{-2}. Here, we have taken $\frac{\gamma_n}{g^2}B = 1$, i.e. magnetic field, B, is in unit of $\frac{g^2}{\gamma_n}$, and the squeezing parameter $r = 1$. Temperature is constant at $\frac{k_B}{g^2\hbar}T = 1$.

Figure (4.10) shows that GP rotates clockwise with a speed $0.95\frac{rad}{g^2t}$. It is slower and smoother in comparison with the one in figure (4.1). To see how the magnetic field affects we looked at GP with changing magnetic field strength.

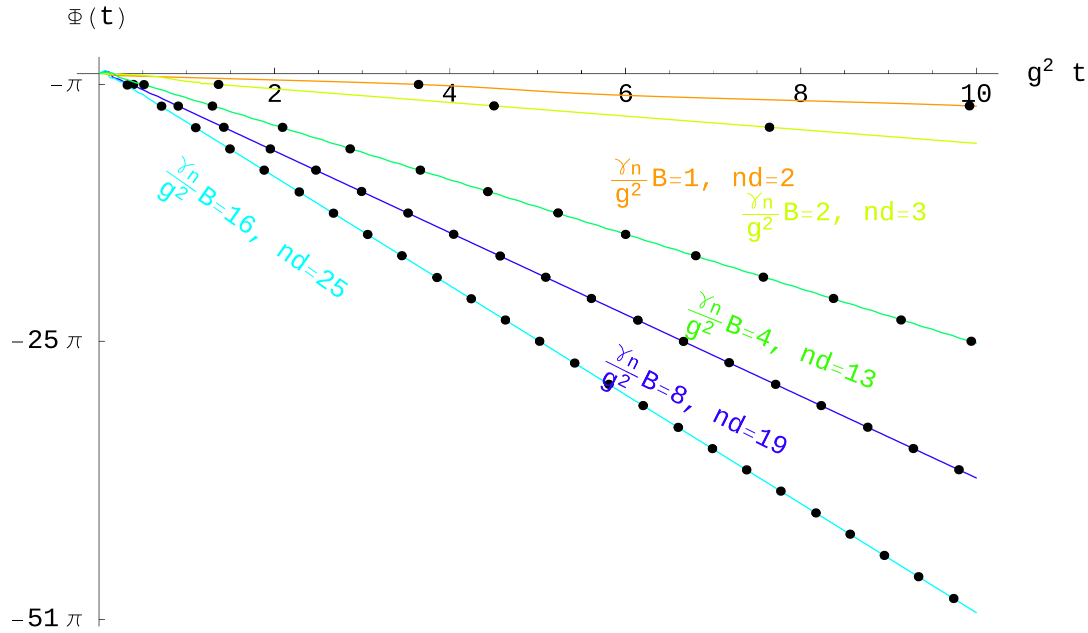


Figure 4.11: GP (Φ_G, in radians), versus time. Time, t, is in unit of g^{-2}. We have changed magnetic field from $B = \frac{g^2}{\gamma_n}$, to $B = 16\frac{g^2}{\gamma_n}$. The squeezing parameter is taken as $r = 1$. Temperature is constant at $\frac{k_B}{g^2\hbar}T = 1$.

In this figure the two behavior attracts the attention. One is the decrease of stability with the increase in magnetic field, which was observed before also. Second, this decrease occurs gradually, in contrast with the environment that was not driven by a squeezed state electromagnetic field. To see whether this is also valid for a different squeezing we increase r twice.

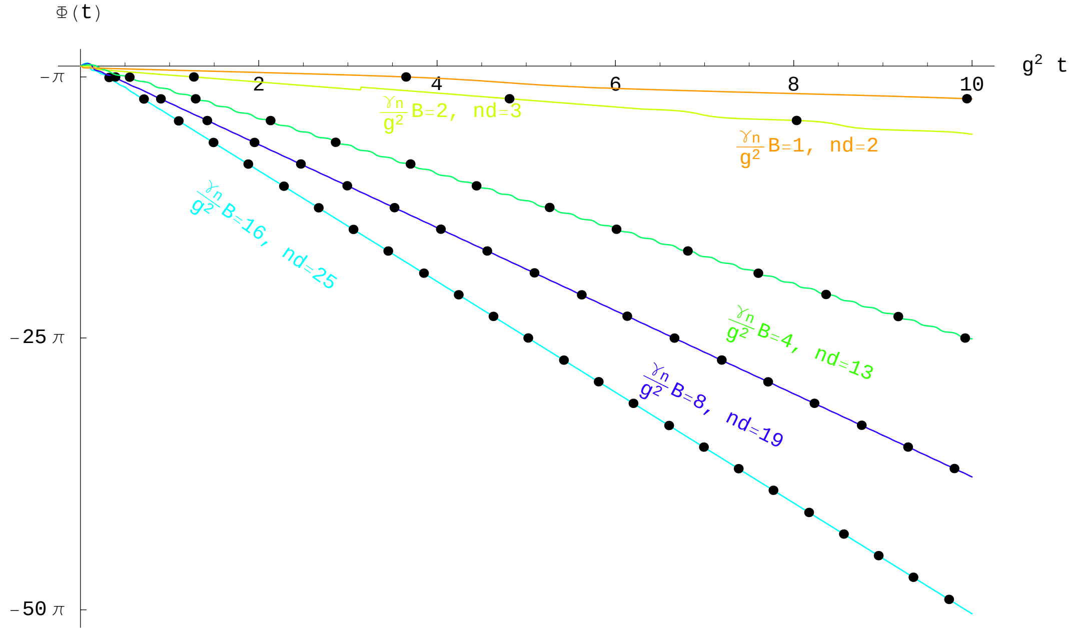


Figure 4.12: GP (Φ_G, in radians), versus time. Time, t, is in unit of g^{-2}. We have changed magnetic field from $B = \frac{g^2}{\gamma_n}$, to $B = 16\frac{g^2}{\gamma_n}$. The squeezing parameter is taken as $r = 2$. Temperature is constant at $\frac{k_B}{g^2\hbar}T = 1$.

This figure is almost the same with the previous one, the graphs are only a little bit less flattened in this case. But these two shows that *the wild increase in the speed of GP can be smoothened with driving the environment a squeezed state electromagnetic field*.

The next figure shows how GP is affected with changing squeezing. It is seen that it changes only the smoothness of GP in its time evolution, for $r > 0$.

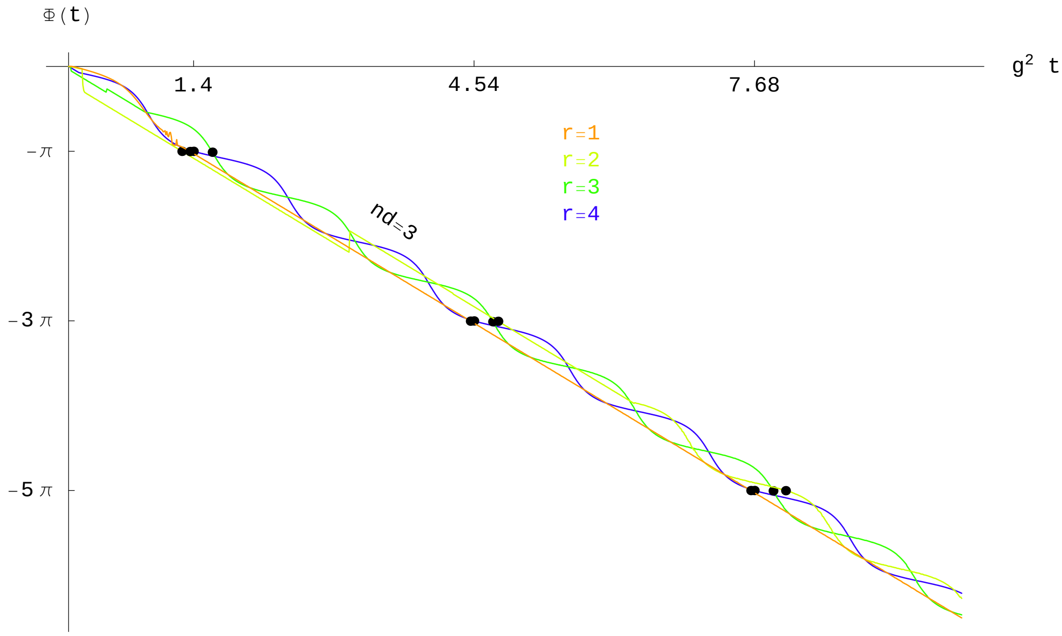


Figure 4.13: GP (Φ_G, in radians), versus time. Time, t, is in unit of g^{-2}. We have fixed magnetic field at $B = 2\frac{g^2}{\gamma_n}$. The squeezing parameter is varied from $r = 1$ to $r = 4$. Temperature is constant at $\frac{k_B}{g^2\hbar}T = 1$.

This completes our analysis on how different types of environment affects the geometric phase, and we see the advantage of radiation in a squeezed state. In the next section we will look at the effects of such an environment on entanglement, as a resource for quantum computing.

4.2 Steady-State Bi-Partite Entanglement Supported By Squeezed Environment

The practical applications in quantum information theory do not demand an arbitrary entanglement. The quantum system which is in an entangled state needed to carry this property long enough to be used in computations. Among the suggestions aiming this purpose few are [61, 64]. In these works robust entangled states in two- and three-level atomic systems is studied. Our study is on a similar system, [65].

We consider a system of three atoms such that one is connected to the others by a dipole-dipole interaction. This atom, occupying the central position in the arrangement is connected with a 'bath' which provides decoherence in the system, (see figure 4.14).

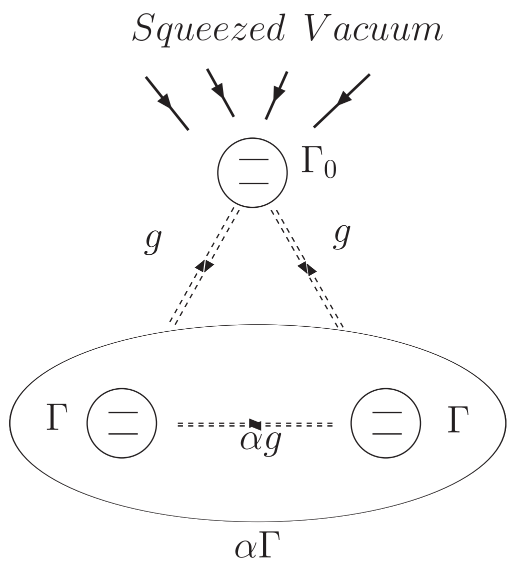


Figure 4.14: Two target atoms with decay rates Γ are in interaction with a source atom of decay rate Γ_0 which is driven by a squeezed vacuum.

The total Hamiltonian of this system has the form

$$\begin{aligned}\hat{H} = \ & \frac{1}{2} \sum_{j=0}^{2} \omega_j \hat{\sigma}_j^z \\ & + \ (g \sum_{f=1}^{2} \hat{\sigma}_0^+ \hat{\sigma}_f^- + \frac{\alpha}{2} g \hat{\sigma}_1^+ \hat{\sigma}_2^- + h.c.),\end{aligned} \tag{4.24}$$

where ω_j denotes the atomic transition frequency, g is the coupling constant of interaction of the "source" atom labeled by the subscript 0 with the "target" atoms, $\alpha/2$ gives the relative strength of interaction between the "target" atoms, and $\hat{\sigma}_j$ denotes the atomic operators as defined in (3.44). For simplicity, we assume that the target atoms have symmetric positions with respect to the source atom and all atoms have the same frequency ω.

The evolution of the density matrix $\hat{\rho}_S$, of the system is described by the equation

$$\dot{\hat{\rho}}_S(t) = -i[\hat{H}, \hat{\rho}_S(t)] + \mathcal{L}(\hat{\rho}_S(t)), \tag{4.25}$$

where the form of the Liouvillean term, $\mathcal{L}$, depends on the specification of the "bath".

In the case of thermal "bath" acting on the source atom, we have

$$\begin{aligned} \mathcal{L} = \mathcal{L}_{thermal} = \frac{\Gamma_0}{2}\{&(\bar{n}+1)(2\hat{\sigma}_0^-\hat{\rho}_S(t)\hat{\sigma}_0^+ - \hat{\sigma}_0^+\hat{\sigma}_0^-\hat{\rho}_S(t) - \hat{\rho}_S(t)\hat{\sigma}_0^+\hat{\sigma}_0^-) \\ &+\bar{n}(2\hat{\sigma}_0^+\hat{\rho}_S(t)\hat{\sigma}_0^- - \hat{\sigma}_0^-\hat{\sigma}_0^+\hat{\rho}_S(t) - \hat{\rho}_S(t)\hat{\sigma}_0^-\hat{\sigma}_0^+)\} \\ + \sum_{j,k=1}^{2}\frac{\Gamma_{jk}}{2}&(2\hat{\sigma}_j^-\hat{\rho}_S(t)\hat{\sigma}_k^+ - \hat{\sigma}_j^+\hat{\sigma}_k^-\hat{\rho}_S(t) - \hat{\rho}_S(t)\hat{\sigma}_j^+\hat{\sigma}_k^-), \end{aligned} \tag{4.26}$$

where $\Gamma_j = \Gamma_{jj}$ denotes the spontaneous decay rate of the j-th atom, $\Gamma_{12} = \Gamma_{21}$ is the collective emission rate of the target atoms, and $\bar{n}$ is the average number of "bath" excitations.

In the case of squeezed vacuum state, we get

$$\mathcal{L} = \mathcal{L}_{thermal} - m(\hat{\sigma}_0^+\hat{\rho}_S(t)\hat{\sigma}_0^+ e^{-2i\omega_s t} + \hat{\sigma}_0^-\hat{\rho}_S(t)\hat{\sigma}_0^- e^{2i\omega_s t}), \tag{4.27}$$

where $2\omega_s$ denotes the frequency of the squeezed mode and m specifies the amount of squeezing.

Since we are looking for the robust entanglement of the target atoms, we restrict our consideration to the steady state solutions of the equation (4.25), i.e. $\dot{\hat{\rho}}_S(t) = 0$. The density matrix $\hat{\rho}_S$ is defined in the eight dimensional Hilbert space of the three 2-level atoms. As the measure of entanglement of the mixed state $\hat{\rho}_S(t)$ we calculate the concurrence ([50], [51]).

The numerical calculations performed in the case of thermal environment with the Liouvillean term of the form (4.26) give the maximum value of concurrence (2.40) of the order of $C = 1.15 \times 10^{-8}$ at $\frac{\Gamma_i}{\Gamma_0} = 1, \frac{g}{\Gamma_0} = 1.5, \bar{n} = 0.5, \alpha = 3$, which is practically next to nothing. Thus, the thermal environment cannot generate entanglement in the system under consideration.

The case of squeezed vacuum state shows much more interesting dependence of concurrence on the parameters of the system shown in figure below.

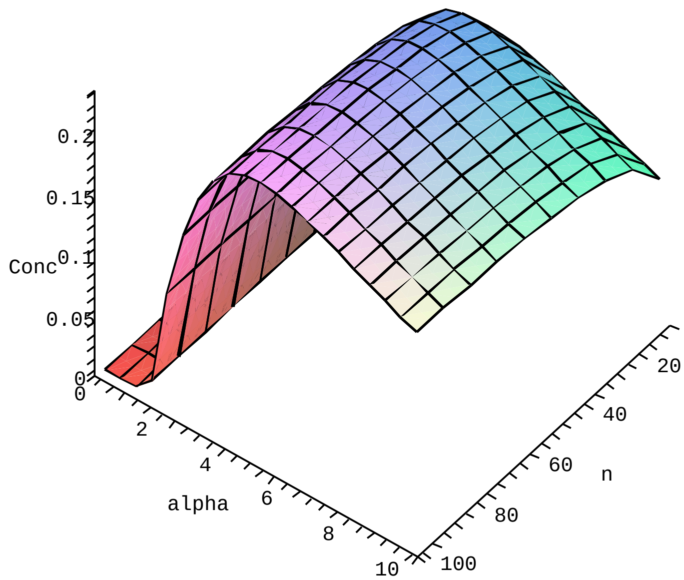


Figure 4.15: Concurrence as a function of α and $\bar{n}$; for $\Gamma/\Gamma_0 = 1$ and $g/\Gamma_0 = 1.5$. The figure indicates that the maximum amount of concurrence occurs at a certain value of α, and increases slowly with $\bar{n}$.

In particular, the maximum value of concurrence $C = 0.227$ is achieved at $\alpha = 5$, and increases with $\bar{n}$. This level of entanglement corresponds to that usually discussed in connection with the practical applications.

CHAPTER 5

CONCLUSION

The availability of easy manipulation with the current technology NMR techniques makes nuclear systems good candidates as carrier systems of the entities that necessary for the quantum information theory (QIT) . With this motivation, we have studied the effects of coupling between a nucleus and a dissipative environment on the geometric phase (GP) that the state of the nucleus gains during its evolution in time. The spin$-1/2$ nucleus is assumed to be under the influence of a static magnetic field of strength B. The environment is taken as a bath of harmonic oscillators, and the temperature, T, is assumed to be held fixed at different values. We have studied the magnetic field effects on GP of the nucleus. It has been shown that B is the primary factor that determines the stability of GP. What we mean by the stability of GP is the speed of change of GP. Since it is the argument of a complex number, [33], the speed of GP is defined as the angle swept in unit time. The increase in magnetic field strength increases the speed of GP, making it less stable. This increase is proportional to B up to a threshold. After this threshold value, increase in the speed of GP becomes wild. Until this point the direction of rotation of GP, as an angle, is constant. However, further increase in B, from this point, suddenly changes the direction of rotations. Together with this change it slows down, and gains speed in the reversed direction with more increase in B. We have shown that the general magnetic field dependence of GP, which we have explained above, does not change with temperature. Change in temperature changes the threshold value of B. It is shown that when the temperature gets higher the wild increase in the speed and the change in the direction of rotation of GP occurs at stronger magnetic field strengths. Thus, a ratio of temperature and magnetic field strength plays a crucial role. This ratio is $\frac{\hbar\gamma_n}{k_B}\frac{B}{T}$, and is related to the number of excitations, or in other words, the dissipation in the environment. At some values, around 1 and 3, of this ratio GP changes so rapidly that it would be interrogated whether it can be used in QIT, and even whether it can be observed or not. Next, we have made similar analysis for an environment having different characteristics. We have assumed that the environment is driven by an electromagnetic field in a squeezed state. We have fixed the temperature at a certain value, and examined the behavior of GP by varying B and the squeezing, r, in the state of driving field. We have shown that the static magnetic field applied to the nucleus affects its GP similar to the previous case with some differences. We have observed that the GP changes smoother, and the decrease in its stability is slower in comparison with that in the previous environment. One other difference we have shown is that, there does not exist a wild change in the speed of GP for the B values at which we observed such a change before. The decrease in its stability occurs gradually. Then, we have looked at whether we can reach further results by changing the squeezing in the state of the driving field. We have shown that for different amounts of r at fixed B, the behavior of GP is almost the same, unless $r = 0$. These differences observed at different environments are crucial for the robustness of the GP.

Then, we have studied the steady-state entanglement of two two-level atoms induced by an external quantum system. This induction is made possible by the interaction of these two target atoms with a third one, which we called as the source and is under the influence of a certain environment. First, we have chosen the environment as a heat bath. As the measure of entanglement between two targets, we have computed the concurrence of their mixed state. The numerical calculations has shown that the thermal environment is not able to create entanglement at least by induction. Next, we have assumed that the source atom is in a squeezed vacuum. In this case , we have found that, depending on the other physical parameters of the system, it is possible to achieve a concurrence value of $\mathcal{C} = 0.227$. Thus, we have shown that the steady-state entanglement can be induced in a system, which has low ability to evolve into an entangled state, through interaction with a strongly fluctuating environment, that is a squeezed vacuum.

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PUBLICATION

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