BARELY TRANSITIVE GROUPS

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ABSTRACT

BARELY TRANSITIVE GROUPS

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A group G is called a barely transitive group if it acts transitively and faithfully on an infinite set and every orbit of every proper subgroup is finite.

A subgroup H of a group G is called a permutable subgroup, if H commutes with every subgroup of G. We showed that if an infinitely generated barely transitive group G has a permutable point stabilizer, then G is locally finite.

We proved that if a barely transitive group G has an abelian point stabilizer H, then G is isomorphic to one of the followings:

(i) G is a metabelian locally finite p-group,

(ii) G is a finitely generated quasi-finite group (in particular H is finite),

(iii) G is a finitely generated group with a maximal normal subgroup N where N is a locally finite metabelian group. In particular, G/N is a quasi-finite simple group.

In all of the three cases, G is periodic.

Keywords: Barely transitive groups, Permutable subgroups.

ÖZ

YALIN GEÇİŞKEN GRUPLAR

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Sonsuz bir küme üzerine, geçişken ve sadık etki eden ve her özalt grubunun yörüngesi sonlu olan gruba yalın geçişken grup denir.

Sonlu eleman tarafından üretilemeyen yalın geçişken bir G grubunun değişili (permutable) nokta stabilizörü varsa, G grubunun lokal-sonlu bir grup olduğunu gösterdik.

Eğer yalın geçişken bir G grubunun değişmeli bir nokta stabilizörü varsa, bu grubun aşağıdaki gruplardan biriyle eş yapılı olduğunu ispatladık.

(i) G sonsuz metabeliyan lokal sonlu bir p-gruptur,

(ii) G sonlu sayıda eleman tarafından üretilen somlumsu (quasi-finite) bir gruptur,

(iii) G sonlu sayıda eleman tarafindan üretilen bir gruptur. G'nin lokal-sonlu metabeliyan maksimal normal bir alt grubu, N, vardır ve G/N sonlumsu basit gruptur.

Her üç durumda da G grubu periyodiktir.

Anahtar Sözcükler: Yalın geçişken gruplar, Değişili (Permutable) altgruplar.

To Ayten and Suat

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CHAPTER 1

INTRODUCTION

In 1947, Kurosh and Chernikov asked the following question; “Is a group satisfying normalizer condition hypercentral?” The negative answer to this question was given by Heineken and Muhammed in 1968 [12]. For each prime p, they constructed a group G with the following properties:

i) G is a locally finite p-group;

ii) $G/G' \cong C_{p^\infty}$;

iii) every proper subgroup is subnormal and nilpotent;

iv) $Z(G) = 1$;

v) G' is an elementary abelian p-group;

vi) the set of all normal subgroups of G contained in G' is linearly ordered by inclusion;

vii) for any proper subgroup K of G, the subgroup KG' is proper in G;

Properties (iii) and (iv) give that these groups satisfy the normalizer condition but they are not hypercentral.

In 1973, Hartley gave another example satisfying (i)-(v) as subgroups of $C_p \wr C_{p^\infty}$ [9]. In 1974, he constructed for each natural number n, a group G_n, satisfying the properties (i)-(iv) and G'_n is an abelian group of exponent p^n [10]. In particular, the exponent of the commutator subgroup of locally finite groups constructed by Hartley could be arbitrarily large. Through these groups, Hartley introduced the concept of bare transitivitiy: a group G has a barely transitive representation if G acts on an infinite set faithfully and transitively and every orbit of every proper subgroup is finite.

The groups Hartley constructed have barely transitive representation. An abstract group is called a barely transitive group, if it has a barely transitive representation. Equivalently, an infinite group G is barely transitive if and only if G has a core-free subgroup H such that for every proper subgroup K of G, the index $|K : K \cap H|$ is finite [18]. In 1997, Belyaev and M. Kuzucuoğlu showed that Heineken-Muhammed groups are also barely transitive [4].

In this dissertation, we investigate the abstract properties of barely transitive groups.

Recall that an infinite group G is called quasi-finite (or Schmidt) group if all of its proper subgroups are finite. An infinite group G is barely transitive in its regular permutation representation if and only if G is quasi-finite [18]. So, the quasi-cyclic group C_{p^∞} is an abelian barely transitive group in its regular permutation representation. There are also non-abelian quasi-finite (hence barely transitive) groups. The first examples of non-abelian quasi-finite groups were given by Ol'sanskii ([24], Theorem 2).

He constructed two generated simple quasi-finite groups. Therefore, there exists periodic simple non-locally finite barely transitive groups. In 1997, it was shown that there exists no simple locally finite barely transitive group [11]. In Proposition 1 of [21] it was shown that every proper normal subgroup of a barely transitive group is locally finite. Then by [21], if there exists a torsion-free barely transitive group G, then it is simple. This is also observed in [1]. But the question "Does there exist a torsion-free barely transitive group ?", raised by Hartley, is still open [10, 17].

A group G is called locally graded if every finitely generated subgroup has a subgroup of finite index. Clearly every locally finite group is a locally graded group. The concept of locally finite barely transitive groups (LFBT-groups) with $G \neq G'$ is well understood [4, 19, 20]. We showed that two properties of LFBT-groups can be generalized to locally graded barely transitive groups (LGBT-groups). The first one is the following; any two proper subgroup of a LGBT-group generate a proper subgroup (Lemma 5.7). Secondly, a LGBT-group G can be generated by an infinite subset M and every infinite subset of M generates G (Lemma 5.8). In addition to these two properties, we have proved a useful lemma on LGBT-groups: A barely transitive group is locally graded if and only if it is infinitely generated (Lemma 5.9).

In general, the class of locally graded groups is not closed under taking homomorphic images (see example 5.5). In Lemma 5.10, we showed that if G is a LGBT-group with a maximal normal subgroup M, then the quotient group G/M is a LGBT-group. It is known that a LFBT-group is a union of an increasing sequence of its proper normal subgroups [21]. Therefore, putting the restriction having a maximal normal subgroup, guarantees that the LGBT-group G above is not locally finite. Under this restriction we have also Lemma 5.14. Namely, if G is a LGBT-group with a maximal normal subgroup M and a point stabilizer H, then the FC-center of H is contained in M. In particular, H is not an FC-group. Another concept that distinguish LGBT-groups from LFBT-groups is simplicity. Recall that a LFBT-group can not be simple [11]. In Lemma 5.20 we showed that if G is a simple LGBT-group with a point stabilizer H, then the Hirsh-Plotkin radical of H is trivial. Note that there is a strong connection between the structural property of a barely transitive group and those of its point stabilizer. So, it is natural to ask that which property should be satisfied by a point stabilizer H so that a LGBT-group G is locally-finite. We show in Lemma 5.11 that a LGBT-group with a point stabilizer of finite exponent is locally finite. Also, if a LGBT-group G has a locally nilpotent-by-solvable point stabilizer, then G is locally finite (Lemma 5.24).

A subgroup H of a group G is called permutable if for any subgroup K of G, the equality $HK = KH$ is satisfied. Let G be LGBT-group with a permutable point stabilizer H. Then G is locally finite (Lemma 5.33).

Another condition for a LGBT-group to be a LFBT-group is to have a splitting automorphism. Let p be a prime number. An automorphism Φ of a group G is called a splitting automorphism of order p if $\Phi^p = 1$ and $gg^{\Phi}g^{\Phi^2}\dots g^{\Phi^{p-1}} = 1$ for all g in G. In Lemma 5.27, it is shown that if a LGBT-group G has a splitting automorphism of order p, then G is locally nilpotent. In particular, it is locally finite.

The main theorem of this dissertation is on relations of structural properties of barely transitive groups and their point stabilizers. We prove, without restricting to the class of locally graded groups, if a barely transitive group G has an abelian point stabilizer, then G is either an infinite metabelian locally finite p-group, or a finitely generated quasi-finite group, or a finitely generated group with a maximal normal subgroup M where M is a locally finite metabelian group and G/M is quasi-finite (Theorem 5.35).

A subset Δ of a G-set Ω is called a block for G if for any element g of G, $\Delta g = \Delta$ or $\Delta g \cap \Delta = \emptyset$. For a transitive group (G, Ω), if Ω has no non-trivial G-block, then G is called primitive; otherwise it is called imprimitive. By definition, a barely transitive group is a group of permutations acting transitively on an infinite set. So, it is natural to investigate the primitivity of a barely transitive group. One can see that Olshanskii groups (of exponent p, for a fixed prime p) given in [26] are primitive barely transitive groups. These Olshanskii groups are simple. Indeed, any primitive barely transitive group is simple (Lemma 3.9). In Lemma 3.17 we showed that if a barely transitive group has a proper block then it is finite.

An imprimitive group is called totally imprimitive if it has no maximal proper block. It is known that a LFBT-group can not be primitive [19]. In fact, any LGBT-group (equivalently; any infinitely generated barely transitive group) is totally imprimitive (Lemma 3.20) and also for a totally imprimitive barely transitive group (G, Ω) we have $\Omega = \bigcup_{i=1}^{\infty} \Delta_i$ and $G = \bigcup G_{\{\Delta_i\}}$ where Δ_i are proper blocks and $G_{\{\Delta_i\}}$ are the set stabilizers (Lemma 3.17).

It is well known that if a group G has a local system of simple subgroups, then it is simple (see Theorem 4.4 of [15] or Lemma 3.1 of [11]). On the other hand, there are non-simple groups which are union of their proper non-abelian simple subgroups (see [7], Theorem C). Let S be the class of groups which are union of their non-abelian simple subgroups. So, "What type of S-groups are simple ?" is a natural question. We showed that any barely transitive S-group is simple (Lemma 4.2).

A group with an infinite derived subgroup in which every proper subgroup has finite derived subgroup is called a Miller-Moreno group or group of Miller-

Moreno type. Olshanskii groups can be given as examples of barely transitive Miller-Moreno group. Recall that Olshanskii groups are two-generated quasi-finite groups. Indeed, in Lemma 4.6 we have shown that every barely transitive group of Miller-Moreno type is a finitely generated quasi-finite group.

We also show that every barely transitive group is countable (Lemma 2.5).

The rest of this dissertation is organized as follows. Chapter 1 is the introduction of the thesis. Chapter 2 is the preliminary. In this chapter, we give some examples and basic properties of barely transitive groups. Also, this chapter contains a brief summary of locally finite barely transitive groups and gives some knowledge about torsion free barely transitive groups. In chapter 3, we focus on barely transitive groups as permutation groups. Here we elaborate their primitivity. In chapter 4, we characterize barely transitive S-groups and barely transitive groups of Miller-Moreno type. In chapter 5, we investigate the similarities and distinctions of locally graded barely transitive groups and locally finite barely transitive groups. Also, we show the effect of a permutable point stabilizer of a barely transitive group. Finally, we give the main theorem of this dissertation in the same chapter.

CHAPTER 2

PRELIMINARIES

In this chapter we give the basic definitions and primary results that play an important role through other chapters.

Definition 2.1. *A group G is called a barely transitive group if it acts transitively and faithfully on an infinite set Ω and every orbit of every proper subgroup is finite.*

Lemma 2.2. [*18*] *An infinite group (G, Ω) is a barely transitive group if and only if G possesses a subgroup H such that $\bigcap_{x \in G} H^x = 1$ and $|K : K \cap H| < \infty$ for every proper subgroup $K < G$.*

Note that in the above lemma, H is a stabilizer of a point.

Example 2.3. *The quasi-cyclic group C_{p^∞} for any prime p is a barely transitive group.*

Definition 2.4. *A group G is called quasi-finite if G is an infinite group all of whose proper subgroups are finite.*

Olshanskii has shown that,

- for every prime $p > 10^{75}$, there is an infinite finitely generated p-group in which every proper subgroup has order p [26]. These groups are infinite 2-generator groups all of whose proper subgroups are cyclic of prime order. (non-abelian simple barely transitive groups with finite exponent p.)
- there exists an infinite 2-generator group all of whose proper subgroups are cyclic of prime order where the set of primes occurring as orders is infinite [25]. (non-abelian simple barely transitive groups with infinite exponent.)

These groups, constructed by Olshanskii, are examples of quasi-finite groups. By lemma 2.2, a quasi-finite group is a barely transitive group in its regular permutation representation.

A group G is called locally finite if every finitely generated subgroup of G is finite. Hartley [9, 10] has given examples of non-perfect locally finite barely transitive groups. These examples are locally finite p-groups satisfying the normalizer condition with trivial center and every proper subgroup of them are subnormal and nilpotent.

Let G be a Heineken-Mohammed group. Then $G^{'}$ is a countable elementary abelian group and the set of normal subgroups of G contained in $G^{'}$ is linearly ordered. Theorem 1 of [4] says that if A be a countable elementary abelian normal subgroup of a group G and if the set of normal subgroups which are contained in A is linearly ordered, then there exists $B \leq A$ such that $\cap_{g \in G} B^g = 1$ and $|A : B|$ is finite. As a corollary, the Heineken-Mohammed groups are barely transitive [4].

The first four of the following properties of barely transitive groups are well known for locally finite case. But the proofs work in general without any change. For convenience of reader we give the proofs of these four properties.

Let G be a barely transitive group and H be a point stabilizer of G.
Then,

1. Every proper subgroup of G is residually finite ([18], Lemma 2.13).

   Proof. Let K be a proper subgroup of G. Then $|K^{x^{-1}} : K^{x^{-1}} \cap H| < \infty$ for all x in G. So, $|K : K \cap H^x| < \infty$ for all x in G. As $\bigcap_{x \in G}(K \cap H^x) \leq K \bigcap(\cap_{x \in G} H^x) = 1$, for all $k \in K$, there exists $K \cap H^y \leq K$ such that $k \notin K \cap H^y$ and $|K : K \cap H^y| < \infty$. Hence we can find a normal subgroup N_k of K such that $|K : N_k| < \infty$ and $N_k \leq K \cap H^y$ as required. $\square$

2. G does not have a proper subgroup of finite index ([18], Lemma 2.2).

   Proof. Let K be a proper subgroup of G of finite index. Then $|G:H| \leq |G:K||K:K\cap H| < \infty$ which is impossible. $\square$

3. If N is a finite normal subgroup of G, then $N \leq Z(G)$ ([18], Lemma 2.2).

   Proof. Let N be a finite normal subgroup of G. Then G acts on N by conjugation. That is there is a homomorphism

   $$\varphi : G \longrightarrow Aut(N)$$
   $$g \longrightarrow \varphi_g$$

   where $$\varphi_g : N \rightarrow N$$
   $$n \rightarrow n^g$$

   and

   $Ker\varphi = \{g \in G : n^g = n \text{ for all } n \in N\} = C_G(N)$.

   So, $G/C_G(N) \leq Aut(N)$. As N is finite, $Aut(N)$ is finite. By property 2, G does not have a proper subgroup of finite index. So, $C_G(N) = G$. Hence, $N \leq Z(G)$.

   $\square$

4. For any proper subgroup K of G, the set KH is a proper subset of G ([18], Lemma 2.2).

   Proof. Assume that $K < G$ and $KH = G$.
   Then $|G:H| = |KH:H| = |K:K\cap H| < \infty$ which is a contradiction. $\square$

5. If G is a simple and if $C_G(x)$ is infinite for some x in G, then $H \nsubseteq C_G(x)$ [21].

6. G is a union of an increasing sequence of proper normal subgroups of G if and only if G is locally finite [21].

7. If G is not perfect, then G is locally finite [5].

8. If G is abelian, then $G \cong C_{p^\infty}$ for some prime p [19].

Lemma 2.5. *Let (G, Ω) be a barely transitive group. Then G is countable.*

Proof. Let (G, Ω) be a barely transitive group. Take an infinite countable subset $\Sigma = \{\alpha_1, \alpha_2, \ldots\}$ of Ω. As G is transitive on Ω, there exists $g_i \in G$ such that $\alpha_i g_i = \alpha_{i+1}$. Set $K_i = \langle g_1, g_2, \ldots, g_i \rangle$. Then we have a chain $K_1 \leq K_2 \leq \ldots$ of countable subgroups of G. If this chain terminates after finitely many steps, say K_n, then K_n is transitive on Σ. As every proper subgroup of G has finite orbits, $K_n = G$. Now, assume that we have an infinite chain. Put $K = \bigcup_{i=1}^{\infty} K_i$. Then K is transitive on Σ and so $G = K$ which is countable as it is countably generated. □

In locally finite case, we have the following:

1. There exists no simple LFBT (Locally Finite Barely Transitive) group [11].

2. Every LFBT-group is a p-group for some prime p [5].

3. Let G be a LFBT-group and let H be its point stabilizer. If G contains an element of order p for some prime p and H satisfies min-p, then $G \cong C_{p^\infty}$ [20]. But note that the groups constructed by Olshanskii are not locally finite. Therefore there exists non-locally-finite barely transitive groups satisfying min-p.

4. If G is a LFBT-group and the point stabilizer H is solvable, then G is solvable [5].

Although the concept of LFBT-groups with $G \neq G'$ is well understood, not much known about the structure of arbitrary barely transitive groups. The question "Does there exist a torsion-free barely transitive group ?", raised by Hartley [10, 17], is still open.

Proposition 2.6. *([19], Proposition 1) If a group G is barely transitive, then every proper normal subgroup of G is locally finite.*

Note that Proposition 2.6 gives the simplicity of torsion-free barely transitive groups (TFBT-groups).

Remark 2.7. *If G is a TFBT-group, then $Z(G) = 1$*

Kuzucuoğlu showed that if G is a TFBT-group and if there exists x in G such that $|H : C_H(x)| < \infty$, where H is the point stabilizer, then G is not simple [21]. So, a TFBT-group can not have a nilpotent point stabilizer.

Let G be a barely transitive group and let H be stabilizer of a point. If $H = 1$, then for any element x in G, $|\langle x\rangle| = |\langle x\rangle : \langle x\rangle \cap H| < \infty$. So, we have the following remark.

Remark 2.8. *In a TFBT-group a point stabilizer H can not be identity. So, TFBT-groups are not regular groups.*

Theorem 2.9. *([27], Theorem 28.3) There is a simple torsion-free group G with generators a_1 and a_2 in which every proper subgroup is infinite cyclic. The extraction of roots in G is unique, that is, it follows from $X^k = Y^k$, $k \neq 0$, that $X = Y$ for any X, Y in G; moreover, it follows from $X^k = Y^l$ $(k, l \neq 0)$ that X and Y are elements of the same cyclic subgroup, whence any two maximal subgroups in G have trivial intersection.*

Remark 2.10. *In Theorem 2.9, the torsion-free group G constructed by Olshanskii has only infinite cyclic proper subgroups. If G is a barely transitive group with a point stabilizer H, then $H = 1$ or H is an infinite cyclic group . By above remark, H is non-trivial. Then, H is an abelian torsion-free group. This gives a contradiction with property 5 (see page 8). So, G is not barely transitive.*

CHAPTER 3

BLOCKS OF BARELY TRANSITIVE GROUPS

Primitivity is a fundamental topic in permutation group theory. In this chapter, we inquire the blocks and consequently the primitivity of barely transitive groups.

3.1 Primitive Groups

Definition 3.1. *Let G be a group and Ω be a G-set. Let Δ be a non-empty subset of Ω.*

- *Δ is called a block (more explicitly a G-block) of Ω, if for every $g \in G$ either $\Delta = \Delta g$ or $\Delta \cap \Delta g = \emptyset$.*
- *If the action of G on Ω is transitive and the only G-blocks are one element subsets of Ω and Ω itself, then G is called primitive (or primitive on Ω)*

Remark 3.2. *([8], page 12) If Ω is a G-set and $\Delta \subseteq \Omega$ is a block for G, then for any g in G, Δg is also a block for G. Furthermore, if G is transitive on Ω, the distinct sets Δg $(g \in G)$ form a partition of Ω.*

Let us give some properties on primitive groups.

Proposition 3.3. *([6], Theorem 4.7) Suppose that $|\Omega| > 1$ and G is transitive on Ω. Then G is primitive on Ω if and only if G_α is a maximal subgroup of G for every $\alpha \in \Omega$.*

Proposition 3.4. *([6], page 37) If G is 2-transitive on Ω, then it is primitive on Ω.*

Proposition 3.5. *([6], Theorem 3.13) Let $n > 1$, $\alpha \in \Omega$. Then G is n-transitive on Ω if and only if G is transitive on Ω and G_α is $(n-1)$-transitive on $\Omega \backslash \{\alpha\}$.*

Example 3.6. *Symmetric groups of order n, where n is a natural number, is a primitive group. Because it is n-transitive.*

Proposition 3.7. *([6], Theorem 4.2 and Theorem 4.4) Suppose that G is transitive on Ω and let $N \lhd G$. Then every N-orbit of Ω is a G-block. In particular, if G is faithful and primitive and $N \neq 1$, then N is transitive.*

Question: "Are there any primitive barely transitive group?"
The answer is yes. The following is an example to primitive barely transitive groups.

Example 3.8. *Let G be an Olshanskii group ([26]) which is 2-generated, infinite, simple p-group for some prime p and every proper subgroup has order p. We already know that G is barely transitive (see Lemma 2.2). So, it is enough to show that G is primitive. The corresponding action is the action of G on $\Omega = \{K^x | x \in G\}$ by conjugation where K is any proper subgroup of G. As G is simple and every proper subgroup has prime order, $N_G(K) = K$. Take any $K^x \in \Omega$ and let $y \in Stab_G(K^x)$. Then*

$$K^{xy} = K^x \Rightarrow y^{x^{-1}} \in N_G(K) = K \Rightarrow y \in K^x \Rightarrow Stab_G(K^x) = K^x.$$

Thus, as every proper subgroup of G is maximal, $Stab_G(K^x)$ is a maximal subgroup of G. So, by Proposition 3.3, G is primitive.

Properties of primitive barely transitive groups:

Lemma 3.9. *Every primitive barely transitive group is simple.*

Proof. Let (G, Ω) be a primitive barely transitive group. Then, by definition, G acts transitively and faithfully on Ω and every orbit of every proper subgroup is finite. Assume that G has a non-trivial normal subgroup say N. Then, by Proposition 3.7, N is transitive on Ω. Since a proper subgroup of G can not have an infinite orbit, $N = G$. □

Proposition 3.10. *([19], Lemma 2.8 and Lemma 2.9) There exists no primitive LFBT-group. In particular, there exists no 2-transitive LFBT-group.*

Proposition 3.10 can be generalized to locally graded groups. A group G is called *locally graded* if every finitely generated subgroup has a proper subgroup of finite index. Locally graded barely transitive groups (LGBT-groups) are studied in Chapter 5 in detail. It is known that in a LGBT-group, no point stabilizer contained in a maximal subgroup (see Proposition 5.6). As every point stabilizer of a primitive group is maximal, above proposition can be extended as in the following remark.

Remark 3.11. *There exists no primitive (in particular 2-transitive) LGBT-group.*

Lemma 3.12. *There exists no 2-transitive barely transitive group.*

Proof. Assume that (G, Ω) is a 2-transitive barely transitive group. Take any element α in Ω. By Proposition 3.5, G_α is transitive on $\Omega \backslash \{\alpha\}$. Then for any point β of $\Omega \backslash \{\alpha\}$, G_α orbit of β is $\Omega \backslash \{\alpha\}$ which contradicts with the finiteness of orbits of proper subgroups.. $\square$

Lemma 3.13. *If G is a primitive barely transitive group, then for any non-trivial element g in G, $|supp\, g|$ is infinite.*

Proof. Let G be a primitive barely transitive group and $1 \neq g \in G$ such that $|supp\, g|$ is finite. Since G is barely transitive, $\bigcap_{\alpha \in \Omega} G_\alpha = 1$. So, there exists $\alpha \in \Omega$ such that g is not in G_α. As G is primitive, by Proposition 3.3, G_α is maximal. Thus, $\langle G_\alpha, g \rangle = G$. Let $\Omega_1, \ldots, \Omega_n$ be G_α orbits of Ω meeting non-trivially with $supp\, g$.

Then $supp\, g (= supp\, g^{-1}) \subseteqq \Omega_1 \cup \ldots \cup \Omega_n$. So, $\Omega_1 \cup \ldots \cup \Omega_n$ is invariant under $\langle G_\alpha, g \rangle = G$. But G is transitive on Ω. Hence $\Omega = \Omega_1 \cup \ldots \cup \Omega_n$. Since G is barely transitive, each Ω_i is finite but this gives a contradiction as Ω is infinite.

$\square$

Definition 3.14. *Let G be a group acting on a set Ω and let S be a subset of G. Then the subset of Ω consisting of points fixed by S is called the fixed points of S and denoted by $Fix(S)$.*

Lemma 3.15. *Let (G, Ω) be a barely transitive group with a point stabilizer G_α. Then, $Fix(G_\alpha)$ is a block for G.*

Proof. Let G be a barely transitive group on a set Ω. Set $Fix(G_\alpha) = \{\beta \in \Omega | \beta x = \beta, \forall x \in G_\alpha\}$. Let $x \in N_G(G_\alpha)$. Then we have $G_\alpha = G_\alpha^x = G_{\alpha x}$ which means that $\alpha x \in Fix(G_\alpha)$. Define $\Phi : \{G_\alpha x | x \in N_G(G_\alpha)\} \mapsto Fix(G_\alpha)$ such that $\Phi(G_\alpha x) = \alpha x$. Suppose that $G_\alpha x = G_\alpha y$ for some $x, y \in N_G(G_\alpha)$. Then, $y = gx$ for some $g \in G_\alpha$ which gives us $\Phi(G_\alpha y) = \alpha y = \alpha g x = \alpha x = \Phi(G_\alpha x)$. Hence, Φ is a well-defined map. Let $\Phi(G_\alpha x) = \Phi(G_\alpha y)$. Then, we have $\alpha x = \alpha y$ from which we obtain $\alpha(xy^{-1}) = \alpha$ that is $xy^{-1} \in G_\alpha$. Now, $G_\alpha xy^{-1} = G_\alpha$ leads to $G_\alpha x = G_\alpha y$ which shows that Φ is a one-to-one map. Let $\beta \in Fix(G_\alpha)$. As G is transitive, there exists $g \in G$ such that $\beta = \alpha g$. Since $\beta G_\alpha = \beta$, we have $\alpha g G_\alpha = \alpha g$. Thus, $\alpha g G_\alpha g^{-1} = \alpha$ which means $G_\alpha^{g^{-1}} \leq G_\alpha$. So, $G_\alpha^{g^{-i-1}} \leq G_\alpha^{g^{-i}} \leq \cdots \leq G_\alpha$ for all $i \in N$. Since G is barely transitive, $|\langle g \rangle : \langle g \rangle \cap G_\alpha| = n$ for some natural number n and so $g^{-n} \in G_\alpha$. Thus we obtain $G_\alpha = G_\alpha^{g^{-n}} \leq G_\alpha^{g^{-n+1}} \leq \cdots \leq G_\alpha^{g^{-1}} \leq G_\alpha$ which gives us $G_\alpha = G_\alpha^{g^{-1}}$. That is $g \in N_G(G_\alpha)$ and hence Φ is onto. Therefore we have $|Fix(G_\alpha)| = |N_G(G_\alpha) : G_\alpha|$.

Set $\Delta = Fix(G_\alpha)$. Assume that $\Delta \cap \Delta g \neq \emptyset$ for some g in G. Then, there exists $x \in G$ such that $\alpha x \in \Delta \cap \Delta g$. So, $\alpha x = \alpha t g$ for some $\alpha t \in \Delta$. Thus, as shown in the above paragraph, $t \in N_G(G_\alpha)$. Now, as $\alpha t g \in \Delta$, $\alpha t g G_\alpha = \alpha t g$. Then we get $G_\alpha^{g^{-1}} \leq G_{\alpha t} = G_\alpha^t = G_\alpha$. By the same argument in the above paragraph, $G_\alpha^{g^{-1}} = G_\alpha$. Now, $\Delta = \Delta G_\alpha = \Delta G_\alpha^{g^{-1}} = \Delta g G_\alpha g^{-1}$ and so $\Delta g G_\alpha = \Delta g$. Therefore $\Delta g \leq \Delta$. Similarly, since $\Delta \cap \Delta g \neq \emptyset$ we have $\Delta g^{-1} \cap \Delta \neq \emptyset$. By the same argument, we get $\Delta g^{-1} \leq \Delta$ which means $\Delta \leq \Delta g$. Thus $\Delta g = \Delta$. Hence, $Fix(G_\alpha)$ is a block for G. □

Lemma 3.16. *Let G be a primitive barely transitive group on Ω. Then any non-trivial point stabilizer G_α fix only α.*

Proof. Let G be a primitive barely transitive group acting on Ω. Then by above paragraph, $Fix(G_\alpha)$ is a block for G. As G is primitive, $Fix(G_\alpha)$ is equal to Ω or a single element. Since G is a barely transitive group its action is faithfull, i.e. the only element fixing every point of Ω is 1. Therefore, if G_α is non-trivial, then $Fix(G_\alpha)$ is a single element set. By definition of G_α, α is in $Fix(G_\alpha)$. Thus, $Fix(G_\alpha) = \{\alpha\}$. □

Lemma 3.17. *Let G be a barely transitive group which acts on Ω and has a chain of proper blocks $\Delta_1 < \Delta_2 < \ldots$. Then each proper block Δ_i is finite. If this chain is infinite, then $\Omega = \bigcup_{i=1}^{\infty} \Delta_i$ and $G = \bigcup_{i=1}^{\infty} G_{\{\Delta_i\}}$.*

Proof. Let α be a fixed element in Δ_1. Then $\alpha \in \Delta_i$ for all Δ_i in the given chain and so $G_\alpha \leq G_{\{\Delta_i\}}$. Since $\Delta_i \neq \Omega$ and G is transitive on Ω, $G_{\{\Delta_i\}} \lneqq G$. Then $|G_{\{\Delta_i\}} : G_\alpha| < \infty$ by bare transitivity.

Let $\alpha, \beta \in \Delta_i$. Then, since G is transitive on Ω, there exists g in G such that $\alpha.g = \beta$. Then $\beta \in \Delta_i \cap \Delta_i.g$. Since Δ_i is a block, we have $\Delta_i = \Delta_i.g$. Hence, $g \in G_{\{\Delta_i\}}$. So, $G_{\{\Delta_i\}}$ acts transitively on Δ_i. Thus, $|\Delta_i| = |G_{\{\Delta_i\}} : G_\alpha| < \infty$.

If this chain is infinite, then $\bigcup_{i=1}^{\infty} \Delta_i$ is an infinite block for G. Since each proper block is finite, $\bigcup_{i=1}^{\infty} \Delta_i = \Omega$. Take $\alpha, \beta \in \Omega$. Then there exists a natural number k such that $\alpha, \beta \in \Delta_k$. So, there exists $g \in G_{\{\Delta_k\}}$ such that $\alpha.g = \beta$. Hence $\bigcup_{i=1}^{\infty} G_{\{\Delta_i\}}$ is transitive on Ω. But, by definition of barely transitive groups, every proper subgroup of G has finite orbits. Hence, $G = \bigcup_{i=1}^{\infty} G_{\{\Delta_i\}}$. □

Definition 3.18. *Let G be a group acting transitively on a set Ω. If G has a proper block (i.e. G is not primitive) then G is called imprimitive.*

As described by Neumann in [23] there are two types of imprimitive groups.

Definition 3.19. *Let G be an imprimitive group. Then*

- *If G has a maximal proper block, then G is called almost primitive.*
- *If G has no maximal proper block, then G is called totally imprimitive.*

Lemma 3.20. *Let G be a barely transitive group acting on Ω. Then, G is locally graded if and only if G is totally imprimitive.*

Proof. Assume that G is a totally imprimitive barely transitive group. Then, by Lemma 3.17, $G = \bigcup_{i=1}^{\infty} G_{\{\Delta_i\}}$ where $\Delta_1 < \Delta_2 < \ldots$ is a chain of blocks for G and the proper subgroup $G_{\{\Delta_i\}}$ is the set stabilizer corresponding to Δ_i. Assume that G is finitely generated. Say $G = \langle x_1, x_2, \ldots, x_n \rangle$ for some $x_i \in G$. As $G = \bigcup_{i=1}^{\infty} G_{\{\Delta_i\}}$, each x_i is contained in $G_{\{\Delta_{k_i}\}}$ for some natural number k_i. Say $k_j = max\{k_1, k_2, \ldots, k_n\}$. Then, $G = \langle x_1, x_2, \ldots, x_n \rangle \leq G_{\{\Delta_{k_j}\}} \lneqq G$. Thus G is infinitely generated and, by Lemma 5.9, G is locally graded.

Conversely, suppose that G is a LGBT-group. If G is locally finite, then $G = \bigcup_{i=1}^{\infty} N_i$ where $N_i \trianglelefteq G$ (see property 6, page 8). Take $\alpha \in \Omega$. By Proposition 3.7, αN_i is a block for G. If $\alpha N_i = \alpha N_j$ for all $j \geq i$, then $\Omega = \alpha G = \alpha N_i$ which is not possible as every proper subgroup has finite orbits. Hence G has a strictly increasing infinite chain of blocks for G, i.e. G is totally imprimitive.

Assume that G is not locally finite. By Remark 3.11, we know that G can not be primitive. Suppose that G is almost primitive. So, G has a finite chain of blocks $\Delta_1 < \Delta_2 < \ldots < \Delta_{n-1} < \Delta_n = \Omega$ such that Δ_{i+1} is the minimal block containing Δ_i for all i in $\{1, 2, \ldots, n-1\}$. By Lemma 3.17, each proper block Δ_i is finite. Set $\Delta = \Delta_{n-1}$ and construct $\Sigma_1 = \{\Delta g | g \in G\}$. As Δ is finite and Ω is infinite, Σ_1 is an infinite set. Since G acts on Ω transitively, so does on Σ_1. Every proper subgroup of G has finite orbits on Ω implies that every proper subgroup of G has finite orbits on Σ_1. If G is simple, then $\bigcap_{x \in G} G_{\{\Delta\}}^x = 1$ and so the action of G on Σ_1 is faithfull. It follows that (G, Σ_1) is a LGBT-group. Suppose that (G, Σ_1) has a proper block say Γ. Without loss of generality, we may assume $\Delta \in \Gamma$ (see Remark 3.2). By Lemma 3.17, Γ is finite; say $\Gamma = \{\Delta, \Delta g_1, \ldots, \Delta g_k\}$ for some natural number k. Then the set $\Delta \bigcup (\cup_{i=1}^{k} \Delta g_i)$ is a proper block for (G, Ω) which contradicts with the maximality of Δ. Hence (G, Σ_1) is a primitive LGBT-group which is not possible by Remark 3.11. Hence (G, Ω) is a LGBT-group which is neither locally finite nor simple. So, G has a non-trivial maximal normal subgroup say M. Let $\alpha \in \Delta_1$. As $M \lhd G$, αM is a block for G which contains α. Since G is almost primitive, we have a finite chain of blocks (containing α) for G say $\Delta_1 < \Delta_2 < \ldots < \Delta_{m-1} < \Delta_m = \Omega$ such that $\Delta_i = \alpha M$ for some $i \leq m-1$. Then, $M \leq \bigcap_{x \in G} G_{\{\Delta_{m-1}\}}^x \lhd G$. Since M is a maximal normal subgroup, $M = \bigcap_{x \in G} G_{\{\Delta_{m-1}\}}^x$. Set $\Sigma_2 = \{\Delta_{m-1} x | x \in G\}$. Now, $(G/M, \Sigma_2)$ is a primitive LGBT-group which is not possible by Remark 3.11. Thus, G is totally imprimitive. $\square$

CHAPTER 4

FURTHER RESULTS

4.1 S-Groups

Definition 4.1. *Set **S** be the class of groups that are the set theoretic union of their non-abelian simple subgroups. If a group G in **S**, then G is called an S-group.*

We know that if a group G has a local system of simple subgroups, then it is simple (see Theorem 4.4 of [15] or Lemma 3.1 of [11]). On the other hand, there are non-simple S-groups (see [7], Theorem C). Then the following question is in order.
Question: "What type of S-groups are simple?"

Lemma 4.2. *Every Barely Transitive S-group(BTS-group) is simple.*

Proof. Assume that G is a BTS-group which is not simple. Then G has a non-trivial normal subgroup N and $G = \cup_{S \in \Sigma} S$ where Σ is a set of non-abelian simple subgroups of G. Take any $S \in \Sigma$. As G is not simple, S is a proper subgroup and so residually finite. A residually finite group is simple only if it is finite. So, S is finite and hence G is periodic.

If N is finite, then $N \leq Z(G)$ (see page 8, property 3). Take any x in N. Then there exists S in Σ such that $x \in S$. So, $N \cap S$ is a non-trivial normal subgroup of S. As S is simple, $S = S \cap N$ but this is not possible as S is non-abelian.

So, N is infinite. Assume that N is a proper subgroup of G. Then N is residually finite. In particular, N is locally graded. As every normal subgroup of an S-group is an S-group [7], N is a periodic locally graded S-group. So, N is simple (see [7], Theorem A). But this is not possible as N is an infinite residually finite group. Thus $N = G$. $\square$

Lemma 4.3. *Let G be a LGBT-group with a point stabilizer H. If H is an S-group, then G is locally finite. In particular, if a LGBT-group G is an S-group, then its point stabilizer H can not be an S-group.*

Proof. Assume that G is a LGBT-group with a point stabilizer H. Suppose that H is an S-group. Then $H = \cup_{S \in \Sigma} S$ where Σ is the set of non-abelian simple subgroups of H. As every element in Σ is a proper subgroup of G, each $S \in \Sigma$ is a residually finite simple group. Therefore each $S \in \Sigma$ is finite. Hence, H is periodic. As every periodic, locally graded S-group is simple ([7], Theorem A), H is simple. But H is residually finite. Hence, H is finite and so, by Lemma 5.11, G is locally finite.

If G is an S-group, by Lemma 4.2, G is simple. But there exists no simple LFBT-group (see page 9).

□

4.2 Miller-Moreno Groups

Definition 4.4. *A group with an infinite derived subgroup in which every proper subgroup has finite derived subgroup is called a Miller-Moreno group or group of Miller-Moreno type.*

The groups given by Olshanskii in [26] and [25] are quasi-finite simple groups. Therefore they can be given as examples of barely transitive Miller-Moreno groups.

In the following lemma we use the below proposition.

Proposition 4.5. *([2], Theorem 1) Let G be a perfect minimal-non-FC-group. Then, we have only three cases:*

1. *$G = \langle a, b\rangle$ and $G/Z(G)$ is simple;*
2. *$G/Z(G)$ is an infinite non-abelian quasi-finite group;*
3. *G is a locally finite group.*

Lemma 4.6. *Every barely transitive group of Miller-Moreno type is a finitely generated quasi-finite group.*

Proof. Let G be a barely transitive group of Miller-Moreno type. Then, by definition, G' is infinite and every proper subgroup of G has finite derived subgroup. Since a group with a finite derived subgroup is an FC-group (see Theorem 1.1 of [30] or 14.5.11 of [28]), either G is an FC-group or it is a minimal-non-FC-group. Assume that G is an FC-group. Then the centralizer of any element of G has finite index in G. But a barely transitive group can not have a proper subgroup of finite index (see page 8, property 2); therefore G is an FC-group if and only if G is abelian. But derived subgroup of an abelian group is the trivial subgroup. Therefore an abelian group can not be a Miller-Moreno group. Hence G is a minimal-non-FC-group.

Let H be a point stabilizer of G. Since $H \cap Z(G) \leq \bigcap_{x\in G} H^x = 1$ we get $|Z(G)| = |Z(G) : 1| = |Z(G) : H \cap Z(G)| < \infty$.

Assume that G is locally finite. Then, by property 6 of barely transitive groups (see page 8), $G = \bigcup_{i=1}^{\infty} N_i$ where N_i is a proper normal subgroup of G. Since G is a Miller-Moreno group, each $N_i^{'}$ is finite. So, by property 3 of barely transitive groups, $N_i^{'} \leq Z(G)$ for all i in natural numbers. Then, $G^{'} = \bigcup_{i=1}^{\infty} N_i^{'} \leq Z(G)$ and so $|G^{'}| \leq |Z(G)| < \infty$ which is a contradiction. Hence G can not be locally finite. So by property 7 of barely transitive groups, G is perfect. Now, by proposition 4.5, either G is two generated such that $G/Z(G)$ simple or $G/Z(G)$ is an infinite non-abelian quasi-finite group.

Assume that G is a two generated group such that $G/Z(G)$ is simple. We need to say that G is quasi-finite. As $G/Z(G)$ is simple, $Z(G)$ is the maximal normal subgroup of G. Then, by Lemma 2.5 of [19], $G/Z(G)$ is a barely transitive group with a point stabilizer $HZ(G)/Z(G)$. Then, $HZ(G)/Z(G)$ is either finite or has trivial FC-radical (see Corollary 5.16). Since H is an FC-group, $HZ(G)/Z(G)$ is an FC-group too. So, the first case namely $HZ(G)/Z(G)$ is finite remains. Thus, $|H| = |H : 1| = |H : H \cap Z(G)| = |HZ(G) : Z(G)|$ is finite. Now let K be any proper subgroup of G. Then, $K \cap H$ and $|K : K \cap H|$ are finite and so K is finite as required.

Consider the last case that $G/Z(G)$ is quasi-finite. We want to show that G is quasi-finite and finitely generated. As $G/Z(G)$ is quasi-finite, $HZ(G)/Z(G)$ is finite. So, $|H| = |H : 1| = |H : H \cap Z(G)| = |HZ(G) : Z(G)|$ is finite. Hence, as H has finite index in any proper subgroup of G, G is quasi finite. Assume that G is infinitely generated. Then, by Lemma 5.9, G is locally graded and so, by Lemma 5.11, G is locally finite. But this is not possible as shown above. $\square$

CHAPTER 5

LOCALLY GRADED BARELY TRANSITIVE GROUPS

In this chapter we investigate the locally graded barely transitive groups (LGBT-groups). The question "How far are the LGBT-groups from the LFBT-groups ? " is the motivation of this chapter. We see the close relation of structural properties of barely transitive groups and those of its point stabilizers.

5.1 Locally Graded Groups

Definition 5.1. *A group G is called locally graded if every finitely generated subgroup has a proper subgroup of finite index.*

Example 5.2. *Here are some examples to locally graded groups.*

- *C_{p^∞} for any prime p.*
- *Locally finite groups. In particular, take $G = \cup_{n=5}^{\infty} A_n$ where A_n is the alternating group of degree n. Then G is a simple locally graded group.*
- *Residually finite groups. In particular, the additive group of the set of integers Z is a residually finite and so locally graded. Note that Z is not locally finite.*
- *Locally solvable groups.*

Lemma 5.3. *[29] The class of locally graded groups are closed under taking subgroups, extensions and cartesian products.*

Proof. Let G be a locally graded group and H be a subgroup of G. Take any finitely generated subgroup K of H. As K is also a subgroup of G it has a proper subgroup of finite index. So, H is also locally graded.

Let G/N and N be locally graded groups. Assume that K is a finitely generated subgroup of G. If $K \leq N$ then, as N is locally graded, K has a proper subgroup of finite index. Hence, we may take $K \nleq N$. As K is finitely generated and $K/(K \cap N) \cong KN/N$, KN/N is a finitely generated subgroup of G/N. So, KN/N has a proper subgroup M/N of finite index. Then, $|KN : M| < \infty$ and $KN \neq M$ and so $|K : K \cap M| = |KN \cap K : K \cap M| < \infty$. If $K \cap M = K$, then $K \leq M$. But this is not possible as $N \leq M$ and $M < KN$. So, $K \cap M$ is a proper subgroup of K of finite index.

Let $\{G_i | i \in I\}$ be a set of locally graded groups. Set $G = Cr_{i \in I} G_i$ and let K be a finitely generated subgroup of G. Say $K = \langle k_1, \ldots, k_n \rangle$ for some non-trivial element $k_j \in K$. Denote k_j as $(g_i^j)_{i \in I}$ where $g_i^j \in G_i$. As $k_1 \neq 1$, there exists i in I such that $g_i^1 \neq 1$. Let $H = \langle g_i^j | 1 \leq j \leq n \rangle$. Then H is a finitely generated subgroup of G_i. As G_i is locally graded, H has a proper subgroup M such that $|H : M|$ is finite. Note that, as H is generated by i^{th} components of the generators of K, we have $K \leq H \times Cr_{t \in I - \{i\}} G_t$ and $(M \times Cr_{t \in I - \{i\}} G_t) \cap K$ is a proper subgroup of K. Since $|H \times Cr_{t \in I - \{i\}} G_t : M \times Cr_{t \in I - \{i\}} G_t| = |H : M|$ is finite, $|K : (M \times Cr_{t \in I - \{i\}} G_t) \cap K|$ is finite. Hence G is locally graded.

□

Remark 5.4. *An image of a locally graded group need not to be locally graded.*

The following is an example of a locally graded group with a non-locally graded homomorphic image.

Example 5.5. *Let G be the two generated quasi-finite infinite simple group which is constructed by Olshanskii. Then, G is not-locally graded. As every group is a homomorphic image of a free group (see 14.1.5 of [14]), there exists a free group F such that $F/N = G$ for some normal subgroup N of F. But every free group is a residually finite group (see 6.1.9 of [28] or 14.2.2 of [14]). So, F is locally graded and G is a non-locally graded homomorphic image of F.*

Proposition 5.6. *([1], Lemma 2.1) Let G be a LGBT (Locally Graded Barely Transitive) group and let H be a point stabilizer of G. Then there exists no maximal subgroup containing H.*

The following lemma use the same technic of Lemma 2.10 of [19].

Lemma 5.7. *Let G be a LGBT-group. Then G has a tower of subgroups $H < H_1 < H_2 < \dots$ such that $G = \bigcup_{i=1}^{\infty} H_i$. Moreover, for any proper subgroup K of G, there exists a natural number n such that $K \leq H_n$. In particular, any two proper subgroup generate a proper subgroup.*

Proof. By Proposition 5.6, H is not maximal. Then, there exists $x_1 \in G \backslash H$ such that $H \gneqq \langle H, x_1 \rangle \gneqq G$. Set $H_1 = \langle H, x_1 \rangle$. Similarly, as H is not contained in a maximal, there exists $x_2 \in G \backslash H_1$ such that $H_1 \gneqq \langle H_1, x_2 \rangle \gneqq G$. Define H_i inductively. So, for all i, $H \gneqq H_i \gneqq G$ and $|H_i : H| = |H_i : H_i \cap H|$ is finite. So, G has a tower $H < H_1 < H_2 < \dots$ such that $|H_{i+1} : H_i|$ is finite. As $|G : H|$ is infinite, this chain does not terminate. Thus, $G = \bigcup_{i=1}^{\infty} H_i$.

Let K be a proper subgroup of G. Then $|K : K \cap H|$ is finite. Let X be the set of right transversals of $K \cap H$ in K. As X is finite, there exists $n \in N$ such that $X \in H_n$. Thus, $K \leqq \langle X, H \rangle \leqq H_n$. Now, let $K, L \gneqq G$. Then $|K : K \cap H|$ and $|L : L \cap H|$ are finite. So, $X \cup Y$ is finite where X and Y are transversal sets of K and L respectively. Thus, there exists $n \in N$ such that $X \cup Y \subseteqq H_n$. Hence, $\langle K, L \rangle \leqq \langle X \cup Y, H \rangle \leqq H_n \gneqq G$. □

The following lemma appeared in [19] for locally finite groups. The same proof works for locally graded groups. For the convenience of the reader we give the proof here.

Lemma 5.8. *Let G be a LGBT-group. Then G has an infinite subset M which generates G and every infinite subset of M generates G.*

Proof. We know that $|G : H|$ is infinite. Let M be a transversal of H in G. Then, $G = \langle H, M \rangle$. By Lemma 5.7, $G = \langle M \rangle$. Let B be any infinite subset of M. As $|\langle H, B \rangle : H|$ is infinite, $G = \langle H, B \rangle$ and so, by Lemma 5.7, $G = \langle B \rangle$. □

Lemma 5.9. *A barely transitive group is locally graded if and only if it is infinitely generated.*

Proof. Let G be an infinitely generated barely transitive group. Then every finitely generated subgroup is residually finite. Thus G is locally graded.

Let G be a LGBT-group generated by finitely many elements. Then by definition of locally gradedness, G has a proper subgroup of finite index but this is a contradiction with property 2 (see page 8). Hence G is infinitely generated. $\square$

Lemma 5.10. *If G is a LGBT-group with a point stabilizer H and M is a maximal normal subgroup of G, then G/M is a LGBT-group with a point stabilizer HM/M.*

Proof. Let G be a LGBT-group and M be a maximal normal subgroup of G. Then by Lemma 2.5 [19], G/M is a barely transitive group with a point stabilizer HM/M. Now, by Lemma 5.9, it is enough to show that G/M is infinitely generated. Suppose that $G/M = \langle x_1M, \ldots, x_nM\rangle$, then $G = \langle x_1, \ldots, x_n, M\rangle$. By Lemma 5.7, $G = \langle x_1, \ldots, x_n\rangle$ but this contradicts with Lemma 5.9. Hence G/M is infinitely generated. $\square$

Lemma 5.11. *Let G be a LGBT-group and let H be a point stabilizer of G.*

(i) *If H is locally finite, then G is locally finite.*

(ii) *If H has a finite exponent, then G is locally finite.*

Proof. Let G be a LGBT-group with a point stabilizer H.

(i) Assume that H is locally finite. Take any a finitely generated subgroup K of G. Then by Lemma 5.9, K is a proper subgroup of G. So, $|K : K \cap H|$ is finite and hence $K \cap H$ is a finitely generated locally finite group. Thus, $K \cap H$ is finite and so is K. Hence G is locally finite.

(ii) Suppose that H has a finite exponent. Since H is a proper subgroup of G, H is residually finite (see property 1, page 7). So, as every finitely generated residually finite group of finite exponent is finite (the celebrated result of Zelmanov [32, 31]), H is locally finite. Thus, by part (i), G is locally finite.

$\square$

Remark 5.12. *A LFBT-group can not be simple (see [11]). So above lemma gives that a point stabilizer of a simple LGBT-group has infinite exponent.*

Lemma 5.13. *([3], Corollary 4.7) Every inert subgroup in a finitely generated simple group either is finite or has trivial FC-radical.*

Lemma 5.14. *Let G be a LGBT-group with a maximal normal subgroup N. Then the FC-center of H is contained in N. In particular, H is not an FC-group.*

Proof. Let G be a LGBT-group and N be a maximal normal subgroup of G. Then, by property 6 (see page 8), G is not locally finite. If H has no FC-element, then we are done. Assume that FC-center of H is non-trivial. Let $M = \{g \in G || H : C_H(g)| < \infty\}$. Then, by assumption, $M \neq 1$. Let $x, y \in M$. Then $C_H(xy) \geqq C_H(x) \cap C_H(y)$
$\Rightarrow \; |H : C_H(xy)| \leqq |H : C_H(x) \cap C_H(y)| < \infty$
$\Rightarrow \; xy \in M.$
Trivially $x^{-1} \in M$. So, $M \leqq G$. Let $x \in M$ and $t \in G$. Then, $|H : C_H(x)| < \infty$
$\Rightarrow \; |H^t : C_{H^t}(x^t)| < \infty$
$\Rightarrow \; |H^t \cap H : C_{H^t}(x^t) \cap H| < \infty$
$\Rightarrow \; |H^t \cap H : C_{H^t \cap H}(x^t)| < \infty$
$\Rightarrow \; |H : C_{H \cap H^t}(x^t)| = |H : H \cap H^t||H \cap H^t : C_{H \cap H^t(x^t)}| < \infty$
$\Rightarrow \; |H : C_H(x^t)| < \infty$
$\Rightarrow \; M \trianglelefteq G.$

Suppose that $M = G$.
Claim: G is a minimal-non-FC group.
As every proper subgroup of G has infinite index, G is not an FC-group. Let K be a proper subgroup of G. Take any element t in K. As $M = G$, we have $|H : C_H(t)| < \infty$
$\Rightarrow \; |H \cap K : C_H(t) \cap K| < \infty$
$\Rightarrow \; |H \cap K : C_K(t) \cap H| < \infty$
$\Rightarrow \; |K : C_K(t)| < \infty.$

So, G is a locally graded minimal-non-FC group. By [30] Lemma 8.14, G is locally finite which is a contradiction.

Thus, $M \neq G$. Then, by Lemma 5.7, $\langle M, N\rangle$ is a proper normal subgroup of G. Hence $M \leqq N$. So, FC-center of H is contained in N. If H is an FC-group, then $H = H \cap M \leqq N$. As N is locally finite, H is locally finite. Then, by Lemma 5.11, G is locally finite which is a contradiction.

□

Corollary 5.15. *Let G be a simple LGBT-group and let H be its point stabilizer. Then the FC-center of H is trivial.*

Proof. Assume that G is a simple LGBT-group. Then the maximal normal subgroup of G is 1. By above lemma, FC-center of H is trivial. □

Corollary 5.16. *Let G be a simple barely transitive group and let H be its point stabilizer. Then either H is finite or FC-center of H is trivial.*

Proof. If G is finitely generated, then the result follows from Lemma 5.13. If G is infinitely generated, then by Lemma 5.9, G is a LGBT-group and the results follows from Lemma 5.14. □

Lemma 5.17. *Let G be a non-locally finite LGBT-group which is not simple. Then G has a maximal normal subgroup M and for all x in $G\backslash M$, $\langle x\rangle^G = G$.*

Proof. Assume that G is a LGBT-group which is not simple and G has no maximal normal subgroup. Let Σ be the set of all proper normal subgroups of G and let $N_1 \lneqq N_2 \lneqq N_3 \lneqq \ldots$ be a chain of proper normal subgroups of G. If this chain stops after finitely many steps, then the last term of the chain is an upper bound of the chain contained in Σ. Assume that the chain is infinite. Set $N = \bigcup_{i=1}^{\infty} N_i$. Then, N is a normal subgroup of G. If $G = N$, then G is locally finite (see property 6 at page 8). So, N is an element of Σ. Then, by Zorn‘s lemma, G has a maximal normal subgroup say M. Take $x \in G\backslash M$. Assume that $\langle x\rangle^G$ be a proper subgroup of G. Then $\langle x\rangle^G M$ is a normal subgroup of G. By Lemma 5.7, $\langle x\rangle^G M$ is proper. So, $\langle x\rangle \in M$ which is a contradiction. Hence, $\langle x\rangle^G = G$. □

Proposition 5.18. *([3], Theorem 1.4) Let H be an inert subgroup in a simple group G and let the locally nilpotent radical $F(H)$ of H be distinct from 1. Then*

1. *$H/F(H)$ is an FC-group;*

2. *if H is not an FC-group then either $F(H)$ is a torsion-free group or the periodic radical of $F(H)$ is a p-group for some prime p.*

Proposition 5.19. *[1] Let G be a non-periodic, simple, LGBT-group with a point stabilizer H. Then H has no non-trivial periodic normal subgroup and Hirsch-Plotkin radical of H is trivial.*

Lemma 5.20. *Let G be a simple LGBT-group and let H be its point stabilizer. Then the Hirsh-Plotkin radical of H is trivial.*

Proof. If G is non-periodic, then it is true by Proposition 5.19. So assume that G is periodic. Let S be the Hirsh-Plotkin radical of H. Suppose that S is not trivial. Then, by Proposition 5.18, H/S is an FC-group. As G is periodic, S is a periodic locally nilpotent group and H/S is a periodic FC-group. Since periodic locally solvable groups are locally finite (see [15], page 3) and periodic FC-groups are locally finite (see 14.5.8 in [28]), both S and H/S are locally finite. So, H is locally finite. Then, by Lemma 5.11, G is locally finite. But there exists no simple LFBT-group (see page 9). □

Lemma 5.21. *Let G be a LGBT-group with a maximal normal subgroup M. Then for any natural number n, the n^{th} term of the derived series of H can not be contained in M. In particular, H is not solvable.*

Proof. Let G be a LGBT-group with a maximal normal subgroup M. By Proposition 2.6, M is locally finite. If $H \leq M$, then H is locally finite and therefore, by Lemma 5.11, G is locally finite. But a LFBT-group can not have a maximal normal subgroup (see property 6, page 8). Hence, $H \nleq M$ and we are done for the case $H' = H$. Assume that $H' < H$.

By Lemma 5.10, G/M is a simple LGBT-group with a point stabilizer HM/M. Therefore, by Lemma 5.20, Hirsch-Plotkin radical of HM/M is trivial. Use induction on n:

(i) If $H^{'} \leq M$, then $HM/M \cong H/(H \cap M)$ is abelian. Since Hirsch-Plotkin radical of HM/M is trivial, $HM/M = M/M \Rightarrow HM = M \Rightarrow H \leq M$ and this is a contradiction.

(ii) Assume $H^{(n-1)} \nleq M$. If $H^{(n)} = H^{(n-1)}$, then we are done. So, suppose that $H^{(n-1)} < H^{(n)}$. If $H^{(n)} \leq M$, then $H^{(n-1)}M/M \cong H^{(n-1)}/(H^{(n-1)} \cap M)$ is an abelian normal subgroup of HM/M. So, $H^{(n-1)}M/M$ is contained in the Hirsch-Plotkin radical of HM/M. As HM/M has trivial Hirsch-Plotkin radical, $H^{(n-1)}M/M = M/M$. Then, $H^{(n-1)} \leq M$ which contradicts with the assumption.

So, $H^{(n)} \nleq M$ for all n in natural numbers. If H is solvable, then $H^{(n)} = 1 \in M$ for some natural number n. Hence, H is not solvable.

□

Proposition 5.22. *[1] Let G be a non-simple non-periodic barely transitive group with a point stabilizer H. Let N be a maximal normal subgroup of G. If N is infinite, then the periodic radical $T(H)$ is contained in N and $|N : T(H)|$ is finite.*

Lemma 5.23. *Let G be a non-periodic LGBT-group with a point stabilizer H and an infinite maximal normal subgroup M. Then for any natural number n, the n^{th} term of the derived series of H is not periodic.*

Proof. If $H^{(n)}$ is periodic, then it is contained in the periodic radical $T(H)$ of H. By Proposition 5.22, we have $H^{(n)}$ is contained in N and Lemma 5.21 gives that this is not possible. □

Lemma 5.24. *Let G be a LGBT-group and let H be a point stabilizer of G. If H is locally nilpotent-by-solvable, then G is locally finite.*

Proof. Let G be a LGBT-group and H be a locally nilpotent-by-solvable point stabilizer of G. Then, G is infinitely generated and H has a locally nilpotent normal subgroup N such that H/N is solvable.

Suppose that G is not locally finite. Then either G is simple or G has a non-trivial maximal normal subgroup (see property 6 at page 8). Assume that G is simple. Then by Lemma 5.20, $N = 1$ and so H is solvable. Set d be the derived length of H. Then $H^{(d-1)}$ is a non-trivial abelian normal subgroup of H which is a contradiction to Lemma 5.20. So, G has a non-trivial maximal normal subgroup say M. By Lemma 5.10, G/M is a simple LGBT-group and HM/M is a point stabilizer of G/M. If $HM/M = M/M$, then $H \leq M$. Then, by Proposition 2.6,

H is locally finite and so, by Lemma 5.11, G is locally finite. Now, assume that HM/M is a non-trivial subgroup of G/M. Suppose that $N \nleq M$. Then, NM/M is a nontrivial normal subgroup of HM/M and NM/M is locally nilpotent as $NM/M \cong N/(N \cap M)$. But this contradicts with Lemma 5.20. So, we may assume $N \leq M$. Then, $HM/M \cong H/(H \cap M)$ is solvable as $N \leq H \cap M$ and H/N is solvable. Put t be the derived length of HM/M. Then $(HM/M)^{(t-1)}$ is a non-trivial abelian normal subgroup of HM/M which is a contradiction due Lemma 5.20. □

Corollary 5.25. *Let G be a LGBT-group and H be its point stabilizer.*

(i) *If H is locally (nilpotent-by-abelian), then G is locally finite.*

(ii) *If H is locally supersolvable, then G is locally finite.*

Proof. Let G be a LGBT-group and H be its point stabilizer. Then by Lemma 5.8, G is countable.

(i) If H is finitely generated, it is nilpotent-by-abelian. Hence, by Lemma 5.24, G is locally finite. Assume that H is infinitely generated. As G is countable, we can write $H = \{h_1, h_2, h_3 \ldots\}$. Set $H_i =< h_1, \ldots, h_i >$. Then each H_i is nilpotent-by-abelian and $H = \bigcup_{i=1}^{\infty} H_i$. Then, H_i' is nilpotent. As $H_i' \leq H_{i+1}'$ we have $H' = (\bigcup_{i=1}^{\infty} H_i)' = \bigcup_{i=1}^{\infty} H_i'$ and so H' is locally nilpotent. Thus, H is locally nilpotent-by-abelian. Hence, by Lemma 5.24, G is locally finite.

(ii) Assume that H is locally supersolvable. Then H is locally (nilpotent-by-abelian) (see [28], 5.4.10). Hence, by part (i), G is locally finite.

□

Definition 5.26. *An automorphism Φ of a group G is called a splitting automorphism of order p if $\Phi^p = 1$ and $gg^{\Phi}g^{\Phi^2} \ldots g^{\Phi^{p-1}} = 1$ for all g in G.*

Lemma 5.27. *Let G be a LGBT-group. If G has a splitting automorphism of order p where p is a prime number, then G is locally nilpotent. In particular, G is locally finite.*

Proof. Let Φ be a splitting automorphism of G of order p. Take a finitely generated subgroup K of G and set $K_1 = \langle K, K^{\Phi}, \ldots, K^{\Phi^{p-1}} \rangle$. Then, K_1 is a Φ invariant finitely generated subgroup of G. As G is an infinitely generated group, K_1 is proper in G and so K_1 is residually finite (see property 1 at page 7). Take any a in K_1. Then, there exists $N_a \trianglelefteq K_1$ such that $a \notin N_a$ and $|K_1/N_a|$ is finite. Set $N_{a_1} = N_a \cap N_a^{\Phi} \cap \ldots \cap N_a^{\Phi^{p-1}}$. Then, N_{a_1} is a Φ invariant normal subgroup of K_1 of finite index. Hence, K_1/N_{a_1} is a finite group which admits a splitting automorphism of order p. Then K_1/N_{a_1} is nilpotent of nilpotency class depends on the number of generators of K_1/N_{a_1} (consequently depends on the number of generators of K_1) and p (see Theorem 2 and Theorem 3 of [13], and 6.4.2 and 7.2.1 of [16]). Set $\Psi : K_1 \hookrightarrow Cr_{a \in K_1} K_1/N_{a_1}$ such that $\Psi(k) = (kN_{a_1})_{a \in K_1}$. Ψ is a homomorphism and kernel of Ψ is equal to $\bigcap_{a \in K} N_{a_1} = 1$. Hence K_1 can be embedded in $Cr_{a \in K_1} K_1/N_{a_1}$ which is nilpotent as each factor is nilpotent with nilpotency class depending on number of generators of K_1 and p. As $K \leq K_1$, K is nilpotent and so G is locally nilpotent. An infinite locally nilpotent group can not be simple (see [28], 12.5.2). So, G has a non-trivial proper normal subgroup say M_1. As G is barely transitive, G/M_1 is infinite. Hence G/M_1 has a non-trivial proper normal subgroup say M_2/M_1. Continue like this we have $G = \bigcup_{i=1}^{\infty} M_i$. By property 6 (page 8), G is locally finite. $\square$

5.2 Permutable Groups

Definition 5.28. *Let H be a subgroup of a group G such that $HK = KH$ for any subgroup K of G. Then H is called a permutable subgroup of G and denoted by $HperG$.*

Example 5.29. *Let G be a group. Any normal subgroup of G is permutable.*

The following two lemma are very well known properties of permutation groups (see [28], page 395).

Lemma 5.30. *Let G be a group. If $HperG$ and $KperG$, then $HKperG$.*

Proof. Let H and K be permutable subgroups of G and let T be an arbitrary subgroup of G. We need to say $T(HK) = (HK)T$. Take an arbitrary element thk in $T(HK)$. As H and K are permutable subgroups, we have $thk = h_1t_1k = h_1k_1t_2$ for some $h_1 \in H$, $k_1 \in K$ and $t_1, t_2 \in T$. So, $T(HK) \leq (HK)T$. Similarly, one can show $T(HK) \geq (HK)T$. □

Lemma 5.31. *Let G be a group. If $HperK \leq G$ and α is a homomorphism of G, then $H^\alpha perK^\alpha$.*

Proof. Let $HperK \leq G$ and α is a homomorphism of G. Then for any subgroup T of K, $HT = TH$. We need to say for any subgroup S^α of K^α, $H^\alpha S^\alpha = S^\alpha H^\alpha$. The inverse image of S^α in G is of the form $Sker\alpha$. Now, as $ker\alpha$ is a normal subgroup of G and H permutes with S, we have $Sker\alpha H = SHker\alpha = HSker\alpha$. So, $S^\alpha H^\alpha = (Sker\alpha H)^\alpha = (HSker\alpha)^\alpha = H^\alpha S^\alpha$.

□

Proposition 5.32. *([22], Theorem 7.1.12) A simple group cannot have a proper non-trivial permutable subgroup.*

Proposition 5.33. *Let G be LGBT-group with a permutable point stabilizer H. Then G is locally finite.*

Proof. Let G be a LGBT-group. Assume that $H^G \lneqq G$. Then $H \leq H^G$ is locally finite as every proper normal subgroup of G is locally finite (see Proposition 2.6). So, by Lemma 5.11, G is locally finite.

Suppose $G = H^G$. As H is permutable, by Lemma 5.31 and Lemma 5.30, for any finite set of elements $\{g_1, \ldots, g_k\}$ of G, the set $H^{g_1}H^{g_2}\ldots H^{g_k}$ is a permutable subgroup of G. Recall that LGBT-group can not be generated by finitely many proper subgroups (see Lemma 5.7). So, $H^{g_1}H^{g_2}\ldots H^{g_k}$ is always a proper subgroup of G. Put $K_0 = H$, then as $K_0 \lneqq G = H^G$ there exists $g_1 \in G$ such that $H^{g_1} \nleqq K_0$. Now, set $K_1 = K_0H^{g_1}$. Then K_1 is a proper permutable subgroup of G. As $H^G = G$ and K_1 is proper, there exists $g_2 \in G$ such that $H^{g_2} \nleq K_1$. Similarly set $K_2 = K_1H^{g_2}$ which is a proper permutable subgroup of G. Continuing the process we obtain $K_1 \lneqq K_2 \lneqq K_3 \lneqq \ldots$ an infinite chain of proper permutable subgroups of G such that $\bigcup_{i=1}^{\infty} K_i = G$. By Proposition 5.32, G can not be simple, and so there exists a non-trivial proper normal subgroup N_1 of G. Since $\bigcup_{i=1}^{\infty} K_i = G \gneqq N_1$, there exists a natural number i_1 such that $K_{i_1} \nleq N_1$. By Lemma 5.31, $K_{i_1}N_1/N_1$ is a non-trivial permutable subgroup of G/N_1. So, G/N_1 has a non-trivial proper normal subgroup say N_2/N_1. Similar to above, as N_2/N_1 is a proper subgroup, there exists a natural number i_2 such that $K_{i_2} \nleq N_2$ and $K_{i_2}N_2/N_2$ is a non-trivial permutable subgroup of G/N_2. Continuing like this, we form a non-trivial strictly increasing sequence of normal subgroups $N_1 \lneqq N_2 \lneqq \ldots$. If we set $N = \bigcup_{j=1}^{\infty} N_j$, then $N \trianglelefteq G$ and G/N has no proper permutable subgroup. Assume that N is a proper subgroup. Then, there exists K_j such that $K_j \nleq N$ which implies that K_jN/N is a non-trivial permutable proper subgroup of G/N and this is a contradiction. Hence $G = \bigcup_{j=1}^{\infty} N_j$ is locally finite (see property 6, page 8). □

5.3 The Main Theorem

There is a strong connection between the structural property of barely transitive groups and those of its point stabilizers. We have presented various properties by conditioning points stabilizers. In this section, we give the main result obtained by a structure of point stabilizers.

First, we give a well-known lemma which will be used in the following theorem. For the convenience of the reader we give the proof.

Lemma 5.34. *Every abelian-by-finite FC-group is a central-by-finite group.*

Proof. Let G be an abelian-by-finite FC-group. Then there exists an abelian subgroup A of G such that $|G : A| < \infty$. Let S be the set of left conjugates of A in G. Then $S = \{x_1A, \ldots, x_nA\}$ for some $n \in \mathbb{N}$ and $x_i \in G$, $i \in \{1, 2, \ldots, n\}$. Since G is an FC-group, $|G : C_G(x_i)| < \infty$ for all i. Set $T = \bigcap_{i=1}^{n} C_G(x_i)$. Then $|G : T| < \infty$ and hence $|G : A \cap T| < \infty$. Therefore, it is enough to show that $A \cap T$ is central. Take $z \in A \cap T$ and $g \in G$. Since $G = \bigcup_{i=1}^{n} x_iA$, $g = x_ia$ for some $a \in A$ and $i \in \{1, \ldots, n\}$. Note that $z \in T = \bigcap_{i=1}^{n} C_G(x_i)$ and also $z, a \in A$. Therefore, z commutes with both x_i and a. Hence, we have $zg = zx_ia = x_iza = x_iaz = gz$ as required. □

Theorem 5.35. *Let G be a barely transitive group with abelian point stabilizer. Then G is isomorphic to one of the followings:*

i) *G is a metabelian locally finite p-group,*

ii) *G is a finitely generated quasi-finite group (in particular H is finite),*

iii) *G is a finitely generated group with a maximal normal subgroup M where M is a locally finite metabelian group. In particular, G is periodic and G/M is a quasi-finite simple group.*

Proof. Let G be a barely transitive group with an abelian point stabilizer H. First assume that G is infinitely generated. Then by Lemma 5.9 and by Lemma 5.24, G is locally finite. So, by Proposition 1 of [11], G is a metabelian locally finite p-group.

Now, assume that G is finitely generated. Set $M = \{x \in G \,|\, |H : C_H(x)| < \infty\}$. Take any x and y in M. As $|H : C_H(x)|$ and $|H : C_H(y)|$ are finite, $|H : C_H(xy)| \leq |H : C_H(x) \cap C_H(y)|$ is finite too and so, $xy \in M$. Also, $C_H(x) = C_H(x^{-1})$. Hence, M is a subgroup of G. For any $g \in G$ and $x \in M$, we have $|H : C_H(x)| < \infty$ and $|H : H \cap H^g| < \infty$. Therefore, $|H^g : C_{H^g}(x^g)| < \infty$ and also $|H : C_H(x^g)| \leq |H : C_{H \cap H^g}(x^g)| = |H : H \cap C_{H^g}(x^g)| = |H : H \cap H^g||H \cap H^g : H \cap C_{H^g}(x^g)| < \infty$. Hence, M is a normal subgroup of G containing H.

Assume $G = M$ and let $g_1, \ldots, g_n$ be elements of G such that $G = \langle g_1, \ldots, g_n \rangle$. Then $|H| = |H : H \cap Z(G)| = |H : H \cap \bigcap_{i=1}^{n} C_G(g_i)| = |H : \bigcap_{i=1}^{n} C_H(g_i)| < \infty$. Let K be any proper subgroup of G. Then $|K| = |K : K \cap H||K \cap H|$ is finite. So, G is quasi-finite.

Suppose that M is a proper subgroup of G. Recall that a barely transitive group is a union of its proper normal subgroups if and only if it is locally finite (see property 6 page 8). Hence, as G is a finitely generated barely transitive group, G has a maximal normal subgroup N containing M. Then, G/N is simple and, as every proper normal subgroup of G is locally finite (see Proposition 2.6), $H \leq M \leq N$ is locally finite. For any element g of G, $|\langle g \rangle : \langle g \rangle \cap H|$ is finite and H is periodic implies that G is periodic.

As $H \leq N$ and as H has finite index in any proper subgroup of G, any subgroup $K/N \leq G/N$ is finite. So, G/N is quasi-finite. Since H is an abelian subgroup of N, for any h in H we have $|N : C_N(h)| \leq |N : H| < \infty$. Hence, $H \leq FC(N)$ and $|N : FC(N)| \leq |N : H| < \infty$. Since $|FC(N) : H| < \infty$ and H is an abelian group, $FC(N)$ is an abelian-by-finite FC-group. So, by Lemma 5.34, it is central-by-finite. As $Z(FC(N))$ char $FC(N)$ char $N \trianglelefteq G$, we have $Z(FC(N)) \trianglelefteq G$. Since $N/FC(N)$ and $FC(N)/Z(FC(N))$ are finite, $N/Z(FC(N))$ is a finite normal subgroup of G/N. Therefore, $G/Z(FC(N))$ acts on $N/Z(FC(N))$ by conjugation with kernel $C_{G/Z(FC(N))}(N/Z(FC(N)))$. Hence, $[G/Z(FC(N))]/[C_{G/Z(FC(N))}(N/Z(FC(N)))]$ can be embedded to $Aut(N/Z(FC(N)))$ which is finite. But G can not have proper subgroup of finite index. Hence, $N/Z(FC(N))$ is in the center of $G/Z(FC(G))$ and so, $[G, N] \leq Z(FC(N))$. In particular, N' is abelian. Hence, $N'' = 1$.

$\square$

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