VALUATION OF LIFE INSURANCE CONTRACTS USING STOCHASTIC MORTALITY RATE AND RISK PROCESS MODELING

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FEBRUARY 2007

VALUATION OF LIFE INSURANCE CONTRACTS USING STOCHASTIC MORTALITY RATE AND RISK PROCESS MODELING

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF APPLIED MATHEMATICS
OF
THE MIDDLE EAST TECHNICAL UNIVERSITY

BY

ŞİRZAT ÇETİNKAYA

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE
IN
THE DEPARTMENT OF FINANCIAL MATHEMATICS

FEBRUARY 2007

Approval of the Graduate School of Applied Mathematics

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Abstract

VALUATION OF LIFE INSURANCE CONTRACTS USING STOCHASTIC MORTALITY RATE AND RISK PROCESS MODELING

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February 2007, 74 pages

In life insurance contracts, actuaries generally value premiums using deterministic mortality rates and interest rates. They have ignored them stochastically in most of the studies. However it is known that neither interest rates nor mortality rates are constant. It is also known that companies may encounter insolvency problems such as ruin, so the ruin probability need to be added to the valuation of the life insurance contracts process. Insurance companies should model their surplus processes to price some types of life insurance contracts and to see risk position. In this study, mortality rates and surplus processes are modeled and financial strength of companies are utilized when pricing life insurance contracts.

Keywords: Stochastic mortality rate, risk process, the Kalman filter, life insurance contract, affine term structure, jump diffusion models, participating life insurance contract, credit rating.

Öz

HAYAT SİGORTASI POLİÇELERİNİN STOKASTİK ÖLÜM ORANINA GÖRE FİYATLANMASI VE RİSK SÜRECİ MODELLEMESİ

ÇETİNKAYA, Şirzat
Yüksek Lisans, Finansal Matematik Bölümü
Tez Yöneticisi: Doç. Dr. Azize HAYFAVİ

Şubat 2007, 74 sayfa

Hayat sigortası poliçelerinin prim hesaplamaları yapılırken genellikle deterministik ölüm oranı ve faiz oranı kullanılır, stokastik olarak modellemeler çoğu zaman yapılmaz. Sabit faiz ve ölüm oranı düşünülerek hesaplamalar yapılmasına rağmen iki oran da sabit yapıya sahip değildir. Sigorta şirketlerinin risk süreçleri modellenirken başlangıç sermayesi, prim ve hasar büyüklükleri kullanılır. Sigorta şirketleri yükümlülüklerini karşılayacakları kadar sermaye tutmalarına rağmen yükümlülüklerini karşılama konusunda bir takım problemlerle karşılaşabilirler. Bu çalışmada ölüm oranı stokastik olarak modellenmeye, risk süreci için stokastik bir model oluşturulmaya ve şirketlerin finansal durumları göz önünde bulundurularak çeşitli poliçelerin fiyatlandırılması yapılmaya çalışılmıştır.

Anahtar kelimeler: Stokastik ölüm oranı, risk süreci, iştirakli hayat poliçeleri, kredi derecesi, hayat sigortası kontratı, Kalman filtresi.

To my family

Acknowledgments

I would like to thank Assoc. Prof. Dr. Azize Hayfavi, Prof. Dr. Hayri Körezlioğlu, Assist. Prof. Dr. Kasırga Yıldırak, Dr. Coşkun Küçüközmen and Prof. Dr. Ömer Gebizlioğlu for guiding, motivating and encouraging me throughout this study.

I would also like to thank all my teachers and friends of Hacettepe University Actuarial Sciences Department.

I would like to thank all my friends and the members of the Institute of Applied Mathematics for their support.

Finally, my greatest thanks are to my admirable father Uğur, my lovely mother Kadriye and my lovely sister Merve for their support and patience. Without their encouragement this master and thesis would not have been possible.

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Chapter 1

INTRODUCTION

1.1 Introduction

An insurance can be considered under two aspects, economic and legal. From an economic aspect, insurance has a financial intermediation function. It covers individuals from events damaging them financially. From this point of view, individuals buy the right to compensate their losses from the pool if the insured contingency occurs. Hence, insurance is a contingent claim contract on the pool's asset. From a legal aspect, insurance is an agreement between insurance company and insurer. Insureds pay a certain amount of money called premium to the insurer to protect him against the losses from unexpected situations. If the covered event occurs whose objects can be life, health or property, the insurer must provide money or service according to the policy [40].

An insurance contract can be classified as a life insurance contract or a non-life insurance contract. In the life insurance contract, the risk assumed by the insurer is the risk of death of the insured. Life insurance contracts have many advantages. Some of them are as follows: a life insurance contract compensates the losses due to the death of insured; it can be used to pay insured's debts after the insured dies; or in many businesses, the death of a person can affect the business life, so a life insurance contract can be used to help the company with benefit payment [2].

In life insurance contracts, actuaries generally value premiums using deterministic mortality rates and interest rates. They are not given stochastically many times. However it is a well known fact that neither the interest rates nor the mortality intensities are deterministic or constant.

There are three important risks for insurance companies [16],

- The financial risk,
- The systematic mortality risk,
- The unsystematic mortality risk.

The financial risk refers to the uncertainty related to financial factors, the systematic mortality risk refers to the future development of the underlying mortality intensity and the unsystematic mortality risk refers to a possible adverse development of the policyholders mortality (insured portfolio). Insurance companies have the advantage of selecting mortality tables or mortality intensities and interest rates, which are appropriate for their contracts and institutions. Therefore they can protect themselves from unexpected situations like extreme interest rate changes or extreme mortality improvements. Because the insurance contracts have generally long maturities, five years, ten years or more long, the interest and the mortality rates selected at the beginning of the contract may not protect them throughout the contract maturity. It is known that financial assets are very volatile, so the financial risk is an immediate problem for the investors or insurance companies, on the other hand mortality intensity is not very volatile but it can cause some solvency problems for the company in the long term, and it becomes more permanent problem for the companies.

Insurance companies may encounter some difficulties to meet their liabilities, although they hold enough capital to cover their losses. In recent years, there are some bad experiences about this issue. One of the ways to see whether companies meet their liabilities or not is to examine companies' ratings. If the companies consider their ratings and then determine the premiums of life insurance contracts, the premiums will be different for every rating class. This satisfies investors to buy the contracts according to their risk profile.

The thesis is organized as follows: In Chapter 2, the basics of mortality modeling are explained. Survival probabilities are modeled by affine term structures. The survival probability and the force of mortality (or mortality intensity) are modeled with a jump process. The Kalman filter technique is used to estimate parameters of the force of mortality, which is modeled by N.M.R.CIR *(non mean reverting Cox Ingersoll Ross)*. In Chapter 3, the basics of the surplus modeling are explained. A jump diffusion process is used to model the surplus process. Some life insurance contracts are explained in Chapter 4. Chapter 5 discusses further applications of the specific models in Chapter 2 and 3. Chapter 6 concludes.

Chapter 2

MORTALITY MODELING

2.1 Literature Review

In recent years, stochastic mortality modeling has been more and more important, because of the following reasons:

- To prevent insolvency problems:

  The change in the mortality structure can cause very important problems. Especially, life insurance and pension foundations should give more attention, because changes in the mortality structure directly affect their premiums and benefits. Predicting the mortality structure, they may prevent their solvency problems.

- To price mortality derivatives:

  Some mortality derivatives are as follows: Survivor swaps, survivor bonds, mortality options, annuity futures, mortality linked securities [13].

The topic of stochastic mortality modeling has been handled by different authors. Some of them and their assumptions are as follows: Milevsky and Promislow in [34] assumed that the mortality intensity has the Gompertz form with a mean reverting diffusion process. Dahl in [16] used the affine forms. Biffis, Denuit and Devolder in [6] specified the behaviour of Lee Carter model. Biffis in [7] used a jump component in mortality assumptions. Luciano and Vigna in [29] used affine processes to model the mortality. Schrager in [39] assumed the mortality intensity as Gaussian Thiele model and used affine term structure.

2.2 The Basics of Mortality Models

The terminologies and the notations commonly used in life insurance are as follows [20]:

- One's future life time is denoted by *T(x)* or for simplicity by T.
- The probability that a person aged x will die within t years is denoted by ${}_tq_x$.
- The probability that a person aged x will attain age $x+t$ is denoted by ${}_tp_x$.
- The person with age x will survive t years and die within the following u years. another words the person with age x will die between ages x+t and x+t+u is denoted by ${}_{t|u}q_x$.
- The mortality intensity (or the instantaneous rate of mortality, the force of mortality) is denoted by $\mu_x(t)$.

The future lifetime T is a r.v. with a probability distribution function,

$$G_x(t) = Pr[T(x) \leq t], t \geq 0, \tag{2.2.1}$$

$$g_x(t) = G'_x(t). \tag{2.2.2}$$

$G_x(t)$ gives the probability that the person with age x will die within t years, for any fixed t. This probability can also be denoted by the symbol ${}_tq_x$,

$$ {}_tq_x = G_x(t) = Pr[T(x) \leq t],\ t \geq 0. \tag{2.2.3}$$

Let ${}_tp_x$ denote the probability that a person aged x will survive at least t years,

$$ {}_tp_x = 1 -_t q_x = 1 - G_x(t) = Pr[T(x) > t],\ t \geq 0.3 \tag{2.2.4}$$

$$ {}_{t|u}q_x = G_x(t+u) - G_x(t) = Pr[t < T(x) \leq t+u] \tag{2.2.5}$$

denotes the person with age x will survive t years and die within the following u years

The approximation,

$$ {}_cq_{x+t} \approx \mu_x(s)c, \tag{2.2.6}$$

is valid for small values of c. At the age of x the mortality intensity for the age x+t is defined as

$$\begin{aligned} \mu_x(t) &= \mu_{x+t} = \frac{g_x(t)}{1-G_x(t)} = -\frac{d}{dt}ln[1-G_x(t)], && (2.2.7)\\ \mu_x(t) &= -\frac{d}{dt}ln_tp_x, && (2.2.8)\\ {}_tp_x &= e^{-\int_0^t \mu_x(s)ds}. && (2.2.9) \end{aligned}$$

In the literature, some authors suggested analytical distributions for $G_x(t)$. Some of them are given here [20]:

De Moivre(1724) postulated the existence of maximum age w, and $g(t)$ was the density of the uniform distributed between the ages of *0* and w-x,

$$\begin{aligned} g_x(t) &= \frac{1}{w-x}, \; 0 < t < w-x, & (2.2.10) \\ \mu_x(t) &= \frac{1}{w-x-t}, & (2.2.11) \end{aligned}$$

where the mortality intensity is an increasing function of t.

Gombertz(1824) suggested the model for the mortality intensity. It is defined as

$$\mu_x(t) = Bc^{x+t}. \tag{2.2.12}$$

This model satisfies that the mortality intensity increases exponentially.

Makeham(1860) generalized Gompertz law. The model is as follows:

$$\mu_x(t) = A + Bc^{x+t}, \tag{2.2.13}$$

where $A > 0$ to satisfy the exponentially growing.

Weibull(1939) postulated that the mortality intensity grows a power of t,

$$\mu_x(t) = k(x+t)^n, \tag{2.2.14}$$

where $k > 0$ and $n > 0$.

Mortality modeling in life insurance and default modeling in the credit literature are very similar [29]. The mortality intensity can be thought of as a hazard rate in the Cox process approach. When N is considered as a doubly stochastic process, i.e. Cox process random time dependent intensity driven by the μ, the counting process N is Poisson inhomogeneous with the parameter $\int_t^{\cdot} \mu_x(s)ds$.

The conditional distribution of N is defined by, $\forall\ T \geq t \geq 0$ and $k \geq 0$ [7],

$$P(N_T - N_t = k|\mathcal{F}_t) = \frac{(\int_t^T \mu_x(s)ds)^k}{k!} e^{-\int_t^T \mu_x(s)ds}. \tag{2.2.15}$$

If k is taken zero, the conditional probability of survival from time t to T, for the age $x{+}t$ at time t, is obtained as

$$P(T > t|T > 0) = e^{-\int_t^T \mu_x(s)ds}. \tag{2.2.16}$$

The time of death of a person is modeled as a stopping time τ with respect to the filtration, $\mathcal{F}_t$ containing both financial and mortality information. $\mathcal{F}_t = \mathcal{F}_t^{\tau} \vee \mathcal{H}_t$ and $\mathcal{F}_t^{\tau}$ is defined as

$$x_t = \{1_{\{\tau_x \leq t\}}, t \geq 0\},$$
$$\mathcal{F}_t^{\tau} = \bigvee_x \mathcal{F}_t^x.$$

$\mathcal{H}_t$ is a sub filtration and contains the all information except a person's life status (alive or death). $\mathcal{F}_t = \mathcal{F}_t^{\tau} \vee \mathcal{H}_t$ and $\mathcal{H}_t = \mathcal{M}_t \vee \mathcal{G}_t$ where $\mathcal{M}_t$ contains the information about mortality market, and $\mathcal{G}_t$ contains the information about financial market.

When insurance companies calculate life insurance premiums and reserves, they generally use a deterministic mortality intensity. This is a drawback for insurance companies. Deterministic rates may cause financial insolvencies due to life insurance contracts generally having longer maturities. If they use deterministic mortality rates, they do not consider mortality risk properly. This situation may negatively affect companies' financial positions in the long run. Some mortality modeling techniques are discussed in the next sections to calculate more accurate premiums and reserves.

2.3 Survival Probability Modeling with Affine Term Structures

The mortality intensity process is defined on the probability space $(\Omega, \mathcal{F}, P)$ and modeled under the objective measure. The P dynamics of the mortality intensity is given by [16],

$$d\mu_x(t) = \alpha_\mu(t, \mu_x(t))dt + \sigma_\mu(t, \mu_x(t))dw_t^{P,\mu}, \tag{2.3.17}$$

where $dw_t^{P,\mu}$ is the increment of the standard Wiener process. Parameters α_μ and σ_μ specify the drift term and diffusion term, respectively. The coefficients of the model are functions of the current value of the mortality intensity, therefore the mortality intensity is a Markov Process under the conditions of existence and uniqueness of the solution [27].

If the mortality intensity is known, the survival probability from time *0* to t, for the age x at time *0*, can be calculated as follows:

$$_tp_x = e^{-\int_0^t \mu_x(s)ds}. \tag{2.3.18}$$

Despite the fact that the future development of the mortality intensity is not known, it is taken as the expectation of the mortality intensity with respect to the development of the current time.

In the interest rate theory some interest rate models can be formulated in an affine form. The survival probabilities can be formulated like a bond price. Instead of using instantaneous interest rate in bond pricing, instantaneous mortality rate is used. Modeling mortality in an affine form is very useful, because when applying the stochastic mortality to different contracts, affine form satisfies analytical tractability.

$$_{T-t}p_{x+t} = E_t^P[e^{-\int_t^T \mu_x(s)ds}] = p(\mu_x(t), t, T) = e^{A(t,T)-B(t,T)\mu_x(t)}. \tag{2.3.19}$$

(2.3.19) gives the survival probability from time t to T for the age $x+t$. In an arbitrage free market, $p(\mu_x(t), t, T)$ will satisfy the term structure equation. Feynman-Kac stochastic representation formula of (2.3.19) is as follows:

$$\frac{\partial p(\mu_x(t), t, T)}{\partial t} + \alpha_\mu \frac{\partial p(\mu_x(t), t, T)}{\partial \mu} + \frac{1}{2}\sigma_\mu^2 \frac{\partial^2 p(\mu_x(t), t, T)}{\partial \mu^2} - \mu_x p(\mu_x(t), t, T) = 0, \tag{2.3.20}$$

$$p(\mu_x(t), T, T) = 1 \tag{2.3.21}$$

and it is obtained:

$$A_t(t,T) - (1 + B_t(t,T))\mu - \alpha_\mu(t, \mu_{[x]+t})B(t,T) + \frac{1}{2}\sigma_\mu^2 B^2(t,T) = 0. \tag{2.3.22}$$

The boundary value of $p_t(\mu_x(t), T, T) = 1$ implies *A(T,T)=0* and *B(T,T)=0*. A, B, α_μ and σ_μ should satisfy (2.3.20) to exist an ATS. If α_μ and σ_μ^2 are both affine functions of μ, then (2.3.22) becomes a separable differential equation for the unknown functions A and B. α_μ and σ_μ^2 are assumed to have the following forms:

$$\alpha_\mu(t,\mu) = \eta(t)\mu + \vartheta(t), \tag{2.3.23}$$

$$\sigma_\mu(t,\mu) = \sqrt{\upsilon(t)\mu + \psi(t)}. \tag{2.3.24}$$

When (2.3.22) is arranged, it transforms into

$$\begin{aligned} A_t(t,T) &- \vartheta(t)B(t,T) + \frac{1}{2}\psi(t)B^2(t,T), \\ &- [1 + B_t(t,T) + \eta(t)B(t,T) - \frac{1}{2}\upsilon(t)B^2(t,T)]\mu = 0. \end{aligned} \tag{2.3.25}$$

This equation holds for all t, T and μ. Because of the equation holding for all values of μ, the coefficients of μ should be zero. Therefore, the below equation is obtained,

$$B_t(t,T) + \eta(t)B(t,T) - \frac{1}{2}\upsilon(t)B^2(t,T) = -1. \tag{2.3.26}$$

Since the coefficient of μ is zero, the other terms must be equal to zero. This gives the equation:

$$A_t(t,T) = \vartheta(t)B(t,T) - \frac{1}{2}\psi(t)B^2(t,T). \tag{2.3.27}$$

In conclusion, the equations are as follows:

$$\begin{cases} \alpha_\mu(t,\mu) = \eta(t)\mu + \vartheta(t), \\ \sigma_\mu(t,\mu) = \sqrt{\upsilon(t)\mu + \psi(t)}. \end{cases} \tag{2.3.28}$$

$$\begin{cases} B_t(t,T) + \eta(t)B(t,T) - \frac{1}{2}\upsilon(t)B^2(t,T) = -1, \\ B(T,T) = 0. \end{cases} \tag{2.3.29}$$

$$\begin{cases} A_t(t,T) = \vartheta(t)B(t,T) - \frac{1}{2}\psi(t)B^2(t,T), \\ A(T,T) = 0. \end{cases} \tag{2.3.30}$$

(2.3.29) is a Ricatti equation for the determination of B not involve A. Having solved (2.3.29) is inserted the solution B into (2.3.30) and integrate to obtain A.

When $A(t,T)$ and $B(t,T)$ are deterministic functions, the model is termed an affine term-structure model. Generally, A and B are demonstrated as functions of two arguments, t and T. Let τ is $T-t$, then it is shown that $A(t,T) = A(\tau)$ and $B(t,T) = B(\tau)$. The survival probability is as follows:

$$p(\mu_x(t), t, T) = e^{A(\tau) - B(\tau)\mu_x(t)}. \tag{2.3.31}$$

This model is for a special age x. If the mortality intensity is assumed the same for all ages then it can be formulated without specific age x, but this assumption will be unrealistic.

When the mortality intensity is modeled by [42],

$$d\mu_x(t) = (\beta - \alpha\mu_x(t))dt + \sigma dw_t^{P,\mu} \tag{2.3.32}$$

$$
\begin{cases} B_t(t,T) - \alpha B(t,T) = -1 \\ B(T,T) = 0 \end{cases} \tag{2.3.33}
$$

$$
\begin{cases} A_t(t,T) = \beta B(t,T) - \frac{1}{2}\sigma^2 B^2(t,T) \\ A(T,T) = 0 \end{cases} \tag{2.3.34}
$$

When $B(t,T)$ is solved and $A_t(t,T)$ is integrated, the following equations are obtained

$$
A(t,T) = A(\tau) = \frac{(B(t,T) - T + t)(\alpha\beta - \frac{1}{2}\sigma^2)}{\alpha^2} - \frac{\sigma^2 B^2(t,T)}{4\alpha}, \tag{2.3.35}
$$

$$
B(t,T) = B(\tau) = \frac{1}{\alpha}[1 - e^{-\alpha(T-t)}]. \tag{2.3.36}
$$

When the mortality intensity is modeled by [15],

$$
d\mu_x(t) = \kappa(\theta - \mu_x(t))dt + \sigma\sqrt{\mu_x(t)}dw_t,^{P,\mu} \tag{2.3.37}
$$

$$
A(t,T) = A(\tau) = \frac{2\kappa\theta}{\sigma^2} ln(\frac{2he^{(\kappa+h)(T-t)/2}}{2h + (\kappa+h)(e^{(T-t)h} - 1)}), \tag{2.3.38}
$$

$$
B(t,T) = B(\tau) = \frac{2(e^{(T-t)h} - 1)}{2h + (\kappa+h)(e^{(T-t)h} - 1)}, \tag{2.3.39}
$$

$$
h = \sqrt{\kappa^2 + 2\sigma^2}. \tag{2.3.40}
$$

More information about affine term structures can be found in [9]. It should be remembered that using affine term structures satisfies some advantages such as analytical tractability, clear interpretation of the factors and compatibility with option pricing.

2.4 Survival Probability and Mortality Modeling With Jump Process

Biffis in [7] explained the use of a discontinuous setup for modeling the intensity of mortality as follows: “The use of a discontinuous setup for modeling the intensity of mortality may sound unfamiliar. However, we note that neither the size nor the frequency of discontinuity shocks need be unreasonably large. Moreover, the gain in flexibility and distributional richness that can be achieved by adding a discontinuous source of risk in the dynamics of μ can justify the abuse made in terms of path by path behaviour.”

Survival probability can change extremely due to the development of the health area, i.e. if one cured AIDS, survival probabilities of people would increase or due to the bird flu pandemic, probabilities may decrease. Due to the reasons mentioned by Biffis in [7], a jump component can be added to the model of the survival probability and the mortality intensity.

The probability that a person aged x at year t will attain age $x+1$ is denoted by p_x. The probability is modeled as a jump diffusion process such that for every fixed $x \geq 0$ the probability has dynamics of the form

$$dp_x(t) = k(\theta - p_x(t))dt + \sigma dw_t^{P,\mu} + Jd\pi_t(h), \qquad (2.4.41)$$

where k is the mean reversion coefficient, θ is a central tendency parameter for the survival probability, t is the time parameter, σ is the diffusion coefficient, and $d\pi$ is the increment of the Poisson process with intensity rate h. J is a random variable independent of $w_t^{P,\mu}$ and π having finite fourth moment.

2.4.1 Estimation of The Jump Model Parameters

The parameters of (2.4.41) can be estimated by the method of moments and maximum likelihood method. Maximum likelihood method can be used when the jumps are normally distributed.

2.4.2 The Method of Moments

In order to derive the T interval characteristic function $F(p, T; s)$ of p_x in the interval [0,T], Kolmogrov backward equation (KBE) is solved subject to the boundary condition which is defined by

$$F(p, T = 0; s) = e^{isp}, \tag{2.4.42}$$

where p is the initial value, $p_x(0)$ and $i = \sqrt{-1}$. The backward equation is as follows [17]:

$$0 = \frac{\partial F}{\partial p} k(\theta - p_x) + \frac{1}{2}\frac{\partial F}{\partial r^2}\sigma^2 - \frac{\partial F}{\partial T} + hE(F(p + J) - F(p)). \tag{2.4.43}$$

The last term comes from the effect of the Poisson shock. The characteristic function is defined as

$$F(p, T; s) = e^{A(T;s)+pB(T;s)}. \tag{2.4.44}$$

Das in [17] developed the stochastic differential equation with jump's moments by differentiating the characteristic function with respect to s and valuing the derivative at $s = 0$. μ_n shows the nth moment, and F_n is the nth derivative of F with respect to s, i.e. $F_n = \frac{\partial^n F}{\partial s^n}$. Based on above explanations, μ_n can be obtained as follows [17]:

$$\mu_n = \frac{1}{s^n} F_n(s = 0). \tag{2.4.45}$$

A_n and B_n are the nth derivative of A and B with respect to s.

$$A(T; s) = \int (k\theta i s e^{-kT} - \frac{1}{2}\sigma^2 s^2 e^{-2kT} + hE[e^{isJe^{-kT}} - 1])dT, \tag{2.4.46}$$

$$\frac{dA}{ds} = \int (k\theta i e^{-kT} - \sigma^2 s e^{-2kT} + hie^{-kT} E[Je^{isJe^{-kT}}])dT, \tag{2.4.47}$$

$$\frac{d^2A}{ds^2} = \int (-\sigma^2 e^{-2kT} - he^{-2kT} E[J^2 e^{isJe^{-kT}}])dT, \tag{2.4.48}$$

$$\frac{d^3A}{ds^3} = \int (-ihe^{-3kT} E[J^3 e^{isJe^{-kT}}])dT, \tag{2.4.49}$$

$$\frac{d^4A}{ds^4} = \int (-ihe^{-4kT} E[J^4 e^{isJe^{-kT}}])dT. \tag{2.4.50}$$

The integrals and $E[.]$ are bounded. It can be obtained that when the derivatives are evaluated at $s = 0$, they are as follows:

$$\left(\frac{dA}{ds}\right)_{s=0} = \int i(k\theta e^{-kT} + he^{-kT} E[J])dT = i(-\theta e^{-kT} - \frac{h}{k} e^{-kT} E[J]) + c_1. \tag{2.4.51}$$

$A(T = 0; s)$ is equal to zero, therefore $c_1 = \theta + \frac{h}{k} E[J]$, when it is substituted this to equation then

$$\left(\frac{dA}{ds}\right)_{s=0} = i((\theta + \frac{h}{k}E[J])(1 - e^{-kT})). \tag{2.4.52}$$

Other derivatives can be obtained similarly;

$$\left(\frac{d^2A}{ds^2}\right)_{s=0} = -(\frac{\sigma^2 + hE[J^2]}{2k})(1 - e^{-2kT}), \tag{2.4.53}$$

$$\left(\frac{d^3A}{ds^3}\right)_{s=0} = -ihE[J^3](\frac{1 - e^{-3kT}}{3k}), \tag{2.4.54}$$

$$\left(\frac{d^4A}{ds^4}\right)_{s=0} = hE[J^4](\frac{1 - e^{-4kT}}{4k}). \tag{2.4.55}$$

These steps can be also done for B. The derivatives of B are obtained as follows:

$$(\frac{dB}{ds}) = ie^{-kT}, \tag{2.4.56}$$

$$(\frac{d^2B}{ds^2}) = (\frac{d^3B}{ds^3}) = (\frac{d^4B}{ds^4}) = 0. \tag{2.4.57}$$

The following derivative is frequently used in the expressions of the moments,

$$\left(\frac{dA}{ds} + p\frac{dB}{ds}\right)_{s=0} = i((\theta + \frac{h}{k}E[J])(1 - e^{-kT}) + pe^{-kT}) = i\mu_1. \tag{2.4.58}$$

The moments for the distribution of the probability are defined as

$$\mu_n = \frac{1}{i^n}F_n(s = 0). \tag{2.4.59}$$

The first, second, third and the fourth moments can be evaluated as follows: The first moment is defined by

$$\mu_1 = \frac{1}{i}(\frac{dF}{ds})_{s=0} = \frac{1}{i}(\frac{dA}{ds} + p\frac{dB}{ds})_{s=0}. \tag{2.4.60}$$

Because of $A(s = 0) = 0$ and $B(s = 0) = 0$, and using above results the first moment can be rewritten as

$$\mu_1 = (\theta + \frac{h}{k}E[J])(1 - e^{-kT}) + pe^{-kT}. \tag{2.4.61}$$

The second moment is defined by

$$\begin{aligned}
\mu_2 &= \frac{1}{i^2}(\frac{d^2F}{ds^2})_{s=0} \\
&= \frac{1}{i^2}(e^{A+pB}[\frac{d^2A}{ds^2} + (\frac{dA}{ds} + p\frac{dB}{ds})^2])_{s=0} \\
&= -(\frac{d^2A}{ds^2})_{s=0} - [(\frac{dA}{ds} + p\frac{dB}{ds})^2]_{s=0} \\
&= \frac{\sigma^2 + hE[J^2]}{2k}(1 - e^{-2kT}) + ((\theta + \frac{h}{k}E[J])(1 - e^{-kT}) + pe^{-kT})^2 \\
&= \frac{\sigma^2 + hE[J^2]}{2k}(1 - e^{-2kT}) + \mu_1^2. \qquad (2.4.62)
\end{aligned}$$

The third moment is defined by

$$\begin{aligned}
\mu_3 &= \frac{1}{i^3}(\frac{d^3A}{ds^3} + 3\frac{d^2A}{ds^2}(\frac{dA}{ds} + p\frac{dB}{ds}) + (\frac{dA}{ds} + p\frac{dB}{ds})^3)_{s=0} \\
&= hE[J^3](\frac{1 - e^{-3kT}}{3k}) + 3\mu_1(\sigma^2 + hE[J^2])(\frac{1 - e^{-2kT}}{2k}) + \mu_1^3. \; (2.4.63)
\end{aligned}$$

Lastly the fourth moment is defined by

$$
\begin{aligned}
\mu_4 &= \frac{1}{i^4}[\frac{d^4A}{ds^4}+3(\frac{d^2A}{ds^2})^2+4\frac{d^3A}{ds^3}(\frac{dA}{ds}+p\frac{dB}{ds})+6\frac{d^2A}{ds^2}(\frac{dA}{ds}+p\frac{dB}{ds})^2 \\
&+ (\frac{dA}{ds}+p\frac{dB}{ds})^4]_{s=0} \\
&= hE[J^4](\frac{1-e^{-4kT}}{4k})+3((\sigma^2+hE[J^2])(\frac{1-e^{-2kT}}{2k}))^2 \\
&+ 4\mu_1 hE[J^3](\frac{1-e^{-3kT}}{3k}) \\
&+ 6\mu_1^2((\sigma^2+hE[J^2])(\frac{1-e^{-2kT}}{2k})) \\
&+ \mu_1^4. \qquad (2.4.64)
\end{aligned}
$$

These results can be used to estimate the model parameters, in other words, for the method of moment estimation.

To sum up, the moments are as follows:

$$\mu_1 = (\theta+\frac{hE[J]}{k})(1-e^{-kT})+pe^{-kT}, \qquad (2.4.65)$$

$$\mu_2 = \frac{\sigma^2+hE[J^2]}{2k}(1-e^{-2kT})+\mu_1^2, \qquad (2.4.66)$$

$$\mu_3 = hE[J^3](\frac{1-e^{-3kT}}{3k})+3\mu_1(\sigma^2+hE[J^2])(\frac{1-e^{-2kT}}{2k})+\mu_1^3, \qquad (2.4.67)$$

$$
\begin{aligned}
\mu_4 &= hE[J^4](\frac{1-e^{-4kT}}{4k}) + 3((\sigma^2 + hE[J^2])(\frac{1-e^{-2kT}}{2k}))^2 \\
&+ 4\mu_1 hE[J^3](\frac{1-e^{-3kT}}{3k}) + 6\mu_1^2((\sigma^2 + hE[J^2])(\frac{1-e^{-2kT}}{2k})) \\
&+ \mu_1^4. \qquad (2.4.68)
\end{aligned}
$$

Diagnostics:

The variance of the jump diffusion process is as follows:

$$
V = \mu_2 - \mu_1^2 = \frac{\sigma^2 + hE[J^2]}{2k}(1 - e^{-2kT}). \qquad (2.4.69)
$$

The skewness is as follows:

$$
\begin{aligned}
S = Skewness &= \frac{E(J-\mu_1)^3}{(\mu_2 - \mu_1^2)^{3/2}} \\
&= \frac{2\sqrt{2k}e^{-kT}(1 + e^{kT} + e^{2kT})hE(J^3)}{3(1+e^{kT})(\sigma^2 + hE(J^2))\sqrt{(1-e^{-2kT})(\sigma^2 + hE(J^2))}}.
\end{aligned}
\qquad (2.4.70)
$$

The kurtosis is as follows:

$$
\begin{aligned}
K = Kurtosis &= \frac{E(J-\mu_1)^4}{(\mu_2 - \mu_1^2)^2} \\
&= \frac{(e^{2kT}-1)(3h^2E(J^2)^2 + 6h\sigma^2E(J^2) + 3\sigma^4) + khE(J^4)(e^{2kT}+1)}{(e^{2kT}-1)(\sigma^2 + hE(J^2))^2}.
\end{aligned}
\qquad (2.4.71)
$$

$$\min_{(\psi)} \left[\begin{pmatrix} V - \hat{V} \\ S - \hat{S} \\ K - \hat{K} \end{pmatrix} \begin{pmatrix} V - \hat{V} \\ S - \hat{S} \\ K - \hat{K} \end{pmatrix}^T \right],$$

where ψ includes k, h, σ, $E(J^2)$, $E(J^3)$, the square of the differences is minimized with respect to ψ. Parameters of the model can be found by this way.

One year survival probabilities are only obtained by (2.4.41). The survival probabilities within small intervals are obtained by the modeling with the mortality intensity. For this purpose, the mortality intensity is modeled by a jump diffusion process. The bond prices for the interest rate model having jump component were modeled and obtained approximated formula for the bond prices in [19]. The mortality intensity has dynamics as the form

$$d\mu_x(t) = k(\theta - \mu_x(t))dt + \sigma dw_t^{P,\mu} + Jd\pi_t(h), \qquad (2.4.72)$$

where k is the mean reversion coefficient, θ is the long term mortality intensity, t is time, σ is the volatility coefficient, $dw_t^{p,\mu}$ is the increment of Brownian motion, J is the jump size, $d\pi$ is the increment in a Poisson process with intensity rate h, and τ is the $(T-t)$. The survival probability is a function of time to maturity and the mortality intensity. It is assumed that all time dependences comes from μ. The dynamics of the survival probability is denoted by using Ito's formula for jump diffusion processes. Ito's formula is as follows:

$$\begin{aligned} dp(\mu_x(t), t, T) &= \left(\frac{\partial p(\mu, t, T)}{\partial \mu} k(\theta - \mu) + \frac{1}{2}\sigma^2 \frac{\partial^2 p(\mu, t, T)}{\partial \mu^2}\right)dt \\ &+ \sigma \frac{\partial p(\mu, t, T)}{\partial \mu} dw_t^{p,\mu} + [p(\mu + J, t) - p(\mu, t)]d\pi(h) \qquad (2.4.73) \end{aligned}$$

The PDDE for the survival probability is as follows:

$$
\begin{aligned}
0 &= [k(\theta-\mu)-\lambda\sigma]\frac{\partial p(\mu,t,T)}{\partial \mu}+\frac{\partial p(\mu,t,T)}{\partial t}+\frac{1}{2}\sigma^2\frac{\partial^2 p(\mu,t,T)}{\partial \mu^2}-\mu p(\mu,t,T)\\
&+ hE_t[p(\mu+J,t)-p(\mu,t)]
\end{aligned}
$$

with the boundary condition, $p(\mu,\tau=0)=1$. The boundary value implies $A(0)=1$ and $B(0)=0$. λ is the market price of mortality risk. The market price of jump risk is unsystematic and diversifiable. Given the proposed affine solution, $p(\mu,t,T)=A(\tau)e^{-B(\tau)\mu}$, the PDDE (partial difference differential equation) becomes

$$
\begin{aligned}
0 &= [k(\theta-\mu)-\lambda\sigma]\frac{\partial p(\mu,\tau)}{\partial \mu}-\frac{\partial p(\mu,\tau)}{\partial \tau}+\frac{1}{2}\sigma^2\frac{\partial^2 p(\mu,\tau)}{\partial \mu^2}-\mu p(\mu,\tau)\\
&+ hp(\mu,\tau)E_t[e^{-B(\tau)J}-1] \qquad (2.4.74)
\end{aligned}
$$

and

$$
\begin{aligned}
1 &= \frac{\partial B}{\partial \tau}+kB, \qquad (2.4.75)\\
\frac{\frac{\partial A}{\partial \tau}}{A} &= (\lambda\sigma-k\theta)B+\frac{1}{2}\sigma^2B^2+hE_t[e^{-B(\tau)J}-1], \qquad (2.4.76)
\end{aligned}
$$

where $E_t[e^{-B(\tau)J}]$ is the moment generating function of the distribution of jump sizes. The jump sizes are assumed normally distributed with mean α and γ^2

variance by [4] and [1]. The PDDE is as follows:

$$\begin{aligned} 0 &= [k(\theta-\mu)-\lambda\sigma]\frac{\partial p(\mu,\tau)}{\partial \mu} - \frac{\partial p(\mu,\tau)}{\partial \tau} + \frac{1}{2}\sigma^2\frac{\partial^2 p(\mu,\tau)}{\partial \mu^2} \\ &- \mu p(\mu,\tau) + hp(\mu,\tau)(e^{(-\alpha B(\tau)+\frac{1}{2}\gamma^2 B^2(\tau))}-1) \end{aligned}$$

with

$$\frac{\partial B}{\partial \tau} + kB = 1$$

and

$$\frac{\frac{\partial A}{\partial \tau}}{A} = (\lambda\sigma - k\theta)B + \frac{1}{2}\sigma^2 B^2 + h(e^{(-\alpha B(\tau)+\frac{1}{2}\gamma^2 B^2(\tau))}-1) \qquad (2.4.77)$$

As mentioned in [19], [4] and [1] used two term Taylor series approximation of the exponential function within (2.4.74), $exp(-B(\tau)J)$. After two term Taylor series approximation, (2.4.74) will be as follows:

$$\begin{aligned} 0 &= [k(\theta-\mu)-\lambda\sigma]\frac{\partial p(\mu,\tau)}{\partial \mu} - \frac{\partial p(\mu,\tau)}{\partial \tau} + \frac{1}{2}\sigma^2\frac{\partial^2 p(\mu,\tau)}{\partial \mu^2} - \mu p(\mu,\tau) \\ &+ hp(\mu,\tau)E_t[(1-JB+\frac{J^2B^2}{2})-1]. \end{aligned} \qquad (2.4.78)$$

There is no assumption about the distribution of jump sizes. First and second moments of the distribution are only needed.

$$
\begin{aligned}
0 &= [k(\theta-\mu)-\lambda\sigma]\frac{\partial p(\mu,\tau)}{\partial\mu}-\frac{\partial p(\mu,\tau)}{\partial\tau}+\frac{1}{2}\sigma^2\frac{\partial^2 p(\mu,\tau)}{\partial\mu^2}-\mu p(\mu,\tau) \\
&+ hp(\mu,\tau)(-\alpha B+\frac{(\mu^2+\gamma^2)B^2}{2}), \qquad (2.4.79)
\end{aligned}
$$

where α and γ^2 are the mean and variance of the distribution, respectively, and

$$
\begin{aligned}
1 &= \frac{\partial B}{\partial\tau}+kB \qquad (2.4.80) \\
\frac{\frac{\partial A}{\partial\tau}}{A} &= (\lambda\sigma-k\theta-h\alpha)B+\frac{1}{2}[\sigma^2+h(\alpha^2+\gamma^2)]B^2, \qquad (2.4.81)
\end{aligned}
$$

where *A(0)=1* and *B(0)=0*, and the approximate closed form solution for survival probability is as follows:

$$
\begin{aligned}
p(\mu,\tau) &= exp\{-\frac{1-e^{-k\tau}}{k}\mu+\frac{M_1k+M_2}{k^2}\tau \\
&+ \frac{M_1k+2M_2}{k^3}(e^{-k\tau}-1)-\frac{M_2}{2k^3}(e^{-2k\tau}-1)\}, \qquad (2.4.82)
\end{aligned}
$$

where $M_1=-k\theta+\lambda\sigma-h\alpha$ and $M_2=\frac{1}{2}(\sigma^2+h(\alpha^2+\gamma^2))$. The condition $M_1k+M_2<0$ should be satisfied to satisfy $\lim\limits_{\tau\to\infty}(p(\mu,\tau))=0$.

2.4.3 Maximum Likelihood Estimation

When (2.4.41) is discretized, the new equation will be as follows:

$$
\triangle p=k(\theta-p)\triangle t+\sigma\triangle w^{P,\mu}+J(\mu,\gamma^2)\triangle\pi(q). \qquad (2.4.83)
$$

If the jumps are normally distributed, the Poisson-Gaussian model can be estimated by using a Bernoulli approximation [17]. Important point is that at most one jump can be occurred in each time interval.

$$\begin{aligned} f[p(s)|p(t)] &= q\exp\left(\frac{-(p(s)-p(t)-k(\theta-p(t))\triangle t-\mu)^2}{2(\sigma^2\triangle t+\gamma^2)}\right)\frac{1}{\sqrt{2\pi(\sigma^2\triangle t+\gamma^2)}} \\ &\quad +(1-q)\exp\left(\frac{-(p(s)-p(t)-k(\theta-p(t))\triangle t)^2}{2\sigma^2\triangle t}\right)\frac{1}{\sqrt{2\pi\sigma^2\triangle t}}, \end{aligned} \tag{2.4.84}$$

where $q = h\triangle t + O(\triangle t)$. This approximates the true Poisson-Gaussian density with a mixture of normal distributions. The aim is to maximize the function L,

$$\max_{\Omega=[k,\sigma,\theta,\mu,\gamma^2,q]}(L) = \max_{\Omega=[k,\sigma,\theta,\mu,\gamma^2,q]}\left(\sum_{t=1}^{T}(\log(f[p(t+\triangle t)|p(t)]))\right). \tag{2.4.85}$$

2.5 The Kalman Filter Technique

This section introduces a very useful tool named after the contributions of R. E. Kalman(1960). The Kalman filter technique is very useful when the underlying state variables are not observable. The one year probability of survival can be generated by a certain group of lives which has been under observation for a certain period. The one year survival probabilities are observable, but the underlying state variables may be unobservable. The Kalman filter technique can be applied to determine parameters of the survival probabilities.

The Kalman filter begins with a measurement system that represents the relationship between logarithmic survival probabilities and a transition system which is the model of state variables.

The measurement and transition equations are the state space form of the model. The Kalman filter uses the state space formulation recursively to make inferences about the unobserved values of the state variables by conditioning on the observed values. Finally, maximum likelihood function is used to find the model parameter by using recursive inferences. Theoretical side of the technique is found in [22].

Application of the Kalman filter technique to the interest rates can be found in the literature, for instance [10]. For example, if the mortality intensity are modeled by CIR, the Kalman filter's steps will be as follows:

Measurement Equation:

$$ln(p(\mu_x(t),t,T)) = a_t + b_t Y_{t|t-1} + v_t. \tag{2.5.86}$$

Transition Equation:

$$\begin{aligned}
Y_{t|t-1} &= \theta(1-e^{(-\kappa dt)}) + e^{-\kappa dt} Y_{t-1|t-1} + u_t, && (2.5.87)\\
Y_{t|t-1} &= d(\psi) + \phi(\psi) Y_{t-1|t-1} + u_t, && (2.5.88)\\
u_t &\sim N(0, Q_t), &&\\
v_t &\sim N(0, R), &&\\
Q_t &= \frac{\theta\sigma^2}{2\kappa}(1-e^{-kdt})^2 + \frac{\sigma^2}{\kappa}(e^{-kdt} - e^{-2kdt})Y_{t-1}, && (2.5.89)\\
d(\psi) &= \theta(1-e^{(-\kappa dt)}), && (2.5.90)\\
\phi(\psi) &= e^{-\kappa dt}, && (2.5.91)
\end{aligned}$$

where ψ is the parameter set.

Step 1: *Initializing the state vector*

The appropriate initial values must be found to start the recursion, for this purpose the unconditional mean and variance of the transition system can be used. The unconditional mean and variance are

$$E(Y_1) = \theta, \tag{2.5.92}$$

$$Var(Y_1) = \frac{\sigma^2\theta}{2\kappa}. \tag{2.5.93}$$

Step 2: *Forecasting the measurement equation*

Transition system is used to estimate mean and variance of the measurement equation. The mean and variance are defined as

$$E(p(\mu_x(t),\tau)|F_{t-1}) = A + BE(Y_t|F_{t-1}), \tag{2.5.94}$$

$$Var(p(\mu_x(t),\tau)|F_{t-1}) = BVar(Y_t|F_{t-1})H^T + R. \tag{2.5.95}$$

Step 3: *Updating the inference about the state vector*

The real value of the measurement system is observed and then the prediction error is evaluated,

$$\zeta_t = p(\mu_x(t),\tau) - E(p(\mu_x(t),\tau)|F_{t-1}). \tag{2.5.96}$$

The prediction error is used to update the inference about the unobserved transition system. The conditional expectation is revised by using prediction error,

$$E(Y_t|F_t) = E(Y_t|F_{t-1}) + K_t \varsigma_t, \tag{2.5.97}$$

$$K_t = Var(Y_t|F_{t-1})B^T Var(Y_t|F_{t-1})^{-1}, \tag{2.5.98}$$

where K_t is called the Kalman gain matrix. It determines the weight is given to the new observation in the updated state system forecast.

The conditional variance of the state system is

$$Var(Y_t|F_t) = (I - K_t B)Var(Y_t|F_{t-1}). \tag{2.5.99}$$

Step 4: *Forecasting the state vector*

The next step is the finding the unknown values of the state system.

The conditional expectation:

$$E(Y_{t+1}|F_t) = d + \phi E(Y_t|F_t). \tag{2.5.100}$$

The conditional variance:

$$Var(Y_{t+1}|F_t) = Var(Y_t|F_{t-1}) - FVar(Y_t|F_t)F^T + Q_t. \tag{2.5.101}$$

Step 5: *Constructing the likelihood function*

Previous four steps must be repeated for each discrete time step in the data sample.

Using $Var(Y_t|F_{t-1})$ and ς_t the log likelihood function is constructed as

$$\begin{aligned} LogL(\psi) &= -\frac{nN}{2}log2\pi - \frac{1}{2}\sum_{t=1}^{N} log(det(Var(Y_t|F_{t-1}))) \\ &- \frac{1}{2}\varsigma_t^T Var(Y_t|F_{t-1})^{-1}\varsigma_t. \end{aligned} \tag{2.5.102}$$

2.6 The Market Price of Mortality Risk

In the literature, the market price of mortality risk is discussed by [16], [13], [39] and [6]. The filtered probability space $(\Omega, \mathcal{F}, F, P)$ is fixed, and $F = (\mathcal{F}_t)_{0 \leqslant t \leqslant T}$ contains all available information. $\mathcal{F}$ is a filtration generated by two independent Brownian motions $w^{P,\mu}$ and w and jump process M supposed independent from Brownian motions. $w^{P,\mu}$ drives the mortality intensity, and w drives the financial market, $\mathcal{M}_t = \sigma(w_u^{P,\mu}, u \leq t)$ and $\mathcal{G}_t = \sigma(w_u, u \leq t)$.

The insurance company experiences both systematic and unsystematic mortality risk. It is known that the life insurance contracts are not fully tradable in the financial markets or in the reinsurance markets. This leads to find an Equivalent Martingale Measure.

The probability measures P and Q are said to be equivalent, if

$\forall A \epsilon \mathcal{F}$, $P(A) = 0 \Leftrightarrow Q(A) = 0$.

An insurance contract is described by the process $N_{x_i}(t)$ which starts with value 1 at time 0 and jumps to zero at τ_i, $N_{x_i}(t) = 1_{\{\tau_i \leq t\}}$ for an insured aged x_i at time 0. The compensated process is defined by

$$M_{x_i}(t) = N_{x_i}(t) - \int_0^t 1_{\{\tau_i < t\}} \mu_{x_i}(s) ds \tag{2.6.103}$$

is a martingale.

The measure could be changed from P to Q by the Radon Nikodym density process $\wedge_t$ characterized by

$$\frac{d\wedge_t}{\wedge_{t-}} = d\ln(\frac{dQ}{dP}) = -k_t^{\mu} dw_t^{\mu} - \sum_{i=1}^{N} \rho_t^i dM_{x_i}(t). \tag{2.6.104}$$

It should be noted that the systematic mortality risk is not diversifiable for the insurance companies, on the other hand the unsystematic mortality risk is diversifiable when the size of portfolio increased. This satisfies to change the measure for all ages, because ρ_t^i does not needed to be specified for every individual. If $k^{\mu} = 0$ and $\rho_t^i = 0$ for all i, the working probability P coincides with the risk neutral probability Q.

Chapter 3

SURPLUS MODELING

Insurance companies may be in difficulty to meet their liabilities. Some of the factors contributing to the financial difficulties are as follows [38]:

- Incorrect pricing
- Insufficient reserving
- Incorrect underwriting and management decisions
- Losses on investment
- Random fluctuation of claims
- The fluctuation of the basic probabilities of the claims and their trends
- Catastrophic events like hurricanes, earthquakes
- The insolvency of the reinsurer
- Guaranteed rates to insureds

There are some bad experiences in history, although insurance companies seemed nondefaultable. As mentioned [14], the topic of insolvency risk in connection with life insurance companies has recently attracted a great deal of attention.

Some companies having solvency problems are as follows:

Company	Year
Pacific Standard Life Ins. Co. (US)	1989
Mutual Security Life Ins. Co. (US)	1990
First Executive Life Ins. Company (US)	1991
Fidelity Bankers Life Ins. Co. (US)	1991
Monarch Life Ins.Co. (US)	1994
Confederation Life Ins. Co. (US)	1994
Nissan Mutual Life (Jap.)	1997
Chiyoda Mutual Life Ins. Co. (Jap.)	2000
Kyoei Life Ins. Co. (Jap.)	2000
Equitable Life (UK)	2000
Tokyo Mutual Life Ins. (Jap.)	2001
HIH Ins. (Aus.)	2001

One of the main aims of modeling the surplus is obtain the ruin probability. It is a good indication of whether the insurer's assets are matched to liabilities of the insurance company sufficiently well [24]. If the company has positive surplus, it can collect less premium or if the company has negative surplus, it should raise some premiums or it should transfer some risks to reinsurance. These decisions can be done by analyzing the surplus (risk) process and these decisions affect competition between the companies.

3.1 Surplus Modeling

The surplus at time t of the insurer with an initial capital of $u > 0$ is given by

$$U_t = u + c_t - S_t, \quad t \geq 0,$$

where U_t denotes the surplus process or risk process at time t, c_t denotes premiums collected through time t, u is the initial surplus and S_t denotes aggregate claims process paid through time t [11]. Fig. 3.1 is an example graph of surplus process, i.e. risk process.

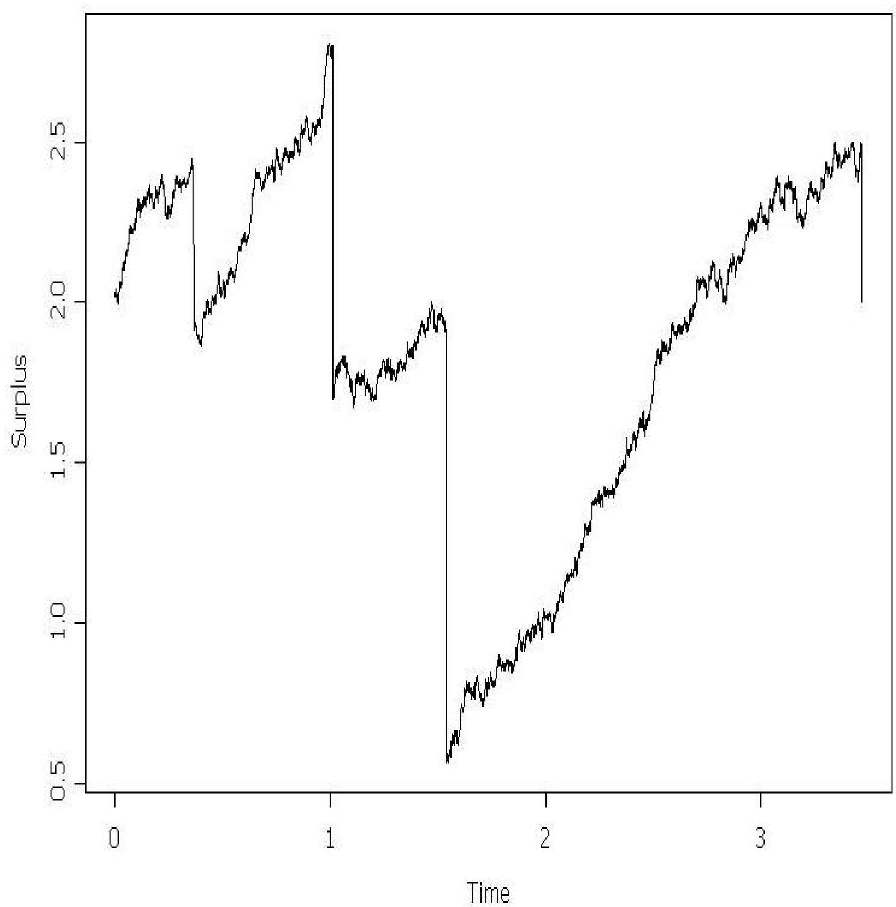

Figure 3.1: Risk process.

For a certain portfolio of insurance, let N_t denote the number of claims, S_t denote aggregate claims up to time t and X_i denote the amount of ith claim. By assumption initial number of claims, N_0, is zero. If N_t is equal to zero, S_t will be zero.

Then the aggregate claim process and the risk process can be written as follows:

$S_t = X_1 + X_2 + X_3 + X_4 + ... + X_{N_t}$,

$U_t = u + c_t - \sum_{i=1}^{N_t} X_i$.

The claim number process generally assumed to be a Poisson process, so it can be said that the aggregate claim process is a compound Poisson process,

$$Pr(N_{t+h} - N_t = k | N_s, s \leq t) = \frac{e^{-\lambda h}(\lambda h)^k}{k!}, \quad k = 0, 1, 2, 3, ... \;\; \forall t \geq 0 \;\; and \;\; h > 0. \tag{3.1.1}$$

The premiums are payable continuously at constant rate c per unit time, ignored interest. The total premium is ct. Premiums with a security loading will satisfy $ct > E(S_t)$ and c is defined as

$$c = (1 + \theta)\lambda\mu.$$

The properties of the Poisson process [11]:

1. The increments are stationary. The distribution of $N_{t+h} - N_t$, which is

Poisson with parameter λh, depends on the length of the interval but not its location, t.

2. For any set of disjoint time intervals, the increments are independent. $t_1 < t_1 + h_1 < t_2 < t_2 + h_2 < t_3$, the increments $N_{t_1+h_1} - N_{t_1}, N_{t_2+h_2} - N_{t_2}, ..., N_{t_n+h_n} - N_{t_n}$ are mutually independent.

3. The probability of simultaneous claims are zero,

   $Pr[N_{t+h} - N_t > 1] = 0, h \longrightarrow 0.$

In the classical Cramer-Lundberg process, the rate of premium income received by the insurance company is assumed to be constant. In real life, inflows of the company are supposed to be regular, but outflows are not. It can be irregular due to unknown events, for example accidents, catastrophic events etc. There are two reasons for why the outflows are stochastic: It is not known when the claim occurs, and how much it will cost.

The time of ruin is defined as

$$T = \min\{t : t \geq 0 \ \ and \ \ U_t < 0\}. \tag{3.1.3}$$

$\psi_u(t) = Pr(T < t)$ is the probability of ruin before time t. The adjustment coefficient, R, is the smallest positive solution to the equation,

$$1 + (1 + \theta)\mu R = M_X(R), \tag{3.1.4}$$

where $M_X(R)$ is the moment generating function of the claim severity random variable X, μ is the mean of claim amounts, θ is the premium loading factor or relative security loading. If R is known, Cramer's asymptotic ruin formula is defined as [26]

$$\psi(u) \approx \frac{\mu\theta e^{-Ru}}{M_X'(R) - \mu(1+\theta)}, \quad u \to \infty. \tag{3.1.5}$$

Details can be found in [26], [11], [31] and [33]. The more realistic approach is that outflows and inflows being stochastic. Surplus data can be obtained directly from the company since the collected premiums and paid benefits are already given. Modeling of the surplus process with using historical data as discussed in this thesis brings some easiness. That is, it is not needed to define the distribution for claim sizes and number of claims, and also simulations are done more realistically and easily. For this purpose, the approach in [30] is used. They introduced nonparametric estimators of the coefficients of a univariate jump diffusion process when observations, U, are recorded discretely. The model parameters, drift, diffusion and intensity, are supposed to be dependent on U. After the time discretization of U, the jumps are detected while the square of the increment (ΔU_t) between successive observations, $(\Delta U_t)^2$, exceeds a predetermined level.

The filtered probability space is $(\Omega, \mathcal{F}, F, P)$ as in paragraph (2.2).

The model is defined as

$$dU_t = \mu(U_t)dt + \sigma(U_t)dw_t^U + dJ_t, \quad t\epsilon[0,T], \tag{3.1.6}$$

where w^U is a Brownian motion and J is a pure jump independent of w^U. It is assumed that $(U_t)_{t\in[0,T]}$ is a real process, and the time interval should be fixed [0, T]. The stochastic process U can describe the evolution of an economic variable, as an interest rate or a logarithmic asset price, as well as any diffusion. In this thesis, U_t is thought as the surplus of an insurance company. The drift, μ, the volatility, σ and the intensity of jumps depend on U_t.

The jump process can be decomposed as the sum of two jumps bigger than one and the sum of the compensated jumps smaller than one. It can be showed as follows $J \equiv J_1 + \bar{J}_2$, and

$$\begin{aligned} J_{1s} &= \int_0^s \int_{|u|>1} um(dt, du), \\ \bar{J}_{2s} &= \int_0^s \int_{|u|\leq 1} u[m(dt, du) - v(du)dt], \end{aligned}$$

where m is the jumps random measure of U, and v is the Levy measure of J. J_{1s} represents the sum of the jumps bigger than one and it is a compound Poisson process which can be written as $J_{1s} = \sum_{l=1}^{N_s^1} X_l^1$.

The method in [30] is followed to model the surplus of an insurance company. The surplus process can be separated into two parts Υ_t and $J_{1,t}$,

$$U_t = \Upsilon_t + J_{1,t}, \tag{3.1.7}$$

where

$$\Upsilon_t = \int_0^t \mu_u du + \int_0^t \sigma_u dw_u, \tag{3.1.8}$$

$$J_{1,t} = \sum_{k=1}^{N_t} X_k, \;\; J_{1,0^-} = 0. \tag{3.1.9}$$

Some related notations are as follows :

- $\triangle_i L = L_{t_i} - L_{t_{i-1}}$ is the increase of L between t_i and t_{i-1}.

- $\triangle L_t = L_t - L_{t-}$ is the size of the jump at any time t.
- $(\tau_j)_{j \in N}$ is the jump instants of J_1 and $\tau^{(i)}$ is the instant of the first jump in $]t_{i-1}, t_i]$, if $\triangle_i N > 0$.
- h is the bandwidth parameter. It has the properties that when n goes to infinity, h goes to zero and $nh \to \infty$.
- $\bar{m}$ is the compensated measure of the jump process; $\bar{m}(dt, du) = m(dt, du) - v(du)dt$.
- $b(u)$ is a real deterministic function such that $\lim\limits_{n\to\infty} b(\delta) = 0$ and $\lim\limits_{n\to\infty} \frac{\delta \log \frac{1}{\delta}}{b(\delta)} = 0$ where $\delta = T/n$.

$$\forall i = 1, ..., n, \quad 1_{\{\triangle_i N = 0\}}(w) = 1_{\{(\triangle_i U)^2 \leq b(\delta)\}}(w) \tag{3.1.10}$$

(3.1.10) shows that if $(\Delta_i U)^2$ is greater than $b(u)$, it can be said that a jump occurred within $]t_{i-1}, t_i]$.

The whole jump process, J, can be estimated by using $\hat{U}_{\tau^{(i)}} = (\Delta_i U) 1_{(\Delta_i U)^2 > b(u)}$, and from the practical perspective $\hat{J}_{1,t} = \sum\limits_{i=1}^{t_i \wedge t} \triangle_i U \ 1_{\{(\triangle_i U)^2 > b(\delta)\}}$ is an appropriate approximation of the jump part of U_t, and then the approximation of the diffusion part will be as $\hat{\Upsilon} = U - \hat{J}_1$.

3.1.1 Estimation of The Jump Process Parameters

As mentioned in [23], the kernel estimation can be used when estimating the model parameters. A kernel function, a map $K(.) : [-1, 1] \to \mathbb{R}$, is generally used with a bandwidth parameter, h. A Gaussian kernel is usually used in estimation of interest rate procedures. A kernel function is set as follows:

$$K_h(z) = \frac{1}{h}K(\frac{z}{h}). \tag{3.1.11}$$

Let h determine the efficient width of the K_h, and the smoothness level. The higher h the more smoothness will be. If h is large, the width of the $K_h : [-h, h] \to \mathbb{R}$ will be large, and this h gives more smoothing. If h is small, the width of the $K_h : [-h, h] \to \mathbb{R}$ will be small, and so the smoothness will be little.

There is a problem to select the appropriate bandwidth parameter, h. It is a very important process to select h. It can be done by a process containing trial and error and experience, but more practical approach is selecting h as follows:

$$h = h_s \sigma N^{-1/5}, \tag{3.1.12}$$

where N is the number of data points in the sample and σ is a constant that practitioners determine.

The model parameters estimated by the kernel estimation are defined as follows [30]:

$$\hat{\sigma}^2_{n,h}(u) = \frac{n\sum_{i=1}^{n} K(\frac{U_{t_i}-\hat{J}_{1,t_i}-u}{h})(\Delta_i U)^2 1_{\{(\Delta_i U)^2 \le b(\delta)\}}}{T\sum_{i=1}^{n} K(\frac{U_{t_i}-\hat{J}_{1,t_i}-u}{h})}, \tag{3.1.13}$$

$$\hat{\mu}_{n,T}(u) = \frac{\sum_{i=1}^{n} K(\frac{U_{t_i}-\hat{J}_{1,t_i}-u}{h})(\Delta_i U) 1_{\{(\Delta_i U)^2 \le b(\delta)\}}}{\delta\sum_{i=1}^{n} K(\frac{U_{t_i}-\hat{J}_{1,t_i}-u}{h})}, \tag{3.1.14}$$

$$\hat{\lambda}_{n,T}(u) = \frac{\sum_{i=1}^{n} K(\frac{U_{t_i}-u}{h}) 1_{\{(\Delta_i U)^2 \ge b(\delta)\}}}{\delta\sum_{i=1}^{n} K(\frac{U_{t_i}-u}{h})}, \tag{3.1.15}$$

and the kernel function is defined as

$$K(z) = \frac{e^{-z^2/2}}{\sqrt{2\pi}}. \tag{3.1.16}$$

Chapter 4

LIFE INSURANCE CONTRACTS

Life insurance companies sell various contracts, such as life annuities, pure endowment contracts. An annuity contract satisfies its owner a series of payments for a fixed period or during the owner's life time. A whole life insurance provides for a payment following the death of the insured regardless of when it occurs. If the policy covers a certain set of time, it is called a term life insurance or an endowment insurance. Term life insurance and endowment are not alike, because term life insurance contracts pay benefit to insured, if he/she dies during the policy term, otherwise nothing pays to insured. Contrary to the term insurance contracts, endowments pay benefit to insureds, if the policy owner dies during the policy term and also pays benefits if the policy owner survives the policy term. A pure endowment pays the benefit, if the insured is living at the end of the maturity. Therefore an endowment insurance is the sum of a term life insurance and a pure endowment.

Besides these simple contracts, life insurance companies offer many other complex contracts such as bonus options, equity linked policies, participating life insurance contracts, guaranteed annuity options. Participating life insurance contracts provides a bonus, which is credited to the mathematical reserve and depends on the performance of a special investment portfolio and guarantees a minimum interest rate. Guaranteed annuity options provide the policyholder with the right to convert a deferred survival benefit into an annuity at a fixed conversion rate [8].

Main difference between the financial and actuarial valuation is the probability of survival. In actuarial valuation, a probability of survival and an interest rate are used. The actuarial discount rate (ADR) is as follows [39]:

$$ADR_x(t) = r_t + \mu_x(t). \tag{4.0.1}$$

This discount rate depends not only on the survival rate but also the interest rate. It is used when evaluating the different insurance contracts. There are a lot of types of insurance contracts. Some of the valuation methods are explained below. When experimenting numerical examples, it is assumed that the market price of mortality risk is zero, so that the cash flows can be priced using real world mortality intensities.

4.1 Pure Endowments

A pure endowment of maturity n provides for payment of the sum insured only if the insured is alive at the end of the contract maturity ([20], [32]). The value of a pure endowment contract, ${}_nE_x(t)$, for a x-year old is given by *(under the risk neutral measure)* [39]

$${}_nE_x(t) = E_t^Q[\exp(-\int_0^n ADR_{x+s}(t+s)ds)]. \tag{4.1.2}$$

If the independence between financial market and mortality is assumed, the expectation can be factorized. The premium for the pure endowment with benefit C_0 is defined as

$$ {}_nE_x(t) = C_0 P(t, t+n) p^Q(\mu_x(t), t, T), \tag{4.1.3} $$

where *P(t,T)* is the price of a zero coupon bond with maturity T at time t.

4.2 Life Annuities

A life annuity provides for periodic payments, usually equal size, each payment contingent upon the survival of a policyholder. In a temporary life annuity, payments cease at the end of a specified period [32]. Let $a_{\overline{x:n}|}$ denote the present value of this payment stream.

$$ a_{\overline{x:n}|} = \sum_{i=1}^{n} {}_iE_x(t) = \sum_{i=1}^{n} P(t, t+i) p^Q(\mu_x(t), t, t+i). \tag{4.2.4} $$

4.3 Participating Life Insurance Contracts

In participating life insurance contracts, at the end of each policy calendar, the insurance company grants a bonus, which is credited to the mathematical reserve and depends on the performance of a special investment portfolio. In other words, the policyholder is guaranteed a fixed interest rate r_g. On top of this fixed rate, the policy holder is entitled to a share δ of the assets of the life insurance company. The guaranteed rate r_g is usually less than the market rate for a risk free asset of the same maturity as the policy.

Some recent contributions in this direction are Briys and Varenne in [12], Grosen and Jorgensen in [21], Bacinello in [3], Miltersen and Persson in [35], Bernard, Olivier Le Courtois and François Quittard-Pinon in [5].

Let U_t denote the value of the underlying fund at time t and r_t denote the riskless short rate process. Under the risk neutral measure Q, U_t follows the jump diffusion process and r_t has the Vasicek dynamics:

$$\begin{aligned} dr_t &= a(\theta - r_t)dt + v dw_{1,t}^{Q}, &\quad (4.3.5)\\ \frac{dU_t}{U_t} &= (r_t - q(t)k)dt + \sigma(U_t)dw_t^{U,Q} + (e^Y - 1)dJ_t. &\quad (4.3.6)\end{aligned}$$

J_t is a Poisson process with intensity function $q(t)$. The term $q(t)k$ is subtracted from the instantaneous mean rate of return in order to adjust the effect from random jumps. Y is normally distributed with mean μ_Y and variance σ_Y^2, and k is defined by

$$k = E[e^Y - 1] = e^{(\mu_Y + \sigma_Y^2/2)} - 1. \quad (4.3.7)$$

J_t, Y and $w_t^{U,Q}$ are independent. $w_t^{U,Q}$ and $w_{1,t}^{Q}$ are correlated Brownian motions with correlation coefficient ρ,

$$dw_t^{U,Q} dw_{1,t}^{Q} = \rho dt. \quad (4.3.8)$$

Under the risk-neutral probability measure Q, the zero-coupon bond price with expiry date T, $P(t,T)$, has the process

$$\begin{aligned} \frac{dP(t,T)}{P(t,T)} &= r_t dt - \sigma_P(t,T)dw_{1,t}^{Q}, &\quad (4.3.9)\\ \sigma_P(t,T) &= \frac{v}{a}(1 - e^{-a(T-t)}). &\quad (4.3.10)\end{aligned}$$

$w_{1,t}^Q$ and $w_{2,t}^Q$ are independent Brownian motions, and $w_t^{U,Q}$ can be defined as

$$dw_t^{U,Q} = \rho dw_{1,t}^Q + \sqrt{(1-\rho^2)}dw_{2,t}^Q. \tag{4.3.11}$$

When (4.3.11) is shifted into (4.3.6), U_t has the dynamics

$$\frac{dU_t}{U_t} = r_t dt + \rho\sigma(U_t)dw_{1,t}^Q + \sigma(U_t)\sqrt{(1-\rho^2)}dw_{2,t}^Q + (e^Y - 1)dJ_t. \tag{4.3.12}$$

The Radon Nikodym derivative of the T-forward-neutral measure can be written as

$$\frac{dQ^T}{dQ} = \exp(-\int_0^T \sigma_P(s,T)dw_{1,s}^Q - 1/2\int_0^T \sigma_P^2(s,T)ds). \tag{4.3.13}$$

Girsanov's theorem states that

$$dw_{1,t}^{Q^T} = dw_{1,t}^Q + \sigma_P(t,T)dt. \tag{4.3.14}$$

Under Q^T the bond price process, $P(t,T)$ follow the stochastic differential equation

$$\frac{dP(t,T)}{P(t,T)} = (r_t + \sigma_P^2(t,T))dt - \sigma_P(t,T)dw_{1,t}^{Q^T}. \tag{4.3.15}$$

Under the T-forward measure Q^T where the discount bond price $P(0,T)$ is used as the numeraire, U_t has the process

$$\begin{aligned}\frac{dU_t}{U_t} &= (r_t - \rho\sigma(U_t)\sigma_P(t,T) - k\lambda(t))dt \\ &+ \rho\sigma(U_t)dw_{1,t}^{Q^T} + \sqrt{1-\rho^2}\sigma(U_t)dw_{2,t}^{Q^T} + (e^Y - 1)dJ_t.\end{aligned} \tag{4.3.16}$$

The instantaneous interest rate process under the forward neutral probability can be written as

$$dr_t = a(\theta_t - r_t)dt + v dw_{1,t}^{Q^T}, \tag{4.3.17}$$

where $\theta_t = \theta - (v^2/a^2)(1 - e^{-a(T-t)})$.

Up to now, relevant theoretical substructures are given. Now, how the participating life insurance contracts are valued will be explained. Let U_0 denote the assets initial value, L_0 denote the policyholders investment which is equal to αU_0 and $E_0 = (1-\alpha)U_0$ denote the initial equity. The insurance company guarantees to invest the policyholder's investment with the rate r_g, so the guaranteed amount at the end of the period is $L_T^g = L_0 e^{r_g T}$. This amount may change the financial position of the firm. If the default occurs this amount could be lower, whereas the company performs better asset management this amount might be higher. If assuming no bankruptcy prior the maturity, policy owner receive at the maturity,

$$\Theta_L(T) = \begin{cases} U_T, \quad U_T < L_T^G \\ L_T^G, \quad L_T^G \leq U_T \leq \frac{L_T^G}{\alpha} \\ L_T^G + \delta(\alpha U_T - L_T^G), \quad U_T > \frac{L_T^G}{\alpha} \end{cases}$$

$$\Theta_L(T) = L_T^G + \delta(\alpha U_T - L_T^G)^+ - (L_T^G - U_T)^+, \tag{4.3.18}$$

where the first term is the guaranteed amount, the second term is the bonus option and the last part is the put option associated with the default risk. Policyholder will receive, if the default occurs,

$$\Theta_L(\tau) = \begin{cases} L_0 e^{r_g \tau}, \quad \lambda_{Thr.} \geq 1 \\ \lambda_{Thr.} L_0 e^{r_g \tau}, \ if \ \lambda_{Thr.} < 1 \end{cases}$$

where $\lambda_{Thr.}$ is the coefficient which the insurance company decides, if it is greater than unity that is good for insureds, since policyholders can take minimum guaranteed amount. The contract value at time t is defined in [5] as follows:

$$\begin{aligned} V_L(t) &= E_Q^t[e^{-\int_t^T r_s ds}[L_T^G + \delta(\alpha U_T - L_T^G)^+ - (L_T^G - U_T)^+]1_{\{\tau \geq T\}} \\ &+ e^{-\int_t^\tau r_s ds} \min(\lambda_{Thr.}, 1) L_\tau^g 1_{\{\tau < T\}}]. \end{aligned} \tag{4.3.19}$$

Given the fact that the relative prices are martingale under the T-forward risk neutral equivalent martingale measure. (4.3.19) can be rewritten as

$$\begin{aligned} V_L(0) &= P(0,T) E_{Q^T}^t[[L_T^G + \delta(\alpha U_T - L_T^G)^+ - (L_T^G - U_T)^+]1_{\{\tau \geq T\}} \\ &+ e^{-\int_\tau^T r_s ds} \min(\lambda_{Thr.}, 1) L_\tau^g 1_{\{\tau < T\}}]. \end{aligned} \tag{4.3.20}$$

Chapter 5

APPLICATIONS

In this section, previously explained models for mortality modeling are used to model the mortality. The market price of mortality risk is assumed to be zero so cash flows can be priced using real world mortality intensities. The method of surplus process modeling is applied to ISE-100 index, USD and Euro series because insurance companies' surplus data set could not be reached.

Some mortality model parameters are determined by minimizing the mean square error,

$$min(\sum_{t=1}^{T}\frac{(\ p_x^o(t) - p_x^e(t))^2}{n}), \tag{5.0.1}$$

where $p_x^o(t)$ is the observed survival probability from age x to age x+t, and $p_x^e(t)$ is the expected survival probability from age x to age x+t.

5.1 Applications of Mortality Modeling

US mortality rates from 1946 to 2003 are used for the estimation of the model parameters, for this aim relevant data is obtained from The Human Mortality Database (www.mortality.org), and the data includes one year death probabilities from the age 0 to 110, but the age 65 is selected for the mortality modeling, because in many countries this is the retirement age.

Fig. 5.1 shows the mortality rates for the age from 0 to 100 for the US population from 1946 to 2003. Fig. 5.2 shows the historical development of the survival probabilities for the age 65.

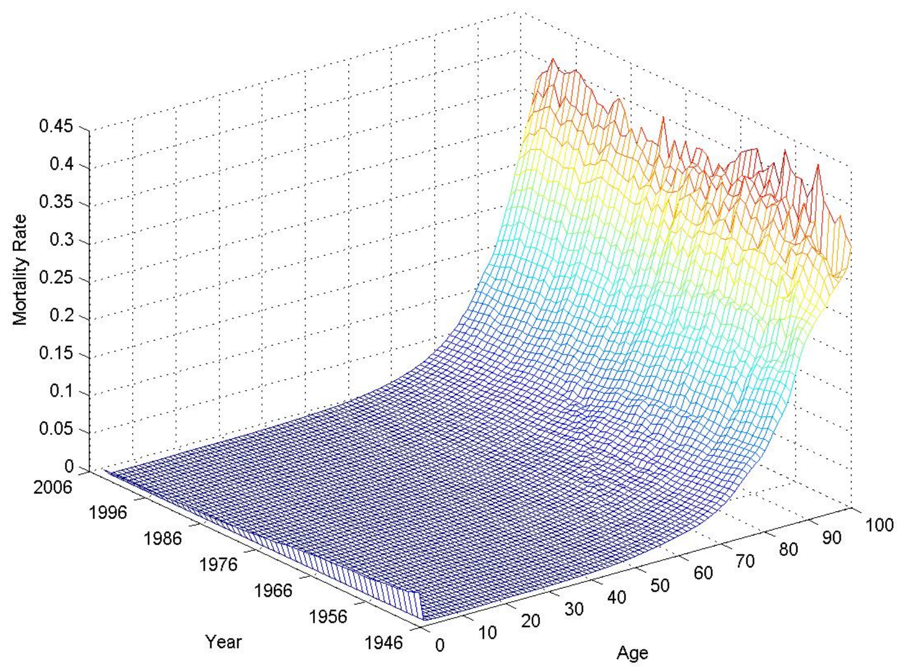

Figure 5.1: Mortality rates of the US population from 1946 to 2003.

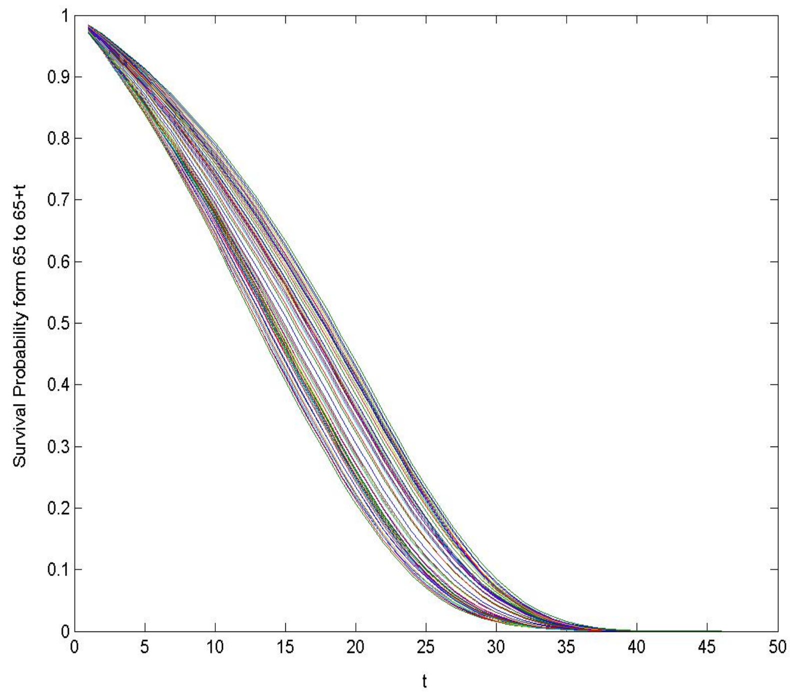

Figure 5.2: Survival probabilities for the age 65 from 1946 to 2003.

The shape of survival probabilities has a time varying property. The mortality trends affecting the shape of the survival probabilities are as follows [36]:

1. An increasing density of deaths around the mode (at old ages) of the curve of deaths is evident; therefore the graph of the survival function looks like a rectangle. The term “rectangularization” denotes this process.

2. The mode of the curve of deaths moves towards very old ages; it is called “expansion” of the survival function.

3. It is seen that higher levels and a larger dispersion of deaths at young ages.

The mortality trends, above mentioned, affect directly life insurance products valuation. (1) and (2) affect living benefits, and (3) affects death benefits. Because of these reasons, the correct estimation of mortality is very important to correct actuarial valuation.

Affine term structures for CIR and Vasicek models are mentioned before. In CIR model, if θ is taken zero, the new model will be non mean reverting CIR (N.M.R.CIR), a non mean reverting model. It gives a smaller mean square error than CIR, so it fits to the observed survival probabilities better than CIR. The estimated parameter set for CIR and N.M.R.CIR can be found in Table 5.1. $\mu_{65}(0)$ is assumed 0.015 for N.M.R.CIR process and 4.1e-5 for the equation (2.4.82) and CIR process. Fig. 5.3 and Fig. 5.4 show expected and observed survival probabilities graph for CIR and N.M.R.CIR respectively.

	k	θ	σ^2	MSE
CIR	0.004782	0.989243	1e-006	0.001114
	(0.042747)	(0.377195)	(0.178630)	
$N.M.R.CIR$	0.090596	0	1e-006	6.307721e-005
	(0.78188)		(0.035747)	

Table 5.1: Estimated parameters of CIR and N.M.R.CIR models.

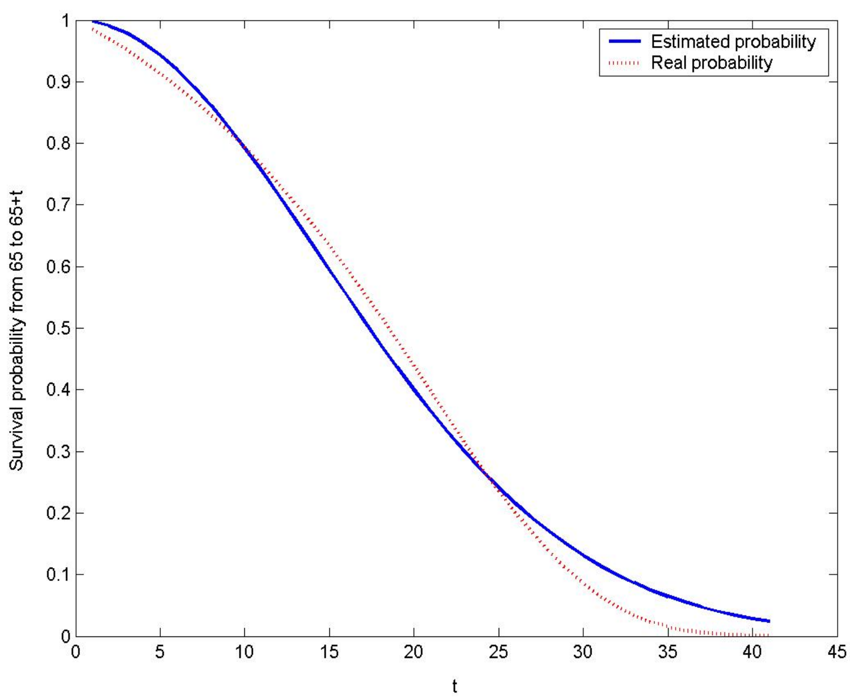

Figure 5.3: Estimated survival probabilities of CIR model for the age 65.

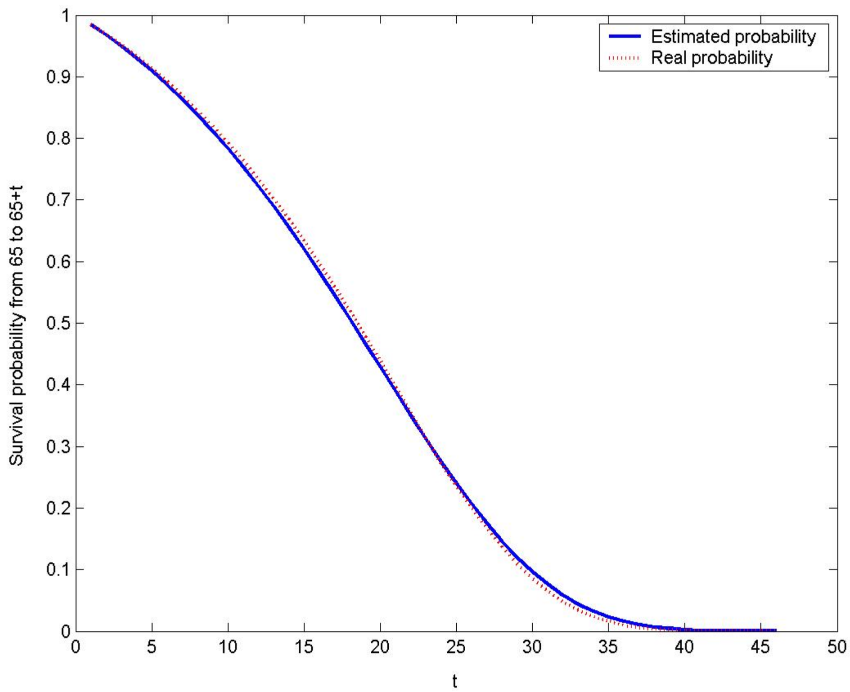

Figure 5.4: Estimated survival probabilities of N.M.R.CIR model for the age 65.

After modeling the mortality intensity with CIR and N.M.R.CIR, it is modeled the processes having a jump component. The results can be found in Table 5.2, Table 5.3 and Table 5.4.

q	k	θ	μ	υ	γ	MSE
0.92512	0.000286	0.060985	0.005126	5.099e-005	1e-006	0.00104
(0.1078)	(0.0054)	(0.1418)	(0.0122)	(0.2824)	(0.3712)	

Table 5.2: Estimated parameters of (2.4.82).

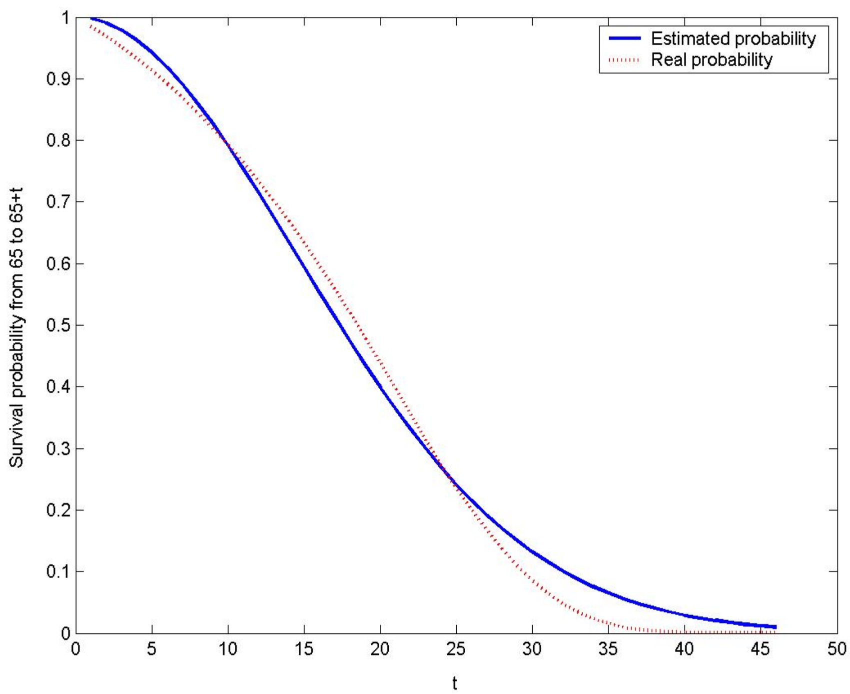

Figure 5.5: Estimated survival probabilities of (2.4.82) for the age 65.

v	h	k	θ	MSE
0.0067905	3.2799e-005	0.99999	0.97695	3.5411e-011
(1.007)	(0.8028)	(0.9621)	(0.1367)	
$E[J]$	$E[J^2]$	$E[J^3]$	$E[J^4]$	
5.3574e-006	0.000286	0.000566	0.00089	
(0.8407)	(1.124)	(1.154)	(1.2708)	

Table 5.3: Estimated parameters of (2.4.41) by using the method of moment.

q	k	θ	$Likelihood\ Value$
0.018203	0.01134	0.99	333.09
(0.29167)	(0.2436)	(0.3856)	
μ	σ	γ	
0.002553	0.000644	1.78e-007	
(0.00088)	(0.001516)	(0.020506)	

Table 5.4: Estimated parameters of the model (2.4.83).

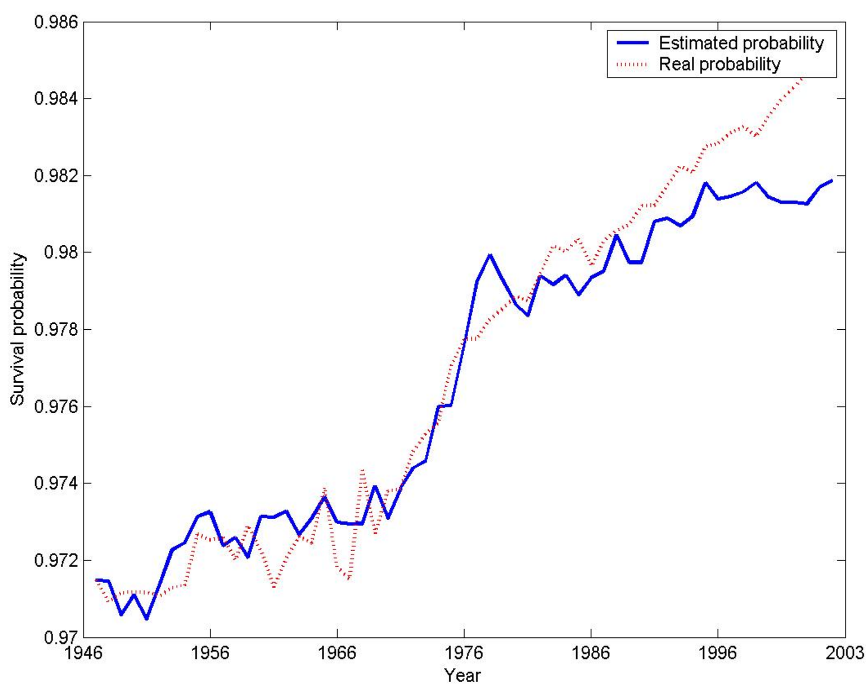

Figure 5.6: One year estimated survival probabilities for the age 65 by using (2.4.83).

	k	σ^2	R	$Likelihood\ Value$
$N.M.R.CIR$	0.094119	9.93e-007	0.001026	6.699e+015
	(1.957e-013)	(9.668e-014)	(1.099e-013)	

Table 5.5: Estimated parameters of N.M.R.CIR model using the Kalman filter.

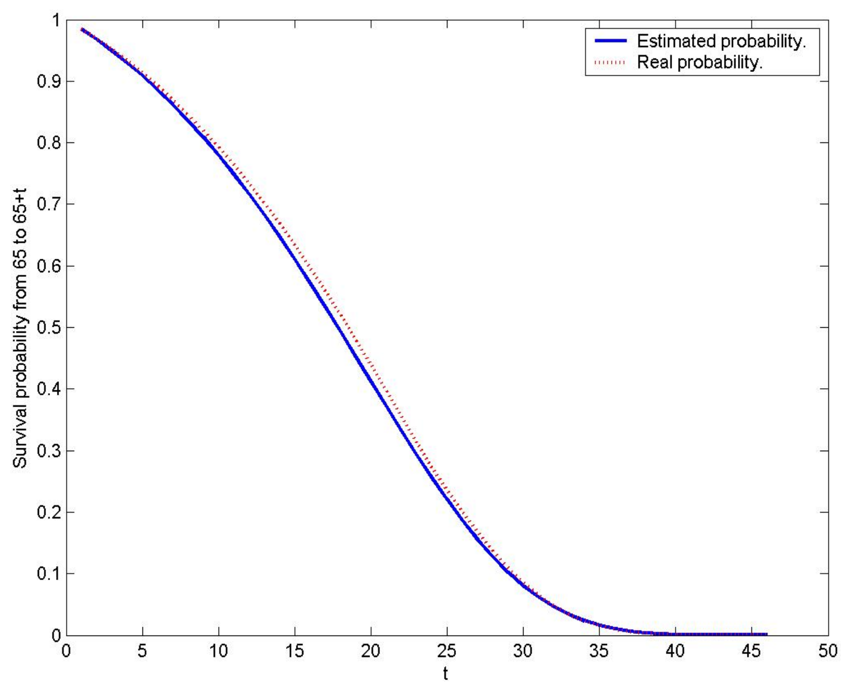

Figure 5.7: Estimated survival probabilities by using the Kalman filter for the age 65.

Some mortality modeling techniques are explained in Chapter 2, and some of these modeling techniques are used to model mortality. Evaluated parameter values and standard deviations are given in Table 5.1, Table 5.2, Table 5.3, Table 5.4 and Table 5.5. N.M.R.CIR process gives the best fit among various modeling techniques. The most important point here is that N.M.R.CIR process is a non mean reverting process. Some parameters are unreliable because of their standard deviations, such as jump size parameters in Table 5.3.

5.2 Applications of Surplus Modeling

Istanbul Stock Exchange 100 composite index, Euro and USD price series are used to show the disentangling of the jumps from the data, since the real data about any insurance company's surplus could not be reached. The related data is obtained from the web site of Central Bank of The Republic of Turkey.

A threshold is needed to define the jumps. The threshold should be a changing one over to obtain more accurate results. Let b_t denote the threshold

$$b_t = c\,\sigma_t^2, \tag{5.2.2}$$

$$\hat{\sigma}_t^2 = K + P\hat{\sigma}_{t-1}^2 + Q\hat{\varepsilon}_t^2, \tag{5.2.3}$$

where c is a constant to be calibrated, and $\hat{\varepsilon}$ estimated innovations. σ_t^2 follows a GARCH(1,1) process. The new observation should be satisfy (5.2.4), if it is a jump.

$$J_t = \{\Delta logU_t | (\Delta logU_t)^2 > b_t\} \tag{5.2.4}$$

In this application, c is selected as 9. Jumps are detected the variations which are three conditional standard deviations away from zero. h_s is selected 3, because the model fits the data better. Table 5.6 reports the estimated parameters.

	μ	σ^2	λ
ISE100	0.00166	0.00032	0.00244
Dollar	0.00131	0.000103	0.00913
Euro	0.000514	9.152e-005	0.00603

Table 5.6: Kernel Parameter Estimation Results for the series ISE-100, USD and Euro.

It is seen that there are *27* jumps in “ISE-100 Index” from 02.01.1997 to 15.11.2006, when Euro series are investigated, there are *13* jumps from 04.01.1999 to 20.11.2006, when USD series are investigated, there are *17* jumps from 26.02.2001 to 16.11.2006. Fig. 5.8, Fig. 5.9 and Fig. 5.10 show the logarithmic asset prices, jumps' sizes and returns for the ISE-100, Euro and USD series respectively.

The model can be used to determine the ruin probability. Monte Carlo method could be used to estimate the ruin probability and to see the insurance companies' risk position over the long term. The number of simulation is the main drawback to find the ruin probability. Many attempts may be needed to find it.

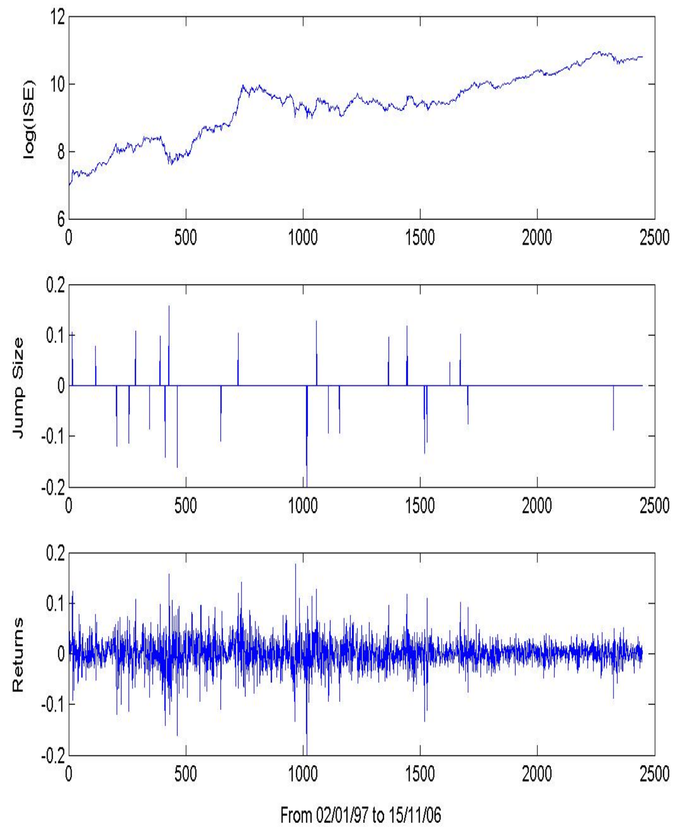

Figure 5.8: Logarithmic asset prices, jump sizes and returns of ISE series.

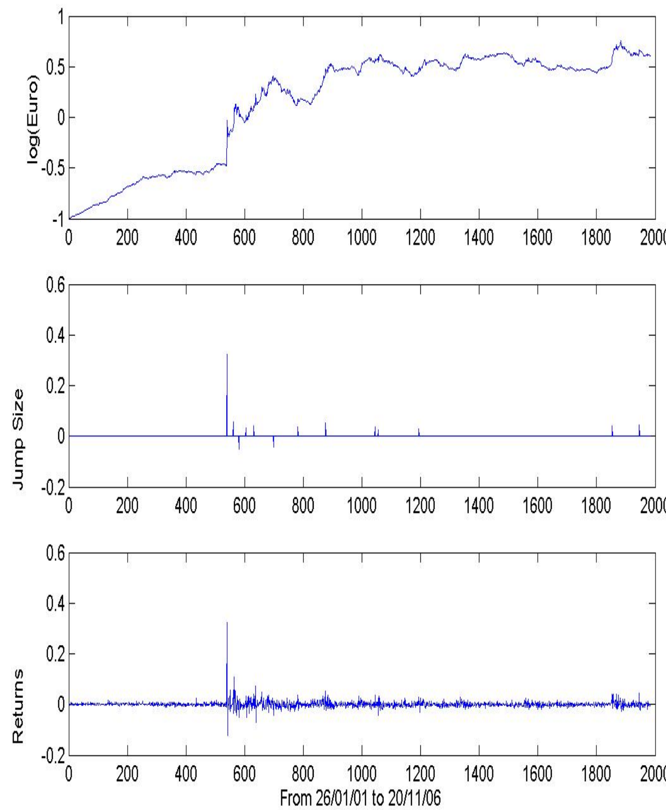

Figure 5.9: Logarithmic asset prices, jump sizes and returns of Euro series.

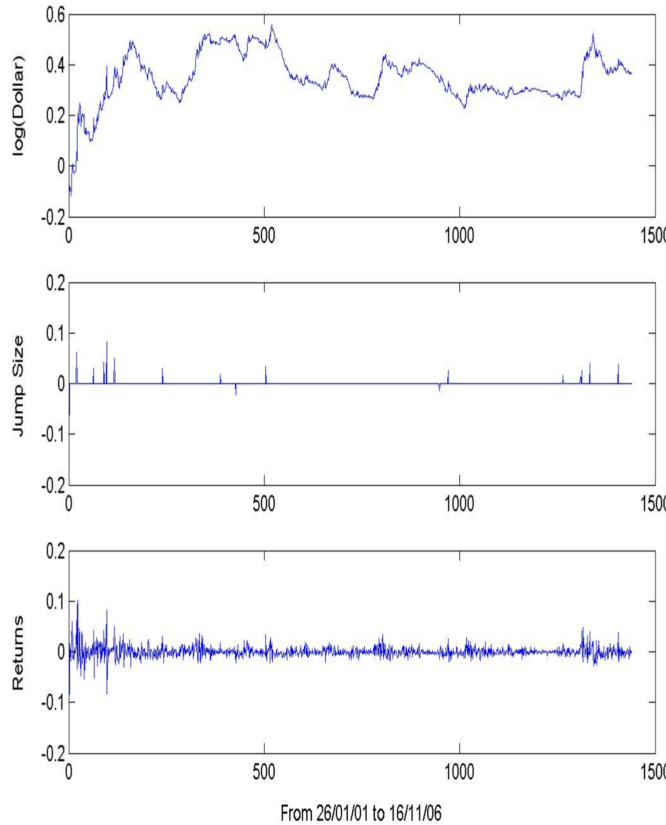

Figure 5.10: Logarithmic asset prices, jump sizes and returns of USD series.

5.3 Applications of Life Insurance Contracts

Every company has a different financial strength, so the best way to show the financial strength is to use credit ratings of companies. Standard and Poors defines the credit rating as follows: "Credit rating is a current opinion of the creditworthiness of an obligor with respect to a specific financial obligation, a specific class of financial obligations, or a specific financial program (including ratings on medium-term note programs and commercial paper programs). It takes into consideration the creditworthiness of guarantors, insurers, or other forms of credit enhancement on the obligation and takes into account the currency in which the obligation is denominated. The issue of credit rating is not a recommendation to purchase, sell, or hold a financial obligation, inasmuch as it does not comment as to market price or suitability for a particular investor." Ratings satisfy some advantages to insurance companies some of them are as follows [41]:

- A high rated firm can borrow more easily and cheaply.
- Reinsurance companies look for highly rated insurance companies.
- Agents and brokers investigate ratings before recommend a firm to their clients.
- Investors investigate the rating reports before investing in an insurance company.

Although insurance companies have enough amount of assets to meet their liabilities, they may have solvency or liquidity problems. So, based on the above results, insurance companies should consider their ratings when determining the value of their life insurance contracts. For this purpose, corporate bond default rates examined by S&P are used, however these default rates includes overall universes of bond ratings not specifically for insurers (Table 5.7).

	1-year default rate	5-year default rate	10-year default rate
AAA	0.00	0.10	0.48
AA+	0.00	0.17	0.38
AA	0.00	0.14	0.71
AA-	0.02	0.46	1.20
A+	0.06	0.58	1.60
A	0.05	0.51	1.67
A-	0.04	0.85	2.31
BBB+	0.35	2.33	4.66
BBB	0.34	2.24	5.46
BBB-	0.43	5.59	10.88
BB+	0.52	7.56	14.00
BB	1.16	10.86	18.73
BB-	2.07	16.30	26.45
B+	3.29	21.45	31.48
B	9.31	31.68	39.56
B-	13.15	40.18	49.23
CCC	27.87	50.46	57.21

Table 5.7: Corporate bond default rates (in percent) by rating class, 1981-2002, [41].

In this section, valuations of pure endowment and participating life insurance contracts are approximately done. Mortality rates obtained by N.M.R.CIR process is used for the pure endowment contract valuation. *P(0,5)* is assumed to be 0.54. Corporate bond default rates are added to valuation. It is assumed that any life insurance companies' ratings and default rates are the same as corporate bond default rates. 5-year default probability is added to (4.1.3) and 5-year pure endowment contract is priced as

$$ {}_5E_x(0) = (1 - Pr(\tau < 5))P(0,5)\,{}_5p_x^Q \tag{5.3.5} $$

where τ is the default time.

	AAA	AA	A	BBB	BB	B	CCC
${}_5E_{65}$	0.4879	0.4877	0.4859	0.4774	0.4353	0.3336	0.2419

Table 5.8: Price of the 5-year pure endowment.

The values of parameters in Table 5.3 are used valuation of the participating life insurance contract. Contract values are obtained by Monte Carlo simulations. The impact of some parameters are examined. The following results can be detected when the graphs are investigated. The contract price and the probability that the value of assets being under the threshold increase, when the guaranteed rate and the volatility are increased. The higher the participating level, the higher the price of the contract. The graphs show that the other parameters do not influence the contract price and the probability that the value of assets being under the threshold.

$r_0 = 0.03$	$\sigma_r = 0.008$	$\rho = -0.02$	$P(0, 10) = 0.27$
$\alpha = 0.85$	$a = 0.4$	$\sigma_j = 0.001$	$U_0 = 100$
$\lambda_U = 10$	$r_g = 0.026$	$\theta = 0.06$	$\lambda_{Thr.} = 0.8$
$\sigma_U = 0.1$	$\mu_j = -0.005$	$V_L(0) = 42.09$	$P(\tau < T) = 0.053$

Table 5.9: Parameter values, the contract value and the default probability.

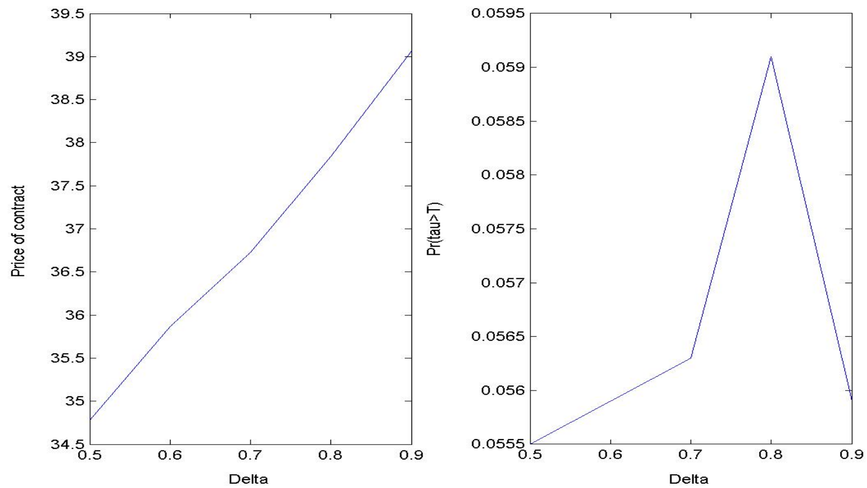

Figure 5.11: Contract values and the probability of being under the threshold (w.r.t δ).

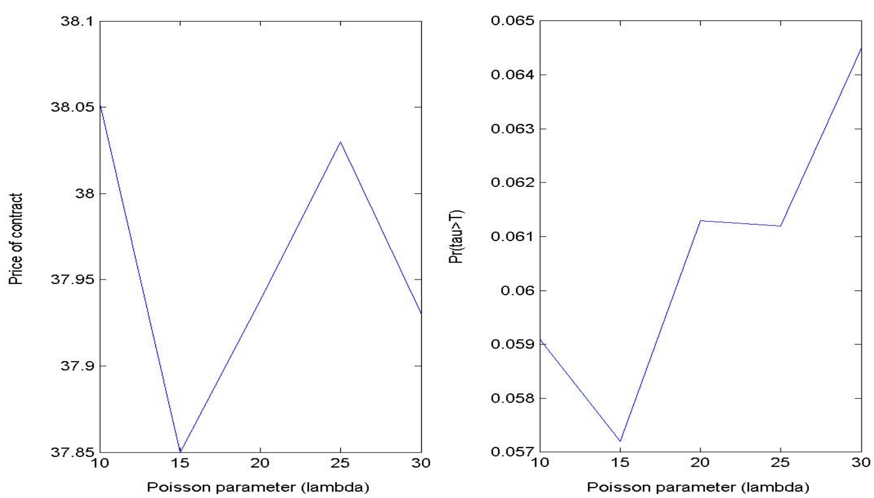

Figure 5.12: Contract values and the probability of being under the threshold (w.r.t. λ_U).

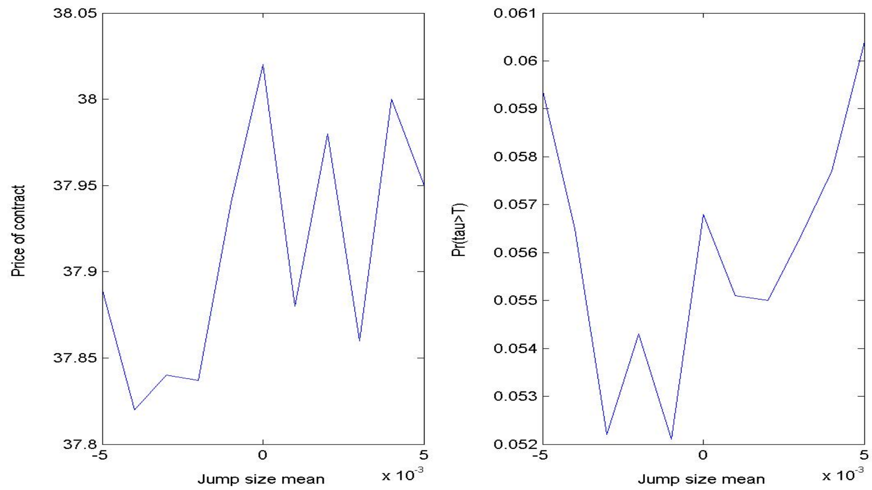

Figure 5.13: Contract values and the probability of being under the threshold (w.r.t. μ_j).

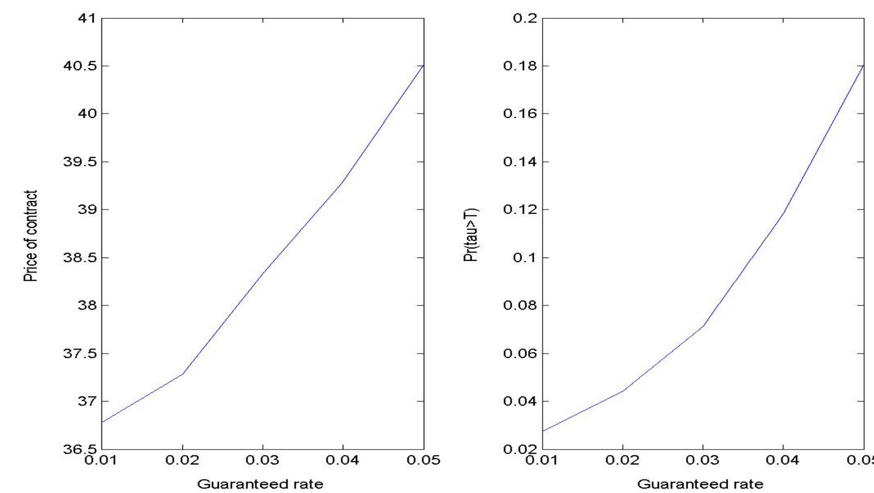

Figure 5.14: Contract values and the probability of being under the threshold (w.r.t. r_g).

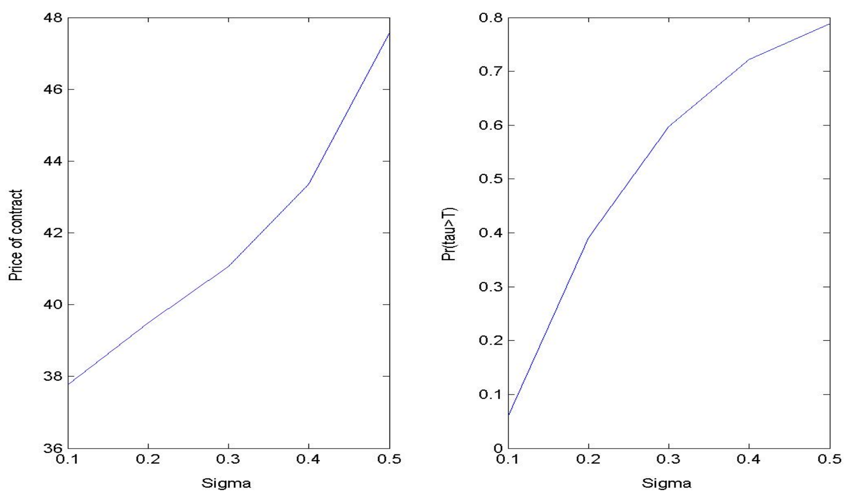

Figure 5.15: Contract values and the probability of being under the threshold (w.r.t. σ_U).

Chapter 6

CONCLUSION

In this thesis, we mainly concentrate on stochastic modeling of mortality, stochastic modeling of the surplus process of an insurance company, and a different approach to valuation of life insurance contracts taking into account the financial strength of the company.

In the literature, some authors discussed the mortality intensity as a stochastic process. Some stochastic models are investigated and model parameters are estimated. Some models which are used to model interest rates are also used to model the mortality. Given parameter values and standard deviations, a non mean reverting process N.M.R.CIR, gives the best fit among various modeling techniques. Some parameters are found to be unreliable because of their standard deviations such as jump size parameters.

Corporate bond ratings are utilized when the pure endowment contract is priced. By this approach, financial strengths of companies are reflected to the contract prices. Participating life insurance contracts are explained and the price of contracts are approximately done by the method in [5]. A jump component is added to the asset process differently from the authors' method, so the asset process includes some unexpected fluctuations. Some sensitivity analyses are performed to see how the prices of participating life insurance contract and default probabilities depend on the volatility of asset, jump intensity, guaranteed rate, participation rate and the mean of jump size. It is concluded that when the guaranteed rate and the volatility are increased, the contract price and the probability that the value of assets being under the threshold increase. The higher the participating level, the higher the price of the contract. Other parameters do not influence the contract price and the probability that the value of assets being under the threshold.

The surplus process of an insurance company is approximated as a jump diffusion process. It is thought that insurance companies can use it easily for different purposes such as determining the minimum amount of capital requirement and pricing some contracts.

The market price of mortality risk is not considered in the applications and the parameters of the surplus process could not be estimated by real data. These drawbacks can be considered and this study can be extended in further research.

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