INVERSE DYNAMICS CONTROL OF FLEXIBLE JOINT PARALLEL MANIPULATORS

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I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

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This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

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ABSTRACT

INVERSE DYNAMICS CONTROL OF FLEXIBLE JOINT PARALLEL MANIPULATORS

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The purpose of this thesis is to develop a position control method for parallel manipulators so that the end effector can follow a desired trajectory specified in the task space where joint flexibility that occurs at the actuated joints is also taken into consideration.

At the beginning of the study, a flexible joint is modeled, and the equations of motion of the parallel manipulators are derived for both actuator variables and joint variables by using the Lagrange formulation under three assumptions regarding dynamic coupling between the links and the actuators. These equations of motion are transformed to an input/output relation between the actuator torques and the actuated joint variables to achieve the trajectory tracking control. Moreover, the singular configurations of the parallel manipulators are explained.

As a case study, a three degree of freedom, two legged planar parallel manipulator is simulated considering joint flexibility. The structural damping of the active joints, viscous friction at the passive joints and the rotor damping are also considered throughout the study. Matlab® and Simulink® softwares are used for the simulations. The results of the simulations reveal that steady state errors are negligibly small and good tracking performances can be achieved.

Keywords: Flexible joint, parallel manipulator, inverse dynamics control, singularity analysis

ÖZ

ESNEK EKLEMLİ PARALEL MANİPÜLATÖRLERİN
TERS DİNAMİK KONTROLÜ

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Bu tezin amacı eklem esnekliği de göz önüne alınarak paralel manipülatörlere ait uç işlemcinin görev uzayında belirlenen bir yörüngeyi izleyebilmesi için bir konum kontrol yöntemi geliştirmektir.

Çalışmanın başında esnek bir eklem modellenmiş ve paralel manipülatörün hareket denklemleri Lagrange formülasyonu kullanılarak hem eyletici hem de eklem değişkenleri cinsinden olmak üzere eyletici ve uzuvlar arasındaki dinamik bağlantı ile ilgili olarak üç varsayım ile türetilmiştir. Bu hareket denklemleri analitik ters dinamik kontrolü yaklaşımıyla eyletici torkları ve aktif eklem değişkenleri arasında bir giriş/çıkış denklemine dönüştürülerek yörünge takip kontrolü sağlanmıştır. Bunun yanı sıra paralel manipülatörlerin tekil durum analizleri anlatılmıştır.

Örnek olarak üç serbestlik dereceli, iki bacaklı bir düzlemsel paralel manipülatör eklem esnekliği ile beraber ele alınmıştır. Aynı zamanda aktif eklemlerdeki yapısal sönüm, pasif eklemlerdeki viskoz sürtünme ve eyletici rotorunun sönümü de dikkate alınmıştır. Benzetim için Matlab® ve Simulink® yazılımları kullanılmıştır. Elde edilen sonuçlarda kararlı hal hataların ihmal edilebilir düzeyde olduğu saptanmış ve iyi bir yörünge takibi sağlanmıştır.

Anahtar Kelimeler: Esnek eklem, paralel manipülatör, ters dinamik kontrol, tekil durum analizi

To My Family

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CHAPTER I

INTRODUCTION

1.1 Literature Survey

Parallel manipulators have been an intensive area of research for over a decade since they have some advantages over the serial ones. Parallel manipulators can carry heavier loads with their closed loop structure more precisely. These mechanisms are most commonly used in industrial applications such as flight simulators and earthquake simulators, micro-motion manipulations where high load capability and high motion accuracy are needed. However these manipulators face with the problem of having relatively small functional workspace and difficulties in design and control. For this reason, the parallel manipulators have become a focus of interest in various fields of research.

On the other hand, joint flexibility is important in manipulator dynamics and control system design since their drives exhibit this behavior. In order to handle high precision manipulations, the joint flexibility should be taken into consideration in the control system.

There are many researchers who studied the control of flexible joint serial manipulators taking their starting point from the flexible joint model derivation of Spong [1].

Among the motion control methods for flexible joints two nonlinear control schemes are put forward which are called the feedback linearization and singular perturbation approaches [1]. The feedback linearization control of flexible-joint robots is also called the analytical inverse dynamics control and is studied by various authors. In this approach the intermediate variables are analytically eliminated and the input torques are found as functions of the end-effector motion up to the fourth derivative.

The elimination requires the differentiation of the equations of motion and the acceleration level task equations twice.

The singular perturbation approach uses the advantage of order reduction by decomposing the system into two subsystems namely a fast subsystem (flexible joints) and a slow subsystem (rigid manipulator). The model order is lowered by first neglecting the fast phenomena. Then corrections due to the fast phenomena are reintroduced by calculating them in a separate time scale in which the slow variables are assumed to be constant. However this approach is limited in applicability since it is valid only when the joint springs are sufficiently stiff.

Forrest-Barlach and Babcock [2] used the inverse dynamics control method for the cylindrical coordinate arm with drive train compliance and actuator dynamics in the radial and each of the revolute degree of freedom.

Jankowski and Van Brussel [3] applied inverse dynamics control in discrete time where solution of the singular sets of differential equations is used to avoid the further differentiations of the system equations of motion.

İder and Özgören [4] utilized inverse dynamics control at the acceleration level by using implicit numerical integration methods that account for the higher order derivative information for solving the singular set of differential equations. The asymptotic stability is achieved by the feedback of joint positions, velocities and rotor velocities.

All of the above studies focused on the control of flexible joint serial manipulators. There are limited numbers of studies in the literature concerning control of parallel manipulators. Most of these studies did not take the joint flexibility into their control strategies.

Dado and Al-Huniti [5] studied dynamic simulation model for mixed-loop planar robots with flexible joint drive. The mathematical model of a five-link, three degree of freedom manipulator was derived using the virtual work method. The drive signal at the motor was based on the error between the desired and actual motions using proper position and velocity gains.

Parallel manipulators possess drive singular positions in addition to the kinematic singular positions that serial manipulators also have. Singularity analysis of parallel manipulators has been the subject of many studies in the last years.

İder [6] examined the singularities that occur in the parallel manipulators and showed that the manipulator can pass through the singular positions while the actuator forces and the system motion remain stable by modifying the system equations of motion.

Ji [7] studied on the singular configurations that planar parallel manipulators have in general.

Liu [8] designed a new spatial parallel manipulator and looked for the singular positions that this parallel manipulator has.

1.2 Objective

This thesis aims at trajectory tracking control of the end effector of a parallel manipulator by using the analytical inverse dynamics approach taking joint flexibility into consideration.

To facilitate the solution, the system equations of motion are transformed to an input/output relation between the actuator torques and the actuated joint variables. System constraints are utilized to eliminate the unactuated joint variables of the system. Since the structural damping of the active joints, viscous friction at the passive joints and the rotor damping characteristics are also included, an additional complexity occurs due to the presence of torque rate in this relation.

Another aim of the study is to find out the singular positions of the parallel manipulators to avoid them in trajectory planning stage. Since the parallel manipulators have additional singularities due to their closed loop structures, it is important to emphasize the existence of these configurations that arise both inside the workspace and at the workspace boundaries.

1.3 Outline of the Study

In this thesis, the following chapters are organized to explain the control theory and the case study.

In Chapter 2, the dynamics of a parallel manipulator is explained when the joint flexibility is added into the analysis. The system equations of motion and system constraint equations are derived.

Chapter 3 is related to the inverse dynamics control approach. The control law and task space equations are introduced. The procedures for the elimination of the unactuated joint variables from the system constraint equations and the elimination of actuator variables from the equations of motion are considered to get the input/output relation

Chapter 4 presents the concept of singularity in parallel manipulators. Types of singularities and the physical results of the singular positions are discussed.

In Chapter 5, a parallel manipulator is analyzed as a case study. All of the theoretical knowledge presented in Chapters 2–4 is applied to this example. The equations of motion are derived, the singular configurations are identified and simulations are performed via the proposed inverse dynamics control method.

Chapter 6 reviews and concludes the comparisons of the simulations and presents recommendations for future work.

CHAPTER II

MANIPULATOR DYNAMICS WITH FLEXIBLE JOINTS

2.1 Overview

Consider an *n* degree of freedom parallel manipulator. Let this system be converted into an open-tree structure by disconnecting a sufficient number of unactuated joints and the degree of freedom of the open-tree system be *m*, i.e., the number of independent loop closure constraints in the parallel manipulator be *m*-*n*. Let the set of the generalized coordinates corresponding to the manipulator joint variables which express the relative joint positions be defined as

$$G_1 = \{\theta_1, \ldots\ldots, \theta_m\} \tag{2.1}$$

Hence the vector of manipulator joint variables of the rigid links that contains both the actuated and unactuated joint variables is

$$\bar{\theta} = [\theta_1, \ldots\ldots, \theta_m]^T \tag{2.2}$$

In the parallel manipulator as many joints as the degree of freedom of the manipulator are actuated. Due to the elasticity of the transmission elements, joint elasticity occurs at the actuated joints. The sources of elasticity at the joints are generally couplings, harmonic drives, thin shafts used in drive trains. Since joint flexibility is the main source contributing to overall robot flexibility as experimentally verified by ref.[9], it is important to take joint flexibility into account in order to get higher performance from the controller.

Joint elasticity and structural damping of the power transmission elements at an actuated joint are modeled as a torsional spring and a torsional damper respectively.

For the ith transmission, K_i stand for the spring constant and D_i is used for the damping constant as seen in Figure 2.1.

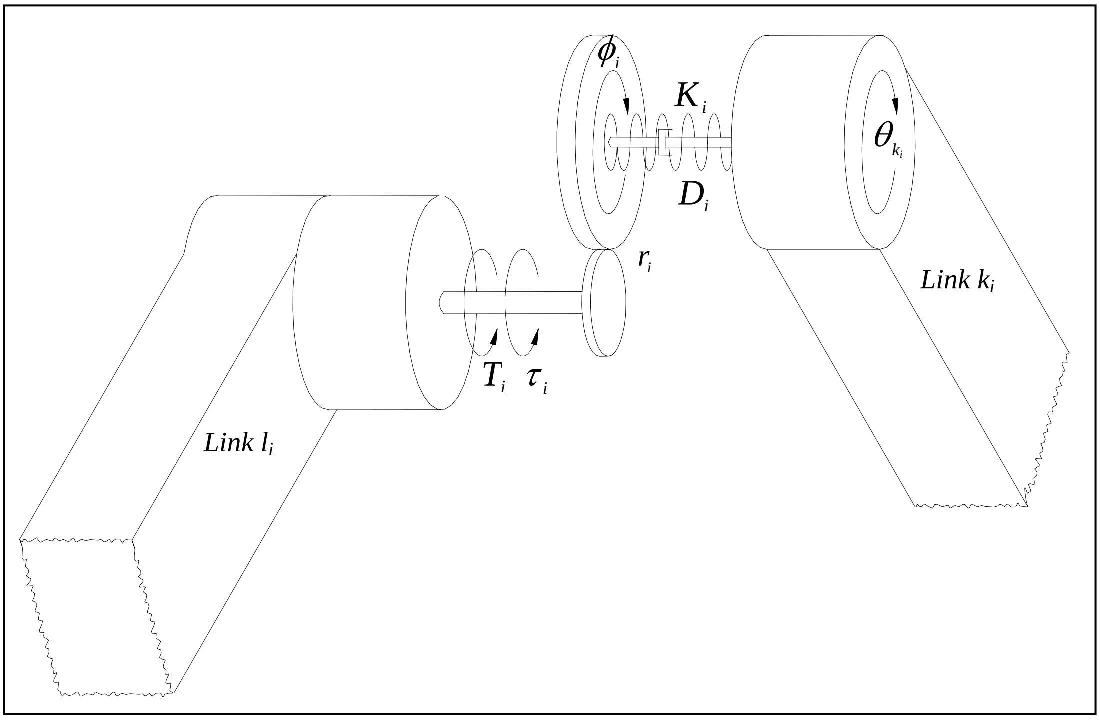

Figure 2.1 A Flexible Joint Dynamic Model

In the figure, θ_{k_i} is the manipulator joint variable which corresponds to angular position of the driven link of the manipulator with respect to the link that the ith actuator is mounted. k_i is link number of the link which is driven by the ith actuator and l_i is link number of the link on which the actuator is mounted.

For this flexible joint model, the second set of generalized coordinates corresponding to the actuator variables are written as

$$G_2 = \{\phi_1, \ldots\ldots, \phi_n\} \tag{2.3}$$

Hence, and the vector of the actuator variables of the flexible joints is

$$\overline{\phi} = [\phi_1, \ldots\ldots, \phi_n]^T \tag{2.4}$$

The angles in the second set of generalized coordinates are obtained as

$$\phi_i = \frac{\tau_i}{r_i} \qquad i = 1, \ldots, n \tag{2.5}$$

where r_i is the speed reduction ratio.

Actuator variable τ_i in Equation 2.5 is the angular position of the i[th] actuator's rotor with respect to the link that the actuator is mounted. On the other hand, the torques supplied by the actuators are denoted by T_i^a and the torques T_i after the speed reduction are

$$T_i = r_i T_i^a \qquad i = 1, \ldots, n \tag{2.6}$$

2.2 Manipulator Dynamics

The dynamic model of parallel manipulators with flexible joints can be derived with the following assumptions which simplify the equations of motion considerably and were first stated by ref. [1].

The assumptions are as follows:

- The links of the parallel manipulator are rigid.
- The kinetic energy of the rotor is due mainly to its own rotation. In other words, the motion of the rotor is a pure rotation with respect to the inertial reference frame provided that the gear ratio is sufficiently large.
- The rotor inertia is symmetric about the rotor axis of rotation so that the velocity of the rotor center of mass is independent of the rotor position.

Elasticity at each of the joints creates an additional degree of freedom to the whole system. Therefore rotors of the actuators are modeled as fictitious rigid links with their own inertial parameters. When an *n* degree of freedom parallel manipulator with *n* number of actuators is considered, the whole system turns out to be a *2n* degree of freedom system.

The equations of motion corresponding to both sets of generalized coordinates that were stated in Equations 2.1 and 2.3 can be derived by using the Lagrange's equations.

The Lagrange's equation for the set of manipulator joint variables

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{\theta}_j}\right)-\frac{\partial K}{\partial \theta_j}+\frac{\partial D}{\partial \dot{\theta}_j}+\frac{\partial U}{\partial \dot{\theta}_j}=\tilde{Q}_j+Q'_j \qquad j=1,...,m \tag{2.7}$$

The Lagrange's equation for the set of actuator variables

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{\phi}_i}\right)-\frac{\partial K}{\partial \phi_i}+\frac{\partial D}{\partial \dot{\phi}_i}+\frac{\partial U}{\partial \dot{\phi}_i}=\tilde{Q}_{i+m} \qquad i=1,...,n \tag{2.8}$$

where $K, D, U, \tilde{Q}, Q'$ stand for kinetic energy, dissipation function, potential energy, generalized non-potentialized force and the generalized constraint force terms respectively.

2.2.1 Kinetic Energy

Kinetic energy of a link can be written as

$$KE_{Li}=\frac{1}{2}m_i^L(\bar{V}_{G_i}^L)^T\bar{V}_{G_i}^L+\frac{1}{2}(\bar{\omega}_i^L)^T\hat{I}_i^L\bar{\omega}_i^L \qquad i=1,...,m \tag{2.9}$$

where

$$\bar{V}_{G_i}^L=\sum_{j=1}^{m}\bar{W}_{ij}^L\dot{\theta}_j \qquad i=1,...,m \tag{2.10}$$

$$\bar{\omega}_i^L=\sum_{j=1}^{m}\bar{\Omega}_{ij}^L\dot{\theta}_j \qquad i=1,...,m \tag{2.11}$$

$$\hat{I}_i^L=\left[\hat{C}^{(0,i)}\right]\underline{\hat{I}}_i^L\left[\hat{C}^{(0,i)}\right]^T \qquad i=1,...,m \tag{2.12}$$

In Equations 2.9 – 2.12,

m_i^L is the mass of the i[th] link.

$\overline{V}_{G_i}^{\,L}$ is the mass center velocity vector of the i[th] link as expressed in fixed reference frame.

$\overline{W}_{ij}^{L}$ is the velocity influence coefficient vector.

$\overline{\omega}_i^{L}$ is the angular velocity of the i[th] link as expressed in fixed reference frame.

$\overline{\Omega}_{ij}^{L}$ is the angular velocity influence coefficient vector.

$\hat{I}_i^{L}$ is the moment of inertia matrix of the i[th] link as expressed in fixed reference frame.

$\hat{C}^{(0,i)}$ is the transformation matrix from the reference frame attached to the i[th] link to the fixed reference frame.

$\underline{\hat{I}}_i^{L}$ is the moment of inertia matrix of the i[th] link as expressed in its body reference frame.

Kinetic energy of a link can be rewritten by combining the Equations 2.9, 2.10 and 2.11 in more compact form as below.

$$KE_{Li} = \sum_{j=1}^{m}\sum_{k=1}^{m} m_{jki}^{L}\dot{\theta}_j\dot{\theta}_k \qquad i = 1,\ldots,m \tag{2.13}$$

where

$$m_{ijk}^{\ L} = \frac{1}{2} m_i^{\,L} (\overline{W}_{ij}^{\,L})^T \overline{W}_{ik}^{\,L} + \frac{1}{2} (\overline{\Omega}_{ij}^{L})^T \hat{I}_i^{\,L} \overline{\Omega}_{ik}^{L} \tag{2.14}$$

In similar manner, kinetic energy of an actuator can be written as:

$$KE_{Ai} = \frac{1}{2} m_i^{\,A} (\overline{V}_{G_i}^{\ A})^T \overline{V}_{G_i}^{\ A} + \frac{1}{2} (\overline{\omega}_i^{\,A})^T \hat{I}_i^{\,A} \overline{\omega}_i^{\,A} \qquad i = 1,\ldots,n \tag{2.15}$$

where

$$\bar{V}_{G_i}{}^{A} = \sum_{j=1}^{m} \bar{W}_{ij}^{A} \dot{\theta}_j \qquad i = 1,...,n \qquad (2.16)$$

$$\bar{\omega}_i{}^{A} = \sum_{j=1}^{n} \bar{\Omega}_{l_i,j}^{L} \dot{\theta}_j + \hat{C}^{(0,l_i)} \bar{u}_i \dot{\tau}_i \qquad i = 1,...,n \qquad (2.17)$$

$$\hat{I}_i{}^{A} = \left[\hat{C}^{(0,i)} \right] \underline{\hat{I}}_i{}^{A} \left[\hat{C}^{(0,i)} \right]^T \qquad i = 1,...,n \qquad (2.18)$$

In Equations 2.15 – 2.18,

$m_i{}^{A}$ is the mass of the i^th^ actuator rotor.

$\bar{V}_{G_i}{}^{A}$ is the mass center velocity vector of the i^th^ actuator rotor as expressed in fixed reference frame.

$\bar{W}_{ij}^{A}$ is the velocity influence coefficient vector.

$\bar{\omega}_i{}^{A}$ is the angular velocity of the i^th^ actuator rotor as expressed in fixed reference frame.

$\bar{\Omega}_{l_i,j}^{A}$ is the angular velocity influence coefficient vector of the previous link.

$\hat{C}^{(0,l_i)}$ is the transformation matrix from the reference frame attached to the l_i ^th^ actuator rotor to the fixed reference frame.

$\bar{u}_i$ is the unit vector along rotation axis of the i^th^ actuator rotor in the link frame on which the actuator is mounted.

$\hat{I}_i{}^{A}$ is the moment of inertia matrix of the i^th^ actuator rotor as expressed in fixed reference frame.

$\hat{C}^{(0,i)}$ is the transformation matrix from the reference frame attached to the i^th^ actuator rotor to the fixed reference frame.

$\underline{\hat{I}}_i{}^{A}$ is the moment of inertia matrix of the i^th^ actuator rotor as expressed in its body reference frame.

At this point, Assumption-2 which was put forward in Section 2.1 is invoked to the formulation and angular velocity expression of the actuator turns out to be as in Equation 2.19.

$$\bar{\omega}_i^A = \hat{C}^{(0,l_i)}\bar{u}_i\dot{\tau}_i \qquad i = 1,...,n \qquad (2.19)$$

Therefore kinetic energy of an actuator can be rewritten in more compact form as

$$KE_{Ai} = \frac{1}{2}m_i^A(\bar{V}_{G_i}{}^A)^T\bar{V}_{G_i}{}^A + \frac{1}{2}I_i^A\dot{\tau}_i^2 \qquad i = 1,...,n \qquad (2.20)$$

Assumption-3 is related to the mass distribution of the rotor. With that assumption symmetric mass distribution is assumed and principality of the axes of rotation is guaranteed. Therefore matrix representation of the inertia dyadic forms a diagonal matrix. In addition, mass center velocity of the actuator rotor becomes independent of the rotor position since mass center velocity is the translational velocity of the fixed point at the l_i th link.

Total kinetic energy of the manipulator is formulated as

$$K = \sum_{i=1}^{m} KE_{Li} + \sum_{i=1}^{n} KE_{Ai} \qquad (2.21)$$

2.2.2 Potential Energy

Potential energy is caused by the gravity for rigid links and can be derived as

$$PE_{Li} = -\bar{g}^T m_i^L \bar{r}_i^L \qquad i = 1,...,m \qquad (2.22)$$

where

$\bar{g}$ is the gravitational acceleration vector

$\bar{r}_i^L$ is the mass center position of the link as expressed in the fixed reference frame.

Potential energy is caused by the gravity and the elastic potential for the actuators and can be derived as

$$PE_{Ai} = -\bar{g}^T m_i {}^A\bar{r}_i^A + \frac{1}{2} K_i (\theta_{k_i} - \phi_i)^2 \qquad i = 1, \ldots, n \tag{2.23}$$

where

$\bar{r}_i^A$ is the mass center position of actuator as expressed in the fixed reference frame.

K_i is the joint spring constant of i^{th} transmission.

Total potential energy of the manipulator is formulated as

$$U = \sum_{i=1}^{m} PE_{Li} + \sum_{i=1}^{n} PE_{Ai} \tag{2.24}$$

2.2.3 Dissipation Function

There exists a positive definite dissipation function when a dissipative force occurs in the system. Causes of dissipative forces in the manipulator system are the structural damping of the actuated joints, the viscous friction that take place at the unactuated joints and the damping at the rotors. The dissipative functions can be formulated for each of the cases in a different way.

Dissipation function due to structural damping at the actuated joints,

$$D^a = \frac{1}{2} \sum_{i=1}^{n} D_i^a (\dot{\theta}_{k_i} - \dot{\phi}_i)^2 \tag{2.25}$$

where

D_i^a is the joint damping constant of i^{th} transmission.

Dissipation function due to damping that occurs at the rotors of the actuators,

$$D^r = \frac{1}{2} \sum_{i=1}^{n} D^r{}_i (r_i \dot{\phi}_i)^2 \tag{2.26}$$

where

D_i^r is the damping constant of $\mathrm{i^{th}}$ actuator.

Dissipation function due to viscous friction at the unactuated joints,

$$D^u = \frac{1}{2}\sum_{i=1}^{m-n} D_i^u \dot{\theta}_{l_i}^{\,2} \tag{2.27}$$

where

D_i^u is the joint damping constant of $\mathrm{i^{th}}$ unactuated joint.

Similarly, the dissipation function due to viscous friction at the disconnected joints can be written and denoted by D^d. As a result, the dissipation function that gives the total dissipative force of the manipulator is apparently equal to the sum of the individual dissipative forces at the rotors and joints.

$$D = D^a + D^r + D^u + D^d \tag{2.28}$$

2.2.4 Generalized Forces

In the system model, generalized forces consist of two types of forces. First type of forces are the non-potentialized ones which are supplied by the drive trains and second type of the forces are the constraint forces that are imposed on the system by disconnecting a sufficient number of the unactuated joints.

Non-potentialized forces are obtained by writing the work done by the manipulator in virtual form. The virtual work done by the manipulator that corresponds to the first set of generalized coordinates, i.e. the manipulator variables is zero if there is no external generalized force applied. On the other hand, the virtual work done by the manipulator that corresponds to the second set of generalized coordinates, i.e. the actuator variables is simply the torques after speed reduction. This fact can be formulated as in Equations 2.29 and 2.30.

$$\delta\tilde{W}_i = 0 \qquad i = 1,...,m \tag{2.29}$$

$$\delta\tilde{W}_{i+m} = T_i\delta\phi_i \qquad i = 1,...,n \tag{2.30}$$

Therefore the non-potentialized force terms obtained from the virtual work equations are found as

$$\tilde{Q}_i = 0 \qquad i = 1,...,m \tag{2.31}$$

$$\tilde{Q}_{i+m} = T_i \qquad i = 1,...,n \tag{2.32}$$

In most of the situations the *m-n* loop closure constraint equations at position level which are obtained by reconnecting the disconnected joints can be expressed as

$$\psi_i(\theta_1,...,\theta_m) = 0 \qquad i = 1,...,(m-n) \tag{2.33}$$

and at velocity level they can be written as,

$$\sum_{j=1}^{m} B_{ij}\dot{\theta}_j = 0 \qquad i = 1,...,(m-n) \tag{2.34}$$

In matrix form,

$$\hat{B}\dot{\overline{\theta}}_j = \overline{0} \qquad j = 1,...,m \tag{2.35}$$

where

$$B_{ij} = \frac{\partial\psi_i}{\partial\theta_j} \qquad i = 1,...,(m-n)\,,\ j = 1,...,m \tag{2.36}$$

Generalized constraint forces are obtained by writing the velocity level constraint equations in virtual form.

$$\sum_{j=1}^{m} B_{ij}\delta\theta_j = 0 \qquad i = 1,...,(m-n) \tag{2.37}$$

One can also write,

$$\sum_{i=1}^{m-n} \lambda_i \left[\sum_{j=1}^{m} B_{ij}\delta\theta_j \right] = 0 \tag{2.38}$$

$$\sum_{j=1}^{m} \left[\sum_{i=1}^{m-n} \lambda_i B_{ij} \right] \delta\theta_j = 0 \tag{2.39}$$

Expression in the brackets of Equation 2.39 leads to the definition of the constraint forces corresponding to the manipulator variables.

$$Q'_j = \sum_{i=1}^{m-n} B_{ij}\lambda_i \qquad j = 1,...,m \tag{2.40}$$

where λ_i, $i = 1,...,m-n$ are the Lagrange multipliers.

2.3 System Equations of Motion

Two sets of equations of motion are derived corresponding to the two sets of generalized coordinates for this system model. First one belongs to manipulator variables while the second one belongs to the actuator variables. After the expressions are substituted into Equations 2.7 and 2.8 and manipulated for each of the sets, the final forms of the equations of motion are obtained as follows.

The equations of motion for the first set of generalized coordinates are

$$\sum_{k=1}^{m} M_{ik}\ddot{\theta}_i + Q_i + Da_i + St_i - \sum_{k=1}^{m-n} \lambda_k B_{ki} = 0 \qquad i = 1,...,m \tag{2.41}$$

The equations of motion for the second set of generalized coordinates are

$$I_i^r r_i^2 \ddot{\phi}_i + D_i^r r_i^2 \dot{\phi}_i - D_i(\dot{\theta}_i - \dot{\phi}_i) - K_i(\theta_i - \phi_i) = T_i \qquad i = 1,...,n \tag{2.42}$$

The Equations (2.41) and (2.42) are rewritten in matrix form as

$$\hat{M}(\overline{\theta})\ddot{\overline{\theta}} + \overline{Q}(\overline{\theta},\dot{\overline{\theta}}) + \overline{D}a(\dot{\overline{\theta}},\dot{\overline{\phi}}) + \overline{S}t(\overline{\theta},\overline{\phi}) - \hat{B}^T(\overline{\theta})\overline{\lambda} = 0 \tag{2.43}$$

where

$\hat{M}(\overline{\theta})$ is the $m \times m$ symmetric positive definite generalized mass matrix,

$\overline{Q}(\overline{\theta},\dot{\overline{\theta}})$ is the $m \times 1$ vector which contains Coriolis, centrifugal and gravitational terms,

$\overline{D}a(\dot{\overline{\theta}},\dot{\overline{\phi}})$ is the $m \times 1$ vector which contains damping terms,

$\overline{S}t(\overline{\theta},\overline{\phi})$ is the $m \times 1$ vector which contains stiffness terms,

$\hat{B}(\overline{\theta})$ is the $(m-n) \times m$ matrix generalized force matrix.

$\overline{\lambda}$ is the $(m-n) \times 1$ vector whose elements consist of the Lagrange multipliers which mean the constraint forces imposed on the disconnected joint and

$$\hat{I}^r \ddot{\overline{\phi}} + \hat{D}^r \dot{\overline{\phi}} - \hat{D}^a(\dot{\overline{q}} - \dot{\overline{\phi}}) - \hat{K}(\overline{q} - \overline{\phi}) = \overline{T} \tag{2.44}$$

where

$\hat{I}^r$ is an $n \times n$ matrix whose elements are the inertial parameters of the links and can be expressed as

$$\hat{I}^r = diag[I_i^r r_i^2] \qquad i = 1,...,n \tag{2.45}$$

$\hat{D}^r$ is an $n \times n$ matrix whose elements are the inertial parameters of the rotors and can be expressed as

$$\hat{D}^r = diag[D_i^r r_i^2] \qquad i = 1,...,n \tag{2.46}$$

$\hat{K}$ is an $n \times n$ matrix whose elements are spring constants of the actuated joints and can be expressed as

$$\hat{K} = diag[K_i] \qquad i = 1,...,n \tag{2.47}$$

$\overline{T}$ is an $n \times 1$ vector whose elements are control torques after speed reduction.

2.4 Closed Loop Constraints

As explained in Section 2.2.4, the system constraint equations are necessary for the derivation of generalized constraint forces which physically mean the net torque applied by the joint forces at the disconnected joint(s) about the rotation axis of the joints.

On the other hand constraint equations are also necessary for writing the unactuated joint coordinates in terms of the actuated ones.

Let the joint coordinates vector of the rigid links $\overline{\theta} = [\theta_1,.....,\theta_m]^T$ be separated into two subvectors, which correspond to variables of the actuated joints $\overline{q}$ $(n \times 1)$ and the variables of the unactuated joints $\overline{\theta}^u$ $[(m-n) \times 1]$, respectively.

After the constraint equations of the manipulator when it is disconnected are written as in Equation 2.34, $(m-n) \times m$ matrix $\hat{B}$ is constructed. However, when the unactuated joint variables are written in terms of the actuated ones, matrix $\hat{B}$ is subdivided into an $(m-n) \times (m-n)$ matrix $\hat{B}^u$ and $(m-n) \times n$ matrix B^a as the coefficient matrices of the joint variables to which they correspond. Hence, Equation 2.35 can be written as

$$\hat{B}^u \dot{\overline{\theta}}^u = -\hat{B}^a \dot{\overline{q}} \tag{2.48}$$

One can solve for $\dot{\bar{\theta}}^u$ as below

$$\dot{\bar{\theta}}^u = \hat{C}\dot{\bar{q}} \tag{2.49}$$

where $\hat{C}$ is an $(m-n)\times n$ matrix and expressed as

$$\hat{C} = -(\hat{B}^u)^{-1}\hat{B}^a \tag{2.50}$$

Further differentiation of Equation 2.49 up to jerk level yields

$$\ddot{\bar{\theta}}^u = \hat{C}\ddot{\bar{q}} + \dot{\hat{C}}\dot{\bar{q}} \tag{2.51}$$

where

$$\dot{\hat{C}} = -\left((\dot{\hat{B}}^u)^{-1}\hat{B}^a + (\hat{B}^u)^{-1}\dot{\hat{B}}^a\right) \tag{2.52}$$

$$\dddot{\bar{\theta}}^u = \hat{C}\dddot{\bar{q}} + 2\dot{\hat{C}}\ddot{\bar{q}} + \ddot{\hat{C}}\dot{\bar{q}} \tag{2.53}$$

where

$$\ddot{\hat{C}} = -\left((\ddot{\hat{B}}^u)^{-1}\hat{B}^a + 2(\dot{\hat{B}}^u)^{-1}\dot{\hat{B}}^a + (\hat{B}^u)^{-1}\ddot{\hat{B}}^a\right) \tag{2.54}$$

CHAPTER III

INVERSE DYNAMICS CONTROL

3.1 Task Space Equations

The control method to be used for the parallel manipulator which has *m* links and *n* actuators at the joints is based on obtaining an equation between the inputs and the outputs. The inputs for the robot manipulators can be joint torques/forces or voltages supplied to the actuators. Since end effector position tracking is aimed in the control problem, the outputs become the joint positions either in task space or in the joint space.

When the commanded motion is specified in task space, then a relation ought to be derived between the joint space coordinates and task space coordinates.

Let $x_i, i=1,...,n$ represent the Cartesian end effector position variables. Then the functions that are used to relate each coordinate of the end effector to the joint coordinates, θ_j, $j=1,...,m$, i.e. so called task equations are written as

$$x_i = f_i(\theta_1,...,\theta_m) \qquad i=1,...,n \tag{3.1}$$

where *m* is the number of coordinates as expressed in the joint space.

Taking one step differentiation of Equation 3.1 yields the following velocity relation.

$$\dot{x}_i = \sum_{j=1}^{m} \Gamma^P{}_{ij} \dot{\theta}_j \qquad i=1,...,n \tag{3.2}$$

where

$$\Gamma^P{}_{ij} = \frac{\partial f_i}{\partial \dot{\theta}_j} \tag{3.3}$$

Equation 3.2 can be written in matrix form as

$$\dot{\vec{x}} = \hat{\Gamma}^{P}\dot{\vec{\theta}} \tag{3.4}$$

where $\hat{\Gamma}^{P}$ is the $n \times m$ manipulator Jacobian matrix.

The manipulator Jacobian matrix derived in Equation 3.4 can be written in terms of only the actuated variables by making use of Equation 2.49. Therefore the same procedure is followed to find $n \times n$ matrix $\hat{\Gamma}^{P^{a}}$ and $n \times (m-n)$ matrix $\hat{\Gamma}^{P^{u}}$ as the coefficient matrices of the joint variables to which they correspond. This can be formulated as follows.

$$\hat{\Gamma}^{P^{a}}\dot{\vec{q}} + \hat{\Gamma}^{P^{u}}\dot{\vec{\theta}}^{u} = \dot{\vec{x}} \tag{3.5}$$

Substituting Equation 2.49 into Equation 3.5 gives

$$\hat{\Gamma}^{P^{a}}\dot{\vec{q}} + \hat{\Gamma}^{P^{u}}\left[-\left(\hat{B}^{u}\right)^{-1}\hat{B}^{a}\dot{\vec{q}}\right] = \dot{\vec{x}} \tag{3.6}$$

Factoring out the joint coordinates vector of the actuated variables gives

$$\dot{\vec{x}} = \hat{J}\dot{\vec{q}} \tag{3.7}$$

where $\hat{J}$ is $n \times n$ manipulator Jacobian matrix expressed as

$$\hat{J} = \hat{\Gamma}^{P^{a}} - \hat{\Gamma}^{P^{u}}\hat{B}^{u^{-1}}\hat{B}^{a} \tag{3.8}$$

The Equation 3.7 is differentiated up to snap level as below.

$$\ddot{\vec{x}} = \dot{\hat{J}}\dot{\vec{q}} + \hat{J}\ddot{\vec{q}} \tag{3.9}$$

$$\dddot{\bar{x}} = \ddot{\hat{J}}\dot{\bar{q}} + 2\dot{\hat{J}}\ddot{\bar{q}} + \hat{J}\dddot{\bar{q}} \tag{3.10}$$

$$\ddddot{\bar{x}} = \dddot{\hat{J}}\dot{\bar{q}} + 3\ddot{\hat{J}}\ddot{\bar{q}} + 3\dot{\hat{J}}\dddot{\bar{q}} + \hat{J}\ddddot{\bar{q}} \tag{3.11}$$

Hence,

$$\ddddot{\bar{q}} = \hat{J}^{-1}(\ddddot{\bar{x}} - \dddot{\hat{J}}\dot{\bar{q}} - 3\ddot{\hat{J}}\ddot{\bar{q}} - 3\dot{\hat{J}}\dddot{\bar{q}}) \tag{3.12}$$

At this point, system equations of motion should be written in terms of the actuated joint variables and the Lagrange multipliers should be eliminated by using the constraint equations.

To realize this, first $\ddot{\theta}_i^u$ and $\dot{\theta}_i^u$, $i = 1,...,(m-n)$ are eliminated by using the Equations 2.49 and 2.51 from the Equation 2.43. Among the m number of scalar equations obtained, the n number of equations that correspond to the actuated joint variables can be written in matrix form as below.

$$\hat{M}^{a*}\ddot{\bar{q}} + \hat{R}^a\dot{\bar{q}} + \bar{Q}^a + \hat{D}^a(\dot{\bar{q}} - \dot{\bar{\phi}}) + \hat{K}(\bar{q} - \bar{\phi}) - \hat{B}^{a^T}\bar{\lambda} = 0 \tag{3.13}$$

where

$\hat{M}^{a*}$ is an $n \times n$ submatrix generated from symmetric generalized mass matrix whose elements consist of all of the joint positions.

$\hat{R}^a$ is an $n \times n$ bias matrix whose elements consist of all of the joint position and velocities.

$\bar{Q}^a$ is an $n \times 1$ subvector of the $m \times 1$ $\bar{Q}$ vector that contains centrifugal, Coriolis and gravitational terms.

$\hat{D}^a$ is an $n \times n$ matrix whose elements consist of the joint damping constants of the actuated joints which can be expressed as

$$\hat{D}^a = diag[D_i] \qquad i = 1...n \tag{3.14}$$

$\hat{K}$ is an $n \times n$ matrix whose elements consist of the joint spring constants of the actuated joints which can be expressed as

$$\hat{K} = diag[K_i] \qquad i = 1...n \tag{3.15}$$

$\hat{B}^a$ is an $(m-n) \times n$ submatrix of the matrix $\hat{B}$.

$\overline{\lambda}$ is the $(m-n) \times 1$ vector of Lagrange multipliers.

The remaining $m-n$ equations that correspond to the unactuated joint variables can be written as below.

$$\hat{M}^{u*}\ddot{\overline{q}} + \hat{R}^{u}\dot{\overline{q}} + \overline{Q}^{u} - \hat{B}^{u^T}\overline{\lambda} = 0 \tag{3.16}$$

where

$\hat{M}^{u*}$ is an $(m-n) \times n$ submatrix generated from symmetric generalized mass matrix $\hat{M}$ whose elements consist of all of the joint positions.

$\hat{R}^u$ is an $(m-n) \times n$ bias matrix whose elements consist of all of the joint position and velocities.

$\overline{Q}^u$ is an $(m-n) \times 1$ subvector of the $m \times 1$ $\overline{Q}$ vector that contains centrifugal, Coriolis and gravitational terms .

$\hat{B}^u$ is an $(m-n) \times (m-n)$ submatrix of the matrix $\hat{B}$.

$\overline{\lambda}$ is the $(m-n) \times 1$ vector of Lagrange multipliers.

As a second part of the elimination procedure, the Lagrange multipliers in the equations are solved from Equation 3.16 as below.

$$\overline{\lambda} = \left(\hat{B}^{u^T}\right)^{-1} \left\{\hat{M}^{u*}\ddot{\overline{q}} + \hat{R}^{u}\dot{\overline{q}} + \overline{Q}^{u}\right\} \tag{3.17}$$

Then, the Equation 3.17 is substituted into Equation 3.13.

$$\hat{M}^{**}\ddot{\overline{q}} + \hat{R}^{*}\dot{\overline{q}} + \overline{Q}^{*} + \hat{D}^{a}(\dot{\overline{q}} - \dot{\overline{\phi}}) + \hat{K}(\overline{q} - \overline{\phi}) = \overline{0} \tag{3.18}$$

In this way, the manipulator system in concern with constraints is transformed to a system without constraints. In this final form of the equations of motion for the first set of generalized coordinates, the unactuated joint variables at the acceleration and velocity levels and the constraint forces disappear.

The matrix form of the equations of motion for the second set of generalized coordinates, i.e. actuator variables are rewritten below for the sake of convenience.

$$\hat{I}^{r}\ddot{\overline{\phi}} + \hat{D}^{r}\dot{\overline{\phi}} - \hat{D}^{a}(\dot{\overline{q}} - \dot{\overline{\phi}}) - \hat{K}(\overline{q} - \overline{\phi}) = \overline{T} \tag{3.19}$$

As far as the inverse dynamics control law is considered, one needs to obtain an input-output relation using the equations of motion stated in Equations 3.18 and 3.19. Here, the inputs are the joint torques applied by the actuators and the outputs are the task space positions of the tip point of the end effector and the orientation of the end effector with respect to the fixed reference frame. In order to get a relation between the task space location of the end effector and the joint torques, the intermediate variables $\overline{\phi}$ and $\overline{q}$ should be eliminated. To facilitate the solution, the following steps are followed.

Step-1 : Factoring out dissipative and inductive part of Equation 3.18.

$$\hat{D}^{a}(\dot{\overline{q}} - \dot{\overline{\phi}}) + \hat{K}(\overline{q} - \overline{\phi}) = -\left[\hat{M}^{**}\ddot{\overline{q}} + \hat{R}^{*}\dot{\overline{q}} + \overline{Q}^{*}\right] \tag{3.20}$$

Step-2 : Substituting Equation 3.20 into Equation 3.19.

$$\hat{I}^{r}\ddot{\overline{\phi}} + \hat{D}^{r}\dot{\overline{\phi}} + \left(\hat{M}^{**}\ddot{\overline{q}} + \hat{R}^{*}\dot{\overline{q}} + \overline{Q}^{*}\right) = \overline{T} \tag{3.21}$$

Step-3 : Taking time derivative of Equation 3.21.

$$\hat{I}^{r}\dddot{\overline{\phi}} + \hat{D}^{r}\ddot{\overline{\phi}} + \dot{\hat{M}}^{**}\ddot{\overline{q}} + \hat{M}^{**}\dddot{\overline{q}} + \dot{\hat{R}}^{*}\dot{\overline{q}} + \hat{R}^{*}\ddot{\overline{q}} + \dot{\overline{Q}}^{*} = \dot{\overline{T}} \tag{3.22}$$

Step-4 : Multiplying Equation 3.21 by $\hat{K}$ and Equation 3.22 by $\hat{D}^a$ and adding them up

$$\hat{K}\hat{I}^r\ddot{\bar{\phi}} + \hat{K}\hat{D}^r\dot{\bar{\phi}} + \hat{K}\hat{M}^{**}\ddot{\bar{q}} + \hat{K}\hat{R}^{*}\dot{\bar{q}} + \hat{K}\bar{Q}^{*} + \hat{D}^a\hat{I}^r\dddot{\bar{\phi}} + \hat{D}^a\hat{D}^r\ddot{\bar{\phi}} + \hat{D}^a\dot{\hat{M}}^{**}\ddot{\bar{q}} + \hat{D}^a\hat{M}^{**}\dddot{\bar{q}}$$
$$+\hat{D}^a\dot{\hat{R}}^{*}\dot{\bar{q}} + \hat{D}^a\hat{R}^{*}\ddot{\bar{q}} + \hat{D}^a\dot{\bar{Q}}^{*} = \hat{D}^a\dot{\bar{T}} + \hat{K}\bar{T} \qquad (3.23)$$

In simplified form,

$$\hat{I}^r(\hat{D}^a\dddot{\bar{\phi}} + \hat{K}\ddot{\bar{\phi}}) + \hat{D}^r(\hat{D}^a\ddot{\bar{\phi}} + \hat{K}\dot{\bar{\phi}}) + \hat{D}^a(\dot{\hat{M}}^{**}\ddot{\bar{q}} + \hat{M}^{**}\dddot{\bar{q}} + \dot{\hat{R}}^{*}\dot{\bar{q}} + \hat{R}^{*}\ddot{\bar{q}} + \dot{\bar{Q}}^{*})$$
$$+\hat{K}\hat{M}^{**}\ddot{\bar{q}} + \hat{K}\hat{R}^{*}\dot{\bar{q}} + \hat{K}\bar{Q}^{*} = \hat{D}^a\dot{\bar{T}} + \hat{K}\bar{T} \qquad (3.24)$$

Step-5: Factoring out dissipative and inductive part of Equation 3.18 associated with actuator variables and taking time derivatives

$$\hat{D}^a\dot{\bar{\phi}} + \hat{K}\bar{\phi} = \hat{M}^{**}\ddot{\bar{q}} + \hat{R}^{*}\dot{\bar{q}} + \bar{Q}^{*} + \hat{D}^a\dot{\bar{q}} + \hat{K}\bar{q} \qquad (3.25)$$

$$\hat{D}^a\ddot{\bar{\phi}} + \hat{K}\dot{\bar{\phi}} = \dot{\hat{M}}^{**}\ddot{\bar{q}} + \hat{M}^{**}\dddot{\bar{q}} + \dot{\hat{R}}^{*}\dot{\bar{q}} + \hat{R}^{*}\ddot{\bar{q}} + \dot{\bar{Q}}^{*} + \hat{D}^a\ddot{\bar{q}} + \hat{K}\dot{\bar{q}} \qquad (3.26)$$

$$\hat{D}^a\dddot{\bar{\phi}} + \hat{K}\ddot{\bar{\phi}} = \ddot{\hat{M}}^{**}\ddot{\bar{q}} + 2\dot{\hat{M}}^{**}\dddot{\bar{q}} + \hat{M}^{**}\ddddot{\bar{q}} + \ddot{\hat{R}}^{*}\dot{\bar{q}} + 2\dot{\hat{R}}^{*}\ddot{\bar{q}} + \hat{R}^{*}\dddot{\bar{q}} + \ddot{\bar{Q}}^{*} + \hat{D}^a\dddot{\bar{q}} + \hat{K}\ddot{\bar{q}} \qquad (3.27)$$

Step-6 : Multiplying Equation 3.24 by $\hat{K}^{-1}$

$$\hat{K}^{-1}\hat{I}^r(\hat{D}^a\dddot{\bar{\phi}} + \hat{K}\ddot{\bar{\phi}}) + \hat{K}^{-1}\hat{D}^r(\hat{D}^a\ddot{\bar{\phi}} + \hat{K}\dot{\bar{\phi}}) + \hat{K}^{-1}\hat{D}^a(\dot{\hat{M}}^{**}\ddot{\bar{q}} + \hat{M}^{**}\dddot{\bar{q}} + \dot{\hat{R}}^{*}\dot{\bar{q}} + \hat{R}^{*}\ddot{\bar{q}} + \dot{\bar{Q}}^{*})$$
$$+\hat{M}^{**}\ddot{\bar{q}} + \hat{R}^{*}\dot{\bar{q}} + \bar{Q}^{*} = (\hat{K}^{-1}\hat{D}^a)\dot{\bar{T}} + \bar{T} \qquad (3.28)$$

Step-7 : Substituting Equations 3.26 and 3.27 into Equation 3.28.

$$\hat{K}^{-1}\hat{I}^r\left[\ddot{\hat{M}}^{**}\ddot{\bar{q}} + 2\dot{\hat{M}}^{**}\dddot{\bar{q}} + \hat{M}^{**}\ddddot{\bar{q}} + \ddot{\hat{R}}^{*}\dot{\bar{q}} + 2\dot{\hat{R}}^{*}\ddot{\bar{q}} + \hat{R}^{*}\dddot{\bar{q}} + \ddot{\bar{Q}}^{*} + \hat{D}^a\dddot{\bar{q}} + \hat{K}\ddot{\bar{q}}\right]$$
$$+\hat{K}^{-1}\hat{D}^r\left[\dot{\hat{M}}^{**}\ddot{\bar{q}} + \hat{M}^{**}\dddot{\bar{q}} + \dot{\hat{R}}^{*}\dot{\bar{q}} + \hat{R}^{*}\ddot{\bar{q}} + \dot{\bar{Q}}^{*} + \hat{D}^a\ddot{\bar{q}} + \hat{K}\dot{\bar{q}}\right]$$
$$+\hat{K}^{-1}\hat{D}^a\left[\dot{\hat{M}}^{**}\ddot{\bar{q}} + \hat{M}^{**}\dddot{\bar{q}} + \dot{\hat{R}}^{*}\dot{\bar{q}} + \hat{R}^{*}\ddot{\bar{q}} + \dot{\bar{Q}}^{*}\right] + \hat{M}^{**}\ddot{\bar{q}} + \hat{R}^{*}\dot{\bar{q}} + \bar{Q}^{*}$$
$$= \left[\hat{K}^{-1}\hat{D}^a\right]\dot{\bar{T}} + \bar{T} \qquad (3.29)$$

In rearranged form

$$\hat{K}^{-1}\hat{I}^{r}\left[\left(\hat{M}^{**}\right)\ddddot{\bar{q}}+\left(2\dot{\hat{M}}^{**}+\hat{R}^{*}+\hat{D}^{a}\right)\dddot{\bar{q}}+\left(\ddot{\hat{M}}^{**}+2\dot{\hat{R}}^{*}+\hat{K}\right)\ddot{\bar{q}}+\left(\ddot{\hat{R}}^{*}\right)\dot{\bar{q}}+\ddot{\bar{Q}}^{*}\right]$$
$$+\hat{K}^{-1}\hat{D}^{r}\left[\left(\hat{M}^{**}\right)\dddot{\bar{q}}+\left(\dot{\hat{M}}^{**}+\hat{R}^{*}+\hat{D}^{a}\right)\ddot{\bar{q}}+\left(\dot{\hat{R}}^{*}+\hat{K}\right)\dot{\bar{q}}+\dot{\bar{Q}}^{*}\right]$$
$$+\hat{K}^{-1}\hat{D}^{a}\left[\left(\hat{M}^{**}\right)\dddot{\bar{q}}+\left(\dot{\hat{M}}^{**}+\hat{R}^{*}\right)\ddot{\bar{q}}+\left(\dot{\hat{R}}^{*}\right)\dot{\bar{q}}+\dot{\bar{Q}}^{*}\right]+\left(\hat{M}^{**}\right)\ddot{\bar{q}}+\left(\hat{R}^{*}\right)\dot{\bar{q}}+\bar{Q}^{*}$$
$$=\left[\hat{K}^{-1}\hat{D}^{a}\right]\dot{\bar{T}}+\bar{T} \tag{3.30}$$

Step-8 : Substituting Equation 3.12 into Equation 3.30.

$$\hat{K}^{-1}\hat{I}^{r}\left[\hat{M}^{**}\left(\hat{J}^{-1}(\ddddot{\bar{x}}-\dddot{\hat{J}}\dot{\bar{q}}-3\ddot{\hat{J}}\ddot{\bar{q}}-3\dot{\hat{J}}\dddot{\bar{q}})\right)+\left(2\dot{\hat{M}}^{**}+\hat{R}^{*}+\hat{D}^{a}\right)\dddot{\bar{q}}+\left(\ddot{\hat{M}}^{**}+2\dot{\hat{R}}^{*}+\hat{K}\right)\ddot{\bar{q}}\right.$$
$$\left.+\ddot{\hat{R}}^{*}\dot{\bar{q}}+\ddot{\bar{Q}}^{*}\right]+\hat{K}^{-1}\hat{D}^{r}\left[\hat{M}^{**}\dddot{\bar{q}}+\left(\dot{\hat{M}}^{**}+\hat{R}^{*}+\hat{D}^{a}\right)\ddot{\bar{q}}+\left(\dot{\hat{R}}^{*}+\hat{K}\right)\dot{\bar{q}}+\dot{\bar{Q}}^{*}\right]$$
$$+\hat{K}^{-1}\hat{D}^{a}\left[\hat{M}^{**}\dddot{\bar{q}}+\left(\dot{\hat{M}}^{**}+\hat{R}^{*}\right)\ddot{\bar{q}}+\dot{\hat{R}}^{*}\dot{\bar{q}}+\dot{\bar{Q}}^{*}\right]+\hat{M}^{**}\ddot{\bar{q}}+\hat{R}^{*}\dot{\bar{q}}+\bar{Q}^{*}$$
$$=\bar{T}+\left[\hat{K}^{-1}\hat{D}^{a}\right]\dot{\bar{T}} \tag{3.31}$$

where $\bar{q}\,,\dot{\bar{q}}\,,\ddot{\bar{q}}\,,\dddot{\bar{q}}$ can be written in terms of $\bar{x}\,,\dot{\bar{x}}\,,\ddot{\bar{x}}\,,\dddot{\bar{x}}$ using the task space equations 3.1, 3.7, 3.9 – 3.11.

3.2 Control Law

Equation 3.31 gives a relation between inputs and outputs after the elimination of the intermediate variables $\bar{\phi}$ and $\bar{q}$. This equation can be written in more compact form as

$$\hat{N}(\bar{x})\ddddot{\bar{x}}+\bar{P}(\dddot{\bar{x}},\ddot{\bar{x}},\dot{\bar{x}},\bar{x})=\bar{T}+\hat{S}\dot{\bar{T}} \tag{3.32}$$

where

$$\hat{N}=\hat{K}^{-1}\hat{I}^{r}\hat{M}^{**}\hat{J}^{-1} \tag{3.33}$$

$$\hat{S} = \hat{K}^{-1}\hat{D}^{a} \tag{3.34}$$

$$\begin{aligned}\overline{P} = {} & \hat{K}^{-1}\hat{I}^{r}\Big[\Big(-\hat{M}^{**}3\hat{J}^{-1}\dot{\hat{J}} + 2\dot{\hat{M}}^{**} + \hat{R}^{*} + \hat{D}^{a}\Big)\dddot{\bar{q}} + \Big(-\hat{M}^{**}3\hat{J}^{-1}\ddot{\hat{J}} + \ddot{\hat{M}}^{**} + 2\dot{\hat{R}}^{*} + \hat{K}\Big)\ddot{\bar{q}} \\ & + \Big(-\hat{M}^{**}\hat{J}^{-1}\dddot{\hat{J}} + \ddot{\hat{R}}^{*}\Big)\dot{\bar{q}} + \ddot{\bar{Q}}^{*}\Big] + \hat{K}^{-1}\hat{D}^{r}\Big[\hat{M}^{**}\dddot{\bar{q}} + \Big(\dot{\hat{M}}^{**} + \hat{R}^{*} + \hat{D}^{a}\Big)\ddot{\bar{q}} + \Big(\dot{\hat{R}}^{*} + \hat{K}\Big)\dot{\bar{q}} + \dot{\bar{Q}}^{*}\Big] \\ & + \hat{K}^{-1}\hat{D}^{a}\Big[\hat{M}^{**}\dddot{\bar{q}} + \Big(\dot{\hat{M}}^{**} + \hat{R}^{*}\Big)\ddot{\bar{q}} + \dot{\hat{R}}^{*}\dot{\bar{q}} + \dot{\bar{Q}}^{*}\Big] + \hat{M}^{**}\ddot{\bar{q}} + \hat{R}^{*}\dot{\bar{q}} + \bar{Q}^{*}\end{aligned} \tag{3.35}$$

The basic principle of inverse dynamics control is to find a control input vector which will linearize and decouple the Equation 3.32. Therefore $\overline{T} + \hat{S}\dot{\overline{T}}$ is chosen as

$$\overline{T} + \hat{S}\dot{\overline{T}} = \hat{N}\bar{u} + \overline{P} \tag{3.36}$$

Then Equation 3.32 yields

$$\ddddot{\bar{x}} = \bar{u} \tag{3.37}$$

where $\bar{u}$ is *nx1* control input vector that represents the command snaps.

In the inverse dynamics solution, matrix $\hat{S}$ has to be inverted. If the matrix $\hat{D}^{a}$ which contains damping constants at the active joints is not invertible, then some entries of the diagonal matrix $\hat{S}$ become zero and this leads matrix $\hat{S}$ to be singular. To overcome this problem, the Equation 3.36 can be written as *n* number of scalar equations. The equations in which the $\hat{S}$ matrix entries are zero give directly the corresponding torques. On the other hand, nonzero entries cause the appearance of the torque rates in the equations. In this case, the torque rates are numerically integrated and the remaining torque values are calculated.

3.3 Position Error Dynamics

New control variable $\overline{u}$ can be chosen by using the error states as

$$\overline{u} = \dddot{\overline{x}}^d + \hat{C}_1(\dddot{\overline{x}}^d - \dddot{\overline{x}}) + \hat{C}_2(\ddot{\overline{x}}^d - \ddot{\overline{x}}) + \hat{C}_3(\dot{\overline{x}}^d - \dot{\overline{x}}) + \hat{C}_4(\overline{x}^d - \overline{x}) + \hat{C}_5\int(\overline{x}^d - \overline{x})dt \qquad (3.38)$$

where the superscript *d* is used for the desired values and $\hat{C}_i$, $i = 1,...,5$ are diagonal feedback gain matrices.

When Equation 3.38 is substituted into Equation 3.37, the following linear error dynamics is obtained after the computed torques are applied on the system without considering any modeling error or disturbance.

$$\ddddot{\overline{e}}_p + \hat{C}_1\dddot{\overline{e}}_p + \hat{C}_2\ddot{\overline{e}}_p + \hat{C}_3\dot{\overline{e}}_p + \hat{C}_4\overline{e}_p + \hat{C}_5\int\overline{e}_p dt = 0 \qquad (3.39)$$

where $\overline{e}_p$ is the vector of errors describing how much the system is deviated from its actual task position and can be expressed as

$$\overline{e}_p = \overline{x}^d - \overline{x} \qquad (3.40)$$

Asymptotic stability is achieved by an appropriate selection of the feedback gains. For this purpose, pole placement technique or some norms like ISE, ITAE, IAE, etc. can be utilized. In this study Integral of Time Multiplied by the Absolute Value of Error (ITAE) performance index is used.

This performance index has such an effect that the weight of the absolute error decreases as the time goes by. Therefore at the small values of time, the errors become large and as time increases, the absolute error gets smaller.

In general, the characteristic equation of a feedback control system has the form $s^n + C_1s^{n-1} + C_2s^{n-2} + ... + C_n$. The coefficients for the fourth and the fifth order systems are given according to ITAE norm as in Table 3.1.

Table 3.1 Feedback Gains

Feedback Gains	Without Integral Control	With Integral Control
C_1	$2.1\omega_o$	$2.8\omega_o$
C_2	$3.4\omega_o^2$	$5.0\omega_o^2$
C_3	$2.7\omega_o^3$	$5.5\omega_o^3$
C_4	ω_o^4	$3.4\omega_o^4$
C_5	-	ω_o^5

where ω_o is a positive constant value.

System poles for the fourth order system are obtained by using ITAE norm as

$$p_{1,2} = -0.4240\omega_o \pm j1.2630\omega_o \tag{3.41}$$

$$p_{3,4} = -0.6260\omega_o \pm j0.4141\omega_o \tag{3.42}$$

Similarly, system poles for the fifth order system are

$$p_{1,2} = -0.3764\omega_o \pm j1.2920\omega_o \tag{3.43}$$

$$p_3 = -0.8955\omega_o \tag{3.44}$$

$$p_{4,5} = -0.5758\omega_o \pm j0.5339\omega_o \tag{3.45}$$

CHAPTER IV

THE CONCEPT OF SINGULARITY

4.1 Singularity in Manipulators

Singularity in both serial and parallel manipulators is a very significant issue since it creates some singular configurations for the manipulators. These are defined to be configurations when manipulator Jacobian matrix in Equation 3.7 has less than full rank. Physically, these configurations correspond to situations where the joints have been aligned in such a way that there is at least one direction of motion for the end effector that physically cannot be achieved by the mechanism just because of the extended or folded positions of the links. Singular configurations occur at workspace boundaries for the serial and parallel manipulators and inside the workspace volume only for the parallel manipulators due to their closed loop structure. When the axes of two or more joints line up and consequently the links are in extended or folded positions, another end effector degree of freedom gets lost.

All robotic manipulators have singular configurations. In other words, the existence of singularities cannot be eliminated even by careful design. For this reason, singularities are a serious cause of drawbacks in robotic analysis and control to be handled only by a proper trajectory generation.

Motions have to be carefully planned in the region of singularities to avoid them. This is not only because at the singularities there will be an unobtainable motion at the end effector, but also because many real time motion planning and control algorithms make use of the manipulator Jacobian. For the reasons above, the analysis of singularities is an important issue in robotics and continues to be the interest of research. In the following sections the types of these singularities will be explained in detail.

4.2 Types of Singularities

Since this study covers the analysis of parallel manipulators, the following two sections will explain the singular cases that the researchers come across most commonly.

4.2.1 Drive Singularity

As previously stated, the closed loop structure produce a special type of degeneracy to the parallel manipulators which can be called *drive singularity,* where the motion control ability becomes lost and the actuator forces grow unboundedly. Since some of the joints are unactuated, at certain positions the actuators of the system may become unable to control the moving platform. As the system approaches to a drive singularity the actuator forces grow without bounds. [6]

Therefore the studies related to the drive singularities mostly aim at finding only the locations of the singular positions for the purpose of avoiding them in the motion planning stage.

Drive singularity occurs while solving for the actuator forces. At a drive singularity the actuators cannot realize the assigned snap values and influence the end effector snaps. Consequently they lose control in one or more degrees of freedom.

In this study, the condition where drive singularity prevails is obtained by using the equation below as stated in [6].

$$\det(\hat{B}^u) = 0 \tag{4.1}$$

where $\hat{B}^u$ is $(m-n)\times(m-n)$ matrix composed of the columns of the coefficient matrix $\hat{B}$ as explained in Chapter II.

4.2.2 Inverse Kinematic Singularity

Inverse kinematic singular configurations exist for both parallel and serial manipulators when the desired motion is expressed in Cartesian reference frame. For the prescribed $\bar{x}$, $\dddot{\bar{q}}$ can be calculated using Equation 3.12, $\dddot{\bar{q}}$ from Equation 3.10, $\ddot{\bar{q}}$ from Equation 3.9, $\dot{\bar{q}}$ from Equation 3.7 and finally $\bar{q}$ from equation 3.1. However during this inverse kinematic solution, this type of singularities takes place.

To implement this fact, let an $m \times m$ matrix $\hat{\Gamma}$ is defined as

$$\hat{\Gamma} = \begin{bmatrix} \hat{B} \\ \hat{\Gamma}^P \end{bmatrix} \tag{4.2}$$

which is composed of $(m-n)\times m$ matrix $\hat{B}$ and $n \times m$ matrix $\hat{\Gamma}^P$ as described in Chapter II and III.

Then the condition where inverse kinematic singularity prevails is obtained by using the equation below as derived in [6].

$$\det(\hat{\Gamma}) = 0 \tag{4.3}$$

The condition above leads to a few singular configurations that affect the links being extended or folded positions. As a result of that some of the joint positions become undistinguishable.

CHAPTER V

CASE STUDY AND SIMULATIONS

5.1 Case Study

In this chapter, a planar parallel manipulator is considered as a case study in order to check out the performance of the control law. Parallel manipulators are generally classified with respect to their number of legs and type of joints that these legs have beginning from the fixed base to the moving platform. The parallel manipulator to be analyzed has two legs and each of them has three revolute joints from the fixed base to the moving platform. Therefore it is said to be 2-RRR planar manipulator which is shown in Figure 5.1.

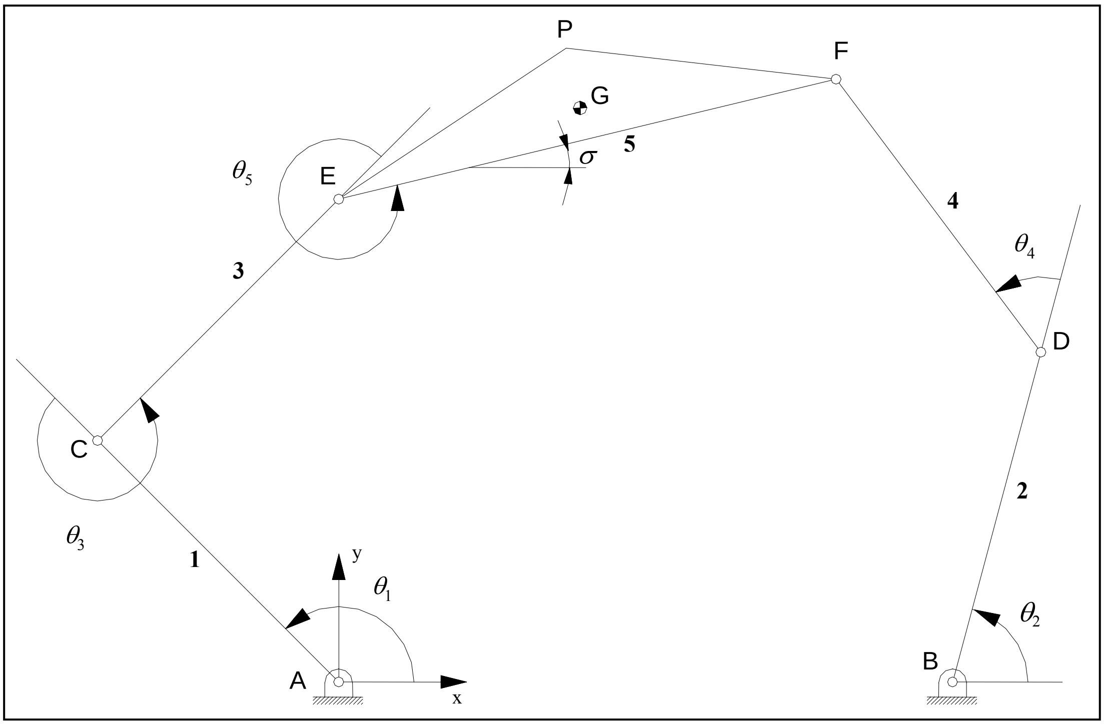

Figure 5.1 2-RRR Planar Manipulator with Three Flexible Joint Actuation

This planar parallel manipulator has six links including the fixed link and six revolute joints. It is actuated by three actuators located at A, B and C whose joint variables are θ_1, θ_2 and θ_3. The rotation axes of the joints are perpendicular to the plane of motion. The manipulator is a three degree of freedom one excluding the additional degrees of freedom that arise due to the flexible joints. Considering three flexible joint actuation, the degree of freedom of the system increases to six.

The viscous damping of the actuators and the torsional damping characteristics of joints are considered in the system.

In the following sections, the Lagrangian formulation of the manipulator is going to be derived to find the system equations of motion as explained in Chapter II. Before that the sets of generalized coordinates need to be defined.

Let the sets of the generalized coordinates corresponding to the manipulator variables and actuator variables be defined respectively as

$$G_1 = \{\theta_1, \theta_2, \theta_3, \theta_4, \theta_5\} \tag{5.1}$$

$$G_2 = \{\phi_1, \phi_2, \phi_3\} \tag{5.2}$$

Then vector of manipulator variables of the rigid links become

$$\bar{\theta} = \begin{bmatrix} \theta_1 & \theta_2 & \theta_3 & \theta_4 & \theta_5 \end{bmatrix}^T \tag{5.3}$$

which is subdivided into vector of actuated and unactuated joint variables as

$$\bar{q} = \begin{bmatrix} \theta_1 & \theta_2 & \theta_3 \end{bmatrix}^T \tag{5.4}$$

$$\bar{\theta}^u = \begin{bmatrix} \theta_4 & \theta_5 \end{bmatrix}^T \tag{5.5}$$

and vector of actuator variables of the flexible joints turn out to be

$$\bar{\phi} = \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 \end{bmatrix}^T \tag{5.6}$$

Since five joint variables are assigned, i.e. $m=5$ and degree of freedom of the manipulator excluding the additional degree of freedom caused by flexibility is three, i.e. $n=3$, one need to disconnect the joint at Point-F and get two open kinematic chains and therefore two constraint equations associated with this disconnection. These open chains with the unit vectors assigned to the links and actuators are illustrated in Figures 5.2 and 5.3.

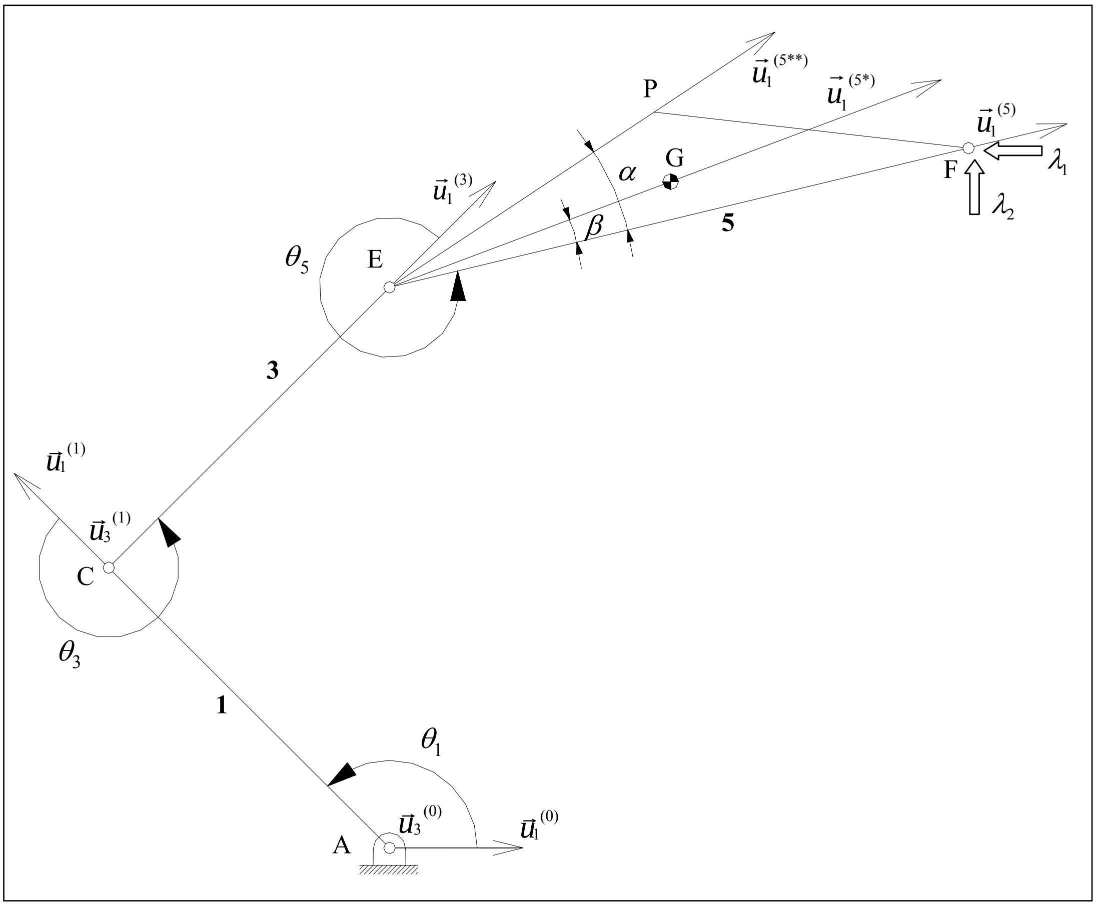

Figure 5.2 Open Kinematic Chain-1

In both of the figures λ_i, $i=1,2$ show the forces applied at the disconnected joint in horizontal and vertical directions physically.

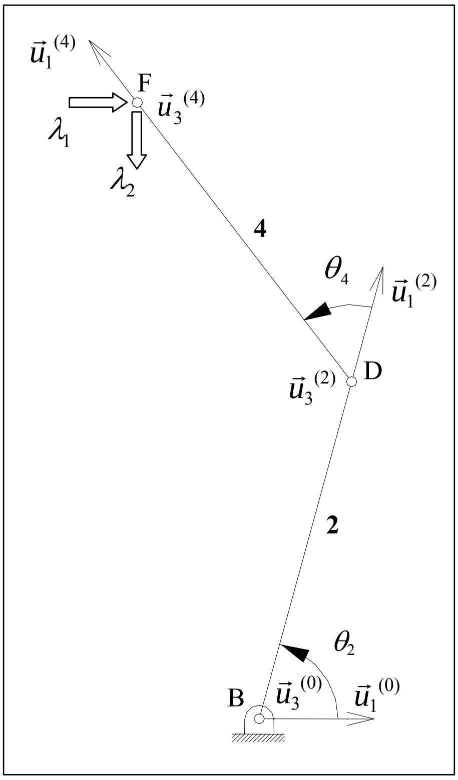

Figure 5.3 Open Kinematic Chain-2

5.1.1 Kinetic Energy Expressions

Angular and translational velocity components that lead to the kinetic energy expressions are going to be written firstly.

Link-1:

$$\overline{r}_{G_1}^{(0)} = \frac{L_1}{2}(\overline{u}_1 c\theta_1 + \overline{u}_2 s\theta_1) \tag{5.7}$$

$$\overline{V}_{G_1}^{(0)} = \dot{\overline{r}}_{G_1}^{(0)} = \frac{L_1}{2}(-\overline{u}_1 s\theta_1\dot{\theta}_1 + \overline{u}_2 c\theta_1\dot{\theta}_1) = \begin{bmatrix} -\frac{L_1}{2}s\theta_1\dot{\theta}_1 \\ \frac{L_1}{2}c\theta_1\dot{\theta}_1 \\ 0 \end{bmatrix} \tag{5.8}$$

$$\overline{\omega}_1^{(0)} = \dot{\theta}_1\overline{u}_3 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \tag{5.9}$$

Link-2:

$$\bar{r}_{G_2}{}^{(0)} = d_0\bar{u}_1 + \frac{L_2}{2}(\bar{u}_1 c\theta_2 + \bar{u}_2 s\theta_2) \tag{5.10}$$

$$\bar{V}_{G_2}{}^{(0)} = \dot{\bar{r}}_{G_2}{}^{(0)} = \frac{L_2}{2}(-\bar{u}_1 s\theta_2\dot{\theta}_2 + \bar{u}_2 c\theta_2\dot{\theta}_2) = \begin{bmatrix} -\frac{L_2}{2}s\theta_2\dot{\theta}_2 \\ \frac{L_2}{2}c\theta_2\dot{\theta}_2 \\ 0 \end{bmatrix} \tag{5.11}$$

$$\bar{\omega}_2{}^{(0)} = \dot{\theta}_2\bar{u}_3 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix} \tag{5.12}$$

Link-3:

$$\bar{r}_{G_3}{}^{(0)} = L_1(\bar{u}_1 c\theta_1 + \bar{u}_2 s\theta_1) + \frac{L_3}{2}(\bar{u}_1 c\theta_{13} + \bar{u}_2 s\theta_{13}) \tag{5.13}$$

$$\bar{V}_{G_3}{}^{(0)} = \dot{\bar{r}}_{G_3}{}^{(0)} = L_1(-\bar{u}_1 s\theta_1\dot{\theta}_1 + \bar{u}_2 c\theta_1\dot{\theta}_1) + \frac{L_3}{2}(-\bar{u}_1 s\theta_{13}\dot{\theta}_{13} + \bar{u}_2 c\theta_{13}\dot{\theta}_{13}) = \begin{bmatrix} -L_1 s\theta_1\dot{\theta}_1 - \frac{L_3}{2}s\theta_{13}\dot{\theta}_{13} \\ L_1 c\theta_1\dot{\theta}_1 + \frac{L_3}{2}c\theta_{13}\dot{\theta}_{13} \\ 0 \end{bmatrix} \tag{5.14}$$

$$\bar{\omega}_3{}^{(0)} = \dot{\theta}_1\bar{u}_3 + \dot{\theta}_3\bar{u}_3 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{13} \end{bmatrix} \tag{5.15}$$

Link-4:

$$\bar{r}_{G_4}{}^{(0)} = d_0\bar{u}_1 + L_2(\bar{u}_1 c\theta_2 + \bar{u}_2 s\theta_2) + \frac{L_4}{2}(\bar{u}_1 c\theta_{24} + \bar{u}_2 s\theta_{24}) \tag{5.16}$$

$$\overline{V}_{G_4}^{(0)} = \dot{\overline{r}}_{G_4}^{(0)} = L_2(-\overline{u}_1 s\theta_2\dot{\theta}_2 + \overline{u}_2 c\theta_2\dot{\theta}_2) + \frac{L_4}{2}(-\overline{u}_1 s\theta_{24}\dot{\theta}_{24} + \overline{u}_2 c\theta_{24}\dot{\theta}_{24}) = \begin{bmatrix} -L_2 s\theta_2\dot{\theta}_2 - \frac{L_4}{2} s\theta_{24}\dot{\theta}_{24} \\ L_2 c\theta_2\dot{\theta}_2 + \frac{L_4}{2} c\theta_{24}\dot{\theta}_{24} \\ 0 \end{bmatrix} \quad (5.17)$$

$$\overline{\omega}_5^{(0)} = \dot{\theta}_2\overline{u}_3 + \dot{\theta}_4\overline{u}_3 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{24} \end{bmatrix} \quad (5.18)$$

Link-5:

$$\overline{r}_{G_5}^{(0)} = L_1(\overline{u}_1 c\theta_1 + \overline{u}_2 s\theta_1) + L_3(\overline{u}_1 c\theta_{13} + \overline{u}_2 s\theta_{13}) + g_5[\overline{u}_1 c(\theta_{135} + \beta) + \overline{u}_2 s(\theta_{135} + \beta)] \quad (5.19)$$

$$\overline{V}_{G_5}^{(0)} = \dot{\overline{r}}_{G_5}^{(0)} = L_1(-\overline{u}_1 s\theta_1\dot{\theta}_1 + \overline{u}_2 c\theta_1\dot{\theta}_1) + L_3(-\overline{u}_1 s\theta_{13}\dot{\theta}_{13} + \overline{u}_2 c\theta_{13}\dot{\theta}_{13})$$
$$+g_5(-\overline{u}_1 s(\theta_{135} + \beta)\dot{\theta}_{135} + \overline{u}_2 c(\theta_{135} + \beta)\dot{\theta}_{135}) = \begin{bmatrix} -L_1 s\theta_1\dot{\theta}_1 - L_3 s\theta_{13}\dot{\theta}_{13} - g_5 s(\theta_{135} + \beta)\dot{\theta}_{135} \\ L_1 c\theta_1\dot{\theta}_1 + L_3 c\theta_{13}\dot{\theta}_{13} + g_5 c(\theta_{135} + \beta)\dot{\theta}_{135} \\ 0 \end{bmatrix} \quad (5.20)$$

$$\overline{\omega}_5^{(0)} = \dot{\theta}_1\overline{u}_3 + \dot{\theta}_3\overline{u}_3 + \dot{\theta}_5\overline{u}_3 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{135} \end{bmatrix} \quad (5.21)$$

Actuator-1:

$$\overline{V}_1^A = \overline{0} \quad (5.22)$$

$$\overline{\omega}_1^A = \begin{bmatrix} 0 \\ 0 \\ r_1\dot{\phi}_1 \end{bmatrix} \quad (5.23)$$

Actuator-2:

$$\bar{V}_2^{\,A} = \bar{0} \tag{5.24}$$

$$\bar{\omega}_2^{\,A} = \begin{bmatrix} 0 \\ 0 \\ r_2\dot{\phi}_2 \end{bmatrix} \tag{5.25}$$

Actuator-3:

$$\bar{V}_3^{\,A} = \begin{bmatrix} -L_1 s\theta_1\dot{\theta}_1 \\ L_1 c\theta_1\dot{\theta}_1 \\ 0 \end{bmatrix} \tag{5.26}$$

$$\bar{\omega}_3^{\,A} = \begin{bmatrix} 0 \\ 0 \\ r_3\dot{\phi}_3 \end{bmatrix} \tag{5.27}$$

The kinetic energy expressions are obtained by substituting the translational and angular components into Equations 2.9 and 2.20 as follows

$$KE_{L1} = \frac{1}{2}[m_1^{\,L}\frac{L_1^{\,2}}{4} + I_{1zz}]\dot{\theta}_1^{\,2} \tag{5.28}$$

$$KE_{L2} = \frac{1}{2}[m_2^{\,L}\frac{L_2^{\,2}}{4} + I_{2zz}]\dot{\theta}_2^{\,2} \tag{5.29}$$

$$KE_{L3} = \frac{1}{2}m_3^{\,L}L_1^{\,2}\dot{\theta}_1^{\,2} + \frac{1}{2}m_3^{\,L}L_1L_3c(\theta_1 - \theta_{13})\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_3) + \frac{1}{2}[m_3^{\,L}\frac{L_3^{\,2}}{4} + I_{3zz}](\dot{\theta}_1 + \dot{\theta}_3)^2 \tag{5.30}$$

In expanded form

$$KE_{L3} = \frac{1}{2} m_3{}^L L_1{}^2 \dot{\theta}_1{}^2 + \frac{1}{2} m_3{}^L L_1 L_3 c(-\theta_3) \dot{\theta}_1{}^2 + \frac{1}{2} m_3{}^L L_1 L_3 c(-\theta_3) \dot{\theta}_1 \dot{\theta}_3 + \frac{1}{2} [m_3{}^L \frac{L_3{}^2}{4} + I_{3zz}] \dot{\theta}_1{}^2$$
$$+ [m_3{}^L \frac{L_3{}^2}{4} + I_{3zz}] \dot{\theta}_1 \dot{\theta}_3 + \frac{1}{2} [m_3{}^L \frac{L_3{}^2}{4} + I_{3zz}] \dot{\theta}_3{}^2 \qquad (5.31)$$

$$KE_{L4} = \frac{1}{2} m_4{}^L L_2{}^2 \dot{\theta}_2{}^2 + \frac{1}{2} m_4{}^L L_2 L_4 c(\theta_2 - \theta_{24}) \dot{\theta}_2 (\dot{\theta}_2 + \dot{\theta}_4) + \frac{1}{2} [m_4{}^L \frac{L_4{}^2}{4} + I_{4zz}] (\dot{\theta}_2 + \dot{\theta}_4)^2 \qquad (5.32)$$

In expanded form

$$KE_{L4} = \frac{1}{2} m_4{}^L L_2{}^2 \dot{\theta}_2{}^2 + \frac{1}{2} m_4{}^L L_2 L_4 c(-\theta_4) \dot{\theta}_2{}^2 + \frac{1}{2} m_4{}^L L_2 L_4 c(-\theta_4) \dot{\theta}_2 \dot{\theta}_4 + \frac{1}{2} [m_4{}^L \frac{L_4{}^2}{4} + I_{4zz}] \dot{\theta}_2{}^2$$
$$+ [m_4{}^L \frac{L_4{}^2}{4} + I_{4zz}] \dot{\theta}_2 \dot{\theta}_4 + \frac{1}{2} [m_4{}^L \frac{L_4{}^2}{4} + I_{4zz}] \dot{\theta}_4{}^2 \qquad (5.33)$$

$$KE_{L5} = \frac{1}{2} m_5{}^L L_1{}^2 \dot{\theta}_1{}^2 + \frac{1}{2} m_5{}^L L_3{}^2 (\dot{\theta}_1 + \dot{\theta}_3)^2 + \frac{1}{2} [m_5{}^L g_5{}^2 + I_{5zz}] (\dot{\theta}_1 + \dot{\theta}_3 + \dot{\theta}_5)^2$$
$$+ m_5{}^L L_1 L_3 c(\theta_1 - \theta_{13}) \dot{\theta}_1 (\dot{\theta}_1 + \dot{\theta}_3) + m_5{}^L L_1 g_5 c(\theta_1 - \theta_{135} - \beta) \dot{\theta}_1 (\dot{\theta}_1 + \dot{\theta}_3 + \dot{\theta}_5)$$
$$+ m_5{}^L L_3 g_5 c(\theta_{13} - \theta_{135} - \beta) (\dot{\theta}_1 + \dot{\theta}_3)(\dot{\theta}_1 + \dot{\theta}_3 + \dot{\theta}_5) \qquad (5.34)$$

In expanded form

$$KE_{L5} = \frac{1}{2} m_5{}^L L_1{}^2 \dot{\theta}_1{}^2 + \frac{1}{2} m_5{}^L L_3{}^2 \dot{\theta}_1{}^2 + m_5{}^L L_3{}^2 \dot{\theta}_1 \dot{\theta}_3 + \frac{1}{2} m_5{}^L L_3{}^2 \dot{\theta}_3{}^2 + \frac{1}{2} [m_5{}^L g_5{}^2 + I_{5zz}] \dot{\theta}_1{}^2$$
$$+ \frac{1}{2} [m_5{}^L g_5{}^2 + I_{5zz}] \dot{\theta}_3{}^2 + \frac{1}{2} [m_5{}^L g_5{}^2 + I_{5zz}] \dot{\theta}_5{}^2 + [m_5{}^L g_5{}^2 + I_{5zz}] \dot{\theta}_1 \dot{\theta}_3 + [m_5{}^L g_5{}^2 + I_{5zz}] \dot{\theta}_1 \dot{\theta}_5$$
$$+ [m_5{}^L g_5{}^2 + I_{5zz}] \dot{\theta}_3 \dot{\theta}_5 + m_5{}^L L_1 L_3 c(-\theta_3) \dot{\theta}_1{}^2 + m_5{}^L L_1 L_3 c(-\theta_3) \dot{\theta}_1 \dot{\theta}_3 + m_5{}^L L_1 g_5 c\left[-(\theta_3 + \theta_5 + \beta)\right] \dot{\theta}_1{}^2$$
$$+ m_5{}^L L_1 g_5 c\left[-(\theta_3 + \theta_5 + \beta)\right] \dot{\theta}_1 \dot{\theta}_3 + m_5{}^L L_1 g_5 c\left[-(\theta_3 + \theta_5 + \beta)\right] \dot{\theta}_1 \dot{\theta}_5 + m_5{}^L L_3 g_5 c\left[-(\theta_5 + \beta)\right] \dot{\theta}_1{}^2$$
$$+ m_5{}^L L_3 g_5 c\left[-(\theta_5 + \beta)\right] \dot{\theta}_1 \dot{\theta}_3 + m_5{}^L L_3 g_5 c\left[-(\theta_5 + \beta)\right] \dot{\theta}_1 \dot{\theta}_5 + m_5{}^L L_3 g_5 c\left[-(\theta_5 + \beta)\right] \dot{\theta}_1 \dot{\theta}_3$$
$$+ m_5{}^L L_3 g_5 c\left[-(\theta_5 + \beta)\right] \dot{\theta}_3{}^2 + m_5{}^L L_3 g_5 c\left[-(\theta_5 + \beta)\right] \dot{\theta}_3 \dot{\theta}_5 \qquad (5.35)$$

$$KE_{A1} = \frac{1}{2}[r_1^2 I^r{}_{1zz}]\dot{\phi}_1^2 \tag{5.36}$$

$$KE_{A2} = \frac{1}{2}[r_2^2 I^r{}_{2zz}]\dot{\phi}_2^2 \tag{5.37}$$

$$KE_{A3} = \frac{1}{2}[m_3{}^A L_1^2 + I^r{}_{3zz}]\dot{\theta}_1^2 + \frac{1}{2}[r_3^2 I^r{}_{3zz}]\dot{\phi}_3^2 \tag{5.38}$$

It is clear that total kinetic energy is the sum of all the kinetic energy contributions of links and actuators and can be formulated in simplified form as below.

$$\begin{aligned}
K = {} & \frac{1}{2}\dot{\theta}_1^2 \Big\{[m_1{}^L \frac{L_1^2}{4} + I_{1zz}] + m_3{}^L L_1^2 + m_3{}^L L_1 L_3 c(-\theta_3) + [m_3{}^L \frac{L_3^2}{4} + I_{3zz}] + m_5{}^L L_1^2 \\
& + m_5{}^L L_3^2 + [m_5{}^L g_5^2 + I_{5zz}] + 2m_5{}^L L_1 L_3 c(-\theta_3) + 2m_5{}^L L_1 g_5 c\big[-(\theta_3 + \theta_5 + \beta)\big] \\
& + 2m_5{}^L L_3 g_5 c\big[-(\theta_5 + \beta)\big] + [m_3{}^A L_1^2 + I^r{}_{3zz}]\Big\} + \frac{1}{2}\dot{\theta}_2^2 \Big\{[m_2{}^L \frac{L_2^2}{4} + I_{2zz}] + m_4{}^L L_2^2 \\
& + m_4{}^L L_2 L_4 c(-\theta_4) + [m_4{}^L \frac{L_4^2}{4} + I_{4zz}]\Big\} + \frac{1}{2}\dot{\theta}_3^2 \Big\{[m_3{}^L \frac{L_3^2}{4} + I_{3zz}] + m_5{}^L L_3^2 \\
& + [m_5{}^L g_5^2 + I_{5zz}] + 2m_5{}^L L_3 g_5 c\big[-(\theta_5 + \beta)\big]\Big\} + \frac{1}{2}\dot{\theta}_4^2 \Big\{[m_4{}^L \frac{L_4^2}{4} + I_{4zz}]\Big\} \\
& + \frac{1}{2}\dot{\theta}_5^2 \Big\{[m_5{}^L g_5^2 + I_{5zz}\Big\} + \dot{\theta}_1\dot{\theta}_3 \Big\{m_3{}^L \frac{1}{2} L_1 L_3 c(-\theta_3) + [m_3{}^L \frac{L_3^2}{4} + I_{3zz}] \\
& + m_5{}^L L_3^2 + [m_5{}^L g_5^2 + I_{5zz}] + m_5{}^L L_1 L_3 c(-\theta_3) + m_5{}^L L_1 g_5 c[-(\theta_3 + \theta_5 + \beta)] \\
& + 2m_5{}^L L_3 g_5 c[-(\theta_5 + \beta)]\Big\} + \dot{\theta}_1\dot{\theta}_5 \Big\{[m_5{}^L g_5^2 + I_{5zz}] + m_5{}^L L_1 g_5 c\big[-(\theta_3 + \theta_5 + \beta)\big] \\
& + m_5{}^L L_3 g_5 c[-(\theta_5 + \beta)]\Big\} + \dot{\theta}_3\dot{\theta}_5 \Big\{[m_5{}^L g_5^2 + I_{5zz}] + m_5{}^L L_3 g_5 c[-(\theta_5 + \beta)]\Big\} \\
& + \dot{\theta}_2\dot{\theta}_4 \Big\{\frac{1}{2} m_4{}^L L_2 L_4 c(-\theta_4) + [m_4{}^L \frac{L_4^2}{4} + I_{4zz}]\Big\} + \frac{1}{2}\dot{\phi}_1^2 \Big\{[r_1^2 I^r{}_{1zz}]\Big\} \\
& + \frac{1}{2}\dot{\phi}_2^2 \Big\{[r_2^2 I^r{}_{2zz}]\Big\} + \frac{1}{2}\dot{\phi}_3^2 \Big\{[r_3^2 I^r{}_{3zz}]\Big\}
\end{aligned} \tag{5.39}$$

Therefore Lagrange components related to kinetic energy are obtained as

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{\theta}_1}\right) = \ddot{\theta}_1 \{[m_1{}^L \frac{L_1{}^2}{4} + I_{1zz}] + m_3{}^L L_1{}^2 + m_3{}^L L_1 L_3 c(-\theta_3) + [m_3{}^L \frac{L_3{}^2}{4} + I_{3zz}] + m_5{}^L L_1{}^2$$
$$+ m_5{}^L L_3{}^2 + [m_5{}^L g_5{}^2 + I_{5zz}] + 2m_5{}^L L_1 L_3 c(-\theta_3) + 2m_5{}^L L_1 g_5 c[-(\theta_3 + \theta_5 + \beta)]$$
$$+ 2m_5{}^L L_3 g_5 c[-(\theta_5 + \beta)] + [m_3{}^A L_1{}^2 + I^r{}_{3zz}]\} + \ddot{\theta}_3 \{m_3{}^L \frac{L_1}{2} L_3 c(-\theta_3) + [m_3{}^L \frac{L_3{}^2}{4} + I_{3zz}]$$
$$+ m_5{}^L L_3{}^2 + [m_5{}^L g_5{}^2 + I_{5zz}] + m_5{}^L L_1 L_3 c(-\theta_3) + m_5{}^L L_1 g_5 c[-(\theta_3 + \theta_5 + \beta)]$$
$$+ 2m_5{}^L L_3 g_5 c[-(\theta_5 + \beta)]\} + \ddot{\theta}_5 \{[m_5{}^L g_5{}^2 + I_{5zz}] + m_5{}^L L_1 g_5 c[-(\theta_3 + \theta_5 + \beta)]$$
$$+ m_5{}^L L_3 g_5 c[-(\theta_5 + \beta)]\} \qquad (5.40)$$

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{\theta}_2}\right) = \ddot{\theta}_2 \left\{[m_2{}^L \frac{L_2{}^2}{4} + I_{2zz}] + m_4{}^L L_2{}^2 + m_4{}^L L_2 L_4 c(-\theta_4) + [m_4{}^L \frac{L_4{}^2}{4} + I_{4zz}]\right\}$$
$$+ \ddot{\theta}_4 \left\{\frac{1}{2} m_4{}^L L_2 L_4 c(-\theta_4) + [m_4{}^L \frac{L_4{}^2}{4} + I_{4zz}]\right\} \qquad (5.41)$$

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{\theta}_3}\right) = \ddot{\theta}_3 \left\{[m_3{}^L \frac{L_3{}^2}{4} + I_{3zz}] + m_5{}^L L_3{}^2 + [m_5{}^L g_5{}^2 + I_{5zz}] + 2m_5{}^L L_3 g_5 c[-(\theta_5 + \beta)]\right\}$$
$$+ \ddot{\theta}_1 \{m_3{}^L \frac{L_1}{2} L_3 c(-\theta_3) + [m_3{}^L \frac{L_3{}^2}{4} + I_{3zz}] + m_5{}^L L_3{}^2 + [m_5{}^L g_5{}^2 + I_{5zz}] + m_5{}^L L_1 L_3 c(-\theta_3)$$
$$+ m_5{}^L L_1 g_5 c[-(\theta_3 + \theta_5 + \beta)] + 2m_5{}^L L_3 g_5 c[-(\theta_5 + \beta)]\} + \ddot{\theta}_5 \{[m_5{}^L g_5{}^2 + I_{5zz}]$$
$$+ m_5{}^L L_3 g_5 c[-(\theta_5 + \beta)]\} \qquad (5.42)$$

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{\theta}_4}\right) = \ddot{\theta}_4 \left\{[m_4{}^L \frac{L_4{}^2}{4} + I_{4zz}]\right\} + \ddot{\theta}_2 \left\{\frac{1}{2} m_4{}^L L_2 L_4 c(-\theta_4) + [m_4{}^L \frac{L_4{}^2}{4} + I_{4zz}]\right\} \qquad (5.43)$$

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{\theta}_5}\right) = \ddot{\theta}_5 \{[m_5{}^L g_5{}^2 + I_{5zz}]\} + \ddot{\theta}_1 \{[m_5{}^L g_5{}^2 + I_{5zz}] + m_5{}^L L_1 g_5 c[-(\theta_3 + \theta_5 + \beta)]$$
$$+ m_5{}^L L_3 g_5 c[-(\theta_5 + \beta)]\} + \ddot{\theta}_3 \{[m_5{}^L g_5{}^2 + I_{5zz}] + m_5{}^L L_3 g_5 c[-(\theta_5 + \beta)]\} \qquad (5.44)$$

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{\phi}_1}\right) = \ddot{\phi}_1 \left\{ r_1^2 I^r{}_{1zz} \right\} \tag{5.45}$$

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{\phi}_2}\right) = \ddot{\phi}_2 \left\{ r_2{}^2 I^r{}_{2zz} \right\} \tag{5.46}$$

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{\phi}_3}\right) = \ddot{\phi}_3 \left\{ r_3{}^2 I^r{}_{3zz} \right\} \tag{5.47}$$

$$\frac{\partial K}{\partial \theta_1} = 0 \tag{5.48}$$

$$\frac{\partial K}{\partial \theta_2} = 0 \tag{5.49}$$

$$\begin{aligned}\frac{\partial K}{\partial \theta_3} &= \frac{1}{2}\dot{\theta}_1^2 \left\{ m_3{}^L L_1 L_3 s(-\theta_3) + 2m_5{}^L L_1 L_3 s(-\theta_3) + 2m_5{}^L L_1 g_5 s[-(\theta_3+\theta_5+\beta)] \right\} \\ &+\dot{\theta}_1\dot{\theta}_3 \left\{ \frac{1}{2} m_3{}^L L_1 L_3 s(-\theta_3) + m_5{}^L L_1 L_3 s(-\theta_3) + m_5{}^L L_1 g_5 s[-(\theta_3+\theta_5+\beta)] \right\} \\ &+\dot{\theta}_1\dot{\theta}_5 \left\{ m_5{}^L L_1 g_5 s[-(\theta_3+\theta_5+\beta)] \right\}\end{aligned} \tag{5.50}$$

$$\frac{\partial K}{\partial \theta_4} = \frac{1}{2}\dot{\theta}_2{}^2 m_4{}^L L_2 L_4 s(-\theta_4) + \dot{\theta}_2\dot{\theta}_4 \frac{1}{2} m_4{}^L L_2 L_4 s(-\theta_4) \tag{5.51}$$

$$\begin{aligned}\frac{\partial K}{\partial \theta_5} &= \dot{\theta}_1^2 \left\{ m_5{}^L L_1 g_5 s[-(\theta_3+\theta_5+\beta)] + m_5{}^L L_3 g_5 s[-(\theta_5+\beta)] \right\} + \dot{\theta}_3^2 \left\{ m_5{}^L L_3 g_5 s[-(\theta_5+\beta)] \right\} \\ &+\dot{\theta}_1\dot{\theta}_3 \left\{ m_5{}^L L_1 g_5 s[-(\theta_3+\theta_5+\beta)] + 2m_5{}^L L_3 g_5 s[-(\theta_5+\beta)] \right\} + \dot{\theta}_1\dot{\theta}_5 \left\{ m_5{}^L L_1 g_5 s[-(\theta_3+\theta_5+\beta)] \right. \\ &\left. +m_5{}^L L_3 g_5 s[-(\theta_5+\beta] \right\} + \dot{\theta}_3\dot{\theta}_5 \left\{ m_5{}^L L_3 g_5 s[-(\theta_5+\beta)] \right\}\end{aligned} \tag{5.52}$$

$$\frac{\partial K}{\partial \phi_1} = 0 \quad (5.53)$$

$$\frac{\partial K}{\partial \phi_2} = 0 \quad (5.54)$$

$$\frac{\partial K}{\partial \phi_3} = 0 \quad (5.55)$$

5.1.2 Potential Energy Expressions

Potential energy contributions for each of the links and actuators are formulated as

$$PE_{L1} = m_1{}^L g(\frac{L_1}{2} s\theta_1) \quad (5.56)$$

$$PE_{L2} = m_2{}^L g(\frac{L_2}{2} s\theta_2) \quad (5.57)$$

$$PE_{L3} = m_3{}^L g(L_1 s\theta_1 + \frac{L_3}{2} s\theta_{13}) \quad (5.58)$$

$$PE_{L4} = m_4{}^L g(L_2 s\theta_2 + \frac{L_4}{2} s\theta_{24}) \quad (5.59)$$

$$PE_{L5} = m_5{}^L g\left[L_1 s\theta_1 + L_3 s\theta_{13} + g_5 s(\theta_{135} + \beta)\right] \quad (5.60)$$

$$PE_{A1} = \frac{1}{2} K_1(\phi_1 - \theta_1)^2 \quad (5.61)$$

$$PE_{A2} = \frac{1}{2} K_2(\phi_2 - \theta_2)^2 \quad (5.62)$$

$$PE_{A3} = \frac{1}{2} K_3(\phi_3 - \theta_3)^2 + m_3{}^A g(L_1 s\theta_1) \quad (5.63)$$

Total potential energy of the system is

$$U = m_1{}^L g(\frac{L_1}{2} s\theta_1) + m_3{}^L g(L_1 s\theta_1 + \frac{L_3}{2} s\theta_{13}) + m_5{}^L g\left[L_1 s\theta_1 + L_3 s\theta_{13} + g_5 s(\theta_{135} + \beta)\right]$$
$$+ m_2{}^L g(\frac{L_2}{2} s\theta_2) + m_4{}^L g(L_2 s\theta_2 + \frac{L_4}{2} s\theta_{24}) + \frac{1}{2} K_1(\phi_1 - \theta_1)^2 + \frac{1}{2} K_2(\phi_2 - \theta_2)^2$$
$$+ \frac{1}{2} K_3(\phi_3 - \theta_3)^2 + m_3{}^A g(L_1 s\theta_1) \quad (5.64)$$

The Lagrange components associated with the total potential energy of the system are written as

$$\frac{\partial U}{\partial \theta_1} = m_1{}^L g \frac{L_1}{2} c\theta_1 + m_3{}^L g(L_1 c\theta_1 + \frac{L_3}{2} c\theta_{13}) + m_5{}^L g\left[L_1 c\theta_1 + L_3 c\theta_{13} + g_5 c(\theta_{135} + \beta)\right]$$
$$- K_1(\phi_1 - \theta_1) + m_3{}^A g L_1 c\theta_1 \quad (5.65)$$

$$\frac{\partial U}{\partial \theta_2} = m_2{}^L g(\frac{L_2}{2} c\theta_2) + m_4{}^L g(L_2 c\theta_2 + \frac{L_4}{2} c\theta_{24}) - K_2(\phi_2 - \theta_2) \quad (5.66)$$

$$\frac{\partial U}{\partial \theta_3} = m_3{}^L g(\frac{L_3}{2} c\theta_{13}) + m_5{}^L g\left[L_3 c\theta_{13} + g_5 c(\theta_{135} + \beta)\right] - K_3(\phi_3 - \theta_3) \quad (5.67)$$

$$\frac{\partial U}{\partial \theta_4} = m_2{}^L g(\frac{L_2}{2} c\theta_2) + m_4{}^L g(L_2 c\theta_2 + \frac{L_4}{2} c\theta_{24}) \quad (5.68)$$

$$\frac{\partial U}{\partial \theta_5} = m_5{}^L g\left[g_5 c(\theta_{135} + \beta)\right] \quad (5.69)$$

$$\frac{\partial U}{\partial \phi_1} = K_1(\phi_1 - \theta_1) \quad (5.70)$$

$$\frac{\partial U}{\partial \phi_2} = K_2(\phi_2 - \theta_2) \quad (5.71)$$

$$\frac{\partial U}{\partial \phi_3} = K_3(\phi_3 - \theta_3) \quad (5.72)$$

5.1.3 Dissipation Function Expressions

As mentioned in Section 2.2.3 dissipation functions for the actuated joints, unactuated joints and rotors are described as

$$D^a = \frac{1}{2}D_1(\dot{\theta}_1 - \dot{\phi}_1)^2 + \frac{1}{2}D_2(\dot{\theta}_2 - \dot{\phi}_2)^2 + \frac{1}{2}D_3(\dot{\theta}_3 - \dot{\phi}_3)^2 \quad (5.73)$$

$$D^u = \frac{1}{2}D_4\dot{\theta}_4^{\,2} + \frac{1}{2}D_5\dot{\theta}_5^{\,2} \quad (5.74)$$

$$D^r = \frac{1}{2}D^r{}_1 r_1^{\,2}\dot{\phi}_1^{\,2} + \frac{1}{2}D^r{}_2 r_2^{\,2}\dot{\phi}_2^{\,2} + \frac{1}{2}D^r{}_3 r_3^{\,2}\dot{\phi}_3^{\,2} \quad (5.75)$$

However there is also viscous friction at the disconnected joint when it is reconnected and the dissipation function for the disconnected joint can be written as

$$D^d = \frac{1}{2}D_6(\dot{\theta}_4 - \dot{\theta}_5)^2 \quad (5.76)$$

Therefore the dissipation function of the whole system is the sum of all of the individual contributions.

$$D = \frac{1}{2}D_1(\dot{\theta}_1 - \dot{\phi}_1)^2 + \frac{1}{2}D_2(\dot{\theta}_2 - \dot{\phi}_2)^2 + \frac{1}{2}D_3(\dot{\theta}_3 - \dot{\phi}_3)^2 + \frac{1}{2}D_4\dot{\theta}_4^{\,2} + \frac{1}{2}D_5\dot{\theta}_5^{\,2} + \frac{1}{2}D_6(\dot{\theta}_4 - \dot{\theta}_5)^2$$
$$+ \frac{1}{2}D^r{}_1 r_1^{\,2}\dot{\phi}_1^{\,2} + \frac{1}{2}D^r{}_2 r_2^{\,2}\dot{\phi}_2^{\,2} + \frac{1}{2}D^r{}_3 r_3^{\,2}\dot{\phi}_3^{\,2} \quad (5.77)$$

Lagrange components are written as

$$\frac{\partial D}{\partial \dot{\theta}_1} = D_1(\dot{\theta}_1 - \dot{\phi}_1) \quad (5.78)$$

$$\frac{\partial D}{\partial \dot{\theta}_2} = D_2(\dot{\theta}_2 - \dot{\phi}_2) \quad (5.79)$$

$$\frac{\partial D}{\partial \dot{\theta}_3} = D_3(\dot{\theta}_3 - \dot{\phi}_3) \tag{5.80}$$

$$\frac{\partial D}{\partial \dot{\theta}_4} = D_4\dot{\theta}_4 + D_6(\dot{\theta}_4 - \dot{\theta}_5) = (D_4 + D_6)\dot{\theta}_4 - D_6\dot{\theta}_5 \tag{5.81}$$

$$\frac{\partial D}{\partial \dot{\theta}_5} = D_5\dot{\theta}_5 - D_6(\dot{\theta}_4 - \dot{\theta}_5) = -D_6\dot{\theta}_4 + (D_5 + D_6)\dot{\theta}_5 \tag{5.82}$$

$$\frac{\partial D}{\partial \dot{\phi}_1} = -D_1(\dot{\theta}_1 - \dot{\phi}_1) + D^r{}_1 r_1{}^2 \dot{\phi}_1 \tag{5.83}$$

$$\frac{\partial D}{\partial \dot{\phi}_2} = -D_2(\dot{\theta}_2 - \dot{\phi}_2) + D^r{}_2 r_2{}^2 \dot{\phi}_2 \tag{5.84}$$

$$\frac{\partial D}{\partial \dot{\phi}_3} = -D_3(\dot{\theta}_3 - \dot{\phi}_3) + D^r{}_3 r_3{}^2 \dot{\phi}_3 \tag{5.85}$$

5.1.4 Closed Loop Constraints and Generalized Force Equations

As mentioned previously, there should be two constraint equations when joint at Point-F is disconnected.

At the position level these constraint equations can be written as in Equation 2.33 as

$$L_1 c\theta_1 + L_3 c\theta_{13} + L_5 c\theta_{135} - L_2 c\theta_2 - L_4 c\theta_{24} - d_0 = 0 \tag{5.86}$$

$$L_1 s\theta_1 + L_3 s\theta_{13} + L_5 s\theta_{135} - L_2 s\theta_2 - L_4 s\theta_{24} = 0 \tag{5.87}$$

which lead to the following velocity level constraint equations.

$$-L_1 s\theta_1 \dot{\theta}_1 - L_3 s\theta_{13}(\dot{\theta}_1 + \dot{\theta}_3) - L_5 s\theta_{135}(\dot{\theta}_1 + \dot{\theta}_3 + \dot{\theta}_5) + L_2 s\theta_2 \dot{\theta}_2 + L_4 s\theta_{24}(\dot{\theta}_2 + \dot{\theta}_4) = 0 \tag{5.88}$$

$$L_1 c\theta_1 \dot{\theta}_1 + L_3 c\theta_{13}(\dot{\theta}_1 + \dot{\theta}_3) + L_5 c\theta_{135}(\dot{\theta}_1 + \dot{\theta}_3 + \dot{\theta}_5) - L_2 c\theta_2 \dot{\theta}_2 - L_4 c\theta_{24}(\dot{\theta}_2 + \dot{\theta}_4) = 0 \tag{5.89}$$

Velocity level constraint equations can be written symbolically as

$$B_{11}\dot{\theta}_1 + B_{12}\dot{\theta}_2 + B_{13}\dot{\theta}_3 + B_{14}\dot{\theta}_4 + B_{15}\dot{\theta}_5 = 0 \tag{5.90}$$

$$B_{21}\dot{\theta}_1 + B_{22}\dot{\theta}_2 + B_{23}\dot{\theta}_3 + B_{24}\dot{\theta}_4 + B_{25}\dot{\theta}_5 = 0 \tag{5.91}$$

where

$$B_{11} = -L_1 s\theta_1 - L_3 s\theta_{13} - L_5 s\theta_{135} \tag{5.92}$$

$$B_{12} = L_2 s\theta_2 + L_4 s\theta_{24} \tag{5.93}$$

$$B_{13} = -L_3 s\theta_{13} - L_5 s\theta_{135} \tag{5.94}$$

$$B_{14} = L_4 s\theta_{24} \tag{5.95}$$

$$B_{15} = -L_5 s\theta_{135} \tag{5.96}$$

$$B_{21} = L_1 c\theta_1 + L_3 c\theta_{13} + L_5 c\theta_{135} \tag{5.97}$$

$$B_{22} = -L_2 c\theta_2 - L_4 c\theta_{24} \tag{5.98}$$

$$B_{23} = L_3 c\theta_{13} + L_5 c\theta_{135} \tag{5.99}$$

$$B_{24} = -L_4 c\theta_{24} \tag{5.100}$$

$$B_{25} = L_5 c\theta_{135} \tag{5.101}$$

Factoring out the unactuated joint variables gives

$$\begin{bmatrix} \dot{\theta}_4 \\ \dot{\theta}_5 \end{bmatrix} = -\begin{bmatrix} B_{14} & B_{15} \\ B_{24} & B_{25} \end{bmatrix}^{-1} \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} \tag{5.102}$$

This leads to

$$\dot{\bar{\theta}}^u = \hat{C}_{2x3}\dot{\bar{q}} \tag{5.103}$$

Therefore generalized constraint forces are found by arranging the velocity level constraint equations according to Equation 2.40

where

$$\hat{B}^T = \begin{bmatrix} -L_1 s\theta_1 - L_3 s\theta_{13} - L_5 s\theta_{135} & L_1 c\theta_1 + L_3 c\theta_{13} + L_5 c\theta_{135} \\ L_2 s\theta_2 + L_4 s\theta_{24} & -L_2 c\theta_2 - L_4 c\theta_{24} \\ -L_3 s\theta_{13} - L_5 s\theta_{135} & L_3 c\theta_{13} + L_5 c\theta_{135} \\ \hdashline L_4 s\theta_{24} & -L_4 c\theta_{24} \\ -L_5 s\theta_{135} & L_5 c\theta_{135} \end{bmatrix} = \begin{bmatrix} \hat{B}^{a^T} \\ \hdashline \hat{B}^{u^T} \end{bmatrix}$$

(5.104)

Non-potentialized forces are found by Equations 2.31 and 2.32 for the link variables and joint variables respectively as

$$\bar{\tilde{Q}}_1 = \bar{0} \tag{5.105}$$

$$\bar{\tilde{Q}}_2 = \bar{T} \tag{5.106}$$

5.1.5 System Equations of Motion

The system equations of motion corresponding to the first set of generalized coordinates in matrix form are obtained as

$$\begin{bmatrix} M_{11} & 0 & M_{13} & 0 & M_{15} \\ 0 & M_{22} & 0 & M_{24} & 0 \\ M_{13} & 0 & M_{33} & 0 & M_{35} \\ 0 & M_{24} & 0 & M_{44} & 0 \\ M_{15} & 0 & M_{35} & 0 & M_{55} \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \\ \ddot{\theta}_4 \\ \ddot{\theta}_5 \end{bmatrix} + \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \end{bmatrix} + \begin{bmatrix} Da_1 \\ Da_2 \\ Da_3 \\ Da_4 \\ Da_5 \end{bmatrix} + \begin{bmatrix} St_1 \\ St_2 \\ St_3 \\ St_4 \\ St_5 \end{bmatrix} - \hat{B}^T \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(5.107)

where

$$M_{11} = [m_1^L \frac{L_1^2}{4} + I_{1zz}] + m_3^L L_1^2 + m_3^L L_1 L_3 c\theta_3 + [m_3^L \frac{L_3^2}{4} + I_{3zz}] + m_5^L L_1^2 + m_5^L L_3^2$$
$$+[m_5^L g_5^2 + I_{5zz}] + 2m_5^L L_1 L_3 c\theta_3 + 2m_5^L L_1 g_5 c(\theta_3 + \theta_5 + \beta) + 2m_5^L L_3 g_5 c(\theta_5 + \beta)$$
$$+[m_3^A L_1^2 + I^r{}_{3zz}] \tag{5.108}$$

$$M_{13} = \frac{1}{2} m_3^L L_1 L_3 c\theta_3 + [m_3^L \frac{L_3^2}{4} + I_{3zz}] + m_5^L L_3^2 + [m_5^L g_5^2 + I_{5zz}] + m_5^L L_1 L_3 c\theta_3$$
$$+m_5^L L_1 g_5 c(\theta_3 + \theta_5 + \beta) + 2m_5^L L_3 g_5 c(\theta_5 + \beta) \tag{5.109}$$

$$M_{15} = [m_5{}^L g_5{}^2 + I_{5zz}] + m_5{}^L L_1 g_5 c(\theta_3 + \theta_5 + \beta) + m_5{}^L L_3 g_5 c(\theta_5 + \beta) \tag{5.110}$$

$$M_{22} = [m_2{}^L \frac{L_2{}^2}{4} + I_{2zz}] + m_4{}^L L_2{}^2 + m_4{}^L L_2 L_4 c\theta_4 + [m_4{}^L \frac{L_4{}^2}{4} + I_{4zz}] \tag{5.111}$$

$$M_{24} = \frac{1}{2} m_4{}^L L_2 L_4 c\theta_4 + [m_4{}^L \frac{L_4{}^2}{4} + I_{4zz}] \tag{5.112}$$

$$M_{33} = [m_3{}^L \frac{L_3{}^2}{4} + I_{3zz}] + m_5{}^L L_3{}^2 + [m_5{}^L g_5{}^2 + I_{5zz}] + 2m_5{}^L L_3 g_5 c(\theta_5 + \beta) \tag{5.113}$$

$$M_{35} = [m_5{}^L g_5{}^2 + I_{5zz}] + m_5{}^L L_3 g_5 c(\theta_5 + \beta) \tag{5.114}$$

$$M_{44} = [m_4{}^L \frac{L_4{}^2}{4} + I_{4zz}] \tag{5.115}$$

$$M_{55} = [m_5{}^L g_5{}^2 + I_{5zz}] \tag{5.116}$$

$$Q_1 = m_1{}^L g \frac{L_1}{2} c\theta_1 + m_3{}^L g (L_1 c\theta_1 + \frac{L_3}{2} c\theta_{13}) + m_5{}^L g \left[L_1 c\theta_1 + L_3 c\theta_{13} + g_5 c(\theta_{135} + \beta) \right]$$
$$+ m_3{}^A g L_1 c\theta_1 \tag{5.117}$$

$$Q_2 = m_2{}^L g (\frac{L_2}{2} c\theta_2) + m_4{}^L g (L_2 c\theta_2 + \frac{L_4}{2} c\theta_{24}) \tag{5.118}$$

$$Q_3 = \frac{1}{2}\dot{\theta}_1^2 \left[m_3{}^L L_1 L_3 s\theta_3 + 2m_5{}^L L_1 L_3 s\theta_3 + 2m_5{}^L L_1 g_5 s(\theta_{35} + \beta) \right]$$
$$+ \dot{\theta}_1 \dot{\theta}_3 \left[\frac{1}{2} m_3{}^L L_1 L_3 s\theta_3 + m_5{}^L L_1 L_3 s\theta_3 + m_5{}^L L_1 g_5 s(\theta_{35} + \beta) \right]$$
$$+ \dot{\theta}_1 \dot{\theta}_5 \left[m_5{}^L L_1 g_5 s(\theta_{35} + \beta) \right] + m_3{}^L g (\frac{L_3}{2} c\theta_{13}) + m_5{}^L g \left[L_3 c\theta_{13} + g_5 c(\theta_{135} + \beta) \right] \tag{5.119}$$

$$Q_4 = \frac{1}{2}\dot{\theta}_2{}^2 m_4{}^L L_2 L_4 s\theta_4 + \dot{\theta}_2 \dot{\theta}_4 \frac{1}{2} m_4{}^L L_2 L_4 s\theta_4 + m_2{}^L g (\frac{L_2}{2} c\theta_2) + m_4{}^L g (L_2 c\theta_2 + \frac{L_4}{2} c\theta_{24}) \tag{5.120}$$

$$Q_5 = \dot{\theta}_1^2 \left[m_5{}^L L_1 g_5 s(\theta_{35} + \beta) + m_5{}^L L_3 g_5 s(\theta_5 + \beta) \right] + \dot{\theta}_3{}^2 m_5{}^L L_3 g_5 s(\theta_5 + \beta)$$
$$+ \dot{\theta}_1 \dot{\theta}_3 \left[m_5{}^L L_1 g_5 s(\theta_{35} + \beta) + 2m_5{}^L L_3 g_5 s(\theta_5 + \beta) \right] + \dot{\theta}_1 \dot{\theta}_5 \left[m_5{}^L L_1 g_5 s(\theta_{35} + \beta) \right.$$
$$\left. + m_5{}^L L_3 g_5 s(\theta_5 + \beta) \right] + \dot{\theta}_3 \dot{\theta}_5 m_5{}^L L_3 g_5 s(\theta_5 + \beta) + m_5{}^L g \left[g_5 c(\theta_{135} + \beta) \right] \tag{5.121}$$

$$Da_1 = D_1(\dot{\theta}_1 - \dot{\phi}_1) \tag{5.122}$$

$$Da_2 = D_2(\dot{\theta}_2 - \dot{\phi}_2) \tag{5.123}$$

$$Da_3 = D_3(\dot{\theta}_3 - \dot{\phi}_3) \tag{5.124}$$

$$Da_4 = (D_4 + D_6)\dot{\theta}_4 - D_6\dot{\theta}_5 \tag{5.125}$$

$$Da_5 = -D_6\dot{\theta}_4 + (D_5 + D_6)\dot{\theta}_5 \tag{5.126}$$

$$St_1 = K_1(\theta_1 - \phi_1) \tag{5.127}$$

$$St_2 = K_2(\theta_2 - \phi_2) \tag{5.128}$$

$$St_3 = K_3(\theta_3 - \phi_3) \tag{5.129}$$

$$St_4 = 0 \tag{5.130}$$

$$St_5 = 0 \tag{5.131}$$

The system equations of motion corresponding to the second set of generalized coordinates in matrix form are obtained as

$$\begin{bmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{bmatrix} \begin{bmatrix} \ddot{\phi}_1 \\ \ddot{\phi}_2 \\ \ddot{\phi}_3 \end{bmatrix} + \begin{bmatrix} D^r{}_1\dot{\phi}_1 \\ D^r{}_2\dot{\phi}_2 \\ D^r{}_3\dot{\phi}_3 \end{bmatrix} - \begin{bmatrix} D_1(\dot{\theta}_1 - \dot{\phi}_1) \\ D_2(\dot{\theta}_2 - \dot{\phi}_2) \\ D_3(\dot{\theta}_3 - \dot{\phi}_3) \end{bmatrix} - \begin{bmatrix} K_1(\theta_1 - \phi_1) \\ K_2(\theta_2 - \phi_2) \\ K_3(\theta_3 - \phi_3) \end{bmatrix} = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} \tag{5.132}$$

where $\hat{I}^r = diag[I_i^r r_i^2]$ and $\hat{D}^r = diag[D_i^r r_i^2]$ for $i = 1,2,3$.

In order to get a generic form of equations of motion corresponding to the first set of generalized coordinates, one needs to eliminate the unactuated joint variables and constraint forces. When constraint equation substitutions and manipulations are done as explained in Chapter-III, equations of motion in matrix form are obtained as in Equations 5.133 and 5.135.

$$\hat{M}^{a*}\ddot{\overline{q}} + \hat{R}^{a}\dot{\overline{q}} + \overline{Q}^{a} + \hat{D}^{a}(\dot{\overline{q}} - \dot{\overline{\phi}}) + \hat{K}(\overline{q} - \overline{\phi}) - \hat{B}^{a^{T}}\overline{\lambda} = 0 \tag{5.133}$$

where

$\hat{M}^{a*}$ and $\hat{R}^{a}$ are 3×3 matrices whose elements are given in Appendix B.

$$\overline{Q}^{a} = \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix} \tag{5.134}$$

and $\hat{D}^{a} = diag[D_i]$, $\hat{K} = diag[K_i]$ for $i = 1, 2, 3$.

$$\hat{M}^{u*}\ddot{\overline{q}} + \hat{R}^{u}\dot{\overline{q}} + \overline{Q}^{u} - \hat{B}^{u^{T}}\overline{\lambda} = 0 \tag{5.135}$$

where

$\hat{M}^{u*}$ and $\hat{R}^{u}$ are 2×3 matrices whose elements are also given in Appendix B.

$$\overline{Q}^{u} = \begin{bmatrix} Q_4 \\ Q_5 \end{bmatrix} \tag{5.136}$$

The vector of Lagrange multipliers in Equation 5.135 is solved and substituted into Equation 5.133 in order to be eliminated. Therefore the final form of the equations of motion for the first set of generalized coordinates are written as

$$\hat{M}^{**}\ddot{\overline{q}} + \hat{R}^{*}\dot{\overline{q}} + \overline{Q}^{*} + \hat{D}^{a}(\dot{\overline{q}} - \dot{\overline{\phi}}) + \hat{K}(\overline{q} - \overline{\phi}) = \overline{0} \tag{5.137}$$

where

$$\hat{M}^{**} = \hat{M}^{a*} - \hat{B}^{a^T} \left(\hat{B}^{u^T}\right)^{-1} \hat{M}^{u*} \tag{5.138}$$

$$\hat{R}^{*} = \hat{R}^{a} - \hat{B}^{a^T} \left(\hat{B}^{u^T}\right)^{-1} \hat{R}^{u} \tag{5.139}$$

$$\overline{Q}^{*} = \overline{Q}^{a} - \hat{B}^{a^T} \left(\hat{B}^{u^T}\right)^{-1} \overline{Q}^{u} \tag{5.140}$$

Finally input/output relation is obtained as the same as in Equation 3.32 when all manipulations are done as described in Chapter-III.

The task equations of the parallel manipulator in concern consist of the position of the tip point as expressed in the fixed reference frame and the orientation of the fifth link with respect to the fixed reference frame.

Tip point position:

$$\overline{r}_P^{(0)} = L_1(\overline{u}_1 c\theta_1 + \overline{u}_2 s\theta_1) + L_3(\overline{u}_1 c\theta_{13} + \overline{u}_2 s\theta_{13}) + d_5(\overline{u}_1 c(\theta_{135} + \alpha) + \overline{u}_2 s(\theta_{135} + \alpha)) \tag{5.141}$$

$$x_P = L_1 c\theta_1 + L_3 c\theta_{13} + d_5 c(\theta_{135} + \alpha) \tag{5.142}$$

$$y_P = L_1 s\theta_1 + L_3 s\theta_{13} + d_5 s(\theta_{135} + \alpha) \tag{5.143}$$

Orientation of link-5:

$$\sigma = \theta_{135} \tag{5.144}$$

The 3×5 manipulator Jacobian matrix is formed by one step differentiation of the position level task equations.

Tip point velocity:

$$\dot{x}_P = -L_1 s\theta_1 \dot{\theta}_1 - L_3 s\theta_{13} \dot{\theta}_{13} - d_5 s(\theta_{135} + \alpha)\dot{\theta}_{135} \tag{5.145}$$

$$\dot{y}_P = L_1 c\theta_1 \dot{\theta}_1 + L_3 c\theta_{13} \dot{\theta}_{13} + d_5 c(\theta_{123} + \alpha)\dot{\theta}_{135} \tag{5.146}$$

which lead to

$$\begin{bmatrix} -L_1 s\theta_1 - L_3 s\theta_{13} - d_5 s(\theta_{135}+\alpha) & 0 & -L_3 s\theta_{13} - d_5 s(\theta_{135}+\alpha) & 0 & -d_5 s(\theta_{135}+\alpha) \\ L_1 c\theta_1 + L_3 c\theta_{13} + d_5 c(\theta_{135}+\alpha) & 0 & L_3 c\theta_{13} + d_5 c(\theta_{135}+\alpha) & 0 & d_5 c(\theta_{135}+\alpha) \end{bmatrix} \dot{\underline{\theta}} = \begin{bmatrix} \dot{x}_P \\ \dot{y}_P \end{bmatrix}$$

(5.147)

Angular velocity of link-5:

$$\dot{\sigma} = \dot{\theta}_{135} \tag{5.148}$$

This leads to

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \\ \dot{\theta}_4 \\ \dot{\theta}_5 \end{bmatrix} = \begin{bmatrix} \dot{\sigma} \end{bmatrix} \tag{5.149}$$

The task equations written for the tip point positions in horizontal and vertical directions constitute the first two rows and the task equation written for the orientation of the fifth link represents the third row of Jacobian matrix. Therefore considering the Equation 3.4 the Jacobian matrix and the vector of task space velocities turn out to be as below.

$$\hat{\Gamma}^P = \begin{bmatrix} -L_1 s\theta_1 - L_3 s\theta_{13} - d_5 s(\theta_{135}+\alpha) & 0 & -L_3 s\theta_{13} - d_5 s(\theta_{135}+\alpha) & 0 & -d_5 s(\theta_{135}+\alpha) \\ L_1 c\theta_1 + L_3 c\theta_{13} + d_5 c(\theta_{135}+\alpha) & 0 & L_3 c\theta_{13} + d_5 c(\theta_{135}+\alpha) & 0 & d_5 c(\theta_{135}+\alpha) \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

(5.150)

$$\dot{\underline{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} \dot{x}_P \\ \dot{y}_P \\ \dot{\sigma} \end{bmatrix} \tag{5.151}$$

5.2 Singularity Analysis

In this section, singular configurations of the parallel manipulator in Figure 5.1 will be derived as described in Chapter IV.

5.2.1 Drive Singular Configurations

Referring to Equation 4.1,

$$\det(\hat{B}^u) = \begin{vmatrix} L_4 s\theta_{24} & -L_5 s\theta_{135} \\ -L_4 c\theta_{24} & L_5 c\theta_{135} \end{vmatrix} = L_4 L_5 \left[s\theta_{24} c\theta_{135} - c\theta_{24} s\theta_{135} \right] = L_4 L_5 s(\theta_{24} - \theta_{135}) \quad (5.152)$$

Thus, singular configurations occur when $s(\theta_{24} - \theta_{135}) = 0$. That means, there exists two cases as

Case-I: $\theta_{24} - \theta_{135} = \pm n\pi \qquad n = 1, 3, 5....$

As a result of the case-I condition, link 4 and link 5 become extended inside the functional workspace as shown in Figure 5.4.

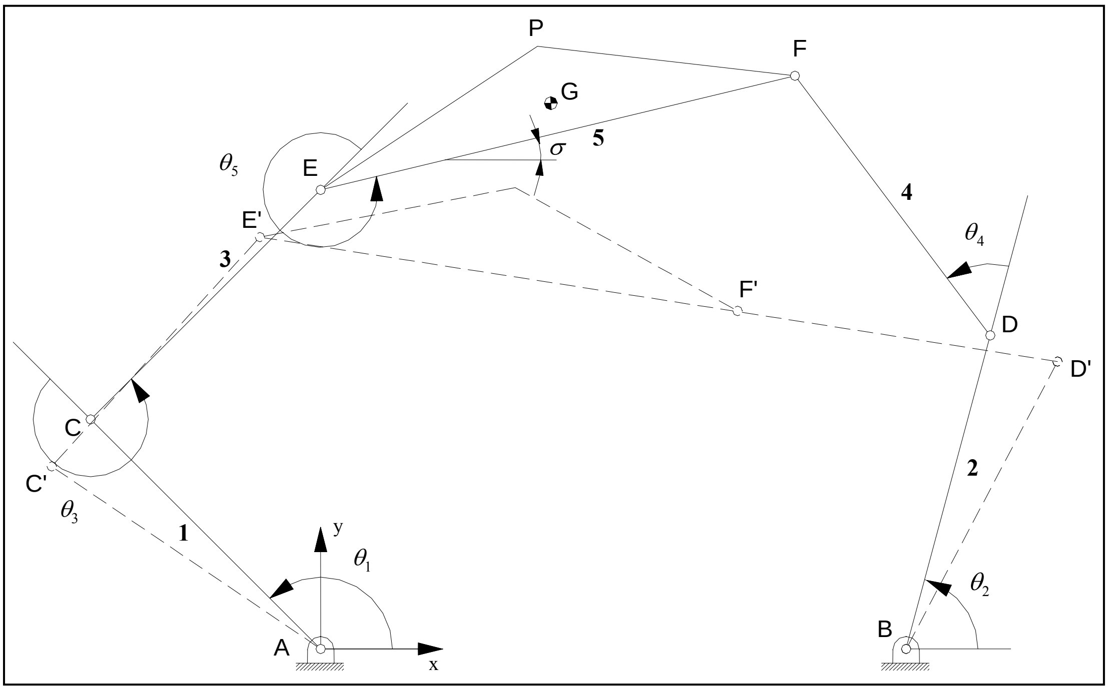

Figure 5.4 Drive Singular Configuration (Case-I)

Case-II: $\theta_{24} - \theta_{135} = \pm n\pi \qquad n = 0, 2, 4....$

As a result of the case-II condition, link 4 and link 5 become folded inside the functional workspace as shown in Figure 5.5.

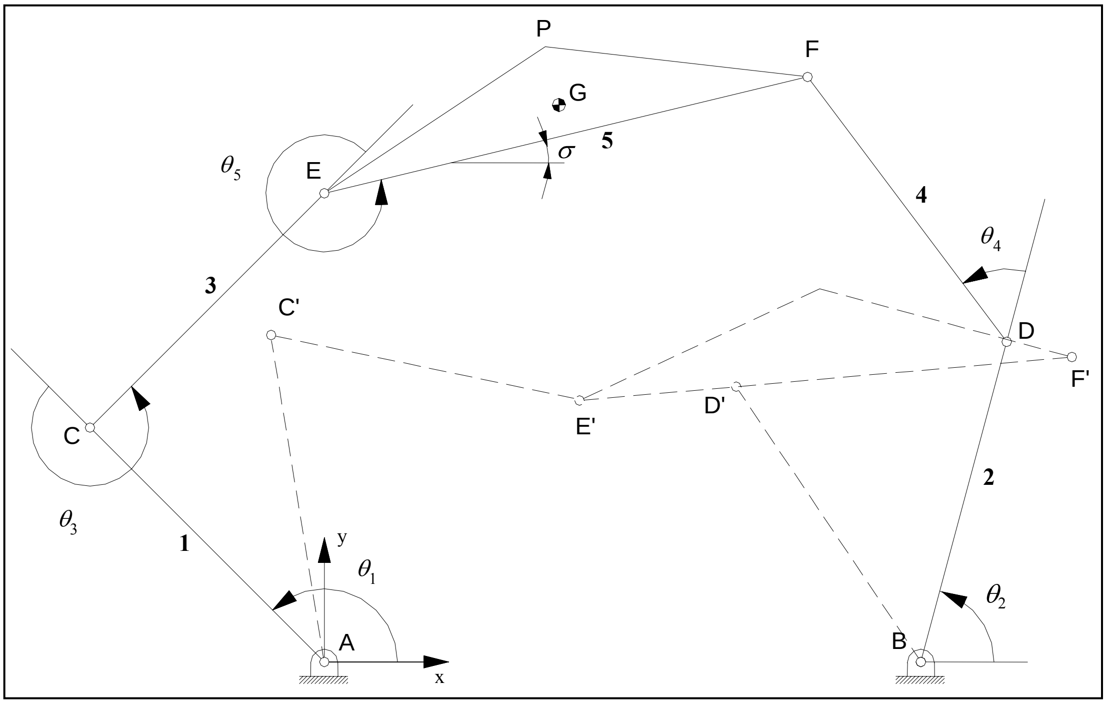

Figure 5.5 Drive Singular Configuration (Case-II)

As it can easily be inspected, drive singular configurations for the parallel manipulator in concern cause that the angular position of the link 4 with respect to the fixed reference frame and the angular position of the link 5 with respect to the fixed reference frame become undistinguishable.

5.2.2 Inverse Kinematic Singular Configurations

Referring to Equations 4.2 and 4.3, matrix $\hat{\Gamma}$ is constructed as

$$\hat{\Gamma} = \begin{bmatrix} -L_1 s\theta_1 - L_3 s\theta_{13} - L_5 s\theta_{135} & L_2 s\theta_2 + L_4 s\theta_{24} & -L_3 s\theta_{13} - L_5 s\theta_{135} & L_4 s\theta_{24} & -L_5 s\theta_{135} \\ L_1 c\theta_1 + L_3 c\theta_{13} + L_5 c\theta_{135} & -L_2 c\theta_2 - L_4 c\theta_{24} & L_3 c\theta_{13} + L_5 c\theta_{135} & -L_4 c\theta_{24} & L_5 c\theta_{135} \\ \hdashline -L_1 s\theta_1 - L_3 s\theta_{13} - d_5 s(\theta_{135} + \alpha) & 0 & -L_3 s\theta_{13} - d_5 s(\theta_{135} + \alpha) & 0 & -d_5 s(\theta_{135} + \alpha) \\ L_1 c\theta_1 + L_3 c\theta_{13} + d_5 c(\theta_{135} + \alpha) & 0 & L_3 c\theta_{13} + d_5 c(\theta_{135} + \alpha) & 0 & d_5 c(\theta_{135} + \alpha) \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} \quad (5.153)$$

$$\det(\hat{\Gamma}) = -\frac{1}{2}L_1L_2L_3L_4\left[c(\theta_1+\theta_{24}-\theta_{13}-\theta_2)-c(-\theta_1+\theta_{24}+\theta_{13}-\theta_2)\right] \quad (5.154)$$

$$\det(\hat{\Gamma}) = -\frac{1}{2}L_1L_2L_3L_4\left[c(\theta_4-\theta_3)-c(\theta_4+\theta_3)\right] \quad (5.155)$$

Using the trigonometric identity $\sin A.\sin B = \left[\cos(A-B)-\cos(A+B)\right]/2$,

$$\det(\hat{\Gamma}) = -\frac{1}{2}L_1L_2L_3L_4\left[2s\theta_4 s\theta_3\right] = -L_1L_2L_3L_4\left[s\theta_4 s\theta_3\right] \quad (5.156)$$

Thus, singular configurations occur when $s\theta_3 = 0$ or $s\theta_4 = 0$. That means, there exists four cases as

Case-I: $\theta_3 = \pm n\pi \qquad n = 1,3,5....$

As a result of the case-I condition, link 1 and link 3 become folded as shown in Figure 5.6.

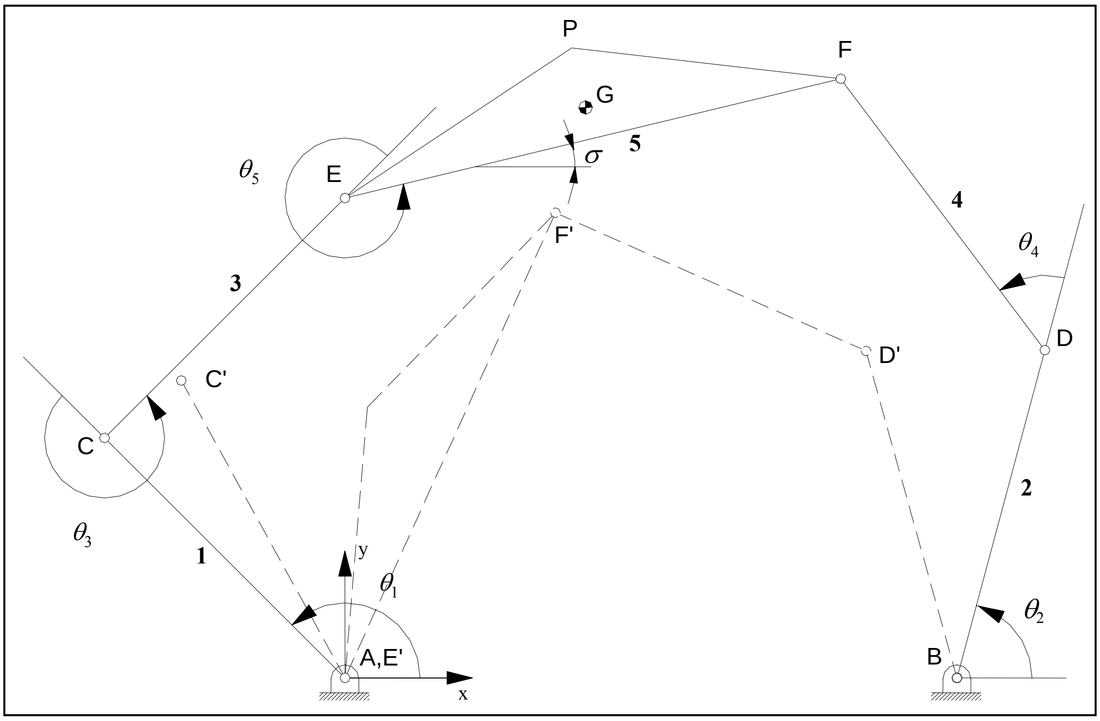

Figure 5.6 Inverse Kinematic Singular Configuration (Case-I)

Case-II: $\theta_3 = \pm n\pi \qquad n = 0, 2, 4....$

As a result of the case-I condition, link 1 and link 3 become extended as shown in Figure 5.7.

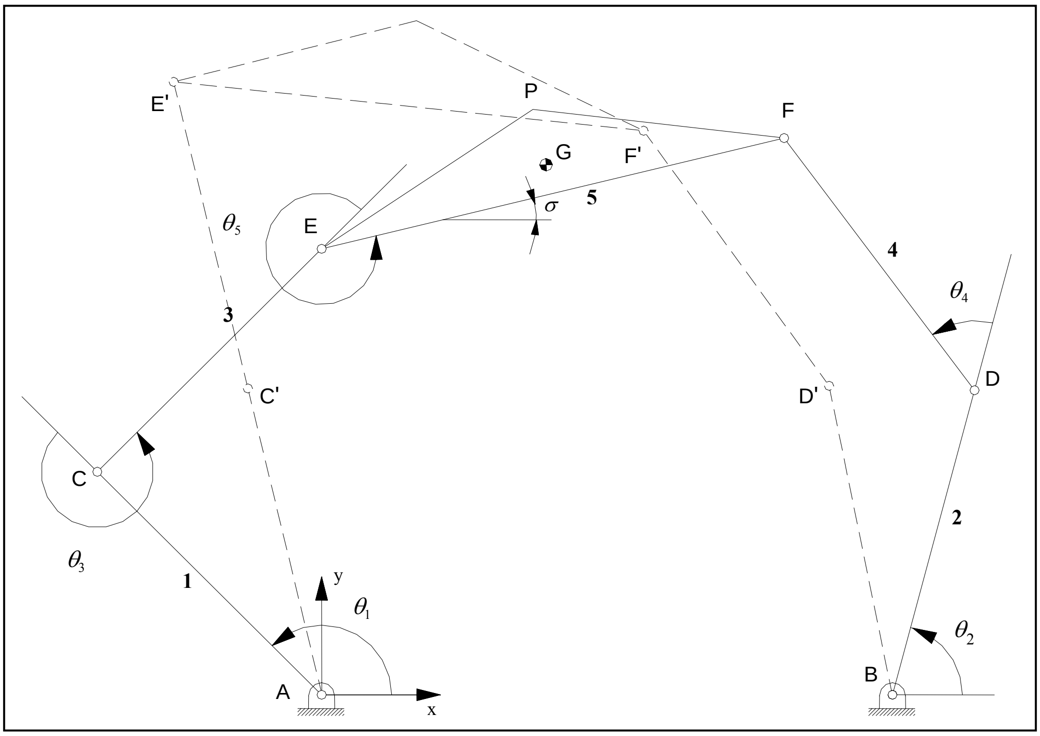

Figure 5.7 Inverse Kinematic Singular Configuration (Case-II)

The conclusion to be drawn from the cases I and II of the inverse kinematic singular configurations for the parallel manipulator in concern is that the angular position of the link 1 with respect to the fixed reference frame and the angular position of the link 3 with respect to the fixed reference frame become undistinguishable.

Case-III: $\theta_4 = \pm n\pi \qquad n = 1, 3, 5....$

As a result of the case-III condition, link 2 and link 4 become folded as shown in Figure 5.8.

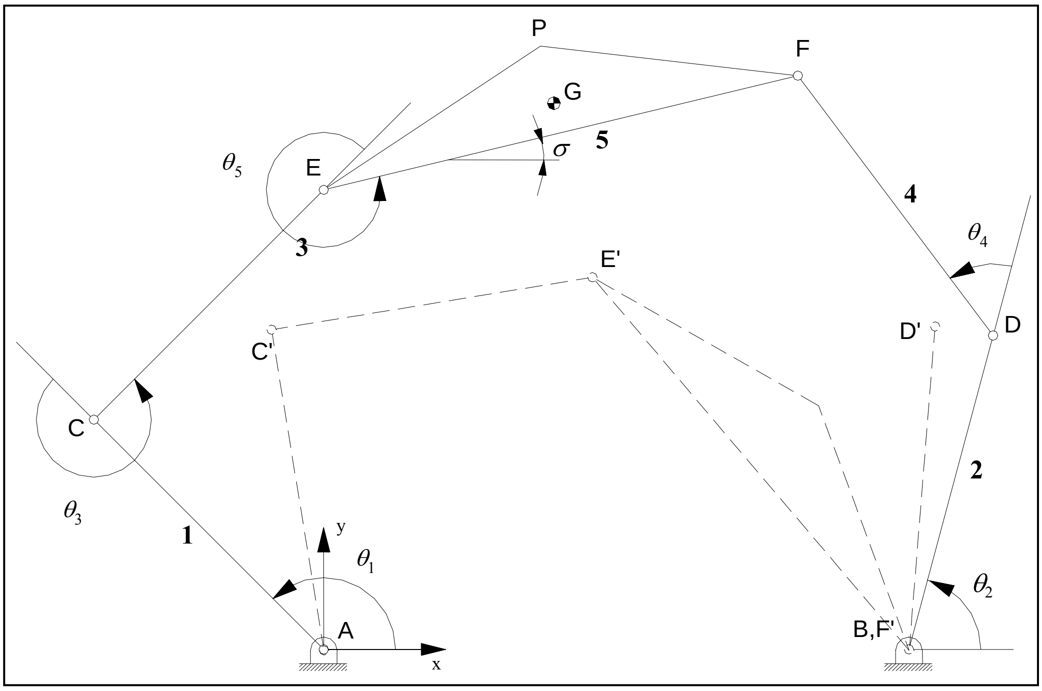

Figure 5.8 Inverse Kinematic Singular Configuration (Case-III)

Case-IV: $\theta_4 = \pm n\pi \qquad n = 0, 2, 4....$

As a result of the case-IV condition, link 2 and link 4 become extended as shown in Figure 5.9

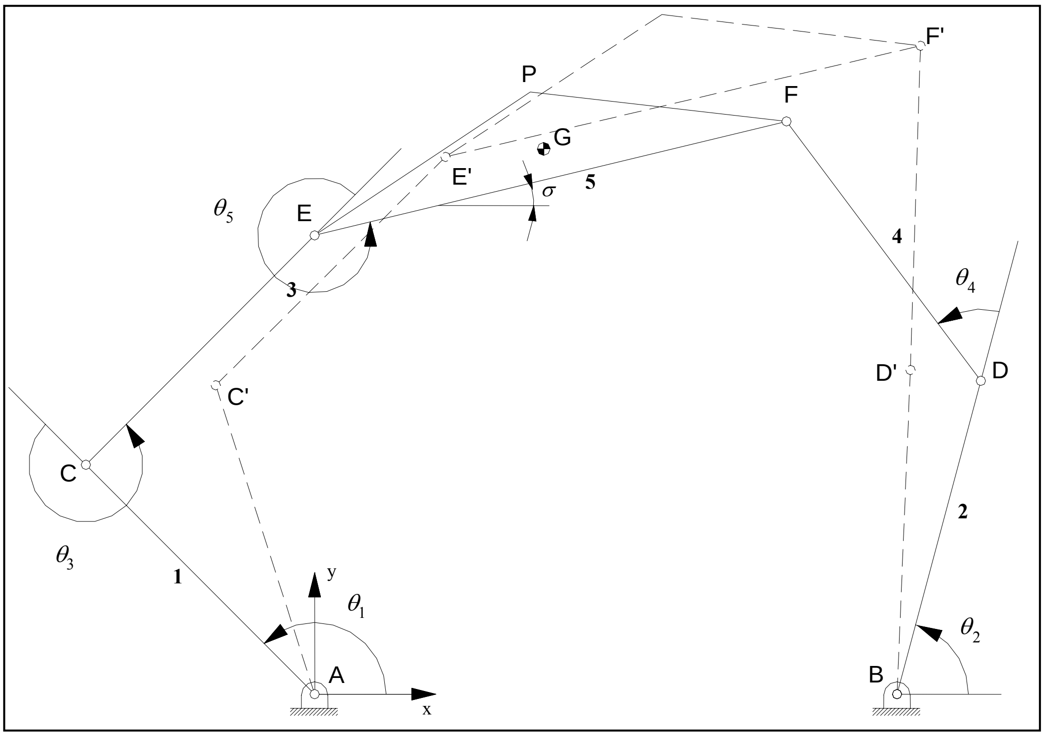

Figure 5.9 Inverse Kinematic Singular Configuration (Case-IV)

The conclusion to be drawn from the cases III and IV of the inverse kinematic singular configurations for the parallel manipulator in concern is that the angular position of the link 2 with respect to the fixed reference frame and the angular position of the link 4 with respect to the fixed reference frame become undistinguishable.

In this study, planar parallel manipulator under analysis tracks a trajectory in task space such that it is not around or at any of the singular configurations that were derived in Section 5.2. Instead of this, they are avoided with a proper trajectory planning.

5.3 Simulation Environment

In this section, the performance of the control law explained in Chapter-III is checked by using Matlab® and its one of the integrated tools Simulink®. Simulink® is a software package for modeling, simulating and analyzing dynamic systems [18]. It is user friendly modeling software and a tool that is adaptable to any problem with its rich feature set and powerful numerical algorithms.

Using Matlab® and Simulink® made almost everything easier in this study since most of the complicated equations in this study have been expressed in matrix form. Moreover, the initial conditions that are defined before the simulation runs can be altered to see the manipulator behavior in different positions.

The Simulink® models consist of some levels in hierarchy. These are arranged from the lower levels to upper ones. Some of the Simulink® blocks make up a subsystem and these subsystems form another subsystem at one step upper level. This situation goes on till finally these subsystems constitute the main system at the top level which governs the lower levels. This feature make the programmer feel free and easy since some of the blocks are directly related to this feature.

When it comes to the simulation, some parameters are necessarily introduced to the system before it runs. These parameters are of two kinds. First type of parameters is the *constant parameters* like damping and spring constants of the joints, inertias of the rotors and feedback gains coming from the control norm. For this kind of parameters an m-file called parameter.m was written and loaded to the workspace at the beginning. This file is changeable depending upon the integral control is included in the analysis or not. Any other constant parameters were written in the relevant m-files. The second type of parameters is called the *configuration parameters* consisting of the parameters required for Simulink® itself. Solver options and simulation time are of this kind.

In the simulations of this study, among the two types of solvers, fixed-step solver was chosen due to the fact that the model has continuous states. Simulink® computes the simulation's next time by adding a fixed-size time step to the current time. In addition, the continuous solvers employ numerical integration to compute the values of a model's continuous states at the current step from the values at the previous step and the values of the state derivatives. Among the various types of fixed-step solvers, one of the most complex numerical integration methods called Fourth Order Runge-Kutta (ode4) method was preferred since more accurate results are aimed. The chosen fixed step size will be discussed later in this section. Once the algorithm is successfully iterated, it may run without any time limit. In other words, the limit of simulation stop time depends on the programmer.

After the outline of the simulation is drawn, the comprehensive usage of Matlab® and Simulink® and the algorithms used in the simulation will be introduced. Furthermore, the main system and the subsystems of the model associated with the main system will be presented.

5.3.1 Main System

The main system of the model is as shown in Figure 5.10 and is composed of four major subsystems: task reference, controller, computed torque block and manipulator dynamics and kinematics.

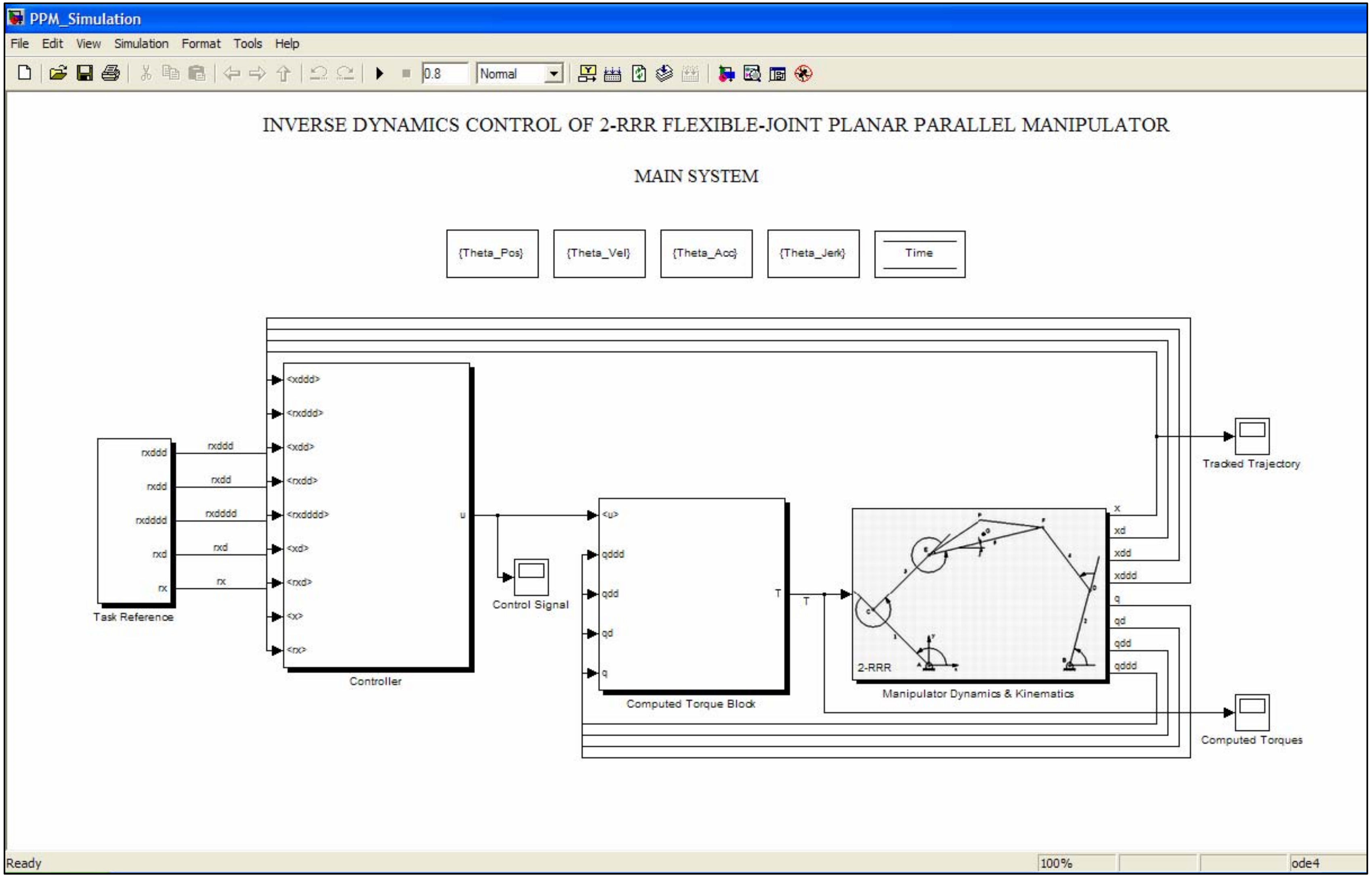

Figure 5.10 Main System of the Model

Simulation begins with the initial conditions and desired motion trajectory specified at $t = 0$. Then this information generates matrix signals of the desired task space values and its derivatives. This matrix signals then go directly into the controller subsystem in order to produce control input signal $\bar{u}$ in accordance with the Equation 3.38. The third major subsystem is responsible from computing torques at each sampling time by control algorithm explained in Chapter-III. This algorithm is the called the inverse dynamics algorithm. Finally, the computed torques are applied to the real system using the forward dynamics solution this time inside manipulator dynamics and kinematics subsystem. Inside this major subsystem, joint angular position variables are computed and from forward kinematic solution task space unknowns are calculated. Generated joint space and task space position signals and all of the derivative signals are then fed back to the relevant subsystems to find the errors in all states.

In the main system, there is a data memory block which stores the time values generated by a counter clock placed inside manipulator dynamics and kinematics subsystem. There are also four more blocks at the top which are called tag visibility blocks and are used to route the signal from where it is located. Since they are located at the main system, i.e., the level at the top, the signal tags carrying the same name as in the main system are routed to all of the levels.

5.3.2 Subsystems

This section covers the four major subsystems and how they were built in more detail. Task reference subsystem is a simple subsystem that is composed of Matlab® function blocks that define the prescribed end effector task space trajectory in all states and data read blocks for time values. Then the generated signals directly go into the controller subsystem as shown in Figure 5.11. Her in this block, real values are subtracted from the prescribed reference values and multiplied by the feedback gains of the ITAE norm with the choice of integral control. Moreover, the errors in all states are transferred to the workspace for the future comments.

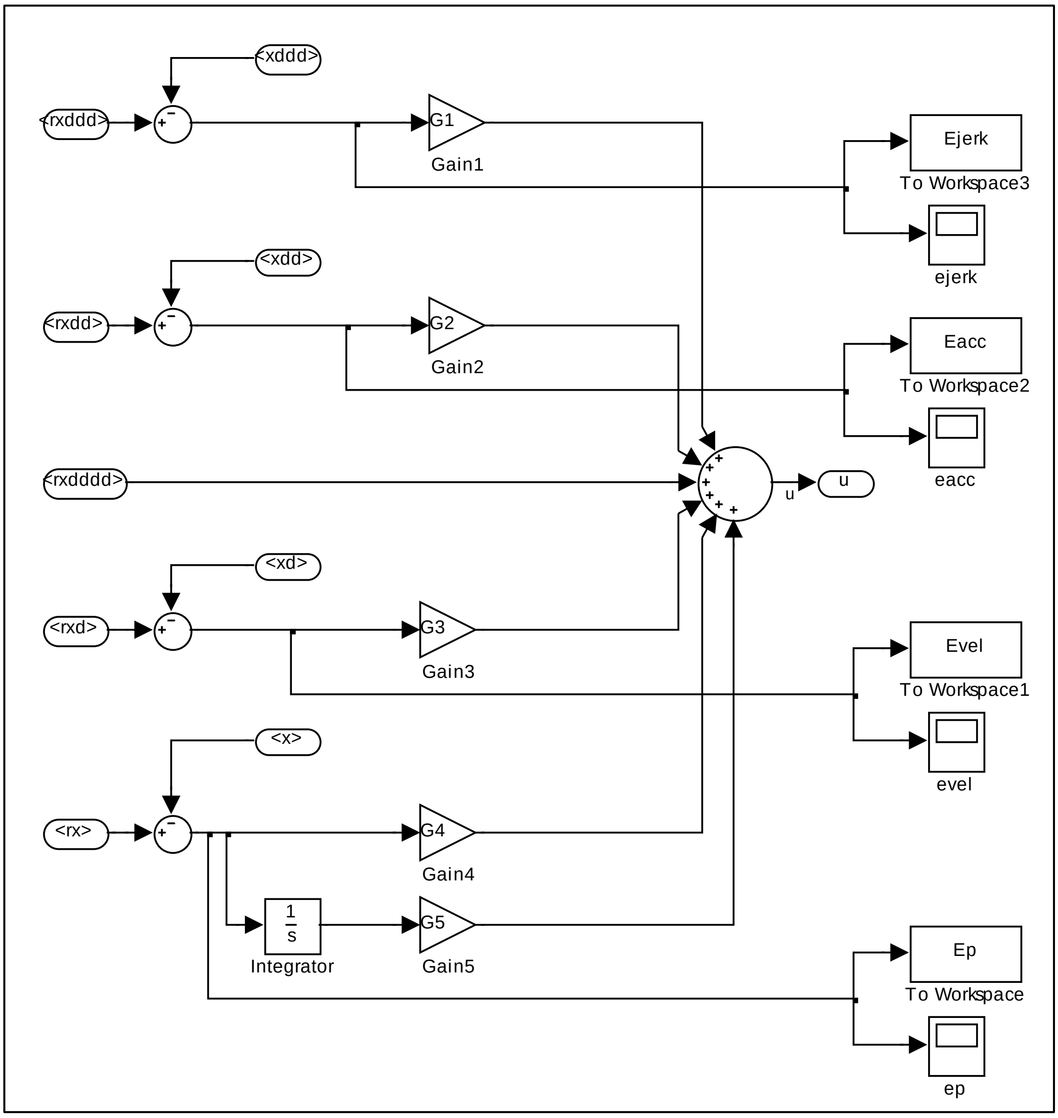

Figure 5.11 Controller Subsystem

The computed torque block which is illustrated in Figure 5.12 uses the inverse dynamics solution by putting the control signal vector $\bar{u}$ into the snap values of Equation 3.32 and calculates the torques to be applied by the actuators for making use of them in the forward dynamics solution. The choice of these gains is going to be explained later in this section. Since numerical integration of the computed torques is necessary, one needs to define the initial torque values that are applied by drive shaft. These initial values are obtained by defining the reference trajectory values as the actual values. By this way, initial torques are determined from the computed torque curves. The initial torques are nonzero because of the gravitational forces applied on the actuators.

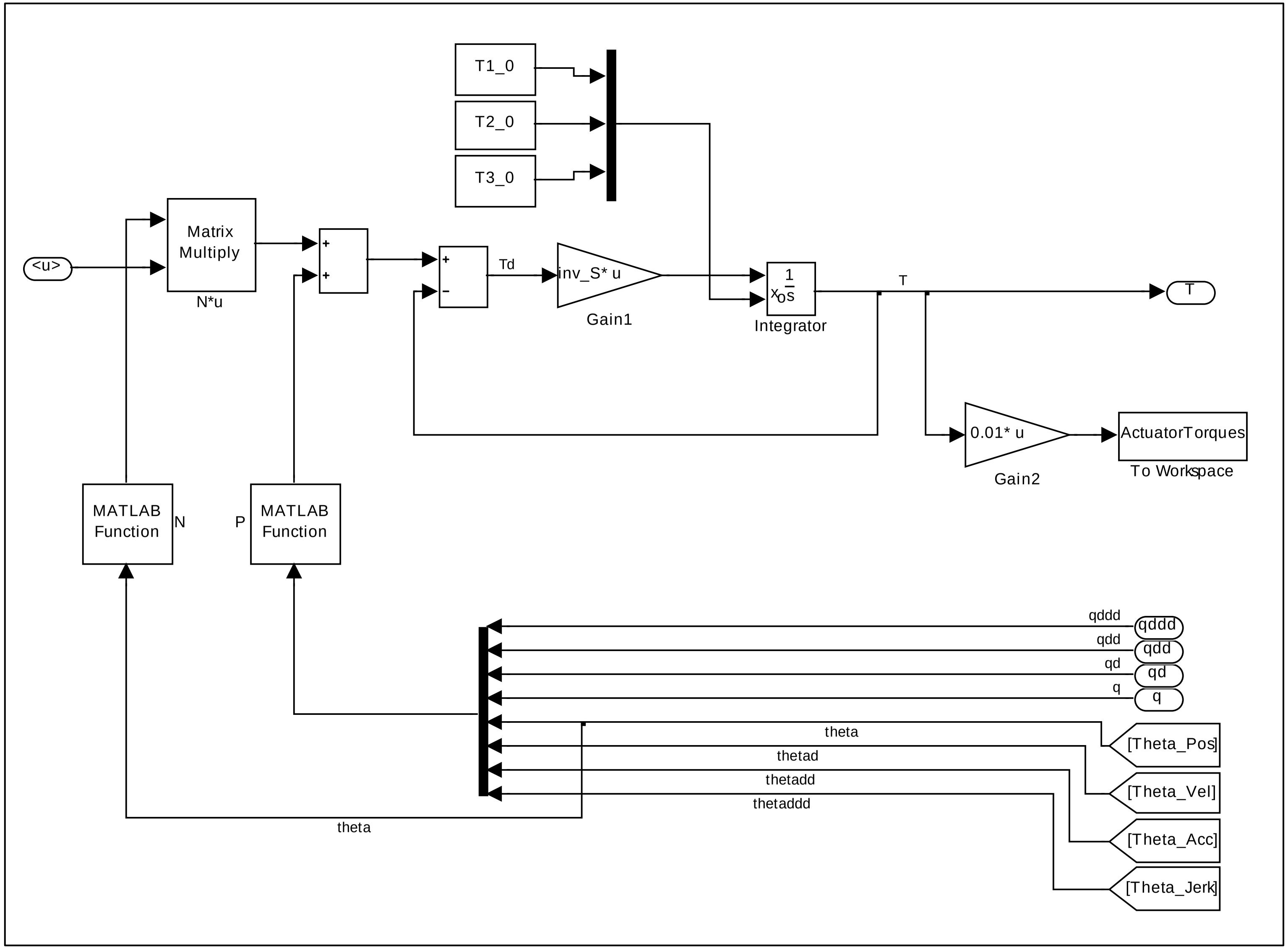

Figure 5.12 Computed Torque Block

After the torques are computed and formed a vector signal, then this signal enters the last major subsystem called Manipulator Dynamics and Kinematics. This subsystem is a bit more complicated than the others since almost all of the unknowns like manipulator joint position values up to snap values in both joint space and task space and actuator positions are computed at each sampling time.

This subsystem has three levels from top to bottom. These levels are named in accordance with their distance to the major subsystem. The level at the bottom is Level-3 for instance meaning that it is three steps far away.

Level-1 as shown in Figure 5.13 contains manipulator dynamics subsystem and a Matlab® function called forward kinematics. This function obtains the task space values of the end effector after the torques are applied to the parallel manipulator in the subsystem and the outgoing signals are then fed back to the controller subsystem in order to find errors.

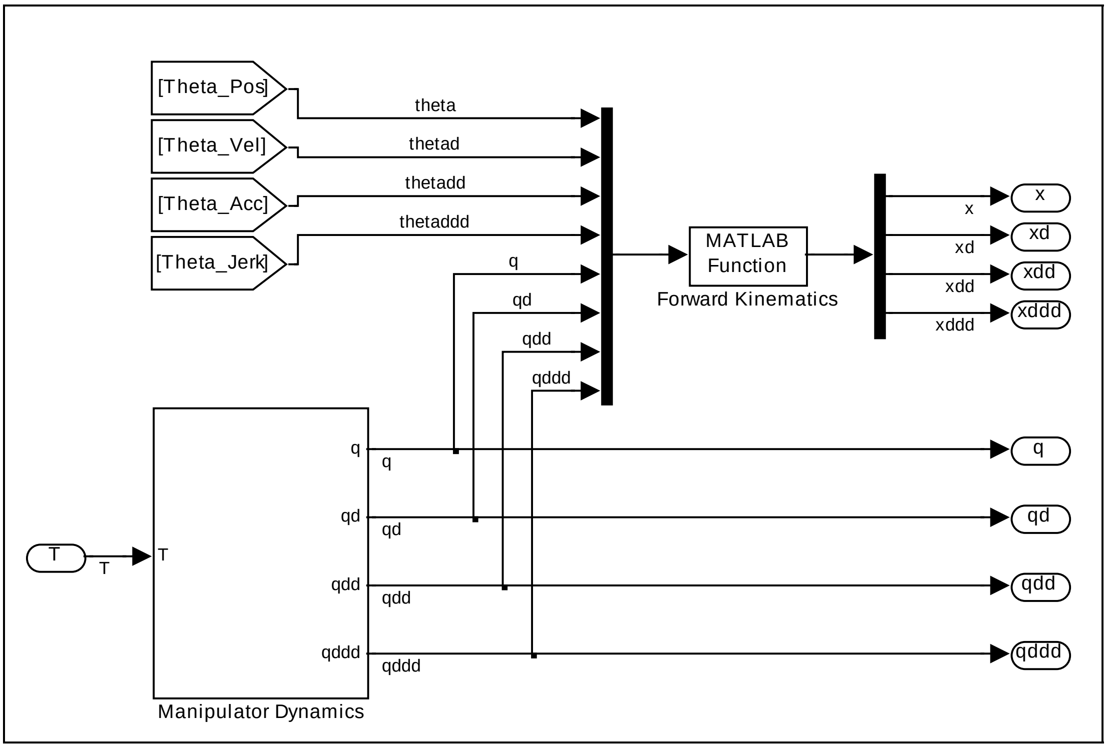

Figure 5.13 Manipulator Dynamics and Kinematics Subsystem

The manipulator dynamics subsystem at level-2 is illustrated in Figure 5.14.

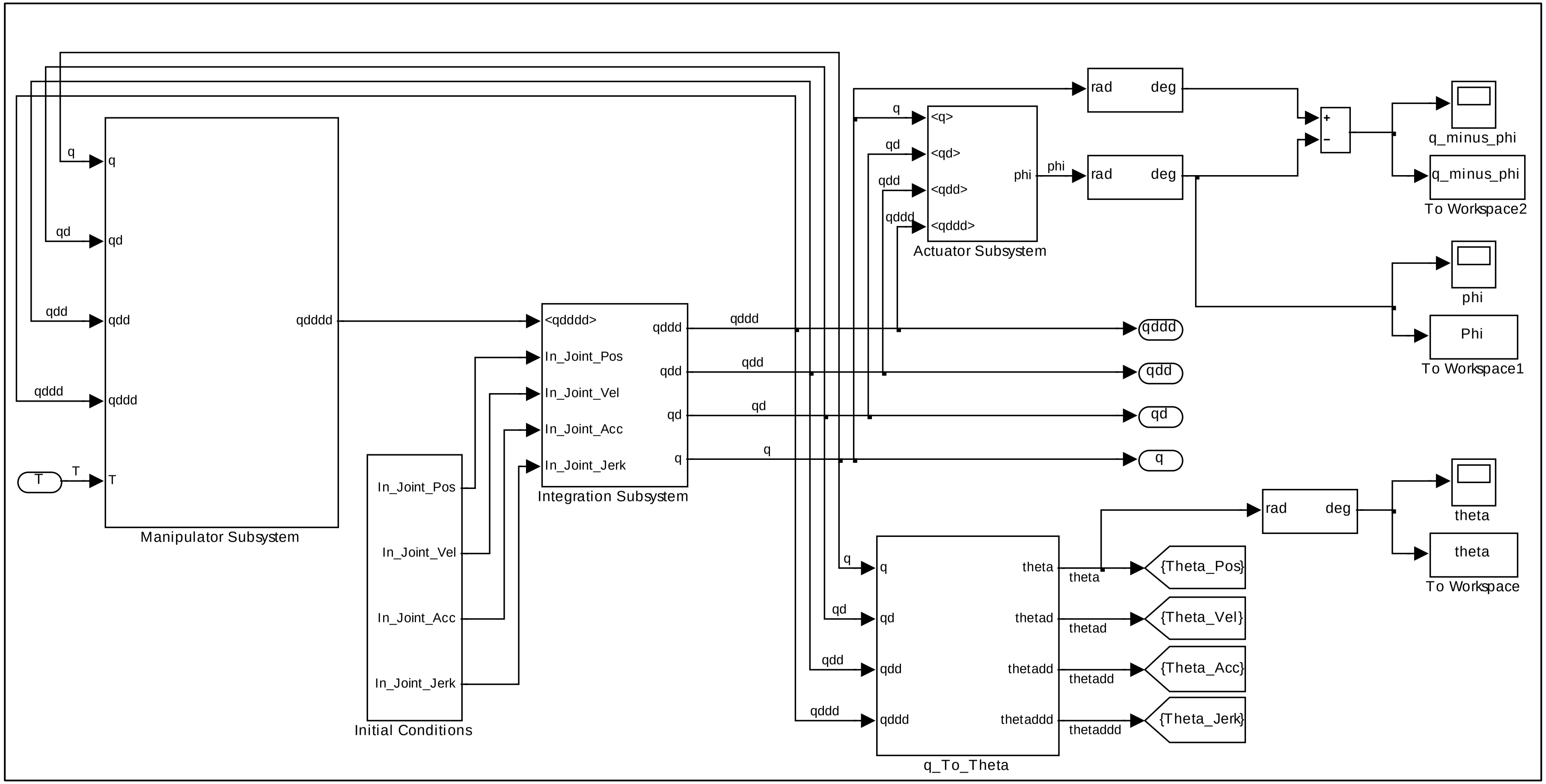

Figure 5.14 Manipulator Dynamics

There exist many subsystems at this level. As it can easily be seen from the figure, control torques are used in the manipulator subsystem that is shown in Figure 5.17 with the feedback signals carrying the information of actuated joint position values up to the jerk values. With this knowledge, manipulator subsystem calculates the snaps of the actuated joins. Furthermore, the clock begins to work here to count the time. Time values are then sent to data write block to be stored in the data memory block of the main system. After that, these values are transferred from the data memory block to the data read block of the task reference major subsystem for the reference trajectory. Time values are also loaded to the workspace for plotting the graphs.

From that point on, generated signal of actuated joint snaps go directly into the integration subsystem as given in Figure 5.16 in order to find the joint states from the jerks to the positions. The integration subsystem consists of four integrators each of which has initial condition source externally. The initial conditions needed for all of the states come from the initial conditions source block.

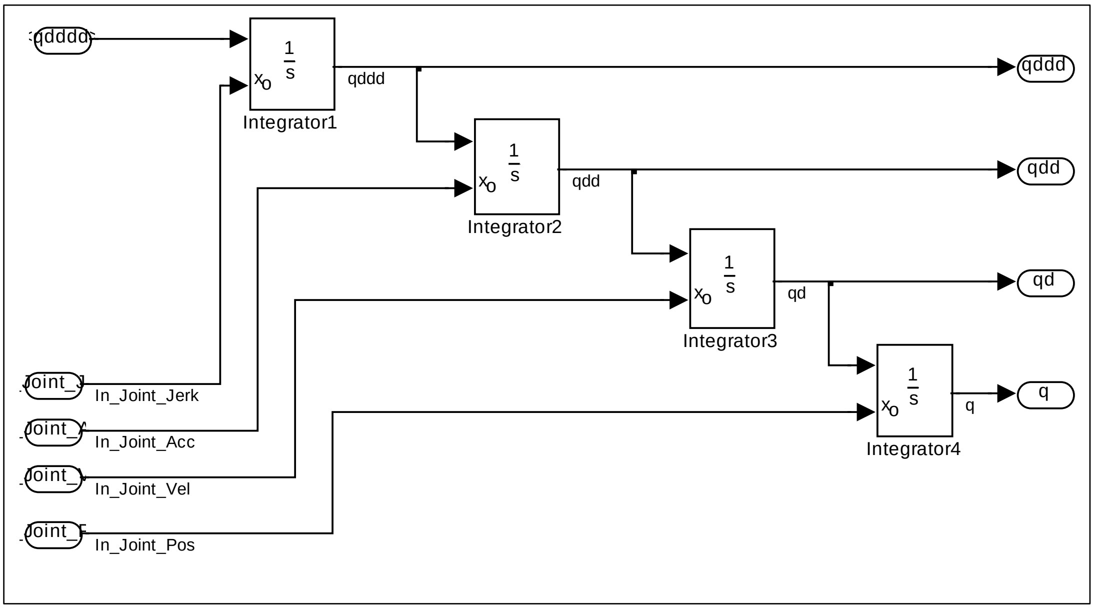

Figure 5.15 Integration Subsystem

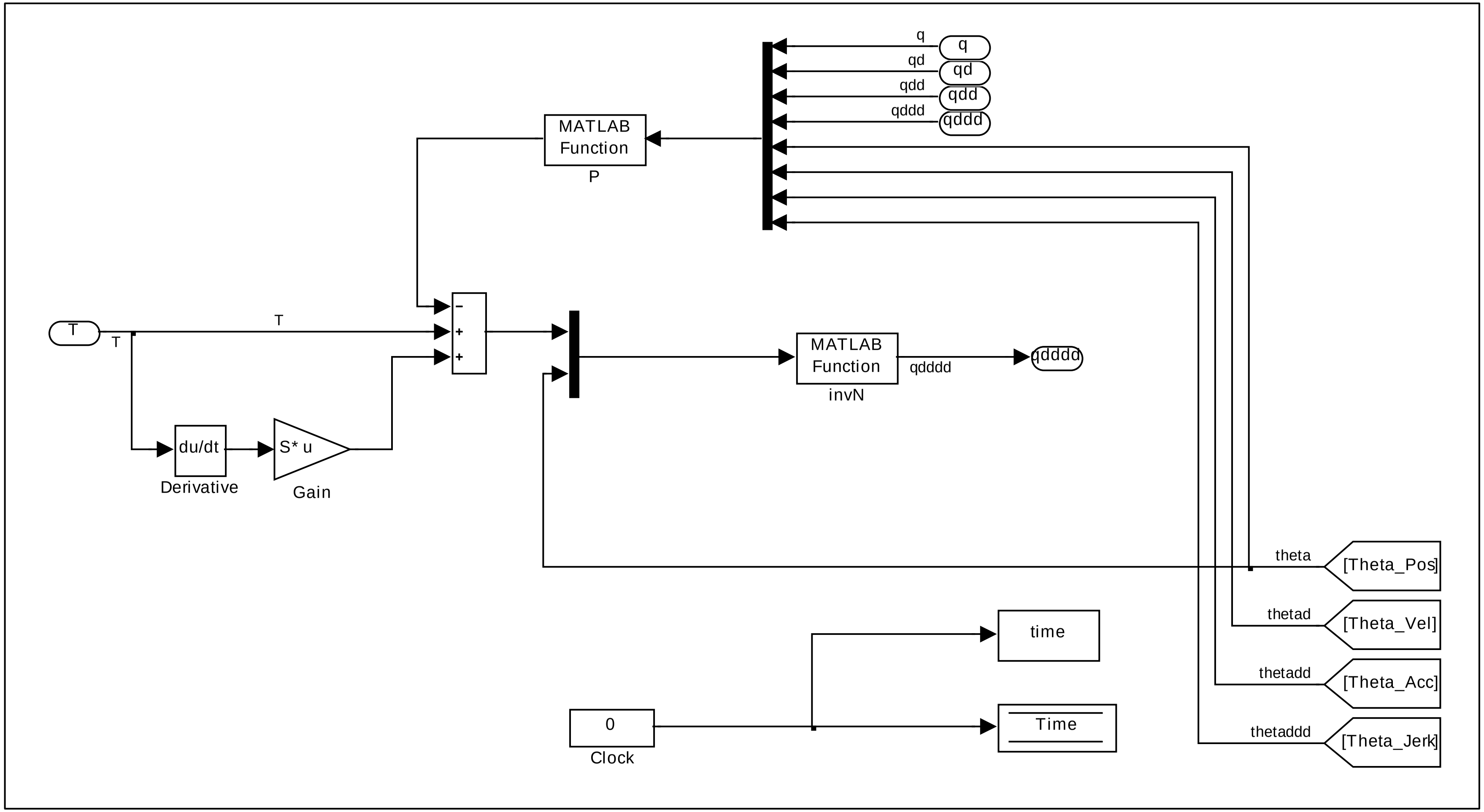

Figure 5.16 Manipulator Subsystem

Another subsystem is q_To_Theta subsystem. This subsystem has a very critical significance and plays a vital role for finding the unactuated joint values from the actuated joint variables via the constraint equations as expressed in Equations 5.86 and 5.87. For this purpose, some of the algorithms are used for the parallel manipulators. Among the various types of algorithms, trust-region dogleg algorithm is preferred to be used for this case study. This is provided by one of the built-in functions of Matlab® called *fsolve*. This function is used to solve systems of nonlinear equations and has an algorithm for systems where the number of equations is equal to the number of unknown variables. The algorithm is based on finding the roots of the constraint equations.

As it can be inspected from Figure 5.18, the vector of actuated joint variables $\overline{q}$ is returned to the vector of unactuated joint variables $\overline{\theta}^u$ by many Matlab® functions that handle the fsolve function and then these two vector signals are concatenated by the multiplexers to construct the vector of joint variables $\overline{\theta}$ vector signal to be routed to any level of the model to be used where necessary.

However, since solving unactuated joints is based on finding a root of the constraint equations, one needs to use smaller sampling times for the model in order that this function converges to a root in region of predefined epsilon value that defines the error. This value was considered to be 1.0e-06. For this reason sampling time of the Simulink® model for this case study was preferred as 1.5e-04 after a lot of trials. That causes a bit longer computation time but more accurate results are obtained.

Last subsystem in this level is the actuator variables subsystem as demonstrated in Figure 5.19. With the known values of joint variables in all states, the actuator variables are calculated by substituting the known values of joint variables into the one of the equations of motion set. Here in this subsystem another numerical integration for the actuator variables is required as far as the Equation 3.25 is concerned. For the numerical integration, one needs to define the initial ϕ_i, $i = 1, 2, 3.$ values.

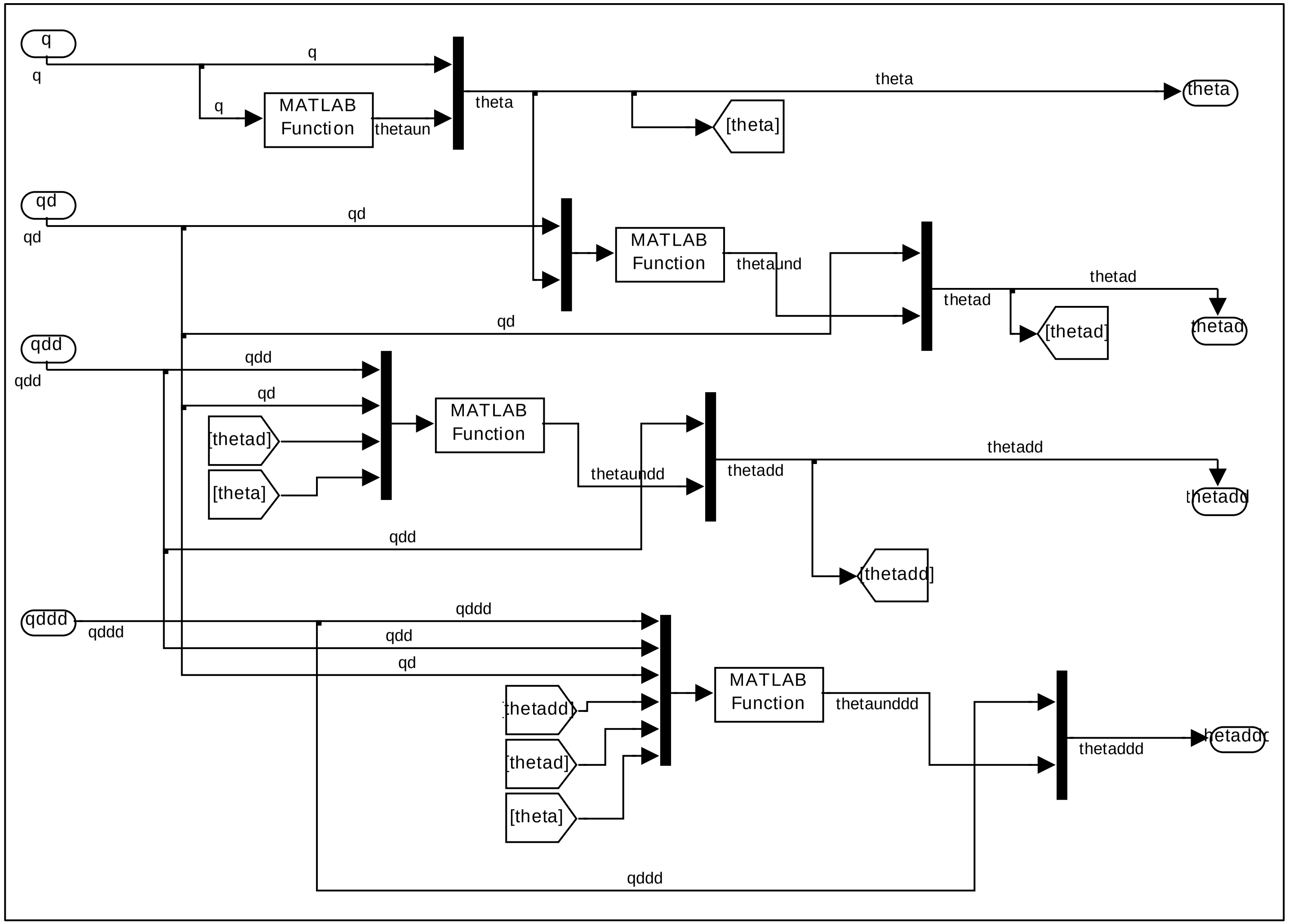

Figure 5.17 q_To_Theta Subsystem

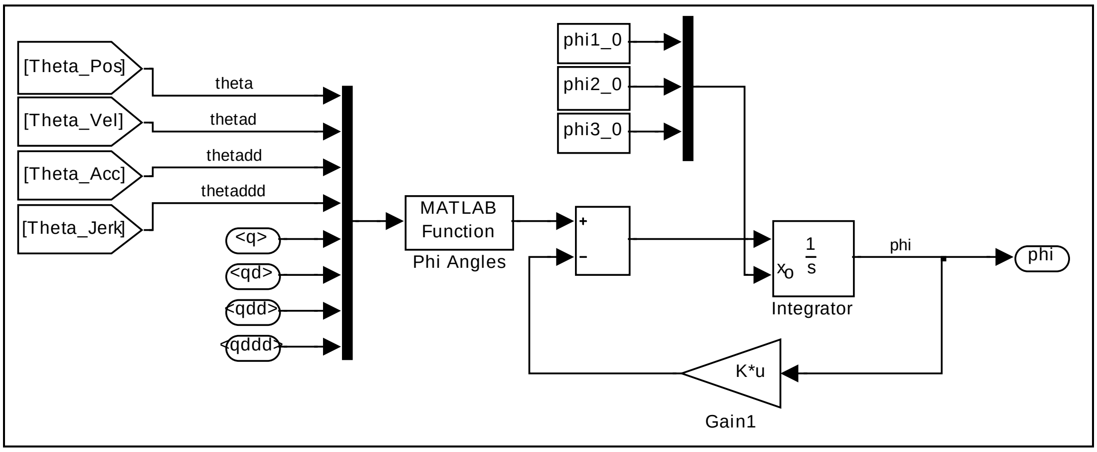

Figure 5.18 Actuator Subsystem

Similarly these initial values are automatically obtained like the initial torques. Running the simulation with the actual initial values with time independency gives the initial values. The initial values are nonzero because of the gravitational forces applied on the actuators.

Consequently, the initial deflections at the actuated joints could be found by simply subtracting actuator positions from active joint position values. It should be noted that initial deflections are also nonzero. The subtraction process is realized in the manipulator dynamics subsystem and the deflection values are loaded to the workspace with the other values like $\overline{\theta}$ and $\overline{\phi}$ angles.

5.4 Control Simulations

Performance of the parallel manipulator in the case study is analyzed basically in three groups of simulations. In some of the simulations modeling error is considered by setting the manipulator inertia and mass properties, the torsional spring constants and the damping constants 10% larger in the model.

Beside that some of the simulations do not contain the integral control. Therefore the feedback gains alter depending on the integral control is included or not. These feedback gains of ITAE norm were tabulated in Chapter III.

In the simulations, the data used are as below.

Table 5.2 The Geometric Data

Symbol	Definition	Value	Symbol	Definition	Value
L_1	$\|\overrightarrow{AC}\|$	1.0 m	d_0	$\|\overrightarrow{AB}\|$	1.8 m
L_2	$\|\overrightarrow{BD}\|$	1.0 m	g_5	$\|\overrightarrow{EG}\|$	0.75 m
L_3	$\|\overrightarrow{CE}\|$	1.0 m	d_5	$\|\overrightarrow{EP}\|$	0.8 m
L_4	$\|\overrightarrow{DF}\|$	1.0 m	α	$\angle PEF$	20 deg
L_5	$\|\overrightarrow{EF}\|$	1.5 m	β	$\angle GEF$	7 deg

The motors and speed reducers at the active joints have small dimensions compared to the links. The masses of both rotors and speed reducers are assumed to be lumped at the joints.

Table 5.3 The Mass/ Inertial Properties and Gear Ratios

Symbol	Value	Symbol	Value
m_1^L	10 kg	m_3^A	1.2 kg
m_2^L	10 kg	I^r_{1zz}	7.0e-05 kg.m^2
m_3^L	10 kg	I^r_{2zz}	8.0e-05 kg.m^2
m_4^L	10 kg	I^r_{3zz}	9.0e-05 kg.m^2
m_5^L	15 kg	r_1	100
m_1^A	1.2 kg	r_2	100
m_2^A	1.2 kg	r_3	100

Structural damping constants of the actuated joints correspond to a 3% damping ratio for the structural vibration of each rotor.

Table 5.4 The Damping and Spring Constants

Symbol	Value	Symbol	Value
D_1	0.0355 N.m.s/rad	D_1^r	0.0003 N.m.s/rad
D_2	0.0379 N.m.s/rad	D_2^r	0.0003 N.m.s/rad
D_3	0.0402 N.m.s/rad	D_3^r	0.0003 N.m.s/rad
D_4	0.0200 N.m.s/rad	K_1	5000 N.m/rad
D_5	0.0200 N.m.s/rad	K_2	5000 N.m/rad
D_6	0.0200 N.m.s/rad	K_3	5000 N.m/rad

The parallel manipulator in Figure 5.1 is assumed to be at rest initially and have the following initial active joint positions as below.

$$\theta_{1_\circ} = 135^\circ \tag{5.157}$$

$$\theta_{2_\circ} = 75^\circ \tag{5.158}$$

$$\theta_{3_\circ} = -90^\circ \tag{5.159}$$

The initial active joint positions lead to the initial passive joint angles by the algorithm explained in 5.3.2 as

$$\theta_{4_\circ} = 51.91^\circ \tag{5.160}$$

$$\theta_{5_\circ} = -31.46^\circ \tag{5.161}$$

These initial joint angles correspond to the following task space initial positions.

$$x_{1_\circ} = 0.6668 \text{ m} \tag{5.162}$$

$$x_{2_\circ} = 1.8563 \text{ m} \tag{5.163}$$

$$x_{3_\circ} = 0.2364 \text{ rad} = 13.55^\circ \tag{5.164}$$

The desired trajectory motion is a deployment motion in task space and can be written as

$$x_1^d = x_P^d = \begin{cases} 0.70 + \dfrac{0.5}{T}\left[t - \dfrac{T}{2\pi}\sin\dfrac{2\pi t}{T}\right] m & 0 \le t \le T \\ 1.20\ m & t > T \end{cases} \tag{5.165}$$

$$x_2^d = y_P^d = \begin{cases} 1.90 - \dfrac{0.5}{T}\left[t - \dfrac{T}{2\pi}\sin\dfrac{2\pi t}{T}\right] m & 0 \le t \le T \\ 1.40\ m & t > T \end{cases} \tag{5.166}$$

$$x_3^d = \sigma^d = \begin{cases} 20 + \dfrac{15}{T}\left[t - \dfrac{T}{2\pi}\sin\dfrac{2\pi t}{T}\right] deg & 0 \le t \le T \\ 35\ deg & t > T \end{cases} \tag{5.167}$$

where T is the period of the deployment motion and selected as

$$T = 0.6\ s \tag{5.168}$$

As far as the initial task space positions and initial desired task space positions are considered, it can be easily examined that system begins its motion with the initial position errors. That means in all of the simulations initial position errors are assumed to be present. In addition to this, when the reference or desired trajectory is given as the actual initial joint values, the initial torques and initial actuator position angles are computed as follows.

Initial torques to be applied after speed reduction:

$$T_{1_\circ} = -128.51\ \text{N.m} \tag{5.169}$$

$$T_{2_\circ} = 50.81\ \text{N.m} \tag{5.170}$$

$$T_{3_\circ} = 69.34\ \text{N.m} \tag{5.171}$$

Initial phi angles:

$$\phi_{1_\circ} = 2.3305 \ \text{rad} = 133.52^\circ \tag{5.172}$$

$$\phi_{2_\circ} = 1.3192 \ \text{rad} = 75.58^\circ \tag{5.173}$$

$$\phi_{3_\circ} = -1.5569 \ \text{rad} = -89.20^\circ \tag{5.174}$$

In the first group of simulations, there is no modeling error and as a result of that no integral control. Therefore the control feedback gains are picked up as in the first column of Table 3.1. The simulations are repeated for various ω_o values. These conditions are named as the first type of conditions and called CT-1 in short hand.

In the second group of simulations, modeling error is considered but integral control is not used. Conditions in this group are named as the second type of conditions and called CT-2, similarly.

Third group of simulations are performed under consideration of modeling error and integral control. This time, the control feedback gains are picked up as in the second column of Table 3.1. The condition type is called CT-3 this time.

5.5 Results

There are four graphs plotted to represent

- the desired and responded displacements of the trajectory in three dimensions,
- the control torques to be supplied by the actuators,
- the position errors,
- the deflections between $\bar{q}$ and $\bar{\phi}$.

Besides, the position errors are also plotted for the three groups of simulations in the same graph to compare the effects of ω_o values.

Simulation Group-1

a) $\omega_o = 30$ rad/s

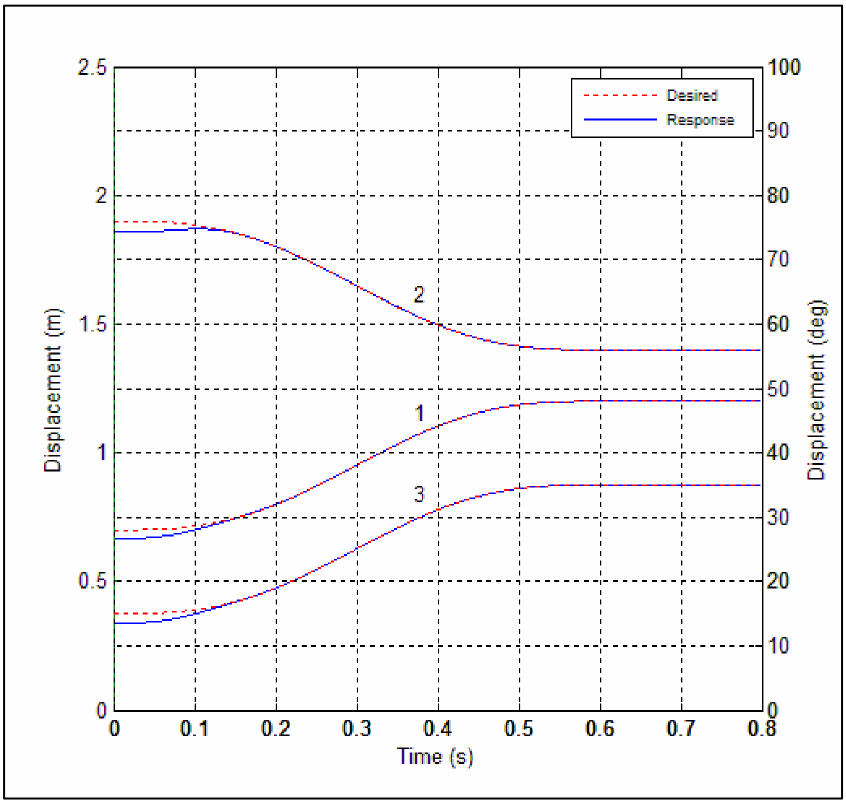

Figure 5.19 Position Response: 1. x_1, 2. x_2, 3. x_3 (CT-1, $\omega_o = 30$ rad/s)

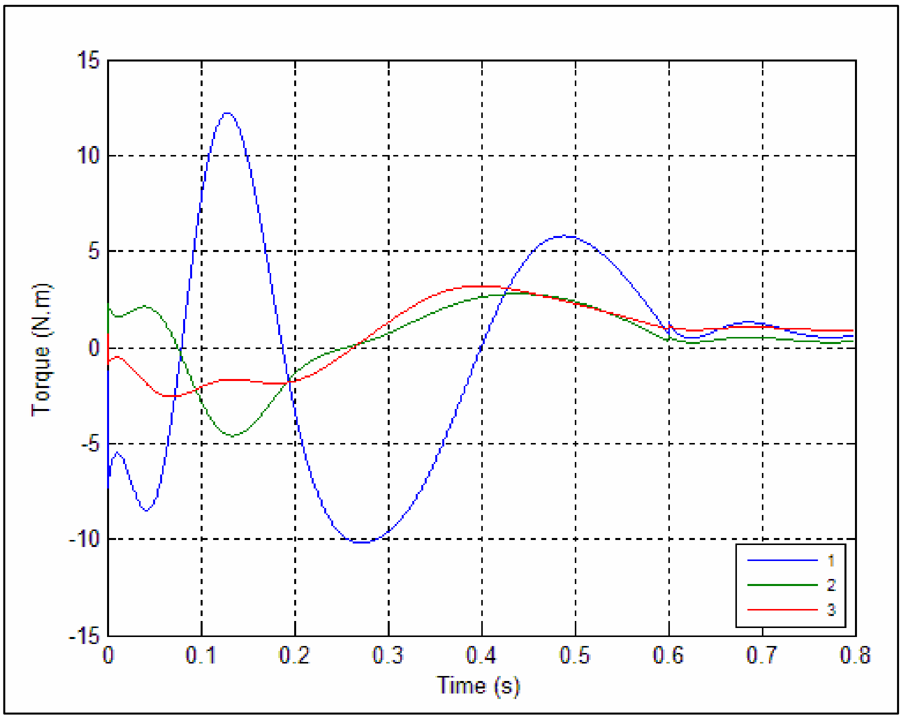

Figure 5.20 Control Torques: 1. T_1^a, 2. T_2^a, 3. T_3^a (CT-1, $\omega_o = 30$ rad/s)

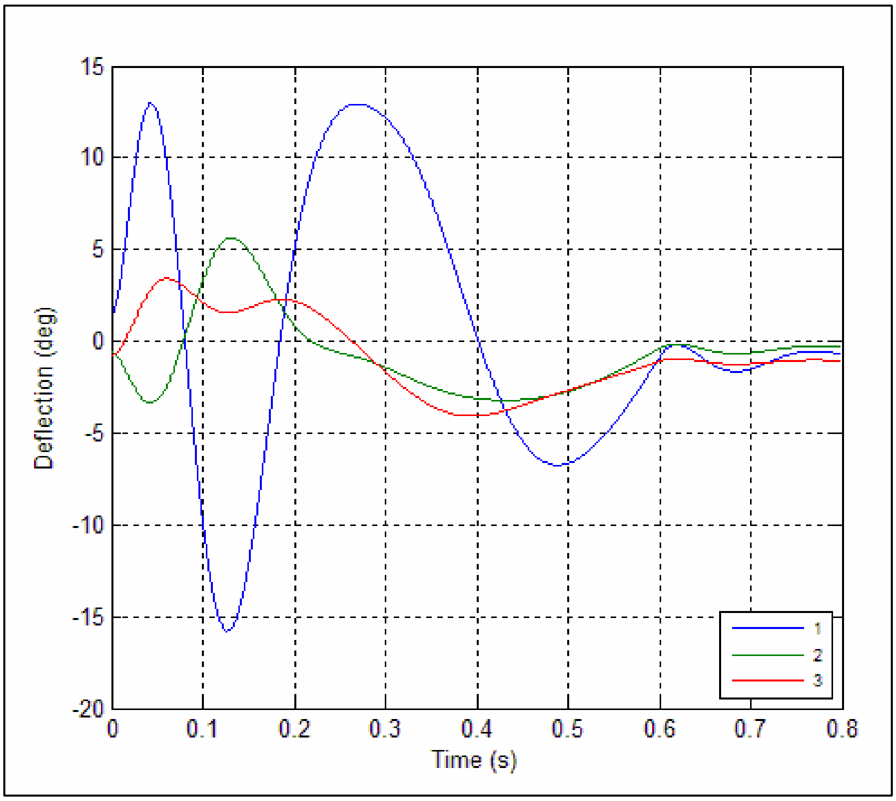

Figure 5.21 Deflections: 1. $\theta_1 - \phi_1$, 2. $\theta_2 - \phi_2$, 3. $\theta_3 - \phi_3$ (CT-1, $\omega_o = 30$ rad/s)

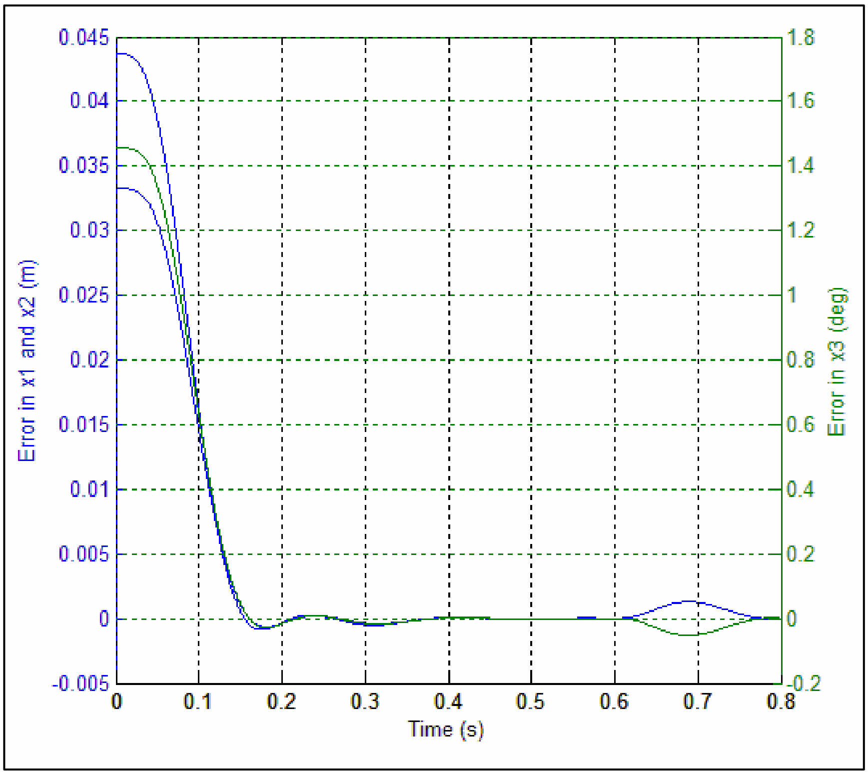

Figure 5.22 Position Errors (CT-1, $\omega_o = 30$ rad/s)

b) $\omega_o = 50$ rad/s

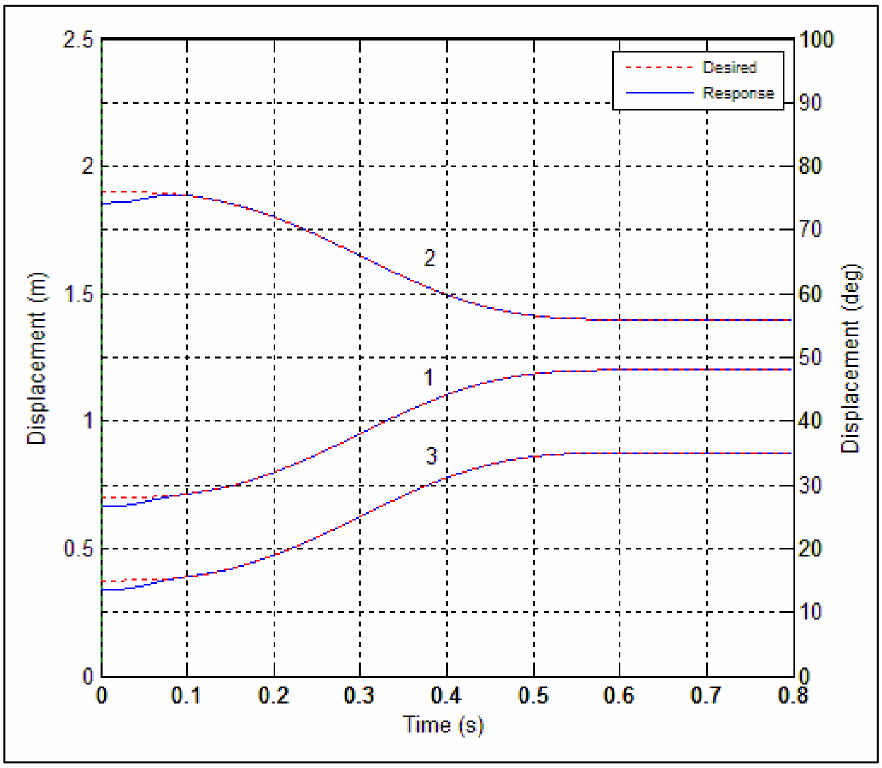

Figure 5.23 Position Response: 1. x_1, 2. x_2, 3. x_3 (CT-1, $\omega_o = 50$ rad/s)

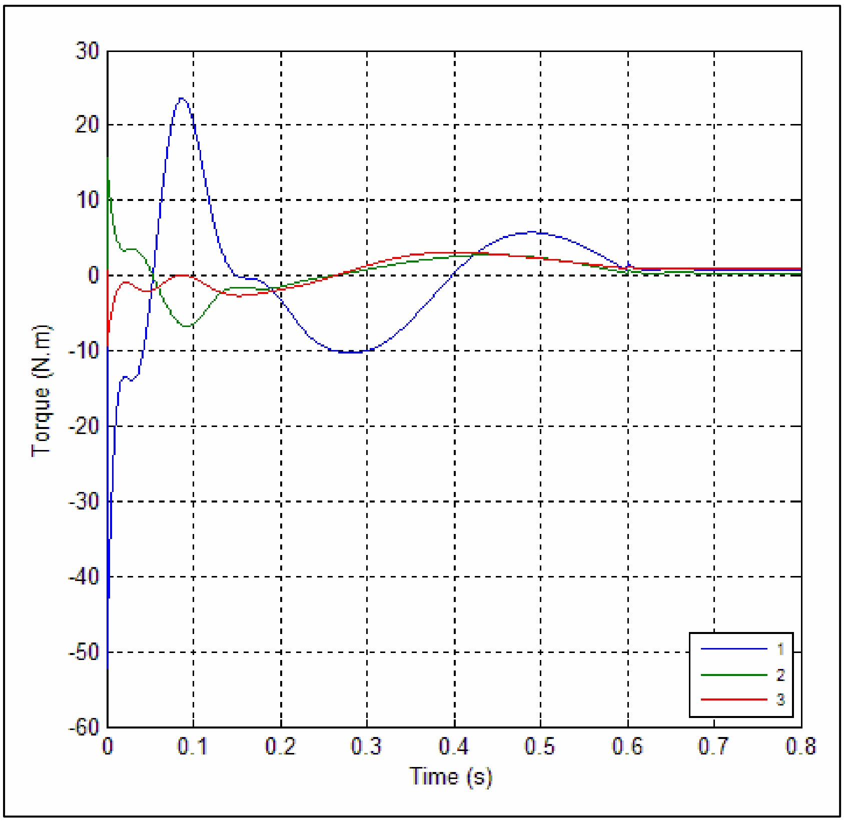

Figure 5.24 Control Torques: 1. T_1^a, 2. T_2^a, 3. T_3^a (CT-1, $\omega_o = 50$ rad/s)

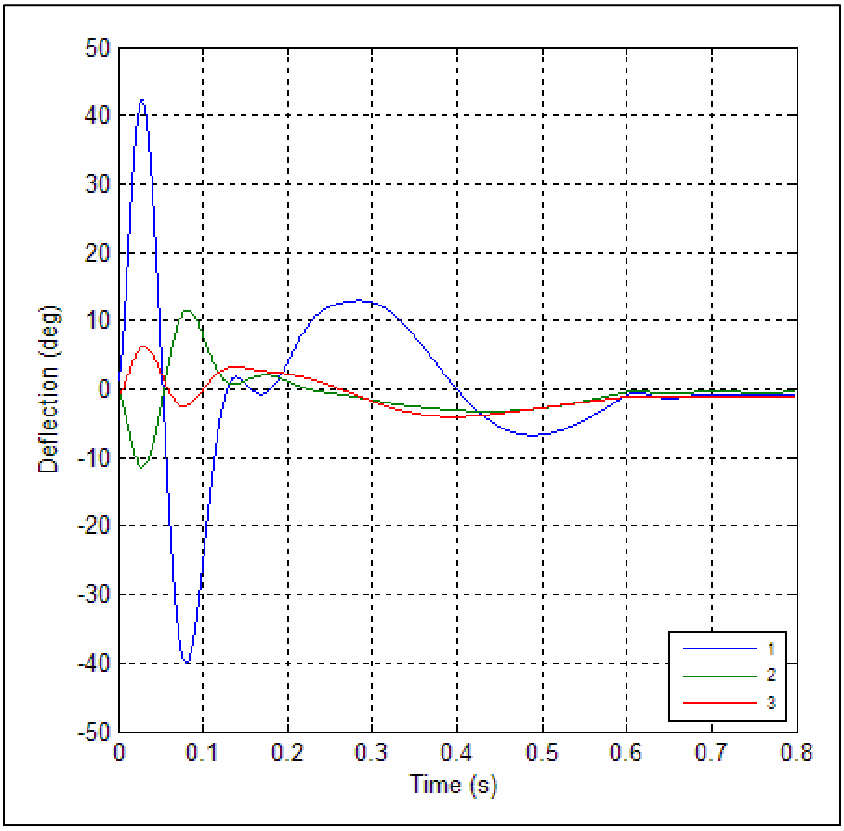

Figure 5.25 Deflections: 1. $\theta_1 - \phi_1$, 2. $\theta_2 - \phi_2$, 3. $\theta_3 - \phi_3$ (CT-1, $\omega_o = 50$ rad/s)

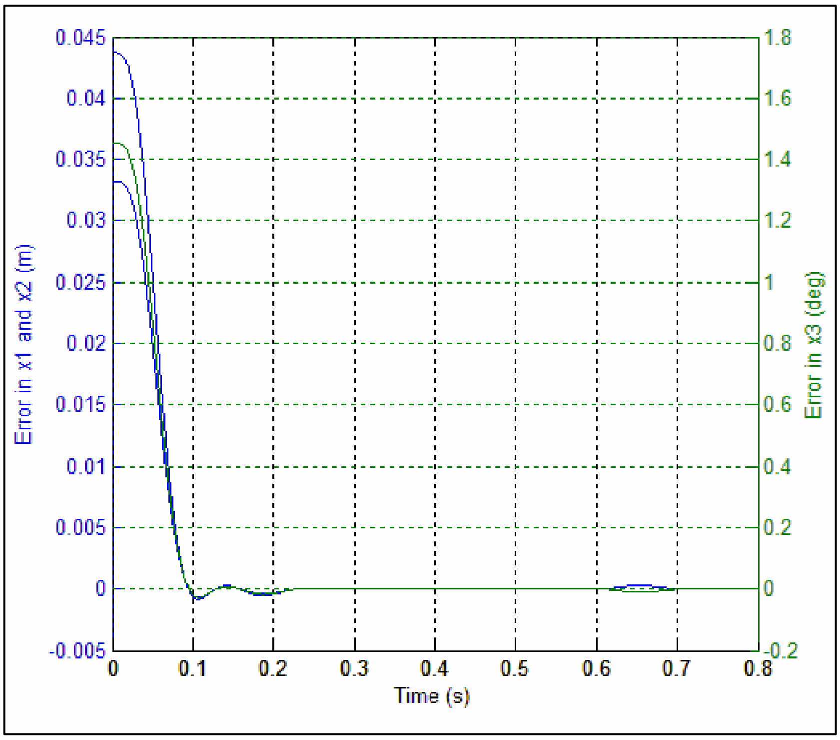

Figure 5.26 Position Errors (CT-1, $\omega_o = 50$ rad/s)

Simulation Group-2

a) $\omega_o = 30$ rad/s

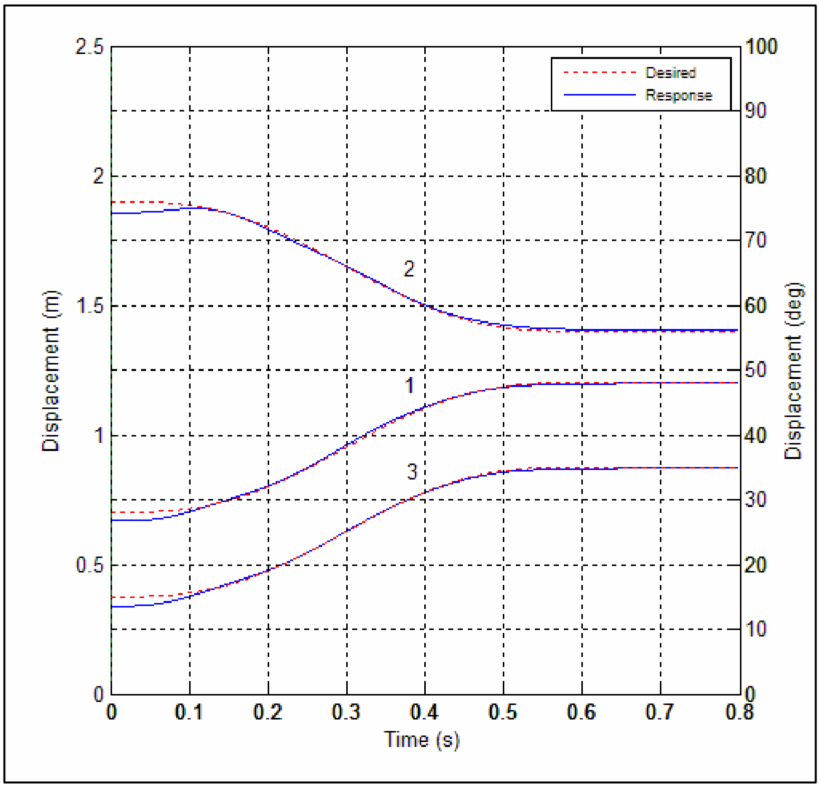

Figure 5.27 Position Response: 1. x_1, 2. x_2, 3. x_3 (CT-2, $\omega_o = 30$ rad/s)

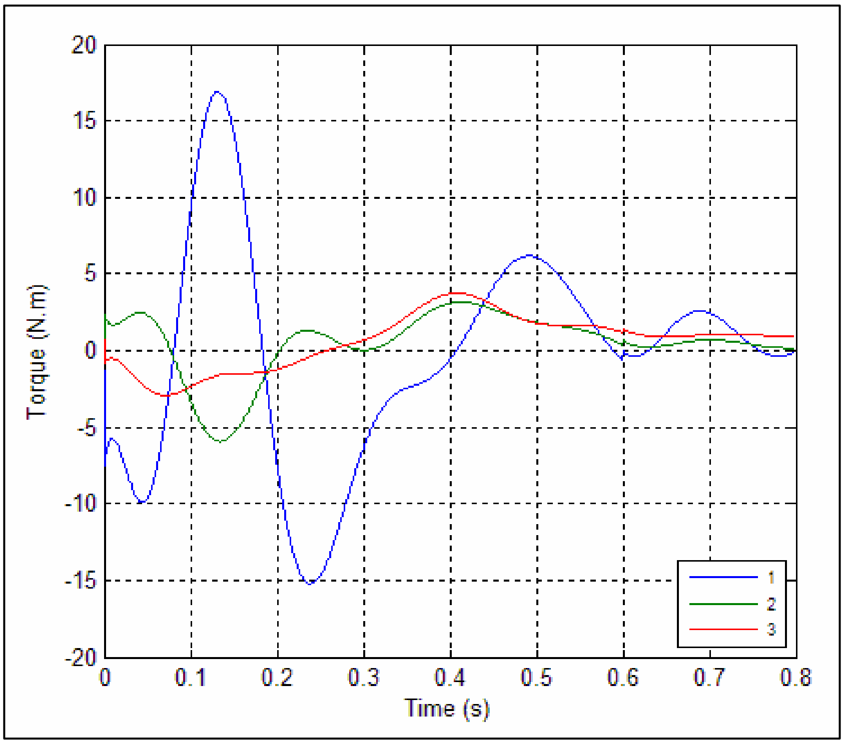

Figure 5.28 Control Torques: 1. T_1^a, 2. T_2^a, 3. T_3^a (CT-2, $\omega_o = 30$ rad/s)

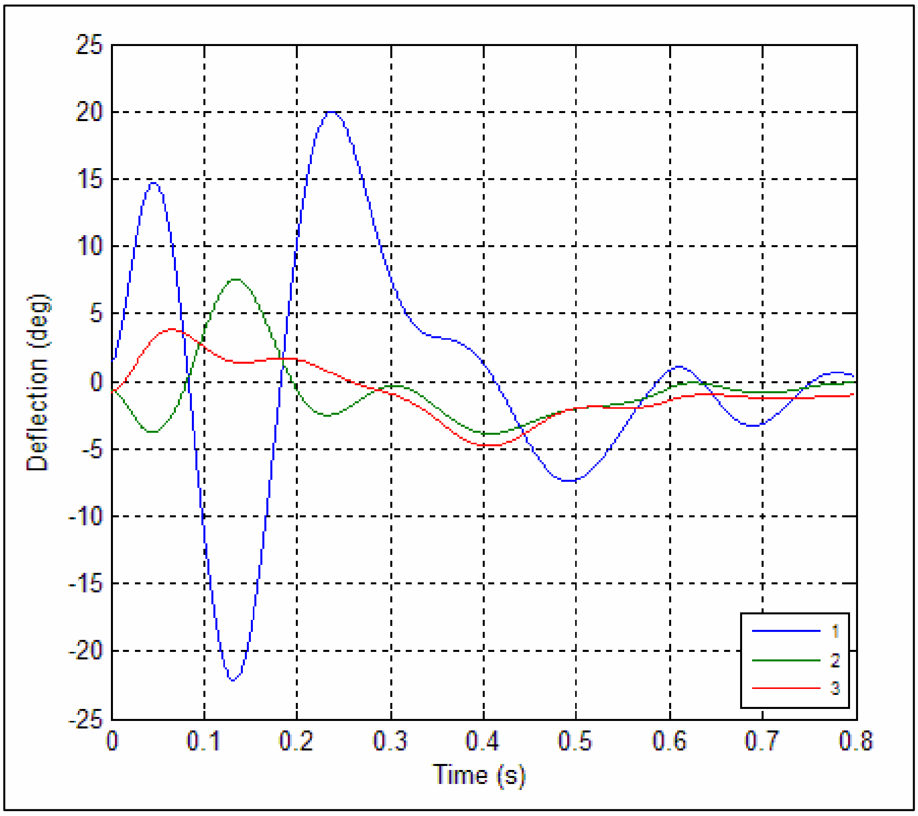

Figure 5.29 Deflections: 1. $\theta_1 - \phi_1$, 2. $\theta_2 - \phi_2$, 3. $\theta_3 - \phi_3$ (CT-2, $\omega_o = 30$ rad/s)

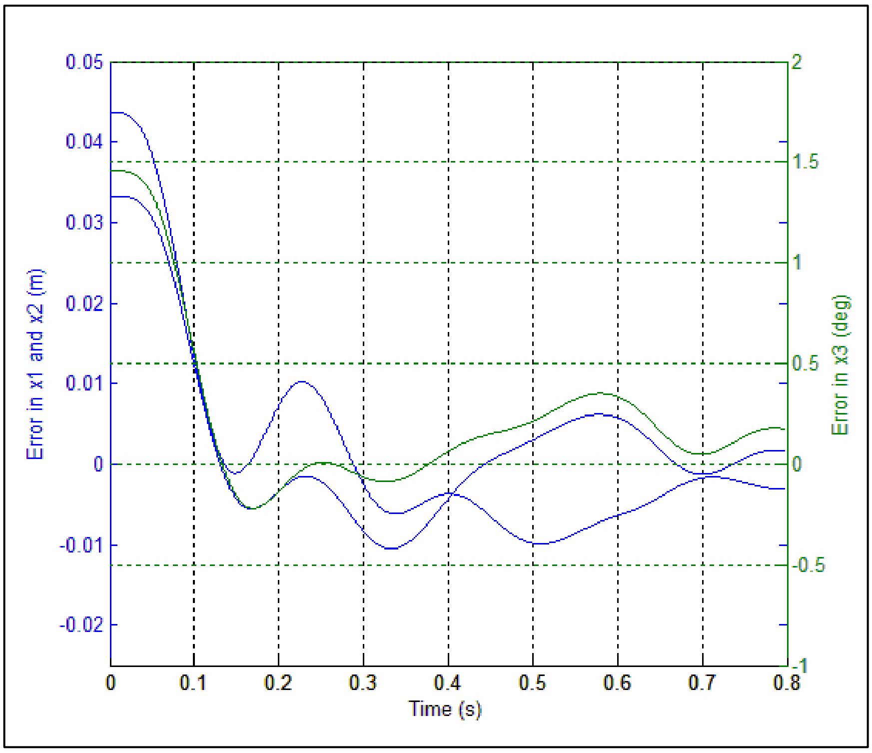

Figure 5.30 Position Errors (CT-2, $\omega_o = 30$ rad/s)

b) $\omega_o = 50$ rad/s

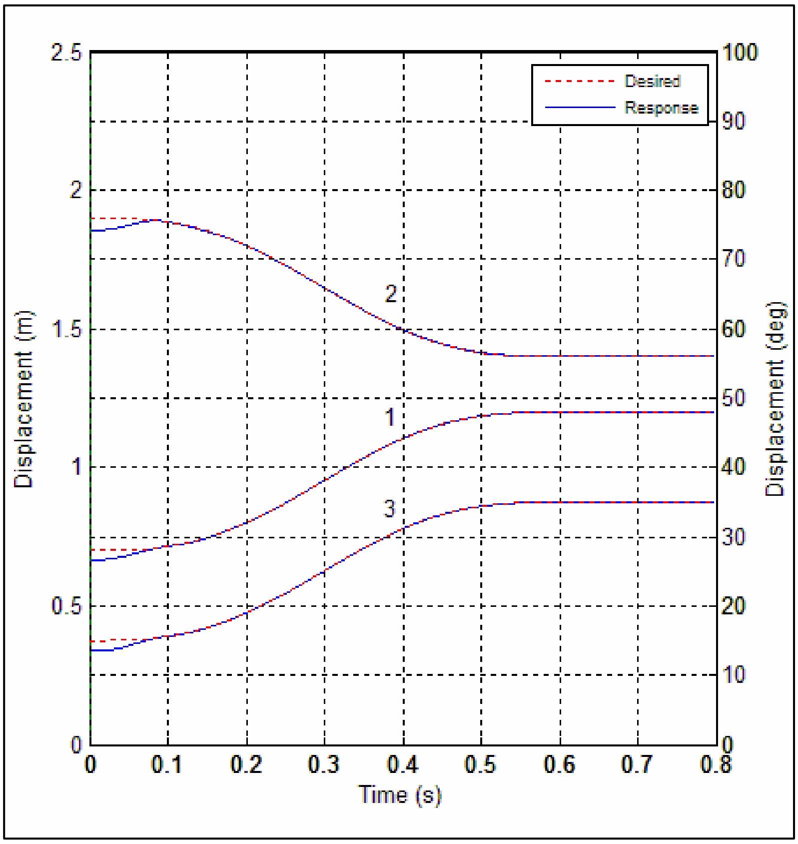

Figure 5.31 Position Response: 1. x_1, 2. x_2, 3. x_3 (CT-2, $\omega_o = 50$ rad/s)

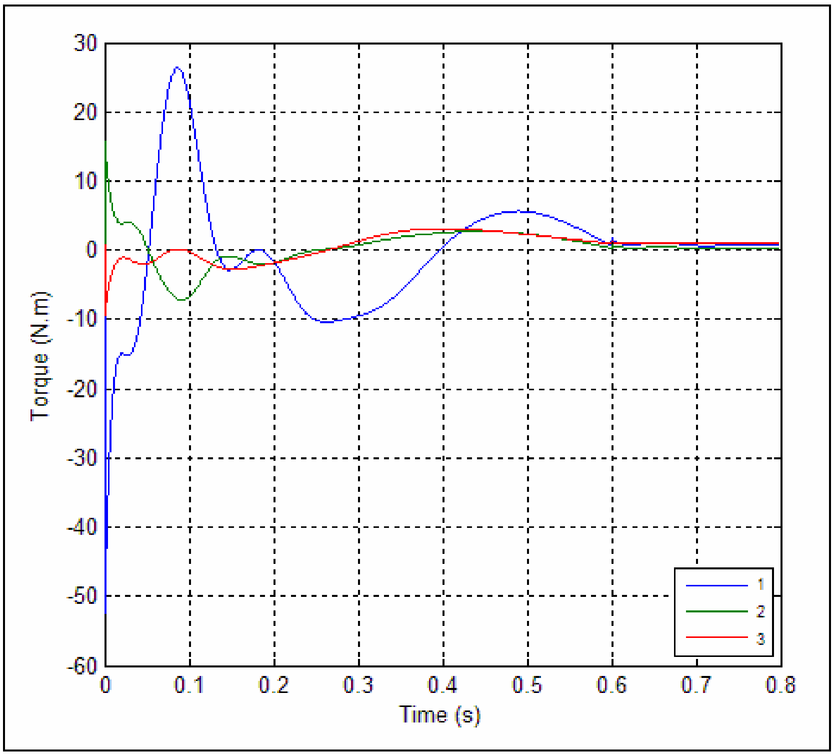

Figure 5.32 Control Torques: 1. T_1^a, 2. T_2^a, 3. T_3^a (CT-2, $\omega_o = 50$ rad/s)

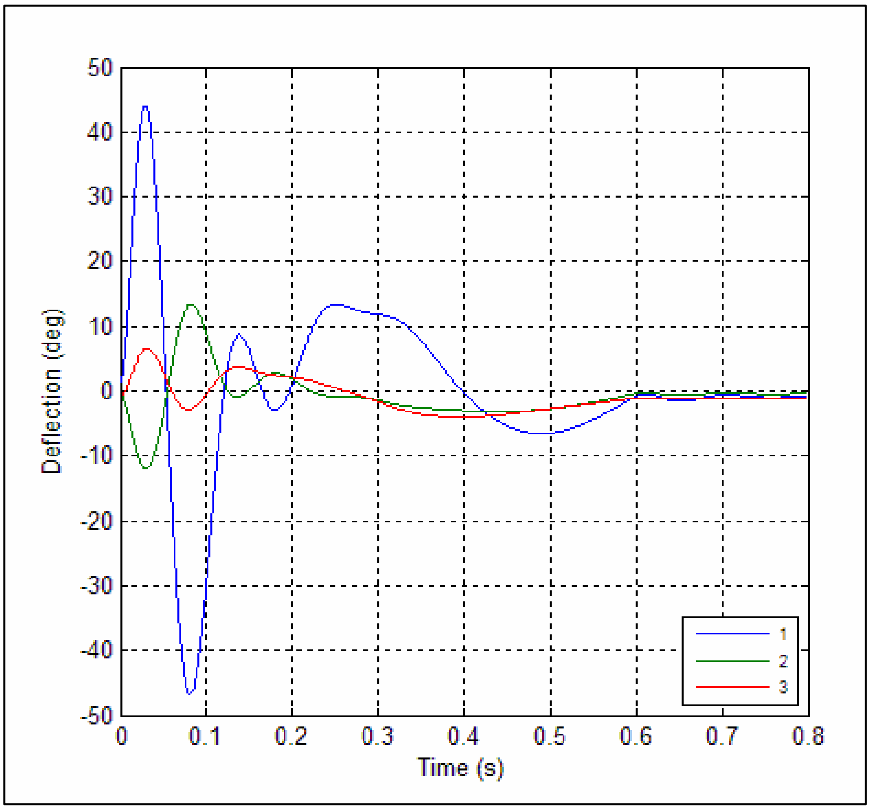

Figure 5.33 Deflections: 1. $\theta_1 - \phi_1$, 2. $\theta_2 - \phi_2$, 3. $\theta_3 - \phi_3$ (CT-2, $\omega_o = 50$ rad/s)

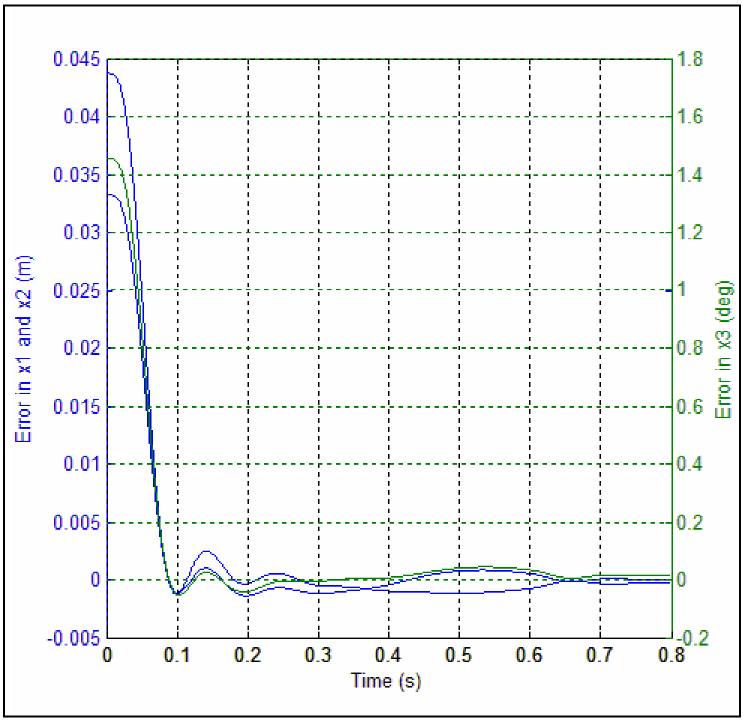

Figure 5.34 Position Errors (CT-2, $\omega_o = 50$ rad/s)

Simulation Group-3

a) $\omega_o = 30$ rad/s

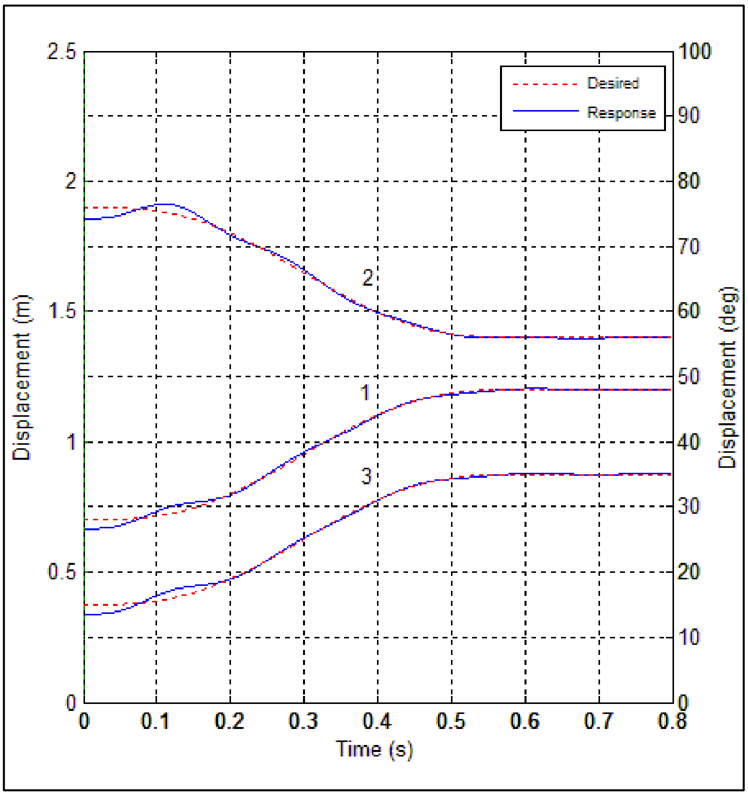

Figure 5.35 Position Response: 1. x_1, 2. x_2, 3. x_3 (CT-3, $\omega_o = 30$ rad/s)

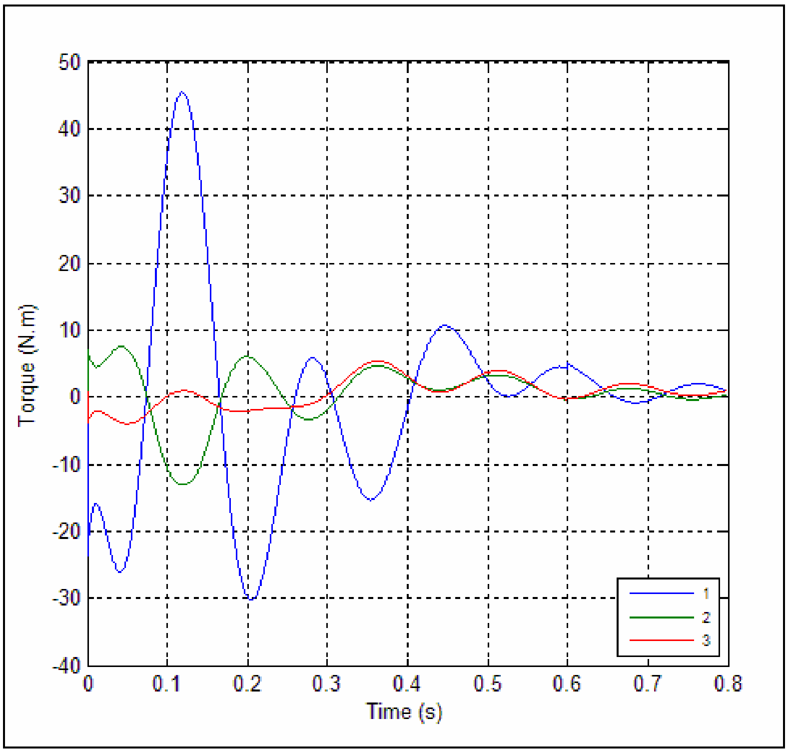

Figure 5.36 Control Torques: 1. T_1^a, 2. T_2^a, 3. T_3^a (CT-3, $\omega_o = 30$ rad/s)

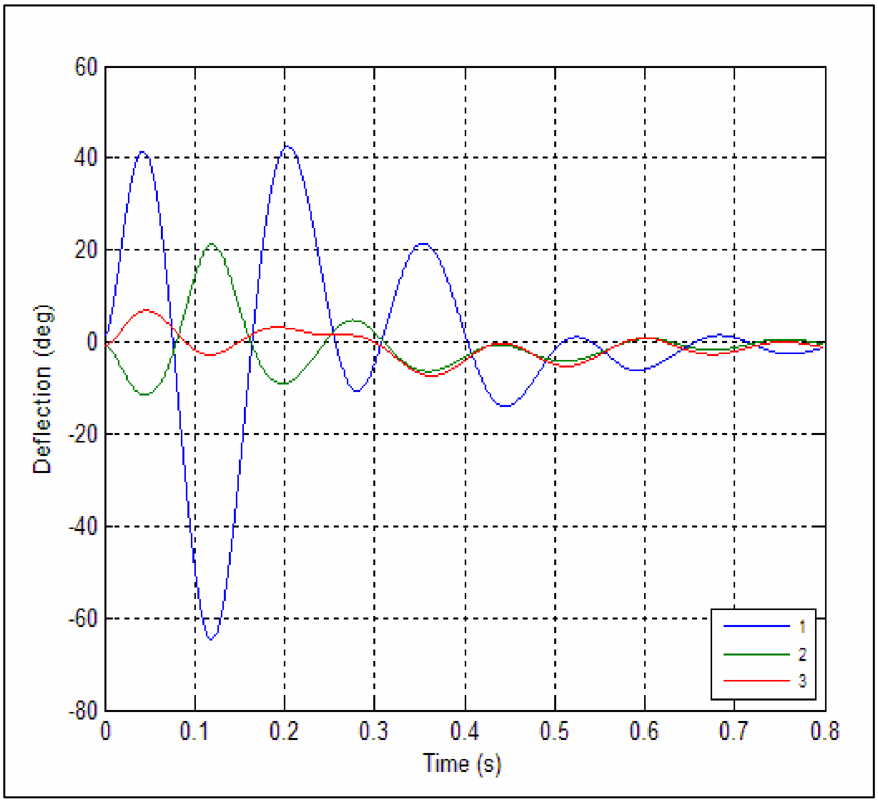

Figure 5.37 Deflections: 1. $\theta_1 - \phi_1$, 2. $\theta_2 - \phi_2$, 3. $\theta_3 - \phi_3$ (CT-3, $\omega_o = 30$ rad/s)

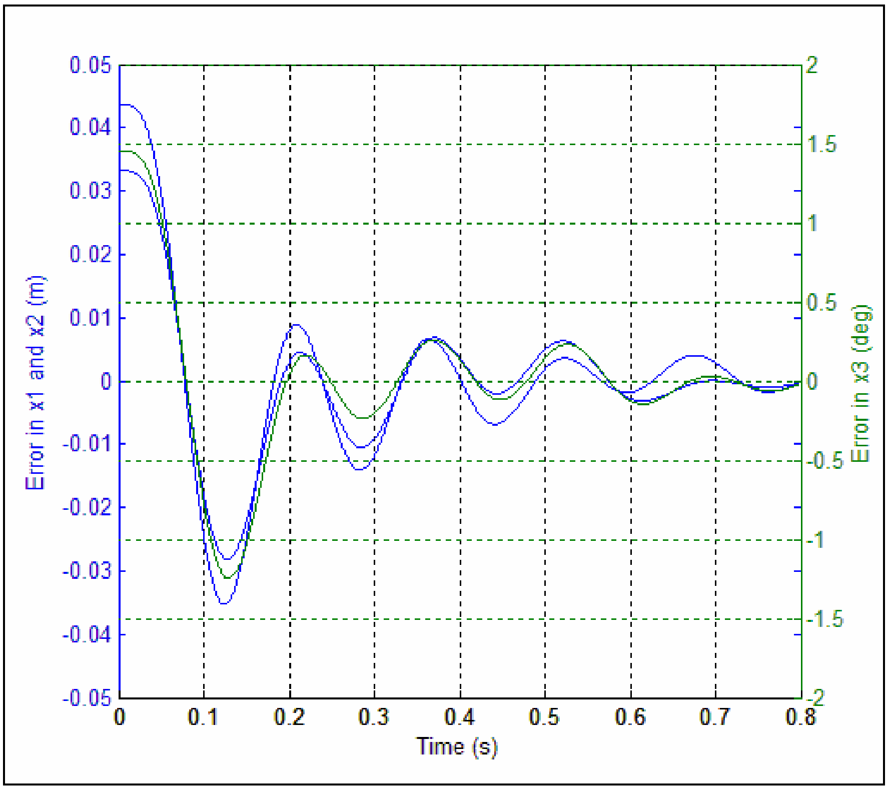

Figure 5.38 Position Errors (CT-3, $\omega_o = 30$ rad/s)

b) $\omega_o = 50$ rad/s

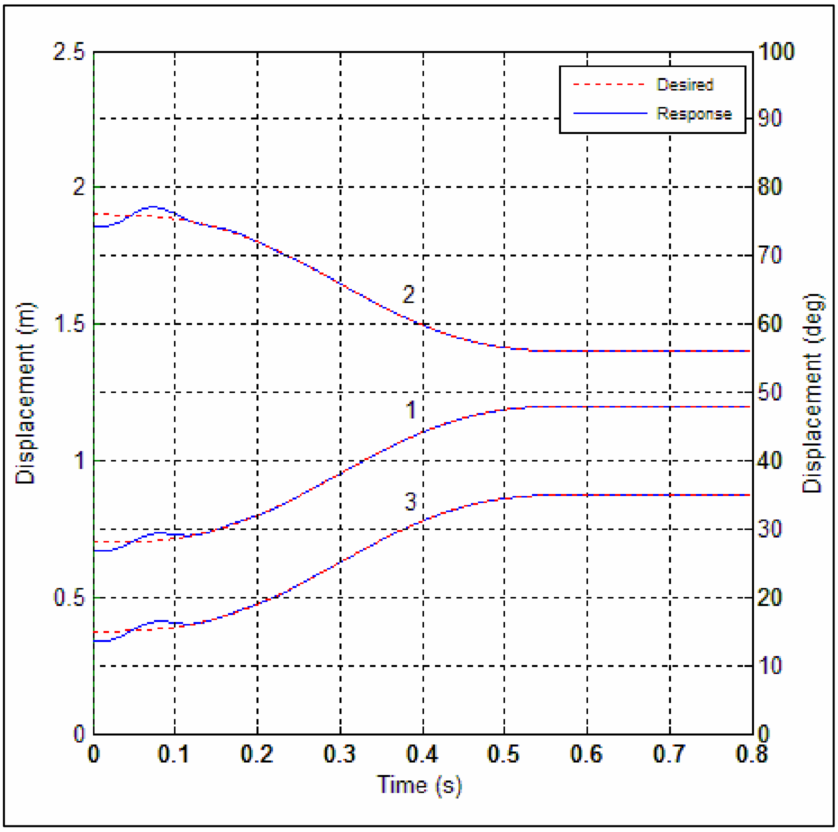

Figure 5.39 Position Response: 1. x_1, 2. x_2, 3. x_3 (CT-3, $\omega_o = 50$ rad/s)

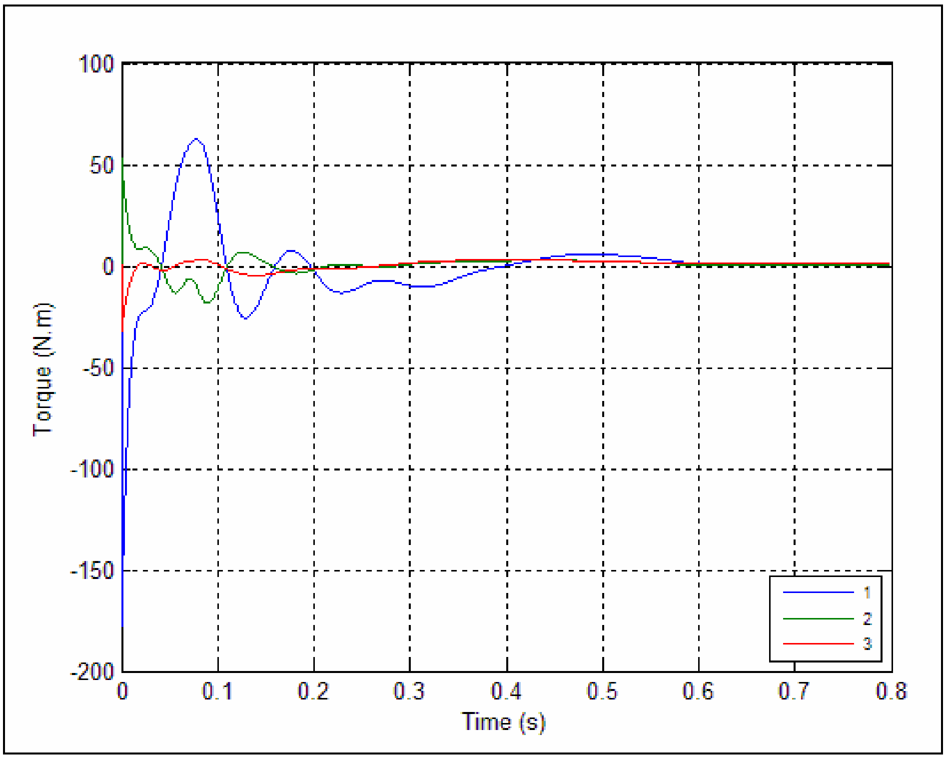

Figure 5.40 Control Torques: 1. T_1^a, 2. T_2^a, 3. T_3^a (CT-3, $\omega_o = 50$ rad/s)

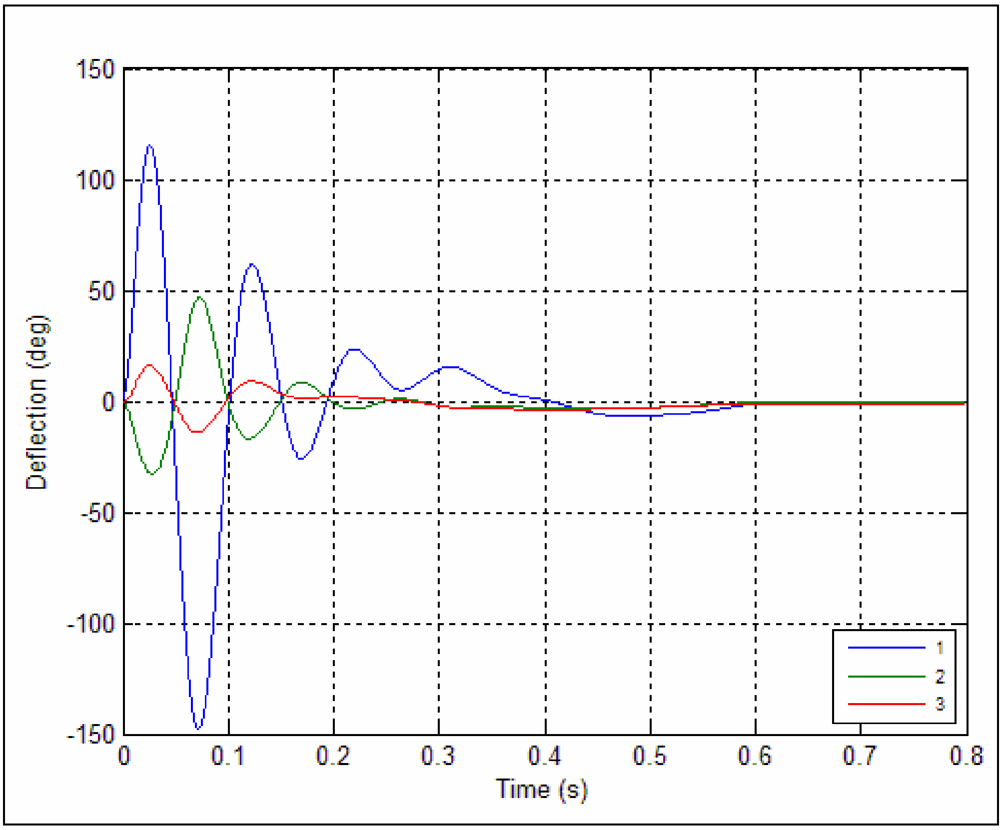

Figure 5.41 Deflections: 1. $\theta_1 - \phi_1$, 2. $\theta_2 - \phi_2$, 3. $\theta_3 - \phi_3$ (CT-3, $\omega_o = 50$ rad/s)

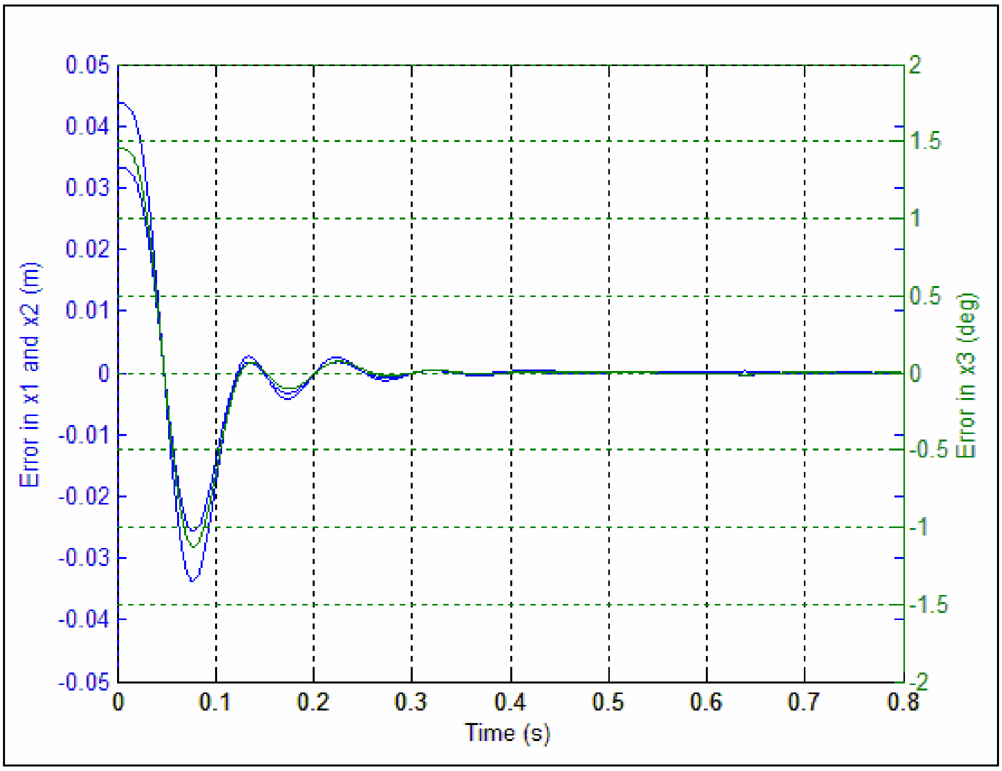

Figure 5.42 Position Errors (CT-3, $\omega_o = 50$ rad/s)

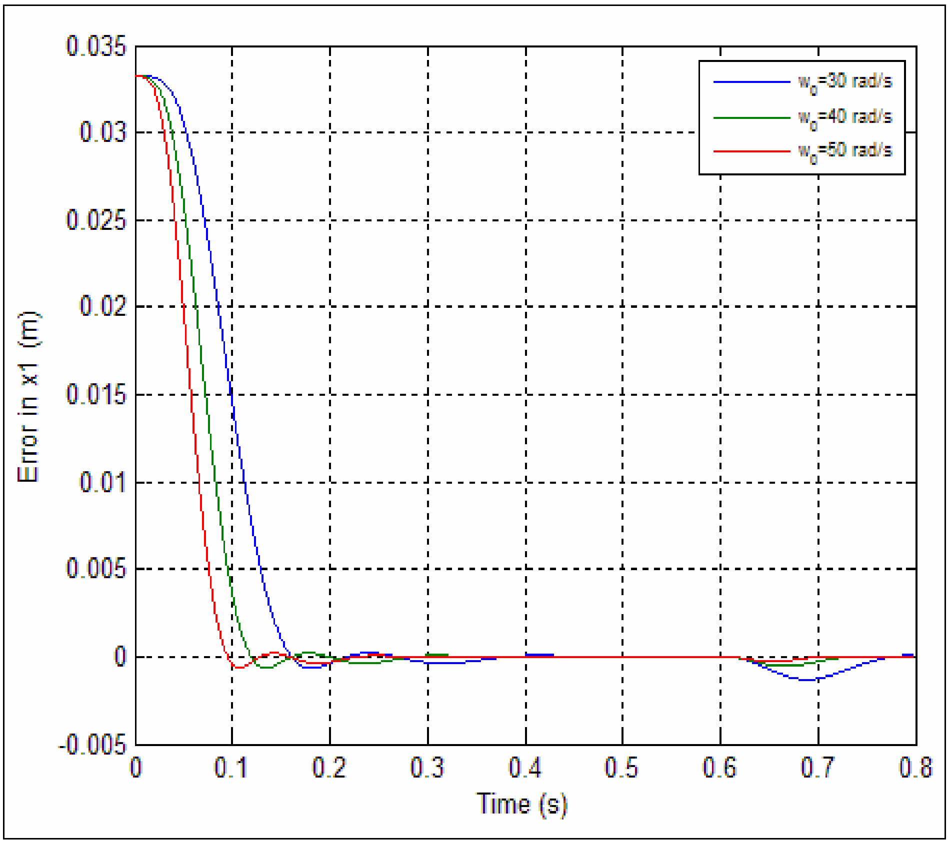

Figure 5.43 Effect of Feedback Gains on x_1 (CT-1)

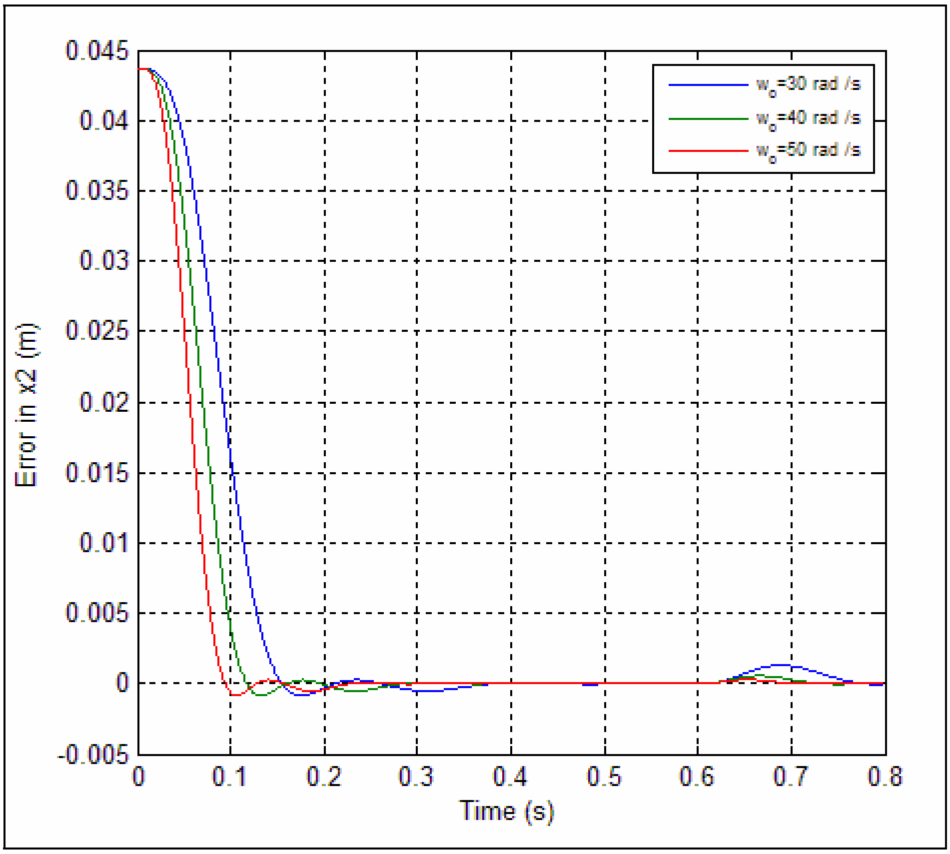

Figure 5.44 Effect of Feedback Gains on x_2 (CT-1)

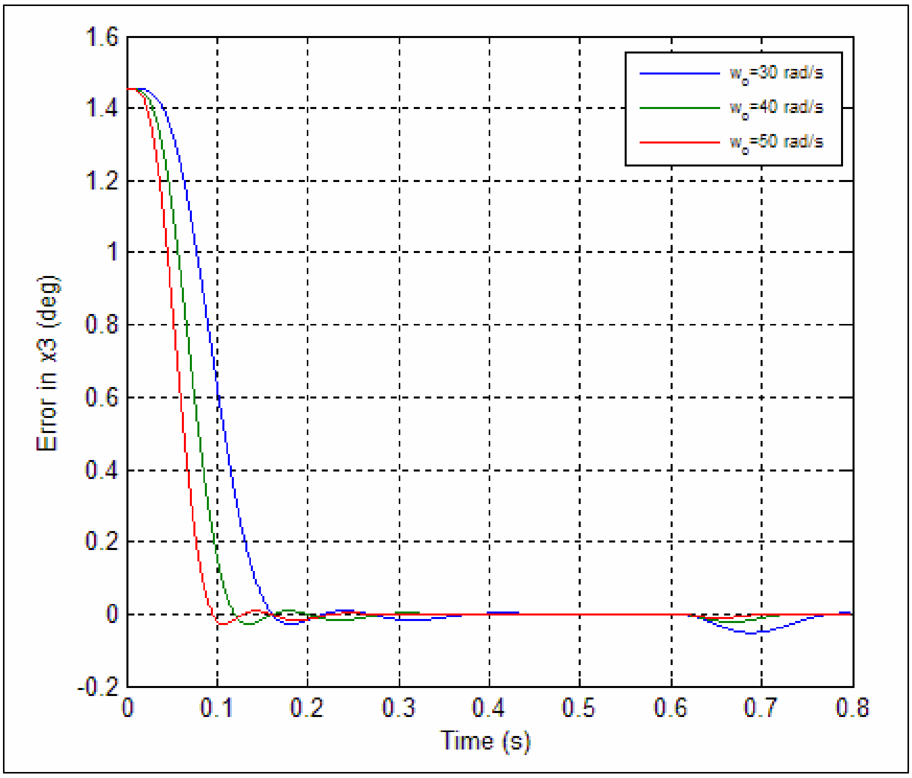

Figure 5.45 Effect of Feedback Gains on x_3 (CT-1)

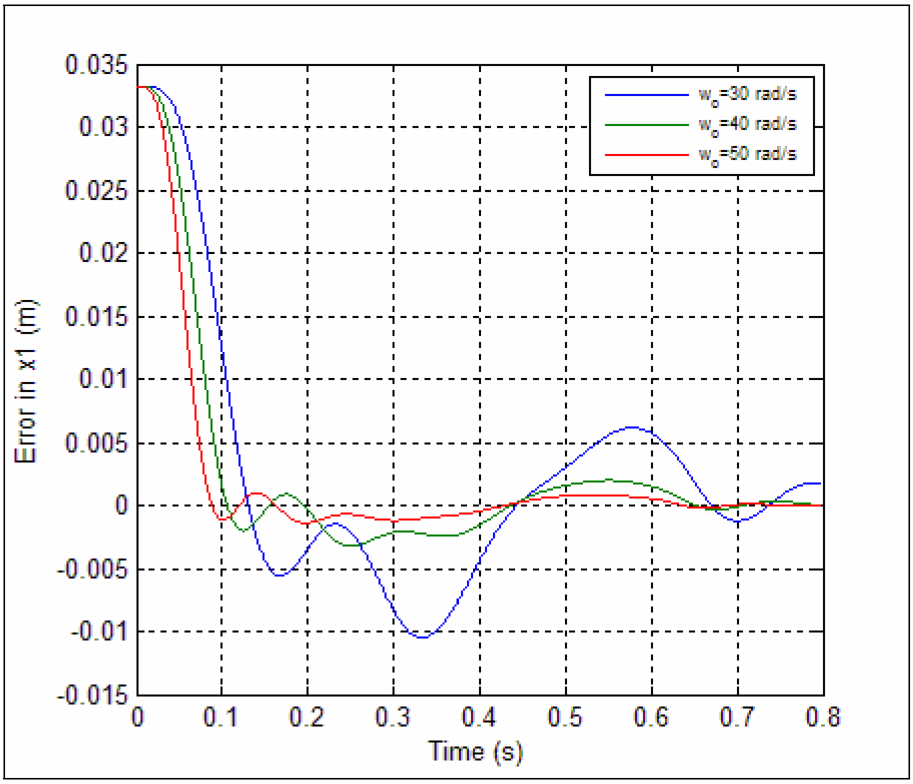

Figure 5.46 Effect of Feedback Gains on x_1 (CT-2)

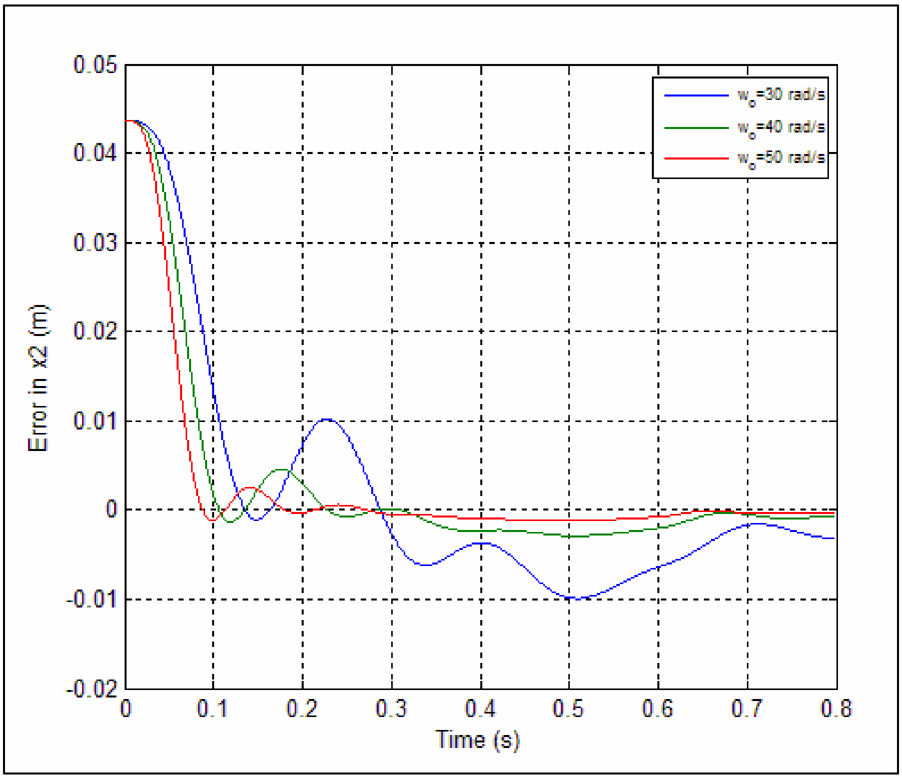

Figure 5.47 Effect of Feedback Gains on x_2 (CT-2)

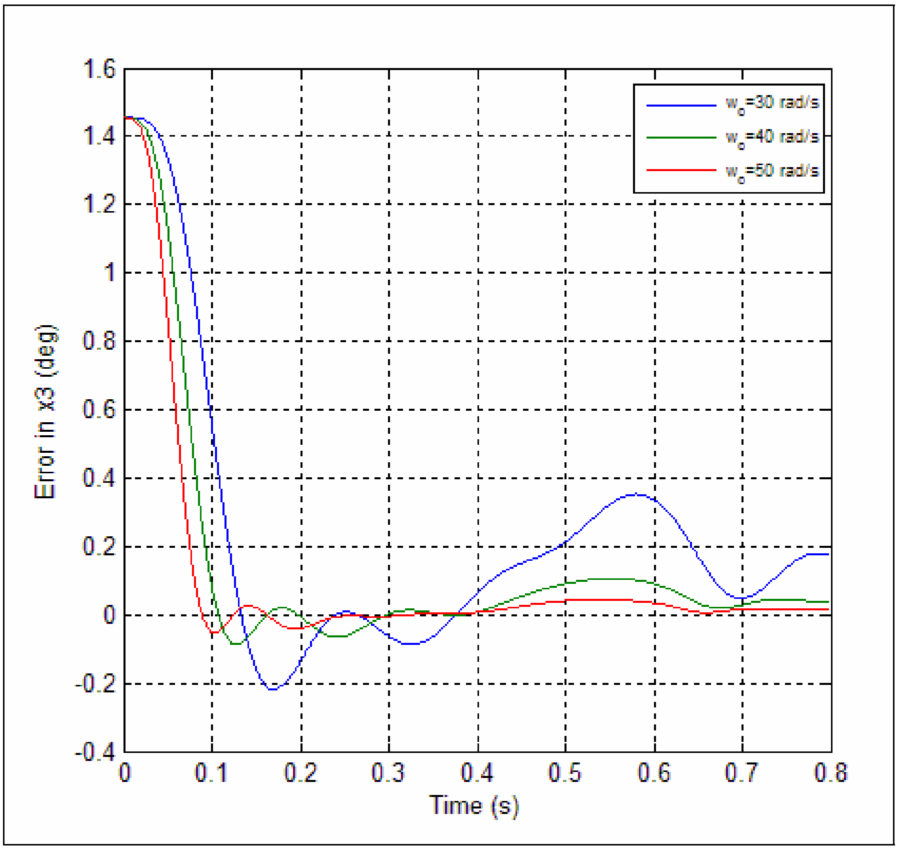

Figure 5.48 Effect of Feedback Gains on x_3 (CT-2)

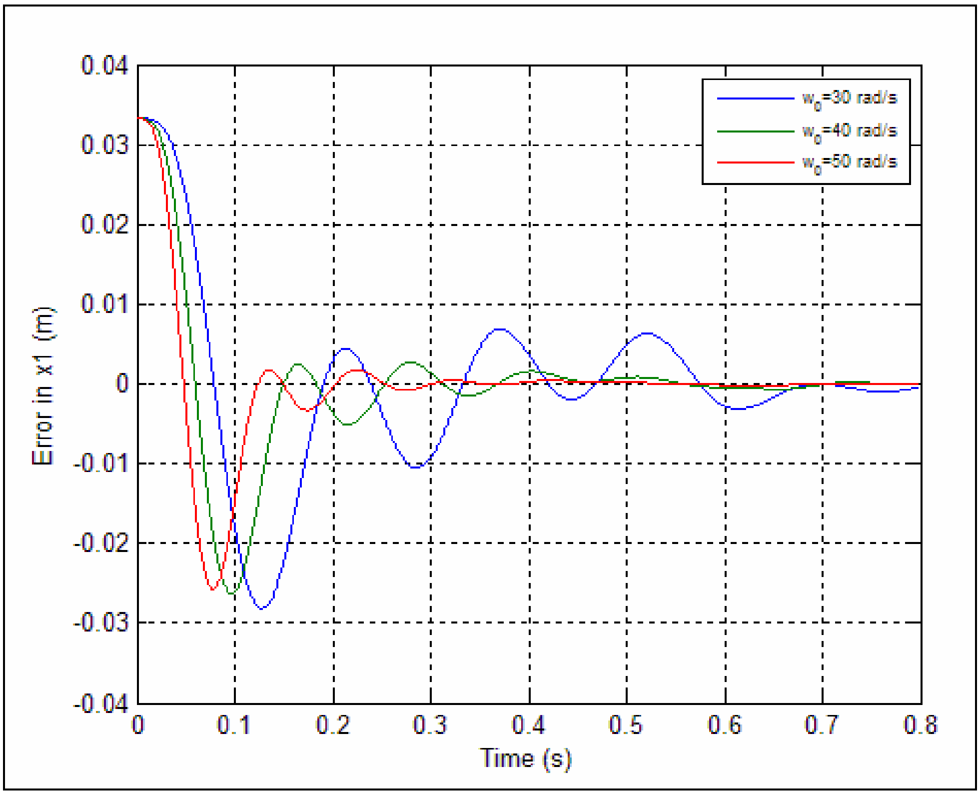

Figure 5.49 Effect of Feedback Gains on x_1 (CT-3)

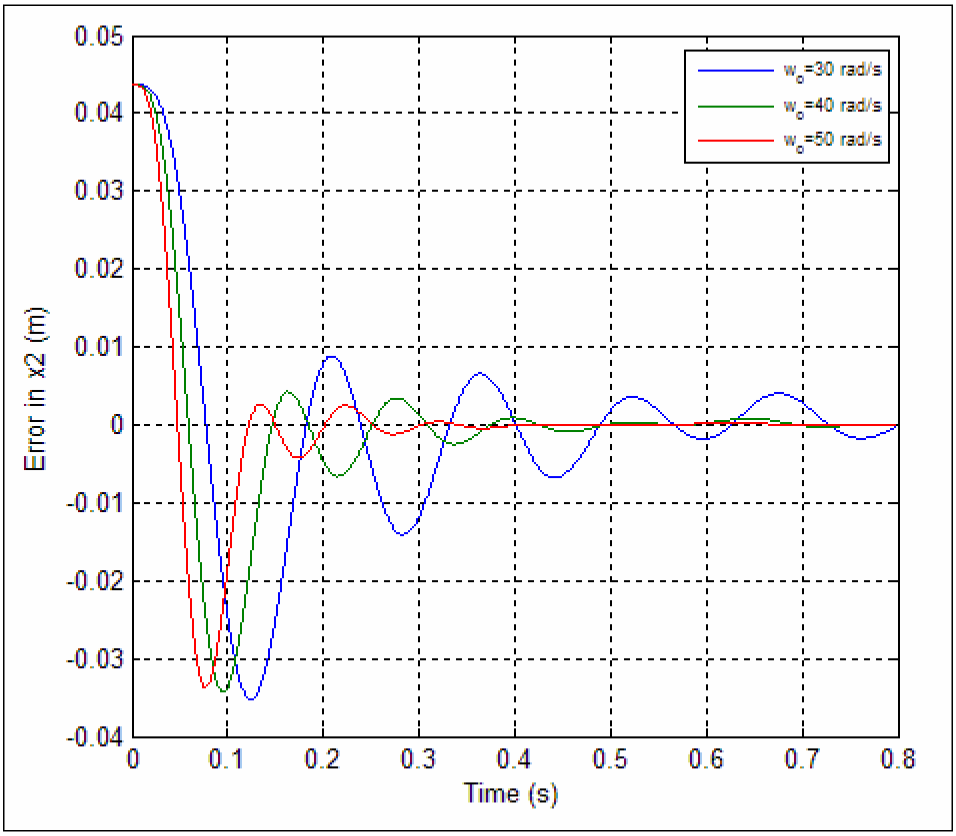

Figure 5.50 Effect of Feedback Gains on x_2 (CT-3)

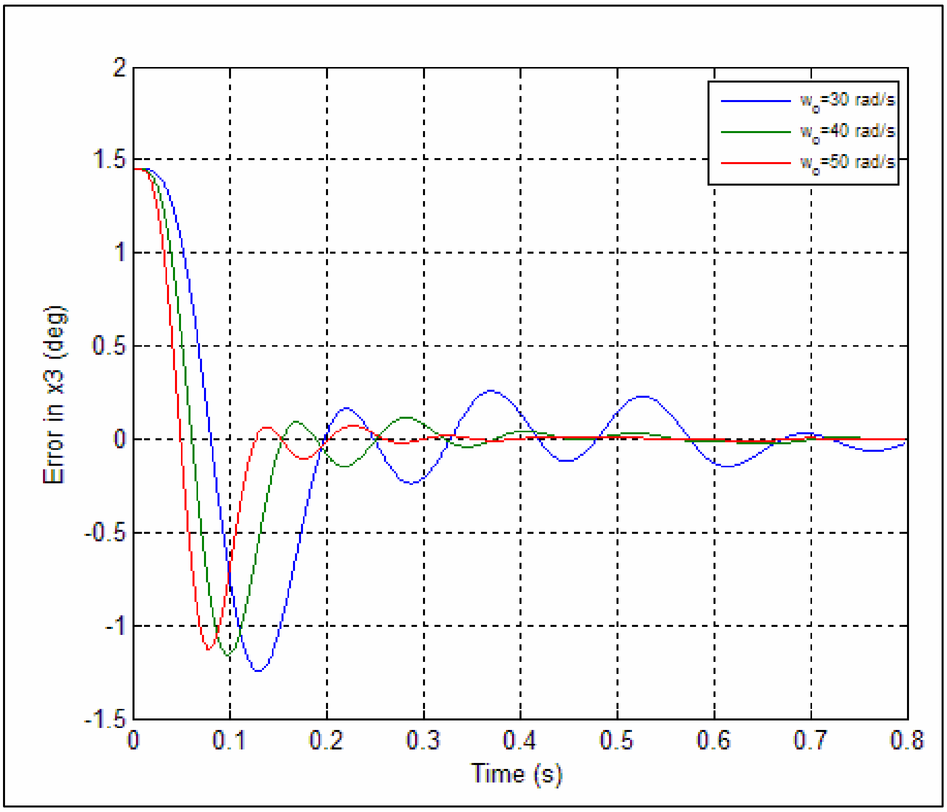

Figure 5.51 Effect of Feedback Gains on x_3 (CT-3)

The results can be tabulated as in Table 5.5 for comparison. This table shows the steady state position errors, maximum torques applied by the actuators and maximum deflections that occur during the motion.

Table 5.5 Simulation Results

CT	Explanation	Steady State Position Error			Maximum Torque Applied by the Actuators (Nm)			Maximum Deflections (deg)		
		x_1 (m)	x_2 (m)	x_3 (deg)	T_1^a	T_2^a	T_3^a	$\theta_1-\phi_1$	$\theta_2-\phi_2$	$\theta_3-\phi_3$
	$\omega_o = 30$ rad/s									
1	with initial error but without modeling error and integral control	-	-	-	12.23	-4.61	3.21	-15.79	5.64	-4.07
2	with initial, model errors but without integral control	4.51E-04	5.24E-04	1.65E-01	16.91	-5.93	3.76	-22.16	7.56	-4.82
3	with initial, model errors and with integral control	8.79E-05	4.65E-05	1.22E-02	45.49	-13.09	5.35	-64.75	21.26	-7.29
	$\omega_o = 50$ rad/s									
1	with initial error but without modeling error and integral control	-	-	-	-52.25	15.72	-9.39	42.13	11.59	6.34
2	with initial, model errors but without integral control	5.45E-05	3.16E-04	1.56E-02	-52.53	15.78	-9.33	-46.79	13.45	6.55
3	with initial, model errors and with integral control	2.63E-05	2.20E-05	1.01E-03	-177.40	53.24	-32.21	-147.90	47.08	16.01

CHAPTER VI

SUMMARY AND CONCLUSIONS

6.1. Summary

This thesis has presented the inverse dynamics algorithm for the position control of parallel manipulators with flexible joints. Joint flexibility is modeled as a torsional spring and the damping characteristics of the actuated joints are considered as the torsional dampers. Rotor damping and viscous frictions at the unactuated joints are also included in the dynamics of the parallel manipulators. Lagrange's equations are used to find the system equations of motion and the unactuated joint variables are eliminated in the set of equations of motion that correspond to the manipulator joint variables.

Since the main idea of the inverse dynamics control algorithm is to seek a control input vector which will linearize and decouple the input/output relation between the control torques and the joint variables, intermediate variables that belong to the actuators are analytically eliminated from the sets of equation of motion. Position control is achieved by the desired end effector snaps and errors in the motion states.

As a case study 2-RRR planar parallel manipulator is considered. Simulations are performed for different conditions depending upon modeling error and/or integral control inclusion. Matlab® and Simulink® are utilized as simulation and technical languages. In the simulations more accurate results are aimed and one of the most complex numerical integration methods called Fourth Order Runge Kutta Method is used to increase the accuracy.

Furthermore, the types of singularities of the parallel manipulators are explained. Analytical expressions that lead to singular configurations are derived and some possible singular positions for the case study are analyzed.

6.2. Discussions and Conclusions

There are three groups of simulations carried out in this thesis. In all of the simulations, the initial error is taken into account.

In the first group of simulations modeling error is not considered and integral terms in the control are not included in the control law. As seen from Figure 5.19, good tracking performances have been obtained. However it is obvious that these initial errors cause larger initial torques and larger tracking errors during the motion. As the ω_o values are increased without altering the simulation conditions, the tracking errors decrease at the cost of increases in torques to be applied by the actuators and elastic deflections as seen in Figures 5.20, 5.21 and 5.43–5.45.

Second group of simulations are performed in the presence of both initial and modeling errors. For this purpose, mass/inertia parameters, spring and damping constants are assumed to be %10 larger in the model. Integral terms are not included in the control law to see the effects of integral control on the system with additional existence of modeling error. As seen from Figures 5.30 and 5.34, the tracking errors and the steady state errors increase considerably. However the elastic deflections and the torques to be supplied by the actuators do not increase significantly as it can easily be seen from Table 5.5. As a general property, the increase in ω_o values provides the decreases in the tracking errors and the steady state errors.

In the final group of simulations, the integral terms are included to the system with initial and modeling error. As a matter of fact, the purpose of utilizing the integral control is to eliminate or at least decrease the tracking and the steady state errors that arise due to modeling error. It is shown in Figures 5.38 and 5.42 that the both tracking and steady state errors are decreased. However tracking errors fluctuate and the initial torques and the initial elastic deflections increase abruptly. This problem is overcome by increasing ω_o values. As ω_o increases the fluctuations in tracking errors decrease and good trajectory tracking performance is achieved in spite of the presence of both initial and modeling errors.

As a result, it is shown that parallel manipulators with significant joint flexibility can follow the specified trajectory with high performance by the proposed control algorithm.

6.3. Future Work

The following studies are strongly recommended.

- The control algorithm can be further extended for the hybrid force and position control of parallel manipulators.

- Control methods for the case when the singular configurations are passed through in the parallel manipulators can be developed.

REFERENCES

[1] **Spong M. W., (1987),** "Modeling and Control of Elastic Joint Robots", Journal of Dynamic Systems, Measurement and Control, Vol. 109, pp. 310-319.

[2] **Forrest-Barlach M. G., Babcock S. M., (1987),** "Inverse Dynamics Position Control of a Compliant Manipulator", IEEE Journal of Robotics and Automation, No.1, Vol. Ra-3, pp. 75-83.

[3] **Jankowski K. P., Van Brussel H., (1992),** "An Approach to Discrete Inverse Dynamics Control of Flexible Joint Robots", IEEE Transactions on Robotics and Automation, No.5, Vol. 8, pp. 651-658.

[4] **İder K. and Özgören K., (2000),** "Trajectory Tracking Control of Flexible Joint Robots", Computers and Structures, Vol. 76, pp. 757-763.

[5] **Dado M. H., Al-Huniti N. S., and Eljabali A. K., (2001),** "Dynamic Simulation Model for Mixed-loop Planar Robots with Flexible Joint Drives", Mechanism and Machine Theory, Vol. 36, pp. 547-559.

[6] **İder K., (2005),** "Inverse Dynamics of Parallel Manipulators in the Presence of Drive Singularities", Mechanism and Machine Theory, Vol. 40, pp. 33-44.

[7] **Ji Z., (2003),** "Study of Planar Three-degree-of-freedom 2-RRR Parallel Manipulators", Mechanism and Machine Theory, Vol. 38, pp. 409-416.

[8] **Liu X., Tang X., and Wang J., (2004),** "Singularity Analysis of a New Parallel Manipulator with Revolute Actuators", Proceedings of the 11th World Congress in Mechanism and Machine Science, Tianjin, China.

[9] **Rivin E. I., (1984),** "Compliance Breakdown for Robot Structures", Proceedings of the Symposium Engineering Applied Mechanics, Toronto.

[10] **İder K., (2004),** "ME528 Flexible Multibody Dynamics Lecture Notes", Mechanical Engineering Department, Middle East Technical University, Ankara, Turkey.

[11] **Özgören K., (2005),** "ME502 Advanced Dynamics Lecture Notes", Mechanical Engineering Department, Middle East Technical University, Ankara, Turkey.

[12] **Özgören K., (2005),** "ME522 Principles of Robotics Lecture Notes", Mechanical Engineering Department, Middle East Technical University, Ankara, Turkey.

[13] **Erdağı, İ., (1995)**, "Trajectory Tracking Control of Robotic Manipulators", M.S. Thesis, Mechanical Engineering Department, Middle East Technical University, Ankara, Turkey.

[14] **İyigün, H. H., (2001)**, "Inverse Dynamics Control of Constrained Flexible Joint Robots", M.S. Thesis, Mechanical Engineering Department, Middle East Technical University, Ankara, Turkey.

[15] **Gardner J. F., (2001),** "Simulations of Machines", Brooks/Cole, California, USA

[16] **Cavallo A. Setola R. and Vasca F., (1996),** "Using Matlab, Simulink and Control Toolbox : a Practical Approach", Prentice Hall, New York, USA.

[17] **Zlajpah L., (1998),** "Simulation of n-R Planar Manipulators", Simulation Practice and Theory, Vol. 6, pp. 305-321.

[18] **The Mathworks Inc., (1999),** "Matlab Simulink User's Guide".

APPENDIX A

TIME DERIVATIVES OF MATRICES

A.1 The First and Second Time Derivatives of Matrix $\hat{M}$

$$\dot{M}_{11} = -(m_3^{\ L} + 2m_5^{\ L})L_1L_3 s\theta_3\dot{\theta}_3 - 2m_5^{\ L}L_1g_5 s(\theta_{35}+\beta)\dot{\theta}_{35} - 2m_5^{\ L}L_3g_5 s(\theta_5+\beta)\dot{\theta}_5$$

$$\dot{M}_{13} = -\left(\frac{1}{2}m_3^{\ L} + m_5^{\ L}\right)L_1L_3 s\theta_3\dot{\theta}_3 - m_5^{\ L}L_1g_5 s(\theta_{35}+\beta)\dot{\theta}_{35} - 2m_5^{\ L}L_3g_5 s(\theta_5+\beta)\dot{\theta}_5$$

$$\dot{M}_{15} = -m_5^{\ L}L_1g_5 s(\theta_{35}+\beta)\dot{\theta}_{35} - m_5^{\ L}L_3g_5 s(\theta_5+\beta)\dot{\theta}_5$$

$$\dot{M}_{22} = -m_4^{\ L}L_2L_4 s\theta_4\dot{\theta}_4$$

$$\dot{M}_{24} = -\frac{1}{2}m_4^{\ L}L_2L_4 s\theta_4\dot{\theta}_4$$

$$\dot{M}_{33} = -2m_5^{\ L}L_3g_5 s(\theta_5+\beta)\dot{\theta}_5$$

$$\dot{M}_{35} = -m_5^{\ L}L_3g_5 s(\theta_5+\beta)\dot{\theta}_5$$

$$\dot{M}_{44} = 0$$

$$\dot{M}_{55} = 0$$

$$\ddot{M}_{11} = -(m_3^{\ L} + 2m_5^{\ L})L_1L_3(c\theta_3\dot{\theta}_3^{\,2} + s\theta_3\ddot{\theta}_3) - 2m_5^{\ L}L_1g_5\left[c(\theta_{35}+\beta)\dot{\theta}_{35}^{\,2} + s(\theta_{35}+\beta)\ddot{\theta}_{35}\right]$$
$$-2m_5^{\ L}L_3g_5\left[c(\theta_5+\beta)\dot{\theta}_5^{\,2} + s(\theta_5+\beta)\ddot{\theta}_5\right]$$

$$\ddot{M}_{13} = -\left(\frac{1}{2}m_3{}^L + m_5{}^L\right)L_1L_3(c\theta_3\dot{\theta}_3{}^2 + s\theta_3\ddot{\theta}_3) - m_5{}^L L_1 g_5\left[c(\theta_{35}+\beta)\dot{\theta}_{35}{}^2 + s(\theta_{35}+\beta)\ddot{\theta}_{35}\right]$$
$$-2m_5{}^L L_3 g_5\left[c(\theta_5+\beta)\dot{\theta}_5{}^2 + s(\theta_5+\beta)\ddot{\theta}_5\right]$$

$$\ddot{M}_{15} = -m_5{}^L L_1 g_5\left[c(\theta_{35}+\beta)\dot{\theta}_{35}{}^2 + s(\theta_{35}+\beta)\ddot{\theta}_{35}\right] - m_5{}^L L_3 g_5\left[c(\theta_5+\beta)\dot{\theta}_5{}^2 + s(\theta_5+\beta)\ddot{\theta}_5\right]$$

$$\ddot{M}_{22} = -m_4{}^L L_2 L_4(c\theta_4\dot{\theta}_4{}^2 + s\theta_4\ddot{\theta}_4)$$

$$\ddot{M}_{24} = -\frac{1}{2}m_4{}^L L_2 L_4(c\theta_4\dot{\theta}_4{}^2 + s\theta_4\ddot{\theta}_4)$$

$$\ddot{M}_{33} = -2m_5{}^L L_3 g_5\left[c(\theta_5+\beta)\dot{\theta}_5{}^2 + s(\theta_5+\beta)\ddot{\theta}_5\right]$$

$$\ddot{M}_{35} = -m_5{}^L L_3 g_5\left[c(\theta_5+\beta)\dot{\theta}_5{}^2 + s(\theta_5+\beta)\ddot{\theta}_5\right]$$

$$\ddot{M}_{44} = 0$$

$$\ddot{M}_{55} = 0$$

A.2 The First and Second Time Derivatives of Vector $\bar{Q}$

$$\dot{Q}_1 = -m_1{}^L g\left(\frac{L_1}{2}s\theta_1\dot{\theta}_1\right) - m_3{}^L g\left(L_1 s\theta_1\dot{\theta}_1 + \frac{L_3}{2}s\theta_{13}\dot{\theta}_{13}\right) - m_5{}^L g\left[L_1 s\theta_1\dot{\theta}_1 + L_3 s\theta_{13}\dot{\theta}_{13}\right.$$
$$\left.+g_5 s(\theta_{135}+\beta)\dot{\theta}_{135}\right] - m_3{}^A g(L_1 s\theta_1\dot{\theta}_1)$$

$$\dot{Q}_2 = -m_2{}^L g\frac{L_2}{2}(s\theta_2\dot{\theta}_2) - m_4{}^L g\left[L_2(s\theta_2\dot{\theta}_2) + \frac{L_4}{2}(s\theta_{24}\dot{\theta}_{24})\right]$$

$$\dot{Q}_3 = \dot{\theta}_1\ddot{\theta}_1\left[(m_3{}^L + 2m_5{}^L)L_1L_3s\theta_3 + 2m_5{}^L L_1 g_5 s(\theta_{35}+\beta)\right] + \frac{1}{2}\dot{\theta}_1{}^2\left[(m_3{}^L + 2m_5{}^L)L_1L_3c\theta_3\dot{\theta}_3\right.$$

$$\left. + 2m_5{}^L L_1 g_5 c(\theta_{35}+\beta)\dot{\theta}_{35}\right] + (\ddot{\theta}_1\dot{\theta}_3 + \dot{\theta}_1\ddot{\theta}_3)\left[(\frac{1}{2}m_3{}^L + m_5{}^L)L_1L_3s\theta_3 + m_5{}^L L_1 g_5 s(\theta_{35}+\beta)\right]$$

$$+\dot{\theta}_1\dot{\theta}_3\left[(\frac{1}{2}m_3{}^L + m_5{}^L)L_1L_3c\theta_3\dot{\theta}_3 + m_5{}^L L_1 g_5 c(\theta_{35}+\beta)\dot{\theta}_{35}\right] + (\ddot{\theta}_1\dot{\theta}_5 + \dot{\theta}_1\ddot{\theta}_5)m_5{}^L L_1 g_5 s(\theta_{35}+\beta)$$

$$+\dot{\theta}_1\dot{\theta}_5\left[m_5{}^L L_1 g_5 c(\theta_{35}+\beta)\dot{\theta}_{35}\right] - m_3{}^L g\frac{L_3}{2}s\theta_{13}\dot{\theta}_{13} - m_5{}^L g\left[L_3 s\theta_{13}\dot{\theta}_{13} + g_5 s(\theta_{135}+\beta)\dot{\theta}_{135}\right]$$

$$\dot{Q}_4 = m_4{}^L L_2L_4(\dot{\theta}_2\ddot{\theta}_2 s\theta_4 + \frac{1}{2}\dot{\theta}_2{}^2 c\theta_4\dot{\theta}_4) + \frac{1}{2}m_4{}^L L_2L_4\left[\ddot{\theta}_2(\dot{\theta}_4 s\theta_4) + \dot{\theta}_2(\ddot{\theta}_4 s\theta_4 + \dot{\theta}_4{}^2 c\theta_4)\right]$$

$$-m_2{}^L g\frac{L_2}{2}s\theta_2\dot{\theta}_2 - m_4{}^L g(L_2 s\theta_2\dot{\theta}_2 + \frac{L_4}{2}s\theta_{24}\dot{\theta}_{24})$$

$$\dot{Q}_5 = 2\dot{\theta}_1\ddot{\theta}_1\left[m_5{}^L L_1 g_5 s(\theta_{35}+\beta) + m_5{}^L L_3 g_5 s(\theta_5+\beta)\right] + \dot{\theta}_1{}^2\left[m_5{}^L L_1 g_5 c(\theta_{35}+\beta)\dot{\theta}_{35}\right.$$

$$\left.+m_5{}^L L_3 g_5 c(\theta_5+\beta)\dot{\theta}_5\right] + 2\dot{\theta}_3\ddot{\theta}_3 m_5{}^L L_3 g_5 s(\theta_5+\beta) + \dot{\theta}_3{}^2 m_5{}^L L_3 g_5 c(\theta_5+\beta)\dot{\theta}_5 + (\ddot{\theta}_1\dot{\theta}_3 + \dot{\theta}_1\ddot{\theta}_3)$$

$$\left[m_5{}^L L_1 g_5 s(\theta_{35}+\beta) + 2m_5{}^L L_3 g_5 s(\theta_5+\beta)\right] + \dot{\theta}_1\dot{\theta}_3\left[m_5{}^L L_1 g_5 c(\theta_{35}+\beta)\dot{\theta}_{35}\right.$$

$$\left.+2m_5{}^L L_3 g_5 c(\theta_5+\beta)\dot{\theta}_5\right] + (\ddot{\theta}_1\dot{\theta}_5 + \dot{\theta}_1\ddot{\theta}_5)\left[m_5{}^L L_1 g_5 s(\theta_{35}+\beta) + m_5{}^L L_3 g_5 s(\theta_5+\beta)\right]$$

$$+\dot{\theta}_1\dot{\theta}_5\left[m_5{}^L L_1 g_5 c(\theta_{35}+\beta)\dot{\theta}_{35} + m_5{}^L L_3 g_5 c(\theta_5+\beta)\dot{\theta}_5\right] + (\ddot{\theta}_3\dot{\theta}_5 + \dot{\theta}_3\ddot{\theta}_5)m_5{}^L L_3 g_5 s(\theta_5+\beta)$$

$$+\dot{\theta}_3\dot{\theta}_5{}^2 m_5{}^L L_3 g_5 c(\theta_5+\beta) - m_5{}^L g\left[g_5 s(\theta_{135}+\beta)\dot{\theta}_{135}\right]$$

$$\ddot{Q}_1 = -m_1{}^L g\frac{L_1}{2}(c\theta_1\dot{\theta}_1{}^2 + s\theta_1\ddot{\theta}_1) - m_3{}^L g\left[L_1(c\theta_1\dot{\theta}_1{}^2 + s\theta_1\ddot{\theta}_1) + \frac{L_3}{2}(c\theta_{13}\dot{\theta}_{13}{}^2 + s\theta_{13}\ddot{\theta}_{13})\right]$$

$$-m_5{}^L g\left[L_1(c\theta_1\dot{\theta}_1{}^2 + s\theta_1\ddot{\theta}_1) + L_3(c\theta_{13}\dot{\theta}_{13}{}^2 + s\theta_{13}\ddot{\theta}_{13}) + g_5(c(\theta_{135}+\beta)\dot{\theta}_{135}{}^2 + s(\theta_{135}+\beta)\ddot{\theta}_{135})\right]$$

$$-m_3{}^A g L_1(c\theta_1\dot{\theta}_1{}^2 + s\theta_1\ddot{\theta}_1)$$

$$\ddot{Q}_2 = -m_2{}^L g \frac{L_2}{2}(c\theta_2\dot{\theta}_2{}^2 + s\theta_2\ddot{\theta}_2) - m_4{}^L g\left[L_2(c\theta_2\dot{\theta}_2{}^2 + s\theta_2\ddot{\theta}_2) + \frac{L_4}{2}(c\theta_{24}\dot{\theta}_{24}{}^2 + s\theta_{24}\ddot{\theta}_{24})\right]$$

$$\begin{aligned}
\ddot{Q}_3 &= (\ddot{\theta}_1^2 + \dot{\theta}_1\dddot{\theta}_1)\left[(m_3{}^L + 2m_5{}^L)L_1L_3s\theta_3 + 2m_5{}^L L_1 g_5 s(\theta_{35}+\beta)\right] \\
&+2\dot{\theta}_1\ddot{\theta}_1\left[(m_3{}^L + 2m_5{}^L)L_1L_3c\theta_3\dot{\theta}_3 + 2m_5{}^L L_1 g_5 c(\theta_{35}+\beta)\dot{\theta}_{35}\right] \\
&+\frac{1}{2}\dot{\theta}_1^2\left[(m_3{}^L + 2m_5{}^L)L_1L_3(-s\theta_3\dot{\theta}_3^2 + c\theta_3\ddot{\theta}_3) + 2m_5{}^L L_1 g_5(-s(\theta_{35}+\beta)\dot{\theta}_{35}{}^2 + c(\theta_{35}+\beta)\ddot{\theta}_{35})\right] \\
&+(\dddot{\theta}_1\dot{\theta}_3 + 2\ddot{\theta}_1\ddot{\theta}_3 + \dot{\theta}_1\dddot{\theta}_3)\left[(\frac{1}{2}m_3{}^L + m_5{}^L)L_1L_3s\theta_3 + m_5{}^L L_1 g_5 s(\theta_{35}+\beta)\right] \\
&+2(\ddot{\theta}_1\dot{\theta}_3 + \dot{\theta}_1\ddot{\theta}_3)\left[(\frac{1}{2}m_3{}^L + m_5{}^L)L_1L_3c\theta_3\dot{\theta}_3 + m_5{}^L L_1 g_5 c(\theta_{35}+\beta)\dot{\theta}_{35}\right] \\
&+(\dddot{\theta}_1\dot{\theta}_5 + 2\ddot{\theta}_1\ddot{\theta}_5 + \dot{\theta}_1\dddot{\theta}_5)\left[m_5{}^L L_1 g_5 s(\theta_{35}+\beta)\right] + 2(\ddot{\theta}_1\dot{\theta}_5 + \dot{\theta}_1\ddot{\theta}_5)\left[m_5{}^L L_1 g_5 c(\theta_{35}+\beta)\dot{\theta}_{35}\right] \\
&+\dot{\theta}_1\dot{\theta}_5\left[m_5{}^L L_1 g_5(-s(\theta_{35}+\beta)\dot{\theta}_{35}{}^2 + c(\theta_{35}+\beta)\ddot{\theta}_{35})\right] - m_3{}^L g\frac{L_3}{2}(c\theta_{13}\dot{\theta}_{13}{}^2 + s\theta_{13}\ddot{\theta}_{13}) \\
&-m_5{}^L g\left[L_3(c\theta_{13}\dot{\theta}_{13}{}^2 + s\theta_{13}\ddot{\theta}_{13}) + g_5(c(\theta_{135}+\beta)\dot{\theta}_{135}{}^2 + s(\theta_{135}+\beta)\ddot{\theta}_{135})\right]
\end{aligned}$$

$$\begin{aligned}
\ddot{Q}_4 &= m_4{}^L L_2L_4\left[\ddot{\theta}_2^2 s\theta_4 + \dot{\theta}_2(\dddot{\theta}_2 s\theta_4 + \ddot{\theta}_2 c\theta_4\dot{\theta}_4) + \dot{\theta}_2\ddot{\theta}_2 c\theta_4\dot{\theta}_4 + \frac{1}{2}\dot{\theta}_2^2(-s\theta_4\dot{\theta}_4{}^2 + c\theta_4\ddot{\theta}_4)\right] \\
&+\frac{1}{2}m_4{}^L L_2L_4\left[\dddot{\theta}_2\dot{\theta}_4 s\theta_4 + 2\ddot{\theta}_2(\ddot{\theta}_4 s\theta_4 + c\theta_4\dot{\theta}_4^2) + \dot{\theta}_2(\dddot{\theta}_4 s\theta_4 + 3\dot{\theta}_4\ddot{\theta}_4 c\theta_4 - \dot{\theta}_4^3 s\theta_4)\right] \\
&-m_2{}^L g\frac{L_2}{2}\left(c\theta_2\dot{\theta}_2^2 + s\theta_2\ddot{\theta}_2\right) - m_4{}^L g\left[L_2(c\theta_2\dot{\theta}_2^2 + s\theta_2\ddot{\theta}_2) + \frac{L_4}{2}(c\theta_{24}\dot{\theta}_{24}{}^2 + s\theta_{24}\ddot{\theta}_{24})\right]
\end{aligned}$$

$$\dddot{Q}_5 = 2(\ddot{\theta}_1^2 + \dot{\theta}_1\dddot{\theta}_1)\left[m_5{}^L L_1 g_5 s(\theta_{35}+\beta) + m_5{}^L L_3 g_5 s(\theta_5+\beta) \right] + 4\dot{\theta}_1\ddot{\theta}_1 \left[m_5{}^L L_1 g_5 c(\theta_{35}+\beta)\dot{\theta}_{35} \right.$$

$$\left. + m_5{}^L L_3 g_5 c(\theta_5+\beta)\dot{\theta}_5 \right] + \dot{\theta}_1^2 \left\{ m_5{}^L L_1 g_5 \left[-s(\theta_{35}+\beta)\dot{\theta}_{35}{}^2 + c(\theta_{35}+\beta)\ddot{\theta}_{35} \right] \right.$$

$$\left. + m_5{}^L L_3 g_5 \left[-s(\theta_5+\beta)\dot{\theta}_5{}^2 + c(\theta_5+\beta)\ddot{\theta}_5 \right] \right\} + 2(\ddot{\theta}_3^2 + \dot{\theta}_3\dddot{\theta}_3) m_5{}^L L_3 g_5 s(\theta_5+\beta)$$

$$+ 4\dot{\theta}_3\ddot{\theta}_3 m_5{}^L L_3 g_5 c(\theta_5+\beta)\dot{\theta}_5 + \dot{\theta}_3{}^2 m_5{}^L L_3 g_5 \left[-s(\theta_5+\beta)\dot{\theta}_5{}^2 + c(\theta_5+\beta)\ddot{\theta}_5 \right]$$

$$+ (\dddot{\theta}_1\dot{\theta}_3 + 2\ddot{\theta}_1\ddot{\theta}_3 + \dot{\theta}_1\dddot{\theta}_3)\left[m_5{}^L L_1 g_5 s(\theta_{35}+\beta) + 2m_5{}^L L_3 g_5 s(\theta_5+\beta) \right]$$

$$+ 2(\ddot{\theta}_1\dot{\theta}_3 + \dot{\theta}_1\ddot{\theta}_3)\left[m_5{}^L L_1 g_5 c(\theta_{35}+\beta)\dot{\theta}_{35} + 2m_5{}^L L_3 g_5 c(\theta_5+\beta)\dot{\theta}_5 \right]$$

$$+ \dot{\theta}_1\dot{\theta}_3 \left\{ m_5{}^L L_1 g_5 \left[-s(\theta_{35}+\beta)\dot{\theta}_{35}{}^2 + c(\theta_{35}+\beta)\ddot{\theta}_{35} \right] + 2m_5{}^L L_3 g_5 \left[-s(\theta_5+\beta)\dot{\theta}_5{}^2 + c(\theta_5+\beta)\ddot{\theta}_5 \right] \right\}$$

$$+ (\dddot{\theta}_1\dot{\theta}_5 + 2\ddot{\theta}_1\ddot{\theta}_5 + \dot{\theta}_1\dddot{\theta}_5)\left[m_5{}^L L_1 g_5 s(\theta_{35}+\beta) + m_5{}^L L_3 g_5 s(\theta_5+\beta) \right]$$

$$+ 2(\ddot{\theta}_1\dot{\theta}_5 + \dot{\theta}_1\ddot{\theta}_5)\left[m_5{}^L L_1 g_5 c(\theta_{35}+\beta)\dot{\theta}_{35} + m_5{}^L L_3 g_5 c(\theta_5+\beta)\dot{\theta}_5 \right]$$

$$+ \dot{\theta}_1\dot{\theta}_5 \left\{ m_5{}^L L_1 g_5 \left[-s(\theta_{35}+\beta)\dot{\theta}_{35}{}^2 + c(\theta_{35}+\beta)\ddot{\theta}_{35} \right] + m_5{}^L L_3 g_5 \left[-s(\theta_5+\beta)\dot{\theta}_5{}^2 + c(\theta_5+\beta)\ddot{\theta}_5 \right] \right\}$$

$$+ (\dddot{\theta}_3\dot{\theta}_5 + 2\ddot{\theta}_3\ddot{\theta}_5 + \dot{\theta}_3\dddot{\theta}_5) m_5{}^L L_3 g_5 s(\theta_5+\beta) + (\ddot{\theta}_3\dot{\theta}_5 + \dot{\theta}_3\ddot{\theta}_5) m_5{}^L L_3 g_5 c(\theta_5+\beta)\dot{\theta}_5$$

$$+ (\ddot{\theta}_3\dot{\theta}_5{}^2 + 2\dot{\theta}_3\dot{\theta}_5\ddot{\theta}_5) m_5{}^L L_3 g_5 c(\theta_5+\beta) - \dot{\theta}_3\dot{\theta}_5{}^2 m_5{}^L L_3 g_5 s(\theta_5+\beta)\dot{\theta}_5$$

$$- m_5{}^L g \left\{ g_5 \left[c(\theta_{135}+\beta)\dot{\theta}_{135}{}^2 + s(\theta_{135}+\beta)\ddot{\theta}_{135} \right] \right\}$$

A.3 The First, Second and Third Time Derivatives of Matrix $\hat{B}$

$$\dot{B}_{11} = -L_1 c\theta_1\dot{\theta}_1 - L_3 c\theta_{13}\dot{\theta}_{13} - L_5 c\theta_{135}\dot{\theta}_{135}$$

$$\dot{B}_{12} = L_2 c\theta_2\dot{\theta}_2 + L_4 c\theta_{24}\dot{\theta}_{24}$$

$$\dot{B}_{13} = -L_3 c\theta_{13}\dot{\theta}_{13} - L_5 c\theta_{135}\dot{\theta}_{135}$$

$$\dot{B}_{14} = L_4 c\theta_{24}\dot{\theta}_{24}$$

$$\dot{B}_{15} = -L_5 c\theta_{135}\dot{\theta}_{135}$$

$$\dot{B}_{21} = -L_1 s\theta_1\dot{\theta}_1 - L_3 s\theta_{13}\dot{\theta}_{13} - L_5 s\theta_{135}\dot{\theta}_{135}$$

$$\dot{B}_{22} = L_2 s\theta_2\dot{\theta}_2 + L_4 s\theta_{24}\dot{\theta}_{24}$$

$$\dot{B}_{23} = -L_3 s\theta_{13}\dot{\theta}_{13} - L_5 s\theta_{135}\dot{\theta}_{135}$$

$$\dot{B}_{24} = L_4 s\theta_{24}\dot{\theta}_{24}$$

$$\dot{B}_{25} = -L_5 s\theta_{135}\dot{\theta}_{135}$$

$$\ddot{B}_{11} = -L_1\left(-s\theta_1\dot{\theta}_1^{\ 2} + c\theta_1\ddot{\theta}_1\right) - L_3\left(-s\theta_{13}\dot{\theta}_{13}^{\ 2} + c\theta_{13}\ddot{\theta}_{13}\right) - L_5\left(-s\theta_{135}\dot{\theta}_{135}^{\ 2} + c\theta_{135}\ddot{\theta}_{135}\right)$$

$$\ddot{B}_{12} = L_2\left(-s\theta_2\dot{\theta}_2^{\ 2} + c\theta_2\ddot{\theta}_2\right) + L_4\left(-s\theta_{24}\dot{\theta}_{24}^{\ 2} + c\theta_{24}\ddot{\theta}_{24}\right)$$

$$\ddot{B}_{13} = -L_3\left(-s\theta_{13}\dot{\theta}_{13}^{\ 2} + c\theta_{13}\ddot{\theta}_{13}\right) - L_5\left(-s\theta_{135}\dot{\theta}_{135}^{\ 2} + c\theta_{135}\ddot{\theta}_{135}\right)$$

$$\ddot{B}_{14} = L_4\left(-s\theta_{24}\dot{\theta}_{24}^{\ 2} + c\theta_{24}\ddot{\theta}_{24}\right)$$

$$\ddot{B}_{15} = -L_5\left(-s\theta_{135}\dot{\theta}_{135}^{\ 2} + c\theta_{135}\ddot{\theta}_{135}\right)$$

$$\ddot{B}_{21} = -L_1\left(c\theta_1\dot{\theta}_1^{\ 2} + s\theta_1\ddot{\theta}_1\right) - L_3\left(c\theta_{13}\dot{\theta}_{13}^{\ 2} + s\theta_{13}\ddot{\theta}_{13}\right) - L_5\left(c\theta_{135}\dot{\theta}_{135}^{\ 2} + s\theta_{135}\ddot{\theta}_{135}\right)$$

$$\ddot{B}_{22} = L_2\left(c\theta_2\dot{\theta}_2^{\ 2} + s\theta_2\ddot{\theta}_2\right) + L_4\left(c\theta_{24}\dot{\theta}_{24}^{\ 2} + s\theta_{24}\ddot{\theta}_{24}\right)$$

$$\ddot{B}_{23} = -L_3\left(c\theta_{13}\dot{\theta}_{13}^{\ 2} + s\theta_{13}\ddot{\theta}_{13}\right) - L_5\left(s\theta_{135}\dot{\theta}_{135}^{\ 2} + s\theta_{135}\ddot{\theta}_{135}\right)$$

$$\ddot{B}_{24} = L_4\left(c\theta_{24}\dot{\theta}_{24}^{\ 2} + s\theta_{24}\ddot{\theta}_{24}\right)$$

$$\ddot{B}_{25} = -L_5\left(s\theta_{135}\dot{\theta}_{135}{}^{2} + s\theta_{135}\ddot{\theta}_{135}\right)$$

$$\dddot{B}_{11} = -L_1\left(-c\theta_1\dot{\theta}_1^3 - 3s\theta_1\dot{\theta}_1\ddot{\theta}_1 + c\theta_1\dddot{\theta}_1\right) - L_3\left(-c\theta_{13}\dot{\theta}_{13}{}^{3} - 3s\theta_{13}\dot{\theta}_{13}\ddot{\theta}_{13} + c\theta_{13}\dddot{\theta}_{13}\right)$$
$$-L_5\left(-c\theta_{135}\dot{\theta}_{135}{}^{3} - 3s\theta_{135}\dot{\theta}_{135}\ddot{\theta}_{135} + c\theta_{135}\dddot{\theta}_{135}\right)$$

$$\dddot{B}_{12} = L_2\left(-c\theta_2\dot{\theta}_2^3 - 3s\theta_2\dot{\theta}_2\ddot{\theta}_2 + c\theta_2\dddot{\theta}_2\right) + L_4\left(-c\theta_{24}\dot{\theta}_{24}{}^{3} - 3s\theta_{24}\dot{\theta}_{24}\ddot{\theta}_{24} + c\theta_{24}\dddot{\theta}_{24}\right)$$

$$\dddot{B}_{13} = -L_3\left(-c\theta_{13}\dot{\theta}_{13}{}^{3} - 3s\theta_{13}\dot{\theta}_{13}\ddot{\theta}_{13} + c\theta_{13}\dddot{\theta}_{13}\right) - L_5\left(-c\theta_{135}\dot{\theta}_{135}{}^{3} - 3s\theta_{135}\dot{\theta}_{135}\ddot{\theta}_{135} + c\theta_{135}\dddot{\theta}_{135}\right)$$

$$\dddot{B}_{14} = L_4\left(-c\theta_{24}\dot{\theta}_{24}{}^{3} - 3s\theta_{24}\dot{\theta}_{24}\ddot{\theta}_{24} + c\theta_{24}\dddot{\theta}_{24}\right)$$

$$\dddot{B}_{15} = -L_5\left(-c\theta_{135}\dot{\theta}_{135}{}^{3} - 3s\theta_{135}\dot{\theta}_{135}\ddot{\theta}_{135} + c\theta_{135}\dddot{\theta}_{135}\right)$$

$$\dddot{B}_{21} = -L_1\left(-s\theta_1\dot{\theta}_1^3 + 3c\theta_1\dot{\theta}_1\ddot{\theta}_1 + s\theta_1\dddot{\theta}_1\right) - L_3\left(-s\theta_{13}\dot{\theta}_{13}{}^{3} + 3c\theta_{13}\dot{\theta}_{13}\ddot{\theta}_{13} + s\theta_{13}\dddot{\theta}_{13}\right)$$
$$-L_5\left(-s\theta_{135}\dot{\theta}_{135}{}^{3} + 3c\theta_{135}\dot{\theta}_{135}\ddot{\theta}_{135} + s\theta_{135}\dddot{\theta}_{135}\right)$$

$$\dddot{B}_{22} = L_2\left(-s\theta_2\dot{\theta}_2^3 + 3c\theta_2\dot{\theta}_2\ddot{\theta}_2 + s\theta_2\dddot{\theta}_2\right) + L_4\left(-s\theta_{24}\dot{\theta}_{24}{}^{3} + 3c\theta_{24}\dot{\theta}_{24}\ddot{\theta}_{24} + s\theta_{24}\dddot{\theta}_{24}\right)$$

$$\dddot{B}_{23} = -L_3\left(-s\theta_{13}\dot{\theta}_{13}{}^{3} + 3c\theta_{13}\dot{\theta}_{13}\ddot{\theta}_{13} + s\theta_{13}\dddot{\theta}_{13}\right) - L_5\left(-s\theta_{135}\dot{\theta}_{135}{}^{3} + 3c\theta_{135}\dot{\theta}_{135}\ddot{\theta}_{135} + s\theta_{135}\dddot{\theta}_{135}\right)$$

$$\dddot{B}_{24} = L_4\left(-s\theta_{24}\dot{\theta}_{24}{}^{3} + 3c\theta_{24}\dot{\theta}_{24}\ddot{\theta}_{24} + s\theta_{24}\dddot{\theta}_{24}\right)$$

$$\dddot{B}_{25} = -L_5\left(-s\theta_{135}\dot{\theta}_{135}{}^{3} + 3c\theta_{135}\dot{\theta}_{135}\ddot{\theta}_{135} + s\theta_{135}\dddot{\theta}_{135}\right)$$

A.4 The First, Second and Third Time Derivatives of Matrix $\hat{\Gamma}^P$

$$\dot{G}_{11} = -L_1 c\theta_1 \dot{\theta}_1 - L_3 c\theta_{13} \dot{\theta}_{13} - d_5 c(\theta_{135} + \alpha)\dot{\theta}_{135}$$

$$\dot{G}_{12} = 0$$

$$\dot{G}_{13} = -L_3 c\theta_{13} \dot{\theta}_{13} - d_5 c(\theta_{135} + \alpha)\dot{\theta}_{135}$$

$$\dot{G}_{14} = 0$$

$$\dot{G}_{15} = -d_5 c(\theta_{135} + \alpha)\dot{\theta}_{135}$$

$$\dot{G}_{21} = -L_1 s\theta_1 \dot{\theta}_1 - L_3 s\theta_{13} \dot{\theta}_{13} - d_5 s(\theta_{135} + \alpha)\dot{\theta}_{135}$$

$$\dot{G}_{22} = 0$$

$$\dot{G}_{23} = -L_3 s\theta_{13} \dot{\theta}_{13} - d_5 s(\theta_{135} + \alpha)\dot{\theta}_{135}$$

$$\dot{G}_{24} = 0$$

$$\dot{G}_{25} = -d_5 s(\theta_{135} + \alpha)\dot{\theta}_{135}$$

$$\dot{G}_{31} = 0$$

$$\dot{G}_{32} = 0$$

$$\dot{G}_{33} = 0$$

$$\dot{G}_{34} = 0$$

$$\dot{G}_{35} = 0$$

$$\ddot{G}_{11} = -L_1\left(-s\theta_1\dot{\theta}_1^2 + c\theta_1\ddot{\theta}_1\right) - L_3\left(-s\theta_{13}\dot{\theta}_{13}^{\ 2} + c\theta_{13}\ddot{\theta}_{13}\right) - d_5\left[-s(\theta_{135}+\alpha)\dot{\theta}_{135}^{\ 2} + c(\theta_{135}+\alpha)\ddot{\theta}_{135}\right]$$

$$\ddot{G}_{12} = 0$$

$$\ddot{G}_{13} = -L_3\left(-s\theta_{13}\dot{\theta}_{13}^{\ 2} + c\theta_{13}\ddot{\theta}_{13}\right) - d_5\left[-s(\theta_{135}+\alpha)\dot{\theta}_{135}^{\ 2} + c(\theta_{135}+\alpha)\ddot{\theta}_{135}\right]$$

$$\ddot{G}_{14} = 0$$

$$\ddot{G}_{15} = -d_5\left[-s(\theta_{135}+\alpha)\dot{\theta}_{135}^{\ 2} + c(\theta_{135}+\alpha)\ddot{\theta}_{135}\right]$$

$$\ddot{G}_{21} = -L_1\left(c\theta_1\dot{\theta}_1^2 + s\theta_1\ddot{\theta}_1\right) - L_3\left(c\theta_{13}\dot{\theta}_{13}^{\ 2} + s\theta_{13}\ddot{\theta}_{13}\right) - d_5\left[c(\theta_{135}+\alpha)\dot{\theta}_{135}^{\ 2} + s(\theta_{135}+\alpha)\ddot{\theta}_{135}\right]$$

$$\ddot{G}_{22} = 0$$

$$\ddot{G}_{23} = -L_3\left(c\theta_{13}\dot{\theta}_{13}^{\ 2} + s\theta_{13}\ddot{\theta}_{13}\right) - d_5\left[c(\theta_{135}+\alpha)\dot{\theta}_{135}^{\ 2} + s(\theta_{135}+\alpha)\ddot{\theta}_{135}\right]$$

$$\ddot{G}_{24} = 0$$

$$\ddot{G}_{25} = -d_5\left[c(\theta_{135}+\alpha)\dot{\theta}_{135}^{\ 2} + s(\theta_{135}+\alpha)\ddot{\theta}_{135}\right]$$

$$\ddot{G}_{31} = 0$$

$$\ddot{G}_{32} = 0$$

$$\ddot{G}_{33} = 0$$

$$\ddot{G}_{34} = 0$$

$$\ddot{G}_{35} = 0$$

$$\dddot{G}_{11} = -L_1\left(-c\theta_1\dot{\theta}_1^3 - 3s\theta_1\dot{\theta}_1\ddot{\theta}_1 + c\theta_1\dddot{\theta}_1\right) - L_3\left(-c\theta_{13}\dot{\theta}_{13}{}^3 - 3s\theta_{13}\dot{\theta}_{13}\ddot{\theta}_{13} + c\theta_{13}\dddot{\theta}_{13}\right)$$
$$-d_5\left[-c(\theta_{135}+\alpha)\dot{\theta}_{135}{}^3 - 3s(\theta_{135}+\alpha)\dot{\theta}_{135}\ddot{\theta}_{135} + c(\theta_{135}+\alpha)\dddot{\theta}_{135}\right]$$

$$\dddot{G}_{12} = 0$$

$$\dddot{G}_{13} = -L_3\left(-c\theta_{13}\dot{\theta}_{13}{}^3 - 3s\theta_{13}\dot{\theta}_{13}\ddot{\theta}_{13} + c\theta_{13}\dddot{\theta}_{13}\right) - d_5\left[-c(\theta_{135}+\alpha)\dot{\theta}_{135}{}^3 - 3s(\theta_{135}+\alpha)\dot{\theta}_{135}\ddot{\theta}_{135}\right.$$
$$\left.+c(\theta_{135}+\alpha)\dddot{\theta}_{135}\right]$$

$$\dddot{G}_{14} = 0$$

$$\dddot{G}_{15} = -d_5\left[-c(\theta_{135}+\alpha)\dot{\theta}_{135}{}^3 - 3s(\theta_{135}+\alpha)\dot{\theta}_{135}\ddot{\theta}_{135} + c(\theta_{135}+\alpha)\dddot{\theta}_{135}\right]$$

$$\dddot{G}_{21} = -L_1\left(-s\theta_1\dot{\theta}_1^3 + 3c\theta_1\dot{\theta}_1\ddot{\theta}_1 + s\theta_1\dddot{\theta}_1\right) - L_3\left(-s\theta_{13}\dot{\theta}_{13}{}^3 + 3c\theta_{13}\dot{\theta}_{13}\ddot{\theta}_{13} + s\theta_{13}\dddot{\theta}_{13}\right)$$
$$-d_5\left[-s(\theta_{135}+\alpha)\dot{\theta}_{135}{}^3 + 3c(\theta_{135}+\alpha)\dot{\theta}_{135}\ddot{\theta}_{135} + s(\theta_{135}+\alpha)\dddot{\theta}_{135}\right]$$

$$\dddot{G}_{22} = 0$$

$$\dddot{G}_{23} = -L_3\left(-s\theta_{13}\dot{\theta}_{13}{}^3 + 3c\theta_{13}\dot{\theta}_{13}\ddot{\theta}_{13} + s\theta_{13}\dddot{\theta}_{13}\right) - d_5\left[-s(\theta_{135}+\alpha)\dot{\theta}_{135}{}^3 + 3c(\theta_{135}+\alpha)\dot{\theta}_{135}\ddot{\theta}_{135}\right.$$
$$\left.+s(\theta_{135}+\alpha)\dddot{\theta}_{135}\right]$$

$$\dddot{G}_{24} = 0$$

$$\dddot{G}_{25} = -d_5\left[-s(\theta_{135}+\alpha)\dot{\theta}_{135}{}^3 + 3c(\theta_{135}+\alpha)\dot{\theta}_{135}\ddot{\theta}_{135} + s(\theta_{135}+\alpha)\dddot{\theta}_{135}\right]$$

$$\dddot{G}_{31} = 0$$

$$\dddot{G}_{32} = 0$$

$$\dddot{G}_{33} = 0$$

$$\dddot{G}_{34} = 0$$

$$\dddot{G}_{35} = 0$$

APPENDIX B

ELEMENTS OF MATRICES

B.1 The Elements of Matrices $\hat{M}^{a*}$ and $\hat{M}^{u*}$

$M^{a*}{}_{11} = M_{11} + M_{15}C_{21}$

$M^{a*}{}_{12} = M_{15}C_{22}$

$M^{a*}{}_{13} = M_{13} + M_{13}C_{23}$

$M^{a*}{}_{21} = M_{24}C_{11}$

$M^{a*}{}_{22} = M_{22} + M_{24}C_{12}$

$M^{a*}{}_{23} = M_{24}C_{13}$

$M^{a*}{}_{31} = M_{13} + M_{35}C_{21}$

$M^{a*}{}_{32} = M_{35}C_{22}$

$M^{a*}{}_{33} = M_{33} + M_{35}C_{23}$

$M^{u*}{}_{11} = M_{44}C_{11}$

$M^{u*}{}_{12} = M_{24} + M_{44}C_{12}$

$$M^{u*}{}_{13} = M_{44}C_{13}$$

$$M^{u*}{}_{21} = M_{15} + M_{55}C_{21}$$

$$M^{u*}{}_{22} = M_{55}C_{22}$$

$$M^{u*}{}_{23} = M_{35} + M_{55}C_{23}$$

B.2 The Elements of Matrices $\hat{\boldsymbol{R}}^a$ and $\hat{\boldsymbol{R}}^u$

$$R^a{}_{11} = M_{15}\dot{C}_{21}$$

$$R^a{}_{12} = M_{15}\dot{C}_{22}$$

$$R^a{}_{13} = M_{15}\dot{C}_{23}$$

$$R^a{}_{21} = M_{24}\dot{C}_{11}$$

$$R^a{}_{22} = M_{24}\dot{C}_{12}$$

$$R^a{}_{23} = M_{24}\dot{C}_{13}$$

$$R^a{}_{31} = M_{35}\dot{C}_{21}$$

$$R^a{}_{32} = M_{35}\dot{C}_{22}$$

$$R^a{}_{33} = M_{35}\dot{C}_{23}$$

$$R^u{}_{11} = M_{44}\dot{C}_{11} + (D_4 + D_6)C_{11} - D_6C_{21}$$

$$R^{u}{}_{12} = M_{44}\dot{C}_{12} + (D_4 + D_6)C_{12} - D_6 C_{22}$$

$$R^{u}{}_{13} = M_{44}\dot{C}_{13} + (D_4 + D_6)C_{13} - D_6 C_{23}$$

$$R^{u}{}_{21} = M_{55}\dot{C}_{21} + (D_5 + D_6)C_{21} - D_6 C_{11}$$

$$R^{u}{}_{22} = M_{55}\dot{C}_{22} + (D_5 + D_6)C_{22} - D_6 C_{12}$$

$$R^{u}{}_{23} = M_{55}\dot{C}_{23} + (D_5 + D_6)C_{23} - D_6 C_{13}$$