SCHWARZ PROBLEM FOR COMPLEX PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT

SCHWARZ PROBLEM FOR COMPLEX PARTIAL DIFFERENTIAL EQUATIONS

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This study consists of four chapters. In the first chapter we give some historical background of the problem, basic definitions and properties. Basic integral operators of complex analysis and and Schwarz problem for model equations are presented in Chapter 2. Chapter 3 is devoted to the investigation of the properties of a class of strongly singular integral operators. In the last chapter we consider the Schwarz boundary value problem for the general partial complex differential equations of higher order.

Keywords: Schwarz problem, strongly singular integral operators, complex partial differential equations.

ÖZ

KISMİ KOMPLEKS DİFERENSİYEL DENKLEMLER İÇİN SCHWARZ PROBLEMİ

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Bu çalışma dört bölümden oluşmaktadır. İlk bölümde problemin tarihçesi, temel tanımlar ve özellikler ele alınmıştır. İkinci bölümde temel integral operatörler ve model denklemler için Schwarz sınır değer problemleri verilmiştir. Üçüncü bölümde bir sınıf kuvvetli tekil integral operatörlerin özellikleri incelenmiştir. Dördüncü bölümde yüksek mertebeden difererensiyel denklemler için Schwarz sınır değer problemi çalışılmıştır.

Anahtar Kelimeler: Schwarz problemi, kuvvetli tekil integral operatörler, kompleks diferensiyel denklemler.

To my family,
Egemen and Doruk AKSOY

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CHAPTER 1

Introduction and Preliminaries

1.1 Introduction

The methods and special features of holomorphic (analytic) functions of a complex variable enable us to apply complex function theory in many branches of theoretical and applied mathematics; in particular, in geometry, algebra, partial differential equations, etc., and in mathematical physics, hydrodynamics, shell theory, elasticity theory, quantum mechanics, etc. Physical problems can be converted to mathematical models including boundary value problems. Thus, the theory of boundary value problems has many applications.

From the beginning of the theory of boundary value problems, two basic boundary value problems, namely Riemann-Hilbert and Riemann problems, have been investigated for analytic functions, Beltrami equations and elliptic first order equations; see [36], [37], [25]. In Riemann-Hilbert problem, a linear combination of the real part and the imaginary part of the function satisfying the given differential equation is prescribed on the boundary of the domain. Riemann problem concerns the piecewise continuous solution of the given equation satisfying a jump discontinuity condition along the boundary of the given domain. In recent years, Schwarz, Dirichlet, Neumann, Robin and mixed boundary value problems have great interest in differential equations of higher order. These problems are particular cases of the Riemann-Hilbert and Riemann problems defined above. The Schwarz problem is a particular case of the Riemann-Hilbert problem. Dirichlet, Neumann and Robin problems are the special cases of the Riemann problem. These boundary value problems have been investigated in simply connected bounded domains and in particular on the unit disc of the complex plane for complex model equations ([9], [10]), since in the unit disc, solutions are given explicitly. By the well-known conformal mapping theorem, any simply connected domain is conformally equivalent to the unit disc. Thus the solutions may also be considered in such domains.

The model equations are simple inhomogeneous partial differential equations, where the differential operator is obtained by the product of some powers of the Cauchy-Riemann operator $\partial_{\bar{z}}$ and some powers of its complex conjugate, that is, the operator ∂_z. These operators are the Laplace operator $\partial_{\bar{z}}\partial_z$, Bitsadze operator $\partial_{\bar{z}}^2$, polyanalytic operator of order k given by $\partial_{\bar{z}}^k$, and polyharmonic operator of order k, $\partial_{\bar{z}}^k\partial_z^k$. Schwarz, Dirichlet, Neumann and some related boundary value problems are solved for the Bitsadze equation in the unit disc of the complex plane in [8]. Second-order equations of special type with some boundary conditions are treated extensively by Begehr in [10]. Also the Schwarz problem is considered for polyanalytic functions and bi-polyanalytic functions, see [16] and [15], and iterated Neumann problem is investigated for higher-order Poisson equations, see [17]. The main aim of the investigations on model equations is to develop a complete theory for elliptic equations of arbitrary order. Using the results obtained for the model equations, one can consider a more general equation of arbitrary order with the same boundary conditions under suitable conditions.

In this study, we are mainly interested in the Schwarz problem for general partial complex differential equations in the unit disc of the complex plane. The boundary value problem is studied by reducing it to a singular integral equation. The idea of this method is based on the work of I. N. Vekua [36] for generalized analytic functions. Integral operators are important for reducing the boundary value problems for differential equations to integral equations. The Pompeiu operator

$$Tf(z) := -\frac{1}{\pi}\int\int_D f(\zeta)\frac{d\xi d\eta}{\zeta - z}\,, \qquad z \in \mathbb{C},$$

and its weak derivative with respect to z,

$$\Pi f := -\frac{1}{\pi}\int\int_D f(\zeta)\frac{d\xi d\eta}{(\zeta - z)^2}\,, \qquad z \in \mathbb{C},$$

are the main tools of Vekua's theory of generalized analytic functions. Investigations on complex first order partial differential equations are based on the properties of these operators, see [22], [12]. In Chapter 2, first we give their important properties. Iterating the Pompeiu integral operator T with itself and with its conjugate leads to a hierarchy of integral operators $T_{m,n}$ given in [13], related to the differential operator $\partial_{\bar{z}}^m\partial_z^n$. These operators are important in dealing with the differential equations of higher order. Secondly in Chapter 2, we present main results on them.

The Pompeiu operator can be modified in accordance with the some boundary condition by adding a holomorphic function. For the unit disc $\mathbb{D} = \{|z| < 1\}$, the following two operators are defined in [6]. The first is the operator $\widetilde{T}$ defined by

$$\widetilde{T}f(z) = -\frac{1}{\pi}\int\int_{\mathbb{D}}\left[\frac{f(\zeta)}{\zeta - z} + \frac{z\overline{f(\zeta)}}{1 - z\bar{\zeta}}\right]d\xi d\eta$$

which satisfies

$$\frac{\partial\widetilde{T}f}{\partial\bar{z}} = f \quad \text{in } \mathbb{D}$$

and

$$\mathrm{Re}\widetilde{T}f = 0 \quad \text{on } \partial\mathbb{D}\,.$$

The second is the operator $\widetilde{T}_1$ defined by

$$\widetilde{T}_1 f(z) := -\frac{1}{2\pi}\int\int_{\mathbb{D}}\left(\frac{f(\zeta)}{\zeta}\frac{\zeta + z}{\zeta - z} + \frac{\overline{f(\zeta)}}{\bar{\zeta}}\frac{1 + z\bar{\zeta}}{1 - z\bar{\zeta}}\right)d\xi d\eta\,,$$

satisfying

$$\frac{\partial \widetilde{T}_1 f}{\partial \bar{z}} = f \quad \text{in } \mathbb{D}$$

and the Schwarz conditions

$$\text{Re}\widetilde{T} f = 0 \quad \text{on } \partial\mathbb{D} \;\text{ and } \quad \text{Im}\widetilde{T}_1 f(0) = 0 \; .$$

These operators are important for treating first-order differential equations in the unit disc, see [12], [22]. Since the operator $\widetilde{T}_1$ satisfies the Schwarz conditions naturally, it is in the center of interest when dealing with the Schwarz problem for complex partial differential equations. Iterating this operator with itself generates the operators $\widetilde{T}_k$, for $k \in \mathbb{N}$, which are investigated extensively in [13], [4], [14] and [5]. The results concerning these operators are reviewed in Chapter 2. Properties of the weakly singular operators $\partial_z^l \widetilde{T}_k$ for $0 \leq l \leq k-1$ are known. But, as Begehr pointed out in [6], [14], and [5], the properties of the strongly singular operators $\partial_z^k \widetilde{T}_k$ are not known in the literature for $k > 1$. For $k = 1$, it is proven in [36] that L^2 norm of $\partial_z \widetilde{T}_1 := \widetilde{\Pi}$ is 1. These operators are important for treating higher order general partial differential equations. One of the aims of this study is to obtain some properties of this strongly singular integral operators of Calderon-Zygmund type. In Chapter 3 we show that they are bounded operators in L^p for $1 < p < \infty$. Also we give a sharper estimate for their L^2 norms. These operators are investigated by decomposing them into two parts. One of the parts is the operator $T_{-k,k}$ which is investigated extensively by Begehr and Hile in [13]. The other part P_k will be handled via Schur's theorem and Forelli-Rudin type estimates. Also, we prove that the P_k operator is compact from L^{p_1} to L^{p_2} for $p_2 < p_1$. This property is shown by defining a compact sequence of operators whose limit is P_k.

In [5], Begehr considers a general complex partial differential equation of order k with homogeneous Schwarz conditions. He reduces the problem into a singular integral equation and he observes that the problem is solvable and gives the form of the solution. The differential equation that he considers does not contain the terms whose transforms to the integral equation yields $\partial_z^k \widetilde{T}_k$ for $k > 1$. After getting some properties of these operators in Chapter 3, we deal with a general complex partial differential equation containing these terms in Chapter 4. In the investigation the problem, we mainly use the Fredholm theory. Basic definitions and theorems concerning Fredholm theory is presented in the preliminary section.

1.2 Preliminaries

1.2.1 Basic Definitions and Inequalities

Theorem 1.2.1. *[33] Let p and q be conjugate exponents, $1 < p < \infty$. Let X be a measure space, with measure μ. Let f and g be measurable functions on X, with range in $[0, \infty)$. Then*

$$\int_X fg d\mu \leq \left(\int_X f^p d\mu \right)^{1/p} \left(\int_X g^q d\mu \right)^{1/q} . \tag{1.1}$$

and

$$\left(\int_X (f+g)^p d\mu \right)^{1/p} \leq \left(\int_X f^p d\mu \right)^{1/p} + \left(\int_X g^p d\mu \right)^{1/p} . \tag{1.2}$$

The inequality (1.1) is Hölder's; (1.2) is Minkowski's. Following theorem gives an analogue of Minkowski's inequality.

Theorem 1.2.2. *[33] Let X be a measure space, with measure μ and Y be a measure space, with measure λ. Let f be measurable functions on $X \times Y$, with range in $[0, \infty)$. Then*

$$\left(\int_X \left[\int_Y f(x,y) d\lambda(y) \right]^p d\mu(x) \right)^{1/p} \leq \int_Y \left[\int_X f^p(x,y) d\mu(x) \right]^{1/p} d\lambda(y) .$$

1.2.2 Fredholm Theory

Following definitions and results are mainly given in [30], [34] and [26] in which $B(X,Y)$ will denote the set of all bounded operators from X to Y, $K(X)$ is the set of all compact operators on X and A' is the adjoint of the operator A.

Definition 1.2.3. *Let X, Y be Banach spaces. An operator $A \in B(X,Y)$ is said to be a Fredholm operator from X to Y if the numbers $\alpha(A) = \dim\ker A$ and $\beta(A) = \mathrm{codim} ImA$ are finite. The set of Fredholm operators from X to Y is denoted by $\Phi(X,Y)$. The index of a Fredholm operator is defined as $i(A) = \alpha(A) - \beta(A)$.*

Theorem 1.2.4. *(Fredholm Alternative) For an index-zero Fredholm operator A either of the following mutually exclusive events takes place.*

(1) The homogeneous equation $Ax = 0$ has only the zero solution. The equation $Ax = y$ is solvable and has a unique solution given an arbitrary right hand side.

(2) The homogeneous equation $Ax = 0$ has a nonzero solution. The homogeneous equation $Ax = 0$ has finitely many linearly independent solutions $x_1, \ldots, x_n$. The conjugate equation $A'y' = 0$ has finitely many linearly independent solutions $y'_1, \ldots, y'_n$. The equation $Tx = y$ is solvable if and only if $y'_1(y) = \cdots = y'_n(y) = 0$ and the general solution is the sum of a particular solution and the general solution of the homogeneous solution.

Theorem 1.2.5. *Let $K \in K(X)$. Then $I - K$ is an index-zero Fredholm operator.*

Definition 1.2.6. *Let $A \in B(X,Y)$. An operator $L \in B(Y,X)$ is a left approximate inverse of A if $LA - I \in K(X)$. An operator $R \in B(Y,X)$ is a right approximate inverse of A if $AR - I \in K(Y)$. An operator $S \in B(Y,X)$ is an approximate inverse of A if S is a left and right approximate inverse of A. If an operator A has an approximate inverse S then A is called approximately invertible.*

Theorem 1.2.7. *(Noether Criterion) An operator is a Fredholm operator if and only if it is approximately invertible.*

Corollary 1.2.8. *The product of finitely many Fredholm operators is itself a Fredholm operator.*

Theorem 1.2.9. *The index of the product of finitely many Fredholm operators equals the sum of the indices of the factors.*

Corollary 1.2.10. *Let* A *be a Fredholm operator and let* S *be an approximate inverse of* A*. Then* $ind(A) = -ind(S)$.

Theorem 1.2.11. *(Bounded Index Stability Theorem) Assume that* $A \in \Phi(X, Y)$*. Then there is an* $\eta > 0$ *such that for any* $T \in B(X, Y)$ *satisfying* $\|T\| < \eta$*, one has* $A + T \in \Phi(X, Y)$ *with* $i(A + T) = i(A)$.

Theorem 1.2.12. *(Nikolskii Criterion) An operator is an index-zero Fredholm operator if and only if it is the sum of an invertible operator and a compact operator.*

CHAPTER 2

Integral Operators and Schwarz Problems

This chapter is devoted to the presentation of the integral operators that are used in further discussions and definition of Schwarz problems for analytic functions and some model differential equations.

2.1 Integral Operators of Complex Analysis

In this section, we discuss main properties of the basic integral operators of the complex analysis. These operators are used in solving the boundary value problems for the complex partial differential equations. Firstly, we begin with Pompeiu operators that are used for first-order differential equations and related boundary value problems. Secondly, higher-order Pompeiu operators which are obtained by iterating the Pompeiu operators are introduced. These operators are important for higher-order partial differential equations. Next, modification of these operators for the unit disc are considered. We start by giving some preliminary results concerning the complex partial differential operators.

A complex valued function $w = u + iv$ where u and v are real valued functions of real variables x and y with

$$z = x + iy \quad \bar{z} = x - iy \ ,$$

will be denoted by $w(z)$. The complex partial differential operators ∂_z and

$\partial_{\bar{z}}$ are defined in terms of the real partial differential operators ∂_x and ∂_y by

$$\partial_z = \frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) , \quad \partial_{\bar{z}} = \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) .$$

They satisfy the basic rules of differentiation and also the relations

$$\frac{\partial \bar{w}}{\partial z} = \overline{\left(\frac{\partial w}{\partial \bar{z}}\right)}$$

and

$$\frac{\partial \bar{w}}{\partial \bar{z}} = \overline{\left(\frac{\partial w}{\partial z}\right)} .$$

If u and v satisfy the Cauchy Riemann system

$$u_x = v_y \ , \quad u_y = -v_x$$

for an analytic (holomorphic) function $w = u + iv$, then w satisfies $\dfrac{\partial w}{\partial \bar{z}} = 0$.

The following lemma gives the complex form of the well known Green-Gauss integral formula for functions of two real variables which are continuously differentiable in a bounded domain D with smooth boundary ∂D and continuous in the closure $\overline{D} = D \cap \partial D$. The set of continuous functions of $z = x + iy$ in $\overline{D}$ is denoted by $C(\overline{D})$ and $C^m(D)$ is the set of functions which are continuous and have continuous partial derivatives of order less than or equal to m in D. Henceforth, we will consider functions assuming complex values.

Lemma 2.1.1. *[36] Let $D \subset \mathbb{C}$ be a bounded domain with smooth boundary and let $w \in C^1(D) \bigcap C(\bar{D})$. Then*

$$\int\int_D w_{\bar{z}}(z) \, dxdy = \frac{1}{2i}\int_{\partial D} w(z)dz$$

and

$$\int\int_D w_z(z) \, dxdy = -\frac{1}{2i}\int_{\partial D} w(z)d\bar{z} .$$

The following theorem gives the Cauchy-Pompeiu representations for functions which are not assumed to be holomorphic.

Theorem 2.1.2. *[36] Let $D \subset \mathbb{C}$ be a bounded domain with smooth boundary and let $w \in C^1(D) \bigcap C(\bar{D})$. Then for $z \in D$ we have*

$$w(z) = \frac{1}{2\pi i} \int_{\partial D} w(\zeta) \frac{d\zeta}{\zeta - z} - \frac{1}{\pi} \int \int_D w_{\bar{\zeta}}(\zeta) \frac{d\xi d\eta}{\zeta - z} \ ,$$

$$w(z) = -\frac{1}{2\pi i} \int_{\partial D} w(\zeta) \frac{d\bar{\zeta}}{\overline{\zeta - z}} - \frac{1}{\pi} \int \int_D w_{\zeta}(\zeta) \frac{d\xi d\eta}{\overline{\zeta - z}} \ .$$

2.1.1 Pompeiu Operators

Definition 2.1.3. *Suppose D is a domain in $\mathbb{C}$. Also suppose that $F(z, \zeta)$ is a bounded function for each $z, \zeta \in D$. Then,*

$$K(z, \zeta) = \frac{F(z, \zeta)}{(\zeta - z)^{\alpha}}, \qquad 0 < \alpha \leq 2$$

is called a kernel with a weak or strong singularity depending on whether $\alpha < 2$ or $\alpha = 2$, respectively. The operator

$$Tu(z) = \int_D K(z, \zeta) u(\zeta) \ d\xi d\eta \ , \quad \zeta = \xi + i\eta$$

is called the corresponding singular integral operator.

Definition 2.1.4. *[36] Let D be a domain in $\mathbb{C}$ and let $f \in L^1(D)$. Then the operator T given by*

$$Tf(z) := -\frac{1}{\pi} \int \int_D f(\zeta) \frac{d\xi d\eta}{\zeta - z} \ , \qquad z \in \mathbb{C},$$

is called the Pompeiu operator. By $\overline{T}$ we denote

$$\overline{T} f(z) := -\frac{1}{\pi} \int \int_D f(\zeta) \frac{d\xi d\eta}{\overline{\zeta - z}} \ , \qquad z \in \mathbb{C}.$$

Thus, the Pompeiu operator T is a weakly singular operator. It is investigated extensively by Vekua [36]. For the differentiability of this operator we need the following definition of generalized derivatives.

Definition 2.1.5. *Let $f, g \in L^1(D)$. Then g is called the generalized (distributional) derivative of f with respect to $\bar{z}$ if for all $\varphi \in C_0^1(D)$*

$$\int\int_D \left(f \frac{\partial \varphi}{\partial \bar{z}} + g\varphi \right) \, dxdy = 0 \, ,$$

where $C_0^1(D)$ denotes the set of complex-valued functions in D continuously differentiable and having compact support in D. This derivative is denoted by $g = f_{\bar{z}} = \partial_{\bar{z}} f$. Analogously the relation

$$\int\int_D \left(f \frac{\partial \varphi}{\partial z} + g\varphi \right) \, dxdy = 0$$

defines a generalized derivative with respect to z.

These derivatives are also called derivatives in Sobolev's sense or weak sense.

Theorem 2.1.6. *[36] If $f \in L^1(D)$ then Tf has generalized first order derivative with respect to $\bar{z}$ equal to f, i.e.,*

$$\frac{\partial}{\partial \bar{z}} Tf = f$$

holds.

Theorem 2.1.7. *[14] If $f \in L^p(D)$, $p > 1$, then Tf has generalized first order derivative with respect to z equal to Πf,i.e.,*

$$\frac{\partial}{\partial z} Tf = \Pi f := -\frac{1}{\pi} \int\int_D f(\zeta) \frac{d\xi d\eta}{(\zeta - z)^2} \, , \qquad z \in \mathbb{C} \, .$$

Theorem 2.1.8. *[4] For $f \in L^p(\mathbb{C})$ we have $\Pi f \in L^p(\mathbb{C})$ and*

$$\|\Pi f\|_{L^p(\mathbb{C})} \leq \Lambda_p \|f\|_{L^p(\mathbb{C})} \quad (p > 1)$$

with $\|\Pi\|_{L^2(\mathbb{C})} = 1$.

2.1.2 Higher Order Pompeiu Operators

Iterating the operators T and $\overline{T}$, a hierarchy of kernel functions and higher-order integral operators are constructed and general higher order Cauchy-Pompeiu representation formulas are developed in [14].

Definition 2.1.9. *For $m, n \in \mathbb{Z}$ satisfying $0 \leq m + n$ and $0 < m^2 + n^2$ let*

$$K_{m,n} = \begin{cases} \dfrac{(-1)^n(-m)!}{(n-1)!\pi}(\zeta - z)^{m-1}(\overline{\zeta - z})^{n-1} & \text{if } m \leq 0, \\ \dfrac{(-1)^m(-n)!}{(m-1)!\pi}(\zeta - z)^{m-1}(\overline{\zeta - z})^{n-1} & \text{if } n \leq 0, \\ \dfrac{(\zeta - z)^{m-1}(\overline{\zeta - z})^{n-1}}{(m-1)!(n-1)!\pi}\left[\log|\zeta - z|^2 - \displaystyle\sum_{\mu=1}^{m-1}\frac{1}{\mu} - \sum_{\nu=1}^{n-1}\frac{1}{\nu}\right] & \text{if } 1 \leq m, n, \end{cases}$$

and for $f \in L^1(D), D \subset \mathbb{C}$ a domain,

$$T_{0,0}f(z) = f(z) \ ,$$

$$T_{m,n}f(z) = \int\int K_{m,n}(z,\zeta)f(\zeta)d\xi d\eta \quad \text{for } (m,n) \neq (0,0).$$

Example 2.1.10. *For the following particular choices of m and n, the operators $T_{m,n}$ are*

$$Tf(z) = T_{0,1} = -\frac{1}{\pi}\int\int_D f(\zeta)\frac{d\xi d\eta}{\zeta - z} \ ,$$

$$\overline{T}f(z) = T_{1,0} = -\frac{1}{\pi}\int\int_D f(\zeta)\frac{d\xi d\eta}{\overline{\zeta - z}} \ ,$$

$$\Pi f = T_{-1,1} = -\frac{1}{\pi}\int\int_D f(\zeta)\frac{d\xi d\eta}{(\zeta - z)^2} \ .$$

The principal attention is given to the strongly singular integral operators of the form $T_{-m,m}$, in our further investigations. They are viewed as a Cauchy principal value integral,

$$T_{-m,m}f(z) := \lim_{\epsilon \to 0}\int\int_{D_\epsilon} K_{-m,m}(z,\zeta)f(\zeta)d\xi d\eta \tag{2.1}$$

where D_ϵ is the domain $D - \{\zeta : |\zeta - z| \le \epsilon\}$. They can be analyzed with the theory of Calderon-Zygmund [19]. Essential properties are given in the following theorems.

Theorem 2.1.11. *[13] Assume* $m + n = 0, (m, n) \ne (0, 0)$, *and let* w *be a complex valued function in* $L^p(\mathbb{C})$ *where* $1 < p < \infty$. *Then* $T_{m,n}w$, *as defined by (2.1) with* $D = \mathbb{C}$, *also belong* $L^p(\mathbb{C})$, *and*

$$\|T_{m,n}\|_{L^p(\mathbb{C})} \le M(p)\|w\|_{L^p(\mathbb{C})} \ . \tag{2.2}$$

Theorem 2.1.12. *[13] For* $f \in L^2(\mathbb{C})$, $T_{-m,m}$ *is a unitary operator from* $L^2(\mathbb{C})$ *into itself, i.e.* $\|T_{-m,m}\|_{L^2(\mathbb{C})} = 1$.

For the differentiability, integrability and many other properties of these operators, see [13]

2.1.3 Operators for the Unit Disc

In the case of the unit disc $\mathbb{D} = \{z : |z| < 1\}$, a modification of the Cauchy-Pompeiu formula is given by the following theorem [4].

Theorem 2.1.13. *[14] Any* $w \in C^1(\mathbb{D}) \bigcap C(\overline{\mathbb{D}})$ *is representable as*

$$\begin{aligned} w(z) &= \frac{1}{2\pi i}\int_{\partial\mathbb{D}} Rew(\zeta)\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta} + \frac{1}{2\pi}\int_{\partial\mathbb{D}} Imw(\zeta)\frac{d\zeta}{\zeta} \\ &\quad -\frac{1}{\pi}\int\int_{\mathbb{D}}\left(\frac{w_{\bar\zeta}(\zeta)}{\zeta-z} + \frac{z\overline{w_{\bar\zeta}(\zeta)}}{1-z\bar\zeta}\right) d\xi d\eta\ , \quad z \in \mathbb{D}. \end{aligned}$$

Subtracting $iImw(0)$ from either side, the following formula is obtained.

Corollary 2.1.14. *[14] Any* $w \in C^1(\mathbb{D}) \bigcap C(\overline{\mathbb{D}})$ *can be represented as*

$$\begin{aligned} w(z) &= \frac{1}{2\pi i}\int_{\partial\mathbb{D}} Rew(\zeta)\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta} \\ &- \frac{1}{2\pi}\int\int_{\mathbb{D}}\left(\frac{w_{\bar\zeta}(\zeta)}{\zeta}\frac{\zeta+z}{\zeta-z} + \frac{\overline{w_{\bar\zeta}(\zeta)}}{\bar\zeta}\frac{1+z\bar\zeta}{1-z\bar\zeta}\right) d\xi d\eta + iImw(0),\ z \in \mathbb{D}. \end{aligned}$$

This formula is called the Cauchy-Schwarz-Poisson-Pompeiu formula.

The operator $\widetilde{T}_1$ which was mentioned in introduction is defined in [4] as follows.

Definition 2.1.15. *[4] For $f \in L^1(\mathbb{D})$,*

$$\widetilde{T}_1 f(z) := -\frac{1}{2\pi} \int\int_{\mathbb{D}} \left(\frac{f(\zeta)}{\zeta} \frac{\zeta + z}{\zeta - z} + \frac{\overline{f(\zeta)}}{\bar{\zeta}} \frac{1 + z\bar{\zeta}}{1 - z\bar{\zeta}} \right) d\xi d\eta.$$

Differentiability properties of the operators $\widetilde{T}_1$ are given in the next lemma.

Lemma 2.1.16. *[4] For $f \in L^1(\mathbb{D})$, the function $\widetilde{T}_1 f$ has generalized first order derivatives*

$$\frac{\partial}{\partial \bar{z}} \widetilde{T}_1 f = f \qquad \frac{\partial}{\partial z} \widetilde{T}_1 f = \Pi_1 f$$

where Π_1 is given by

$$\Pi_1 f := -\frac{1}{\pi} \int\int_{\mathbb{D}} \left[\frac{f(\zeta)}{(\zeta - z)^2} + \frac{\overline{f(\zeta)}}{(1 - z\bar{\zeta})^2} \right] d\xi d\eta \;, \qquad z \in \mathbb{D} \;.$$

The operator Π_1 is a strongly singular integral operator from $L^p(\mathbb{D})$ into itself, satisfying $\|\Pi_1\|_{L^2(\mathbb{D})} = 1$, which is proved in [36], page 210.

The following definitions and theorems are given in [4], [13], [5], [14] and in [6].

Definition 2.1.17. *Let $k \in \mathbb{N}$, $f \in L^p(\mathbb{D})$, $1 \le p$. Then for $z \in \mathbb{D}$,*

$$\widetilde{T}_k f(z) := \frac{(-1)^k}{2\pi(k-1)!} \int\int_{\mathbb{D}} (\overline{\zeta - z} + \zeta - z)^{k-1} \left[\frac{f(\zeta)}{\zeta} \frac{\zeta + z}{\zeta - z} + \frac{\overline{f(\zeta)}}{\bar{\zeta}} \frac{1 + z\bar{\zeta}}{1 - z\bar{\zeta}} \right] d\xi d\eta \;.$$

Moreover, $\widetilde{T}_0 f = f$.

Theorem 2.1.18. *For $k \in \mathbb{N}$, $f \in L^p(\mathbb{D})$, $1 \le p$ and $z \in \mathbb{D}$, $\widetilde{T}_k f(z) = \widetilde{T}_1^k f(z)$.*

Proof. If $k = 1$, there is nothing to prove. Let $k \ge 2$ and $\{f_n\}$ be sequence in $C_0^\infty(\mathbb{D})$ converging to f in the norm of $L^1(\mathbb{D})$. Then

$$\widetilde{T}_k f_n \to \widetilde{T}_k f \quad \text{in } L^1(\mathbb{D})$$

We have for $k \geq 2$,

$$\frac{\partial \widetilde{T}_k f_n}{\partial \bar{z}} = \frac{(-1)^{k-1}}{2\pi(k-2)!} \int\int_{\mathbb{D}} (\overline{\zeta - z} + \zeta - z)^{k-2} \left[\frac{f_n(\zeta)}{\zeta} \frac{\zeta + z}{\zeta - z} + \frac{\overline{f_n(\zeta)}}{\bar{\zeta}} \frac{1 + z\bar{\zeta}}{1 - z\bar{\zeta}} \right] d\xi d\eta$$

$$= \widetilde{T}_{k-1} f_n(z) \ ,$$

since there is no singularity at $\zeta = z$ and the differentiation under the integral sign is allowed. For $z \in \partial\mathbb{D}$, $z\bar{z} = 1$ is satisfied and thus,

$$\frac{\overline{1 + z\bar{\zeta}}}{\bar{z}} = \zeta + z \quad \text{and} \quad -\frac{\overline{1 - z\bar{\zeta}}}{\bar{z}} = \zeta - z$$

hold. Now for $z \in \partial\mathbb{D}$,

$$\widetilde{T}_k f_n(z) = \frac{(-1)^k}{2\pi(k-1)!} \int\int_{\mathbb{D}} (2Re(\zeta - z))^{k-1} \left[\frac{f_n(\zeta)}{\zeta} \frac{\zeta + z}{\zeta - z} - \frac{\overline{f_n(\zeta)}}{\bar{\zeta}} \frac{\overline{\zeta + z}}{\overline{\zeta - z}} \right] d\xi d\eta$$

$$= \frac{(-1)^k i}{\pi(k-1)!} \int\int_{\mathbb{D}} (2Re(\zeta - z))^{k-1} Im \left(\frac{f_n(\zeta)}{\zeta} \frac{\zeta + z}{\zeta - z} \right) d\xi d\eta$$

is obtained. Thus, $Re\widetilde{T}_k f_n(z) = 0$ follows. If $z = 0$,

$$\widetilde{T}_k f_n(0) = \frac{(-1)^k}{2\pi(k-1)!} \int\int_{\mathbb{D}} (2Re(\zeta))^{k-1} \left[\frac{f_n(\zeta)}{\zeta} + \frac{\overline{f_n(\zeta)}}{\bar{\zeta}} \right] d\xi d\eta$$

$$= \frac{(-1)^k}{\pi(k-1)!} \int\int_{\mathbb{D}} (2Re(\zeta))^{k-1} Re \left(\frac{f_n(\zeta)}{\zeta} \right) d\xi d\eta$$

gives $Im\widetilde{T}_k f_n(0) = 0$. Now applying the Cauchy-Schwarz-Poisson-Pompeiu formula to $\widetilde{T}_k f_n$ gives

$$\widetilde{T}_k f_n(z) = -\frac{1}{2\pi} \int\int_{\mathbb{D}} \left(\frac{\widetilde{T}_{k-1} f_n(\zeta)}{\zeta} \frac{\zeta + z}{\zeta - z} + \frac{\overline{\widetilde{T}_{k-1} f_n(\zeta)}}{\bar{\zeta}} \frac{1 + z\bar{\zeta}}{1 - z\bar{\zeta}} \right) d\xi d\eta$$

$$= \widetilde{T}_1(\widetilde{T}_{k-1}) f_n(z) \ , z \in \mathbb{D}.$$

Since $\widetilde{T}_k f_n \to \widetilde{T}_k f \quad$ in $L^1(\mathbb{D})$, the result follows. □

Theorem 2.1.19. *[4] For $f \in L^p(\mathbb{D})$, $p \geq 1$ and $k \in \mathbb{N}$, $\widetilde{T}_k f$ has the following properties:*

$$\frac{\partial^l}{\partial \bar{z}^l} \widetilde{T}_k f = \widetilde{T}_{k-l} f, \ 1 \leq l \leq k, \tag{2.3}$$

$$Re \, \frac{\partial^l}{\partial \bar{z}^l} \widetilde{T}_k f = 0 \ on \ \partial\mathbb{D}, \ 0 \leq l \leq k-1, \tag{2.4}$$

$$Im \, \frac{\partial^l}{\partial \bar{z}^l} \widetilde{T}_k f(0) = 0 \ , \ 0 \leq l \leq k-1 \ . \tag{2.5}$$

Moreover,

$$\frac{\partial^l}{\partial z^l} \widetilde{T}_k f(z) = \sum_{\lambda=0}^{l} \left(\begin{array}{c} l \\ \lambda \end{array} \right) \frac{(-1)^{k-\lambda}(l-\lambda)!}{(k-\lambda-1)!} \frac{1}{\pi} \int\int_{\mathbb{D}} (\zeta - z + \overline{\zeta - z})^{k-\lambda-1}$$

$$\left[\frac{f(\zeta)}{(\zeta - z)^{l-\lambda+1}} + \frac{\bar{\zeta}^{l-\lambda-1}\overline{f(\zeta)}}{(1 - z\bar{\zeta})^{l-\lambda+1}} \right] d\xi d\eta +$$

$$+ \frac{(-1)^{k-l-1}}{(k-l-1)!} \frac{1}{2\pi} \int\int_{\mathbb{D}} (\zeta - z + \overline{\zeta - z})^{k-l-1} \left[\frac{f(\zeta)}{\zeta} + \frac{\overline{f(\zeta)}}{\bar{\zeta}} \right] d\xi d\eta \tag{2.6}$$

is a weakly singular integral if $0 \leq l \leq k-1$, while

$$\frac{\partial^k}{\partial z^k} \widetilde{T}_k f(z) = \frac{(-1)^k k}{\pi} \int\int_{\mathbb{D}} \left[\left(\frac{\overline{\zeta - z}}{\zeta - z} \right)^{k-1} \frac{f(\zeta)}{(\zeta - z)^2} \right.$$

$$\left. + \left(\frac{\zeta - z + \overline{\zeta - z}}{1 - z\bar{\zeta}} \bar{\zeta} - 1 \right)^{k-1} \frac{\overline{f(\zeta)}}{(1 - z\bar{\zeta})^2} \right] d\xi d\eta \tag{2.7}$$

is a strongly singular integral.

Proof. The equalities (2.3),(2.4) and (2.5) follow directly from the Theorem 2.1.17. To obtain (2.6) and (2.7), Leibniz rule is used. □

As Begehr pointed out in [5], [6] and in [14], the properties of the operators (2.7) are not known in the literature. One of the aims of this thesis is to prove the boundedness of these operators on L^p spaces for $1 < p < \infty$ and particularly, estimate the L^2 norm of these operators. Their proofs are given in Chapter 3.

2.2 Schwarz Boundary Value Problems for Model Equations in the Unit Disc

In this section, we mainly present the Schwarz problems for the basic model differential equations in the unit disc. Firstly, we present the Schwarz problem for analytic functions with unique solution. Then, the Schwarz problem is discussed for first and second-order equations. Finally we present the Schwarz problem for inhomogeneous polyanalytic functions.

2.2.1 Schwarz Boundary Value Problem for Analytic Functions

The following problem is discussed by Schwarz in [35].

Schwarz boundary value problem. Find an analytic function w in the unit disc $\mathbb{D}$, i.e. a solution to $w_{\overline{z}} = 0$ in $\mathbb{D}$, satisfying

$$\text{Re } w = \gamma \text{ on } \partial\mathbb{D} \ , \ \text{Im } w(0) = c$$

for $\gamma \in C(\partial\mathbb{D}; \mathbb{R}), c \in \mathbb{R}$ given.

Theorem 2.2.1. *This Schwarz problem is uniquely solvable. The solution is given by the Schwarz formula*

$$w(z) = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \gamma(\zeta) \frac{\zeta + z}{\zeta - z} \frac{d\zeta}{\zeta} + ic \ .$$

2.2.2 Schwarz Boundary Value Problem for First-Order Equations

Theorem 2.2.2. *[16] The Schwarz problem for the inhomogeneous Cauchy-Riemann equation in the unit disc $\mathbb{D}$ defined by*

$$w_{\overline{z}} = f \ in \ \mathbb{D} \ , \ Re \ w = \gamma \ on \ \partial\mathbb{D} \ , \ Im \ w(0) = c$$

for $f \in L^1(\mathbb{D}), \gamma \in C(\partial\mathbb{D};\mathbb{R}), c \in \mathbb{R}$ is uniquely solvable in the distributional sense by the Cauchy-Schwarz-Poisson-Pompeiu formula

$$w(z) = \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} \gamma(\zeta)\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta} + ic - \frac{1}{2\pi}\int\int_{\mathbb{D}}\Big[\frac{f(\zeta)}{\zeta}\frac{\zeta+z}{\zeta-z} + \frac{\overline{f(\zeta)}}{\overline{\zeta}}\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}\Big]d\xi d\eta\ .$$

Remark 2.2.3. *In the above theorem, the second integral is $\widetilde{T}_1 f(z)$. If homogeneous boundary conditions are given, it is the only solution to the Schwarz problem for the inhomogeneous Cauchy-Riemann equation.*

2.2.3 Schwarz Boundary Value Problem for Second-Order Equations

There are two basic second-order differential operators. One of them is the Laplace operator $\partial_z\partial_{\bar{z}}$, the other is the Bitsadze operator $\partial_{\bar{z}}^2$. The Schwarz problem is considered for both operators in the sequel. Since the operator ∂_z^2 is the complex conjugate of the Bitsadze operator, all of the results can be applied to that operator [16], [10].

Theorem 2.2.4. *[10] The Schwarz problem for the Poisson equation, in the unit disc $\mathbb{D}$, defined by*

$$w_{z\bar{z}} = f \text{ in } \mathbb{D}\ ,\ Re\, w = \gamma_0\ ,\ Re\, w_{\bar{z}} = \gamma_1 \text{ on } \partial\mathbb{D}\ ,\ Im\, w(0) = c_0\ ,\ Im\, w_{\bar{z}}(0) = c_1\ ,$$

is uniquely solvable in the distributional sense for $f \in L^1(\mathbb{D}), \gamma_0, \gamma_1 \in C(\partial\mathbb{D};\mathbb{R}), c_0, c_1 \in \mathbb{R}$. The solution is

$$w(z) = ic_0 + ic_1(z+\bar{z}) - \frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma_0(\zeta)\overline{\frac{\zeta+z}{\zeta-z}}\frac{d\bar{\zeta}}{\bar{\zeta}}$$

$$+\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma_1(\zeta)[\zeta\log(1-z\bar{\zeta})^2 - \bar{\zeta}\log(1-\bar{z}\zeta)^2 + z - \bar{z}]\frac{d\zeta}{\zeta}$$

$$+\frac{1}{\pi}\int\int_{\mathbb{D}}\{f(\zeta)[\log|\zeta-z|^2 - \log(1-z\bar{\zeta})] - \overline{f(\zeta)}\log(1-z\bar{\zeta})\}d\xi d\eta$$

$$-\frac{1}{\pi}\int\int_{\mathbb{D}}\Big\{f(\zeta)\Big[\frac{\log(1-z\overline{\zeta})}{\overline{\zeta}^2}+\log|\zeta|\Big]-\overline{f(\zeta)}\Big[\frac{\log(1-\overline{z}\zeta)}{\zeta^2}+\log|\zeta|\Big]\Big\}d\xi d\eta$$

$$-\frac{1}{\pi}\int\int_{\mathbb{D}}\Big[\frac{f(\zeta)}{\zeta}+\frac{\overline{f(\zeta)}}{\overline{\zeta}}\Big]\frac{z-\overline{z}}{2}\ d\xi d\eta\ .$$

The result follows from solving the Schwarz problem for inhomogeneous Cauchy-Riemann equations

$$w_{\overline{z}} = g \ in\ \mathbb{D}\ ,\ \mathrm{Re}\ w = \gamma_0\ on\ \partial\mathbb{D}\ ,\ \mathrm{Im}\ w(0) = c_0$$

and

$$g_z = f\ in\ \mathbb{D}\ ,\ \mathrm{Re}\ g = \gamma_1\ on\ \partial\mathbb{D}\ ,\ \mathrm{Im}\ g(0) = c_1\ .$$

Also a dual result can be obtained for the same equation with the boundary conditions

$$\mathrm{Re}\ w = \gamma_0\ ,\ \mathrm{Re}\ w_{\overline{z}} = \gamma_1\ on\ \partial\mathbb{D}\ ,\ \mathrm{Im}\ w(0) = c_0\ ,\ \mathrm{Im}\ w_{\overline{z}}(0) = c_1\ .$$

Theorem 2.2.5. *[8],[10] The Schwarz problem for the inhomogeneous Bitsadze equation in the unit disc* $\mathbb{D}$ *defined by*

$$w_{\overline{z}\overline{z}} = f\ in\ \mathbb{D}\ ,\ Re\, w = \gamma_0\ ,\ Re\, w_{\overline{z}} = \gamma_1\ on\ \partial\mathbb{D}\ ,\ Im\, w(0) = c_0\ ,\ Im\, w_{\overline{z}}(0) = c_1\ ,$$

for $f \in L^1(\mathbb{D}), \gamma_0, \gamma_1 \in C(\partial\mathbb{D};\mathbb{R}), c_0, c_1 \in \mathbb{R}$ *is uniquely solvable in the distributional sense by*

$$w(z) = ic_0 + i(z+\overline{z}) + \frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma_0(\zeta)\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta}$$

$$-\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma_1(\zeta)\frac{\zeta+z}{\zeta-z}\ (\zeta-z+\overline{\zeta-z})\frac{d\zeta}{\zeta}+$$

$$\frac{1}{2\pi}\int\int_{\mathbb{D}}\Big(\frac{f(\zeta)}{\zeta}\frac{\zeta+z}{\zeta-z}+\frac{\overline{f(\zeta)}}{\overline{\zeta}}\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}\Big)(\zeta-z+\overline{\zeta-z})d\xi d\eta\ .$$

The result follows from solving the Schwarz problem for the inhomogeneous Cauchy-Riemann equations

$$w_{\overline{z}} = g \ in \ \mathbb{D} \ , \ \mathrm{Re} \ w = \gamma_0 \ on \ \partial\mathbb{D} \ , \ \mathrm{Im} \ w(0) = c_0$$

and

$$g_{\bar{z}} = f \ in \ \mathbb{D} \ , \ \mathrm{Re} \ g = \gamma_1 \ on \ \partial\mathbb{D} \ , \ \mathrm{Im} \ g(0) = c_1 \ .$$

In [17] and [10], the Dirichlet, Neumann and mixed boundary value problems are also considered for these model equations.

2.2.4 Schwarz Boundary Value Problem for Inhomogeneous Polyanalytic Equations

We begin with the definition of the polyanalytic functions, see [3].

Definition 2.2.6. *A function of the form*

$$w(z) = \sum_{k=0}^{n-1} \phi_k(z)\bar{z}^k \ ,$$

where $\phi_k(z)$ are analytic functions, is called a polyanalytic function of the n-th order.

Thus if $w(z)$ is a polyanalytic function of order k, it satisfies the equation

$$\frac{\partial^k w}{\partial \bar{z}^k} = 0 \ .$$

Theorem 2.2.7. *[10] The Schwarz problem for the inhomogeneous polyanalytic equation in the unit disc $\mathbb{D}$ defined by*

$$\partial_{\bar{z}}^k w = f \ in \ \mathbb{D} \ , \ Re \ \partial_{\bar{z}}^l w = \gamma_l \ on \ \partial\mathbb{D} \ , \ Im \ \partial_{\bar{z}}^l w(0) = c_l \ , \ 0 \le l \le k-1 \ ,$$

is uniquely solvable in the distributional sense for $f \in L^1(\mathbb{D}), \gamma_l \in C(\partial\mathbb{D};\mathbb{R}), c_l \in \mathbb{R}, 0 \le l \le k-1$. The solution is

$$w(z) = i\sum_{l=0}^{k-1} \frac{c_l}{l!} \ (z+\overline{z})^l + \sum_{l=0}^{k-1} \frac{(-1)^l}{2\pi i l!} \int_{\partial\mathbb{D}} \gamma_l(\zeta)\frac{\zeta+z}{\zeta-z} \ (\zeta - z + \overline{\zeta - z})^l \frac{d\zeta}{\zeta}$$

$$+\frac{(-1)^k}{2\pi(k-1)!}\int\int_{\mathbb{D}}\Big(\frac{f(\zeta)}{\zeta}\frac{\zeta+z}{\zeta-z}+\frac{\overline{f(\zeta)}}{\overline{\zeta}}\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}\Big)(\zeta-z+\overline{\zeta-z})^{k-1}d\xi d\eta\ .$$

In this section we have considered the Schwarz problem for model equations. In Chapter 4, we consider the Schwarz problem for general higher-order complex partial differential equations.

CHAPTER 3

The Properties of $\partial_z^k \widetilde{T}_k$

In this chapter, we discuss some properties of the linear operators

$$\begin{aligned}\frac{\partial^k}{\partial z^k}\widetilde{T}_k f(z) &= \frac{(-1)^k k}{\pi}\int\int_{\mathbb{D}}\left[\left(\frac{\overline{\zeta - z}}{\zeta - z}\right)^{k-1}\frac{f(\zeta)}{(\zeta - z)^2}\right. \\ &\quad \left. + \left(\frac{\zeta - z + \overline{\zeta - z}}{1 - z\bar{\zeta}}\bar{\zeta} - 1\right)^{k-1}\frac{\overline{f(\zeta)}}{(1 - z\bar{\zeta})^2}\right] d\xi d\eta\end{aligned}$$

in L^p space. We call these operators Π_k. These operators are Calderon-Zygmund type operators. It is known that, Π_1 has L^2 norm 1, see [36] where it is called $\widetilde{\Pi}$,. Our aim is to prove the boundedness of the operators Π_k in L^p and particularly, to estimate the L^2 norms of them. Firstly, these operators are decomposed into two parts. One of the parts is the higher-order strongly singular Pompeiu operator $T_{-k,k}$, which is extensively studied by Begehr and Hile in [14]. As given in Chapter 2, these operators are known to be bounded in L^p $(1 < p < \infty)$, and unitary in the space L^2. The other part, denoted by P_k, is investigated in the space L^p and it is proved that they are bounded for $1 < p < \infty$, using Forelli-Rudin estimates and the well-known Schur's theorem. Next we give an estimate for the L^2 norms of Π_k. Then, we discuss the compactness of the operators P_k from L^{p_1} to L^{p_2} for $p_2 < p_1$.

3.1 Norm Estimates

3.1.1 Preliminaries

In the proof of our norm estimates, one of the two facts we will use is given by the following theorem, usually called Schur's test, which is a very effective tool in proving the L^p boundedness of integral operators.

Theorem 3.1.1. *(Schur) Let μ be a positive measure on a measure space X. Let $K(x,y)$ be a positive measurable function on $X \times X$, and let $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. If there exists a positive constant C and a positive measurable function h on X such that*

$$\int_X K(x,y)h(y)^q d\mu(y) \leq Ch(x)^q$$

and

$$\int_X K(x,y)h(x)^p d\mu(x) \leq Ch(y)^p$$

for all x and y in X, then the integral operator

$$Tf(x) = \int_X K(x,y)f(y)d\mu(y)$$

is bounded on $L^p(X, d\mu)$ with $\|T\|_{L^p(X)} \leq C$.

For the proof, see [38], [27] for example.

The second result we will use is due to Forelli-Rudin [24]. In their paper the above Schur's test is seen with $C = a^{1/p}$ and $C = b^{1/q}$ in the integral estimates, respectively. Before giving the following proposition, we introduce some general notation. Let $\mathbb{C}^n$ denote the n dimensional complex space of all ordered n-tuples $z = (z_1, \ldots, z_n)$ of complex numbers with the inner product

$$\langle z, w \rangle = z_1\bar{w}_1 + \ldots + z_n\bar{w}_n \ .$$

$\mathbb{B} = \{z \in \mathbb{C}^n : |z| = \langle z, z\rangle^{1/2} < 1\}$ is the unit ball of $\mathbb{C}^n$. The letter ν shall denote the normalized Lebesgue measure on $\mathbb{C}^n$, so that $\nu(\mathbb{B}) = 1$.

Corresponding to a complex number

$$s = \sigma + it \; (\sigma > -1, -\infty < t < \infty) \,,$$

a kernel K_s is defined by

$$K_s(z,w) = \frac{(1-|w|^2)^s}{(1-\langle z,w\rangle)^{n+1+s}}$$

where $z, w \in \mathbb{B}$. Note that, the kernel K_0 is the Bergman kernel for $\mathbb{B}$. For $z \in \mathbb{C}^n$, the letter h will be used for the function $h(z) = 1 - |z|^2$. The following proposition gives estimates for certain integrals on $\mathbb{B}$, which are denoted as Forelli-Rudin estimates in the literature. The generalizations of these type of estimates are given in the book of Rudin [32] as Proposition 1.4.10.

Proposition 3.1.2. *[24]* ***(a)*** *If* $0 < c < 1 + \sigma$, *then*

$$\sup_{z\in\mathbb{B}} h^c(z) \int_{\mathbb{B}} |K_s(z,w)| h^{-c}(w) \, d\nu(w) = A(c,s,n)$$

where $A(c,s,n) = \dfrac{n!\Gamma(1+\sigma-c)\Gamma(c)}{|\Gamma\left(\frac{n+1+s}{2}\right)|^2}$.

(b) *If* $0 < c + \sigma < 1 + \sigma$, *then*

$$\sup_{w\in\mathbb{B}} h^c(w) \int_{\mathbb{B}} |K_s(z,w)| h^{-c}(z) \, d\nu(z) = A(c+\sigma, s, n).$$

(c) *If* $\sigma > 0$, *then*

$$\sup_{w\in\mathbb{B}} \int_{\mathbb{B}} |K_s(z,w)| \, d\nu(z) = \frac{n!\Gamma(\sigma)}{|\Gamma\left(\frac{n+1+s}{2}\right)|^2} \,.$$

(d) *If* $-\infty < t < \infty$, *then*

$$\sup \frac{1}{1+|\log h(z)|} \int_{\mathbb{B}} |K_{it}(z,w)| dv(w) = A_0 \,,$$

where $A_0 = \dfrac{n!}{|\Gamma\left(\frac{n+1+it}{2}\right)|^2}$.

In [24], Forelli and Rudin investigate a class of operators T_s, defined by

$$T_s f(z) = \begin{pmatrix} n+s \\ n \end{pmatrix} \int_{\mathbb{B}} K_s(z,w) f(w) d\nu(w) \qquad (z \in \mathbb{B}) \ ,$$

which are induced by K_s. These operators are known as Bergman-type projections. They prove that, for $1 \le p < \infty$, T_s is a bounded linear operator on $L^p(\mathbb{B})$ if and only if $\mathrm{Re}(1+s) > \frac{1}{p}$. This property is also observed by Begehr and Dzhuraev [11]. It is shown that [24], the theorem does not extend to the case $p = \infty$. Kolaski [29] prove that theorem for the weighted case and Choe [20] generalized the weighted case. Kaptanoğlu has defined and characterized extended Bergman projections from Lebesgue classes onto all Besov spaces, see [28].

3.1.2 Main results

Before giving the proofs of the main results, we start with decomposing the operator Π_k

$$\Pi_k f(z) = T_{-k,k} f(z) + P_k f(z) \ ,$$

where

$$T_{-k,k} f(z) = \frac{(-1)^k k}{\pi} \int\int_{\mathbb{D}} \left(\frac{\overline{\zeta - z}}{\zeta - z} \right)^{k-1} \frac{f(\zeta)}{(\zeta - z)^2} d\xi d\eta \tag{3.1}$$

and

$$P_k f(z) = \frac{(-1)^k k}{\pi} \int\int_{\mathbb{D}} \left(\frac{\overline{\zeta - z} + \zeta - z}{1 - z\bar{\zeta}} \bar{\zeta} - 1 \right)^{k-1} \frac{\overline{f(\zeta)}}{(1 - z\bar{\zeta})^2} d\xi d\eta \ . \tag{3.2}$$

Our main results are the following theorems.

Theorem 3.1.3. *Let f be a complex valued function in $L^p(\mathbb{D})$ where $1 < p < \infty$. Then $\Pi_k f$ also belongs to $L^p(\mathbb{D})$, and there exists a constant $C(k,p) > 0$ depending on p and k such that*

$$\|\Pi_k f\|_{L^p(\mathbb{D})} \le C(k,p) \|f\|_{L^p(\mathbb{D})}$$

holds.

Proof. Let us start with the operator P_k,

$$P_k f(z) = \frac{(-1)^k k}{\pi} \int\int_{\mathbb{D}} \left(\frac{\overline{\zeta - z} + \zeta - z}{1 - z\bar{\zeta}} \bar{\zeta} - 1 \right)^{k-1} \frac{\overline{f(\zeta)}}{(1 - z\bar{\zeta})^2} d\xi d\eta$$

$$= \frac{(-1)^k k}{\pi} \int\int_{\mathbb{D}} \left(\frac{|\zeta|^2 - 1 + \bar{\zeta}(\bar{\zeta} - \bar{z})}{1 - z\bar{\zeta}} \right)^{k-1} \frac{\overline{f(\zeta)}}{(1 - z\bar{\zeta})^2} d\xi d\eta \, .$$

Taking the absolute value of both sides,

$$|P_k f(z)| \leq \frac{k}{\pi} \int\int_{\mathbb{D}} \frac{||\zeta|^2 - 1 + \bar{\zeta}(\bar{\zeta} - \bar{z})|^{k-1}}{|1 - z\bar{\zeta}|^{k+1}} |f(\zeta)| d\xi d\eta$$

$$\leq \frac{k}{\pi} \int\int_{\mathbb{D}} \frac{((1 - |\zeta|^2) + |\zeta||\zeta - z|)^{k-1}}{|1 - z\bar{\zeta}|^{k+1}} |f(\zeta)| d\xi d\eta$$

holds. Using the binomial theorem we have

$$|P_k f(z)| \leq \frac{k}{\pi} \int\int_{\mathbb{D}} \frac{\sum_{m=0}^{k-1} \binom{k-1}{m} (1 - |\zeta|^2)^m (|\zeta||\zeta - z|)^{k-1-m}}{|1 - z\bar{\zeta}|^{k+1}} |f(\zeta)| \, d\xi d\eta \, ,$$

and by the fact that $\left| \frac{\zeta - z}{1 - z\bar{\zeta}} \right| \leq 1$ for $|z| \leq 1$, $|\zeta| < 1$, we obtain

$$|P_k f(z)| \leq \frac{k}{\pi} \int\int_{\mathbb{D}} \frac{\sum_{m=0}^{k-1} \binom{k-1}{m} (1 - |\zeta|^2)^m |1 - z\bar{\zeta}|^{k-1-m}}{|1 - z\bar{\zeta}|^{k+1}} |f(\zeta)| \, d\xi d\eta$$

$$= \frac{k}{\pi} \sum_{m=0}^{k-1} \binom{k-1}{m} \int\int_{\mathbb{D}} \frac{(1 - |\zeta|^2)^m}{|1 - z\bar{\zeta}|^{m+2}} |f(\zeta)| \, d\xi d\eta \, . \tag{3.3}$$

Letting $0 < a < \min\{1 + m, \frac{1}{p-1}\}$, $m = 0, 1, \ldots, k-1$ we have

$$|P_k f(z)|$$

$$\leq \frac{k}{\pi} \sum_{m=0}^{k-1} \binom{k-1}{m} \int\int_{\mathbb{D}} \frac{(1 - |\zeta|^2)^m (1 - |\zeta|^2)^{-a/q} (1 - |\zeta|^2)^{a/q}}{|1 - z\bar{\zeta}|^{m+2}} |f(\zeta)| \, d\xi d\eta \, ,$$

where $\frac{1}{p}+\frac{1}{q}=1$. Applying Hölder's inequality to the integral we get

$$|P_k f(z)| \le k \sum_{m=0}^{k-1} \begin{pmatrix} k-1 \\ m \end{pmatrix} \frac{1}{\pi} \left(\int\int_{\mathbb{D}} \frac{(1-|\zeta|^2)^m (1-|\zeta|^2)^{-a}}{|1-z\bar{\zeta}|^{m+2}} \, d\xi d\eta \right)^{1/q}$$

$$\times \left(\int\int_{\mathbb{D}} \frac{(1-|\zeta|^2)^m (1-|\zeta|^2)^{pa/q}}{|1-z\bar{\zeta}|^{m+2}} |f(\zeta)|^p \, d\xi d\eta \right)^{1/p} .$$

Using the Hölder's inequality for the summation leads to

$$|P_k f(z)| \le k \left(\sum_{m=0}^{k-1} \begin{pmatrix} k-1 \\ m \end{pmatrix} \frac{1}{\pi} \int\int_{\mathbb{D}} \frac{(1-|\zeta|^2)^m (1-|\zeta|^2)^{-a}}{|1-z\bar{\zeta}|^{m+2}} \, d\xi d\eta \right)^{1/q}$$

$$\times \left(\sum_{m=0}^{k-1} \begin{pmatrix} k-1 \\ m \end{pmatrix} \frac{1}{\pi} \int\int_{\mathbb{D}} \frac{(1-|\zeta|^2)^m (1-|\zeta|^2)^{pa/q}}{|1-z\bar{\zeta}|^{m+2}} |f(\zeta)|^p \, d\xi d\eta \right)^{1/p} ,$$

or

$$|P_k f(z)| \le k \left(\sum_{m=0}^{k-1} \begin{pmatrix} k-1 \\ m \end{pmatrix} \frac{1}{\pi} \int\int_{\mathbb{D}} \frac{(1-|\zeta|^2)^{m-a}}{|1-z\bar{\zeta}|^{m+2}} \, d\xi d\eta \right)^{1/q}$$

$$\times \left(\sum_{m=0}^{k-1} \begin{pmatrix} k-1 \\ m \end{pmatrix} \frac{1}{\pi} \int\int_{\mathbb{D}} \frac{(1-|\zeta|^2)^{m+pa/q}}{|1-z\bar{\zeta}|^{m+2}} |f(\zeta)|^p \, d\xi d\eta \right)^{1/p} .$$

Since $\frac{1}{\pi}\, dxdy$ is the normalized measure in $\mathbb{D}$, using Proposition 3.2.2 part (a) with

$$0 < a < 1+m \ , \ m = 0, 1, \ldots, k-1$$

we get

$$|P_k f(z)|^p \le k^p \left(\sum_{m=0}^{k-1} \begin{pmatrix} k-1 \\ m \end{pmatrix} C_1(m,a)(1-|z|^2)^{-a} \right)^{p/q}$$

$$\times \left(\sum_{m=0}^{k-1} \begin{pmatrix} k-1 \\ m \end{pmatrix} \frac{1}{\pi} \int\int_{\mathbb{D}} \frac{(1-|\zeta|^2)^{m+pa/q}}{|1-z\bar{\zeta}|^{m+2}} |f(\zeta)|^p \, d\xi d\eta \right)$$

$$= k^p \left(\sum_{m=0}^{k-1} \begin{pmatrix} k-1 \\ m \end{pmatrix} C_1(m,a) \right)^{p/q} (1-|z|^2)^{-ap/q}$$

$$\times \left(\sum_{m=0}^{k-1} \binom{k-1}{m} \frac{1}{\pi} \int\int_{\mathbb{D}} \frac{(1-|\zeta|^2)^{m+pa/q}}{|1-z\bar{\zeta}|^{m+2}} |f(\zeta)|^p \, d\xi d\eta \right) ,$$

where $C_1(m,a) = \dfrac{\Gamma(1+m-a)\Gamma(a)}{\Gamma^2(\frac{m+2}{2})}$. Integrating both sides of this inequality we have

$$\int\int_{\mathbb{D}} |P_k f(z)|^p \, dxdy \le k^p \left(\sum_{m=0}^{k-1} \binom{k-1}{m} C_1(m,a) \right)^{p/q}$$

$$\times \left(\sum_{m=0}^{k-1} \binom{k-1}{m} \int\int_{\mathbb{D}} (1-|z|^2))^{-pa/q} \right.$$

$$\left. \times \frac{1}{\pi} \int\int_{\mathbb{D}} \frac{(1-|\zeta|^2)^{m+pa/q}}{|1-z\bar{\zeta}|^{m+2}} |f(\zeta)|^p \, d\xi d\eta \, dxdy \right) ,$$

and utilizing Fubini Theorem we obtain

$$\int\int_{\mathbb{D}} |P_k f(z)|^p \, dxdy \le k^p \left(\sum_{m=0}^{k-1} \binom{k-1}{m} C_1(m,a) \right)^{p/q}$$

$$\times \left(\sum_{m=0}^{k-1} \binom{k-1}{m} \int\int_{\mathbb{D}} (1-|\zeta|^2)^{pa/q} |f(\zeta)|^p \right.$$

$$\left. \times \frac{1}{\pi} \int\int_{\mathbb{D}} \frac{(1-|\zeta|^2)^m (1-|z|^2)^{-pa/q}}{|1-z\bar{\zeta}|^{m+2}} \, dxdy \, d\xi d\eta \right) .$$

Applying Proposition 3.2.2 part (b) for

$$0 < \frac{pa}{q} + m < 1 + m \ \ , m = 0, 1, \ldots, k-1 ,$$

we get

$$\begin{aligned} \int\int_{\mathbb{D}} & \ |P_k f(z)|^p \, dxdy \\ \le & \ \ k^p \left(\sum_{m=0}^{k-1} \binom{k-1}{m} C_1(m,a) \right)^{p/q} \\ \times & \ \ \left(\sum_{m=0}^{k-1} \binom{k-1}{m} C_2(m,a) \right) \int\int_{\mathbb{D}} |f(\zeta)|^p \, d\xi d\eta , \end{aligned}$$

where $C_2(m,a) = \dfrac{\Gamma(1-\frac{pa}{q})\Gamma(\frac{pa}{q}+m)}{\Gamma^2(\frac{m+2}{2})}$. Thus

$$\left(\int\int_{\mathbb{D}} |P_k f(z)|^p \, dxdy\right)^{\frac{1}{p}} \leq k \left(\sum_{m=0}^{k-1} \begin{pmatrix} k-1 \\ m \end{pmatrix} C_1(m,a)\right)^{1/q}$$

$$\times \left(\sum_{m=0}^{k-1} \begin{pmatrix} k-1 \\ m \end{pmatrix} C_2(m,a)\right)^{1/p} \left(\int\int_{\mathbb{D}} |f(z)|^p \, d\xi d\eta\right)^{1/p} .$$

Now, for

$$N(k,p) = k \left(\sum_{m=0}^{k-1} \begin{pmatrix} k-1 \\ m \end{pmatrix} C_1(m,a)\right)^{1/q} \left(\sum_{m=0}^{k-1} \begin{pmatrix} k-1 \\ m \end{pmatrix} C_2(m,a)\right)^{1/p}$$

we have

$$\|P_k f\|_{L^p(\mathbb{D})} \leq N(k,p)\|f\|_{L^p(\mathbb{D})} . \tag{3.4}$$

Since

$$\|\Pi_k f\|_{L^p(\mathbb{D})} \leq \|T_{-k,k} f\|_{L^p(\mathbb{D})} + \|P_k f\|_{L^p(\mathbb{D})} ,$$

using the Theorem 2.1.10, we obtain

$$\begin{aligned} \|\Pi_k f\|_{L^p(\mathbb{D})} &\leq (M(p) + N(k,p))\|f\|_{L^p(\mathbb{D})} \\ &= C(k,p)\|f\|_{L^p(\mathbb{D})} . \end{aligned}$$

□

Theorem 3.1.4. Π_k *is a bounded operator on* $L^2(\mathbb{D})$ *with norm less than or equal to*

$$1 + k\Gamma\left(\frac{1}{2}\right) \sum_{m=0}^{k-1} \begin{pmatrix} k-1 \\ m \end{pmatrix} \frac{\Gamma(m+\frac{1}{2})}{\Gamma^2(\frac{m+2}{2})} .$$

Proof. For $p = 2$, we have $0 < a < 1$, $m = 0, 1, \ldots, k-1$. Repeating the computations above, we find

$$\int\int_{\mathbb{D}} |P_k f(z)|^2 \, dxdy$$

$$\leq \left(k\sum_{m=0}^{k-1}\binom{k-1}{m}A(m,a)\right)\left(k\sum_{m=0}^{k-1}\binom{k-1}{m}B(m,a)\right)$$
$$\times \int\int_{\mathbb{D}}|f(\zeta)|^2 d\xi d\eta\ ,$$

where

$$A(m,a)=\frac{\Gamma(a)\Gamma(1+m-a)}{\Gamma^2(\frac{m+2}{2})} \quad \text{and} \quad B(m,a)=\frac{\Gamma(1-a)\Gamma(m+a)}{\Gamma^2(\frac{m+2}{2})}\ .$$

If $a=\dfrac{1}{2}$, then $A(m,\frac{1}{2})=B(m,\frac{1}{2})$ and

$$\begin{aligned}(C(k))^2 &= \left(k\sum_{m=0}^{k-1}\binom{k-1}{m}A(m,\frac{1}{2})\right)^2\\ &= \left(k\sum_{m=0}^{k-1}\binom{k-1}{m}\frac{\Gamma(\frac{1}{2})\Gamma(m+\frac{1}{2})}{\Gamma^2(\frac{m+2}{2})}\right)^2\end{aligned}$$

assumes its minimum value. Thus

$$\|P_k f\|_{L^2(\mathbb{D})} \leq C(k)\|f\|_{L^2(\mathbb{D})}$$

which completes the proof. □

3.2 Compactness of the Operators P_k

In this section we prove that the operators

$$P_k f(z)=\frac{(-1)^k k}{\pi}\int\int_{\mathbb{D}}\left(\frac{\overline{\zeta-z}+\zeta-z}{1-z\bar{\zeta}}\bar{\zeta}-1\right)^{k-1}\frac{\overline{f(\zeta)}}{(1-z\bar{\zeta})^2}d\xi d\eta\ .$$

are compact from $L^{p_1}(\mathbb{D})$ to $L^{p_2}(\mathbb{D})$ where $p_2<p_1$.

Let us define the operators

$$P_{k,r}f(z)=\frac{(-1)^k k}{\pi}\int\int_{r\mathbb{D}}\left(\frac{\overline{\zeta-z}+\zeta-z}{1-z\bar{\zeta}}\bar{\zeta}-1\right)^{k-1}\frac{\overline{f(z)}}{(1-z\bar{\zeta})^2}d\xi d\eta\ ,$$

for $f \in L^p(\mathbb{D})$, $z \in \mathbb{D}$, where $r\mathbb{D} = \{z \in \mathbb{C} : |z| < r\}$ for fixed $r \in (0,1)$ and $k \in \mathbb{N}$. Since

$$\|P_{k,r}f\|_{L^p(\mathbb{D})} \leq \|P_k f\|_{L^p(\mathbb{D})} \leq N(k,p)\|f\|_{L^p(\mathbb{D})}$$

holds for $1 < p < \infty$ in (3.4), the operators $P_{k,r}$ is a sequence of bounded operators in $L^p(\mathbb{D})$ for $1 < p < \infty$.

In the sequel we will utilize the following lemma from Conway [21], Page 177 Ex. 7.

Lemma 3.2.1. *Suppose $1 < p < \infty$ and $K(z,w)$ is a measurable function on $\mathbb{D} \times \mathbb{D}$ such that*

$$\int\int_{\mathbb{D}} \left(\int\int_{\mathbb{D}} |K(z,w)|^p dA(w) \right)^{q-1} dA(z) < \infty$$

for $1/p + 1/q = 1$. Then the integral operator T defined by

$$Tf(w) = \int\int_{\mathbb{D}} K(z,w) f(z) dA(z)$$

is compact on $L^p(\mathbb{D})$, where dA denote the Lebesque measure on the unit disc $\mathbb{D}$.

It can be easily seen that for the original operator P_k, Lemma 3.2.1 is not applicable. That is why we first prove that the operators $P_{k,r}$ are compact in $L^p(\mathbb{D})$ for $1 < p < \infty$, $0 < r < 1$.

Lemma 3.2.2. *The operators $P_{k,r}$ are compact on $L^p(\mathbb{D})$ for $1 < p < \infty$ and for fixed $r \in (0,1)$.*

Proof. The operator $P_{k,r}f$ is of the form

$$P_{k,r}f(z) = \frac{1}{\pi} \int\int_{\mathbb{D}} K(z,\zeta) \overline{f(\zeta)} d\xi d\eta \ ,$$

where

$$K(z,\zeta) = (-1)^k k \left(\frac{\overline{\zeta - z} + \zeta - z}{1 - z\bar{\zeta}} \bar{\zeta} - 1 \right)^{k-1} \frac{\chi_{r\mathbb{D}}(\zeta)}{(1 - z\bar{\zeta})^2}$$

with the characteristic function

$$\chi_{r\mathbb{D}}(\zeta) = \begin{cases} 1 & \text{if } \zeta \in r\mathbb{D}, \\ 0 & \text{if } \zeta \notin r\mathbb{D}\,. \end{cases}$$

Making the calculations similar to those in Theorem 3.1.3 and using Hölder's inequality, we have the following estimate for $|K(z,\zeta)|^p$.

$$\begin{aligned} |K(z,\zeta)|^p &\leq \left(k \sum_{m=0}^{k-1} \binom{k-1}{m} \frac{(1-|\zeta|^2)^m}{|1-z\bar{\zeta}|^{m+2}} \right)^p \\ &\leq k^p \left(\sum_{m=0}^{k-1} \binom{k-1}{m} \right)^{p/q} \left(\sum_{m=0}^{k-1} \binom{k-1}{m} \frac{(1-|\zeta|^2)^{mp}}{|1-z\bar{\zeta}|^{mp+2p}} \right) \\ &= C(k) \left(\sum_{m=0}^{k-1} \binom{k-1}{m} \frac{(1-|\zeta|^2)^{mp}}{|1-z\bar{\zeta}|^{mp+2p}} \right) \end{aligned}$$

for a generic constant $C(k)$. Then we have

$$\frac{1}{\pi} \int\int_{\mathbb{D}} \left(\frac{1}{\pi} \int\int_{\mathbb{D}} |K(z,\zeta)|^p dxdy \right)^{q-1} d\xi d\eta$$

$$= \frac{1}{\pi} \int\int_{r\mathbb{D}} \left(\frac{1}{\pi} \int\int_{\mathbb{D}} |K(z,\zeta)|^p dxdy \right)^{q-1} d\xi d\eta$$

$$\leq C(k) \frac{1}{\pi} \int\int_{r\mathbb{D}} \left(\sum_{m=0}^{k-1} \binom{k-1}{m} \frac{1}{\pi} \int\int_{\mathbb{D}} \frac{(1-|\zeta|^2)^{mp}}{|1-z\bar{\zeta}|^{mp+2p}} dxdy \right)^{q-1} d\xi d\eta$$

$$= C(k) \frac{1}{\pi} \int\int_{r\mathbb{D}} \left(\sum_{m=0}^{k-1} \binom{k-1}{m} (1-|\zeta|^2)^{2-2p} \right.$$

$$\left. \times \frac{1}{\pi} \int\int_{\mathbb{D}} \frac{(1-|\zeta|^2)^{mp+2p-2}}{|1-z\bar{\zeta}|^{mp+2p}} dxdy \right)^{q-1} d\xi d\eta$$

Now by part (c) of the Proposition 3.1.2 to the inner integral we get,

$$\frac{1}{\pi} \int\int_{\mathbb{D}} \left(\frac{1}{\pi} \int\int_{\mathbb{D}} |K(z,\zeta)|^p dxdy \right)^{q-1} d\xi d\eta \leq$$

$$\leq C(k)\int\int_{r\mathbb{D}}\left(\sum_{m=0}^{k-1}\begin{pmatrix} k-1 \\ m \end{pmatrix} C_1(m)(1-|\zeta|^2)^{2-2p}\right)^{q-1} d\xi d\eta$$

$$\leq C(k)\int\int_{r\mathbb{D}}(1-|\zeta|^2)^{\frac{2-2p}{p-1}} d\xi d\eta$$

since $q-1=1/(p-1)$. Thus the inequality

$$\frac{1}{\pi}\int\int_{\mathbb{D}}\left(\frac{1}{\pi}\int\int_{\mathbb{D}}|K(z,\zeta)|^p dxdy\right)^{q-1} d\xi d\eta \leq C(k)\int_0^{2\pi}\int_0^r \frac{tdtd\theta}{\pi(1-t^2)^2} < \infty$$

is obtained which shows that $P_{k,r}$ are compact on $L^p(\mathbb{D})$ by Lemma 3.2.1. $\square$

Lemma 3.2.3. *The operators $P_{k,r}$ are compact operators from $L^{p_1}(\mathbb{D})$ to $L^{p_2}(\mathbb{D})$ whenever $p_2 < p_1$.*

Proof. It is known that the canonical imbedding $i_{p_1p_2} : L^{p_1}(\mathbb{D}) \to L^{p_2}(\mathbb{D})$ with $p_2 < p_1$ is a bounded linear operator [30], and the operator $P_{k,r} : L^{p_1}(\mathbb{D}) \to L^{p_2}(\mathbb{D})$ can be decomposed into the composition of a compact operator $P_{k,r} : L^{p_1}(\mathbb{D}) \to L^{p_1}(\mathbb{D})$ and bounded operator $i_{p_1p_2} : L^{p_1}(\mathbb{D}) \to L^{p_2}(\mathbb{D})$, it follows that [26], $P_{k,r} : L^{p_1}(\mathbb{D}) \to L^{p_2}(\mathbb{D})$ is a compact operator. $\square$

Now, using the compactness of the operators $P_{k,r}$ we prove the compactness of P_k.

Theorem 3.2.4. *The operators $P_k : L^{p_1}(\mathbb{D}) \to L^{p_2}(\mathbb{D})$ are compact if $1 < p_2 < \dfrac{2p_1}{p_1+1} < \infty$.*

Proof. In order to prove the compactness of P_k, it suffices to establish that P_k is a uniform limit of the sequence $\{P_{k,r}\}$ of compact operators, that is, $\|P_k - P_{k,r}\| \to 0$ as $r \to 1^-$ [34]. Using the estimate (3.3) for the operators P_k we have

$$\|P_kf - P_{k,r}f\|_{L^{p_2}(\mathbb{D})} = \left(\int\int_{\mathbb{D}}|P_kf - P_{k,r}f|^{p_2} dxdy\right)^{1/p_2}$$

$$\leq \left(\int\int_{\mathbb{D}}\left(\frac{k}{\pi}\sum_{m=0}^{k-1}\begin{pmatrix} k-1 \\ m \end{pmatrix}\int\int_{\mathbb{D}-r\mathbb{D}}\frac{(1-|\zeta|^2)^m}{|1-z\bar{\zeta}|^{m+2}}|f(\zeta)|d\xi d\eta\right)^{p_2} dxdy\right)^{1/p_2}.$$

Using Minkowski's inequality in Theorem 1.2.2, we get

$$\|P_k f - P_{k,r} f\|_{L^{p_2}(\mathbb{D})}$$

$$\leq \int\int_{\mathbb{D}-r\mathbb{D}} \left(\int\int_{\mathbb{D}} \left(\frac{k}{\pi} \sum_{m=0}^{k-1} \binom{k-1}{m} \frac{(1-|\zeta|^2)^m}{|1-z\bar{\zeta}|^{m+2}} \right)^{p_2} |f(\zeta)|^{p_2} dxdy \right)^{1/p_2} d\xi d\eta$$

$$\leq C(k) \int\int_{\mathbb{D}-r\mathbb{D}} |f(\zeta)| \left(\sum_{m=0}^{k-1} \binom{k-1}{m} \frac{1}{\pi} \int\int_{\mathbb{D}} \frac{(1-|\zeta|^2)^{mp_2}}{|1-z\bar{\zeta}|^{mp_2+2p_2}} dxdy \right)^{1/p_2} d\xi d\eta \,.$$

Applying the Proposition 3.1.2 part (c) to the inner integral we have

$$\|P_k f - P_{k,r} f\|_{L^{p_2}(\mathbb{D})} \leq C(k) \int\int_{\mathbb{D}-r\mathbb{D}} |f(\zeta)|(1-|\zeta|^2)^{\frac{2-2p_2}{p_2}} d\xi d\eta$$

$$\leq \left(\int\int_{\mathbb{D}-r\mathbb{D}} (1-|\zeta|^2)^{\frac{(2-2p_2)p_1}{p_2(p_1-1)}} d\xi d\eta \right)^{\frac{p_1-1}{p_1}} \left(\int\int_{\mathbb{D}-r\mathbb{D}} |f(\zeta)|^{p_1} d\xi d\eta \right)^{1/p_1} .$$

Now, if $p_2 < \dfrac{2p_1}{p_1+1}$, then $d = \dfrac{(2-2p_2)p_1}{p_2(p_1-1)} > -1$ and

$$\|P_k f - P_{k,r} f\|_{L^{p_2}(\mathbb{D})} \leq C(k) \left(\int_r^1 (1-t^2)^d 2t dt \right) \|f\|_{L^{p_1}(\mathbb{D})}$$

$$= \frac{(1-r^2)^{d+1}}{d+1} \|f\|_{L^{p_1}(\mathbb{D})}$$

Hence,

$$\|P_k - P_{k,r}\| \leq \frac{(1-r^2)^{d+1}}{d+1} \,.$$

Since $\dfrac{(1-r^2)^{d+1}}{d+1} \to 0$ as $r \to 1^-$, then $\|P_k - P_{k,r}\| \to 0$ as $r \to 1^-$, which completes the proof. □

Note 1. The idea of defining a sequence of compact operators and taking the limit to get a compact operator is also used in proving the compactness of Toeplitz operators on the Bergman space of the unit disc in [2] and [31].

CHAPTER 4

Schwarz Boundary Value Problem for Higher-Order Differential Equations

In this chapter, we investigate a general higher-order complex partial differential equation in the unit disc with homogeneous and nonhomogeneous Schwarz boundary conditions.

4.1 Higher-Order Equations with Homogeneous Schwarz Boundary Conditions

We begin with the following theorem of Begehr [5] about the homogeneous Schwarz problem for a higher-order differential equation. The nonhomogeneous case of this theorem will be considered at the end of the next section.

Theorem 4.1.1. *[5] The Schwarz problem*

$$Re\frac{\partial^l w}{\partial \bar{z}^l} = 0 \quad on\ \partial\mathbb{D}, \qquad Im\frac{\partial^l w}{\partial \bar{z}^l}(0) = 0, \quad 0 \leq l \leq k-1,$$

for the k-th order equation

$$\frac{\partial^k w}{\partial \bar{z}^k} + q_1(z)\frac{\partial^k w}{\partial \bar{z}^{k-1}\partial z} + q_2(z)\frac{\partial^k \overline{w}}{\partial \bar{z}\partial z^{k-1}}$$
$$+\sum_{l=0}^{k-1}\sum_{j=0}^{l}\left(a_{lj}(z)\frac{\partial^l w}{\partial \bar{z}^{l-j}\partial z^j} + b_{lj}(z)\overline{\frac{\partial^l w}{\partial \bar{z}^{l-j}\partial z^j}}\right) = f(z) \tag{4.1}$$

is solvable for given $a_{ij}, b_{ij} \in C^{\alpha}(\overline{\mathbb{D}}), 0 < \alpha < 1, f \in L^p(\mathbb{D})$, q_1, q_2 *measurable satisfying* $|q_1(z)| + |q_2(z)| \le q_0 < 1$, $q_0\|\Pi_1\|_{L^p(\mathbb{D})} < 1$. *Any solution is given in the form* $w = \widetilde{T}_k g$, *where* $g \in L^p(\mathbb{D})$ *is a solution of the singular integral equation*

$$\begin{aligned} g+ \quad & q_1(z)\Pi_1 g \quad +q_2(z)\overline{\Pi_1}g+ \\ + \quad & \sum_{l=0}^{k-1}\sum_{j=0}^{l}\left(a_{lj}(z)\frac{\partial^j \widetilde{T}_{k-l+j}g}{\partial z^j} + b_{lj}(z)\overline{\frac{\partial^j \widetilde{T}_{k-l+j}g}{\partial z^j}}\right) = f \ . \end{aligned}$$

Proof. See [5]. □

Remark 4.1.2. *For the case* $k = 1$, *(4.1) becomes the generalized Beltrami equation [7]. For* $k = 2$, *the equation (4.1) becomes a general second-order equation of elliptic type. The principal part of (4.1) does not involve all the highest order derivatives for* $k > 2$.

As we have pointed out previously, because of not knowing the properties of the integral operators Π_k for $k > 1$, the differential equation in the previous theorem is not taken in its general form. Now, we consider a more general complex differential equation with homogeneous and nonhomogeneous Schwarz conditions. In the sequel, $W^{p,k}(\mathbb{D})$ will denote the Sobolev space consisting of the functions which itself and their k-th order weak derivatives belong to the space $L^p(\mathbb{D})$.

Problem (H). Find $w \in W^{p,k}(\mathbb{D})$ as a solution to a k-th order complex differential equation equation

$$\begin{aligned} \frac{\partial^k w}{\partial \bar{z}^k} + \sum_{j=1}^{k} q_{1j}(z)\frac{\partial^k w}{\partial \bar{z}^{k-j}\partial z^j} + \sum_{j=1}^{k} q_{2j}(z)\frac{\partial^k \overline{w}}{\partial z^{k-j}\partial \bar{z}^j} + \\ + \sum_{l=0}^{k-1}\sum_{m=0}^{l}\left[a_{ml}(z)\frac{\partial^l w}{\partial \bar{z}^{l-m}\partial z^m} + b_{ml}(z)\frac{\partial^l \overline{w}}{\partial z^{l-m}\partial \bar{z}^m}\right] = f(z) \text{ in } \mathbb{D} \end{aligned} \qquad (4.2)$$

satisfying the homogeneous Schwarz boundary conditions

$$\mathrm{Re}\frac{\partial^l w}{\partial \bar{z}^l} = 0 \ \text{ on } \partial\mathbb{D}\ , \qquad\qquad \mathrm{Im}\frac{\partial^l w}{\partial \bar{z}^l}(0) = 0\ ,\ 0 \le l \le k-1\ , \tag{4.3}$$

where

$$a_{ml}, b_{ml} \in L^p(\mathbb{D}), f \in L^p(\mathbb{D}), \tag{4.4}$$

and q_{1j} and q_{2j}, $j = 1, \ldots, k$, measurable bounded functions satisfing

$$\sum_{j=1}^{k} (|q_{1j}(z)| + |q_{2j}(z)|) \le q_0 < 1\ . \tag{4.5}$$

Remark 4.1.3. a) *The condition (4.5) on q_{1j} and q_{2j} implies the ellipticity of the given differential equation, see [1], [22], [23].*

b) *In [1] and [23], Riemann-Hilbert type problem is considered for the higher-order differential equations not containing $\overline{w}$ and its derivatives.*

Lemma 4.1.4. *The Schwarz problem (4.2) and (4.3) is equivalent to the singular integral equation*

$$(I + \hat{\Pi} + \hat{K})g = f\ , \tag{4.6}$$

where $w = \widetilde{T}_k g$,

$$\hat{\Pi}g = \sum_{j=1}^{k} q_{1j}\Pi_j g + \sum_{j=1}^{k} q_{2j}\overline{\Pi_j g} \tag{4.7}$$

and

$$\hat{K}g = \sum_{l=0}^{k-1}\sum_{m=0}^{l}\left(a_{ml}\frac{\partial^m \widetilde{T}_{k-l+m}g}{\partial z^m} + b_{ml}\frac{\partial^m \overline{\widetilde{T}_{k-l+m}g}}{\partial \bar{z}^m}\right)\ . \tag{4.8}$$

Proof. By Theorem 2.2.7, it is known that, $w = \widetilde{T}_k g$ is the only solution to the equation

$$\frac{\partial^k w}{\partial \bar{z}^k} = g$$

satisfying the conditions

$$\operatorname{Re} \frac{\partial^l w}{\partial \bar{z}^l} = 0 \text{ on } \partial\mathbb{D},\ 0 \le l \le k-1,$$

$$\operatorname{Im} \frac{\partial^l w}{\partial \bar{z}^l}(0) = 0\ ,\ 0 \le l \le k-1\ .$$

If we differentiate $w = \widetilde{T}_k g$ using the differentiability properties of the operators $\widetilde{T}_k$ given in Theorem 2.1.15, the equations

$$\frac{\partial^k w}{\partial \bar{z}^{k-j} \partial z^j} = \frac{\partial^k \widetilde{T}_k g}{\partial \bar{z}^{k-j} \partial z^j} = \frac{\partial}{\partial z^j} \widetilde{T}_{k-(k-j)} g = \frac{\partial}{\partial z^j} \widetilde{T}_j g = \Pi_j g$$

and similarly

$$\frac{\partial^k \bar{w}}{\partial z^{k-j} \partial \bar{z}^j} = \frac{\partial^k \overline{\widetilde{T}_k g}}{\partial z^{k-j} \partial \bar{z}^j} = \frac{\partial}{\partial \bar{z}^j} \overline{\left(\frac{\partial^k \widetilde{T}_k g}{\partial \bar{z}^{k-j}} \right)} = \frac{\partial}{\partial \bar{z}^j} \overline{\widetilde{T}_j g} = \overline{\Pi_j g}$$

are obtained for $1 \le j \le k$. Also we have

$$\frac{\partial^l w}{\partial \bar{z}^{l-m} \partial z^m} = \frac{\partial^l \widetilde{T}_k g}{\partial \bar{z}^{l-m} \partial z^m} = \frac{\partial^m \widetilde{T}_{k-l+m} g}{\partial z^m}$$

and

$$\frac{\partial^l \bar{w}}{\partial z^{l-m} \partial \bar{z}^m} = \frac{\partial^l \overline{\widetilde{T}_k g}}{\partial z^{l-m} \partial \bar{z}^m} = \frac{\partial}{\partial \bar{z}^m} \overline{\left(\frac{\partial^{l-m} \widetilde{T}_k g}{\partial z^{l-m}} \right)} = \frac{\partial^m \overline{\widetilde{T}_{k-l+m} g}}{\partial \bar{z}^m}\ .$$

Therefore, g satisfies equation (4.6) if and only if $w = \widetilde{T}_k g$ satisfies equation (4.2) with the conditions (4.3). Thus the result follows. □

Therefore, the equation (4.2) with the boundary conditions (4.3) is transformed into a singular integral equation of the form (4.6). In the following section we will analyze that singular integral equation by means of the Fredholm theory.

4.1.1 Analysis of the Integral Equation of Problem (H)

In this section we discuss two different points of view in analyzing the singular integral equation (4.6).

(Case 1) First, we impose a condition on q_0 to make the operator $I+\hat{\Pi}$.

Lemma 4.1.5. *If*

$$q_0\|\Pi_k\|_{L^p(\mathbb{D})} < 1 \tag{4.9}$$

for $p>1$, then the operator $I+\hat{\Pi}$ is invertible.

Before starting with the proof of this lemma, we state the following well-known fact [34].

Theorem 4.1.6. *Let X be a Banach space and $T \in B(X)$. If $\|T\|<1$, then $I-T$ is invertible, i.e., the inverse correspondence $(I-T)^{-1}$ is a bounded linear operator. Moreover, $\|(I-T)^{-1}\| \leq (1-\|T\|)^{-1}$.*

Proof. (of Lemma 4.1.5) The norm of the operator $\hat{\Pi}$ satisfies the inequality

$$\|\hat{\Pi}\|_{L^p(\mathbb{D})} \leq \sum_{j=1}^{k}(|q_{1j}|+|q_{2j}|)\|\Pi_j\|_{L^p(\mathbb{D})} \leq q_0\|\Pi_k\|_{L^p(\mathbb{D})} \ .$$

If condition (4.9) holds, then we have $\|\hat{\Pi}\|_{L^p(\mathbb{D})}<1$. In that case, by Theorem 4.1.6 the operator $I+\hat{\Pi}$ is invertible. □

Lemma 4.1.7. *Under the conditions (4.4), $\hat{K}$ is a compact operator in $L^p(\mathbb{D})$ for $p>2$.*

Proof. Consider first the operators

$$\frac{\partial^l}{\partial z^l}\widetilde{T}_k g(z) = \sum_{\lambda=0}^{l}\begin{pmatrix} l \\ \lambda \end{pmatrix}\frac{(-1)^{k-\lambda}(l-\lambda)!}{(k-\lambda-1)!}\frac{1}{\pi}\int\int_{\mathbb{D}}(\zeta-z+\overline{\zeta-z})^{k-\lambda-1}$$

$$\left[\frac{g(\zeta)}{(\zeta-z)^{l-\lambda+1}}+\frac{\bar{\zeta}^{l-\lambda-1}\overline{g(\zeta)}}{(1-z\bar{\zeta})^{l-\lambda+1}}\right]d\xi d\eta +$$

$$+\frac{(-1)^{k-l-1}}{(k-l-1)!}\frac{1}{2\pi}\int\int_{\mathbb{D}}(\zeta - z + \overline{\zeta - z})^{k-l-1}\left[\frac{g(\zeta)}{\zeta} + \frac{\overline{g(\zeta)}}{\bar{\zeta}}\right] d\xi d\eta$$

for $0 \le l \le k-1$. Define

$$Ag(z) = \int\int_{\mathbb{D}}(\zeta - z + \overline{\zeta - z})^{k-\lambda-1}\left[\frac{g(\zeta)}{(\zeta - z)^{l-\lambda+1}} + \frac{\bar{\zeta}^{l-\lambda-1}\overline{g(\zeta)}}{(1-z\bar{\zeta})^{l-\lambda+1}}\right] d\xi d\eta$$

for $0 \le \lambda \le l$ and

$$Bg(z) = \int\int_{\mathbb{D}}(\zeta - z + \overline{\zeta - z})^{k-l-1}\left[\frac{g(\zeta)}{\zeta} + \frac{\overline{g(\zeta)}}{\bar{\zeta}}\right] d\xi d\eta \; .$$

Now,

$$|Ag(z)| \le \int\int_{\mathbb{D}}(2|\zeta - z|)^{k-\lambda-1}\left[\frac{1}{|\zeta - z|^{l-\lambda+1}} + \frac{|\zeta|^{l-\lambda-1}}{|1-z\bar{\zeta}|^{l-\lambda-1}}\right] |g(\zeta)| d\xi d\eta \; .$$

Using the fact that $\left|\frac{\zeta - z}{1 - z\bar{\zeta}}\right| \le 1$ when $|\zeta| < 1$ and $|z| \le 1$ and Hölder's inequality, we get

$$|Ag(z)| \le C\left(\int\int_{\mathbb{D}}|z-\zeta|^{q(k-l-2)} d\xi d\eta\right)^{1/q} \|g\|_{L^p(\mathbb{D})} \; .$$

Since $k - l - 2 \ge 1$, then for $1 < q < 2$,

$$|Ag(z)| \le C\|g\|_{L^p(\mathbb{D})}$$

holds.

For $B(z)$ we have

$$|Bg(z)| \le \left(\int\int_{\mathbb{D}}|\zeta - z|^{q(k-l-1)}|\zeta|^{-q} d\xi d\eta\right)^{1/q} \|g\|_{L^p(\mathbb{D})}$$

which gives $|B(z)| \le C\|g\|_{L^p(\mathbb{D})}$ for $p > 2$. Thus

$$\left|\frac{\partial^l \widetilde{T}_k g}{\partial \bar{z}^l}\right| \le C\|g\|_{L^p(\mathbb{D})} \tag{4.10}$$

for $0 \le l \le k-1$ is obtained. Therefore, $\dfrac{\partial^l \widetilde{T}_k g}{\partial \bar{z}^l}$ is bounded on $\mathbb{D}$. For $z_1, z_2 \in \overline{\mathbb{D}}$, we have

$$|Ag(z_1) - Ag(z_2)| \le \int\int_{\mathbb{D}} (|\zeta - z_1|^{k-l-2} - |\zeta - z_2|^{k-l-2}) |g(\zeta)| d\xi d\eta \ ,$$

$$|Bg(z_1) - Bg(z_2)| \le \int\int_{\mathbb{D}} (|\zeta - z_1|^{k-l-1} - |\zeta - z_2|^{k-l-1}) \frac{|g(\zeta)|}{|\zeta|} d\xi d\eta \ .$$

Using Lemma 4.4 in [13] and Hölder's inequality, we get

$$\left| \frac{\partial^l \widetilde{T}_k g(z_1)}{\partial \bar{z}^l} - \frac{\partial^l \widetilde{T}_k g(z_2)}{\partial \bar{z}^l} \right| \le C|z_1 - z_2|^{\alpha} \|g\|_{L^p(\mathbb{D})} \ , \qquad (4.11)$$

where $0 < \alpha < 1$ and $p > 2$. Thus, in particular, $\dfrac{\partial^l \widetilde{T}_k g}{\partial \bar{z}^l}$ is uniformly continuous. Arzela-Ascoli theorem [21] implies that the operators $\dfrac{\partial^l \widetilde{T}_k g}{\partial \bar{z}^l}$ for $0 \le l \le k-1$ are compact operators from the space $L^p(\mathbb{D})$ to the space of Hölder continuous functions $C^{\alpha}(\overline{\mathbb{D}})$. Now, the operator $\hat{K}$ satisfies

$$|\hat{K} g|^p \le \left(\sum_{l=0}^{k-1} \sum_{m=0}^{l} (|a_{ml}(z)| + |b_{ml}(z)|) \left| \frac{\partial^m \widetilde{T}_{k-l+m} g}{\partial z^m} \right| \right)^p \ ,$$

and by (4.10) we have

$$\|\hat{K} g\|^p_{L^p(\mathbb{D})} \le C \|g\|^p_{L^p(\mathbb{D})} \int\int_{\mathbb{D}} \left(\sum_{l=0}^{k-1} \sum_{m=0}^{l} |a_{ml}(z)| \right)^p dxdy$$

$$+ C \|g\|^p_{L^p(\mathbb{D})} \int\int_{\mathbb{D}} \left(\sum_{l=0}^{k-1} \sum_{m=0}^{l} |b_{ml}(z)| \right)^p dxdy \ .$$

Using the Minkowski inequality we get

$$\|\hat{K} g\|_{L^p(\mathbb{D})} \le M \|g\|_{L^p(\mathbb{D})} \ .$$

Let $\{g_n\}$ be a bounded sequence in $L^p(\mathbb{D})$; then $\{\dfrac{\partial^l \widetilde{T}_k g_n}{\partial \bar{z}^l}\}$ is bounded in $C^{\alpha}(\mathbb{D})$. By the Arzela-Ascoli theorem there is a convergent subsequence

$\{\frac{\partial^l \widetilde{T}_k g_{n_k}}{\partial \bar{z}^l}\}$. Then

$$\{a_{ml}\frac{\partial^m \widetilde{T}_{k-l+m} g_{n_k}}{\partial z^m} + b_{ml}\frac{\partial^m \overline{\widetilde{T}_{k-l+m} g_{n_k}}}{\partial \bar{z}^m}\}$$

is a convergent sequence in $L^p(\mathbb{D})$ for any $0 \leq m \leq l$ and $0 \leq l \leq k-1$ since its elements satisfy

$$\|a_{ml}\frac{\partial^m \widetilde{T}_{k-l+m} g_{n_k}}{\partial z^m} + b_{ml}\frac{\partial^m \overline{\widetilde{T}_{k-l+m} g_{n_k}}}{\partial \bar{z}^m}\|^p_{L^p(\mathbb{D})}$$
$$\leq \int\int_{\mathbb{D}} (|a_{ml}(z)| + |b_{ml}(z)|)^p \left|\frac{\partial^m \widetilde{T}_{k-l+m} g_{n_k}}{\partial z^m}\right|^p dxdy \ .$$

Therefore, $\hat{K}$ is a compact operator in $L^p(\mathbb{D})$ into itself. □

Thus, we are ready to give the following theorem which gives the condition for the solution of the problem (4.2) and (4.3).

Theorem 4.1.8. *If condition (4.9) is satisfied, then equation (4.2) with the conditions (4.3) has a solution of the form* $w = \widetilde{T}_k g$*, where* $g \in L^p(\mathbb{D})$*,* $p > 2$*, is a solution of the singular integral equation (4.6).*

Proof. If (4.9) holds then by Lemma 4.1.5 and Lemma 4.1.7, the operator $I+\hat{\Pi}+\hat{K}$ becomes the sum of an invertible operator and a compact operator. Nikolskii Criterion (Theorem 1.2.12) implies that the operator $I+\hat{\Pi}+\hat{K}$ is a Fredholm operator with index zero. By Theorem 1.2.4, the singular integral equation (4.6) has the Fredholm alternative. Thus, if g is a solution to the integral equation (4.6), then $w = \widetilde{T}_k g$ is a solution to the given equation (4.2) with the conditions (4.3). Therefore, the differential equation (4.2) with the boundary conditions (4.3) has a solution of the form $w = \widetilde{T}_k g$, where g is a solution of the integral equation (4.6). □

(Case 2) For the second point of view, we separate the principal singular part $\hat{\Pi}$ of the singular integral equation (4.6) as

$$\hat{\Pi} = \hat{T} + \hat{P}$$

with

$$\hat{T}g = \sum_{j=1}^{k} q_{1j} T_{-j,j} g + \sum_{j=1}^{k} q_{2j} \overline{T_{-j,j} g}$$

and

$$\hat{P}g = \sum_{j=1}^{k} q_{1j} P_j g + \sum_{j=1}^{k} q_{2j} \overline{P_j g} \ ,$$

where the operators $T_{-k,k}$ and P_k are defined in (3.1) and (3.2), respectively. Then the integral equation (4.6) is rewritten in the form

$$(I + \hat{T} + \hat{P} + \hat{K})g = f \ .$$

Now we are in a position to prove that for p close enough to 2, the operator $I + \hat{T}$ is invertible in $L^p(\mathbb{D})$. To prove this, we need the following well known Riesz-Thorin theorem of interpolation theory.

Theorem 4.1.9. (The Riesz-Thorin interpolation theorem) *[18] Assume that $p_0 \neq p_1$ and $q_0 \neq q_1$ and that $T : L^{p_0}(U) \to L^{q_0}(U)$ with norm M_0, and that $T : L^{p_1}(U) \to L^{q_1}(U)$ with norm M_1. Then $T : L^p(U) \to L^q(U)$ with norm $M \leq M_0^{1-\theta} M_1^{\theta}$ provided that $0 < \theta < 1$ and*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \ , \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1} \ .$$

Lemma 4.1.10. *The operator $I + \hat{T}$ is invertible in $L^p(\mathbb{D})$ for $0 < p - 2 < \epsilon$.*

Proof. Since $\|\hat{T}\|_{L^p(\mathbb{D})} \leq \sum_{j=1}^{k} (|q_{1j}| + |q_{2j}|) \|T_{-j,j}\|_{L^p(\mathbb{D})} \leq q_0 \|T_{-k,k}\|_{L^p(\mathbb{D})}$ and by the property that $\|T_{-j,j}\|_{L^2(\mathbb{D})} = \|T_{-k,k}\|_{L^2(\mathbb{D})} \leq 1$ for $j = 1, \ldots, k$, the Riesz-Thorin Theorem implies that $q_0 \|T_{-k,k}\|_{L^p(\mathbb{D})} < 1$ holds for $p > 2$ and sufficiently close to 2. Therefore, the inequality $\|\hat{T}\|_{L^p(\mathbb{D})} < 1$ is satisfied. Then by Theorem 4.1.6, the operator $I + \hat{T}$ is invertible in $L^p(\mathbb{D})$ with $0 < p - 2 < \epsilon$. □

Now, we can prove the following theorem.

Theorem 4.1.11. *If the inequality*

$$q_0 \|P_k\|_{L^p(\mathbb{D})} \|(I+\hat{T})^{-1} - K_1\|_{L^p(\mathbb{D})} < 1 \tag{4.12}$$

is satisfied for some $K_1 \in K(L^p(\mathbb{D}))$, $0 < p-2 < \epsilon$, *then equation (4.2) with the boundary conditions (4.3) has a solution of the form* $w = \widetilde{T}_k g$, *where* $g \in L^p(\mathbb{D})$ *is a solution of the singular integral equation (4.6).*

Proof. Since $I+\hat{T}$ is invertible for p close enough to 2 by Lemma 4.1.10, and $\hat{K}$ is compact in L^p by Lemma 4.1.7, Nikolskii Criterion implies that $I+\hat{T}+\hat{K}$ is a Fredholm operator with index zero. Thus by Theorem 1.2.11 (Bounded Index Stability Theorem) given in [34], there is an $\eta > 0$ such that $\hat{P}$ satisfying $\|\hat{P}\|_{L^p(\mathbb{D})} < \eta$ will imply that the operator $I+\hat{T}+\hat{P}+\hat{K}$ is also a Fredholm operator. Now, we will determine the condition to have this property. Since $I+\hat{T}+\hat{K}$ is a Fredholm operator, Theorem 1.2.7 (Noether Criterion) implies that it is approximately invertible. Then there exists $A_0 \in B(L^p(\mathbb{D}))$ and $F_1, F_2 \in K(L^p(\mathbb{D}))$ such that the equations

$$A_0(I+\hat{T}+\hat{K}) = I - F_1 \quad \text{and} \quad (I+\hat{T}+\hat{K})A_0 = I - F_2 \tag{4.13}$$

are satisfied, i.e., A_0 is the approximate inverse of the operator $I+\hat{T}+\hat{K}$. Now by Corollary 1.2.10, we have $A_0 \in \Phi(L^p(\mathbb{D}))$ with index zero and therefore, it has the form

$$A_0 = (I+\hat{T})^{-1} - K_1 \ ,$$

where $K_1 \in K(L^p(\mathbb{D}))$ by Theorem 1.2.12 with

$$F_1 = (I+\hat{T})^{-1}\hat{K} - K_1(I+\hat{T}) - K_1\hat{K}$$

and

$$F_2 = \hat{K}(I+\hat{T})^{-1} - (I+\hat{T})K_1 - \hat{K}K_1 \ .$$

The choice of the compact operator K_1 is presented in the note below. Now, take

$$\eta = \frac{1}{\|A_0\|} = \frac{1}{\|(I+\hat{T})^{-1} - K_1\|_{L^p(\mathbb{D})}} \ .$$

If the inequality $q_0\|P_k\|_{L^p(\mathbb{D})}\|(I+\hat{T})^{-1}-K_1\|_{L^p(\mathbb{D})}<1$ is satisfied, then the inequality

$$\begin{aligned}\|A_0\hat{P}\|_{L^p(\mathbb{D})} &\leq \|A_0\|_{L^p(\mathbb{D})}\|\hat{P}\|_{L^p(\mathbb{D})}\\ &\leq \|(I+\hat{T})^{-1}-K_1\|_{L^p(\mathbb{D})}\sum_{j=1}^{k-1}(|q_{1j}|+|q_{2j}|)\|P_j\|_{L^p(\mathbb{D})}\\ &\leq q_0\|P_k\|_{L^p(\mathbb{D})}\|(I+\hat{T})^{-1}-K_1\|_{L^p(\mathbb{D})}<1\end{aligned}$$

is obtained, and thus $I+A_0\hat{P}$ and $I+\hat{P}A_0$ are invertible. Since the equalities

$$A_0(I+\hat{T}+\hat{K}+\hat{P})=I-F_1+A_0\hat{P}$$

and

$$(I+\hat{T}+\hat{K}+\hat{P})A_0=I-F_2+\hat{P}A_0$$

hold, they imply

$$(I+A_0\hat{P})^{-1}A_0(I+\hat{T}+\hat{K}+\hat{P})=I-(I+A_0\hat{P})^{-1}F_1\ ,$$

$$(I+\hat{T}+\hat{K}+\hat{P})A_0(I+\hat{P}A_0)^{-1}=I-F_2(I+\hat{P}A_0)^{-1}\ .$$

Since $(I+A_0\hat{P})^{-1}$ and $(I+\hat{P}A_0)^{-1}$ are bounded operators and $(I+A_0\hat{P})^{-1}F_1$ and $F_2(I+\hat{P}A_0)^{-1}$ are compact, being the composition of bounded and compact operators, the operator $I+\hat{T}+\hat{K}+\hat{P}$ is a Fredholm operator. By Theorem 1.2.9 we get

$$i(I+\hat{T}+\hat{K}+\hat{P})+i(A_0)+i(I+\hat{P}A_0)^{-1}=i(I-F_2(I+\hat{P}A_0)^{-1})\ .$$

Taking into consideration Theorem 1.2.12,

$$i(A_0)=i(I+\hat{P}A_0)^{-1}=i(I-F_2(I+\hat{P}A_0))^{-1}=0$$

holds. Thus, $i(I+\hat{T}+\hat{K}+\hat{P})=0$ is obtained, i.e., $I+\hat{T}+\hat{K}+\hat{P}$ is a Fredholm operator with index zero. Therefore the Fredholm alternative applies to the operator $I+\hat{T}+\hat{K}+\hat{P}$. Thus, if g is a solution of the index-zero Fredholm operator $I+\hat{\Pi}+\hat{K}$, then $w=\widetilde{T}_k g$ is a solution of the Schwarz problem (4.2) and (4.3). □

Note 2. In Theorem 4.1.11, K_1, F_1 and F_2 satisfy

$$(I+T)^{-1}(I-(I+T)K_1)(I+K(I+T)^{-1})(I+T) = I - F_1 \tag{4.14}$$

and

$$(I+T)(I+(I+T)^{-1}K)(I-K_1(I+T))(I+T)^{-1} = I - F_2\ . \tag{4.15}$$

The choice of the compact operator K_1 can be done as follows. It is known that the sum or difference and composition of compact operators and of bounded and compact operators yield compact operators. Since the operator $\hat{K}$ is compact and the operator $(I+\hat{T})^{-1}$ is bounded, we can represent K_1 as

$$(I+T)K_1 = \hat{K}(I+\hat{T})^{-1} - (\hat{K}(I+\hat{T})^{-1})^2 + \cdots + (-1)^n(\hat{K}(I+\hat{T})^{-1})^n$$

for some n. In this case the compact operators F_1 and F_2 satisfying (4.14) and (4.15), respectively, are in the form

$$F_1 = [(I+\hat{T})^{-1}\hat{K}]^n$$

and

$$F_2 = [\hat{K}(I+\hat{T})^{-1}]^n\ .$$

4.2 Higher-Order Equations with Nonhomogeneous Schwarz Boundary Conditions

In this section, we consider the general differential equation (4.2) with the nonhomogeneous Schwarz conditions. A related problem is the following.

Problem (NH) Find w as a solution to a k-th order complex differential equation equation (4.2) in $\mathbb{D}$, with the conditions (4.4) and (4.5), satisfying the nonhomogeneous Schwarz boundary conditions

$$\mathrm{Re}\frac{\partial^l w}{\partial \bar{z}^l} = \gamma_l \ \text{ on } \partial\mathbb{D}\ , \qquad \mathrm{Im}\frac{\partial^l w}{\partial \bar{z}^l}(0) = c_l\ ,\ 0 \le l \le k-1\ , \tag{4.16}$$

where $\gamma_l \in C(\partial\mathbb{D};\mathbb{R}), c_l \in \mathbb{R}, 0 \le l \le k-1$.

We denote a solution of the equation (4.2) with the conditions (4.16) as $w = w_1 + w_2$ where w_1 is the solution of the problem

$$\frac{\partial^k w_1}{\partial \bar{z}^k} = 0 \ , \quad \mathrm{Re}\frac{\partial^l w_1}{\partial \bar{z}^l} = \gamma_l \text{ on } \partial\mathbb{D} \ , \qquad \mathrm{Im}\frac{\partial^l w_1}{\partial \bar{z}^l}(0) = c_l \ , \ 0 \le l \le k-1 \tag{4.17}$$

and w_2 is a solution of the problem

$$\frac{\partial^k w_2}{\partial \bar{z}^k} + Lw_2 = f - Lw_1 \ , \tag{4.18}$$

$$\mathrm{Re}\frac{\partial^l w_2}{\partial \bar{z}^l} = 0 \ \text{ on } \partial\mathbb{D} \ , \qquad \mathrm{Im}\frac{\partial^l w_2}{\partial \bar{z}^l}(0) = 0 \ , \ 0 \le l \le k-1 \ , \tag{4.19}$$

where

$$Lw_2 := \sum_{j=1}^{k} q_{1j}(z)\frac{\partial^k w_2}{\partial \bar{z}^{k-j}\partial z^j} + \sum_{j=1}^{k} q_{2j}(z)\frac{\partial^k \overline{w}_2}{\partial \bar{z}^{j}\partial z^{k-j}} +$$
$$\sum_{l=0}^{k-1}\sum_{m=0}^{l}\left[a_{ml}(z)\frac{\partial^l w_2}{\partial \bar{z}^{l-m}\partial z^m} + b_{ml}(z)\frac{\partial^l \overline{w}_2}{\partial \bar{z}^{m}\partial z^{l-m}}\right] .$$

Now, let us consider. problem (4.17) first. By Theorem 2.2.7, the unique solution is given as

$$w_1 = i\sum_{l=0}^{k-1}\frac{c_l}{l!}\ (z+\overline{z})^l + \sum_{l=0}^{k-1}\frac{(-1)^l}{2\pi i l!}\int_{\partial\mathbb{D}}\gamma_l(\zeta)\frac{\zeta+z}{\zeta-z}\ (\zeta - z + \overline{\zeta - z})^l\frac{d\zeta}{\zeta} \ .$$

Let us denote $w_1 = f_1 + f_2$ where

$$f_1(z) = \sum_{l=0}^{k-1}\frac{c_l}{l!}\ (z+\overline{z})^l$$

and

$$f_2(z) = \sum_{l=0}^{k-1}\frac{(-1)^l}{2\pi i l!}\int_{\partial\mathbb{D}}\gamma_l(\zeta)\frac{\zeta+z}{\zeta-z}\ (\zeta - z + \overline{\zeta - z})^l\frac{d\zeta}{\zeta} \ .$$

Now,

$$\frac{\partial^k f_1}{\partial \bar{z}^{k-j} z^j} = \frac{\partial^k \bar{f}_1}{\partial \bar{z}^j z^{k-j}} = 0$$

and

$$\frac{\partial^l f_1}{\partial \bar{z}^{l-m} z^m} = \frac{\partial^l \bar{f}_1}{\partial \bar{z}^m z^{l-m}} = i\sum_{j=l}^{k-1} \frac{c_j}{(j-l)!}(z+\bar{z})^{j-l}$$

for $l < k$ hold. Then the problem (4.18), (4.19) becomes

$$\frac{\partial^k w_2}{\partial \bar{z}^k} + Lw_2 = f - \sum_{l=0}^{k-1}\sum_{m=0}^{l}(a_{ml}+b_{ml})i\sum_{j=l}^{k-1}\frac{c_j}{(j-l)!}(z+\bar{z})^{j-l} - Lf_2 \quad (4.20)$$

$$\mathrm{Re}\frac{\partial^l w_2}{\partial \bar{z}^l} = 0 \text{ on } \partial\mathbb{D}\ , \qquad \mathrm{Im}\frac{\partial^l w_2}{\partial \bar{z}^l}(0) = 0\ ,\ 0 \le l \le k-1\ . \quad (4.21)$$

If we denote

$$\tilde{f} = f - \sum_{l=0}^{k-1}\sum_{m=0}^{l}(a_{ml}+b_{ml})i\sum_{j=l}^{k-1}\frac{c_j}{(j-l)!}(z+\bar{z})^{j-l} - Lf_2\ ,$$

we conclude that the problem (4.20), (4.21) is equivalent to the singular integral equation

$$(I + \hat{\Pi} + \hat{K})g_1 = \tilde{f} \quad (4.22)$$

where

$$w_2 = \widetilde{T}_k g_1\ .$$

Thus, if g_1 is a solution of the singular integral equation (4.22), then $w_2 = \widetilde{T}_k g_1$ is a solution of the problem (4.20) and (4.21). Therefore, the following theorems can be stated.

Theorem 4.2.1. *If condition (4.9) is satisfied, then equation (4.2) with the conditions (4.16) has a solution of the form*

$$w = \widetilde{T}_k g_1 + i\sum_{l=0}^{k-1}\frac{c_l}{l!}\ (z+\overline{z})^l + \sum_{l=0}^{k-1}\frac{(-1)^l}{2\pi i l!}\int_{\partial\mathbb{D}}\gamma_l(\zeta)\frac{\zeta+z}{\zeta-z}\ (\zeta - z + \overline{\zeta - z})^l\frac{d\zeta}{\zeta}\ , \quad (4.23)$$

where $g_1 \in L^p(\mathbb{D}$, $p > 2$, is a solution of the singular integral equation (4.22).

Theorem 4.2.2. *If the inequality*

$$q_0 \|P_k\|_{L^p(\mathbb{D})} \|(I+\hat{T})^{-1} - K_1\|_{L^p(\mathbb{D})} < 1$$

is satisfied for some $K_1 \in K(L^p)$, $0 < p-2 < \epsilon$, *then equation (4.2) with the boundary conditions (4.16) has a solution of the form*

$$w = \widetilde{T}_k g_1 + i \sum_{l=0}^{k-1} \frac{c_l}{l!}\ (z+\overline{z})^l + \sum_{l=0}^{k-1} \frac{(-1)^l}{2\pi i l!} \int_{\partial\mathbb{D}} \gamma_l(\zeta) \frac{\zeta+z}{\zeta-z}\ (\zeta - z + \overline{\zeta - z})^l \frac{d\zeta}{\zeta}\ , \tag{4.24}$$

where $g_1 \in L^p(\mathbb{D})$ *is a solution of the singular integral equation (4.22).*

Note 3. The proofs of the above theorems can be done similar to the proofs of the Theorems 4.1.8 and 4.1.11. When the right hand side of the singular integral equation (4.6) is zero, if (4.6) has only the trivial solution, then for any $f(z)$ problems **(H)** and **(NH)** are uniquely solvable. On the contrary, if the homogeneous singular integral equation has a nontrivial solution, using the Fredholm alternative gives two different conditions on the function f to be satisfied.

Now, we consider the differential equation given in Theorem 4.1.1 with the nonhomogeneous Schwarz conditions.

Theorem 4.2.3. *The Schwarz problem*

$$Re \frac{\partial^l w}{\partial \bar{z}^l} = \gamma_l \quad on\ \partial\mathbb{D}, \qquad Im \frac{\partial^l w}{\partial \bar{z}^l}(0) = c_l, \quad 0 \le l \le k-1,$$

for the k-th order equation (4.1) is solvable for given $a_{ij}, b_{ij} \in L^p(\mathbb{D}), f \in L^p(\mathbb{D})$, q_1, q_2 *measurable satisfying* $|q_l(z)| + |q_2(z)| \le q_0 < 1$, $q_0 \|\Pi_1\|_{L^p(\mathbb{D})} < 1$ *and*
$\gamma_l \in C(\partial\mathbb{D}; \mathbb{R}), c_l \in \mathbb{R}, 0 \le l \le k-1$. *Any solution is given in the form*

$$w = \widetilde{T}_k g + i \sum_{l=0}^{k-1} \frac{c_l}{l!}\ (z+\overline{z})^l + \sum_{l=0}^{k-1} \frac{(-1)^l}{2\pi i l!} \int_{\partial\mathbb{D}} \gamma_l(\zeta) \frac{\zeta+z}{\zeta-z}\ (\zeta - z + \overline{\zeta - z})^l \frac{d\zeta}{\zeta}\ ,$$

where $g \in L^p(\mathbb{D})$ is a solution of the singular integral equation

$$\begin{aligned} g + \quad & q_1(z)\Pi_1 g \quad + q_2(z)\overline{\Pi_1 g} + \\ + \quad & \sum_{l=0}^{k-1}\sum_{j=0}^{l}\left(a_{lj}(z)\frac{\partial^j \widetilde{T}_{k-l+j}g}{\partial z^j} + b_{lj}(z)\overline{\frac{\partial^j \widetilde{T}_{k-l+j}g}{\partial z^j}}\right) = \tilde{f}\ , \end{aligned}$$

where

$$\begin{aligned} \tilde{f} = f & - q_1(z)\frac{\partial^k f_2}{\partial \bar{z}^{k-1}\partial z} - q_2(z)\frac{\partial^k \overline{f_2}}{\partial z^{k-1}\partial \bar{z}} \\ & - \sum_{l=0}^{k-1}\sum_{j=0}^{l}\left(a_{lj}(z)\frac{\partial^l f_2}{\partial \bar{z}^{l-j}\partial z^j} + b_{lj}(z)\overline{\frac{\partial^l f_2}{\partial \bar{z}^{l-j}\partial z^j}}\right) \\ & - \sum_{l=0}^{k-1}\sum_{j=0}^{l}(a_{lj}(z) + b_{lj}(z))i\sum_{n=l}^{k-1}\frac{c_n}{(n-l)!}(z+\bar{z})^{n-l} \end{aligned}$$

with

$$f_2(z) = \sum_{l=0}^{k-1}\frac{(-1)^l}{2\pi i l!}\int_{\partial\mathbb{D}}\gamma_l(\zeta)\frac{\zeta+z}{\zeta-z}\ (\zeta - z + \overline{\zeta - z})^l\frac{d\zeta}{\zeta}\ .$$

Proof. In the singular integral equation, only $\widetilde{\Pi}_1$ and $\overline{\widetilde{\Pi}_1}$ as the singular part appears. It is known that, $\|\Pi_1\|_{L^2(\mathbb{D})} = 1$. Thus for $0 < p - 2 < \epsilon$,

$$\|q_1(z)\Pi_1 + q_2(z)\overline{\Pi_1}\|_{L^2(\mathbb{D})} \leq q_0\|\Pi_1\|_{L^2(\mathbb{D})} < 1$$

is satisfied for any $q_0 < 1$. The remaining operator in the singular integral equation is compact, so that the Fredholm alternative applies to that equation. If g is a solution of the equation, then

$$w = \widetilde{T}_k g + i\sum_{l=0}^{k-1}\frac{c_l}{l!}\ (z+\overline{z})^l + \sum_{l=0}^{k-1}\frac{(-1)^l}{2\pi i l!}\int_{\partial\mathbb{D}}\gamma_l(\zeta)\frac{\zeta+z}{\zeta-z}\ (\zeta - z + \overline{\zeta - z})^l\frac{d\zeta}{\zeta}$$

is a solution of the differential equation (4.1) with the nonhomogeneous Schwarz condition. □

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