A BOUNDARY ELEMENT FORMULATION FOR AXI-SYMMETRIC PROBLEMS IN PORO-ELASTICITY

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF THE NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

MEHMET H. ÖZYAZICIOĞLU

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY
IN
CIVIL ENGINEERING

JULY 2006

Approval of the Graduate School of Natural and Applied Sciences

____
Prof. Dr. Canan Özgen
Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Doctor of Philosophy.

____
Prof. Dr. Erdal Çokça
Head of Department

This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Doctor of Philosophy.

____________________	____________________
Prof. Dr. Münevver Tezer	Prof. Dr. M. Yener Özkan
Co-Supervisor	Supervisor

Examining Committee Members

Prof. Dr. Mehmet Utku	(METU, CE)	____________________
Prof. Dr. M. Yener Özkan	(METU, CE)	____________________
Prof. Dr. Yalçın Mengi	(METU, ES)	____________________
Prof. Dr. Münevver Tezer	(METU, MATH)	____________________
Yard. Doç. Dr. Hakan Argeşo	(Başkent Univ., ME)	____________________

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Mehmet H. Özyazıcıoğlu

ABSTRACT

A BOUNDARY ELEMENT FORMULATION FOR AXI-SYMMETRIC PROBLEMS IN PORO-ELASTICITY

Özyazıcıoğlu , Mehmet H.

Ph. D., Department of Civil Engineering

Supervisor : Prof. Dr. M. Yener Özkan

Co-Supervisor : Prof. Dr. Münevver Tezer

July 2006, 222 pages

A formulation is proposed for the boundary element analysis of poro-elastic media with axi-symmetric geometry. The boundary integral equation is reduced to a set of line integral equations in the generating plane for each of the Fourier coefficients, through complex Fourier series expansion of boundary quantities in circumferential direction. The method is implemented into a computer program, where the fundamental solutions are integrated by Gaussian Quadrature along the generator, while Fast Fourier Transform algorithm is employed for integrations in circumferential direction. The strongly singular integrands in boundary element equations are regularized by a special technique. The Fourier transform solution is then inverted in to Rθz space via inverse FFT. The success of the method is assessed by problems with analytical solutions. A good fit is observed in each case, which indicates effectiveness and reliability of the present method.

Key Words: Poro-Elasticity, Boundary Element Method, Axi-symmetric, Fast Fourier Transform, Wave Propagation

ÖZ

POROELASTİSİTEDE EKSENEL SİMETRİK PROBLEMLER İÇİN BİR SINIR ELEMAN FORMÜLASYONU

Özyazıcıoğlu , Mehmet H.

Doktora, İnşaat Mühendisliği Bölümü

Tez Yöneticisi : Prof. Dr. M. Yener Özkan

Ortak Tez Yöneticisi : Prof. Dr. Münevver Tezer

Temmuz 2006, 222 sayfa

Eksenel simetrik geometriye sahip poro-elastik ortamlar için bir sınır eleman formülasyonu önerilmektedir. Sınır değişkenleri açısal yönde karmaşık Fourier serisine açılarak sınır integral denklemleri meridyen düzleminde her bir Fourier bileşeni için yazılan eğrisel integral denklemlerine indirgenmektedir. Sınır elemen denklemlerinde tezahür eden çekirdek fonksiyonları dönel cismi üreten eğri üzerinde Gauss metodu, açısal yönde ise Hızlı Fourier Dönüşüm algoritması kullanılarak integre edilmektedir. Sınır eleman denklemlerindeki tekil integraller, özel bir dönüşüm yolu ile düzenlenmektedir. Fourier uzayında elde edilen çözüm ters Hızlı Fourier Dönüşümü yolu ile Rθz uzayına taşınmaktadır. Yöntemin başarısı analitik çözümü bulunan örnek problemlerde test edilmiştir. Çözümlemelerin her durumda analitik ifadelerle iyi uyum göstermesi yöntemin etkin ve güvenirliliğini ortaya koymaktadır.

Anahtar Kelimeler: Poro-Elastisite, Sınır Eleman Metodu, Eksenel Simetrik, Hızlı Fourier Dönüşümü, Dalga Yayılışı

To My Parents

ACKNOWLEDGEMENTS

I wonder what I own, after so many have contributed so much to this study. First, I would like to acknowledge my supervisors Prof. Dr. M. Yener Özkan and Prof. Dr. Münevver Tezer for their guidance throughout the research. Also, my sincere appreciations go to Prof. Dr. Yalçın Mengi for his help, guidance and moral support in shaping this study, he always well received questions and was ready to help, I personally gained much insight from his lectures where he always communicated an enthusiasm for computational mechanics. Prof. Dr. Mehmet Utku has always been so much attentive and spared time to discus with me, beside his administrative work. The author would also like to thank Asst. Prof. Dr. Hakan Argeşo for his suggestions and comments. I ought to acknowledge the fellowship provided by Prof. Dr. Rer. Nat. Heinz Antes and Prof. Dr. ing. Martin Schanz (currently in Technical University of Graz) of Technical University of Carolo Wilhelmina at Braunschweig, and other friends in the institute of Angewandte Mechanik, in particular, Dr. Jens Uwe Böhrnsen, Dobromil Pryl and Ms. Barbara Ulrich, who were always kind enough and patient to help me. Thanks go to Prof. Dr. H. F. Wang, whose book I found of great usefulness in understanding poro-elasticity. A special thank is due to Prof. Dr. N. Ortner who deigned to send me a copy of his work.I also extend my thanks to Ms. Mary Chibnall, assistant librarian of RAS of London; she never refrained from providing me with photocopies of journal articles.

This study was supported by the State Planning Organization (DPT) Grant No: BAP-08-11-DPT2002K120510-DK10.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

FIGURE

LIST OF SYMBOLS AND ABBREVIATIONS

Material Constants

λ	drained Lame modulus
λ_u	undrained Lame modulus
K	drained bulk modulus
K_u	undrained bulk modulus
K_f	bulk modulus of the pore fluid
K_s	bulk modulus of individual (intact) grains
μ	shear modulus of the solid bulk material
ν	drained Poisson's ratio
ν_u	un-drained Poisson's ratio
Q	Biot modulus, reciprocal of constrained storage modulus
α	Biot effective stress coefficient
κ	permeability coefficient
η	dynamic viscosity of pore fluid
V	bulk material volume, total volume
V_f	volume of fluid in bulk material
V_p	volume of pores in bulk volume
ρ	bulk mass density
ρ_d	dry mass density of bulk material
ρ_g	mass density of individual solid grains
ρ_f	fluid mass density

Kinematic Variables

n	porosity, ratio of poro volume to bulk volume

u_i	components of solid (skeleton) displacement vector
U_i	components of fluid displacement vector
w_i	relative fluid displacement, $U_i - u_i$
q_i	components of fluid flux vector, $n \cdot \dot{w}_i$
θ	increment of fluid content, for small displacement theory $\theta = \frac{\delta m_f}{\rho_f}$
ε_{ij}	components of solid strain tensor
r	distance
r_i or $r_{,i}$	components of the gradient of distance, a unit vector along the distance from point A (source) to point P (field)
R, θ, z	coordinate designations for cylindrical coordinate system

Kinetic Variables

M_f	mass of fluid in bulk volume at current configuration
m_f	fluid mass content at current configuration, $\frac{M_f}{V}$
δm_f	change in fluid mass content, $m_f - m_{f_0}$
m_{f_0}	fluid mass content at reference configuration
p	pore pressure
σ_{ij}^f	components of stress in the fluid, $-p\delta_{ij}$
τ_{ij}	components of total stress tensor

Other variables/operators

ω	angular frequency corresponding to time in Fourier transform space
δ_{ij}	Kronecker delta
$\Delta(A,P)$	Dirac delta function in 3-D
$a(t)$	fluid injection rate
B	Skempton’s pore pressure coefficient
∇^2	Laplacian operator, $\nabla \bullet \nabla$
$\lambda_{1,2,3}$	roots of the characteristic equation of poro-elastic adjoint operator

λ_4	an abbreviation, $\lambda_4^2 = -\dfrac{\omega^2(\rho + \beta\rho_f)}{\lambda + 2\mu}$

Abbreviations

ACS	Actual System
AXS	Auxiliary System
BE	Boundary Element
BEM	Boundary Element Method
BIE	Boundary Integral Equation
EHS	Elastic Half Space
FFT	Fast Fourier Transform
FS	Fundamental solution
FT	Fourier Transform
GIT	Gauss Integral Theorem
HS	Half Space
ND	Non-Dimensional
PE	Poro-Elastic

CHAPTER 1

INTRODUCTION

Axi-symmetric boundary element formulations for elasto-dynamic (Brebbia and Dominguez 1992, Wang and Banerjee 1990, Becker 1992) and acoustic analysis (Juhl 1993, Soenarko, Pozrikidis 2002) are available in the literature. However, these formulations are either based on integration of the fundamental solutions in circumferential direction, hence fully axi-symmetric (both geometry and boundary conditions are axi-symmetric) or they expand the boundary quantities into regular (real) Fourier series separating the boundary excitations into symmetric and anti-symmetric modes, the final response is obtained by combining solutions for each of these modes. The first approach suffers from the accurate evaluation of Elliptic Integrals, while in the second approach care ought to be given to integrations in circumferential direction where Gaussian Quadrature along with segmentation is proposed (Brebbia and Dominguez, 1992), for accurate evaluation; the segmentation procedure, on the other hand, lead to extended run times.

Following the second approach described above, an axi-symmetric formulation for poro-elasticity is given by Dargush and Chopra (1996). Their formulation involves the symmetric modes only (symmetric boundary conditions), thus fully axi-symmetric.

An alternative method based on complex Fourier series expansion of boundary quantities is developed by Özkan (1995). In this method the integrations in circumferential direction is accomplished by the Fast Fourier Transform algorithm (Brigham, 1988), which is shown to bring in considerable savings in computer run times over integration using segmentation along with Gaussian Quadrature.

The formulation of Tsepora and Polyzos (2003) for gradient elasticity is similar to Özkan, 1995; however, they used the so called non-periodic FFT (Press, et. al., 1992) for integrations in θ-direction when the source point lies in the integration element. According to this algorithm, the FFT is applied with the value at θ=0 is set to zero (actually the algorithm is to be modified for numerical stability as discussed by Press, et. al. 1992). The integrals when θ=0 are computed separately for each frequency by using advanced integration techniques (Guiggiani, 1992).

In this study, we extend the method proposed in (Özkan, 1995) to axi-symmetric poro-elastodynamic analysis with arbitrary boundary conditions. To summarize, we first write the boundary integral equations in cylindrical coordinate system by transforming boundary variables from Cartesian (xyz) to cylindrical coordinate (Rθz) system; the z-coordinate of the Rθz-system coincides with the axis of revolution of the body. The boundary variables (generalised displacement and traction vectors) are then expanded in complex Fourier series with respect to θ-coordinate; thereby we reduce the surface (3-D) boundary integral equations to a set of N' boundary integral equations in the generating plane (2-D), where the kernels appear in the form of Fourier integrals of fundamental solutions; where, N' is the number of terms in the truncated complex Fourier series (N' must be a positive power of 2). The kernels now appearing in line integral equations can be integrated effectively by FFT algorithm. The solution of N' line (2-D) integral equations establishes the coefficients of the boundary variables in the complex Fourier expansions; the combination of these coefficients using again FFT algorithm yields the solution in Rθz space.

The formulation developed in this study has two major advantages: First, the use of FFT algorithm in integrations over θ-direction increases computational performance considerably compared to Gaussian Quadrature for the same purpose. Second, using complex Fourier expansion obviates the analysis for symmetric and anti-symmetric modes separately (Özkan, 1995); analysis for a general boundary condition is accomplished in a single run.

The formulation is developed in frequency domain, yet solutions in time domain can be obtained by inverse FFT after solution vectors have been obtained for a sufficient number of frequencies.

As a part of this work, a general purpose computer program, named AxiPoro, is implemented in ANSI standard C++ language. The program compiles equally well under both Windows and Unix, compilation under Linux should bring no problems.

The method is assessed in four problems, these are, (i) the poro-elastic column problem, (ii) cylindrical cavity problem, (iii) spherical cavity problem and (iv) the vertical compliance function for a rigid circular foundation. The first three problems possess exact analytical solutions, for the last problem the results are compared with the numerical solution given by Apsel and Luco (1987). A good fit is observed in each case, except for the last problem where the largest relative difference in absolute value of the compliance is found to be %7.5; however, it should be noted that in the solution we provide an ideal elastic problem is simulated by a poro-elastic formulation, hence, a perfect match is not expected.

The organization of this thesis is as follows: In Chapter 2, we make a review of theory of poro-elasticity including a brief history. The governing equations are derived using the methods of continuum mechanics and a short discussion of waves in infinite poro-elastic media is given. Chapter 3 is a review of published boundary element work concerning poro-elasticity only. In Chapter 4, we develop the boundary integral equations for poro-elasticity. Fundamental solutions of dynamic poro-elasticity are developed in Chapter 5. In chapter 6, boundary element formulation for axi-symmetric geometry using complex Fourier expansion of boundary variables is presented. The treatment for strongly singular integrals encountered when $\theta=0$ is described. The computer implementation and manual of input instructions are described in Chapter 7. Assessment problems and results are presented in Chapter 8. The thesis ends in Chapter 9 with conclusions and suggestions for further study.

CHAPTER 2

REVIEW OF THE THEORY OF PORO-ELASTICITY

2.1 Background Information

A porous medium is a material matrix composed of a deformable solid skeleton with interconnected pores and fluid residing in these pores. The pores are interconnected, as mentioned, in the form of small irregular arcs so as to provide fluid mass exchanges within the body or with the outside.

Although occurrence of more than one type of fluid, either miscible or immiscible, partially filling the pores is possible; this study addresses the presence of a single fluid fully saturating the pores. In this regard, a porous medium is the superposition in time and space of two media, the solid skeleton and the pore fluid (Coussy, 1995).

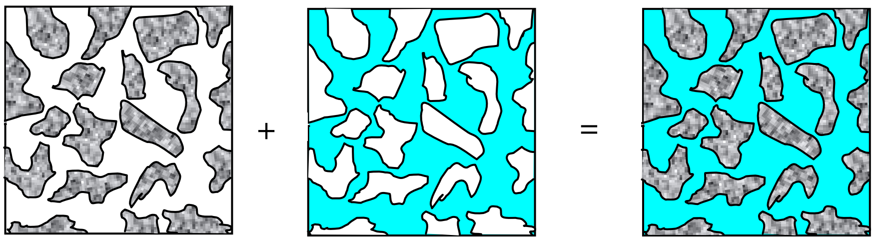

Figure 2.1 Two phase *porous medium* as the superposition of a granular solid matrix and a fluent medium (after Coussy, 1995).

Examples of such media are foundation soils saturated with ground water and porous rock containing hydrocarbons and the bone; porous metals and plastics are also produced. The phenomenon may involve interaction of disparate materials like soil and water, oil or natural gas and porous rock, etc., but the general framework of study is the same, "pore-fluid solid interaction".

The mechanics of fluid filled porous media, as the solid went through deformation or fluid flowed in or out, has captivated the attention of many scientists and engineers for centuries, in connection with, for instance, the settlement of land surface in long term owing to a civil structure, underground water flow, flow through earth structures (e.g. dams), fluctuation of water levels in wells when trains pass nearby, land subsidence after extensive utilization of underground reservoirs and of course soil liquefaction during large earthquakes.

It is immediately observed that the deformation of the porous solid skeleton and flow of the pore-fluid are not independent (one affects the other), and coupling always occurs among fluid mass content, fluid pressure (pore pressure), deformations in the bulk solid and applied stresses.

The word solid is often used to refer to the skeletal framework of bulk porous-material, and the terms fluid pressure and pore-pressure are used interchangeably. The bulk material will be referred to as "porous medium".

2.1.1 The Continuum Assumption

If we were to the view the porous medium on a length scale on the order of the pore size, the pore space will appear extremely detailed and chaotic. If we hope to model the flow and transport of the fluid on a macro or mega volume of the porous medium, it is necessary to average the description of the system over a *representative elementary volume* (REV) that is large compared to the sizes of the pores but small compared to the macroscopic dimensions of the porous medium.

The properties of the system will be described by continuum variables such as porosity and permeability.

Fundamental definitions:

The term *porosity* is used to demonstrate how porous the bulk material is, in other words, what percent of the total bulk material volume is constituted by interconnected pores. Therefore, porosity, designated by n is the ratio of the interconnected pore volume to the total volume. Although considerations about the presence of two degrees of porosity (of pores and of fissures/cracks) exist in the literature, we stick to single porosity formulation.

Permeability (after Dullien 1992), designated in this text by κ; refers to the conductivity of the porous medium to permeation by a fluid. A definition of such a general sense is of limited usefulness; because this permeability with respect to a particular porous medium changes with different permeating fluids and flow field. It would be more instructive and more practical as well as more scientific to single out the material parameter that gives the contribution of only the porous medium to the conductivity; and is independent of both fluid properties and flow field. This last quantity is called the *intrinsic permeability* "k^d", which is uniquely determined by the pore structure only.
The definitions of permeability will gain more clarity when Darcy's Law is discussed in the course.

2.2 Brief History of Porous Media Theories

In the following, we make a brief of the history of poro-elasticity following Wang (2000) and Chen (1992). Extensive review of the history of the development of porous media theories are covered in the works of de Boer (1996, 2000).

Although attempts to understand and model the behaviour of fluid-filled porous media start earlier, e.g. Telford in 1821(after Chen, 1992) drew attention to squeezing out water in the phenomenon of consolidation of soil, it is generally deemed that the study of fluid filled porous media begins with Darcy (1856) who studied one dimensional flow of water in homogenous sands in connection with the fountains of the city of Dijon, France.

In the period 1900-1930, utilization of underground hydrocarbon reservoirs as well as heavy civil constructions (high rise buildings, dams, etc.) necessitated an improved understanding of fluid flow in porous media and deformational behaviour of soils and rock (Wang, 2000). In relation to the state of pore fluid, the principal problems were discharge and elastic storage in confined aquifers and in relation to deformation the problem was soil consolidation (Wang, 2000).

First, mathematical model on deformation of a porous medium was established by Terzaghi in 1923 (after Detournay and Cheng 1993), who realizing that soil consolidation was in effect a diffusion process (escape of water from high pressure zone) set up his famous one-dimensional consolidation equation. Later an attempt was made by Rendulic in 1936 (after Detournay and Cheng 1993) to extend the Terzaghi's 1-D theory to 3-D by replacing the spatial derivative for the Laplacian, however, in both cases this theory uncoupled the deformation from stresses and fluid pressure and flow; a fully coupled theory was yet to be developed.

The era 1930-1970 witnessed development of constitutive theories for porous media. A real coupled quasi-static 3-D theory was established by Biot (1941.a). Other contributions were due to Kosten and Zwikker (1941), Frenkel (1944), Lubinski (1954 after Chen, 1992), Brutsaert (1964). Biot's theory has gained the widest acceptance. Kosten and Zwikker and Frenkel, deserve a special note; Kosten and Zwikker (1941) were perhaps the first researchers to notice the existence of second compressional wave and Frenkel (1944) was the first to work out the first complete dynamic theory. Brutsaert's unsaturated dynamic model carries features similar to Biot's theory for liquid saturation and to Kosten and Zwikker's for gas saturation. More complex models using the *theory of mixtures* have also been developed (Steel 1967, Bowen 1976, 1980, 1982, Prevost 1980, Burridge and Keller 1981, de Boer 2005). These complex models reduce to Biot's theory under certain simplifying assumptions (Schanz and Diebels, 2003); hence confirm the soundness of this theory. Some analytical solutions were also obtained during this period (Biot, 1941.b, 1941.c, 1942, 1956.a).

Starting with 1960's problems on complex domains with arbitrary boundary conditions were targeted at by powerful numerical methods (Finite Element Method, Boundary Element Method, Method of Characteristics (Mengi and McNiven, 1977), etc.) on digital computers.

2.3 Biot's Theory of Linear Poro-elasticity

Biot's poro-elasticity theory (Biot 1941.a, 1955, 1956.d, 1962.a, 1962.b) is a generalization of elasticity theory to materials with fluid filled pore spaces and it includes Terzaghi's 1-D consolidation theory as a special case. Quite remarkably, this theory was developed by him only, over a deliberation of approximately 40 years. During those years, Biot considered the theory from almost every angle, laid out the thermodynamical foundations to cover various dissipative effects and relaxation mechanisms (1962.a, b). His theory led to mathematical formulation of problems in soil mechanics, geophysics, acoustics and biomechanics (Cowin 1999, Lim and Hong 2000) predicting behaviour beyond that conceivable by classical elasticity theory. The predictions of the theory have been substantially verified experimentally (Plona 1980, Berryman 1980, Ogushwitz 1985, Bonnet and Auriault 1985, Klimentos and McCann 1988, Gurevich et. al. 1999). The material constants (Biot and Willis 1957) involved are easily discernible, physically meaningful and experimentally measurable. The theory redefines the concept of effective stress, which is a fundamental principle in soil mechanics.

In the following, the discussion will be restricted to homogenous and isotropic poro-elastic media.

2.3.1 Governing Equations

A note on the notation:

In what follows, indicial notation (Cartesian tensor) together with (Einstein's) summation convention will be used; all tensor quantities are to be resolved in Cartesian frames. Hence for example, a term like "v_i" represents i-th component of the vector $\underline{\mathbf{v}}$ in Cartesian coordinates. In summation convention, any (twice) repeated index implies automatic summation over its range. The following are equivalent designations for partial differentiation (of a vector $\underline{\mathbf{v}}$ say):

$$\frac{\partial v_i}{\partial x_j} \equiv \partial_j v_i \equiv v_{i,j}$$

All variables (displacements, fluid pressure, stresses, etc.) are functions of both space and time.

The displacement field is defined by the displacements of the solid (skeleton) "u_i" and the displacements of the fluid "U_i". Then the relative displacement of the fluid with respect to solid is

$$w_i = U_i - u_i \,.$$

Assuming that the porosity of a material plane is the same as volume porosity "n", the net flux of fluid through unit area in unit time is then given by

$$q_i = n\dot{w}_i$$

where "q_i" denotes flux in the direction of i-th coordinate axis and $\dot{w}$ is relative velocity of the pore-fluid.

The strains in the solid are defined as usual

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

One more variable, increment of fluid content "θ" introduced by Biot (1941.a), is needed to define the change in the amount of fluid in a unit volume of bulk material. This variable measures the volume of fluid that has flowed in or out per unit volume of bulk material. Hence,

$$\theta = -nw_{i,i}$$

Drained and un-drained conditions:

Although, one almost never encounters true drained as well as un-drained condition in reality, we still define a condition where all internal fluid is prevented from motion by impermeable boundaries as *un-drained*. The condition during or shortly after a rapid process (deformation, loading, injection, etc.) such that the pore-pressure induced cannot dissipate is also considered to be un-drained. On the other hand, if, free drainage of the fluid is permitted or the process is so slow that the excess pore-pressure can dissipate, the porous medium is said to be under *drained* condition.

Mechanically, the un-drained condition corresponds to no relative motion:

$$w_i = U_i - u_i = 0 \qquad \rightarrow \quad \text{un-drained.}$$

Whereas, drained condition corresponds to pore pressure keeping constant:

$$\begin{matrix} p = 0 \\ w_i \neq 0 \end{matrix} \qquad \rightarrow \quad \text{drained.}$$

2.3.1. a Constitutive Equations

Assuming that (1) the material is linear-elastic and isotropic (2) strains are small, Biot (1941.a) arrived at the following constitutive equations:

$$\tau_{ij} = 2\mu\varepsilon_{ij} + \lambda\delta_{ij}\varepsilon_{kk} - \alpha\delta_{ij}p$$
$$\theta = \alpha\varepsilon_{kk} + \frac{1}{Q}p \quad \text{...} (2.1)$$

$\tau_{ij} \rightarrow$ components of total stress tensor

The four elastic constants,

$\lambda \rightarrow$ drained Lame's modulus (dimension $\equiv F/L^2$)
$\mu \rightarrow$ drained shear modulus (dimension $\equiv F/L^2$)
$\alpha \rightarrow$ Biot's effective stress coefficient (dimensionless)
$Q \rightarrow$ Biot deformation modulus, corresponds to the reciprocal of constrained storage coefficient in hydrogeology (dimension $\equiv F/L^2$)

characterize the poro-elastic material. The material constants and their measurement techniques will be discussed in the next section.

Effective Stress Principle (a measure of inter-granular stress):
When the deformation (or strength) response of a porous medium (soil for instance) is considered, it is immediately discerned that the behaviour will not be the same with zero and nonzero pore-pressure. As first realized by Terzaghi, the strain in a soil element in a tri-axial cell should be governed by the inter-granular stresses. Then, which of the stress fields (fluid pore-pressure and solid) or what linear combination of them is directly responsible for the strength and deformational characteristics of the porous medium? This question (and other similar questions) leads naturally to the concept of *effective stress*" (Berrymann, 1992). The effective stress principle is discussed extensively in soil mechanics

books (Lamb and Withman 1979, Suklje 1969) and there are scientific papers discussing this principle (Berrymann 1992, 1993; de Boer and Ehlers 1990) in many articles, here we choose the definition, which follows naturally from the constitutive equation (2.1) and generalizes the Terzaghi's effective stress principle:

$$\tau'_{ij} = \tau_{ij} + \alpha p \delta_{ij}$$

where, τ'_{ij} is effective stress. Then, it is clear that Terzaghi's definition

$$\tau'_{ij} = \tau_{ij} + p \delta_{ij}$$

is the case when $\alpha = 1$.

Hence, in Biot's definition of effective stress, a fraction α of fluid stress ($\sigma_{ij} = -p\delta_{ij}$) is subtracted from total stress. There is experimental evidence that this definition better describe the volumetric strain under confining stress (Wang 2000, p.45).

The constitutive equations (2.1) can be rewritten for effective stresses in the following form:

$$\tau'_{ij} = \tau_{ij} + p\delta_{ij} = 2\mu\varepsilon_{ij} + \lambda\delta_{ij}\varepsilon_{kk}$$

or,

$$\tau'_{ij} = C_{ijkl}\varepsilon_{kl}$$

where,

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)$$

Sign convention:

In equation (2.1) the stress components "τ_{ij}" represent total stresses and are positive when tensile, on the other hand, "p" represents the pore-fluid pressure and is positive when compressive.

An alternative form of constitutive equations is possible (in fact there are four alternatives, see Wang, 2000) if pressure is isolated in the second equation in (2.1)

$$\begin{aligned} \tau_{ij} &= 2\mu\varepsilon_{ij} + \lambda_u \delta_{ij}\varepsilon_{kk} - \alpha Q \delta_{ij}\theta \\ p &= -\alpha Q \varepsilon_{kk} + Q\theta \end{aligned} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2.2)$$

where

$$\lambda_u = \lambda + \alpha^2 Q$$

is the undrained Lame's modulus. We know, on physical grounds that $\lambda_u > \lambda$ hence, Q must be positive (refer to the discussion before equation 2.11). In fact, thermodynamic considerations also stipulate positiveness of Q (Biot, 1962.a).

2.3.1. b Poro-elastic Material Constants

In order to gain insight into the material constants, it is useful to consider spherical stress volumetric strain relations, thus applying contraction over the index "i" in (2.1) and (2.2) above

$$\begin{aligned} \sigma &= Ke - \alpha p \\ \theta &= \alpha e + \frac{p}{Q} \end{aligned} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2.3)$$

and

$$\begin{aligned}\sigma &= K_u e - \alpha Q\theta \\ p &= -\alpha Q e + Q\theta\end{aligned} \quad \text{...} \quad (2.4)$$

where,

$$\sigma = \frac{\tau_{kk}}{3} \rightarrow \text{mean normal stress}$$
$$e = \varepsilon_{kk} \rightarrow \text{volumetric strain}$$
$$K = \lambda + \frac{2}{3}\mu \rightarrow \text{drained bulk modulus}$$
$$K_u = \lambda_u + \frac{2}{3}\mu = K + \alpha^2 Q \rightarrow \text{undrained bulk modulus}$$

Skempton's Coefficient "B" (for pore-pressure built up during confined loading):

Skempton's coefficient B (Skempton, 1954) is defined to be the ratio of the induced pore water pressure to applied all round isotropic stress increment under un-drained conditions,

$$B = -\left.\frac{\partial p}{\partial \sigma}\right|_{\theta} \quad \text{..} \quad (2.5)$$

A negative sign is necessary because positive (tensile) σ gives rise to negative (tensile) p. As can be seen B is non-dimensional.

From (2.3),

$$\frac{\partial p}{\partial \sigma} \xrightarrow{\text{gives}} \quad 1 = K\frac{\partial e}{\partial \sigma} - \alpha\frac{\partial p}{\partial \sigma}$$
$$\frac{\partial \theta}{\partial \sigma} \xrightarrow{\text{gives}} \quad 0 = \alpha\frac{\partial e}{\partial \sigma} + \frac{1}{Q}\frac{\partial p}{\partial \sigma}$$

combining

$$B = \frac{\alpha Q}{K_u} \quad \text{.......} \quad (2.6)$$

recalling the relation $K_u = K + \alpha^2 Q$

$$K_u = \frac{K}{1 - \alpha B} \quad \text{.......} \quad (2.7)$$

From (2.5) it is clear that

$$0 \le B \le 1 \quad \text{.......} \quad (2.8)$$

Skempton's B may also be defined as

$$B = -\left.\frac{\partial e}{\partial \theta}\right|_{\sigma}$$

The effective stress coefficient (Biot's "α"):

"α" is a dimensionless coefficient. From (2.3)

$$\alpha = \left.\frac{\partial \theta}{\partial e}\right|_{p} \quad \text{.......} \quad (2.9)$$

For a solid matrix of single material (individual grains are made of identical material), a restriction on the range of values that α is given by the following

$$n \le \alpha \le 1 \quad \text{.......} \quad (2.10)$$

This conclusion is established in (Biot, 1957). For solid grains of mixed material the lower bound on α has to be relaxed to zero (Berrymann, 1992).

Now from (2.7) with (2.8) and (2.10) one infers that $K_u \geq K$. Since,

$$K_u = K + \alpha^2 Q$$

one has

$$K + \alpha^2 Q \geq K$$

or

$$\alpha^2 Q \geq 0$$

thus

$$Q \geq 0 \quad \text{.......} \quad (2.11)$$

Methods to measure "α", "K" and "K_u"

i) Unjacketed Test

In an unjacketed test, a specimen is tested under a constant cell pressure "σ_c" while pore-pressure is also maintained at the same level (Fig 2.2). This is equivalent to immersing the bare (or with a perforated membrane) specimen in a fluid, till a confining pressure "σ_c" is attained; the same fluid pressure develops in the pore space.

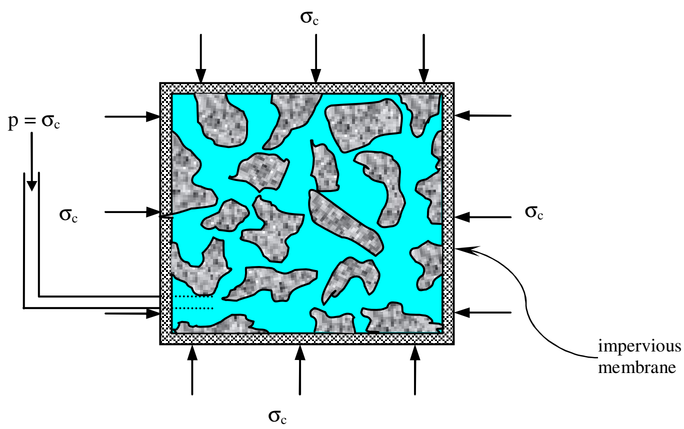

Figure 2.2 Specimen under unjacketted test, pore-pressure is kept equal to cell (surrounding) pressure, as if the membrane were missing (after Wang, 2000).

If now, *unjacketed bulk modulus* is defined as

$$\mathrm{K}_s' = \left.\frac{\partial\sigma}{\partial \mathrm{e}}\right|_{p=-\sigma}$$

from (2.3),

$$\frac{\partial}{\partial \mathrm{e}}(\sigma = \mathrm{Ke} - \alpha(-\sigma))$$

$$\underbrace{\frac{\partial\sigma}{\partial \mathrm{e}}}_{\mathrm{K}_s'} = \mathrm{K} + \alpha\underbrace{\frac{\partial\sigma}{\partial \mathrm{e}}}_{\mathrm{K}_s'}$$

Hence,

$$\alpha = 1 - \frac{K}{K_s'} \quad \text{.......................................} \quad (2.12)$$

For an incompressible solid skeleton $\mathrm{K}_s' \to \infty$, when $\alpha = 1$. In this case, Biot's effective stress reduces to Terzaghi's effective stress definition.

It may be noted that, though K_s' appears to be the same as the bulk modulus of solid grains, i.e. K_s, this is true only when the grains are of the same material; when the solid phase is composed of two types of minerals, e.g. a sandy clay, there is still one K_s', although, one cannot speak of a single grain modulus K_s.

ii) Jacketed test

This test is the usual drained test; a specimen is tested under a constant cell pressure while pore-pressure is kept constant by allowing free drainage of the pore-fluid.

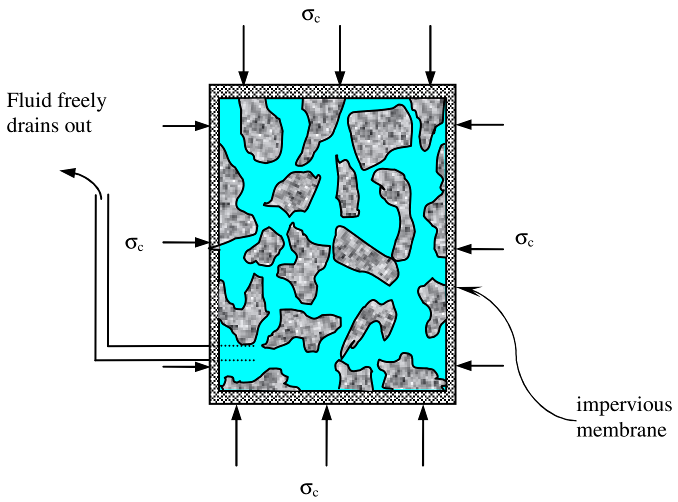

Figure 2.3 Specimen under jacketed test, i.e. pore-pressure is kept constant, normally at zero level. Fluid freely drains out (after Wang, 2000).

If pore pressure is kept at zero value (p=0), equation (2.3) provides Biot's "α" and drained bulk modulus "K":

$$-\sigma_c = Ke - \alpha \underbrace{p}_{=0} \xrightarrow{\text{gives}} K = -\frac{\sigma_c}{e}$$

$$\theta = \alpha e + \frac{p(=0)}{Q} \xrightarrow{\text{gives}} \alpha = \frac{\theta}{e}$$

Poisson's ratio:

Poisson's ratio is defined, in solid mechanics, as the ratio of the lateral to longitudinal strain under uniaxial loading in longitudinal direction.

$$\nu = -\left.\frac{\varepsilon_{\underline{jj}}}{\varepsilon_{\underline{ii}}}\right|_{\sigma_{\underline{jj}}} \qquad (i \neq j)$$

The underline is to mean summation is discarded.

There are two possible definitions for Poisson's ratio in poroelasticity

Drained Poisson's ratio:

It is the Poisson's ratio under drained conditions (p=0)

$$\nu = -\left.\frac{\varepsilon_{\underline{jj}}}{\varepsilon_{\underline{ii}}}\right|_{\sigma_{\underline{jj}};p=0}$$

It can be related to un-drained shear and Lame's moduli through

$$\nu = \frac{\lambda}{2(\lambda+\mu)} \quad \text{...} \quad (2.13)$$

Un-drained Poisson's ratio:

It is the Poisson's ratio under un-drained conditions (θ=0)

$$\nu_u = -\left.\frac{\varepsilon_{\underline{jj}}}{\varepsilon_{\underline{ii}}}\right|_{\sigma_{\underline{jj}};\theta=0}$$

Similarly,

$$\nu_u = \frac{\lambda_u}{2(\lambda_u + \mu)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(2.14)}$$

Recalling the relations

$$\lambda_u = \lambda + \alpha^2 Q$$
$$B = \frac{\alpha Q}{K_u}$$
$$K_u = \frac{K}{1 - \alpha B}$$
$$K = \lambda + \frac{2}{3}\mu$$

One finally derives,

$$\nu_u = \frac{3\nu + \alpha B(1 - 2\nu)}{3 - \alpha B(1 - 2\nu)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(2.15)}$$

from which, it is clear that $\nu_u > \nu$. (2.15) can also be used to relate Skempton's "B" to "ν_u, ν and α" as,

$$B = \frac{3(\nu_u - \nu)}{\alpha(1 - 2\nu)(1 + \nu_u)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(1.26)}$$

Another *useful formula* can be obtained from (2.14) if the relation

$$\lambda_u = \lambda + \alpha^2 Q \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(2.17)}$$

is substituted and "Q" is solved for,

$$Q = \frac{2\mu(\nu_u - \nu)}{\alpha^2(1-2\nu_u)(1-2\nu)} \quad \text{.................................... (2.18)}$$

Rearranging (2.17) as $Q = \frac{\lambda_u - \lambda}{\alpha^2}$

or

$$Q = \frac{K_u - K}{\alpha^2} \quad \text{.. (2.19)}$$

Remark 1: Although various combinations are possible, an isotropic poro-elastic material is fully defined by four independent constants, others can be derived from the relationships given above.

Remark 2: The poro-elastic constants of various earth materials have been measured in the laboratory. A summary table can be found in Wang (2000). The reader is also referred to Yew and Jogi (1978) and Fatt (1959).

A sample procedure to determine poro-elastic constants:

1. From a jacketed (drained) test (isotropic consolidation); determine {α, K}.
2. From an un-drained test (isotropic compression), determine {B or K_u}. B can also be determined, if suitable, by measuring volume expansion of the bulk material per unit volume of fluid injection under back pressure, while cell pressure is zero (see expression 2.9).
3. Measure Young's modulus "E" in a drained test,

$$E = 2 * \mu(1+\nu) \quad \text{.. (2.20)}$$

4. Compute K_u from (2.7) and Q from (2.19) or (2.6).
5. Using (2.13, 2.14, 2.17 and 2.20) compute ν, ν_u, μ, λ, λ_u.

2.3.1. c Field Equations (Balance Laws)

Conservation of Linear Momentum (LME or SEM):

Newton's second law of motion states that in an *inertial reference frame*, material rate of change of momentum of a body is equal to the resultant of applied forces. Hence,

$$\int_\Gamma t_i d\Gamma + \int_\Omega f_i d\Omega = \frac{d}{dt}\int_\Omega \rho_d \dot{u}_i d\Omega + \frac{d}{dt}\int_\Omega n\rho_f \dot{U}_i d\Omega \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(2.21)}$$

where,

t_i → components of the traction vector

f_i → components of the body force vector (defined per unit volume of the bulk material)

ρ_s → density of the solid material

ρ_f → density of the pore fluid

ρ_d → dry density of the bulk material, which is equal to $(1-n)\rho_s$

and, Ω designates the body (domain) and Γ the surface, boundary of Ω.

Inserting, $U_i = u_i + w_i$ and recalling, the Cauchy's stress formula

$$t_i = n_j \tau_{ji} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(2.22)}$$

(2.21) becomes

$$\int_\Gamma n_j \tau_{ji} d\Gamma + \int_\Omega f_i d\Omega = \frac{d}{dt}\int_\Omega \rho_d \dot{u}_i d\Omega + \frac{d}{dt}\int_\Omega n\rho_f \dot{u}_i d\Omega + \frac{d}{dt}\int_\Omega n\rho_f \dot{w}_i d\Omega \quad \ldots\ldots \text{(2.23)}$$

The first integral on the left can be converted to a volume integral by Gauss Integral Theorem as,

$$\int_S n_j \tau_{ji} d\Gamma = \int_V \tau_{ji,j} d\Omega \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. \text{(2.24)}$$

hence, (2.23) becomes

$$\int_\Omega \tau_{ji,j} d\Omega + \int_\Omega f_i d\Omega = \int_\Omega (\rho_d + n\rho_f)\ddot{u}_i d\Omega + \int_\Omega n\rho_f \ddot{w}_i d\Omega$$

But,

$$\rho \mathrel{\hat{=}} \text{bulk density} \mathrel{\hat{=}} \frac{\text{total mass (fluid + solid)}}{\text{total (bulk) volume}}$$

$$\rho = n\rho_f + \underbrace{(1-n)\rho_s}_{\rho_d}$$

therefore,

$$\int_\Omega \left[\tau_{ji,j} + f_i - \rho\ddot{u}_i - n\rho_f \ddot{w}_i\right] d\Omega = 0$$

this implies,

$$\tau_{ji,j} + f_i = \rho\ddot{u}_i + n\rho_f \ddot{w}_i \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(2.25)}$$

which is the LME or Stress Equation of Motion (SEM).

Equation of Continuity:

Consider, an infinitesimal cube of PE material with faces parallel to coordinate planes. Assuming that percent cross-sectional area occupied by the fluid is equal to the porosity, we have for net fluid flux per unit time through i-th face :

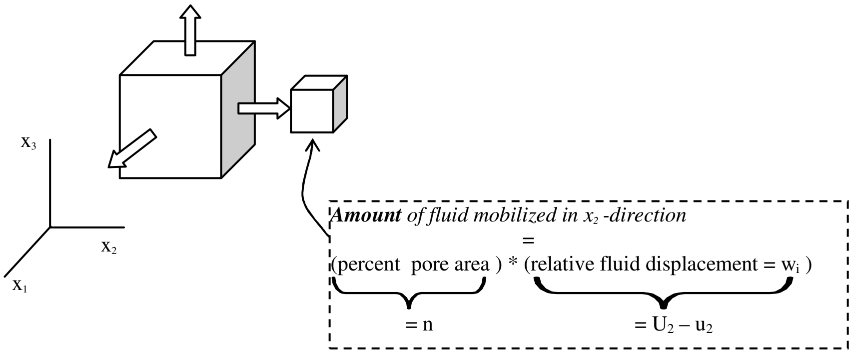

Figure 2.4 Definition of flux vector.

If a(t) represents fluid source per unit volume, law of conservation of mass can be written as follows:

$$\underbrace{\begin{pmatrix}\text{Change in the amount}\\ \text{of fluid inside the}\\ \text{volume in unit time}\end{pmatrix}}_{\int_V \frac{\partial\theta}{\partial t}dV} = \underbrace{\begin{pmatrix}\text{amount of fluid}\\ \text{generated inside}\\ \text{in unit time}\end{pmatrix}}_{\int_V a(t)dV} + \underbrace{\begin{pmatrix}\text{amount of fluid}\\ \text{passing through}\\ \text{the surface in unit}\\ \text{time}\end{pmatrix}}_{-\int_S \vec{n}.\vec{q}dS} \quad \text{.........} \quad (2.26)$$

(2.26) can concisely be stated in words as “accumulation equals net influx plus net generation”. Hence,

$$\int_V \frac{\partial\theta}{\partial t}dV = \int_V a(t)dV - \int_S \underbrace{n_k q_k}_{\substack{q_n \\ \text{normal component of} \\ \text{fluid flux vector through} \\ \text{the boundary}}} dV \quad \text{..} \quad (2.27)$$

using Gauss Integral Theorem (GIT) the second integral (surface integral) on the right can be converted in to a volume integral,

$$\int_V \frac{\partial\theta}{\partial t} dV = \int_V a(t)dV - \int_V q_{k,k} dS$$

collecting the terms

$$\int_V \left[\frac{\partial\theta}{\partial t} + q_{k,k} - a(t)\right] dV = 0$$

hence (since V is arbitrary);

$$\frac{\partial\theta}{\partial t} + q_{k,k} = a(t) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2.28)$$

Darcy's Law:

We interpret Darcy's Law as the *linear momentum equation* for the fluid phase, in disguise; otherwise the so called *Generalized Darcy's Law* (Chen, 1992, 1994a, 1994b) sometimes causes confusion. Thus, writing the LME for the fluid phase per unit volume of the bulk material;

$$\sigma^f_{ji,j} + (\text{body force term}) = \text{inertial terms} + \text{frictional drag}$$

but, $\sigma^f_{ij} = -p\delta_{ij}$ (stress in the fluid)

$$-p_{,i} = \rho_f \ddot{u}_i + \overline{m}\ddot{w}_i + \underbrace{\rho_f g \frac{1}{k} n\dot{w}_i}_{\substack{\text{seepage force} \\ \text{per unit volume}}} - g_i^{\,f} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2.29)$$

where k is the *"hydraulic conductivity"* or Darcy's coefficient in the units of velocity, $[k] \equiv LT^{-1}$ and $\overline{m} = \frac{n\rho_f + \rho_a}{n}$. ρ_a is added mass density, introduced by Biot (Biot 1956.b) to describe the interaction between the internal fluid and the solid skeleton, some authors (e.g. Zienkiewicz et. al., 1980, 1999) discard this effect and take $\rho_a = 0$; and g_i^f is body force acting per unit volume of fluid representing such effects as gravity or magnetism.

Rearranging equation (2.29)

$$n\dot{w}_i = -\kappa\left[p_{,i} + \rho_f \ddot{u}_i + \frac{n\rho_f + \rho_a}{n}\ddot{w}_i - g_i^f\right] \quad \text{.................................... (2.30)}$$

where, $\kappa = \frac{k}{\rho_f g}$ now is the coefficient of *permeability,* with units $[\kappa] \equiv \frac{L^3 T}{M}$. In terms of ***intrinsic permeability*** k^d ($[k^d] \equiv L^2$) and dynamic viscosity of the fluid η ($[\eta] \equiv ML^{-1}T^{-1}$) permeability coefficient can be written as

$$\kappa = \frac{k^d}{\eta} \quad \text{.. (2.31)}$$

The relative velocity w_i can be related to *specific flux* q_i by

$$q_i = n\dot{w}_i$$

thus,

$$q_i = -\kappa\left[p_{,i} + \rho_f \ddot{u}_i + \frac{n\rho_f + \rho_a}{n}\ddot{w}_i - g_i^f\right] \quad \text{.. (2.32)}$$

which is the Generalized Darcy's Law for an isotropic medium. In the absence of dynamic effects and body forces, (2.32) reduces to the form of Darcy's flow equation as it is used in soil mechanics and well hydraulics. In what follows, the body force term g_i will be discarded in this work also.

(alternatively*, we can write per unit volume of bulk material:*

$$n\sigma^f_{ji,j} + ng^f_i = n\rho_f \ddot{U}_i + \rho_a \ddot{w}_i + b\dot{w}_i \quad \text{.............} \; (*)$$

where "b" is the drag coefficient and " ρ_a *" is added mass density both defined per unit volume of bulk material due to fluid resistance to moving particles. Note that, when the fluid is inviscid drag coefficient vanishes, however the fluid continues to develop against accelerating particles. Hence, added mass density* ρ_a *remains regardless of the viscosity of the internal fluid.*

If we insert in (*): $\sigma^f_{ij} = -p\delta_{ij}$ *and* $U_i = u_i + w_i$ *and divide by "n"*

$$-\frac{b}{n}\dot{w}_i = p_{,i} + \rho_f \ddot{u}_i + \rho_f \ddot{w}_i + \frac{\rho_a}{n}\ddot{w}_i - g^f_i$$

or

$$n\dot{w}_i = -\frac{n^2}{b}\left[p_{,i} + \rho_f \ddot{u}_i + \rho_f \ddot{w}_i + \frac{\rho_a}{n}\ddot{w}_i - g^f_i\right]$$

finally the Darcy's Law follows

$$q_i = -\kappa\left[p_{,i} + \rho_f \ddot{u}_i + \frac{n\rho_f + \rho_a}{n}\ddot{w}_i - g^f_i\right]$$

note the relation $\kappa = \frac{n^2}{b}$.)

Conservation of Angular Momentum (ANME):

Conservation of angular momentum yields the symmetry of the total and hence, effective stress tensors, i.e. $\tau_{ij} = \tau_{ji}$ and $\tau'_{ij} = \tau'_{ji}$ (Coussy, 1995).

2.3.2 Boundary and Initial Conditions

A well posed problem in mechanics is not fully defined unless a proper set of initial and boundary conditions are specified. In poro-elasticity, these conditions are defined in terms of the following pairs:

$$(u_i, t_i)$$
$$(p, q_n) \quad \text{...} \quad (2.33.a, b)$$

For each boundary condition, either one term from the above pairs is specified or a combination for each separate pair is defined over part of or the whole boundary. In other words, from (2.33.a)

$$\left.\begin{array}{ll} \text{either,} & u_i\big|_\Gamma = \bar{u}_i \ (\text{Dirichlet}) \\ \text{or,} & t_i\big|_\Gamma = \bar{t}_i \ (\text{Neuman}) \end{array}\right\} (\text{solid B.C.'s}) \quad \text{..........................} \quad (2.34.a, b)$$

and from (2.33.b),

$$\left.\begin{array}{ll} \text{either,} & p\big|_\Gamma = \bar{p} (\text{Dirichlet}) \\ \text{or,} & q_n\big|_\Gamma = \bar{q}_n (\text{Neuman}) \end{array}\right\} (\text{fluid B.C.'s}) \quad \text{..........................} \quad (2.35.a, b)$$

must be specified; alternatively combination type boundary conditions (Robin type) of

$$t_i\big|_\Gamma = k_{ij} u_j\big|_\Gamma \quad \text{...} \quad (2.36)$$

or/and

$$q_n\big|_\Gamma = \beta p\big|_\Gamma \quad \text{...} \quad (2.37)$$

may be specified. Boundary conditions may be different on different parts of the boundary and on some part (2.34) and (2.37) or (2.35) and (2.36) types of boundary conditions (mixed type) may be given.

2.3.3 Governing Equations in Fourier Transform Domain

The Fourier and inverse Fourier transform of a function $f(t)$ is defined by the following pair

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \quad \leftrightarrow \quad f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega$$

Taking Fourier transform of equations (2.1), (2.25), (2.28), (2.32) and assuming zero initial conditions for velocity and displacement one gets the frequency domain expressions,

CE's →

$$\tau_{ij}^F = 2\mu\varepsilon_{ij}^F + \lambda\varepsilon_{kk}^F\delta_{ij} - \alpha p^F\delta_{ij}$$
$$\theta^F = \alpha\varepsilon_{kk}^F + \frac{1}{Q}p^F \qquad \text{.................................. (2.38.a, b)}$$

SDR →

$$\varepsilon_{ij}^F = \frac{1}{2}\left[u_{i,j}^F + u_{j,i}^F\right] \qquad \text{... (2.39)}$$

SEM →

$$\tau_{ji,j}^F + f_i^F = -\omega^2\rho u_i^F - \omega^2 n\rho_f w_i^F \qquad \text{........................... (2.40)}$$

CONT →

$$i\omega\theta^F + q_{k,k}^F = a^F \qquad \text{.. (2.41)}$$

Darcy's Law →

$$q_i^F = -\kappa\left[p_{,i}^F - \omega^2\rho_f u_i^F - \omega^2\frac{\rho_a + n\rho_f}{n}w_i^F\right] \qquad \text{.................... (2.42)}$$

but, since $q_i = n\dot{w}_i$

$$q_i^F = i\omega n w_i^F \qquad \text{.. (2.43)}$$

hence, we get

$$w_i^F = \frac{n\kappa}{i\omega n^2 - \omega^2 \kappa(\rho_a + n\rho_f)} \left[-p_{,i}^F + \omega^2 \rho_f u_i^F \right] \qquad (2.44)$$

or designating

$$\beta = \frac{n^2 \kappa \rho_f \omega^2}{i\omega n^2 - \omega^2 \kappa(\rho_a + n\rho_f)} \qquad (2.45)$$

(2.44) becomes →

$$w_i^F = -\beta \frac{1}{n\rho_f \omega^2} \left[p_{,i}^F - \omega^2 \rho_f u_i^F \right] \qquad (2.46)$$

alternatively

$$q_i^F = i\omega \left[\frac{-\beta}{\omega^2 \rho_f} p_{,i}^F + \beta u_i^F \right] \qquad (2.47)$$

Inserting (2.39) into (2.38.a) one gets

$$\tau_{ji}^F = \mu\left[u_{i,j}^F + u_{j,i}^F\right] + \lambda u_{k,k}^F \delta_{ji} - \alpha p^F \delta_{ji} \qquad (2.48)$$

insert (2.48) and (2.46) into (2.40) we obtain the first three of GE's in terms of skeleton isplacements and pore fluid pressure in frequency domain,

$$\boxed{\mu u_{i,jj}^F + \mu u_{j,ij}^F + \lambda u_{j,ji}^F - (\alpha + \beta) p_{,i}^F + f_i^F = -\omega^2 (\rho + \beta\rho_f) u_i^F} \qquad (2.49)$$

Next, inserting (2.39) into (2.38.b)

$$\theta^F = \alpha u^F_{k,k} + \frac{1}{Q} p^F \qquad (2.50)$$

and (2.50) into (2.41)

$$i\omega\left(\alpha u^F_{k,k} + \frac{1}{Q} p^F\right) + q^F_{k,k} = a^F \qquad (2.51)$$

and recalling (2.43) hence

$$q^F_{k,k} = i\omega n w^F_{k,k} \qquad (2.52)$$

and from (2.46)

$$w^F_{k,k} = -\beta \frac{1}{n\rho_f \omega^2}\left[p^F_{,kk} - \omega^2 \rho_f u^F_{k,k}\right] \qquad (2.53)$$

inserting (2.53) into (2.52) and the result into (2.51) one gets as a result the fourth GE,

$$\boxed{-\beta \frac{1}{\omega^2 \rho_f} p^F_{,kk} + (\alpha+\beta) u^F_{k,k} + \frac{1}{Q} p^F = \frac{1}{i\omega} a^F} \qquad (2.54)$$

In operator form (2.49) and (2.54) are

$$\mu \nabla^2 \underline{\mathbf{u}}^F + \mu \underline{\nabla}.(\underline{\mathbf{u}}^F \underline{\nabla}) + \lambda \underline{\nabla}(\underline{\nabla}.\underline{\mathbf{u}}^F) - (\alpha+\beta)\underline{\nabla} p^F + \underline{\mathbf{f}}^F = -\omega^2(\rho + \beta\rho_f)\underline{\mathbf{u}}^F \qquad \text{(A)}$$

$$-\beta \frac{1}{\omega^2 \rho_f} \nabla^2 p^F + (\alpha+\beta)\underline{\nabla}.\underline{\mathbf{u}}^F + \frac{1}{Q} p^F = \frac{1}{i\omega} a^F \qquad \text{(B)}$$

In matrix form, equations (A) and (B) would read,

$$\begin{bmatrix} \left[\mu\nabla^2+\omega^2(\rho+\beta\rho_f)\right]\underline{\mathbf{I}}\bullet+(\lambda+\mu)\underline{\nabla}\,\underline{\nabla}\bullet & -(\alpha+\beta)\underline{\nabla} \\ -i\omega(\alpha+\beta)\underline{\nabla}\bullet & -\dfrac{\beta}{i\omega\rho_f}\nabla^2-\dfrac{i\omega}{Q} \end{bmatrix}\begin{bmatrix}\underline{u}^F \\ p^F\end{bmatrix}+\begin{bmatrix}\underline{f}^F \\ a^F\end{bmatrix}=\underline{0}$$

…………………......... (2.55)

or, alternatively

$$\begin{bmatrix} \left[\mu\,\partial_{kk}+\omega^2(\rho+\beta\rho_f)\right]\delta_{ij}+(\lambda+\mu)\partial_{ij} & -(\alpha+\beta)\partial_i \\ -i\omega(\alpha+\beta)\partial_j & -\dfrac{\beta}{i\omega\rho_f}\partial_{kk}-\dfrac{i\omega}{Q} \end{bmatrix}\begin{bmatrix}u_j^F \\ p^F\end{bmatrix}+\begin{bmatrix}f_i^F \\ a^F\end{bmatrix}=\underline{0}$$

……………………….. (2.56)

2.3.4 Wave Propagation in Infinite Poro-elastic Media

The preceding section dealt with governing equations of Biot's poro-elasticity theory. In this section we investigate the types of body waves that exist in poro-elastic media. Contrary to the classical elasticity theory, Biot's theory infers the existence three body waves; two dilatational waves and one shear wave (Biot 1956.b, c).

To simplify the discussion, we consider the case of infinite permeability; i.e. $\kappa\to\infty$ (or wave propagation in the absence of dissipation). In this case the governing equations of poro-elasticity (2.56) become,

$$\left(\alpha-\frac{n\rho_f}{\overline{m}}\right)\ddot{u}_{j,j}+\frac{1}{Q}\ddot{p}-\frac{n}{\overline{m}}p_{,jj}=0 \quad \text{……………………………………...... (2.57)}$$

$$\mu u_{i,jj}+(\lambda+\mu)u_{j,ij}=\left(\rho-\frac{n\rho_f^2}{\overline{m}}\right)\ddot{u}_i+\left(\alpha-\frac{n\rho_f}{\overline{m}}\right)p_i \quad \text{………..………….... (2.58)}$$

Now, we introduce the potentials for the displacement vector $\underline{u}$ as

$$u_i = \phi_{,i} + \varepsilon_{ijk}\psi_{k,j} \quad \text{...} (2.59)$$

Where, ϕ and $\underline{\psi}$ are the irrotational and solenoidal (divergence free, $\psi_{j,j} = 0$) parts, respectively. Inserting (2.59) into (2.57) and (2.58), we get

$$\left(\alpha - \frac{n\rho_f}{\overline{m}}\right)\ddot{\phi}_{,kk} + \frac{1}{Q}\ddot{p} - \frac{n}{\overline{m}}p_{,kk} = 0 \quad \text{.......................................} (2.60)$$

$$(\lambda + 2\mu)\phi_{,kk} - \left(\rho - \frac{n\rho_f^2}{\overline{m}}\right)\ddot{\phi} = \left(\alpha - \frac{n\rho_f}{\overline{m}}\right)p \quad \text{..............................} (2.61)$$

$$\mu\psi_{k,mm} - \left(\rho - \frac{n\rho_f^2}{\overline{m}}\right)\ddot{\psi}_k = 0 \quad \text{..} (2.62)$$

It is obvious that equations (2.60) and (2.61) govern the dilatational waves (p-waves), while (2.62) governs the rotational (shear) waves. From (2.62), we obtain the propagation velocity of the shear waves as

$$c_s = \sqrt{\frac{\mu}{\left(\rho - \frac{n\rho_f^2}{\overline{m}}\right)}} \quad \text{...} (2.63)$$

This shows that the shear waves in poro-elastic media propagate slightly faster than an idealized elastic body with same shear modulus and density (bulk density). Equation (2.63) is slightly different from that obtained by Biot (1956.b), this stems from the fact that Biot disregards equation of continuity in his investigation.

Equations (2.60) and (2.61) are two coupled wave equations for which we assume plane harmonic wave solutions of the form

$$\phi = Ae^{i(\underline{k}\cdot\underline{x} - \omega t)}, \; p = Ae^{i(\underline{k}\cdot\underline{x} - \omega t)} \quad \text{..} (2.64)$$

Substituting (2.64) into (2.60) and (2.61), we get the dispersion relation

$$c_p^4 - \underbrace{\left[\frac{\lambda+2\mu}{\rho - \frac{n\rho_f^2}{\overline{m}}} + \frac{nQ}{\overline{m}} + \frac{Q\left(\alpha - \frac{n\rho_f}{\overline{m}}\right)^2}{\rho - \frac{n\rho_f^2}{\overline{m}}}\right]}_{B} c_p^2 + \underbrace{\frac{nQ}{\overline{m}}\left(\frac{\lambda+2\mu}{\rho - \frac{n\rho_f^2}{\overline{m}}}\right)}_{C} = 0 \quad \text{.........} \quad (2.65)$$

where, $c_p = \frac{\omega}{|k|}$ is the propagation velocity of dilatational waves. It is obvious that (2.65) gives two wave velocities corresponding to the slow and fast compressional waves respectively. Investigating the coefficients B (>0) and C (≥0), we observe that (2.65) always have two positive roots for c_p^2 (C=$c_{p_1}^2 * c_{p_2}^2 > 0 \rightarrow c_{p_1}^2$ and $c_{p_2}^2$ have the same sign, B=$c_{p_1}^2 * c_{p_2}^2 > 0 \rightarrow c_{p_1}^2$ and $c_{p_2}^2$ both positive).

An interesting case occurs when we assume dry soil, i.e. Q=0 and ρ_a=0. In that case the dispersion relation (2.65) reduces to

$$c_p^4 - \frac{\lambda+2\mu}{\rho - n\rho_f} c_p^2 = 0 \quad \text{..} \quad (2.66)$$

this gives

$$c_{p_1}^2 = \frac{\lambda+2\mu}{\rho - n\rho_f} \; ; \; c_{p_2}^2 = 0 \quad \text{..} \quad (2.67)$$

as expected.

For the case of infinite permeability, the waves are non-dispersive, however for finite permeability (2.65) become frequency dependent and therefore the p-waves become dispersive. The slow wave is difficult to observe in experiments (Yew and Yogi 1976, Klimentos and McCann 1988); nevertheless, Plona (1980) and Gurevich et. al. (1999) verified the existence of this wave experimentally.

CHAPTER 3

STATE OF THE ART - BEM IN PORO-ELASTICITY

The basic idea of boundary element method consists of transforming the partial differential equation in to a boundary integral via the use of method of weighted residuals or theorem of reciprocal work together with the fundamental solutions (free space Green's functions) of the adjoint differential operator. Then a system of algebraic equations is formed by bringing the source point to the boundary and numerically integrating the resulting boundary integrals after proper discretization of the boundary. Since correct fundamental solutions are essential for successful deployment of BEM, the first applications of BEM in poro-elasticity had to wait until they are established. We can investigate the BEM studies in poro-elasticity in two eras. Quite naturally, in the early era the focus was on quasi-statics, and in the latter on dynamic applications.

BEM in Quasi-static Poro-elasticity:

For linear quasi-static poro-elasticity an indirect BEM in the Laplace domain was developed by Cleary (1977) without application, where he proved the reciprocity theorem for poro-elasticity and introduced the quasi-static fundamental solutions. Banerjee and Butterfield (1981) give a time marching scheme to solve Biot consolidation problems. Aramaki (1986) extends this scheme for soil profiles including thin layers of high permeability.

Predeleanu (1981) presents a time domain BIE formulation aiming at applications in Biot consolidation problems; he suggests that the fundamental solutions be found by using the analogy between thermo-elasticity and poro-elasticity. The paper adjourns with recommendations for possible future extensions.

Cheng and Ligget (1984 a, b) proposed both 2 and 3-D Laplace domain poro-elastic BEM and numerical results were presented for 2-D only. The procedure was reformulated later by Cheng and Detournay (1988) for new material parameters which were physically more meaningful.

In 1985, Nishimure (after Chen 1992) published a 2-D time domain BEM for soil consolidation. Dargush (1987) and Dargush and Banerjee (1989) developed time domain BE formulations for both 2-D and 3-D quasi-static poro-elasticity. These BEM formulations were extended to axisymmetric consolidation (Dargush and Banerjee, 1991); in this work full (both geometry and loading) axi-symmetry was assumed.

A 3-D BEM formulation via the reciprocal work theorem in Laplace-transform domain was given by Badmus et. al. (1993). Numerical results for one dimensional soil consolidation of a column and 3-D consolidation of a finite soil layer were compared to the existing exact analytical solutions, the comparisons display excellent agreement. The solution for the 3-D Mandel problem by the BEM model was also compared to an available FEM solution from the older literature. The quasi-static fundamental solutions were listed in the appendix.

Cavalcanti and Telles (2003) discus the application of time independent fundamental solutions to solve Biot's plane strain consolidation equations.

BEM in Dynamic Poro-elasticity:

In dynamic poro-elasticity, no progress was made until the first fundamental solutions were published (Bonnet 1987 and Boutin 1987); although the first attempt to derive fundamental solutions had been made as early as 1975, the work of Burridge and Vargas (1979) (based on an earlier work (Vargas, 1975)) did not contain all of the fundamental solutions. Norris (1985) derived point load fundamental solutions only. Bonnet (1987) showed that the six displacement components (3 solid and 3 fluid) in poro-elasticity were not independent and a formulation with solid displacements and the pore-pressure was sufficient; noting also the analogy in frequency domain to thermoelasticity, he pointed out that 3-D fundamental solutions were available in Kupradze, 1979. Bonnet then described the procedure to derive the 2-D fundamental solutions; however, he did not present all fundamental solutions.

The Laplace domain BEM formulation by Manolis and Beskos (1989) contained six independent variables (3 skeleton and 3 relative fluid displacements).

Based on the reciprocity theorem by Cleary, Wiebe and Antes (1991), presented a direct time domain BEM formulation for dynamic poro-elasticity using the fluid and solid displacements as the state variables (six variables); for the case of inviscid fluid, they worked out the first true time domain fundamental solution in closed form.

Cheng, et. al. (1991) derived the integral equations of poro-elasticity from the reciprocity relation in frequency domain. They obtained the 3-D as well as 2-D fundamental solutions by the frequency domain thermoelastic analogy. Although the fundamental solutions were given in differential form only, they are listed in explicit form in the dissertation by Badmus (1990) and Cheng and Detournay (1998). An analytical solution to the one dimensional poro-elastic column, harmonically excited either from the top by a vertical solid stress/fluid pressure or from bottom by specified displacement was provided and to verify the model numerical results from a two dimensional BEM implementation were compared to these solutions. A preliminary Coulomb failure analysis in plain strain was provided.

Dominguez (1991, 1992) published an application of BEM in 2-D dynamic poro-elasticity. He used Bonnet's fundamental solutions with corrections.

The first complete set of fundamental solutions in Laplace transform domain was published by Chen (1992). Chen derived time domain fundamental as well, but they are not in closed form and contain some integrals.

Approximate time domain fundamental solutions were derived by Kaynia (1992) and Gatmiri and Kamalian (2002).

A full axi-symmetric formulation for dynamic analysis of foundations on poro-elastic media by BEM was accomplished by Dargush and Chopra (1995).

Antes and Wiebe (1996), discuss both time and Laplace domain BEM formulations for wave propagation problems, using two formulations of poro-elasticity, namely the solid-fluid displacement (u-U formulation, 6 unknowns) and solid displacement-pore pressure (u-p formulation, 4 unknowns). They present the boundary integral equations for the 4 different formulations and discuss the forms of fundamental solutions in each case. The authors' claim that time domain fundamental solutions, in explicit form, can only be obtained in the idealized case of infinite permeability. The methods of deriving fundamental solutions are briefly mentioned and the 3-D fundamental solutions for displacements in Laplace domain are given in explicit form. Numerical results from a 3-D BEM solution for 1-D column problem (classical bar theory) are presented in graphics; the results compare well to analytical solution (Cheng et. al. 1991). The paper discusses the importance of finite permeability in connection to the superfluous resonances occurring in the case of infinite permeability. The authors compare poro-elastic waves to purely elastic waves and conclude that for short observation times a one phase elastic solution with bulk material properties would be sufficient. They remark also that the problem may best be formulated in transform domain using four variables (u-p). A nice review of formulation of integral equations and fundamental solutions is given by Cheng and Detournay (1998).

Chopra (2001) gives a review of theoretical background of poro-elasticity and the use of BEM in related problems.

A time domain poro-elastodynamic boundary element formulation using a special time stepping procedure, called Convolution Quadrature Method by Lubic, which requires the Laplace transform fundamental solutions only, has been published (Schanz, 2001a, b).

Fundamental solutions of poro-elasticity in the dynamic rage for both u_i-p (solid displacements-pore pressure) and u_i-U_i (solid displacements-fluid displacements) formulations are investigated by Pryl and Schanz (2004) for incompressible constituents.

More recently, a simplified BE model is developed for low relative velocity behaviour by Schanz and Struckmeier (2005).

CHAPTER 4

BIE FORMULATION FOR PORO-ELASTICITY

4.1 Boundary Integral Equations (BIE):

Boundary integral equations of poro-elastodynamics can be obtained from either the *weighted residual statement* or the *theorem of reciprocal work* (Schanz 2001.a, b). Here, both methods will be presented to show that they yield the same BIE's.

In what follows every variable is in Fourier transform space, but to simplify the notation, the superscript $()^F$ designating Fourier transform will be omitted later.

4.1.1 BIE by the Method of Weighted Residuals

The equations (2.49) and (2.54) are multiplied by two different weight functions and integrated over the problem domain. Since (2.49) is a vector equation it must be multiplied by a vector function and (2.54) with a scalar function

The technique essentially consists in taking the inner product of the system of PDE's (2.55) with a vector of weighting functions $\left[U_i \quad P\right]^T$, i.e., let $[B]$ represent the partial differential operator pertaining to equations (2.55) of the chapter 2, then one can write these equations in matrix form as,

$$[B]\begin{bmatrix} u_i \\ p \end{bmatrix} + \{F\} = 0 \quad \text{...} \quad (4.1)$$

where,

$$\{F\} = \begin{bmatrix} f_i \\ a \end{bmatrix} \quad \text{..} \quad (4.2)$$

Then the weighted residual statement becomes,

$$\int_{\Omega} \begin{bmatrix} U_i \\ P \end{bmatrix}^T \left([\mathbf{B}] \begin{bmatrix} u_i \\ p \end{bmatrix} + \{F\} \right) d\Omega = 0 \quad \text{..} \quad (4.3)$$

Then,

$$\int_{\Omega} (\text{eqn. } 2.49) * U_i \, d\Omega = 0 \;\rightarrow \text{gives}$$

$$\int_{\Omega} (\underbrace{\mu u^F_{i,jj}}_{1} + \underbrace{\mu u^F_{j,ij}}_{2} + \underbrace{\lambda u^F_{j,ji}}_{3} - \underbrace{(\alpha + \beta) p^F_{,i}}_{4} + \underbrace{\omega^2 (\rho + \beta \rho_f) u^F_i}_{5} + \underbrace{f_i}_{6}) * U_i \, d\Omega = 0 \quad \text{.............} \quad (4.4)$$

Now, each integral will be evaluated by parts separately,

$$(1) \rightarrow \int_{\Omega} \mu u_{i,jj} U_i \, d\Omega = \int_{\Omega} \mu \left(u_{i,j} U_i \right)_{,j} d\Omega - \int_{\Omega} \mu u_{i,j} U_{i,j} d\Omega$$

the first integral on the right can now be transformed to surface integral by applying GIT. Hence,

$$\int_{\Omega} \mu u_{i,jj} U_i \, d\Omega = \int_{\Gamma} \mu \left(u_{i,j} U_i \right) n_j d\Gamma - \int_{\Omega} \mu u_{i,j} U_{i,j} d\Omega$$

applying integration by parts and GIT to the second integral,

$$\int_{\Omega} \mu u_{i,jj} U_i \, d\Omega = \int_{\Gamma} \mu\left(u_{i,j} U_i\right) n_j d\Gamma - \int_{\Gamma} \mu\left(u_i U_{i,j}\right) n_j d\Gamma + \int_{\Omega} \mu u_i U_{i,jj} d\Omega$$

similarly,

(2) → with GIT twice

$$\int_{\Omega} \mu u_{j,ij} U_i \, d\Omega = \int_{\Gamma} \mu n_j u_{j,i} U_i \, d\Gamma - \int_{\Gamma} \mu n_i u_j U_{i,j} d\Gamma + \int_{\Omega} \mu u_j U_{i,ji} d\Omega$$

(3) → with GIT twice

$$\int_{\Omega} \lambda u_{j,ji} U_i \, d\Omega = \int_{\Gamma} \lambda n_i u_{j,j} U_i \, d\Gamma - \int_{\Gamma} \lambda n_j u_j U_{i,i} d\Gamma + \int_{\Omega} \lambda u_j U_{i,ij} d\Omega$$

(4) → $\int_{\Omega} -(\alpha+\beta) p_{,i} U_i \, d\Omega = \int_{\Gamma} -(\alpha+\beta) n_i p U_i \, d\Gamma - \int_{\Omega} -(\alpha+\beta) p U_{i,i} d\Omega$

(5) → $\int_{\Omega} -\omega^2 (\rho + \beta\rho_f) u_i U_i \, d\Omega$

(6) → $\int_{\Omega} f_i U_i \, d\Omega$

Finally, (1) + (2) + (3) + (4) + (5) + (6) → give

$$\int_{\Gamma} \left[\mu\left(u_{i,j} U_i - u_i U_{i,j}\right) n_j + \mu n_j \left(u_{j,i} U_i - U_{j,i} u_i\right) + \lambda n_i \left(u_{j,j} U_i - u_i U_{j,j}\right) - (\alpha+\beta) n_i p U_i\right] d\Gamma$$
$$+ \int_{\Omega} u_i \left[\mu U_{i,jj} + \mu U_{j,ij} + \lambda U_{j,ji} + \omega^2 (\rho + \beta\rho_f) U_i\right] d\Omega + \int_{\Omega} (\alpha+\beta) p U_{i,i} d\Omega + \int_{\Omega} f_i U_i \, d\Omega = 0$$

…….......... (4.5)

collecting the terms,

$$\int_{\Gamma}\left[\mu\left(u_{i,j}+u_{j,i}\right)U_i n_j-\mu\left(U_{i,j}+U_{j,i}\right)u_i n_j+\lambda\underbrace{n_i}_{n_j\delta_{ji}}\left(\underbrace{u_{j,j}}_{u_{k,k}}U_i-u_i\underbrace{U_{j,j}}_{U^S_{k,k}}\right)-(\alpha+\beta)\underbrace{n_i}_{n_j\delta_{ji}}pU_i\right]d\Gamma$$
$$+\int_{\Omega}u_i\left[\mu U_{i,jj}+\mu U_{j,ij}+\lambda U_{j,ji}+\omega^2(\rho+\beta\rho_f)U_i\right]d\Omega+\int_{\Omega}(\alpha+\beta)pU_{i,i}d\Omega+\int_{\Omega}f_iU_i\,d\Omega=0$$

………..... (4.6)

or,

$$\int_{\Gamma}\left[\left(\mu\left(u_{i,j}+u_{j,i}\right)+\lambda u_{k,k}\delta_{ji}\right)U_i n_j-\left(\mu\left(U_{i,j}+U_{j,i}\right)+\lambda U_{k,k}\delta_{ji}\right)u_i n_j-(\alpha+\beta)n_j\delta_{ji}pU_i\right]d\Gamma$$
$$+\int_{\Omega}u_i\left[\mu U_{i,jj}+\mu U_{j,ij}+\lambda U_{j,ji}+\omega^2(\rho+\beta\rho_f)U_i\right]d\Omega+\int_{\Omega}(\alpha+\beta)pU_{i,i}d\Omega+\int_{\Omega}f_iU_i\,d\Omega=0$$

………... (4.7)

Next for equation (2.54),

$$\int_{\Omega}(\text{eqn. }2.54)*P\,d\Omega\rightarrow\text{gives,}$$

$$\int_{\Omega}\left[\underbrace{\frac{-\beta}{\omega^2\rho_f}p_{,ii}}_{1}+\underbrace{(\alpha+\beta)u_{i,i}}_{2}+\underbrace{\frac{1}{Q}p}_{3}\right]P\,d\Omega=\underbrace{\int_{\Omega}\frac{1}{i\omega}a\;P\;d\Omega}_{4}\quad\text{……………….......................... (4.8)}$$

Again employing GIT

$$(1)\rightarrow\int_{\Omega}\frac{-\beta}{\omega^2\rho_f}p_{,ii}P\;d\Omega=\int_{\Gamma}\frac{-\beta}{\omega^2\rho_f}n_ip_{,i}P\;d\Gamma-\int_{\Gamma}\frac{-\beta}{\omega^2\rho_f}n_ipP_{,i}d\Gamma+\int_{\Omega}\frac{-\beta}{\omega^2\rho_f}pP_{,ii}d\Gamma$$

$$(2) \rightarrow \int_{\Omega} (\alpha+\beta) u_{i,i} P \, d\Omega = \int_{\Gamma} (\alpha+\beta) n_i u_i P \, d\Gamma - \int_{\Omega} (\alpha+\beta) u_i P_{,i} \, d\Omega$$

$$(3) \rightarrow \int_{\Omega} \frac{1}{Q} pP \, d\Omega$$

$$(4) \rightarrow \int_{\Omega} \frac{1}{i\omega} a^F P \, d\Omega$$

Finally, (1)+ (2) + (3) = (4) → gives,

$$\begin{aligned} &\int_{\Gamma} \frac{-\beta}{\omega^2 \rho_f} n_i \left[p_{,i} P \; - pP_{,i} \right] + (\alpha+\beta) n_i u_i P \, d\Gamma \\ &+ \int_{\Omega} \left[\frac{-\beta}{\omega^2 \rho_f} pP_{,ii} - (\alpha+\beta) u_i P_{,i} + \frac{1}{Q} pP \right] d\Omega = \int_{\Omega} \frac{1}{i\omega} a \; P \, d\Omega \end{aligned} \quad \text{.............................} \; (4.9)$$

Now, adding (4.7) and (4.9)

$$\begin{aligned} &\int_{\Gamma} \left[\left(\mu \left(u_{i,j} + u_{j,i} \right) + \lambda u_{k,k} \delta_{ji} \right) U_i n_j - \left(\mu \left(U_{i,j} + U_{j,i} \right) + \lambda U_{k,k} \delta_{ji} \right) u_i n_j - (\alpha+\beta) n_j \delta_{ji} p U_i \right] d\Gamma \\ &+ \int_{\Omega} u_i \left[\mu U_{i,jj} + \mu U_{j,ij} + \lambda U_{j,ji} + \omega^2 (\rho + \beta \rho_f) U_i \right] d\Omega + \int_{\Omega} (\alpha+\beta) p U_{i,i} d\Omega + \int_{\Omega} f_i U_i d\Omega \\ &+ \int_{\Gamma} \left\{ \frac{-\beta}{\omega^2 \rho_f} n_i \left[p_{,i} P \; - pP_{,i} \right] + (\alpha+\beta) \underbrace{n_i u_i}_{n_j \delta_{ji} u_i} P \right\} d\Gamma \\ &+ \int_{\Omega} \left[\frac{-\beta}{\omega^2 \rho_f} pP_{,ii} - (\alpha+\beta) u_i P_{,i} + \frac{1}{Q} pP \right] d\Omega = \int_{\Omega} \frac{1}{i\omega} a \; P \, d\Omega \end{aligned}$$

.... (4.10)

collecting terms,

$$\int_\Gamma \left[\left(\mu\left(u_{i,j}+u_{j,i}\right)+\lambda u_{k,k}\delta_{ji}-\alpha p\delta_{ji}\right)n_j U_i-\left(\mu\left(U_{i,j}+U_{j,i}\right)+\lambda U_{k,k}\delta_{ji}-\alpha P\ \delta_{ji}\right)n_j u_i\right]d\Gamma+$$

$$+\int_\Gamma \left\{\left[\frac{-\beta}{\omega^2\rho_f}p_{,i}+\beta u_i\right]n_i P\ -\left[\frac{-\beta}{\omega^2\rho_f}P_{,i}+\beta U_i\right]n_i p\right\}d\Gamma+$$

$$+\int_\Omega u_i\left[\mu U_{i,jj}+\mu U_{j,ij}+\lambda U_{j,ji}-\left(\alpha+\beta\right)P_{,i}+\omega^2(\rho+\beta\rho_f)U_i\right]d\Omega$$

$$+\int_\Omega p\left[\frac{-\beta}{\omega^2\rho_f}P_{,ii}+(\alpha+\beta)U_{i,i}+\frac{1}{Q}P\right]d\Omega=\int_\Omega\frac{1}{i\omega}a\ \ P\ d\Omega-\int_\Omega f_i U_i\,d\Omega$$

………………………... (4.11)

Equation (4.11) is the basic integral equation (BIE) of poro-elasticity. Recalling that stress tensor is related to displacements by equation (2.48)

$$\tau_{ji}=\left[\mu\left(u_{i,j}+u_{j,i}\right)+\lambda u_{k,k}\delta_{ji}-\alpha p\delta_{ji}\right]$$

traction (stress) vector to stress tensor by Cauchy's stress formula (2.22)

$$t_i=\tau_{ji}n_j$$

components of the flux vector to pressure and skeleton displacements by (2.47)

$$\frac{q_i}{i\omega}=\left[\frac{-\beta}{\omega^2\rho_f}p_{,i}+\beta u_i\right]$$

and normal component of flux vector is

$$q_n=q_i n_i$$

one can then identify the following in eqn. (4.11),

$$\left[\mu\left(u_{i,j}+u_{j,i}\right)+\lambda u_{k,k}\delta_{ji}-\alpha p\delta_{ji}\right]n_j\rightarrow t_i$$ ……………………………………. (4.12.a)

$$\left[\mu\left(U_{i,j}+U_{j,i}\right)+\lambda U_{k,k}\delta_{ji}-\alpha P\ \delta_{ji}\right]n_j \rightarrow t_i^* \quad \text{………………………… (4.12.b)}$$

$$\left[\frac{-\beta}{\omega^2\rho_f}p_{,i}+\beta u_i\right]n_i \rightarrow \frac{q_n}{i\omega} \quad \text{………………………………………… (4.13.a)}$$

$$\left[\frac{-\beta}{\omega^2\rho_f}P_{,i}+\beta U_i\right]n_i \rightarrow \frac{q_n^*}{i\omega} \quad \text{……………………………………….. (4.13.b)}$$

then BIE abbreviates to the following form

$$\int_\Gamma\left[\underbrace{\left(\mu\left(u_{i,j}+u_{j,i}\right)+\lambda u_{k,k}\delta_{ji}-\alpha p\delta_{ji}\right)n_j}_{t_i}U_i-\underbrace{\left(\mu\left(U_{i,j}+U_{j,i}\right)+\lambda U_{k,k}\delta_{ji}-\alpha P\ \delta_{ji}\right)n_j}_{t_i^*}u_i\right]d\Gamma+$$

$$+\int_\Gamma\left\{\underbrace{\left[\frac{-\beta}{\omega^2\rho_f}p_{,i}+\beta u_i\right]n_i}_{\frac{q_n}{i\omega}}P-\underbrace{\left[\frac{-\beta}{\omega^2\rho_f}P_{,i}+\beta U_i\right]n_i}_{\frac{q_n^*}{i\omega}}p\right\}d\Gamma+$$

$$+\int_\Omega u_i\left[\mu U_{i,jj}+\mu U_{j,ij}+\lambda U_{j,ji}-(\alpha+\beta)P_{,i}+\omega^2(\rho+\beta\rho_f)U_i\right]d\Omega$$

$$+\int_\Omega p\left[\frac{-\beta}{\omega^2\rho_f}P_{,ii}+(\alpha+\beta)U_{i,i}+\frac{1}{Q}P\right]d\Omega=\int_\Omega\frac{1}{i\omega}a^F P\ d\Omega-\int_\Omega f_i U_i\,d\Omega$$

……………….. (4.14)

$$\int_\Gamma\left[t_iU_i-t_i^*u_i\right]d\Gamma+\frac{1}{i\omega}\int_\Gamma\left\{q_nP-q_n^*p\right\}d\Gamma+$$

$$+\int_\Omega u_i\left[\mu U_{i,jj}+\mu U_{j,ij}+\lambda U_{j,ji}-(\alpha+\beta)P_{,i}+\omega^2(\rho+\beta\rho_f)U_i\right]d\Omega \quad \text{……….. (4.15)}$$

$$+\int_\Omega p\left[\frac{-\beta}{\omega^2\rho_f}P_{,ii}+(\alpha+\beta)U_{i,i}+\frac{1}{Q}P\right]d\Omega=\int_\Omega\frac{1}{i\omega}a^F P\ d\Omega-\int_\Omega f_i U_i\,d\Omega$$

If there are no body forces ($f_i^F = 0$) and no internal fluid sources ($a^F = 0$), eqn. (4.15) further simplifies to,

$$\int_\Gamma [t_i U_i - t_i^* u_i] d\Gamma + \frac{1}{i\omega}\int_\Gamma \{q_n P - q_n^* p\} d\Gamma +$$
$$+ \int_\Omega u_i \underbrace{[\mu U_{i,jj} + \mu U_{j,ij} + \lambda U_{j,ji} - (\alpha+\beta) P_{,i} + \omega^2(\rho + \beta\rho_f) U_i]}_{E1} d\Omega \quad \ldots\ldots\ldots\ldots\ldots.. (4.16)$$
$$+ \int_\Omega p \underbrace{\left[\frac{-\beta}{\omega^2 \rho_f} P_{,ii} + (\alpha+\beta) U_{i,i} + \frac{1}{Q} P\right]}_{E2} d\Omega = 0$$

It is seen that the terms designated by E1 and E2 in equation (4.16) are the same form of PDE's as equations (2.49) and (2.54), the governing equations of poro-elasticity.

From (4.16), two boundary integral equations result in the following way

(1) Choose U_i and P such that they satisfy

$$\underbrace{\mu U_{i,jj}^{1\ S} + \mu U_{j,ij}^{1\ S} + \lambda U_{j,ji}^{1\ S} - (\alpha+\beta) P_{,i}^{1S} + \omega^2(\rho + \beta\rho_f) U_i^{1S}}_{E1} = -\delta_{il}\Delta(A,P)$$

and $\ldots$ (4.17)

$$\underbrace{\frac{-\beta}{\omega^2 \rho_f} P_{,ii}^{1\ S} + (\alpha+\beta) U_{i,i}^{1\ S} + \frac{1}{Q} P^{1S} = 0}_{E2}$$

then (4.16) gives

$$\boxed{u_l(A) = \int_\Gamma [t_i U_i^{1S} - t_i^{1^*} u_i] d\Gamma + \frac{1}{i\omega}\int_\Gamma [q_n P^{1S} - q_n^{1\ *} p] d\Gamma} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.18)$$

if it is noted, (4.17) are the same equations as (2.49) and (2.54), the GE's of poro-elasticity, with body force per unit volume $f_i^F = \delta_{il} \Delta(A,P)$ and fluid source $a_i^F = 0$. Then, (4.17) can be interpreted as another poro-elastic medium under the influence of a unit load at point "A" in "l" direction and no fluid source. Schematically, (4.17) defines,

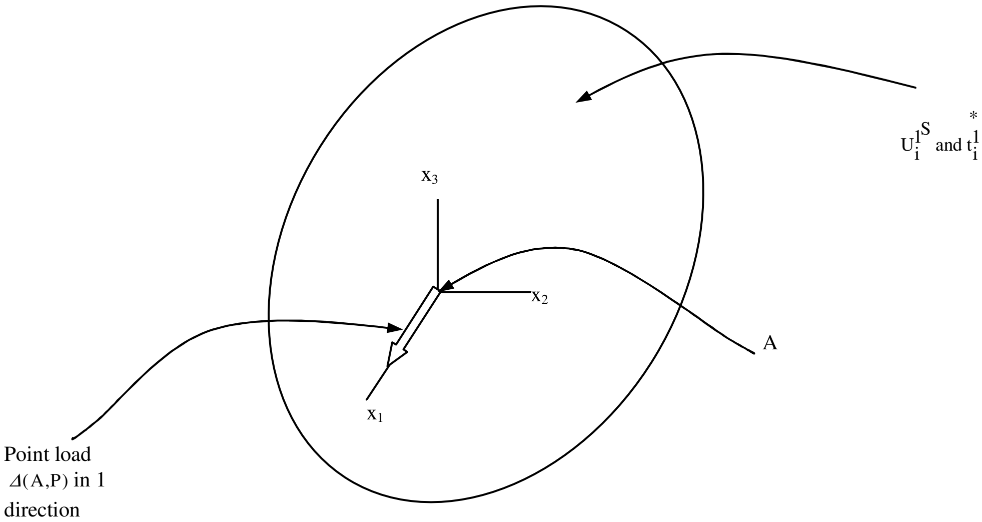

Figure 4.1 Interpretation of fundamental solutions in weighted residual method as suddenly applied point loads in poro-elastic full space.

U_i^{lS}'s and P^{lS}'s are found by applying the unit load in 1, 2, 3 directions in succession.

(2) Choose U_i and P such that they satisfy

$$\underbrace{\mu U^L_{i,jj} + \mu U^L_{j,ij} + \lambda U^L_{j,ji} - (\alpha+\beta) P^L_{,i} + \omega^2 (\rho + \beta\rho_f) U^L_i}_{E1} = 0$$

and ……….…….. (4.19)

$$\underbrace{\frac{-\beta}{\omega^2 \rho_f} P^L_{,ii} + (\alpha+\beta) U^L_{i,i} + \frac{1}{Q} P^L}_{E2} = \frac{\Delta(A,P)}{i\omega}$$

then (4.16) gives,

$$\boxed{\frac{1}{i\omega}p(A) = \int_{\Gamma}\left[t_i^{*L}u_i - t_iU_i^L\right]d\Gamma + \frac{1}{i\omega}\int_{\Gamma}\left\{q_n^{*L}p - q_nP^L\right\}d\Gamma} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(4.20)}$$

again (4.19) are the same equations as (2.49), (2.54) with zero body forces and unit impulsive fluid source at point "A".

Solutions to (4.17) and (4.19) give the ***fundamental solutions*** of 3-D poro-elasticity in Fourier Transform Space (FTS). It is to be noted that *fundamental solutions* correspond to *unit **Dira**c sources in **time and space***, for both body force and fluid source terms in governing equations (2.55). That is in time domain, fundamental solutions correspond to solutions of the governing equations for

$$f_i = \begin{cases} \delta(t)\Delta(A,P)\vec{e}_i \\ \text{or} \\ 0 \end{cases}$$

and

$$a = \begin{cases} 0 \\ \text{or} \\ \delta(t)\Delta(A,P) \end{cases}$$

where, $\vec{e}_i$ designates a unit vector in the direction of i-th coordinate axis.

4.1.2 BIE by Reciprocal Work Theorem for Poro-elasticity

Define two poroelastic systems :

1. Actual System (ACS), actual poro-elastic medium to be analysed.
2. Auxillary System (AXS), fictitious infinite medium of same PE material

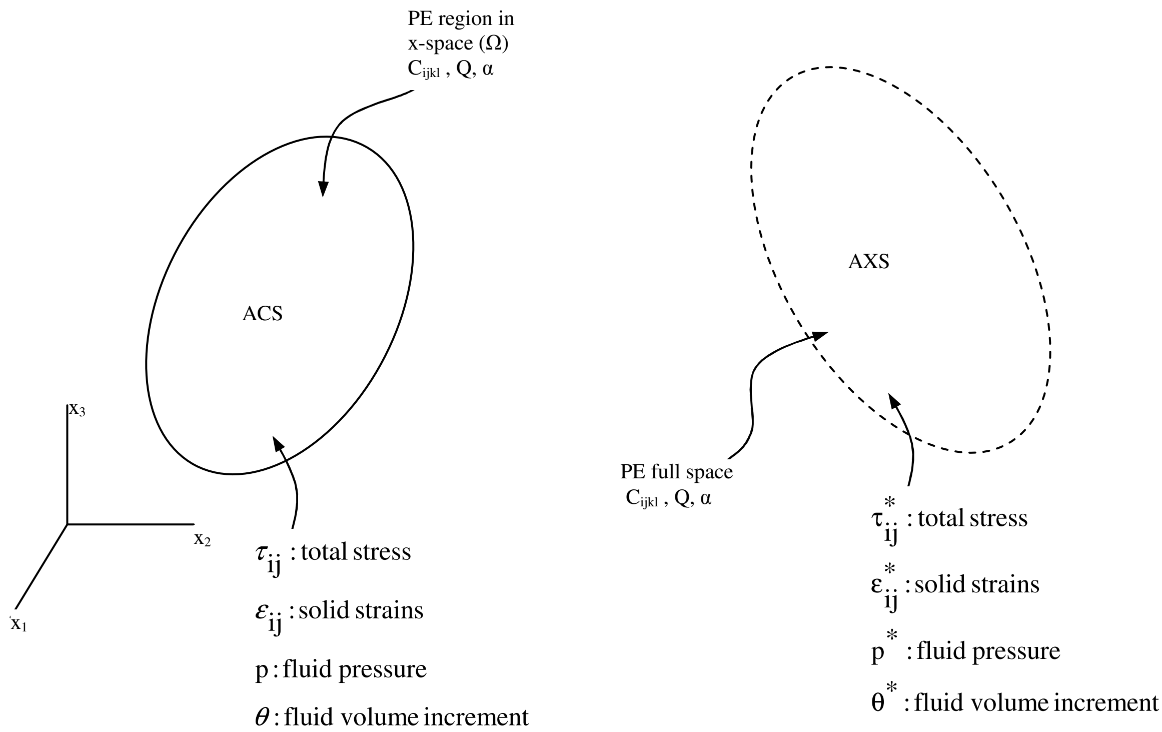

Figure 4.2 Definition of the actual (ACS) and auxiliary (AXS) poro-elastic systems in reciprocal work theorem.

The *reciprocity theorem* states (in FTS) that,

$$\tau_{ij}\varepsilon^*_{ij} + p\theta^* = \tau^*_{ij}\varepsilon_{ij} + p^*\theta \quad \text{...} \quad (4.21)$$

where $(\ )^*$ designates AXS variables.

Proof of the reciprocity (in FTS) (Schanz 2001 a, b):

Recall → Constitutive equations of poro-elasticity in FTS (eqn. 2.38),

$$\left.\begin{aligned} \tau_{ij} &= \underbrace{2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij}}_{C_{ijkl}\varepsilon_{kl}} - \alpha p\delta_{ij} \\ \theta &= \alpha\varepsilon_{kk} + \frac{1}{Q}p \end{aligned}\right\} \quad \text{.............} \quad \text{for ACS}$$

and

$$\left.\begin{aligned}\tau^*_{ij} &= \underbrace{2\mu\varepsilon^*_{ij} + \lambda\varepsilon^*_{kk}\delta_{ij}}_{C_{ijkl}\varepsilon^*_{kl}} - \alpha p^*\delta_{ij}\\ \theta^* &= \alpha\varepsilon^*_{kk} + \frac{1}{Q}p^*\end{aligned}\right\} \quad \text{.................. for AXS}$$

Thus,

$$\tau_{ij}\varepsilon^*_{ij} + p\theta^* = \left[C_{ijkl}\varepsilon_{kl} - \alpha p\delta_{ij}\right]\varepsilon^*_{ij} + p\left[\alpha\varepsilon^*_{kk} + \frac{1}{Q}p^*\right]$$

$$= \underbrace{\underbrace{C_{ijkl}}_{C_{klij}}\varepsilon^*_{ij}}_{\tau^*_{kl}+\alpha p^*\delta_{kl}}\varepsilon_{kl} - \alpha p\underbrace{\varepsilon^*_{ij}\delta_{ij}}_{\varepsilon^*_{ij}=\varepsilon^*_{kk}} + \alpha p\varepsilon^*_{kk} + \underbrace{\frac{1}{Q}p}_{\theta-\alpha\varepsilon_{kk}}p^*$$

$$= \tau^*_{kl}\varepsilon_{kl} + \alpha p\underbrace{\varepsilon_{kl}\delta_{kl}}_{\varepsilon_{kk}} + \theta p^* - \alpha p^*\varepsilon_{kk}$$

$$= \tau^*_{kl}\varepsilon_{kl} + \theta p^*$$

which proves the reciprocity theorem for PE in FTS. Now, to obtain the boundary integral equation integrate (4.21) over the domain of ACS ;

$$\int_\Omega\left[\underbrace{\tau_{ij}\varepsilon^*_{ij}}_{(1)} + \underbrace{p\theta^*}_{(2)}\right]d\Omega = \int_\Omega\left[\underbrace{\tau^*_{ij}\varepsilon_{ij}}_{(3)} + \underbrace{p^*\theta}_{(4)}\right]d\Omega \quad \text{....................................} \quad (4.22)$$

(1) →

$$\int_\Omega \tau_{ij}\varepsilon^*_{ij}d\Omega = \int_\Omega \tau_{ij}\left[\frac{1}{2}\left(u^*_{i,j} + u^*_{j,i}\right)\right]d\Omega = \int_\Omega \frac{1}{2}\left[\tau_{ij}u^*_{i,j} + \underbrace{\tau_{ij}u^*_{j,i}}_{\tau_{ji}u^*_{i,j}}\right]d\Omega$$

$$= \int_\Omega \frac{1}{2}\left[\underbrace{\tau_{ij}u^*_{i,j} + \underbrace{\tau_{ji}}_{=\tau_{ij}}u^*_{i,j}}_{2\tau_{ij}u^*_{i,j}}\right]d\Omega$$

$$= \int_\Omega \tau_{ij}u^*_{i,j}d\Omega$$

$$= \int_\Omega\left[\left(\tau_{ij}u^*_i\right)_{,j} - \tau_{ij,j}u^*_i\right]d\Omega$$

$$= \int_\Omega\left(\tau_{ij}u^*_i\right)_{,j}d\Omega - \int_\Omega \tau_{ij,j}u^*_i d\Omega$$

Now, applying GIT to the first integral on the right and employing SEM,

$$\tau_{ij,j} = -f_i - \omega^2 \rho u_i - n\omega^2 \rho_f w_i \quad (\text{Eqn. } 2.40)$$

then,

$$(1) \xrightarrow{\text{gives}} \int_\Gamma \underbrace{n_j \tau_{ij}}_{t_i} u_i^* d\Gamma + \int_\Omega \left[f_i + \omega^2 \rho u_i + n\omega^2 \rho_f w_i \right] u_i^* d\Omega$$

$$= \int_\Gamma t_i u_i^* d\Gamma + \int_\Omega \left[f_i + \omega^2 \rho u_i + n\omega^2 \rho_f w_i \right] u_i^* d\Omega$$

$$(2) \xrightarrow{\text{gives}} \int_\Omega p \underbrace{\theta^*}_{\frac{(-q_{i,i}^* + a^*)}{i\omega}} d\Omega = \int_\Omega -\frac{1}{i\omega} q_{i,i}^* p d\Omega + \int_\Omega \frac{1}{i\omega} a^* p d\Omega$$

apply GIT to the first integral,

$$= \int_\Gamma -\frac{1}{i\omega} \underbrace{n_i q_i^*}_{q_n^*\text{:normal component of flux vector}} p d\Gamma + \int_\Omega \frac{1}{i\omega} q_i^* p_{,i} d\Omega + \int_\Omega \frac{1}{i\omega} a^* p d\Omega$$

$$= \int_\Gamma -\frac{1}{i\omega} q_n^* p d\Gamma + \int_\Omega \frac{1}{i\omega} q_i^* p_{,i} d\Omega + \int_\Omega \frac{1}{i\omega} a^* p d\Omega$$

$$(3) \xrightarrow{\text{gives}} \int_\Omega \tau_{ij}^* \varepsilon_{ij} d\Omega = \int_\Omega \tau_{ij}^* u_{i,j} d\Omega = \int_\Gamma \underbrace{n_j \tau_{ij}^*}_{t_i^*} u_i d\Gamma - \int_\Omega \underbrace{\tau_{ij,j}^*}_{-f_i^* - \omega^2 \rho u_i^* - \omega^2 n \rho_f w_i^*} u_i d\Omega$$

$$= \int_\Gamma t_i^* u_i d\Gamma - \int_\Omega \left[-f_i^* - \omega^2 \rho u_i^* - \omega^2 n \rho_f w_i^* \right] u_i d\Omega$$

$$(4) \xrightarrow{\text{gives}} \int_\Omega p^* \theta d\Omega = \int_\Omega p^* \frac{1}{i\omega} \left[-q_{i,i} + a \right] d\Omega$$

$$= \int_\Omega -\frac{1}{i\omega} p^* q_{i,i} d\Omega + \int_\Omega \frac{1}{i\omega} p^* a d\Omega$$

$$= \int_\Gamma -\frac{1}{i\omega} p^* \underbrace{n_i q_i}_{q_n} d\Gamma + \int_\Omega \frac{1}{i\omega} p_{,i}^* q_i d\Omega + \int_\Omega \frac{1}{i\omega} p^* a d\Omega$$

$$= \int_\Gamma -\frac{1}{i\omega} p^* q_n d\Gamma + \int_\Omega \frac{1}{i\omega} p_{,i}^* q_i d\Omega + \int_\Omega \frac{1}{i\omega} p^* a d\Omega$$

$(1) + (2) = (3) + (4) \rightarrow$ gives

$$\int_\Gamma t_i u_i^* d\Gamma + \int_\Omega \left[f_i + \omega^2 \rho u_i + n\omega^2 \rho_f w_i\right] u_i^* d\Omega + \int_\Gamma -\frac{1}{i\omega} q_n^* p d\Gamma + \int_\Omega \frac{1}{i\omega} q_i^* p_{,i} d\Omega + \int_\Omega \frac{1}{i\omega} a^* p d\Omega$$
$$= \int_\Gamma t_i^* u_i d\Gamma + \int_\Omega \left[f_i^* + \omega^2 \rho u_i^* + \omega^2 n \rho_f w_i^*\right] u_i d\Omega + \int_\Gamma -\frac{1}{i\omega} p^* q_n d\Gamma + \int_\Omega \frac{1}{i\omega} p_{,i}^* q_i d\Omega + \int_\Omega \frac{1}{i\omega} p^* a d\Omega$$

………... (4.23)

making the following subtitutions in the above,

From Darcy's Law (Eqn. 2.42):

$$p_{,i} = -\frac{q_i}{\kappa} + \omega^2 \rho_f u_i + \omega^2 \frac{\rho_a + n\rho_f}{n} w_i$$

and

$$q_i = i\omega n w_i$$

one obtains

$$\int_\Gamma t_i u_i^* d\Gamma + \int_\Omega \left[f_i + \omega^2 \rho u_i + n\omega^2 \rho_f w_i\right] u_i^* d\Omega +$$
$$\int_\Gamma -\frac{1}{i\omega} q_n^* p d\Gamma + \int_\Omega \frac{1}{i\omega} \Bigg(\underbrace{i\omega n w_i^*}_{q_i^*} \Bigg) \Bigg[\underbrace{-\frac{q_i}{\kappa} + \omega^2 \rho_f u_i + \omega^2 \frac{\rho_a + n\rho_f}{n} w_i}_{p_{,i}} \Bigg] d\Omega + \int_\Omega \frac{1}{i\omega} a^* p d\Omega$$
$$= \int_\Gamma t_i^* u_i d\Gamma + \int_\Omega \left[f_i^* + \omega^2 \rho u_i^* + \omega^2 n \rho_f w_i^*\right] u_i d\Omega + \int_\Gamma -\frac{1}{i\omega} p^* q_n d\Gamma +$$
$$\int_\Omega \frac{1}{i\omega} \Bigg[\underbrace{-\frac{q_i^*}{\kappa} + \omega^2 \rho_f u_i^* + \omega^2 \frac{\rho_a + n\rho_f}{n} w_i^*}_{p_{,i}^*} \Bigg] \Bigg(\underbrace{i\omega n w_i}_{q_i} \Bigg) d\Omega + \int_\Omega \frac{1}{i\omega} p^* a d\Omega$$

several terms cancel and the following integral equation will result,

$$\int_{\Gamma} t_i u_i^* d\Gamma + \int_{\Gamma} \frac{1}{i\omega} q_n p^* d\Gamma + \int_{\Omega} f_i u_i^* d\Omega + \int_{\Omega} \frac{1}{i\omega} p a^* d\Omega$$
$$= \int_{\Gamma} t_i^* u_i d\Gamma + \int_{\Gamma} \frac{1}{i\omega} q_n^* p d\Gamma + \int_{\Omega} f_i^* u_i d\Omega + \int_{\Omega} \frac{1}{i\omega} p^* a d\Omega \quad \text{.............} (4.24)$$

In the absence of body forces ($f_i = 0$) and fluid source (a = 0) in ACS, (4.24) becomes

$$\int_{\Gamma} t_i u_i^* d\Gamma + \int_{\Gamma} \frac{1}{i\omega} q_n p^* d\Gamma + \int_{\Omega} \frac{1}{i\omega} p a^* d\Omega = \int_{\Gamma} t_i^* u_i d\Gamma + \int_{\Gamma} \frac{1}{i\omega} q_n^* p d\Gamma + \int_{\Omega} f_i^* u_i d\Omega \quad \text{......} \quad (4.25)$$

From (4.25), two integral equations will result if ;

1. f_i^* is taken as point load with unit stregth (unit Dirac source) at the source point A in x_l -direction, and a^* is zero; i.e., $f_i^{l^*} = \Delta(A,P)\delta_{il}$ and $a^* = 0$
   In which case, u_i^* and p^* are the solution of the system :

$$\mu u_{i,jj}^{l\ *} + \mu u_{j,ij}^{l\ *} + \lambda u_{j,ji}^{l\ *} - (\alpha + \beta) p_{,i}^{l\ *} + \omega^2 (\rho + \beta \rho_f) u_i^{l^*} + \underbrace{\delta_{il} \Delta(A,P)}_{f_i^{l^*}} = 0$$

and

$$\frac{-\beta}{\omega^2 \rho_f} p_{,ii}^{l\ *} + (\alpha + \beta) u_{i,i}^{l\ *} + \frac{1}{Q} p^{l^*} = \underbrace{0}_{a^*}$$

and t_i^* is given by

$$t_i^{l^*} = n_j \tau_{ji}^{l^*} = n_j \left[\mu \left(u_{i,j}^{l\ *} + u_{j,i}^{l\ *} \right) + \lambda \delta_{ji} u_{k,k}^{l\ *} - \alpha \delta_{ji} p^{l^*} \right]$$

where, $u_i^{l^*}$ and p^{l^*} are the ***fundamental solutions***. There are 9 (for $u_i^{l^*}$) + 3 (for p^{l^*}) = 12 (total) fundamental solutions for this set.

2. f_i^* are zero and a^* is a point fluid source of unit strength (unit Dirac source), i.e., $f_i^{1^*} = 0$ and $a^* = \Delta(A,P)$

In which case, u_i^* and p^* are the solution of the system :

$$\mu u^*_{i,jj} + \mu u^*_{j,ij} + \lambda u^*_{j,ji} - (\alpha+\beta)p^*_{,i} + \omega^2(\rho+\beta\rho_f)u^*_i = 0$$

and

$$\frac{-\beta}{\omega^2\rho_f}p^*_{,ii} + (\alpha+\beta)u^*_{i,i} + \frac{1}{Q}p^* = \frac{\overbrace{\Delta(A,P)}^{a^*}}{i\omega}$$

and t_i^* is given by

$$t_i^* = n_j\tau^*_{ji} = n_j\left[\mu\left(u^*_{i,j} + u^*_{j,i}\right) + \lambda\delta_{ji}u^*_{k,k} - \alpha\delta_{ji}p^*\right]$$

where, u_i^* and p^* are the ***fundamental solutions***. There are 3 (u_i^*) + 1 (p^*) = 4 fundamental solutions for this set.

Then one obtains the following set of boundary integral equations,

$$\boxed{\begin{aligned}&(1) \;\rightarrow\; u_1(A) = \int_\Gamma t_i u_i^{1^*} d\Gamma + \int_\Gamma \frac{1}{i\omega} q_n p^{1^*} d\Gamma - \int_\Gamma t_i^{1^*} u_i d\Gamma - \int_\Gamma \frac{1}{i\omega} q_n^{1^*} p d\Gamma \\ &(2) \;\rightarrow\; \frac{1}{i\omega} p(A) = \int_\Gamma t_i^* u_i d\Gamma + \int_\Gamma \frac{1}{i\omega} q_n^* p d\Gamma - \int_\Gamma t_i u_i^* d\Gamma - \int_\Gamma \frac{1}{i\omega} q_n p^* d\Gamma\end{aligned}} \quad \text{.............} \quad (4.26)$$

The BIE's in (4.26) contain a set of 16 fundamental solutions in total. Note that, eqn.'s (4.26) are the same BIE obtained earlier by method of weighted residuals (eqn's (4.18) and (4.20)).

CHAPTER 5

FUNDAMENTAL SOLUTIONS (3-D) OF DYNAMIC PORO-ELASTICITY

5.1 Fundamental Solutions

The early attempts to find the fundamental solutions for poro-elastodynamics are due to Burridge and Vargas (1979), Bonnet (Bonnet, 1987) and Boutin (Boutin, 1987). However, the solutions they provided were incomplete or contained errors (Chen, 1994.a). The first successful derivation therefore is known to have provided by Chen (1992) who followed the method utilized by the Soviet applied mathematician Kupradze in his monograph (Kupradze, 1979) in connexion with thermoelasticity (see also Nowacki, 1975). The method is deemed to be due to Swiss mathematician Hörmander (Kupradze 1979). In the following we lay out this method, and give full derivations of the fundamental solutions of the governing equations of poro-elastodynamics.

5.1.1 An Operator Method for Finding Fundamental Solutions of Systems of Differential Operators

A linear partial differential operator with *constant coefficients* follows the same rules of *linear algebra*, i.e. it can be added, multiplied, inverted much the same as an ordinary matrix.

For instance, consider the following linear ordinary differential equation,

$$-\frac{d^2y}{dx^2}+y=q(x)$$

or in operator form

$$(1 - D^2)y = q(x)$$

where, $L = 1 - D^2$ is an ordinary differential operator, where $D = \frac{d}{dx}$.

In terms of this operator "L" the solution can be written as

$$y = \frac{1}{1-D^2}q(x)$$

since "L" is a linear operator with *constant* coefficients, it can be expanded to power series much as we do with algebraic functions, that is:

$$\frac{1}{1-D^2} = 1 + D^2 + D^4 + ...$$

The solution is then,

$$y = (1 + D^2 + D^4 + ...)q(x)$$

Same idea applies to systems of ODE also, e.g. consider the system,

$$2(D-2)y(x) + (D-1)z(x) = e^x$$
$$(D+3)y(x) + z(x) = 0$$

which can be written in matrix form as,

$$\begin{bmatrix} 2(D-2) & D-1 \\ D+3 & 1 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} e^x \\ 0 \end{bmatrix}$$

then, the particular integral is given by

$$\begin{bmatrix} y \\ z \end{bmatrix}_p = \begin{bmatrix} 2(D-2) & D-1 \\ D+3 & 1 \end{bmatrix}^{-1}\begin{bmatrix} e^x \\ 0 \end{bmatrix}$$

evaluating the inverse operator, one gets

$$\begin{bmatrix} y \\ z \end{bmatrix}_p = \frac{1}{\Delta}\mathrm{Adj}\left(\begin{bmatrix} 2(D-2) & D-1 \\ D+3 & 1 \end{bmatrix}\right) = \frac{1}{-(D^2+1)}\begin{bmatrix} 1 & -(D-1) \\ -(D+3) & 2(D-2) \end{bmatrix}\begin{bmatrix} e^x \\ 0 \end{bmatrix}$$

where, $\Delta = -(D^2+1) = \begin{vmatrix} 2(D-2) & D-1 \\ D+3 & 1 \end{vmatrix}$ is the determinant of the matrix operator and "Adj" represents the matrix adjoint.

First operating $\frac{-1}{D^2+1}$ on $\begin{bmatrix} e^x \\ 0 \end{bmatrix}$ (recall that $\frac{1}{f(D)}e^{ax} = \frac{e^{ax}}{f(a)}$),

$$\begin{bmatrix} y \\ z \end{bmatrix}_p = \begin{bmatrix} 1 & -(D-1) \\ -(D+3) & 2(D-2) \end{bmatrix}\begin{bmatrix} \frac{-e^x}{2} \\ 0 \end{bmatrix}$$

and finally,

$$\begin{bmatrix} y \\ z \end{bmatrix}_p = \begin{bmatrix} \frac{-e^x}{2} \\ 2e^x \end{bmatrix}$$

is the required particular integral of the system.

The same techniques can be applied to find *the fundamental solutions* for many *ordinary differential equations* as well. For instance, it can be shown that the fundamental solution to the first order operator

$$(D-\lambda)U^{(1)}(x-a,\lambda)=\delta(x-a)$$

is

$$U^{(1)}(x-a,\lambda)=\frac{1}{2}\text{sign}(x-a)e^{\lambda(x-a)}$$

Although, it is possible to derive the above fundamental solution via Fourier (or Laplace transform) we present a method based on a transformation over the dependent variable, as

$$V(x-a,\lambda)=U^{(1)}(x-a,\lambda).e^{-\lambda(x-a)}$$

This reduces the differential equation to

$$D[V(x-a,\lambda)]=\delta(x-a)$$

on integration, we get

$$V(x-a,\lambda)=\frac{1}{2}\text{sign}(x-a)$$

Thus, on back substitution the fundamental solution for $U^{(1)}$ becomes,

$$U^{(1)}(x-a,\lambda)=\frac{1}{2}\text{sign}(x-a).e^{\lambda(x-a)}$$

It is to be noted that above fundamental solution reduces in the limit $\lambda \rightarrow 0$ to

$$U^{(1)}(x-a,0)=\frac{1}{2}\operatorname{sign}(x-a)$$

which is the fundamental solution of $DU(x-a)=\delta(x-a)$

In view of this information one can derive the fundamental solution to the 1-D Helmholtz equation:

$$\left(D^2-\lambda^2\right)U^{(2)}(x-a,\lambda)=\delta(x-a) \quad \rightarrow \text{ 1-D Helmholtz equation}$$

But the 1-D Helmholtz operator can algebraically be factored to read

$$(D-\lambda)(D+\lambda)U^{(2)}(x-a,\lambda)=\delta(x-a)$$

then the fundamental solution is given by

$$U^{(2)}(x-a,\lambda)=\frac{\delta(x-a)}{(D-\lambda)(D+\lambda)}$$

applying partial fractions expansion the inverse operator becomes

$$U^{(2)}(x-a,\lambda)=\frac{1}{2\lambda}\left[\frac{1}{(D-\lambda)}-\frac{1}{(D+\lambda)}\right]\delta(x-a)$$

then the fundamental solution of the 1-D Helmholtz equation can be written as the sum of those of two first order operator's as

$$U^{(2)}(x-a,\lambda)=\frac{1}{2\lambda}\left[U^{(1)}(x-a,\lambda)-U^{(1)}(x-a,-\lambda)\right]$$

more explicitly

$$
\begin{aligned}
U^{(2)}(x-a,\lambda) &= \frac{1}{2\lambda}\left[\frac{1}{2}\operatorname{sign}(x-a)e^{\lambda(x-a)} - \frac{1}{2}\operatorname{sign}(x-a)e^{-\lambda(x-a)}\right] \\
&= \frac{\operatorname{sign}(x-a)}{2\lambda}\underbrace{\left[\frac{e^{\lambda(x-a)} - e^{-\lambda(x-a)}}{2}\right]}_{\sinh\lambda(x-a)} \\
&= \frac{\operatorname{sign}(x-a)}{2\lambda}\sinh\lambda(x-a) \\
&= \frac{1}{2\lambda}\sinh\lambda|x-a|
\end{aligned}
$$

which is the same as given in many references, e.g. Rashed 2002, Pozrikidis 2002, and Kythe 1995.

Using this technique, it is possible to derive the fundamental solutions of higher order operators, to mention e.g.

$$
\left(D^4 - \lambda^4\right)U^{(4)}(x-a,\lambda) = \delta(x-a)
$$

which governs beam bending for instance, by the above method has the fundamental solution

$$
\begin{aligned}
U^{(4)}(x-a,\lambda) &= \frac{1}{\left(D^4 - \lambda^4\right)}\delta(x-a) \\
&= \frac{1}{2\lambda^2}\left[\frac{1}{D^2-\lambda^2} - \frac{1}{\underbrace{D^2+\lambda^2}_{D^2-(i\lambda)^2}}\right]\delta(x-a) \\
&= \frac{1}{2\lambda^2}\left[\frac{\sinh\lambda|x-a|}{2\lambda} - \frac{\sinh i\lambda|x-a|}{2i\lambda}\right] \\
&= \frac{1}{4\lambda^3}\left[\sinh\lambda|x-a| - \sin\lambda|x-a|\right]
\end{aligned}
$$

this checks with that given by Schanz 2001.b.

Now, these ideas will be extended to partial differential equations with *constant* coefficients, and the method will then be applied to obtain fundamental solutions of poroelasticity as particular integrals of the governing equations with Dirac sources.

A set of coupled partial differential equations is written in matrix operator form as,

$$\underline{B}\underline{u} = \underline{0} \quad \text{(5.1)}$$

where $\underline{B}$ is the matrix partial differential operator and $\underline{u}$ is the vector of unknowns. A matrix differential operator is a matrix whose elements are partial differential operators. A multiplication of such a matrix with a vector of functions means that the differential operators as the elements of the matrix are to be applied on the functions in the vector following the rules of normal matrix multiplication, the following example is provided by M. Schanz (Schanz 2001 b),

let $\underline{B} = \begin{bmatrix} \partial_1 & \partial_2 & 0 \\ \partial_t & 4 & \partial_2 \\ 0 & \partial_1 & 2\partial_2 \end{bmatrix}$ and $\underline{u} = \{u_i(x_j,t)\} = \begin{bmatrix} u_1(x_j,t) \\ u_2(x_j,t) \\ u_3(x_j,t) \end{bmatrix}$

Then, $\underline{B}\underline{u} = 0 \rightarrow$ means,

$$\begin{bmatrix} \partial_1 & \partial_2 & 0 \\ \partial_t & 4 & \partial_2 \\ 0 & \partial_1 & 2\partial_2 \end{bmatrix} \begin{bmatrix} u_1(x_j,t) \\ u_2(x_j,t) \\ u_3(x_j,t) \end{bmatrix} = 0 \quad \Leftrightarrow \quad \begin{aligned} \partial_1 u_1(x_j,t) + \partial_2 u_2(x_j,t) &= 0 \\ \partial_t u_1(x_j,t) + 4u_2(x_j,t) + \partial_2 u_3(x_j,t) &= 0 \\ \partial_1 u_2(x_j,t) + 2\partial_2 u_3(x_j,t) &= 0 \end{aligned}$$

The rules of matrix algebra apply analogously. The elements B_{ij}^{cof} of the cofactor matrix of $\underline{B}$ are then computed by the determinant of the sub matrix of $\underline{B}$ with the row “i” and column “j” deleted multiplied by $(-1)^{i+j}$. Then,

$$\underline{B}^{cof} = \begin{bmatrix} 8\partial_1 - \partial_1\partial_2 & -2\partial_t\partial_2 & \partial_t\partial_1 \\ -2\partial_2\partial_2 & 2\partial_1\partial_2 & -\partial_1\partial_1 \\ \partial_2\partial_2 & -\partial_1\partial_2 & 4\partial_1 - \partial_t\partial_2 \end{bmatrix} \quad \text{(5.2)}$$

The determinant of $\underline{B}$ is defined similarly;

$$\Delta = |\underline{B}| = \begin{vmatrix} \partial_1 & \partial_2 & 0 \\ \partial_t & 4 & \partial_2 \\ 0 & \partial_1 & 2\partial_2 \end{vmatrix} = \partial_1(8\partial_2 - \partial_1\partial_2) - \partial_2(2\partial_2\partial_t) \quad \ldots\ldots\ldots\ldots (5.3)$$

The cofactor and determinant provides definition of the inverse operator because,

$$\underline{B}\left(\underline{B}^{cof}\right)^T = \begin{bmatrix} \partial_1 & \partial_2 & 0 \\ \partial_t & 4 & \partial_2 \\ 0 & \partial_1 & 2\partial_2 \end{bmatrix} \begin{bmatrix} 8\partial_1 - \partial_1\partial_2 & -2\partial_t\partial_2 & \partial_t\partial_1 \\ -2\partial_2\partial_2 & 2\partial_1\partial_2 & -\partial_1\partial_1 \\ \partial_2\partial_2 & -\partial_1\partial_2 & 4\partial_1 - \partial_t\partial_2 \end{bmatrix} =$$

$$\begin{bmatrix} \partial_1(8\partial_1 - \partial_1\partial_2) + \partial_2(-2\partial_t\partial_2) & 0 & 0 \\ 0 & \partial_t(-2\partial_2\partial_2) + 4(2\partial_1\partial_2) + \partial_2(-\partial_1\partial_1) & 0 \\ 0 & 0 & \partial_1(-\partial_1\partial_2) + 2\partial_2(4\partial_1 - \partial_t\partial_2) \end{bmatrix}$$

$$= \Delta\, \underline{\mathbf{I}}$$

Therefore, the inverse operator is given by

$$\underline{B}^{-1} = \frac{1}{\Delta}\left(\underline{B}^{cof}\right)^T \quad \ldots\ldots\ldots\ldots (5.4)$$

The operator method applied to find the fundamental solutions of poro-elastodynamics:

For the governing equations of poro-elastodynamics, the differential operator is (recall equation (2.55))

$$\underline{B} = \begin{bmatrix} \mu\nabla^2\delta_{ij} + (\lambda+\mu)\partial_i\partial_j + \omega^2(\rho + \beta\rho_f)\delta_{ij} & -(\alpha+\beta)\partial_i \\ -i\omega(\alpha+\beta)\partial_j & -\dfrac{\beta}{i\omega\rho_f}\nabla^2 - \dfrac{i\omega}{Q} \end{bmatrix} \quad \ldots\ldots\ldots\ldots (5.5)$$

or in extended form,

$$\underline{B} = \begin{bmatrix} \mu\nabla^2 + (\lambda+\mu)\partial_1\partial_1 + \omega^2(\rho+\beta\rho_f) & (\lambda+\mu)\partial_1\partial_2 & (\lambda+\mu)\partial_1\partial_3 & -(\alpha+\beta)\partial_1 \\ (\lambda+\mu)\partial_2\partial_1 & \mu\nabla^2 + (\lambda+\mu)\partial_2\partial_2 + \omega^2(\rho+\beta\rho_f) & (\lambda+\mu)\partial_2\partial_3 & -(\alpha+\beta)\partial_2 \\ (\lambda+\mu)\partial_3\partial_1 & (\lambda+\mu)\partial_3\partial_2 & \mu\nabla^2 + (\lambda+\mu)\partial_3\partial_3 + \omega^2(\rho+\beta\rho_f) & -(\alpha+\beta)\partial_3 \\ -i\omega(\alpha+\beta)\partial_1 & -i\omega(\alpha+\beta)\partial_2 & -i\omega(\alpha+\beta)\partial_3 & -\dfrac{\beta}{i\omega\rho_f}\nabla^2 - \dfrac{i\omega}{Q} \end{bmatrix} \quad \text{.............. (5.6)}$$

then the governing equations in matrix extended form are

$$\begin{bmatrix} \mu\nabla^2 + (\lambda+\mu)\partial_1\partial_1 + \omega^2(\rho+\beta\rho_f) & (\lambda+\mu)\partial_1\partial_2 & (\lambda+\mu)\partial_1\partial_3 & -(\alpha+\beta)\partial_1 \\ (\lambda+\mu)\partial_2\partial_1 & \mu\nabla^2 + (\lambda+\mu)\partial_2\partial_2 + \omega^2(\rho+\beta\rho_f) & (\lambda+\mu)\partial_2\partial_3 & -(\alpha+\beta)\partial_2 \\ (\lambda+\mu)\partial_3\partial_1 & (\lambda+\mu)\partial_3\partial_2 & \mu\nabla^2 + (\lambda+\mu)\partial_3\partial_3 + \omega^2(\rho+\beta\rho_f) & -(\alpha+\beta)\partial_3 \\ -i\omega(\alpha+\beta)\partial_1 & -i\omega(\alpha+\beta)\partial_2 & -i\omega(\alpha+\beta)\partial_3 & -\dfrac{\beta}{i\omega\rho_f}\nabla^2 - \dfrac{i\omega}{Q} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ p \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ a \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

.... (5.7)

The fundamental solutions are the 4 set of solutions for ($\underline{u}$, p) when one of {f_1, f_2, f_3, a} in (5.7) is set equal to the Dirac delta function and others to zero. Designating the fundamental solution matrix by $\underline{G}$, one has for the fundamental solutions

$$\underline{B}\underline{G} + \Delta(A, P)\underline{I} = \underline{0} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5.8)$$

where,

$$\underline{G} = \begin{bmatrix} u^{*}_{lk} & u^{*}_{l} \\ p^{*}_{k} & p^{*} \end{bmatrix}$$

is a 4x4 matrix of fundamental solutions.

If we let $$\underline{G} = \left(\underline{B}^{cof}\right)^{T} \Phi \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5.9)$$

$$\underline{B}\left(\underline{B}^{cof}\right)^{T} \Phi + \Delta(A, P)\underline{I} = \underline{0}$$

but since

$$\underline{B}\left(\underline{B}^{cof}\right)^{T} = \det(\underline{B})\underline{I}$$

$$\det(\underline{B})\Phi + \Delta(A, P) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5.10)$$

If now, (5.10) can be solved for Φ then fundamental solutions can be evaluated from (5.9). It remains to evaluate $\underline{B}^{cof}$ and $\det(\underline{B})$, solve (5.10) for Φ then evaluate the matrix of fundamental solutions by $\underline{G} = \left(\underline{B}^{cof}\right)^{T} \Phi$

Before proceeding to find determinant and the cofactor transpose of $\underline{B}$ matrix, we introduce the abbreviations,

$$
\begin{aligned}
&A = \mu\nabla^2 + \omega^2(\rho + \beta\rho_f) \\
&B = \lambda + \mu \\
&C = \alpha + \beta \qquad \qquad \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(5.11)} \\
&D = \frac{\beta}{i\omega\rho_f}\nabla^2 + \frac{i\omega}{Q} \\
&F = BD + i\omega C^2
\end{aligned}
$$

then det($\underline{B}$) is written as,

$$
\det(\underline{B}) = \begin{vmatrix} A + B\partial_1\partial_1 & B\partial_1\partial_2 & B\partial_1\partial_3 & -C\partial_1 \\ B\partial_1\partial_2 & A + B\partial_2\partial_2 & B\partial_2\partial_3 & -C\partial_2 \\ B\partial_1\partial_3 & B\partial_2\partial_3 & A + B\partial_3\partial_3 & -C\partial_3 \\ -i\omega C\partial_1 & -i\omega C\partial_2 & -i\omega C\partial_3 & -D \end{vmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(5.12)}
$$

multiplying the first row by $\partial_2\partial_3$, second row by $\partial_1\partial_3$ and the third by $\partial_1\partial_2$,

$$
\det(\underline{B}) = \frac{1}{\partial_{112233}} \begin{vmatrix} A\partial_{23} + B\partial_{1123} & B\partial_{1223} & B\partial_{1233} & -C\partial_{123} \\ B\partial_{1123} & A\partial_{13} + B\partial_{1223} & B\partial_{1233} & -C\partial_{123} \\ B\partial_{1123} & B\partial_{1223} & A\partial_{12} + B\partial_{1233} & -C\partial_{123} \\ -i\omega C\partial_1 & -i\omega C\partial_2 & -i\omega C\partial_3 & -D \end{vmatrix}
$$

where, operations like $\partial_1\partial_2\partial_3$ have been abbreviated as ∂_{123}, etc.

The determinant can then be simplified by row operations to yield

$$
\det(\underline{B}) = \frac{-A^2}{\partial_{11122333}} \begin{vmatrix} \partial_{23} & -\partial_{13} & 0 & 0 \\ 0 & \partial_{13} & -\partial_{12} & 0 \\ \left(B + i\omega\frac{C^2}{D}\right)\partial_{1123} & \left(B + i\omega\frac{C^2}{D}\right)\partial_{1223} & A\partial_{12} + \left(B + i\omega\frac{C^2}{D}\right)\partial_{1233} & 0 \\ i\omega C\partial_{1123} & i\omega C\partial_{1223} & i\omega C\partial_{1233} & D\partial_{123} \end{vmatrix}
$$

factoring $\left(B + i\omega\frac{C^2}{D}\right)\partial_{12}$ out and expanding about the last column,

$$\det(\underline{B}) = \frac{-A^2D\left(B+i\omega\frac{C^2}{D}\right)}{\partial_{1233}}\begin{vmatrix} \partial_{23} & -\partial_{13} & 0 \\ 0 & \partial_{13} & -\partial_{12} \\ \partial_{13} & \partial_{23} & \frac{A}{B+i\omega\frac{C^2}{D}}+\partial_{33} \end{vmatrix}$$

Finally,

$$\det(\underline{B}) = -A^2\left[\left(BD + i\omega C^2\right)\nabla^2 + AD\right] \quad \text{.. (5.13)}$$

(5.13) when extended reads,

$$\det(\underline{B}) = -\frac{\beta(\lambda+2\mu)\mu^2}{i\omega\rho_f}\left(\nabla^2 + \frac{\omega^2(\rho+\beta\rho_f)}{\mu}\right)^2 \left[\nabla^4 - \left(\frac{\omega^2\rho_f}{\beta Q} + \frac{\omega^2\rho_f(\alpha+\beta)^2}{\beta(\lambda+2\mu)} - \frac{\omega^2(\rho+\beta\rho_f)}{(\lambda+2\mu)}\right)\nabla^2 - \frac{\omega^4\rho_f(\rho+\beta\rho_f)}{Q\beta(\lambda+2\mu)}\right]$$

which can be factored to read,

$$\det(\underline{B}) = -\frac{\beta(\lambda+2\mu)\mu^2}{i\omega\rho_f}\left(\nabla^2-\lambda_3^2\right)^2\left(\nabla^2-\lambda_1^2\right)\left(\nabla^2-\lambda_2^2\right) \quad \text{.................... (5.14)}$$

where,

$$\lambda_3^2 = -\frac{\omega^2(\rho+\beta\rho_f)}{\mu}$$

$$\lambda_{1,2}^2 = \frac{1}{2}\left[\left(\frac{\omega^2\rho_f}{\beta Q} + \frac{\omega^2\rho_f(\alpha+\beta)^2}{\beta(\lambda+2\mu)} - \frac{\omega^2(\rho+\beta\rho_f)}{(\lambda+2\mu)}\right) \pm \sqrt{\left(\frac{\omega^2\rho_f}{\beta Q} + \frac{\omega^2\rho_f(\alpha+\beta)^2}{\beta(\lambda+2\mu)} - \frac{\omega^2(\rho+\beta\rho_f)}{(\lambda+2\mu)}\right)^2 + 4\frac{\omega^4\rho_f(\rho+\beta\rho_f)}{Q\beta(\lambda+2\mu)}}\right] \quad \text{........ (5.15)}$$

Then, according to (5.10) one has

$$-\frac{\beta(\lambda+2\mu)\mu^2}{i\omega\rho_f}\left(\nabla^2-\lambda_3^2\right)^2\left(\nabla^2-\lambda_1^2\right)\left(\nabla^2-\lambda_2^2\right)\Phi+\Delta(A,P)=0 \quad \text{……………} (5.16)$$

Let

$$\Psi=-\frac{\beta(\lambda+2\mu)\mu^2}{i\omega\rho_f}\left(\nabla^2-\lambda_3^2\right)\Phi \quad \text{……..………………………………} (5.17)$$

Then,

$$\left(\nabla^2-\lambda_3^2\right)\left(\nabla^2-\lambda_1^2\right)\left(\nabla^2-\lambda_2^2\right)\Psi+\Delta(A,P)=0 \quad \text{…………………………} (5.18)$$

<u>Algebraic Reduction - The difference trick:</u> (Ortner, 1989)

Introduce the following,

$$\begin{aligned}\Psi_1&=\left(\nabla^2-\lambda_3^2\right)\left(\nabla^2-\lambda_2^2\right)\Psi\\ \Psi_2&=\left(\nabla^2-\lambda_3^2\right)\left(\nabla^2-\lambda_1^2\right)\Psi \quad \text{………………………………………..} (5.19)\\ \Psi_3&=\left(\nabla^2-\lambda_1^2\right)\left(\nabla^2-\lambda_2^2\right)\Psi\end{aligned}$$

Then from (5.18)

$$\begin{aligned}&\left(\nabla^2-\lambda_1^2\right)\Psi_1+\Delta(A,P)=0\\ &\left(\nabla^2-\lambda_2^2\right)\Psi_2+\Delta(A,P)=0 \quad \text{……………………………………………..} (5.20)\\ &\left(\nabla^2-\lambda_3^2\right)\Psi_3+\Delta(A,P)=0\end{aligned}$$

These equations are well known Helmholtz equations, thus the solutions are

$$\Psi_1 = \frac{1}{4\pi r} e^{-\lambda_1 r}$$
$$\Psi_2 = \frac{1}{4\pi r} e^{-\lambda_2 r} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(5.21)}$$
$$\Psi_3 = \frac{1}{4\pi r} e^{-\lambda_3 r}$$

where,

$$r = \text{distance between points A and P}$$
$$r = \left\| \overrightarrow{PA} \right\| = \left\| \vec{r}_P - \vec{r}_A \right\| = \sqrt{(x_i - a_i)(x_i - a_i)} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(5.22)}$$

From (5.19)

$$\Psi_2 - \Psi_1 = (\lambda_2^2 - \lambda_1^2)(\nabla^2 - \lambda_3^2)\Psi$$
$$\Psi_3 - \Psi_2 = (\lambda_3^2 - \lambda_2^2)(\nabla^2 - \lambda_1^2)\Psi \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(5.23)}$$

hence,

$$(\nabla^2 - \lambda_3^2)\Psi = \frac{\Psi_2 - \Psi_1}{\lambda_2^2 - \lambda_1^2}$$
$$(\nabla^2 - \lambda_1^2)\Psi = \frac{\Psi_3 - \Psi_2}{\lambda_3^2 - \lambda_2^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(5.24.a, b)}$$

subtracting (5.24.b from 5.24.a)

$$\Psi = \frac{1}{\lambda_3^2 - \lambda_1^2}\left(\frac{\Psi_3 - \Psi_2}{\lambda_3^2 - \lambda_2^2} - \frac{\Psi_2 - \Psi_1}{\lambda_2^2 - \lambda_1^2} \right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(5.25)}$$

Substituting Ψ_1, Ψ_3, Ψ_3 from (5.21) into (5.25)

$$\Psi = \frac{1}{4\pi r}\frac{1}{\lambda_3^2 - \lambda_1^2}\left(\frac{e^{-\lambda_3 r} - e^{-\lambda_2 r}}{\lambda_3^2 - \lambda_2^2} - \frac{e^{-\lambda_2 r} - e^{-\lambda_1 r}}{\lambda_2^2 - \lambda_1^2}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5.26)$$

rearranging,

$$\Psi = \frac{1}{4\pi r}\left(\frac{e^{-\lambda_1 r}}{(\lambda_1^2 - \lambda_2^2)(\lambda_1^2 - \lambda_3^2)} + \frac{e^{-\lambda_2 r}}{(\lambda_2^2 - \lambda_3^2)(\lambda_2^2 - \lambda_1^2)} + \frac{e^{-\lambda_3 r}}{(\lambda_3^2 - \lambda_1^2)(\lambda_3^2 - \lambda_2^2)}\right) \quad \ldots\ldots\ldots (5.27)$$

The fundamental solutions can now be determined from (5.9):

$$\underline{G} = \left(\underline{B}^{cof}\right)^T \Phi .$$

where,

$$\left(\underline{B}^{cof}\right)^T = -A\begin{bmatrix} AD + F(\partial_{22} + \partial_{33}) & -F\partial_{12} & -F\partial_{13} & -AC\partial_1 \\ -F\partial_{12} & AD + F(\partial_{11} + \partial_{33}) & -F\partial_{23} & -AC\partial_2 \\ -F\partial_{13} & -F\partial_{23} & AD + F(\partial_{22} + \partial_{11}) & -AC\partial_3 \\ -i\omega AC\partial_1 & -i\omega AC\partial_2 & -i\omega AC\partial_3 & -\left(A^2 + AB\nabla^2\right) \end{bmatrix}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5.28)$$

Where, the factor "F" is defined in (5.11). We also recall (5.17),

$$\psi = -\frac{\beta(\lambda + 2\mu)\mu^2}{i\omega\rho_f}\left(\nabla^2 - \lambda_3^2\right)\Phi$$

from which Φ can be obtained by inversion as,

$$\Phi = -\frac{i\omega\rho_f}{\beta(\lambda + 2\mu)\mu^2}\frac{1}{\left(\nabla^2 - \lambda_3^2\right)}\Psi \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5.29)$$

Recall also the factor A in (5.28) is given by $A = \mu(\nabla^2 - \lambda_3^2)$, then the fundamental solution matrix in extended form is

$$\underline{G} = -\mu\left(\nabla^2 - \lambda_3^2\right)\begin{bmatrix} AD + F\left(\partial_{22} + \partial_{33}\right) & -F\partial_{12} & -F\partial_{13} & -AC\partial_1 \\ -F\partial_{12} & AD + F\left(\partial_{11} + \partial_{33}\right) & -F\partial_{23} & -AC\partial_2 \\ -F\partial_{13} & -F\partial_{23} & AD + F\left(\partial_{22} + \partial_{11}\right) & -AC\partial_3 \\ -i\omega\omega A\partial_1 & -i\omega\omega A\partial_2 & -i\omega\omega A\partial_3 & -\left(A^2 + AB\nabla^2\right) \end{bmatrix}$$

$$* \left(-\frac{i\omega\rho_f}{\beta(\lambda + 2\mu)\mu^2} \frac{1}{\left(\nabla^2 - \lambda_3^2\right)} \Psi \right)$$

.... (5.30)

simplifying,

$$\underline{G} = \frac{i\omega\rho_f}{\beta(\lambda + 2\mu)\mu}\begin{bmatrix} \left(AD + F\nabla^2\right) - F\partial_{11} & -F\partial_{12} & -F\partial_{13} & -AC\partial_1 \\ -F\partial_{12} & \left(AD + F\nabla^2\right) - F\partial_{22} & -F\partial_{23} & -AC\partial_2 \\ -F\partial_{13} & -F\partial_{23} & \left(AD + F\nabla^2\right) - F\partial_{33} & -AC\partial_3 \\ -i\omega AC\partial_1 & -i\omega AC\partial_2 & -i\omega AC\partial_3 & -\left(A^2 + AB\nabla^2\right) \end{bmatrix}\Psi$$

.... (5.31)

or more simply in index notation

$$\underline{G} = \frac{i\omega\rho_f}{\beta(\lambda + 2\mu)\mu}\begin{bmatrix} \left(AD + F\nabla^2\right)\delta_{ij} - F\partial_{ij} & -AC\partial_i \\ -i\omega AC\partial_j & -\left(A^2 + AB\nabla^2\right) \end{bmatrix}\Psi \quad \text{.............................. (5.32)}$$

Thus, the 16 fundamental solutions in four groups are given by

$$G_{ij} = u_{ij}^* = \frac{i\omega\rho_f}{\beta(\lambda + 2\mu)\mu}\left[\left(AD + F\nabla^2\right)\delta_{ij} - F\partial_{ij}\right]\Psi \quad \text{... (5.33)}$$

$$G_{4j} = p_j^* = \frac{i\omega\rho_f}{\beta(\lambda+2\mu)\mu}\left[-i\omega AC\partial_j\right]\Psi \quad \text{...} \quad (5.34)$$

$$G_{i4} = u_i^* = \frac{i\omega\rho_f}{\beta(\lambda+2\mu)\mu}\left[-AC\partial_i\right]\Psi \quad \text{...} \quad (4.35)$$

$$G_{44} = p^* = -\frac{i\omega\rho_f}{\beta(\lambda+2\mu)\mu}\left[A^2 + AB\nabla^2\right]\Psi \quad \text{...} \quad (4.36)$$

5.2 Derivation of Fundamental Solutions of Poro-elastodynamics

In this section, explicit expressions for the **1st and 2nd fundamental solutions** (FS's) of the **dynamic poro-elasticity** in **frequency domain** will be derived.

1st FS's can be obtained by inserting the expression for ψ from (5.27) into (5.33), (5.34), 5.35) and (5.36).

2nd FS's, however are obtained from the 1st FS's via the relationships for the traction vector (Cauchy's Stress Formula, eqn. 2.22) and fluid flux (Darcy's Law, eqn. 2.47)

Before derivation, we repeat the expression for ψ (5.27) below.

$$\Psi = \frac{1}{4\pi r}\left(\frac{e^{-\lambda_1 r}}{(\lambda_1^2-\lambda_2^2)(\lambda_1^2-\lambda_3^2)} + \frac{e^{-\lambda_2 r}}{(\lambda_2^2-\lambda_3^2)(\lambda_2^2-\lambda_1^2)} + \frac{e^{-\lambda_3 r}}{(\lambda_3^2-\lambda_1^2)(\lambda_3^2-\lambda_2^2)}\right)$$

Also, the following identities will prove to be useful and will be used frequently in the course:

Recall that r designates the distance from the source point "A" to the field point "P", (5.22)

$$r = \left\|\overrightarrow{PA}\right\| = \left\|\vec{r}_P - \vec{r}_A\right\| = \sqrt{(x_i - a_i)(x_i - a_i)}$$

$r = r(A,P)$ is a two point scalar function. Then, the gradient of "r",

$$\underline{\nabla} r = \begin{bmatrix} \frac{\partial r}{\partial x_1} & \frac{\partial r}{\partial x_2} & \frac{\partial r}{\partial x_3} \end{bmatrix} = (\partial_i r) = (r_{,i}) \quad \text{..} \ (5.37)$$

will be a vector, when the differentiation is evaluated in (5.37)

$$r_{,i} = \frac{x_i - a_i}{r} \quad \text{..} \ (5.38)$$

which are identified to be the components of the unit vector $\underline{r} = (r_i)$ pointing in the direction from "A" to "P".

Similarly,

$$r_{i,j} = r_{,ij} = \frac{1}{r}\left(\delta_{ij} - r_i r_j\right) \quad \text{..} \ (5.39)$$

Also, since the function ψ contains terms like $\frac{e^{-\lambda r}}{r}$, we further needs;

$$\begin{aligned}
\partial_{ij}\left(\frac{e^{-\lambda r}}{r}\right) &= \partial_i\left(-\lambda e^{-\lambda r} r_j \frac{1}{r} - \frac{1}{r^2} r_j e^{-\lambda r}\right) \\
&= \left[\left(\lambda^2 + \frac{3\lambda}{r} + \frac{3}{r^2}\right) r_i r_j - \frac{1}{r}\left(\lambda + \frac{1}{r}\right)\delta_{ij}\right]\frac{e^{-\lambda r}}{r} \\
&= \left[\frac{1}{r^2}\left(3 r_i r_j - \delta_{ij}\right) + \frac{\lambda}{r}\left(3 r_i r_j - \delta_{ij}\right) + \lambda^2 r_i r_j\right]\frac{e^{-\lambda r}}{r} \\
&= R_{ij}^{(\lambda)} \frac{e^{-\lambda r}}{r}
\end{aligned} \quad \text{..................................} \ (5.40)$$

Here, we introduce the abbreviation

$$R_{ij}^{\lambda} = \frac{1}{r^2}\left(3 r_i r_j - \delta_{ij}\right) + \frac{\lambda}{r}\left(3 r_i r_j - \delta_{ij}\right) + \lambda^2 r_i r_j$$

because it will often recur in the development to follow. By (5.40) one can infer the following easily,

$$\nabla^2\left(\frac{e^{-\lambda r}}{r}\right) = \partial_{kk}\left(\frac{e^{-\lambda r}}{r}\right) = \underbrace{R^{\lambda}_{kk}}_{\lambda^2}\frac{e^{-\lambda r}}{r} = \lambda^2\frac{e^{-\lambda r}}{r} \quad \text{.......................................} (5.41)$$

and,

$$\begin{aligned}\nabla^2\partial_{ij}\left(\frac{e^{-\lambda r}}{r}\right) &= \lambda^2\frac{e^{-\lambda r}}{r}\left[\left(\lambda^2+\frac{3\lambda}{r}+\frac{3}{r^2}\right)r_i r_j - \frac{1}{r}\left(\lambda+\frac{1}{r}\right)\delta_{ij}\right] \\ &= \lambda^2\frac{e^{-\lambda r}}{r}\left[\frac{1}{r^2}\left(3r_i r_j-\delta_{ij}\right)+\frac{\lambda}{r}\left(3r_i r_j-\delta_{ij}\right)+\lambda^2 r_i r_j\right]\end{aligned} \quad \text{.........................} (5.42)$$

hence,

$$\nabla^4\left(\frac{e^{-\lambda r}}{r}\right) = \nabla^2\nabla^2\left(\frac{e^{-\lambda r}}{r}\right) = \nabla^2\partial_{kk}\left(\frac{e^{-\lambda r}}{r}\right) = \lambda^4\frac{e^{-\lambda r}}{r} \quad \text{......................................} (5.43)$$

Now, we start deriving the components of G matrix.

5.2.1 First (Displacement) Fundamental Solutions

The terms u^*_{ij} *: Solid Displacement components at the field point "P" due to unit point load in j-direction at the source point "A":*

$$G_{ij} = u^*_{ij} = \frac{i\omega\rho_f}{\beta(\lambda+2\mu)\mu}\left[\left(AD+F\nabla^2\right)\delta_{ij} - F\partial_{ij}\right]\Psi \quad \text{...} (5.44)$$

where, (recall 5.11)

$$\mathrm{AD} = \left[\mu\nabla^2 + \omega^2(\rho + \beta\rho_f)\right]\left[\frac{\beta}{i\omega\rho_f}\nabla^2 + \frac{i\omega}{Q}\right]$$

$$= \frac{\mu\beta}{i\omega\rho_f}\nabla^4 + \left(\frac{i\omega\mu}{Q} + \frac{\beta\omega^2(\rho + \beta\rho_f)}{i\omega\rho_f}\right)\nabla^2 + \frac{i\omega^3(\rho + \beta\rho_f)}{Q}$$

and

$$F = BD + i\omega C^2 = (\lambda + \mu)\left[\frac{\beta}{i\omega\rho_f}\nabla^2 + \frac{i\omega}{Q}\right] + i\omega(\alpha + \beta)^2$$

$$= \frac{\beta(\lambda + \mu)}{i\omega\rho_f}\nabla^2 + \frac{i\omega(\lambda + \mu)}{Q} + i\omega(\alpha + \beta)^2$$

Then,

$$\mathrm{AD} + F\nabla^2 = (\lambda + 2\mu)\frac{\beta}{i\omega\rho_f}\nabla^4 + \left[(\lambda + 2\mu)\frac{i\omega}{Q} + \frac{\beta\omega^2(\rho + \beta\rho_f)}{i\omega\rho_f} + i\omega(\alpha + \beta)^2\right]\nabla^2 + \frac{i\omega^3(\rho + \beta\rho_f)}{Q}$$

.... (5.45)

(1) → Evaluate $(\mathrm{AD} + F\nabla^2)\Psi$

Recall (5.41) and (5.43), then

$$(\mathrm{AD} + F\nabla^2)\Psi = \frac{(\lambda + 2\mu)\beta}{i\omega\rho_f}\frac{1}{4\pi r}\left[\frac{\lambda_1^4 e^{-\lambda_1 r}}{(\lambda_1^2 - \lambda_2^2)(\lambda_1^2 - \lambda_3^2)} + \frac{\lambda_2^4 e^{-\lambda_2 r}}{(\lambda_2^2 - \lambda_3^2)(\lambda_2^2 - \lambda_1^2)} + \frac{\lambda_3^4 e^{-\lambda_3 r}}{(\lambda_3^2 - \lambda_1^2)(\lambda_3^2 - \lambda_2^2)}\right]$$

$$+ \left[(\lambda + 2\mu)\frac{i\omega}{Q} + \frac{\beta\omega^2(\rho + \beta\rho_f)}{i\omega\rho_f} + i\omega(\alpha + \beta)^2\right]\frac{1}{4\pi r} *$$

$$\left[\frac{\lambda_1^2 e^{-\lambda_1 r}}{(\lambda_1^2 - \lambda_2^2)(\lambda_1^2 - \lambda_3^2)} + \frac{\lambda_2^2 e^{-\lambda_2 r}}{(\lambda_2^2 - \lambda_3^2)(\lambda_2^2 - \lambda_1^2)} + \frac{\lambda_3^2 e^{-\lambda_3 r}}{(\lambda_3^2 - \lambda_1^2)(\lambda_3^2 - \lambda_2^2)}\right]$$

$$+ \frac{i\omega^3(\rho + \beta\rho_f)}{Q}\frac{1}{4\pi r}\left[\frac{e^{-\lambda_1 r}}{(\lambda_1^2 - \lambda_2^2)(\lambda_1^2 - \lambda_3^2)} + \frac{e^{-\lambda_2 r}}{(\lambda_2^2 - \lambda_3^2)(\lambda_2^2 - \lambda_1^2)} + \frac{e^{-\lambda_3 r}}{(\lambda_3^2 - \lambda_1^2)(\lambda_3^2 - \lambda_2^2)}\right]$$

Collecting terms for each $e^{\lambda_i r}$, and taking the factor $\frac{(\lambda+2\mu)\beta}{i\omega\rho_f}\frac{1}{4\pi r}$ out of the parenthesis, we get

$$
\begin{aligned}
\left(AD+F\nabla^2\right)\Psi = \frac{(\lambda+2\mu)\beta}{i\omega\rho_f}\frac{1}{4\pi r}\Bigg\{ & \frac{e^{-\lambda_1 r}}{(\lambda_1^2-\lambda_2^2)(\lambda_1^2-\lambda_3^2)}\underbrace{\left[\lambda_1^4-\left(\frac{\omega^2\rho_f}{\beta Q}-\frac{\omega^2(\rho+\beta\rho_f)}{\lambda+2\mu}+\frac{\omega^2\rho_f(\alpha+\beta)^2}{\beta(\lambda+2\mu)}\right)\lambda_1^2-\frac{\omega^4\rho_f(\rho+\beta\rho_f)}{\beta Q(\lambda+2\mu)}\right]}_{I_1} \\
& +\frac{e^{-\lambda_2 r}}{(\lambda_2^2-\lambda_3^2)(\lambda_2^2-\lambda_1^2)}\underbrace{\left[\lambda_2^4-\left(\frac{\omega^2\rho_f}{\beta Q}-\frac{\omega^2(\rho+\beta\rho_f)}{\lambda+2\mu}+\frac{\omega^2\rho_f(\alpha+\beta)^2}{\beta(\lambda+2\mu)}\right)\lambda_2^2-\frac{\omega^4\rho_f(\rho+\beta\rho_f)}{\beta Q(\lambda+2\mu)}\right]}_{I_2} \\
& +\frac{e^{-\lambda_3 r}}{(\lambda_3^2-\lambda_1^2)(\lambda_3^2-\lambda_2^2)}\underbrace{\left[\lambda_3^4-\left(\frac{\omega^2\rho_f}{\beta Q}-\frac{\omega^2(\rho+\beta\rho_f)}{\lambda+2\mu}+\frac{\omega^2\rho_f(\alpha+\beta)^2}{\beta(\lambda+2\mu)}\right)\lambda_3^2-\frac{\omega^4\rho_f(\rho+\beta\rho_f)}{\beta Q(\lambda+2\mu)}\right]}_{I_3}\Bigg\}
\end{aligned}
\quad \text{.................... (5.46)}
$$

this expression is substantially simplified on simplifying the multipliers I_1, I_2, I_3,

Recall that, (refer to equation 5.15)

$$\lambda_1^2+\lambda_2^2=\frac{\omega^2\rho_f}{\beta Q}-\frac{\omega^2(\rho+\beta\rho_f)}{\lambda+2\mu}+\frac{\omega^2\rho_f(\alpha+\beta)^2}{\beta(\lambda+2\mu)} \quad \text{and} \quad \lambda_1^2\lambda_2^2=-\frac{\omega^4\rho_f(\rho+\beta\rho_f)}{\beta Q(\lambda+2\mu)}$$

$$I_1 = \lambda_1^4 - \Bigg(\underbrace{\frac{\omega^2\rho_f}{\beta Q} - \frac{\omega^2(\rho+\beta\rho_f)}{\lambda+2\mu} + \frac{\omega^2\rho_f(\alpha+\beta)^2}{\beta(\lambda+2\mu)}}_{\lambda_1^2+\lambda_2^2}\Bigg)\lambda_1^2 - \underbrace{\frac{\omega^4\rho_f(\rho+\beta\rho_f)}{\beta Q(\lambda+2\mu)}}_{-\lambda_1^2\lambda_2^2}$$

$$= \lambda_1^4 - (\lambda_1^2+\lambda_2^2)\lambda_1^2 + \lambda_1^2\lambda_2^2$$

$$= 0$$

$$I_2 = \lambda_2^4 - \Bigg(\underbrace{\frac{\omega^2\rho_f}{\beta Q} - \frac{\omega^2(\rho+\beta\rho_f)}{\lambda+2\mu} + \frac{\omega^2\rho_f(\alpha+\beta)^2}{\beta(\lambda+2\mu)}}_{\lambda_1^2+\lambda_2^2}\Bigg)\lambda_2^2 - \underbrace{\frac{\omega^4\rho_f(\rho+\beta\rho_f)}{\beta Q(\lambda+2\mu)}}_{-\lambda_1^2\lambda_2^2}$$

$$= \lambda_2^4 - (\lambda_1^2+\lambda_2^2)\lambda_2^2 + \lambda_1^2\lambda_2^2$$

$$= 0$$

$$I_3 = \lambda_3^4 - \Bigg(\underbrace{\frac{\omega^2\rho_f}{\beta Q} - \frac{\omega^2(\rho+\beta\rho_f)}{\lambda+2\mu} + \frac{\omega^2\rho_f(\alpha+\beta)^2}{\beta(\lambda+2\mu)}}_{\lambda_1^2+\lambda_2^2}\Bigg)\lambda_3^2 - \underbrace{\frac{\omega^4\rho_f(\rho+\beta\rho_f)}{\beta Q(\lambda+2\mu)}}_{-\lambda_1^2\lambda_2^2}$$

$$= \lambda_3^4 - (\lambda_1^2+\lambda_2^2)\lambda_3^2 + \lambda_1^2\lambda_2^2$$

$$= \lambda_3^4 - \lambda_1^2\lambda_3^2 - \lambda_2^2\lambda_3^2 + \lambda_1^2\lambda_2^2$$

$$= \lambda_3^2(\lambda_3^2-\lambda_1^2) - \lambda_2^2(\lambda_3^2-\lambda_1^2)$$

$$= (\lambda_3^2-\lambda_1^2)(\lambda_3^2-\lambda_2^2)$$

Hence,

$$(AD + F\nabla^2)\Psi = \frac{\beta(\lambda+2\mu)}{i\omega\rho_f}\frac{1}{4\pi r}e^{-\lambda_3 r} \quad \text{.. (5.47)}$$

(2) Evaluate $F\partial_{ij}\Psi \rightarrow$

$$
\begin{aligned}
F\partial_{ij}\Psi = &\left[\frac{\beta(\lambda+\mu)}{i\omega\rho_f}\nabla^2 + \frac{i\omega(\lambda+\mu)}{Q} + i\omega(\alpha+\beta)^2\right]\partial_{ij} \\
&\frac{1}{4\pi r}\left(\frac{e^{-\lambda_1 r}}{(\lambda_1^2-\lambda_2^2)(\lambda_1^2-\lambda_3^2)} + \frac{e^{-\lambda_2 r}}{(\lambda_2^2-\lambda_3^2)(\lambda_2^2-\lambda_1^2)} + \frac{e^{-\lambda_3 r}}{(\lambda_3^2-\lambda_1^2)(\lambda_3^2-\lambda_2^2)}\right) \\
= &\frac{1}{4\pi(\lambda_1^2-\lambda_2^2)(\lambda_1^2-\lambda_3^2)}\left[\frac{\beta(\lambda+\mu)}{i\omega\rho_f}\lambda_1^2 + \frac{i\omega(\lambda+\mu)}{Q} + i\omega(\alpha+\beta)^2\right]\partial_{ij}\frac{e^{-\lambda_1 r}}{r} \\
&+ \frac{1}{4\pi(\lambda_2^2-\lambda_3^2)(\lambda_2^2-\lambda_1^2)}\left[\frac{\beta(\lambda+\mu)}{i\omega\rho_f}\lambda_2^2 + \frac{i\omega(\lambda+\mu)}{Q} + i\omega(\alpha+\beta)^2\right]\partial_{ij}\frac{e^{-\lambda_2 r}}{r} \\
&+ \frac{1}{4\pi(\lambda_3^2-\lambda_1^2)(\lambda_3^2-\lambda_2^2)}\left[\frac{\beta(\lambda+\mu)}{i\omega\rho_f}\lambda_3^2 + \frac{i\omega(\lambda+\mu)}{Q} + i\omega(\alpha+\beta)^2\right]\partial_{ij}\frac{e^{-\lambda_3 r}}{r}
\end{aligned}
$$

.... (5.48)

Referring to (5.40) and (5.42), i.e.

$$
\begin{aligned}
\partial_{ij}\frac{e^{-\lambda_k r}}{r} &= \left[\frac{1}{r^2}(3r_i r_j - \delta_{ij}) + \frac{\lambda_k}{r}(3r_i r_j - \delta_{ij}) + \lambda_k^2 r_i r_j\right]\frac{e^{-\lambda_k r}}{r} \\
&= R_{ij}^{(k)}\frac{e^{-\lambda_k r}}{r}
\end{aligned}
$$

Then,

$$F\partial_{ij}\Psi=\frac{1}{4\pi\left(\lambda_1^2-\lambda_2^2\right)\left(\lambda_1^2-\lambda_3^2\right)}\underbrace{\left[\frac{\beta(\lambda+\mu)}{i\omega\rho_f}\lambda_1^2+\frac{i\omega(\lambda+\mu)}{Q}+i\omega(\alpha+\beta)^2\right]}_{K_1}R_{ij}^{(1)}\frac{e^{-\lambda_1 r}}{r}+\frac{1}{4\pi\left(\lambda_2^2-\lambda_3^2\right)\left(\lambda_2^2-\lambda_1^2\right)}\underbrace{\left[\frac{\beta(\lambda+\mu)}{i\omega\rho_f}\lambda_2^2+\frac{i\omega(\lambda+\mu)}{Q}+i\omega(\alpha+\beta)^2\right]}_{K_2}R_{ij}^{(2)}\frac{e^{-\lambda_2 r}}{r}$$
$$+\frac{1}{4\pi\left(\lambda_3^2-\lambda_1^2\right)\left(\lambda_3^2-\lambda_2^2\right)}\underbrace{\left[\frac{\beta(\lambda+\mu)}{i\omega\rho_f}\lambda_3^2+\frac{i\omega(\lambda+\mu)}{Q}+i\omega(\alpha+\beta)^2\right]}_{K_3}R_{ij}^{(3)}\frac{e^{-\lambda_3 r}}{r} \tag{5.49}$$

$$G_{ij}=u_{ij}^{*}=\frac{i\omega\rho_f}{\beta(\lambda+2\mu)\mu}\left[\left(AD+F\nabla^2\right)\delta_{ij}-F\partial_{ij}\right]\Psi$$
$$=\frac{i\omega\rho_f}{\beta(\lambda+2\mu)\mu}\frac{1}{4\pi r}\left\{\begin{array}{l}\frac{\beta(\lambda+2\mu)}{i\omega\rho_f}\delta_{ij}e^{-\lambda_3 r}-\frac{1}{\left(\lambda_1^2-\lambda_2^2\right)\left(\lambda_1^2-\lambda_3^2\right)}\left[\frac{\beta(\lambda+\mu)}{i\omega\rho_f}\lambda_1^2+\frac{i\omega(\lambda+\mu)}{Q}+i\omega(\alpha+\beta)^2\right]R_{ij}^{(1)}e^{-\lambda_1 r}\\ \qquad -\frac{1}{\left(\lambda_2^2-\lambda_3^2\right)\left(\lambda_2^2-\lambda_1^2\right)}\left[\frac{\beta(\lambda+\mu)}{i\omega\rho_f}\lambda_2^2+\frac{i\omega(\lambda+\mu)}{Q}+i\omega(\alpha+\beta)^2\right]R_{ij}^{(2)}e^{-\lambda_2 r}\\ \qquad -\frac{1}{\left(\lambda_3^2-\lambda_1^2\right)\left(\lambda_3^2-\lambda_2^2\right)}\left[\frac{\beta(\lambda+\mu)}{i\omega\rho_f}\lambda_3^2+\frac{i\omega(\lambda+\mu)}{Q}+i\omega(\alpha+\beta)^2\right]R_{ij}^{(3)}e^{-\lambda_3 r}\end{array}\right\}$$
$$=\frac{1}{4\pi\mu r}\left\{\begin{array}{l}e^{-\lambda_3 r}\delta_{ij}-\frac{1}{\left(\lambda_1^2-\lambda_2^2\right)\left(\lambda_1^2-\lambda_3^2\right)}\underbrace{\frac{i\omega\rho_f}{\beta(\lambda+2\mu)}\left[\frac{\beta(\lambda+\mu)}{i\omega\rho_f}\lambda_1^2+\frac{i\omega(\lambda+\mu)}{Q}+i\omega(\alpha+\beta)^2\right]}_{K_1}R_{ij}^{(1)}e^{-\lambda_1 r}\\ \qquad -\frac{1}{\left(\lambda_2^2-\lambda_3^2\right)\left(\lambda_2^2-\lambda_1^2\right)}\underbrace{\frac{i\omega\rho_f}{\beta(\lambda+2\mu)}\left[\frac{\beta(\lambda+\mu)}{i\omega\rho_f}\lambda_2^2+\frac{i\omega(\lambda+\mu)}{Q}+i\omega(\alpha+\beta)^2\right]}_{K_2}R_{ij}^{(2)}e^{-\lambda_2 r}\\ \qquad -\frac{1}{\left(\lambda_3^2-\lambda_1^2\right)\left(\lambda_3^2-\lambda_2^2\right)}\underbrace{\frac{i\omega\rho_f}{\beta(\lambda+2\mu)}\left[\frac{\beta(\lambda+\mu)}{i\omega\rho_f}\lambda_3^2+\frac{i\omega(\lambda+\mu)}{Q}+i\omega(\alpha+\beta)^2\right]}_{K_3}R_{ij}^{(3)}e^{-\lambda_3 r}\end{array}\right\} \tag{5.50}$$

The above expression for $G_{ij} = u^*_{ij}$ can be shortened by algebraic manipulations:

Denote → $\lambda_4^2 = -\dfrac{\omega^2(\rho + \beta\rho_f)}{\lambda + 2\mu}$ .... (5.51)

and recall $\lambda_1^2 + \lambda_2^2 = \dfrac{\omega^2\rho_f}{\beta Q} \underbrace{- \dfrac{\omega^2(\rho + \beta\rho_f)}{\lambda + 2\mu}}_{\lambda_4^2} + \dfrac{\omega^2\rho_f(\alpha+\beta)^2}{\beta(\lambda + 2\mu)}$

also noting that, $\dfrac{\omega^2\rho_f}{\beta Q} = \dfrac{\lambda_1^2\lambda_2^2}{\lambda_4^2}$ and $\dfrac{\lambda + \mu}{\lambda + 2\mu} = 1 - \dfrac{\lambda_4^2}{\lambda_3^2}$

$$
\begin{aligned}
K_1 &= \frac{i\omega\rho_f}{\beta(\lambda + 2\mu)}\left[\frac{\beta(\lambda+\mu)}{i\omega\rho_f}\lambda_1^2 + \frac{i\omega(\lambda+\mu)}{Q} + i\omega(\alpha+\beta)^2\right] \\
&= \frac{(\lambda+\mu)}{(\lambda+2\mu)}\lambda_1^2 - \frac{\omega^2\rho_f}{\beta Q}\frac{(\lambda+\mu)}{(\lambda+2\mu)} \underbrace{- \frac{\omega^2\rho_f(\alpha+\beta)^2}{\beta(\lambda+2\mu)}}_{\frac{\omega^2\rho_f}{\beta Q} + \lambda_4^2 - (\lambda_1^2 + \lambda_2^2)} \\
&= \left(1 - \frac{\lambda_4^2}{\lambda_3^2}\right)\lambda_1^2 - \frac{\lambda_1^2\lambda_2^2}{\lambda_4^2}\left(1 - \frac{\lambda_4^2}{\lambda_3^2}\right) + \frac{\lambda_1^2\lambda_2^2}{\lambda_4^2} + \lambda_4^2 - \lambda_1^2 - \lambda_2^2 \\
&= -\frac{1}{\lambda_3^2}\left(\lambda_4^2 - \lambda_2^2\right)\left(\lambda_1^2 - \lambda_3^2\right)
\end{aligned}
$$

Similarly,

$$K_2 = -\frac{1}{\lambda_3^2}\left(\lambda_4^2 - \lambda_1^2\right)\left(\lambda_2^2 - \lambda_3^2\right)$$

$$K_3 = \frac{1}{\lambda_3^2}\left(\lambda_3^2 - \lambda_1^2\right)\left(\lambda_3^2 - \lambda_2^2\right)$$

Hence,

$$u^*_{ij} = \frac{1}{4\pi\mu\lambda_3^2 r}\left[\frac{\lambda_4^2 - \lambda_2^2}{\lambda_1^2 - \lambda_2^2} R^{(1)}_{ij} e^{-\lambda_1 r} + \frac{\lambda_4^2 - \lambda_1^2}{\lambda_2^2 - \lambda_1^2} R^{(2)}_{ij} e^{-\lambda_2 r} + \left(\lambda_3^2\delta_{ij} - R^{(3)}_{ij}\right)e^{-\lambda_3 r}\right] \quad \text{.................... (5.52)}$$

or

$$u_{ij}^{*} = -\frac{1}{4\pi\omega^2(\rho+\beta\rho_f)r}\left[\frac{\lambda_4^2-\lambda_2^2}{\lambda_1^2-\lambda_2^2}R_{ij}^{(1)}e^{-\lambda_1 r} - \frac{\lambda_4^2-\lambda_1^2}{\lambda_1^2-\lambda_2^2}R_{ij}^{(2)}e^{-\lambda_2 r} + \left(\lambda_3^2\delta_{ij} - R_{ij}^{(3)}\right)e^{-\lambda_3 r}\right] \text{ .... (5.53)}$$

The terms u_i^* *: Solid Displacement components at the field point "P" due to unit fluid injection rate at the source point "A":*

$$G_{i4} = u_i^* = \frac{i\omega\rho_f}{\beta(\lambda+2\mu)\mu}\left[-AC\partial_i\right]\Psi \text{ ………………...................................... (5.54)}$$

Recall → $A = \mu\nabla^2 + \omega^2(\rho+\beta\rho_f) = \mu\left(\nabla^2 - \lambda_3^2\right)$
$C = \alpha + \beta$

Then,

$$G_{i4} = -\frac{i\omega\rho_f(\alpha+\beta)}{\beta(\lambda+2\mu)}\partial_i\left[\nabla^2\psi - \lambda_3^2\psi\right]$$

$$= -\frac{i\omega\rho_f(\alpha+\beta)}{\beta(\lambda+2\mu)}\partial_i\left[\frac{1}{4\pi r}\left(\frac{\lambda_1^2 e^{-\lambda_1 r}}{(\lambda_1^2-\lambda_2^2)(\lambda_1^2-\lambda_3^2)} + \frac{\lambda_2^2 e^{-\lambda_2 r}}{(\lambda_2^2-\lambda_3^2)(\lambda_2^2-\lambda_1^2)} + \frac{\lambda_3^2 e^{-\lambda_3 r}}{(\lambda_3^2-\lambda_1^2)(\lambda_3^2-\lambda_2^2)} - \frac{\lambda_3^2 e^{-\lambda_1 r}}{(\lambda_1^2-\lambda_2^2)(\lambda_1^2-\lambda_3^2)} - \frac{\lambda_3^2 e^{-\lambda_2 r}}{(\lambda_2^2-\lambda_3^2)(\lambda_2^2-\lambda_1^2)} - \frac{\lambda_3^2 e^{-\lambda_3 r}}{(\lambda_3^2-\lambda_1^2)(\lambda_3^2-\lambda_2^2)}\right)\right]$$

.... (5.55)

upon simplification

$$G_{i4} = -\frac{i\omega\rho_f(\alpha+\beta)}{\beta(\lambda+2\mu)(\lambda_1^2-\lambda_2^2)}\partial_i\left[\frac{1}{4\pi r}\left(e^{-\lambda_1 r} - e^{-\lambda_2 r}\right)\right] \text{ .... (5.56)}$$

carrying out the differentiation,

$$G_{i4} = -\frac{i\omega\rho_f(\alpha+\beta)}{\beta(\lambda+2\mu)(\lambda_1^2-\lambda_2^2)}\left[-\frac{r_i}{4\pi r^2}\left(e^{-\lambda_1 r} - e^{-\lambda_2 r}\right) + \frac{1}{4\pi r}\left(-\lambda_1 e^{-\lambda_1 r}r_i + \lambda_2 e^{-\lambda_2 r}r_i\right)\right] \text{ ... (5.57)}$$

finally,

$$G_{i4} = u_i^* = \frac{i\omega\rho_f(\alpha+\beta)}{4\pi\beta(\lambda+2\mu)(\lambda_1^2-\lambda_2^2)}\left[\left(\lambda_1+\frac{1}{r}\right)e^{-\lambda_1 r} - \left(\lambda_2+\frac{1}{r}\right)e^{-\lambda_2 r}\right]\frac{r_i}{r} \text{ .... (5.58)}$$

The terms p_j^* *: Fluid Pressure at the field point "P" due to unit load in j-direction at the source point "A":*

$$G_{4j} = p_j^* = \frac{i\omega\rho_f}{\beta(\lambda+2\mu)\mu}\left[-i\omega AC\partial_j\right]\Psi \text{ .... (5.59)}$$

note that,

$$G_{4j} = p_j^* = \frac{i\omega\rho_f}{\beta(\lambda+2\mu)\mu}\left[-i\omega AC\partial_j\right]\Psi = i\omega u_j^* = i\omega G_{j4} \text{ .... (5.60)}$$

hence,

$$G_{4j} = p_j^* = -\frac{\omega^2\rho_f(\alpha+\beta)}{4\pi\beta(\lambda+2\mu)(\lambda_1^2-\lambda_2^2)}\left[\left(\lambda_1+\frac{1}{r}\right)e^{-\lambda_1 r} - \left(\lambda_2+\frac{1}{r}\right)e^{-\lambda_2 r}\right]\frac{r_j}{r} \text{ .... (5.61)}$$

The term p^* *: Fluid Pressure at the field point "P" due to unit fluid injection at the source point "A":*

$$G_{44} = p^* = -\frac{i\omega\rho_f}{\beta(\lambda+2\mu)\mu}\left[A^2 + AB\nabla^2\right]\Psi \text{ ……………................................... (5.62)}$$

where,

$$A^2 = \left[\mu\nabla^2 + \omega^2(\rho+\beta\rho_f)\right]^2 = \mu^2\nabla^4 + 2\mu\omega^2(\rho+\beta\rho_f)\nabla^2 + \omega^4(\rho+\beta\rho_f)^2 \text{ ... (*)}$$

$$AB\nabla^2 = \left[\mu\nabla^2 + \omega^2(\rho + \beta\rho_f)\right](\lambda + \mu)\nabla^2 = \mu(\lambda + \mu)\nabla^4 + (\lambda + \mu)\omega^2(\rho + \beta\rho_f)\nabla^2 \quad \text{..........} (**)$$

inserting (*) and (**) in G_{44},

$$G_{44} = -\frac{i\omega\rho_f}{\beta(\lambda + 2\mu)\mu}\left[\mu(\lambda + 2\mu)\nabla^4 + \omega^2(\rho + \beta\rho_f)(\lambda + 3\mu)\nabla^2 + \omega^4(\rho + \beta\rho_f)^2\right]\Psi \quad \text{..................} (5.63)$$

where, using the identities (5.41) and (5.43)

$$\nabla^4\Psi = \frac{1}{4\pi r}\left[\lambda_1^4\frac{e^{-\lambda_1 r}}{(\lambda_1^2 - \lambda_2^2)(\lambda_1^2 - \lambda_3^2)} + \lambda_2^4\frac{e^{-\lambda_2 r}}{(\lambda_2^2 - \lambda_3^2)(\lambda_2^2 - \lambda_1^2)} + \lambda_3^4\frac{e^{-\lambda_3 r}}{(\lambda_3^2 - \lambda_1^2)(\lambda_3^2 - \lambda_2^2)}\right]$$

$$\nabla^2\Psi = \frac{1}{4\pi r}\left[\lambda_1^2\frac{e^{-\lambda_1 r}}{(\lambda_1^2 - \lambda_2^2)(\lambda_1^2 - \lambda_3^2)} + \lambda_2^2\frac{e^{-\lambda_2 r}}{(\lambda_2^2 - \lambda_3^2)(\lambda_2^2 - \lambda_1^2)} + \lambda_3^2\frac{e^{-\lambda_3 r}}{(\lambda_3^2 - \lambda_1^2)(\lambda_3^2 - \lambda_2^2)}\right]$$

hence,

$$G_{44} = -\frac{i\omega\rho_f}{\beta(\lambda + 2\mu)\mu}\frac{1}{4\pi r}\left[\begin{array}{l} \underbrace{\frac{\left(\lambda_1^4\mu(\lambda + 2\mu) + \lambda_1^2\omega^2(\rho + \beta\rho_f)(\lambda + 3\mu)\nabla^2 + \omega^4(\rho + \beta\rho_f)^2\right)}{(\lambda_1^2 - \lambda_2^2)(\lambda_1^2 - \lambda_3^2)}}_{T_1} e^{-\lambda_1 r} + \\ \underbrace{\frac{\left(\lambda_2^4\mu(\lambda + 2\mu) + \lambda_2^2\omega^2(\rho + \beta\rho_f)(\lambda + 3\mu)\nabla^2 + \omega^4(\rho + \beta\rho_f)^2\right)}{(\lambda_2^2 - \lambda_3^2)(\lambda_2^2 - \lambda_1^2)}}_{T_2} e^{-\lambda_2 r} + \\ \underbrace{\frac{\left(\lambda_3^4\mu(\lambda + 2\mu) + \lambda_3^2\omega^2(\rho + \beta\rho_f)(\lambda + 3\mu)\nabla^2 + \omega^4(\rho + \beta\rho_f)^2\right)}{(\lambda_3^2 - \lambda_1^2)(\lambda_3^2 - \lambda_2^2)}}_{T_3} e^{-\lambda_3 r} \end{array}\right]$$

.... (5.64)

The expressions T_1, T_2, T_3 can be simplified by algebraic manipulations as follows,

$$T_1 = \frac{\left(\lambda_1^4\mu(\lambda+2\mu)+\lambda_1^2\omega^2(\rho+\beta\rho_f)(\lambda+3\mu)\nabla^2+\omega^4(\rho+\beta\rho_f)^2\right)}{\left(\lambda_1^2-\lambda_2^2\right)\left(\lambda_1^2-\lambda_3^2\right)}$$

$$= \frac{\mu(\lambda+2\mu)}{\left(\lambda_1^2-\lambda_2^2\right)\left(\lambda_1^2-\lambda_3^2\right)}\left[\lambda_1^4+\lambda_1^2\underbrace{\frac{\omega^2(\rho+\beta\rho_f)}{\mu}}_{-\lambda_3^2}\underbrace{\frac{(\lambda+3\mu)}{(\lambda+2\mu)}}_{1+\frac{\lambda_4^2}{\lambda_3^2}}\nabla^2+\underbrace{\frac{\omega^2(\rho+\beta\rho_f)}{\mu}}_{-\lambda_3^2}\underbrace{\frac{\omega^2(\rho+\beta\rho_f)}{(\lambda+2\mu)}}_{-\lambda_4^2}\right]$$

$$= \frac{\mu(\lambda+2\mu)}{\left(\lambda_1^2-\lambda_2^2\right)\left(\lambda_1^2-\lambda_3^2\right)}\left[\underbrace{\lambda_1^4-\lambda_1^2\lambda_3^2-\lambda_1^2\lambda_4^2+\lambda_3^2\lambda_4^2}_{\left(\lambda_1^2-\lambda_3^2\right)\left(\lambda_1^2-\lambda_4^2\right)}\right]$$

$$= \mu(\lambda+2\mu)\frac{\left(\lambda_1^2-\lambda_4^2\right)}{\left(\lambda_1^2-\lambda_2^2\right)}$$

Similarly,

$$T_2 = \frac{\left(\lambda_2^4\mu(\lambda+2\mu)+\lambda_2^2\omega^2(\rho+\beta\rho_f)(\lambda+3\mu)\nabla^2+\omega^4(\rho+\beta\rho_f)^2\right)}{\left(\lambda_2^2-\lambda_3^2\right)\left(\lambda_2^2-\lambda_1^2\right)}$$

$$= \mu(\lambda+2\mu)\frac{\left(\lambda_2^2-\lambda_4^2\right)}{\left(\lambda_1^2-\lambda_2^2\right)}$$

$$T_3 = \frac{\left(\lambda_3^4\mu(\lambda+2\mu)+\lambda_3^2\omega^2(\rho+\beta\rho_f)(\lambda+3\mu)\nabla^2+\omega^4(\rho+\beta\rho_f)^2\right)}{\left(\lambda_3^2-\lambda_1^2\right)\left(\lambda_3^2-\lambda_2^2\right)}$$

$$= 0$$

Since, $G_{44} = p^*$ is the solution for the fluid part, which cannot sustain shear waves. As a result, we have

$$G_{44} = p^* = -\frac{i\omega\rho_f}{4\pi\beta\left(\lambda_1^2-\lambda_2^2\right)r}\left[\left(\lambda_1^2-\lambda_4^2\right)e^{-\lambda_1 r}-\left(\lambda_2^2-\lambda_4^2\right)e^{-\lambda_2 r}\right] \quad \text{.......................... (5.65)}$$

5.2.2 Second Fundamental Solutions (FS's for Tractions and Net Flux)

The terms t_{ij}^* $\rightarrow$ *FS for tractions due to unit force at the source pt. "A" in "j" direction:*

First we recall Cauchy's Stress Formula (2.22) for the traction vector,

$$t_i^l = \tau_{ji}^l n_j$$

where,

$$\tau_{ji}^l = 2\mu\varepsilon_{ji}^l + \lambda\delta_{ji}\varepsilon_{kk}^l - \alpha p^l \delta_{ji}$$

or, noting that, $\varepsilon_{ji}^l = \frac{1}{2}\left(u_{j,i}^l + u_{i,j}^l\right)$

$$t_i^l = \left[\mu\left(u_{i,k}^l + u_{k,i}^l\right) + \lambda\delta_{ik}u_{s,s}^l - \alpha p^l \delta_{ik}\right]n_k \quad \text{...} \quad (5.66)$$

Hence, replacing the index "l" by "j"

$$t_{ij}^* = \left[\mu\left(u_{ij,k}^* + u_{kj,i}^*\right) + \lambda\delta_{ik}u_{sj,s}^* - \alpha p_j^* \delta_{ik}\right]n_k \quad \text{...} \quad (5.67)$$

Recall → (5.52)

$$u_{ij}^* = \frac{1}{4\pi\mu\lambda_3^2 r}\left[\frac{\lambda_4^2 - \lambda_2^2}{\lambda_1^2 - \lambda_2^2} R_{ij}^{(1)} e^{-\lambda_1 r} + \frac{\lambda_4^2 - \lambda_1^2}{\lambda_2^2 - \lambda_1^2} R_{ij}^{(2)} e^{-\lambda_2 r} + \left(\lambda_3^2 \delta_{ij} - R_{ij}^{(3)}\right) e^{-\lambda_3 r}\right]$$

To simplify the notation, we denote

$$A_1 = \frac{\lambda_4^2 - \lambda_2^2}{\lambda_1^2 - \lambda_2^2} \qquad \text{and} \qquad A_2 = \frac{\lambda_4^2 - \lambda_1^2}{\lambda_2^2 - \lambda_1^2}$$

Then,

$$u^*_{ij,l} = \frac{1}{4\pi\mu\lambda_3^2}\left[\begin{array}{l} -\frac{r_l}{r^2}\left[A_1R^{(1)}_{ij}e^{-\lambda_1 r} + A_2R^{(2)}_{ij}e^{-\lambda_2 r} + \left(\lambda_3^2\delta_{ij} - R^{(3)}_{ij}\right)e^{-\lambda_3 r}\right] + \\ \qquad \frac{1}{r}\left[\begin{array}{r} A_1R^{(1)}_{ij,l}e^{-\lambda_1 r} - \lambda_1 A_1 R^{(1)}_{ij}e^{-\lambda_1 r}r_l + A_2R^{(2)}_{ij,l}e^{-\lambda_2 r} - \lambda_2 A_2 R^{(2)}_{ij}e^{-\lambda_2 r}r_l \\ - R^{(3)}_{ij,l}e^{-\lambda_3 r} - \lambda_3\left(\lambda_3^2\delta_{ij} - R^{(3)}_{ij}\right)e^{-\lambda_3 r}r_l \end{array}\right] \end{array}\right]$$

.... (5.68)

$$\begin{aligned} R^{(\lambda_k)}_{ij,l} &= \frac{\partial}{\partial x_l}\left\{\frac{1}{r^2}(3r_ir_j - \delta_{ij}) + \frac{\lambda_k}{r}(3r_ir_j - \delta_{ij}) + \lambda_k^2 r_ir_j\right\} \\ &= -\frac{2r_l}{r^3}(3r_ir_j - \delta_{ij}) + \frac{1}{r^2}3(r_ir_j)_{,l} - \frac{\lambda_k r_l}{r^2}(3r_ir_j - \delta_{ij}) + \frac{\lambda_k}{r}3(r_ir_j)_{,l} + \lambda_k^2(r_ir_j)_{,l} \end{aligned}$$

.... (5.69)

$$\begin{aligned} (r_ir_j)_{,l} &= r_{i,l}r_j + r_ir_{j,l} \\ &= \frac{1}{r}(\delta_{il} - r_ir_l)r_j + \frac{1}{r}(\delta_{jl} - r_jr_l)r_i \\ &= \frac{1}{r}(\delta_{il}r_j + \delta_{jl}r_i - 2r_ir_jr_l) \end{aligned}$$

.... (5.70)

Therefore,

$$\begin{aligned} R^{(\lambda_k)}_{ij,l} &= \frac{1}{r^3}(2\delta_{ij}r_l - 6r_ir_jr_l) + \frac{1}{r^3}(3\delta_{il}r_j + 3\delta_{jl}r_i - 6r_ir_jr_l) + \frac{\lambda_k}{r^2}(2\delta_{ij}r_l - 3r_ir_jr_l) \\ &\qquad + \frac{\lambda_k}{r^2}(3\delta_{il}r_j + 3\delta_{jl}r_i - 6r_ir_jr_l) + \frac{\lambda_k^2}{r}(\delta_{il}r_j + \delta_{jl}r_i - 2r_ir_jr_l) \\ &= \frac{1}{r^3}(2\delta_{ij}r_l + 3\delta_{jl}r_i + 3\delta_{il}r_j - 12r_ir_jr_l) + \frac{\lambda_k}{r^2}(\delta_{ij}r_l + 3\delta_{jl}r_i + 3\delta_{il}r_j - 9r_ir_jr_l) \\ &\qquad + \frac{\lambda_k^2}{r}(3\delta_{jl}r_i + 3\delta_{il}r_j - 2r_ir_jr_l) \end{aligned}$$

.... (5.71)

and,

$$\begin{aligned} \lambda_k R^{(\lambda_k)}_{ij}r_l &= \frac{\lambda_k}{r^2}(3r_ir_jr_l - \delta_{ij}r_l) + \frac{\lambda_k^2}{r}(3r_ir_jr_l - \delta_{ij}r_l) + \lambda_k^3 r_ir_jr_l \\ R^{(\lambda_k)}_{ij}\frac{r_l}{r} &= \frac{1}{r^3}(3r_ir_jr_l - \delta_{ij}r_l) + \frac{\lambda_k}{r^2}(3r_ir_jr_l - \delta_{ij}r_l) + \frac{\lambda_k^2}{r}r_ir_jr_l \end{aligned}$$

.... (5.72)

Then,

$$u^*_{ij,l} = \frac{1}{4\pi\mu\lambda_3^2 r}\left[\underbrace{\left(-R^{(1)}_{ij}\frac{r_l}{r} + R^{(1)}_{ij,l} - \lambda_1 R^{(1)}_{ij} r_l\right)}_{I^{(1)}_{ijl}} A_1 e^{-\lambda_1 r} + \underbrace{\left(-R^{(2)}_{ij}\frac{r_l}{r} + R^{(2)}_{ij,l} - \lambda_2 R^{(2)}_{ij} r_l\right)}_{I^{(2)}_{ijl}} A_2 e^{-\lambda_2 r} + \underbrace{\left(-\lambda_3^2\frac{r_l}{r}\delta_{ij} + R^{(3)}_{ij}\frac{r_l}{r} - R^{(3)}_{ij,l} - \lambda_3^3\delta_{ij} r_l + \lambda_3 R^{(3)}_{ij} r_l\right)}_{I^{(3)}_{ijl}} e^{-\lambda_3 r}\right]$$

.... (5.73)

$$I^{(1)}_{ijl} = -R^{(1)}_{ij}\frac{r_l}{r} + R^{(1)}_{ij,l} - \lambda_1 R^{(1)}_{ij} r_l$$
$$= \frac{3}{r^3}\left(\delta_{ij} r_l + \delta_{il} r_j + \delta_{jl} r_i - 5 r_i r_j r_l\right) + \frac{3\lambda_1}{r^2}\left(\delta_{ij} r_l + \delta_{il} r_j + \delta_{jl} r_i - 5 r_i r_j r_l\right) + \frac{\lambda_1^2}{r}\left(\delta_{ij} r_l + \delta_{il} r_j + \delta_{jl} r_i - 6 r_i r_j r_l\right) - \lambda_1^3 r_i r_j r_l$$

.... (5.74)

$$I^{(2)}_{ijl} = -R^{(2)}_{ij}\frac{r_l}{r} + R^{(2)}_{ij,l} - \lambda_2 R^{(2)}_{ij} r_l$$
$$= \frac{3}{r^3}\left(\delta_{ij} r_l + \delta_{il} r_j + \delta_{jl} r_i - 5 r_i r_j r_l\right) + \frac{3\lambda_2}{r^2}\left(\delta_{ij} r_l + \delta_{il} r_j + \delta_{jl} r_i - 5 r_i r_j r_l\right) + \frac{\lambda_2^2}{r}\left(\delta_{ij} r_l + \delta_{il} r_j + \delta_{jl} r_i - 6 r_i r_j r_l\right) - \lambda_2^3 r_i r_j r_l$$

.... (5.75)

$$I^{(3)}_{ijl} = -\lambda_3^2\frac{r_l}{r}\delta_{ij} + R^{(3)}_{ij}\frac{r_l}{r} - R^{(3)}_{ij,l} - \lambda_3^3\delta_{ij} r_l + \lambda_3 R^{(3)}_{ij} r_l$$
$$= \frac{3}{r^3}\left(5 r_i r_j r_l - \delta_{ij} r_l - \delta_{il} r_j - \delta_{jl} r_i\right) + \frac{3\lambda_3}{r^2}\left(5 r_i r_j r_l - \delta_{ij} r_l - \delta_{il} r_j - \delta_{jl} r_i\right) + \frac{\lambda_3^2}{r}\left(6 r_i r_j r_l - 2\delta_{ij} r_l - \delta_{il} r_j - \delta_{jl} r_i\right) + \lambda_3^3\left(r_i r_j r_l - \delta_{ij} r_l\right)$$

.... (5.76)

therefore,

$$u^*_{ij,l} = \frac{1}{4\pi\mu\lambda_3^2 r}\left[A_1 I^{(1)}_{ijl} e^{-\lambda_1 r} + A_2 I^{(2)}_{ijl} e^{-\lambda_2 r} + I^{(3)}_{ijl} e^{-\lambda_3 r}\right]$$ .... (5.77)

and then,

$$u^*_{sj,s} = \frac{1}{4\pi\mu\lambda_3^2 r}\left[A_1 I^{(1)}_{sjs} e^{-\lambda_1 r} + A_2 I^{(2)}_{sjs} e^{-\lambda_2 r} + I^{(3)}_{sjs} e^{-\lambda_3 r}\right]$$ .... (5.78)

where,

$$I_{sjs}^{(1)} = \frac{3}{r^3}\left(\underbrace{\underbrace{\delta_{sj}r_s}_{r_j} + \underbrace{\delta_{ss}r_j}_{3r_j} + \underbrace{\delta_{js}r_s}_{r_j} - \underbrace{5r_sr_jr_s}_{5r_j}}_{=0}\right) + \frac{3\lambda_1}{r^2}\left(\underbrace{\delta_{sj}r_s + \delta_{ss}r_j + \delta_{js}r_s - 5r_sr_jr_s}_{=0}\right) + \frac{\lambda_1^2}{r}\left(\underbrace{\delta_{sj}r_s + \delta_{ss}r_j + \delta_{js}r_s - 6r_sr_jr_s}_{-r_j}\right) - \lambda_1^3\underbrace{r_sr_jr_s}_{r_j}$$

$$= -\lambda_1^2\left(\lambda_1 + \frac{1}{r}\right)r_j \quad \text{.. (5.79)}$$

similarly,

$$I_{sjs}^{(2)} = -\lambda_2^2\left(\lambda_2 + \frac{1}{r}\right)r_j. \quad \text{.................................... (5.80)}$$

$$I_{sjs}^{(3)} = 0. \quad \text{.................................... (5.81)}$$

Therefore,

$$u_{sj,s}^{*} = \frac{-r_j}{4\pi\mu\lambda_3^2 r}\left[A_1\lambda_1^2\left(\lambda_1 + \frac{1}{r}\right)e^{-\lambda_1 r} + A_2\lambda_2^2\left(\lambda_2 + \frac{1}{r}\right)e^{-\lambda_2 r}\right] \quad \text{.. (5.82)}$$

$$\left(u_{ij,l}^{*} + u_{lj,i}^{*}\right) = \frac{1}{4\pi\mu\lambda_3^2 r}\left[A_1\left(I_{ijl}^{(1)} + I_{lji}^{(1)}\right)e^{-\lambda_1 r} + A_2\left(I_{ijl}^{(2)} + I_{lji}^{(2)}\right)e^{-\lambda_2 r} + \left(I_{ijl}^{(3)} + I_{lji}^{(3)}\right)e^{-\lambda_3 r}\right] \quad \text{............. (5.83)}$$

evaluate → $\left(I_{ijl}^{(1)} + I_{lji}^{(1)}\right)$, $\left(I_{ijl}^{(2)} + I_{lji}^{(2)}\right)$ and $\left(I_{ijl}^{(3)} + I_{lji}^{(3)}\right)$

$I_{lji}^{(1)} = I_{ijl}^{(1)}$, symmetric with respect to the indices “i” and “l”

$I_{lji}^{(2)} = I_{ijl}^{(2)}$, symmetric with respect to the indices “i” and “l”

$$\left(I_{ijl}^{(1)} + I_{lji}^{(1)}\right) = 2 * I_{ijl}^{(1)} \quad \text{.................................... (5.84)}$$

$$\left(I_{ijl}^{(2)} + I_{lji}^{(2)}\right) = 2 * I_{ijl}^{(2)} \quad \text{.................................... (5.85)}$$

$I_{lji}^{(3)} \neq I_{ijl}^{(3)}$ **not** symmetric with respect to the indices “i” and “l”

$$\left(I^{(3)}_{lji} + I^{(3)}_{ijl}\right) = \frac{6}{r^3}\left(5r_ir_jr_l - \delta_{ij}r_l - \delta_{il}r_j - \delta_{jl}r_i\right) + \frac{6\lambda_3}{r^2}\left(5r_ir_jr_l - \delta_{ij}r_l - \delta_{il}r_j - \delta_{jl}r_i\right)$$
$$+\frac{\lambda_3^2}{r}\left(12r_ir_jr_l - 3\delta_{ij}r_l - 2\delta_{il}r_j - 3\delta_{jl}r_i\right) + \lambda_3^3\left(2r_ir_jr_l - \delta_{ij}r_l - \delta_{jl}r_i\right)$$
.... (5.86)

$$t^*_{ij} = \mu\left(u^*_{kj,i} + u^*_{ij,k}\right)n_k + \lambda u^*_{sj,s}\delta_{ki}n_k - \alpha\delta_{ki}p^*_jn_k \quad \text{... (5.87)}$$

$$\left(u^*_{kj,i} + u^*_{ij,k}\right)n_k = \frac{1}{4\pi\mu\lambda_3^2 r}\left[A_1\left(I^{(1)}_{kji} + I^{(1)}_{ijk}\right)n_k e^{-\lambda_1 r} + A_2\left(I^{(2)}_{kji} + I^{(2)}_{ijk}\right)n_k e^{-\lambda_2 r} + \left(I^{(3)}_{kji} + I^{(3)}_{ijk}\right)n_k e^{-\lambda_3 r}\right]$$

$$\left(I^{(1)}_{kji} + I^{(1)}_{ijk}\right)n_k = \frac{6}{r^3}\left(r_in_j + n_ir_j + \frac{\partial r}{\partial n}\left(\delta_{ij} - 5r_ir_j\right)\right) + \frac{6\lambda_1}{r^2}\left(r_in_j + n_ir_j + \frac{\partial r}{\partial n}\left(\delta_{ij} - 5r_ir_j\right)\right)$$
$$+\frac{2\lambda_1^2}{r}\left(r_in_j + n_ir_j + \frac{\partial r}{\partial n}\left(\delta_{ij} - 6r_ir_j\right)\right) - 2\lambda_1^3 r_ir_j\frac{\partial r}{\partial n}$$

Denote, $R^{(5)}_{ij} = r_in_j + n_ir_j + \frac{\partial r}{\partial n}\left(\delta_{ij} - 5r_ir_j\right)$ and $R^{(6)}_{ij} = r_in_j + n_ir_j + \frac{\partial r}{\partial n}\left(\delta_{ij} - 6r_ir_j\right)$ ...... (5.88)

$$\left(I^{(1)}_{kji} + I^{(1)}_{ijk}\right)n_k = \frac{6}{r^3}R^{(5)}_{ij} + \frac{6\lambda_1}{r^2}R^{(5)}_{ij} + \frac{2\lambda_1^2}{r}R^{(6)}_{ij} - 2\lambda_1^3 r_ir_j\frac{\partial r}{\partial n}$$

$$\left(I^{(2)}_{kji} + I^{(2)}_{ijk}\right)n_k = \frac{6}{r^3}R^{(5)}_{ij} + \frac{6\lambda_2}{r^2}R^{(5)}_{ij} + \frac{2\lambda_2^2}{r}R^{(6)}_{ij} - 2\lambda_2^3 r_ir_j\frac{\partial r}{\partial n}$$

$$\left(I^{(3)}_{kji} + I^{(3)}_{ijk}\right)n_k = -\frac{6}{r^3}R^{(5)}_{ij} - \frac{6\lambda_3}{r^2}R^{(5)}_{ij} - \frac{\lambda_3^2}{r}\left(3\frac{\partial r}{\partial n}\left(\delta_{ij} - 4r_ir_j\right) + 2n_ir_j + 3n_jr_i\right) - \lambda_3^3\left(\frac{\partial r}{\partial n}\left(\delta_{ij} - 2r_ir_j\right) + r_in_j\right)$$

$$u^*_{sj,s}\underbrace{\delta_{ki}n_k}_{n_i} = \frac{1}{4\pi\mu\lambda_3^2 r}\left[A_1 I^{(1)}_{sjs}n_i e^{-\lambda_1 r} + A_2 I^{(2)}_{sjs}n_i e^{-\lambda_2 r} + \underbrace{I^{(3)}_{sjs}}_{=0}n_i e^{-\lambda_3 r}\right]$$

$$I^{(1)}_{sjs}n_i = -\lambda_1^2\left(\lambda_1 + \frac{1}{r}\right)r_jn_i$$

$$I^{(2)}_{sjs}n_i = -\lambda_2^2\left(\lambda_2 + \frac{1}{r}\right)r_jn_i$$

Finally,

$$t_{ij}^{*} = \mu \frac{1}{4\pi\mu\lambda_3^2 r}\left[A_1\left(\frac{6}{r^3}R_{ij}^{(5)} + \frac{6\lambda_1}{r^2}R_{ij}^{(5)} + \frac{2\lambda_1^2}{r}R_{ij}^{(6)} - 2\lambda_1^3 r_i r_j \frac{\partial r}{\partial n}\right)e^{-\lambda_1 r} + A_2\left(\frac{6}{r^3}R_{ij}^{(5)} + \frac{6\lambda_2}{r^2}R_{ij}^{(5)} + \frac{2\lambda_2^2}{r}R_{ij}^{(6)} - 2\lambda_2^3 r_i r_j \frac{\partial r}{\partial n}\right)e^{-\lambda_2 r} \right.$$

$$\left. + e^{-\lambda_3 r}\left(-\frac{6}{r^3}R_{ij}^{(5)} - \frac{6\lambda_3}{r^2}R_{ij}^{(5)} - \frac{\lambda_3^2}{r}\left(3\frac{\partial r}{\partial n}\left(\delta_{ij} - 4r_i r_j\right) + 2n_i r_j + 3n_j r_i\right) - \lambda_3^3\left(\frac{\partial r}{\partial n}\left(\delta_{ij} - 2r_i r_j\right) + r_i n_j\right)\right)\right] \quad \text{.. (5.89)}$$

$$+ \lambda \frac{1}{4\pi\mu\lambda_3^2 r}\left[-A_1\lambda_1^2\left(\lambda_1 + \frac{1}{r}\right)r_j n_i e^{-\lambda_1 r} - A_2\lambda_2^2\left(\lambda_2 + \frac{1}{r}\right)r_j n_i e^{-\lambda_2 r}\right] + \alpha\frac{\omega^2\rho_f(\alpha+\beta)}{4\pi\beta r(\lambda+2\mu)\left(\lambda_1^2 - \lambda_2^2\right)}\left[\left(\lambda_1 + \frac{1}{r}\right)e^{-\lambda_1 r} - \left(\lambda_2 + \frac{1}{r}\right)e^{-\lambda_2 r}\right]r_j n_i$$

or,

$$t_{ij}^{*} = \frac{1}{4\pi\lambda_3^2 r}\left[A_1 e^{-\lambda_1 r}\left(\frac{2R_{ij}^{(6)}}{r}\lambda_1^2 - 2r_i r_j\frac{\partial r}{\partial n}\lambda_1^3 + \left(\lambda_1 + \frac{1}{r}\right)\left(\frac{6R_{ij}^{(5)}}{r^2} - \lambda_1^2\frac{\lambda}{\mu}r_j n_i + \alpha\frac{\omega^2\rho_f(\alpha+\beta)\lambda_3^2}{\beta(\lambda+2\mu)\left(\lambda_4^2 - \lambda_2^2\right)}r_j n_i\right)\right)\right.$$

$$+ A_2 e^{-\lambda_2 r}\left(\frac{2R_{ij}^{(6)}}{r}\lambda_2^2 - 2r_i r_j\frac{\partial r}{\partial n}\lambda_2^3 + \left(\lambda_2 + \frac{1}{r}\right)\left(\frac{6R_{ij}^{(5)}}{r^2} - \lambda_2^2\frac{\lambda}{\mu}r_j n_i + \alpha\frac{\omega^2\rho_f(\alpha+\beta)\lambda_3^2}{\beta(\lambda+2\mu)\left(\lambda_4^2 - \lambda_1^2\right)}r_j n_i\right)\right)$$

$$\left. - e^{-\lambda_3 r}\left(\frac{2R_{ij}^{(6)}}{r}\lambda_3^2 - 2r_i r_j\frac{\partial r}{\partial n}\lambda_3^3 + \left(\lambda_3 + \frac{1}{r}\right)\left(\frac{6R_{ij}^{(5)}}{r^2} + \lambda_3^2\left(r_i n_j + \delta_{ij}\frac{\partial r}{\partial n}\right)\right)\right)\right]$$

……………………..………………………………………… (5.90)

The terms q_n^{*j} → FS for normal fluid flux due to unit force at the source pt. "A" in "j" direction:

$$q_n^{*j}(A,P) = q_i^{*j} n_i \quad \text{......} (5.91)$$

Recall → Equation 2.47 (Darcy's Law in FTS)

$$q_i^{*j} = \frac{\beta}{i\omega\rho_f}\left[p_{,i}^{*j} - \omega^2 \rho_f u_i^{*j}\right] \quad \text{......} (5.92)$$

where,

$$u_{ij}^{*} = \frac{1}{4\pi\mu\lambda_3^2 r}\left[\frac{\lambda_4^2 - \lambda_2^2}{\lambda_1^2 - \lambda_2^2} R_{ij}^{(1)} e^{-\lambda_1 r} + \frac{\lambda_4^2 - \lambda_1^2}{\lambda_2^2 - \lambda_1^2} R_{ij}^{(2)} e^{-\lambda_2 r} + \left(\lambda_3^2 \delta_{ij} - R_{ij}^{(3)}\right) e^{-\lambda_3 r}\right]$$

and

$$p^{*j} = \underbrace{-\frac{\omega^2 \rho_f(\alpha+\beta)}{4\pi\beta(\lambda+2\mu)(\lambda_1^2 - \lambda_2^2)}}_{A_3}\left[\left(\lambda_1 + \frac{1}{r}\right)e^{-\lambda_1 r} - \left(\lambda_2 + \frac{1}{r}\right)e^{-\lambda_2 r}\right]\frac{r_j}{r}$$

Then,

$$p_{,i}^{*j} = A_3\left[-\frac{r_i}{r^2}e^{-\lambda_1 r} - \left(\lambda_1 + \frac{1}{r}\right)\lambda_1 r_i e^{-\lambda_1 r} + \frac{r_i}{r^2}e^{-\lambda_2 r} + \left(\lambda_2 + \frac{1}{r}\right)\lambda_2 r_i e^{-\lambda_2 r}\right]\frac{r_j}{r}$$
$$+ A_3\left[\left(\lambda_1 + \frac{1}{r}\right)e^{-\lambda_1 r} - \left(\lambda_2 + \frac{1}{r}\right)e^{-\lambda_2 r}\right]\left(\frac{r_{j,i}}{r} - \frac{r_j r_i}{r^2}\right)$$

.... (5.93)

Recalling that, $r_{j,i} = \frac{1}{r}\left(\delta_{ji} - r_j r_i\right)$ and collecting terms,

$$p_{,i}^{*j} = \frac{A_3}{r}\left[e^{-\lambda_2 r}\underbrace{\left(\frac{1}{r^2}\left(3r_i r_j - \delta_{ij}\right) + \frac{\lambda_2}{r}\left(3r_i r_j - \delta_{ij}\right) + \lambda_2^2 r_i r_j\right)}_{R_{ij}^{(2)}} - e^{-\lambda_1 r}\underbrace{\left(\frac{1}{r^2}\left(3r_i r_j - \delta_{ij}\right) + \frac{\lambda_1}{r}\left(3r_i r_j - \delta_{ij}\right) + \lambda_1^2 r_i r_j\right)}_{R_{ij}^{(1)}}\right]$$

$$p^{*j}_{,i} = \frac{A_3}{r}\left[R^{(2)}_{ij} e^{-\lambda_2 r} - R^{(1)}_{ij} e^{-\lambda_1 r}\right] \quad \text{.......} \quad (5.94)$$

Inserting in (5.91)

$$q^{*j}_n = \frac{\beta}{i\omega\rho_f}\left[p^{*j}_{,i} - \omega^2 \rho_f u^{*j}_i\right] n_i$$

$$q^{*j}_n = \frac{\beta n_i}{i\omega\rho_f}\left[\frac{A_3}{r}\left[R^{(2)}_{ij} e^{-\lambda_2 r} - R^{(1)}_{ij} e^{-\lambda_1 r}\right] - \frac{\omega^2 \rho_f}{4\pi\mu\lambda_3^2 r}\left[\frac{\lambda_4^2 - \lambda_2^2}{\lambda_1^2 - \lambda_2^2} R^{(1)}_{ij} e^{-\lambda_1 r} + \frac{\lambda_4^2 - \lambda_1^2}{\lambda_2^2 - \lambda_1^2} R^{(2)}_{ij} e^{-\lambda_2 r} + \left(\lambda_3^2 \delta_{ij} - R^{(3)}_{ij}\right) e^{-\lambda_3 r}\right]\right]$$

$$q^{*j}_n = \frac{\beta n_i}{i\omega\rho_f}\left[R^{(1)}_{ij} e^{-\lambda_1 r} \underbrace{\left[-\frac{A_3}{r} - \frac{\omega^2 \rho_f}{4\pi\mu\lambda_3^2 r}\frac{\lambda_4^2 - \lambda_2^2}{\lambda_1^2 - \lambda_2^2}\right]}_{T_1} + R^{(2)}_{ij} e^{-\lambda_2 r} \underbrace{\left[\frac{A_3}{r} - \frac{\omega^2 \rho_f}{4\pi\mu\lambda_3^2 r}\frac{\lambda_4^2 - \lambda_2^2}{\lambda_2^2 - \lambda_1^2}\right]}_{T_2} - \frac{\omega^2 \rho_f}{4\pi\mu\lambda_3^2 r}\left(\lambda_3^2 \delta_{ij} - R^{(3)}_{ij}\right) e^{-\lambda_3 r}\right]$$

.... (5.95)

Recall→ $A_3 = -\dfrac{\omega^2 \rho_f (\alpha + \beta)}{4\pi\beta(\lambda + 2\mu)(\lambda_1^2 - \lambda_2^2)}$

Then,

$$T_1 = \frac{\omega^2 \rho_f (\alpha + \beta)}{4\pi\beta(\lambda + 2\mu)(\lambda_1^2 - \lambda_2^2) r} - \frac{\omega^2 \rho_f}{4\pi\mu\lambda_3^2 r}\frac{\lambda_4^2 - \lambda_2^2}{\lambda_1^2 - \lambda_2^2}$$

$$= \frac{\omega^2 \rho_f}{4\pi\mu\lambda_3^2 (\lambda_1^2 - \lambda_2^2) r}\left[\frac{(\alpha + \beta)}{\beta(\lambda + 2\mu)}\mu\lambda_3^2 - \left(\lambda_4^2 - \lambda_2^2\right)\right]$$

Recall→ $\lambda_3^2 = -\dfrac{\omega^2 (\rho + \beta\rho_f)}{\mu}$

$$T_1 = \frac{\omega^2 \rho_f}{4\pi\mu\lambda_3^2 (\lambda_1^2 - \lambda_2^2) r}\left[-\frac{(\alpha + \beta)}{\beta(\lambda + 2\mu)}\omega^2 (\rho + \beta\rho_f) - \left(\lambda_4^2 - \lambda_2^2\right)\right]$$

Recall→ $\lambda_4^2 = -\dfrac{\omega^2 (\rho + \beta\rho_f)}{\lambda + 2\mu}$

$$T_1 = \frac{\omega^2\rho_f}{4\pi\mu\lambda_3^2(\lambda_1^2-\lambda_2^2)r}\left[\frac{(\alpha+\beta)}{\beta}\lambda_4^2 - (\lambda_4^2-\lambda_2^2)\right]$$

$$T_1 = \frac{\omega^2\rho_f}{4\pi\beta\mu\lambda_3^2(\lambda_1^2-\lambda_2^2)r}\left[\alpha\lambda_4^2+\beta\lambda_2^2\right]$$

Similarly,

$$T_2 = -\frac{\omega^2\rho_f}{4\pi\beta\mu\lambda_3^2(\lambda_1^2-\lambda_2^2)r}\left[\alpha\lambda_4^2+\beta\lambda_1^2\right]$$

Thus,

$$q_n^{*j} = \frac{i\omega n_i}{4\pi\mu\lambda_3^2 r}\left[\frac{\alpha\lambda_4^2+\beta\lambda_2^2}{\lambda_2^2-\lambda_1^2}R_{ij}^{(1)}e^{-\lambda_1 r} + \frac{\alpha\lambda_4^2+\beta\lambda_1^2}{\lambda_1^2-\lambda_2^2}R_{ij}^{(2)}e^{-\lambda_2 r} + \beta\left(\lambda_3^2\delta_{ij}-R_{ij}^{(3)}\right)e^{-\lambda_3 r}\right] \quad \text{......... (5.96)}$$

t_i^* → *FS for tractions due to unit fluid injection rate at the source pt. "A":*

$$t_i^* = \tau_{ji}^* n_j$$

$$\tau_{ji}^* = \mu\left(u_{j,i}^* + u_{i,j}^*\right) + \lambda\delta_{ji}u_{k,k}^* - \alpha\delta_{ji}p^*$$

Recall →
$$u_i^* = \underbrace{\frac{i\omega\rho_f(\alpha+\beta)}{4\pi\beta(\lambda+2\mu)(\lambda_1^2-\lambda_2^2)}}_{A_4}\left[\left(\lambda_1+\frac{1}{r}\right)e^{-\lambda_1 r} - \left(\lambda_2+\frac{1}{r}\right)e^{-\lambda_2 r}\right]\frac{r_i}{r}$$

$$p^* = -\frac{i\omega\rho_f}{4\pi\beta(\lambda_1^2-\lambda_2^2)r}\left[(\lambda_1^2-\lambda_4^2)e^{-\lambda_1 r} - (\lambda_2^2-\lambda_4^2)e^{-\lambda_2 r}\right]$$

and

$$u_{i,j}^* = \frac{A_4}{r}\left[-R_{ij}^{(1)}e^{-\lambda_1 r} + R_{ij}^{(2)}e^{-\lambda_2 r}\right] \quad \text{.. (5.97)}$$

(note, derivation of (5.97) follows the same outline as given for $p_{,i}^{*j}$ previously (5.94))

$$\tau_{ij}^* = \mu\left[\frac{A_4}{r}\left[-R_{ij}^{(1)}e^{-\lambda_1 r} + R_{ij}^{(2)}e^{-\lambda_2 r}\right] + \frac{A_4}{r}\left[-R_{ji}^{(1)}e^{-\lambda_1 r} + R_{ji}^{(2)}e^{-\lambda_2 r}\right]\right] + \lambda\delta_{ij}\frac{A_4}{r}\left[-R_{kk}^{(1)}e^{-\lambda_1 r} + R_{kk}^{(2)}e^{-\lambda_2 r}\right]$$
$$+ \alpha\delta_{ij}\frac{i\omega\rho_f}{4\pi\beta(\lambda_1^2 - \lambda_2^2)r}\left[(\lambda_1^2 - \lambda_4^2)e^{-\lambda_1 r} - (\lambda_2^2 - \lambda_4^2)e^{-\lambda_2 r}\right]$$

……................... (5.98)

note, $R_{ji}^{(m)} = R_{ij}^{(m)}$ and

$$R_{kk}^{(m)} = \frac{1}{r^2}\left(3\underbrace{r_k r_k}_{1} - \underbrace{\delta_{kk}}_{3}\right) + \frac{\lambda_m}{r}(3r_k r_k - \delta_{kk}) + \lambda_m^2 r_k r_k$$
$$= \lambda_m^2$$

thus,

$$\tau_{ij}^* = \frac{2\mu A_4}{r}\left[-R_{ij}^{(1)}e^{-\lambda_1 r} + R_{ij}^{(2)}e^{-\lambda_2 r}\right] + \frac{A_4\lambda\delta_{ij}}{r}\left[-\lambda_1^2 e^{-\lambda_1 r} + \lambda_2^2 e^{-\lambda_2 r}\right]$$
$$+ \frac{i\omega\rho_f\alpha\delta_{ij}}{4\pi\beta(\lambda_1^2 - \lambda_2^2)r}\left[(\lambda_1^2 - \lambda_4^2)e^{-\lambda_1 r} - (\lambda_2^2 - \lambda_4^2)e^{-\lambda_2 r}\right]$$
$$= \frac{2\mu A_4}{r}\left[-R_{ij}^{(1)}e^{-\lambda_1 r} + R_{ij}^{(2)}e^{-\lambda_2 r}\right] + e^{-\lambda_1 r}\left[-\frac{\lambda\lambda_1^2 A_4}{r} + \frac{i\omega\rho_f\alpha\delta_{ij}}{4\pi\beta(\lambda_1^2 - \lambda_2^2)r}(\lambda_1^2 - \lambda_4^2)\right]\delta_{ij}$$
$$+ e^{-\lambda_2 r}\left[\frac{\lambda\lambda_2^2 A_4}{r} - \frac{i\omega\rho_f\alpha\delta_{ij}}{4\pi\beta(\lambda_1^2 - \lambda_2^2)r}(\lambda_2^2 - \lambda_4^2)\right]\delta_{ij}$$

$$\tau_{ij}^* = -\frac{i\omega\rho_f}{4\pi\beta(\lambda_1^2 - \lambda_2^2)r}\left\{\left[\frac{2\mu(\alpha+\beta)}{\lambda+2\mu}R_{ij}^{(1)} + \left(\frac{\lambda(\alpha+\beta)\lambda_1^2}{\lambda+2\mu} - \alpha(\lambda_1^2 - \lambda_4^2)\right)\delta_{ij}\right]e^{-\lambda_1 r}\right.$$
$$\left. - \left[\frac{2\mu(\alpha+\beta)}{\lambda+2\mu}R_{ij}^{(2)} + \left(\frac{\lambda(\alpha+\beta)\lambda_2^2}{\lambda+2\mu} - \alpha(\lambda_2^2 - \lambda_4^2)\right)\delta_{ij}\right]e^{-\lambda_2 r}\right\}$$

.... (5.99)

$t_i^* = \tau_{ji}^* n_j \longrightarrow$

$$t_i^* = -\frac{i\omega\rho_f}{4\pi\beta(\lambda_1^2-\lambda_2^2)r}\left\{\left[\frac{2\mu(\alpha+\beta)}{\lambda+2\mu}R_{ij}^{(1)}n_j + \left(\frac{\lambda(\alpha+\beta)\lambda_1^2}{\lambda+2\mu} - \alpha(\lambda_1^2-\lambda_4^2)\right)n_i\right]e^{-\lambda_1 r}\right.$$
$$\left. - \left[\frac{2\mu(\alpha+\beta)}{\lambda+2\mu}R_{ij}^{(2)}n_j + \left(\frac{\lambda(\alpha+\beta)\lambda_2^2}{\lambda+2\mu} - \alpha(\lambda_2^2-\lambda_4^2)\right)n_i\right]e^{-\lambda_2 r}\right\}$$

.... (5.100)

$$R_{ij}^{(m)}n_j = \frac{1}{r^2}(3r_ir_j-\delta_{ij})n_j + \frac{\lambda_m}{r}(3r_ir_j-\delta_{ij})n_j + \lambda_m^2 r_ir_jn_j$$
$$= \frac{1}{r^2}\left(3r_i\frac{\partial r}{\partial n}-n_i\right) + \frac{\lambda_m}{r}\left(3r_i\frac{\partial r}{\partial n}-n_i\right) + \lambda_m^2 r_i\frac{\partial r}{\partial n}$$

.... (5.101)

q_n^* → *FS for normal component of fluid flux vector due to unit fluid injection rate at the source pt. "A":*

Recall → (Darcy's Law in FTS)

$$q_n^*(A,P) = \frac{\beta}{i\omega\rho_f}\left[p_{,i}^* - \omega^2\rho_f u_i^*\right]n_i$$

where,

$$u_i^* = \underbrace{\frac{i\omega\rho_f(\alpha+\beta)}{4\pi\beta(\lambda+2\mu)(\lambda_1^2-\lambda_2^2)}}_{A_4}\left[\left(\lambda_1+\frac{1}{r}\right)e^{-\lambda_1 r} - \left(\lambda_2+\frac{1}{r}\right)e^{-\lambda_2 r}\right]\frac{r_i}{r}$$

and

$$p^* = -\frac{i\omega\rho_f}{4\pi\beta(\lambda_1^2-\lambda_2^2)r}\left[(\lambda_1^2-\lambda_4^2)e^{-\lambda_1 r} - (\lambda_2^2-\lambda_4^2)e^{-\lambda_2 r}\right]$$

therefore,

$$p_{,i}^* = \frac{i\omega\rho_f}{4\pi\beta(\lambda_1^2-\lambda_2^2)}\frac{r_i}{r^2}\left[(\lambda_1^2-\lambda_4^2)e^{-\lambda_1 r} - (\lambda_2^2-\lambda_4^2)e^{-\lambda_2 r}\right] + \frac{i\omega\rho_f}{4\pi\beta(\lambda_1^2-\lambda_2^2)}\frac{r_i}{r}\left[(\lambda_1^2-\lambda_4^2)\lambda_1 e^{-\lambda_1 r} - (\lambda_2^2-\lambda_4^2)\lambda_2 e^{-\lambda_2 r}\right]$$

$$p^*_{,i} = \frac{i\omega\rho_f}{4\pi\beta(\lambda_1^2-\lambda_2^2)}\frac{r_i}{r}\left[\left(\lambda_1+\frac{1}{r}\right)(\lambda_1^2-\lambda_4^2)e^{-\lambda_1 r}-\left(\lambda_2+\frac{1}{r}\right)(\lambda_2^2-\lambda_4^2)e^{-\lambda_2 r}\right] \text{ .... } (5.102)$$

$$q^*_n = \frac{\beta}{i\omega\rho_f}\left[\frac{i\omega\rho_f}{4\pi\beta(\lambda_1^2-\lambda_2^2)}\frac{r_i}{r}\left[\left(\lambda_1+\frac{1}{r}\right)(\lambda_1^2-\lambda_4^2)e^{-\lambda_1 r}-\left(\lambda_2+\frac{1}{r}\right)(\lambda_2^2-\lambda_4^2)e^{-\lambda_2 r}\right]\right.$$
$$\left.-\omega^2\rho_f\frac{i\omega\rho_f(\alpha+\beta)}{4\pi\beta(\lambda+2\mu)(\lambda_1^2-\lambda_2^2)}\left[\left(\lambda_1+\frac{1}{r}\right)e^{-\lambda_1 r}-\left(\lambda_2+\frac{1}{r}\right)e^{-\lambda_2 r}\right]\frac{r_i}{r}\right]n_i$$

$$q^*_n = \frac{r_i n_i}{4\pi(\lambda_1^2-\lambda_2^2)r}\left\{\underbrace{\left[(\lambda_1^2-\lambda_4^2)-\frac{\omega^2\rho_f(\alpha+\beta)}{\lambda+2\mu}\right]}_{T_1}\left(\lambda_1+\frac{1}{r}\right)e^{-\lambda_1 r}-\underbrace{\left[(\lambda_2^2-\lambda_4^2)-\frac{\omega^2\rho_f(\alpha+\beta)}{\lambda+2\mu}\right]}_{T_2}\left(\lambda_2+\frac{1}{r}\right)e^{-\lambda_2 r}\right\}$$

.... (5.103)

$$T_1 = (\lambda_1^2-\lambda_4^2)-\frac{\omega^2\rho_f(\alpha+\beta)}{\lambda+2\mu}$$

Recall $\rightarrow$ $\lambda_4^2 = -\frac{\omega^2(\rho+\beta\rho_f)}{\lambda+2\mu}$

Then,

$$T_1 = (\lambda_1^2-\lambda_4^2)+\frac{\rho_f(\alpha+\beta)\lambda_4^2}{\rho+\beta\rho_f}$$
$$= \lambda_1^2+\frac{-\lambda_4^2(\rho+\beta\rho_f)+\rho_f(\alpha+\beta)\lambda_4^2}{\rho+\beta\rho_f}$$
$$= \lambda_1^2-\lambda_4^2\frac{\rho-\alpha\rho_f}{\rho+\beta\rho_f}$$

similarly,

$$T_2 = (\lambda_2^2-\lambda_4^2)+\frac{\rho_f(\alpha+\beta)\lambda_4^2}{\rho+\beta\rho_f}$$
$$= \lambda_2^2+\frac{-\lambda_4^2(\rho+\beta\rho_f)+\rho_f(\alpha+\beta)\lambda_4^2}{\rho+\beta\rho_f}$$
$$= \lambda_2^2-\lambda_4^2\frac{\rho-\alpha\rho_f}{\rho+\beta\rho_f}$$

As a result,

$$q^*_n = \frac{1}{4\pi(\lambda_1^2-\lambda_2^2)r}\frac{\partial r}{\partial n}\left[\left(\lambda_1^2-\lambda_4^2\frac{\rho-\alpha\rho_f}{\rho+\beta\rho_f}\right)\left(\lambda_1+\frac{1}{r}\right)e^{-\lambda_1 r}-\left(\lambda_2^2-\lambda_4^2\frac{\rho-\alpha\rho_f}{\rho+\beta\rho_f}\right)\left(\lambda_2+\frac{1}{r}\right)e^{-\lambda_2 r}\right] \quad (5.104)$$

5.3 Investigation of Singularities of the Fundamental Solutions

i. $$u_{ij}^{*} = \frac{1}{4\pi\mu\lambda_3^2 r}\left[\underbrace{\frac{\lambda_4^2-\lambda_2^2}{\lambda_1^2-\lambda_2^2}}_{A_1} R_{ij}^{(1)} e^{-\lambda_1 r} + \underbrace{\frac{\lambda_4^2-\lambda_1^2}{\lambda_2^2-\lambda_1^2}}_{A_2} R_{ij}^{(2)} e^{-\lambda_2 r} + \left(\lambda_3^2\delta_{ij} - R_{ij}^{(3)}\right)e^{-\lambda_3 r}\right]$$

Recall → $R_{ij}^{(m)} = \frac{1}{r^2}\left(3r_i r_j - \delta_{ij}\right) + \frac{\lambda_m}{r}\left(3r_i r_j - \delta_{ij}\right) + \lambda_m^2 r_i r_j$

and Taylor's expansion of $e^{-\lambda_m r}$:

$$\boxed{e^{-\lambda_m r} = 1 - \lambda_m r + \frac{1}{2}\lambda_m^2 r^2 - \frac{1}{6}\lambda_m^3 r^3 + \ldots} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5.105)$$

inserting above:

$$u_{ij}^{*}=\frac{1}{4\pi\mu\lambda_3^2 r}\left[A_1\left(\frac{1}{r^2}\left(3r_i r_j-\delta_{ij}\right)+\frac{\lambda_1}{r}\left(3r_i r_j-\delta_{ij}\right)+\lambda_1^2 r_i r_j\right)\left(1-\lambda_1 r+\frac{1}{2}\lambda_1^2 r^2-\frac{1}{6}\lambda_1^3 r^3+\ldots\right)\right.$$

$$+A_2\left(\frac{1}{r^2}\left(3r_i r_j-\delta_{ij}\right)+\frac{\lambda_2}{r}\left(3r_i r_j-\delta_{ij}\right)+\lambda_2^2 r_i r_j\right)\left(1-\lambda_2 r+\frac{1}{2}\lambda_2^2 r^2-\frac{1}{6}\lambda_2^3 r^3+\ldots\right)+$$

$$\left.+\left(\lambda_3^2\delta_{ij}-\left(\frac{1}{r^2}\left(3r_i r_j-\delta_{ij}\right)+\frac{\lambda_3}{r}\left(3r_i r_j-\delta_{ij}\right)+\lambda_3^2 r_i r_j\right)\right)\left(1-\lambda_3 r+\frac{1}{2}\lambda_3^2 r^2-\frac{1}{6}\lambda_3^3 r^3+\ldots\right)\right]$$

$$u_{ij}^{*}=\frac{1}{4\pi\mu\lambda_3^2 r}\left[\frac{1}{r^2}\left(3r_i r_j-\delta_{ij}\right)\left\{A_1\left(1-\lambda_1 r+\frac{1}{2}\lambda_1^2 r^2\right)+A_2\left(1-\lambda_2 r+\frac{1}{2}\lambda_2^2 r^2\right)-\left(1-\lambda_3 r+\frac{1}{2}\lambda_3^2 r^2\right)\right\}\right.$$

$$\left.+\frac{1}{r}\left(3r_i r_j-\delta_{ij}\right)\left\{A_1\left(1-\lambda_1 r\right)+A_2\left(1-\lambda_2 r\right)-\left(1-\lambda_3 r\right)\right\}+\left(A_1\lambda_1^2+A_2\lambda_2^2-\lambda_3^2\right)r_i r_j+\lambda_3^2\delta_{ij}+O(r)\right]$$

$$u_{ij}^{*}=\frac{1}{4\pi\mu\lambda_3^2 r}\left[\frac{1}{r^2}\left(3r_i r_j-\delta_{ij}\right)\underbrace{\left(A_1+A_2-1\right)}_{=0}-\frac{1}{r^2}\left(3r_i r_j-\delta_{ij}\right)\left(A_1\lambda_1+A_2\lambda_2-\lambda_3\right)r+\frac{1}{r^2}\left(3r_i r_j-\delta_{ij}\right)\left(A_1\lambda_1^2+A_2\lambda_2^2-\lambda_3^2\right)\frac{r^2}{2}\right.$$

$$\left.+\frac{1}{r}\left(3r_i r_j-\delta_{ij}\right)\left(A_1\lambda_1+A_2\lambda_2-\lambda_3\right)-\frac{1}{r}\left(3r_i r_j-\delta_{ij}\right)\left(A_1\lambda_1^2+A_2\lambda_2^2-\lambda_3^2\right)r+\left(A_1\lambda_1^2+A_2\lambda_2^2-\lambda_3^2\right)r_i r_j+\lambda_3^2\delta_{ij}\right]+O(r^0)$$

some terms cancel, and

$$u_{ij}^* = \frac{1}{4\pi\mu\lambda_3^2 r}\left[-\frac{1}{2}(3r_i r_j - \delta_{ij})(A_1\lambda_1^2 + A_2\lambda_2^2 - \lambda_3^2) + (A_1\lambda_1^2 + A_2\lambda_2^2 - \lambda_3^2)r_i r_j + \lambda_3^2\delta_{ij}\right] + O(r^0)$$

$$= \frac{1}{4\pi\mu\lambda_3^2 r}\left[\underbrace{(A_1\lambda_1^2 + A_2\lambda_2^2)}_{\lambda_4^2}\underbrace{\left(-\frac{3}{2}r_i r_j + \frac{1}{2}\delta_{ij} + r_i r_j\right)}_{\frac{1}{2}(\delta_{ij} - r_i r_j)} + \lambda_3^2\underbrace{\left(\frac{3}{2}r_i r_j - \frac{1}{2}\delta_{ij} - r_i r_j + \delta_{ij}\right)}_{\frac{1}{2}(r_i r_j + \delta_{ij})}\right] + O(r^0)$$

$$= \frac{1}{8\pi\mu\lambda_3^2 r}\left[\lambda_4^2(\delta_{ij} - r_i r_j) + \lambda_3^2(\delta_{ij} + r_i r_j)\right] + O(r^0)$$

$$= \frac{1}{8\pi\mu\lambda_3^2 r}\left[(\lambda_4^2 + \lambda_3^2)\delta_{ij} - (\lambda_4^2 - \lambda_3^2)r_i r_j\right] + O(r^0)$$

$$(\lambda_4^2 + \lambda_3^2) = \lambda_3^2\left(\frac{\lambda_4^2}{\lambda_3^2} + 1\right) = \lambda_3^2\left(\frac{\mu}{\lambda + 2\mu} + 1\right) = \lambda_3^2\,\frac{3 - 4\nu}{2(1-\nu)}$$

$$(\lambda_4^2 - \lambda_3^2) = -\lambda_3^2\left(1 - \frac{\lambda_4^2}{\lambda_3^2}\right) = -\lambda_3^2\left(1 - \frac{\mu}{\lambda + 2\mu}\right) = -\lambda_3^2\,\frac{1}{2(1-\nu)}$$

$$u_{ij}^* = \underbrace{\frac{1}{16\pi\mu(1-\nu)r}\left[(3 - 4\nu)\delta_{ij} + r_i r_j\right]}_{1^{st}\ \text{FS of Elastostatics}} + O(r^0) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5.106)$$

$$O(u_{ij}^*) \propto \frac{1}{r} \rightarrow (\text{weak singularity}) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5.107)$$

ii.

$$p_j^* = -\frac{\omega^2\rho_f(\alpha+\beta)}{4\pi\beta(\lambda+2\mu)(\lambda_1^2 - \lambda_2^2)}\left[\left(\lambda_1 + \frac{1}{r}\right)e^{-\lambda_1 r} - \left(\lambda_2 + \frac{1}{r}\right)e^{-\lambda_2 r}\right]\frac{r_j}{r}$$

$$p_j^* = -\frac{\omega^2\rho_f(\alpha+\beta)}{4\pi\beta(\lambda+2\mu)(\lambda_1^2 - \lambda_2^2)}\left[\left(\lambda_1 + \frac{1}{r}\right)(1 - \lambda_1 r + \ldots) - \left(\lambda_2 + \frac{1}{r}\right)(1 - \lambda_2 r + \ldots)\right]\frac{r_j}{r}$$

$$= -\frac{\omega^2\rho_f(\alpha+\beta)}{4\pi\beta(\lambda+2\mu)(\lambda_1^2 - \lambda_2^2)}\left[\lambda_1 + \frac{1}{r} - \lambda_1 - \lambda_2 - \frac{1}{r} + \lambda_2 + O(r)\right]\frac{r_j}{r}$$

$$= O(r^0)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots (5.108)$$

$$O(p_j^*) \propto r^0 \rightarrow \text{(non singular)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5.109)$$

iii. $$u_j^* = \frac{i\omega\rho_f(\alpha+\beta)}{4\pi\beta(\lambda+2\mu)(\lambda_1^2-\lambda_2^2)}\left[\left(\lambda_1+\frac{1}{r}\right)e^{-\lambda_1 r}-\left(\lambda_2+\frac{1}{r}\right)e^{-\lambda_2 r}\right]\frac{r_i}{r}$$

by the preceding reasoning

$$O(u_i^*) \propto r^0 \rightarrow \text{(non singular)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5.110)$$

iv. $$p^* = -\frac{i\omega\rho_f}{4\pi\beta(\lambda_1^2-\lambda_2^2)r}\left[(\lambda_1^2-\lambda_4^2)e^{-\lambda_1 r}-(\lambda_2^2-\lambda_4^2)e^{-\lambda_2 r}\right]$$

$$p^* = -\frac{i\omega\rho_f}{4\pi\beta(\lambda_1^2-\lambda_2^2)r}\left[(\lambda_1^2-\lambda_4^2)(1-\lambda_1 r+\ldots)-(\lambda_2^2-\lambda_4^2)(1-\lambda_1 r+\ldots)\right]$$
$$= -\frac{i\omega\rho_f}{4\pi\beta(\lambda_1^2-\lambda_2^2)r}\left[(\lambda_1^2-\lambda_4^2)-(\lambda_2^2-\lambda_4^2)\right]+O(r^0)$$

$$p^* = -\frac{i\omega\rho_f}{4\pi\beta r}+O(r^0) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5.111)$$

$$O(p^*) \propto \frac{1}{r} \rightarrow \text{(weak singularity)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5.112)$$

v.

$$t^*_{ij} = \underbrace{\frac{1}{4\pi\lambda_3^2 r}}_{D}\Bigg[\left(A_1 e^{-\lambda_1 r} + A_2 e^{-\lambda_2 r} - e^{-\lambda_3 r}\right)\frac{6R^{(5)}_{ij}}{r^3} + \left(A_1\lambda_1 e^{-\lambda_1 r} + A_2\lambda_2 e^{-\lambda_2 r} - \lambda_3 e^{-\lambda_3 r}\right)\frac{6R^{(5)}_{ij}}{r^2} + \left(A_1\lambda_1^2 e^{-\lambda_1 r} + A_2\lambda_2^2 e^{-\lambda_2 r} - \lambda_3^2 e^{-\lambda_3 r}\right)\frac{2R^{(6)}_{ij}}{r} +$$

$$-2r_i r_j\frac{\partial r}{\partial n}\left(A_1\lambda_1^3 e^{-\lambda_1 r} + A_2\lambda_2^3 e^{-\lambda_2 r} - \lambda_3^3 e^{-\lambda_3 r}\right) - \lambda_3^2\left(\lambda_3 + \frac{1}{r}\right)e^{-\lambda_3 r}\left(r_i n_j + \delta_{ij}\frac{\partial r}{\partial n}\right)\Bigg]$$

$$+\underbrace{\frac{1}{4\pi\lambda_3^2 r}}_{D}\left[\underbrace{\left(-A_1\lambda_1^2\frac{\lambda}{\mu} + \alpha\frac{\omega^2\rho_f(\alpha+\beta)\lambda_3^2}{\beta(\lambda+2\mu)(\lambda_1^2-\lambda_2^2)}\right)}_{T_1}\left(\lambda_1 + \frac{1}{r}\right)e^{-\lambda_1 r} - \underbrace{\left(A_2\lambda_2^2\frac{\lambda}{\mu} + \alpha\frac{\omega^2\rho_f(\alpha+\beta)\lambda_3^2}{\beta(\lambda+2\mu)(\lambda_1^2-\lambda_2^2)}\right)}_{T_2}\left(\lambda_2 + \frac{1}{r}\right)e^{-\lambda_2 r}\right]r_j n_i$$

$$t^*_{ij} = D\left[\underbrace{\left(A_1\left(\lambda_1 + \frac{1}{r}\right)e^{-\lambda_1 r} + A_2\left(\lambda_2 + \frac{1}{r}\right)e^{-\lambda_2 r} - \left(\lambda_3 + \frac{1}{r}\right)e^{-\lambda_3 r}\right)\frac{6R^{(5)}_{ij}}{r^3}}_{K_1} + \underbrace{\left(A_1\lambda_1^2 e^{-\lambda_1 r} + A_2\lambda_2^2 e^{-\lambda_2 r} - \lambda_3^2 e^{-\lambda_3 r}\right)\frac{2R^{(6)}_{ij}}{r}}_{K_2} \underbrace{-\lambda_3^2\left(\lambda_3 + \frac{1}{r}\right)\left(r_i n_j + \delta_{ij}\frac{\partial r}{\partial n}\right)e^{-\lambda_3 r}}_{K_3}\right.$$

$$\left.\underbrace{-2r_i r_j\frac{\partial r}{\partial n}\left(A_1\lambda_1^3 e^{-\lambda_1 r} + A_2\lambda_2^3 e^{-\lambda_2 r} - \lambda_3^3 e^{-\lambda_3 r}\right)}_{K_4}\right] + D\left[\underbrace{T_1\left(\lambda_1 + \frac{1}{r}\right)e^{-\lambda_1 r} - T_2\left(\lambda_2 + \frac{1}{r}\right)e^{-\lambda_2 r}}_{K_5}\right]r_j n_i$$

Recall $\rightarrow$ $R^{(5)}_{ij} = r_i n_j + r_j n_i + \left(\delta_{ij} - 5r_i r_j\right)\frac{\partial r}{\partial n}$ and $R^{(6)}_{ij} = r_i n_j + r_j n_i + \left(\delta_{ij} - 6r_i r_j\right)\frac{\partial r}{\partial n}$

$$K_1 = \frac{6R^{(5)}_{ij}}{r^2}\left[A_1\left(\lambda_1 + \frac{1}{r}\right)\left(1 - \lambda_1 r + \frac{\lambda_1^2}{2}r^2 - \frac{\lambda_1^3}{6}r^3 + \ldots\right) + A_2\left(\lambda_2 + \frac{1}{r}\right)\left(1 - \lambda_2 r + \frac{\lambda_2^2}{2}r^2 - \frac{\lambda_2^3}{6}r^3 + \ldots\right) - \left(\lambda_3 + \frac{1}{r}\right)\left(1 - \lambda_3 r + \frac{\lambda_3^2}{2}r^2 - \frac{\lambda_3^3}{6}r^3 + \ldots\right)\right]$$

$$K_1 = \frac{6R_{ij}^{(5)}}{r^2}\left[\underline{\underline{(A_1\lambda_1 + A_2\lambda_2 - \lambda_3)}} - \left(\underbrace{A_1\lambda_1^2 + A_2\lambda_2^2}_{\lambda_4^2} - \lambda_3^2\right)r + (A_1\lambda_1^3 + A_2\lambda_2^3 - \lambda_3^3)\frac{r^2}{2} + \underbrace{(A_1 + A_2 - 1)}_{=0}\frac{1}{r} - \underline{\underline{(A_1\lambda_1 + A_2\lambda_2 - \lambda_3)}}\frac{1}{r}r + \left(\underbrace{A_1\lambda_1^2 + A_2\lambda_2^2}_{\lambda_4^2} - \lambda_3^2\right)\frac{1}{r}\frac{r^2}{2} \right.$$
$$\left. - (A_1\lambda_1^3 + A_2\lambda_2^3 - \lambda_3^3)\frac{1}{r}\frac{r^3}{6} + O(r^3)\right]$$
$$= \frac{6R_{ij}^{(5)}}{r^2}\left[-(\lambda_4^2 - \lambda_3^2)\frac{r}{2} + (A_1\lambda_1^3 + A_2\lambda_2^3 - \lambda_3^3)\frac{r^2}{3}\right] + O(r)$$
$$= \frac{R_{ij}^{(5)}}{r}\left[-3(\lambda_4^2 - \lambda_3^2) + (A_1\lambda_1^3 + A_2\lambda_2^3 - \lambda_3^3)2r\right] + O(r)$$

$$K_2 = \left(A_1\lambda_1^2 e^{-\lambda_1 r} + A_2\lambda_2^2 e^{-\lambda_2 r} - \lambda_3^2 e^{-\lambda_3 r}\right)\frac{2R_{ij}^{(6)}}{r}$$
$$= \frac{2R_{ij}^{(6)}}{r}\left[A_1\lambda_1^2(1 - \lambda_1 r + \ldots) + A_2\lambda_2^2(1 - \lambda_2 r + \ldots) - \lambda_3^2(1 - \lambda_3 r + \ldots)\right]$$
$$= \frac{2R_{ij}^{(6)}}{r}\left[(A_1\lambda_1^2 + A_2\lambda_2^2 - \lambda_3^2) - (A_1\lambda_1^3 + A_2\lambda_2^3 - \lambda_3^3)r\right] + O(r)$$

$$K_1 + K_2 = \frac{\lambda_4^2 - \lambda_3^2}{r}\underbrace{\left(-3R_{ij}^{(5)} + 2R_{ij}^{(6)}\right)}_{-r_i n_j - r_j n_i - \left(\delta_{ij} - 3r_i r_j\right)\frac{\partial r}{\partial n}} + 2\left(A_1\lambda_1^3 + A_2\lambda_2^3 - \lambda_3^3\right)\underbrace{\left(R_{ij}^{(5)} - R_{ij}^{(6)}\right)}_{r_i r_j \frac{\partial r}{\partial n}} + O(r)$$

$$= -\frac{\lambda_4^2 - \lambda_3^2}{r}\left(r_i n_j + r_j n_i + \left(\delta_{ij} - 3r_i r_j\right)\frac{\partial r}{\partial n}\right) + 2\left(A_1\lambda_1^3 + A_2\lambda_2^3 - \lambda_3^3\right) r_i r_j \frac{\partial r}{\partial n} + O(r)$$

$$K_3 = \lambda_3^2\left(\lambda_3 + \frac{1}{r}\right)\left(r_i n_j + \delta_{ij}\frac{\partial r}{\partial n}\right)e^{-\lambda_3 r}$$

$$= \lambda_3^2\left(\lambda_3 + \frac{1}{r}\right)\left(r_i n_j + \delta_{ij}\frac{\partial r}{\partial n}\right)\left(1 - \lambda_3 r + \ldots\right)$$

$$= \underline{\underline{\lambda_3^3\left(r_i n_j + \delta_{ij}\frac{\partial r}{\partial n}\right)}} + \lambda_3^2\left(r_i n_j + \delta_{ij}\frac{\partial r}{\partial n}\right)\frac{1}{r} - \underline{\underline{\lambda_3^3\left(r_i n_j + \delta_{ij}\frac{\partial r}{\partial n}\right)}} + O(r)$$

$$= \lambda_3^2\left(r_i n_j + \delta_{ij}\frac{\partial r}{\partial n}\right)\frac{1}{r} + O(r)$$

$$K_4 = 2r_i r_j \frac{\partial r}{\partial n}\left[A_1\lambda_1^3 e^{-\lambda_1 r} + A_2\lambda_2^3 e^{-\lambda_2 r} - \lambda_3^3 e^{-\lambda_3 r}\right]$$

$$= 2r_i r_j \frac{\partial r}{\partial n}\left[A_1\lambda_1^3\left(1 - \lambda_1 r + \ldots\right) + A_2\lambda_2^3\left(1 - \lambda_2 r + \ldots\right) - \lambda_3^3\left(1 - \lambda_3 r + \ldots\right)\right]$$

$$= 2r_i r_j \frac{\partial r}{\partial n}\left[A_1\lambda_1^3 + A_2\lambda_2^3 - \lambda_3^3\right] + O(r)$$

$$
\begin{aligned}
K_5 &= T_1\left(\lambda_1 + \frac{1}{r}\right)e^{-\lambda_1 r} - T_2\left(\lambda_2 + \frac{1}{r}\right)e^{-\lambda_2 r} \\
&= T_1\left(\lambda_1 + \frac{1}{r}\right)(1-\lambda_1 r + \ldots) - T_2\left(\lambda_2 + \frac{1}{r}\right)(1-\lambda_2 r + \ldots) \\
&= (T_1\lambda_1 - T_2\lambda_2) + (T_1 - T_2)\frac{1}{r} - (T_1\lambda_1 - T_2\lambda_2)\frac{1}{r}r + O(r) \\
&= (T_1 - T_2)\frac{1}{r} + O(r)
\end{aligned}
$$

$$
\begin{aligned}
T_1 - T_2 &= \left(-A_1\lambda_1^2\frac{\lambda}{\mu} + \alpha\frac{\omega^2\rho_f(\alpha+\beta)\lambda_3^2}{\beta(\lambda+2\mu)(\lambda_1^2-\lambda_2^2)}\right) - \left(A_2\lambda_2^2\frac{\lambda}{\mu} + \alpha\frac{\omega^2\rho_f(\alpha+\beta)\lambda_3^2}{\beta(\lambda+2\mu)(\lambda_1^2-\lambda_2^2)}\right) \\
&= -\underbrace{(A_1\lambda + A_2\lambda_2^2)}_{\lambda_4^2}\frac{\lambda}{\mu} \\
&= -\lambda_4^2\frac{\lambda}{\mu}
\end{aligned}
$$

therefore,

$$
K_5 = -\lambda_4^2\frac{\lambda}{\mu}\frac{1}{r} + O(r)
$$

$$
\begin{aligned}
t_{ij}^{*} &= \frac{1}{4\pi\lambda_3^2 r}\left[-\left(\lambda_4^2-\lambda_3^2\right)\left(r_i n_j + r_j n_i + \left(\delta_{ij}-3r_i r_j\right)\frac{\partial r}{\partial n}\right)\frac{1}{r} + \underline{\underline{2\left(A_1\lambda_1^3 + A_2\lambda_2^3-\lambda_3^3\right)r_i r_j\frac{\partial r}{\partial n}}} - \lambda_3^2\left(r_i n_j + \delta_{ij}\frac{\partial r}{\partial n}\right)\frac{1}{r} - \underline{\underline{2\left[A_1\lambda_1^3 + A_2\lambda_2^3-\lambda_3^3\right]r_i r_j\frac{\partial r}{\partial n}}} - \lambda_4^2\frac{\lambda}{\mu}n_i r_j\frac{1}{r} + O(r)\right] \\
&= \frac{1}{4\pi\lambda_3^2 r^2}\left[-\left(\lambda_4^2-\lambda_3^2\right)\left(r_i n_j + r_j n_i + \left(\delta_{ij}-3r_i r_j\right)\frac{\partial r}{\partial n}\right) - \lambda_3^2\left(r_i n_j + \delta_{ij}\frac{\partial r}{\partial n}\right) - \lambda_4^2\frac{\lambda}{\mu}n_i r_j\right] + O(r^0) \\
&= \frac{1}{4\pi\lambda_3^2 r^2}\left[-\lambda_4^2 n_i r_j - \lambda_4^2\left(r_i n_j + \left(\delta_{ij}-3r_i r_j\right)\frac{\partial r}{\partial n}\right) + \lambda_3^2\left(\underline{\underline{r_i n_j + \delta_{ij}\frac{\partial r}{\partial n}}} - 3r_i r_j\frac{\partial r}{\partial n}\right) + \lambda_3^2 n_i r_j - \underline{\underline{\lambda_3^2\left(r_i n_j + \delta_{ij}\frac{\partial r}{\partial n}\right)}} - \lambda_4^2\frac{\lambda}{\mu}n_i r_j\right] + O(r^0) \\
&= \frac{1}{4\pi\lambda_3^2 r^2}\left[-\lambda_4^2 n_i r_j\left(1+\frac{\lambda}{\mu}\right) - \lambda_4^2\left(r_i n_j + \left(\delta_{ij}-3r_i r_j\right)\frac{\partial r}{\partial n}\right) + \lambda_3^2\left(n_i r_j - 3r_i r_j\frac{\partial r}{\partial n}\right)\right] + O(r^0) \\
&= \frac{1}{4\pi r^2}\left[-\underbrace{\frac{\lambda_4^2}{\lambda_3^2}}_{\frac{\mu}{\lambda+2\mu}}\left(1+\frac{\lambda}{\mu}\right)n_i r_j - \underbrace{\frac{\lambda_4^2}{\lambda_3^2}}_{\frac{\mu}{\lambda+2\mu}}\left(r_i n_j + \left(\delta_{ij}-3r_i r_j\right)\frac{\partial r}{\partial n}\right) + \left(n_i r_j - 3r_i r_j\frac{\partial r}{\partial n}\right)\right] + O(r^0)
\end{aligned}
$$

Note that; $\frac{\mu}{\lambda+2\mu}=\frac{1-2\nu}{2(1-\nu)}$ $\quad 1+\frac{\lambda}{\mu}=\frac{1}{1-2\nu}$

$$
\begin{aligned}
t_{ij}^{*} &= \frac{1}{8\pi(1-\nu)r^2}\left[-n_i r_j - (1-2\nu)\left(r_i n_j + \left(\delta_{ij}-3r_i r_j\right)\frac{\partial r}{\partial n}\right) + 2(1-\nu)n_i r_j - 6(1-\nu)r_i r_j\frac{\partial r}{\partial n}\right] + O(r^0) \\
&= \frac{1}{8\pi(1-\nu)r^2}\left[-(1-2\nu)\left(r_i n_j - n_i r_j\right) - (1-2\nu)\left(\delta_{ij}-3r_i r_j\right)\frac{\partial r}{\partial n} - 6(1-\nu)r_i r_j\frac{\partial r}{\partial n}\right] + O(r^0)
\end{aligned}
$$

finally,

$$t_{ij}^{*} = \underbrace{\frac{1}{8\pi(1-\nu)r^{2}}\left[(1-2\nu)(n_i r_j - r_i n_j) - \left[(1-2\nu)\delta_{ij} + 3 r_i r_j\right]\frac{\partial r}{\partial n}\right]}_{2^{nd}\ \text{FS of Elastostatics}} + O(r^{0}) \qquad \text{…………………………………………….. (5.113)}$$

$$O\left(t_{ij}^{*}\right) \propto \frac{1}{r^{2}} \rightarrow (\text{strong singularity}) \qquad \text{……………………………………………………………………… (5.114)}$$

vi. $$q_n^{*j} = \frac{i\omega n_i}{4\pi\mu\lambda_3^2 r}\left[\underbrace{\frac{\alpha\lambda_4^2 + \beta\lambda_2^2}{\lambda_2^2 - \lambda_1^2}}_{C_1} R_{ij}^{(1)} e^{-\lambda_1 r} + \underbrace{\frac{\alpha\lambda_4^2 + \beta\lambda_1^2}{\lambda_1^2 - \lambda_2^2}}_{C_2} R_{ij}^{(2)} e^{-\lambda_2 r} + \beta\left(\lambda_3^2\delta_{ij} - R_{ij}^{(3)}\right)e^{-\lambda_3 r}\right]$$

$$q_n^{*j} = \frac{i\omega n_i}{4\pi\mu\lambda_3^2 r}\left[\begin{array}{l} C_1\underbrace{\left(\frac{1}{r^2}(3r_ir_j - \delta_{ij}) + \frac{\lambda_1}{r}(3r_ir_j - \delta_{ij}) + \lambda_1^2 r_i r_j\right)\left(1 - \lambda_1 r + \frac{\lambda_1^2 r^2}{2} - \ldots\right)}_{T_1} + C_2\underbrace{\left(\frac{1}{r^2}(3r_ir_j - \delta_{ij}) + \frac{\lambda_2}{r}(3r_ir_j - \delta_{ij}) + \lambda_2^2 r_i r_j\right)\left(1 - \lambda_2 r + \frac{\lambda_2^2 r^2}{2} - \ldots\right)}_{T_2} \\ + \underbrace{\beta\lambda_3^2\delta_{ij}(1 - \lambda_3 r + \ldots)}_{T_3} - \beta\underbrace{\left(\frac{1}{r^2}(3r_ir_j - \delta_{ij}) + \frac{\lambda_3}{r}(3r_ir_j - \delta_{ij}) + \lambda_3^2 r_i r_j\right)\left(1 - \lambda_3 r + \frac{\lambda_3^2 r^2}{2} - \ldots\right)}_{T_4} \end{array}\right]$$

$$
\begin{aligned}
T_1 &= \left(\frac{1}{r^2}\left(3r_ir_j-\delta_{ij}\right)+\frac{\lambda_1}{r}\left(3r_ir_j-\delta_{ij}\right)+\lambda_1^2r_ir_j\right)\left(1-\lambda_1r+\frac{\lambda_1^2r^2}{2}-\ldots\right)\\
&= \frac{1}{r^2}\left(3r_ir_j-\delta_{ij}\right)\underline{\underline{-\frac{\lambda_1}{r}\left(3r_ir_j-\delta_{ij}\right)}}+\frac{\lambda_1^2}{2}\left(3r_ir_j-\delta_{ij}\right)\underline{\underline{+\frac{\lambda_1}{r}\left(3r_ir_j-\delta_{ij}\right)}}-\lambda_1^2\left(3r_ir_j-\delta_{ij}\right)+\lambda_1^2r_ir_j\\
&= \frac{1}{r^2}\left(3r_ir_j-\delta_{ij}\right)+\frac{\lambda_1^2}{2}\left(\delta_{ij}-r_ir_j\right)+O(r)
\end{aligned}
$$

Similarly,

$$T_2=\frac{1}{r^2}\left(3r_ir_j-\delta_{ij}\right)+\frac{\lambda_2^2}{2}\left(\delta_{ij}-r_ir_j\right)+O(r)$$

and

$$T_4=\frac{1}{r^2}\left(3r_ir_j-\delta_{ij}\right)+\frac{\lambda_3^2}{2}\left(\delta_{ij}-r_ir_j\right)+O(r)$$

$$T_3=\beta\lambda_3^2\delta_{ij}+O(r)$$

$$q_n^{*j}=\frac{i\omega n_i}{4\pi\mu\lambda_3^2 r}\left[C_1\left(\frac{1}{r^2}(3r_ir_j-\delta_{ij})+\frac{\lambda_1^2}{2}(\delta_{ij}-r_ir_j)\right)+C_2\left(\frac{1}{r^2}(3r_ir_j-\delta_{ij})+\frac{\lambda_2^2}{2}(\delta_{ij}-r_ir_j)\right)+\beta\lambda_3^2\delta_{ij}-\beta\left(\frac{1}{r^2}(3r_ir_j-\delta_{ij})+\frac{\lambda_3^2}{2}(\delta_{ij}-r_ir_j)\right)+O(r)\right]$$

$$=\frac{i\omega n_i}{4\pi\mu\lambda_3^2 r}\left[\left(C_1\frac{\lambda_1^2}{2}+C_2\frac{\lambda_2^2}{2}-\beta\frac{\lambda_3^2}{2}\right)(\delta_{ij}-r_ir_j)+\beta\lambda_3^2\delta_{ij}+\frac{1}{r^2}(3r_ir_j-\delta_{ij})\underbrace{(C_1+C_2-\beta)}_{=0}\right]+O(r^0)$$

Note that, $C_1\frac{\lambda_1^2}{2}+C_2\frac{\lambda_2^2}{2}-\beta\frac{\lambda_3^2}{2}=-\frac{\alpha\lambda_4^2+\beta\lambda_2^2}{\lambda_1^2-\lambda_2^2}\frac{\lambda_1^2}{2}+\frac{\alpha\lambda_4^2+\beta\lambda_1^2}{\lambda_1^2-\lambda_2^2}\frac{\lambda_2^2}{2}-\beta\frac{\lambda_3^2}{2}=-\frac{\alpha\lambda_4^2+\beta\lambda_3^2}{2}$

$$q_n^{*j}=\frac{i\omega n_i}{4\pi\mu\lambda_3^2 r}\left[-\frac{\alpha\lambda_4^2+\beta\lambda_3^2}{2}(\delta_{ij}-r_ir_j)+\beta\lambda_3^2\delta_{ij}\right]+O(r^0)$$

$$=\frac{i\omega}{4\pi\mu r}\left[\frac{\beta-\alpha\frac{\lambda_4^2}{\lambda_3^2}}{2}n_j+\frac{\beta+\alpha\frac{\lambda_4^2}{\lambda_3^2}}{2}\frac{\partial r}{\partial n}r_j\right]+O(r^0) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5.115)$$

$$=\frac{i\omega}{4\pi\mu r}\left[\frac{\beta-\alpha\frac{1-2\nu}{2(1-\nu)}}{2}n_j+\frac{\beta+\alpha\frac{1-2\nu}{2(1-\nu)}}{2}\frac{\partial r}{\partial n}r_j\right]+O(r^0)$$

$$=\frac{i\omega}{16\pi\mu(1-\nu)r}\left[2\beta(1-\nu)\left(n_j+\frac{\partial r}{\partial n}r_j\right)-\alpha(1-2\nu)\left(n_j-\frac{\partial r}{\partial n}r_j\right)\right]+O(r^0)$$

$$O\left(q_n^{*j}\right) \propto \frac{1}{r} \;\rightarrow\; \left(\text{weak singularity}\right) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5.116)$$

vii.

$$t_i^* = -\frac{i\omega\rho_f n_j}{4\pi\beta\left(\lambda_1^2-\lambda_2^2\right)r}\left\{\underbrace{\left[\underbrace{\frac{2\mu(\alpha+\beta)}{\lambda+2\mu}}_{C_1}R_{ij}^{(1)} + \underbrace{\left(\frac{\lambda(\alpha+\beta)\lambda_1^2}{\lambda+2\mu}-\alpha\left(\lambda_1^2-\lambda_4^2\right)\right)}_{C_2}\delta_{ij}\right]e^{-\lambda_1 r}}_{T_1} - \underbrace{\left[\underbrace{\frac{2\mu(\alpha+\beta)}{\lambda+2\mu}}_{C_1}R_{ij}^{(2)} + \underbrace{\left(\frac{\lambda(\alpha+\beta)\lambda_2^2}{\lambda+2\mu}-\alpha\left(\lambda_2^2-\lambda_4^2\right)\right)}_{C_3}\delta_{ij}\right]e^{-\lambda_2 r}}_{T_2}\right\}$$

$$\begin{aligned}
T_1 &= \left[C_1 R_{ij}^{(1)} + C_2\delta_{ij}\right]e^{-\lambda_1 r} \\
&= C_1\left(\frac{1}{r^2}\left(3r_i r_j - \delta_{ij}\right) + \frac{\lambda_1}{r}\left(3r_i r_j - \delta_{ij}\right) + \lambda_1^2 r_i r_j\right)\left(1-\lambda_1 r + \frac{\lambda_1^2 r^2}{2} - \ldots\right) + C_2\delta_{ij}\left(1-\lambda_1 r + \ldots\right) \\
&= C_1\left[\frac{1}{r^2}\left(3r_i r_j - \delta_{ij}\right) - \underline{\underline{\frac{\lambda_1}{r}\left(3r_i r_j - \delta_{ij}\right)}} + \frac{\lambda_1^2}{2}\left(3r_i r_j - \delta_{ij}\right) + \underline{\underline{\frac{\lambda_1}{r}\left(3r_i r_j - \delta_{ij}\right)}} - \lambda_1^2\left(3r_i r_j - \delta_{ij}\right) + \lambda_1^2 r_i r_j\right] + C_2\delta_{ij} + O(r) \\
&= C_1\left[\frac{1}{r^2}\left(3r_i r_j - \delta_{ij}\right) - \frac{\lambda_1^2}{2}\left(r_i r_j - \delta_{ij}\right)\right] + C_2\delta_{ij} + O(r)
\end{aligned}$$

Similarly,

$$T_2 = C_1\left[\frac{1}{r^2}\left(3r_i r_j - \delta_{ij}\right) - \frac{\lambda_2^2}{2}\left(r_i r_j - \delta_{ij}\right)\right] + C_3\delta_{ij} + O(r)$$

$$\begin{aligned} t_i^* &= \frac{-i\omega\rho_f n_j}{4\pi\beta\left(\lambda_1^2 - \lambda_2^2\right)r}\left\{C_1\left[\underline{\underline{\frac{1}{r^2}\left(3r_i r_j - \delta_{ij}\right)}} - \frac{\lambda_1^2}{2}\left(r_i r_j - \delta_{ij}\right)\right] + C_2\delta_{ij} - C_1\left[\underline{\underline{\frac{1}{r^2}\left(3r_i r_j - \delta_{ij}\right)}} - \frac{\lambda_2^2}{2}\left(r_i r_j - \delta_{ij}\right)\right] - C_3\delta_{ij}\right\} + O(r^0) \\ &= \frac{i\omega\rho_f n_j}{8\pi\beta\left(\lambda_1^2 - \lambda_2^2\right)r}\left[C_1\left(r_i r_j - \delta_{ij}\right)\left(\lambda_1^2 - \lambda_2^2\right) - 2\left(C_2 - C_3\right)\delta_{ij}\right] + O(r^0) \end{aligned}$$

$$\begin{aligned} C_2 - C_3 &= \frac{\lambda(\alpha + \beta)}{\lambda + 2\mu}\left(\lambda_1^2 - \lambda_2^2\right) - \alpha\left(\lambda_1^2 - \lambda_2^2\right) \\ &= \left(\underbrace{\frac{\lambda}{\lambda + 2\mu}}_{\frac{\nu}{1-\nu}}(\alpha + \beta) - \alpha\right)\left(\lambda_1^2 - \lambda_2^2\right) \\ &= \frac{\nu\beta - (1 - 2\nu)\alpha}{1 - \nu}\left(\lambda_1^2 - \lambda_2^2\right) \end{aligned}$$

$$t_i^* = \frac{i\omega\rho_f n_j}{8\pi\beta\underline{\underline{(\lambda_1^2-\lambda_2^2)}}\, r}\left[C_1\left(r_i r_j - \delta_{ij}\right)\underline{\underline{(\lambda_1^2-\lambda_2^2)}} - 2\frac{\nu\beta-(1-2\nu)\alpha}{1-\nu}\underline{\underline{(\lambda_1^2-\lambda_2^2)}}\delta_{ij}\right] + O(r^0)$$

$$= \frac{i\omega\rho_f n_j}{8\pi\beta\, r}\left[\underbrace{\frac{2\mu}{\lambda+2\mu}}_{\frac{1-2\nu}{1-\nu}}(\alpha+\beta)r_i r_j - \underbrace{\frac{2\mu}{\lambda+2\mu}}_{\frac{1-2\nu}{1-\nu}}(\alpha+\beta)\delta_{ij} - 2\frac{\nu\beta-(1-2\nu)\alpha}{1-\nu}\delta_{ij}\right] + O(r^0)$$

$$= \frac{i\omega\rho_f n_j}{8\pi\beta(1-\nu)\, r}\left[(\alpha+\beta)(1-2\nu)r_i r_j - \{(\alpha+\beta)(1-2\nu) + 2(\nu\beta-(1-2\nu)\alpha)\}\delta_{ij}\right] + O(r^0)$$

$$= \frac{i\omega\rho_f}{8\pi\beta(1-\nu)\, r}\left[\beta\left((1-2\nu)\frac{\partial r}{\partial n}r_i - n_i\right) + \alpha(1-2\nu)\left(\frac{\partial r}{\partial n}r_i + n_i\right)\right] + O(r^0)$$

or

$$t_i^* = \frac{i\omega\rho_f}{8\pi\beta(1-\nu)\, r}\left[(\alpha+\beta)(1-2\nu)\frac{\partial r}{\partial n}r_i + [\alpha(1-2\nu)-\beta]n_i\right] + O(r^0) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (5.117)$$

$$O(t_i^*) \propto \frac{1}{r} \quad \rightarrow \quad (\text{ weak singularity }) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (5.118)$$

viii. $$q_n^* = \frac{1}{4\pi\left(\lambda_1^2 - \lambda_2^2\right)r}\frac{\partial r}{\partial n}\left[\underbrace{\underbrace{\left(\lambda_1^2 - \lambda_4^2\frac{\rho - \alpha\rho_f}{\rho + \beta\rho_f}\right)}_{C_1}\left(\lambda_1 + \frac{1}{r}\right)e^{-\lambda_1 r}}_{T_1} - \underbrace{\underbrace{\left(\lambda_2^2 - \lambda_4^2\frac{\rho - \alpha\rho_f}{\rho + \beta\rho_f}\right)}_{C_2}\left(\lambda_2 + \frac{1}{r}\right)e^{-\lambda_2 r}}_{T_2}\right]$$

$$T_1 = C_1\left(\lambda_1 + \frac{1}{r}\right)\left(1 - \lambda_1 r + \ldots\right) = C_1\left(\lambda_1 + \frac{1}{r} - \lambda_1\right) + O(r) = C_1\frac{1}{r} + O(r) \quad \text{and} \quad T_2 = C_2\frac{1}{r}$$

thus,

$$q_n^* = \frac{1}{4\pi\left(\lambda_1^2 - \lambda_2^2\right)r}\frac{\partial r}{\partial n}\left(C_1 - C_2\right)\frac{1}{r} + O(r^0)$$

$$C_1 - C_2 = \left(\lambda_1^2 - \lambda_4^2\frac{\rho - \alpha\rho_f}{\rho + \beta\rho_f}\right) - \left(\lambda_2^2 - \lambda_4^2\frac{\rho - \alpha\rho_f}{\rho + \beta\rho_f}\right) = \left(\lambda_1^2 - \lambda_2^2\right)$$

$$q_n^* = \underbrace{\frac{1}{4\pi r^2}\frac{\partial r}{\partial n}}_{2^{nd}\ \text{FS of Acoustics}} + O(r^0) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(5.119)}$$

$$O\left(q_n^*\right) \propto \frac{1}{r^2} \rightarrow \left(\text{strong singularity}\right) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(5.120)}$$

5.4 BIE in Matrix Form and the Free Term Coefficient

We recall the BIE of PE in the absence of body sources (Eqn. 4.26)

$$
\begin{aligned}
&(1) \rightarrow \; u_l(A) = \int_\Gamma t_i u_{il}^* d\Gamma + \int_\Gamma \frac{1}{i\omega} q_n p_l^* d\Gamma - \int_\Gamma t_{il}^* u_i d\Gamma - \int_\Gamma \frac{1}{i\omega} q_n^{l\,*} p d\Gamma \\
&(2) \rightarrow \; \frac{1}{i\omega} p(A) = \int_\Gamma t_i^* u_i d\Gamma + \int_\Gamma \frac{1}{i\omega} q_n^* p d\Gamma - \int_\Gamma t_i u_i^* d\Gamma - \int_\Gamma \frac{1}{i\omega} q_n p^* d\Gamma
\end{aligned}
\quad \text{........................ (4.26)}
$$

Introducing the notation,

$$
\underline{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ p \end{bmatrix} \quad \text{and} \quad \underline{t} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ q_n \end{bmatrix} \quad \text{.. (5.121)}
$$

where, $\underline{u}$ represents a generalized displacement and $\underline{t}$ represents a generalized traction vector, but equation (4.26) gives the solid displacement and the fluid pressure for a point inside the domain, when the source point "A" is brought to the boundary "Γ" the BIE (3.26) can be expressed in matrix form as

$$
\boxed{\underline{cu}(A) = \int_\Gamma \underline{G'}(A,P)\underline{t}(P) d\Gamma - \int_\Gamma \underline{H'}(A,P)\underline{u}(P) d\Gamma} \quad \text{.. (5.122)}
$$

where,

$$
\underline{G'} = \begin{bmatrix} \left[u_{ij}^*\right]^T & \frac{1}{i\omega}\left\{p_j^*\right\} \\ -i\omega\left\{u_j^*\right\}^T & -p^* \end{bmatrix} = \begin{bmatrix} u_{11}^* & u_{21}^* & u_{31}^* & \frac{p_1^*}{i\omega} \\ u_{12}^* & u_{22}^* & u_{32}^* & \frac{p_2^*}{i\omega} \\ u_{13}^* & u_{23}^* & u_{33}^* & \frac{p_3^*}{i\omega} \\ -i\omega u_1^* & -i\omega u_2^* & -i\omega u_3^* & -p^* \end{bmatrix} \quad \text{................................ (5.123)}
$$

and

$$\underline{H}' = \begin{bmatrix} \left[t^*_{ij}\right]^T & \frac{1}{i\omega}\left\{q_n^{*j}\right\} \\ -i\omega\left\{t^*_j\right\}^T & -q_n^* \end{bmatrix} = \begin{bmatrix} t^*_{11} & t^*_{21} & t^*_{31} & \frac{q_n^{*1}}{i\omega} \\ t^*_{12} & t^*_{22} & t^*_{32} & \frac{q_n^{*2}}{i\omega} \\ t^*_{13} & t^*_{23} & t^*_{33} & \frac{q_n^{*3}}{i\omega} \\ -i\omega t^*_1 & -i\omega t^*_2 & -i\omega t^*_3 & -q_n^* \end{bmatrix} \qquad (5.124)$$

The constant “c” is called the free term coefficient and is equal to 0.5 for a smooth boundary.

CHAPTER 6

BOUNDARY ELEMENT FORMULATION FOR PORO-ELASTIC SOLIDS WITH AXI-SYMMETRIC GEOMETRY

We closely follow the basic outline of the formulation given in Özkan, 1995. Consider a poro-elastic isotropic axi-symmetric body of boundary S, referred to a cylindrical coordinate system R-θ-z as shown in Figure 6.1 where the z-axis is the axis of revolution of the body. It will be assumed that the boundary conditions are not axi-symmetric. In this section, the 3-D boundary integral equation developed previously for poro-elastodynamics will be expressed in cylindrical coordinates through a coordinate transformation and the method presented in Özkan, 1995 will be extended to poro-elastodynamic boundary element formulation. This method is based on complex Fourier series expansion of the boundary quantities (displacements, pore-pressure, tractions and normal component of fluid flux vector) in circumferencal direction.

There are two main advantages of this method (Özkan, 1995) over others available in the literature (Brebbia and Dominguez 1992, Dargush and Chopra 1996) :

i) the evaluation of integrals in θ direction is accomplished by FFT algorithm, which reduces the computational load,

ii) the need for differentiating symmetric and anti-symmetric modes in the analysis is eliminated, which facilitates computer programming.

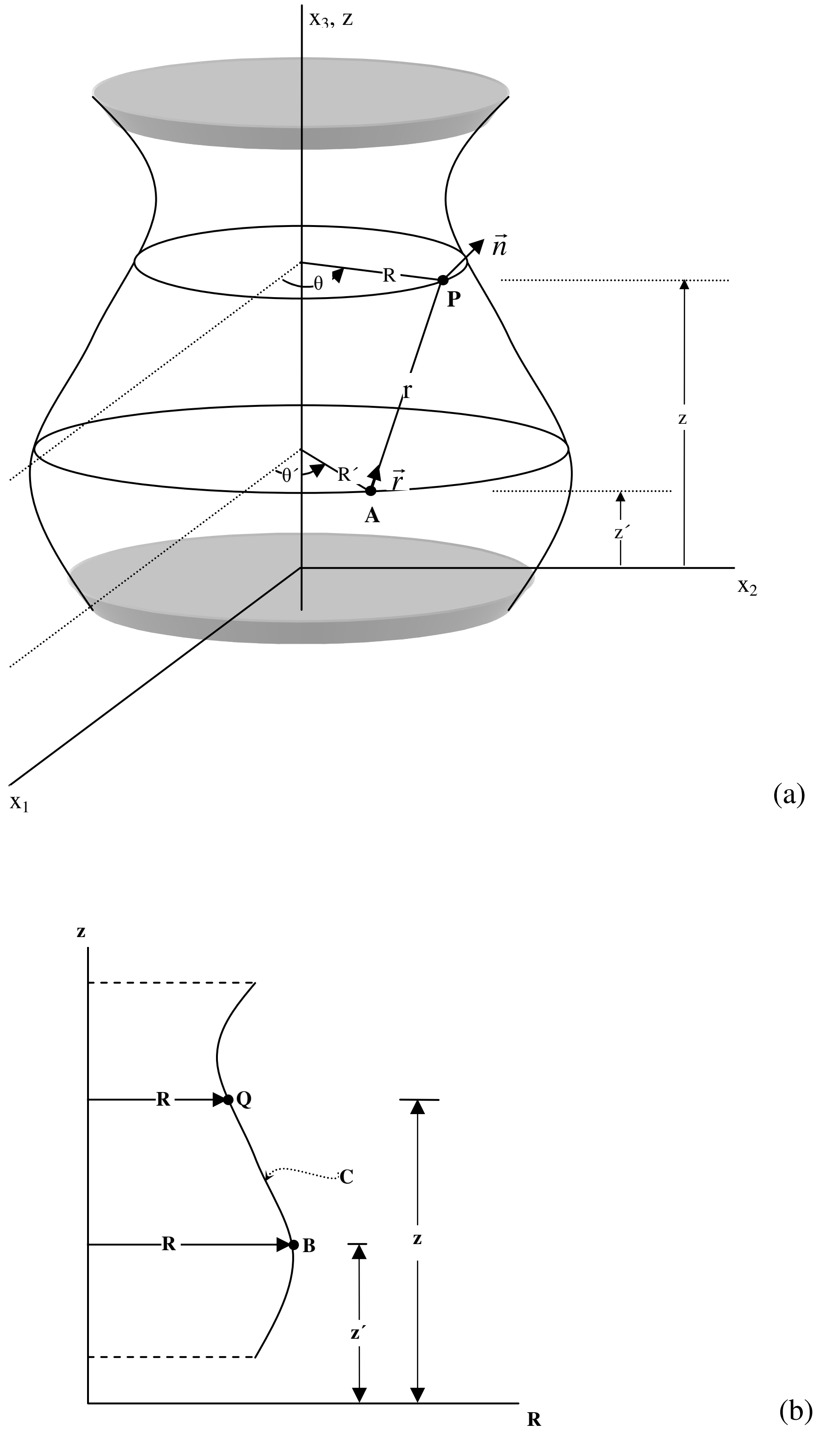

Figure 6.1 An axi-symmetric body referred to Rθz coordinate system: a) three dimensional body, b) x-section on R-Z plane

6.1 Boundary Integral Equations in Cylindrical Coordinates

We start by recalling the 3-D boundary integral formulation in FTS when the body is referred to a rectangular coordinate system, in matrix form:

$$\underline{c}\underline{u}(A) = \int_{\Gamma} \underline{G}'(A,P)\underline{t}(P)d\Gamma - \int_{\Gamma} \underline{H}'(A,P)\underline{u}(P)d\Gamma \quad \text{...} \quad (6.1)$$

where the terms involving body sources have been disregarded as previously. We again note that an underline designates a matrix quantity; $\underline{G}'$ and $\underline{H}'$ matrices (4x4) contain first and second fundamental solutions of poro-elastodynamics, respectively. We recall that these solutions are two point (source point A and field point P) functions and associated with an infinite medium with either a point force in only one coordinate direction in turn or a unit fluid injection rate at a point "A" (called the source point). The point "P" in the Eqn. (6.1) is the integration point on the boundary "S". $\underline{t}$ and $\underline{u}$ are (4x1) column matrices representing generalized traction and displacement vectors at the boundary points, respectively; as may be recalled, $\underline{c}$ is a (4x4) matrix which has the form

$$\underline{c} = \frac{1}{2}\underline{I} \quad \text{...} \quad (6.2)$$

if the boundary is smooth at the source point "A". Equation (6.2) holds in our formulation, as we shall use constant elements in the analysis.

The fundamental solutions in Eqn. (6.1) are functions of the positions of the source point A and the integration point P, and on the angular frequency ω; and they involve the variables (recall 5.57, 58, 61 65, 90, 98, 100, 104) :

r	:	the distance between "A" and "P"
$\underline{r}$	:	unit vector in $\overrightarrow{AP}$ direction
$\underline{n}$	:	outer unit normal vector at "P"

r_n : derivative of r in n direction at P (n is the normal axis at "P")

where,

$$
\begin{aligned}
& r = \sqrt{[x_i - a_i](x_i - a_i)} \\
& r_i = \frac{x_i - a_i}{r} \qquad \text{.. (6.3)} \\
& r_n = \frac{\partial r}{\partial n} = r_i n_i
\end{aligned}
$$

x_j, a_j : coordinates of "P" and "A" , respectively in rectangular coordinate system j runs 1-3.

In cylindrical coordinates, we refer the point "P" by (R, θ, z) and point "A" by (R´, θ´, z´), see **Figure 6.1**.

One can infer that the resolution of the generalized displacements and tractions in cylindrical and rectangular coordinate frames obey the following transformation rule.

$$\underline{t} = \underline{Q}\,\underline{t}_c \quad \text{and} \quad \underline{u} = \underline{Q}\,\underline{u}_c \qquad \text{……………………………….................. (6.4)}$$

where, $\underline{u_c} = [u_r \;\; u_\theta \;\; u_z \;\; p]^T$ and $\underline{t_c} = [t_r \;\; t_\theta \;\; t_z \;\; q_n]^T$ are the generalized displacement and traction vectors resolved in cylindrical coordinate frame. The transformation (rotation) matrix $\underline{Q}$ is defined at a certain point "D" by

$$\underline{Q} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{…………………………………………… (6.5)}$$

where "α" is the angular (θ) coordinate of the point "D".

Then the transformation rule for the generalized displacement and traction vectors at points "A" and "P" can be written as

$$\begin{aligned} \underline{u}(A) &= \underline{Q}(A)\,\underline{u}_c(A) \\ \underline{u}(P) &= \underline{Q}(P)\,\underline{u}_c(P) \\ \underline{t}(A) &= \underline{Q}(A)\,\underline{t}_c(A) \\ \underline{t}(P) &= \underline{Q}(P)\,\underline{t}_c(P) \end{aligned} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (6.6)$$

on substituting (6.6) in (6.1) we have for the BIE

$$\underline{c}\,\underline{Q}(A)\underline{u}_c(A) = \int_\Gamma \underline{G}'(A,P)\underline{Q}(P)\underline{t}_c(P)d\Gamma - \int_\Gamma \underline{H}'(A,P)\underline{Q}(P)\underline{u}_c(P)d\Gamma \quad \ldots\ldots\ldots\ldots \; (6.7)$$

on multiplying (6.7) from right by $\underline{Q}^T(A)$, the superscript "T" designates the matrix transpose, we get

$$\underbrace{\underline{Q}^T(A)\,\underline{c}\,\underline{Q}(A)}_{\underline{c}_c}\,\underline{u}_c(A) = \int_\Gamma \underbrace{\underline{Q}^T(A)\underline{G}'(A,P)\underline{Q}(P)}_{\underline{G}_c}\,\underline{t}_c(P)d\Gamma - \int_\Gamma \underbrace{\underline{Q}^T(A)\underline{H}'(A,P)\underline{Q}(P)}_{\underline{H}_c}\,\underline{u}_c(P)d\Gamma$$

or, more simply

$$\boxed{\underline{c}_c\,\underline{u}_c(A) = \int_\Gamma \underline{G}_c(A,P)\,\underline{t}_c(P)d\Gamma - \int_\Gamma \underline{H}_c(A,P)\,\underline{u}_c(P)d\Gamma} \ldots\ldots\ldots\ldots \quad (6.8)$$

The surface integrals in (6.8) can be written as two iterated integrals, over the circumferential direction and over the generator "C", if one notes that the surface differential element in cylindrical coordinates is $d\Gamma = R\,d\theta\,ds$, i.e.

$$\int_\Gamma (\ldots..)d\Gamma \xrightarrow{\text{becomes}} \int_C \int_0^{2\pi} (\ldots..)R\,d\theta\,ds$$

hence (5.8) becomes,

$$\underline{c}_c\,\underline{u}_c(A)=\int_C\int_0^{2\pi}\underline{G}_c(A,P)\underline{t}_c(P)Rd\theta ds-\int_C\int_0^{2\pi}\underline{H}_c(A,P)\underline{u}_c(P)Rd\theta ds \quad \text{......................} (6.9)$$

where the differential element along the generator "C" is

$$ds=\sqrt{dr^2+dz^2}$$

Equation (6.9) is the expression of BIE of poro-elastodynamics in cylindrical coordinates. Here $\underline{G}_c(A,P)$ and $\underline{H}_c(A,P)$ represent the fundamental solution matrices in cylindrical coordinates and are given by

$$\begin{aligned}\underline{G}_c(A,P)&=\underline{Q}^T(A)\underline{G}'(A,P)\underline{Q}(P)\\ \underline{H}_c(A,P)&=\underline{Q}^T(A)\underline{H}'(A,P)\underline{Q}(P)\end{aligned} \quad \text{..} (6.10)$$

also, the free term coefficient $\underline{c}_c$ in cylindrical coordinates is related to $\underline{c}$ by

$$\underline{c}_c=\underline{Q}^T(A)\underline{c}\,\underline{Q}(A) \quad \text{...} (6.11)$$

In view of the orthogonality of $\underline{Q}$ and the form of $\underline{c}$, we have

$$\underline{c}_c=\frac{1}{2}\underline{I} \quad \text{...} (6.12)$$

In (6.11) it is understood that the boundary is smooth at the source point "A".
Now, the variables $r,\vec{r},\vec{n}$ which appear in $\underline{G}_c(A,P)$ and $\underline{H}_c(A,P)$ have the expressions, in cylindrical coordinates, as

$$r=\sqrt{\left[(R\cos\theta-R'\cos\theta')^2+(R\sin\theta-R'\sin\theta')^2+(z-z')^2\right]} \quad \text{...................} (6.13)$$

$$\underline{r} = (r_1, r_2, r_3) = \frac{1}{r}[(R\cos\theta - R'\cos\theta'), (R\sin\theta - R'\sin\theta'), (z - z')] \quad \text{……….} \quad (6.14)$$

$$\underline{n} = (n_1, n_2, n_3) = (n_R \cos\theta, n_R \sin\theta, n_z) \quad \text{……………………………...} \quad (6.15)$$

where n_R, n_z are the cylindrical components of the outer unit normal vector $\underline{n}$ at the integration point "P". It may be noted that due to the axi-symmetry of the body, the θ-component of the normal vanishes, i.e. $n_\theta = 0$.

6.2 Expansion of Field Variables in Complex Fourier Series

The expansion of boundary quantities in complex Fourier series stems from the awareness that these quantities (u_i, p, t_i, q_n) are *periodic in angular direction.* Consider, for instance, the solid displacement components in cylindrical coordinates. From the axi-symmetry, it is immediately realized that

$$u_i(R, \theta, z) = u_i(R, \theta + 2\pi, z) \quad \text{……………………………………..….} \quad (6.16)$$

hence, the boundary displacements are ***"2π" periodic*** in angular direction, the same is true for other boundary quantities. Any periodic function $f(\theta) = f(\theta + T)$, where T is the period, can be expanded into complex Fourier series as

$$f(\theta) = \sum_{k=-\infty}^{\infty} \tilde{f}^k e^{i\omega_k\theta} \quad \text{…………………………………………...……} \quad (6.17)$$

where,

$$\omega_k = \frac{2\pi k}{T} \quad \text{…………………………………………………} \quad (6.18)$$

$$\tilde{f}^k = \frac{1}{T}\int_0^T f(\theta) e^{-i\omega_k\theta} d\theta \quad \text{………………………………………………} \quad (6.19)$$

Here, $\tilde{f}^k$ is the Fourier coefficient at the frequency "k" (k=0, ±1, ±2, ±3,…).

Then, for f to represent a typical element of $\underline{u}$ or $\underline{t}$ which are "2π" periodic, we can write in discrete form

$$\tilde{f}^k = \frac{1}{N}\sum_{n=0}^{N-1} f^n e^{-i2\pi\frac{kn}{N}} \quad , \quad (k = 0,1,2,....., N-1)$$
$$f^n = \sum_{k=0}^{N-1} \tilde{f}^k e^{i2\pi\frac{kn}{N}} \quad , \quad (n = 0,1,2,....., N-1) \qquad \text{..................... (6.20)}$$

The formulas in Eqn.'s (6.20) can be obtained by subdividing the interval $[0, 2\pi]$ into "N" (N is an even integer) equal parts and by taking into account the periodicity of the function "f" with the period 2π. f^n in Eqn.'s (6.20) designate the value of the function at $\theta = \theta_n = n\Delta\theta$, with $\Delta\theta = \frac{2\pi}{N}$. The two formulas in Eqn.'s (6.20) are referred to as "discrete" and "inverse discrete Fourier transform" formulas, respectively. The frequency $k = \frac{N}{2} = k_c$ corresponds to the cut-off frequency, which is the highest frequency that can be considered in the analysis. We note that the discrete and inverse discrete Fourier transforms can effectively be computed by FFT algorithm (Brigham 1988), if the subdivision "N" is chosen to be $N = 2^M$, where "M" is a positive integer.

When the generalized displacement $\underline{u}_c$ and traction $\underline{t}_c$ are expanded in θ-direction in complex Fourier series,

$$\underline{u}_c(R',\theta',z') = \sum_{k=-\infty}^{\infty} \underline{u}_c^k(R',z')e^{ik\theta'}$$
$$\underline{u}_c(R,\theta,z) = \sum_{k=-\infty}^{\infty} \underline{u}_c^k(R,z)e^{ik\theta} \qquad \text{.. (6.21)}$$
$$\underline{t}_c(R,\theta,z) = \sum_{k=-\infty}^{\infty} \underline{t}_c^k(R,z)e^{ik\theta}$$

then the equation (6.9) looks

$$\underline{c}_c \sum_{k=-\infty}^{\infty} \underline{\tilde{u}}_c^k(R',z')e^{ik\theta'} = \sum_{k=-\infty}^{\infty} \left[\begin{array}{l} \int_C \int_0^{2\pi} \underline{G}_c(A,P)\underline{\tilde{t}}_c^k(R,z)e^{ik\theta} R d\theta ds \\ \qquad - \int_C \int_0^{2\pi} \underline{H}_c(A,P)\underline{\tilde{u}}_c^k(R,z)e^{ik\theta} R d\theta ds \end{array} \right] \quad \ldots\ldots\ldots (6.22)$$

since $e^{ik\theta'}e^{-ik\theta'} = 1$, we can play the following trick,

$$\underline{c}_c \sum_{k=-\infty}^{\infty} \underline{\tilde{u}}_c^k(R',z')e^{ik\theta'} = \sum_{k=-\infty}^{\infty} \left[\begin{array}{l} \int_C \int_0^{2\pi} \underline{G}_c(A,P)\underline{\tilde{t}}_c^k(R,z)e^{ik\theta}\left(e^{ik\theta'}e^{-ik\theta'}\right) R d\theta ds \\ \qquad - \int_C \int_0^{2\pi} \underline{H}_c(A,P)\underline{\tilde{u}}_c^k(R,z)e^{ik\theta}\left(e^{ik\theta'}e^{-ik\theta'}\right) R d\theta ds \end{array} \right]$$

and,

$$\underline{c}_c \sum_{k=-\infty}^{\infty} \underline{\tilde{u}}_c^k(R',z')e^{ik\theta'} = \sum_{k=-\infty}^{\infty} e^{ik\theta'} \left[\begin{array}{l} \int_C \int_0^{2\pi} \underline{G}_c(A,P)\underline{\tilde{t}}_c^k(R,z)e^{ik(\theta-\theta')} R d\theta ds \\ \qquad - \int_C \int_0^{2\pi} \underline{H}_c(A,P)\underline{\tilde{u}}_c^k(R,z)e^{ik(\theta-\theta')} R d\theta ds \end{array} \right]$$

collecting all the terms to one side and combining under one summation,

$$\sum_{k=-\infty}^{\infty} e^{ik\theta'} \left\{ \begin{array}{l} \underline{c}_c \underline{\tilde{u}}_c^k(R',z') - \int_C \int_0^{2\pi} \underline{G}_c(A,P)\underline{\tilde{t}}_c^k(R,z)e^{ik(\theta-\theta')} R d\theta ds \\ \qquad + \int_C \int_0^{2\pi} \underline{H}_c(A,P)\underline{\tilde{u}}_c^k(R,z)e^{ik(\theta-\theta')} R d\theta ds \end{array} \right\} = 0$$

since harmonics $e^{ik\theta'}$ are linearly independent, for the above summation to be zero we must have

$$\left[\underline{c}_c \underline{\tilde{u}}_c^k(R',z') - \int_C \int_0^{2\pi} \underline{G}_c(A,P)\underline{\tilde{t}}_c^k(R,z)e^{ik(\theta-\theta')} R d\theta ds + \int_C \int_0^{2\pi} \underline{H}_c(A,P)\underline{\tilde{u}}_c^k(R,z)e^{ik(\theta-\theta')} R d\theta ds \right] = 0$$

or,

$$\underline{c}_c\, \underline{\tilde{u}}_c^{\,k}(R',z') = \int_C \int_0^{2\pi} \underline{G_c}(A,P) e^{ik(\theta-\theta')} d\theta\, \underline{\tilde{t}}_c^{\,k}(R,z) R ds - \int_C \int_0^{2\pi} \underline{H_c}(A,P) e^{ik(\theta-\theta')} d\theta\, \underline{\tilde{u}}_c^{\,k}(R,z) R ds \quad \text{.... (6.23)}$$

since $\underline{\tilde{t}}_c^{\,k}(R,z)$ and $\underline{\tilde{u}}_c^{\,k}(R,z)$ no longer are functions of "θ".

It can be shown that the fundamental solutions $\underline{G}_c(A,P)$ and $\underline{H}_c(A,P)$ are functions of the form

$$\underline{G}_c(r,\theta-\theta',z-z'),\ \underline{H}_c(r,\theta-\theta',z-z') \quad \text{.. (6.24)}$$

When (6.23) is observed along with the form in (6.24), one can identify the inner integrals in (6.23)

$$\begin{aligned} \underline{G}_c^k(R',z';R,z) &= \int_0^{2\pi} \underline{G}_c(A,P) e^{ik(\theta-\theta')} d(\theta-\theta') \\ \underline{H}_c^k(R',z';R,z) &= \int_0^{2\pi} \underline{H}_c(A,P) e^{ik(\theta-\theta')} d(\theta-\theta') \end{aligned} \quad \text{.................................. (6.25)}$$

since θ' is unvarying under the integral sign; or equivalently we can write

$$\begin{aligned} \underline{G}_c^k(R',z';R,z) &= \int_0^{2\pi} \underline{G}_c(A,P)\big|_{\theta'=0} e^{ik\theta} d\theta \\ \underline{H}_c^k(R',z';R,z) &= \int_0^{2\pi} \underline{H}_c(A,P)\big|_{\theta'=0} e^{ik\theta} d\theta \end{aligned} \quad \text{.. (6.26)}$$

Then for the k-th Fourier component of boundary quantities, we have

$$\underline{c}_c \underline{\tilde{u}}_c^k(R',z') = \int_C \underline{G}_c^k(R',z';R,z)\underline{\tilde{t}}_c^k(R,z)R\,ds - \int_C \underline{H}_c^k(R',z';R,z)\underline{\tilde{u}}_c^k(R,z)R\,ds \quad \text{............ (6.27)}$$

Again, it may be noted that "C" is a generating curve of the axi-symmetric body passing through point "P → (R, θ, z)". Because the integrals in (6.27) are line integrals (in stead of surface integrals in 6.8), this procedure reduces the dimensionality of the boundary integral equations by one.

6.3 Spatial Discretization and Boundary Element Equations

The first step in our aim is to solve (6.27) for a number of Fourier coefficients of the unknown boundary quantities, i.e. $\underline{\tilde{u}}_c^k, \underline{\tilde{t}}_c^k$ once the Fourier coefficients are available, the solution in (R, θ, z) system can then be produced numerically by an inverse FFT.

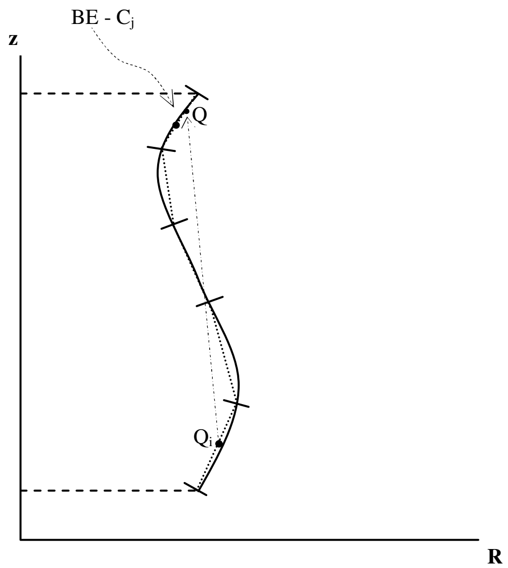

Figure 6.2 Boundary Element discretization of the generator in R-z plane

To solve (6.27) we introduce the constant element formulation, where the curve "C" is approximated by straight line segments (called boundary elements) over which the boundary quantities are assumed to be constant. The node of a boundary element will be taken as its mid point. Let the node and boundary of the j-th element be $Q_j \rightarrow (R', z')$ and C_j respectively, see **Figure 6.2**.

When we introduce the spatial discretization equation (6.27) reads

$$\underline{c}_c \underline{\tilde{u}}_c^k(Q_i) = \sum_j \underbrace{\left(\int_{C_j} \underline{G}_c^k(Q_i\,;Q) R ds \right)}_{\underline{\tilde{G}}_{ij}^k} \underline{\tilde{t}}_{c\,j}^k - \sum_j \underbrace{\left(\int_{C_j} \underline{H}_c^k(Q_i\,;Q) R ds \right)}_{\underline{\tilde{H}}_{ij}^k} \underline{\tilde{u}}_{c\,j}^k \quad \ldots\ldots\ldots (6.28)$$

$$\boxed{\underline{c}_c \underline{\tilde{u}}_{c\,i}^k = \sum_j \underline{\tilde{G}}_{ij}^k \underline{\tilde{t}}_{c\,j}^k - \sum_j \underline{\tilde{H}}_{ij}^k \underline{\tilde{u}}_{c\,j}^k} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.29)$$

where, $\underline{\tilde{u}}_{c\,j}^k$ and $\underline{\tilde{t}}_{c\,j}^k$ are the values of $\underline{\tilde{u}}_c^k$ and $\underline{\tilde{t}}_c^k$ over the element "j", and

$$\begin{aligned} \underline{\tilde{G}}_{ij}^k &= \int_{C_j} \underline{G}_c^k(Q_i\,;Q) R ds \\ \underline{\tilde{H}}_{ij}^k &= \int_{C_j} \underline{H}_c^k(Q_i\,;Q) R ds \end{aligned} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.30)$$

When the definitions of $\underline{G}_c^k(Q_i\,;Q)$ and $\underline{H}_c^k(Q_i\,;Q)$ are inserted in (6.30) and change the order of integration is reversed, we get

$$\begin{aligned} \underline{\tilde{G}}_{ij}^k &= \int_0^{2\pi} \underbrace{\left(\int_{C_j} \underline{G}_c \left(\underbrace{R', z'}_{Q_i}; \underbrace{R, z}_{Q}; \theta - \theta' \right) \Bigg|_{\theta'=0} R\, ds \right)}_{\underline{G}_{ij}(\theta)} e^{ik\theta} d\theta \\ \underline{\tilde{H}}_{ij}^k &= \int_0^{2\pi} \underbrace{\left(\int_{C_j} \underline{H}_c \left(\underbrace{R', z'}_{Q_i}; \underbrace{R, z}_{Q}; \theta - \theta' \right) \Bigg|_{\theta'=0} R\, ds \right)}_{\underline{H}_{ij}(\theta)} e^{ik\theta} d\theta \end{aligned} \quad \ldots\ldots\ldots\ldots\ldots (6.31)$$

here we introduce,

$$\underline{G}_{ij}(\theta) = \int_{C_j} \underline{G}_c\left(\underbrace{R', z'}_{Q_i}; \underbrace{R, z}_{Q}; \theta - \theta'\right)\Bigg|_{\theta'=0} R\, ds$$
$$\underline{H}_{ij}(\theta) = \int_{C_j} \underline{H}_c\left(\underbrace{R', z'}_{Q_i}; \underbrace{R, z}_{Q}; \theta - \theta'\right)\Bigg|_{\theta'=0} R\, ds \quad \text{.......................................} \quad (6.32)$$

hence (6.31) abbreviates to

$$\tilde{\underline{G}}_{ij}^{k} = \int_0^{2\pi} \underline{G}_{ij}(\theta) e^{ik\theta} d\theta$$
$$\tilde{\underline{H}}_{ij}^{k} = \int_0^{2\pi} \underline{H}_{ij}(\theta) e^{ik\theta} d\theta \quad \text{...} \quad (6.33)$$

The integrals in (6.32) with respect to the integration point Q ≡ (R, z) in R-z plane can be computed numerically for a given θ, hence $\underline{G}_{ij}(\theta)$ and $\underline{H}_{ij}(\theta)$ are determined at a series of angles θ; then the integrals in (6.33), which are in the form of Fourier integrals, can be evaluated effectively by FFT algorithm.

For that we write,

$$\tilde{\underline{G}}_{ij}^{k} = \frac{2\pi}{N} \sum_{n=0}^{N-1} \underline{G}_{ij}^{n} e^{i2\pi\frac{kn}{N}}$$
$$\tilde{\underline{H}}_{ij}^{k} = \frac{2\pi}{N} \sum_{n=0}^{N-1} \underline{H}_{ij}^{n} e^{i2\pi\frac{kn}{N}} \quad \text{...} \quad (6.34)$$

where $\underline{G}_{ij}^{n}$ and $\underline{H}_{ij}^{n}$ are the values of $\underline{G}_{ij}(\theta)$ and $\underline{G}_{ij}(\theta)$ at $\theta = \theta_n$, i.e.

$$\underline{G}_{ij}^{n} = \underline{G}_{ij}(\theta)\Big|_{\theta_n}$$
$$\underline{H}_{ij}^{n} = \underline{H}_{ij}(\theta)\Big|_{\theta_n} \quad \text{...} \quad (6.35)$$

where,

$$\theta_n = n\Delta\theta \qquad \Delta\theta = \frac{2\pi}{N} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.36)$$

To summarize, $\underline{\tilde{G}}_{ij}^{k}$ and $\underline{\tilde{H}}_{ij}^{k}$ in (6.33) can be computed numerically in two steps:

- Take $\theta = \theta_n = \dfrac{2\pi n}{N}$ and compute the line integrals in (6.32) by Gaussian quadrature (or by a special Gaussian rule if the integral is singular), which determines N values of $\underline{G}_{ij}^{n}$ and $\underline{H}_{ij}^{n}$, (n=0 … N-1).
- Insert these values in (6.34) to compute the sums, i.e. $\underline{\tilde{G}}_{ij}^{k}$ and $\underline{\tilde{H}}_{ij}^{k}$ for the frequency range k = 0 … N-1, for this step FFT algorithm can be used to facilitate the computations.

Recall now, the discretized boundary element equation (6.29), this equation can be written in matrix form as

$$\underline{\tilde{H}}^{k}\underline{\tilde{u}}^{k} = \underline{\tilde{G}}^{k}\underline{\tilde{t}}^{k}, \qquad k = 0, 1, 2, ..., N' - 1 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.37)$$

In (6.37) the Fourier expansion is truncated to $N'=2^{M'}$ terms, it may be noted that the number of subdivisions $N=2^{M}$ for integration in θ-direction does not have to be equal to the number of terms in truncated complex Fourier series sum; for more accurate integrations $N \geq N'$. The solution of (6.37) for unknown boundary quantities determines the Fourier coefficients of all the boundary quantities $\underline{\tilde{u}}^{k}$ and $\underline{\tilde{t}}^{k}$, the solution in (R, θ, z) system is then found through inverse FFT over all “k”. In constant element formulation, $\underline{\tilde{H}}^{k}$ and $\underline{\tilde{G}}^{k}$ are (4M x 4M) dimensional matrices, where M is the number of boundary elements, with

$$\underline{\tilde{G}}^{k} = \left[\underline{\tilde{G}}_{ij}^{k}\right] \qquad \underline{\tilde{H}}^{k} = \left[\underline{\tilde{H}}_{ij}^{k} + \frac{1}{2}\underline{I}\delta_{ij}\right] \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.38)$$

$\tilde{\underline{u}}^k$ and $\tilde{\underline{t}}^k$ on the other hand are (3Mx1) column matrices with

$$\tilde{\underline{u}}^k = \left[\tilde{\underline{u}}_i^k\right] \qquad \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(6.39)}$$
$$\tilde{\underline{t}}^k = \left[\tilde{\underline{t}}_i^k\right]$$

Finally the formulation is complete and we can summarise the procedure as follows:

1. Choose $N = 2^{MM}$ (N is the number of subdivisions in θ - direction), and compute the increment $\Delta\theta = 2\pi / N$. N must be a number in power of "2" for FFT to be applied.

2. Discretize the generating curve "C" in (R, z) plane, let the number of boundary elements be M.

3. Choose the number of terms $N' = 2^{MMp}$ to be retained in complex Fourier expansion of boundary variables. Compute the Fourier coefficients of the boundary excitations in θ-direction at frequencies $k = 0, 1, 2, ..., N' - 1$ either analytically if they have a simple analytical form or by FFT algorithm if their analytical form is complicated or they are specified in discrete form.

4. Compute $\tilde{\underline{G}}_{ij}^s$ and $\tilde{\underline{H}}_{ij}^s$ (s=0…N-1), and form the system matrices $\tilde{\underline{H}}^k$ and $\tilde{\underline{G}}^k$ for $k = 0 ... N' - 1$. A frequency shift is necessary when assembling $\tilde{\underline{G}}_{ij}^s$ and $\tilde{\underline{H}}_{ij}^s$ into $\tilde{\underline{G}}^k$ and $\tilde{\underline{H}}^k$, as "s" and "k" run through different ranges.

5. Solution of the complex algebraic system of equations in (6.37) together with the specified boundary conditions yields the Fourier coefficients $\tilde{\underline{u}}^k$ and $\tilde{\underline{t}}^k$ at frequency points $k = 0, 1, 2, ..., N' - 1$.

6. By an inverse FFT evaluate the boundary quantities in (R, θ, z) space.

6.4 Computation of $\underline{G}_{ij}(\theta)$ and $\underline{H}_{ij}(\theta)$

When the collocation (source) point Q_i does not belong to the boundary element, over which the integration is performed, the integrals in (6.32) are all ordinary (non singular) integrals and can be computed to high precision by Gaussian Integration. On the other hand, when the collocation point (source point) belongs to the boundary element (i.e. when, when i = j and θ = 0 in (6.32)), these expressions become singular and require special treatment.

6.4.1 The Non-Singular Integrals

Here, we start by repeating (6.32),

$$\underline{G}_{ij}(\theta) = \int_{C_j} \underline{G}_c\left(\underbrace{R', z'}_{Q_i}; \underbrace{R, z}_{Q}; \theta - \theta'\right)\Bigg|_{\theta'=0} R\,ds$$

$$\underline{H}_{ij}(\theta) = \int_{C_j} \underline{H}_c\left(\underbrace{R', z'}_{Q_i}; \underbrace{R, z}_{Q}; \theta - \theta'\right)\Bigg|_{\theta'=0} R\,ds$$

Recall that,

$$\underline{G}_c(A,P) = \underline{Q}^T(A)\underline{G}'(A,P)\underline{Q}(P)$$
$$\underline{H}_c(A,P) = \underline{Q}^T(A)\underline{H}'(A,P)\underline{Q}(P)$$

then, (6.32) becomes

$$\underline{G}_{ij}(\theta) = \int_{C_j} \left(\underline{Q}^T(\theta')\underline{G}'(Q_i; Q; \theta', \theta)\underline{Q}(\theta)\right)\Big|_{\theta'=0} R\,ds$$

$$\underline{H}_{ij}(\theta) = \int_{C_j} \left(\underline{Q}^T(\theta')\underline{H}'(Q_i; Q; \theta', \theta)\underline{Q}(\theta)\right)\Big|_{\theta'=0} R\,ds \qquad \text{.............................. (6.40)}$$

or

$$\underline{G}_{ij} = \int_{C_j} \underline{G}'(Q_i;Q;\theta',\theta)\Big|_{\theta'=0} R\,ds * \underline{Q}(\theta)$$
$$\underline{H}_{ij} = \int_{C_j} \underline{H}'(Q_i;Q;\theta',\theta)\Big|_{\theta'=0} R\,ds * \underline{Q}(\theta) \qquad \text{.................................... (6.41)}$$

when $i \neq j$ or $\theta \neq 0$, $\underline{G}_{ij}$ and $\underline{H}_{ij}$ will be computed by standard Gaussian quadrature, else the integrands become singular whose treatment is considered in the next section. In (6.41), $Q_i = (R', z')$ represents R-z coordinates of the source point as the node 'i' in $\theta' = 0$ plane, and $Q = (R, z)$ represents R-z coordinates of the integration point in the element 'j' which is in θ-plane.

Over a typical boundary element, Figure 6.3, the coordinates of the mid-node are given by

$$a_R^{(k)} = \frac{R_2^{(k)} + R_1^{(k)}}{2}$$
$$a_z^{(k)} = \frac{z_2^{(k)} + z_1^{(k)}}{2} \qquad \text{.. (6.42)}$$

where, $R_i^{(k)}, z_i^{(k)}$ are the coordinates of the end-points of the BE in Rz-plane.

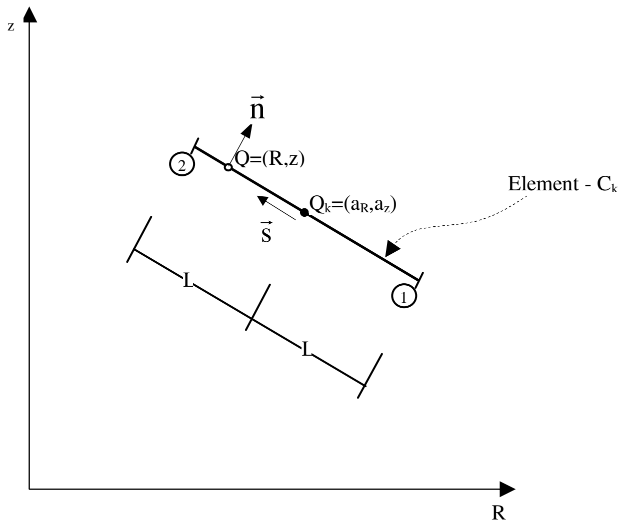

Figure 6.3 Typical boundary element, unit tangent and normal vectors

We define

$$b_R^{(k)} = \frac{R_2^{(k)} - R_1^{(k)}}{2}$$
$$b_z^{(k)} = \frac{z_2^{(k)} - z_1^{(k)}}{2} \quad \text{(6.43)}$$

Then, one has for the parametric description of the integration point Q,

$$R^{(k)} = a_R^{(k)} + b_R^{(k)} t$$
$$z^{(k)} = a_z^{(k)} + b_z^{(k)} t \quad \text{(6.44)}$$

where t = -1 .. 1 is the parameter. With this parametric form, one has $ds = L\,dt$, L is the half element length (Figure 6.3). Then (6.41) becomes

$$\underline{G}_{ij} = \int_{-1}^{1} \underline{G}'\left(a_R^{(i)}, a_z^{(i)}; R^{(j)}, z^{(j)}; \theta', \theta\right)\Big|_{\theta'=0} \left(a_R^{(j)} + b_R^{(j)} t\right) L\,dt \; * \; \underline{Q}(\theta)$$
$$\underline{H}_{ij} = \int_{-1}^{1} \underline{H}'\left(a_R^{(i)}, a_z^{(i)}; R^{(j)}, z^{(j)}; \theta', \theta\right)\Big|_{\theta'=0} \left(a_R^{(j)} + b_R^{(j)} t\right) L\,dt \; * \; \underline{Q}(\theta) \quad \text{(6.45)}$$

The kernels in (6.45) contain variables, such as

$$r = \sqrt{\left(R^{(j)}\cos(\theta) - a_R^{(i)}\cos(\theta')\right)^2 + \left(R^{(j)}\sin(\theta) - a_R^{(i)}\sin(\theta')\right)^2 + \left(z^{(j)} - a_z^{(i)}\right)^2}$$

or, when $\theta' = 0$

$$r = \sqrt{\left(R^{(j)}\cos(\theta) - a_R^{(i)}\right)^2 + \left(R^{(j)}\sin(\theta)\right)^2 + \left(z^{(j)} - a_z^{(i)}\right)^2} \quad \text{(6.46)}$$

also,

$$r_1 = \frac{R^{(j)}\cos(\theta) - a_R^{(i)}}{r}$$
$$r_2 = \frac{R^{(j)}\sin(\theta)}{r} \qquad \text{..} \; (6.47)$$
$$r_3 = \frac{z^{(j)} - a_z^{(i)}}{r}$$

the unit vector parallel to the element in the direction from (1) to (2):

$$s_R^{(j)} = \frac{R_2^{(j)} - R_1^{(j)}}{2*L} = \frac{b_R^{(j)}}{L} = \bar{b}_R$$
$$s_\theta^{(j)} = 0 \qquad \text{..} \; (6.48)$$
$$s_z^{(j)} = \frac{z_2^{(j)} - z_1^{(j)}}{2*L} = \frac{b_z^{(j)}}{L} = \bar{b}_z$$

and the unit normal vector to the BE has components,

$$n_R = s_z \quad \text{and} \quad n_z = -s_R \quad \text{..} \; (6.49)$$

6.4.2 Treatment of Singular Integrals

The singular terms in (6.45) become singular when $i = j$ and $\theta(= \theta') = 0$. The kernels in (6.41) and (6.45) are the fundamental solutions, which can be written as the sum of a regular and a singular part ($\underline{G}' = \underline{G}'_{Reg} + \underline{G}'_{Sing}$ and $\underline{H}' = \underline{H}'_{Reg} + \underline{H}'_{Sing}$). The singular parts of fundamental solutions were previously investigated in section 5.3. Therefore, the singularity problem in equations (6.41 - 6.45) essentially reduce to the integration of the singular parts of the fundamental solutions. To illustrate, consider e.g. the fundamental solution for the displacements due to unit load on the solid that is

$$u_{ij}^* = \frac{1}{4\pi\mu\lambda_3^2 r}\left[\frac{\lambda_4^2 - \lambda_2^2}{\lambda_1^2 - \lambda_2^2} R_{ij}^{(1)} e^{-\lambda_1 r} + \frac{\lambda_4^2 - \lambda_1^2}{\lambda_2^2 - \lambda_1^2} R_{ij}^{(2)} e^{-\lambda_2 r} + \left(\lambda_3^2 \delta_{ij} - R_{ij}^{(3)}\right) e^{-\lambda_3 r}\right]$$

whose singular part as shown earlier is the elastostatic fundamental solution, i.e.

$$\left(u^*_{ij}\right)_{Sing} = \frac{1}{16\pi\mu(1-\nu)}\left[(3-4\nu)\delta_{ij} + r_i r_j\right]\frac{1}{r} \quad \text{......................................} \quad (6.50)$$

Therefore, the singular part in (6.45) is

$$\left(G'_{mn}\right)^{Sing}_{ij} = \frac{1}{16\pi\mu(1-\nu)}\left\{(3-4\nu)\delta_{mn}\int_{-1}^{1}\frac{1}{r}\bigg|_{\substack{\theta'=0\\ \theta=0}}\left(a_R^{(j)} + b_R^{(j)}t\right)L\,dt + \int_{-1}^{1}\frac{r_m r_n}{r}\bigg|_{\substack{\theta'=0\\ \theta=0}}\left(a_R^{(j)} + b_R^{(j)}t\right)L\,dt\right\} * \underline{Q}(\theta)\big|_{\theta=0}$$

$$\text{............................} \quad (6.51)$$

As an example, we may consider the term $\left(G_{11}\right)^{Sing}_{\underline{i}\underline{i}}$,

$$\left(G'_{11}\right)^{Sing}_{\underline{i}\underline{i}} = \frac{1}{16\pi\mu(1-\nu)}\left\{(3-4\nu)\int_{-1}^{1}\frac{1}{r}\bigg|_{\substack{\theta'=0\\ \theta=0}}\left(a_R^{(i)} + b_R^{(i)}t\right)L\,dt + \int_{-1}^{1}\frac{r_1^{\,2}}{r}\bigg|_{\substack{\theta'=0\\ \theta=0}}\left(a_R^{(i)} + b_R^{(i)}t\right)L\,dt\right\} \quad \text{....} \quad (6.52)$$

then, from (6.46) and (6.47) we have for $i = j$ and $\theta = 0$

$$\begin{aligned} r &= \sqrt{\left(R^{(i)}\cos(\theta) - a_R^{(i)}\right)^2 + \left(R^{(i)}\sin(\theta)\right)^2 + \left(z^{(i)} - a_z^{(i)}\right)^2} \\ &= \sqrt{\left[\left(a_R^{(i)} + b_R^{(i)}t\right)*1 - a_R^{(i)}\right]^2 + \left[\left(a_R^{(i)} + b_R^{(i)}t\right)*0\right]^2 + \left[\left(a_z^{(i)} + b_z^{(i)}t\right) - a_z^{(i)}\right]^2} \quad \text{................} \quad (6.53) \\ &= L\,|\,t\,| \end{aligned}$$

$$r_1 = \frac{R^{(i)}\cos(0) - a_R^{(i)}}{r} = \frac{b_R^{(i)}t}{L\,|\,t\,|} = \frac{b_R^{(i)}}{L}\mathrm{sign}(t) \quad \text{.......................................} \quad (6.54)$$

Thus, for the term $\left(G'_{11}\right)_{\underline{i}\underline{i}}$ we have

$$(G'_{11})_{\underline{ii}} = \frac{1}{16\pi\mu(1-\nu)}\left\{(3-4\nu)\int_{-1}^{1}\frac{1}{L|t|}\left(a_R^{(i)}+b_R^{(i)}t\right)L\,dt + \int_{-1}^{1}\frac{\left(b_R^{(i)}\right)^2}{L^2(L|t|)}\left(a_R^{(i)}+b_R^{(i)}t\right)L\,dt\right\}$$

$$= \frac{1}{16\pi\mu(1-\nu)}\left\{(3-4\nu)\left[a_R^{(i)}\int_{-1}^{1}\frac{1}{|t|}dt + b_R^{(i)}\underbrace{\int_{-1}^{1}\frac{t}{|t|}dt}_{=0}\right] + \left(\frac{b_R^{(i)}}{L}\right)^2\left[a_R^{(i)}\int_{-1}^{1}\frac{1}{|t|}dt + b_R^{(i)}\underbrace{\int_{-1}^{1}\frac{t}{|t|}dt}_{=0}\right]\right\}$$

$$= \frac{a_R^{(i)}}{16\pi\mu(1-\nu)}\left\{(3-4\nu)+\left(\frac{b_R^{(i)}}{L}\right)^2\right\}\underbrace{\int_{-1}^{1}\frac{1}{|t|}dt}_{\text{divergent}}$$

………………….……………..…. (6.55)

Therefore, $(G'_{mn})_{ij}^{Sing}$ and $(H'_{mn})_{ij}^{Sing}$ are *not* generally *integrable* in the form (6.45).

Now, consider a surface element S_k formed around the axi-symmetric BE C_k and subtended by the angle $\Delta\theta$, Figure 6.4. If the angle $\Delta\theta$ is sufficiently small, one can write (by the mean value theorem) for the surface integral of some function $f(P)$ over S_k

$$\int_{S_k} f(P)\,dA = \int_{-\frac{\Delta\theta}{2}}^{\frac{\Delta\theta}{2}}\left\{\int_{C_k} f(\underbrace{Q}_{(R,z)},\theta)R\,ds\right\}d\theta \approx \Delta\theta\int_{C_k} f(Q,\theta=0)R\,ds \quad \text{……………..….. (6.56)}$$

Hence, we propose that (after Özkan 1995) the singular parts of integrals in (6.45) when $i = j$ and $\theta = 0$ can be approximated by the following surface integrals (see Figure 6.4),

$$\boxed{\begin{aligned} \underline{G}_0 &= \frac{1}{\Delta\theta}\int_{S_k}\underline{G}_c\left(\underbrace{R',z'}_{Q_k};R,z;\theta-\theta'\right)\Bigg|_{\theta'=0} dA \\ \underline{H}_0 &= \frac{1}{\Delta\theta}\int_{S_k}\underline{H}_c\left(R',z';R,z;\theta-\theta'\right)\Big|_{\theta'=0} dA \end{aligned}}$$

………………………………….. (6.57)

when $\Delta\theta \to 0$, $\underline{G}_0 \to \underline{G}_{kk}(\theta = 0)$ and $\underline{H}_0 \to \underline{H}_{kk}(\theta = 0)$, where subscript 'k' refers to the constant boundary element 'C_k' along the generator and its node.

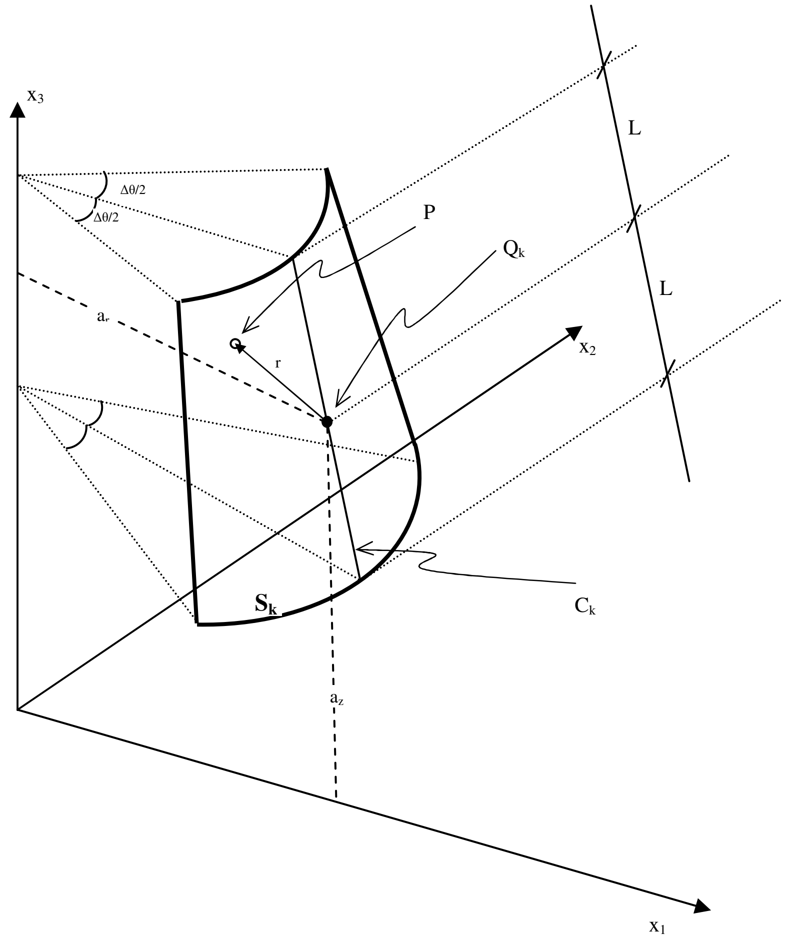

Figure 6.4 Surface S_k about element C_k for singular integration.

Since, θ is small, we can use the approximations

$$\sin(\theta) \approx \theta \quad \text{and} \quad \cos(\theta) = 1 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(6.58)}$$

In (6.57), the coordinates of the node Q_k are

$$R' = a_R^{(k)} \quad ; \quad z' = a_z^{(k)} \qquad \text{.. (6.59)}$$

and for the integration point 'P', we can write,

$$\begin{aligned} x_1 &= R\underbrace{\cos(\theta)}_{\approx 1} = \left(a_R^{(k)} + b_R^{(k)}u\right) \\ x_2 &= R\sin(\theta) = \left(a_R^{(k)} + b_R^{(k)}u\right)\theta \\ x_3 &= z = a_z^{(k)} + b_z^{(k)}u \\ \theta &= \frac{\Delta\theta}{2}v \end{aligned} \qquad \text{.. (6.60)}$$

where,

$$-1 \le u, v \le 1 \qquad \text{.. (6.61)}$$

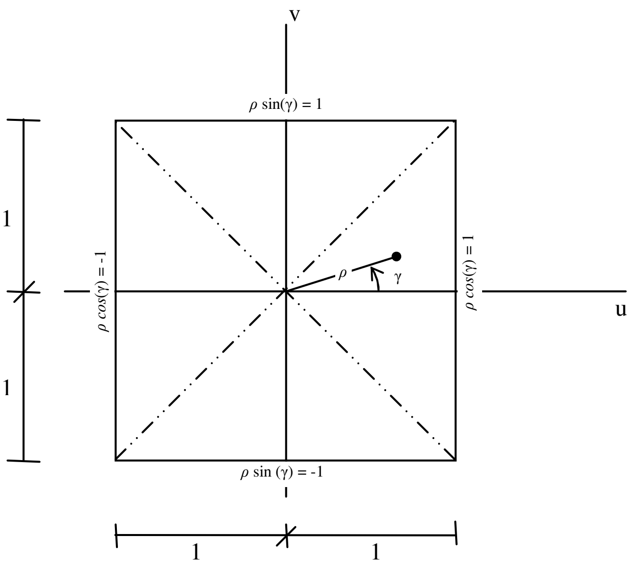

Figure 6.5 The image of surface element S_k in u-v space and the polar coordinates.

Equations (6.60) parameterize the curved surface element S_k and define a coordinate transformation from Rθz system to u-v plane.

When (6.60) is used in (6.46) and (6.47), we have

$$r = \sqrt{L^2u^2 + \left(\frac{\Delta\theta}{2}\right)^2 \left(a_R^{(k)} + b_R^{(k)}u\right)^2 v^2} = L\sqrt{u^2 + \left(\frac{\Delta\theta}{2}\right)^2 \left(\bar{a}_R^{(k)} + \bar{b}_R^{(k)}u\right)^2 v^2}$$

or with the abbreviation

$$D(u,v) = \sqrt{u^2 + \left(\frac{\Delta\theta}{2}\right)^2 \left(\bar{a}_R^{(k)} + \bar{b}_R^{(k)}u\right)^2 v^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(6.62)}$$

$$r = L * D(u,v) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(6.63)}$$

and

$$\begin{aligned} r_1 &= \frac{x_1 - a_1}{r} = \frac{a_R + b_R u - a_R}{L * D(u,v)} = \bar{b}_R \frac{u}{D(u,v)} \\ r_2 &= \frac{x_2 - a_2}{r} = \frac{(a_R + b_R u)\frac{\Delta\theta}{2} v - 0}{L * D(u,v)} = \frac{(\bar{a}_R + \bar{b}_R u)v}{D(u,v)} \frac{\Delta\theta}{2} \\ r_3 &= \frac{x_3 - a_3}{r} = \frac{b_z u}{L * D(u,v)} = \bar{b}_z \frac{u}{D(u,v)} \end{aligned} \quad \ldots\ldots\ldots\ldots\ldots \text{(6.64)}$$

where,

$$\begin{aligned} \bar{a}_R &= \frac{a_R}{L} \quad ; \quad \bar{b}_R = \frac{b_R}{L} \\ \bar{a}_z &= \frac{a_z}{L} \quad ; \quad \bar{b}_z = \frac{b_z}{L} \end{aligned} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(6.65)}$$

Note that,

$$\begin{aligned} a_1 &= R'\cos(\theta') = R' = a_R \\ a_2 &= R'\sin(\theta') = 0 \\ a_3 &= z' = a_z \end{aligned}$$

and the normal vector to the surface element is

$$\begin{aligned} \underline{n} &= (n_1, n_2, n_3) \\ &= (n_R \cos\theta,\ \ n_R \sin\theta,\ \ n_z) \end{aligned}$$

since θ is small,

$$\underline{n} = \left(n_R,\ \ n_R\theta,\ \ n_z\right) = \left(n_R,\ \ n_R\frac{\Delta\theta}{2}v,\ \ n_z\right) \quad \text{.......... (6.66 a)}$$

since the surface element S^k is generated by a straight line, n_R and n_z components of the normal vector does not change with position and are given by (see 6.49)

$$n_R = \bar{b}_z \qquad ; \qquad n_z = -\bar{b}_R$$

therefore

$$\underline{n} = \left(\bar{b}_z,\ \ \underbrace{\bar{b}_z\frac{\Delta\theta}{2}v}_{\approx 0},\ \ -\bar{b}_R\right) \quad \text{.......... (6.66 b)}$$

Also, the differential surface element 'dA' becomes

$$dA = R\ d\theta ds = R\ J\ dudv \quad \text{.......... (6.67)}$$

where,

$$J = L\left(\frac{\Delta\theta}{2}\right) \quad \text{.......... (6.68)}$$

It may be noted that the jacobian in (6.68) is given by

$$J = \frac{\partial(s,\theta)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial s}{\partial u} & \frac{\partial s}{\partial v} \\ \frac{\partial \theta}{\partial u} & \frac{\partial \theta}{\partial v} \end{vmatrix} = \begin{vmatrix} L & 0 \\ 0 & \frac{\Delta\theta}{2} \end{vmatrix}$$

or ……………………………………..…… (6.69)

$$J = \left| \vec{G} \right|$$

where $\vec{G}$ is the infinitesimal area vector and in this case,

$$\vec{G} = -\det\begin{bmatrix} \vec{e}_R & \vec{e}_\theta & \vec{e}_z \\ \frac{\partial R}{\partial u} & \frac{\partial \theta}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial R}{\partial v} & \frac{\partial \theta}{\partial v} & \frac{\partial z}{\partial v} \end{bmatrix} \quad \text{……………………………………………} \quad (6.70)$$

Now, the singular parts of the fundamental solutions are to be integrated according to the formula (6.57). The singular parts of the fundamental solutions have been derived in chapter 5, section (5.3). Recall the form of the singular part of the first fundamental solution matrix

$$\underline{G}^{(\sin g)} = \begin{bmatrix} \left(u_{11}^*\right)^{(\sin g)} & \left(u_{21}^*\right)^{(\sin g)} & \left(u_{31}^*\right)^{(\sin g)} & \frac{\left(p_1^*\right)^{(\sin g)}}{i\omega} \\ \left(u_{12}^*\right)^{(\sin g)} & \left(u_{22}^*\right)^{(\sin g)} & \left(u_{32}^*\right)^{(\sin g)} & \frac{\left(p_2^*\right)^{(\sin g)}}{i\omega} \\ \left(u_{13}^*\right)^{(\sin g)} & \left(u_{23}^*\right)^{(\sin g)} & \left(u_{33}^*\right)^{(\sin g)} & \frac{\left(p_3^*\right)^{(\sin g)}}{i\omega} \\ -i\omega\left(u_1^*\right)^{(\sin g)} & -i\omega\left(u_2^*\right)^{(\sin g)} & -i\omega\left(u_3^*\right)^{(\sin g)} & -\left(p^*\right)^{(\sin g)} \end{bmatrix} \quad \text{…………………} \quad (6.71)$$

Hence according to (6.57)

$$\underline{G}^0 = \frac{1}{\Delta\theta}\int_{S^k} \begin{bmatrix} \left(u_{11}^*\right)^{(\sin g)} & \left(u_{21}^*\right)^{(\sin g)} & \left(u_{31}^*\right)^{(\sin g)} & \frac{\left(p_1^*\right)^{(\sin g)}}{i\omega} \\ \left(u_{12}^*\right)^{(\sin g)} & \left(u_{22}^*\right)^{(\sin g)} & \left(u_{32}^*\right)^{(\sin g)} & \frac{\left(p_2^*\right)^{(\sin g)}}{i\omega} \\ \left(u_{13}^*\right)^{(\sin g)} & \left(u_{23}^*\right)^{(\sin g)} & \left(u_{33}^*\right)^{(\sin g)} & \frac{\left(p_3^*\right)^{(\sin g)}}{i\omega} \\ -i\omega\left(u_1^*\right)^{(\sin g)} & -i\omega\left(u_2^*\right)^{(\sin g)} & -i\omega\left(u_3^*\right)^{(\sin g)} & -\left(p^*\right)^{(\sin g)} \end{bmatrix} R\, dA \quad \text{…………………} \quad (6.72)$$

Now, consider $\left(u_{ij}^{*}\right)^{(Sing)}$ terms in (6.72),

Recall → $\left(u_{ij}^{*}\right)_{Sing} = \frac{1}{16\pi\mu(1-\nu)}\left[(3-4\nu)\delta_{ij} + r_i r_j\right]\frac{1}{r}$

$$G_{11}^{0} = \frac{1}{\Delta\theta}\int_{S^k} u_{11}^{*\,(\sin g)}\, R\, dA \quad \text{.. (6.73)}$$

where

$$\left(u_{11}^{*}\right)^{(\sin g)} = \frac{1}{16\pi\mu(1-\nu)}\left[(3-4\nu) + (r_1)^2\right]\frac{1}{r} \quad \text{.. (6.74)}$$

$$\begin{aligned}
G_{11}^{0} &= \frac{1}{\Delta\theta}\int_{S^k} \frac{1}{16\pi\mu(1-\nu)}\left[(3-4\nu)+(r_1)^2\right]\frac{1}{r} R\, dA \\
&= \frac{1}{\Delta\theta}\frac{1}{16\pi\mu(1-\nu)}\int_{-1}^{1}\int_{-1}^{1}\left[(3-4\nu)+\left(\bar{b}_R \frac{u}{D(u,v)}\right)^2\right]\frac{1}{L*D(u,v)}\left(a_R + b_R u\right)L\left(\frac{\Delta\theta}{2}\right)dudv \\
&= \frac{1}{2}\frac{1}{16\pi\mu(1-\nu)}\left\{(3-4\nu)\int_{-1}^{1}\int_{-1}^{1}\frac{\left(a_R + b_R u\right)}{D(u,v)}dudv + \left(\bar{b}_R\right)^2\int_{-1}^{1}\int_{-1}^{1}\frac{u^2\left(a_R + b_R u\right)}{D^3(u,v)}dudv\right\}
\end{aligned}$$

or

$$G_{11}^{0} = \frac{(3-4\nu)}{16\pi\mu(1-\nu)}\frac{a_R}{2}\int_{-1}^{1}\int_{-1}^{1}\frac{\left(1+\frac{\bar{b}_R}{\bar{a}_R}u\right)}{D(u,v)}dudv + \frac{1}{16\pi\mu(1-\nu)}\frac{a_R}{2}\left(\bar{b}_R\right)^2\int_{-1}^{1}\int_{-1}^{1}\frac{u^2\left(1+\frac{\bar{b}_R}{\bar{a}_R}u\right)}{D^3(u,v)}dudv$$

Let,

$$A = \frac{(3-4\nu)}{16\pi\mu(1-\nu)} \text{ and } B = \frac{1}{16\pi\mu(1-\nu)} \quad \text{.. (6.75)}$$

$$G_{11}^{0} = \frac{a_R}{2}\left\{ A \underbrace{\int_{-1}^{1}\int_{-1}^{1} \frac{\left(1+\frac{\bar{b}_R}{\bar{a}_R}u\right)}{D(u,v)} dudv}_{I_1} + B\left(\bar{b}_R\right)^2 \underbrace{\int_{-1}^{1}\int_{-1}^{1} \frac{u^2\left(1+\frac{\bar{b}_R}{\bar{a}_R}u\right)}{D^3(u,v)} dudv}_{I_2} \right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(6.76)}$$

or

$$G_{11}^{0} = \frac{a_R}{2}\left\{ A * I_1 + B\left(\bar{b}_R\right)^2 * I_2 \right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.... \text{(6.77)}$$

Similarly,

$$G_{12}^{0} = \frac{b_R}{2}\left(a_R\right)^2 \frac{B}{L^2}\left(\frac{\Delta\theta}{2}\right) \underbrace{\int_{-1}^{1}\int_{-1}^{1} \frac{uv\left(1+\frac{\bar{b}_R}{\bar{a}_R}u\right)^2}{D^3(u,v)} dudv}_{I_4}$$

$$G_{12}^{0} = \frac{b_R}{2}\left(a_R\right)^2 \frac{B}{L^2}\left(\frac{\Delta\theta}{2}\right) I_4 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. \text{(6.78)}$$

However, this term vanishes in the limit when $\Delta\theta \rightarrow 0$

Therefore,

$$G_{12}^{0} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.... \text{(6.79)}$$

$$G_{13}^{0} = \frac{a_R}{2}\bar{b}_R \bar{b}_z B \int_{-1}^{1}\int_{-1}^{1} \frac{u^2\left(1+\frac{\bar{b}_R}{\bar{a}_R}u\right)}{D^3(u,v)} dudv$$

$$G_{13}^{0} = \frac{a_R}{2}\bar{b}_R \bar{b}_z B \;\; I_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(6.80)}$$

$$G_{21}^{0} = G_{12}^{0} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. \text{(6.81)}$$

$$G_{22}^0 = \frac{a_R}{2}\left\{ A\int_{-1}^{1}\int_{-1}^{1}\frac{\left(1+\frac{\bar{b}_R}{\bar{a}_R}u\right)}{D(u,v)}dudv + B(a_R)^2\left(\frac{\Delta\theta}{2}\right)^2 \underbrace{\int_{-1}^{1}\int_{-1}^{1}\frac{v^2\left(1+\frac{\bar{b}_R}{\bar{a}_R}u\right)^3}{D^3(u,v)}dudv}_{I_5} \right\} \quad \text{……... (6.82)}$$

$$= \frac{a_R}{2}\left\{ A * I_1 + B(a_R)^2\left(\frac{\Delta\theta}{2}\right)^2 I_5 \right\}$$

$$G_{23}^0 = \frac{b_z}{2}\frac{B}{L^2}(a_R)^2\left(\frac{\Delta\theta}{2}\right)\underbrace{\int_{-1}^{1}\int_{-1}^{1}\frac{uv\left(1+\frac{\bar{b}_R}{\bar{a}_R}u\right)^2}{D^3(u,v)}dudv}_{I_4}$$

$$= \frac{b_z}{2}\frac{B}{L^2}(a_R)^2\left(\frac{\Delta\theta}{2}\right) I_4 \quad \text{………………………………. (6.83)}$$

$$= 0$$

$$G_{31}^0 = G_{13}^0 \quad \text{………………………………………………………….. (6.84)}$$

$$G_{32}^0 = G_{23}^0 = 0 \quad \text{……………………………………………………….. (6.85)}$$

$$G_{33}^0 = \frac{a_R}{2}\left\{ A\underbrace{\int_{-1}^{1}\int_{-1}^{1}\frac{\left(1+\frac{\bar{b}_R}{\bar{a}_R}u\right)}{D(u,v)}dudv}_{I_1} + B(\bar{b}_z)^2 \underbrace{\int_{-1}^{1}\int_{-1}^{1}\frac{u^2\left(1+\frac{\bar{b}_R}{\bar{a}_R}u\right)}{D^3(u,v)}dudv}_{I_2} \right\} \quad \text{……………… (6.86)}$$

$$= \frac{a_R}{2}\left[A * I_1 + B(\bar{b}_z)^2 * I_2 \right]$$

The terms

$$G_{4j}^0 = 0 \quad \text{………………………………………………………... (6.87)}$$

and

$$G_{i4}^0 = 0 \quad \text{………………………………………..………………. (6.88)}$$

since $u_j^* \sim O(r^0)$ and $p_j^* \sim O(r^0)$ are non-singular.

The term $G^0_{44} \rightarrow$

$$\left(p^*\right)^{(Sing)} = -\frac{i\omega\rho_f}{4\pi\beta}\frac{1}{r} \qquad \text{.. (6.89)}$$

therefore,

$$G^0_{44} = \frac{1}{\Delta\theta}\int_{S^k} -\frac{i\omega\rho_f}{4\pi\beta}\frac{1}{r} R\, dA$$

$$= -\frac{i\omega\rho_f}{4\pi\beta}\frac{1}{\Delta\theta}\int_{S^k}\frac{1}{L * D(u,v)}\left(a_R + b_R u\right)L\left(\frac{\Delta\theta}{2}\right)dudv$$

$$G^0_{44} = -\frac{i\omega\rho_f}{4\pi\beta}\frac{a_R}{2}\underbrace{\int_{-1}^{1}\int_{-1}^{1}\frac{\left(1+\frac{\bar{b}_R}{\bar{a}_R}u\right)}{D(u,v)}dudv}_{I_1} \qquad \text{...................................... (6.90)}$$

$$G^0_{44} = \frac{a_R}{2}\left(-\frac{i\omega\rho_f}{4\pi\beta}\right) * I_1 \qquad \text{.. (6.91)}$$

The final expression for (6.72) is then

$$\underline{G}^0 = \frac{a_R}{2}\begin{bmatrix} A * I_1 + B\left(\bar{b}_R\right)^2 * I_2 & 0 & \bar{b}_R\bar{b}_z B * I_2 & 0 \\ 0 & A * I_1 & 0 & 0 \\ \bar{b}_R\bar{b}_z B * I_2 & 0 & A * I_1 + B\left(\bar{b}_z\right)^2 * I_2 & 0 \\ 0 & 0 & 0 & -\frac{i\omega\rho_f}{4\pi\beta} * I_1 \end{bmatrix} \qquad \text{............... (6.92)}$$

Now, consider the singular part of the second fundamental solution matrix

$$\underline{H}^{(\sin g)} = \begin{bmatrix} \left(t^*_{11}\right)^{(\sin g)} & \left(t^*_{21}\right)^{(\sin g)} & \left(t^*_{31}\right)^{(\sin g)} & \frac{\left(q_n^{*1}\right)^{(\sin g)}}{i\omega} \\ \left(t^*_{12}\right)^{(\sin g)} & \left(t^*_{22}\right)^{(\sin g)} & \left(t^*_{32}\right)^{(\sin g)} & \frac{\left(q_n^{*2}\right)^{(\sin g)}}{i\omega} \\ \left(t^*_{13}\right)^{(\sin g)} & \left(t^*_{23}\right)^{(\sin g)} & \left(t^*_{33}\right)^{(\sin g)} & \frac{\left(q_n^{*3}\right)^{(\sin g)}}{i\omega} \\ -i\omega\left(t^*_1\right)^{(\sin g)} & -i\omega\left(t^*_2\right)^{(\sin g)} & -i\omega\left(t^*_3\right)^{(\sin g)} & -\left(q_n^{\ *}\right)^{(\sin g)} \end{bmatrix} \qquad \text{.............................. (6.93)}$$

Therefore, according to (6.57) we have

$$\underline{H}^{(0)} = \frac{1}{\Delta\theta}\int_{S^k}\begin{bmatrix} \left(t_{11}^*\right)^{(\sin g)} & \left(t_{21}^*\right)^{(\sin g)} & \left(t_{31}^*\right)^{(\sin g)} & \frac{\left(q_n^{*1}\right)^{(\sin g)}}{i\omega} \\ \left(t_{12}^*\right)^{(\sin g)} & \left(t_{22}^*\right)^{(\sin g)} & \left(t_{32}^*\right)^{(\sin g)} & \frac{\left(q_n^{*2}\right)^{(\sin g)}}{i\omega} \\ \left(t_{13}^*\right)^{(\sin g)} & \left(t_{23}^*\right)^{(\sin g)} & \left(t_{33}^*\right)^{(\sin g)} & \frac{\left(q_n^{*3}\right)^{(\sin g)}}{i\omega} \\ -i\omega\left(t_1^*\right)^{(\sin g)} & -i\omega\left(t_2^*\right)^{(\sin g)} & -i\omega\left(t_3^*\right)^{(\sin g)} & -\left(q_n^{*}\right)^{(\sin g)} \end{bmatrix} R\,d\theta\,dr \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.94)$$

The expressions for the singular parts are given in section 5.3 of Chapter 5. Recall that,

$$\left(t_{ij}^*\right)^{(\sin g)} = \frac{1}{8\pi(1-\nu)r^2}\left[(1-2\nu)\left(n_i r_j - r_i n_j\right) - \left[(1-2\nu)\delta_{ij} + 3r_i r_j\right]\frac{\partial r}{\partial n}\right]$$

In view of (6.63)-(6.66) we have for the normal derivative of the distance vector

$$\begin{aligned} \frac{\partial r}{\partial n} &= \underline{r} \bullet \underline{n} \\ &= \left(\bar{b}_R \frac{u}{D(u,v)}, \frac{\left(\bar{a}_R + \bar{b}_R u\right)v}{D(u,v)}\frac{\Delta\theta}{2}, \bar{b}_z \frac{u}{D(u,v)}\right) \bullet \left(n_R,\ n_R \frac{\Delta\theta}{2} v,\ n_z\right) \\ &= n_R \bar{b}_R \frac{u}{D(u,v)} + n_R\left(\frac{\Delta\theta}{2}\right)^2 \frac{\left(\bar{a}_R + \bar{b}_R u\right)v^2}{D(u,v)} + n_z \bar{b}_z \frac{u}{D(u,v)} \\ &= \underbrace{\left(n_R \bar{b}_R + n_z \bar{b}_z\right)}_{=0}\frac{u}{D(u,v)} + n_R\left(\frac{\Delta\theta}{2}\right)^2 \frac{\left(\bar{a}_R + \bar{b}_R u\right)v^2}{D(u,v)} \end{aligned}$$

$$\begin{aligned} \frac{\partial r}{\partial n} &= n_R\left(\frac{\Delta\theta}{2}\right)^2 \frac{\left(\bar{a}_R + \bar{b}_R u\right)v^2}{D(u,v)} \\ &= \bar{b}_z\left(\frac{\Delta\theta}{2}\right)^2 \frac{\left(\bar{a}_R + \bar{b}_R u\right)v^2}{D(u,v)} \end{aligned} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.95)$$

Now, we consider each term of the matrix separately.

The term $H_{11}^0 \rightarrow$

$$\left(t_{11}^{*}\right)^{(\sin g)} = \frac{-1}{8\pi(1-\nu)r^2}\left[(1-2\nu)+3(r_1)^2\right]\frac{\partial r}{\partial n}$$

$$H_{11}^{0} = \frac{1}{\Delta\theta}\int_{S_k}\frac{-1}{8\pi(1-\nu)r^2}\left[(1-2\nu)+3(r_1)^2\right]\frac{\partial r}{\partial n}RdA$$

$$= \frac{-1}{16\pi(1-\nu)}\left(\frac{\Delta\theta}{2}\right)^2 n_R\left\{(1-2\nu)\int_{-1}^{1}\int_{-1}^{1}\frac{\left(\bar{a}_R+\bar{b}_R u\right)^2 v^2}{D^3(u,v)}dudv + 3\left(\bar{b}_R\right)^2\int_{-1}^{1}\int_{-1}^{1}\frac{\left(\bar{a}_R+\bar{b}_R u\right)^2 u^2 v^2}{D^5(u,v)}dudv\right\}$$

the result tends to zero as $\Delta\theta \rightarrow 0$, therefore we take

$$H_{11}^{0} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(6.96)}$$

The term $H_{12}^{0} \rightarrow$

$$\left(t_{12}^{*}\right)^{(\sin g)} = \frac{1}{8\pi(1-\nu)r^2}\left[(1-2\nu)(n_1 r_2 - r_1 n_2) - 3r_1 r_2\frac{\partial r}{\partial n}\right]$$

$$\left(t_{12}^{*}\right)^{(\sin g)} = \frac{(1-2\nu)}{8\pi(1-\nu)}\frac{\left(\bar{b}_z\frac{\left(\bar{a}_R+\bar{b}_R u\right)v}{D(u,v)}\frac{\Delta\theta}{2} - \bar{b}_R\frac{u}{D(u,v)}\bar{b}_z\frac{\Delta\theta}{2}v\right)}{L^2 D^2(u,v)} - \frac{3}{8\pi(1-\nu)}\frac{\bar{b}_R\frac{u}{D(u,v)}\frac{\left(\bar{a}_R+\bar{b}_R u\right)v}{D(u,v)}\frac{\Delta\theta}{2}\bar{b}_z\left(\frac{\Delta\theta}{2}\right)^2\frac{\left(\bar{a}_R+\bar{b}_R u\right)v^2}{D(u,v)}}{L^2 D^2(u,v)}$$

$$= \frac{(1-2\nu)}{8\pi(1-\nu)L^2}\frac{\Delta\theta}{2}\bar{b}_z\frac{\bar{a}_R v}{D^3(u,v)} - \frac{3\bar{b}_R\bar{b}_z}{8\pi(1-\nu)L^2}\left(\frac{\Delta\theta}{2}\right)^3\frac{\left(\bar{a}_R+\bar{b}_R u\right)^2 uv^3}{D^5(u,v)}$$

$$\left(t_{12}^{*}\right)^{(\sin g)} = \frac{(1-2\nu)}{8\pi(1-\nu)L^2}\frac{\Delta\theta}{2}\bar{b}_z\frac{\bar{a}_R v}{D^3(u,v)} - \frac{3\bar{b}_R\bar{b}_z}{8\pi(1-\nu)L^2}\left(\frac{\Delta\theta}{2}\right)^3\frac{\left(\bar{a}_R+\bar{b}_R u\right)^2 uv^3}{D^5(u,v)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(6.97)}$$

$$H_{21}^{0}=\frac{1}{\Delta\theta}\int_{S_k}\left(t_{12}^{*}\right)^{(sing)}RdA$$

$$=\frac{1}{\Delta\theta}\left\{\begin{array}{l}\frac{(1-2\nu)}{8\pi(1-\nu)L^2}\frac{\Delta\theta}{2}\bar{b}_z\int_{-1}^{1}\int_{-1}^{1}\frac{\bar{a}_R v}{D^3(u,v)}(a_R+b_R u)L\left(\frac{\Delta\theta}{2}\right)dudv-\\ \qquad\frac{3\bar{b}_R\bar{b}_z}{8\pi(1-\nu)L^2}\left(\frac{\Delta\theta}{2}\right)^3\int_{-1}^{1}\int_{-1}^{1}\frac{(\bar{a}_R+\bar{b}_R u)^2 uv^3}{D^5(u,v)}(a_R+b_R u)L\left(\frac{\Delta\theta}{2}\right)dudv\end{array}\right\}$$

$$=\frac{(1-2\nu)\bar{b}_z}{16\pi(1-\nu)}\frac{\Delta\theta}{2}\int_{-1}^{1}\int_{-1}^{1}\frac{\bar{a}_R(\bar{a}_R+\bar{b}_R u)v}{D^3(u,v)}dudv-\frac{3\bar{b}_R\bar{b}_z}{16\pi(1-\nu)}\left(\frac{\Delta\theta}{2}\right)^3\int_{-1}^{1}\int_{-1}^{1}\frac{(\bar{a}_R+\bar{b}_R u)^3 uv^3}{D^5(u,v)}dudv$$

integrands of both integrals on the right are odd functions of "v", hence the integrals evaluate to zero. We have,

$$H_{21}^{0}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. (6.98)$$

The term $H_{31}^{0}\rightarrow$

$$\left(t_{13}^{*}\right)^{(sing)}=\frac{1}{8\pi(1-\nu)r^2}\left[(1-2\nu)(n_1r_3-r_1n_3)-3r_1r_3\frac{\partial r}{\partial n}\right]$$

$$\left(t_{13}^{*}\right)^{(sing)}=\frac{(1-2\nu)}{8\pi(1-\nu)}\frac{\left(\bar{b}_z\bar{b}_z\frac{u}{D(u,v)}-\bar{b}_R\frac{u}{D(u,v)}\left(-\bar{b}_R\right)\right)}{L^2D^2(u,v)}-\frac{3}{8\pi(1-\nu)}\frac{\bar{b}_R\frac{u}{D(u,v)}\bar{b}_z\frac{u}{D(u,v)}}{L^2D^2(u,v)}\bar{b}_z\left(\frac{\Delta\theta}{2}\right)^2\frac{(\bar{a}_R+\bar{b}_R u)v^2}{D(u,v)}$$

$$=\frac{(1-2\nu)}{8\pi(1-\nu)L^2}\frac{\left((\bar{b}_z)^2+(\bar{b}_R)^2\right)u}{D^3(u,v)}-\frac{3}{8\pi(1-\nu)}\frac{\bar{b}_R(\bar{b}_z)^2}{L^2}\left(\frac{\Delta\theta}{2}\right)^2\frac{(\bar{a}_R+\bar{b}_R u)v^2u^2}{D^5(u,v)}$$

but, $(\bar{b}_z)^2+(\bar{b}_R)^2=1$

$$\left(t_{13}^{*}\right)^{(sing)}=\frac{(1-2\nu)}{8\pi(1-\nu)L^2}\frac{u}{D^3(u,v)}-\frac{3}{8\pi(1-\nu)}\frac{\bar{b}_R(\bar{b}_z)^2}{L^2}\left(\frac{\Delta\theta}{2}\right)^2\frac{(\bar{a}_R+\bar{b}_R u)v^2u^2}{D^5(u,v)} \quad \ldots\ldots\ldots\ldots (6.99)$$

therefore, H_{31}^{0} becomes

$$H_{31}^{0} = \frac{1}{\Delta\theta}\left\{\begin{aligned}&\frac{(1-2\nu)}{8\pi(1-\nu)L^2}\int_{-1}^{1}\int_{-1}^{1}\frac{u}{D^3(u,v)}(a_R + b_R u)L\left(\frac{\Delta\theta}{2}\right)dudv - \\ &\qquad \frac{3\bar{b}_R(\bar{b}_z)^2}{8\pi(1-\nu)L^2}\left(\frac{\Delta\theta}{2}\right)^2\int_{-1}^{1}\int_{-1}^{1}\frac{(\bar{a}_R + \bar{b}_R u)v^2u^2}{D^5(u,v)}(a_R + b_R u)L\left(\frac{\Delta\theta}{2}\right)dudv\end{aligned}\right\}$$

$$= \frac{(1-2\nu)}{16\pi(1-\nu)}\int_{-1}^{1}\int_{-1}^{1}\frac{(\bar{a}_R + \bar{b}_R u)u}{D^3(u,v)}dudv - \frac{3\bar{b}_R(\bar{b}_z)^2}{16\pi(1-\nu)}\left(\frac{\Delta\theta}{2}\right)^2\int_{-1}^{1}\int_{-1}^{1}\frac{(\bar{a}_R + \bar{b}_R u)^2 v^2u^2}{D^5(u,v)}dudv$$

the second term tends to zero as $\Delta\theta \rightarrow 0$, as a result

$$H_{31}^{0} = \frac{\bar{a}_R}{2}\frac{(1-2\nu)}{8\pi(1-\nu)}\underbrace{\int_{-1}^{1}\int_{-1}^{1}\frac{\left(1 + \frac{\bar{b}_R}{\bar{a}_R}u\right)u}{D^3(u,v)}dudv}_{I_3} \quad \text{………..…..………………….} \quad (6.100)$$

Similarly, the other terms of the $\underline{H}^0$ can be obtained and one has as a result

$$\underline{H}^0 = \frac{\bar{a}_R}{2}\begin{bmatrix} 0 & 0 & -\frac{1-2\nu}{8\pi(1-\nu)}\frac{\bar{b}_R}{\bar{a}_R}I_3 & \frac{\bar{b}_Z L[2\beta(1-\nu)-\alpha(1-2\nu)]}{16\pi\mu(1-\nu)}I_1 \\ 0 & 0 & 0 & 0 \\ \frac{1-2\nu}{8\pi(1-\nu)}\frac{\bar{b}_R}{\bar{a}_R}I_3 & 0 & 0 & -\frac{\bar{b}_R L[2\beta(1-\nu)-\alpha(1-2\nu)]}{16\pi\mu(1-\nu)}I_1 \\ \frac{\omega^2\rho_f[\alpha(1-2\nu)-\beta]L\bar{b}_Z}{8\pi\beta(1-\nu)}I_1 & 0 & -\frac{\omega^2\rho_f[\alpha(1-2\nu)-\beta]L\bar{b}_R}{8\pi\beta(1-\nu)}I_1 & 0 \end{bmatrix}$$

……… (6.101)

$\underline{G}^0$ and $\underline{H}^0$ contain the following integral expressions:

$$I_1 = \int_{-1}^{1}\int_{-1}^{1}\frac{\left(1+\frac{\bar{b}_R}{\bar{a}_R}u\right)}{D(u,v)}dudv$$

$$I_2 = \int_{-1}^{1}\int_{-1}^{1}\frac{\left(1+\frac{\bar{b}_R}{\bar{a}_R}u\right)u^2}{D^3(u,v)}dudv \quad \text{……………………………………………} \quad (6.102)$$

$$I_3 = \int_{-1}^{1}\int_{-1}^{1}\frac{u^2}{D^3(u,v)}dudv$$

where, $D(u,v)=\sqrt{u^2+\left(\frac{\Delta\theta}{2}\right)^2\left(\bar{a}_R^{(k)}+\bar{b}_R^{(k)}u\right)^2 v^2}$ ……………………………...… (6.103)

It may be noted that in writing I_3 in equation 6.102, we dropped the term $\int_{-1}^{1}\int_{-1}^{1}\frac{u}{D^3(u,v)}dudv$. Because, this term vanishes as $\Delta\theta \rightarrow 0$:

$$\lim_{\Delta\theta\to 0}\int_{-1}^{1}\int_{-1}^{1}\frac{u}{D^3(u,v)}dudv=\lim_{\Delta\theta\to 0}\int_{-1}^{1}\int_{-1}^{1}\frac{u}{\left(\sqrt{u^2+\left(\frac{\Delta\theta}{2}\right)^2\left(\bar{a}_R^{(k)}+\bar{b}_R^{(k)}u\right)^2 v^2}\right)^3}dudv$$

$$=\int_{-1}^{1}\int_{-1}^{1}\lim_{\Delta\theta\to 0}\frac{u}{\left(\sqrt{u^2+\left(\frac{\Delta\theta}{2}\right)^2\left(\bar{a}_R^{(k)}+\bar{b}_R^{(k)}u\right)^2 v^2}\right)^3}dudv$$

$$=\int_{-1}^{1}\int_{-1}^{1}\frac{u}{\left(\sqrt{u^2}\right)^3}dudv$$

$$=\int_{-1}^{1}\int_{-1}^{1}\frac{u}{(|u|)^3}dudv$$

$$=\int_{-1}^{1}\int_{-1}^{1}\underbrace{\frac{\operatorname{sign}(u)}{|u|^2}}_{\text{odd fnct. of "u"}}dudv=0$$

However, the integrals (6.102) are still singular at the origin. We regularize them further by the following polar transformation in u-v plane (**Figure 6.5**):

$$\begin{aligned}u&=\rho\cos(\gamma)\\ v&=\rho\sin(\gamma)\end{aligned} \qquad \text{……………………………………………….... (6.104)}$$

Accordingly,

$$du\,dv=\rho\,d\rho\,d\gamma \qquad \text{………..………………………….……… (6.105)}$$

and the integrals (6.102) become

$$I_1 = \int_0^{2\pi}\int_{\rho(\gamma)} \frac{1+\frac{\overline{b}_R}{\overline{a}_R}\rho\cos(\gamma)}{\sqrt{(\rho\cos(\gamma))^2+\left(\frac{\Delta\theta}{2}\right)^2\left(\overline{a}_R^{(k)}+\overline{b}_R^{(k)}\rho\cos(\gamma)\right)^2(\rho\sin(\gamma))^2}}\rho d\rho d\gamma$$

$$= \int_0^{2\pi}\int_{\rho(\gamma)} \frac{1+\frac{\overline{b}_R}{\overline{a}_R}\rho\cos(\gamma)}{\sqrt{\cos^2(\gamma)+\left(\frac{\Delta\theta}{2}\right)^2\left[\overline{a}_R^{(k)}+\overline{b}_R^{(k)}\rho\cos(\gamma)\right]^2\sin^2(\gamma)}} d\rho d\gamma$$

or with the short hand,

$$\overline{D}(\rho,\gamma) = \sqrt{\cos^2(\gamma)+\left(\frac{\Delta\theta}{2}\right)^2\left[\overline{a}_R^{(k)}+\overline{b}_R^{(k)}\rho\cos(\gamma)\right]^2\sin^2(\gamma)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.106)$$

$$I_1 = \int_0^{2\pi}\int^{\rho(\gamma)} \frac{1+\frac{\overline{b}_R}{\overline{a}_R}\rho\cos(\gamma)}{\overline{D}(\rho,\gamma)} d\rho d\gamma$$

$$I_2 = \int_0^{2\pi}\int_0^{\rho(\gamma)} \frac{\cos^2(\gamma)\left(1+\frac{\overline{b}_R}{\overline{a}_R}\rho\cos(\gamma)\right)}{\overline{D}^3(\rho,\gamma)} d\rho d\gamma \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.107)$$

$$I_3 = \int_0^{2\pi}\int_0^{\rho(\gamma)} \frac{\cos^2(\gamma)}{\overline{D}^3(\rho,\gamma)} d\rho\, d\gamma$$

Referring to the **Figure 6.5** and considering that $\overline{D}(\rho,\gamma)$ is an even function of 'γ', the integrals in (6.107) can be put in the form

$$I_i = 2\times\left[\int_0^{\frac{\pi}{4}}\int_0^{\frac{1}{\cos(\gamma)}} F_i\, d\rho\, d\gamma + \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\int_0^{\frac{1}{\sin(\gamma)}} F_i\, d\rho\, d\gamma + \int_{\frac{3\pi}{4}}^{\pi}\int_0^{\frac{-1}{\cos(\gamma)}} F_i\, d\rho\, d\gamma\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.108)$$

where,

$$F_1 = \frac{1 + \frac{\bar{b}_R}{\bar{a}_R}\rho\cos(\gamma)}{\bar{D}(\rho,\gamma)}$$

$$F_2 = \frac{\cos^2(\gamma)\left(1 + \frac{\bar{b}_R}{\bar{a}_R}\rho\cos(\gamma)\right)}{\bar{D}^3(\rho,\gamma)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.109)$$

$$F_3 = \frac{\cos^2(\gamma)}{\bar{D}^3(\rho,\gamma)}$$

The integrands (6.109) are non-singular and can be computed by standard Gaussian quadrature. It may be of interest to note that to accelerate computations the integrals in (6.108) can be evaluated analytically over "ρ"; then the resulting forms would be as follows:

$$\begin{aligned} I_1 = 4*\Bigg\{ &\int_0^{\pi/4} \frac{\sqrt{4\cos^2(\gamma) + (\bar{a}_R + \bar{b}_R)^2(\Delta\theta)^2\sin^2(\gamma)} - \sqrt{4\cos^2(\gamma) + \bar{a}_R^{\,2}(\Delta\theta)^2\sin^2(\gamma)}}{\bar{a}_R\bar{b}_R(\Delta\theta)^2\sin^2(\gamma)\cos(\gamma)} d\gamma \\ &+ \int_{\pi/4}^{3\pi/4} \frac{\sqrt{4\cos^2(\gamma) + (\bar{a}_R\sin(\gamma) + \bar{b}_R\cos(\gamma))^2(\Delta\theta)^2} - \sqrt{4\cos^2(\gamma) + \bar{a}_R^{\,2}(\Delta\theta)^2\sin^2(\gamma)}}{\bar{a}_R\bar{b}_R(\Delta\theta)^2\sin^2(\gamma)\cos(\gamma)} d\gamma \\ &+ \int_{3\pi/4}^{\pi} \frac{\sqrt{4\cos^2(\gamma) + (\bar{a}_R - \bar{b}_R)^2(\Delta\theta)^2\sin^2(\gamma)} - \sqrt{4\cos^2(\gamma) + \bar{a}_R^{\,2}(\Delta\theta)^2\sin^2(\gamma)}}{\bar{a}_R\bar{b}_R(\Delta\theta)^2\sin^2(\gamma)\cos(\gamma)} d\gamma \Bigg\} \end{aligned}$$

$$\begin{aligned} I_2 = -16*\Bigg\{ &\int_0^{\pi/4} \frac{\cos(\gamma)}{\bar{a}_R\bar{b}_R(\Delta\theta)^2\sin^2(\gamma)} \left[\frac{1}{\sqrt{4\cos^2(\gamma) + (\bar{a}_R + \bar{b}_R)^2(\Delta\theta)^2\sin^2(\gamma)}} - \frac{1}{\sqrt{4\cos^2(\gamma) + \bar{a}_R^{\,2}(\Delta\theta)^2\sin^2(\gamma)}} \right] d\gamma \\ &+ \int_{\pi/4}^{3\pi/4} \frac{\cos(\gamma)}{\bar{a}_R\bar{b}_R(\Delta\theta)^2\sin^2(\gamma)} \left[\frac{1}{\sqrt{4\cos^2(\gamma) + (\bar{a}_R\sin(\gamma) + \bar{b}_R\cos(\gamma))^2(\Delta\theta)^2}} - \frac{1}{\sqrt{4\cos^2(\gamma) + \bar{a}_R^{\,2}(\Delta\theta)^2\sin^2(\gamma)}} \right] d\gamma \\ &+ \int_{3\pi/4}^{\pi} \frac{\cos(\gamma)}{\bar{a}_R\bar{b}_R(\Delta\theta)^2\sin^2(\gamma)} \left[\frac{1}{\sqrt{4\cos^2(\gamma) + (\bar{a}_R - \bar{b}_R)^2(\Delta\theta)^2\sin^2(\gamma)}} - \frac{1}{\sqrt{4\cos^2(\gamma) + \bar{a}_R^{\,2}(\Delta\theta)^2\sin^2(\gamma)}} \right] d\gamma \Bigg\} \end{aligned}$$

$$I_3 = 4*\left\{\int_0^{\pi/4} \frac{1}{\bar{b}_R \cos(\gamma)}\left[\frac{\bar{a}_R+\bar{b}_R}{\sqrt{4\cos^2(\gamma)+\left(\bar{a}_R+\bar{b}_R\right)^2(\Delta\theta)^2\sin^2(\gamma)}}-\frac{\bar{a}_R}{\sqrt{4\cos^2(\gamma)+\bar{a}_R{}^2(\Delta\theta)^2\sin^2(\gamma)}}\right]d\gamma\right.$$

$$+\int_{\pi/4}^{3\pi/4} \frac{1}{\bar{b}_R \cos(\gamma)}\left[\frac{\bar{a}_R\sin(\gamma)+\bar{b}_R\cos(\gamma)}{\sin(\gamma)\sqrt{4\cos^2(\gamma)+\left(\bar{a}_R\sin(\gamma)+\bar{b}_R\cos(\gamma)\right)^2(\Delta\theta)^2}}-\frac{\bar{a}_R}{\sqrt{4\cos^2(\gamma)+\bar{a}_R{}^2(\Delta\theta)^2\sin^2(\gamma)}}\right]d\gamma$$

$$\left.+\int_{3\pi/4}^{\pi} \frac{1}{\bar{b}_R \cos(\gamma)}\left[\frac{\bar{a}_R-\bar{b}_R}{\sqrt{4\cos^2(\gamma)+\left(\bar{a}_R-\bar{b}_R\right)^2(\Delta\theta)^2\sin^2(\gamma)}}-\frac{\bar{a}_R}{\sqrt{4\cos^2(\gamma)+\bar{a}_R{}^2(\Delta\theta)^2\sin^2(\gamma)}}\right]d\gamma\right\}$$

It is found that the computer subroutines that use above forms execute much faster than those employing forms (6.108). We, in addition, note that further dividing the angular interval at $\gamma=\frac{\pi}{2}$ improves numerical accuracy considerably, since

$\bar{D}(\rho,\gamma)=\sqrt{\cos^2(\gamma)+\left(\frac{\Delta\theta}{2}\right)^2\left[\bar{a}_R^{(k)}+\bar{b}_R^{(k)}\rho\cos(\gamma)\right]^2\sin^2(\gamma)}$ tends to zero at $\gamma=\frac{\pi}{2}$ when $\Delta\theta \rightarrow 0$.

Also, it has to be noted that, the preceding expressions all contain "$\bar{b}_R$" in the denominator which can become zero; therefore a further regularization is required to avoid division by zero error for vertical elements.

6.5 Computation of Stress Resultants

The force and moment resultants are of use when checking the solution for equilibrium or they can be used as impedance functions to be used in sub-structure methods in soil-structure interaction analysis.

The general expressions for the resultants of the surface tractions over a part S′ of the body (considering the axi-symmetry of the geometry) in integral form are

$$F_i=\int_{C'}\left(\int_0^{2\pi} t_i\, d\theta\right)R\,ds \qquad \text{and} \qquad M_i=\int_{C'}\left(\int_0^{2\pi}\varepsilon_{ijk}\,x_j\,t_k\, d\theta\right)R\,ds \quad \ldots\ldots\ldots\ldots\ldots\ (6.110)$$

where,

C′ : the generator of the surface part S′

t_i : Cartesian traction components

x_i : components of the distance vector b/w the origin and the surface

F_i : force resultant in x_i-direction

M_i : moment resultant in x_i-direction

The Cartesian components of the tractions t_i are related to the cylindrical components as

$$t_1 = t_r \cos\theta - t_\theta \sin\theta$$
$$t_2 = t_r \sin\theta + t_\theta \cos\theta \quad \ldots\ldots (6.111)$$
$$t_3 = t_z$$

and, as done previously, we proceed by expanding t_r, t_θ and t_z in complex Fourier series

$$t_r = \sum_{k=-\infty}^{\infty} \tilde{t}_r^{\,k} e^{ik\theta}$$
$$t_\theta = \sum_{k=-\infty}^{\infty} \tilde{t}_\theta^{\,k} e^{ik\theta} \quad \ldots\ldots (6.112)$$
$$t_z = \sum_{k=-\infty}^{\infty} \tilde{t}_z^{\,k} e^{ik\theta}$$

To obtain the expression for the resultants in terms of Fourier coefficients of the tractions, we insert (6.112) and (6.111) in (6.110). The following integrals repeatedly appear in the expressions and it is worth noting the results here,

$$\int_0^{2\pi} e^{ik\theta} \cos\theta d\theta = \begin{cases} \pi & \text{if } k = -1,1 \\ 0 & \text{otherwise} \end{cases}$$
$$\ldots\ldots (6.113)$$
$$\int_0^{2\pi} e^{ik\theta} \sin\theta d\theta = \begin{cases} -i\pi & \text{if } k = -1 \\ i\pi & \text{if } k = 1 \\ 0 & \text{otherwise} \end{cases}$$

For example, for "F_1", we have

$$F_1 = \int_{C'} \left(\int_0^{2\pi} (t_1) d\theta \right) R ds = \int_{C'} \left(\int_0^{2\pi} (t_r \cos\theta - t_\theta \sin\theta) d\theta \right) R ds \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. \; (6.114)$$

$$F_1 = \int_{C'} \left(\sum_{k=-\infty}^{\infty} \left(\tilde{t}_r^{\,k} \int_0^{2\pi} e^{ik\theta} \cos\theta d\theta - \tilde{t}_\theta^{\,k} \int_0^{2\pi} e^{ik\theta} \sin\theta d\theta \right) \right) R ds \quad \ldots\ldots\ldots\ldots\ldots\ldots.. \; (6.115)$$

in view of (6.113) we have

$$F_1 = \int_{C'} \left[\pi \left(\tilde{t}_t^{-1} + \tilde{t}_t^{1} \right) - i\pi \left(\tilde{t}_\theta^{1} - \tilde{t}_\theta^{-1} \right) \right] R ds \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \; (6.116)$$

But, in discrete Fourier Transform, the Fourier amplitudes are circular (periodic) with period "N′" and therefore, the amplitudes in the negative Frequency region $-\frac{N'}{2} < k < 0$ are the same as those to the right of the cut-off (Nyquist) frequency $k_c = \frac{N'}{2}$; thus,

$$\tilde{t}_r^{-1} = \tilde{t}_r^{N'-1} \quad ; \quad \tilde{t}_\theta^{-1} = \tilde{t}_\theta^{N'-1} \quad ; \quad \tilde{t}_z^{-1} = \tilde{t}_z^{N'-1} \quad \ldots\ldots\ldots\ldots \; (6.117)$$

Hence, we write

$$F_1 = \pi \int_{C'} \left[\left(\tilde{t}_t^{N'-1} + \tilde{t}_t^{1} \right) - i \left(\tilde{t}_\theta^{1} - \tilde{t}_\theta^{N'-1} \right) \right] R ds \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \; (6.118)$$

The line integral over C′ can be evaluated after discretization, in view of constant element formulation

$$F_1 = \pi \sum_j \left[\left(\tilde{t}_t^{N'-1} + \tilde{t}_t^{1} \right)_{C'_j} - i \left(\tilde{t}_\theta^{1} - \tilde{t}_\theta^{N'-1} \right)_{C'_j} \right] \int_{C'_j} R ds$$

where summation is over all the elements comprising C'.

The integral on the right hand side can be evaluated after noting that, over an element

$$R\big|_j = (a_r)_j + (b_r)_j t \qquad \text{and} \qquad ds = L_j dt$$

where, L_j is the half length of the BE.

Thus,

$$\int_{C'_j} R ds = \int_{-1}^{1} \left[(a_R)_j + (b_R)_j t \right] L_j dt = (a_R)_j (2L_j)$$

Finally,

$$F_1 = \pi \sum_j \left[\left((\tilde{t}_r)_j^{N'-1} + (\tilde{t}_r)_j^1 \right) - i\left((\tilde{t}_\theta)_j^1 - (\tilde{t}_\theta)_j^{N'-1} \right) \right] (a_R)_j (2L_j) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.119)$$

Similarly,

$$F_2 = \pi \sum_j \left[i\left[(\tilde{t}_r)_j^1 - (\tilde{t}_r)_j^{N'-1} \right] + \left[(\tilde{t}_\theta)_j^1 + (\tilde{t}_\theta)_j^{N'-1} \right] \right] (a_R)_j (2L_j) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.120)$$

$$F_3 = 2\pi \sum_j (\tilde{t}_z)_j^0 (a_R)_j (2L_j) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.121)$$

Similarly, for the moment resultants,

$$M_1 = \pi \sum_j \left[i\left[(\tilde{t}_z)_j^1 - (\tilde{t}_z)_j^{N'-1} \right] \alpha_j - i\left[(\tilde{t}_R)_j^1 - (\tilde{t}_R)_j^{N'-1} \right] \beta_j - \left[(\tilde{t}_\theta)_j^1 + (\tilde{t}_\theta)_j^{N'-1} \right] \beta_j \right] \quad \ldots\ldots (6.122)$$

$$M_2 = \pi \sum_j \left[\left[\left(\tilde{t}_R\right)_j^1 + \left(\tilde{t}_R\right)_j^{N'-1} \right] \beta_j - i \left[\left(\tilde{t}_\theta\right)_j^1 - \left(\tilde{t}_\theta\right)_j^{N'-1} \right] \beta_j - \left[\left(\tilde{t}_z\right)_j^1 + \left(\tilde{t}_z\right)_j^{N'-1} \right] \alpha_j \right] \quad \text{........ (6.123)}$$

$$M_3 = 2\pi \sum_j \left(\tilde{t}_\theta\right)_j^0 \alpha_j \quad \text{.. (6.124)}$$

where, α_j and β_j are defined by

$$\alpha_j = \int_{C_j} R^2 ds \qquad \beta_j = \int_{C_j} R\, z\, ds \quad \text{.. (6.125)}$$

For a constant element, above integrals for α_j and β_j can be evaluated analytically, considering that a typical BE C_j (**Figure 6.3**) can be parameterized by

$$R = a_R^{(j)} + b_R^{(j)} t \qquad z = a_z^{(j)} + b_z^{(j)} t \qquad -1 \le t \le 1 \quad \text{... (6.126)}$$

where a_R, b_R, a_z, b_z are as defined in (6.42) and (6.43), then one finds

$$\alpha_j = 2 * L \left[\left(a_R^{(j)}\right)^2 + \frac{\left(b_R^{(j)}\right)^2}{3} \right] \qquad \beta_j = 2 * L \left[\left(a_R^{(j)}\right)\left(a_z^{(j)}\right) + \frac{\left(b_R^{(j)}\right)\left(b_z^{(j)}\right)}{3} \right] \quad \text{...................................... (6.127)}$$

6.6 Solutions at Interior Points

Once the Fourier coefficients of the boundary quantities $\tilde{u}_c^k$ and $\tilde{t}_c^k$ are determined, the *displacements - pore pressure,* and *total stress (or effective stress) - fluid flux* components can be computed at interior points of interest, if desired.

6.6.1 Determination of Displacements and Pore-Pressure at Internal Points

We again start by writing the boundary integral equation (6.8) in cylindrical coordinates,

$$\boxed{\underline{u}_c(A) = \int_{\Gamma} \underline{G}_c(A,P)\underline{t}_c(P)d\Gamma - \int_{\Gamma} \underline{H}_c(A,P)\underline{u}_c(P)d\Gamma} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(6.128)}$$

which relates the displacements and pore-pressure at an interior point "A" to the boundary quantities, which have already been determined in the preceding analysis procedure, note that, for an interior point the free term coefficient $\underline{c}_c = \underline{I}$. As before, we expand the boundary quantities (only) in to complex Fourier series in circumferential direction and write,

$$\underline{u}_c = \sum_{k=-\infty}^{\infty}\left[\int_C \int_0^{2\pi} \underline{G}_c(A,P)e^{ik\theta}d\theta\,\underline{\tilde{t}}_c^{\,k}(R,z)Rds - \int_C \int_0^{2\pi} \underline{H}_c(A,P)e^{ik\theta}d\theta\,\underline{\tilde{u}}_c^{\,k}(R,z)Rds\right] \quad \ldots\ldots \text{(6.129)}$$

where the surface integral has been decomposed into two iterated integrals.

With the shorthand,

$$\begin{aligned} \underline{\hat{G}}^{k}(Q',Q) &= \int_0^{2\pi} \underline{G}_c(A,P)e^{ik\theta}d\theta \\ \underline{\hat{H}}^{k}(Q',Q) &= \int_0^{2\pi} \underline{H}_c(A,P)e^{ik\theta}d\theta \end{aligned} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(6.130)}$$

we have

$$\underline{u}_c = \sum_{k=-\infty}^{\infty}\left[\int_C \underline{\hat{G}}^{k}\,\underline{\tilde{t}}_c^{\,k}(R,z)Rds - \int_C \underline{\hat{H}}^{k}\,\underline{\tilde{u}}_c^{\,k}(R,z)Rds\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(6.131)}$$

After truncating the series in (6.131) to N′ terms, and introducing boundary discretization with *constant* elements,

$$\underline{u}_c(A) = \sum_{k=0}^{N'-1}\sum_j \left[\left(\int_{C_j} \hat{\underline{G}}^k (Q_j, Q) R ds\right)\tilde{\underline{t}}^k_{c\,j} - \left(\int_{C_j} \hat{\underline{H}}^k (Q_j, Q) R ds\right)\tilde{\underline{u}}^k_{c\,j}\right] \quad \text{………..} \; (6.132)$$

or

$$\boxed{\underline{u}_c(A) = \sum_{k=0}^{N'-1}\sum_j \left[\hat{\underline{G}}^k_j \, \tilde{\underline{t}}^k_{c\,j} - \hat{\underline{H}}^k_j \, \tilde{\underline{u}}^k_{c\,j}\right]} \quad \text{………...…………………………} \; (6.133)$$

where we introduced,

$$\begin{aligned} \hat{\underline{G}}^k_j &= \int_{C_j} \hat{\underline{G}}^k (Q_j, Q) R ds \\ \hat{\underline{H}}^k_j &= \int_{C_j} \hat{\underline{H}}^k (Q_j, Q) R ds \end{aligned} \quad \text{……………………….…………………….} \; (6.134)$$

for brevity. Equation (6.133) determines displacement components and pore-pressure at an interior point "A" in cylindrical coordinates in terms of the complex Fourier coefficients $\tilde{\underline{t}}^k_c$ and $\tilde{\underline{u}}^k_c$. Since no singularity is involved, the matrices $\hat{\underline{G}}^k_j$ and $\hat{\underline{H}}^k_j$ in (6.133) can be computed by employing Gaussian Integration along the generator, while FFT algorithm along θ-direction.

CHAPTER 7

COMPUTER IMPLEMENTATION

As a part of this study, a computer program has been developed for the elastodynamic analysis of porous solids with axi-symmetric geometry, following the formulation derived and outlined in the previous chapters. The implementation uses ANSI-99 standard C++ language instructions. The program has been developed under WINDOWS environment (successful compilation are achieved using gnu C++ compiler versions 3 and 4, Dev-C++ (uses MingW compiler) versions 4 and 5, MS-C compiler version 2003, BORLAND compiler V5.2). Compilation under UNIX (Linux) systems brings no problems; the author managed to compile the program under IBM-AIX using both xlc and g++ compilers without modifications.

7.1 Organization of the Computer Program

The flow chart of the program AxiPoro is given in Figure 7.1. The main steps of the program are as follows:

a) First the system matrices $\underline{G}^k$ and $\underline{H}^k$ are formed. We do that by performing integrations in r-z plane by Gaussian Quadrature while that in circumferential direction is obtained via FFT algorithm.

b) Then the BE equation 6.37 is solved for each frequency, in view of the given boundary conditions, which establish the Fourier coefficients of the boundary quantities.

c) Trough inverse FFT, the boundary quantities are computed in rθz space.

d) Finally, if desired, the resultant forces and moments are compute via 6.119-6.123, and displacement at interior points by 6.133.

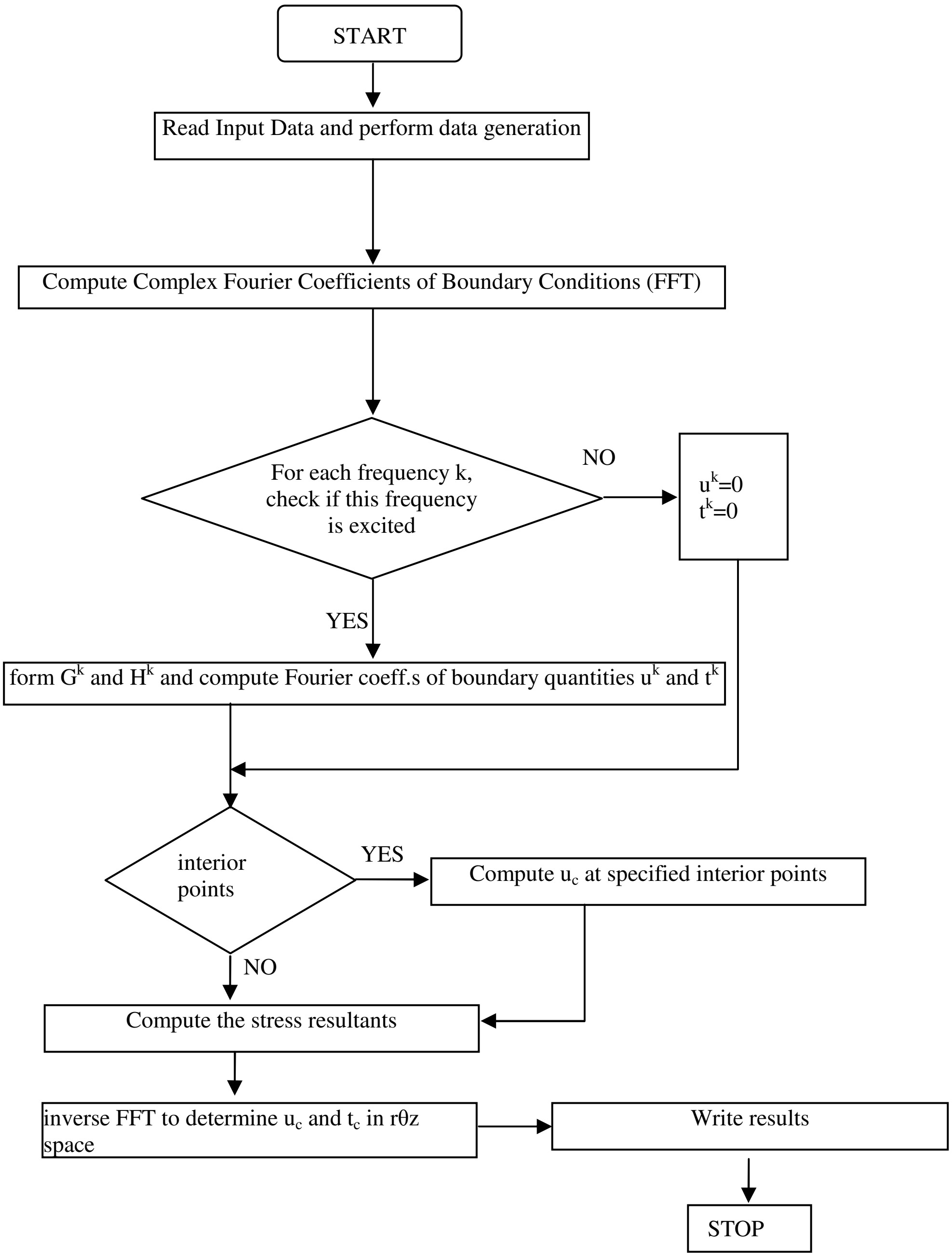

Figure 7.1 Flow chart diagram for the program AxiPoro

7.2 Description of Functions in the Computer Program AxiPoro

The main steps of the program are as follows;

- Read the input file
- Form system matrices $\underline{\tilde{G}}^k$ and $\underline{\tilde{H}}^k$ (equation 6.36)
- Solve the system (6.37) for Fourier coefficients of boundary quantities
- Compute solution at interior points (6.133), and the stress resultants (6.119-124 and) if desired
- Compute the solution in Rθz-space through inverse FFT

The C functions that fulfill the required tasks are described below:

main :

Performs partial input and organizes function calls in the order described above

initialize :

Allocates space for matrices used in the analysis.

Mult :

Performs multiplication of two complex matrices; this function is called by RRot and RL_Rot functions

Form_Rot_Mat :

Forms rotation matrix for a given angle of rotation, equation (6.5).

RRot :

Multiplies element matrices by rotation matrix before FFT

II123 :

Computes the integrals in equation (6.102)

GS0, HS0 :

Fills in $\underline{G}^0$ and $\underline{H}^0$ matrices, equations (6.92) and (6.101).

fund3D_singular_part:

Computes singular parts of fundamental solutions.

fund3D :

Computes complete dynamic fundamental solutions.

integrate1 :

Integrates the fundamental solutions when source point falls in the integration element (singular case).

integrate2 :

Integrates the fundamental solutions when source point falls outside the integration element (non-singular case).

AssembleElement :

For a particular element forms element matrices by proper calls to functions integrate1 or integrate2, then assembles the element matrices in system matrices $\underline{\tilde{G}}^k$ and $\underline{\tilde{H}}^k$. This function is called by Assemble.

ReadBoundaryConditions :

Reads 2 blocks of boundary condition data.

interpolate :

Interpolates a given vector of boundary condition data at $\frac{2\pi}{N'}$ intervals.

WriteInterpolatedBoundaryConditionData:

Write interpolated boundary conditions in output file.

TransformInterpolatedBoundaryConditions :

Transforms boundary condition data via FFT.

RearrangeSystem :

Rearranges the system equation (6.37) into $\underline{A}\,\underline{x} = \underline{b}$ form.

ImposeBoundaryConditions :

Forms the right hand side vector in the rearranged system equation $\underline{A}\,\underline{x} = \underline{b}$.

Assemble :

Performs the task of forming system matrices by calling AssembleElement function successively.

Solve :

Solves the linear system $\underline{A}\,\underline{x} = \underline{b}$ by calling lu_dec and forw_back functions.

lu_dec :

Performs LU decomposition of a square matrix by partial pivot changes.

forw_back :

Solves a linear system for a given right hand side vector after the coefficient matrix has been LU decomposed.

Compute_Solution_at_an_interior_point :

Computes solution at a given interior point.

RL_Rot :

Performs rotation of element matrices before integration in circumferential direction by FFT, this function is called by Compute_Solution_at_an_interior_point.

BackTransformBoundaryQuantities :

Transforms boundary quantities in to rθz-space, after all unknown Fourier coefficients have been found.

WriteSolution :

Prints boundary quantities in the output file.

Write_Solution_Interior_Points :

Prints solutions computed at requested interior points.

Stress_Resultants :

Computes stress resultants over given elements, if requested.

Write_Stress_Resultants :

Prints stress resultants in output file.

CleanUpMemory :

De-allocates space for all matrices.

fft :

Performs Fast Fourier Transform of a vector of complex numbers.

7.3 Input Instructions for Program AxiPoro

All the data related to program execution is to be written in a single input file. The name of the input file is interactively specified by the user; the input file name may be any valid file name (with or without extensions), accepted by the system.
The input file is free format and is composed of the following blocks of data:

<u>TITLE :</u>
The program expects to read a single line of input at top. Write a <u>single line</u> of any thing of the form

TITLE

A descriptive sentence up to 255 characters can be written; otherwise a blank line at top must be left. TITLE will be re-printed in the output file.

<u>GENERAL INFORMATION DATA BLOCK:</u>
This block has the following information; this information can be input in a single line or broken into multiple lines as appropriate; individual data values are separated by any number of spaces:

```
OME    PORO    KAPP    RO   ROf    ROa    MU    NU    HYS
ALF    Q
N_NODES    N_ELEMS    N_INT_PTS    N_ELEM_RES
MM    MMp    N_GAUSS    CODE
```

where,

OME	:	the angular frequency
PORO	:	porosity
KAPP	:	permeability coefficient (κ) in equation 2.32

RO	:	bulk mass density
ROf	:	mass density of fluid
ROa	:	added mass density
MU	:	shear modulus
NU	:	drained Poisson's ratio
HYS	:	hysteretic damping ratio, (values in the range 0.05-0.15 are common for granular earth materials like soil)
ALF	:	α in equation (2.1)
Q	:	Q in equation (2.1)
N_NODES	:	number of nodes
N_ELEMS	:	number of elements
N_INT_PTS	:	number of internal points
N_ELEM_RES	:	number of elements for stress resultants
MM	:	2's power ($N=2^{MM}$) which defines number of angular subdivisions for integrations in θ-direction
MMp	:	2's power ($N'=2^{MMp}$) which defines the number of terms to be retained in truncated complex Fourier series
N_GAUSS	:	number of Gauss points used in integrations along the generator
CODE	:	a control parameter, either "1" or "0", which specifies whether the interpolated boundary conditions are to be printed or not, if "1" is entered program prints the interpolated boundary condition values at angular deviations $\theta_n = n\frac{2\pi}{N'}$; n=0…N′-1.

<u>Remarks:</u>

1. No upper limits are set for the size of the problem (N_NODES, N_ELEMS, MM, MMp, etc.); the problem size is limited only by the hardware memory.
2. Any number of Gauss points can be specified between 1 and 22.

3. MM must be greater than or equal to MMp.

4. The program uses two different N (N and N′) values for integrations in θ direction and for the complex Fourier series sum; this stems from the fact that one needs refined angular sub-divisions (MM=7, 8 is usually good) for accurate integration of fundamental solutions in θ-direction while a small number of terms (MMp=2-4) in complex Fourier series is sufficient in many cases. Jumps in the boundary conditions may necessitate number of terms in the complex Fourier series to be increased.

5. For element as well as system matrices the program allocates matrices with three indices. These matrices can be conceived as sheets of 2-D matrices each corresponding to a particular Fourier amplitude in equation (6.27). The use of different N values in the computation of integrals and in the number terms in Fourier series results in element matrices to contain more frequencies for θ variation than the system matrices. When assembling the element matrices, the program makes a frequency adjustment to match element matrix sheets to system matrix sheets properly.

6. The program divides elements into two when integrating, thus uses twice as many Gauss points as specified by N_GAUSS. A special cubic transformation (Kahaner et. al. 1989, Telles 1987) is also implemented to smooth the kernels for end point singularities.

<u>BOUNDARY POINTS DATA BLOCK:</u>

This block defines the nodal coordinates of boundary elements in the form:

NO R Z

where,

NO	:	node number
R	:	R coordinate of boundary node

Z : z coordinate of boundary node

The nodes need not be entered in successive order.

ELEMENT DEFINITION DATA BLOCK:

Defines the element connectivity. Input N_ELEMS lines in the following form:

ENO node1 node2

where,

ENO	:	element number
node1	:	beginning node of the element
node2	:	end node of the element

BOUNDARY CONDITIONS DATA BLOCK:

The program expects to read two blocks of boundary condition data; one block for generalized displacement boundary conditions and one block for generalized traction boundary conditions. There is no definitive order for each block. The two blocks are in the following format:

(1) TYPE N_BC
(2) EL_NO COMP N_VALS
(3) 0 val ………. … angle val …………. 360 val

where,

TYPE	:	boundary condition type for this block, enter either "D" for displacement boundary conditions and "T" for traction boundary conditions
N_BC	:	number of boundary conditions of type TYPE
EL_NO	:	element number for boundary condition input

COMP	:	component no of generalized vector (u or t), ranges from 1 to 4. For displacement boundary conditions 4 specifies pore-pressure while for tractions the net outward flux.
val	:	boundary condition value
angle	:	angular deviation (in degrees) for boundary condition value

Repeat (2) and (3) N_BC times for boundary condition of type TYPE.

Remark 1: The boundary condition values are to be the cylindrical components of the boundary variables.

Remark 2: If a boundary condition specified by (2) and (3) above must contain 0 and 360 degree angular deviations; for instance if boundary condition is uniform in θ-direction (axi-symmetric), it suffices to enter value at zero and 360 degree angular deviation only.

If there are no boundary conditions of a particular type (displacement or traction), write only line (1) and enter 0 for N_BC.

Remark 3: If no boundary condition (either displacement or traction) is specified for a boundary variable component, the program automatically assumes traction free boundary condition; which may significantly reduce the input requirements.

INTERIOR POINTS DATA BLOCK:

Contains N_INT_PTS lines of input for the rectangular coordinates of interior points where displacement and pore-pressure output is requested. Input N_INT_PTS lines of the form

X Y Z

where,

X	:	x-coordinate of interior point

Y	:	y-coordinate of interior point
Z	:	z-coordinate of interior point

If N_INT_PTS is 0 skip this block.

ELEMENTS FOR STRESS RESULTANTS DATA BLOCK:

Contains the element no's of those elements over which the force and moment resultants are to be computed. Write N_ELEM_RES element no's separated with spaces in the following form:

el_1 el_2 ………….. $el_{N\_ELEM\_RES}$

If N_ELEM_RES specified in the "general information data block" is 0, then skip this block.

7.4 Output File

The name of the output file is entered interactively by the user, the file name can be any valid name (with or without extensions) accepted by the system. The output file is self descriptive and consists of the following sections.

INPUT ECHO:

The input information, such as the title, material data, element information and boundary conditions, is copied for input checking purposes.

RESULTS AT THE BOUNDARY:

First generalized displacements then generalized tractions at every boundary element are printed at angular deviations $\theta_n = n\frac{2\pi}{N'}$; n=0…N′-1. It should be noted that both displacement and traction components are referred to the cylindrical coordinate system.

RESULTS AT INTERIOR POINTS:

Generalized displacement components are printed at desired interior points. Displacement components are referred to cylindrical coordinate system.

FORCE AND MOMENT RESULTANTS:

The force and moment resultants of the tractions acting on the prescribed part of the boundary (defined in the elements for stress resultants data block of the input file) are printed in this section. The values are the Cartesian components of the forces and moments, where moment resultants are computed with respect to the origin of the coordinate system.

7.5 Convergence

In this section we make a simple convergence check for the method of integration employed in the computer implementation. For that purpose, we compute one term from each of the two BE matrices using the proposed method and compare them with the values obtained by usual methods of calculus. We start by repeating (6.33a), where the coefficients of the tractions are given by

$$\underline{\tilde{G}}_{ij}^{k} = \int_{0}^{2\pi} \underline{G}_{ij}(\theta) e^{ik\theta} d\theta \quad \ldots\ldots \quad (7.1)$$

where,

$$\underline{G}_{ij}(\theta) = \int_{C_j} \underline{G}_c \left(\underbrace{R', z'}_{Q_i} ; \underbrace{R, z}_{Q} ; \theta - \theta' \right) \Bigg|_{\theta'=0} R\, ds \quad \ldots\ldots \quad (7.2)$$

and

$$\underline{G}_c(Q_i, Q, \theta - \theta') = \underline{Q}^T(\theta') \underline{G}'(Q_i, Q, \theta', \theta) \underline{Q}(\theta) \quad \ldots\ldots \quad (7.3)$$

when (7.3) and (7.2) are substituted in (7.1), we get

$$\underline{\tilde{G}}_{ij}^{k} = \int_0^{2\pi} \left\{ \int_{C_j} \underline{G}'(Q_i, Q, \theta' = 0, \theta) R \, ds \right\} \underline{Q}(\theta) e^{ik\theta} d\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7.4)$$

Now consider that we want to compute (7.4) for the top surface of a cylindrical poro-elastic (PE) body of unit base radius, as shown in **Figure 7.2**.

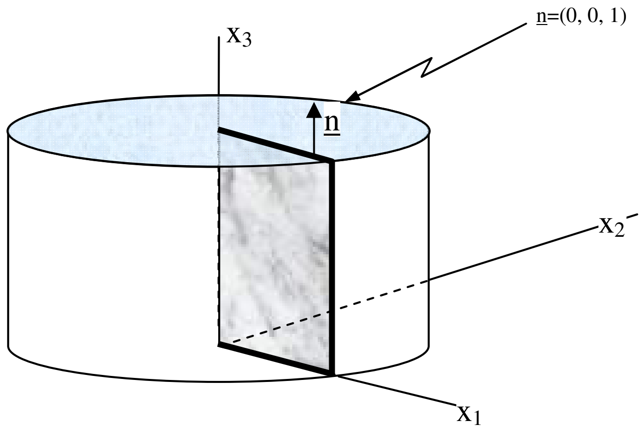

Figure 7.2 A cylindrical PE body; the meridional plane and the generating curve are highlighted.

The top surface "S" of the PE body is depicted in more detail in **Figure 7.3**. Since, we discretize only the generator; for the top surface we introduce only one boundary element along x_1 – axis and take the source point (A) on the same element in order to enforce the integration to be a singular case.

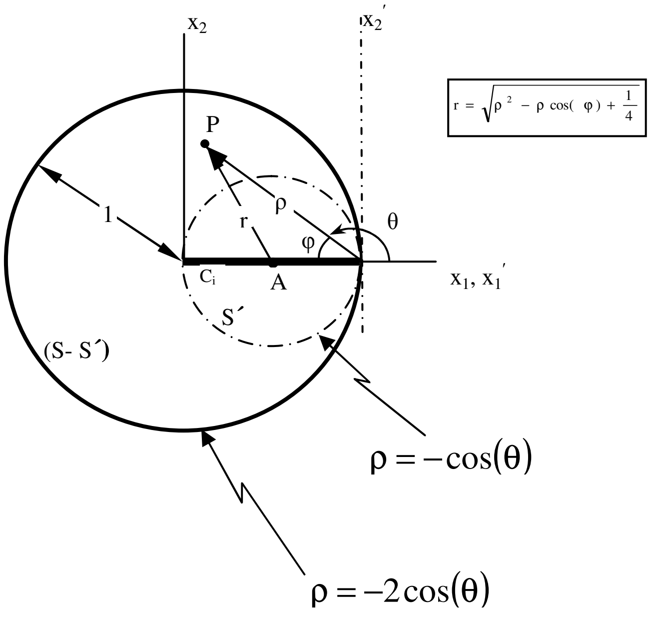

Figure 7.3 The top surface of the previous cylindrical PE body.

We further simplify by taking ω=0 (static problem) and k=0 (zeroeth frequency in angular direction). Therefore, we have

$$\underset{\sim}{\tilde{G}}_{ii}^{k=0} = \int_0^{2\pi} \left\{ \int_{C_i} \underline{G}'(Q_i, Q, \theta' = 0, \theta) R \, ds \right\} \underline{Q}(\theta) d\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(7.5)}$$

but, this integral is simply a surface integral over the top surface “S”. Hence, we can write

$$\underset{\sim}{\tilde{G}}_{ii}^{k=0} = \int_S \underline{G}'(A, P) \, dA \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(7.6)}$$

We want to compute the third diagonal element in (7.6), we shall do this first by usual methods of calculus and than by the procedure outlined in Sections 6.3 and 6.4, and we would like to check if the results from the latter will converge to the value we obtain from the first when the number of subdivisions in angular direction is increased.

Taking the material parameters $\mu = 1.$ and $\nu = 0.25$, the third diagonal element of 1st FS matrix reduces (in static case) for the element in **Figure 7.2** to

$$G'_{33} = \frac{1}{8\pi\mu(1-\nu)}\left[(3-4\nu)\delta_{33} + r_3 r_3\right] = \frac{1}{6\pi r} \quad \text{......................................} (7.7)$$

since $r_3 = 0$ on the top surface. Therefore,

$$\tilde{G}_{33}^{k=0} = \frac{1}{6\pi}\int_S \frac{1}{r}\, dA \quad \text{..} (7.8)$$

where we removed the indices "i" for clarity. The integral in (7.8) can best be integrated by dividing the region "S" into two: (i) a smaller circular region " S´ " of radius 0.5 about the singular point "A", (ii) the remaining region " S- S´ " where the kernel is never singular. Therefore, we continue by writing

$$\tilde{G}_{33}^{0} = \frac{1}{6\pi}\left\{\underbrace{\int_{S'} \frac{1}{r}\, dA}_{K_1} + \underbrace{\int_{S-S'} \frac{1}{r}\, dA}_{K_2}\right\} \quad \text{..} (7.9)$$

We integrate the terms on r.h.s. of (7.9) separately. The first term K_1 can be integrated exactly if one converts to polar coordinates over S´

$$K_1 = \frac{1}{6\pi}\int_0^{2\pi}\int_0^{1/2} \frac{1}{r}\, r\, dr d\theta = \frac{1}{6} \quad \text{..} (7.10)$$

For the second term, by referring to the **Figure 7.2** and noting the polar representations of the inner and outer circles in the shifted coordinate frame $x_1' - x_2'$, we write,

$$K_2 = \frac{1}{6\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\cos(\theta)}^{-2\cos(\theta)} \frac{1}{r}\,\rho\ d\rho d\theta \quad \ldots\ldots (7.11)$$

The distance "r" can be expressed in terms of "ρ" via cosine theorem:

$$r^2 = \rho^2 + \left(\frac{1}{2}\right)^2 - 2\rho\frac{1}{2}\cos(\pi-\theta) = \rho^2 - \rho\cos(\pi-\theta) + \frac{1}{4} \quad \ldots\ldots (7.12)$$

hence,

$$K_2 = \frac{1}{6\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\cos(\theta)}^{-2\cos(\theta)} \frac{\rho}{\sqrt{\rho^2 - \rho\cos(\pi-\theta) + \frac{1}{4}}}\ d\rho d\theta \quad \ldots\ldots (7.13)$$

Although (7.13) cannot be evaluated analytically, it can be computed to any desired accuracy (by any numerical integration method), since it does not contain the singular point. Here, we present the value produced by the program MAPLE (MATHCAD 2001 computes exactly the same result as above):

K_2=0.1447384859 ……………………………………………………... (7.14)

The second term K_2 can of course be integrated more conventionally as follows;

$$K_2 = \frac{1}{6\pi} 2 \left\{ \int_{-1}^{0} \int_{0}^{\sqrt{1-x_1^2}} \frac{1}{\sqrt{\left(x_1 - \frac{1}{2}\right)^2 + (x_2)^2}} \, dx_2 dx_1 + \int_{0}^{1} \int_{\sqrt{\left(\frac{1}{2}\right)^2 - \left(x_1 - \frac{1}{2}\right)^2}}^{\sqrt{1-x_1^2}} \frac{1}{\sqrt{\left(x_1 - \frac{1}{2}\right)^2 + (x_2)^2}} \, dx_2 dx_1 \right\} \quad \text{.. (7.15)}$$

in which case MAPLE produces the result,

K_2=0.1447363638 ……………………………………………….... (7.16)

(MATHCAD 2001, in this case disagrees with MAPLE and computes K_2=0.144738519468)

Hence, we get the following half analytical result for $\tilde{G}_{33}^{0}$:

$\boxed{\tilde{G}_{33}^{0} = K_1 + K_2 = 0.31140515}$ (using 7.14)

or …………………...….. (7.17)

$\boxed{\tilde{G}_{33}^{0} = K_1 + K_2 = 0.31140303}$ (using 7.16)

Similarly, from (6.33 - b)

$$\underline{\tilde{H}}_{ij}^{k} = \int_{0}^{2\pi} \underline{H}_{ij}(\theta) e^{ik\theta} d\theta \quad \text{…………………….…………………………… (7.18)}$$

where,

$$\underline{H}_{ij}(\theta) = \int_{C_j} \underline{H}_c \left(\underbrace{R', z'}_{Q_i} ; \underbrace{R, z}_{Q} ; \theta - \theta' \right) \Bigg|_{\theta'=0} R \, ds \quad \text{…………………………………….. (7.19)}$$

and

$$\underline{H}_c(Q_i, Q, \theta-\theta') = \underline{Q}^T(\theta')\underline{H}'(Q_i, Q, \theta', \theta)\underline{Q}(\theta) \quad \text{.................................} \quad (7.20)$$

when (7.20) and (7.19) are substituted in (7.18), we get

$$\underline{\tilde{H}}_{ij}^k = \int_0^{2\pi} \left\{ \int_{C_j} \underline{H}'(Q_i, Q, \theta'=0, \theta) R \, ds \right\} \underline{Q}(\theta) e^{ik\theta} d\theta \quad \text{.................................} \quad (7.21)$$

Again, we want to compute one element of (6.23) for the top surface of a cylindrical poro-elastic (PE) body of unit radius, as shown in **Figure 6.1**. To simplify, we set ω=k=0

Therefore, we have

$$\underline{\tilde{H}}_{\underline{ii}}^{k=0} = \int_0^{2\pi} \left\{ \int_{C_i} \underline{H}'(Q_i, Q, \theta'=0, \theta) R \, ds \right\} \underline{Q}(\theta) d\theta \quad \text{.................................} \quad (7.22)$$

but, this integral is simply a surface integral over the top surface "S". Hence, we can write

$$\underline{\tilde{H}}_{\underline{ii}}^{k=0} = \int_S \underline{H}'(Q_i, Q, \theta'=0, \theta) dA \quad \text{.................................} \quad (7.23)$$

We want to compute (1, 3) element of (7.23), we shall do this first by usual methods of calculus and than by the procedure outlined in sections 6.3 and 6.4.

We again select the material parameters $\mu = 1.$ and $\nu = 0.25$, the third element of first row of 2nd FS matrix reduces (in static case) for the element in **Figure 7.2** to

$$H'_{13} = \frac{1}{12\pi}\frac{r_1}{r^2} = \frac{1}{12\pi}\frac{(x_1 - a_1)}{r^3} \qquad \text{(7.24)}$$

Since, $x_1 = R\cos(\theta)$ and $a_1 = R\cos(0)$

$$\tilde{H}_{13}^{k=0} = \frac{1}{12\pi}\int_S \frac{x_1 - a_1}{r^3}\,dA \qquad \text{(7.25)}$$

where we removed the indices "i" for clarity. We again divide the region "S" into two as shown in **Figure 7.2,** hence,

$$\tilde{H}_{13}^{0} = \frac{1}{12\pi}\left\{\underbrace{\int_{S'} \frac{x_1 - a_1}{r^3}\,dA}_{KK_1} + \underbrace{\int_{S-S'} \frac{x_1 - a_1}{r^3}\,dA}_{KK_2}\right\} \qquad \text{(7.26)}$$

We integrate the terms on r.h.s. of (7.26) separately. The first term KK_1 can be integrated exactly if one converts to polar coordinates over S´, we note that (7.26) is strongly singular and must be understood in "*Cauchy Principle Value*" sense, thus

$$\begin{aligned} KK_1 &= \frac{1}{12\pi}\int_0^{2\pi}\int_0^{1/2} \frac{r\cos(\theta)}{r^3}\, r\,dr d\theta \\ &= \frac{1}{12\pi}\int_0^{2\pi} \cos(\theta)\,d\theta \int_0^{1/2} \frac{1}{r}\,dr \\ &= 0 \end{aligned} \qquad \text{(7.27)}$$

One can convince oneself about this result, by writing the integral KK_1 alternatively as

$$KK_1 = \frac{1}{12\pi}\int_{-\pi/2}^{\pi/2}\int_{0}^{\cos(\alpha)} \frac{\rho\cos(\theta) - 0.5}{r^3}\,\rho\,d\rho d\alpha \quad \ldots \quad (7.28)$$

and MATHCAD 2001 produces

$$KK_1 = -1.952 \times 10^{-9} \quad \ldots \quad (7.29)$$

Still, another approach would be to have MATHCAD compute the following,

$$KK_1 = \frac{1}{12\pi}\int_{0}^{2\pi}\int_{\varepsilon}^{1/2} \frac{r\cos(\theta)}{r^3}\,r\,drd\theta \quad \ldots \quad (7.30)$$

For $\varepsilon = 10^{-15}$ MATHCAD 2001 computes $KK_1 = \frac{1}{12\pi}\int_{0}^{2\pi}\int_{10^{-15}}^{1/2} \frac{r\cos(\theta)}{r^3}\,r\,drd\theta = 0$

For the second term, by referring to the **Figure 7.2** and noting the polar representations of the inner and outer circles in the shifted coordinate frame x_1' - x_2' , we write,

$$KK_2 = \frac{1}{12\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\cos(\theta)}^{-2\cos(\theta)} \frac{x_1 - 0.5}{r^3}\,\rho\ d\rho d\theta \quad \ldots \quad (7.31)$$

Alternatively, one can write

$$KK_2 = \frac{2}{12\pi}\left[\int_{-1}^{0}\int_{0}^{\sqrt{1-x_1^2}} \frac{x_1 - 0.5}{\left[(x_1 - 0.5)^2 + x_2^2\right]^{3/2}}\,dx_2dx_1 + \int_{0}^{1}\int_{\sqrt{(0.5)^2-(x_1-0.5)^2}}^{\sqrt{1-x_1^2}} \frac{x_1 - 0.5}{\left[(x_1 - 0.5)^2 + x_2^2\right]^{3/2}}\,dx_2dx_1\right]$$

$$\ldots \quad (7.32)$$

MATHCAD 2001 computes $KK_2 = -0.046322183$ for (7.31) and $KK_2 = -0.046322205$ for (7.32), respectively. Finally,

$$\boxed{\tilde{H}_{13}^{0} = -0.046322183} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (7.33)$$

The computed values for $\tilde{G}_{33}^{0}$ and $\tilde{H}_{13}^{0}$, by the procedure we described earlier, are summarized in the following table,

Table 7.1 Convergence computations for the proposed method

N= 2^M	Gauss Rule	$\tilde{G}_{33}^{0}$	$\tilde{H}_{13}^{0}$
8 (M=3)	8	0.304332	-0.005506
16	8	0.308021	-0.027872
32	8	0.309817	-0.037362
64	8	0.310637	-0.041865
128	8	0.311026	-0.044095
256 (M=8)	8	0.311216	-0.045208
512 (M=9)	8	0.311310	-0.045767
1024 (M=10)	8	0.311359	-0.046032
2048 (M=11)	8	0.311393	-0.046225
4096 (M=12)	**8**	**0.3114056**	**-0.046264**
8192 (M=13)	8	0.3113876	-0.046237
8192 (M=14)	8	0.3113613	-0.046242
8192 (M=14)	***10***	*0.3113848*	*-0.046267*
8192 (M=14)	***14***	*0.3114006*	*-0.046294*
16384 (M=15)	***8***	*0.3113428*	*-0.046256*
16384 (M=15)	***15***	*0.3113956*	*-0.046300*
16384 (M=15)	***20***	*0.3114019*	*-0.046312*
MATHCAD2001		**0.3114052**	**-0.0463222**

The values in **Table 7.1** are computed by the computer program developed in this work. In the computations, line elements are divided into six sub-elements (-1.0

.. -0.2 .. -0.1 .. 0.0 .. 0.1 ..0.2 .. 1.0) and to compute expressions in (5.102) 17 point Gauss rule is used in both ρ and γ directions. As seen these values converge to the half-analytical value (MATHCAD 2001) as the number of divisions in angular direction is increased. Unfortunately, the numerical accuracy is spoiled "*slightly*" when angular divisions are too much refined, i.e. N>12, but this can be amended by using more Gauss points at increased computational cost.

CHAPTER 8

ASSESSMENT OF THE FORMULATION

8.1 One Dimensional Wave Propagation in a PE Layer

This problem, in the context of poro-elasticity is first mentioned in Cheng et. al. (1991), an analytical solution is also provided in the same reference. We briefly work out the theory here in cylindrical coordinates.

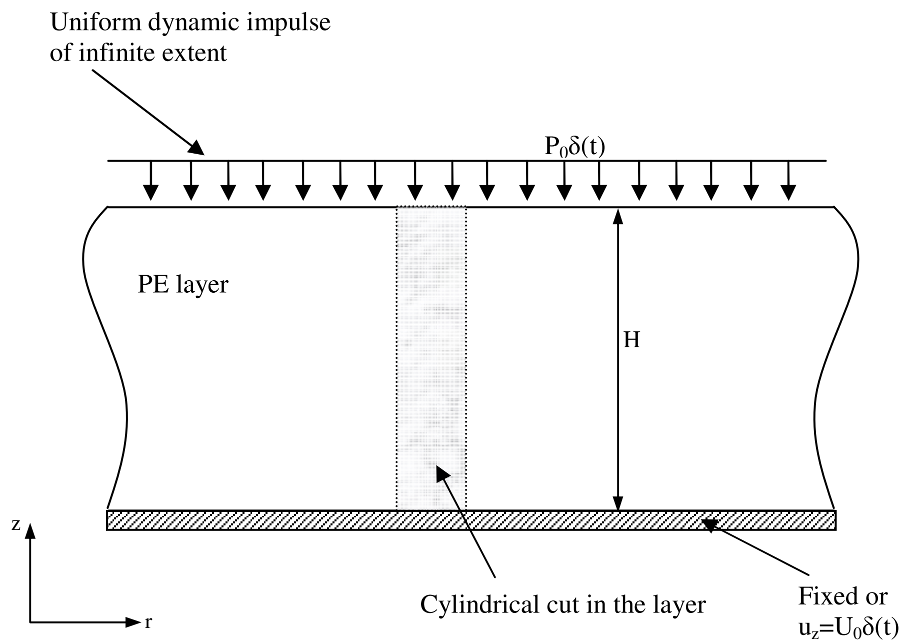

Figure 8.1 One dimensional wave propagation in a layer.

In this problem we consider a PE layer of infinite extent loaded at the top surface by a uniform pressure (infinitely wide) which is suddenly applied and removed, the fluid is assumed to drain freely from the top surface while the bottom surface is impermeable; displacement excitation at the bottom is considered separately. Since the surface load is infinitely wide, one can readily assume that lateral displacements are zero, thus the governing equations of 3-D poro-elasticity, equations (2.55), in this case reduce to the following:

$$\begin{aligned} &(\lambda+2\mu)u_{z,zz} - (\alpha+\beta)p_{,z} + \omega^2(\rho+\beta\rho_f)u_z = 0 \\ &-\frac{\beta}{\omega^2\rho_f}p_{,zz} + \frac{1}{Q}p + (\alpha+\beta)u_{z,z} = 0 \end{aligned} \quad \text{.. (8.1)}$$

together with the boundary conditions (in FTS)

$$\begin{aligned} u_z\big|_{z=0} &= 0 \quad ; \quad q_n\big|_{z=0} = 0 \\ t_z\big|_{z=H} &= -P_0 \quad ; \quad p\big|_{z=H} = 0 \end{aligned} \quad \text{... (8.2)}$$

The solution to this system is given in Cheng et. al. (1991) and Schanz (2001.b), the following solution from Schanz (2001.b) is reproduced here (set, s=iω),

$$u_z(\omega,z) = \frac{P_0}{i\omega(\lambda+2\mu)(d_1\lambda_2 - d_2\lambda_1)}\left[d_2\frac{e^{-i\omega\lambda_1(H-y)} - e^{-i\omega\lambda_1(H+y)}}{1+e^{-2i\omega\lambda_1 H}} - d_1\frac{e^{-i\omega\lambda_2(H-y)} - e^{-i\omega\lambda_2(H+y)}}{1+e^{-2i\omega\lambda_2 H}}\right] \text{........ (8.3)}$$

$$p(\omega,z) = \frac{P_0 d_1 d_2}{(\lambda+2\mu)(d_1\lambda_2 - d_2\lambda_1)}\left[\frac{e^{-i\omega\lambda_1(H-y)} + e^{-i\omega\lambda_1(H+y)}}{1+e^{-2i\omega\lambda_1 H}} - \frac{e^{-i\omega\lambda_2(H-y)} + e^{-i\omega\lambda_2(H+y)}}{1+e^{-2i\omega\lambda_2 H}}\right] \text{................ (8.4)}$$

where, λ_1 and λ_2 are the positive roots of the following characteristic equation;

$$-\beta\frac{\lambda+2\mu}{\rho_f}\lambda^4 - \left(\frac{\lambda+2\mu}{Q} - (\rho+\beta\rho_f)\frac{\beta}{\rho_f} + (\alpha+\beta)^2\right)\lambda^2 + \frac{\rho+\beta\rho_f}{Q} = 0 \text{ .... (8.5)}$$

and

$$d_i = \frac{(\lambda+2\mu)\lambda_i^2 - (\rho+\beta\rho_f)}{(\alpha+\beta)\lambda_i} \text{ .... (8.6)}$$

In solving this problem by BEM, we model the PE "layer" by a PE "column" of unit diameter as shown in figure 8.2.

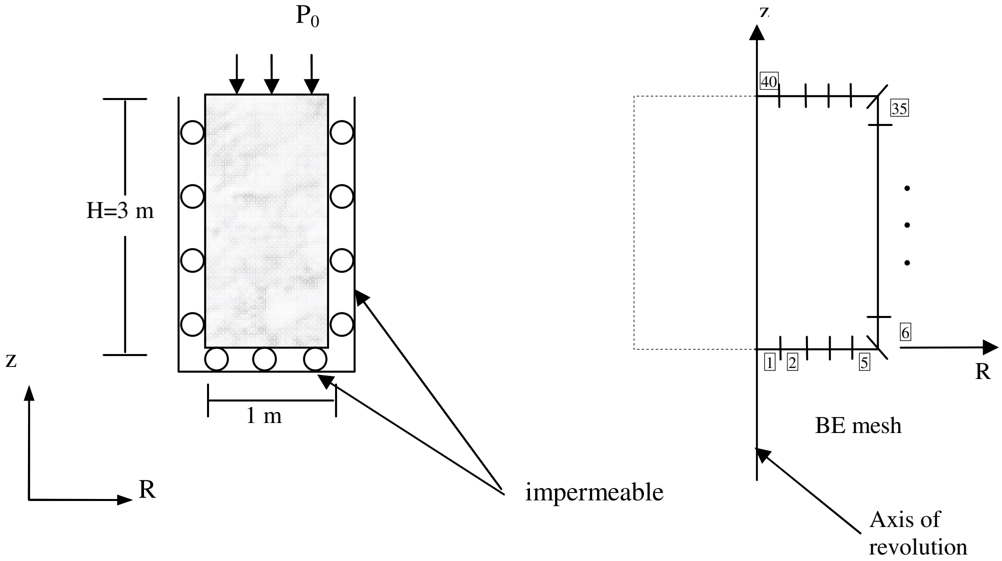

Figure 8.2 Equivalent column model of PE layer for BE analysis.

Although, the PE column in Figure 8.2 is not an exact model for the layer in Figure 8.1, the impermeable rigid walls can sufficiently prevent lateral displacement and flow, in which case the behaviour of PE column closely approximates that of PE layer. The material properties are given in Table 8.1 below.

Table 8.1 Material data for Berea sand stone

n	α	Q (Pa)	μ(Pa)	ν	κ(m^4/N/s)	ρ(kg/m^3)	ρ_f(kg/m^3)	ρ_a(kg/m^3)
0.19	0.778	$1.353*10^{10}$	$6*10^9$	0.2	$1.9*10^{-10}$	2458	1000	125.4

The poro-elastic column problem is solved by program AxiPoro. The column is modelled by 40 axi-symmetric boundary elements. N=128 (2^7) and N=32 (2^5) sub-divisions for circumferential integrations were used for convergence checking. A slight hysteretic damping is introduced with z_H=0.003. The results for top displacement are plotted in Figures 8.3-8.5 together with the analytical solution, equation 8.3.

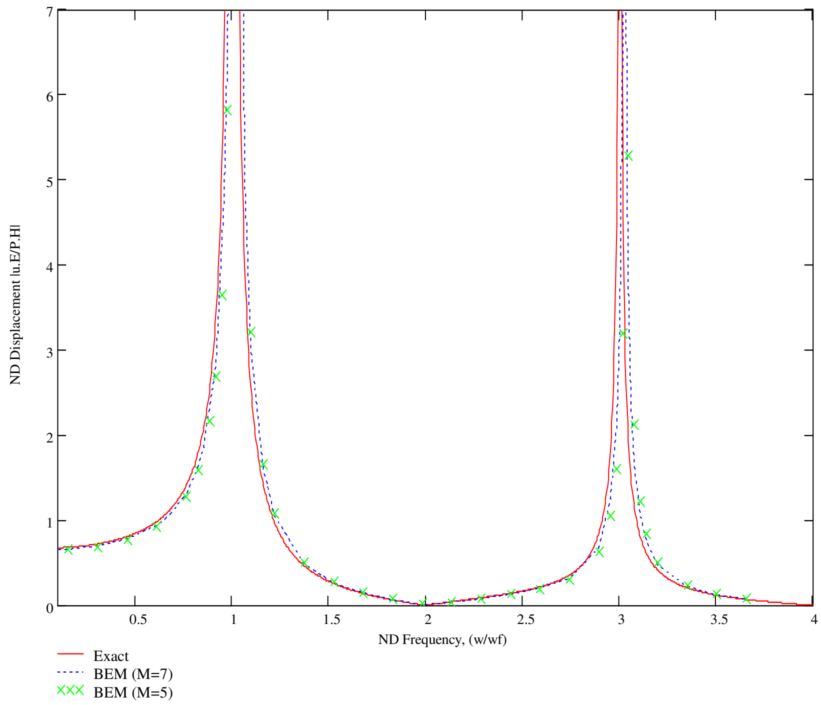

Figure 8.3 PE column: Top displacement amplitude spectra (traction B.C. at top), BEM vs. analytical solution; E=2λ+μ.

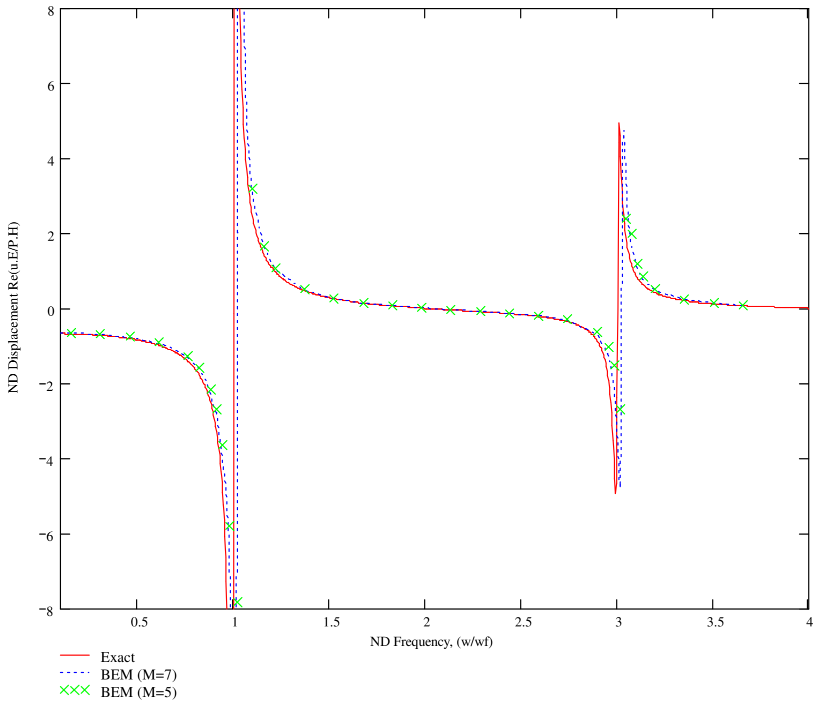

Figure 8.4 PE column: Real part of top displacement (traction B.C. at top), BEM vs. analytical solution, E=2λ+μ.

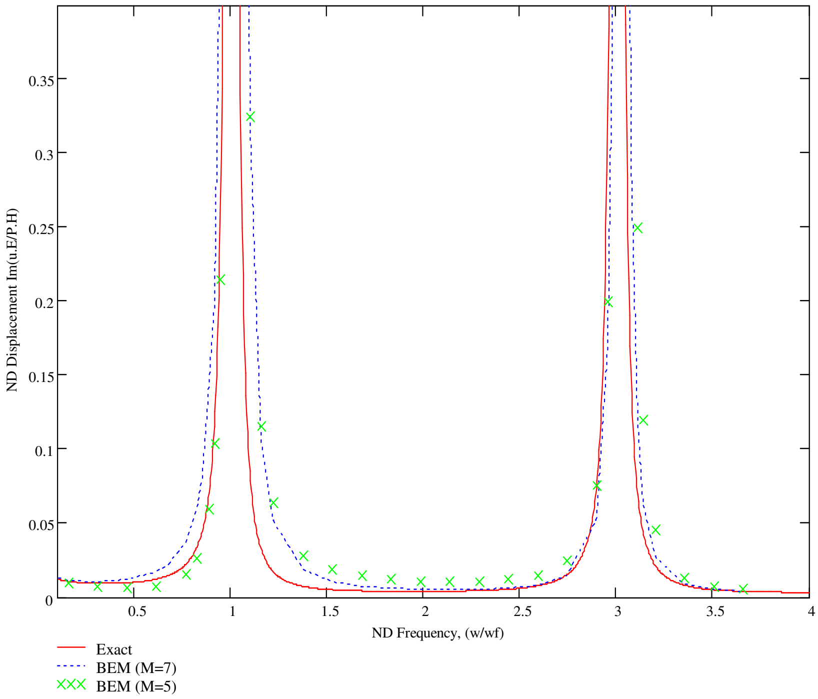

Figure 8.5 PE column: Imaginary part of top displacement (traction B.C. at top), BEM vs. analytical solution, E=2λ+μ.

The circular frequency is non-dimensionalized (ND) with respect to fundamental vibration frequency of the dynamically impermeable porous material

$$\omega_f = 2 \cdot \pi \cdot \frac{V_u}{4 \cdot H} \quad \text{……………………………………………………..………. (8.7)}$$

where,

$$V_u = \sqrt{\frac{\lambda_u + 2 \cdot \mu}{\rho}} \quad \text{……………………………………………………….…. (8.8)}$$

is the p-wave velocity for dynamically impermeable material.

The results for pore-pressure at the bottom of layer are displayed in the Figures 8.6-8.8 below.

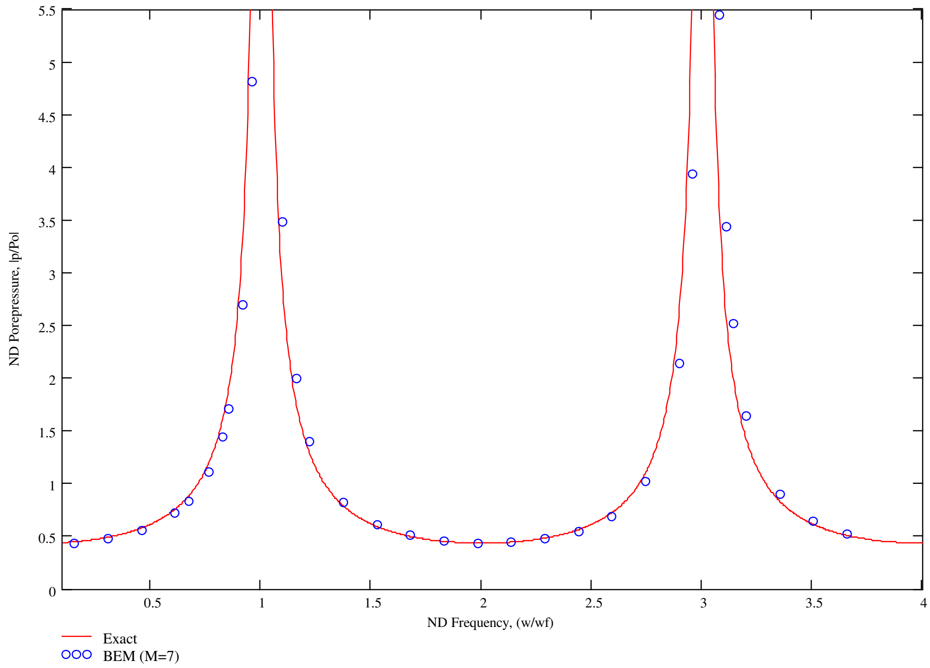

Figure 8.6 PE column: Pore-pressure amplitude at the bottom, (traction B.C. at top), BEM vs. analytical solution.

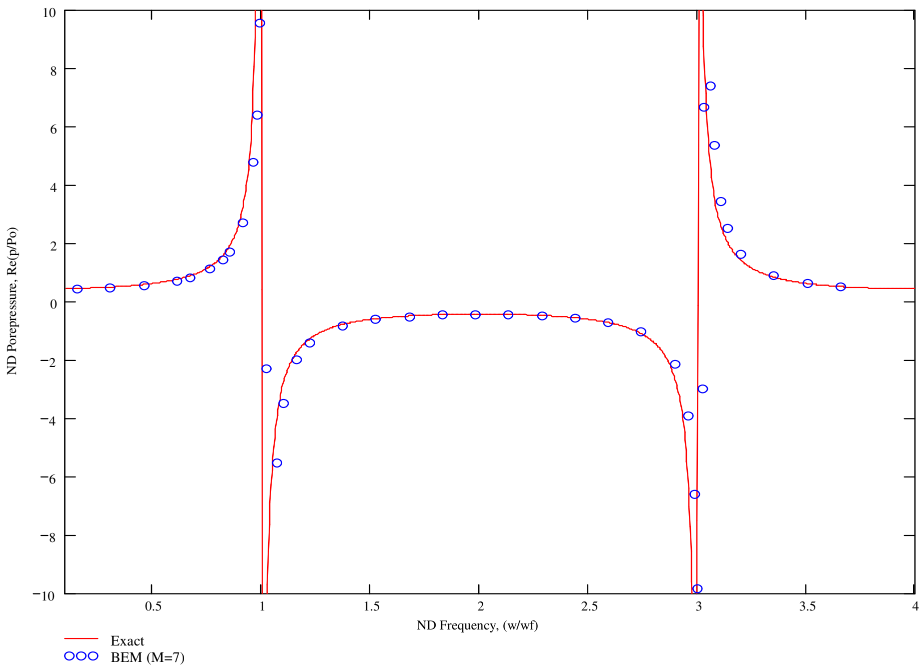

Figure 8.7 PE column: Real part of pore-pressure at the bottom (traction B.C. at top), BEM vs. analytical solution.

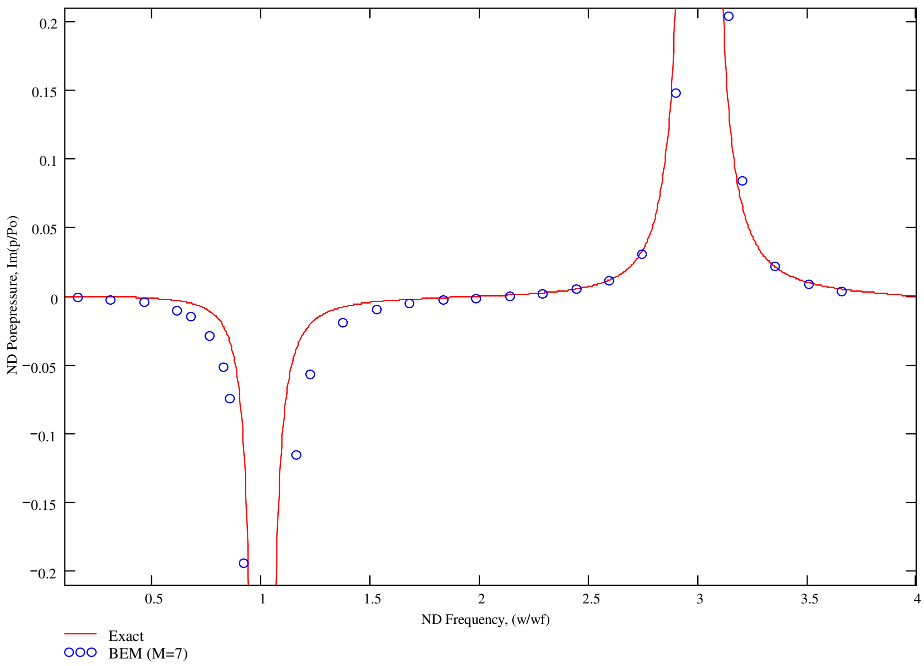

Figure 8.8 PE column: Imaginary part of pore-pressure at the bottom (traction B.C. at top), BEM vs. analytical solution.

Next, we consider the same body for displacement boundary conditions at the bottom; the top surface is free of tractions. This time the boundary conditions read as

$$u_z\big|_{z=0} = U_0 \quad ; \quad q_n\big|_{z=0} = 0$$
$$t_z\big|_{z=H} = 0 \quad ; \quad p\big|_{z=H} = 0 \qquad \text{.. (8.9)}$$

Again from Schanz (2001.b) the solution is,

$$u_z(\omega, z) = \frac{U_0}{(\lambda+2\mu)(\lambda_2^2-\lambda_1^2)}\left[D_2\frac{e^{-i\omega\lambda_1(2*H-y)}+e^{-i\omega\lambda_1 y}}{1+e^{-2i\omega\lambda_1 H}} - D_1\frac{e^{-i\omega\lambda_2(2*H-y)}+e^{-i\omega\lambda_2 y}}{1+e^{-2i\omega\lambda_2 H}}\right]$$

.... (8.10)

$$p(\omega, z) = \frac{i\omega U_0}{(\lambda+2\mu)(\lambda_2^2-\lambda_1^2)}\left[d_1 D_2\frac{e^{-i\omega\lambda_1(2*H-y)}-e^{-i\omega\lambda_1 y}}{1+e^{-2i\omega\lambda_1 H}} - d_2 D_1\frac{e^{-i\omega\lambda_2(2*H-y)}-e^{-i\omega\lambda_2 y}}{1+e^{-2i\omega\lambda_2 H}}\right]$$

.... (8.11)

where,

$$D_i = (\lambda + 2\mu)\lambda_i^2 + \alpha\rho_f - \rho \quad \text{...} \quad (8.12)$$

Again the column is modelled by 40 axi-symmetric boundary elements. The angular divisions for circumferential integrations were $N=2^8 = 256$. A hysteretic damping of 0.5% is introduced. The results for top displacement are plotted in Figures 8.9-8.11 together with the analytical solution, equation 8.8.

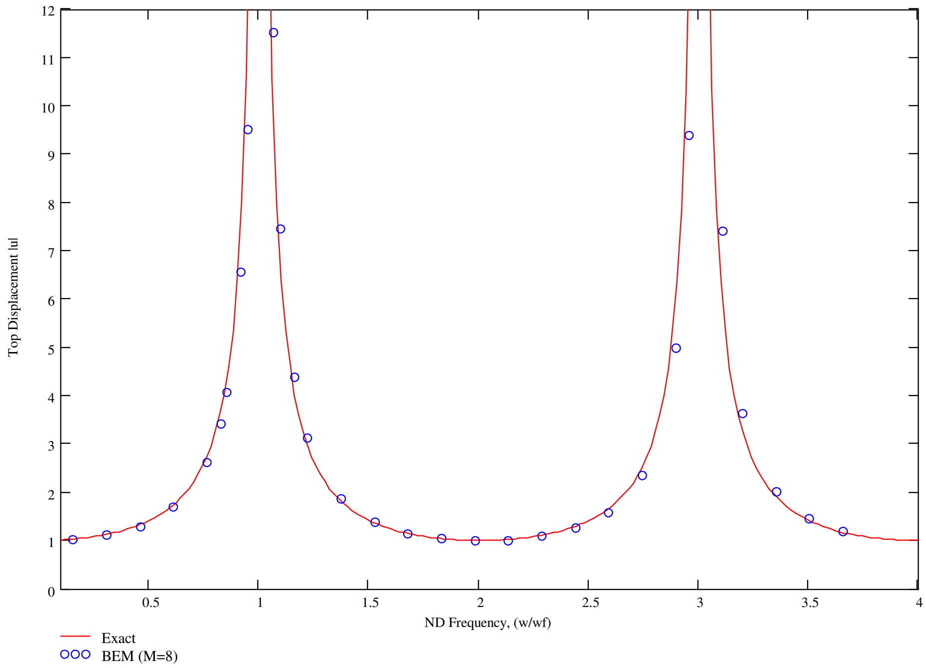

Figure 8.9 PE column: Top displacement amplitude spectra, (displacement B.C. at the bottom) BEM vs. analytical solution, E=2λ+μ.

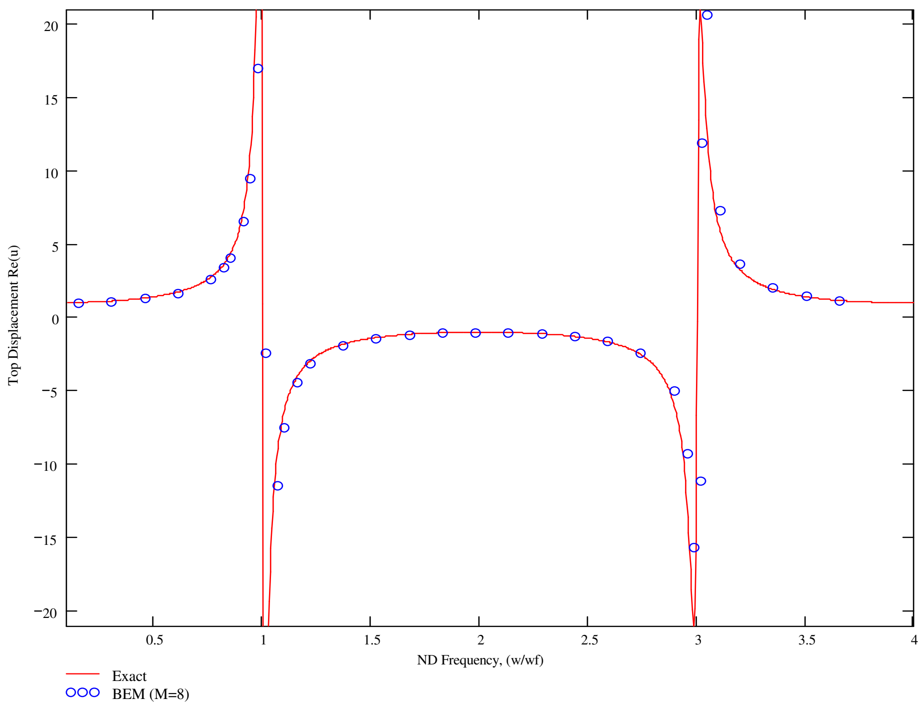

Figure 8.10 PE column: Real part of top displacement, (displacement B.C. at the bottom) BEM vs. analytical solution, E=2λ+μ.

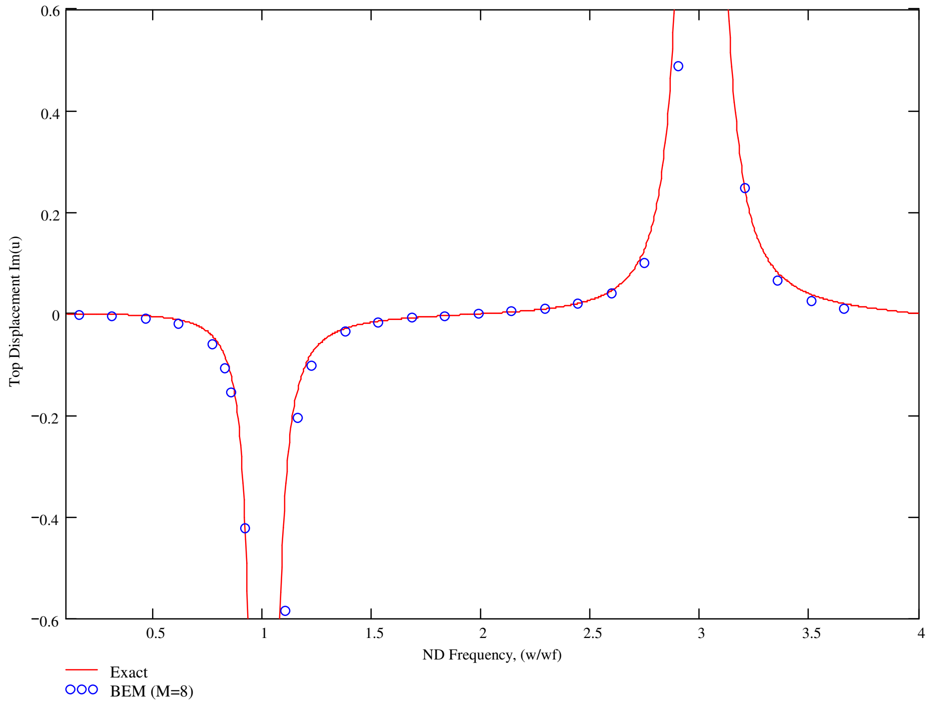

Figure 8.11 PE column: Imaginary part of top displacement, (displacement B.C. at the bottom) BEM vs. analytical solution, E=2λ+μ.

8.2 Sudden Pressurization of a Circular Cavity (Infinitely long cylinder in a PE full space)

This problem was investigated by Senjuntichai and Rajapakse (1993). They considered step load, gradually applied step load and triangular pulse load type pressurizations for either drainage free or impermeable wall conditions. However, the governing equations they solved did not include the "continuity equation", therefore, to comply with our BEM formulation, we re-work the solution for sudden pressurization (Dirac loading in time) of the circular cavity with permeable wall condition below, following the outline in Senjuntichai and Rajapakse (1993).

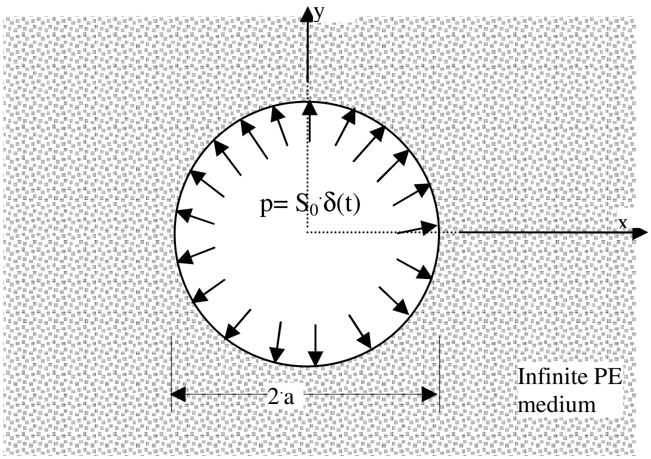

Figure 8.12 Circular cavity (infinite cylinder) in a poro-elastic full space suddenly pressurized.

We consider an infinitely long cylindrical cavity, whose axis of revolution coincides with the vertical axis (z-axis), Figure 8.12, in a PE medium of infinite extent. Because the displacement conditions are essentially that of plane strain and the boundary conditions are also axi-symmetric, we write

$$u_z = \frac{\partial}{\partial z}(.) = 0 \text{ and } u_\theta = \frac{\partial}{\partial \theta}(.) = 0 \quad \text{......................................} (8.13)$$

(u_r, u_t, u_z) are the components of the displacement vector in cylindrical coordinates. Under these conditions the governing equations of 3-D poro-elasticity, equations (2.55), in this case reduce to the following:

$$(\lambda + 2\mu)\frac{d}{dr}\left(\frac{1}{r}\left(\frac{d}{dr}(ru)\right)\right) - (\alpha + \beta)\frac{dp}{dr} + \omega^2(\rho + \beta\rho_f)u = 0$$

$$(\alpha + \beta)\frac{1}{r}\left(\frac{d}{dr}(ru)\right) - \frac{\beta}{\omega^2\rho_f}\frac{1}{r}\frac{d}{dr}\left(r\frac{dp}{dr}\right) + \frac{p}{Q} = 0 \quad \text{............ (8.14)}$$

where, we substituted u for u_r for simplicity.

If we introduce a displacement potential ϕ, such that $u = \frac{d\phi}{dr}$, then (8.14) become

$$(\lambda + 2\mu)\nabla_r^2\phi - (\alpha + \beta)p + \omega^2(\rho + \beta\rho_f)\phi = 0 \quad \text{.............................. (8.15)}$$

$$(\alpha + \beta)\nabla_r^2\phi - \frac{\beta}{\omega^2\rho_f}\nabla_r^2 p + \frac{p}{Q} = 0 \quad \text{.. (8.16)}$$

where, $\nabla_r^2 = \frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}(.)\right)$

Eliminating p between (8.15) and (8.16), we get

$$(\nabla_r^2 - \lambda_1^2)(\nabla_r^2 - \lambda_2^2)\phi = 0 \quad \text{.. (8.17)}$$

where,

$$\lambda_{1,2}^2 = \frac{1}{2}\left[B \mp \sqrt{B^2 - 4c}\right] \quad \text{... (8.18)}$$

$$B = \frac{\omega^2 \rho_f}{\beta Q} + \frac{\omega^2 \rho_f (\alpha + \beta)^2}{\beta(\lambda + 2\mu)} - \frac{\omega^2 (\rho + \beta\rho_f)}{(\lambda + 2\mu)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (8.19)$$

$$c = \frac{\omega^4 \rho_f (\rho + \beta\rho_f)}{\beta Q(\lambda + 2\mu)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (8.20)$$

It is well known that the solution to (8.17) can be written in the form,

$$\phi = \phi_1 + \phi_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (8.21)$$

where,

$$\begin{aligned} \phi_1 &= A \cdot I_0(\lambda_1 r) + C \cdot K_0(\lambda_1 r) \\ \phi_2 &= B \cdot I_0(\lambda_2 r) + D \cdot K_0(\lambda_2 r) \end{aligned} \quad \ldots\ldots\ldots\ldots\ldots\ldots (8.22)$$

where, I_0 and K_0 are modified Bessel functions of order zero (Abramowitz and Stegun, 1964). For (8.22) to be regular at infinity, the constants A and B must be zero. Thus,

$$\phi = C \cdot K_0(\lambda_1 r) + D \cdot K_0(\lambda_1 r) \quad \ldots\ldots\ldots\ldots\ldots\ldots (8.23)$$

Thus, the radial displacement function is

$$u = \frac{d\phi}{dr} = -\lambda_1 \cdot C \cdot K_1(\lambda_1 r) - \lambda_2 \cdot D \cdot K_1(\lambda_2 r) \quad \ldots\ldots\ldots\ldots\ldots\ldots (8.24)$$

Similarly, one obtains for the pore-pressure solution

$$p = d_1 C \cdot K_0(\lambda_1 r) + d_2 C \cdot K_0(\lambda_2 r) \quad \ldots\ldots\ldots\ldots\ldots\ldots (8.25)$$

where,

$$d_i = \frac{\lambda + 2\mu}{\alpha + \beta}\lambda_i^2 + \frac{\omega^2(\rho + \beta\rho_f)}{\alpha + \beta} \quad \text{.. (8.26)}$$

The radial stress is obtained from

$$\tau_{rr} = \lambda\left(\frac{du}{dr} + \frac{u}{r}\right) + 2\mu\frac{du}{dr} - \alpha p \quad \text{....................................... (8.27)}$$

Te unknown coefficients are determined from boundary conditions (B.C.'s). The B.C.'s for circular (cylindrical) cavity problem are the following:

$$\begin{aligned} p\big|_{r=a} &= 0 \\ \tau_{rr}\big|_{r=a} &= -S_0 \end{aligned} \quad \text{.. (8.28)}$$

From (8.27) one finds for the unknown constants "C" and "D"

$$C = \frac{-S_0}{Det} d_2 \cdot K_0(\lambda_2 \cdot a) \quad \text{... (8.29)}$$

$$D = \frac{S_0}{Det} d_1 \cdot K_0(\lambda_1 \cdot a) \quad \text{... (8.30)}$$

where,

$$Det = d_2 \cdot N(\lambda_1 a) \cdot K_0(\lambda_2 a) - d_1 \cdot N(\lambda_2 a) \cdot K_0(\lambda_1 a) \quad \text{.................... (8.31)}$$

$$N(\varsigma, x) = (\lambda + 2\mu) \cdot \varsigma^2 \cdot K_0(\varsigma \cdot x) + \frac{2\mu}{a} \cdot \varsigma \cdot K_1(\varsigma \cdot x) \quad \text{..................... (8.32)}$$

Finally, we obtain

$$u = \frac{S_0}{Det}\left(\lambda_1 \cdot d_2 \cdot K_0(\lambda_2 \cdot a) \cdot K_1(\lambda_1 r) - \lambda_2 \cdot d_1 \cdot K_0(\lambda_1 \cdot a) \cdot K_1(\lambda_2 r)\right) \quad (8.33)$$

$$p = \frac{S_0 \cdot d_1 \cdot d_2}{Det}\left(-K_0(\lambda_2 \cdot a) \cdot K_1(\lambda_1 r) + K_0(\lambda_1 \cdot a) \cdot K_1(\lambda_2 r)\right) \quad (8.34)$$

The poro-elastic circular cavity problem is solved by program AxiPoro, the material data are again that of Berea sandstone's (Table 8.1). A finite cylindrical cavity of 10 metres height is modelled by 25 axi-symmetric boundary elements. The angular divisions for circumferential integrations were $N = 2^7 = 128$. A slight hysteretic damping is introduced with z_H=0.05. The results for radial surface displacements (compliance) at mid-height of the cavity are plotted in Figures 8.13-8.15 together with the analytical solution (equation 8.33). Some noise is observed in the figures due to waves generated at the ends of the cavity.

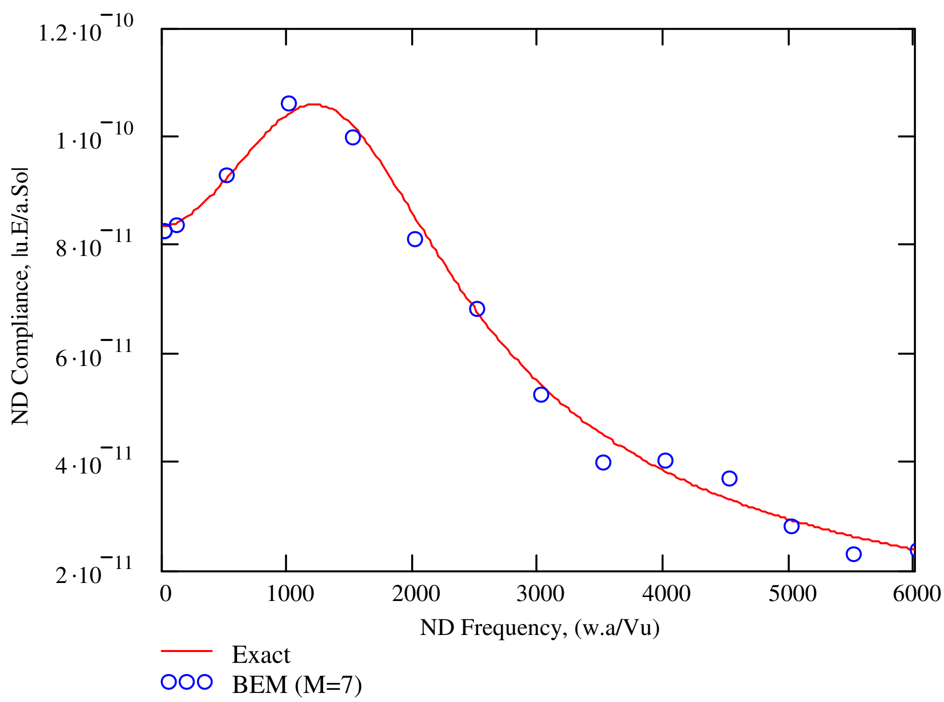

Figure 8.13 Circular cavity: Absolute value of ND compliance, BEM vs. analytical solution, E=2λ+μ.

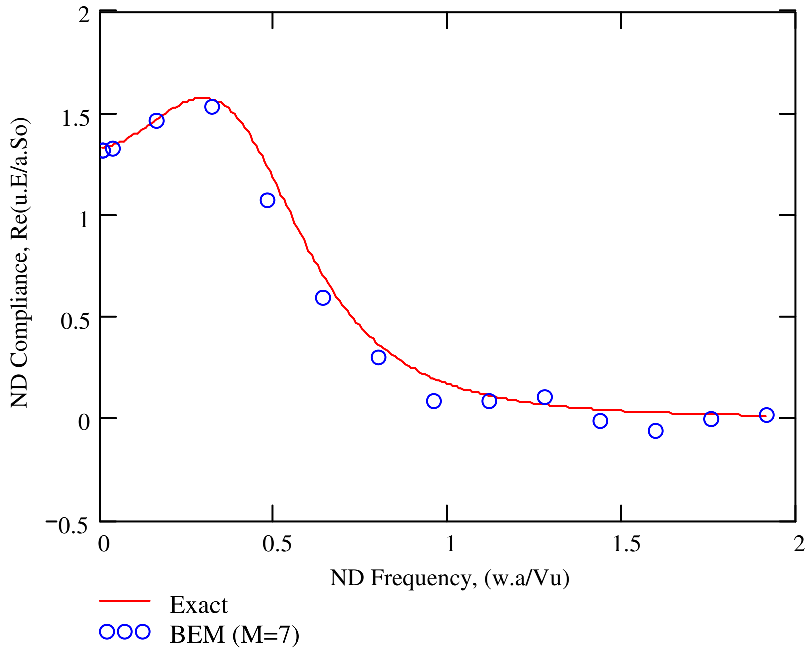

Figure 8.14 Circular cavity: Real part of ND compliance, BEM vs. analytical solution, E=2λ+μ.

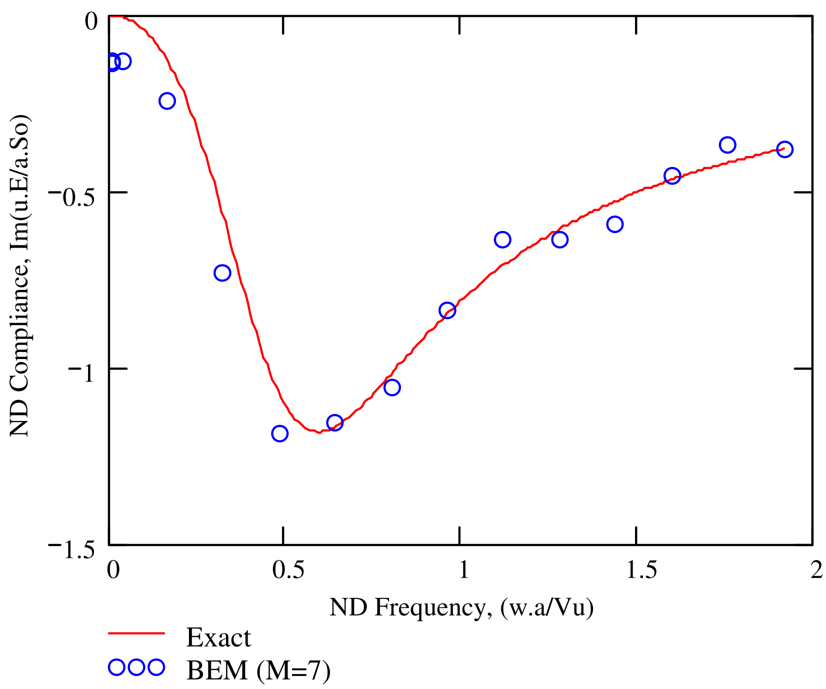

Figure 8.15 Circular cavity: Imaginary part of ND compliance, BEM vs. analytical solution, E=2λ+μ.

8.3 Sudden Pressurization of a Spherical Cavity in an Infinite PE Medium

This problem is solved for this study, an extensive literature survey revealed no earlier solution available in the literature. We consider a spherical cavity in an infinite PE medium; in the absence of body forces and fluid source, the governing equations of 3-D poro-elasticity in FTS (eqn.'s 2.55) in this case reduce to the following:

$$(\lambda+2\mu)\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{du_r}{dr}\right)-\frac{2}{r^2}u_r\right]-(\alpha+\beta)\frac{dp}{dr}+\omega^2(\rho+\beta\rho_f)u_r=0 \quad \text{..................... (8.35)}$$

$$-\frac{\beta}{\omega^2\rho_f}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dp}{dr}\right)+(\alpha+\beta)\frac{1}{r^2}\frac{d}{dr}\left(r^2u_r\right)+\frac{p}{Q}=0 \quad \text{................................. (8.36)}$$

$$u_\theta=u_\phi=0 \quad \text{.. (8.37)}$$

If one introduces a displacement potential Φ such that, $u_r=\frac{\partial\Phi}{\partial r}$ then (8.35) and (8.36) become,

$$\frac{d^2}{dr^2}(\Psi)-\frac{\alpha+\beta}{\lambda+2\mu}(\chi)-\lambda_4^2(\Psi)=0 \quad \text{.. (8.38)}$$

$$\frac{d^2}{dr^2}(\chi)-\frac{\omega^2\rho_f(\alpha+\beta)}{\beta}\frac{d^2}{dr^2}(\Psi)-\frac{\omega^2\rho_f}{\beta Q}(\chi)=0 \quad \text{.................................... (8.39)}$$

where,

$$\begin{aligned}\Psi&=r\Phi\\ \chi&=rp\end{aligned} \quad \text{.. (8.40)}$$

and, as defined before, $\lambda_4^2=-\frac{\omega^2(\rho+\beta\rho_f)}{\lambda+2\mu}$.

Now, we assume solutions of the form,

$$\begin{aligned}\Psi &= \Psi_0 e^{im\omega r} \\ \chi &= \chi_0 e^{im\omega r}\end{aligned} \qquad \text{…………………………………………………………… (8.41)}$$

Substituting (8.13) in (8.10) and (8.11), one obtains the eigenvalue problem:

$$\begin{bmatrix} -m^2\omega^2 - \lambda_4{}^2 & -\dfrac{\alpha+\beta}{\lambda+2\mu} \\ \dfrac{m^2\omega^4\rho_f(\alpha+\beta)}{\beta} & -\dfrac{\omega^2\rho_f}{\beta Q} - m^2\omega^2 \end{bmatrix} \begin{bmatrix} \Psi_0 \\ \chi_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{……………………………….. (8.42)}$$

hence the characteristic equation:

$$\left(m^2\omega^2 + \lambda_4{}^2\right)\left(\frac{\omega^2\rho_f}{\beta Q} + m^2\omega^2\right) + \frac{\alpha+\beta}{\lambda+2\mu}\frac{m^2\omega^4\rho_f(\alpha+\beta)}{\beta} = 0 \qquad \text{……………………… (8.43)}$$

Introduce the following abbreviations,

$$B = \frac{\rho+\beta\rho_f}{\lambda+2\mu} - \frac{\rho_f}{\beta Q} - \frac{\rho_f(\alpha+\beta)^2}{\beta(\lambda+2\mu)} \quad ; \quad C = \frac{\rho_f(\rho+\beta\rho_f)}{\beta Q(\lambda+2\mu)} \qquad \text{……………………….… (8.44)}$$

Thus, the roots of the characteristic equation are

$$\begin{aligned} m_1 &= \sqrt{\frac{1}{2}\left[B - \sqrt{B^2+4C}\right]} \quad ; \quad m_2 = \sqrt{\frac{1}{2}\left[B + \sqrt{B^2+4C}\right]} \\ m_3 &= -m_1 \quad ; \quad m_4 = -m_2 \end{aligned} \qquad \text{…………….. (8.45)}$$

Then the solution of the system (8.38) and (8.39) becomes,

$$\begin{aligned}\Psi &= \Psi_1 e^{-im_1\omega r} + \Psi_2 e^{-im_2\omega r} + \Psi_3 e^{im_1\omega r} + \Psi_4 e^{im_2\omega r} \\ \chi &= \chi_1 e^{-im_1\omega r} + \chi_2 e^{-im_2\omega r} + \chi_3 e^{im_1\omega r} + \chi_4 e^{im_2\omega r}\end{aligned} \qquad \text{……………….....…………… (8.46)}$$

It is clear that the terms $e^{im_k\omega r}$ represent waves propagating inwards. In an infinite medium no waves propagate from infinity, hence we discard these terms and write,

$$\begin{aligned}\Psi &= \Psi_1 e^{-im_1\omega r} + \Psi_2 e^{-im_2\omega r} \\ \chi &= \chi_1 e^{-im_1\omega r} + \chi_2 e^{-im_2\omega r}\end{aligned} \qquad (8.47)$$

Moreover, the amplitudes Ψ_k, χ_k are related through (8.42) as

$$\chi_k = \frac{-(\lambda+2\mu)\left(m_k{}^2\omega^2+\lambda_4{}^2\right)}{\alpha+\beta}\Psi_k \qquad (8.48)$$

Finally, one obtains

$$\begin{aligned}\Psi &= \Psi_1 e^{-im_1\omega r} + \Psi_2 e^{-im_2\omega r} \\ \chi &= -\frac{\lambda+2\mu}{\alpha+\beta}\left[\Psi_1\left(m_1{}^2\omega^2+\lambda_4{}^2\right)e^{-im_1\omega r} + \Psi_2\left(m_2{}^2\omega^2+\lambda_4{}^2\right)e^{-im_2\omega r}\right]\end{aligned} \qquad (8.49)$$

Furthermore, since $\Psi = r\Phi$ and $u_r = \frac{\partial\Phi}{\partial r}$

$$\Phi = \Psi_1\frac{e^{-im_1\omega r}}{r} + \Psi_2\frac{e^{-im_2\omega r}}{r} \qquad (8.50)$$

Thus,

$$\begin{aligned}u &= -\Psi_1\left[\frac{1}{r}+i\omega m_1\right]\frac{e^{-i\omega m_1 r}}{r} - \Psi_2\left[\frac{1}{r}+i\omega m_2\right]\frac{e^{-i\omega m_2 r}}{r} \\ p &= -\frac{\lambda+2\mu}{\alpha+\beta}\left[\Psi_1\left(m_1{}^2\omega^2+\lambda_4{}^2\right)\frac{e^{-i\omega m_1 r}}{r} + \Psi_2\left(m_2{}^2\omega^2+\lambda_4{}^2\right)\frac{e^{-i\omega m_2 r}}{r}\right]\end{aligned} \qquad (8.51)$$

Where, we introduced $u=u_r$ for simplicity. The integration constants Ψ_1 and Ψ_2 are to be evaluated from boundary conditions at $r = a$:

$$\begin{aligned}\tau_{rr}\big|_{r=a} &= -S_0 \\ p &= 0\end{aligned} \qquad (8.52)$$

Finally,

$$\Psi_3 = -\frac{aS_0}{\Delta}\left(m_2^2\omega^2 + \lambda_4^2\right)e^{i\omega m_1 a} \quad \text{.. (8.53)}$$

$$\Psi_4 = \frac{aS_0}{\Delta}\left(m_1^2\omega^2 + \lambda_4^2\right)e^{i\omega m_2 a} \quad \text{.. (8.54)}$$

where,

$$\Delta = \left[\frac{4\mu}{a}\left(i\omega m_1 + \frac{1}{a}\right) - (\lambda + 2\mu)\omega^2 m_1\right]\left(m_2^2\omega^2 + \lambda_4^2\right) - \left[\frac{4\mu}{a}\left(i\omega m_2 + \frac{1}{a}\right) - (\lambda + 2\mu)\omega^2 m_2\right]\left(m_1^2\omega^2 + \lambda_4^2\right) \quad \text{........... (8.55)}$$

The poro-elastic spherical cavity problem is solved by program AxiPoro, the material data are again that of Berea sandstone's (Table 8.1). The cavity is modelled by 35 axi-symmetric boundary elements. The angular divisions for circumferential integrations were $N = 2^7 = 128$. No hysteretic damping is introduced for this problem. The results for radial displacement at the surface (compliance) are plotted in Figures 8.16-8.18 together with the analytical solution, equation 8.51. The pore-pressure at ND radial distance R=1.5 is plotted in figures 8.19-8.21. The circular frequency is non-dimensionalized with a/V_u, where V_u is given by (8.8).

To obtain distribution of displacement and pore-pressure along the radius, 8 interior output points are specified along R-axis at coordinates (R,z) = (1.5,0), (2.0,0), (2.5,0), (3.0,0), (5.0,0), (8.0,0), (12.0,0), (20.0,0). The distribution of displacement (radial) and pore-pressure along the radius for varying frequencies are displayed in Figures 8.22 - 8.26.

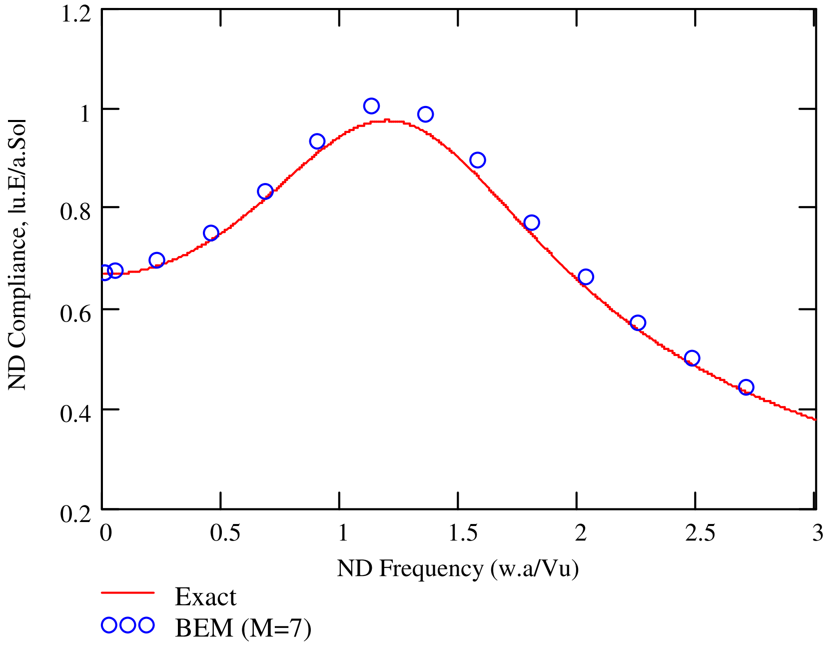

Figure 8.16 Spherical cavity: Absolute value of ND compliance, BEM vs. analytical solution, E=2λ+μ.

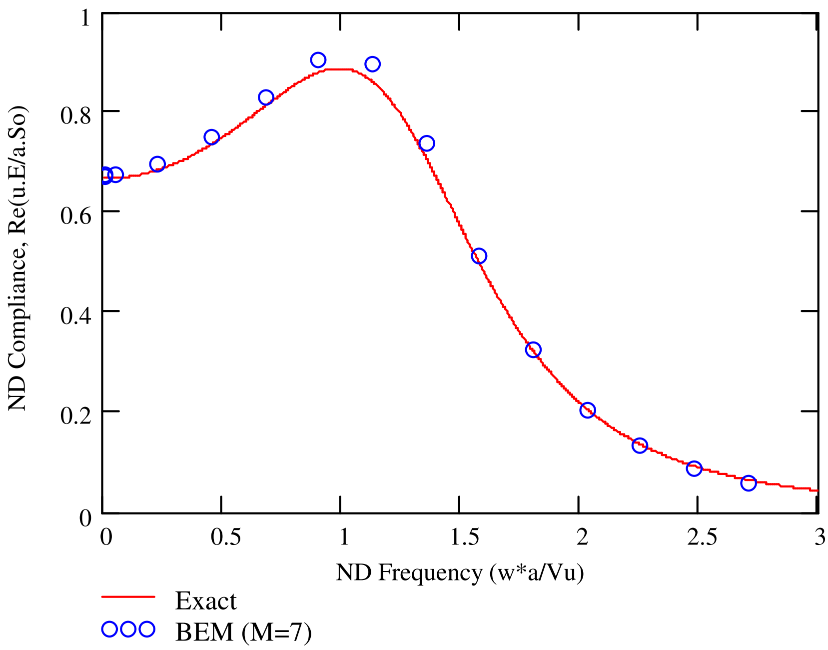

Figure 8.17 Spherical cavity: Real part of ND compliance, BEM vs. analytical solution, E=2λ+μ.

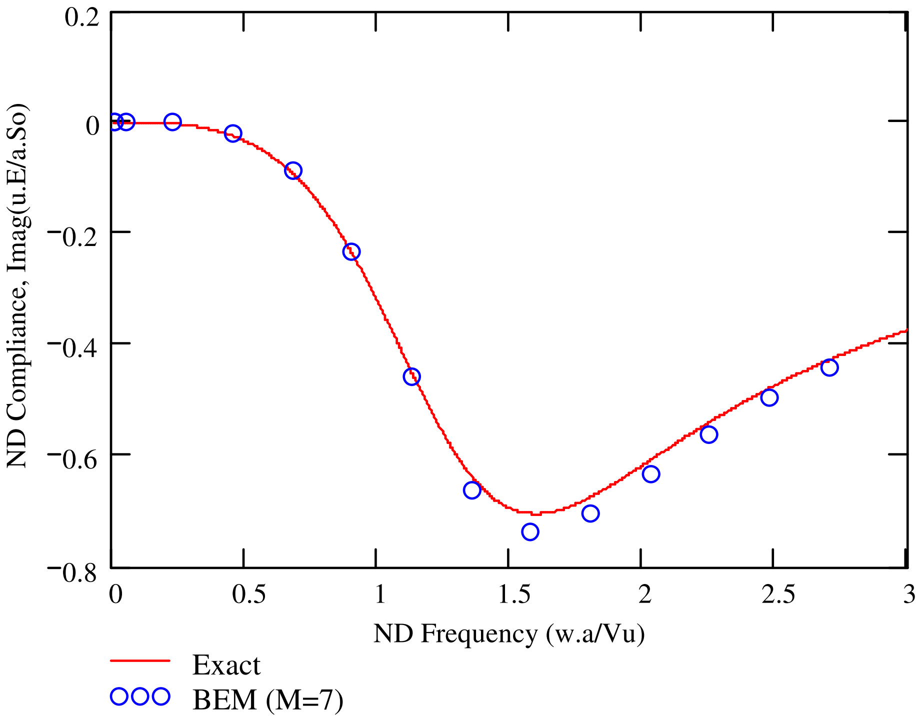

Figure 8.18 Spherical cavity: Imaginary part of ND compliance, BEM vs. analytical solution, E=2λ+μ.

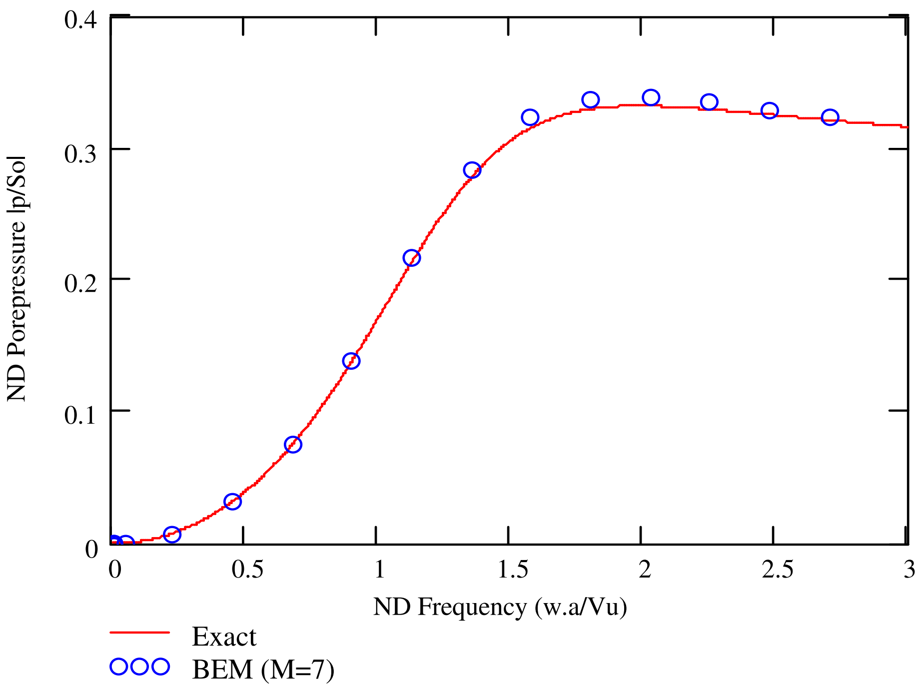

Figure 8.19 Spherical cavity: Absolute value of ND pore-pressure at R=1.5.

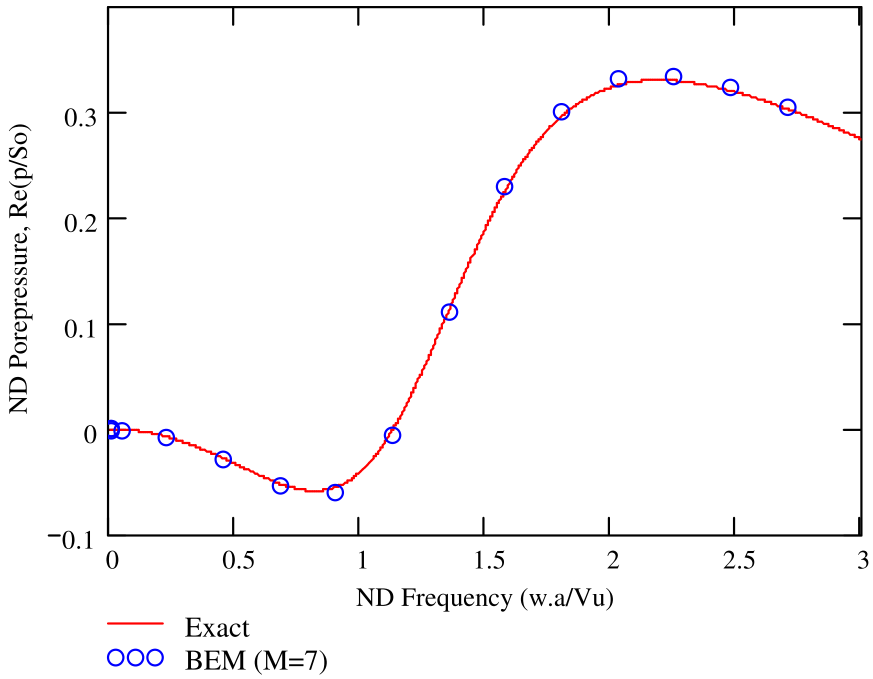

Figure 8.20 Spherical cavity: Real part of ND pore-pressure at R=1.5.

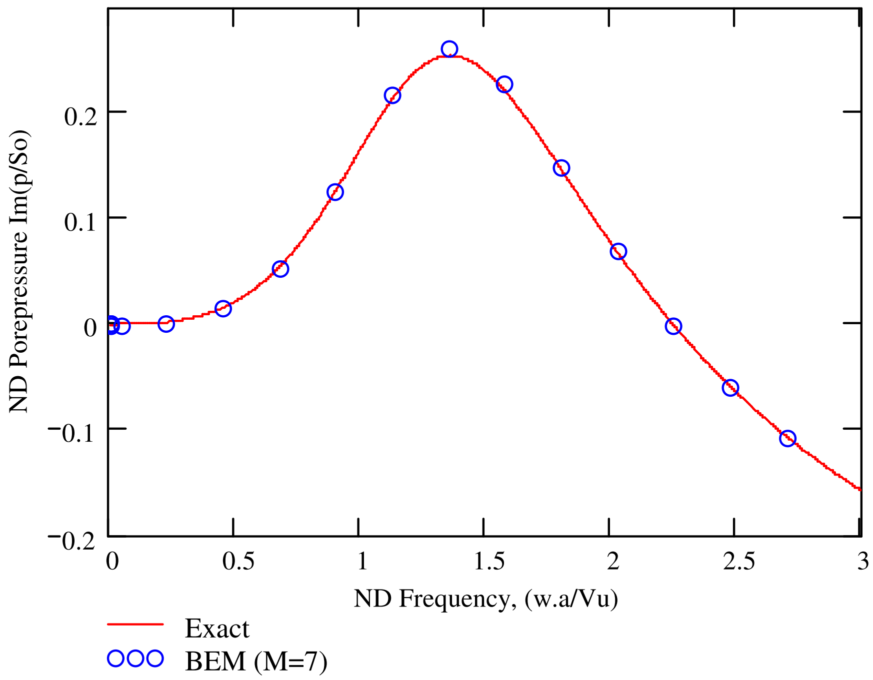

Figure 8.21 Spherical cavity: Imaginary part of ND pore-pressure at R=1.5.

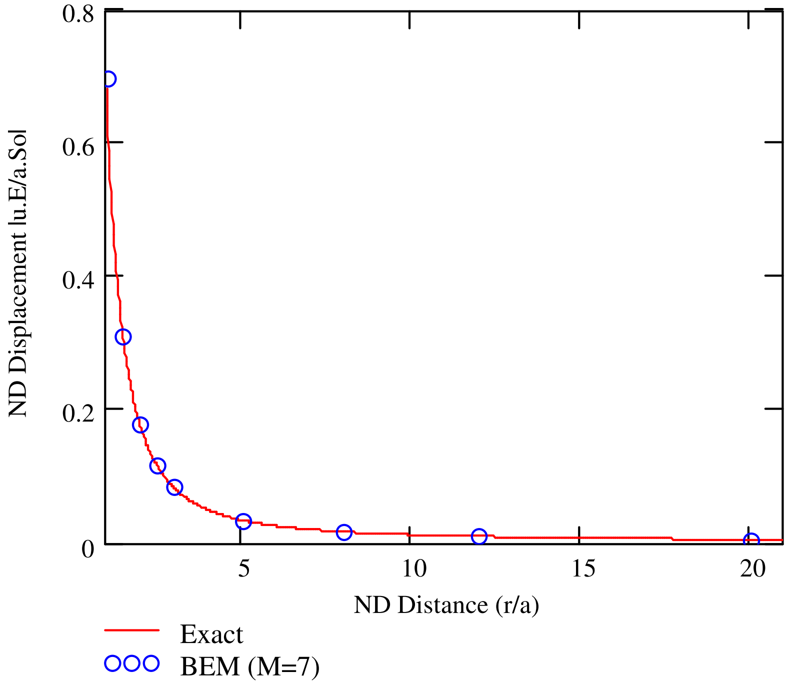

Figure 8.22 Spherical cavity: Radial distribution of ND displacement at $\overline{\varpi} = 0.225$, E=2λ+μ.

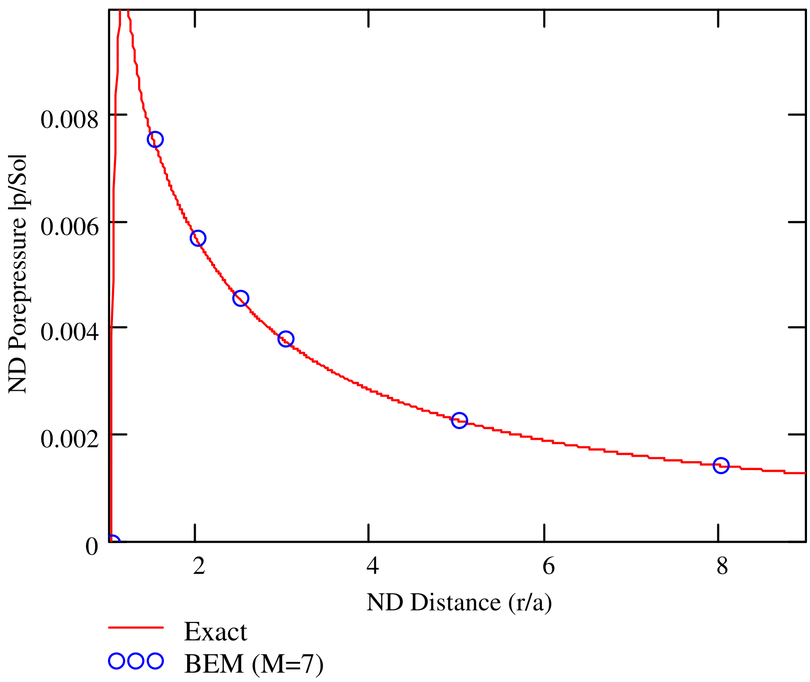

Figure 8.23 Spherical cavity: Radial distribution of ND pore-pressure at $\overline{\varpi} = 0.225$.

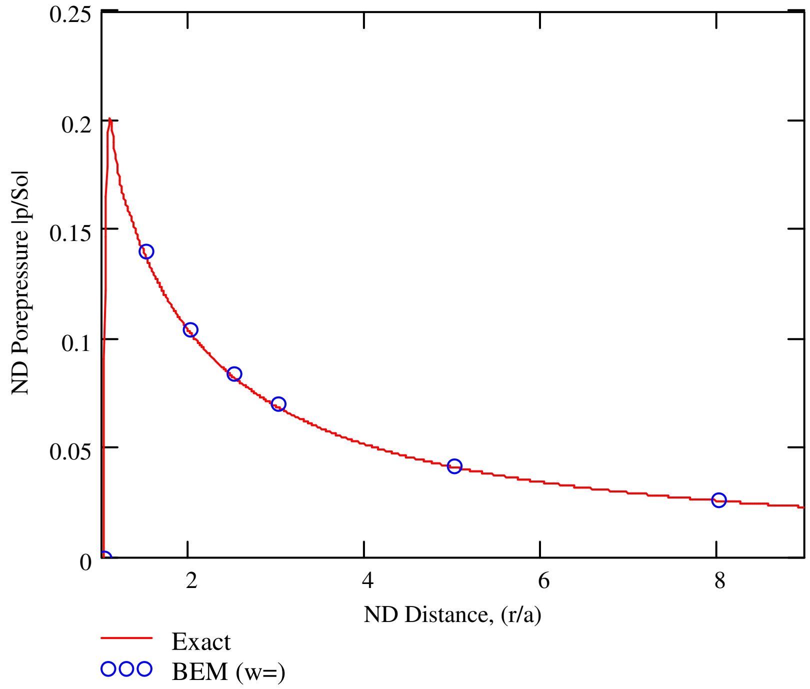

Figure 8.24 Spherical cavity: Radial distribution of ND pore-pressure at $\overline{\varpi} = 0.45$.

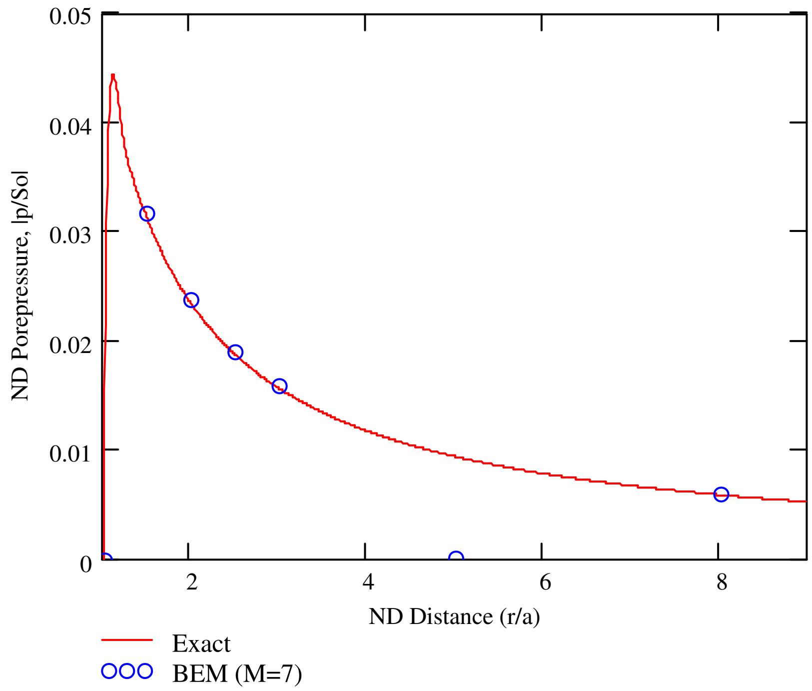

Figure 8.25 Spherical cavity: Radial distribution of ND pore-pressure at $\overline{\varpi} = 0.9$.

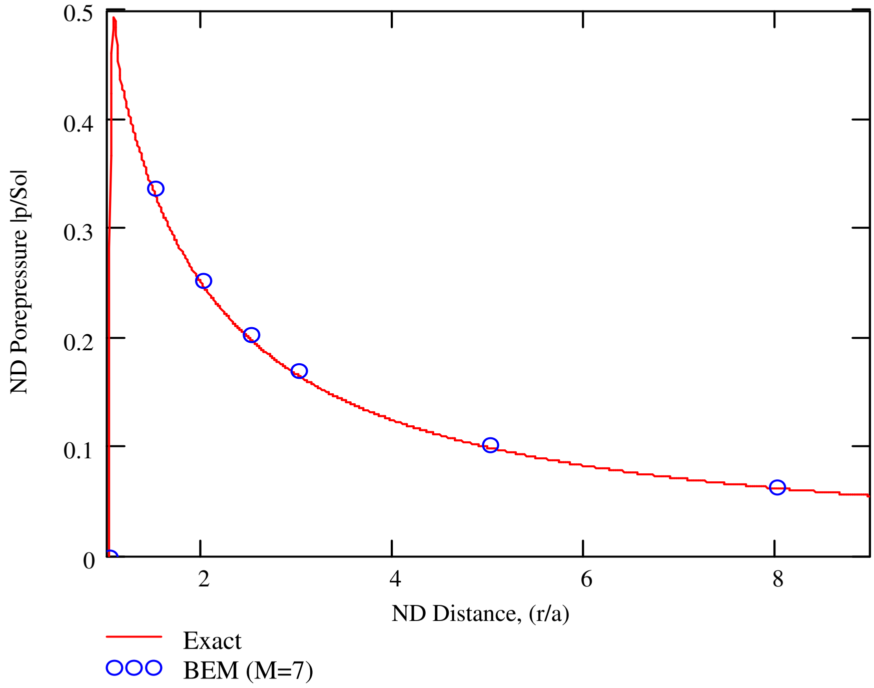

Figure 8.26 Spherical cavity: Radial distribution of ND pore-pressure at $\overline{\omega} = 1.8$.

8.4 Vertical Compliance for Rigid Circular Foundation on Elastic Half Space

Compliance (or impedance) functions are needed in soil-structure interaction analysis. Here, we show how program AxiPoro can be used to determine compliance (or impedance) relations for rigid axi-symmetric foundations; we consider a rigid circular disc resting on an elastic half space. This problem was considered earlier by Apsel and Luco (1987). The geometry and the BE mesh used are described in figures 8.27 and 8.28, the angular sub-divisions used in this analysis is N = 2^7 = 128. We solve the problem in non-dimensional (ND) space, appropriate ND variables and parameters

$$\overline{x}_i = \frac{x_i}{a};\ \overline{u}_i = \frac{u_i}{U_o};\ \overline{t}_i = \frac{t_i}{\mu \cdot U_0};\ \overline{q}_n = \frac{q_n \cdot a}{c_s \cdot U_0};\ \overline{p} = \frac{p \cdot a}{\mu \cdot U_0}\ \overline{\omega} = \frac{\omega \cdot a}{c_s};$$

$$\overline{n} = n;\ \overline{\kappa} = \frac{\mu \cdot \kappa}{a \cdot c_s};\ \overline{\rho} = 1;\ \overline{\rho}_f = \frac{\rho_f}{\rho};\ \overline{\mu} = 1;\ \overline{\nu} = \nu;\ \overline{Q} = \frac{Q}{\mu};\ \overline{F}_i = \frac{F}{\mu \cdot a \cdot U_0}$$

Where over-bar denotes a ND variable and F_i are resultant forces, $c_s = \sqrt{\frac{\mu}{\rho}}$ .is the shear wave velocity for ideal elastic material. The problem we consider is an ideal elastic problem, yet our formulation involves poro-elastic media; in order to simulate elastic behaviour, we set $\bar{\kappa} = 1$ and other poro-elastic material parameters are given negligibly small values, namely, $\bar{Q}, \bar{\rho}_f, \bar{\rho}_a, \alpha$ are set to 10^{-4}.

Impedance at a particular frequency is the force acting on the foundation for unit displacement; the compliance is the algebraic reciprocal of the impedance. When requested, the program AxiPoro computes the stress resultants for specified elements and prints at the end of the output file. In this case the impedance is the vertical stress resultant for elements 1 through 10. The results are plotted in Figures 8.29 and 8.30. There is a good agreement between the results of Apsel and Luco (1987) and the program AxiPoro, the maximum absolute difference being 7.5%; it should however be noted that, Apsel and Luco's results, based on a numerical method, were not exact either.

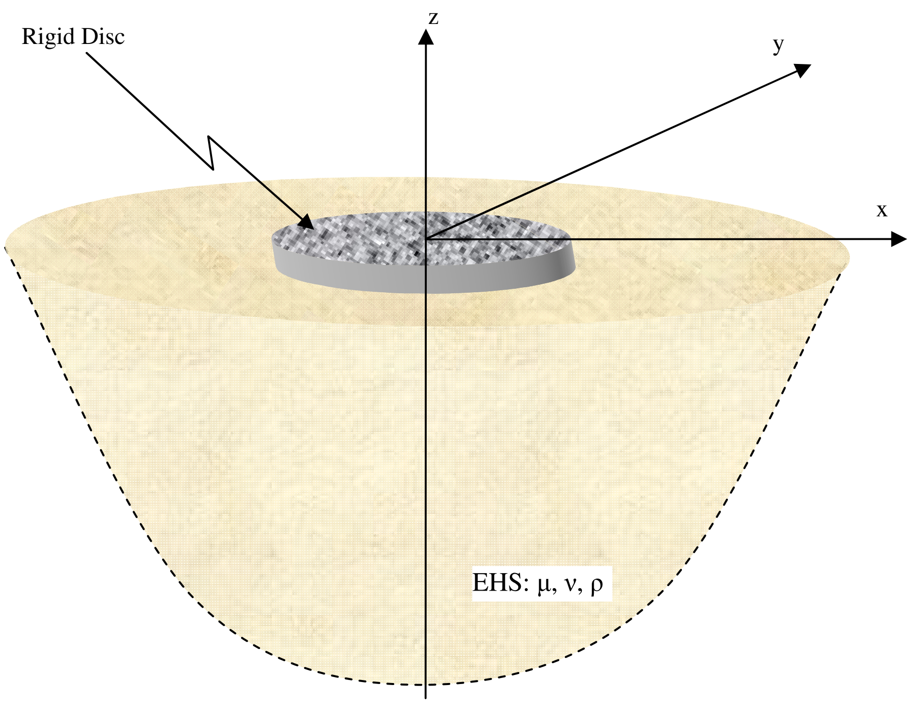

Figure 8.27 Rigid circular foundation resting on an elastic half space.

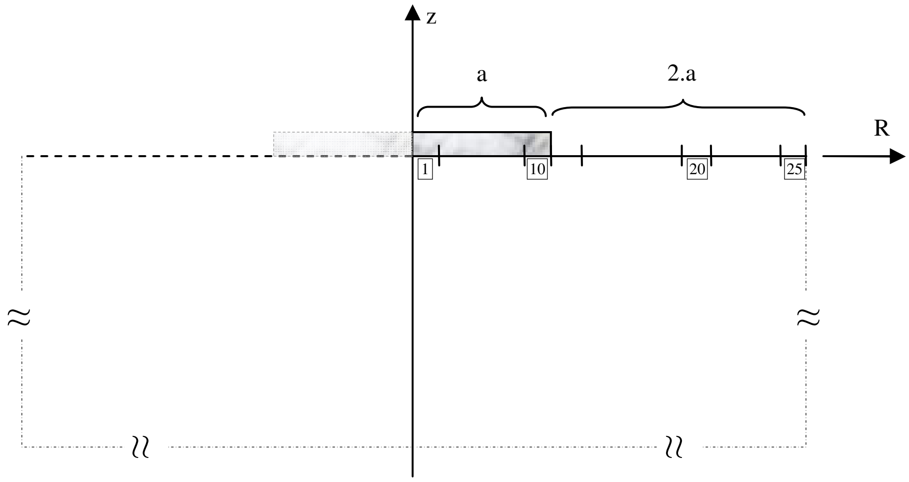

Figure 8.28 Rigid circular foundation on EHS, axi-symmetric BE mesh.

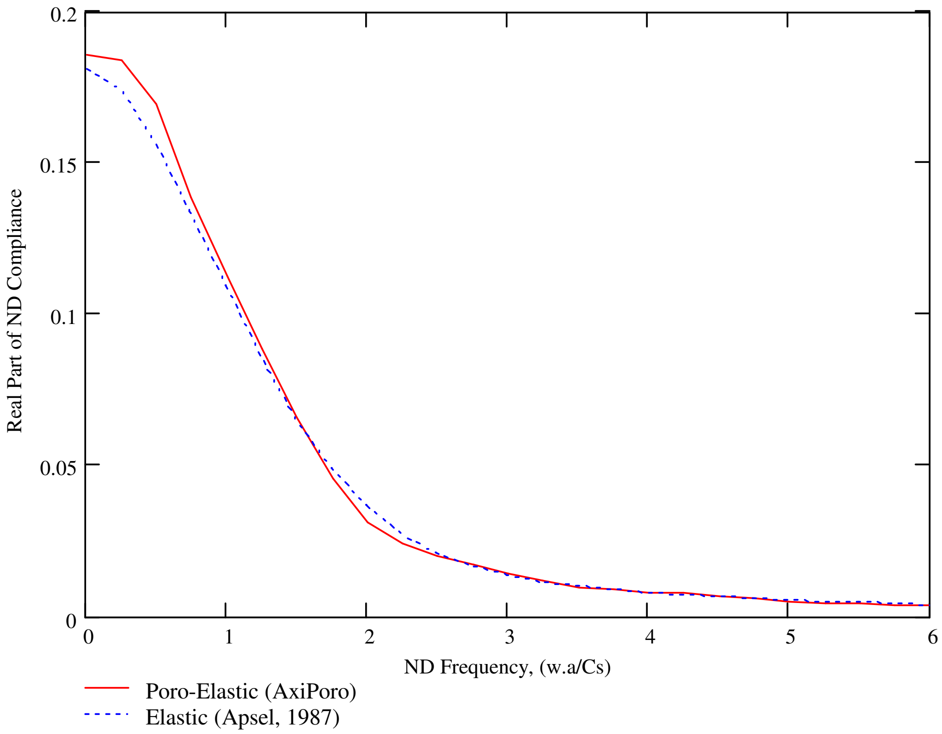

Figure 8.29 Rigid circular foundation on EHS, real part of ND vertical compliance.

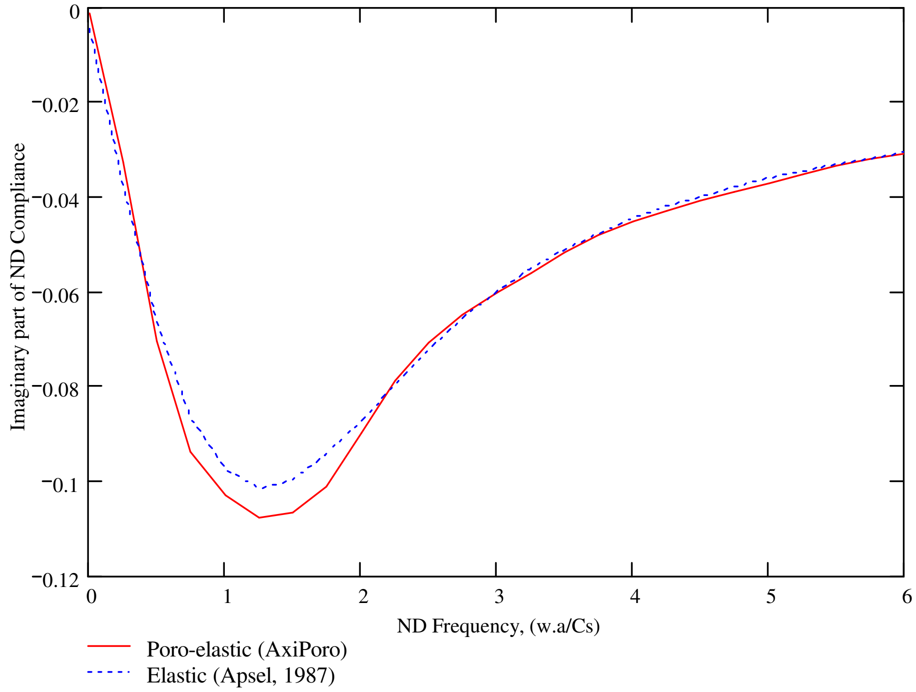

Figure 8.30 Rigid circular foundation on EHS, imaginary part of ND vertical compliance.

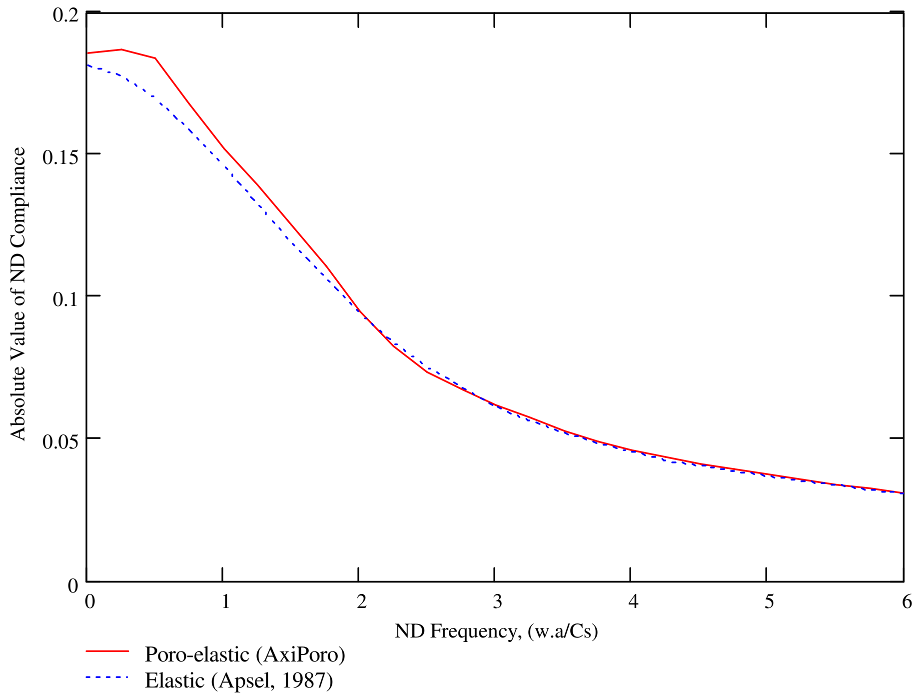

Figure 8.31 Rigid circular foundation on EHS, absolute value of ND vertical compliance.

CHAPTER 9

CONCLUSIONS

In this study, the axi-symmetric formulation proposed in Özkan, 1995 is extended for dynamic poro-elasticity. The formulation is explained in Chapter 6 and can be summarized as follows:

- expand the traction and displacements in complex Fourier series in circumferential direction, this reduces surface BI equations to line integral equations along the generator, for each frequency of the series expansion
- the kernels appearing in the reduced BIE are now in the form of Fourier integrals, which can be computed efficiently by FFT algorithm
- introduce boundary discretization along the generator and solve the reduced BE equations for each frequency of the complex Fourier series expansion, thereby determine the Fourier coefficients of the unknown boundary quantities. The integrations along the generator can be performed via Gaussian Quadrature; however a special treatment is necessary when the source point is on the integration element.
- combine the complex Fourier coefficients of the boundary variables through inverse FFT to compute the variation of unknown boundary quantities in circumferential direction.

The proposed formulation has features superior to the other methods described in the literature. To be precise,

1. The expansion of boundary variables in complex Fourier series, instead of real Fourier series, obviates the need for differentiating symmetric and anti-symmetric modes in the analysis. This further provides for easier coding in the case of arbitrary boundary conditions.
2. The integrations in circumferential direction are performed by FFT algorithm, thereby the reduced BE equations at all frequencies are obtained in one roll simultaneously. In other methods these integrals are computed by Gaussian Quadrature (Brebbia and Dominguez 1992) at each frequency separately. The use of FFT algorithm, therefore, increases the computational performance and accuracy remarkably. Still other formulations entail numerical evaluation of complete elliptic integrals (Guiggiani and Cassalini, 1986), which is both computationally involved and problematic when the modulus "k" approaches unity.
3. When the source point is on the integration element, the line integration at θ=0 pose a particular hyper-singular behaviour. This difficulty is circumvented by a special technique described in Chapter 6. In alternative formulations complete elliptic integrals has to be computed numerically, which may lead to numerical instabilities at high frequencies.
4. The convergence of the method for a given BE mesh is controlled by the number of subdivisions in circumferential direction as shown in section 7.6. Good accuracy is obtained for $N = 2^8$ angular sub-divisions while perfect convergence is observed at $N = 2^{12}$ sub-divisions.

The proposed formulation is coded in C language as a part of this work. The program computes unknown boundary quantities, force and moment resultants over specified elements, as well as the displacements and pore-pressure at prescribed interior points. The boundary conditions in circumferential direction are input at arbitrary angular positions; the program can interpolate to obtain variation of boundary values at appropriate angular deviations for FFT computations. In the current implementation only real boundary conditions are to be input, for complex boundary conditions analytical expressions can be entered in the code, otherwise a slight modification in the implementation is still possible to accommodate for input of complex boundary conditions.

In the example problems presented in Chapter 8, a good fit is observed between the solutions obtained by the proposed method and the analytical solutions, which shows the effectiveness of this formulation.

The accuracy is improved as the number of angular sub-divisions ($N = 2^{MM}$) and the number of terms in the complex Fourier series ($Np = 2^{MMp}$) are increased, sufficient accuracy is obtained for MM = 7 and MMp = 2 for uniform boundary conditions (axi-symmetric boundary conditions). For good results MM = 8 is recommended. The parameter MMp should be increased for non axi-symmetric boundary conditions, especially when there are jumps in the variation of boundary conditions in θ-direction. It should be noted that boundary element mesh refinement may also be needed to rectify the results.

Recommendations for Further Study:

The axi-symmetric bodies having two different materials cannot be handled with the program developed in this study. The program can be modified to handle multi-domain problems.

Similarly, the formulation can be extended for self weight and similar body force effects by exploiting the derivable from a potential nature of these effects, or otherwise dual-reciprocity approach is always within easy reach.

Stress and flux vector components can be computed after stress kernels of 3-D poro-elastodynamics are implemented in the computer program. These kernels are available in Badmus, 1990; however, to author's knowledge, the kernels have not been verified and should be used with caution.

The evaluation of singular integrals discussed in Chapter 6 can alternatively be performed using asymptotic expansion of kernels for θ tending to zero. This will be the subject of a further study.

REFERENCES:

1. Abramowitz, M. and Stegun, I. A.; 1964. Handbook of Mathematical Functions, Applied Mathematics Series, Vol. 55, National Bureau of Standards, Washington, USA.
2. Antes, H. and Wiebe, T. (1996). Analysis of waves in 3-d poroelastic media, in Mechanics of Poroelastic Media: A.P.S Selvadurai (Ed.), Kluwer Academic Publishers, Netherlands.
3. Apsel, R. J. and Luco, J. E. (1987). Impedance functions for foundations embedded in a layered medium: an integral equation approach, Earthquake Eng. Struct. Dyn., V.15, pp.213-231.
4. Aramaki, G. (1986). Boundary elements for thin layers with high permeability in Biot's consolidation analysis, Appl. Math. Modelling, V.10, pp.82-86.
5. Badmus, T. (1990). Boundary Integral Formulation for Quasi-Static and Dynamic Problems of Porous-Elasticity, Thesis presented to the University of Delaware, in partial fulfilment of the requirements for the Doctor of Philosophy, Newark.
6. Badmus, T.; Cheng, A. H-D.; Grilli, S. (1993). A Laplace-Transform based three-dimensional BEM for poroelasticity, Int. J. Num. Meth. In Engng., Vol. 36, pp. 67-85.
7. Banerjee, P. K. and Butterfield, R. (1981). Boundary Element Methods in Engineering Science, McGraw Hill Book Comp. Ltd., USA.
8. Becker, A. A. (1992). The Boundary Element Method in Engineering, McGraw Hill, London.
9. Berryman, J. G., (1980). Confirmation of Biot's theory, Appl. Phys. Lett., V.37, pp.382-384.

10. Berrymann, J. G. (1992). Effective stress for transport properties of inhomogeneous porous rock, J. Geophys. Res., V.97, pp.17409-17424.
11. Berrymann, J. G. (1993). Effective-stress rules for pore-fluid transport in rocks containing two minerals, Int. J. Rock Mech. Min. Sci. Geomech. Abstr. 30, 1165-1168.
12. Biot, M. A. (1941.a). General theory of three dimensional consolidations, J Appl. Phys., V.12, pp. 155-164.
13. Biot, M. A. (1941.b). Consolidation settlement under a rectangular load distribution, J Appl. Phys., V.12, pp. 426-430.
14. Biot, M. A. and Flingan, C. M. (1941.c). Consolidation settlement of a soil with impervious top surface, J Appl. Phys., V.12, pp. 578-581.
15. Biot, M. A. Flingan, C. M. (1942). Bending settlement of a slab resting on a consolidating foundation, J Appl. Phys., V.13, pp. 35-40.
16. Biot, M. A. (1955). Theory of elasticity and consolidation for a porous anisotropic solid, J Appl. Phys., V.26, pp. 183-185.
17. Biot, M. A. (1956.a). General solutions of the equations of elasticity and consolidation for a porous material, J Appl. Phys., V.23 (1), pp. 91-96.
18. Biot, M. A. (1956.b). The theory of propagation of elastic waves in a fluid saturated porous solid, I. Low frequency range, J Acoust. Soc. Am., V.28, pp. 168-178.
19. Biot, M. A. (1956.c). The theory of propagation of elastic waves in a fluid saturated porous solid, II. high frequency range, J Acoust. Soc. Am., V.28, pp. 179-191.
20. Biot, M. A. (1956.d). Theory of deformation of a porous viscoelastic anisotropic solid, J Appl. Phys., V.27, pp. 459-467.
21. Biot, M. A. and Willis, (1957). The elastic coefficients of the theory of consolidation, J Appl. Mech., V.14, pp. 954-962.
22. Biot, M. A. (1962.a). Mechanics of deformation and acoustic propagation in porous media, J Appl. Phys., V.33, pp. 1482-1498.
23. Biot, M. A. (1962.b). Generalized theory of acoustic propagation in porous dissipative media, J Acoust. Soc. Am., V.34, pp. 1256-1264.

24. Bonnet, G. and Auriault, J.-L. (1985). Dynamics of saturated and deformable porous media: Homogenization theory and determination of the solid-liquid coupling coefficients. In: Physics of Finely Divided Matter, N Boccara and M. Daoud Eds., Springer Verlag, Berlin, Germany, pp. 306-316.
25. Bonnet, G. (1987). Basic singular solutions for a poroelastic medium in the dynamic range, J. Acoust Soc. Am., 82(5),1758-1762.
26. Boutin, C., Bonnet, G., and Bard, P. Y. (1987). Green functions and associated sources in infinite and stratified poroelastic media, Geophys. J. Roy. Astr. Soc., V.90, pp.521-550.
27. Bowen, R.M. (1976). "Theory of Mixtures," Continuum Physics, Ed. Eringen, A.C., 3, 1-127.
28. Bowen, R.M. (1980). Incompressible porous media models by use of the theory of mixtures, Int. J. Eng. Sci., V.18, No.9, pp.1129-1148.
29. Bowen, R.M. (1982). Compressible porous media models by use of the theory of mixtures, Int. J. Eng. Sci., V.20, No.6, pp.697-735.
30. Brebbia, C. A. and Dominguez, J. (1992). Boundary Elements An Introductory Course, Second Edition, Computational Mechanics Publications, UK.
31. Brigham, E. O. (1988), The Fast Fourier Transform, Prentice Hall Inc., Englewood Cliffs, New Jersey, USA.
32. Brutsaert, W. (1964). The propagation of elastic waves in unconsolidated unsaturated granular mediums, J. Geophys. Res., V.69, No.2, pp.243-257.
33. Burridge, R. and Vargas, C. A. (1979). The fundamental solution in dynamic poroelasticity, Geophys. J. Roy. Astr. Soc., V.58, pp.61-90.
34. Burridge, R. and Keller, J. B. (1981). Poroelasticity equations derived from microstructure, J. Acoust. Soc. Am., V.70, No.4, pp.1140-1146.
35. Cavalcanti, M. C. and Telles, J. C. F. (2003). Biot's consolidation theory—application of BEM with time independent fundamental solutions for poro-elastic saturated media, Engineering Analysis with Boundary Elements, V.27, No.2, pp. 145-157.

36. Chen, J. (1992). Fundamental solutions and boundary element formulations for dynamic problems of poro-elasticity and thermo-elasticity, PhD. Dissertation, State University of New York at Buffalo.
37. Chen, J. (1994.a). Time domain fundamental solutions to Biot's complete equations of dynamic poroelasticity, part I: two-dimensional solution, Int. J. Solids Structures, V.31, No.10, pp.1447-1490.
38. Chen, J. (1994.b). Time domain fundamental solutions to Biot's complete equations of dynamic poroelasticity, part II: three-dimensional solution, Int. J. Solids Structures, V.31, No.2, pp.169-202.
39. Cheng, A. H-D. and Ligget, J. A. (1984.a). Boundary integral equation method for linear porous-elasticity with applications to soil consolidation, Int. J. Num. Meth. In Engng., Vol. 20, pp. 255-278.
40. Cheng, A. H-D. and Ligget, J. A. (1984.b). Boundary integral equation method for linear porous-elasticity with applications to fracture propagation, Int. J. Num. Meth. In Engng., Vol. 20, pp. 279-296.
41. Cheng, A. H. D. and Detournay, E. (1988). A direct boundary element method for plane strain poroelasticity, Int. J. Numer. Anal. Meth. Geomech., V.12, pp.551-572.
42. Cheng, A. H-D.; Badmus, T.; Beskos, D. E. (1991). Integral equation for dynamic poroelasticity in frequency domain with BEM solution, J. Eng. Mech., ASCE, Vol. 117, No. 5, pp. 1136-1157.
43. Cheng, A. H. D. and Detournay, E. (1998). On singular integral equations and fundamental solutions of poroelasticity, In. J. Solids Structures, V.35, pp. 4521-4555.
44. Chopra, M. (2001). Poro-elasticity using the BEM, Chapter 7, in Coupled Field Problems (Advances in Boundary Elements), Editors: A. J. Kassab, M. H. Aliabadi, Computational Mechanics, UK.
45. Cleary, M. P. (1977). Fundamental solutions for fluid saturated porous solid, Int. J. Solids Structures, V.13, pp.785-806.
46. Coussy, O. (1995). Mechanics of Porous Continua, John Wiley and Sons, West Sussex, UK.

47. Cowin, S. C., (1999). Bone poroelasticity, J. Biomech., V.32, pp.217-238.
48. Darcy, H. (1856). Les fontaines publique de la ville de Dijon, Paris: Victor Dalmont, Editeur, Libraire Des Corps Impériaux Des Ponts Et Chaussées Et Des Mines, http://biosystems.okstate.edu/darcy/French/index.htm, Last Accessed : 15 June 2006.
49. Dargush, G. F. (1987). Boundary Element Methods for The Analogous Problems of Thermo-mechanics and Soil Consolidation, Ph. D. Dissertation, State University of New York at Buffalo, New York, USA.
50. Dargush G. F. and Banerjee, P. K. (1989). A time domain BEM for poroelasticity, Int. J. Num. Meth. Eng., V.28, pp. 2423- 2449.
51. Dargush G. F. and Banerjee, P. K. (1991). A boundary element method for axisymmetric soil consolidation, Int. J. Solids Structures, V.28, pp. 897-915.
52. Dargush, G. F. and Chopra, M. B. (1996). Dynamic analysis of axisymmetric foundations on poroelastic media, J. Eng. Mech., ASCE, V.122, No.7, pp. 623-632.
53. de Boer, R. and Ehlers, W. (1990). The development of the concept of effective stresses, Acta Mechanica, V.83, pp.77-92.
54. de Boer, R. (1996). Highlights in the historical development of the porous media theory, Appl. Mech. Rev., V.49, pp.201-262.
55. de Boer, R. (2000). Theory of Porous Media: Highlights in Historical Development and Current State, Springer-Verlag, Berlin, Germany.
56. de Boer, R. (2005). Trends in Continuum Mechanics of Porous Media (Theory and Applications of Transport in Porous Media), Springer, Dordrecht, the Netherlands.
57. Detournay, E. and Cheng, A. H.-D., (1993). Fundamentals of Poroelasticity, Chapter 5 in Comprehensive Rock Engineering: Principles, Practice and Projects, Vol II, Analysis and Design Method, ed. C. Fairhurst, Pergamon Press, pp.113-171.
58. Dominguez, J. (1991). An integral formulation for dynamic poroelasticity, J. Appl. Mech, ASME, V. 58, pp. 588-591.

59. Dominguez, J. (1992). Boundary element approach for dynamic poroelastic problems, Int. J. Num. Meth. Eng., V. 35, pp. 307-324.
60. Dullien, F. A. L. (1992). Porous Media, 2nd Edition, Academic Press, USA.
61. Fatt, I. (1959). The Biot-Willis elastic coefficients for sandstone, J. App. Mech., V.26, pp.296-297.
62. Frenkel, J., (1944). "On the theory of seismic and seismoelectric phenomena in moist soil," J. Phys., V.8, pp.230-241.
63. Gatmiri, B. and Kamalian, M. (2002). On the fundamental solution of dynamic poroelastic boundary integral equations in the time domain, Int. J. Geomech., V.2, No.4, pp.381-398.
64. Guiggiani, M. (1992). Computing principal value integrals in 3-D BEM for time-harmonic elastodynamics-a direct approach, Comm. Appl. Numer. Meth., V.8, pp.141-149.
65. Guiggiani, M. and Cassalini, P., (1986). On the numerical implementation of BEM for axisymmetric elasticity, in Boundary Elements VIII Conference, Editors: M. Tanaka and C. A. Brebbia, Springer, pp. 695-706.
66. Gurevich, B., Kelder, O., Smeulders, D. M. J., (1999). Validation of the Slow compressional wave in porous media: comparison of experiments and numerical simulations, Transport in Porous Media, V.36, pp.149-160.
67. Juhl, P. (1993). An axisymmetric integral equation formulation for free space non-axisymmetric radiation and scattering of a known incident wave, J. Sound and Vibration, V. 163, pp. 397-406.
68. Kaynia, A. M. (1992). Transient Green's functions of fluid-saturated porous media, Computers and Structures, V.44, pp.19-27.
69. Kahaner, D. K., Moler, C. and Nash, S.G. (1989). Numerical Methods and Software, Prentice-Hall, Englewood Cliffs, NJ.
70. Klimentos, T. and McCann, C., (1988). Why is the Biot's slow compressional wave not observed in real rocks?, Geophysics, V.53, No.12, pp.1605-1609.

71. Kosten, C. W. and Zwikker, C. (1941). Extended theory of the absorption of sound by compressible wall-coverings, Physica, V.8, No.9, pp.968-978.
72. Kupradze, V. D. (1979). Three Dimensional Problems of Mathematical Theory of Elasticity and Thermo-elasticity, North-Holland.
73. Kythe, P. An Introduction to Boundary Element Method, CRC Press, April 13 1995, New York.
74. Lamb, T. W. and Withman, R. V. (1979). Soil Mechanics (SI Version), John Wiley & Sons, Singapore.
75. Lim, T. and Hong, J. H., (2000). Poroelastic properties of bovine vertebral trabecular bone, J. Orthopaedic Res., V.18, pp.671-677.
76. Manolis, G. D. and Beskos, D. E. (1989). Integral formulation and fundamental solutions of dynamic poroelasticity and thermoelasticity, Acta Mechanica, V.76, No.1-2, pp.89-104.
77. Mengi, Y. and McNiven, H. D. (1977). Response of fluid-filled porous media to transient input, J. Acoust. Soc. Am., V.61, No.1, pp.84-94.
78. Norris, A. N. (1985). Radiation from a point source and scattering theory in a fluid-saturated porous solid, J. Acoust. Soc. Am, V.77, No.5, pp.2012-2023.
79. Nowacki, W. (1975). Dynamic Problems of Thermo-elasticity, Noordhoff International Publishing, Leyden, The Netherlands.
80. Ogushwitz, P. R. (1985). Applicability of the Biot theory. I. Low porosity materials, J. Acoust. Soc. Am., V.77, pp. 429-440.
81. Ortner, Norbert (1989) ; Methods of constructing fundamental solutions of decomposable linear differential operators (Invited Contribution), in Boundary Elements IX, Vol.1 (Mathematical and Computational Aspects), Eds. Brebbia, C. A. et. al, Computational Mechanics Publications, Southampton Boston, Springer Verlag.
82. Özkan, Gonca (1995). A Boundary Element Method for Axisymmetric Elastodynamic Analysis, Ph. D. Thesis, Middle East Technical University, Dept. of Eng. Sci., Ankara, Turkey.

83. Plona, T. J. (1980). Observation of a second bulk compressional wave in a porous medium at ultrasonic frequencies, Appl. Phys. Lett., V.36, pp. 259-261.
84. Pozrikidis, C (2002), A Practical Guide to Boundary Element Methods with the Software Library BEMLIB, CRC Press, May 15 2002, New York.
85. Predeleanu, M. (1981). Boundary integral method for porous media. In: Boundary Element Methods (Brebbia, C. A., ed.), pp. 325-334, Springer, Berlin.
86. Press, W. H., Teukolsky, S. A., Vetterling, W. T., Flannery, B. P. (1992). Numerical Recipes in FORTRAN, 2nd Edition, Cambridge University Press, Cambridge.
87. Prevost, J. H. (1980). Mechanics of continuous porous media, Int. J. Eng. Sci., V.18, No.5, pp.787-800.
88. Rashed, Youssef R. (2002), Boundary Element Premier, Tutorial 4: Fundamental solutions I, simple and compound operators, Boundary Element Communications, V.13, No.1, pp. 38-46.
89. Schanz, M. (2001.a). Application of 3D time domain boundary element formulation to wave propagation in poroelastic solids, Engineering Analysis with Boundary Elements, V.25, pp. 365-376.
90. Schanz, M. (2001.b). Wave Propagation in Viscoelastic and Poroelastic Continua: A BEM Approach, Lecture Notes in Applied Mechanics V.2, Springer Verlag, April 2001.
91. Schanz, M. and Diebels, S. (2003). A comparative study of Biot's theory and the linear Theory of Porous Media for wave propagation problems, Acta Mechanica, V.161, No.3-4, pp.213-235.
92. Schanz, M. and Pryl, D. (2004). Dynamic fundamental solutions for compressible and incompressible modeled poroelastic continua, Int. J. Solids and Struct., V.41, pp. 4047-4073.
93. Schanz, M. and Struckmeier, V. (2005). Wave propagation in a simplified modeled poroelastic continuum: Fundamental solutions and time domain

boundary element formulation, Int. J. Numer. Meth. Engng., V.64, pp.1816-1839.

94. Senjuntichai, T. and Rajapakse, R. K. N. D., (1993). Transient response of a circular cavity in a poroelastic medium, Int. J. Num. Analyt. Meth. Geomech., V.17, pp.357-383.

95. Skempton, A. W. (1954). The pore pressure coefficients A and B, Geotechnique, V.4, pp.143-147.

96. Soenarko, B. (1993). A boundary element formulation for radiation of acoustic waves from axisymmetric bodies with arbitrary boundary conditions, J. Acoust. Soc. Am., V.93, pp.631-639.

97. Steel, T. R. (1967). Applications of a theory of interacting continua, Quart. J. Mech. and Appl. Math., V.20, pp.57-72.

98. Suklje, L. (1969). Rheological Aspects of Soil Mechanics, Wiley-Interscience, New York.

99. Telles, J. C. F. (1987). A self-adaptive co-ordinate transformation for efficient numerical evaluation of general boundary element integrals, Int. J. Numer. Meth. Engng., V.24, No.5, pp.959-973.

100. Tsepoura, K. G. and Polyzos, D. (2003). Static and harmonic BEM solutions of gradient elasticity problems with axisymmetry, Computational Mechanics, V.32, pp.89-103.

101. Vargas, C. A. (1975). Dynamic Poroelasticity, Ph. D. Dissertation, submitted to The Faculty of Graduate School of Arts and Science in Partial Fulfillment of the requirements for the Degree of Doctor of Philosophy at New York University, New York, USA.

102. Wang, H. C. and Banerjee, P. K. (1990). Generalised axi-symmetric elastodynamic analysis by boundary element methods, Int. J. Num. Meth. Eng., V.30, pp. 115-131.

103. Wang, H. F., (2000). Theory of Linear Poroelasticity with Applications to Geomechanics and Hydrogeology, Princeton University Press, Princeton, New Jersey, USA.

104. Wiebe, Th. and Antes, H. (1991). A time domain integral formulation of dynamic poroelasticity, Acta Mechanica, 90, pp. 125-137.

105. Yew, C. H. and Jogi, P. N. (1976), Study of wave motions in fluid-saturated porous rocks, J. Acoust. Soc. Am., V.60, No., pp.2-8.

106. Yew, C. H. and Jogi, P. N. (1978). The determination of Biot parameters for sandstones part 1: static tests, Experimental Mechanics, 18, pp.167-172.

107. Zienkiewicz, O. C., Chang, C. T., and Bettes, P. (1980). Drained, undrained, consolidating and dynamic behaviour assumptions in soils, Geotechnique, V.30, No.4, pp. 385-395.

108. Zienkiewicz, O. C., Chan, A.H.C., Pastor, M., Schrefler, B.A., Shiomi, T. (1999). Computational Geomechanics with Special Reference to Earthquake Engineering, John Wiley and Sons, UK.

VITA

Mehmet Özyazıcıoğlu was graduated from Department of Civil Engineering of METU, Turkey in 1992. He received M.S. degree from the same department in 1999. His maintains an interest in analytic and asymptotic methods in wave mechanics, fundamental solutions and green functions for partial/ordinary differential equations and treatment of singular integrals in Boundary Element Method.