PUBLIC DEBT MANAGEMENT IN TURKEY WITH STOCHASTIC OPTIMIZATION APPROACH

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PUBLIC DEBT MANAGEMENT IN TURKEY WITH STOCHASTIC OPTIMIZATION APPROACH

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Approval of the Institute of Applied Mathematics

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This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

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Abstract

PUBLIC DEBT MANAGEMENT IN TURKEY WITH STOCHASTIC OPTIMIZATION APPROACH

Çelebi, Nuray

M.Sc., Department of Financial Mathematics

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The Prime Ministry of Undersecretariat of Treasury which maintains the financial administration of Republic of Turkey has several tasks to handle, one of which is to manage the government's debt in a manner that minimizes the cost regarding risk. The debt management problem is choosing the right instrument and maturity composition that has the least cost and risk which is affected by many stochastic factors.

The objective of this thesis is the optimization of the debt management problem of the Turkish Government via a stochastic simulation framework under the constraints of changes in portfolio positions. Value-at-Risk of the optimal portfolio is calculated to measure market risk.

Macroeconomic variables in the optimization problem are modeled with econometric models like autoregressive processes (AR), autoregressive integrated moving average processes (ARIMA) and generalized autoregressive conditionally heteroscedastic (GARCH) processes. The simulation horizon is 2005-2015. Debt portfolio is optimized at 2005 and 2015 where the representative scenarios for the optimization are obtained by clustering the previously generated 25,000 scenarios into 30 groups at each stage.

Keywords: Clustering, Scenario Generation, Stochastic Programming, Value-at-Risk.

Öz

RASSAL OPTİMİZASYON YAKLAŞIMIYLA TÜRKİYE'DE KAMU BORÇ YÖNETİMİ

Çelebi, Nuray

Yüksek Lisans, Finansal Matematik Bölümü

Tez Yöneticisi: Yard. Doç. Esma GAYGISIZ

Aralık 2005, 92 sayfa

Riski göz önünde bulundurarak devletin borcunun maliyetini en küçüklemek Türkiye Cumhuriyeti'nin finansal yönetimini sürdürmekle yükümlü olan Hazine Müsteşarlığı'nın görevlerinden birisidir. En ucuz maliyet ve en düşük riske sahip uygun araçların bulunması problemi olan borç yönetimi rassal etkenlerden etkilenmektedir.

Bu tezin amacı, Türkiye'nin borç yönetim problemini borç portföyü araçlarının pozisyonlarındaki değişim kısıtları altında rassal benzetim yöntemi çerçevesinde en iyi şekilde çözmesini sağlamaktır. Piyasa riskini ölçmek amacıyla en iyi çözüme karşılık gelen Riske Maruz Değer hesaplanmıştır.

En iyileme problemindeki makroekonomik değişkenler autoregressive (AR) süreçleri, autoregressive integrated moving average (ARIMA) süreçleri, generalized autoregressive conditionally heteroscedastic (GARCH) süreçleri gibi ekonometrik modeller kullanılarak modellenmiştir. Benzetim dönemi 2005-2015 olarak alınmıştır. Borç portföyü daha önce üretilen 25,000 senaryonun her evrede 30 gruba ayrılmasıyla belirlenen temsilci senaryolar kullanılarak 2005 ve 2015 yıllarında en iyilenmiştir.

Anahtar Kelimeler: Clustering, Rassal Programlama, Riske Maruz Değer, Senaryo Üretimi.

To my family,

ACKNOWLEDGMENTS

I would like to express my gratitude to my Thesis Committee especially to Assist. Prof. Esma Gaygısız for her support and encouragement, Prof. Dr. Gerhard-Wilhelm Weber for his friendship and suggestions and Dr. C. Coşkun Küçüközmen for his inspirations and motivations throughout this study.

I would also like to thank Uygar Pekerten for being by my side all this way.

Finally and most gratefully, I would like to thank my family for their endless support.

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Chapter 1

INTRODUCTION

Allocating debt in a fashion that minimizes risk or cost while building an acceptable maturity profile and currency composition is a complex problem. Public debt management is one of the problems in the heart of asset allocation and strategic risk management.

In recent years, there is an increasing tendency in many government agencies, that is to combine academic knowledge with practice on public debt management problem. Countries like New Zealand and Sweden set up debt management offices to identify the uncertainty inherent in the macroeconomic variables that play a crucial role in debt management problem of the country. The variables are modeled and uncertainties in the variables are incorporated into a mathematical programming framework which is known as *stochastic programming*. Stochastic programming is an applicable tool for asset liability management problems since it does not only model the uncertainties inherent in the variables, but also takes into account the risk and cost constraints and rebalancing strategies both from regulatory and corporate side.

In order to be used in calculations, the uncertainties in decision making problems should be represented in a form suitable for quantitative models. The uncertainties have to be approximated by an appropriate number of discrete outcomes. The outcomes should at the same time satisfy predetermined statistical properties. The method to obtain these discrete outcomes is referred to as *scenario generation*. New scenarios are branched from old to create a multistage scenario tree. [1]

The main objective of the debt management problem is to minimize cost or risk taking into account the debt management objectives of the government. From the academicians' point of view, portfolio variance was considered to be the main indicator of measuring risk. However, it is not an easily applicable measure of risk for practitioners because of the fact that variance-based risk management is based on the computation of the variance of the portfolios which may consist of huge numbers of instruments in different currencies, maturities and risk profiles. Thus, more easily applied risk measures like *Value-at-Risk* (VaR) or *Conditional Value-at-Risk* (CVaR) are preferred as means of risk management.

Prime Ministry Undersecreteriat of Treasury is the main resource of debt and its management in Turkey. As stated in the Republic of Turkey Prime Ministry Undersecreteriat of Treasury's Public Debt Management Report in 2005, domestic debt portfolio of Turkey is governed not only by one benchmark rule but by a set of debt management objectives and principals. These objectives are in accordance with the risk and cost objectives of the government.

The basic principles of debt and risk management in Turkey are as follows:[2]

1. Considering the macro-economic balances and monetary policies, a sustainable, transparent and accountable borrowing policy should be maintained and

2. Considering the level of risk in the global market conditions and the cost factors, the cost must be minimized in the medium and long-term maturities.

In this study, we approach the public debt management problem from practitioners' point of view while integrating the most recent academic work in it. We minimize the cost of government debt with due regard to change in portfolio positions and risk constraints. Debt portfolio of Treasury is too big to measure the inherent risk by the variance of the portfolio. Thus, a VaR or CVaR risk

[1] Gülpınar et al., 2003

[2] Republic of Turkey Prime Ministry Undersecreteriat of Treasury Public Debt Management Report February 2005

management approach would be more applicable. Although VaR lacks subadditivity [3] and is difficult to implement in the optimization problem, we use it as the risk measure of the government debt portfolio.

The objective of this thesis is the optimization of the debt management problem of the Turkish Government via stochastic programming approach and using a simulation model to generate scenarios for a period of ten years under constraints of changes in portfolio positions and Value-at-Risk. The study proceeds as follows: A brief overview of Turkish public debt management problem and econometric modeling of the macroeconomic variables in the optimization problem are given in Chapter 2. Chapter 3 refers to scenario generation procedure. Clustering and its algorithm are described in Chapter 4 whereas Chapter 5 explains the financial optimization problem using a stochastic programming approach. Chapter 6 introduces the results of the optimization problem and Chapter 7 concludes the study while giving a general idea about the future work on this study.

[3] VaR of a portfolio consisting of two portfolios can be greater than the sum of the VaRs of the individual portfolios.

Chapter 2

GOVERNMENT DEBT STRUCTURE IN TURKEY

In this chapter, first it will be given a brief overview of the public debt management and its components in Turkey, especially for the period between 2003 and 2005. The underlining information, which is received from public debt management reports of Treasury between 2003 and February 2005, is important in the sense that it will help us understand the borrowing behavior of Treasury that is important in trying to visualize the borrowing strategies in future. Then, the risk management objective of the Treasury will be explained.

2.1 DEBT ANALYSIS IN TURKEY

The Prime Ministry of Undersecretariat of Treasury which maintains the financial administration of Republic of Turkey has several tasks to handle, one of which is to manage the government's debt in a way that minimizes the cost regarding risk. The debt management problem is to choose the right instrument and maturity composition that has the least cost and risk which is affected by many stochastic factors. In our study, we try to model and optimize the borrowing behavior of Treasury. But, lack of data on external borrowing instruments like FX-denominated bonds force us to use a borrowing strategy based on domestic borrowing instruments, like T-Bills and G-Bonds. CPI-indexed bonds are

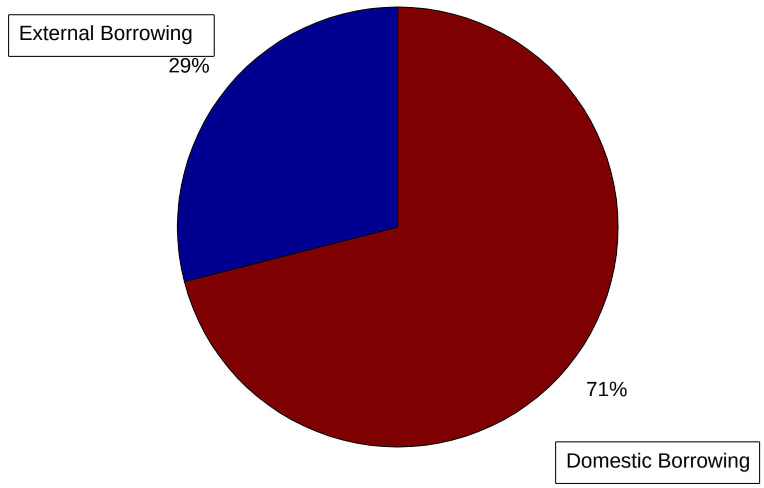

Figure 2.1: Borrowing Ratios in 2004

not included in our optimization problem as well because of the few number of CPI-indexed bonds issued in Turkey for the last twenty years.

Borrowing, which is not determined arbitrarily, is done in foreign and domestic currencies while domestic borrowing ratio is tried to be kept higher than external borrowing. It is based on rules determined by the Treasury. As stated in the Republic of Turkey Prime Ministry Undersecreteriat of Treasury's Public Debt Management Report in February 2005, domestic debt portfolio of Turkey is governed not only by one benchmark rule but by a set of debt management objectives and principals. These objectives are in accordance with the risk and cost objectives of the government.

The basic principles of debt and risk management in Turkey are as follows:[1]

1. Considering the macro-economic balances and monetary policies, a sustainable, transparent and accountable borrowing policy should be maintained

[1]Republic of Turkey Prime Ministry Undersecreteriat of Treasury Public Debt Management Report February 2005

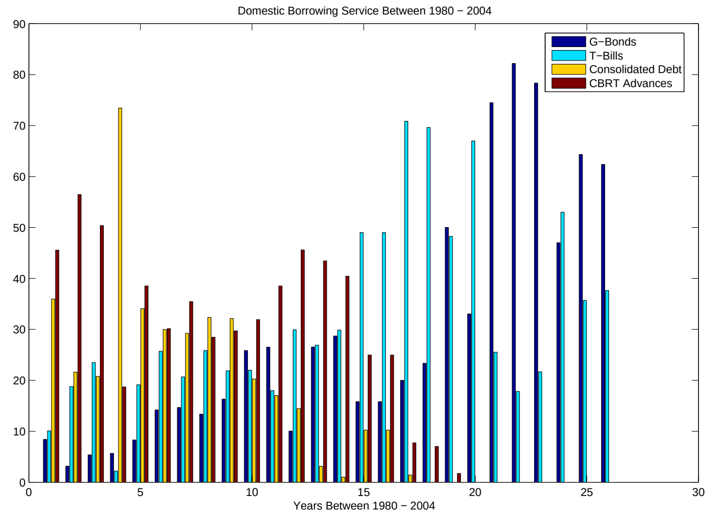

Figure 2.2: Domestic Borrowing Service Between 1980 - 2004

and

2. Considering the level of risk in the global market conditions and the cost factors, the cost must be minimized in the medium and long-term maturities.

The issuance methods of domestic borrowing through the market that the Turkish Government uses are:

1. Zero-coupon YTL auctions
2. FX-denominated auctions
3. Floating-rate auctions
4. Fixed-coupon TL auctions
5. Direct sales and taps

Although Treasury issues fixed or floating rate, domestic or FX-denominated instruments either with short or long maturities, it tries to create a debt portfolio in which

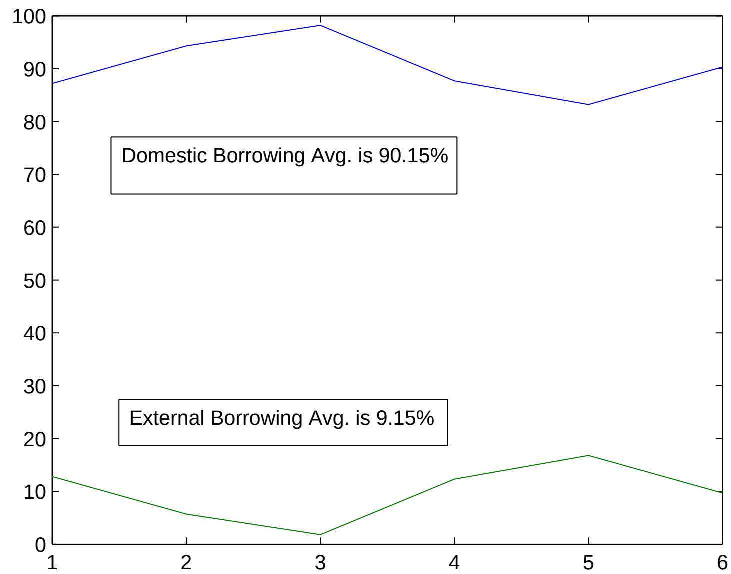

Figure 2.3: Currency Composition of Borrowing Between 2003 Q3 and 2004 Q4

- domestic debt instruments are preferred to FX-denominated instruments,
- fixed rate auctions have a relatively higher ratio than floating rate auctions,
- the maturity is tried to be kept as long as possible

	2003	2004 Q1	2004 Q2	2004 Q3	2004 Q4	2005 Q1
Fixed Rate	94	75.4	88.3	81.1	92.1	84
Floating Rate	6	24.6	11.7	18.9	7.9	16
YTL-Denominated	87.2	94.3	98.2	87.7	83.2	90.3
FX-Denominated	12.8	5.7	1.8	12.3	16.8	9.7

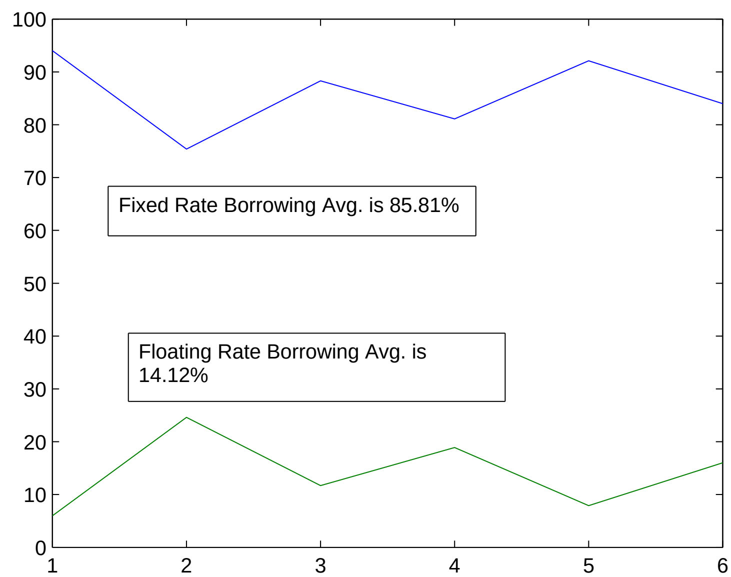

Figure 2.4: Interest Rate Composition of Domestic Borrowing Between 2003 Q3 and 2004 Q4

Determinants of domestic borrowing in Turkey between 2003 and 2004: [2]

Domestic Borrowing	**2003**	**2004**
Auctions, TAP, Public offer, Direct placement	75.51	92.64
Compulsory saving scheme	10.94	0
Switching auctions+Post auction switching	4.65	3.27
Restructuring of securities issued to public banks	3.84	2.64
Private placement onlent	4.54	1.46
Private placement	0.05	0
Total	**100**	**100**

As the table reveals, most of the borrowing is done through Treasury auctions. Figure 2.5 shows the ratios of issuance methods in which zero coupon YTL auctions have the highest proportion and FX-denominated, floating rate note and coupon paying instrument auctions have comparatively small percentages.

One of the most important considerations is the repayment of the money borrowed. It is not an unexpected situation in Turkey that the budget revealing a surplus exhibits a cash deficit as long as the interest repayments are considered. Hence interest rates play a crucial role in the borrowing policies. Treasury prefers to borrow with fixed rates rather than floating rates in order to prevent itself from unexpected interest rate movements. Every year, nearly one third of the payments to borrowers is comprised of interest payments. It is shown in Figure 2.6 that interest payments are not small enough to neglect neither in debt management nor in our optimization problem.

Maturity profile of the instruments in the debt portfolio is an important factor to keep in balance for debt management problem. Treasury tries to keep the maturity of borrowing as long as possible. It sometimes issues debt with long maturities in order to lengthen the current maturity profile. Since the main reason of this is to lengthen the maturity, the least costly and risky instruments are issued, which are zero coupon YTL denominated bonds. [3]

[2] PDMR February 2005
[3] PDMR February 2005

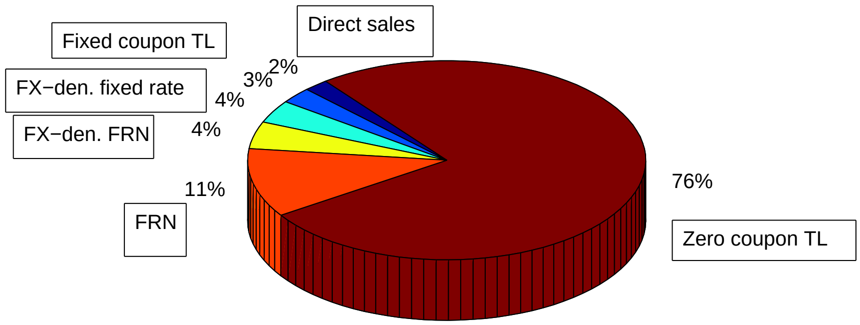

Figure 2.5: Breakdown of Domestic Borrowing by Issue Type in 2004

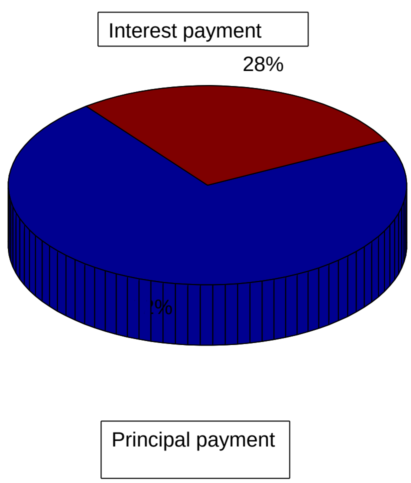

Figure 2.6: Domestic Debt Service in 2004

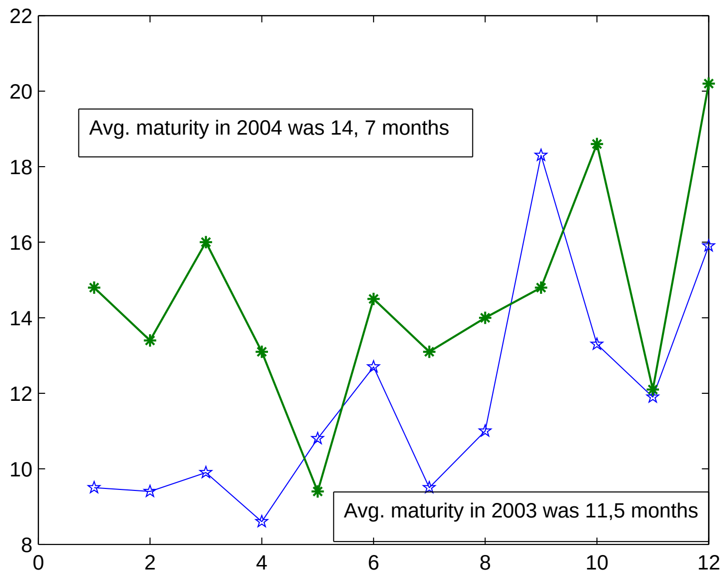

Figure 2.7: Maturity Profile in 2003 and 2004

Considering the maturity profile and currency composition, Treasury incurs costs related to borrowing strategies. The cost profile of zero coupon TL denominated debt of Turkey between 2003 and 2004 is shown in Figure 2.8.

Annual borrowing limit is the difference between the total initial appropriations and the estimated revenues indicated in the budget law.[4] Net borrowing is the difference between domestic and external borrowing:

- Net domestic borrowing
  - Borrowing
  - Repayment
- Net external borrowing
  - Borrowing
  - Repayment

[4] Article 5 of Law no.4749 pertaining to "Public Financing and Debt Management"

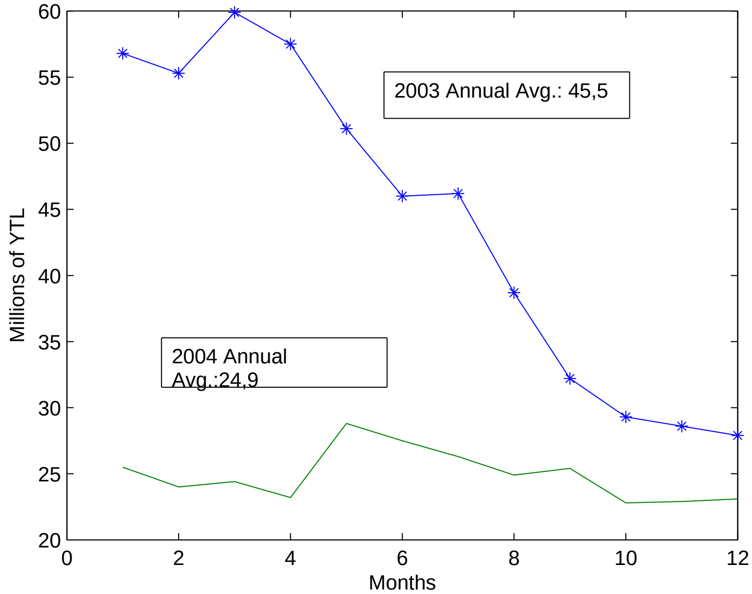

Figure 2.8: Weighted Average Cost of Zero-Coupon TL-Denominated Borrowing in 2003 and 2004

Net borrowing in 2004 in quadrillian TL was

Net Borrowing	**33.8**
Net Domestic Borrowing	30.8
Borrowing	158.1
Repayment	127.4
Net External Borrowing	3.1
Borrowing	12.9
Repayment	9.8

The change in the domestic debt stock between two periods is defined as the net borrowing requirement. Domestic debt stock comprises of securitized and unsecuritized debt which have subtitles :

- Securitized debt
  - Government bonds

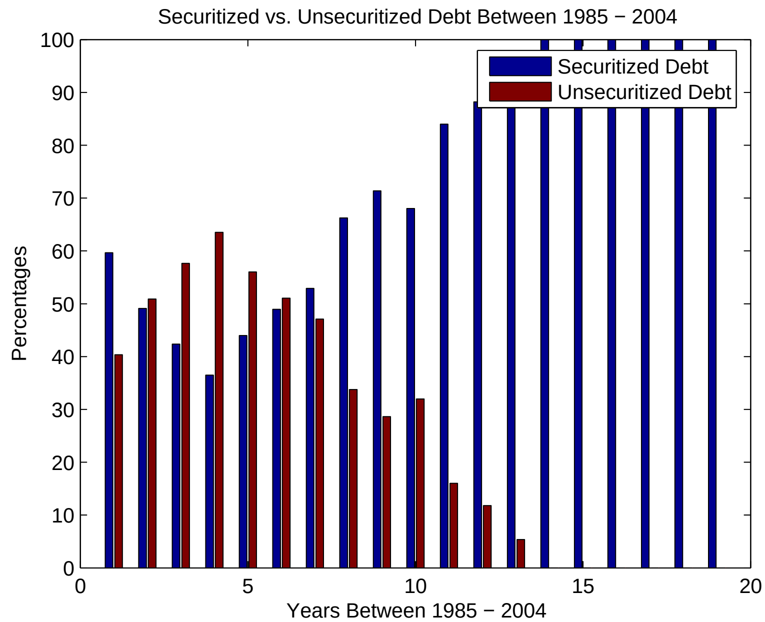

Figure 2.9: Securitized vs. Unsecuritized Debt of Turkey Between 1985-2004

    * Cash
    * Non-cash
  - Treasury bills
    * Cash
    * Non-cash
- Unsecuritized debt
  - FX difference
  - Short term CBRT advances

The highest proportion of domestic debt comes from the securitized debt. Although Treasury auctions vary in the instrument and the way they are issued, discounted YTL denominated T-Bills and G-Bonds with maturities 3 months to 5 years constitute the greatest amount of domestic debt.

Besides from discounted YTL denominated auctions, although very few in the last 10 years, Treasury issues

- Floating Rate Notes (FRN)
- FX denominated G-Bonds and
- CPI indexed G-Bonds

An overview of Treasury FRN, G-Bond and T-Bill auctions between 1998 and 2005 is:

1. Between 1998 and 1999, Treasury issued 10 CPI indexed bonds with maturities 12 to 14 months.
2. 66 FRN auctions with maturities 18 to 24 months took place between 1999 and 2005:
    - 57 issues of TL and YTL denominated FRNs with maturities 18 months to 3 years took place between 1999 and 2005. 28 of those were paying coupons quarterly and 1 was semi-annually.
    - USD denominated 13 months discounted G-Bonds were issued in 2004.
    - 6 USD denominated FRNs with maturities 3 to 5 years, paying coupons semi-annually and having interest $LIBOR + 1.60$ were issued in 2005.
    - 2 EURO denominated FRNs with maturity 5 years and semi-annual coupon payments were issued in 2005.
3. Between 2001 and 2003, 22 discounted FX denominated G-bonds and T-Bills were issued:
    (a) 17 of them were USD and EURO denominated G-Bonds and T-Bills with maturities 11 to 18 months.
    (b) 5 of the auctions were fixed couponed, USD and EURO denominated, 2 year bonds with semi-annual coupon payments.

Turkey borrows from external markets as well as from domestic market. External debt disbursement of consolidated budget in 2003 to 2005 Q1 is

	2003	2004$Q1$	2004$Q2$	2004$Q3$	2004$Q4$	2005$Q1$
Total	**100**	**100**	**100**	**100**	**100**	**100**
Program Financing	78.47	90.39	83.82	78	64.77	79.78
Bond Issuance	59.5	90.39	36.44	30.47	64.77	58.16
International Institiutions	18.97	0	47.38	47.53	0	21.62
Project Financing	21.53	9.61	16.18	22	35.23	21.84

In 2004, program financing through international capital markets via Eurobond issuances was approximately USD5.75 billion. The shortest maturing Eurobond was a five year bond while the longest maturity was 31 years.[5] The average maturity has risen to 17.6 years while it was 8.6 years in 2003. Regarding the borrowing costs, cost of external borrowing has fallen compared to 2003. Average borrowing cost of USD-denominated debt decreased to 8.1% while it was 10.1% in 2003. Average borrowing cost of Euro, just like Dollar, has sloped downward since it has fallen from 9.9% in 2003 to 6.3% in 2004. Compared to leading emerging market countries, Turkey has a relatively high Eurobond issuance ratio:

[5] PDMR February 2005

(billion USD)	1996	1997	1998	1999	2000	2001	2002	2003	2004	Total
Argentina	9.9	10.2	10.6	11.6	9.2	2.1	0	0	0	58.1
Mexico	13.5	7.2	1.5	3.6	4.9	4.7	4	7.4	7.7	59.3
Brazil	1.3	4.9	2.7	4.7	6.8	6.7	3.9	5.8	5.7	44.1
Turkey	2.8	2.9	2.7	5.0	7.5	2.2	3.3	5.3	5.8	40.0
Russia	1	3.2	11.7	0	0	0	0	0	0	15.9
Columbia	1.3	1	1.4	1.5	1.5	4.2	1	1	1.4	14.8
Philippines	1.1	0	0.5	2.9	1.7	0.6	2.9	3.2	4.1	17
Venezuela	0.4	4.3	0.5	0.2	0.5	1.2	0	3.7	4	15.2
Poland	0.2	0.4	0	0	0.6	0.9	2.7	4.3	3.7	13.1
South Africa	0.8	0.8	0	1.3	0.8	0.9	1.3	1.4	1	8.6
Panama	0	1.2	0.3	0.5	0.4	1.2	0.6	0.3	1.2	5.7
Ukraine	0	0	0.6	0.3	1.1	0	0.4	1	1.1	4.5
Peru	0	0	0	0	0	0	0.9	1.3	1.3	4.5
Total	32.3	36.1	32.5	31.6	35	24.7	22	34.7	37	300.8
Turkey's %	8.7	8	8.3	15.8	21.4	8.9	15	15.3	13.6	13.3

Besides from Euro and Dollar bonds, Turkey borrows in SAMURAI, GLOBAL and YANKEE bonds from external markets. SOEs[6] and other public sectors prefer EURO bonds while SAMURAI bonds are preferred by municipalities and other public sector.

2.2 RISK MANAGEMENT APPLICATIONS OF PUBLIC DEBT IN TURKEY

Considering the borrowing policies defined above, the basic principles of debt and risk management are defined as follows:[7]

[6] KÎTler

[7] Law no.4749 on "Public Financing and Debt Management" and "Regulation on the Principles and Procedures for the Coordination and Administration of Debt and Risk Management", March 2002

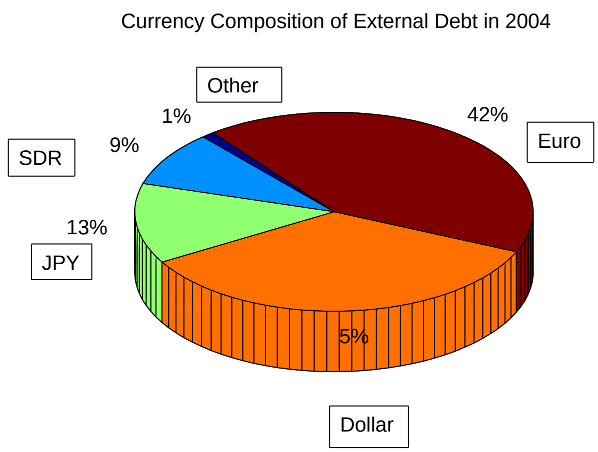

Figure 2.10: Currency Composition of External Debt in 2004

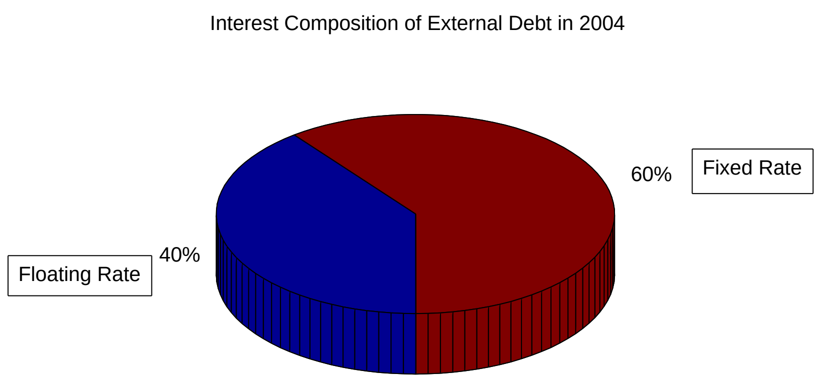

Figure 2.11: Interest Composition of External Debt in 2004

- On account of the macro-economic balances and monetary policies, a sustainable, transparent and accountable borrowing policy should be maintained and

- The financing requirements at the lowest possible cost in the medium and long term under a risk level constraint which is determined in consideration of domestic and external market conditions and cost factors should be fulfilled.

The borrowing structure of the government is shaped by a set of benchmark strategies which target a cost and risk structure implied by the debt management objectives of the government. These benchmarks are obligatory for borrowers and are used for performance measurement.

Undersecretariat of Treasury has begun to implement strategic benchmarks since 2004 in order to ensure the transparency of public debt management, and at the same time to borrow effectively at minimum cost and a prudent level of risk.[8] Strategic benchmark rules for the period 2005 - 2007 are:[9]

- Raising the funds, especially in YTL.

- Accomplishing a mainly YTL-denominated, fixed rate domestic borrowing strategy.

- Lengthening the average maturity of domestic borrowing on account of the market conditions.

- Keeping a certain level of cash throughout the year to get rid of the liquidity risk associated with cash and debt management.

2.3 CONCLUSION

Debt management policy of Treasury which is based on three items, has been elucidated throughout the chapter. First, ratio of domestic currency denominated

[8]Treasury uses a Cost-at-Risk approach to determine these strategic benchmarks.
[9]PDMR February 2005

debt to FX-denominated debt is aimed to be kept as high as possible. Second, fixed rate debt is preferred to floating rate debt to prevent unexpected fluctuations in interest rates. Third, the longer the maturity the more suitable it is to Treasury's debt management policy.

Having explained the debt management policy of Treasury, we will try to identify the behaviors of the macroeconomic variables in the optimization problem in the coming chapter.

Chapter 3

MACROECONOMIC MODELING OF GOVERNMENT DEBT IN TURKEY

After a brief overview of public debt management policy and how it is performed in the last two years, it comes to define the variables in our optimization problem and identify their behaviors.

The optimization problem is mainly based on the idea to minimize expected cost and find the optimal portfolio positions which result in the minimum expected cost. Then, the risk of the optimal portfolios will be calculated by VaR-Historical Simulation approach.

Different portfolio strategies in terms of maturities are confronted in the optimization problem. The stochastic components in the problem are modeled in this chapter. Actually, we have first aimed to work on portfolio strategies with different currencies and maturity profiles. But, lack of data on external borrowing instruments like FX-denominated bonds force us to use a borrowing strategy based on domestic borrowing instruments like T-Bills and G-Bonds. Secondly, although CPI- indexed bonds with different maturities and durations have been included as a means of borrowing in Swedish National Debt Office's Government Debt Analysis Report which we built our study on, we did not use CPI-indexed bonds in our borrowing strategies because of the fact that Turkey is a country with highly volatile inflation and would rather prefer to borrow with fixed rate instruments.[1] The aim of modeling the macroeconomic variables in the optimization problem is to explain the borrowing behavior of Treasury as explicitly as possible. However as the lack of data and complexity of the problem is considered, the variables in the model are tried to be kept at a minimum level possible. Hence, we decided to use four variables in our optimization problem:

- Short term interest rates
- Long term interest rates
- Real GDP growth
- Net borrowing requirement

Existence of a unit root force us to use $ARIMA(1,1,1)$ and $GARCH(1,1)$ to model short and long term interest rates. Real GDP growth follows the states of the economy and is modeled with a regime switching $AR(1)$ process when the economy is in one of the two states, *boom* or *recession*. The last variable in the optimization problem which is the net borrowing requirement, is modeled with $ARIMA(1,1,1)$ and $GARCH(1,1)$ due to the existence of a unit root.

3.1 INTEREST RATES

The exchange rate fluctuations give rise to investors' to expect higher returns on assets and higher risk premiums. Then, it can be concluded that there is a positive relationship between exchange rate risk and interest rates. Hence, we first thought to explain interest rate behaviors by exchange rate movements.

To my best knowledge, *Berument* and *Günay* (2001) have modeled nominal interest rates, (**R**) in Turkey with autoregressive processes via inflation and exchange rate risks:

$$R_t = \gamma_0 + \gamma_1 \mathrm{E}(ER_t) + \gamma_2 \sigma_t + \gamma_3 h_t + \eta_t$$

[1] *Consumer Price Index* (*CPI*) is considered as a measure of inflation in many countries.

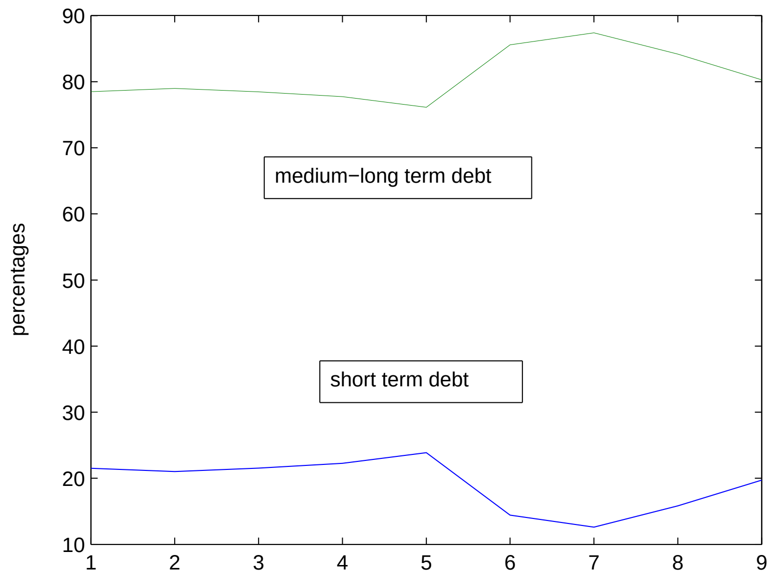

Figure 3.1: Interest Rate Composition of Debt Between 1983 Q3 - 2004 Q4

where E(ER_t) is the expected value of the exchange rate in domestic currency[2] [3] and σ_t is the exchange rate risk which is modeled as $GARCH(1, 1)$; h_t is the inflation risk and is modeled with $GARCH(2, 1)$ when inflation is modeled with

[2]Exchange rate is measured as the logarithmic first difference of the foreign exchange basket values where Basket=$\begin{cases} 1USD + 1.5DM, & \text{until passing to Euro currency;} \\ 1USD + 0.67Euro, & \text{after Euro currency.} \end{cases}$.

[3]ER_t is modeled with AR(p):

$$ER_t = \beta_0 + \sum_{i=1}^{p} \beta_i ER_{t-i} + \varepsilon_t$$

where ε_t is the residual term with zero mean and time varying conditional variance.

$ARIMA(1,1,1)$[4] Then,[5]

$$R_t = \underset{(14.83928)}{102.7432} - \underset{(-6.03055)}{20.6395}\mathrm{E}(ER_t) + \underset{(1.83465)}{87.3514405}\sigma_t - \underset{(-4.21967)}{7.9865}h_t + \eta_t$$

The t-statistics of the interest rate equation points that we can model interest rates following *Berument* and *Günay* (2001). But, this does not give us any indication about the individual behaviors of short or long term interest rates. Hence, we try to model short and long interest rates as separate variables.

Interest rate of a quarter is found to be the weighted average of the interest rates of the treasury auctions in the corresponding quarter with weights being the TL amount of the auction divided by the total amount of auctions in that quarter. Then, short term interest rates are taken as the treasury auction rates with maturities less than or equal to six months. Long term interest rates, on the other hand are the ones with maturities greater than or equal to 12 months.

3.1.1 Short Term Interest Rates

When working with time series, one of the first things that need to be checked before going further is the stationarity of the data. If the series is non-stationary, i.e exhibits unit root(s), then the shocks to the variable will not decay with time and the variable need to be modeled with *integrated* models. Thus, we first check the stationarity of the variables in the problem by Dickey Fuller(DF) test. There

[4] Let π_t denote the inflation and h_t the inflation risk, then

$$\pi_t = \underset{(1.425)}{0.27885}\pi_{t-1} + \underset{(8.397)}{0.78403}\varepsilon_{t-1} + \varepsilon_t$$

and

$$h_t^2 = 4.89 + \underset{(13.327)}{1.10086}h_{t-1}^2 - \underset{(-1.756)}{0.14579}h_{t-2}^2 + \underset{(76.995)}{0.98503}\varepsilon_{t-1}$$

The t-statistics are all at acceptable values except for the lagged parameter of π_t. However, it is going to be admissible if we use a 16% confidence level by which we can model CPI data with ARIMA(1,1,1) and the conditional variance of inflation with GARCH(2,1).

[5] RATS and OX programs are used to model the macroeconomic variables in the problem.

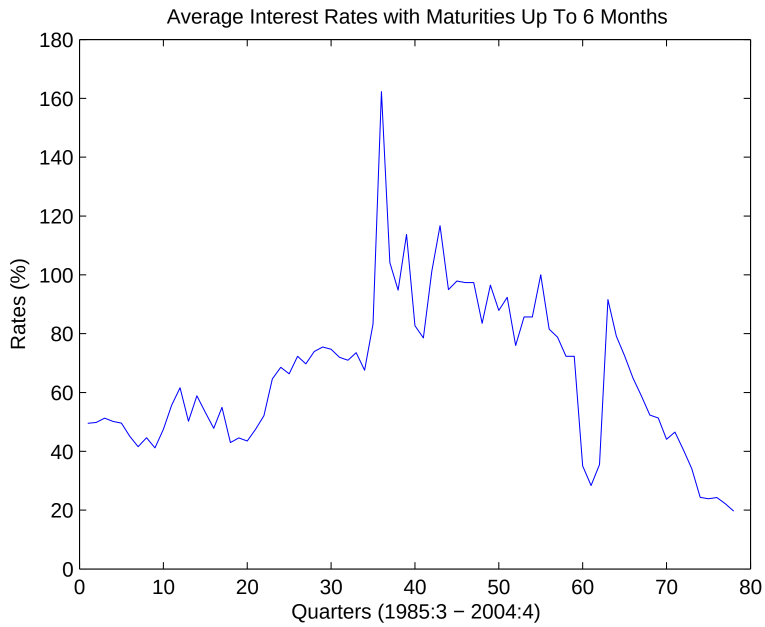

Figure 3.2: Short Term Interest Rates Between 1985 Q3 - 2004 Q4

are two ways to determine the number of lags used in the Dickey Fuller test. First and less used is to choose the number of lags which minimizes the information criteria that the regression analysis program uses. We used the second way which is to choose the frequency of the data as the number of lags. Since our data is quarterly, we looked at the results of the Dickey Fuller test with four lags. The critical values for 1%, 5% and 10% confidence levels are below:[6]

Significance Level	1%	5%	10%
Critical Value	-3.43	-2.86	-2.57

Applying Dickey Fuller test with four lags to short term interest rate data between period $1986Q1$ and $2004Q4$, we obtain the test statistic -1.24506. This statistic is above the critical values of $-3.43, -2.86, -2.57$ which are the values corresponding to 1%, 5% and 10% confidence levels respectively. This means that our short rate data exhibits a unit root and should be modeled with an *integrated*

[6]Chris Brooks, "Introductory Econometrics For Finance"

model like $ARIMA(p, d, q)$ instead of an $AR(p)$ process. An $ARIMA(p, d, q)$ model is equivalent to an $ARMA(p, q)$ model in which the variable is differenced d-times to get rid of the unit root(s). Among the time series' which contain unit root, most of them have a unique unit root while it is infrequent to have more than one. Then estimation is done on the differenced data.

Our short term interest rate series is differenced once to discard the unit root. Then ARMA(1,1) is applied to the differenced series. Thus, it can be modeled with $ARIMA(1, 1, 1)$

$$\begin{aligned} \Delta R_s = -&0.15256\Delta R_{s(t-1)} - 0.21071\varepsilon_{t-1} + \varepsilon_t \\ &(-1.624) \qquad (-1.436) \end{aligned}$$

where R_s denotes the short term interest rates and ε_t is $IID(0, \sigma_\varepsilon^2)$. The t-statistics are admissible at approximately 16% confidence level.

Disturbance terms reveal heteroscedasticity and can be modeled with $GARCH(1, 1)$

$$\begin{aligned} \sigma_{\varepsilon_t}^2 = 3.3513 + &1.78047\sigma_{\varepsilon_{t-1}}^2 + 0.45589\varepsilon_{t-1}^2 \\ &(3.394) \qquad (4.108) \end{aligned}$$

The t-statistics are acceptable even at 1% confidence level.

3.1.2 Long Term Interest Rates

Next comes to identify the behavior of long term interest rates which are taken to be the interest rates of the treasury auctions with maturities greater than or equal to 12 months between 1986Q1 and 2004Q4.

The first thing to investigate is the existence of unit root(s). Applying DF test with 4 lags, the test statistics is -1.09421 which is in the non-rejection region even for 1% confidence level. Hence, our series has unit root and need to be modeled with an integrated model.

We model long term interest rates with $ARMA(1, 1)$ after taking the first difference to expel the unity. Let R_l denote the long term interest rates, then

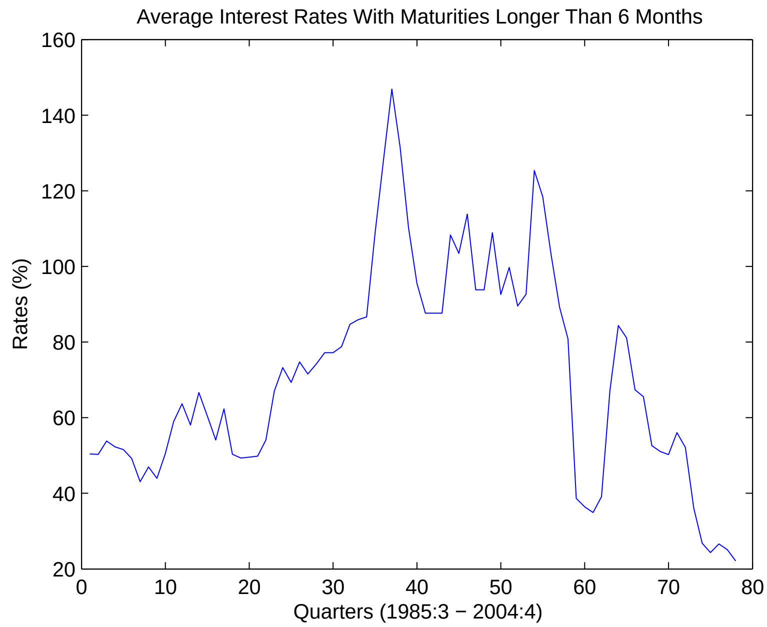

Figure 3.3: Long Term Interest Rates Between 1985 Q3 - 2004 Q4

R_l can be modeled with $ARIMA(1,1,1)$ while the conditional variance of the disturbance term is modeled with $GARCH(1,1)$:

$$\Delta R_l = -0.18546\Delta R_{l-1} - 0.36682\varsigma_{t-1} + \varsigma_t$$
$$(-3.387) \qquad (-3.783)$$

and

$$\sigma^2_{\varsigma_t} = 0.95034\sigma^2_{\varsigma_{t-1}} + 0.80857\varsigma^2_{t-1}$$
$$(13.312) \qquad (13.001)$$

where the constant term of $GARCH(1,1)$ is approximately zero and is highly significant. As the t-statistics of both models show, they are far in the non-rejection region and hence are acceptable.

Real interest rates are obtained by subtracting expected inflation from nomi-

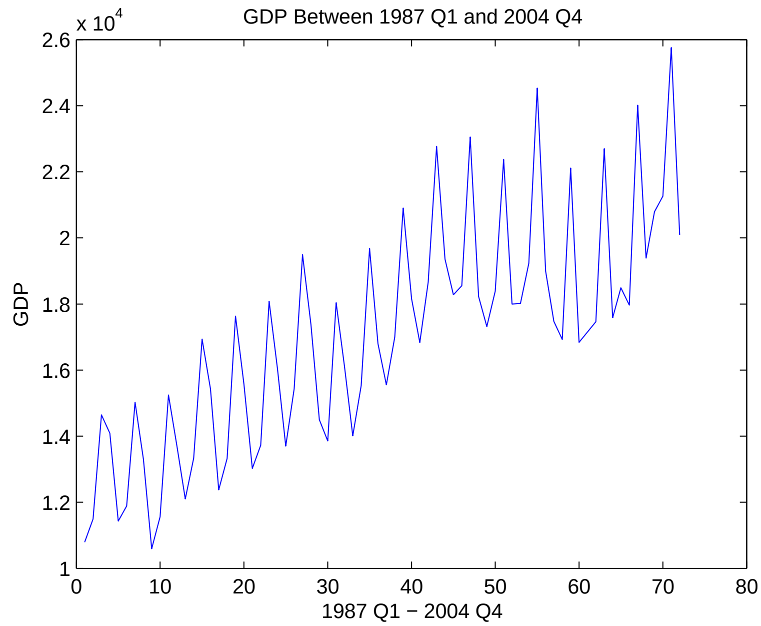

Figure 3.4: GDP between 1987 Q1 - 2004 Q4

nal interest rates.

$$RR_t = R_t - \pi_t^e$$

where RR_t is the real interest rate at time t and π^e is the expected inflation for the corresponding time period.

3.2 REAL GDP GROWTH

The real GDP growth is another variable that we need to model to use in the optimization problem. GDP growth data follows business cycles, changing according to the states of the economy. Thus, current economic regimes and discrete shifts in the variables of the underlining model should be taken into consideration when modeling real GDP growth.

Hamilton (1990) states that parameters of autoregressions are subject to occasional discrete shifts. Markov chain regime switching processes are used to

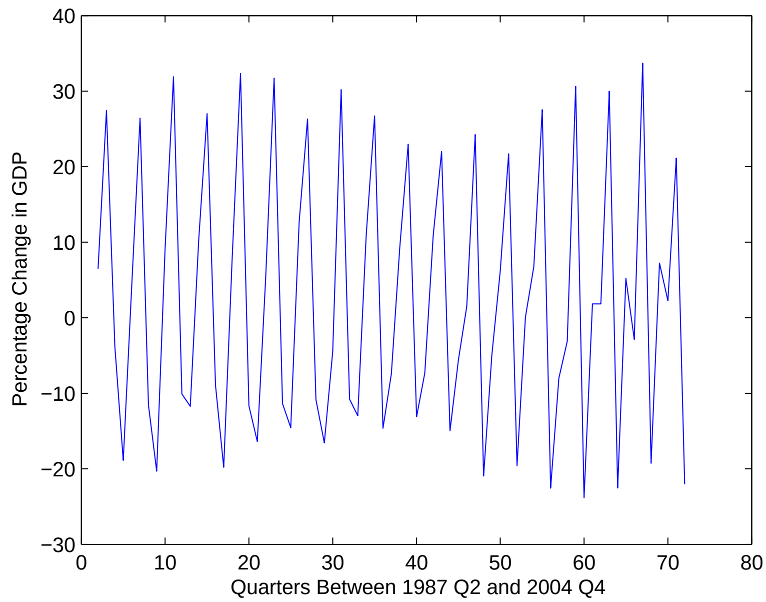

Figure 3.5: Percentage Change in GDP between 1987 Q1 - 2004 Q4

estimate the parameters characterizing the different regimes and the transitions between states of the economy.

States of the economy are supposed to capture the current economic regime of the country. Assuming that Turkish economy is in one of the two states, *boom* or *recession*, a two-state Markov chain regime switching process is going to be used to model real GDP growth data.

If we take the GDP growth, g, as the percentage of the first logarithmic difference of quarterly GDP between $1987Q3$ to $2004Q4$:

$$g_t = 100 \log(GDP_t - GDP_{t-1})$$

then GDP is modeled with a two-state regime switching AR(1) process:

$$g_t = \mu_s + \beta g_{t-1} + \epsilon_t$$

where ϵ_t is $IID(0, \sigma_\epsilon^2)$,

$$\mu_s = \begin{cases} \mu_1, & \text{s=boom;} \\ \mu_2, & \text{s=recession.}^7 \end{cases}$$

Then

$$g_t = \mu_s - 0.23074 g_{t-1} + \epsilon_t$$

and

$$\mu_s = \begin{cases} 3.55996, & \text{s=boom;} \\ 0.30837, & \text{s=recession.} \end{cases}$$

In a two-state Markov chain regime switching process, the current state, s_t, depends on the most recent value of the state s_{t-1}. The transition probability of moving from state i to state j is given by

$$P\{s_t = j | s_{t-1} = i, s_{t-2} = k, \ldots,\} = P\{s_t = j | s_{t-1} = i\} = p_{ij}$$

and p_{ij} are determined by a 2×2 matrix

$$P = \begin{pmatrix} p_{BB} & p_{BR} \\ p_{RB} & p_{RR} \end{pmatrix}$$

where p_{BB} is the probability that a boom state will be followed by a boom state, p_{BR} is the probability that a recession will follow a boom *etc.*

The transition matrix of two-state Markov chain regime switching process for real GDP growth is

$$P = \begin{pmatrix} 0.70130 & 0.30130 \\ 0.29870 & 0.69870 \end{pmatrix}$$

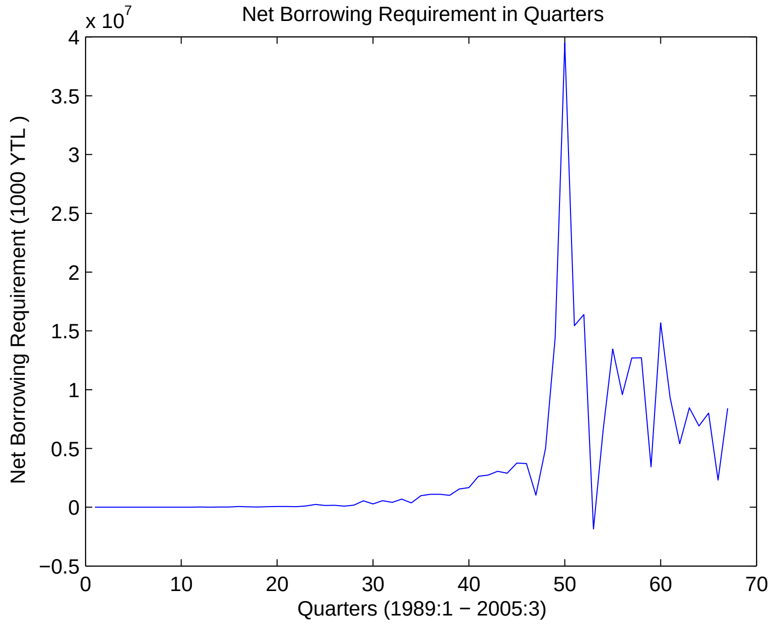

Figure 3.6: Net Borrowing Requirement between 1989Q1-2005Q3

3.3 NET BORROWING REQUIREMENT

Net borrowing requirement is another variable in the optimization problem that should be modeled. We define the **net borrowing requirement** at time t as the difference of domestic debt positions at time t and $t-1$.[8] If (NBR) is the net borrowing requirement and DD is the domestic debt position, then

$$(NBR)_t = DD_t - DD_{t-1}$$

To model the net borrowing requirement, the existence of unit root(s) is checked first. Taking the data between 1989 Q1 to 2005 Q5, t-statistics of the Dickey Fuller test with 4 lags is -1.53328 which is in the non-rejection region for 1%, 5% and 10% confidence levels. Hence our series has a unit root.

$ARMA(1,1)$ is applied after the unity is removed by differencing the data

[8]It is stated in Public Financing and Debt Management Law that the difference between the total initial allocations and the revenues proposed in the budget law contribute to the annual borrowing limit which we do not use because of simplicity.

once. $ARIMA(1,1,1)$ gives admissible results for most of the time the series which exhibit a unit root when the number of the data in the series is less than 100. Thus, we apply $ARIMA(1,1,1)$ to our series:

$$\Delta(NBR)_t = \underset{(1.402)}{0.3419}\Delta(NBR)_{t-1} + \underset{(7.767)}{0.8649}\epsilon_{t-1} + \epsilon_t$$

where ϵ_t is $IID(0, \sigma_\epsilon^2)$

The t-statistics in parenthesis is acceptable at 17% confidence level for $(NBR)_{t-1}$ and 1% confidence level for ϵ_{t-1}.

Since the confidence level is high, $ARIMA(1,1,1)$ and $GARCH(1,1)$ is tried in case of existence of ARCH effects:

$$\Delta(NBR)_t = \underset{(3.597)}{0.767}\Delta(NBR)_{t-1} + \underset{(6.002)}{0.893}\epsilon_{t-1} + \epsilon_t$$

and

$$\sigma_{\varepsilon_t}^2 = 1739.59 + \underset{(4.645)}{2.11984}\sigma_{\varepsilon_{t-1}}^2 + \underset{(5.316)}{0.55515}\varepsilon_{t-1}^2$$

The t-statistics are acceptable even at 1% confidence level and hence our net borrowing requirement series is well modeled with $ARIMA(1,1,1)$ and $GARCH(1,1)$.

3.4 CONCLUSION

Throughout the chapter, we have tried to model the stochasticity inherent in the variables of the optimization problem that are short and long term interest rates, real GDP growth and net borrowing requirement. Short and long term interest rates are modeled with ARIMA(1,1,1) and GARCH(1,1) which indicate that the series are non-stationary and heteroscedasticity is revealed. Excluding the data during the crisis will remove this non-stationarity. GDP is assumed to follow the economic cycle of the country, thus is modeled with an AR(1) regime switching process. The last variable in the problem, which is the net borrowing requirement is modeled with ARIMA(1,1,1) and GARCH(1,1). As in the case of interest rates, data exhibits non-stationarity and heteroscedasticity. Including the data during the crisis lead to this non-stationarity.

Next chapter will be a step further in the optimization problem after the determination of the macroeconomic variables in the model and will present the scenario generation and clustering procedures. The econometric models of the variables will be used to simulate possible future movements of these variables.

Chapter 4

SCENARIO GENERATION AND CLUSTERING

After being determined and modeled, the macroeconomic variables need to be set so that possible future movements of the variables will be reflected. Then, these movements should be calibrated in order to be implemented into the optimization problem.

This chapter will give the basics of the scenario generation and clustering procedures. Scenario generation enables us to visualize the future movements of the variables while clustering allows us to implement these movements into the optimization problem by grouping them with *K-Means Clustering Approach*.

4.1 SCENARIO GENERATION

Scenario generation procedure which is necessary for the better management of future behaviors of the variables in the optimization problem will be explained in this chapter which is followed by clustering of the previously generated scenarios to be able to use them in our optimization problem.

In order for our model to reflect the future, we need to take into account more or less every movement that can happen in the variables of the model. Each movement of the variable is given by a scenario. Hence, as many scenarios as possible should be generated to give a better representation of the future.

The uncertainties in a decision-making problem have to be approximated by a limited number of discrete outcomes in order to make a decision. Random variables are the uncertain return values of the instruments in a decision-making problem.

A multistage financial optimization problem with decision periods $1, 2, \ldots, T$ is based on[1]

- a stochastic model of the future economic environment (prices, interest,cash-flows,etc.). The scenario process is expressed as a stochastic process $\xi_1, \xi_2, \ldots, \xi_T$;
- a decision model which specifies the actions to be taken at different states of the decision making problem. The decisions at time t are $x_1, x_2(\xi_1), x_3(\xi_1, \xi_2), \ldots, x_T(\xi_1, \xi_2, \ldots, \xi_{T-1})$ and may depend on the previous observations, $\xi_1, \xi_2, \ldots, \xi_{T-1}$;
- an objective function which expresses the long-term expectations of the decision maker.

The most common way of solving the problem is to discretize the random vectors ξ_i into $\xi_1, \xi_2, \ldots, \xi_T$ finitely many values. Then the process $\{\xi_1\}, \{\xi_1, \xi_2\}, \{\xi_1, \xi_2, \xi_3\}, \ldots, \{\xi_1, \xi_2, \ldots, \xi_T\}$, can be expressed as a scenario tree.

A **scenario tree** is composed of *states*, *nodes* and *arcs* linking the nodes. States are usually the time periods of the problem. The decisions in a stochastic programming problem are made at the nodes. The arcs linking the nodes are the realizations of the random variables. The scenario tree branches off for each possible realization of the random variables $\xi = \{\xi_1, \xi_2, \ldots, \xi_T\}$. **Scenario tree generation** is a method to generate discrete outcomes for the random variables. The random variables are discretized in order to determine the factors of the risky events which are then approximated by a set of scenarios. Given the event history up to time *t*, the uncertainty at time *t+1* is characterized by a set of discretizations of the observations at time *t+1*.

A scenario tree is constructed by branching new scenarios from old at each time period in a multi-stage model. The nodes in a scenario tree represent the

[1] Pflug, 2000

events of the world. *Today* is represented by the root node to which the value of observed data is attached. As we branch down the tree, the nodes are conditional on their predecessors. A well generated scenario tree should represent all possible outcomes of the random variables, the bad ones as well as the good ones.

Assume an investment horizon T on a portfolio of assets. Let $\xi^t \equiv \{\xi_1, \ldots, \xi_t\}$ denote the stochastic events at $t = 1, \ldots, T$ and $\{\Omega, \mathcal{F}, \mathcal{P}\}$ be the probabilistic specifications of the future uncertainty. Then we can say that the decision-process is $\mathcal{F}_{t-1}$ adapted, i.e the decision at a given stage does only depend on the past information, not to the future realizations of the random events.

A possible realization of the stochastic variables $\xi^t \equiv \{\xi_1, \ldots, \xi_T\}$ represents a scenario. The set of the leaves of the scenario tree constitutes the set of scenarios and the nodes at time $t \geq 1$ correspond a possible realization of ξ^t. We should note that ξ_t can take finitely many values.

Discrete outcomes are sampled from true or assumed probability spaces with pre-determined statistical properties. However, the samples may not reflect the distribution if the number of outcomes is small. The scenario tree should be big enough to represent both optimistic and pessimistic future movements. Hence, the number of scenarios should be increased as long as it preserves computational tractability.

4.1.1 Computations

Scenario generation was the easiest part of this study. We have modeled the economic variables in our problem by econometric models. Every econometric model has a disturbance term which we assume to be an independent and identically distributed random variable with constant means. Although variances of some of the error terms exhibit heteroscedasticity, they are modeled via models like GARCH and its extensions which also do contain the error term itself.

The scenario generation procedure is based on the idea of generating IID random variables. We have generated 25,000 independent and identically distributed random variables for each variable in the optimization problem by using $MATLAB7.0$ and inserted the generated values to the corresponding econometric models. For example, let the model be an AR(1) process

$$Y_t = 1.5 + 2.33Y_{t-1} + \varepsilon_t$$

where ε_t is $N(0, \sigma^2)$ and we want to generate, let say 5 scenarios. Then, we generate 5 ε_t values ε_t^i, $i = 1, 2, 3, 4, 5$ with a random number generator using the normal distribution with mean 0 and variance σ^2 . These values are then inserted into our model to get 5 different Y_t values which are

$$Y_t = 1.5 + 2.33Y_{t-1} + \varepsilon_t^1$$

$$Y_t = 1.5 + 2.33Y_{t-1} + \varepsilon_t^2$$

$$Y_t = 1.5 + 2.33Y_{t-1} + \varepsilon_t^3$$

$$Y_t = 1.5 + 2.33Y_{t-1} + \varepsilon_t^4$$

$$Y_t = 1.5 + 2.33Y_{t-1} + \varepsilon_t^5$$

It should be noted that the greater the number of scenarios, the better the variables will mirror the future. However, increasing the number of scenarios will lead to more time consumption in the optimization part. Hence, we found it enough to use 25,000 scenarios as a good representative of future.

After having explained the scenario generation procedure, it comes to make scenarios meaningful in our computations by grouping them according to their similarities. The coming section describes the clustering procedure. The generated scenarios have been clustered into 30 clusters by using K-Means clustering approach with $K = 30$.

4.2 CLUSTERING

Grouping objects into groups according to their similarities is called **clustering**. In a clustering process, the elements of the same group (cluster) are expected to have higher similarity than the elements of the other groups (clusters).

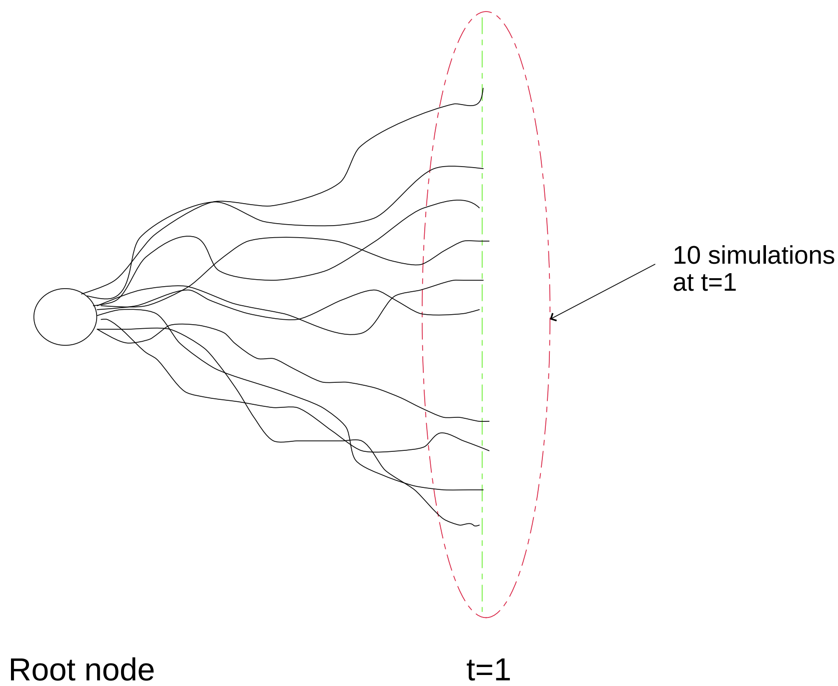

Figure 4.1: Scenarios at t=1

Clustering can be done in many ways among which the most widely used techniques are *hierarchial clustering* and *partitional clustering* (often called *k-means*). In hierarchical clustering, every cluster with size greater than one is composed of subclusters. Given a set P of objects and a number k, a $k-clustering$ algorithm will partition P into k subgroups $P_1, P_2, \ldots, P_k$. In k-means clustering, every element belongs to just one cluster according to some predetermined optimization criteria. This criteria is often the minimization of an error or cost function among the group members.

4.2.1 K-Means Clustering Algorithm

Given a predetermined number K of clusters and $\mathcal{L} = \{l_n\}_{n=1}^{N}$ where $l_n \in \mathbb{R}$ is the set of d-dimensional data points.

Define

$$f : \{P_1, P_2, \ldots, P_K \mid P_K \subseteq \mathcal{L}\} \longrightarrow \mathcal{R}$$

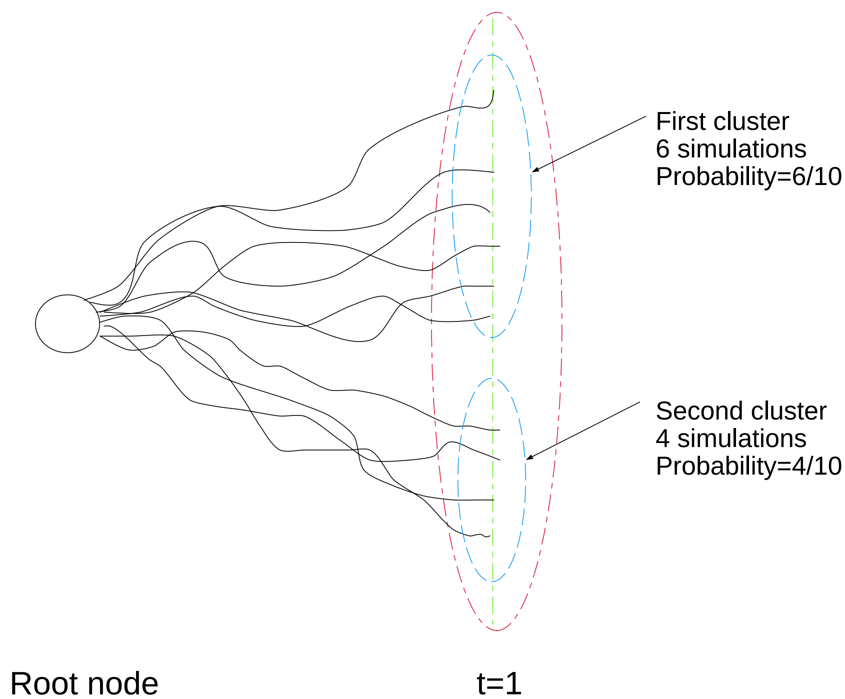

Figure 4.2: Clustering at t=1

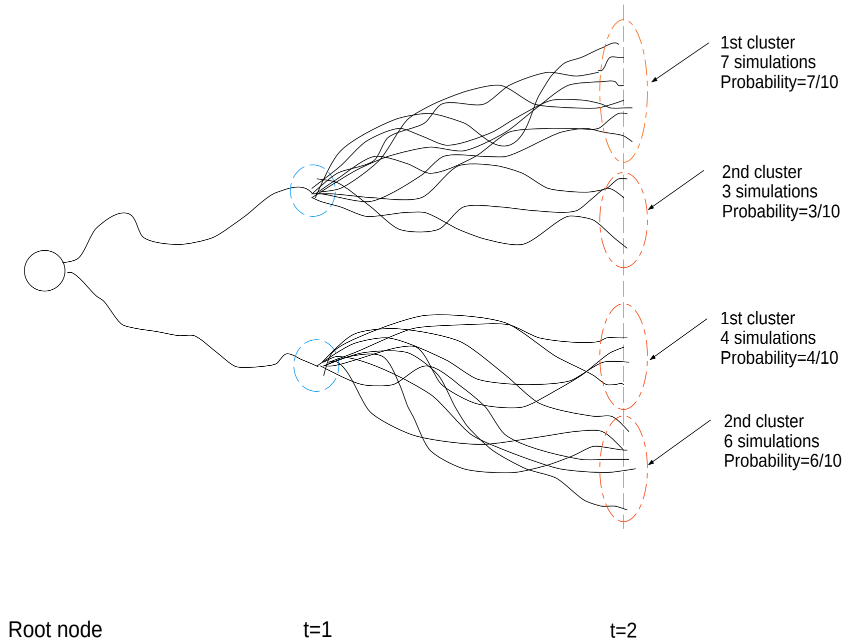

Figure 4.3: Clustering at t=2

which inputs the data from $\mathcal{L}$ and gives the sum of distances of the data to the centroid of the cluster that they belong to.

Given K and N data points $\mathcal{L} = \{l_1, l_2, \ldots, l_N\} \in \mathcal{L}$, the $K-means$ method minimizes the cost or error function while partitioning P into K clusters $P = \{P_1, P_2, \ldots, P_K\}$.

The mathematical expression for the clustering problem is

$$minf(P) = \sum_{k=1}^{K} \sum_{l \in P_k} d(l, \hat{z_k}),$$

such that

$$\mathcal{P} = \{P_1, P_2, \ldots, P_K \mid \bigcup_{k=1}^{K} P_k = \mathcal{L} \text{ and } P_i \cap P_j = \emptyset, \forall i, j \; i \neq j\},$$

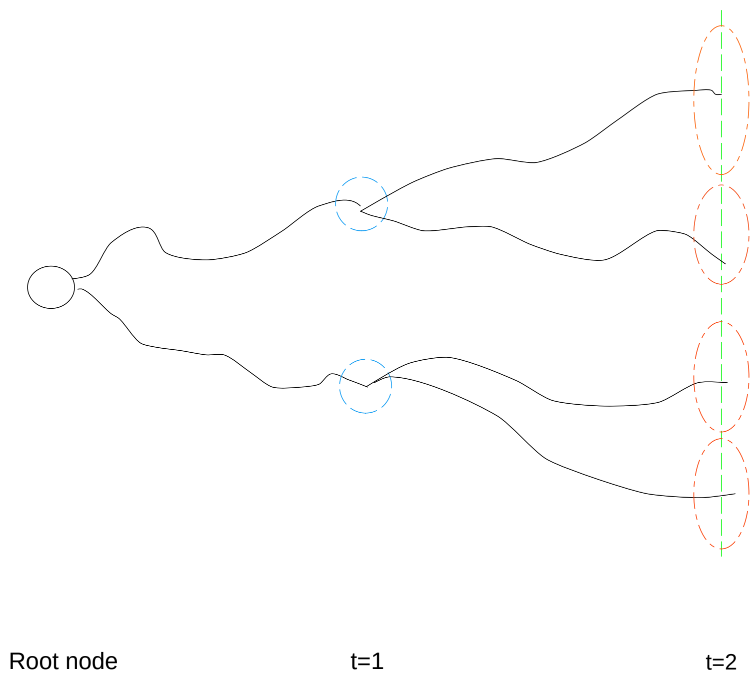

Figure 4.4: Results of Clustering at t=2

$$\hat{z_k} := \frac{1}{\mid P_k \mid} \sum_{j=1}^{|P_k|} l_{ji}$$

for $i = 1, 2, \ldots, K$.

The K-means algorithm to partition the given data set can be stated as follows:

Step 1: Choose the number of clusters K that you want to assign your data.

Step 2: Assign the data points to one of K clusters, form K initial clusters and compute their centroids.

Step 3: Assign each object to its nearest centroid.

Step 4: Compute the centroids again.

Step 5: Compute steps $2, 3$ and 4 until the termination criteria is satisfied. The termination criteria can be

- a given maximum number of iterations reached or
- not being able to re-assign the data anymore.

If the termination criteria is not satisfied, go to Step 2.

The termination of the algorithm means that we have reached the optimal allocation of the data points and the centroids we found cluster our data set properly.

4.2.2 Finding K: The Number of Clusters

It is really a difficult process to determine the *true* number of clusters K and the mechanics and applications of this process is beyond the scope of this thesis. However, it should be noted that the greater the number of clusters, the better will the optimization problem perform. This holds only when the number of simulations are enough to cover the number of clusters without a need to add many dummy nodes. The reason behind this is that the model strays from reality as long as the number of scenarios are not enough and many dummy nodes having probability zero are added to the system. Besides, the greatest handicap of increasing the number of clusters is that as long as the number is increased, the total number of nodes in the scenario tree will increase exponentially. The total number of scenarios N_{total} is

$$| N_{total} |= \sum_{s=0}^{S-1} K^s$$

where S is the number of stages in the optimization problem. Hence, one should be careful with the number of scenarios used to generate the scenario tree.

We find the number of clusters by *trial and error* method. This means that we increase the number of clusters as long as our scenario set consisting of 25,000 scenarios for each instrument in our portfolio, do not show inefficiency both in terms of the number of scenarios and the time spent on calculations. We have begun with 10 clusters and increased the number until 30 clusters. The scenarios at year 1 are clustered into 30 clusters each of which are clustered into 30 clusters at year 10. Hence our scenario tree consists of 930 clusters of scenarios, 30 of which are at year 1 and 900 clusters at year 10.

The connections between clusters are found based on the idea that the paths are created between the scenarios with the shortest distance.

Our clustering variable is four dimensional, one dimension for each variable in the optimization problem. Thus, we can not use general methods to make a sketch of the clusters. However, $MATLAB7.0$ gives an overall picture of the scheme. At t=1, we cluster the scenarios into 30 clusters. Figure 4.1 shows a sketch of the first cluster at the first year. We have 29 more clusters at t=1. The circles in the figure are the centroid scenarios. In Figure 4.2, first cluster of the 13^{th} cluster at t=2 is shown, one of the 900 clusters at 10th year.

4.2.3 Convergence of the Algorithm

It is shown in Appendix that K-means clustering approach converges after a finite number of iterations. But the greater the K and the number of variables in the problem, the greater the consumption of time is. Hence, we choose to use a *move ratio* ξ which measures the number of re-assigned data to each cluster.

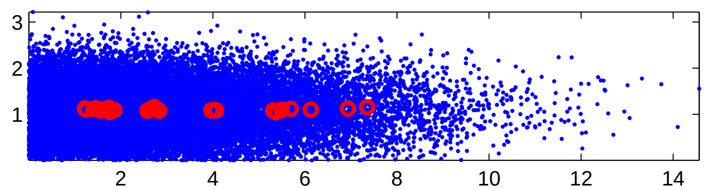

Figure 4.5: First Cluster of the First Year

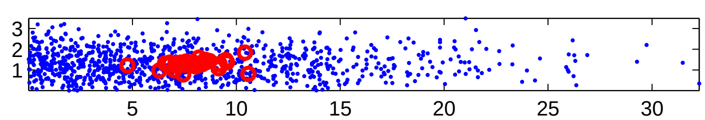

Figure 4.6: First Cluster of the Thirteenth Cluster at Year 10

$\xi = 5\%$ is used indicating that the algorithm stops when at least 95% of the variables are assigned to true clusters.

4.3 CONCLUSION

In this chapter, we have first explained the scenario generation procedure used to approximate the uncertainties inherent in the variables and reflect their future movements. Next, 25,000 scenarios have been generated and clustered into 30 clusters at each stage to use them in our optimization problem. The number of scenarios and clusters are chosen to effectively imitate the future while allowing computational tractability.

The next step that follows clustering is optimization which will be explained in Chapter 5.

Chapter 5

GOVERNMENT DEBT MANAGEMENT IN TURKEY USING STOCHASTIC OPTIMIZATION

Before coming to optimization step, first we have modeled the variables in the optimization problem which are short and long term interest rates, real GDP growth and net borrowing requirement. Next came to reproduce the future movements of theses variables by generating scenarios. 25,000 scenarios have been generated by $MATLAB7.0$ and turned into a three stage scenario tree by clustering the scenarios. $2005Q1$ is $t = 0$ and represents the root node. $t = 1$ is one year from today while $t = 2$ is ten years from the root node. Thus, the problem is optimized at 2006 and 2015. We have chosen a ten year period since we assume that the economy of Turkey repeats itself more or less at every ten years.

The next step after the construction of the scenario tree and clustering is the optimization procedure whose results will help us determine an optimal strategy. The objective function will be minimized subject to a couple of constraints and the decision variables which result in the minimum objective function will be taken to be the optimal decisions.

Multi-period financial optimization is a vital concept regarding both the institutions and the investors. Probable market movements and the associated risk factors are most of the time, difficult to predict. Mathematical programming tools are used to handle this problem. But stochastic factors existing in the mathematical programming problems make it difficult to solve the problems. Thus, stochastic programming is used to model the stochastic factors inherent in the problem. Given the underlying stochastic factors of the problem, the market movements are modeled and solved via stochastic programming tools.

Asset liability management problem has various stochastic factors that need to be modeled. Stochastic programming is a prevailing modeling paradigm for this problem.

Before starting to stochastic programming, we had better give some fundamental information about stochastic programming.

5.1 BASICS OF STOCHASTIC PROGRAMMING

5.1.1 Recourse Models

The *recourse problem* tries to find a policy by which the decision maker takes *recourse decisions* after the realization of the uncertain observations while anticipating future observations. In other words, mathematical and adaptive models are combined to form a common mathematical framework.

5.1.2 Decisions and Stages

Stochastic linear programs are types of linear programs whose input data have some uncertainty inherent in it. The uncertainty is represented by random variables. As the uncertainty is disclosed, the recourse actions take place.

The set of decisions in stochastic linear programs is divided into two groups:

- The *first stage decisions* are the decisions which take place before the realization of the observations. The period in which the decisions are made is called the *first stage*.

- Some of the decisions take place after the realization of the observations. These decisions are called *second stage decisions* and the period in which the decisions take place is the *second period.*

Let x be a vector which represent the first stage decisions. Then the second stage decisions are expressed by the vector y either as $y(w)$ or $y(x, w)$ if the second stage decisions are expected to exhibit the outcomes of both the random variable and first stage decisions where $\xi = \xi(w)$ represents the random observation. To sum up, the observations and related decisions can be shown as follows:

$$x \longrightarrow \xi(w) \longrightarrow y(x, w)$$

5.1.3 Two-Stage Stochastic Linear Programs with Recourse

The corrective action taken after the realization of the random variable representing the uncertainty is called the *recourse action.* The stochastic programming problems depend on three basic features:

- The decision x made today,
- The occurrence of a random variable $w \in \Omega$,
- The recourse action y taken after the realization of the random variable $w \in \Omega$.

The mathematical representation of the two-stage stochastic LP with recourse is

$$\begin{aligned}
&\min_x c^T x + \mathrm{E}_\xi[\mathcal{Q}(\mathrm{x}, \mathrm{w})] \qquad (5.1.1)\\
&\textit{such that } Ax = b\\
&x \geq 0\\
&\textit{where}\\
&\qquad \mathcal{Q}(x, w) = \min_y q(w)^T y(w)\\
&\qquad\qquad s.t.\ T(w)x + W(w)y(w) = h(w)
\end{aligned}$$

where SP^i denote the stochastic problem solved at the i-the stage for $i = 1, 2$, E_ξ is the mathematical expectation with respect to ξ and Ξ is the support for $\xi(w)$ for $w \in \Omega$ (*Support* is the smallest closed subset in R^N such that $\mathcal{P}(\xi \in \Xi) = 1)$[1]. We also know that $c \in \mathrm{R}^{n_1}$, $b \in \mathrm{R}^{m_1}$, A is an $m_1 \times n_1$ matrix.

$$T(w) = the\, technology\, matrix\, of\,\, size\, m_2 \times n_1$$

$$W = the\, recourse\, matrix\, of\,\, size\, m_2 \times n_2$$

$q(w) \in \mathrm{R}^{n_2}$ and $h(w) \in \mathrm{R}^{m_2}$.

Stochastic programming problems can be solved by decomposing with Bender's or L-Shaped decomposition methods. But, we choose to solve the problem by programming with $MATLAB$ and $Mathematica$ rather than using these techniques.

5.2 PROBLEM DEFINITION: FORMULATING GOVERNMENT DEBT MANAGEMENT PROBLEM

The aim of the optimization problem is to minimize the expected average annual real cost of the borrowing policies. The borrowing instruments are short and long term YTL-denominated T-Bills and G-Bonds. The maturities vary from 3 months to 5 years. Every node represents a quarter. Cost of decisions, maturing debt and covering the net borrowing requirement are calculated at every node as well as the net borrowing requirement itself. Average annual real GDP is calculated at $t = 1$ and $t = 2$. The problem is optimized over the changes in portfolio positions subject to debt balance, market, borrowing requirement and non-negativity constraints and a default buyback limit. After the optimal borrowing policy is determined, the risk related to the optimal portfolio is calculated with $RiskMetrics^{TM}$ VaR-Historical Simulation approach.

[1]Birge, Louveaux

To make calculations simpler, we have made some assumptions. The assumptions that underly our problem are:

1. No transaction costs due to issue or buy-backs are considered in the model.
2. Derivatives and other debt management instruments are not included in the model.
3. Decision limits on the variables are used instead of price spreads to describe the illiquidity of some market instruments.
4. Trading in fractions of instruments is allowed. For example 0.75 T-Bonds or 1.99 T-Bills may exist in our portfolio.
5. The outstanding debt is bought at the market price in the leaf nodes, hence mark-to-market costs will be incurred.
6. There is no amortization plan that should be followed by the government.
7. Premiums/discounts which occur as a result of not trading at par are not considered as part of the net borrowing requirement. Instead, they are taken to be the costs that should be financed by the model.
8. If the gross borrowing requirement is positive (negative), it is allowed to buy-back (issue) debt.

5.3 INTRODUCTION OF VARIABLES

5.3.1 Definitions

The first coming notations[2] are those related to *nodes* which represent the center scenarios of the clusters:

N	A set of non-dummy variables
N_{Total}	A set of non-dummy variables

[2] see Grill and Östberg, 2003

S	Number of time stages, which is 2 in our model
s(n)	Time stage that node $n \in N$ belongs to, $s \in [0, S-1]$
N_n^+	A set of successors to node $n \in N$
n_{--}	Predecessor node to $n \in N$
N_n^{--}	A set of all predecessors to node $n \in N$
N_{Leaf}	A set of terminal nodes
$N_{NonLeaf}$	A set of non-leaf nodes
N_{Root}	The root node (today)
$N_{NonRoot}$	A set of non-root nodes
$N_{NonRoot-NonLeaf}$	A set of internal nodes
$N_{FirstLevel}$	A set of successors to the root node
t_s	Number of years from root node to time stage $s \in [0, S-1]$
p_n	The cumulative probability of reaching node $n \in N$

The parameters embraced in our model are

- instrument parameters
- price and cost parameters
- decision limit parameters

5.3.2 Instrument Parameters

A set of financial instruments, including short term and long term T-Bills and G-Bonds are used in our optimization model. The following are used to represent the instrument parameters:

I	A set of financial instruments
D={YTL}	Set of debt types
$I_d \subseteq I$	Set of financial instruments of debt type $d \in D$
$trans_{ijn}$	One unit of instrument $i \in I$ transforms into $trans_{ijn}$ units of instrument $j \in I$ between nodes $n_- \in N_{NonLeaf}$ and $n \in N_{NonRoot-NonLeaf}$
P_{ni}	Price of the i-th instrument at node $n \in N$

5.3.3 Price and Cost Parameters

Price and cost parameters are those related to net borrowing requirement, cost of variables and GDP. The parameters defined below are in YTL currency.

BR_n^{NBR}	Government net borrowing requirement at node $n \in N_{NonLeaf}$
$BR_n^{NBR,mat.}$	Debt stemming from covering the net borrowing requirement between nodes n_- and n
$BR_{ni}^{mat.}$	Amount maturing at node $n \in N_{NonRoot-NonLeaf}$ if one unit of instrument $i \in I$ was held at node $n_- \in N_{NonLeaf}$
$BR_{initial}^{mat.}$	Debt maturing at root node
C_{ni}^{debt}	The cost of debt from instrument $i \in I$ which matures at node $n \in N_{NonRoot}$. The cost is accumulated from previous nodes until node $n \in N_{Root}$
$C_n^{NBR,debt}$	Cost of covering the net borrowing requirement between nodes $n_- \in N_{NonLeaf}$ and $n \in N$
$C_{ni}^{dec.}$	Cost of decision for one unit of instrument $i \in I$ at node $n \in N_{NonLeaf}$ (issuance has a positive cost while cost of buy-back is negative)
$C_{initial}^{dec.}$	Not yet realized mark-to-market costs at root node
$\overline{GDP}_n^{nom.}$	Average nominal GDP from root node to node $n \in N$

5.3.4 Decision Limit Parameters

Decision limit parameters are set to control the debt balance and the market constraints. Debt balance is satisfied by equating the number of instruments at time $t+1$ to the number of instruments at time t plus the number of instruments issued at t, minus the number of instruments bought back at t for each instrument $i \in I$. Market constraints ensure that the interest rates will not be influenced by high demand or low supply by putting limits on the issuance and buybacks.

The parameters are :

$Limit_i^{dec.,k}$	k is either issue or buyback decision for instrument

	$i \in I$ for which this is the decision limit (in millions of YTL)
$Limit_d^{debt.,(l,u)}$	Lower and upper bounds for post-decision debt of type $d \in D$ (in millions of YTL)
$debt_i^{initial}$	Initial debt in instrument $i \in I$ (in millions of local currency)

5.4 OPTIMIZATION PROBLEM

The basic principle underlying any kind of financial management problem, including the debt management problem, is to buy and sell assets. In general, these assets can be consumption or investment assets. However in government debt management problem, we consider investment assets which are T-Bills and G-Bonds.

We now define the state and decision variables, state transition functions, objective and constraints in our model.

5.4.1 Variables

These are state and decision variables which are used in the optimization problem:

x_{ni}	Millions of YTL debt in instrument $i \in I$ at node $n \in N_{NonLeaf}$
u_{ni}	Millions of YTL debt change in instrument $i \in I$ at node $n \in N_{NonLeaf}$ as a result of issue(+) or buybacks(-)

5.4.2 State Transition Functions

State transition functions relate the variables of the nodes $n_- \in N_{NonLeaf}$ to node $n \in N$.

Total cost of debt from root node to any node $n \in N$ is $C_n^{tot.,debt}$:

$$C_n^{tot.,debt} = \begin{cases} 0, & n \in N_{Root}; \\ C_{n_-}^{tot.,debt} + \sum_{i \in I} \frac{(x_{n_-,i} + u_{n_-,i})}{P_{n_-,i}} C_{ni}^{debt} + C_n^{NBR,debt}, & n \in N_{NonRoot}. \end{cases}$$

Total cost of decisions from root node to node $n \in N$ is $C_N^{tot.,dec.}$:

$$C_n^{tot.,dec.} = \begin{cases} C_{initial}^{dec.} + \sum_{i \in I}(u_{ni}C_{ni}^{dec.}), & n \in N_{Root}; \\ C_{n_-}^{tot.,dec.} + \sum u_{ni}C_{ni}^{dec.}, & n \in N_{NonRoot-NonLeaf}; \\ C_{n_-}^{tot.,dec.}, & n \in N_{Leaf}. \end{cases}$$

The next transition function is $\overline{C_n^{ann.}}$ which is the average annual nominal cost from the root node to node $n \in N_{FirstLevel} \cup N_{Leaf}$

$$\overline{C_n^{ann.}} = \frac{(C_n^{tot.,debt} + C_n^{tot.,dec.})/(t_{s(n)})}{\overline{GDP}_n^{nom.}}$$

and is simply the ratio of nominal costs to nominal GDP.

There are two transition functions on the borrowing requirement. $BR_n^{tot.,mat.}$ is the borrowing requirement stemming from maturing debt at node $n \in N_{NonLeaf}$:

$$BR_n^{tot.,mat.} = \begin{cases} BR_{initial}^{mat.}, & n \in N_{Root}; \\ \sum_{i \in I}(x_{n_-,i} + u_{n_-,i})BR_{ni}^{mat.}, & n \in N_{NonRoot-NonLeaf}. \end{cases}$$

Our second transition function on borrowing requirement is $BR_n^{tot.}$ which is the total borrowing requirement at node $n \in N$:

$$BR_n^{tot.} = BR_n^{NBR} + BR_n^{tot.,mat.} + BR_n^{NBR,mat.}$$

5.4.3 Objective

It is common to use some utility function by maximizing which a trade-off between risk and return is obtained. But, the aim of the government is not to maximize the return on debt instruments. Rather, it is to minimize the cost with due regard to an acceptable level of risk. Hence, our optimization problem will minimize the expected average annual real cost from root node to $n \in N_{FirstLevel} \cup N_{Leaf}$ where the instrument set is $I = \{3m, 6m, 12m, 18m, 3y, 5y\}$ T-Bills and G-Bonds.

If E$[\overline{C_n^{ann.}}]$ is the expected value of average annual real cost at node $n \in$

$N_{FirstLevel} \cup N_{Leaf}$, then

$$\mathrm{E}[\,\overline{C_n^{ann.}}\,] = \sum_{n \in S} p_n \overline{C_n^{ann.}}$$

where $S = N_{FirstLevel} \cup N_{Leaf}$

Thus, our objective is to minimize expected average annual real cost with respect to changes in portfolio positions:

$$\min_{u_{ni}} \mathrm{E}[\overline{C_n^{ann.}}]$$

5.4.4 Constraints

The objective function is minimized with respect to the changes in portfolio positions of instruments while introducing a couple of constraints and decision limits to the problem. These constraints are debt balance, market, borrowing requirement and non-negativity constraints. Debt balance constraint helps to keep in balance the number of instruments at each stage. Severe fluctuations in interest rates are prevented by imposing a market constraint. Non-negativity constraint guarantees that there is no investing. Borrowing requirement constraint is placed to meet the borrowing requirement targets of the government. Decision limits, on the other hand, are put to ensure that the issue or buybacks will not go beyond these limits. After the optimization problem is solved under the given constraints, the risk inherent in the optimal strategy is calculated with a *Value-at-Risk* (VaR) approach.

Constraints of the problem are defined to be:

Debt balance constraint

- $x_{nj} = \begin{cases} debt_j^{initial}, & n \in N_{Root}; \\ \sum_{i \in I} (x_{n_{-},i} + u_{n_{-},i}) trans_{ijn}, & n \in N_{NonRoot}, \forall j \in I. \end{cases}$

Market costraint

- $u_{ni}.P_{ni} \leq Limit_i^{dec.,issue} \quad \forall n \in N_{NonLeaf}, \forall i \in I$

  $u_{ni}.P_{ni} \geq -Limit_i^{dec.,buyback} \; \forall n \in N_{NonLeaf}, \quad \forall i \in I$

Default buyback limit

- $x_{ni} + u_{ni} \geq 0 \quad \forall n \in N_{NonLeaf} \quad \forall i \in I$

Borrowing requirement constraint

- $\sum_{i \in I} u_{ni} P_{ni} = BR_n^{tot.} \; \forall n \in N_{NonLeaf}$

Non-negativity constraint

- $x_{ni} \geq 0 \; \forall n \in N_{NonRoot-NonLeaf}$ and $\forall i \in I$

The first constraint (the debt balance constraint) ensures that the number of instruments at time $t+1$ are equal to the number of instruments at time t plus the number of instruments issued minus the number of instruments bought back.

Market constraint guarantees that the extreme demand or supply will not be allowed. Hence, the interest rates will not be affected from Treasury auction rates.

Borrowing requirement constraint is taken into consideration in order to stabilize the government borrowing requirement which must be met in each decision node.

Negative debt is in fact investing and investing is not allowed in our optimization model. So we impose a non-negativity constraint.

As we get the result of the problem with the given objective and constraints, we then measure the risk inherent in the optimal portfolio strategy with a Value-at-Risk approach. In order to prevent non-convexity of the optimization problem, VaR is not integrated into the problem as a constraint or part of the objective function. Instead, first we find the optimal solution of the optimization problem and then calculate the VaR of the optimal portfolio by $RiskMetrics^{TM}$ VaR- Historical Simulation approach. Thus, our model uses VaR as a measure of risk which is calculated using the VaR Historical-Simulation approach of $RiskMetrics^{TM}$.[3]

5.4.5 Government Debt Optimization Problem For Turkey

After defining the objective and constraints of the government debt optimization problem for Turkey, we do need to consider the initial debt amounts in current portfolio. Since the end of our data is 2004Q4, we use the not-yet-matured total value of each debt type in 2004 as our initial values. Our portfolio contains 3 and 6 month T-Bills as short rate instruments. Long rate instruments, on the other hand, are 12 and 18 months T-Bills and G-Bonds, 3 and 5 years G-Bonds. Their initial values are

1. $debt_{3m}^{initial} = 14,527,885$ YTL

2. $debt_{6m}^{initial} = 5,617,697$ YTL

3. $debt_{12m}^{initial} = 18,374,608$ YTL

4. $debt_{18m}^{initial} = 48,584,970$ YTL

5. $debt_{3y}^{initial} = 1,250,345$ YTL

6. $debt_{5y}^{initial} = 4,109,642$ YTL

One other parameter that should be computed initially is $C_{initial}^{dec.}$. It is the sum of not-yet-realized mark-to-market cost of each instrument in the portfolio and is computed as follows: The costs related to all instruments maturing in 2004Q4

[3]Details related to the calculation of Historical Simulation VaR approach can be found in Appendix.

are calculated. Then, the costs of not-yet-matured instruments are added to the previous costs as if they have matured. Hence we find the following values:

1. $C^{dec.}_{initial,3m} = 410,342\,YTL$

2. $C^{dec.}_{initial,6m} = 1,259,063\,YTL$

3. $C^{dec.}_{initial,12m} = 4,015,140\,YTL$

4. $C^{dec.}_{initial,18m} = 4,864,414\,YTL$

5. $C^{dec.}_{initial,3y} = 358,861\,YTL$

6. $C^{dec.}_{initial,5y} = 0\,YTL$

The not-yet-realized cost of 5-year instruments is 0 since there is no 5-year T-Bills or G-Bonds which matured in the last quarter of 2004 or which will mature sometime in 2005 or later. Then, the total cost of decision at root node is

$$C^{dec.}_{initial} = \sum_{i \in I} C^{dec.}_{initial,i}$$

$$C^{dec.}_{initial} = 10,907,820\,YTL$$

There are two more unknowns in our optimization problem. They are the limits on issue and buyback decisions. We assume that Treasury will repeat the past more or less and not issue more than the maximum amount in the last twenty years. Hence we take the decision limit on issues to be the maximum value of total income from Treasury auctions between 30.05.1985 and 22.11.2004:

$$Limit^{dec.,issue}_{i} = 5,826,151,890 \;\; YTL$$

Although buyback auctions are very rare in Turkey, buybacks have taken place through switching auctions in 2003 and 2004. As the lower bound on buyback decisions, we use the minimum of the nominal amount bought back through switching auctions.[4] Hence

$$Limit_i^{dec.,buyback} = 2,899,092,065\ YTL$$

As the objective, constraints, decision variables and risk measure of the problem are delineated, government debt optimization problem for Turkey can be written as follows:

[4]Data can be found in www.tcmb.gov.tr

$\min_{u_{ni}} \mathrm{E}[\overline{C_n^{ann.}}]$

subject to

- $x_{nj} = \begin{cases} debt_j^{initial}, & n \in N_{Root}; \\ \sum_{i \in I}(x_{n_-,i} + u_{n_-,i}) trans_{ijn}, & n \in N_{NonRoot}, \forall j \in I. \end{cases}$

- $u_{ni}.P_{ni} \leq 5,826,151,890 \quad \forall n \in N_{NonLeaf}, \forall i \in I$

  $u_{ni}.P_{ni} \geq -2,899,092,065 \; \forall n \in N_{NonLeaf}, \quad \forall i \in I$

- $\sum_{i \in I} u_{ni} P_{ni} = BR_n^{tot.} \; \forall n \in N_{NonLeaf}$
- $x_{ni} + u_{ni} \geq 0 \quad \forall n \in N_{NonLeaf} \quad \forall i \in I$

- $x_{ni} \geq 0 \; \forall n \in N_{NonRoot-NonLeaf}$ and $\forall i \in I$

where

1. $debt_{3m}^{initial} = 14,527,885$ YTL

2. $debt_{6m}^{initial} = 5,617,697$ YTL

3. $debt_{12m}^{initial} = 18,374,608$ YTL

4. $debt_{18m}^{initial} = 48,584,970$ YTL

5. $debt_{3y}^{initial} = 1,250,345$ YTL

6. $debt_{5y}^{initial} = 4,109,642$ YTL

5.5 PRICE AND CASH FLOW CALCULATIONS

The formula for the price of instruments has been given in the Appendix. The price and cost calculations at nodes are nothing but a matter of computation since the scenario tree has been generated and the price formulas are available. After having calculated the prices, cost is just the difference between the prices at nodes $n_{-} \in N_{NonLeaf}$ and $n \in N$. For example, if the price of the instrument issued at node $n_{-} \in N_{NonLeaf}$ is 93 YTL and is 90 YTL at node $n \in N$, then the cost is 3 YTL per instrument. This is because you receive 90 YTL for the instrument which you have received 93 YTL before. The idea is similar for buyback operations.

5.5.1 Calculations at Nodes

Our scenario tree consists of two stages $t = 0, 1, 2$. Today is $t = 0$ and $t = 1$ is one year from today. Unlike $t = 1$, $t = 2$ is 10 years from now. We have chosen a ten years horizon because Turkish economy is very fragile and has an economic cycle of nearly ten years.[5] Hence, it is not meaningful to use a longer horizon whose estimates are suspicious in how they really reflect the future.

Every node on the scenario tree represents a quarter. Decisions on issues and buybacks are made at every node except the decision node, which is $t = 1$. Then mark-to-market costs due to this issue and buyback decisions occur at every node except the decision node $t = 1$. At $t = 1$, the outstanding result is integrated into the optimization problem as an input. Furthermore $t = 2$ is the end of our horizon, thus all outstanding debt is bought back at the market price which again results in mark-to-market costs.[6]

[5] Ertuğrul and Selçuk, "A Brief Account of the Turkish Economy, 1980-2000"

[6] We do not take into account the costs due to coupon payments and FX rate movements since the coupon paying and FX-denominated instruments are not included in our model. But, the procedure is the same for them, too. Hence, the model can simply be extended to a problem where FX-denominated and coupon paying instruments are used as a means of debt management problem.

Every node on the scenario tree has two components, a destination node and a departure node. The departure node is the node on the scenario tree where you came from. The destination node, on the other hand, is the node on the scenario tree that you are going to and at the same time represents $t = 1$.

Since decisions are made at every $n \in N_{NonLeaf}$, mark-to-market costs due to price movements occur at departure nodes. If the debt matures before the destination node, it is refinanced by an automatic refinancing algorithm. This algorithm is based on the idea that the instrument is refinanced by an instrument whose maturity is closest to the maturing one. But, there is one point that should be carefully dealt with. Since we do not use one single instrument as a long term or short term instrument, then how are we going to determine which instrument to choose? The idea is simple: If the maturing debt is a short term debt, then it is going to be refinanced by a short term instrument. But, 3 months and 6 months T-Bills both exist as short term instruments in our problem. To get over this problem, each instrument is attained 0.5 probability and the refinancing is done arbitrarily between them. The same logic applies for long term instruments as well. If the maturing debt is a long term debt, then it is going to be refinanced by a long term instrument. Long term instruments are 12 months, 18 months, 3 years and 5 years T-Bills and G-Bonds. The probabilities of each financing strategy are calculated as the ratio of debt in each instrument to debt in all long term instruments. With this approach we assume that the Treasury will follow its long term debt strategy in the past. Then the probabilities of the long term instruments are

- $p_{12m} = 0,50$
- $p_{18m} = 0,30$
- $p_{3y} = 0,10$
- $p_{5y} = 0,10$

If the destination node is a leaf node, $n \in N_{Leaf}$, then no automatic refinancing is done. Instead, all outstanding debt is bought back at the market price which again results in mark-to-market costs.

5.5.2 Covering of Net Borrowing Requirement in Between Nodes

Net borrowing requirement at node $n \in N$ is taken to be the difference of debt between nodes $n_- \in N_{NonLeaf}$ and $n \in N$. The change in total debt at any node $n_- \in N_{NonLeaf}$ due to issue or buyback decisions and costs resulting from these decisions and the refinancing algorithm are considered as the net borrowing requirement at node $n \in N$.

5.6 CONCLUSION

The basic principle of any optimization problem is the maximization of a utility function or minimization of a cost or loss function. Since Treasury is responsible for the minimization of cost of debt rather than maximization of return on debt, we have chosen to minimize expected average annual real cost from root node to node $n \in N_{FirstLevel} \cup N_{Leaf}$ where the instrument set is $I = \{3m, 6m, 12m, 18m, 3y, 5y\}$ T-Bills and G-Bonds. The objective is minimized with respect to changes in portfolio positions while taking into account a couple of constraints. These are debt balance, market, default buyback, borrowing requirement and non-negativity constraints. As the optimal solution of the problem is found, the inbuilt risk of the portfolio is calculated with $RiskMetrics^{TM}$ Historical Simulation VaR approach.

After the identification of the optimization problem, the coming chapter will give an evaluation of the results of the problem while presenting a comparison with Sweden whose study we have taken as an initiative.

CHAPTER 6

RESULTS

One of the aims of our study is to try to improve the debt management policies of Treasury by suggesting a new approach, namely debt management by stochastic optimization. We have modeled the macroeconomic variables that affect the debt management problem, generated scenarios to reflect possible future movements of these variables and then clustered these variables in order to use in the quantitative analysis of debt management. The optimization problem is solved by using the programming tools of *MATLAB* and *Mathematica*. The last thing that remains is to comment on the appropriateness of the results to Turkey.

In the preceding chapter, the results of the optimization problem for Turkey at each cluster at t=1 (year 1) and t=2 (year 10) will be given while evaluating the results of the optimization problems for Turkey and Sweden whose study we have taken as the first step.

6.1 RESULTS FOR TURKEY

At t=1, an optimal position change in each instrument and a corresponding VaR is calculated for 30 clusters meanwhile the minimum of them is taken to be the optimal solution at t=1. At t=2, all current debt is bought back because of which we can not talk about optimality of changes in portfolio positions. Instead, a minimum VaR value is to be calculated.

Tables below show the optimal changes in portfolio positions and the corresponding VaR values for each cluster at 95% confidence level and a quarterly time horizons at t=1.

Cluster No	u_{3m}	u_{6m}	u_{12m}	u_{18m}	u_{3y}	u_{5y}
1	-31205000	-33589000	-11298000	0	0	0
2	-23302000	-23239000	-11298000	0	0	0
3	-27883000	-28227000	-11298000	0	0	0
4	-20530000	-20019000	-11298000	0	0	0
5	-17198000	-17060000	-11298000	0	0	0
6	-7584000	-7491000	-7037800	0	0	0
7	-11581000	-11672000	-10662000	0	0	0
8	-34716000	-39411000	-11298000	0	0	0
9	-32188000	-35739000	-11298000	0	0	0
10	-9114100	-8992500	-8697600	0	0	0
11	-32382000	-36169000	-11298000	0	0	0
12	-32268000	-35914000	-11298000	0	0	0
13	-24615000	-24628000	-11298000	0	0	0
14	-32305000	-35997000	-11298000	0	0	0
15	-13816000	-13951000	-10662000	0	0	0
16	-31206000	-33590000	-11298000	0	0	0
17	-13462000	-13364000	-11298000	0	0	0
18	-1865000	-1804100	-1772100	0	0	0
19	-33581000	-33151000	-11298000	0	0	0
20	-35325000	-43042000	-11298000	0	0	0
21	-5453000	-5424100	-5299800	0	0	0
22	-20619000	-20467000	-11298000	0	0	0
23	-31007000	-31946000	-11298000	0	0	0
24	-17325000	-17154000	-11298000	0	0	0
25	-31339000	-33877000	-11298000	0	0	0
26	-31077000	-33314000	-11298000	0	0	0
27	-32116000	-35579000	-11298000	0	0	0
28	-2385300	-2480000	-1586600	0	0	0
29	**-32269000**	**-35917000**	**-11298000**	**0**	**0**	**0**
30	-27845000	-28019000	-11298000	0	0	0

Cluster No	VaR
1	-4222600000000
2	-3406800000000
3	-4512200000000
4	-3794400000000
5	-3023900000000
6	-2073100000000
7	-263700000000
8	-4236100000000
9	-5892400000000
10	-1995500000000
11	-5705300000000
12	-8290200000000
13	-3536900000000
14	-6062300000000
15	-3131600000000
16	-5512900000000
17	-2799900000000
18	-1099200000000
19	-468100000000
20	-4976700000000
21	-1551800000000
22	-3147700000000
23	-3941600000000
24	-2622500000000
25	-4991800000000
26	-4547100000000
27	-6325600000000
28	-130900000000
29	**-8519500000000**
30	-3697800000000

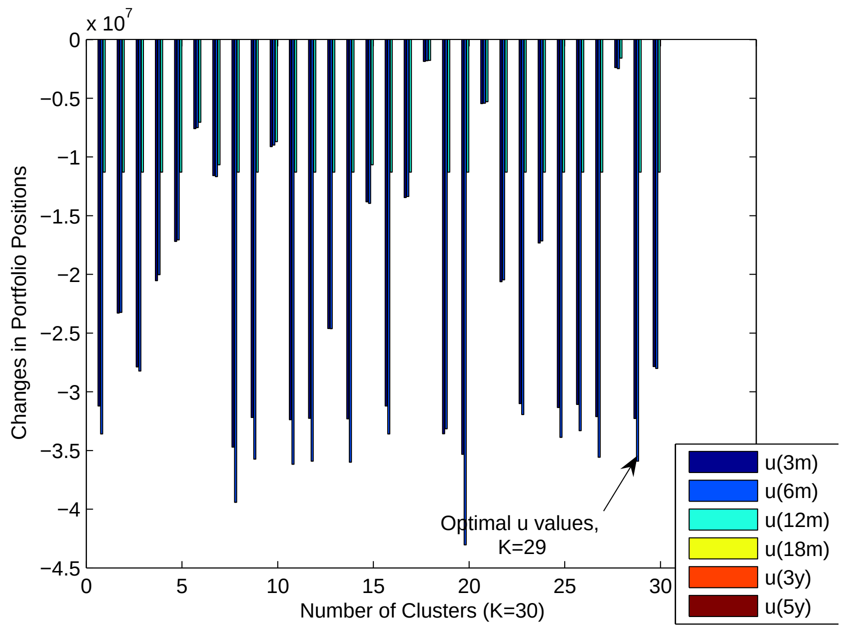

Figure 6.1: Optimal Values of Changes in Portfolio Positions at t=1

Although the results above are those of the optimization problem at each cluster, we take the cluster which creates the minimum expected cost among 30 clusters. The minimum VaR is taken to be the value corresponding to the one at cluster where the expected cost is minimum which is obtained at 29^{th} cluster when VaR is $8,519,500,000,000$ YTL. The optimal strategy consists of the changes in portfolio positions of $3m$, $6m$ and $12m$ instruments which is not surprising since the average maturity of domestic debt in Turkey in 2003 and 2004 was about 12 months indicating that although the maturity is tried to be kept as long as possible, it is approximately one year.

At t=2, no change in debt positions occur since all debt is bought back at the current market value. However, there is still market risk that the government should protect against. Hence, VaR is of the government debt portfolio at t=2 also do need to be calculated.

The optimization problem after t=1 proceeds with the 29^{th} cluster which gives the minimum expected cost and VaR value. Since there is no decision on

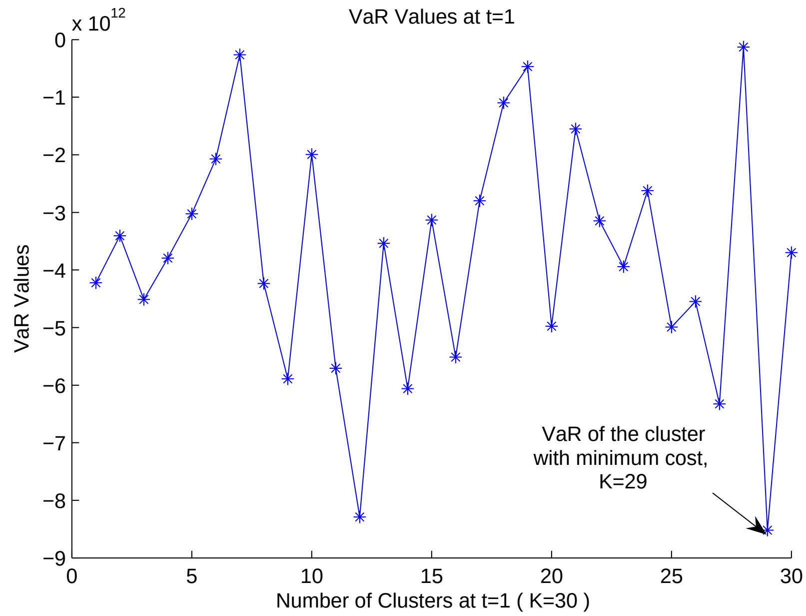

Figure 6.2: VaR Values at t=1

instruments at year 10, we just calculate the VaR which stems from the VaR of the clusters originated from 29^{th} cluster. The table below gives the VaR values of the 30 clusters originated from 29^{th} cluster calculated for a 95% confidence interval and quarterly time horizons.

Cluster No	VaR
1	-896,207,669,123,796
2	-1.211508154015e+015
3	-1.211508154015e+015
4	-1.11522457816996e+015
5	-1.211508154015e+015
6	-1.60660957177572e+015
7	-1.60660957177572e+015
8	-1.11156740499856e+015
9	-1.20490468561543e+015
10	-1.20490468561543e+015
11	-1.20490468561543e+015
12	-1.20490468561543e+015
13	-1.61885487559389e+015
14	-751,954,423,171,818
15	-1.61885487559389e+015
16	-1.61885487559389e+015
17	-1.61885487559389e+015
18	-1.61885487559389e+015
19	-1.11009541652658e+015
20	-1.06516762921777e+015
21	-1.08480455005267e+015
22	-1.61885487559389e+015
23	-1.45840647927873e+015
24	-1.211508154015e+015
25	-1.02443252008972e+015
26	-854536214548876
27	-1.08480455005267e+015
28	-1.02443252008972e+015
29	-848,545,248,653,893
30	**-602,417,623,217,780**

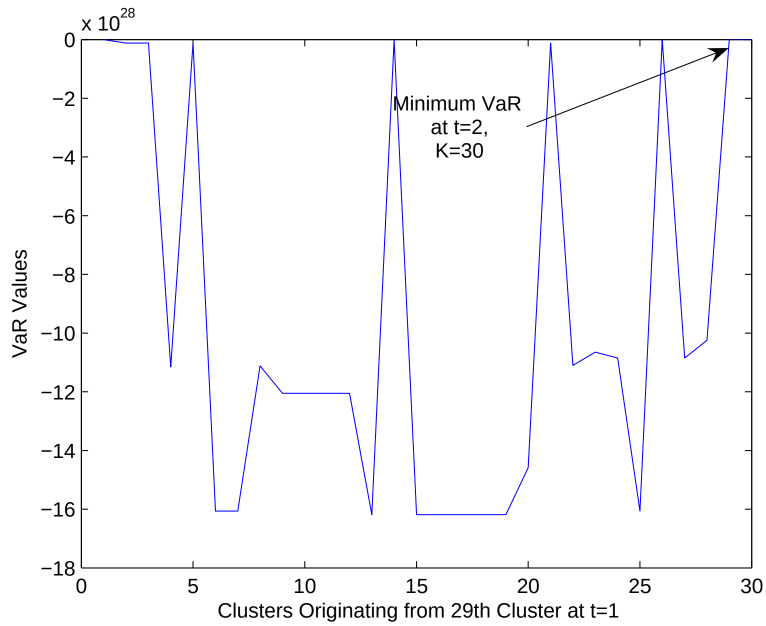

Figure 6.3: VaR Values at t=2

1

As the Table shows, VaR values at 2015 have grown drastically compared to VaR values at 2006 which is due to the noticeable increase in net borrowing requirement.[2] This result is predicted in the sense that Turkey faces crisis nearly every ten years and huge amounts of money need to be kept in order to protect against. This enormous amounts reflect the incremental shifts in the net borrowing requirement and the effects of the existing or approaching crises in the near future.

[1]"e" stands for exponential.

[2]This great shift in net borrowing requirement in the next 10 years is a result of non-stationarity of the data. Thus excluding the data during the crisis to remove this non-stationarity will discard unit roots and lead to less VaR values.

6.2 CONCLUSION

The aim of the study was to optimize the expected average annual real cost of borrowing strategies of the Treasury under the constraints of changes in portfolio positions which we think is met as the results of problem are obtained. Although Treasury borrows in domestic and foreign currencies, it prefers YTL-denominated instruments to FX-denominated instruments. As a result of lack of data on FX-denominated and CPI-indexed instruments, our model contains only YTL-denominated T-Bills and G-Bonds. Their maturities vary from 3 months to 5 years. The average maturity of domestic borrowing in 2004 was 14,7 months while it was 11,5 months in 2003. Hence we expect our results to reflect a similar maturity profile as the most recent ones. One more thing that needs to pay attention is that the net borrowing requirement has an increasing trend since 2000. Thus, it is not an unexpected situation that it will reach huge amounts by 2015. After the optimal borrowing policy is determined, the risk related to the optimal portfolio is calculated with $RiskMetrics^{TM}$ VaR-Historical Simulation approach. The dramatical increase in the net borrowing requirement will cause an enormous VaR value. Thus, we can say that our model reflects the borrowing behaviors of Treasury more or less although we did not take the complete actions of Treasury.

We should also point out that our results are very different from those obtained by Swedish National Debt Management Office whose work was a starting point for this study. Here are a couple of basic reasons. First of all, the model assumptions and variables are very different from each other. In contrary to the highly volatile economy of Turkey, Sweden has a very stable economy. Hence, variables of their model neither exhibit unit roots nor heteroscedasticity which result in less risk. Secondly, their borrowing policies depend not only on fixed rate domestic instruments, but also external borrowing instruments as well as CPI-indexed bonds. Hence, it is foreseen that their portfolio is more diversified and optimal strategy will include instruments which ours do not. Thirdly, they incorporate self-derived risk coefficients in the optimization problem and use scenario and time series risks to measure risk. Unlike Sweden, Treasury does not have a well developed risk management strategy based on works for years. It has based the public debt management problem on a risk management framework since 2004 which is still improving. That is why we preferred to use a more widely used and easily applicable risk management approach, namely Value-at-Risk. Finally, we did not take into account the durations to make our calculations simpler. Hence, it is straightforward to notice that the results of the optimization problems for Turkey and Sweden are dissimilar.

Final chapter is the conclusion part of this thesis and also contains the work to be done in the near future.

Chapter 7

CONCLUSION

In this study, a stochastic simulation framework similar to the one used by Swedish National Debt Management Office, is used to optimize the debt management problem of the Turkish Government under the constraints of changes in portfolio positions. A Historical-Simulation VaR approach is used to calculate the market risk of the debt portfolio. Although the debt portfolio is comprised of external and domestic borrowing instruments, FX-denominated and inflation-linked instruments are not included in the model due to the lack of data. Simulation horizon is ten years and covers 2005-2015 period. The macroeconomic variables in the optimization problem are short and long term interest rates, real GDP growth and net borrowing requirement. Short term interest rates, long term interest rates and net borrowing requirement are modeled with first order autoregressive integrated moving average processes, ARIMA(1,1,1), and the heteroscedasticity inherent in the variables is modeled with generalized autoregressive conditionally heteroscedastic, GARCH(1,1), processes whereas a first order regime switching autoregressive process, AR(1), is used to model real GDP growth. 25,000 scenarios were generated depending on the econometric modeling of the variables to represent possible future movements of the variables. These scenarios were then transformed into a form suitable for quantitative applications by clustering. K-Means clustering approach was used to cluster the scenarios. Increasing the number of clusters without ruining the problem by adding dummy scenarios lead us to choose $K = 30$ and the variables were clustered into 30 groups at each stage. Costs and risks of each cluster were calculated. Value-at-Risk is used as the risk measure. VaR is the amount that an investor can lose with a predetermined probability over a given time horizon. We used a 95% confidence level and quarterly time horizons to calculate VaR. The optimal changes in positions of instruments were found to be the one which cost least to the government. Then the VaR values corresponding to the optimal strategies at each cluster were calculated and the minimum of these was chosen to be the optimal risk allocation corresponding to the optimal changes in portfolio positions at $t = 1$. The optimal strategy and risk profile is in 29^{th} cluster at $t = 1$. Since there is no debt allocation at $t = 2$, all debt is bought back at the ongoing market price and an optimal VaR was calculated from the clusters generated from the 29^{th} cluster at $t = 1$. The VaR values at $t = 2$ have shown a radical increase due to the drastic shifts in the net borrowing requirement. Excluding the net borrowing requirement data during the crisis may expel the unit roots in the net borrowing requirement and lead to solve the problem of great VaRs.

Debt Management Offices in well developed countries like Sweden and New Zealand work merely on the government debt management problem. They try to model the macroeconomic variables that affect the debt management problem while introducing self-derived or efficiency proved risk measures. On contrary to these, Turkey neither has a debt management office nor any model that it bases its debt management problem. Instead, Treasury is responsible for managing debt while keeping the risk in balance. Hence, we hope that our study will be an initiative to the debt management policies of Treasury. Besides from being the first study of this type in Turkey, it suggests a method to manage government debt management problem. Although inflation and interest rates in Turkey have been modeled in similar fashions before, this is going to be the first study which incorporates econometric modeling to stochastic optimization approach. Thus, we hope that the contents and results of our study which we have made a great effort on, will be a mile stone on the studies on Turkish economy.

7.1 FUTURE WORK

Throughout the study, we have made a few assumptions which made our calculations simpler. Removing or improving the assumptions below, the study will cover more in the future.

Our first assumption was that Turkey will repeat more or less the same economic cycle in the last twenty years. Depending on this, we did not filter the data during the crisis which result in the existence of unit roots in the macroeconomic variables and increase in costs and VaR values. Hence, filtering and discarding the data during the crisis will result in strategies with less cost and risk. But this is prone to questions like "How sure are we that the country will not face crisis in the near future?" Thus, besides from filtering, alternative models can be chosen to describe the variables before and after crisis.

We have used K-Means clustering approach because of its applicability to very large categorical data sets without assessing how well other methods perform. Optimizing different clustering approaches to choose the best suiting one will give more efficient and reliable results.

Although we have modeled exchange rates and consumer price index before, we did not include FX-denominated and CPI-linked instruments in our model. Despite the few number of data on these instruments which prevents us from using econometric models, they can be incorporated into the optimization problem if their behaviors can be explained by trustworthy methods other than econometric modeling.

Chapter 8

APPENDIX

Throughout the chapter, basics of stochastic processes which are useful concepts in macroeconomic simulation, the algorithm used for clustering and VaR risk measure will be given.

8.1 Basic Definitions

8.1.1 Autoregressive Processes

An *autoregressive process* of order p, **AR(p)** is a process where the current value of the variable, Y_t depends on the previous values of the variable plus an error term.[1] Then AR(p) process is defined as

$$Y_t = c + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \ldots + \phi_p Y_{t-p} + \varepsilon_t$$

where $\phi_1, \phi_2, \ldots, \phi_p$ are constants, c the autoregressive constant and ε_t is the error term which is distributed $N(0, \sigma^2)$.

The most general AR(p) process is a *first-order autoregression*, denoted AR(1) which is

$$Y_t = c + \phi Y_{t-1} + \varepsilon_t$$

[1] Chris Brooks, Introductory Econometrics for Finance

8.1.2 Autoregressive Moving Average Processes

An *autoregressive moving average process* of order p, q, abbreviated (**ARMA(p,q)**) is the combination of an AR (p) process and a moving average process of order q (**MA(q)**).

A MA(p) process is nothing but a linear combination of current and previous values of white noise processes. Then Y_t depends on the current and previous values of the white noise disturbance terms: [2]

$$Y_t = \mu + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \ldots + \theta_p \varepsilon_{t-q} + \varepsilon_t$$

where ε_t is $IID(0, \sigma^2)$.

In an ARMA(p,q) process, the current value of the variable depends on the previous values of the variable plus a linear combination of the white noise disturbance terms:

$$Y_t = \mu + \sum_{i=1}^{p} \phi_i Y_{t-i} + \sum_{j=1}^{q} \theta_j \varepsilon_{t-j} + \varepsilon_t$$

where ε_t is $N(0, \sigma^2)$

8.1.3 Autoregressive Integrated Moving Average Processes

An *autoregressive integrated moving average* process of order (p,d,q), (ARIMA(p,d,q)) is the same as an ARMA(p,q) except for the point that the data reveals non-stationarity which means that the data contains unit root(s). The integration factor in ARIMA(p,d,q) indicates that the variable is differenced d-times to get rid of the unit root(s).

If Y_t is the variable to be modeled and Δ is the difference operator, then

[2] Chris Brooks, Introductory Econometrics for Finance

$$
\begin{aligned}
\Delta Y_t &= Y_t - Y_{t-1}, \\
\Delta^2 Y_t &= \Delta(\Delta Y_t), \\
&\vdots \\
\Delta^d Y_t &= \underbrace{\Delta(\Delta(\Delta \ldots (\Delta Y_t)))}_{d-times}
\end{aligned}
$$

An ARIMA(p,d,q) process is nothing but an ARMA(p,q) process applied to $\Delta^d Y_t$.

Autoregressive and moving average processes are *homoscedastic* which means that the variance of the error term does not change depending on time. However, most of the financial data, especially those belonging to underdeveloped or emerging countries, exhibit *heteroscedastic effects* which means that the conditional variance of the disturbance terms change with time. The changing conditional variance of the error term captures the effects of the shocks to the variable.

8.1.4 Autoregressive Conditionally Heteroscedastic Processes

Let us first define an *autoregressive conditionally heteroscedastic* process of order q (ARCH(q)):

$$h_t^2 = h_0 + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2$$

where ε_t is $IID(0, h_t^2)$

8.1.5 Generalized Autoregressive Conditionally Heteroscedastic Processes

ARCH model has some limitations which make it difficult to apply in some cases. First, as the number of lags of the squared errors increase, it becomes difficult to capture all of them. Second, adding more parameters to the model makes it more complicated and the non-negativity constraint of the ARCH model which tells that $h_0 \geq 0$, $\alpha_i \geq 0$ in might be violated. Bollerslev and Taylor (1986) developed **GARCH** (*Generalized ARCH*) models which overcome some of the limitations of ARCH models. GARCH models allow the conditional variance of the error term to depend both on its previous lags and the error term itself. Following the AR model previously defined, GARCH(p,q) is

$$h_t^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{i=1}^{p} \beta_j h_{t-j}^2$$

where ε_t is $IID(0, h_t^2)$.

Even though they are thought to model many financial data, GARCH models have some disadvantages. First, there is a non-negativity condition on the regression parameters which can sometimes be violated. Second, GARCH models are assumed to give the same reaction to positive and negative shocks since the sign of the shock is lost by taking the squares of the lagged errors. Positive and negative shocks are supposed to affect the model at the same level.

To prevent this kind of discrepancies, *asymmetric GARCH models* are developed. **GJR** named after the authors Glosten, Jagannathan and Runkle (1993) and **Exponential GARCH** (EGARCH)developed by Nelson (1991) are two of the most popular asymmetric GARCH extensions.

8.1.6 The GJR Model

The conditional variance is modeled as in the general GARCH models except that there is an additional term that accounts for possible asymmetries. The conditional variance is given by

$$h_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \beta_1 h_{t-1}^2 + \eta \varepsilon_{t-1}^2 I_{t-1}$$

where

$$I_{t-1} = \begin{cases} 1, & \varepsilon_{t-1} < 0; \\ 0, & \text{otherwise.} \end{cases}$$

8.1.7 The EGARCH Model

The EGARCH model has several representations one of which is

$$\ln(h_t^2) = \kappa + \beta \ln(h_{t-1}^2) + \zeta \frac{\varepsilon_{t-1}}{\sqrt{h_{t-1}^2}} + \alpha[\frac{|h_{t-1}|}{\sqrt{h_{t-1}^2}} - \sqrt{\frac{2}{\pi}}] \tag{8.1.1}$$

EGARCH is superior to GARCH in that there is no need to impose non-negativity constraints since $\ln h_t^2$ is always positive. Second, the relationship between volatility and returns is modelled via $\zeta \frac{\varepsilon_{t-1}}{\sqrt{h_{t-1}^2}}$ where $\zeta < 0$ indicates a negative relationship.

8.1.8 Markov Chains

For a finite set of states indexed by $\mathcal{S} = \{1, 2, \ldots, N\}$ with each state having a transition probability $p_{ij} \geq 0$ that gives the probability of moving from state i to state j, we have

$$P\{s_t = j | s_{t-1} = i, s_{t-2} = k, \ldots\} = P\{s_t = j | s_{t-1} = i\} = p_{ij}$$

where s_t denotes the state at time t. Such a process is called a **Markov Chain**. The current state in a Markov Chain is affected only by the previous state.

Note that $p_{i1} + p_{i2} + \ldots + p_{iN} = 1$ for all $i \in \mathcal{S}$ and the transition probabilities are collected in a matrix P called the **transition matrix**.

$$P = \begin{bmatrix} p_{11} & p_{12} & \dots & p_{1N} \\ p_{21} & p_{22} & \dots & p_{2N} \\ \vdots & \vdots & \dots & \vdots \\ p_{N1} & p_{N2} & \dots & p_{NN} \end{bmatrix}$$

In a Markov Chain, the path that followed in the previous states has no influence on moving from one state to another.

8.1.9 Markov Chain Regime Switching Process

Dealing with models which follow the same time series process is easy but may not reflect the real time events all the time. Some variables follow different time series over different subsamples. **Markov Chain Regime Switching Processes** are used to cope with this kind of variables. A Markov Chain Regime Switching Process is as follows

$$Y_t = \begin{cases} c_A + \phi_A Y_{t-1} + \varepsilon_t, & s = A; \\ c_B + \phi_B Y_{t-1} + \varepsilon_t, & s = B. \end{cases}$$

where ϕ will take on a property in state A and another in state B, so the value of Y_t will change.

8.2 Derivation of the Centroid Function for K-Means Clustering Algorithm

When we are given the number of clusters, in other words K is known, finding the centroids of the clusters is an easy process. Clustering error is minimized after being defined as a function of the centroid.

Let K be the number of clusters, m_k is the centroid and f_k is the error function of the k^{th} cluster. Then

$$f_k(P_k) = \sum_{i \in P_k} \|i - m_k\|^2 = \sum_{i \in P_k} \|(i - m_k)^T (i - m_k)\|$$

$$= \sum_{i \in P_k} (i^T i - 2i^T m_k - m_k^T m_k) \ \forall k \in [0, K]$$

The clustering error is minimized

$$\frac{\partial f_k}{\partial m_k} = 0$$

$$\frac{\partial f_k}{\partial m_k} = -2 \sum_{i \in P-k} i^T + 2m_k.|P_k|$$

$$m_k = \frac{\sum_{i \in P_k} i}{|P_k|}$$

where $|P_k|$ is the number of elements in cluster k.

As it can be seen from the formula of the m_k, it is the sample mean of the k^{th} cluster.

8.2.1 Convergence of the Algorithm

If a point is moved from one cluster to another, this means that the latter clustering error is less than the former one. Let i be an element that is moved from cluster k to cluster l at j^{th} iteration. Then we can write

$$\|i - m_l^{(j)}\|^2 < \|i - m_k^{(j)}\|^2$$

If f_k is the clustering error function for the k-th cluster, then it is a strictly decreasing function since every value of the error function comes from a cluster where it is less than the error in the previous iteration:

$$f_k^j < f_k^{j-1}$$

This implies that given a finite number, N, of data points to be clustered and a predetermined number K, it takes finitely many iterations to cluster N into K clusters.

8.3 Fixed Income Securities

A **bond** is a long term contract under which the borrower makes a cash payment(*coupon*) at specific dates and a final cash payment(*principal*) at the end of the holding period to the holder. For example, a 100 dollar bond with 10 percent monthly payment issued for one year will pay the 10 dollars to the holder every month. At the end of one year, the holder will receive 110 dollars.

8.3.1 Characteristics of Bonds:

Par value is the face value of the bond stated by the borrower. **Maturity date** is the time that the par value of the bond is repaid to the holder of the bond.

8.3.2 Bond Valuation

The value of a bond today is the sum of present values of payments made by the borrower. Bonds can be divided into two groups according to their way of making payments:

- Bonds which make coupon payments and
- Bonds which do not pay any coupons,

Bonds paying no coupons only have one cash flow, principal, that needs to be discounted. If P is the par value, T is the maturity date and r_T is the coupon market rate of interest, then the value of the bond is

$$V = \frac{P}{(1 + r_T)^T}$$

If the bond is paying c percent coupon, then the value of the bond is

$$V = \sum_{t=1}^{T} \frac{cP}{(1 + r_t)^t} + \frac{P}{(1 + r_T)^T}$$

which we are not going to use in our model since Treasury has not issued coupon paying bonds enough to use in optimization model.

8.4 Risk Measures

The concept of portfolio optimization developed by Markowitz[3] has been under spotlight since its introduction. To optimize a portfolio, two points should be taken into consideration:

- modeling utility functions, risks and constraints
- efficiency of handling large number of instruments and scenarios

Markowitz used a quadratic programming approach to handle portfolio optimization problem. But, recent advances, both in mathematical programming and portfolio optimization theory have shown that linear programming performs better than quadratic programming in the case of portfolio optimization.

When it comes to risk, the most widely used approach was the one known as mean-variance approach. But, the tremendous developments in finance industry and the financial instruments made it crucial to use more detailed and easily applicable measures of risk. **Value-at-Risk** (**VaR**) takes stage at this point. Value-at-Risk is the measure of maximum potential loss of a portfolio of financial instruments with a given probability and a pre-determined horizon.[4] Although

[3] Portfolio Selection, 1952

[4] $RiskMetrics^{TM}$, 1996, Technical Document 4^{th} Edition

VaR answers the question 'how much can an investor lose with x% probability over a given time horizon?', it is not a *coherent* measure of risk.[5] It is not coherent in the sense that it lacks sub-additivity. VaR of a portfolio of two instruments can be *greater* than the sum of the VaRs' of the instruments individually, hence diversification does not work.[6] It should also be noted that VaR is undesirable in optimization problems with VaR objectives and non-convex constraints. Most of the time, it is difficult to find the optimum solution and there exists more than one local solution. Then the optimization algorithm stops when it finds the first local value which may not be the optimal one.

The Basel Accord[7] states that banks should preserve capital reserves that are multiples of VaR characterizing their portfolios in order to protect them from solvency. Thus, many financial institutions including banks use VaR as a measure of risk. Although it is rare, bankruptcy events occur as a result of non-protection against market risk. Since Turkey has a history of crises leading to extreme losses, we use VaR as our risk measure with 95% confidence interval and quarterly time horizons.

We use $RiskMetrics^{TM}$ Historical Simulation VaR approach. The historical simulation VaR approach is based on the idea that portfolio instrument series are transformed into percentage price return series. These return series are then used to *re-value* the portfolio for the maturing instruments. The portfolio re-valuation series is sorted and VaR is calculated from this series depending on the confidence interval given. If the confidence interval is 95%, the number of elements in the revaluation series is divided by 100 and multiplied by 5. The resulting element of the series is the VaR of the portfolio at that stage.

Depending on the VaR Historical-Simulation approach, we first create the series on short and long term interest rates containing the rates up to 2004Q4 and their simulated values until 2015. These simulated values are clustered at $t = 1$ and $t = 2$. 30 different interest rate series at $t = 1$ and 900 different series at $t = 2$ are obtained as a result of the clustering process. Then the returns on

[5] Artzner *et al.* 1999 "Coherent Measures of Risk"

[6] VaR is coherent only when it is based on the standard deviation of normal distributions.

[7] Basel Committee on Banking Supervision, International Convergence of Capital Management and Capital Standards, July 1998

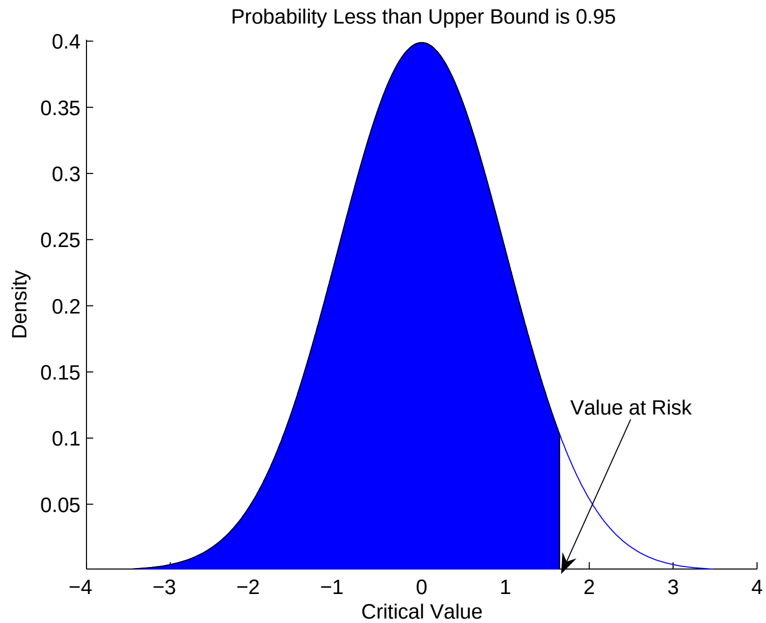

Figure 8.1: Value-at-Risk for 95% Confidence Level

interest rate instruments are calculated by the percentage changes in prices of the instruments:

$$Return_t = 100 * \frac{P_t - P_{t-1}}{P_{t-1}}$$

where P_t is the price of the instrument at time t.

Next is calculated and sorted the resultant change in portfolio value which is equal to returns on the instruments times amount of the instruments maturing at that stage. This is the heart of the historical simulation approach since the portfolio is re-valued for each quarter using the percentage price changes of the corresponding quarter. 30 series with 81 data (81 quarters between 1985Q3 and 2004Q4)[8] are sorted at $t = 1$. Then 5^{th} element[9] of the sorted series is taken to be the VaR of the portfolio at $t = 1$.[10]

[8]900 series with 121 data are sorted at $t = 2$

[9]$\frac{81}{100} * 5 = 4.05$.

[10]The same logic applies at $t = 2$ with 900 series sorted which contain 121 data.

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