KERR BLACK HOLES AND THEIR GENERALIZATIONS

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
THE MIDDLE EAST TECHNICAL UNIVERSITY

BY

HAKAN CEBECİ

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

IN

THE DEPARTMENT OF PHYSICS

OCTOBER 2003

Approval of the Graduate School of Natural and Applied Sciences.

Prof. Dr. Canan Özgen
Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Doctor of Philosophy.

Prof. Dr. Sinan Bilikmen
Head of Department

This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Doctor of Philosophy.

Prof. Dr. Tekin Dereli
Co-Supervisor

Assoc. Prof. Dr. Ayşe Karasu
Supervisor

Examining Committee Members

Prof. Dr. Tekin Dereli

Assoc. Prof. Dr. Ayşe Karasu

Prof. Dr. Atalay Karasu

Prof. Dr. Metin Gürses

Assist. Prof. Dr. Özgür Sarıoğlu

ABSTRACT

KERR BLACK HOLES AND THEIR GENERALIZATIONS

Cebeci, Hakan

Ph.D, Department of Physics

Supervisor: Assoc. Prof. Dr. Ayşe Karasu

Co-Supervisor: Prof. Dr. Tekin Dereli

October 2003, 115 pages.

The scalar tensor theory of gravitation is constructed in D dimensions in all possible geometries of spacetime. In Riemannian geometry, theory of gravitation involves a spacetime metric g with a torsion-free, metric compatible connection structure. If the geometry is non-Riemannian, then the gauge theory of gravitation can be constructed with a spacetime metric g and a connection structure with torsion. In non-Riemannian theory, connections may be metric compatible or non-metric compatible. It is shown that theory of gravitation which involves non-metric compatible connection and torsion, can be rewritten in terms of torsion-free theory. It is also shown that scalar tensor theory can be reformulated in Einstein frame by applying a conformal transformation. By adding an antisymmetric axion field, the axi-dilaton theory is studied in Riemannian and non-Riemannian geometries. Motion of massive test particles is examined in all these geometries. The static, spherically symmetric and stationary, Kerr-type axially symmetric solutions of the scalar tensor and axi-dilaton theories are presented. As an application, the geodesic elliptical orbits based on a torsion-free connection and the autoparallel orbits based on a connection with a torsion, are examined in Kerr Brans-Dicke geometry. Perihelion shift of the elliptical orbit is calculated in both cases and the results are compared.

Keywords: connection, metric-compatibility, torsion field, singularity, Kerr black holes, autoparallel orbits

ÖZ

KERR KARA DELİKLERİ VE GENELLEMELERİ

Cebeci, Hakan
Doktora , Fizik Bölümü
Tez Yöneticisi: Assoc. Prof. Dr. Ayşe Karasu
Ortak Tez Yöneticisi: Prof. Dr. Tekin Dereli

Ekim 2003, 115 sayfa.

Skaler tensör kütleçekim kuramı olası tüm uzay-zaman geometrilerinde D boyutta kuruldu. Riemann geometrisinde, kütleçekim kuramı, burulma alanı olmayan, metrik uyumlu bağlantı yapısına sahip uzay-zaman metriği içerir. Eğer geometri Riemann değilse, o halde kütle çekim ayar kuramı, g uzay-zaman metriği ve burulma alanına sahip bağlantı yapısıyla kurulabilir. Riemann olmayan geometride, bağlantı yapıları metrik uyumlu veya metrik uyumsuz olabilir. Metrik uyumsuz ve burulma alanlı bağlantı yapısına sahip bir kütleçekim kuramının, burulma alanına sahip olmayan kurama dayanarak yeniden yazılabileceği gösterildi. Ayrıca, bir konformal dönüşüm uygulanarak, skaler tensör kuramının Einstein referans sisteminde de matematiksel olarak ifade edilebileceği ispatlandı. Anti-simetrik bir aksiyon alanı eklenerek, Riemann ve Riemann olmayan geometrilerde aksi-dilaton kuramı çalışıldı. Bütün bu geometrilerde, kütleli test parçacıklarının hareketi incelendi. Skaler tensör ve aksi-dilaton kuramlarının statik, küresel simetrik ve durağan Kerr tipi, eksenel simetriye sahip çözümleri sunuldu. Bir uygulama olarak, Kerr Brans-Dicke geometrisinde, burulma alanına sahip olmayan bağlantı yapısına dayanan geodesik eliptik yörüngeler ve burulma alanlı bağlantı yapısına sahip eliptik otoparalel yörüngeler incelendi. Her iki durumda eliptik yorüngenin perihelyon kayması hesaplandı ve sonuçlar karşılaştırıldı.

Anahtar Sözcükler: bağlantı, metrik uyumluluk, burulma alanı, tekillik, Kerr kara delikleri, otoparalel yörüngeler

To my family

ACKNOWLEDGMENTS

I would like to express my deepest gratitudes and thanks to Prof. Dr. Tekin Dereli for his excellent supervision, guidance, continuous encouragement and support throughout this work. We have been working together under the supervision of Prof. Dr. Dereli for more than six years now. During that time, he has introduced me to the beautiful areas of theoretical physics by motivating perfectly. I also would like to express my sincere gratitudes and thank Assoc. Prof. Dr. Ayşe Karasu and Prof. Dr. Atalay Karasu for helping me personally in many situations and also for their friendliness. I am also grateful to Assoc. Prof. Dr. Ayşe Karasu and Assist. Prof. Dr. Özgür Sarıoğlu for their help in arranging a personal computer for my use from the department. I also would like to thank Assoc. Prof. Dr. Ayşe Karasu, Prof. Dr. Metin Gürses and Prof. Dr. Atalay Karasu for their enlighting comments during the thesis progress meetings. I want to send my thanks to Prof. Dr. Robin W. Tucker for his help and hospitality during my stay at the Physics Department of Lancaster University through Tübitak BDP-BAYG program fellowship provided by Prof. Dr. Tekin Dereli. I especially thank my close friends İsmail Turan and Nejat Tevfik Yılmaz for their help in latex, for useful physical and mathematical discussions and also for their morale support. I am also thankful to my dear friends İsmet Yurduşen, Barış Akaoğlu, Koray Karaca, Oben Sezer, Tahir Çolakoğlu, Halil Gamsızkan and Murat Taş for their morale support and for their help in computer-related problems. Finally, I am very grateful to my family. This thesis would not have been possible without their morale support, interest and continuous affection.

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

Gravitation is one of the fundamental forces in nature. Its description can be made in an elegant way geometrically. The mathematical or geometric formulation of classical gravitation in 4 or higher dimensions is based on spacetime manifold equipped with a Lorentzian structure and a connection on the bundle of linear orthonormal spacetime frames [1]. Physically, gravitation is a gauge theory such that connections are $so(n, 1)$ valued gauge fields in $D = n + 1$ dimensions. The Einstein formulation of gravitation is based on a spacetime structure with a metric compatible Levi-Civita (torsion-free) connection determined by the metric. Metric compatibility means vanishing of the covariant derivative of the metric tensor field along any vector X on the spacetime manifold. Einstein identified the gravitational force with the spacetime curvature associated with a metric g [2]. There exists more general connection structures of gravitation, which offer a framework to describe interaction of matter fields with gravitation geometrically [1]. These general structures may involve metric compatible connection fields with torsion as proposed by Cartan or non-metric compatible connections with a zero-torsion as suggested by Weyl [1], or there also exist non-metric compatible connection structures with non-vanishing torsion [3].

In 1961, Brans and Dicke [4] proposed a new model of gravitation by including a scalar field. Their starting point was the idea of Mach which states that the phenomenon of inertia should arise from accelerations with respect to the general mass distribution of the universe. Therefore, according to this idea, inertial masses of the various elementary particles cannot be fundamental constants, but rather they represent interaction of particles with some cosmic (scalar) field. However, the absolute scale of elementary particle masses (such as electrons) can be measured by measuring the gravitational accelerations $\frac{Gm}{r^2}$, [5] thus, equivalently, rather than varying the fundamental particle masses, the gravitational constant G should vary, by identifying G with the inverse of a scalar field ϕ, i.e $G \sim \phi^{-1}$. This new theory is also known as scalar tensor theory of gravitation. Brans and Dicke formulated the theory on a 4 dimensional spacetime manifold with a Levi-Civita (torsion-free) connection. It is shown in [6], that the scalar tensor theory of gravity can be formulated on a spacetime with a connection with a dynamical torsion that is proportional to the gradient of the scalar field. It is also shown that the scalar tensor theory with torsion can be rewritten in terms of the torsion-free theory. Such a reformulation does shift the Brans-Dicke (scalar field) coupling parameter ω. Interestingly, the motion of massive test particles in these geometries should be interpreted accordingly. It was originally assumed by Brans and Dicke that the histories of test particles to be Levi-Civita geodesics associated with the metric derived from the field equations. However, if the theory is formulated with a torsion, then this assumption should be changed and therefore a new interpretation should be made. In [2], it is shown that worldlines of test particles in a geometry with torsion are autoparallels of a connection with torsion. Therefore, according to whether the geometry is torsion-free or not, or even it is metric-compatible or not, the histories of particles are either Levi-Civita autoparallels (geodesics) or autoparallels of a non-Riemannian connection. Autoparalel worldline equations with a torsion and with a non-metric field (non-metricity) can be rewritten in terms of Levi-Civita worldlines, such that new terms can be interpreted as the force or acceleration terms due to non-metric fields and torsion. If the torsion is induced by a scalar field, then this new force can be interpreted as the force produced by the scalar field.

By adding an antisymmetric gauge field H interaction to the scalar tensor theory, axi-dilaton gravity theory is obtained. This gauge field can be derived from a gauge potential such that $H = dA$. Generally, the gauge field is represented by a $(p+2)$-form tensor field called the axion. If $p = 0$, then the theory becomes that of a scalar tensor interacting with a Maxwell field F. If $p = 1$, then it represents the bosonic part of some low energy action of strings, which are the extended objects in one spatial dimension in a spacetime manifold. The low energy limits of some string theories are known as the bosonic parts of supergravity actions. It is stated as a conjecture that all (super)string models belong to an eleven dimensional M-theory which accommodates some dualities. In a general setting, a $(p + 2)$-form axion field represents the interactions of a p-brane which are the extended objects in p spatial dimensions. Just as Maxwell potential couples to a particle, a $(p + 1)$-form gauge potential A can couple to p branes.

In the first part of this work, we construct the scalar tensor theory of gravitation in D dimensions in the metric-compatible and non-metric compatible space-time geometries mentioned above. The theory is reformulated in the so-called Einstein frame by applying a conformal transformation on the fields. Axi-dilaton theory is studied in Brans-Dicke frame in all possible geometric structures. The motion of free massive test particles is examined within a similar framework. In the second part, static, spherically symmetric and stationary, Kerr (rotating) type solutions of scalar tensor and axi-dilaton theories are presented. In the third part, as an application, geodesic elliptical orbit equations and autoparallel elliptical orbit equations (with torsion) are examined in Kerr Brans-Dicke geometry, which is the stationary, axially symmetric solutions of scalar tensor theory in 4 dimensions. Kerr Brans-Dicke geometry represents the external gravitational field of a spinning source, identified with its scalar charge, mass and angular momentum. Perihelion shift of the elliptical orbits is calculated in both cases and compared.

CHAPTER 2

SCALAR TENSOR THEORIES OF GRAVITATION

Scalar tensor theories of gravitation are considered to be a possible alternative to Einstein's theory of gravity. The scalar field couples to the gravitational field such that for large values of the coupling constant ω, the theory reduces to Einstein's theory of gravitation. Scalar tensor theories were first proposed by Brans and Dicke [4]. Hence, it is also called the Brans-Dicke theory in the literature. Geometrically, the theory was firstly formulated by a space-time manifold in which $so(3,1)$ valued connections were Levi-Civita (torsion-free) and metric compatible. It can be seen that, the theory can also be formulated in terms of a spacetime geometry in which connections are not Levi-Civita but metric compatible [6]. It can also be formulated in a spacetime in which connections are neither Levi-Civita nor metric compatible. In the latter case, interacting fields can be rescaled under a conformal group such that the corresponding action possesses a Weyl symmetry. This chapter is organised as follows. First, possible formulations of scalar tensor theories are developed in $D = n + 1$ dimensions. Next, axi-dilaton gravity theories are formulated within a similar framework. Finally, the motion of spinless massive test particles in such spacetimes are studied.

2.1 Scalar Tensor Theory Of Gravity In Metric-Compatible Spacetimes

In a spacetime geometry equipped with a metric-compatible connection structure, the scalar tensor theory of gravity in $D = n + 1$ dimensions, is described by the action density D-form,

$$\mathcal{L} = \frac{1}{2}\phi R^{ab} \wedge *(e_a \wedge e_b) - \frac{\omega}{2\phi} d\phi \wedge *d\phi. \tag{2.1}$$

Here the basic field variables are the co-frame 1-forms e^a, in terms of which the spacetime metric can be written as $g = \eta_{ab} e^a \otimes e^b$ where $\eta_{ab} = diag(-+++++\cdots)$. Hodge $*$ map is defined so that the oriented volume form is defined as $*1 = e^0 \wedge e^1 \wedge \cdots \wedge e^n$. ϕ is the massless scalar field usually called as dilaton in string theories. Physically, it describes the inverse of the locally varying gravitational coupling constant G ($\phi \sim G^{-1}$). Since spacetime is metric-compatible i.e $\nabla g = 0$, the connection 1-form fields satisfy $\omega^{ab} = -\omega^{ba}$ and the Cartan structure equations

$$de^a + \omega^a{}_b \wedge e^b = T^a \tag{2.2}$$

where T^a denotes the torsion 2-forms. The corresponding curvature 2-forms are obtained from,

$$R^{ab} = d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb}. \tag{2.3}$$

First, we construct the theory under the constraint that the connections are Levi-Civita, i.e. they satisfy the structure equation

$$de^a + \omega^a{}_b \wedge e^b = 0. \tag{2.4}$$

In that case, the constrained Lagrangian $\mathcal{L}'$ is obtained by adding the constraint term to $\mathcal{L}$:

$$\mathcal{L}^{'} = \frac{1}{2}\phi R^{ab} \wedge *(e_a \wedge e_b) - \frac{\omega}{2\phi} d\phi \wedge *d\phi + (de^a + \omega^a{}_b \wedge e^b) \wedge \lambda_a, \tag{2.5}$$

where λ_a are Lagrange multiplier $(n-1)$-forms. The field equations are obtained by making variations of $\mathcal{L}^{'}$ with respect to e^a, $\omega^a{}_b$, ϕ and λ_a. So the variation of the Lagrangian density $\mathcal{L}^{'}$ with respect to the fields e^a, $\omega^a{}_b$, ϕ and λ_a leads to

$$\begin{aligned}
\delta\mathcal{L}^{'} &= \delta e^c \wedge \{\frac{1}{2}\phi R^{ab} \wedge *(e_a \wedge e_b \wedge e_c) + \frac{\omega}{2\phi}(\iota_c d\phi \wedge *d\phi + d\phi \wedge \iota_c(*d\phi)) \\
&\quad + D\lambda_c\} + \delta\phi\{\frac{1}{2}R^{ab} \wedge *(e^a \wedge e^b) + \omega d(\frac{*d\phi}{\phi}) + \frac{\omega}{2\phi^2} d\phi \wedge *d\phi\} \\
&\quad + \delta\omega^a{}_b \wedge \{D(\frac{1}{2}\phi * (e_a \wedge e^b)) + \frac{1}{2}(e^b \wedge \lambda_a - e_a \wedge \lambda^b)\} \\
&\quad + (de^a + \omega^a{}_b \wedge e^b) \wedge \delta\lambda_a + mod(d),
\end{aligned}$$

where $mod(d)$ are closed forms. Since it is assumed that the fields vanish on the boundary of the spacetime manifold, the closed forms do not contribute to the field equations, i.e. they satisfy

$$\int_M d\Omega = \int_{\partial M} \Omega = 0.$$

Hence, the variational principle

$$\int_M \delta\mathcal{L}^{'} = 0$$

implies the following field equations:

$$\frac{1}{2}\phi^{(0)} R^{ab} \wedge *(e_a \wedge e_b \wedge e_c) = -\frac{\omega}{2\phi}(\iota_c d\phi \wedge *d\phi + d\phi \wedge \iota_c(*d\phi)) -^{(0)} D\lambda_c, \tag{2.6}$$

$$\frac{1}{2}\,^{(0)}R^{ab} \wedge *(e_a \wedge e_b) = -\omega d\left(\frac{*d\phi}{\phi}\right) - \frac{\omega}{2\phi^2} d\phi \wedge *d\phi, \tag{2.7}$$

$$ {}^{(0)}D\left(\frac{1}{2}\phi * (e^a \wedge e^b)\right) = \frac{1}{2}(e^a \wedge \lambda^b - e^b \wedge \lambda^a) \tag{2.8} $$

with the constraint equation $de^a +^{(0)} \omega^a{}_b \wedge e^b = 0$. Here ${}^{(0)}$ implies that the connections are torsion-free. Lagrange multiplier $(n-1)$-forms, λ_a, are calculated from equation (2.8). Since $T^a = 0$, the left hand side of equation (2.8) becomes

$$ {}^{(0)}D\left(\frac{\phi}{2} * (e^a \wedge e^b)\right) = \frac{d\phi}{2} \wedge *(e^a \wedge e^b). $$

Defining

$$ \Lambda^{ab} = d\phi \wedge *(e^a \wedge e^b) \tag{2.9} $$

equation (2.8) becomes,

$$ \Lambda^{ab} = e^a \wedge \lambda^b - e^b \wedge \lambda^a. \tag{2.10} $$

Taking the inner product of both sides of (2.10) with respect to the frame vector X_a, we obtain,

$$ \iota_a \Lambda^{ab} = \lambda^b + e^b \wedge \iota_a \lambda^a, \tag{2.11} $$

where we have used the identities,

$$ \iota_b\, e^a = \delta_b{}^a, \qquad \iota^b\, e^a = \eta^{ba}, $$

together with

$$ \iota_a e^a = (n+1) $$

which is the dimension of spacetime and

$$ e^a \wedge \iota_a \Omega = p\Omega $$

satisfied for any p-form Ω. If we further apply the inner product operator ι_b to (2.11), we obtain

$$ \iota_b \iota_a \Lambda^{ab} = 4\iota_a \lambda^a. $$

Therefore,

$$\begin{aligned}\lambda^a &= \iota_c\Lambda^{ca} - e^a \wedge \iota_b\lambda^b \\ &= \iota_c\Lambda^{ca} - e^a \wedge \frac{1}{4}\iota_c\iota_a\Lambda^{ac}.\end{aligned}$$

We calculate

$$\iota_a\Lambda^{ac} = \iota^c(*d\phi)$$

and use

$$\iota^c\iota_c(*d\phi) = 0$$

to obtain

$$\iota_c\iota_a\Lambda^{ac} = 0.$$

Hence,

$$\lambda^a = \iota^a(*d\phi). \tag{2.12}$$

So, the Einstein field equation (2.6) becomes

$$\frac{1}{2}\phi\,^{(0)}R^{ab}\wedge*(e_a\wedge e_b\wedge e_c) = -\frac{\omega}{2\phi}\{\iota_c d\phi\wedge*d\phi+d\phi\wedge\iota_c(*d\phi)\}-^{(0)}D(\iota_c(*d\phi)). \tag{2.13}$$

To obtain the scalar field equation, we consider the exterior multiplication of equation (2.13) by e^a and multiply equation (2.7) by $(n-1)\phi$ and then subtract two equations side by side. After some algebra, we obtain the equation satisfied by the scalar field:

$$\{n + (n-1)\omega\}d(*d\phi) = 0. \tag{2.14}$$

We conclude from this equation that

$$d(*d\phi) = 0$$

provided that $\omega \neq -\frac{n}{n-1}$.

Now we consider the following action density in $D = n+1$ dimensions in which the co-frames e^a and the connection 1-form fields $\omega^a{}_b$ are varied independently without any constraint:

$$\mathcal{L} = \frac{1}{2}\phi R^{ab} \wedge *(e_a \wedge e_b) - \frac{c}{2\phi} d\phi \wedge *d\phi. \tag{2.15}$$

Independent variations of $\mathcal{L}$ with respect to $\omega^a{}_b$, ϕ and e^a yields

$$\begin{aligned} \delta\mathcal{L} &= \delta e^c \wedge \{\frac{1}{2}\phi R^{ab} \wedge *(e_a \wedge e_b \wedge e_c) + \frac{c}{2\phi}(\iota_c d\phi \wedge *d\phi + d\phi \wedge \iota_c(*d\phi))\} \\ &+ \delta\phi\{\frac{1}{2}R^{ab} \wedge *(e_a \wedge e_b) + cd\left(*\frac{d\phi}{\phi}\right) + \frac{c}{2\phi^2} d\phi \wedge *d\phi\} \\ &+ \delta\omega^{ab} \wedge \{D\left(\frac{1}{2}\phi * (e_a \wedge e_b)\right)\} + mod(d). \end{aligned}$$

The variational principle,

$$\int_M \delta\mathcal{L} = 0$$

implies the following field equations:

$$\frac{1}{2}\phi R^{ab} \wedge *(e_a \wedge e_b \wedge e_c) = -\frac{c}{2\phi}\{\iota_c d\phi \wedge *d\phi + d\phi \wedge \iota_c(*d\phi)\}, \tag{2.16}$$

$$\frac{1}{2}R^{ab} \wedge *(e_a \wedge e_b) = -cd\left(*\frac{d\phi}{\phi}\right) - \frac{c}{2\phi^2} d\phi \wedge *d\phi, \tag{2.17}$$

$$D\left(\frac{1}{2}\phi * (e_a \wedge e_b)\right) = 0. \tag{2.18}$$

Equation (2.18) can be simplified as

$$D\left(\phi * (e_a \wedge e_b)\right) = d\phi \wedge *(e_a \wedge e_b) + \phi D\left(*(e_a \wedge e_b)\right).$$

Here

$$D\left(*(e_a \wedge e_b)\right) = T^c \wedge *(e_a \wedge e_b \wedge e_c),$$

where the torsion 2-forms T^c are defined by the structure equations,

$$T^a = de^a + \omega^a{}_b \wedge e^b.$$

Then we obtain

$$d\phi \wedge *(e_a \wedge e_b) = -\phi T^c \wedge *(e_a \wedge e_b \wedge e_c). \tag{2.19}$$

Algebraic field equations (2.19) can be solved uniquely to obtain

$$T^c = e^c \wedge \frac{d\phi}{(n-1)\phi}. \tag{2.20}$$

The connection one forms $\omega^a{}_b$ can be decomposed according to

$$\omega^a{}_b = {}^{(0)}\omega^a{}_b + K^a{}_b, \tag{2.21}$$

where the contorsion 1-forms $K^{ab} = -K^{ba}$ satisfy

$$K^a{}_b \wedge e^b = T^a. \tag{2.22}$$

Substituting equation (2.20) into (2.22) gives

$$K^a{}_b = \frac{1}{(n-1)\phi}\{e^a \iota_b d\phi - e_b \iota^a d\phi\}. \tag{2.23}$$

On the other hand, the curvature 2-forms R^{ab} are calculated from

$$d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb} = R^{ab}. \tag{2.24}$$

Substitution of (2.21) into (2.24) results in the decomposition of the curvature 2-forms as

$$R^{ab} = {}^{(0)}R^{ab} + {}^{(0)}DK^{ab} + K^a{}_c \wedge K^{cb}, \tag{2.25}$$

where

$$ {}^{(0)}DK^{ab} = dK^{ab} + {}^{(0)}\omega^b{}_c \wedge K^{ac} + {}^{(0)}\omega^a{}_c \wedge K^{cb}. \tag{2.26}$$

We calculate $R^{ab} \wedge *(e_a \wedge e_b \wedge e_c)$ in terms of ${}^{(0)}R^{ab} \wedge *(e_a \wedge e_b \wedge e_c)$ and simplify it by using equations (2.23), (2.25) and (2.26) and some identities. After some algebra, it yields

$$\begin{aligned} R^{ab} \wedge *(e_a \wedge e_b \wedge e_c) &= {}^{(0)}R^{ab} \wedge *(e_a \wedge e_b \wedge e_c) + \frac{2}{\phi}\, {}^{(0)}D(\iota_c(*d\phi)) \\ &\quad - \frac{2n}{(n-1)\phi^2} d\phi \wedge \iota_c(*d\phi) \\ &\quad - \frac{n}{(n-1)\phi^2} \iota_c(d\phi \wedge *d\phi). \end{aligned} \tag{2.27}$$

Then we calculate,

$$\begin{aligned} R^{ab} \wedge *(e_a \wedge e_b) &= {}^{(0)}R^{ab} \wedge *(e_a \wedge e_b) - \frac{n}{(n-1)\phi^2} d\phi \wedge *d\phi \\ &\quad - \frac{2n}{(n-1)} d\left(*\frac{d\phi}{\phi}\right). \end{aligned} \tag{2.28}$$

If we substitute (2.28) into the action density (2.15), it reduces to

$$\mathcal{L} = \frac{1}{2}\phi\, {}^{(0)}R^{ab} \wedge *(e_a \wedge e_b) - \frac{\left(c - \frac{n}{n-1}\right)}{2\phi} d\phi \wedge *d\phi + mod(d). \tag{2.29}$$

Substituting (2.27) and (2.28) into the field equations (2.16) and (2.17), one obtains

$$\begin{aligned} \frac{1}{2}\phi\, {}^{(0)}R^{ab} \wedge *(e_a \wedge e_b \wedge e_c) &= -\frac{\left(c - \frac{n}{n-1}\right)}{2\phi} \{\iota_c d\phi \wedge *d\phi + d\phi \wedge \iota_c(*d\phi)\} \\ &\quad - {}^{(0)}D(\iota_c(*d\phi)), \end{aligned} \tag{2.30}$$

and

$$\frac{1}{2}\, {}^{(0)}R^{ab} \wedge *(e_a \wedge e_b) = \frac{\left(c - \frac{n}{n-1}\right)}{2\phi^2} d\phi \wedge *d\phi - \left(c - \frac{n}{n-1}\right) \frac{1}{\phi} d(*d\phi). \tag{2.31}$$

We consider the exterior product of (2.16) by e^c (the trace of the Einstein field equation) and multiply equation (2.17) by $(n-1)\phi$ and then subtract two resulting equations side by side. This yields the scalar field equation

$$(n-1)cd(*d\phi) = 0 \tag{2.32}$$

provided that $c \neq 0$. We would like to note that, if one defines

$$\omega = c - \frac{n}{n-1}, \tag{2.33}$$

equations (2.30) and (2.31) become equivalent to equations (2.13) and (2.7).

Now we consider the conformal rescalings of the metric induced by the following co-frame rescaling

$$e^a \to e^{\sigma(x)} e^a, \tag{2.34}$$

where $\sigma(x)$ is any dimensionless scalar field. Under this rescaling, the connection 1-forms ${}^{(0)}\omega^{ab}$ transform as

$${}^{(0)}\omega^{ab} \to {}^{(0)} \omega^{ab} + \iota^b d\sigma e^a - e^b \iota^a d\sigma. \tag{2.35}$$

Hence, under the conformal rescaling of the field variables

$$\begin{aligned} e^a &\to e^{\sigma(x)} e^a, \\ \phi &\to e^{-(n-1)\sigma(x)} \phi \end{aligned} \tag{2.36}$$

the torsion-free action (2.1) is conformally scale invariant for the parameter value $\omega = -\frac{n}{n-1}$. Under conformal scaling rules stated above, contorsion 1-forms $K^a{}_b$ transform as

$$K^a{}_b \to K^a{}_b + e_b \iota^a d\sigma - \iota_b d\sigma e^a. \tag{2.37}$$

Therefore in a geometry with torsion specified by $T^a = e^a \wedge \frac{d\phi}{\phi(n-1)}$, connection 1-forms do not transform, i.e.

$$\omega^a{}_b \longrightarrow \omega^a{}_b.$$

Hence $R^{ab} \rightarrow R^{ab}$. Therefore under the conformal rescaling of the fields, action density (2.15) is conformally scale invariant for $c = 0$. In the scale invariant limit ($c = 0$), the field equations (2.16) and (2.17) admit solutions with an arbitrarily chosen ϕ.

We can reformulate the scalar tensor theory in Einstein frame by applying a conformal transformation in both Riemannian and non-Riemannian cases. By adopting

$$\tilde{e}^a = \left(\frac{\phi}{\phi_0}\right)^{\frac{1}{n-1}} e^a, \tag{2.38}$$

where ϕ_0 is a constant, new co-frame fields $\tilde{e}^a$ become orthonormal with respect to the spacetime metric $\tilde{g}$ such that

$$\tilde{g} = \left(\frac{\phi}{\phi_0}\right)^{\frac{2}{n-1}} g. \tag{2.39}$$

In terms of this metric, associated Hodge dual is denoted by $\tilde{*}$. For an arbitrary frame independent p-form Ω

$$*\Omega = \left(\frac{\phi}{\phi_0}\right)^{\frac{2p-(n+1)}{(n-1)}} \tilde{*}\Omega \tag{2.40}$$

is satisfied.

Proof: We can write Ω in $\{e^a\}$ basis as

$$\Omega = \frac{1}{p!}\Omega_{abc\cdots p} e^a \wedge e^b \wedge e^c \cdots \wedge e^p.$$

Then

$$*\Omega = \frac{1}{p!}\Omega_{abc\cdots p}\frac{1}{(n+1-p)!}\epsilon^{abc\cdots p}{}_{r\cdots u}e^r \wedge e^s \wedge e^t \cdots \wedge e^u.$$

In terms of $\{\tilde{e}^a\}$ basis, we can write

$$*\Omega = \frac{1}{p!}\Omega_{abc\cdots p}\left(\frac{\phi}{\phi_0}\right)^{-\frac{(n+1-p)}{(n-1)}}\frac{1}{(n+1-p)!}\epsilon^{abc\cdots p}{}_{r\cdots u}\tilde{e}^r \wedge \tilde{e}^s \wedge \tilde{e}^t \cdots \wedge \tilde{e}^u.$$

Since

$$\tilde{*}(\tilde{e}^a \wedge \tilde{e}^b \wedge \tilde{e}^c \cdots \wedge \tilde{e}^p) = \frac{1}{(n+1-p)!}\epsilon^{abc\cdots p}{}_{r\cdots u}\tilde{e}^r \wedge \tilde{e}^s \wedge \tilde{e}^t \cdots \wedge \tilde{e}^u,$$

we can write

$$*\Omega = \frac{1}{p!}\Omega_{abc\cdots p}\left(\frac{\phi}{\phi_0}\right)^{\frac{p-(n+1)}{n-1}}\tilde{*}(\tilde{e}^a \wedge \tilde{e}^b \wedge \tilde{e}^c \cdots \wedge \tilde{e}^p).$$

This becomes

$$*\Omega = \frac{1}{p!}\Omega_{abc\cdots p}\left(\frac{\phi}{\phi_0}\right)^{\frac{p-(n+1)}{n-1}}\left(\frac{\phi}{\phi_0}\right)^{\frac{p}{n-1}}\tilde{*}(e^a \wedge e^b \wedge e^c \cdots \wedge e^p).$$

Using $\Omega = \frac{1}{p!}\Omega_{abc\cdots p}e^a \wedge e^b \wedge e^c \cdots \wedge e^p$, we obtain

$$*\Omega = \left(\frac{\phi}{\phi_0}\right)^{\frac{2p-(n+1)}{n-1}}\tilde{*}\Omega.$$

In the reformulation of (2.1) in terms of $\tilde{g}$, the new connection fields $\tilde{\omega}^a{}_b$ can be decomposed in terms of ${}^{(0)}\omega^a{}_b$ as

$$\tilde{\omega}^a{}_b = {}^{(0)}\omega^a{}_b + \Omega^a{}_b, \tag{2.41}$$

where $\tilde{\omega}^a{}_b$ satisfy structure equations

$$d\tilde{e}^a + \tilde{\omega}^a{}_b \wedge \tilde{e}^b = 0. \tag{2.42}$$

Substituting (2.38) and (2.41) into (2.42) solves $\Omega^a{}_b$ uniquely. Since

$$de^a +^{(0)} \omega^a{}_b \wedge e^b = 0,$$

we obtain

$$\Omega^a{}_b = \frac{1}{(n-1)\phi}\{e^a \iota_b d\phi - e_b \iota^a d\phi\}. \tag{2.43}$$

We can determine the transformed curvature two-forms $\tilde{R}^{ab}$ that satisfy

$$\tilde{R}^{ab} = d\tilde{\omega}^{ab} + \tilde{\omega}^a{}_c \wedge \tilde{\omega}^{cb} \tag{2.44}$$

in terms of ${}^{(0)}R^{ab}$. Substitution of (2.41) into (2.44) yields

$$\tilde{R}^{ab} =^{(0)} R^{ab} +^0 D(\Omega^{ab}) + \Omega^a{}_c \wedge \Omega^{cb}. \tag{2.45}$$

Substituting (2.45) and (2.43) into the action density D-form (2.1) and using identity (2.40), we obtain the following action density in Einstein frame:

$$\mathcal{L} = \frac{1}{2}\phi_0 \tilde{R}^{ab} \wedge \tilde{*}(\tilde{e}_a \wedge \tilde{e}_b) - \frac{\left(\omega + \frac{n}{n-1}\right)}{2}\phi_0 \frac{1}{\phi^2} d\phi \wedge \tilde{*} d\phi + mod(d). \tag{2.46}$$

We define a new (dimensionless) scalar field

$$\Phi = \ln\left(\frac{\phi}{\phi_0}\right) \tag{2.47}$$

such that $d\Phi = \frac{d\phi}{\phi}$, to obtain Einstein-Klein Gordon action density

$$\mathcal{L} = \frac{1}{2}\phi_0 \tilde{R}^{ab} \wedge \tilde{*}(\tilde{e}_a \wedge \tilde{e}_b) - \frac{\tilde{k}}{2}\phi_0 d\Phi \wedge \tilde{*} d\Phi + mod(d) \tag{2.48}$$

where we define

$$\tilde{k} = \omega + \frac{n}{n-1}. \tag{2.49}$$

The field equations obtained from this action density are:

$$\frac{\phi_0}{2}\tilde{R}^{ab}\wedge\tilde{*}(\tilde{e}_a\wedge\tilde{e}_b\wedge\tilde{e}_c) = -\frac{\tilde{k}}{2}\phi_0\{\tilde{\imath}_c d\Phi\wedge\tilde{*}d\Phi + d\Phi\wedge\tilde{\imath}_c(\tilde{*}d\Phi)\} \tag{2.50}$$

and

$$d(\tilde{*}d\Phi) = 0. \tag{2.51}$$

On the other hand, in the reformulation of the action density (2.15) in Einstein frame (by applying the transformation (2.38)), we can assume a similar decomposition in connection fields

$$\tilde{\omega}^a{}_b = \omega^a{}_b + \Gamma^a{}_b. \tag{2.52}$$

However, since $\omega^a{}_b$ satisfy structure equation

$$de^a + \omega^a{}_b\wedge e^b = T^a,$$

with

$$T^a = e^a\wedge\frac{d\phi}{\phi(n-1)},$$

substitution of (2.52) into structure equation (2.42) will give us

$$\Gamma^a{}_b = 0. \tag{2.53}$$

Therefore,

$$\tilde{\omega}^a{}_b = \omega^a{}_b \tag{2.54}$$

and hence

$$\tilde{R}^{ab} = R^{ab}. \tag{2.55}$$

Thus the action density (2.15) can be reformulated in the Einstein frame in terms of the new scalar field Φ as:

$$\mathcal{L} = \frac{1}{2}\phi_0 \tilde{R}^{ab} \wedge (\tilde{e}_a \wedge \tilde{e}_b) - \frac{c}{2}\phi_0 d\Phi \wedge \tilde{*} d\Phi + mod(d). \tag{2.56}$$

The field equations are the same as equations (2.50) and (2.51) except that the coupling constant $\tilde{k}$ is replaced with c. Thus we see that when we reformulate the scalar tensor gravity in Riemannian geometry into Einstein frame, coupling parameter ω shifts. However, coupling parameter stays fixed when the theory is reformulated in a geometry with torsion. In both cases, the independent connection variations of the transformed actions lead to

$$D(\tilde{\omega})\{\tilde{*}(\tilde{e}^a \wedge \tilde{e}^b)\} = 0 \tag{2.57}$$

from which we can obtain $\tilde{T}^a = 0$. Therefore, although the torsion associated with the metric g may not be zero, the torsion in the geometry of the transformed metric $\tilde{g}$ is zero. Under local conformal scale transformation such that,

$$e^a \rightarrow \exp(\sigma(x)) e^a,$$

$$\phi \rightarrow \exp(-(n-1)\sigma(x))\phi$$

new co-frames are not affected, i.e.

$$\tilde{e}^a \rightarrow \tilde{e}^a.$$

Hence the metric $\tilde{g}$ is identified as the scale invariant atomic metric [6].

2.2 Scalar Tensor Theory Of Gravity In Spacetimes With Non-Metricity

In the previous section, we assumed that $\nabla g = 0$, i.e. covariant derivative of metric tensor is zero. In this section, we consider $\nabla g \neq 0$. Non-metricity tensor is defined by

$$S = \nabla g. \tag{2.58}$$

A geometry with $S = 2Qg$ (with zero torsion) was first suggested by Weyl [3, 10, 11]. Q is introduced as dimensionless Weyl connection one form field which transforms as

$$Q \rightarrow Q + d\sigma, \tag{2.59}$$

where σ is arbitrary dimensionless real scalar field.

In field-particle interactions including gravity, if any field element Φ transforms as,

$$\Phi \rightarrow \exp(-q\sigma)\Phi \tag{2.60}$$

then such a transformation is identified as Weyl transformation and q is called dimensionless Weyl charge or scale charge of the Weyl group representation [3]. If under such transformations, any action describing related interactions remains invariant or if the action changes by a total divergence, then the action possesses a Weyl symmetry. In gravitational interactions, scale charges are relative on representation carried by a class of metric tensors $[g]$, elements of which are equivalent [3] under the transformation

$$g \rightarrow \exp(2\sigma)g. \tag{2.61}$$

Before constructing a locally Weyl covariant theory of gravitation in D dimensions, we introduce some concepts related to Weyl geometry. In terms of Q, the exterior Weyl covariant derivative of a p-form field Φ^p_q with scale (Weyl) charge

q is defined as [3]

$$\mathcal{D}\Phi^p_q = D\Phi^p_q + qQ \wedge \Phi^p_q, \tag{2.62}$$

where D is the exterior covariant derivative defined for the tensor field. Under the transformation

$$\Phi^p_q \to \exp(-q\sigma)\Phi^p_q, \tag{2.63}$$

the exterior Weyl covariant derivative transforms as,

$$\mathcal{D}\Phi^p_q \to \exp(-q\sigma)\mathcal{D}\Phi^p_q. \tag{2.64}$$

If we introduce the Hodge map $*$ associated with metric g, we can verify the following rules for any p-form field Φ^p_q with Weyl charge q. Under the transformation,

$$\tilde{g} = \exp(2\sigma)g,$$

$$*\Phi^p_q = \tilde{*}\exp\{(2p-n-1)\sigma\}\Phi^p_q,$$

where $\tilde{*}$ is the Hodge map associated with metric $\tilde{g}$. Therefore, together with transformation (2.63), it requires that,

$$*\Phi^p_q \to \exp(n+1-2p-q) * \Phi^p_q. \tag{2.65}$$

Hence,

$$\mathcal{D}(*\Phi^p_q) = D(*\Phi^p_q) + (q-(n+1-2p))\, Q \wedge *\Phi^p_q. \tag{2.66}$$

If one denotes Φ^p_q by $\{{}^p_q\}\Phi$ then the following rules generalise the results of [3] to

$(n+1)$ dimensions:

$$*\left(\{^p_q\}\Phi\right) = \{^{n+1-p}_{q-(n+1-2p)}\}(*\Phi),$$

$$\mathcal{D}\left(\{^p_q\}\Phi\right) = \{^{p+1}_q\}\mathcal{D}\Phi,$$

$$\left(\{^{p_1}_{q_1}\}\Phi_1\right) \wedge \left(\{^{p_2}_{q_2}\}\Phi_2\right) = \{^{p_1+p_2}_{q_1+q_2}\}(\Phi_1 \wedge \Phi_2).$$

In order to construct a locally Weyl invariant action, we take the dynamic Weyl connection Q proportional to metric trace of the non-metricity tensor $S = \nabla g$ [3]. In a geometry (g, M) with $\nabla g = S$, we introduce connection one forms Λ_{ab}, such that

$$\Lambda_{ab} = \Omega_{ab} + Q_{ab}, \tag{2.67}$$

where $Q^a{}_b$ is the symmetric part of the connection (called non-metricity one forms) and $\Omega^a{}_b$ is the antisymmetric part of the connection. Since $\eta_{ab} = g(X_a, X_b)$, where X_a is the dual frame vector to the co-frame one form e^a such that $e^a(X_b) = \delta^a_b$, and

$$(D\eta_{ab})(X) = (\nabla_X g)(X_a, X_b) \tag{2.68}$$

for all X, we calculate

$$D\eta_{ab} = d\eta_{ab} - \Lambda^c{}_a\eta_{cb} - \Lambda^c{}_b\eta_{ac} = -\Lambda_{ab} - \Lambda_{ba}.$$

Since $\Lambda_{ab} + \Lambda_{ba} = 2Q_{ab}$,

$$D\eta_{ab} = -2Q_{ab}. \tag{2.69}$$

If we specify a geometry in which the non-metricity tensor is proportional to Q, with $S = \nabla_X g = 2Q \otimes g$, it follows that[1]

$$Q_{ab} = -Q\eta_{ab}. \tag{2.70}$$

Similarly,

$$D\eta^{ab} = 2Q^{ab}. \tag{2.71}$$

Under the local Weyl transformation such that

$$Q \to Q + d\sigma,$$

Q_{ab} transforms as

$$Q_{ab} \to Q_{ab} - d\sigma\, \eta_{ab}. \tag{2.72}$$

Torsion 2-forms are defined by

$$de^a + \Lambda^a{}_b \wedge e^b = T^a. \tag{2.73}$$

Therefore, we conclude that

$$De^a = T^a.$$

However

$$D(e_a) = D(e^c\eta_{ca}) = (D\eta_{ca}) \wedge e^c + (De^c)\eta_{ca},$$

and hence

$$De_a = 2Q \wedge e_a + T_a.$$

It can be noted that, under Weyl scaling

$$e^a \to \exp(\sigma)e^a,$$

[1] In general, in a geometry with $\nabla_X g = 2Q(X)\, g + S_{ab}(X^a, X^b, X)$ symmetric part Q_{ab} of the connection $\Lambda^a{}_b$ can be split into the metric trace component (diagonal part) $\hat{Q}_{ab}$ and the trace-free component (non-diagonal part) $\tilde{Q}_{ab}$ such that $\tilde{Q}_{ab}\, \eta^{ab} = 0$ [12, 13]. In our case the trace-free component is zero and $\hat{Q}_{ab} = -Q\eta_{ab}$.

torsion 2-forms transform as

$$T^a \to \exp(\sigma) T^a.$$

The antisymmetric part $\Omega^a{}_b$ of connection $\Lambda^a{}_b$ can be further decomposed as:

$$\Omega^a{}_b = \omega^a{}_b + K^a{}_b + q^a{}_b. \tag{2.74}$$

Here, $\omega^a{}_b$ are the Levi-Civita (torsion-free) connection one forms which satisfy the structure equations

$$de^a + \omega^a{}_b \wedge e^b = 0.$$

Contorsion one forms $K^a{}_b$ fix the torsion

$$K^a{}_b \wedge e^b = T^a,$$

and

$$q^a{}_b = \iota_b(Q^a{}_c)e^c - e^c \iota^a(Q_{bc}). \tag{2.75}$$

Curvature two forms $R^a{}_b(\Lambda)$ of the connections $\Lambda^a{}_b$ are calculated from

$$R^a{}_b = d\Lambda^a{}_b + \Lambda^a{}_c \wedge \Lambda^c{}_b. \tag{2.76}$$

In terms of Hodge map $*$, following identity holds:

$$\begin{aligned} D\left(*(e_a \wedge e_b \wedge e_c \wedge \cdots \wedge e_p)\right) &= T^r \wedge *(e_a \wedge e_b \wedge e_c \wedge \cdots \wedge e_p \wedge e_r) \\ &+ (n+1)Q \wedge *(e_a \wedge e_b \wedge e_c \wedge \cdots \wedge e_p) \end{aligned} \tag{2.77}$$

Before proving this identity, we should show that

$$D(\epsilon_{abcde\cdots u}) = (n+1)Q\epsilon_{abcde\cdots u}, \tag{2.78}$$

where $\epsilon_{abcde\cdots u}$ is totally antisymmetric tensor in $D = (n+1)$ dimensions defined by:

$$\epsilon_{abcde\cdots u} = \begin{cases} -1 & \text{if } (a,b,c,d,e,\cdots u) \text{ is an odd permutation of } (0,1,2,\cdots n) \\ 0 & \text{if any two indices are equal} \\ 1 & \text{if } (a,b,c,d,e,\cdots u) \text{ is an even permutation of } (0,1,2,\cdots n)\ . \end{cases}$$

Proof:

$$D(\epsilon_{abc\cdots u}) = d\epsilon_{abc\cdots u} - \Lambda^l{}_a \epsilon_{lbc\cdots u} \cdots - \Lambda^l{}_u \epsilon_{abc\cdots l}.$$

We separate the connection $\Lambda^l{}_a = \Omega^l{}_b + Q^l{}_a$ into its symmetric and antisymmetric parts and consider the combinations $\Omega^l{}_a \epsilon_{lbc\cdots u}$ and $Q^l{}_a \epsilon_{lbc\cdots u}$. If $l = a$, $\Omega^l{}_a = 0$ since it is antisymmetric. If $l \neq a$, then l can be any of indices $(b, c \cdots u)$. Therefore $\epsilon_{lbc\cdots u}$ becomes zero. Hence antisymmetric combinations $\Omega^l{}_a \epsilon_{lbc\cdots u}$ have no contribution. Thus,

$$D\epsilon_{abc\cdots u} = -Q^l{}_a \epsilon_{lbc\cdots u} \cdots - Q^l{}_u \epsilon_{abc\cdots l}.$$

Since $Q^l{}_a = -Q\eta^l{}_a$,

$$Q^l{}_a \epsilon_{lbc\cdots u} = -Q\eta^l{}_a \epsilon_{lbc\cdots u} = -Q\epsilon_{abc\cdots u}$$

and since there are $(n+1)$ such terms, we obtain

$$D\epsilon_{abc\cdots u} = (n+1)Q\epsilon_{abc\cdots u}.$$

To prove identity (2.77), we should also calculate $D\{\epsilon_{abc\cdots p}{}^{rs\cdots u}\}$. Since

$$\epsilon_{abcde\cdots p}{}^{rs\cdots u} = \epsilon_{abcde\cdots pr_1 s_1 \cdots u_1} \eta^{rr_1} \eta^{ss_1} \cdots \eta^{uu_1},$$

we obtain,

$$\begin{aligned} D\{\epsilon_{abcde\cdots p}{}^{rs\cdots u}\} &= D(\epsilon_{abcde\cdots pr_1s_1\cdots u_1})\eta^{rr_1}\eta^{ss_1}\cdots\eta^{uu_1} \\ &+\epsilon_{abcde\cdots pr_1s_1\cdots u_1}D(\eta^{rr_1})\eta^{ss_1}\cdots\eta^{uu_1} \\ &+\epsilon_{abcde\cdots pr_1s_1\cdots u_1}\eta^{rr_1}D(\eta^{ss_1})\cdots\eta^{uu_1} \\ &+\cdots+\epsilon_{abcde\cdots pr_1s_1\cdots u_1}\eta^{rr_1}\eta^{ss_1}\cdots D(\eta^{uu_1}) \end{aligned}$$

Now we use

$$D(\epsilon_{abcde\cdots pr_1s_1\cdots u_1}) = (n+1)Q\epsilon_{abcde\cdots pr_1s_1\cdots u_1}$$

and

$$D(\eta^{rr_1}) = 2Q^{rr_1} = -2Q\eta^{rr_1}.$$

Since we have $((n+1)-p)$ such D acting on η terms, we obtain

$$D\{\epsilon_{abcde\cdots p}{}^{rs\cdots u}\} = (2p-(n+1))Q\epsilon_{abcde\cdots p}{}^{rs\cdots u}. \tag{2.79}$$

With $p = (n+1)$, (2.79) reduces to (2.78). Interestingly substituting $p = 0$, we obtain

$$D(\epsilon^{abcde\cdots u}) = -(n+1)Q\epsilon^{abcde\cdots u}. \tag{2.80}$$

Proof of identity (2.77):

$$D\{*(e_a\wedge e_b\wedge e_c\wedge\cdots\wedge e_p)\} = D\{\frac{1}{(n+1-p)!}\epsilon_{abc\cdots p}{}^{rs\cdots u}e_r\wedge e_s\wedge\cdots\wedge e_u\}.$$

Using equation (2.79) and the antisymmetry of $(e_r\wedge e_s\wedge\cdots\wedge e_u)$, we have

$$\begin{aligned} D\{\epsilon_{abc\cdots p}{}^{rs\cdots u}e_r\wedge e_s\cdots\wedge e_u\} &= D(\epsilon_{abc\cdots p}{}^{rs\cdots u})\wedge e_r\wedge e_s\cdots e_u \\ &+(n+1-p)\epsilon_{abc\cdots p}{}^{rs\cdots u}De_r\wedge e_s\cdots e_u. \end{aligned}$$

Since

$$De_r = T_r + 2Q \wedge e_r,$$

$D\{*(e_a \wedge e_b \wedge e_c \cdots e_p)\}$ becomes

$$\begin{aligned} D\{*(e_a \wedge e_b \wedge e_c \cdots e_p)\} &= \frac{(2p-(n+1))}{(n+1-p)!}\epsilon_{abc\cdots p}{}^{rs\cdots u}Q \wedge e_r \wedge e_s \cdots e_u \\ &+ \frac{1}{(n-p)!}\epsilon_{abc\cdots p}{}^{rs\cdots u}(T_r + 2Q \wedge e_r) \wedge e_s \cdots e_u. \end{aligned}$$

This can be simplified as:

$$\begin{aligned} D\{*(e_a \wedge e_b \wedge e_c \cdots e_p)\} &= (2p-(n+1))Q \wedge *(e_a \wedge e_b \wedge e_c \cdots \wedge e_p) \\ &+ T_r \wedge *(e_a \wedge e_b \wedge e_c \cdots e_p \wedge e^r) \\ &+ 2Q \wedge e_r \wedge *(e_a \wedge e_b \wedge e_c \cdots e_p \wedge e^r). \end{aligned}$$

At this stage, we can use the identity

$$\begin{aligned} e_r \wedge *(e_a \wedge e_b \wedge e_c \cdots e_p \wedge e^r) &= e_r \wedge \iota^r\{*(e_a \wedge e_b \wedge e_c \cdots e_p)\} \\ &= (n+1-p) * (e_a \wedge e_b \wedge e_c \cdots e_p). \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} D\{*(e_a \wedge e_b \wedge e_c \cdots \wedge e_p)\} &= T_r \wedge *(e_a \wedge e_b \wedge e_c \cdots e_p \wedge e^r) \\ &+ (n+1)Q \wedge *(e_a \wedge e_b \wedge e_c \cdots e_p). \end{aligned}$$

An action $S = \int_M \mathcal{L}$ is locally Weyl invariant for $(n+1)$-form action density $\mathcal{L}$, if $\mathcal{L}$ transforms under Weyl scalings as:

$$\mathcal{L} \to \mathcal{L} + d\Omega, \tag{2.81}$$

where Ω is n-form.

Now, we consider following (dimensionless) action density in $D = (n+1)$ dimensions:

$$\mathcal{L}[e^a, \phi, \Lambda^a{}_b] = \frac{1}{2}\phi R^a{}_b \wedge *(e_a \wedge e^b) - \frac{\tilde{\omega}}{2\phi}\mathcal{D}\phi \wedge *\mathcal{D}\phi - \frac{1}{2}\phi^{\frac{n-3}{n-1}} dQ \wedge *dQ. \quad (2.82)$$

Here $\tilde{\omega}$ is a real dimensionless coupling constant, since it is obvious from the form of the action that the dimension of the scalar field ϕ, $[\phi] = L^{-(n-1)}$ where L denotes the dimension of length. Therefore Weyl covariant derivative of ϕ is

$$\mathcal{D}\phi = d\phi + (n-1)\phi Q. \quad (2.83)$$

We can note that, action S can have physical dimensions as in [3] by writing it in the form

$$S = \int_M \Lambda_0 \, \mathcal{L}, \quad (2.84)$$

where Λ_0 is a constant with the dimensions of the action. But for simplicity, we choose the physical constants such that $\Lambda_0 = 1$ and $[\Lambda_0] = [1]$ (e.g we take $c = 1$ and $G = 1$). In the action density (2.82), we see that there is a scalar field-Weyl field strength (dQ) coupling in higher dimensions. It arises naturally to provide scale invariance of the action density in higher dimensions. In 4 dimensions, when $n = 3$ this coupling is not observed. The field equations are found by independent variations of the action with respect to e^a, ϕ and $\Lambda^a{}_b$. Co-frame e^a variations yield the Einstein field equation:

$$\frac{1}{2}\phi R^a{}_b \wedge *(e_a \wedge e^b \wedge e_c) = -\frac{\tilde{\omega}}{2\phi}\tau_c[\phi] - \frac{1}{2}\phi^{\frac{n-3}{n-1}}\tau_c[dQ] \quad (2.85)$$

where

$$\tau_c[\phi] = \{\iota_c\mathcal{D}\phi * \mathcal{D}\phi + \mathcal{D}\phi \wedge \iota_c(*\mathcal{D}\phi)\} \quad (2.86)$$

is n-form stress energy tensor of scalar field and

$$\tau_c[dQ] = \{\iota_c dQ \wedge *dQ - dQ \wedge \iota_c(*dQ)\} \tag{2.87}$$

is the stress energy tensor of Weyl field strength. The variation of action density (2.82) with respect to the scalar field ϕ yields

$$\begin{aligned} \frac{1}{2}R^a{}_b \wedge *(e_a \wedge e^b) &= -\frac{\tilde{\omega}}{2\phi^2}\mathcal{D}\phi \wedge *\mathcal{D}\phi - \tilde{\omega}\left\{d\left(\frac{*\mathcal{D}\phi}{\phi}\right) - \frac{(n-1)}{\phi}Q \wedge *\mathcal{D}\phi\right\} \\ &\quad + \frac{(n-3)}{2(n-1)}\phi^{-\frac{2}{n-1}}dQ \wedge *dQ. \end{aligned} \tag{2.88}$$

Using (2.83) and

$$d\left(\frac{*\mathcal{D}\phi}{\phi}\right) = \frac{1}{\phi}d(*\mathcal{D}\phi) - \frac{1}{\phi^2}d\phi \wedge *\mathcal{D}\phi, \tag{2.89}$$

we can simplify (2.88) to obtain,

$$\frac{1}{2}R^a{}_b \wedge *(e_a \wedge e^b) = \frac{\tilde{\omega}}{2\phi^2}\mathcal{D}\phi \wedge *\mathcal{D}\phi - \frac{\tilde{\omega}}{\phi}d(*\mathcal{D}\phi) + \frac{(n-3)}{2(n-1)}\phi^{-\frac{2}{(n-1)}}dQ \wedge *dQ. \tag{2.90}$$

As before, we take the trace of (2.85) by considering its exterior multiplication with e^c and multiply equation (2.90) by $(n-1)\phi$. If we subtract the resulting equations, we obtain the equation satisfied by the scalar field

$$(n-1)\,\tilde{\omega}\, d(*\mathcal{D}\phi) = 0, \tag{2.91}$$

provided that the coupling constant $\tilde{\omega} \neq 0$. However, since Weyl charge of ϕ is $(n-1)$ and Weyl charge of $*\mathcal{D}\phi$ is $(n-1) - (n+1-2) = 0$, we can rewrite the scalar field equation (2.91) as

$$(n-1)\,\tilde{\omega}\,\mathcal{D}\,(*\,\mathcal{D}\,\phi) = 0. \tag{2.92}$$

The connection variations of the action density (2.82) can be performed under constraint that $Q^a{}_b = -Q\,\eta^a{}_b$. Therefore we can add $\lambda^b{}_a \wedge (Q^a{}_b + Q\eta^a{}_b)$ term to action density (2.82), where $\lambda^b{}_a$ are Lagrange multiplier n-forms. Now using

$$\delta\, Q^a{}_b = \frac{1}{2}\{\delta\,\Lambda^a{}_b + \delta\,\Lambda_b{}^a\},$$

the independent variations of action density (2.82) with respect to $\Lambda^a{}_b$ yields

$$\frac{1}{2}D(\phi * (e_a \wedge e^b)) + (-1)^n \lambda^b{}_a = 0. \tag{2.93}$$

On the other hand, the independent Q-field variation results in,

$$\lambda^b{}_a\,\eta^a{}_b - (n-1)\,\tilde{\omega}\,(*\mathcal{D}\,\phi) - d(\phi^{\frac{n-3}{n-1}} * dQ) = 0. \tag{2.94}$$

Equations (2.93) and (2.94) are combined to yield the equation

$$\frac{1}{2}D(\phi * (e_a \wedge e^b)) + \frac{\eta^b{}_a}{(n+1)}\{d(\phi^{\frac{n-3}{n-1}} * dQ) + (n-1)\,\tilde{\omega}\, * \mathcal{D}\,\phi\} = 0. \tag{2.95}$$

We should mention that equation (2.95) can also be obtained directly from the action density (2.82) by noting that $\delta\, Q = -\frac{1}{(n+1)}\eta^b{}_a\,\delta\,\Lambda^a{}_b$. We multiply equation (2.95) by η_{bc} and equate the antisymmetric and the symmetric parts of resulting equation to zero. Equation (2.95) multiplied by η_{bc} takes the form

$$\frac{1}{2}\eta_{bc}D(\phi * (e_a \wedge e^b)) + \frac{\eta_{bc}\eta^b{}_a}{(n+1)}\{d(\phi^{\frac{n-3}{n-1}} * dQ) + (n-1)\,\tilde{\omega}\, * \mathcal{D}\,\phi\} = 0. \tag{2.96}$$

The first term can be simplified as follows: We consider

$$D(\eta_{bc}\phi * (e_a \wedge e^b)) = D\,\eta_{bc} \wedge \phi * (e_a \wedge e^b) + \eta_{bc}D(\phi * (e_a \wedge e^b)).$$

Therefore,

$$\eta_{bc}D(\phi * (e_a \wedge e^b)) \quad = \quad D(\eta_{bc}\phi * (e_a \wedge e^b)) - D\,\eta_{bc} \wedge \phi * (e_a \wedge e^b)$$

$$= D(\phi * (e_a \wedge e_c)) - (-2Q_{bc}) \wedge \phi * (e_a \wedge e^b)$$

$$= D(\phi * (e_a \wedge e_c)) - 2Q \wedge \phi * (e_a \wedge e_c).$$

Hence (2.96) becomes

$$\frac{1}{2}\{D(\phi * (e_a \wedge e_c)) - 2Q\phi \wedge *(e_a \wedge e_c)\} + \frac{\eta_{ac}}{(n+1)}\{d(\phi^{\frac{n-3}{n-1}} * dQ) + (n-1)\,\tilde{\omega} \, * \mathcal{D}\phi\} = 0. \qquad (2.97)$$

Now we equate the antisymmetric and the symmetric parts of equation (2.97) to zero. Using

$$D(*(e_a \wedge e_c)) = *(e_a \wedge e_c \wedge e_b) \wedge T^b + (n+1)Q \wedge *(e_a \wedge e_c), \qquad (2.98)$$

the antisymmetric part yields

$$d\phi \wedge *(e_a \wedge e_c) + \phi \, * (e_a \wedge e_c \wedge e_b) \wedge T^b + (n-1)\phi Q \wedge *(e_a \wedge e_c) = 0. \qquad (2.99)$$

On the other hand, the symmetric part produces

$$d(\phi^{\frac{n-3}{n-1}} * dQ) + (n-1)\tilde{\omega} \, * \mathcal{D}\,\phi = 0. \qquad (2.100)$$

Taking the exterior derivative of both sides of (2.100) will give the scalar field equation (2.91)

$$(n-1)\,\tilde{\omega}\, d(*\mathcal{D}\,\phi) = 0.$$

Equation (2.99) can be solved uniquely for torsion 2-forms T^a as

$$T^a = e^a \wedge \frac{d\phi}{\phi(n-1)} - Q \wedge e^a. \qquad (2.101)$$

Then from $K^a{}_b \wedge e^b = T^a$, we can determine the corresponding contorsion 1-forms

$$K^a{}_b = e^a \iota_b \left(\frac{d\phi}{(n-1)\phi} + Q\right) - e_b \iota^a \left(\frac{d\phi}{(n-1)\phi} + Q\right). \qquad (2.102)$$

Therefore, using

$$\Lambda^{ab} = \omega^{ab} + K^{ab} + q^{ab} + Q^{ab}$$

with

$$q^{ab} = \iota^b Q^a{}_c e^c - \iota^a Q^b{}_c e^c$$

connection one forms Λ^{ab} become

$$\Lambda^{ab} = \omega^{ab} + \frac{1}{(n-1)\phi}\{e^a \iota^b d\phi - e^b \iota^a d\phi\} - Q\eta^{ab}. \tag{2.103}$$

Now, we can discuss the conformal rescaling of the action density (2.82). Under local rescalings

$$e^a \to \exp(\sigma(x))e^a,$$

$$\phi \to \exp(-(n-1)\sigma(x))\phi,$$

$$Q \to Q + d\sigma(x),$$

where $\sigma(x)$ is a dimensionless scalar field variable, the connection components transform as:

$$\begin{aligned}
\omega^a{}_b &\to \omega^a{}_b + e^a \iota_b d\sigma - e_b \iota^a d\sigma, \\
q^a{}_b &\to q^a{}_b + \iota^a d\sigma e_b - \iota_b d\sigma e^a, \\
K^a{}_b &\to K^a{}_b + e^a \iota_b d\sigma - e_b \iota^a d\sigma + \iota^a d\sigma e_b - \iota_b d\sigma e^a = K^a{}_b, \\
Q^a{}_b &\to Q^a{}_b - d\sigma \eta^a{}_b.
\end{aligned} \tag{2.104}$$

Therefore the connection $\Lambda^a{}_b$ transform as

$$\Lambda^a{}_b \to \Lambda^a{}_b - d\sigma \eta^a{}_b. \tag{2.105}$$

Under these rescalings

$$R^a{}_b \to R^a{}_b.$$

Hence, action density (2.82) is conformally scale invariant for any dimensionless coupling constant $\tilde{\omega}$. This shows the difference from metric-compatible scalar-tensor gravity in which the conformal scale invariance is attained for specific values of the coupling constant. Using (2.76), we can express the curvature 2-forms $R^a{}_b(\Lambda)$ of the connection $\Lambda^a{}_b$ and the action in terms of the curvatures $R^a{}_b(\omega)$ of Levi-Civita connections $\omega^a{}_b$. Defining

$$\Delta^{ab} = \frac{1}{(n-1)\phi}\{e^a \iota^b d\phi - e^b \iota^a d\phi\},$$

we obtain

$$R^a{}_b(\Lambda) = R^a{}_b(\omega) + D(\omega)(\Delta^a{}_b) + \Delta^a{}_c \wedge \Delta^c{}_b - dQ\eta^a{}_b, \tag{2.106}$$

where $D(\omega)$ denotes the exterior covariant derivative with respect to the connections $\omega^a{}_b$. We can evaluate

$$\Delta^a{}_c \wedge \Delta^c{}_b = \frac{1}{\{(n-1)\phi\}^2}\{e^a \wedge d\phi\, \iota_b d\phi + d\phi \wedge e_b\, \iota^a d\phi - *(d\phi \wedge *d\phi) e_b \wedge e^a\}. \tag{2.107}$$

If we substitute equations (2.106) and (2.107) into the action density (2.82), we obtain

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\phi R^a{}_b(\omega) \wedge *(e_a \wedge e^b) - \frac{\left(\tilde{\omega} - \frac{n}{n-1}\right)}{2\phi} d\phi \wedge *d\phi - (n-1)\,\tilde{\omega}\, d\phi \wedge *Q \\ &\quad - \frac{(n-1)^2}{2}\tilde{\omega}\,\phi Q \wedge *Q - \frac{1}{2}\phi^{\frac{n-3}{n-1}} dQ \wedge *dQ + mod(d). \end{aligned} \tag{2.108}$$

With $Q = 0$, this action density is the same as the action density (2.29) provided that we identify $\tilde{\omega} = c$. When $\tilde{\omega} = 0$, it is the scale-invariant limit of the action discussed in metric-compatible torsion-free theory. Interestingly when

$\phi = constant$, it reduces to Einstein-Proca system [3]. We can express the field equations (2.85) and (2.88) in terms of $R^a{}_b(\omega)$. Einstein field equation becomes,

$$\begin{aligned}\frac{1}{2}R^a{}_b(\omega)\wedge *(e_a\wedge e^b\wedge e_c) \;=\; & -\frac{\left(\tilde{\omega}-\frac{n}{n-1}\right)}{2\phi}\{\iota_c d\phi * d\phi + d\phi\wedge\iota_c(*d\phi)\} \\ & -\frac{(n-1)\,\tilde{\omega}}{2}\{\iota_c d\phi * Q + \iota_c Q * d\phi + d\phi\wedge\iota_c(*Q) \\ & +Q\wedge\iota_c(*d\phi)\} - \frac{(n-1)^2}{2}\tilde{\omega}\,\phi\tau_c[Q] \\ & -\frac{1}{2}\phi^{\frac{n-3}{n-1}}\tau_c[dQ] - D(\omega)(\iota_c(*d\phi)), \end{aligned} \tag{2.109}$$

where

$$\tau_c[Q] = \{\iota_c Q\wedge *Q + Q\wedge\iota_c(*Q)\}. \tag{2.110}$$

Similarly, equation (2.88) becomes,

$$\begin{aligned}\frac{1}{2}R^a{}_b(\omega)\wedge *(e_a\wedge e^b) \;=\; & \frac{\left(\tilde{\omega}-\frac{n}{n-1}\right)}{2\phi^2}d\phi\wedge *d\phi + \frac{(n-1)^2}{2}\tilde{\omega}Q\wedge *Q \\ & -\frac{\left(\tilde{\omega}-\frac{n}{n-1}\right)}{\phi}d(*d\phi) + \frac{(n-3)}{2(n-1)}\phi^{-\frac{2}{n-1}}dQ\wedge *dQ \\ & -(n-1)\,\tilde{\omega}\,d(*Q). \end{aligned} \tag{2.111}$$

The action density (2.82) and the field equations (2.85) and (2.88) can also be expressed in terms of curvature two forms $R^a{}_b(\Gamma)$ of torsion-free Weyl connection 1-forms $\Gamma^a{}_b$ defined as

$$\Gamma^a{}_b = \Lambda^a{}_b - K^a{}_b. \tag{2.112}$$

Therefore

$$\Gamma^a{}_b = \omega^a{}_b + \iota^a Q e_b - \iota_b Q e^a - Q\eta^a{}_b. \tag{2.113}$$

We can see that,

$$D(\Gamma)(e^a) = de^a + (\Lambda^a{}_b - K^a{}_b)\wedge e^b = 0. \tag{2.114}$$

However,

$$\begin{aligned} D(\Gamma)(e_b) &= de_b - (\Lambda^c{}_b - K^c{}_b) \wedge e_c \\ &= T_b + 2Q \wedge e_b - T_b \\ &= 2Q \wedge e_b. \end{aligned} \tag{2.115}$$

Using (2.76), we obtain

$$R^a{}_b(\Lambda) = R^a{}_b(\Gamma) + D(\Gamma)K^a{}_b + K^a{}_c \wedge K^c{}_b. \tag{2.116}$$

Using (2.114) and (2.115), we can write (2.116) explicitly as

$$\begin{aligned} R^a{}_b(\Lambda) &= R^a{}_b(\Gamma) - 2Q \wedge e_b\, \iota^a \left(\frac{d\phi}{(n-1)\phi} + Q \right) \\ &+ D(\Gamma) \left(\iota_b \left(\frac{d\phi}{(n-1)\phi} + Q \right) \right) \wedge e^a \\ &+ e_b \wedge D(\Gamma) \left(\iota^a \left(\frac{d\phi}{(n-1)\phi} + Q \right) \right) \\ &+ e^a \wedge \left(\frac{d\phi}{(n-1)\phi} + Q \right) \iota_b \left(\frac{d\phi}{\phi(n-1)} + Q \right) \\ &+ (e^a \wedge e_b) * \left(\left(\frac{d\phi}{(n-1)\phi} + Q \right) \wedge * \left(\frac{d\phi}{(n-1)\phi} + Q \right) \right) \\ &+ \left(\frac{d\phi}{(n-1)\phi} + Q \right) \wedge e_b\, \iota^a \left(\frac{d\phi}{(n-1)\phi} + Q \right). \end{aligned} \tag{2.117}$$

In expressing the action density (2.82) in terms of $R^a{}_b(\Gamma)$, we need the following identities:

$$\begin{aligned} D(\Gamma)\{*(e_a \wedge e^b \wedge e_c)\} &= (n-1)Q \wedge *(e_a \wedge e^b \wedge e_c), \\ D(\Gamma)\{*(e_a \wedge e^b)\} &= (n-1)Q \wedge *(e_a \wedge e^b), \\ D(\Gamma)\{*(e_b)\} &= (n+1)Q \wedge *(e_b). \end{aligned} \tag{2.118}$$

These identities can be shown easily by noting that,

$$
\begin{aligned}
D(\Gamma)\eta^{ab} &= 2Q^{ab}, \\
D(\Gamma)\eta_{ab} &= -2Q_{ab},
\end{aligned} \tag{2.119}
$$

and

$$D(\Gamma)\{\epsilon^{abc\cdots s}\} = -(n+1)Q\epsilon^{abc\cdots s}, \tag{2.120}$$

$$D(\Gamma)\{\epsilon_{abc\cdots s}\} = (n+1)Q\epsilon_{abc\cdots s} \tag{2.121}$$

hold. The last two identities can be proved in the same way as before by noting that $\Gamma^a{}_b = \Lambda^a{}_b - K^a{}_b$ and therefore, $\Gamma^a{}_b = \omega^a{}_b + q^a{}_b - Q\eta^a{}_b$. Hence, in calculating the exterior covariant derivative of the antisymmetric tensor $\epsilon^{abc\cdots s}$, the antisymmetric combinations of the connection $\omega^a{}_l\epsilon^{lbc\cdots s}$ and $q^a{}_l\epsilon^{lbc\cdots s}$ have no contribution. Since the dimension of space-time is $(n+1)$, from the symmetric combinations $Q^a{}_l\epsilon^{lbc\cdots s}$ we obtain the results of the equations (2.120) and (2.121). As a final remark, by using (2.119), (2.120) and (2.121), one can easily show that

$$D(\Gamma)\{\epsilon_a{}^b{}_c{}^{de\cdots s}\} = (3-n)Q\epsilon_a{}^b{}_c{}^{de\cdots s} \tag{2.122}$$

and

$$D(\Gamma)\{\epsilon_b{}^{cde\cdots s}\} = (1-n)Q\epsilon_b{}^{cde\cdots s}. \tag{2.123}$$

One can use (2.122) and (2.123) to prove (2.118). But the proof will not be illustrated, since a similar one (the proof of (2.77)) is given before.

Now we can express the action density (2.82) in terms of the curvature 2-forms $R^a{}_b(\Gamma)$ of the connection $\Gamma^a{}_b$. It becomes

$$\mathcal{L} = \frac{1}{2}\phi R^a{}_b(\Gamma) \wedge *(e_a \wedge e^b) - \frac{\left(\tilde{\omega} - \frac{n}{n-1}\right)}{2\phi}\mathcal{D}\phi \wedge *\mathcal{D}\phi$$

$$-\frac{1}{2}\phi^{\frac{n-3}{n-1}} dQ \wedge *dQ + mod(d). \tag{2.124}$$

This action was considered by Dirac with a potential term ($\lambda\phi^2 * 1$ with $\phi = \alpha^2$) in $(3+1)$ dimensions [8]. Similarly, in $(n+1)$ dimensions, we can also add a scale invariant potential term $V(\phi) * 1 = \lambda\phi^{\frac{n+1}{n-1}} * 1$ with a dimensionless coupling constant λ. But it is not of our interest in this work.

Similarly, Einstein field equation (2.85) becomes,

$$\begin{aligned} \frac{1}{2}\phi R^a{}_b(\Gamma) \wedge *(e_a \wedge e^b \wedge e_c) &= -\frac{\left(\tilde{\omega} - \frac{n}{n-1}\right)}{2\phi}\tau_c[\phi] - \frac{1}{2}\phi^{\frac{n-3}{n-1}}\tau_c[dQ] \\ &\quad - D(\Gamma)\{\iota_c(*\mathcal{D}\,\phi)\}, \end{aligned} \tag{2.125}$$

where $\tau_c[\phi]$ and $\tau_c[dQ]$ are given by (2.86) and (2.87), respectively. Furthermore, scalar field variational equation (2.90) becomes,

$$\begin{aligned} \frac{1}{2}R^a{}_b(\Gamma) \wedge *(e_a \wedge e^b) &= \frac{\left(\tilde{\omega} - \frac{n}{n-1}\right)}{2\phi^2}\mathcal{D}\phi \wedge *\mathcal{D}\phi - \frac{\left(\tilde{\omega} - \frac{n}{n-1}\right)}{\phi} d(*\mathcal{D}\,\phi) \\ &\quad + \frac{(n-3)}{2(n-1)}\phi^{-\frac{2}{(n-1)}}\, dQ \wedge *dQ. \end{aligned} \tag{2.126}$$

We can consider the trace of the equation (2.125), and multiply (2.126) by $(n-1)\phi$. If we subtract the resulting equations, once we obtain the equation (2.92) satisfied by the scalar field.

2.3 Axi-Dilaton Gravity In D Dimensions

We can add gauge field couplings to the scalar tensor gravity theory. We consider that a $(p+2)$-form gauge field H called the *axion* , interacts with the gravitational field. The resulting theory is called axi-dilaton gravity in D dimensions. Such an effective gravitational field theory constitutes the bosonic part of higher dimensional supergravity theories. Hence for $p = 1$, axi-dilaton gravity action can be considered as the low energy limit of effective string theory in Brans-Dicke frame. In general, axion potential $(p+1)$-form A can minimally couple to p-branes, which are extended objects in p spatial dimensions, in a background metric field. In that case, it becomes the low energy limit of effective p-brane theory. If $p = 0$ (point particle), H field becomes identical to electromagnetic, Maxwell 2-form field F. In this case, action describes Einstein-Maxwell theory with a massless scalar field in Brans-Dicke frame. Axi-dilaton gravity can be conveniently studied in Einstein frame as most of the researchers have traditionally done. However, the study of the theory in Brans-Dicke frame provides one to determine the connection structure geometrically on which the theory can be formulated. Therefore, either we can impose the connections to be Levi-Civita (torsion-free) as a constraint, or we can formulate the theory in a geometry with a connection with torsion by making independent variations of the action with respect to the connection fields. We can even formulate axi-dilaton theory in a geometry in which both torsion and non-metricity exists. In what follows, we examine all possible formulations. We assume that, the axion field also couples to the scalar field. Hence, we first consider the following action density in $D = (n+1)$ dimensions in Brans-Dicke frame in a geometry in which connections are constrained to be Levi-Civita:

$$\mathcal{L} = \frac{1}{2}\phi\, {}^{(0)}R^{ab} \wedge *(e_a \wedge e_b) - \frac{\omega}{2\phi} d\phi \wedge *d\phi - \frac{1}{2}\phi^k H \wedge *H. \tag{2.127}$$

Levi-Civita ($T^a = 0$) metric-compatible connection 1-forms ${}^{(0)}\omega^a{}_b$ satisfy structure equations

$$de^a +^{(0)} \omega^a{}_b \wedge e^b = 0,$$

and ${}^{(0)}\omega_{ab} = -{}^{(0)}\omega_{ba}$. The corresponding curvature 2-forms are obtained from

$${}^{(0)}R^{ab} = d\,{}^{(0)}\omega^{ab} +^{(0)} \omega^a{}_c \wedge^{(0)} \omega^{cb}.$$

ϕ is the dilaton 0-form field. H is a $(p+2)$-form field that is derived from a $(p+1)$-form axion potential A such that $H = dA$. ω and k are real dimensionless coupling parameters. The co-frame e^a variation of (2.127) leads to Einstein field equation

$$\begin{aligned} \frac{1}{2}\phi\,{}^{(0)}R^{ab} \wedge *(e_a \wedge e_b \wedge e_c) &= -\frac{\omega}{2\phi}(\iota_c d\phi \wedge *d\phi + d\phi \wedge \iota_c(*d\phi)) \\ &\quad -\frac{1}{2}\phi^k \tau_c[H] -^{(0)} D(\iota_c(*d\phi)) \end{aligned} \tag{2.128}$$

where

$$\tau_c[H] = \{\iota_c H \wedge *H - (-1)^p H \wedge \iota_c(*H)\}. \tag{2.129}$$

We notice the improvement term ${}^{(0)}D(\iota_c(*d\phi))$ in the torsion-free formulation. The scalar field variation of the action density (2.127) results in

$$\frac{1}{2}\,{}^{(0)}R^{ab} \wedge *(e_a \wedge e_b) = \frac{\omega}{2\phi^2} d\phi \wedge *d\phi + \frac{1}{2}k\phi^{k-1} H \wedge *H - \frac{\omega}{\phi} d(*d\phi). \tag{2.130}$$

If we consider the trace of (2.128) and multiply (2.130) by $(n-1)\phi$ and then subtract the two equations, we obtain the equation satisfied by the scalar field,

$$\left(\omega + \frac{n}{n-1}\right) d(*d\phi) = \frac{1}{2}\phi^k \alpha H \wedge *H, \tag{2.131}$$

where

$$\alpha = \frac{2p-(n-3)}{n-1} + k. \tag{2.132}$$

On the other hand, independent A field variation gives

$$d(\phi^k * H) = 0 \tag{2.133}$$

with $dH = 0$. Under local conformal rescalings of field variables such that

$$e^a \rightarrow \exp(\sigma(x))e^a,$$

$$\phi \rightarrow \exp(-(n-1)\sigma(x))\phi,$$

the action density (2.127) is conformally scale invariant for the parameter values of $\omega = -\frac{n}{(n-1)}$ and $k = -\frac{(2p-(n-3))}{(n-1)}$.

The action density (2.127) may be rewritten in terms of the $(D-p-2)$-form field

$$G \equiv \phi^k \ * \ H \tag{2.134}$$

that is dual to axion $(p+2)$-form field H. Then, we have in terms of G,

$$\mathcal{L} = \frac{1}{2}\phi\,{}^{(0)}R^{ab} \wedge *(e_a \wedge e_b) - \frac{\omega}{2\phi} d\phi \wedge *d\phi + \frac{1}{2}\phi^{-k} G \wedge *G. \tag{2.135}$$

Einstein field equation obtained from the action density (2.135) is

$$\begin{aligned} \frac{1}{2}\phi\,{}^{(0)}R^{ab} \wedge *(e_a \wedge e_b \wedge e_c) \quad &= \quad -\frac{\omega}{2\phi}\{\iota_c d\phi \wedge *d\phi + d\phi \wedge \iota_c(*d\phi)\} \\ &\quad + \frac{1}{2}\phi^{-k}\tau_c[G] -{}^{(0)} D(\iota_c(*d\phi)), \end{aligned} \tag{2.136}$$

where

$$\tau_c[G] = \{\iota_c G \wedge *G - (-1)^{(n+1-p)} G \wedge \iota_c(*G)\}. \tag{2.137}$$

The scalar field variation of (2.135) leads to,

$$\frac{1}{2}{}^{(0)}R^{ab} \wedge *(e_a \wedge e_b) = \frac{\omega}{2\phi^2} d\phi \wedge *d\phi + \frac{1}{2}k\phi^{-k-1} G \wedge *G - \frac{\omega}{\phi} d(*d\phi). \tag{2.138}$$

Taking the trace of (2.136) and multiplying equation (2.138) by $(n-1)\phi$ and then subtracting the two equations will give the scalar field equation:

$$\left(\omega+\frac{n}{n-1}\right)d(*d\phi)=\frac{1}{2}\phi^{-k}\alpha G\wedge *G. \tag{2.139}$$

Hence given any solution $\{g,\phi,H\}$ of the field equations derived from action density (2.127), we may write down a dual solution $\{g,\phi,G\}$ to the field equations derived from action density (2.135).

Interestingly we can formulate the theory in Einstein frame by adopting the transformation,

$$\tilde{e}^a=\left(\frac{\phi}{\phi_0}\right)^{\frac{1}{n-1}}e^a,$$

where ϕ_0 is a constant. In terms of Klein-Gordon field $\Phi=\ln\left(\frac{\phi}{\phi_0}\right)$, action density (2.127) becomes

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\phi_0\tilde{R}^{ab}\wedge\tilde{*}(\tilde{e}_a\wedge\tilde{e}_b)-\frac{1}{2}\left(\omega+\frac{n}{n-1}\right)\phi_0 d\Phi\wedge\tilde{*}d\Phi\\ &\quad -\frac{1}{2}(\phi_0)^k\exp(\alpha\Phi)H\wedge\tilde{*}H+mod(d).\end{aligned} \tag{2.140}$$

Einstein field equation obtained from this action density is

$$\begin{aligned}\frac{1}{2}\phi_0\tilde{R}^{ab}\wedge\tilde{*}(\tilde{e}_a\wedge\tilde{e}_b\wedge\tilde{e}_c) &= -\frac{1}{2}\left(\omega+\frac{n}{n-1}\right)\phi_0\tilde{\tau}_c[\Phi]\\ &\quad -\frac{1}{2}(\phi_0)^k\exp(\alpha\Phi)\tilde{\tau}_c[H],\end{aligned} \tag{2.141}$$

where the stress-energy n-forms $\tilde{\tau}_c[\Phi]$ and $\tilde{\tau}_c[H]$ are defined as

$$\tilde{\tau}_c[\Phi]=\{\tilde{\iota}_c d\Phi\wedge\tilde{*}d\Phi+d\Phi\wedge\tilde{\iota}_c(\tilde{*}d\Phi)\} \tag{2.142}$$

and

$$\tilde{\tau}_c[H]=\{\tilde{\iota}_c H\wedge\tilde{*}H-(-1)^p H\wedge\tilde{\iota}_c(\tilde{*}H)\}, \tag{2.143}$$

respectively. Similarly the scalar field equation becomes

$$\left(\omega + \frac{n}{n-1}\right)\phi_0 d(\tilde{*}\, d\Phi) = \frac{1}{2}(\phi_0)^k \alpha \exp(\alpha\Phi) H \wedge \tilde{*} H. \tag{2.144}$$

Interestingly, by applying a conformal transformation on the co-frames in Einstein frame, we obtain the so-called string frame action. Applying the transformation

$$\hat{e}^a = \exp(\beta\Phi)\tilde{e}^a \tag{2.145}$$

action density (2.140) becomes

$$\begin{aligned}\mathcal{L} \;=\;& \exp((1-n)\beta\Phi)\left\{\frac{1}{2}\phi_0 \hat{R}^{ab} \wedge \hat{*}(\hat{e}_a \wedge \hat{e}_b) - \frac{1}{2}\phi_0 \hat{k} d\Phi \wedge \hat{*} d\Phi\right\} \\ & -\frac{1}{2}(\phi_0)^k \exp(\alpha_0 \Phi) H \wedge \hat{*} H + mod(d),\end{aligned} \tag{2.146}$$

where

$$\alpha_0 = (2p - (n-3))\beta + \alpha \tag{2.147}$$

and

$$\hat{k} = \tilde{k} + \beta^2 n(1-n). \tag{2.148}$$

with $\tilde{k} = \omega + \frac{n}{n-1}$. Choosing $\beta = \frac{2}{n-1}$ gives the string frame action in $D = (n+1)$ dimensions. The independent connection $\hat{\omega}^{ab}$ variation of (2.146) yields

$$D(\hat{\omega})\left(\exp((1-n)\beta\Phi)\hat{*}(\hat{e}_a \wedge \hat{e}_b)\right) = 0 \tag{2.149}$$

from which we can obtain the torsion 2-forms $\hat{T}^a$ as $\hat{T}^a = \beta d\Phi \wedge \hat{e}^a$ [14]. The co-frame $\hat{e}^a$ variation gives

$$\begin{aligned}\frac{1}{2}\phi_0 e^{(1-n)\beta\Phi} \hat{R}^{ab} \wedge \hat{*}(\hat{e}_a \wedge \hat{e}_b \wedge \hat{e}_c) \;=\;& -\frac{1}{2}\phi_0 \hat{k} e^{(1-n)\beta\Phi} \hat{\tau}_c[\Phi] \\ & -\frac{1}{2}(\phi_0)^k e^{\alpha_0 \Phi} \hat{\tau}_c[H],\end{aligned} \tag{2.150}$$

where

$$\hat{\tau}_c[\Phi] = \{\hat{\imath}_c d\Phi \wedge \hat{*} d\Phi + d\Phi \wedge \hat{\imath}_c(\hat{*} d\Phi)\} \tag{2.151}$$

and

$$\hat{\tau}_c[H] = \{\hat{\imath}_c H \wedge \hat{*} H - (-1)^p H \wedge \hat{\imath}_c(\hat{*} H)\}. \tag{2.152}$$

The scalar field Φ variation yields

$$\begin{aligned} \frac{1}{2}(1-n)\beta\phi_0 e^{(1-n)\beta\Phi}\hat{R}^{ab}\wedge\hat{*}(\hat{e}_a\wedge\hat{e}_b) &= \frac{1}{2}(1-n)\beta\phi_0\hat{k}e^{(1-n)\beta\Phi}d\Phi\wedge\hat{*}d\Phi \\ &\quad -\hat{k}\phi_0 d\left(e^{(1-n)\beta\Phi}\hat{*}d\Phi\right) \\ &\quad -\frac{1}{2}(\phi_0)^k\alpha_0 e^{\alpha_0\Phi}H\wedge\hat{*}H. \end{aligned} \tag{2.153}$$

We consider the exterior multiplication of (2.150) by $\hat{e}^c$ and then multiply the equation by $(-\beta)$. If we subtract the resulting equation from (2.153) and use (2.147), we obtain the scalar field equation

$$\phi_0\hat{k}d\left(e^{(1-n)\beta\Phi}\hat{*}d\Phi\right) = \frac{1}{2}(\phi_0)^k\alpha e^{\alpha_0\Phi}H\wedge\hat{*}H. \tag{2.154}$$

Finally, the gauge field A variation results in

$$d\left(e^{\alpha_0\Phi}\hat{*}H\right) = 0. \tag{2.155}$$

The action density (2.146) and the field equations (2.150) and (2.153) can also be expressed in terms of the curvature 2-forms ${}^{(0)}R^{ab}$ of the torsion-free connections ${}^{(0)}\omega^{ab}$. Since a similar calculation is given in the first section, there is no need to present the result of the calculations.

In non-Riemannian formulation of the axi-dilaton gravity in Brans-Dicke frame, we consider the following action density in which the co-frame 1-forms e^a and the connection gauge field 1-forms ω^{ab} are varied independently:

$$\mathcal{L} = \frac{1}{2}\phi R^{ab} \wedge *(e_a \wedge e_b) - \frac{c}{2\phi} d\phi \wedge *d\phi - \frac{1}{2}\phi^k H \wedge *H. \qquad (2.156)$$

Co-frame variation of this action density yields

$$\frac{1}{2}R^{ab} \wedge *(e_a \wedge e_b \wedge e_c) = -\frac{c}{2\phi}\{\iota_c d\phi \wedge *d\phi + d\phi \wedge \iota_c(*d\phi)\} - \frac{1}{2}\phi^k \tau_c[H], \quad (2.157)$$

where $\tau_c[H]$ is given by the equation (2.129). We see that, in this formulation, no improvement term exists since the connections are not constrained to be Levi-Civita. The scalar field variation of (2.156) leads to

$$\frac{1}{2}R^{ab} \wedge *(e_a \wedge e_b) = \frac{c}{2\phi^2} d\phi \wedge *d\phi + \frac{1}{2}k\phi^{k-1} H \wedge *H - \frac{c}{\phi} d(*d\phi). \qquad (2.158)$$

Taking the trace of Einstein field equation (2.157) and multiplying (2.158) by $(n-1)\phi$ and then subtracting the two equations, gives the scalar field equation

$$cd(*d\phi) = \frac{1}{2}\phi^k \alpha H \wedge *H. \qquad (2.159)$$

On the other hand, independent connection variations of (2.156) leads to

$$D(\frac{\phi}{2} * (e^a \wedge e^b)) = 0, \qquad (2.160)$$

from which we can readily obtain the torsion 2-forms as

$$T^a = e^a \wedge \frac{d\phi}{(n-1)\phi}.$$

As before, the gauge field A variation leads to

$$d(\phi^k * H) = 0.$$

We have shown in section (1.1) that we can rewrite the field equations and the action in a spacetime with torsion, in terms of the curvature 2-forms of the Levi-Civita connections. It results in a shift of the coupling parameter c. We should state that under the conformal scalings of the field variables such that

$$e^a \rightarrow \exp(\sigma)e^a,$$

$$\phi \rightarrow \exp(-(n-1)\sigma)\phi,$$

the action density (2.156) is conformally scale invariant for the parameters $c = 0$ and $k = -\frac{(2p-(n-3))}{(n-1)}$.

Now we formulate the action density (2.156) in Einstein frame. Applying the conformal transformation

$$\tilde{e}^a = \left(\frac{\phi}{\phi_0}\right)^{\frac{1}{n-1}} e^a,$$

the action density (2.156) becomes in terms of Klein-Gordon field Φ,

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\phi_0 \tilde{R}^{ab} \wedge \tilde{*}(\tilde{e}_a \wedge \tilde{e}_b) - \frac{1}{2}c\phi_0 d\Phi \wedge \tilde{*}d\Phi \\ &\quad -\frac{1}{2}(\phi_0)^k \exp(\alpha\Phi) H \wedge \tilde{*}H + mod(d). \end{aligned} \tag{2.161}$$

The field equations are the same as (2.141) and (2.144) except that we replace $\left(\omega + \frac{n}{n-1}\right)$ by c, i.e. Einstein field equation becomes

$$\frac{1}{2}\phi_0 \tilde{R}^{ab} \wedge \tilde{*}(\tilde{e}_a \wedge \tilde{e}_b \wedge \tilde{e}_c) = -\frac{1}{2}c\phi_0 \tilde{\tau}_c[\Phi] - \frac{1}{2}(\phi_0)^k \exp(\alpha\Phi)\tilde{\tau}_c[H], \tag{2.162}$$

where $\tilde{\tau}_c[\Phi]$ and $\tilde{\tau}_c[H]$ are given by (2.142) and (2.143), respectively. Similarly the scalar field equation can be written as

$$\phi_0 c d(\tilde{*}d\Phi) = \frac{1}{2}(\phi_0)^k \alpha \exp(\alpha\Phi) H \wedge \tilde{*}H. \tag{2.163}$$

The gauge field potential A variation yields

$$d(\exp(\alpha\Phi)\tilde{*}H) = 0. \tag{2.164}$$

We notice that in the Einstein frame formulation of axi-dilaton theory with torsion, the coupling parameter c does not shift.

2.4 Axi-Dilaton Gravity In Weyl Geometry

In this section, we construct axi-dilaton gravity in a geometry in which connections are not metric compatible. So we take our scalar tensor action (2.82) and add a $(p+2)$-form gauge field H interaction to that action to obtain axi-dilaton theory in Weyl geometry. We require that the constructed action is to be Weyl symmetric such that under rescalings of the field variables

$$e^a \rightarrow \exp(\sigma(x))e^a,$$

$$\phi \rightarrow \exp(-(n-1)\sigma(x))\phi,$$

the action remains invariant. Therefore, to preserve Weyl symmetry of the total action, we impose the following constraint between p and the gauge field coupling parameter k:

$$(n-1)k + 2p - (n-3) = 0. \tag{2.165}$$

Now we consider the following action density with k satisfying (2.165):

$$\mathcal{L} = \frac{1}{2}\phi R^a{}_b(\Lambda)\wedge*(e_a\wedge e^b) - \frac{\tilde{\omega}}{2\phi}\mathcal{D}\phi\wedge*\mathcal{D}\phi - \frac{1}{2}\phi^{\frac{n-3}{n-1}}dQ\wedge*dQ - \frac{1}{2}\phi^k H\wedge*H, \tag{2.166}$$

where $\mathcal{D}\phi$ is given by (2.83). The co-frame variations of this action give us Einstein field equation

$$\frac{1}{2}\phi R^a{}_b \wedge *(e_a \wedge e^b \wedge e_c) = -\frac{\tilde{\omega}}{2\phi}\tau_c[\phi] - \frac{1}{2}\phi^{\frac{n-3}{n-1}}\tau_c[dQ] - \frac{1}{2}\phi^k\tau_c[H], \tag{2.167}$$

where the stress energy n-forms $\tau_c[\phi]$, $\tau_c[dQ]$ and $\tau_c[H]$ are defined by (2.86), (2.87) and (2.129), respectively. The scalar field variation of (2.166) leads to,

$$\begin{aligned}\frac{1}{2}R^a{}_b \wedge *(e_a \wedge e^b) &= \frac{\tilde{\omega}}{2\phi^2}\mathcal{D}\phi \wedge *\mathcal{D}\phi + \frac{(n-3)}{2(n-1)}\phi^{-\frac{2}{n-1}}dQ \wedge *dQ \\ &+\frac{1}{2}k\phi^{k-1}H \wedge *H - \frac{\tilde{\omega}}{\phi}d(*\mathcal{D}\,\phi). \end{aligned} \tag{2.168}$$

As before, we take the trace of (2.167) and multiply (2.168) by $(n-1)\phi$. Subtracting the two resulting equations and using (2.165), we obtain the scalar field equation which reads

$$(n-1)\tilde{\omega}d(*\mathcal{D}\,\phi) = 0. \tag{2.169}$$

On the other hand, independent connection variations lead to (2.95), i.e.

$$\frac{1}{2}D(\phi * (e_a \wedge e^b)) + \frac{\eta^b{}_a}{(n+1)}\{d(\phi^{\frac{n-3}{n-1}} * dQ) + (n-1)\,\tilde{\omega}\, * \mathcal{D}\,\phi\} = 0.$$

Antisymmetric part of this equation determines torsion

$$T^a = e^a \wedge \frac{d\phi}{(n-1)\phi} - Q \wedge e^a.$$

Symmetric part yields

$$d(\phi^{\frac{n-3}{n-1}} * dQ) = -(n-1)\,\tilde{\omega}\, * \mathcal{D}\,\phi. \tag{2.170}$$

Exterior differentiation of (2.170) reproduces the scalar field equation (2.169). Interestingly, we can formulate axi-dilaton gravity in Weyl geometry, in torsion-free Einstein frame by applying a conformal transformation on the co-frames e^a. With the transformation

$$\tilde{e}^a = \left(\frac{\phi}{\phi_0}\right) e^a,$$

new connection fields $\tilde{\omega}^a{}_b$ can be written in terms of $\Lambda^a{}_b$ as

$$\tilde{\omega}^a{}_b = \Lambda^a{}_b + \Omega^a{}_b \tag{2.171}$$

where

$$\Omega^a{}_b = \iota^a Q\, e_b - \iota_b Q\, e^a. \tag{2.172}$$

Therefore new curvature two forms become

$$\tilde{R}^a{}_b = R^a{}_b(\Lambda) + D(\Lambda)(\Omega^a{}_b) + \Omega^a{}_c \wedge \Omega^c{}_b. \tag{2.173}$$

Substituting (2.173) into (2.166), one obtains the action density in Einstein frame in terms of Klein-Gordon field $\Phi = \ln\left(\frac{\phi}{\phi_0}\right)$ as

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}\phi_0 \tilde{R}^a{}_b \wedge \tilde{*}(\tilde{e}_a \wedge \tilde{e}^b) - \frac{1}{2}\phi_0\tilde{\omega} d\Phi \wedge \tilde{*} d\Phi - (n-1)\phi_0\tilde{\omega} d\Phi \wedge \tilde{*} Q \\
& -\frac{1}{2}\phi_0(n-1)\{(n-1)\tilde{\omega} - n\} Q \wedge \tilde{*} Q - \frac{1}{2}(\phi_0)^{\frac{n-3}{n-1}} dQ \wedge \tilde{*} dQ \\
& -\frac{1}{2}(\phi_0)^k H \wedge \tilde{*} H + mod(d).
\end{aligned} \tag{2.174}$$

Then Einstein field equation becomes

$$\begin{aligned}
\frac{1}{2}\phi_0 \tilde{R}^a{}_b \wedge \tilde{*}(\tilde{e}_a \wedge \tilde{e}^b \wedge \tilde{e}_c) &= -\frac{1}{2}(\phi_0)^{\frac{n-3}{n-1}} \tilde{\tau}_c[dQ] - \frac{1}{2}\phi_0\tilde{\omega}\tilde{\tau}_c[\Phi] \\
&- \frac{1}{2}(n-1)\phi_0\tilde{\omega}\{\tilde{\iota}_c d\Phi \wedge \tilde{*} Q + Q \wedge \tilde{\iota}_c(\tilde{*} d\Phi) \\
&+ \tilde{\iota}_c Q \wedge \tilde{*} d\Phi + d\Phi \wedge \tilde{\iota}_c(\tilde{*} Q)\} \\
&- \frac{1}{2}\phi_0(n-1)\{(n-1)\tilde{\omega} - n\}\tilde{\tau}_c[Q] \\
&- \frac{1}{2}(\phi_0)^k \tilde{\tau}_c[H],
\end{aligned} \tag{2.175}$$

where $\tilde{\tau}_c[\Phi]$ and $\tilde{\tau}_c[H]$ are given by (2.142) and (2.143), respectively and $\tilde{\tau}_c[Q]$ and $\tilde{\tau}_c[dQ]$ are expressed by

$$\tilde{\tau}_c[Q] = \{\tilde{\iota}_c Q \wedge \tilde{*} Q + Q \wedge \tilde{\iota}_c(\tilde{*} Q)\} \tag{2.176}$$

and

$$\tilde{\tau}_c[dQ] = \{\tilde{\iota}_c dQ \wedge \tilde{*}dQ - dQ \wedge \tilde{\iota}_c(\tilde{*}dQ)\}. \tag{2.177}$$

The scalar field variation of the action density (2.174) results in

$$\tilde{\omega}\phi_0 d(\tilde{*}d\Phi) = -(n-1)\tilde{\omega}\phi_0 d(\tilde{*}Q). \tag{2.178}$$

Q field variations give

$$(\phi_0)^{\frac{n-3}{n-1}} d(\tilde{*}dQ) = -(n-1)\phi_0\{(n-1)\tilde{\omega} - n\}\tilde{*}Q - (n-1)\tilde{\omega}\phi_0\tilde{*}d\Phi. \tag{2.179}$$

Finally the gauge field A variation yields

$$d(\tilde{*}H) = 0. \tag{2.180}$$

We notice that when we make a conformal transformation of the field variables in axi-dilaton theory in a geometry in which connections are not metric-compatible, from the so-called Brans-Dicke frame to Einstein frame, the scalar field coupling with axion field is removed. This is not the case in metric-compatible theories. In fact it is a consequence of Weyl symmetry of the action. We can state that the geometrical structure on which a theory is constructed, does affect the interactions.

2.5 The Motion Of (Spinless) Massive Test Particles

In this section, we study the motion of (spinless) massive test particles in the geometries mentioned in the previous sections. Before we continue, we should give the definition of parallelism of a vector field along a curve C. A vector field

Y along a curve is said to be parallel along C if it satisfies

$$\nabla_{\dot{C}} Y = 0, \tag{2.181}$$

where $\dot{C}$ is tangent vector to curve C and ∇ is a linear connection on a manifold M. A curve is an autoparallel of a connection ∇ if its tangent vector field is parallel or covariantly constant along C. Therefore autoparallel curves are given as solutions to equation

$$\nabla_{\dot{C}} \dot{C} = 0. \tag{2.182}$$

We can rewrite this equation in coordinate form. If an autoparallel C is parametrised in terms of parameter τ in a local chart such that $C : \tau \mapsto x^\mu(\tau)$, then these coordinate functions satisfy

$$\frac{d}{d\tau}\left(\frac{dx^\mu}{d\tau}\right) + \Gamma_{\nu\lambda}{}^{\mu} \frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0, \tag{2.183}$$

where connection coefficients $\Gamma_{\nu\lambda}{}^{\mu}$ are defined in any convenient basis $\{X_\mu\}$ according to the equation

$$\nabla_{X_\nu} X_\lambda = \Gamma_{\nu\lambda}{}^{\mu} X_\mu. \tag{2.184}$$

The parameter τ is to be interpreted as the proper time. We note that if we reparametrise C in terms of another parameter t such that $t = C_1\tau + C_2$ then reparametrised curve $C(t)$ is also an autoparallel, i.e

$$\nabla_{\dot{C}} \dot{C} = 0. \tag{2.185}$$

where in this case $\dot{C}$ denotes $dC(t)/dt$.

Acceleration of a particle moving along an arbitrary curve C is defined to be the vector field $\nabla_{\dot{C}} \dot{C}$ on C [9]. Therefore a free particle of mass m moves along the trajectory of an autoparallel of natural connection ∇ parametrised by a proper time τ, i.e. the equation of motion of a free particle is an autoparallel

$$m\nabla_{\dot{C}}\dot{C} = 0. \tag{2.186}$$

A point particle of inertial mass m moving in a non-autoparallel curve C parametrised by the Newtonian time, experiences a force $\mathcal{F}$ on C and the equation of motion is given by

$$\mathcal{F} = m\nabla_{\dot{C}}\dot{C}. \tag{2.187}$$

For example, if C describes the curve or trajectory of a particle of electric charge q with a mass m moving in an electromagnetic field described by two form field F, then the force on the particle is given by $\tilde{\mathcal{F}} = q\iota_{\dot{C}}F$ where one form $\tilde{\mathcal{F}}$ denotes the dual of the force vector $\mathcal{F}$ and we choose the units such that the speed of light c is taken as unity ($c = 1$). In that case, the equation of motion becomes

$$\nabla_{\dot{C}}\widetilde{(m\dot{C})} = q\iota_{\dot{C}}F. \tag{2.188}$$

In the following we deal with the autoparallel equations of motion of free particles in the geometries discussed in the previous sections. We restrict our spacetime to $(3+1)$ dimensions. Before examining equations, we give the idea to evaluate the connection coefficients in respective geometries. Given three arbitrary vector fields, X, Y and Z, consider the action of X on $g(Y,Z)$ where g is $(2,0)$ symmetric metric tensor field specified on spacetime manifold M. $X(g(Y,Z))$ can be written as

$$X(g(Y,Z)) = \nabla_X g(Y,Z) + g(\nabla_X Y, Z) + g(Y, \nabla_X Z), \tag{2.189}$$

where $\nabla_X g(Y,Z)$ is defined as the non-metricity $(3,0)$ tensor $S(X,Y,Z)$, which is symmetric in its last two arguments. Hence (2.189) becomes

$$X(g(Y,Z)) = S(X,Y,Z) + g(\nabla_X Y, Z) + g(Y, \nabla_X Z). \tag{2.190}$$

We consider cyclic permutations between X, Y, and Z and obtain the following expressions:

$$Z(g(X,Y)) = S(Z,X,Y) + g(\nabla_Z X, Y) + g(X, \nabla_Z Y) \tag{2.191}$$

and

$$Y(g(Z,X)) = S(Y,Z,X) + g(\nabla_Y Z, X) + g(Z, \nabla_Y X). \tag{2.192}$$

Adding (2.190) and (2.192) and subtracting (2.191) gives

$$\begin{aligned} X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) &= S(X,Y,Z) - S(Z,X,Y) \\ &+ S(Y,Z,X) + g(\nabla_X Y, Z) \\ &+ g(Y, \nabla_X Z) + g(\nabla_Y Z, X) \\ &+ g(Z, \nabla_Y X) - g(\nabla_Z X, Y) \\ &- g(X, \nabla_Z Y). \end{aligned} \tag{2.193}$$

To simplify expression (2.193), we use the definition of (2-antisymmetric,1) torsion tensor field T [9],

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \tag{2.194}$$

which satisfies $T(X,Y) = -T(Y,X)$. T is a type (2,1) tensor field. Torsion two forms T^a can be written in terms of the torsion tensor T as

$$T^a\,(X,Y) = \frac{1}{2} e^a(T(X,Y)). \tag{2.195}$$

In terms of torsion two forms T^a, torsion tensor T can be written as

$$T = 2T^a \otimes X_a. \tag{2.196}$$

Now, using (2.194), the equation (2.193) can be rewritten as

$$\begin{aligned} 2g(Z, \nabla_X Y) \;=\;& X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) \\ & - g(X,[Y,Z]) - g(Y,[X,Z]) - g(Z,[Y,X]) \\ & - g(X,T(Y,Z)) - g(Y,T(X,Z)) - g(Z,T(Y,X)) \\ & + S(Z,X,Y) - S(Y,Z,X) - S(X,Y,Z). \end{aligned} \tag{2.197}$$

To calculate the connection coefficients, we choose vectors

$$X = \partial_\mu, \qquad Y = \partial_\nu, \qquad Z = \partial_\beta. \tag{2.198}$$

Then $\nabla_X Y$ becomes

$$\nabla_X Y = \nabla_{\partial_\mu} \partial_\nu = \Gamma_{\mu\nu}{}^{\alpha}\, \partial_\alpha.$$

i-) Autoparallel equation of motion with $S = 0$ and $T = 0$:

With the choice (2.198) of vectors, since

$$g(\partial_\mu, \partial_\nu) = g_{\mu\nu}, \tag{2.199}$$

we can evaluate e.g.

$$g(Z, \nabla_X Y) = g(\partial_\beta, \Gamma_{\mu\nu}{}^{\alpha}\, \partial_\alpha) = \Gamma_{\mu\nu}{}^{\alpha} g_{\alpha\beta}$$

and

$$X(g(Y,Z)) = \partial_\mu\, g_{\nu\beta} = g_{\nu\beta\,,\mu}.$$

Also, all the commutations vanish i.e $[X,Y]=0$, $[X,Z]=0$ and $[Y,Z]=0$. Therefore, in the absence of torsion and non-metricity, equation (2.197) becomes

$$2\Gamma_{\mu\nu}{}^{\alpha}\, g_{\alpha\beta} = g_{\nu\beta\,,\mu} + g_{\mu\beta\,,\nu} - g_{\mu\nu\,,\beta}. \tag{2.200}$$

Multiplying (2.200) by $g^{\sigma\beta}$ and using $g_{\alpha\beta}g^{\beta\sigma} = \delta_{\alpha}{}^{\sigma}$, we can obtain Levi-Civita connection coefficients

$$\Gamma_{\mu\nu}{}^{\sigma} = \frac{1}{2} g^{\sigma\beta}\{g_{\nu\beta\,,\mu} + g_{\mu\beta\,,\nu} - g_{\mu\nu\,,\beta}\}. \tag{2.201}$$

Denoting the Levi-Civita connection coefficients by $\{{}^{\sigma}_{\mu\nu}\}$, we can write the Levi-Civita autoparallel equation of motion

$$\hat{\nabla}_{\dot{C}}\, \dot{C} = 0 \tag{2.202}$$

in cooordinate representation

$$\frac{d}{d\tau}\left(\frac{dx^{\mu}}{d\tau}\right) + \{{}^{\mu}_{\nu\lambda}\}\frac{dx^{\nu}}{d\tau}\frac{dx^{\lambda}}{d\tau} = 0. \tag{2.203}$$

This is the so-called geodesics equation of motion for massive spinless test particles in which the connection coefficients depend only on the spacetime metric.

ii-) Autoparallel equation of motion with $S = 0$ but $T \neq 0$:

Now, we evaluate the connection coefficients with torsion. The torsion tensor can be expressed in terms of the torsion 2-forms T^a through (2.196), such that the torsion 2-forms T^a are given by the gradient of the scalar field, i.e. $T^a = e^a \wedge \frac{d\phi}{2\phi}$. Then T becomes

$$T = 2\frac{1}{2\phi}(e^a \wedge d\phi) \otimes X_a = \frac{1}{\phi}(e^a \wedge d\phi) \otimes X_a.$$

This can be expressed as a tensor product,

$$T = \frac{1}{2\phi}(e^a \otimes d\phi - d\phi \otimes e^a) \otimes X_a. \tag{2.204}$$

Therefore,

$$T = \frac{1}{2\phi}\{e^a \otimes d\phi \otimes X_a - d\phi \otimes e^a \otimes X_a\}. \tag{2.205}$$

Since this can be written in any frame (orthonormal or not), we can also write

$$T = \frac{1}{2\phi}\{dx^\gamma \otimes d\phi \otimes X_\gamma - d\phi \otimes dx^\gamma \otimes X_\gamma\}. \tag{2.206}$$

Now we can evaluate e.g. $T(Y, Z)$ and $g(X, T(Y, Z))$,

$$\begin{aligned} T(Y,Z) &= T(\partial_\nu, \partial_\beta) \\ &= \frac{1}{2\phi}\{dx^\gamma(\partial_\nu) \otimes d\phi(\partial_\beta) \otimes X_\gamma - d\phi(\partial_\nu) \otimes dx^\gamma(\partial_\beta) \otimes X_\gamma\}. \end{aligned}$$

Using $d\phi(\partial_\nu) = \partial_\nu \phi$ and $dx^\gamma(\partial_\nu) = \delta^\gamma_\nu$, $T(Y, Z) = T(\partial_\nu, \partial_\beta)$ becomes

$$\begin{aligned} T(\partial_\nu, \partial_\beta) &= \frac{1}{2\phi}\{\delta^\gamma_\nu\, \partial_\beta \phi\, X_\gamma - \partial_\nu \phi\, \delta^\gamma_\beta\, X_\gamma\} \\ &= \frac{1}{2\phi}\{X_\nu \partial_\beta \phi - \partial_\nu \phi X_\beta\}. \end{aligned}$$

Now, since $X_\nu = \partial_\nu$,

$$\begin{aligned} g(X, T(Y,Z)) &= \frac{1}{2\phi}\{\partial_\beta \phi g(\partial_\mu, \partial_\nu) - \partial_\nu \phi g(\partial_\mu, \partial_\beta)\} \\ &= \frac{1}{2\phi}\{\partial_\beta \phi g_{\mu\nu} - \partial_\nu \phi g_{\mu\beta}\}. \end{aligned}$$

We calculate the other components of torsion tensor in a similar way. We substitute all the results into (2.197). After simplifications, we obtain

$$\Gamma_{\mu\nu}{}^{\sigma} = \frac{1}{2} g^{\sigma\beta}\{g_{\mu\beta,\nu} + g_{\nu\beta,\mu} - g_{\mu\nu,\beta}\} - \frac{1}{2\phi}\{g^{\sigma\beta}\partial_\beta \phi g_{\mu\nu} - \delta^\sigma_\mu \partial_\nu \phi\}. \tag{2.207}$$

If C is parametrised in terms of the proper time τ, the autoparallel equation of motion with connection ∇

$$\nabla_{\dot{C}}\dot{C} = 0$$

can be expressed in local coordinates $x^\sigma(\tau)$ as

$$\frac{d^2x^\sigma}{d\tau^2} + \{{}^{\sigma}_{\mu\nu}\}\frac{dx^\nu}{d\tau}\frac{dx^\mu}{d\tau} - \frac{1}{2\phi}\{g^{\sigma\beta}\partial_\beta\phi g_{\mu\nu} - \delta^\sigma_\mu\partial_\nu\phi\}\frac{dx^\nu}{d\tau}\frac{dx^\mu}{d\tau} = 0. \tag{2.208}$$

In the notation of exterior forms, one can realise that $\iota_{\dot{C}}d\phi = \partial_\mu\phi\dot{x}^\mu$ and $\iota_{\dot{C}}\dot{C} = g(\dot{C},\dot{C}) = g_{\mu\nu}\dot{x}^\nu\dot{x}^\mu$. Denoting the dual one form of the tangent four-velocity C by $\tilde{\dot{C}}$, one can express (2.208) in exterior differential form notation in terms of Levi-Civita connection $\hat{\nabla}$, i.e.

$$\widetilde{\hat{\nabla}_{\dot{C}}\dot{C}} = -\frac{1}{2\phi}\iota_{\dot{C}}\,(d\phi\wedge\tilde{\dot{C}}). \tag{2.209}$$

We see that, the right hand side of this equation can be interpreted as a torsion acceleration field which is analogous to Lorentz force equation on charged particles discussed above. However, it can be seen that torsion force produces same acceleration on all massive particles; on the contrary, acceleration produced by Lorentz force depends on the charge of the particles [2], and it produces different accelerations for different masses. For a timelike autoparallel, four-velocity $\dot{C}$ is normalized with

$$\mathbf{g}(\dot{C},\dot{C}) = -1, \tag{2.210}$$

where we choose the units such that the speed of light is taken as unity ($c = 1$). Then multiplying equation (2.208) by $\phi^{1/2}$ and noting that $\frac{d(\phi^{1/2})}{d\tau} = \frac{1}{2\phi^{1/2}}\partial_\mu\phi\,\dot{x}^\mu$

and $g_{\mu\nu}\dot{x}^\nu\dot{x}^\mu = -1$, one can further simplify the equation (2.208) and obtain

$$\frac{d}{d\tau}\left(\phi^{1/2}\frac{dx^\sigma}{d\tau}\right) + \phi^{1/2}\{^{\sigma}_{\mu\nu}\}\frac{dx^\nu}{d\tau}\frac{dx^\mu}{d\tau} = -\frac{1}{2\phi^{1/2}}g^{\beta\sigma}\,\partial_\beta\phi. \tag{2.211}$$

Interestingly, (2.211) may be expressed in exterior forms as

$$\hat{\nabla}_{\dot{C}}\,(\widetilde{\phi^{1/2}\dot{C}}) = -d(\phi^{1/2}). \tag{2.212}$$

iii-)Autoparallel equation of motion with $S \neq 0$ but $T = 0$:

In this section, we evaluate the connection coefficients and discuss the autoparallel motion with non-metricity but zero torsion. In the geometry specified by $S = \nabla_X\, g = 2Q(X)g$, S can be expressed as a tensor product by

$$S = 2Q \otimes g \tag{2.213}$$

where $g = g_{\nu\mu}dx^\nu \otimes dx^\mu$. Then $(3,0)$ S tensor components with X, Y, Z vectors given by (2.198), can be evaluated as

$$\begin{aligned} S(X,Y,Z) &= 2Q(\partial_\mu)\otimes g(\partial_\nu,\partial_\beta) = 2Q_\mu g_{\nu\beta}, \\ S(Y,Z,X) &= 2Q(\partial_\nu)\otimes g(\partial_\beta,\partial_\mu) = 2Q_\nu g_{\beta\mu}, \\ S(Z,X,Y) &= 2Q(\partial_\beta)\otimes g(\partial_\mu,\partial_\nu) = 2Q_\beta g_{\mu\nu}. \end{aligned} \tag{2.214}$$

Then substituting these tensor components into (2.197), we obtain the connection coefficient expression,

$$\Gamma_{\mu\nu}{}^{\sigma} = \frac{1}{2}g^{\sigma\beta}\{g_{\mu\beta\,,\nu} + g_{\nu\beta\,,\mu} - g_{\mu\nu\,,\beta}\} + g^{\sigma\beta}\{g_{\mu\nu}Q_\beta - g_{\beta\mu}Q_\nu - g_{\nu\beta}Q_\mu\}. \tag{2.215}$$

Then, the timelike autoparallel equation of motion

$$\nabla_{\dot{C}}\,\dot{C} = 0$$

becomes after rearrangements,

$$\frac{d}{d\tau}\left(\frac{dx^{\sigma}}{d\tau}\right)+\left\{{}^{\sigma}_{\mu\nu}\right\}\frac{dx^{\nu}}{d\tau}\frac{dx^{\mu}}{d\tau}=g^{\sigma\beta}Q_{\beta}+2Q_{\mu}\dot{x}^{\mu}\dot{x}^{\sigma} \tag{2.216}$$

where we use $g_{\mu\nu}\dot{x}^{\nu}\dot{x}^{\mu}=-1$. Equation (2.216) can be expressed in exterior forms as

$$\widetilde{\hat{\nabla}_{\dot{C}}\dot{C}}=Q+2\left(\iota_{\dot{C}}Q\right)\tilde{\dot{C}}. \tag{2.217}$$

We can interpret the right hand side of this equation as acceleration produced by the non-metric connection field Q which yields the same acceleration on all massive test particles [15].

iv-)Autoparallel equation of motion with $S\neq 0$ and $T\neq 0$:

In this section, we examine the case in which massive test particles move along the autoparallel of a connection with non-metricity and torsion. Torsion two forms are given by

$$T^{a}=e^{a}\wedge\frac{d\phi}{2\phi}+e^{a}\wedge Q. \tag{2.218}$$

Then the torsion tensor defined by $T=2T^{a}\otimes X_{a}$ can be written as

$$T=T_{I}+T_{II}, \tag{2.219}$$

where

$$T_{I}=\frac{1}{2\phi}\{e^{a}\otimes d\phi\otimes X_{a}-d\phi\otimes e^{a}\otimes X_{a}\} \tag{2.220}$$

and

$$T_{II}=e^{a}\otimes Q\otimes X_{a}-Q\otimes e^{a}\otimes X_{a}. \tag{2.221}$$

Now we can evaluate e.g. $T(Y,Z)$ and $g(X,T(Y,Z))$ with X, Y and Z given by (2.198). We have calculated $T_{I}(Y,Z)$ in part (ii). Then writing T_{II} in terms of non-inertial basis

$$T_{II} = \{dx^\gamma \otimes Q \otimes X_\gamma - Q \otimes dx^\gamma \otimes X_\gamma\}, \tag{2.222}$$

we calculate

$$\begin{aligned} T_{II}(Y,Z) &= T_{II}(\partial_\nu, \partial_\beta) \\ &= \{dx^\gamma(\partial_\nu) \otimes Q(\partial_\beta) \otimes X_\gamma - Q(\partial_\nu) \otimes dx^\gamma(\partial_\beta) \otimes X_\gamma\} \\ &= Q_\beta X_\nu - Q_\nu X_\beta, \end{aligned} \tag{2.223}$$

and

$$\begin{aligned} g(X, T_{II}(Y,Z)) &= Q_\beta g(\partial_\mu, \partial_\nu) - Q_\nu g(\partial_\mu, \partial_\beta) \\ &= Q_\beta g_{\mu\nu} - Q_\nu g_{\mu\beta}. \end{aligned} \tag{2.224}$$

We can evaluate other components similarly. Components of the non-metricity tensor given by $S = 2Q \otimes g$ have been calculated in part (iii). Therefore, we substitute all the results into equation (2.197) and simplify to obtain

$$\Gamma_{\mu\nu}{}^{\sigma} = \frac{1}{2} g^{\sigma\beta}\{g_{\mu\beta\,,\nu} + g_{\nu\beta\,,\mu} - g_{\mu\nu\,,\beta}\} - \frac{1}{2\phi}\{g^{\sigma\beta}\,\partial_\beta\phi\, g_{\mu\nu} - \delta^\sigma_\mu\,\partial_\nu\phi\} - \delta^\sigma_\nu Q_\mu. \tag{2.225}$$

Then the autoparallel equation

$$\nabla_{\dot{C}}\dot{C} = 0$$

becomes

$$\begin{aligned} \frac{d}{d\tau}\left(\frac{dx^\sigma}{d\tau}\right) + \{^{\sigma}_{\mu\nu}\}\frac{dx^\nu}{d\tau}\frac{dx^\mu}{d\tau} &= Q_\mu \frac{dx^\mu}{d\tau}\frac{dx^\sigma}{d\tau} + \frac{1}{2\phi}\{g^{\sigma\beta}\partial_\beta\phi g_{\mu\nu}\frac{dx^\nu}{d\tau}\frac{dx^\mu}{d\tau} \\ &\quad - \partial_\nu\phi\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau}\}. \end{aligned} \tag{2.226}$$

Noting that $\iota_{\dot{C}} Q = Q_\mu \dot{x}^\mu$, equation (2.226) can be written in exterior forms as

$$\widetilde{\hat{\nabla}_{\dot{C}}\,\dot{C}} = \iota_{\dot{C}} Q\,\tilde{\dot{C}} - \frac{1}{2\phi}\iota_{\dot{C}}(d\phi \wedge \tilde{\dot{C}}). \tag{2.227}$$

As before, the right hand side of this expression can be interpreted as the acceleration produced by both torsion and non-metric connection field. We note that these yield same acceleration on all massive test particles. As in part (ii), we can multiply equation (2.226) by $\phi^{1/2}$. The resulting timelike ($g_{\mu\nu}\dot{x}^\nu \dot{x}^\mu = -1$) autoparallel equation of motion is

$$\frac{d}{d\tau}\left(\phi^{1/2}\frac{dx^\sigma}{d\tau}\right) + \phi^{1/2}\left\{{}^{\,\sigma}_{\mu\nu}\right\}\frac{dx^\nu}{d\tau}\frac{dx^\mu}{d\tau} = \phi^{1/2} Q_\mu \frac{dx^\mu}{d\tau}\frac{dx^\sigma}{d\tau} - \frac{1}{2\phi^{1/2}} g^{\sigma\beta}\partial_\beta \phi. \tag{2.228}$$

This in turn can be expressed in exterior forms as

$$\hat{\nabla}_{\dot{C}}\,(\widetilde{\phi^{1/2}\,\dot{C}}) = \phi^{1/2}\iota_{\dot{C}} Q\,\tilde{\dot{C}} - d(\phi^{1/2}). \tag{2.229}$$

We see that, the autoparallel equations depend on the connection structure of spacetime geometry. Writing the autoparallel equations in terms of torsion-free connections results in equations of motion with forcing terms. In some cases, we can study actions in which non-metric compatible connection field $Q^a{}_b$ is constrained to depend on scalar matter and torsion field is constrained to depend on contraction of some antisymmetric tensor field as in [16] and [17]. In this sense, it is possible to obtain the action of some fundamental interactions from pure gravity action with such constraints. Therefore, Einstein-Hilbert action density

$$\mathcal{L} = R^a{}_b(\Lambda) \wedge *(e_a \wedge e^b) \tag{2.230}$$

in $D = (n+1)$ dimensions with non-metric compatible connections with torsion, subject to constraints $Q^a{}_b = \frac{d\Phi}{2}\eta^a{}_b$ and $T^a = \exp(\frac{\alpha}{2}\Phi)\iota^a H$ where Φ is a scalar matter field and H is a 3-form string field, produces low energy bosonic string action in $(n+1)$ dimensions

$$\mathcal{L} = R^a{}_b(\omega)\wedge*(e_a\wedge e^b) - \frac{(n-1)n}{4}d\Phi\wedge*d\Phi - \frac{3\exp(\alpha\Phi)}{2}H\wedge*H + mod(d), \quad (2.231)$$

where $R^a{}_b(\omega)$ describes the curvature two forms associated with Levi Civita connections. This shows that, the gravitational interactions with matter couplings can be simply reformulated in terms of connections with non-metricity and non-zero torsion. Therefore matter can be geometrised. This provides the framework for the geometrical unification of interactions. It may be even possible to geometrise the supergravity interactions with fermionic fields by adjusting the constraints accordingly.

Now we consider the autoparallel motion of a massive test particle based on a geometry represented by a torsion-free connection $\nabla^{(\tilde{g},0)}$ with metric field $\tilde{g}$ where $\tilde{g} = \phi g$. g can be conventionally identified as the Brans-Dicke frame metric and $\tilde{g}$ can be identified as the Einstein frame metric. Then we can show that the autoparallel motion based on $\nabla^{(g,T)}$, with torsion T given by $T = 2T^a \otimes X_a$ and $T^a = e^a\wedge\frac{d\phi}{2\phi}$, is identical to the autoparallel motion based on the connection $\nabla^{(\tilde{g},0)}$. First, we develop a relation between the connection coefficients in two frames. The connection coefficient related with connection $\nabla^{(g,T)}$ with $T^a = e^a \wedge \frac{d\phi}{2\phi}$, has been obtained in part (ii). The connection coefficient related with connection $\nabla^{(\tilde{g},0)}$

can be written as

$$\tilde{\Gamma}_{\mu\nu}{}^{\sigma} = \frac{1}{2}\tilde{g}^{\sigma\beta}\{\tilde{g}_{\mu\beta\,,\nu} + \tilde{g}_{\nu\beta\,,\mu} - \tilde{g}_{\mu\nu\,,\beta}\}. \tag{2.232}$$

We can express the right hand side of equation (2.232) in terms of the metric g. Using $\tilde{g} = \phi g$ and noting that $\tilde{g}^{\mu\nu} = \phi^{-1}g^{\mu\nu}$, we can write

$$\begin{aligned}\tilde{\Gamma}_{\mu\nu}{}^{\sigma} &= \frac{1}{2}\phi^{-1}g^{\sigma\beta}\{\phi g_{\mu\beta\,,\nu} + \partial_\nu\phi g_{\mu\beta} + \phi g_{\nu\beta\,,\mu} \\ &\quad +\partial_\mu\phi g_{\nu\beta} - \partial_\beta\phi g_{\mu\nu} - \phi g_{\mu\nu\,,\beta}\}. \end{aligned} \tag{2.233}$$

We can simplify (2.233) and express it in terms of the connection coefficients $\Gamma_{\mu\nu}{}^{\sigma}$ given in part (ii) (equation (2.207)) to obtain

$$\tilde{\Gamma}_{\mu\nu}{}^{\sigma} = \Gamma_{\mu\nu}{}^{\sigma} + \frac{1}{2\phi}\delta^{\sigma}_{\nu}\partial_\mu\phi. \tag{2.234}$$

Therefore, we conclude that

$$\nabla^{(\tilde{g},0)} = \frac{d\phi}{2\phi} + \nabla^{(g,T)}. \tag{2.235}$$

Denoting the tangent four-velocity by V_g in g frame, timelike autoparallel of the connection $\nabla^{(g,T)}$ can be written as

$$\nabla^{(g,T)}_{V_g} V_g = 0, \tag{2.236}$$

where V_g is normalised according to

$$g(V_g, V_g) = -1. \tag{2.237}$$

Now we consider the autoparallel equation of $\nabla^{(\tilde{g},0)}$. Denoting tangent four-velocity by $V_{\tilde{g}}$ in $\tilde{g}$ frame, one can write related time-like autoparallel as

$$\nabla^{(\tilde{g},0)}_{V_{\tilde{g}}} V_{\tilde{g}} = 0. \tag{2.238}$$

where $V_{\tilde{g}}$ is normalised with $c = 1$ according to

$$\tilde{g}(V_{\tilde{g}}, V_{\tilde{g}}) = -1. \tag{2.239}$$

In order that equations (2.236) and (2.238) be identical, parametrisations for two autoparallels should be different [1]. Thus, we assume that the autoparallel curve C of $\nabla^{(g,T)}$ is parametrised such that $C : \tau \mapsto x^\sigma(\tau)$ in any coordinates x^σ and in the same coordinates we assume that the autoparallel curve $\tilde{C}$ of $\nabla^{(\tilde{g},0)}$ is parametrised such that $\tilde{C} : \tilde{\tau} \mapsto x^\sigma(\tilde{\tau})$. Equation (2.237) implies that

$$g_{\mu\nu}\frac{dx^\nu}{d\tau}\frac{dx^\mu}{d\tau} = -1.$$

Using $\tilde{g} = \phi g$, and assuming that the two parametrisations are functions of each other, we can write

$$\phi^{-1}\tilde{g}_{\nu\mu}\frac{dx^\nu}{d\tilde{\tau}}\frac{dx^\mu}{d\tilde{\tau}}\left(\frac{d\tilde{\tau}}{d\tau}\right)^2 = -1. \tag{2.240}$$

Using (2.239), we obtain

$$\left(\frac{d\tilde{\tau}}{d\tau}\right)^2 = \phi. \tag{2.241}$$

This implies that $\frac{d\tilde{\tau}}{d\tau} = \phi^{1/2}$. Noting that

$$\frac{dx^\sigma}{d\tau} = \frac{d\tilde{\tau}}{d\tau}\frac{dx^\sigma}{d\tilde{\tau}} = \phi^{1/2}\frac{dx^\sigma}{d\tilde{\tau}},$$

and

$$\begin{aligned}\frac{d}{d\tau}\left(\phi^{1/2}\frac{dx^\sigma}{d\tilde{\tau}}\right) &= \phi^{1/2}\frac{d\tilde{\tau}}{d\tau}\frac{d}{d\tilde{\tau}}\left(\frac{dx^\sigma}{d\tilde{\tau}}\right) + \frac{1}{2}\phi^{-1/2}\partial_\nu\phi\frac{d\tilde{\tau}}{d\tau}\frac{dx^\nu}{d\tilde{\tau}}\frac{dx^\sigma}{d\tilde{\tau}} \\ &= \frac{d}{d\tilde{\tau}}\left(\frac{dx^\sigma}{d\tilde{\tau}}\right)\phi + \frac{1}{2}\partial_\nu\phi\frac{dx^\nu}{d\tilde{\tau}}\frac{dx^\sigma}{d\tilde{\tau}},\end{aligned}$$

and using (2.234) and (2.241), the autoparallel equation of the connection $\nabla^{(g,T)}$

$$\frac{d}{d\tau}\left(\frac{dx^\sigma}{d\tau}\right) + \Gamma_{\mu\nu}{}^\sigma\frac{dx^\nu}{d\tau}\frac{dx^\mu}{d\tau} = 0$$

transforms into,

$$\phi\frac{d}{d\tilde{\tau}}\left(\frac{dx^{\sigma}}{d\tilde{\tau}}\right)+\frac{1}{2}\partial_{\nu}\phi\frac{dx^{\nu}}{d\tilde{\tau}}\frac{dx^{\sigma}}{d\tilde{\tau}}+\tilde{\Gamma}_{\mu\nu}{}^{\sigma}\phi\frac{dx^{\nu}}{d\tilde{\tau}}\frac{dx^{\mu}}{d\tilde{\tau}}-\frac{1}{2\phi}\partial_{\mu}\phi\frac{dx^{\mu}}{d\tilde{\tau}}\delta^{\sigma}_{\nu}\phi\frac{dx^{\nu}}{d\tilde{\tau}}=0,$$

which implies that

$$\frac{d}{d\tilde{\tau}}\left(\frac{dx^{\sigma}}{d\tilde{\tau}}\right)+\tilde{\Gamma}_{\mu\nu}{}^{\sigma}\frac{dx^{\nu}}{d\tilde{\tau}}\frac{dx^{\mu}}{d\tilde{\tau}}=0. \tag{2.242}$$

Therefore, we have shown that autoparallel equation of the connection $\nabla^{(g,T)}$ is equivalent to autoparallel (geodesics) of the connection $\nabla^{(\tilde{g},0)}$. It can be seen that, autoparallels of $\nabla^{(g,T)}$ are parametrised with the proper time (τ) according to g while autoparallels (geodesics) of $\nabla^{(\tilde{g},0)}$ are parametrised with proper time ($\tilde{\tau}$) according to $\tilde{g}$.

CHAPTER 3

STATIC AND STATIONARY KERR TYPE SOLUTIONS OF SCALAR TENSOR THEORIES

In this chapter, we present metric-compatible ($Q^{ab} = 0$) solutions of scalar tensor and axi-dilaton gravity theories discussed in Chapter 2. In the first section, we discuss static, spherically symmetric solutions which have a Killing symmetry in the time coordinate t, while in the second part, we present axially symmetric, stationary Kerr (rotating) type solutions. We discuss the singularities in both cases. We consider the following action in $D = (n + 1)$ dimensions in which the gravitational field interacts with the scalar field and $(n - 1)$-form antisymmetric gauge field H (we substitute $p = n-3$ in Chapter 2) and the related field equations derived from that action,

$$\mathcal{L} = \frac{1}{2}\phi\, {}^{(0)}R^{ab} \wedge *(e_a \wedge e_b) - \frac{\omega}{2\phi} d\phi \wedge *d\phi - \frac{1}{2}\phi^k H \wedge *H, \tag{3.1}$$

where the connections are constrained to be Levi-Civita (torsion-free). Then the field equations obtained from the action density (3.1) takes the following form:

$$\frac{1}{2}\phi\, {}^{(0)}R^{ab} \wedge *(e_a \wedge e_b \wedge e_c) = -\frac{\omega}{2\phi}\tau_c[\phi] - \frac{1}{2}\phi^k \tau_c[H] - {}^{(0)}D(\iota_c(*d\phi)), \tag{3.2}$$

$$\tilde{k} d(*d\phi) = \frac{1}{2}\alpha\phi^k H \wedge *H, \tag{3.3}$$

$$d(\phi^k * H) = 0, \tag{3.4}$$

where $\tilde{k} = \frac{n}{n-1} + \omega$ and $\alpha = \frac{n-3}{n-1} + k$, and the stress-energy n-forms are given by

$$\tau_c[\phi] = \{\iota_c d\phi \wedge *d\phi + d\phi \wedge \iota_c(*d\phi)\} \tag{3.5}$$

and

$$\tau_c[H] = \{\iota_c H \wedge *H - (-1)^{n-3} H \wedge \iota_c(*H)\} \tag{3.6}$$

respectively. Also, since $H = dA$, it satisfies $dH = 0$. We note that one can easily find dual solutions from this action by just substituting

$$G = \phi^k * H. \tag{3.7}$$

Since G is a 2-form, action density (3.1) with dual field G represents Einstein-Maxwell theory coupled with massless scalar field in higher dimensions. Then, according to the given ansatz, if one solution represents solutions of magnetic-type then its dual formulation represents solutions of electric-type.

3.1 Static, Spherically Symmetric Solutions

In this section, we present the most general static, spherically symmetric solutions to field equations (3.2), (3.3) and (3.4). The ansatz for the solution, is given by

$$g = -f^2(r)dt \otimes dt + h^2(r)dr \otimes dr + R^2(r)d\Omega_{n-1} \tag{3.8}$$

for the metric tensor ($D = n + 1$),

$$\phi = \phi(r) \tag{3.9}$$

for the dilaton 0-form and

$$H = g(r)e^1 \wedge e^2 \wedge e^3 \cdots \wedge e^{n-1} \tag{3.10}$$

for the antisymmetric gauge (axion) field $(n-1)$-form. We see that the solutions are magnetic type. Before giving the general solution, we examine the special cases:

i-) For the charge $Q = 0$ and $\phi = constant$, we obtain the Tangherlini solution [18], which is the generalisation of the Schwarzschild solution in (3+1) dimensions to $D = (n+1)$ dimensions,

$$g = -\left(1 - \frac{2M}{r^{n-2}}\right) dt^2 + \left(1 - \frac{2M}{r^{n-2}}\right)^{-1} dr^2 + r^2 d\Omega_{n-1}. \tag{3.11}$$

From the Einstein field equation, we can calculate the curvature scalar as

$$\mathcal{R} = 0. \tag{3.12}$$

Calculation of other curvature invariants such as $*(R_{ab} \wedge *R^{ab})$ and $*(P_a \wedge *P^a)$, where $P^a = \iota_b R^{ba}$ are Ricci 1-forms, shows that the singularity at $r = 0$ is essential, e.g.

$$*(R_{ab} \wedge *R^{ab}) \sim \frac{C}{r^{2n}}$$

(C is proportional to M) which shows that there exists a curvature singularity at $r = 0$. The event horizon at $r^{n-2} = 2M$ is regular. Therefore, the solutions describe a black hole with an essential singularity at $r = 0$. The mass M_b of the black hole is defined to be

$$M_b \equiv \lim_{r\to\infty} (1 - f^2) r^{n-2} = 2M. \tag{3.13}$$

ii-) For $k = 0$ and $\phi = constant$, we obtain the $D = (n+1)$-dimensional generalisation of the Reissner-Nordström metric. In this case the solution is given by

$$R(r) = r, \tag{3.14}$$

$$f(r) = \left\{1 + \frac{Q^2}{(n-1)(n-2)r^{2(n-2)}} - \frac{2M}{r^{n-2}}\right\}^{1/2}, \tag{3.15}$$

$$h(r) = \frac{1}{f(r)}, \tag{3.16}$$

with the source

$$H = \frac{Q}{r^{n-1}} e^1 \wedge e^2 \wedge e^3 \cdots \wedge e^{n-1}. \tag{3.17}$$

From the Einstein field equation, we obtain the curvature scalar as

$$\mathcal{R} = -\left(\frac{n-3}{n-1}\right) * \{H \wedge *H\} \tag{3.18}$$

which vanishes in $(3+1)$-dimensions. In terms of the function $g(r)$ ($g(r) = \frac{Q}{r^{n-1}}$), this can be written as

$$\mathcal{R} = -\left(\frac{n-3}{n-1}\right) \frac{Q^2}{r^{2(n-1)}}. \tag{3.19}$$

We can deduce that the event horizons at $r^{n-2} = \{M \mp \sqrt{M^2 - \frac{Q^2}{(n-2)(n-1)}}\}$ are regular provided that $M^2 \geq \frac{Q^2}{(n-2)(n-1)}$. Therefore, solutions describe magnetically charged black holes. Calculation of $*(R_{ab} \wedge *R^{ab})$ yields

$$*(R_{ab} \wedge *R^{ab}) \sim \{\frac{E_1}{r^{2(n-1)}} + \frac{E_2}{r^n}\}^2,$$

(The constants E_1 and E_2 depend on M, Q and the dimension of spacetime) which again shows that the singularity at $r = 0$ is a curvature singularity. We can easily obtain the electrical dual of this solution by replacing $F = *H$. In higher dimensions, it was also given by Tangherlini.

iii-) For $Q=0$, we obtain the solutions that generalise the Janis-Newman-Winicour solutions in $(3+1)$ dimensions, to the soltions of Einstein-massless scalar field in $D=(n+1)$ dimensions [19]. In this case, the solution is given as

$$R(r) = rh(r), \tag{3.20}$$

$$f(r) = \left\{ \frac{r^{n-2} - r_0^{n-2}}{r^{n-2} + r_0^{n-2}} \right\}^{-\frac{\beta}{2} - \frac{\beta_0}{2(n-2)(n-1)}}, \tag{3.21}$$

$$h(r) = \left\{ 1 - \left(\frac{r_0}{r} \right)^{2(n-2)} \right\}^{\frac{1}{n-2}} \left\{ \frac{r^{n-2} - r_0^{n-2}}{r^{n-2} + r_0^{n-2}} \right\}^{\frac{1}{2(n-2)}\left(\beta - \frac{\beta_0}{(n-1)}\right)} \tag{3.22}$$

and the scalar field

$$\phi(r) = \phi_0 \left\{ \frac{r^{n-2} - r_0^{n-2}}{r^{n-2} + r_0^{n-2}} \right\}^{\frac{\beta_0}{2(n-2)}}, \tag{3.23}$$

where r_0 is an integration constant; and β_0 and β satisfy

$$(n-2)(\beta^2 - 4) = -\frac{\beta_0^2 \tilde{k}}{(n-1)} \tag{3.24}$$

with $\tilde{k} = \omega + \frac{n}{n-1}$. From Einstein equation, we obtain the curvature scalar $\mathcal{R}$

$$\mathcal{R} = -\frac{\omega}{\phi^2} * \{d\phi \wedge *d\phi\} \tag{3.25}$$

which can be calculated in terms of the solution given above as

$$\begin{aligned} \mathcal{R} &= -\omega \phi_0^2 \beta_0^2 r_0^{2(n-2)} \frac{r^{2(n-3)}}{(r^{n-2} + r_0^{n-2})^4} \left\{ \frac{r^{n-2} - r_0^{n-2}}{r^{n-2} + r_0^{n-2}} \right\}^{-2 - \frac{1}{(n-2)}\left(\beta - \frac{\beta_0}{(n-1)}\right)} \\ &\quad \times \left\{ 1 - \left(\frac{r_0}{r} \right)^{2(n-2)} \right\}^{-\frac{2}{(n-2)}}, \end{aligned} \tag{3.26}$$

which shows that $\mathcal{R}$ is singular at r_0. The other curvature invariant $*(R_{ab} \wedge *R^{ab})$ behaves as

$$*(R_{ab} \wedge *R^{ab}) \sim \frac{C}{r^{4(n-1)}} \left\{ 1 - \left(\frac{r_0}{r} \right)^{2(n-2)} \right\}^{-2 - \frac{2}{n-2}},$$

which shows that $r = 0$ and r_0 are curvature (essential) singularities. r_0 singularity is also called a naked singularity. Therefore the solution presented above cannot be classified as a black hole solution. The solution is asymptotically flat, i.e. as $r \to \infty$, $R(r) \to r$, $f(r) \to 1$, $h(r) \to 1$. At spatial infinity, the scalar field approaches a constant, i.e. $\phi \to \phi_0$.

In the literature, there exist static, spherically symmetric solutions of both magnetic and electric type in $D = (3+1)$ and $D = (n+1)$ dimensions [20, 21, 22, 23, 24, 25, 26]. We now present the complete solution when $\phi \neq$ *constant* and $H \neq 0$. Our solution is given as [27]:

$$\begin{aligned} R(r) &= r\left(1 - \left(\tfrac{C_1}{r}\right)^{n-2}\right)^{\alpha_3}, \\ f(r) &= \left(1 - \left(\tfrac{C_2}{r}\right)^{n-2}\right)^{\alpha_4}\left(1 - \left(\tfrac{C_1}{r}\right)^{n-2}\right)^{\alpha_5}, \\ h(r) &= \left(1 - \left(\tfrac{C_2}{r}\right)^{n-2}\right)^{\alpha_2}\left(1 - \left(\tfrac{C_1}{r}\right)^{n-2}\right)^{\alpha_1}, \end{aligned} \tag{3.27}$$

for the metric tensor field, and

$$\phi(r) = \left(1 - \left(\frac{C_1}{r}\right)^{n-2}\right)^{\frac{2\gamma}{\alpha}}, \tag{3.28}$$

for the scalar field and the axion field is:

$$H = g(r) e^1 \wedge e^2 \wedge e^3 ... \wedge e^{n-1},$$

where

$$g(r) = \frac{Q}{R^{n-1}}. \tag{3.29}$$

The exponents are:

$$\alpha_1 = \gamma\left(\frac{1}{(n-2)} - \frac{2}{(n-1)\alpha}\right) - \frac{1}{2}, \qquad \alpha_2 = -\frac{1}{2},$$

$$\alpha_3 = \left(\frac{1}{(n-2)} - \frac{2}{(n-1)\alpha}\right)\gamma,$$

$$\alpha_4 = \frac{1}{2}, \qquad \alpha_5 = \frac{1}{2} - \left(1 + \frac{2}{(n-1)\alpha}\right)\gamma$$

with

$$\gamma = \frac{(n-1)\alpha^2}{4\tilde{k}(n-2) + (n-1)\alpha^2}, \tag{3.30}$$

where

$$\tilde{k} = \omega + \frac{n}{(n-1)},$$

and $(C_1)^{n-2}$ and $(C_2)^{n-2}$ satisfy

$$Q^2 = \frac{4\tilde{k}(C_1 C_2)^{n-2}(n-2)^2}{\alpha^2}. \tag{3.31}$$

Now we can examine the singularity of the solutions. The curvature scalar $\mathcal{R}$ is calculated from the trace of Einstein field equation. It is given by

$$\mathcal{R} = *\{\left(\frac{n-3}{n-1} - \frac{n\alpha}{\tilde{k}(n-1)}\right)\phi^{k-1} H \wedge *H - \frac{1}{\phi^2}\omega d\phi \wedge *d\phi\}. \tag{3.32}$$

In terms of the solutions, $\mathcal{R}$ can be evaluated as

$$\mathcal{R} = \frac{1}{r^{2(n-1)}}\{\left(\frac{n-3}{n-1} - \frac{n\alpha}{\tilde{k}(n-1)}\right) Q^2 \left(1 - \left(\frac{C_1}{r}\right)^{n-2}\right)^{\{\frac{2(k-1)\gamma}{\alpha} - 2(n-1)\alpha_3\}}$$
$$-\omega\left(\frac{2\gamma}{\alpha}(C_1)^{n-2}(n-2)\right)^2 \left(1 - \left(\frac{C_1}{r}\right)^{n-2}\right)^{-2-2\alpha_1}\left(1 - \left(\frac{C_2}{r}\right)^{n-2}\right)\}. \tag{3.33}$$

Assume that $(C_2)^{n-2} > (C_1)^{n-2}$. Then at $r^{n-2} = (C_2)^{n-2}$, $\mathcal{R}$ is finite. Hence $r = C_2$ surface is a regular event horizon. The calculation of the other curvature invariant $*(R_{ab} \wedge *R^{ab})$ yields

$$*(R_{ab} \wedge *R^{ab}) \sim C\left\{1 - \left(\frac{C_1}{r}\right)^{n-2}\right\}^{-4-4\alpha_1} r^{-4(n-1)}, \tag{3.34}$$

which shows that $r = 0$ is an essential singularity. So the solutions describe a black hole with an event horizon located at C_2. We can define the mass of the black hole as

$$2M \equiv \lim_{r\to\infty} r^{n-2}(1-f^2) = (\tilde{\gamma} - 2\gamma)(C_1)^{n-2} + (C_2)^{n-2} \tag{3.35}$$

where $\tilde{\gamma} = 1 - \frac{4\gamma}{(n-1)\alpha}$. The scalar charge is defined as

$$\Sigma \equiv \lim_{r\to\infty} \frac{\phi'}{\phi} r^{n-1} = 2(n-2)(C_1)^{n-2}\frac{\gamma}{\alpha}. \tag{3.36}$$

The magnetic charge can be found from

$$Q \equiv \lim_{r\to\infty} g r^{n-1} = Q. \tag{3.37}$$

Therefore, by eliminating the integration constants $(C_1)^{n-2}$ and $(C_2)^{n-2}$ above, we can obtain the following relationship between these 3 physical parameters:

$$Q^2 = \frac{2(n-2)\Sigma}{\alpha}\tilde{k}\{(2\gamma - \tilde{\gamma})\frac{\Sigma\alpha}{2(n-2)\gamma} + 2M\}. \tag{3.38}$$

From this relation, we can determine the BPS bound. Since Σ is a real parameter, we have the following inequality satisfied by the mass and the charge of the black hole:

$$M \geq \frac{1}{2\tilde{k}^{1/2}(n-2)(n-1)}\sqrt{4\tilde{k}(n-2) + 4\alpha - (n-1)\alpha^2}|Q| \tag{3.39}$$

provided

$$\alpha^2(n-1) - 4\alpha \leq 4\tilde{k}(n-2). \tag{3.40}$$

It is interesting to note that, the curvature scalar $\mathcal{R}$ and the curvature invariant $*(R_{ab} \wedge *R^{ab})$ also become finite (regular horizon) at $r = C_1$ if the following inequality is satisfied:

$$2\gamma\left(\frac{2}{\alpha(n-1)}-\frac{1}{n-2}\right)-1\geq 0. \tag{3.41}$$

Together with the inequality (3.40), it requires that α should satisfy

$$0\leq\alpha\leq\frac{4(n-2)}{(n-1)^2}. \tag{3.42}$$

Otherwise, $r=C_1$ surface becomes singular and it shows a naked singularity.

Finally, the solutions (3.27), (3.28) and (3.29) are also the solutions of the field equations obtained from the following action density in which independent variation of the connections ω^{ab} produces torsion $T^a=e^a\wedge\frac{d\phi}{\phi(n-1)}$:

$$\mathcal{L}=\frac{1}{2}\phi R^{ab}\wedge *(e_a\wedge e_b)-\frac{\tilde{k}}{2\phi}d\phi\wedge *d\phi-\frac{1}{2}\phi^k H\wedge *H. \tag{3.43}$$

3.2 Stationary, Axially Symmetric, Kerr Type Solutions

The most general, axially symmetric, stationary solutions to vacuum Einstein equations in $(3+1)$ dimensions were given by Kerr in 1963 in Kerr-Schild form [28]. Solutions found by Kerr are called Kerr black holes, or rotating (spinning) black holes characterised by two parameters: the mass of the black hole M and its angular momentum per unit mass l. Later, the solutions were put into Boyer-Lindquist form. These solutions are the uncharged solutions. If the electromagnetic field couples to gravity, then the solutions become charged. In that case, Kerr metric modifies to Kerr-Newman metric. Rotating vacuum uncharged solutions are generalised to $D=(n+1)$ dimensions by Myers and Perry [29]. On the other hand, the rotating type, stationary solutions to Einstein field equations coupled with a scalar field are given by McIntosh in $(3+1)$ dimensions [31]. But these solutions do not describe a black hole. In the literature, there exist rotating solutions of Einstein-Maxwell theory coupled with a scalar field. These solutions are generated from static solutions by using Kaluza-Klein method and by boosting the static solutions [33, 34]. In the work of Horne and Horowitz [33], rotating black string solutions of Einstein theory coupled with a scalar field and a 3-form string field H, are given. There are also rotating black hole solutions of heterotic string theory in which gravity couples to a Maxwell field F, a dilaton field Φ, a string 3-form field H and a matrix valued scalar field M. In the following, we present all possible stationary solutions to field equations (3.2). We take $p=0$, such that H field becomes Maxwell 2-form field.

i-) Vacuum stationary (rotating) solutions, when $\phi = constant$ and $H = F = 0$:

In $D = (n+1)$ dimensions, rotating solutions are characterized by $[\frac{n}{2}]+1$ parameters: The mass and $[\frac{n}{2}]$ spin angular momentum parameters where $[x]$ denotes the integer part of x. In the following we present the solution with one spin parameter. If φ is the angle on the plane in which the black hole is spinning, vacuum (uncharged) solution is given by the metric [29]

$$\begin{aligned} g &= -dt^2 + (r^2+l^2)\sin^2\theta d\varphi^2 + \rho^2 d\theta^2 + \frac{\rho^2}{(r^2+l^2-2Mr^{4-n})}dr^2 \\ &\quad + \frac{2M}{\rho^2 r^{n-4}}(dt - l\sin^2\theta d\varphi)^2 + r^2\cos^2\theta d\Omega_{n-3}, \end{aligned} \tag{3.44}$$

where M is the mass of black hole and l is its intrinsic spin angular momentum per unit mass. $d\Omega_{n-3}$ is the line element of the unit $(n-3)$-sphere. ρ^2 is defined as

$$\rho^2 = r^2 + l^2 \cos^2\theta. \tag{3.45}$$

We can express the metric (3.44) in terms of orthonormal co-frame 1-forms defined as,

$$e^0 = \tfrac{(r^2+l^2-2Mr^{4-n})}{\rho}(dt - l\sin^2\theta d\varphi),$$

$$e^1 = \tfrac{\rho}{(r^2+l^2-2Mr^{4-n})}dr,$$

$$e^2 = \rho d\theta,$$

$$e^3 = \tfrac{\sin\theta}{\rho}\{(r^2-l^2)d\varphi - ldt\},$$

$$e^{i+3} = r\cos\theta\sigma^i, \qquad i = 1, 2, 3, \cdots, (n-3)$$

where σ^i are the 1-forms containing intrinsic coordinates of $(n-3)$ sphere. In terms of the intrinsic coordinates $(\theta_1, \theta_2, \theta_3, \cdots \theta_{n-3})$, $\{\sigma^i\}$'s can be expressed as

$$\sigma^1 = d\theta_1,$$

$$\sigma_2 = \sin\theta_1 d\theta_2,$$

$$\sigma_3 = \sin\theta_1 \sin\theta_2 d\theta_3,$$

$$\vdots$$

$$\sigma^{n-3} = \sin\theta_1 \sin\theta_2 \sin\theta_3 \cdots \sin\theta_{n-4} d\theta_{n-3}.$$

In that case, the metric g can be written as

$$g = -e^0 \otimes e^0 + e^1 \otimes e^1 + e^2 \otimes e^2 + e^3 \otimes e^3 + \sum_{i=1}^{n-3} r^2 \cos^2\theta\sigma^i \otimes \sigma^i. \tag{3.46}$$

We can see that the metric solution (3.44) is axially symmetric, stationary and it is asymptotically flat. Also, it is Ricci flat i.e. all Ricci one forms defined by $P^a = \iota_b R^{ba}$ are zero. When $l = 0$, it reduces to Tangherlini solution in $(n+1)$

dimensions. We can see that metric is not defined on the region where

$$r^2 + l^2 - 2Mr^{4-n} = 0 \tag{3.47}$$

and

$$\rho^2 = 0. \tag{3.48}$$

From the Einstein field equation, the curvature scalar

$$\mathcal{R} = 0.$$

On the other hand, the calculation of other curvature invariant $*\{R_{ab} \wedge *R^{ab}\}$ yields

$$*\{R_{ab} \wedge *R^{ab}\} \sim \frac{C}{\rho^{12}}. \tag{3.49}$$

Therefore, there is an event horizon at r_H where r_H satisfies (3.47), and (3.49) states that there is a curvature (essential) singularity at the points where $\rho = 0$. This means that the region where both $r = 0$ and $\cos\theta = 0$ ($\theta = \frac{\pi}{2}$) are satisfied, forms a singularity. In the literature it is called a ring singularity. Therefore the event horizon at r_H encloses this ring singularity. When $n = 3$, the solution (3.44) reduces to the well-known uncharged Kerr solution in the Boyer-Lindquist coordinates:

$$g = -dt^2 + \rho^2 d\sigma^2 + \frac{2Mr}{\rho^2}(dt - l\sin^2\theta d\varphi)^2 + l^2\sin^4\theta d\varphi^2 + \frac{\rho^2}{\Delta}dr^2, \tag{3.50}$$

where $d\sigma^2 = d\theta^2 + \sin^2\theta d\varphi^2$ and

$$\Delta = r^2 + l^2 - 2Mr. \tag{3.51}$$

In that case, the event horizon forms at $r^2 + l^2 - 2Mr = 0$. This requires that there exist two regular horizon surfaces at $r = M \mp \sqrt{M^2 - l^2}$ (inner and outer horizon surfaces), provided that $M^2 \geq l^2$. As a definition stated in the literature [30], when $M^2 \leq l^2$, Kerr spacetime is called rapidly rotating. When $M^2 = l^2$, the solution is called extreme Kerr spacetime. If $l^2 \leq M^2$, it is called slowly rotating. As a remark, when $l = 0$ the metric (3.50) reduces to Schwarzschild metric.

ii-) Rotating charged solutions when $\phi = constant$ and $F \neq 0$:

In that case, there exist axially symmetric, stationary charged solutions in $(3+1)$ dimensions. These are called Kerr-Newman solutions. They are given by the metric

$$\begin{aligned} g &= -\left(\frac{\tilde{\Delta} - l^2 \sin^2\theta}{\rho^2}\right) dt^2 - \frac{2l \sin^2\theta (r^2 + l^2 - \tilde{\Delta})}{\rho^2} d\varphi dt \\ &\quad + \left\{\frac{(r^2 + l^2)^2 - l^2 \tilde{\Delta} \sin^2\theta}{\rho^2}\right\} \sin^2\theta d\varphi^2 + \rho^2 d\theta^2 + \frac{\rho^2}{\tilde{\Delta}} dr^2, \end{aligned} \quad (3.52)$$

where

$$\tilde{\Delta} = r^2 + Q^2 + l^2 - 2Mr \quad (3.53)$$

with Q being the charge. Maxwell field potential one form A $(F = dA)$ is

$$A = \frac{Qr}{\rho^2} dt - \frac{l \sin^2\theta Qr}{\rho^2} d\varphi. \quad (3.54)$$

From the trace of Einstein field equation in $(3+1)$ dimensions, we obtain the curvature scalar

$$\mathcal{R} = 0.$$

Also, the other curvature invariant $*(R_{ab} \wedge *R^{ab})$ is again of the form

$$*(R_{ab} \wedge *R^{ab}) \sim \frac{C_0}{\rho^{12}}.$$

Therefore, the solution (3.53) describes a charged rotating (Kerr) black hole with event horizons at $r = M \mp \sqrt{M^2 - Q^2 - l^2}$ enclosing a ring singularity, provided that

$$M^2 \geq Q^2 + l^2. \tag{3.55}$$

iii-) Kerr Brans-Dicke solutions ($F = 0$ and $\phi \neq constant$):

The rotating type, axially symmetric, stationary type of solutions of Einstein field equations coupled with Brans-Dicke scalar field are given in (3+1) dimensions by Mcintosh. They are called Kerr Brans-Dicke solutions. We can write the solution in Brans-Dicke frame in Boyer-Lindquist coordinates (t, θ, φ, r) as:

$$\begin{aligned} g &= \phi_0^{-1} \left(\frac{r - (M + \sqrt{M^2 - l^2})}{r - (M - \sqrt{M^2 - l^2})} \right)^{-\frac{1}{2}A} \{ -\frac{\rho^2 \Delta}{P} dt^2 + \frac{P \sin^2\theta}{\rho^2} (d\varphi - \frac{2Mlr}{P} dt)^2 \\ &+\rho^2 \left(\frac{\Delta}{(r-M)^2 - (M^2 - l^2)\cos^2\theta} \right)^{2kA^2} (d\theta^2 + \frac{dr^2}{\Delta}) \}, \end{aligned} \tag{3.56}$$

where Δ is defined by the equation (3.51) and

$$P = \rho^2 \Delta + 2Mr(r^2 + l^2). \tag{3.57}$$

k is given by $k = \omega + \frac{3}{2}$ in terms of the Brans-Dicke parameter ω. The scalar field is

$$\phi = \phi_0 \left(\frac{r - (M + \sqrt{M^2 - l^2})}{r - (M - \sqrt{M^2 - l^2})} \right)^{\frac{A}{2}}, \tag{3.58}$$

where the constant A determines the strength of the scalar field. A calculation of the curvature scalar from Einstein field equation yields

$$\mathcal{R} = -\frac{\omega}{\phi^2} * (d\phi \wedge *d\phi). \tag{3.59}$$

In terms of the given functions, it can be written as

$$\begin{aligned}\mathcal{R} &= A^2\omega\phi_0\frac{\Delta}{\rho^2}\left(\frac{\Delta}{(r-M)^2-(M^2-l^2)\cos^2\theta}\right)^{-2kA^2} \\ &\quad \times\frac{(M^2-l^2)}{(r-(M-\sqrt{M^2-l^2}))^4}\left(\frac{r-(M+\sqrt{M^2-l^2})}{r-(M-\sqrt{M^2-l^2})}\right)^{\frac{A}{2}-2}, \end{aligned} \tag{3.60}$$

which shows that the surfaces at $r = M \mp \sqrt{M^2-l^2}$ are singular. The calculation of other curvature invariant

$$*(R_{ab}\wedge *R^{ab}) \sim \frac{1}{\rho^{12}\Delta^{4kA^2}}\left(\frac{r-(M+\sqrt{M^2-l^2})}{r-(M-\sqrt{M^2-l^2})}\right)^{A} \tag{3.61}$$

implies that the ring singularity is an essential singularity and the surfaces at $r = M \mp \sqrt{M^2-l^2}$ are naked singularities. Therefore, we can state that the solution (3.56) does not describe a black hole. However, the solution is asymptotically flat.

Finally, there exist rotating solutions of the low energy limit of heterotic string theories given by Horowitz and Sen [32]. There are also solutions to the field equations obtained from the action in (3+1) dimensions in which gravity couples to scalar matter and Maxwell fields. Namely rotating solutions of the action density

$$\mathcal{L} = R^{ab}\wedge *(e_a\wedge e_b) - 2d\Phi\wedge *d\Phi - \exp(-2\beta\Phi)F\wedge *F \tag{3.62}$$

is obtained in (3+1) dimensions for the value of the coupling parameter $\beta = \sqrt{3}$ by the Kaluza-Klein method and by boosting the static solution. The corresponding field equations are

$$\begin{aligned}R^{ab}\wedge *(e_a\wedge e_b\wedge e_c) &= -2(\iota_c d\Phi\wedge *d\Phi + d\Phi\wedge\iota_c(*d\Phi)) \\ &\quad - e^{-2\beta\Phi}(\iota_c F\wedge *F - F\wedge\iota_c(*F)), \end{aligned} \tag{3.63}$$

$$d(*d\Phi) = -2\beta e^{-2\beta\Phi} F \wedge *F \tag{3.64}$$

and

$$d(e^{-2\beta\Phi} * F) = 0. \tag{3.65}$$

The solution is given by [33, 34]

$$\begin{aligned} g &= \frac{1-Z}{B} dt^2 - \frac{2\sin^2\theta l Z}{B\sqrt{1-v^2}} dt d\varphi + B\frac{\rho^2}{\Delta} dr^2 \\ &+ \left\{ B(r^2+l^2) + l^2 \sin^2\theta \frac{Z}{B} \right\} \sin^2\theta d\varphi^2 + \rho^2 B d\theta^2, \end{aligned} \tag{3.66}$$

where

$$B = \sqrt{1 + \frac{v^2 Z}{1-v^2}}, \qquad Z = \frac{2Mr}{\rho^2} \tag{3.67}$$

with

$$\Delta = r^2 + l^2 - 2Mr, \qquad \rho^2 = r^2 + l^2 \cos^2\theta.$$

v is the velocity of boost. Maxwell field potential is

$$A = \frac{Qr}{B^2} dt - l(\sin 2\theta) \frac{v}{2\sqrt{1-v^2}} \frac{Z}{B^2} d\varphi \tag{3.68}$$

and the scalar field is

$$\Phi = -\frac{\sqrt{3}}{2} \ln B. \tag{3.69}$$

In this solution, the physical mass M_p, the charge Q and the angular momentum J are given in terms of the boost velocity v and the parameters M and l of rotating solution, as

$$\begin{aligned} J &= \frac{Ml}{\sqrt{1-v^2}}, \\ Q &= \frac{Mv}{1-v^2}, \\ M_p &= M\left(1 + \frac{v^2}{2(1-v^2)}\right). \end{aligned} \tag{3.70}$$

When v is zero, the solution reduces to the original Kerr solution in $(3+1)$ dimensions. We can evaluate the curvature scalar from the trace of the Einstein field equation as

$$\mathcal{R} = -2 * (d\Phi \wedge *d\Phi). \tag{3.71}$$

In terms of the given solution, it is calculated as

$$\mathcal{R} = \frac{3}{2} \frac{M^2 V^4 \Delta^2 (l^2 \cos^2\theta - r^2)^2}{\left((1-v^2) + v^2 Z\right)^2 B^2 \rho^{12}}. \tag{3.72}$$

The calculation of the other curvature invariant $*(R_{ab} \wedge *R^{ab})$ yields

$$*(R_{ab} \wedge *R^{ab}) \sim \frac{1}{B^5 \rho^{10}} \tag{3.73}$$

which implies that the ring singularity is an essential singularity with regular inner and outer event horizons at $r = M - \sqrt{M^2 - l^2}$ and $r = M + \sqrt{M^2 - l^2}$, respectively [33].

CHAPTER 4

AUTOPARALLEL ORBITS IN KERR BRANS-DICKE SPACETIMES

In this chapter, as an application to section about the autoparallel motion of massive test particles in a space-time geometry with torsion, we study the orbital motion of a massive test particle in Kerr Brans-Dicke geometry with torsion. We compare autoparallel orbits based on a geometry with torsion with those based on the assumption that worldlines are geodesic (torsion-free). In [2], the geodesic orbits based on torsion-free connection and autoparallel orbits based on a connection with torsion are compared by considering a spherically symmetric and static source of scalar tensor gravity. In this work, we consider a rotating (spinning) gravitational source and take the Kerr Brans-Dicke metric as a background. The solution describes a stationary and axially symmetric metric and depends on the parameters that may be identified with the scalar charge, the mass and the angular momentum of a localised source [35].

4.1 The Motion Of Massive Test Particles

The Kerr Brans-Dicke solution in a spacetime geometry with torsion is obtained from the independent variations of the following action density with respect to the co-frame fields e^a, the scalar field ϕ and the connection fields $\omega^a{}_b$. So, the variation of the action density

$$\mathcal{L} = \frac{1}{2}\phi R^{ab} \wedge *(e_a \wedge e_b) - \frac{\left(\omega + \frac{3}{2}\right)}{2\phi} d\phi \wedge *d\phi \tag{4.1}$$

yields the field equations

$$\frac{1}{2}\phi R^{ab} \wedge *(e_a \wedge e_b \wedge e_c) = -\frac{c}{2\phi}\{\iota_c d\phi * d\phi + d\phi \wedge \iota_c(*d\phi)\}, \tag{4.2}$$

$$cd(*d\phi) = 0, \tag{4.3}$$

with the torsion 2-forms $T^a = e^a \wedge \frac{d\phi}{2\phi}$, where $c = \omega + \frac{3}{2}$ in terms of the Brans-Dicke parameter ω. The solution to these field equations [31] in Brans-Dicke frame can be written in Boyer-Lindquist coordinates (t, r, θ, φ) as

$$\begin{aligned} g \;=\; & \phi_0^{-1} \left(\frac{r - (M + \sqrt{M^2 - l^2})}{r - (M - \sqrt{M^2 - l^2})} \right)^{-\frac{A}{2}} \{-\frac{\Sigma\Delta}{P} dt^2 + \frac{P \sin^2\theta}{\Sigma}(d\varphi - \frac{2Mlr}{P} dt)^2 \\ & + \Sigma \left(\frac{(r - M)^2 - (M^2 - l^2)}{(r - M)^2 - (M^2 - l^2)\cos^2\theta} \right)^{2cA^2} (d\theta^2 + \frac{dr^2}{\Delta})\}, \end{aligned} \tag{4.4}$$

with the scalar field solution

$$\phi = \phi_0 \left(\frac{r - (M + \sqrt{M^2 - l^2})}{r - (M - \sqrt{M^2 - l^2})} \right)^{\frac{A}{2}}, \tag{4.5}$$

where M denotes the source mass and l denotes its angular momentum per unit mass. It is assumed that $M > l$. The constant A determines the strength of the scalar field. We define

$$\begin{aligned} \Sigma &= r^2 + l^2 \cos^2\theta, \\ \Delta &= r^2 + l^2 - 2Mr, \\ P &= \Delta\Sigma + 2Mr(r^2 + l^2). \end{aligned} \tag{4.6}$$

We first examine the orbits C in this background, for massive spinless particles based on the assumption that the worldline is Levi-Civita autoparallel (geodesic). The equations of motion are

$$\hat{\nabla}_{\dot{C}} \dot{C} = 0$$

in terms of the torsion-free Levi-Civita connection $\hat{\nabla}$ and 4-velocity $\dot{C}$ is normalised according to

$$\mathbf{g}(\dot{C}, \dot{C}) = -1. \tag{4.7}$$

Throughout, we adopt units such that the speed of light $c = 1$ and the gravitational coupling constant $G = 1$. If $C : \tau \mapsto x^\mu(\tau)$ in terms of the proper time τ, these yield equations

$$\frac{d}{d\tau}\left(\frac{dx^\mu}{d\tau}\right) + \left\{{}^{\mu}_{\nu\lambda}\right\} \frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0. \tag{4.8}$$

The metric above has two independent Killing vectors ∂_t and ∂_φ. These generate two constants of motion: the particle energy E and the orbital angular momentum L. Since the orbits are planar, we take $\theta = \frac{\pi}{2}$ and set [35]

$$\bar{L} = m(\dot{\varphi} g_{\varphi\varphi} + \dot{t} g_{\varphi t}), \tag{4.9}$$

$$\bar{E} = m(\dot{\varphi} g_{\varphi t} + \dot{t} g_{tt}), \tag{4.10}$$

where m is the mass of the particle. One can express $\dot{r}$ in (4.7) in terms of $\dot{t}$ and $\dot{\varphi}$ and the metric components on the orbit ($\theta = \frac{\pi}{2}$). On the orbital plane ($\theta = \frac{\pi}{2}$), equation (4.7) becomes

$$\dot{t}^2 g_{tt} + \dot{\varphi}^2 g_{\varphi\varphi} + 2\dot{\varphi}\dot{t} g_{\varphi t} + g_{rr}\dot{r}^2 = -1. \tag{4.11}$$

Therefore,

$$\dot{r}^2 = -\frac{1}{g_{rr}}\{1 + 2g_{t\varphi}\dot{\varphi}\dot{t} + g_{\varphi\varphi}\dot{\varphi}^2 + g_{tt}\dot{t}^2\}. \tag{4.12}$$

Dividing (4.12) by $\dot{\varphi}^2$, we obtain

$$\frac{\dot{r}^2}{\dot{\varphi}^2} = -\frac{1}{g_{rr}\dot{\varphi}^2}\{1 + 2g_{t\varphi}\dot{\varphi}\dot{t} + g_{\varphi\varphi}\dot{\varphi}^2 + g_{tt}\dot{t}^2\}. \tag{4.13}$$

We can eliminate $\dot{\varphi}$ and $\dot{t}$ from the equations (4.9) and (4.10) and substitute into (4.13). Since

$$\left(\frac{dr}{d\varphi}\right)^2 = \frac{\dot{r}^2}{\dot{\varphi}^2} \tag{4.14}$$

the orbit equation may be written as

$$\left(\frac{dr}{d\varphi}\right)^2 = \frac{-\Delta\phi^{-2}}{g_{rr}(g_{\varphi t}\tilde{E} - g_{tt}\tilde{L})^2}\left\{\phi^{-2}\phi_0\Delta + 2g_{\varphi t}\tilde{E}\tilde{L} - g_{\varphi\varphi}\tilde{E}^2 - g_{tt}\tilde{L}^2\right\} \tag{4.15}$$

where $\tilde{E} = \frac{\bar{E}(\phi_0)^{1/2}}{m}$ and $\tilde{L} = \frac{\bar{L}(\phi_0)^{1/2}}{m}$. We define,

$$A(r) = \frac{4M^2l^2 - \Delta r^2}{P_1}, \tag{4.16}$$

$$B(r) = -\frac{2Ml}{r}, \tag{4.17}$$

$$C(r) = \frac{P_1}{r^2}, \tag{4.18}$$

$$\phi_1(r) = \left(\frac{r - (M + \sqrt{M^2 - l^2})}{r - (M - \sqrt{M^2 - l^2})}\right)^{-\frac{A}{2}}, \tag{4.19}$$

$$P_1(r) = 2Ml^2r + (r^2 + l^2)r^2, \tag{4.20}$$

$$G(r) = \frac{r^2}{\Delta}\left(\frac{r^2 + l^2 - 2Mr}{r^2 + M^2 - 2Mr}\right)^{2cA^2}. \tag{4.21}$$

We introduce the variable $u = \frac{1}{r}$. In terms of the functions defined above and the variable u, the orbit equation (4.15) becomes,

$$\begin{aligned}\left(\frac{du}{d\varphi}\right)^2 &= \frac{-u^4\Delta(1/u)}{G(1/u)\left(\tilde{E}B(1/u) - \tilde{L}A(1/u)\right)^2}\{\phi_1(1/u)\Delta(1/u) - \tilde{L}^2A(1/u) \\ &\quad +2B(1/u)\tilde{E}\tilde{L} - C(1/u)\tilde{E}^2\}.\end{aligned} \tag{4.22}$$

Before analysing this equation, we note that in geometrised units [38] where the coupling constant G, ($[G] = [1]$) and speed of light c, ($[c] = [1]$) are dimensionless quantities, both M and l take dimension of the length $[m]$, i.e. $[M] = [m]$ and $[l] = [m]$. Therefore the dimension of the energy and the orbital angular momentum become, $[E] = [m]$ and $[L] = [m]^2$, respectively. Also since $\phi_0 \sim \frac{1}{G}$, $[\phi_0] = [1]$. Hence, $[\tilde{E}] = [1]$ (dimensionless) and $[\tilde{L}] = [m]$. Furthermore, since $[r] = [m]$, $[u] = [m]^{-1}$.

To analyse equation (4.22), we employ the physically motivated approximations discussed in [2]. Therefore, if the radius of a (weak field) Newtonian orbit is much larger than the corresponding Schwarzschild radius of the source, one may expand this orbit equation around $u = 0$ up to third order in order to compare its solutions with those in a Schwarzschild background. Thus, up to third order in u, orbit equation can be written as,

$$\left(\frac{du}{d\varphi}\right)^2 \simeq S_0 + S_1 u + S_2 u^2 + S_3 u^3 \tag{4.23}$$

where the constants are:

$$S_0 = \frac{1}{\tilde{L}^2}(\tilde{E}^2 - 1), \tag{4.24}$$

$$S_1 = \frac{1}{\tilde{L}^3} 4Ml\tilde{E}(\tilde{E}^2 - 1) + \frac{1}{\tilde{L}^2}(2M - \sqrt{M^2 - l^2}A), \tag{4.25}$$

$$\begin{aligned} S_2 \;=\; & -1 + \frac{1}{\tilde{L}^2}\{3l^2(\tilde{E}^2 - 1) - \frac{1}{2}(M^2 - l^2)A^2 \\ & +2c(M^2 - l^2)(\tilde{E}^2 - 1)A^2 + M\sqrt{M^2 - l^2}A\} \\ & +\frac{1}{\tilde{L}^3}\{8M^2\tilde{E}^3 l - 4Ml\tilde{E}\sqrt{M^2 - l^2}A\} \\ & +\frac{12M^2\tilde{E}^2 l^2}{\tilde{L}^4}(\tilde{E}^2 - 1), \end{aligned} \tag{4.26}$$

$$
\begin{aligned}
S_3 \;=\;& 2M + \frac{1}{\tilde{L}^2}\{-\frac{1}{3}(4M^2 - l^2)\sqrt{M^2 - l^2}A - 3l^2\sqrt{M^2 - l^2}A \\
& +6M\tilde{E}^2 l^2 + 4c\tilde{E}^2 M(M^2 - l^2)A^2 + 2M^2\sqrt{M^2 - l^2}A \\
& -(2c + \frac{1}{6})\left(M^2 - l^2\right)^{3/2} A^3\} + \frac{1}{\tilde{L}^3}\{-4M^2\tilde{E}\sqrt{M^2 - l^2}lA \\
& -2M\tilde{E}(M^2 - l^2)lA^2 + 16M^3\tilde{E}^3 l \\
& +8c\tilde{E}Ml(\tilde{E}^2 - 1)(M^2 - l^2)A^2 + 12M\tilde{E}(\tilde{E}^2 - 1)l^3\} \\
& +\frac{12}{\tilde{L}^4}\{2M^3\tilde{E}^2(2\tilde{E}^2 - 1)l^2 - \tilde{E}^2 M^2\sqrt{M^2 - l^2}l^2 A\} \\
& +\frac{1}{\tilde{L}^5}32M^3\tilde{E}^3(\tilde{E}^2 - 1)l^3. \qquad (4.27)
\end{aligned}
$$

All the terms in S_2 except -1 and all the terms in S_3 give general relativistic corrections to the Newtonian orbital equation.

By contrast, we now compare this orbit equation with the one obtained by assuming that the worldline is a timelike autoparallel of a particular connection with torsion specified by the gradient of a scalar field, as $T^a = e^a \wedge \frac{d\phi}{2\phi}$. In that case, autoparallel equation is given by

$$\nabla_{\dot{C}}\dot{C} = 0,$$

where ∇ denotes the connection with torsion. 4-velocity $\dot{C}$ is again normalised with

$$\mathbf{g}(\dot{C}, \dot{C}) = -1.$$

We can express the autoparallel worldline equation in terms of Levi-Civita connection $\hat{\nabla}$ as

$$\widetilde{\hat{\nabla}_{\dot{C}}\dot{C}} = -\frac{1}{2\phi}\iota_{\dot{C}}(d\phi \wedge \tilde{\dot{C}}),$$

where for any vector field V, $\tilde{V} = \mathbf{g}(V, -)$ is the metric related 1-form. This may be further simplified to

$$\hat{\nabla}_{\dot{C}}\,(\widetilde{\phi^{1/2}\dot{C}}) = -d\phi^{1/2}.$$

In local coordinates, this can be written as

$$\frac{d}{d\tau}\left(\phi^{1/2}\frac{dx^\mu}{d\tau}\right) + \phi^{1/2}\{^{\mu}_{\nu\lambda}\}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = -g^{\mu\nu}\frac{\partial_\nu\,\phi}{2\phi^{1/2}}. \tag{4.28}$$

For any Killing vector K with $K\phi = 0$, the expression

$$\gamma_K = \phi^{1/2}\mathbf{g}(K, \dot{C}) \tag{4.29}$$

is constant along the worldline of the particle.

Proof: We can write γ_K in exterior forms as

$$\gamma_K = - * (\phi^{1/2} k \wedge *\tilde{\dot{C}}),$$

where k is the dual 1-form of the Killing vector K. Now we apply $\hat{\nabla}$ with respect to $\dot{C}$ on both sides. Since, the connections are metric-compatible, $*$ and $\hat{\nabla}$ commute. Therefore,

$$\hat{\nabla}_{\dot{C}}\,\gamma_K = - * (\hat{\nabla}_{\dot{C}}\,k \wedge \phi^{1/2} * \tilde{\dot{C}} + k \wedge *\hat{\nabla}_{\dot{C}}\,(\phi^{1/2}\tilde{\dot{C}})). \tag{4.30}$$

Now,

$$\hat{\nabla}_{\dot{C}}\,k \wedge \phi^{1/2} * \tilde{\dot{C}} = \iota_{\dot{C}}\,(\hat{\nabla}_{\dot{C}}\,k)\phi^{1/2} * 1.$$

From the defining Killing equation $\mathcal{L}_K\,g = 0$, it follows that

$$(\hat{\nabla}_Z\,k)(Y) + (\hat{\nabla}_Y\,k)(Z) = 0$$

for any vectors Y and Z. By choosing $Y = Z = \dot{C}$, this implies that

$$(\hat{\nabla}_{\dot{C}}\, k)(\dot{C}) = 0.$$

In exterior forms, it becomes $(\hat{\nabla}_{\dot{C}}\, k)(\dot{C}) = \iota_{\dot{C}}\,(\hat{\nabla}_{\dot{C}}\, k) = 0$. Therefore the first term in (4.30) is zero. On the other hand, from the equation of motion, the second term becomes,

$$k \wedge *\hat{\nabla}_{\dot{C}}\,(\phi^{1/2}\tilde{\dot{C}}) = k \wedge * \left(-\frac{d\phi}{2\phi^{1/2}}\right).$$

Using $k \wedge *d\phi = d\phi \wedge *k$ and $d\phi \wedge *k = \iota_K d\phi * 1 = d\phi(K) * 1$, the second term in (4.30) also becomes zero, since we have assumed $K\phi = d\phi(K) = 0$.

Alternatively, since $\phi^{1/2} g(K, \dot{C}) = g(K, \phi^{1/2}\dot{C})$ and the connection is metric compatible,

$$\hat{\nabla}_{\dot{C}}\, \gamma_K = g(\hat{\nabla}_{\dot{C}}\, K, \phi^{1/2}\dot{C}) + g(K, \hat{\nabla}_{\dot{C}}\,(\phi^{1/2}\dot{C})).$$

The first term is zero due to Killing condition $g(\hat{\nabla}_{\dot{C}}\, K, \dot{C}) = \iota_{\dot{C}}\,(\hat{\nabla}_{\dot{C}}\, k) = 0$. Using the equation of motion, the second term can be written as

$$g(K, \hat{\nabla}_{\dot{C}}\,(\phi^{1/2}\dot{C})) = -g(K, \frac{\tilde{d\phi}}{2\phi^{1/2}}) = -\frac{1}{2\phi^{1/2}} g(K, \tilde{d\phi}),$$

where $\tilde{d\phi}$ is the dual vector of $d\phi$. However, since $g(K, \tilde{d\phi}) = \iota_K d\phi$ and $\iota_K d\phi = d\phi(K) = K\phi = 0$, the second term is also zero. Therefore, γ_K is constant along the worldline of the particle.

As before, the Killing vectors $K_\varphi = \partial_\varphi$ and $K_t = \partial_t$ generate two constants of motion E and L, corresponding to the energy and the orbital angular momentum, respectively. Therefore,

$$L = m \left(\frac{\phi}{\phi_0}\right)^{1/2} (\dot{\varphi} g_{\varphi\varphi} + \dot{t} g_{\varphi t}), \tag{4.31}$$

$$E = m\left(\frac{\phi}{\phi_0}\right)^{1/2}(\dot{\varphi}g_{\varphi t} + \dot{t}g_{tt}) \tag{4.32}$$

in terms of the metric functions evaluated on planar orbits ($\theta = \frac{\pi}{2}$). Eliminating $\dot{\varphi}$ and $\dot{t}$ from the equations (4.31) and (4.32), and substituting in (4.7), one obtains the new autoparallel orbit equation as

$$\left(\frac{dr}{d\varphi}\right)^2 = \frac{\phi^{-2}\Delta}{g_{rr}(g_{t\varphi}\hat{E} - g_{tt}\hat{L})^2}\left\{\phi^{-1}\Delta + 2g_{t\varphi}\hat{E}\hat{L} - g_{\varphi\varphi}\hat{E}^2 - g_{tt}\hat{L}^2\right\}, \tag{4.33}$$

where $\hat{E} = \frac{E(\phi_0)^{1/2}}{m}$ and $\hat{L} = \frac{L(\phi_0)^{1/2}}{m}$. Expressed in terms of the variable $u = \frac{1}{r}$ and the functions $A(r)$, $B(r)$, $C(r)$ and $G(r)$, the orbit equation becomes

$$\begin{aligned}\left(\frac{du}{d\varphi}\right)^2 &= \frac{-u^4\Delta(1/u)}{G(1/u)\left(B(1/u)\hat{E} - \hat{L}A(1/u)\right)^2}\{\Delta(1/u) - \hat{L}^2A(1/u)\\ &\quad + 2B(1/u)\hat{E}\hat{L} - C(1/u)\hat{E}^2\}.\end{aligned} \tag{4.34}$$

Expanding equation (4.34) to third order in u as before, one obtains

$$\left(\frac{du}{d\varphi}\right)^2 \simeq C_0 + C_1u + C_2u^2 + C_3u^3 \tag{4.35}$$

in terms of the constants:

$$C_0 = \frac{1}{\hat{L}^2}(\hat{E}^2 - 1), \tag{4.36}$$

$$C_1 = \frac{4M\hat{E}l}{\hat{L}^3}(\hat{E}^2 - 1) + 2\frac{M}{\hat{L}^2}, \tag{4.37}$$

$$\begin{aligned}C_2 &= \frac{1}{\hat{L}^2}\{3l^2 + 2c(M^2 - l^2)A^2\}(\hat{E}^2 - 1) + \frac{1}{\hat{L}^3}8M^2\hat{E}^3l\\ &\quad + \frac{1}{\hat{L}^4}12M^2\hat{E}^2(\hat{E}^2 - 1)l^2 - 1,\end{aligned} \tag{4.38}$$

$$\begin{aligned}C_3 &= 2M + \frac{1}{\hat{L}^2}\{(6M - 4McA^2)l^2 + 4cM^3A^2\}\hat{E}^2\\ &\quad + \frac{1}{\hat{L}^3}\{[(12M - 8cMA^2)l^3 + 8cM^3lA^2]\hat{E}(\hat{E}^2 - 1) + 16M^3\hat{E}^3l\}\\ &\quad + \frac{1}{\hat{L}^4}\{24M^3\hat{E}^2(2\hat{E}^2 - 1)l^2\} + \frac{1}{\hat{L}^5}\{32M^3\hat{E}^3(\hat{E}^2 - 1)l^3\}.\end{aligned} \tag{4.39}$$

The first three terms of C_2 (terms except -1) and all terms in C_3 imply corrections to Newtonian orbits.

We note that both orbit equations have been written in the form

$$\left(\frac{du}{d\varphi}\right)^2 \simeq g(u) = L_0 + L_1 u + L_2 u^2 + L_3 u^3, \tag{4.40}$$

so their solutions can be analysed in terms of the corresponding constants according to the roots of the equation $g(u) = 0$. Suppose first that all three roots are real. This corresponds that following inequality should be satisfied:

$$4L_1^3 L_3 + 4L_0 L_2^3 - L_2^2 L_1^2 + 27L_0^2 L_3^2 - 18L_0 L_1 L_2 L_3 < 0. \tag{4.41}$$

Suppose further that the roots are distinct and ordered to satisfy $u_1 < u_2 < u_3$. Then

$$u_1 + u_2 + u_3 = -\frac{L_2}{L_3}. \tag{4.42}$$

From the orbit equation, $g(u) \geq 0$ throughout the motion. Thus, $g(u)$ will have a local maximum between u_1 and u_2. Hence, for a bounded orbit, u_1 corresponds to the aphelion and u_2 corresponds to the perihelion. We consider the following cases:

i. If $u_1 > 0$, one obtains bounded orbits of *elliptic* type. This requires that both L_0 and L_2 be negative provided that $L_3 > 0$. Then the particle is confined to the interval $u_1 < u < u_2$. (If $u_1 = u_2$, one obtains circular orbits.)

ii. If $u_1 = 0$, one obtains open orbits of *parabolic* type. This requires that $L_0 = 0$. This is possible for orbits associated with both Levi-Civita and torsional connections provided $E^2 = m^2\phi_0^{-1}$.

iii. If $u_1 < 0$, one obtains open orbits of *hyperbolic* type. This requires that $E^2 > m^2\phi_0^{-1}$ provided that $L_3 > 0$ where $L_3 = C_3$ if the orbit is associated with an autoparallel of the torsional connection and $L_3 = S_3$ if it is associated with the Levi-Civita connection.

4.2 The Analysis Of Bounded Orbits

We are interested in the (bounded) elliptical type of orbits. In that case, from the requirement that $L_0 < 0$ and $L_2 < 0$, we obtain some restrictions on the energy and the constant A. If the orbit is a geodesic of a Levi-Civita connection, then it requires that $\tilde{E}^2 - 1 < 0$ and the constant A should satisfy $A < A_2$ or $A > A_1$, where

$$A_1 = \frac{\Gamma + \sqrt{\Delta_0}}{2\Sigma} \tag{4.43}$$

and

$$A_2 = \frac{\Gamma - \sqrt{\Delta_0}}{2\Sigma} \tag{4.44}$$

with

$$\Gamma = \frac{1}{\tilde{L}^2}\sqrt{M^2 - l^2}\left\{M - \frac{1}{\tilde{L}}4M\tilde{E}l\right\}, \tag{4.45}$$

$$\begin{aligned}\Delta_0 \;=\; & \frac{1}{\tilde{L}^6}\Big\{(M^2 - l^2)M^2\tilde{L}^2 - 8(M^2 - l^2)\tilde{E}^2M^2l^2 \\ & -8M^2\tilde{E}(M^2 - l^2)(1 - 2\tilde{E}^2)l\tilde{L} \\ & -8c(M^2 - l^2)(1 - \tilde{E}^2)\{\tilde{L}^4 - 12M^2\tilde{E}^2(\tilde{E}^2 - 1)l^2 \\ & -3(\tilde{E}^2 - 1)l^2\tilde{L}^2 - 8M^2\tilde{E}^3l\tilde{L}\} \\ & -2(M^2 - l^2)\{\tilde{L}^4 - 12M^2\tilde{E}^4l^2 - 3l^2(\tilde{E}^2 - 1)\tilde{L}^2\}\Big\}\end{aligned} \tag{4.46}$$

and

$$\Sigma = \frac{1}{\tilde{L}^2}(M^2 - l^2)\{2c(1 - \tilde{E}^2) + \frac{1}{2}\}. \tag{4.47}$$

We assume that $\Delta_0 > 0$. We note that if $\Delta_0 < 0$, we conclude that $S_2 < 0$ for all A, since Σ is positive. It means that in that case we obtain elliptical geodesic orbits for all A.

On the other hand, if the orbit is associated with an autoparallel of a connection with torsion, then we obtain the inequality $\hat{E}^2 - 1 < 0$ for the energy. Since C_2 has to be negative for elliptical orbits, then for constant A,

$$\begin{aligned} A^2 \quad > \quad & \frac{1}{2c(M^2 - l^2)(1 - \hat{E}^2)}\{\frac{1}{\hat{L}}8M^2\hat{E}^3 l + \frac{1}{\hat{L}^2}12M^2\hat{E}^2(\hat{E}^2 - 1)l^2 \\ & +3(\hat{E}^2 - 1)l^2 - \hat{L}^2\} \end{aligned} \tag{4.48}$$

should be satisfied. Before giving the general solution of equation (4.40), we can discuss Newtonian orbits. Newtonian orbit in geodesic case is obtained from

$$\left(\frac{du}{d\varphi}\right)^2 \simeq S_0 + S_1 u - u^2, \tag{4.49}$$

i.e. we neglect corrections. Let us define $\bar{L} = m\bar{h}$ with $\bar{h} = \sqrt{M\bar{r}_0}$ $(G = 1, c = 1)$. We can identify $\bar{r}_0$ as standard Newtonian Kepler orbit parameter obtained when $\phi = \phi_0 = constant$ and $l = 0$. Then $\bar{r}_0 = \frac{\tilde{L}^2}{M\phi_0}$ in terms of $\tilde{L}$. We can write constant S_1 in terms of $\bar{r}_0$ as

$$S_1 = \left\{\frac{1}{\tilde{L}}4\tilde{E}(\tilde{E}^2 - 1)l + 2 - \sqrt{1 - \left(\frac{l^2}{M^2}\right)A}\right\}\frac{1}{\phi_0\bar{r}_0}. \tag{4.50}$$

Let us further define a new orbit parameter $\tilde{r}_0$ as $\tilde{r}_0 = \frac{2}{S_1} = \frac{1}{\tilde{u}_0}$. Then by redefining a new variable z as $z = u - \tilde{u}_0$, Newtonian orbit equation (4.49) becomes

$$\left(\frac{dz}{d\varphi}\right)^2 \simeq (S_0 + \tilde{u}_0^2) - z^2 \tag{4.51}$$

whose solution is given by the closed ellipse equation

$$\frac{1}{r} = \frac{1}{\tilde{r}_0}(1 + \tilde{\varepsilon}\cos(\varphi + B)) \tag{4.52}$$

where the constant B determines the initial orientation of the orbit and it can be chosen as zero. The eccentricity of the elliptical orbit is given by

$$\tilde{\varepsilon} = \sqrt{1 + S_0\tilde{r}_0^2} \tag{4.53}$$

in terms of S_0 and $\tilde{r}_0$. If, on the other hand, the worldline is an autoparallel of a connection with torsion, then the Newtonian orbit is obtained from

$$\left(\frac{du}{d\varphi}\right)^2 \simeq C_0 + C_1 u - u^2, \tag{4.54}$$

i.e. we neglect general relativistic corrections as in the geodesic case. We define $r_0 = \frac{L^2}{m^2 M}$ as standard orbit parameter obtained when $\phi = \phi_0$ and $l = 0$. It can be noted that actually $\bar{r}_0 = r_0$ since when $\phi = \phi_0$, the geodesic and autoparallel orbits are equivalent. However, since we define different energy and orbital angular momentum quantities in each case, we take these orbital parameters in distinct notations to prevent confusion. Similarly we can express C_1 in terms of r_0:

$$C_1 = \left\{\frac{1}{\hat{L}}4\hat{E}(\hat{E}^2 - 1)l + 2\right\}\frac{1}{\phi_0 r_0}. \tag{4.55}$$

As before, let us further define a new orbit parameter $\hat{r}_0$ as $\hat{r}_0 = \frac{2}{C_1} = \frac{1}{\hat{u}_0}$. Then by redefining a new variable z as $z = u - \hat{u}_0$, autoparallel Newtonian orbit equation (4.54) becomes

$$\left(\frac{dz}{d\varphi}\right)^2 \simeq (C_0 + \hat{u}_0^2) - z^2 \tag{4.56}$$

whose Newtonian Kepler orbit solution can be written in terms of $r(\varphi)$ as

$$\frac{1}{r} = \frac{1}{\hat{r}_0}\{1 + \hat{\varepsilon}\cos(\varphi + B)\}, \tag{4.57}$$

where the eccentricity $\hat{\varepsilon}$ of the orbit is defined as $\hat{\varepsilon} = \{1 + C_0\hat{r}_0^2\}^{1/2}$.

The general solution of the orbit equation (4.40) can be expressed in terms of Jacobian elliptic functions. By introducing the variables,

$$x = \frac{1}{2}\varphi\sqrt{L_3(u_3 - u_1)}, \qquad y = \sqrt{\frac{u - u_1}{u_2 - u_1}},$$

(4.40) becomes

$$\left(\frac{dy}{dx}\right)^2 = (1 - y^2)(1 - p^2y^2) \tag{4.58}$$

with $p = \sqrt{\frac{u_2 - u_1}{u_3 - u_1}}$. Its general solution is

$$y = sn(x + \delta), \tag{4.59}$$

where δ is an arbitrary constant. Hence for both connections yielding orbits with perihelia,

$$u - u_1 = (u_2 - u_1)sn^2\left(\frac{1}{2}\varphi\sqrt{L_3(u_3 - u_1)} + \delta\right). \tag{4.60}$$

This elliptic solution does not describe a closed elliptical orbit. Therefore, we observe a perihelion shift when we complete one revolution. The periodicity of these solutions enables one to calculate a perihelion shift per revolution. The increase in φ between successive perihelia is given precisely by

$$\Delta\varphi = 2\int_{u_1}^{u_2} \frac{du}{\sqrt{L_3(u - u_1)(u - u_2)(u - u_3)}}. \tag{4.61}$$

With the transformation $y = \sqrt{\frac{u - u_1}{u_2 - u_1}}$, this becomes

$$\Delta\varphi = \frac{4K}{\sqrt{(u_3 - u_1)L_3}}, \tag{4.62}$$

where

$$K = \int_0^1 \frac{dy}{\sqrt{(1-y^2)(1-p^2y^2)}}. \tag{4.63}$$

Depending on the circumstances, one may be able to approximate this integral. This is possible if one is interested in non-relativistic bounded orbits in which the dimensionless quantity Mu remains small compared with unity and the orbital speed is small compared with the speed of light. This would, for example, arise for the motion of the planet Mercury regarded as a test particle in orbit about the Sun as a source. Even at the Sun's surface, where $M_\odot = 1.477 \times 10^3 m$ and $R_\odot = 7 \times 10^8 m$, $Mu = 2.11 \times 10^{-6}$. Since in both cases $\frac{L_3}{L_2}$ is of the order of M, both $-\frac{L_3}{L_2}u_2$ and $-\frac{L_3}{L_2}u_1$ are small quantities at the perihelion and the aphelion [36]. Thus in the following we approximate $-\frac{L_3}{L_2}u_3 \simeq 1$. This means that p^2 can also be considered small, so:

$$K \simeq \frac{1}{2}\pi\left(1+\frac{1}{4}p^2\right)$$

and since the ratios $\frac{u_2}{u_3}$ and $\frac{u_1}{u_3}$ are small, we expand

$$p^2 \simeq \frac{(u_2-u_1)}{u_3} \simeq -\frac{(u_2-u_1)L_3}{L_2}.$$

We note that since $L_2 < 0$, we can replace $-L_2$ by $|L_2|$. Then we further approximate the term

$$\frac{1}{\sqrt{(u_3-u_1)L_3}} = \frac{1}{\sqrt{|L_2|\left(1+\frac{L_3}{L_2}(u_2+2u_1)\right)}} \simeq \frac{1}{\sqrt{|L_2|}}\left(1-\frac{L_3}{2L_2}(u_2+2u_1)\right).$$

After a little algebra, one finds that the increase in φ per revolution becomes

$$\Delta\varphi \simeq \frac{2\pi}{\sqrt{|L_2|}}\left(1-\frac{3L_3}{4L_2}(u_1+u_2)\right). \tag{4.64}$$

The advance of the perihelion (the perihelion shift) per revolution would be

$$\Sigma = \Delta\varphi - 2\pi. \tag{4.65}$$

We can express the perihelion distance and the aphelion distance in terms of elliptical orbit parameters. Then, in terms of the semi-axis major r_0 and orbit eccentricity ε

$$r_1 = (1+\varepsilon)r_0, \qquad r_2 = (1-\varepsilon)r_0,$$

where $r_2 = \frac{1}{u_2}$ corresponds to perihelion distance and $r_1 = \frac{1}{u_1}$ corresponds to aphelion distance. The perihelion shift may be expressed in terms of these orbit parameters for the limiting Newtonian Kepler ellipse. Once a set of Kepler orbit parameters have been ascertained then these formulae permit one to match them to a relativistic orbit in terms of M, ω, l, A and the constants of motion. The perihelion shift of the orbit determined by the Levi-Civita connection can be written in terms of the limiting Newtonian elliptical orbit parameters $\tilde{\varepsilon}$ and $\tilde{r}_0$, as

$$\tilde{\Sigma} = \frac{2\pi}{\sqrt{|S_2|}}\left(1 + \frac{3}{2(1-\tilde{\varepsilon}^2)\tilde{r}_0}\frac{S_3}{|S_2|}\right) - 2\pi. \tag{4.66}$$

On the other hand, the perihelion shift of the orbit determined by the connection with a torsion, can be expressed again in terms of its limiting Newtonian orbit parameters $\hat{\varepsilon}$ and $\hat{r}_0$:

$$\hat{\Sigma} = \frac{2\pi}{\sqrt{|C_2|}}\left(1 + \frac{3}{2(1-\hat{\varepsilon}^2)\hat{r}_0}\frac{C_3}{|C_2|}\right) - 2\pi. \tag{4.67}$$

We note that, even with the same constants of motion and the same limiting Kepler orbits, these two shifts will differ. Interestingly, when $A = 0$ (i.e. the scalar field ϕ is constant) both orbit equations describe geodesic motion given by the Levi-Civita connection in a background Kerr geometry. If one further sets $l = 0$, they both describe geodesic motion in a background Schwarzschild geometry. In this case the constants reduce to $C_2 = S_2 = -1$ and $C_3 = S_3 = 2M$ and the perihelion shift reduces to the classical value [36]

$$\Sigma = \frac{6\pi M}{(1-\varepsilon^2)r_0}. \tag{4.68}$$

Interestingly, we can include fourth order term to see its effect on the perihelion shift. In that case, in both cases orbit equation is of the form,

$$\left(\frac{du}{d\varphi}\right)^2 \simeq p(u) = L_0 + L_1 u + L_2 u^2 + L_3 u^3 + L_4 u^4. \tag{4.69}$$

Assume that the roots of this equation are ordered as $u_4 < u_3 < u_2 < u_1$. Then we can write (4.69) in the following form:

$$\left(\frac{du}{d\varphi}\right)^2 \simeq p(u) = L_4(u-u_1)(u-u_2)(u-u_3)(u-u_4). \tag{4.70}$$

There are two cases in which $p(u) \geq 0$ provided that $L_4 > 0$ [37]:

i-) $u \leq u_4$ or $u \geq u_1$,

ii-) $u_3 \leq u \leq u_2$.

We further assume that the second case holds. Therefore, u_3 will correspond to aphelion and u_2 will correspond to perihelion. Then by making the transformation

$$x = \sqrt{L_4}\varphi, \qquad \frac{u-u_3}{u-u_4} = \frac{u_2-u_3}{u_2-u_4}y^2,$$

equation (4.70) becomes

$$\left(\frac{dy}{dx}\right)^2 = \frac{1}{4}(u_2-u_4)(u_1-u_3)(1-y^2)(1-k^2y^2) \tag{4.71}$$

with $k^2 = \frac{(u_1-u_4)(u_2-u_3)}{(u_2-u_4)(u_1-u_3)}$. Its general solution can be written as

$$y = \frac{1}{4}(u_2 - u_4)(u_1 - u_3)sn(x + \eta), \tag{4.72}$$

where η is an arbitrary constant.

In this case, advance of perihelion in both cases is given exactly by

$$\Delta\varphi = 2\int_{u_3}^{u_2} \frac{du}{\sqrt{L_0 + L_1 u + L_2 u^2 + L_3 u^3 + L_4 u^4}} \tag{4.73}$$

and the perihelion shift is calculated from $\Sigma = \Delta\varphi - 2\pi$. However, we cannot employ the previous third order calculations and assumptions to compute the perihelion shift. On the other hand, we can use the method outlined in [38]. The method is to write or expand the orbit equation around its Newtonian Kepler parameter. First, we write the geodesic fourth order orbit equation:

$$\left(\frac{du}{d\varphi}\right)^2 \simeq S_0 + S_1 u + S_2 u^2 + S_3 u^3 + S_4 u^4, \tag{4.74}$$

where constants S_0, S_1, S_2, S_3 are given by the equations (4.24), (4.25), (4.26) and (4.27), respectively, and the constant S_4 is

$$\begin{aligned} S_4 \;=\; & -2c(M^2 - l^2)A^2 - 2l^2 + \frac{1}{\tilde{L}^2}\{2(M^4 + l^4)(\tilde{E}^2 - 1)c^2A^4 \\ & +3(\tilde{E}^2 - 1)l^4 + 12M^2\tilde{E}^2l^2 + (M^4 + 5l^4)cA^2 \\ & -6M^2l^2cA^2 - 4M^2(\tilde{E}^2 - 1)l^2c^2A^4 \\ & +\tilde{E}^2(7M^4 - 5l^4 - 2M^2l^2)cA^2 - (M^2 - l^2)^2cA^4 \\ & +\frac{1}{48}\{(2M^2 - l^2)(\frac{3}{2}l^2 - M^2) + l^4\}A^4 - \frac{1}{6}M(M^2 - l^2)^{3/2}A^3 \\ & +\{(M^2 - l^2)(2M^2 - \frac{3}{2}l^2) + \frac{7}{24}l^4 - \frac{1}{6}(2M^2 - l^2)\left(\frac{11}{2}M^2 - \frac{15}{4}l^2\right)\}A^2 \\ & +\frac{2}{3}M\sqrt{M^2 - l^2}(M^2 + 5l^2)A - 2M\sqrt{M^2 - l^2}cA^3\} \end{aligned}$$

$$+\frac{1}{\tilde{L}^3}\{16M^2\tilde{E}(M^2-l^2)(2\tilde{E}^2-1)lcA^2+48M^2\tilde{E}(\tilde{E}^2-\frac{1}{2})l^3$$
$$+32M^4\tilde{E}^3l-4M^2\tilde{E}(M^2-l^2)lA^2-12M\sqrt{M^2-l^2}\tilde{E}l^3A$$
$$-\frac{2}{3}M\tilde{E}\sqrt{M^2-l^2}\{2(4M^2-l^2)+(M^2-l^2)A^2\}lA$$
$$-8M\tilde{E}\sqrt{M^2-l^2}(M^2-l^2)lcA^3\}+\frac{1}{\tilde{L}^4}\{36M^2\tilde{E}^2(\tilde{E}^2-1)l^4$$
$$+144M^4\tilde{E}^4l^2-6M^2\tilde{E}^2(M^2-l^2)l^2A^2-36M^3\tilde{E}^2\sqrt{M^2-l^2}l^2A$$
$$+24M^2\tilde{E}^2(M^2-l^2)(\tilde{E}^2-1)l^2cA^2-48M^4\tilde{E}^2l^2\}$$
$$+\frac{1}{\tilde{L}^5}\{128M^4\tilde{E}^3\left(\frac{3}{2}\tilde{E}^2-1\right)l^3-32M^3\tilde{E}^3\sqrt{M^2-l^2}l^3A\}$$
$$+\frac{1}{\tilde{L}^6}\{80M^4\tilde{E}^4(\tilde{E}^2-1)l^4\}. \tag{4.75}$$

Now, we define a new variable as $z=u-\tilde{u}_0$ where $\tilde{r}_0=\frac{2}{S_1}=\frac{1}{\tilde{u}_0}$ is Newtonian Kepler orbit parameter, and we substitute z into (4.74). We assume that the orbit is nearly circular, so that z is small. Therefore we neglect the terms z^4 and z^3 in the resulting orbit equation. Then we obtain the following nearly circular orbit equation:

$$\left(\frac{dz}{d\varphi}\right)^2\simeq\tilde{S}_0+\tilde{S}_1z-\tilde{S}_2z^2, \tag{4.76}$$

where

$$\tilde{S}_0=S_0+S_1\tilde{u}_0+S_2\tilde{u}_0^2+S_3\tilde{u}_0^3+S_4\tilde{u}_0^4, \tag{4.77}$$

$$\tilde{S}_1=S_1+2S_2\tilde{u}_0+3S_3\tilde{u}_0^2+4S_4\tilde{u}_0^3 \tag{4.78}$$

and

$$\tilde{S}_2=|S_2|-3S_3\tilde{u}_0-6S_4\tilde{u}_0^2. \tag{4.79}$$

The solution to the orbit equation (4.76) can be written as:

$$z = \frac{\tilde{S}_1}{2\tilde{S}_2} + \left\{ \frac{\tilde{S}_0}{\tilde{S}_2} + \left(\frac{\tilde{S}_1}{2\tilde{S}_2} \right)^2 \right\}^{1/2} \cos(\sqrt{\tilde{S}_2}\,\varphi + B), \tag{4.80}$$

where B is a constant. We see that, the orbit returns to same r, when $\sqrt{\tilde{S}_2}\Delta\varphi = \sqrt{\tilde{S}_2}(\varphi_2 - \varphi_1) = 2\pi$ is satisfied. Therefore, the change in φ from one perihelion to the next is

$$\Delta\varphi = \frac{2\pi}{\sqrt{\tilde{S}_2}}. \tag{4.81}$$

Thus, the perihelion shift becomes

$$\tilde{\Sigma} = \frac{2\pi}{\sqrt{\tilde{S}_2}} - 2\pi. \tag{4.82}$$

We can express all the expansion constants in terms of the ratio $\frac{r_s}{\tilde{r}_0}$ where $r_s = 2M$ is the Schwarzschild radius, and $\tilde{r}_0$ is the Newtonian orbit parameter. Since we have assumed that Newtonian orbit is much larger than the Schwarzschild radius and the speed of orbiting object is nonrelativistic, using equation (4.79) and writing S_2 as $S_2 = -1 + S_2'$, ($|S_2| = 1 - S_2'$), we can approximate

$$\frac{1}{\sqrt{\tilde{S}_2}} \simeq 1 + \frac{1}{2}\{S_2' + 3S_3\tilde{u}_0 + 6S_4\tilde{u}_0^2\}. \tag{4.83}$$

Then perihelion shift can be written as

$$\tilde{\Sigma} \simeq \pi\{S_2' + 3S_3\tilde{u}_0 + 6S_4\tilde{u}_0^2\}. \tag{4.84}$$

We note that, we can express change in φ (4.81) in the form,

$$\tilde{\Sigma} = \frac{2\pi}{\sqrt{\tilde{S}_2}} = \frac{2\pi}{\sqrt{|S_2|\left(1 - \frac{3S_3\tilde{u}_0}{|S_2|} - \frac{6S_4\tilde{u}_0^2}{|S_2|}\right)}} \simeq \frac{2\pi}{\sqrt{|S_2|}}\{1 + \frac{3S_3\tilde{u}_0}{2|S_2|} + \frac{3S_4\tilde{u}_0^2}{|S_2|}\}.$$

Then if we consider the third order orbit equation ($S_4 = 0$), since we can approximate $\frac{1}{1-\tilde{\varepsilon}^2} \simeq 1 + \tilde{\varepsilon}^2 \cdots$, by subtracting 2π, this reduces to the perihelion shift (4.66) at the zeroth order ($(\tilde{\varepsilon}^2)^0 = 1$ order).

Similarly, we write the autoparallel fourth order orbit equation related with a connection with torsion, as

$$\left(\frac{du}{d\varphi}\right)^2 \simeq C_0 + C_1 u + C_2 u^2 + C_3 u^3 + C_4 u^4 \tag{4.85}$$

with the expansion constants C_0, C_1, C_2 and C_3 given by equations (4.36), (4.37), (4.38) and (4.39), respectively, and the constant C_4 is:

$$\begin{aligned} C_4 \quad = \quad & -2c(M^2 - l^2)A^2 - 2l^2 + \frac{1}{\hat{L}^2}\{2(M^4 + l^4)(\hat{E}^2 - 1)c^2 A^4 \\ & + M^2(M^2 - 6l^2)cA^2 + 12M^2\hat{E}^2 l^2 - 2cM^2\hat{E}^2 l^2 A^2 \\ & + 7M^4\hat{E}^2 cA^2 - 4M^2 l^2(\hat{E}^2 - 1)c^2 A^4 - 5(\hat{E}^2 - 1)l^4 cA^2 \\ & + 3(\hat{E}^2 - 1)l^4\} + \frac{1}{\hat{L}^3}\{32M^4\hat{E}^3 l + 48M^2\hat{E}(\hat{E}^2 - \frac{1}{2})l^3 \\ & + 16M^2\hat{E}(M^2 - l^2)(2\hat{E}^2 - 1)lcA^2\} \\ & + \frac{1}{\hat{L}^4}\{24M^2\hat{E}^2(M^2 - l^2)(\hat{E}^2 - 1)l^2 cA^2 \\ & + 144M^4\hat{E}^4 l^2 + 36M^2\hat{E}^2(\hat{E}^2 - 1)l^4 - 48M^4\hat{E}^2 l^2\} \\ & + \frac{1}{\hat{L}^5}\{128M^4\hat{E}^3\left(\frac{3}{2}\hat{E}^2 - 1\right)l^3\} + \frac{1}{\hat{L}^6}\{80M^4\hat{E}^4(\hat{E}^2 - 1)l^4\}. \end{aligned} \tag{4.86}$$

As in the geodesic case, we write orbit equation (4.85) in terms of new variable z, defining $z = u - \hat{u}_0$, where $\hat{u}_0$ is inverse of Newtonian Kepler orbit parameter, i.e. $\hat{r}_0 = \frac{2}{C_1} = \frac{1}{\hat{u}_0}$. Again, we consider that the orbit is nearly circular, so that z is small and we neglect terms in y^4 and y^3. Then in terms of variable z, the orbit equation (4.85) becomes

$$\left(\frac{dz}{d\varphi}\right)^2 \simeq \hat{C}_0 + \hat{C}_1 z - \hat{C}_2 z^2, \tag{4.87}$$

where

$$\hat{C}_0 = C_0 + C_1\hat{u}_0 + C_2\hat{u}_0^2 + C_3\hat{u}_0^3 + C_4\hat{u}_0^4, \tag{4.88}$$

$$\hat{C}_1 = C_1 + 2C_2\hat{u}_0 + 3C_3\hat{u}_0^2 + 4C_4\hat{u}_0^3 \tag{4.89}$$

and

$$\hat{C}_2 = |C_2| - 3C_3\hat{u}_0 - 6C_4\hat{u}_0^2. \tag{4.90}$$

The solution to the equation (4.87) is

$$z = \frac{\hat{C}_1}{2\hat{C}_2} + \left\{\frac{\hat{C}_0}{\hat{C}_2} + \left(\frac{\hat{C}_1}{2\hat{C}_2}\right)^2\right\}^{1/2} \cos(\sqrt{\hat{C}_2}\,\varphi + B). \tag{4.91}$$

The periodicity of this solution requires that, the change in φ from one perihelion to the next can be written as

$$\Delta\varphi = \frac{2\pi}{\sqrt{\hat{C}_2}}. \tag{4.92}$$

Then the perihelion shift becomes

$$\hat{\Sigma} = \frac{2\pi}{\sqrt{\hat{C}_2}} - 2\pi. \tag{4.93}$$

As in the geodesic case, we can express constants C_0, C_1, C_2, C_3 and C_4 in terms of the ratio $\frac{r_s}{\hat{r}_0}$. Since Newtonian orbit is much larger than Schwarzschild radius $r_s = 2M$ and orbital speed of orbiting object is nonrelativistic, using (4.90) and writing $|C_2| = 1 - C_2'$, we can express the perihelion shift as

$$\hat{\Sigma} \simeq \pi\{C_2' + 3C_3\hat{u}_0 + 6C_4\hat{u}_0^2\}. \tag{4.94}$$

As a note, by this method, one obtains the results of perihelion shift in autoparallel orbit in static Brans-Dicke metric background presented in [2]. Similarly, we note that, we can express equation (4.92) as,

$$\Delta\varphi = \frac{2\pi}{\sqrt{\hat{C}_2}} = \frac{2\pi}{\sqrt{|C_2|\left(1 - \frac{3C_3\hat{u}_0}{|C_2|} - \frac{6C_4\hat{u}_0^2}{|C_2|}\right)}} \simeq \frac{2\pi}{\sqrt{|C_2|}}\{1 + \frac{3C_3\hat{u}_0}{2|C_2|} + \frac{3C_4\hat{u}_0^2}{|C_2|}\}.$$

Then if we consider third order equation ($C_4 = 0$), since we can approximate $\frac{1}{1-\hat{\varepsilon}^2} \simeq 1 + \hat{\varepsilon}^2 \cdots$, by subtracting 2π, this reduces to the perihelion shift (4.67) at the zeroth order ($(\hat{\varepsilon}^2)^0 = 1$ order).

Therefore, we see that even with the same constants of motion and same limiting Kepler parameters, geodesic orbits and autoparallel orbits based on a connection with torsion, differ. Hence, if there exists Brans-Dicke scalar field coupling, we should consider possible formulations of the theory. If scalar theory is specified with a (torsion-free) Levi-Civita connection, then test particles move along the geodesics. If the space-time geometry is equipped with a connection with torsion, then they follow autoparallels of a connection with a spacetime torsion.

CHAPTER 5

CONCLUSION

In this work, we have constructed the scalar tensor theory of gravitation in D dimensions in all possible geometric structures. First we have formulated the theory on metric-compatible, torsion-free connection structure, by considering that, the connections and orthonormal frame vectors are not independent. Then, we have constructed the same theory in a spacetime with torsion, without using any constraint. It is shown that, scalar tensor theory with torsion can be reformulated in terms of torsion-free theory. Result is the shift of the Brans-Dicke coupling parameter. We have then constructed the scalar tensor theory in a spacetime with non-metric compatible connection structure. In this case, we have formulated theory in such a way that corresponding action is Weyl symmetric under conformal Weyl transformations of interacting field elements. We have also rewritten the scalar tensor theory with non-metric compatible connections, in terms of scalar theory with Levi-Civita connections. By adding an antisymmetric axion field we have also constructed axi-dilaton gravity in all possible geometries. We have reformulated both the scalar tensor theory and axi-dilaton theory in Einstein frame. Then in the geometries mentioned, we have examined the motion of massive test particles. We have seen that, worldlines of the particles are nothing but the autoparallels of a specified connection. We have also shown that, Levi-Civita autoparallels in Einstein frame are equivalent to autoparallels of a connection with torsion in Brans-Dicke frame.

We have presented the static spherically symmetric and stationary Kerr (rotating) type axially symmetric solutions of the scalar tensor and axi-dilaton gravity theories. It is also shown that, the static and stationary solutions of pure scalar tensor gravitation theory where the scalar field interacts only with the gravitational field, do not describe a black hole. Although the solutions are asymptotically flat, event horizon surface becomes singular.

As an application to autoparallel motion of massive test particles, we have examined the geodesic elliptical orbits and autoparallel elliptical orbits with a torsion depending on gradient of scalar field, in the Kerr Brans-Dicke spacetime describing a rotating gravitational source. We have presented Newtonian Kepler limit of the solutions. We have seen that even with the same constants of motion and the same limiting Kepler parameters, geodesic orbits and autoparallel orbits with a torsion differ [2],[35]. Therefore, the connection or the gauge structure (whether the connections of theory are constrained to be Levi-Civita or not) should be specified in all interactions including gravity.

REFERENCES

[1] T. Dereli, R. W. Tucker, *On The Motion Of Matter In Spacetime*, gr-qc/0107017.

[2] T. Dereli, R. W. Tucker, Mod. Phys. Lett. **A17**, (2002), 421.

[3] T. Dereli, R. W. Tucker, JHEP **0203**, (2002), 041.

[4] C. H. Brans, R. Dicke, Phys. Rev. **124**, (1961), 925.

[5] S. Weinberg, **Gravitation And Cosmology: Principles And Applications Of The General Theory Of Relativity** (John Wiley & Sons, Inc., 1972).

[6] T. Dereli, R. W. Tucker, Phys. Lett. **B110**, (1982), 206.

[7] H. Cebeci, **M.Sc. Thesis**, *Magnetic Black Branes*, (METU, Ankara, 1999).

[8] P. A. M. Dirac, Proc. Roy. Soc. **A333**, (1973), 403.

[9] I. M. Benn, R. W. Tucker, **An Introduction To Spinors And Geometry With Applications In Physics** (IOP Publishing Ltd, 1987).

[10] H. Weyl, Preuss. Akad. Wiss., Sitzungsber (1918), 465.

[11] H. Weyl, Z. Phys. **56**, (1929), 330.

[12] M. Adak, T. Dereli, L. Ryder, Int. J. Mod. Phys. **D12**, (2003), 145.

[13] T. Dereli, M. Önder, J. Schray, R. W. Tucker, C. Wang, Class. Quantum Grav. **13**, (1996), L103.

[14] T. Dereli, R. W. Tucker, Class. Quantum Grav. **4**, (1987), 791.

[15] M. Gasperini, Class. Quantum Grav. **5**, (1988), 521.

[16] T. Dereli, R. W. Tucker, Class. Quantum Grav. **11**, (1994), 2575.

[17] T. Dereli, R. W. Tucker, Class. Quantum Grav. **12**, (1995), L31.

[18] F. R. Tangherlini, Nuovo Cimento **27**, (1963), 636.

[19] B. C. Xanthopoulos, T. Zannias, Phys. Rev. **D40**, (1989), 2564.

[20] T. Dereli, A. Eriş, A. Karasu, Nuovo Cimento **B93**, (1986), 102.

[21] T. Dereli, A. Eriş, Turk. J. Physics (Doğa) **A9**, (1985), 30.

[22] T. Dereli, M. Önder, R. W. Tucker, Class. Quantum Grav. **12**, (1995), L25.

[23] R. G. Cai, Y. S. Myung, Phys. Rev. **D56**, (1997), 3466.

[24] M. Gürses, E. Sermutlu, Class. Quantum Grav. **12**, (1995), 2799.

[25] G. W. Gibbons, K. Maeda, Nucl. Phys. **B298**, (1988), 74.

[26] G. T. Horowitz, A. Strominger, Nucl. Phys. **B360**, (1991), 197.

[27] H. Cebeci, T. Dereli, Phys. Rev. **D65**, (2002), 047501.

[28] R. P. Kerr, Phys. Rev. Lett., **11**, (1963), 237.

[29] M. J. Perry, Ann. Phys. **172**, (1986), 304.

[30] Barrett O'Neill, **The Geometry Of Kerr Black Holes** (Hamilton Printing Company, 1995).

[31] C. B. G. McIntosh, Comm. Math. Physics, **37**, (1974), 335.

[32] G. T. Horowitz, A. Sen, Phys. Rev. **D53**, (1996), 808.

[33] J. Horne, G. T. Horowitz, Phys. Rev. **D46**, (1992), 1340.

[34] U. Bleyer, V. Frolov, A. Zelnikov, Ann. Physics (Leipzig), **44**, (1987), 371.

[35] H. Cebeci, T. Dereli, R. W. Tucker, Preprint, *Autoparallel Orbits In Kerr Brans-Dicke Spacetimes*, (2003), sent to Int. J. Mod. Phys. D.

[36] J. L. Synge, **Relativity: The General Theory**, (North-Holland Publishing, 1960).

[37] D. R. Guo, Z. X. Wang, **Special Functions**, (World Scientific Publishing, 1989).

[38] Bernard F. Schutz, **A First Course In General Relativity**, (Cambridge University Press, 1990).

[39] M. Nakahara, **Geometry Topology And Physics**, (IOP Publishing, 1990).

APPENDIX A

EXTERIOR ALGEBRA

In this appendix we explain the exterior algebra notation and operators on differential forms which is used throughout the work [39].

A.1 Differential Forms On The Manifolds

Let M be a differentiable manifold of dimension n. Suppose $\{U_i\}$ is a family of open sets which covers M, called submanifolds of M. Introduce a map φ_i from U_i onto an open subset U_i' of R^n. The pair (U_i, φ_i) is called a chart. φ_i is represented by n functions $\{x^i\}$ which are the local coordinate functions. The cotangent space at $y \in U$, where U is a submanifold of an n-dimensional manifold, is defined to be $(T_y\, R^n)^*$ which is the n-dimensional vector space of linear forms on the tangent space at y. The elements of $(T_y\, R^n)^*$ are called the cotangent vectors or simply 1-forms. We can define a basis for $(T_y\, R^n)^*$ as $\{dx^1, dx^2, dx^3, \cdots dx^n\}$. A differential form of order p or a p-form on the manifold M is a totally antisymmetric tensor field of type $(0, p)$. We can denote the vector space of p-forms by $\Lambda^p(T^*M)$. Any p-form can be expressed as a wedge product or an antisymmetric tensor product of the basis one forms, i.e. if w is a p-form,

$$w = \frac{1}{p!} w_{\mu_1\mu_2\mu_3\cdots\mu_p} dx^{\mu_1} \wedge dx^{\mu_2} \wedge dx^{\mu_3} \cdots dx^{\mu_p}, \tag{A.1}$$

where

$$dx^{\mu_1} \wedge dx^{\mu_2} \wedge dx^{\mu_3} \cdots \wedge dx^{\mu_p} = \sum_{P \in S_p} sgn(P) dx^{\mu_{P(1)}} \otimes \cdots dx^{\mu_{P(p)}}, \tag{A.2}$$

where $sgn(P) = +1$ for even permutations and -1 for odd permutations. Also S_p is the symmetric group of order p. We can define the exterior product of a q-form μ and a p-form w as,

$$\begin{aligned}(\mu \wedge w)(X_1, \cdots, X_{q+p}) &= \frac{1}{q!p!} \sum_{P \in S_{q+p}} sgn(P) \mu(X_{P(1)}, \cdots, X_{P(q)}) \\ &\quad w(X_{P(q+1)}, \cdots, X_{P(q+p)}),\end{aligned} \tag{A.3}$$

where $X_i \in T_pM$, which is the tangent space at point p. $(\mu \wedge w) \in \Lambda^{q+p}(T^*M)$ which is zero if $p + q > n$ for an n-dimensional manifold. Some properties of the wedge product are as follows:

i) The wedge product is bilinear,

$$(\varphi_1 + \varphi_2) \wedge \psi = \varphi_2 \wedge \psi + \varphi_1 \wedge \psi$$

and

$$\varphi \wedge (a\psi) = a(\varphi \wedge \psi), \quad a \in R.$$

ii) It is associative,

$$(\varphi \wedge \psi) \wedge \xi = \varphi \wedge (\psi \wedge \xi).$$

iii) It is graded commutative,

$$\varphi \wedge \psi = (-1)^{qp} \psi \wedge \varphi,$$

for $\varphi \in \Lambda^p(T^*M)$ and $\psi \in \Lambda^q(T^*M)$.

iv) It has the following identity,

$$\varphi \wedge \varphi = 0 \quad \text{for} \quad \varphi \in \Lambda^p(T^*M) \quad \text{and} \quad p \quad \text{odd}.$$

In the coordinate basis, T_pM is spanned by $\{\partial/\partial x^\mu\}$ and T_p^*M by $\{dx^\mu\}$. We can define a non-coordinate basis for T_pM and T_p^*M if M is given a metric g. We can define $X_a = X_a^\mu \frac{\partial}{\partial x^\mu}$ as a non-coordinate basis for T_pM and X_a are called the frame vectors, and $e^a = e_\mu^a dx^\mu$ as a non-coordinate basis for T_p^*M and e^a are called the co-frame 1-forms.

A.2 Exterior Derivatives

The exterior derivative d is a map $d : \Lambda^p(T^*M) \rightarrow \Lambda^{p+1}(T^*M)$. Hence it raises the degree of a p-form by one. Consider any p-form w

$$w = \frac{1}{p!} w_{\mu_1\mu_2\mu_3\cdots\mu_p} dx^{\mu_1} \wedge dx^{\mu_2} \wedge dx^{\mu_3} \cdots dx^{\mu_p},$$

then

$$dw = \frac{1}{p!}(\frac{\partial}{\partial x^\mu} w_{\mu_1\cdots\mu_p}) dx^\mu \wedge dx^{\mu_1} \wedge dx^{\mu_2} \wedge dx^{\mu_3} \cdots dx^{\mu_p}. \tag{A.4}$$

The exterior derivative possesses the following properties:

i) For a q-form ξ and a p-form w,

$$d(\xi \wedge w) = d\xi \wedge w + (-1)^q \xi \wedge dw. \tag{A.5}$$

ii) It satisfies

$$d^2 w = 0. \tag{A.6}$$

A.3 Interior (Inner) Derivative

Another operator on differential forms is the inner derivative. It is a map $\iota_X : \Lambda^p(T^*M) \to \Lambda^{p-1}(T^*M)$, hence the degree of the differential form decreases by one. For $X \in T_pM$ and a p-form w

$$w = \frac{1}{p!} w_{\mu_1\mu_2\mu_3\cdots\mu_p} dx^{\mu_1} \wedge dx^{\mu_2} \wedge dx^{\mu_3} \cdots dx^{\mu_p}$$

and $X = X^\mu \frac{\partial}{\partial x^\mu}$,

$$\begin{aligned} \iota_X w &= \frac{1}{(p-1)!} X^\mu w_{\mu\mu_2\mu_3\cdots\mu_p} dx^{\mu_2} \wedge \cdots dx^{\mu_p} \\ &= \frac{1}{p!} \sum_{s=1}^{p} X^{\mu_s} w_{\mu_1\cdots\mu_s\cdots\mu_p} (-1)^{s-1} dx^{\mu_1} \wedge \cdots \wedge \hat{dx}^{\mu_s} \wedge \cdots \wedge dx^{\mu_p}, \end{aligned} \quad \text{(A.7)}$$

where the entry below a hat ˆ has been omitted. Some properties of the inner derivative are:

i) ι_X is an antiderivation. For a p-form w,

$$\iota_X(w \wedge \xi) = \iota_X w \wedge \xi + (-1)^p w \wedge \iota_X \xi.$$

ii) It is nilpotent, i.e.

$$(\iota_X)^2 w = 0.$$

A.4 Hodge Map

Given an n-dimensional manifold with a metric g defined over it, the Hodge map operator $*$ is a linear map $* : \Lambda^p(T^*M) \to \Lambda^{n-p}(T^*M)$ whose action on any

p-form is defined as

$$*(dx^{\mu_1} \wedge dx^{\mu_2} \wedge dx^{\mu_3} \cdots dx^{\mu_p}) = \frac{\sqrt{|g|}}{(n-p)!} \varepsilon^{\mu_1 \mu_2 \mu_3 \cdots \mu_p}{}_{\mu_{p+1} \cdots \mu_n} dx^{\mu_{p+1}} \cdots \wedge dx^{\mu_n}$$

where

$$\varepsilon_{\mu_1 \mu_2 \mu_3 \cdots \mu_n} = \begin{cases} -1 & \text{if } (\mu_1 \mu_2 \mu_3 \cdots \mu_n) \text{ is an odd permutation of } (123 \cdots n), \\ 0 & \text{if any of two indices are equal,} \\ 1 & \text{if } (\mu_1 \mu_2 \mu_3 \cdots \mu_n) \text{ is an even permutation.} \end{cases}$$

For a p-form w,

$$w = \frac{1}{p!} w_{\mu_1 \mu_2 \mu_3 \cdots \mu_p} dx^{\mu_1} \wedge dx^{\mu_2} \wedge dx^{\mu_3} \cdots \wedge dx^{\mu_p},$$

$$*w = \frac{\sqrt{|g|}}{p!(n-p)!} w_{\mu_1 \cdots \mu_p} \varepsilon^{\mu_1 \cdots \mu_p}{}_{\mu_{p+1} \cdots \mu_n} dx^{\mu_{p+1}} \wedge \cdots \wedge dx^{\mu_n}. \qquad \text{(A.8)}$$

If we take the co-frame 1-form field e^a, we can define

$$*(e^a \wedge e^b \wedge e^c \cdots e^p) = \frac{1}{(n-p)!} \varepsilon^{abc \cdots p}{}_{p+1 \cdots n} e^{p+1} \cdots \wedge e^n. \qquad \text{(A.9)}$$

$*$ map has the following properties:

i) For co-frame one forms e^a,

$$*(e^a \wedge e^b \wedge e^c) = \iota^c * (e^a \wedge e^b),$$

where ι^c is the inner derivative with respect to the field X^c.

ii) For any p-form w,

$** (w) = (-1)^{p(n-p)} w$ if (M, g) is Riemannian and,

$** (w) = (-1)^{1+p(n-p)} w$ if it is Lorentzian.

iii) The volume n-form is defined as

$$*1 = \frac{1}{n!} \varepsilon_{abc \cdots n} e^a \wedge e^b \wedge e^c \cdots \wedge e^n.$$

A.5 Linear Connections And Covariant Exterior Derivative

A linear connection [9] on a manifold M is a map $\nabla : \Gamma TM \times \Gamma TM \mapsto \Gamma TM$ that satisfies the following for all $f, g \in \mathcal{F}(M)$ and for all $X, Y, Z \in \Gamma TM$:

$$\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$$

and

$$\nabla_X (fY + gZ) = X(f)Y + f\nabla_X Y + X(g)Z + g\nabla_X Z.$$

Therefore, ∇_X is a linear mapping on vector fields which is also $\mathcal{F}$-linear in X. It is called covariant differentiation with respect to X. From these properties, we can specify ∇ by giving the components of the vector $\nabla_{X_b} X_c$ in any convenient basis $\{X_c\}$:

$$\nabla_{X_b} X_c = \Gamma_{bc}{}^{a} X_a.$$

The n^3 functions $\Gamma_{bc}{}^{a}$, where n is dimension of M, are known as the connection components or connection coefficients in this basis. These coefficients can be used to define a set of 1-forms, called as the connection 1-forms, by

$$\omega^{a}{}_{b} = \Gamma_{cb}{}^{a} e^{c}, \tag{A.10}$$

where $\{e^a\}$ is the co-frame dual to $\{X_a\}$. Equivalently, we can write

$$\nabla_{X_b} X_c = \omega^{a}{}_{c}(X_b) X_a.$$

Given a mixed tensor which is totally antisymmetric in some subset of r vectors, we can associate a set of r-forms with any basis $\{X_j\}$ with its dual basis

$\{e^j\}$. We can suppose that S is a tensor of type $(r+q,p)$. We define a set of r-forms $S^{i_1 i_2 i_3 \cdots i_p}{}_{j_1 \cdots j_q}$ by

$$S^{i_1 i_2 i_3 \cdots i_p}{}_{j_1 \cdots j_q}(X_1, \cdots, X_r) = S(X_1, \cdots, X_r, X_{j_1}, \cdots X_{j_q}, e^{i_1}, \cdots, e^{i_p}).$$

We define the covariant exterior derivative D of $S^{i_1 i_2 i_3 \cdots i_p}{}_{j_1 \cdots j_q}$ in terms of a connection 1-form ω by,

$$\begin{aligned} DS^{i_1 i_2 i_3 \cdots i_p}{}_{j_1 \cdots j_q} &= dS^{i_1 i_2 i_3 \cdots i_p}{}_{j_1 \cdots j_q} + \omega^{i_1}{}_{i_s} \wedge S^{i_s \cdots i_p}{}_{j_1 \cdots j_q} + \cdots \\ &\quad + \omega^{i_p}{}_{i_s} \wedge S^{i_1 \cdots i_s}{}_{j_1 \cdots j_q} - \omega^{j_s}{}_{j_1} \wedge S^{i_1 \cdots i_p}{}_{j_s \cdots j_q} \\ &\quad - \cdots - \omega^{j_s}{}_{j_q} \wedge S^{i_1 \cdots i_p}{}_{j_1 \cdots j_s} \end{aligned}$$

e.g.

$$DS^a{}_b = dS^a{}_b + \omega^a{}_c \wedge S^c{}_b - \omega^d{}_b \wedge S^a{}_d.$$

The exterior covariant derivative satisfies the following identity, for any r-form S

$$D(S^I \wedge T^J) = DS^I \wedge T^J + (-1)^r S^I \wedge DT^J.$$

VITA

The author of this work was born in Merzifon in 1968. He graduated from the Mechanical Engineering Department of Middle East Technical University (METU) with a Bachelor of Science (B.S) degree in 1992. After completing his military service, he started his physics major and obtained his B.S Degree in Physics at METU in 1997. He took his Master of Science Degree in Physics in 1999, also at METU. His research interests include the gauge theories of gravitation in higher dimensions, the geometrical structure of fundamental interactions, black holes, supersymmetry and supergravity theories.