MODELLING AND FORECASTING AGE-SEGMENTED MORTALITY:
EVALUATION OF LEE-CARTER METHOD AND ITS EXTENSIONS

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ABSTRACT

MODELLING AND FORECASTING AGE-SEGMENTED MORTALITY: EVALUATION OF LEE-CARTER METHOD AND ITS EXTENSIONS

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Mortality dynamics deal with the human mortality from birth to death giving insights for the population in different aspects; age, year, gender and enable to ascertain crucial mortality trends as well. Over the last decades, ample new methods have been developed and the mortality modelling has been evolved into more effective ways. Among these models, the pioneering and the most seminal one is Lee-Carter model (Lee & Carter, 1992). From its development, Lee-Carter model has been intensively studied and its variants with different structures have been proposed. While the performance of mortality models has been examined under various issues so far, the modelling of mortality over age patterns for different stages of human life has yet to be studied. Therefore, in this study, our purpose is to utilise a new approach differing from literature by investigating the performance of piecewise mortality models via structural breakpoints for age dimension. Accordingly, the mortality data of three different countries for the eighty-one years averagely; Australia, Japan and Portugal are examined by Lee-Carter model and its extensions with regards to the age structure of corresponding countries. The results reveal that each country has different age structures for female and male subpopulations and the age intervals within these

structures have different characteristics regarding significance of accuracy of these methods. The findings imply the need for a use of different age breaks for female and male subpopulations and the selection of best suited methods for each corresponding age intervals regarding accuracy of mortality forecasts.

Keywords: Piecewise Mortality Modelling, Lee-Carter Model, Functional Demographic Model, Structural Breakpoints of Age, Mortality Forecasting

ÖZ

YAŞA GÖRE AYRILMIŞ MORTALİTENİN MODELLENMESİ VE TAHMİN EDİLMESİ: LEE-CARTER YÖNTEMİ VE TÜREVLERİNİN İNCELENMESİ

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Mortalite dinamikleri, insanların doğumdan ölüme kadar olan süreçte populasyonun yaşı, ilgili yıllar, cinsiyet ve başka diğer birçok alt-grup vasıtasıyla bilgiyi sunarak; oldukça önemli ölümlülük trendleri hakkında da sonuç çıkarmayı sağlamaktadır. Son on yılda, mortalitenin modellenmesi amacıyla birçok yeni yöntemler geliştirilmiş ve mortalitenin modellenmesi çok daha farklı ve etkili bir şekilde sağlanmaya başlanılmıştır. Söz konusu modeller içerisinde, öncül ve en çığır açan model ise Lee-Carter modeli olmuştur (Lee & Carter, 1992). Bu süreçten itibaren, Lee-Cater modeli üzerinde yoğun bir şekilde çalışılmış ve ilgili modelin farklı yapılara sahip türevleri önerilmiştir. Mortalite modellerinin performansı, bugüne kadar birçok farklı problem bağlamında araştırılmış olmasına karşın; mortalitenin insan hayatının farklı dönemlerine özgü yaş aralıklarına göre modellenmesi, araştırılan bir husus olmamıştır. Dolayısıyla bu çalışmanın amacı, yapısal yaşsal kırılım noktaları baz alınarak, parçalı mortalite modellerinin tahmin performansının, literatürden farklı olarak yeni bir yaklaşım kullanılarak incelenmesidir. Bu doğrultuda çalışmada Avustralya, Japonya ve Portekiz olmak üzere üç farklı ülkenin ortalama seksen-bir

yıllık mortalite verileri, ülkelerin sahip olduğu farklı yaşsal yapılar bağlamında Lee-Carter modeli ve türevleri yardımıyla incelenmiştir. Sonuçlar, her bir ülkenin kadın ve erkek alt populasyonlarının farklı yaşsal yapılara sahip olduğunu ve söz konusu yaşsal yapıların mevcut olduğu yaş aralıklarındaki mortalitenin ilgili yöntemlerle tahmin edilmesinde tahmin doğruluklarının anlamlı özelliklere sahip olduğunu göstermektedir. Bulgular, kadın ve erkek alt populasyonlarının sahip olduğu farklı yaşsal kırılımların göz önünde bulundurulması ve gelecek süreçler için mortalitenin tahmin edilmesinde, söz konusu yaş aralıklarına en uygun ve optimistik olan yöntemlerin seçilmesi gerekliliğini ortaya koymaktadır.

Anahtar Kelimeler: Parçalı Mortalite Modellemesi, Lee-Carter Model, Fonksiyonel Demografik Model, Yaşsal Yapısal Kırılım, Mortalitenin Tahmini

To My Father

ACKNOWLEDGEMENTS

TABLE OF CONTENTS

LIST OF TABLES

TABLES

LIST OF FIGURES

FIGURES

LIST OF ABBREVIATIONS

ABBREVIATIONS

APC	Age-period-cohort model
BMS	Booth-Maindonald-Smith model
CBD	Cairns-Blake-Dowd model
HMD	Human Mortality Database
HU	Hyndman-Ullah model
HU50	Hyndman-Ullah method data with fitting period starting from 1950
HUw	Weighted Hyndman-Ullah method
LC	Lee-Carter model
LCnone	Lee-Carter method without adjustment
LM	Lee-Miller model
MAFE	Mean absolute forecast error
MLE	Maximum Likelihood Estimation
OLS	Ordinary Least Square
SVD	Singular value decomposition

CHAPTER 1

INTRODUCTION

1.1. Background of the Thesis

Over the past two centuries, depending on the advancements in health status and medical science, more qualified working conditions and improvement of life standards pertaining to very different aspects; life expectancy has sharply increased both globally and at nationally. Thereby, mortality reduction has been the most crucial development of this era. As a result, several patterns describing human mortality dynamics has changed. While this alteration affects human lives intrinsically, it also transforms the basic concepts for actuarial extent in many terms which put the longevity of human life at the center.

Mortality dynamics deals with the human mortality from birth to death and gives insights for the population at different aspects; age, year, gender, varied sub-groups, etc. Main sources for human mortality data are generally population censuses, official statistics and population registers (Camarda, 2008). This data enables to ascertain mortality and longevity trends of countries pertaining to segregated information.

In the literature, several mortality models emphasize mortality from different concepts. Until 1980's, mortality modelling was widely expressed by on mathematical functions which actually fit observed mortality rates. However, over the last decades, ample productive new methods have taken place and the mortality modelling has been evolving into different and more effective structural and statistical techniques. One of these methods and probably change the mortality forecasting concept is Lee-Carter method (Lee & Carter, 1992). The principle of the Lee-Carter models is based on extrapolation of the past rends of mortality in age and time by forecasting mortality risk, and hedging longevity risk (Liu et al., 2019; Booth et al, 2006). It is still the most

used method for mortality projections of official data of several countries due to its simplicity and reliability of the results.

After the process of Lee-Carter model was proposed, several extensions and variants of this model were presented. These extensions have different characteristics in terms of improvement or additional features for certain aspects of Lee-Carter model (Booth & Tickle, 2008; Cairns et al., 2009; Lee & Miller, 2001; Hyndman & Ullah, 2007). Since then, also several issues with regards to mortality modelling were investigated.

Mortality rates are intensely affected by diseases, epidemic illnesses, wars etc. at certain years, ages and gender groups. These conditions make some shifts at mortality rates of populations. Along with several aspects examined, one of them is the problem of structural change in mortality rates at time dimension. Some studies were conducted to handle the structural change in time (van Berkum et al., 2014; Coelho & Nunes, 2011); however, the performance of mortality models in the case of structural change in age dimension is an area which has yet to be studied.

1.2. Purpose of the Thesis

Although the performance of mortality models in the case of structural change in time has been studied in the literature (van Berkum et al., 2014; Coelho & Nunes, 2011), there are no concrete examples for mortality modeling in terms of structural breakpoints for age. From this point of view, the purpose of this study is to propose a new methodology for mortality modelling by constructing models to specific age intervals which are chosen with regards to divided breakpoints of age and to build piecewise mortality modeling perspective within the same population. Accordingly, Lee-Carter model and its extensions and functional demographic models are utilized within the perspective of age breakpoints. We take into consideration demographic data for Australia, Japan and Portugal which is available in the Human Mortality Database (2019) that ensures the number of deaths and the exposure to risk for individuals age x in the calendar year t specifically. Using this data of countries, the

2

purpose of this study is to model and forecast log-mortality rates for female and male populations independently with regards to breakpoints of age with specific age intervals.

1.3. Organization of the Thesis

In accordance with the purpose of the thesis, this study comprises of five chapters. The first chapter gives brief introduction about the study in terms of background and scope. Chapter 2 which explains fundamentals of mortality concepts in terms of basic terms and notations. In Chapter 3, development of mortality modelling is explained and early methods are stated. In this chapter, classification of different models as parametric and stochastic mortality methods is clarified and principal methods are examined in detail. The application process of the thesis and results pertaining to methodology are presented in Chapter 4. In this chapter, a new approach from the related literature is explained and corresponding steps are conducted. The mortality data of selected countries is examined by mortality curves of each population. From this point, breakpoints of age for female and male subpopulations are determined and significance of mortality rates within the age intervals divided by the age breaks are investigated. By these age breaks, piecewise mortality models which are Lee-Carter method and its several extensions are performed for each population. The accuracy of the models is validated by out-of-sample approach and evaluation of the methods are investigated. As a final chapter, Chapter 5 explains conclusions for the study and gives recommendations of future practices.

CHAPTER 2

FUNDAMENTAL CONCEPTS OF MORTALITY LAWS

2.1. Nature of Mortality Data

Mortality dynamics deals with the human mortality from birth to death and gives insights for the population at different aspects such as age, year, gender and varied sub-groups. Main sources for human mortality data are generally population censuses, official statistics and population registers (Camarda, 2008). This data enables to ascertain mortality and longevity trends of countries pertaining to segregated information.

The whole information of human mortality data is tooled by a table called as "mortality (or life) table". Mortality table for a specific population for a specific year is defined by descending l_x which describes the number of people living at age x. Age holds the values $x = 0, 1, \ldots , n$; where n is a natural upper bound or maximum age such that $l_x > 0$ and $l_x + 1 = 0$.

Mortality tables can be obtained by two methods. First approach is gathering the table by cohorts where a cohort is defined as an actual group of individuals whose birth date is the same year or decade (Wilmoth, 2005). The cohort of people born in year t is longitudinally observed and the number of people living in year t is calculated through the following years for the same cohort. In the long run, l_x for $x = 0, 1, \ldots , n$ are obtained. Second approach is period life table where a period represents time interval. Within the time interval, it assumes that a hypothetical cohort of people experiencing demographic events, in other words mortality is observed only for chosen period of time (Danesi, 2014).

A standard tool for summarizing such data is called Lexis diagram (Lexis, 1875). This diagram contains vital mortality dynamics such as births and deaths requiring changes according to time, age or cohort. The diagram consists of coordinate system where time (period) is shown at *x*-axis and age on *y*-axis (Rau et al., 2017). It is a diagram representing demographic events. Via Lexis diagrams, the information of age at death, time of death and the time of birth cohort can be easily obtained (Camarda, 2008).

In Figure 2.1, diagram presents life dynamics for cohorts between time interval *t - 1* and *t*. Here each 45º line represents each individual who was born at specific time at age zero. Along with the line, red points represent the death of an individual at specific time and the age.
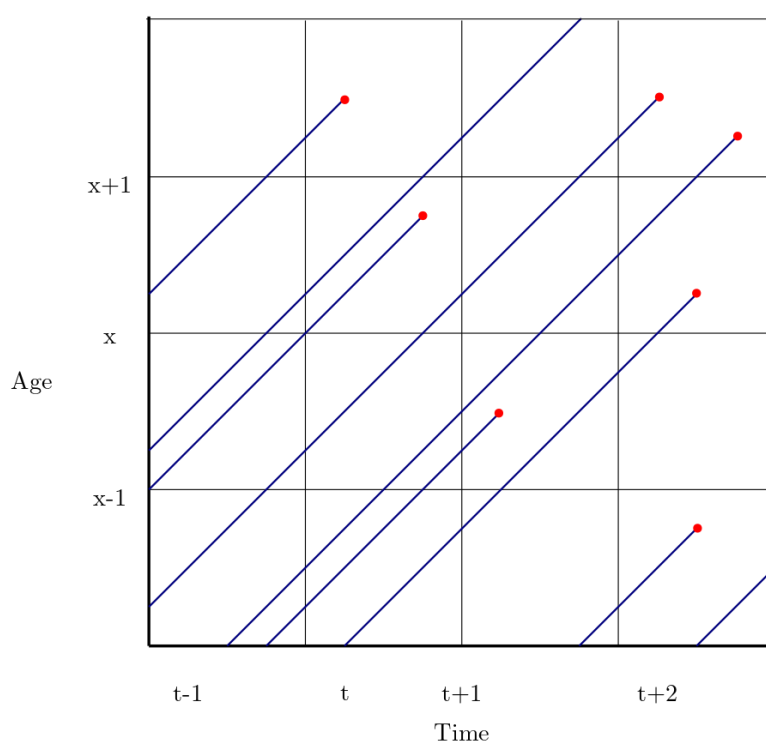
Figure 2.1. An example of a Lexis diagram

6

Another visualization of Lexis diagram is presented at Figure 2.2. Here, at between time *t - 1* and *t* for birth cohorts of ten individuals, the number of deaths from relevant cohort is one; at age one the number of survived individual is nine for time *t* and *t + 1*.
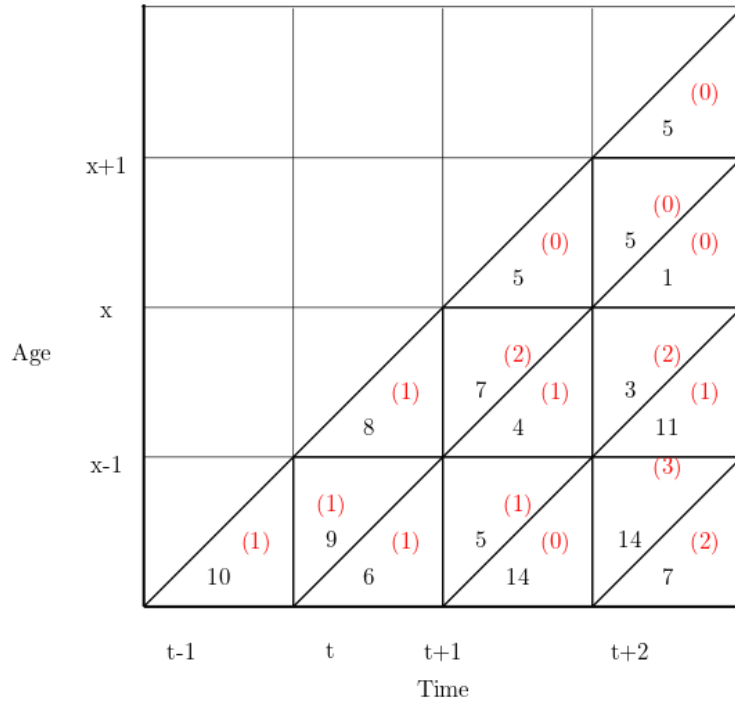
Figure 2.2. An alternative example of a Lexis diagram

2.2. Measures of Mortality

"Mortality" describes the numbers of deaths in a given specific cause and specific period of time. Here, "time" indicates the interval in terms of time when the death of an individual occurs. Time, which is denoted by $T(x)$, is a non-negative continuous random variable (when the opposite is not mentioned) and refers the interval [t, t + 1). Death density function of random variable of time, which is denoted by $f(t)$, is always greater or equal than zero.

7

$$\left.\begin{array}{l} \text{pdf: } f(t) \\ \text{cdf: } F(t) = P(T \le t) \end{array}\right\}$$

$$\int_{-\infty}^{\infty} f(t)dt = 1$$

Since it is only measured for positive values;

$$f(t) = 0 \text{ for } t < 0 \text{ then } \int_{0}^{\infty} f(t)dt = 1$$

2.2.1. Central Death Rate

Central death rate or simply (age-specific) death rate is defined as rate of deaths for specific age x in specific time period. Here, l_x expresses the (average) population of individual aged for time t and d_x represents number of deaths for time t for those whose last birthday age is x; while X is age-at-death random variable. d_x also means as the total number of deaths between ages x and x+1.

$$d_x = l_x - l_{x+1} \tag{2.1}$$

Accordingly,

$$m(t, x) = \frac{d_x}{l_x} \tag{2.2}$$

d_x: total number of deaths for time t for those whose last birthday age is x

l_x: (average) population size of individuals aged x for time t

Central death rate is sometimes cited as crude death rate m_x where the average population is generally calculated by an estimate of the population whose last birthday age is x at the middle of the calendar year t (Cairns et al., 2019).

2.2.2. Mortality Rate

Mortality rate is the probability that an individual aged exactly x at exact time between t and $t + 1$ dies; or survives until time $t + 1$.

$$q(t,x) = P[T(x) \leq t], \ t \geq 0 \tag{2.3}$$

This is the probability that an individual aged x dies within t years. Mortality rate can also be denoted as $_t q_x$ while $_t q_x$ represents the probability of an individual dying; $_t p_x$ corresponds the probability that an individual aged x lives t years more and attains age $x + t$.

2.2.3. Survival Function

As T is a non-negative random variable for time t; survival function is the probability that a survival time of an individual is greater than time t. This function is denoted by $S(t)$. The formula for the survivor function is,

$$S(t) = P(T > t) = 1 - P(T \leq t) = 1 - F(t) \tag{2.4}$$

where;

$$F(t) = P(T \leq t), \ t \geq 0$$

$$\text{and } F(t) = \int_0^t f(t)dt$$

Since the density function $f(t)$ is the derivative of the cumulative distribution function; the notation below can also be used:

$$f(t) = F'(t) = -S'(t) \tag{2.5}$$

9

Note that survivor function has properties as:

i) $S(0) = 1$

ii) $S(\infty) = 0$

iii) If $t_1 < t_2$ then $S(t_1) < S(t_2)$

2.2.4. Force of Mortality (Hazard Rate)

Force of mortality (also known as force of failure, hazard rate or hazard function in different areas) indicates instantaneous rate for an individual who dies at exact time t and exact age x. Force mortality at age x is defined as follows where Δt is denoted by infinitely small period of time:

$$\mu(t, x) = \lim_{\Delta t \to 0} \left[\frac{1}{\Delta t} P(X \leq t + \Delta t \mid X > t) \right] \qquad (2.6)$$

or more simply, force of mortality can be written as:

$$\mu(t, x) = \lim_{t \to 0} \frac{P\left[T(x) \leq t\right]}{t} \qquad (2.7)$$

As a non-negative function μ_x must have properties as below:

i) For all $x \geq 0$, $\mu_x \geq 0$

ii) $\int_0^\infty \mu_x dx = \infty$

There is a strict relationship between the force of mortality and survival function. Therefore, μ_x can be formulated in terms of survival function $S(x)$:

$$P(T < t) = \frac{P(x < T \leq x + t)}{P(T > 0)} = P(x < T \leq x + t) = \frac{F(x+t) - F(x)}{S(x)}$$

then

$$\mu_x = \lim_{t \to 0} \frac{F(x+t) - F(x)}{t S(x)} = \frac{1}{S(x)} \lim_{t \to 0} \frac{F(x+t) - F(x)}{t}$$

$$= \frac{1}{S(x)} \frac{d}{dx} F(x) = -\frac{1}{S(x)} \frac{d}{dx} S(x)$$

and we will eventually obtain survival function as:

$$S(x) = \exp\left(-\int_0^x \mu_y dy \right) \tag{2.8}$$

It is also possible to formulate force of mortality (or hazard ratio), mortality rate and death rate as:

$$\mu(t, x) = \frac{f(t)}{S(t)} = \frac{f(t)}{1 - F(t)} = \frac{-S'(t)}{S(t)}, \; t \geq 0 \tag{2.9}$$

and

$$_t q_x = \int_0^1 S(t) \mu_{x+t} dt \tag{2.10}$$

$$_t m_x = \frac{_t q_x}{\int_0^1 S(t) dt} \tag{2.11}$$

We should also notice that for very small values of Δt, $\mu_x \Delta t$ can be expressed as the probability that an individual who attained age x, dies between the times x and $\mu_x \Delta t$:

$$\mu_x \Delta t \approx P\left(t < X \leq t + \Delta t \mid X > t \right) \tag{2.12}$$

2.2.5. Approximation Assumptions

Population censuses and other data resources comprise number of deaths given stratified in a specific calendar year, age, and gender. Intrinsically, we use the equation 2.2 to calculate death rates from the original data sources. However, to estimate the

mortality rate and force of mortality from the pure death rate, we need to make assumptions to clarify the relative terms (Cairns, 2013). Accordingly, in actuarial practice, these approximations are needed to get the life tables for all actual ages. As a result, death rate and mortality rate can be used as similar terms; both can be used interchangeably in regards to their values. The principal lying behind this perception requires two assumptions (Cairns et al., 2007):

(1) Because of the reason that the force of mortality of human populations $\mu(t,x)$ fluctuates with a slow pace over time; force of mortality is assumed stationary and standing as constant for each integer age and calendar year:

for all $0 \leq s, u < 1$ $\mu(t+s, x+u) = \mu(t,x)$

This assumption expresses that the force of mortality is constant for each of integer age and calendar year.

(2) Population is stationary and the size of the population for all ages stands constant over time of the calendar year.

From the equations 2.10 and 2.11 these assumptions indicate:

i) $m(t,x) = \mu(t,x)$

ii) $q(t,x) = 1 - \exp[-\mu(t,x)] = 1 - \exp[-m(t,x)]$

while relationship (a) is generally used for analysis of death rate (Brouhns et al., 2002) and relationship (b) is appropriate for the analysis of mortality modelling.

Additional assumption requires treating d_x as a random variable. $\boldsymbol{D}_{x,t}$ is a matrix where $D_{x,t} = (d_{x,t})$ and $e_{x,t}$ is a measure of the average population size aged x last birthday in calendar year t, the so-called central exposure to risk. For fixed $e_{x,t}$ values, then $D_{x,t}$ has a Poisson distribution (Currie, 2016):

$$D_{x,t} \sim Poisson(e_{x,t}\mu_{x,t}) \tag{2.13}$$

12

If we approximate the initial value of exposed to risk as:

$$E_{x,t} \approx e_{x,t} + \frac{1}{2} d_{x,t} \qquad (2.14)$$

and $E = (E_{x,t})$ as the matrix of initial exposures, then $D_{x,t}$ has a Binomial distribution (Currie, 2016):

$$D_{x,t} \sim Binomial(E_{x,t}, q_{x,t})$$

CHAPTER 3

MORTALITY MODELLING

3.1. General Overview

Over the past two centuries, depending on the advancements in health status and medical science, more qualified working conditions and improvement of life standards pertaining to very different aspects; life expectancy has sharply increased both globally and at nationally. Thereby, mortality reduction has been the most crucial development of this era. As a result, several patterns describing human mortality dynamics has changed. While this alteration affects human lives intrinsically, it also transforms the basic concepts for actuarial extent in many terms which put the longevity of human life at the center.

Longer life expectancy has also brought about specific mortality-related risks. As a result of the changes in life dynamics; the terms of mortality, longevity and short-term catastrophic mortality risks have become such complex issues that increasingly occupied annuity providers' and life insurers' attention (Cairns et al., 2006). As a result, ability to drawing a specific outline on what is beyond, and anticipating the implications and the projections of the future for the next generations have become a critical topic. Specifically, modelling mortality for human populations has become important especially in health and actuarial sciences with regarding elderly care, provision of pensions, social planning, and governmental policies for changed life dynamics. One can infer that this also means encountering the possibility of some serious economic and social concerns before they occur.

In the related literature, several mortality models emphasize mortality from different concepts. Until 1980's, mortality modelling was widely expressed by on mathematical functions which actually fit observed mortality rates. However, over the last decades,

ample productive new methods have taken place and the mortality modelling has been evolving into different and more effective structural and statistical techniques.

Although there are some reviews to classify models for mortality (Booth & Tickle, 2008; Cairns et al., 2009; Hunt & Blake, 2018); to our knowledge, there is not a single and broadly adopted classification for the models globally. There are many types of different mortality models. Booth and Tickle (2008) classified mortality models have been mainly united under three approaches: (i) expectation, (ii) explanation and (iii) extrapolation. In expectation approach, mortality forecasting is entirely based on the expert's opinion where demographic or specific information are taken in to account. Since this method depends on the subjective manners, it may highly have bias. While explanatory methods rely on structural or causal epidemiological models which contain risk factors and causes of death; extrapolative methods seem to be the most promising approach that take into account past mortality trends for future structures.

Despite the fact that classifications vary from one author to another and it may seem subjective at some points; there are also basic and natural classifications pertaining to the intrinsic properties of the models. Therefore, the mortality models that are critical, common and have broader application in scientific research will be presented under the following titles of this chapter.

3.2. Parametric Mortality Modelling

Before reaching up to today's level of development, mortality research has been very significant area since the very beginning of the 19th century. In the literature, the first theoretical model for mortality was proposed by De Moivre in 1725. After this discovery, many models were generated and formulations were made. Among the parametric models, one can draw attention to generalizations for force of mortality notations. The main reason behind this utilization is that force of mortality simply detects the change or fluctuation of risk of death over the specific age and time (Pascariu et al., 2019).

3.2.1. De Moivre's Law of Mortality

De Moivre suggested the deaths occur in accordance with uniform distribution (De Moivre, 1725) which reflects $T(x)$ distributes uniformly between 0 and $\omega - x$. Therefore,

$$f(x) = \frac{1}{\omega} \text{ for } 0 \leq x < \omega$$

and force mortality,

$$\mu_x = \frac{1}{\omega - x} \tag{3.1}$$

3.2.2. Gompertz's Law of Mortality

Gompertz (1825) proposed a model where force of mortality (hazard rate) μ_x at age x has exponential growth form with the initial degree of mortality and the rate where mortality increases along with age x. Hence,

$$\mu_x = ae^{bx} \tag{3.2}$$

for $a > 0$ and $b > 0$

a: parameter for adult mortality level

b: accelerating parameter

3.2.3. Makeham's Law of Mortality

Afterwards Makeham (1867) modified the Gompertz model where he added a parameter representing age-independent death risk and exponential form of mortality for different ages. The Makeham model predicts that at the beginning of the individual's life, the mortality slowly increases from infancy to childhood and young adulthood depending on age. After the adulthood, it has almost linear shape in accordance with increasing age (Cohen et al., 2018). Force of mortality is

17

$$\mu_x = c + ae^{bx} \tag{3.3}$$

$$\text{for } c > 0$$

and probability density function as

$$f(x) = ae^{bx} \exp\left[-cx + \frac{a}{b}(1 - e^{bx})\right]$$

a: adult mortality level

b: accelerating parameter

c: non-senescent mortality independent of age

Note that both Gompertz and Makeham models considers mortality for only period of adult ages (Camarda, 2008).

3.2.4. Perks Model

Known as one of the logistic models, Perks (1932) proposed a model where force mortality is written as:

$$\mu_x = \frac{e^{\alpha + \beta x}}{1 + e^{\alpha + \beta x}} \tag{3.4}$$

$$\text{for } \alpha, \beta > 0$$

α: infant mortality rate

β: senescence mortality rate

where he noted that the highest ages, mortality deceleration occurs (Gavrilova & Gavrilov, 2014).

3.2.5. Weibull Model

Another parametric model was suggested by Weibull (1951). He proposed this model to describe the durability and failure of some technical components in engineering. According to Weibull model, force of mortality is described as:

$$\mu_x = \alpha x^{\beta} \tag{3.5}$$

for

$$\alpha, \beta > 0$$

α: scale parameter

β: shape parameter

3.2.6. Siler Model

In 1983, Siler proposed a three-component model developed for survival data of animals. According to Siler model, the force mortality is written as the sum of constants representing age independent infancy and old age mortality (Siler, 1983):

$$\mu_x = a_1 e^{-b_1 x} + a_2 + a_3 e^{-b_3 x} \tag{3.6}$$

for $a_1, a_2, a_3 > 0$ and $b_1, b_3 > 0$

a: level of decline

b: rate of decline

3.2.7. Heligman-Pollard Model

Among the parameterization models, Heligman-Pollard model has been stated to be the most well known one. In 1980, Heligman and Pollard suggested a model consisting of eight-parameter and three terms covering human life-span ages which represents mortality patterns in childhood, young adulthood and elderliness. However, the

parameters are limited in use because of the instability and interdependencies (Booth & Tickle, 2008). The Heligman-Pollard model is denoted by:

$$\mu_x = \frac{q_x}{p_x} = A^{(x+B)^C} + De^{-E(lnx+\ln xF)^2} + \frac{GH^x}{1+GH^x} \qquad (3.7)$$

A to *H*: eight parameters referring different spans for ages

$p_x = 1 - q_x$

3.3. Stochastic Mortality Modelling

Today, with the help of improvements in mortality modelling of countries using reliable data, it is clear that the most widely accepted approach for mortality modelling has the nature of stochastic processes (Cairns et al., 2009). This approach is an alternative way of modelling mortality which enables to predict the uncertainty via prediction of expected errors at specific time (Alho, 1998). They represent the advantage of randomness of mortality via probability assumptions (Hahn, 2014).

Among the mortality models and especially stochastic models, one in particular has changed the whole concept as a milestone and been receiving great deal of attention, which Lee and Carter proposed it in 1992 (Lee & Carter, 1992). To forecast the mortality, they used only a two-factor model, which has good predictive power and simple structure (Deaton & Paxson, 2004). Since then many extensions has been studied and proposed based on Lee-Carter model (Renshaw & Haberman, 2003; Cairns et al, 2006; Plat, 2009).

3.3.1. Lee-Carter (LC) Model

In 1992 Ronald D. Lee and Lawrence R. Carter suggested a method that models human mortality simply by log-bilinear form for the central death rate. This model has become a pioneering model which estimates and forecasts age-specific death rates and

life expectancy (Lee & Carter, 1992). Today, it is considered as a milestone in the stochastic modelling and is still the most widely accepted model with its developed extensions (Carfora et al., 2017).

Lee and Carter have constructed a powerful one-parameter model which allows time trend with age-specific loadings to fit and forecast death rates of the US population between years of 1900 and 1987. The model is:

$$log(m_{xt}) = \alpha_x + \beta_x \kappa_t + \varepsilon_{xt} \qquad (3.8)$$

$$\varepsilon_{xt} \sim N(0, \sigma_\varepsilon^2)$$

where

m_{xt} is the central death rate at age x in year t

α_x is a set of age-specific constants representing the general pattern of mortality by age, in other word differences in mortality by age,

β_x is a set of age-specific constants representing the relative speed of change at each age, or differences in relative rates of change by age,

$\beta_x \kappa_t$ is tendency of age-specific death rates to move together; κ_t is an index (a time trend) of level of mortality, or year-to-year changes in the general level of mortality, and

ε_{xt} is the error term at age x in year t

Note that the parameters:

$$\alpha_x = \frac{1}{T} \sum_t log(m_{xt}) \qquad (3.9)$$

$$\sum_t \kappa_t = 0$$

$$\sum_x \beta_x = 1$$

The value of central death rate depends on the overall mortality index κ_t modulated by age β_x. The shape of β_x indicates for rates declining over time related with κ_t (Groot, 2011). The period effect κ_t is often modelled as random-walk process or especially as an ARIMA(0,1,0) process. Here for almost all applications of LC, the random walk drift is used:

$$\kappa_t = \kappa_{t-1} + \theta + \omega_t \qquad (3.10)$$

$$\omega_t \sim N(0, \sigma_\omega^2)$$

where θ is the drift parameter and ω_t is an error term while ε_{xt} and ω_t are independent.

The error terms are assumed to distribute $N(0, \sigma_\varepsilon^2)$.

3.3.1.1. Parameter Estimation

Lee and Carter used singular value decomposition (SVD) in order to get a least squares solution for the estimation of β_x and κ_t. The SVD (Trefethen & Bau, 1997) splits the matrix of $log(m_{xt})$ into the product of three matrices; (i) age, (ii) singular values and (iii) time components (Booth et al., 2002).

The parameter vector α_x is calculated as the mean of $log(m_{xt})$ over time t. Therefore,

$$\hat{\alpha}_x = \frac{1}{T} \sum_t log(\hat{m}_{xt})$$

and

$$Y_{xt} = log(m_{xt}) - \hat{\alpha}_x$$

$$\beta_x \kappa_t = Y_{xt}$$

If we denote Z matrix with $p \times q$ dimensions of rank r; then according to equation below; U is an orthogonal matrix with $p \times p$ dimensions, V is an orthogonal matrix with $q \times q$ dimensions and d is an diagonal matrix with $p \times q$ such that:

$$Z = UdV'$$ (3.11)

where

$V' = (\upsilon_{ji})$ is the inverse matrix of $V = (\upsilon_{ij})$. For $m < n$;

$$A = \begin{bmatrix} u_{1,1} & \cdots & \cdots & u_{1,p} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ u_{p,1} & \cdots & \cdots & u_{p,p} \end{bmatrix} x \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & d_p \end{bmatrix} x \begin{bmatrix} v_{1,1} & \cdots & \cdots & v_{1,q} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ v_{q,1} & \cdots & \cdots & v_{q,q} \end{bmatrix}$$

If we denote $Z = Y_{xt}$ for $x = 1, 2, \ldots, X$ and $t = 1, 2, \ldots, T$ then we obtain first approximation of Y_{xt} as:

$$\hat{Y}_{xt}^1 = d_1 U_1(x) V_1(x) = b_x^1 \kappa_t^1$$

and

$$\hat{\beta}_x^1 = (u_{1,1} \ u_{1,2} \ldots u_{1,X})'$$

$$\hat{\kappa}_x^1 = d_1 \times (v_{1,1} \ v_{1,2} \ldots v_{T,1})'$$

By the equation 3.9, we get β_x and κ_t as below:

$$\hat{\beta}_x^1 = \frac{1}{\sum_x u_{x,1}} (u_{1,1} \ u_{1,2} \ldots u_{1,X})'$$

$$\hat{\kappa}_x^1 = \frac{1}{\sum_x u_{x,1}} \times d_1 \times (v_{1,1} \ v_{1,2} \ldots v_{T,1})'$$

Note that here $U(x)$ represents the age; d singular value and $V(t)$ time components (Coffie, 2015).

The model is fitted by matrix $\hat{Y}_{xt}$ from estimated parameters $\hat{\beta}_x$ and $\hat{\kappa}_x$ as:

$$\log(\hat{m}_{xt}) = \hat{\alpha}_x + \begin{bmatrix} \hat{Y}_{xt_1} \\ \hat{Y}_{xt_2} \\ \dots \\ \hat{Y}_{xt_n} \end{bmatrix} \hat{\beta}_x \qquad (3.12)$$

3.3.2. Lee-Miller (LM) Model

The Lee-Miller method is classified among the variants of Lee-Carter model as properties below:

1. Fitting period commences from 1950.
2. Improvement of κ_t comprises of fitting by life expectancy *e(0)* for year t.
3. The jump-off rates are considered as actual rates.

Lee and Miller (2001) obtained incompliance between fitted mortality rates of the final year of corresponding period and actual mortality rates. Thus, error of jump-off was equaled to 0.6 years in life expectancy as pooled. The jump-off error was eliminated by performing actual rates for jump-off year (Shang et al. 2011).

3.3.3. Booth-Maindonald-Smith (BMS) Model

Booth-Maindonald-Smith (BMS) Model is also classified as one of the extensions of LC method. It has different aspects from LC method by the properties as below (Booth, Maindonald, & Smith, 2002):

1. The period of fitting is based on the statistical 'goodness of fit' perception; with the assumption that κ_t is linear.
2. Adjustment of κ_t is based on the fitting for the distribution of age of deaths instead of total numbers of deaths.

24

In BMS method, appropriate fitting period is obtained by lowest ratio for the mean deviances (Shang et al. 2011).

3.3.4. Poisson Log-Bilinear Model

After Lee-Carter (LC) model was proposed, several extensions of the model were established. One of them is poisson log-linear model developed by Brouhns et al. (2002) which is also called as Poisson LC model. Since the original Lee-Carter model assumes homoscedasticity for the error terms; this assumption might be unrealistic at some aspects (Brouhns et al., 2002; Renshaw & Haberman, 2003). Brouhns et al. (2002) argued that observed number of deaths for advanced ages differ from early ages. It is expressed as "the logarithm of the observed force of the mortality is more variable at older ages than at younger ages because of the much smaller absolute number of deaths at older ages" (p. 378).

Accordingly, they proposed a log-bilinear Poisson regression model for SDV in the original Lee-Carter approach. They modelled time as a factor and considered the model the same as LC model which is:

$$\ln(m_{xt}) = \alpha_x + \beta_x \kappa_t \qquad (3.13)$$

but without error term, as the error is contained by Poisson random variant. Accordingly,

$$D_{x,t} \sim Poisson(E_{x,t} m_{x,t})$$

where E_{xt} death exposure and m_{xt} is death rate which is written as $m_{xt} = e^{\alpha_x + \beta_x \kappa_t}$, and

$$\sum_x \beta_x = 1$$

$$\sum_t \kappa_t = 0$$

Here, instead of using Ordinary Last Squares Estimation (OLS) by SVD; parameters α_x, β_x and κ_t are obtained by Maximum Likelihood Estimation (MLE) approach.

3.3.5. Renshaw-Haberman Model

The Renshaw-Haberman model is classified among the variants of LC model that has an additional cohort effect (Renshaw & Haberman, 2006). Renshaw and Haberman found that there was a significant improvement over the Lee-Carter model (Alijean & Narsoo, 2018). The most crucial improvement was that the standardized error terms displayed very low dependence on birth year, contrast to Lee-Carter method.

Renshaw-Haberman model also handles the number of deaths with Poisson distribution. The model has the same constraints as the previous models:

$$\ln(m_{xt}) = \beta_x^{(1)} + \beta_x^{(2)}\kappa_t^{(2)} + \beta_x^{(3)}\gamma_{t-x}^{(3)} + \varepsilon_{xt} \tag{3.14}$$

where $\gamma_{t-x}^{(3)}$ is the parameter which denotes the cohort effect for persons aged x in year t-x; and restrictions,

$$\sum_t \kappa_t^{(2)} = 0$$

$$\sum_x \beta_x^{(2)} = 1$$

$$\sum_{x,t} \gamma_{t-x}^{(3)} = 0$$

$$\sum_x \beta_x^{(3)} = 1$$

In terms of the estimation of parameters, Renshaw-Haberman model can be stated to be more difficult than the LC model (Sweeting, 2017).

3.3.6. P-Splines

Mortality data characteristically have outlier values. That is especially valid for older ages because of their limited number of individuals. This case takes an effect when the estimation procedure for mortality rates of people aged 90 and older (D'Amato & Russolillo, 2011). The smoothing methods might be effective in order to avoid this inconvenient data structure originated in high variation of these ages (Delwarde, Denuit and Eilers, 2007).

Accordingly, Currie et al. (2004) present a technique of B-splines and P-splines to get better estimation:

$$\log[m(x,t)] = \sum_{i,j} \theta_{ij} B_{ij}^{ay}(x,t) \tag{3.15}$$

with smoothing of the θ_{ij} in the age and cohort.

3.3.7. Age-Period-Cohort (APC) Model

Currie (2006) proposes Age-Period-Cohort (APC) model which has the equation as below:

$$\log[m(x,t)] = \beta_x^{(1)} + \frac{1}{n_a}\kappa_t^{(2)} + \frac{1}{n_a}\gamma_{t-x}^{(3)} \tag{3.16}$$

Here, n_a is total number of ages existing in the data. The model is also a special version of Renshaw-Haberman model when the parameters are taken as:

$$\beta_x^{(2)} = \frac{1}{n_a}$$

$$\beta_x^{(3)} = \frac{1}{n_a}$$

The model is more efficient and robust than the Renshaw-Haberman model. Additionally, Currie (2006) operates P-splines to fit parameters smoothly (Bozikas & Pitselis, 2018).

3.3.8. Cairns-Blake-Dowd (CBD) Model

The Cairns-Blake-Dowd (CBD) model uses logit transformation for each probability of specific year as (Cairns et al, 2009):

$$Y_{tx} = \text{logit}[q(t,x)] = \kappa_t^{(1)} + (x - \bar{x})\kappa_t^{(2)} + \varepsilon_{t,x} \qquad (3.17)$$

where

$\kappa_t^{(1)}$ is a parameter for level of mortality in year t,

$\kappa_t^{(2)}$ is a parameter displaying the effect between age and mortality,

$\bar{x}$ is the mean age for specific age interval,

$\varepsilon_{t,x}$ is error term distributed normally with zero mean and constant variance.

3.3.9. Hyndman-Ullah (HU) Model

In LC model and other variants of it, there is a problem of age smoothness and heterogenic deaths over years (Girosi & King, 2007). In addition to this condition, LC variants only consider one principal component. Hyndman and Ullah (2007) improves LC model in several ways. They presented a functional data model which utilises second and higher order principal components to get additional variation in mortality rates. This model operates penalized regression spline with partial monotonic constraint in order to smooth log-mortality rates. HU method which is also known as 'functional demographic (data) model as (Rabbi, 2018):

$$m_t(x_i) = f_t(x_i) + \sigma_t(x_i)\varepsilon_{t,i} \qquad (3.18)$$

28

$$i = 1, ..., p, \ t = 1, ..., n$$

where,

$m_t(x_i)$ is the observed log-mortality rate for age x_i in year t

$\sigma_t(x_i)$ is amount of noise changing with x_i in year t

$\varepsilon_{t,i}$ is an independent and identically distributed random variable with normal distribution.

3.3.10. Weighted Hyndman-Ullah Model

Weighted Hyndman-Ullah method uses the same smoothing technique as HU method, however this model considers geometrically lessening weights for estimation of parameters by using weighted penalized regression splines. This allow model is based on more recent data rather than distant past (Shang at al., 2011).

$$f_t(x) = a(x) + \sum_{j=1}^{J} b_j(x)k_{t,j} + e_t(x) \tag{3.19}$$

for $t = 1, ..., n$

where

$a(x)$ is mean function

$b_j(x)$ is the set for first J functional principal components

$k_{t,j}$ denotes set of uncorrelated principal component score

$e_t(x)$ is error function

CHAPTER 4

APPLICATION AND RESULTS

4.1. Description of the Data

The data used in this study is provided from Human Mortality Database (HDM). It is an online database, funded mainly by the Department of demography at the University of California, Berkeley and the Max Planck Institute for Demographic Research. The database includes mortality data for 37 countries or areas. The datasets are segregated by either period data or cohort data accordingly (HMD, 2019). The countries' data is available for ages 0 to 110 and beyond and time periods of countries may differ according to the recording years.

For this study, the data of three countries; Australia, Japan and Portugal are considered in regard to mortality rates. These countries are chosen based on the criteria of median age for the population which represents young, average and advanced population age (Table 4.1). The data involves the number of deaths and births, exposures to risk, mortality rates and life-expectancy at birth. For analyses, the ages 0 – 89 are considered where age 89 was selected as a cutoff point. The reason for choosing this cutoff point for age is the fact that older ages display inconsistency and fluctuations (Shang et al., 2010). In general, after age 90, the size of the population turns into discernibly narrower, causing less credible results.

Table 4.1. *The median ages and time span for the chosen countries*

Country	Median Age	Time Span
Australia	37.3 years	1921-2016
Japan	46.7 years	1947-2017
Portugal	44.3 years	1940-2015

4.2. General Characteristics of Mortality Curves

The term "mortality curve" is visualization for the progression of mortality of a specific population over both the age and time dimensions. Mortality curves describe age-specified (log) death (mortality) rates over the time span which are years. It gives significant insights about mortality trends as well as its development over time (Haldrup & Rosenskjold, 2019).

In the last century, significant changes in general mortality trends have been observed; especially for life expectancy in developed countries. Accordingly, with regards to mortality rates, mortality curves display a similar structure over the ages. Additionally, mortality levels tend to decline by time for all ages. Also, for both genders, the shape of curves shows similar features across the countries. These declines are mostly valid at younger ages (Vaupel 2010; Christensen et al. 2009). In early childhood, infant mortality shows the declining characteristics while at the late teens and at the beginning of age 20's, mortality display rapid increase which is called as "accident hump" (Heligman & Pollard 1980). After this process, mortality rates slowly increase by age almost log-linearly (Haldrup & Rosenskjold, 2019). Especially beyond ages 80 – 85, mortality levels have a very slow pace.

In the review of Wong-Fupuy & Haberman (2004), there are remarks pertain to age-time for log age-specific death rates. These are: *"There is a broad consensus across the resulting projections: (i) an approximately log-linear relationship between mortality rates and time, (ii) decreasing improvements according to age"*.

4.2.1. Comparison of Mortality Curves for Selected Countries

When investigating mortality dynamics for selected three countries; Australia, Japan and Portugal, the general mortality patterns explained in previous title are also valid. In Figures 4.3, 4.4. and 4.5; top panels represent age-specific log-mortality rates over corresponding years as a series of smoothed curves (functional observations). Here, multiple years were plotted by the rainbow palette, which displays the most recent years as the color of purple. In other words, the first years are in red which is followed by orange, yellow, green, blue, and indigo, while the most recent one plotted in purple (Hyndman & Shang 2010). These figures typically display age-specific mortality curves by progressively increasing mortality rates as age increases.

The bottom panels represent the progress of death rates over time. This structure can be held as a time series of ages 0 to 89. Here, in general, older ages are seen by bottom lines which start from the colors of purple to younger ages by red colors. It can be inferred from the figures that decreasing improvements for mortality corresponding age over time. This feature can be seen by a slope of the log-death rate plots decreasing with age. Figure 4.3, 4.4 and 4.5 bottom panels show log death rates for males and females for selected ages as univariate time series.

Mortality rates are intensely affected by diseases, epidemic illnesses, wars etc. at certain age and gender groups. Mortality curves display the effects of these circumstances explicitly. Among the three countries; Portugal has the highest general mortality rate and this case is also valid for both female and male subpopulations. On the other hand, Japanese population has the lowest mortality level at total and also at female and male subpopulations. In general, mortality rates are an extremely high level at the early ages of life around 0-1 years, then dropped at a low level between 8-10 years. From this point, at the very beginning of the life, at very young ages; Portuguese population has the highest mortality rate among the three countries, while Japanese population has the lowest mortality rate. Accident hump of each country for male subpopulations have sharp structure relative to female subpopulations.

Additional information can be derived from the corresponding mortality curves of the countries. The effects of World War II can be clearly seen for both Japanese female and male subpopulations by high level of mortality rate at corresponding years.

If the country profiles examined uniquely, in Australia for both female and male subpopulations; it is clearly inferred that mortality rates have exponentially decreasing pattern until the early 10's. From these ages on, mortality rates have increasing form up to late 20's. The accident hump is much more explicit and severe for male subpopulation and the location of it has slightly older age for males. For Australian female subpopulation, mortality rates almost linearly increase from the beginning of 30's. Decreasing of mortality rates with regards to mean value occur especially for ages around 31, 41, 51, 55, 71 and 77 while there is an unusual increase at the age of 11. For Australian male subpopulation, mortality rates almost linearly increase from the beginning of 40's. Mortality rates with regards to mean value decrease especially for ages around 27, 41, 51, 66 and 81 while there is an increase at the age of 10.

As mentioned above, Japanese population has the lowest mortality rates in both subpopulations. Mortality rates for Japanese both female and male subpopulations have smooth exponential decreasing profile through the early 10's. From these ages on, mortality rates have increasing structure up to around the early years of 20's. The accident hump is more apparent and wider for male subpopulation as well. Male subpopulation has different increasing form around the age 29. Both populations have almost linearly increasing mortality rates with regards to mean value from the late 20's (Figure 4.2).

In Portugal for both female and male subpopulations; mortality rates have exponentially decreasing pattern until the early 10's. The accident hump is very explicit and severe for male subpopulation and the location of it has slightly older age for males while for female subpopulation hump is almost unclear. Decreasing of female mortality rates with regards to mean value occur especially for ages around 29,

31, 41, 49, 51 and 71 while male subpopulation, mortality rates with regards to mean value decrease especially for ages around 29, 31, 41, 61 and 81.
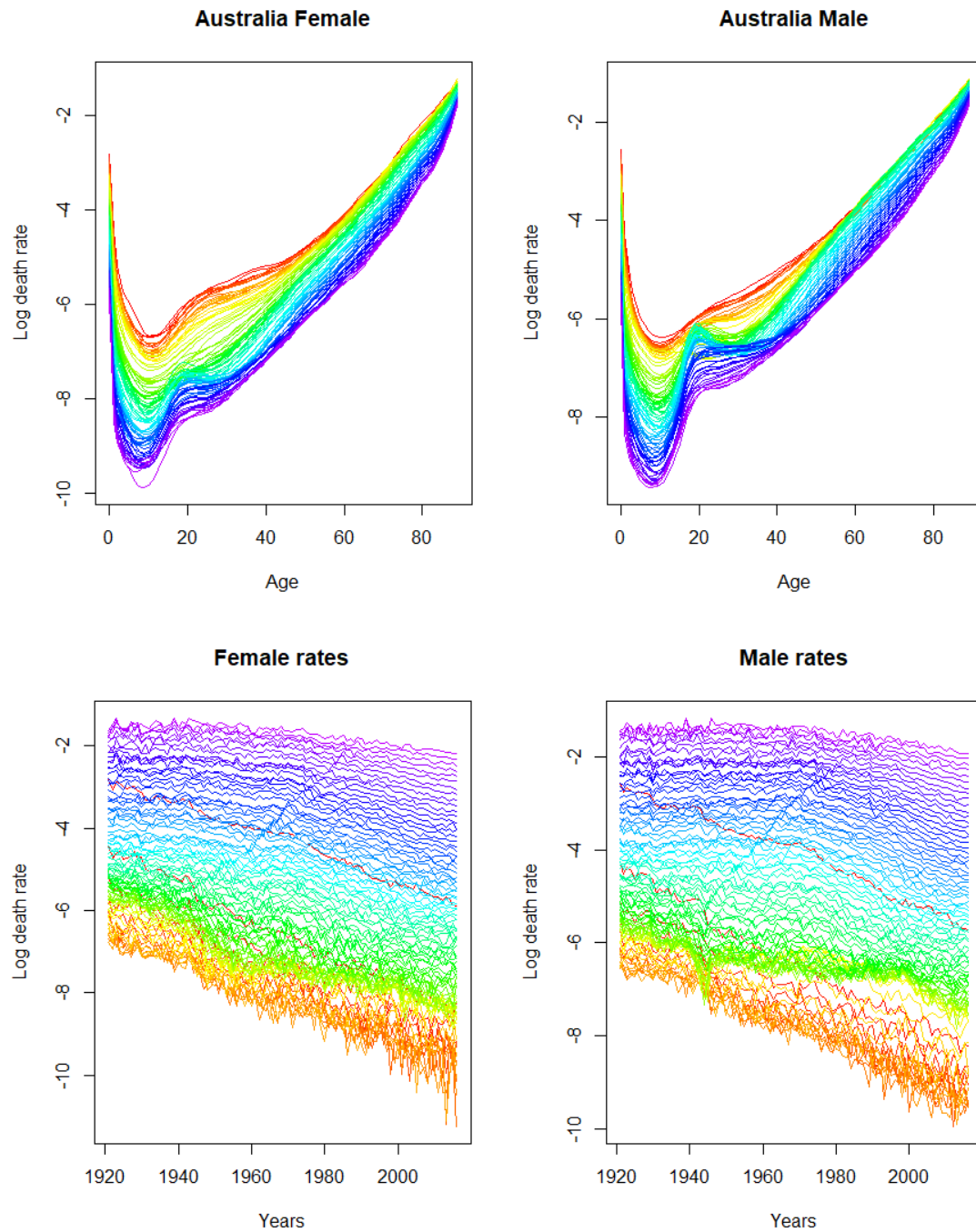

Figure 4.1. Smoothed log-mortality rates for years 1921 to 2016 and log-mortality rates for ages 0 to 89 in Australia
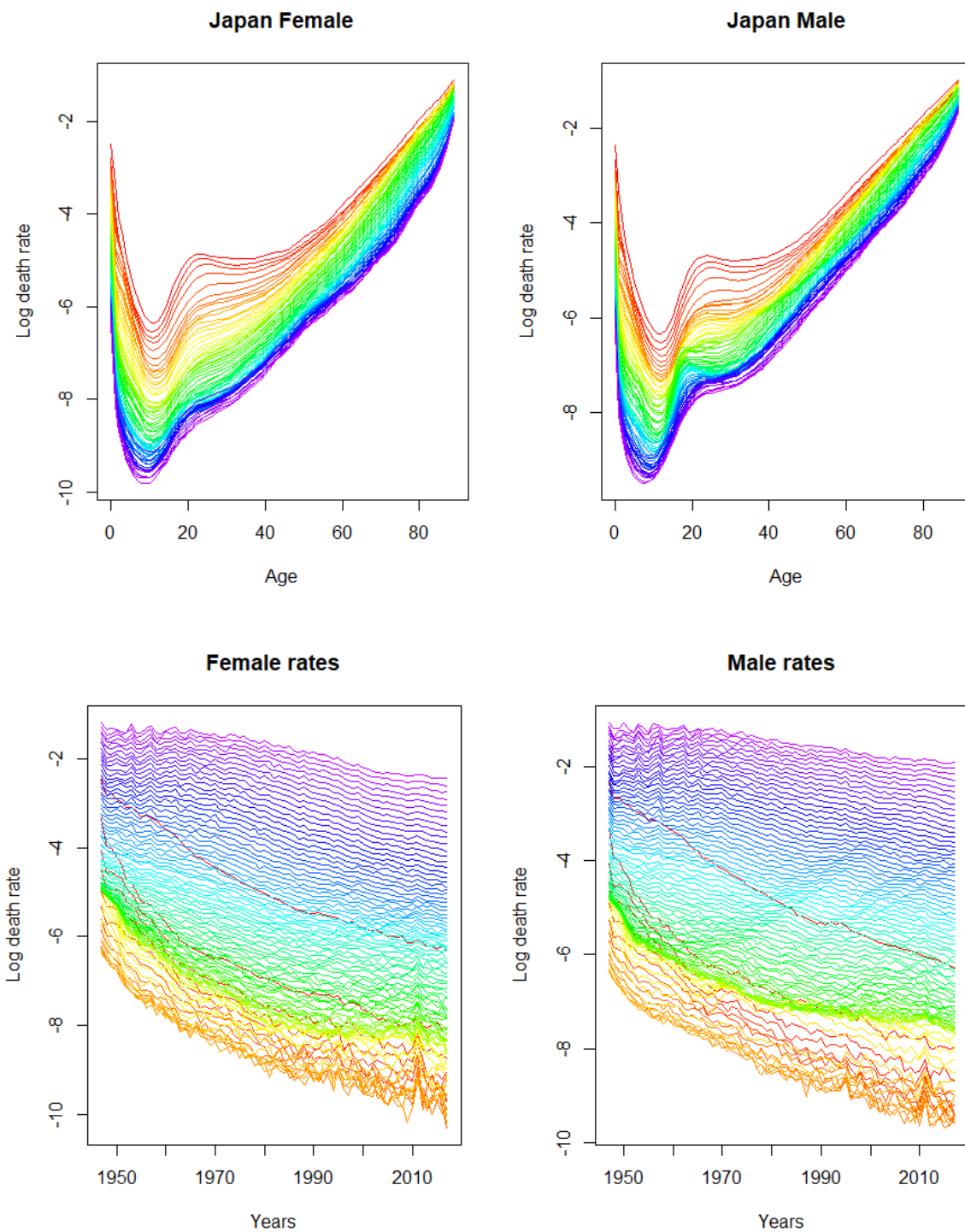
Figure 4.2. Smoothed log-mortality rates for years 1947 to 2017 and log-mortality rates for ages 0 to 89 in Japan
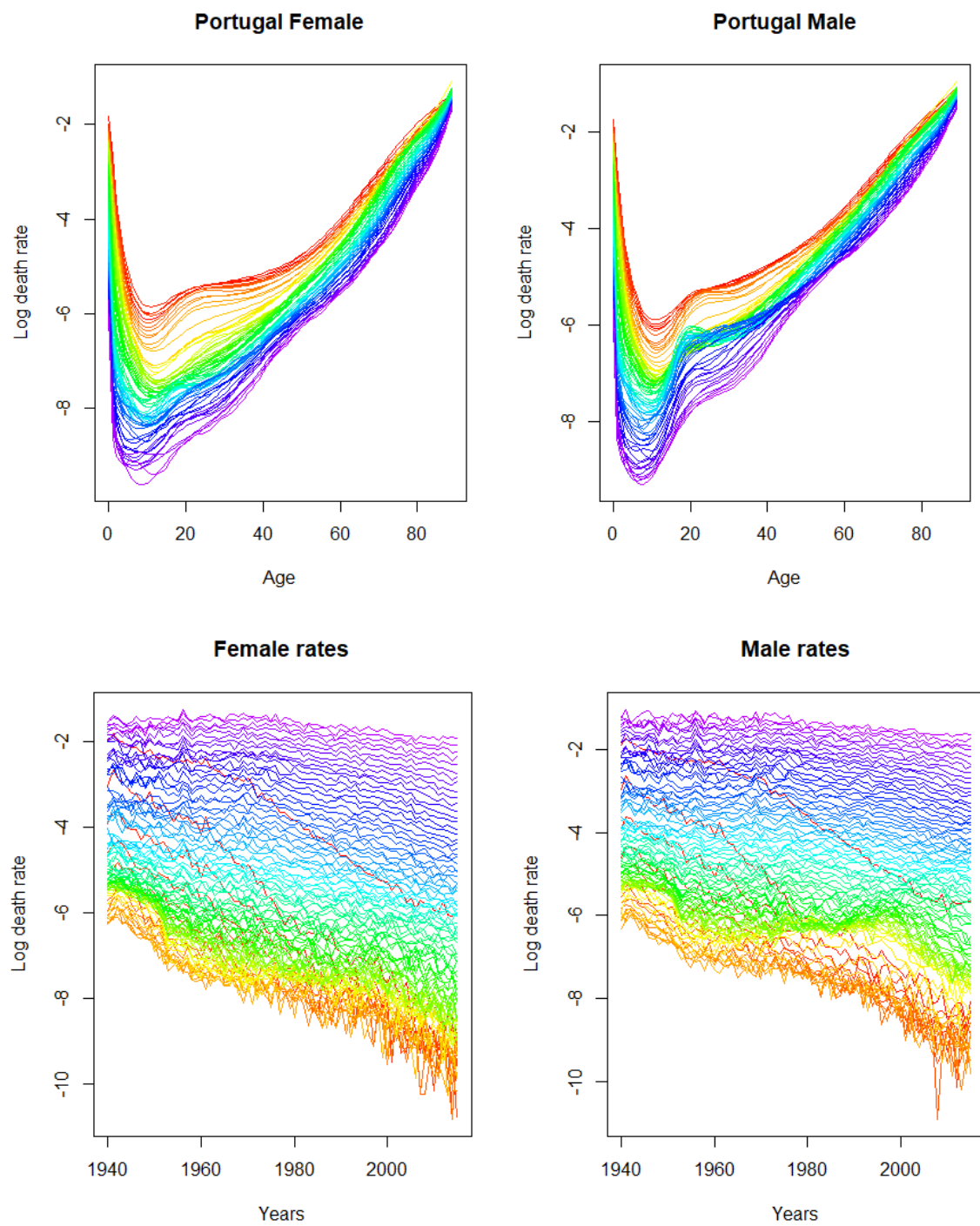
Figure 4.3. Smoothed log-mortality rates for years 1940 to 2015 and log-mortality rates for ages 0 to 89 in Portugal

4.3. Proposed Methodology: Piecewise Mortality Modelling

In this study, different approach from the related literature is proposed. This methodology provides the piecewise mortality modelling by segmented age intervals for the first time in the literature. The proposed approach enables to determine different age structures with regards to significant variations of mortality rate of female and male subpopulations by divided breakpoints of age; and to build piecewise mortality models within the same population. This approach puts forward a new methodology for mortality modelling field by constructing models to specific age intervals which are chosen with regards to forecast accuracy. Within this scope, the application process comprises of sequential of complementary two steps:

i. Before investigating the data for different mortality methods; it is needed to consider the mortality – age relation structure. Accordingly, structural breaks of ages for distribution of mortality rates are obtained. Using determined breakpoints of age; before – between – after the breakpoints, the data are separated into three parts.

ii. By using the age-segmented data intervals, different mortality forecasting methods are applied for log age-specific mortality rates to obtain point forecasts and to enable the comparison among the methods for selected countries for female and male subpopulations. These methods are evaluated in terms of their accuracy. As a continuation of this step, best forecasted models are determined for; (a) different age intervals and (b) female and male subpopulations for each country.

4.4. Determining Structural Breaks for Ages

In mortality field, a general approach is employed especially at modelling process which takes using the logarithmic scale of mortality rates. The logic behind this approach is that by using logarithmic scale of mortality rates, detection of possible differences in mortality between age groups becomes straightforward. At this point,

to determine the structural breakpoints of age for each time span, the log-mortality rates of each country are considered. Specifically, female and male total populations for each country are examined.

There are fruitful research showing changes that have been evidence for the declining mortality rates (Booth et al., 2002; Shang et al., 2010). Several studies showed the breaking years at which significant shifts for mortality rates (Perron, 2008; Coelho, 2012). However, there is a lack of scientific information for statistically determined age shifts in terms of mortality levels. For this purpose, the analyses for structural breaks for ages 0 to 89 of different three countries for female and male populations are conducted. After the breakpoints for ages are obtained, the significance of age-segmented mortality rates for different age parts is validated.

In order to gather the structural breakpoints at ages for mortality rates, the standardized procedure of the Bai-Perron test for unknown multiple breakpoints was performed (Bai & Perrron, 2003; 1998). This procedure was explained by Bai & Perron (2003) as: "The problem of testing for multiple structural breaks is addressed by tests with null hypothesis of no break versus an alternative hypothesis of an arbitrary number of breaks, which allows for a specific to general modeling strategy in consistent determination of appropriate number of breaks". These breakpoints divide data when significant deviation exists in related series. The estimation process consists of regression analysis with constant as regressor explaining potential serial correlation with nonparametric adjustments.

Bai-Perron method considers multiple structural break model with m breaks ($m+1$ regimes) as below:

$$
\begin{aligned}
y_k &= x_k'\beta + z_k'\theta_1 + u_k \quad k = 1,...,K_1 \\
y_k &= x_k'\beta + z_k'\theta_2 + u_k \quad k = K_1+1,...,K_2 \\
&\quad\vdots \qquad\qquad\qquad \vdots \\
y_k &= x_k'\beta + z_k'\theta_{m+1} + u_k \quad k = K_m+1,...,K
\end{aligned}
\tag{4.1}
$$

39

where, y_k is observed dependent variable at series k, x_k and z_k are vectors of covariates and u is the residual term, while θ's are subjects to change (1 to $m+1$).

For the analysis, the mean of log-mortality rates via pooled related time span for three countries with regards to female and male populations are utilised. The sequential testing framework showed at least two significant age breaks for each six subpopulations. The significant breakpoints of ages for the countries and subpopulations are presented in Table 4.2.

Table 4.2. *The estimates of breakpoints at ages for countries and subpopulations*

Country	Subpopulation	Estimates breakpoints at age*	F-statistics
Australia	Female	16, 46	136.406; 38.080
Australia	Male	16, 36	100.657; 35.866
Japan	Female	16, 46	142.888; 9.458
Japan	Male	16, 38	142.442; 36.706
Portugal	Female	16, 45	163.728; 40.222
Portugal	Male	16, 40	143.850; 19.604

* Significant at the 0.05 level

As shown in figures 4.4, 4.5 and 4.6 for each country respectively, after obtaining structural breakpoints of ages for mean of log-mortality rates, the age span of subpopulations for each country are divided into three related age intervals accordingly:

- Australia: Female: (i) Ages 0 – 16, (ii) Ages 17 – 46 and (ii) Ages 47 – 89
  Male: (i) Ages 0 – 16, (ii) Ages 17 – 36 and (ii) Ages 37 – 89
- Japan: Female: (i) Ages 0 – 16, (ii) Ages 17 – 46 and (ii) Ages 47 – 89
  Male: (i) Ages 0 – 16, (ii) Ages 17 – 38 and (ii) Ages 39 – 89
- Portugal: Female: (i) Ages 0 – 16, (ii) Ages 17 – 45 and (ii) Ages 46 – 89
  Male: (i) Ages 0 – 16, (ii) Ages 17 – 40 and (ii) Ages 41 – 89
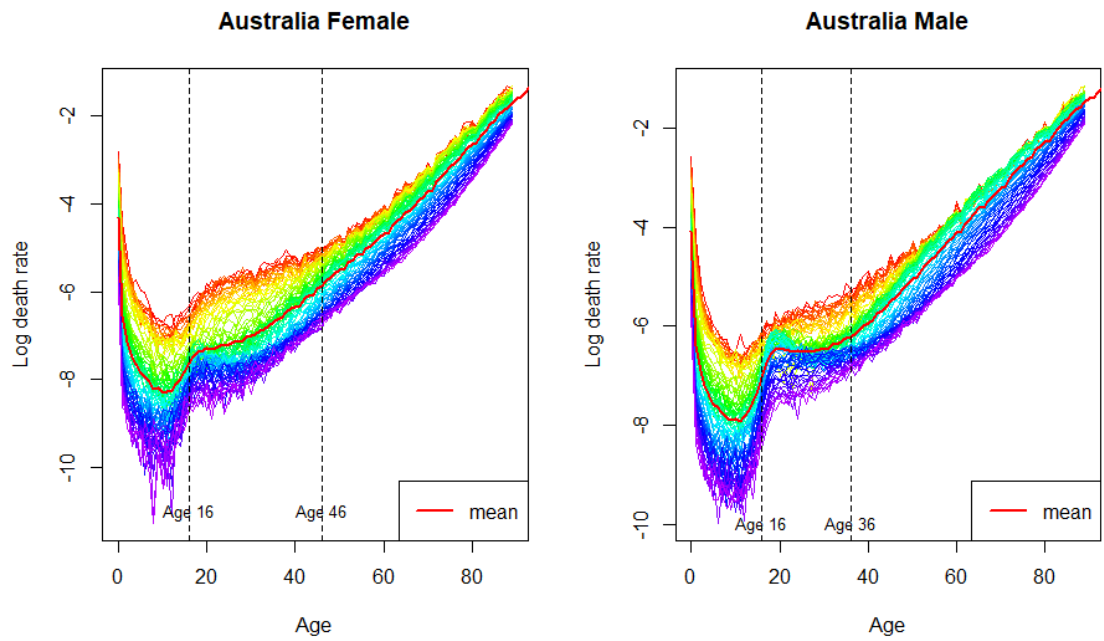
Figure 4.4. The breakpoints of age for Australian subpopulations with corresponding mean distribution
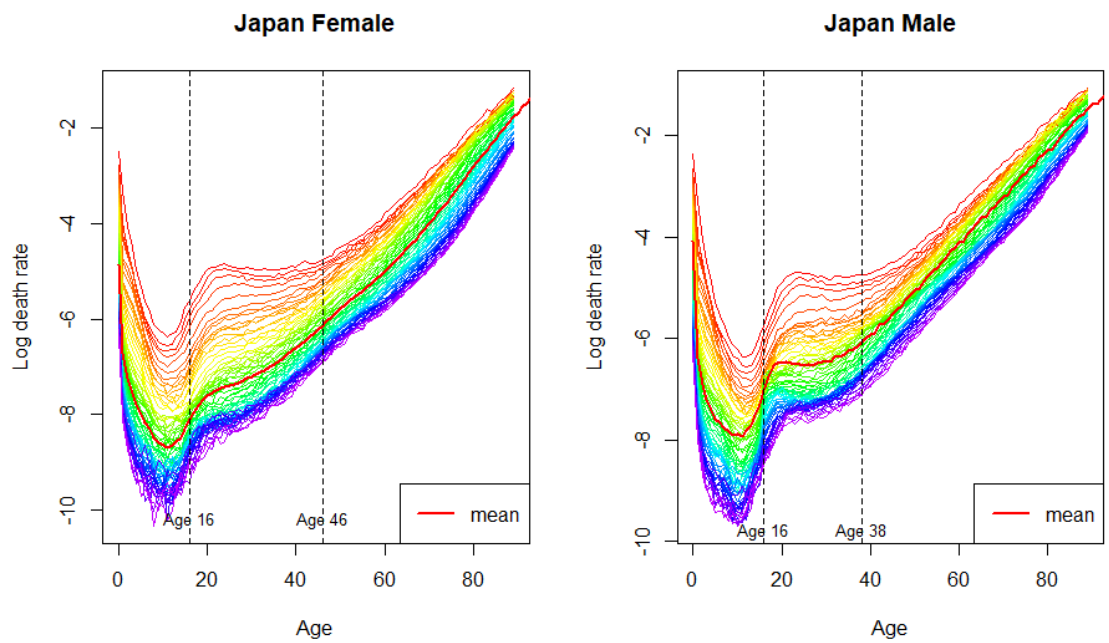
Figure 4.5. The breakpoints of age for Japanese subpopulations with corresponding mean distribution
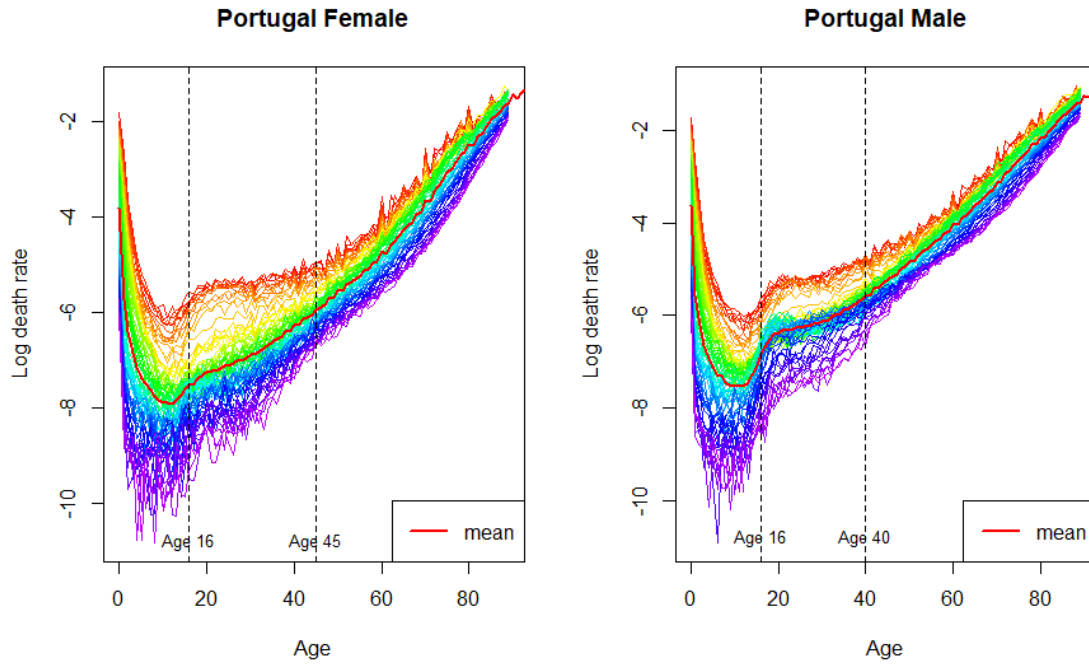
Figure 4.6. The breakpoints of age for Portuguese subpopulations with corresponding mean distribution

Additional comparative analyses are performed to differentiate the age-segmented log-mortality rates in terms of their significance across countries and subpopulations. Kruskal-Wallis statistics for whether three age intervals under 5% significance level have the same mean value are conducted. The related means are shown in Table 4.3. Performing six different Kruskal-Wallis tests, all of the null hypotheses are rejected ($p < 0.05$). The results indicate that the distributions of log-mortality rates are not the same for different age intervals for all selected countries and subpopulations. Accordingly, it can be concluded that different age groups which can be classified as young, adult and advanced ages affect the mortality (Figures 4.7, 4.8 and 4.9).

42

Table 4.3. *Means of mortality rates for segmented ages over countries and subpopulations*

Country	Female			Male		
	Age Group	*Mean*	*SD*	Age Group	*Mean*	*SD*
Australia	0-16	-3.134	0.429	0-16	-2.998	0.408
	17-46	-2.842	0.188	17-36	-2.763	0.052
	47-89	-1.673	0.525	37-89	-1.636	0.603
Japan	0-16	-3.185	0.4623	0-16	-3.086	0.439
	17-46	-2.890	0.1618	17-38	-2.799	0.087
	47-89	-1.703	0.5539	39-89	-1.652	0.603
Portugal	0-16	-2.860	0.5392	0-16	-2.773	0.515
	17-45	-2.813	0.1422	17-40	-2.605	0.109
	46-89	-1.655	0.5740	41-89	-1.522	0.546

Figure 4.7. Boxplots of mortality rates for segmented age groups over Australian subpopulations
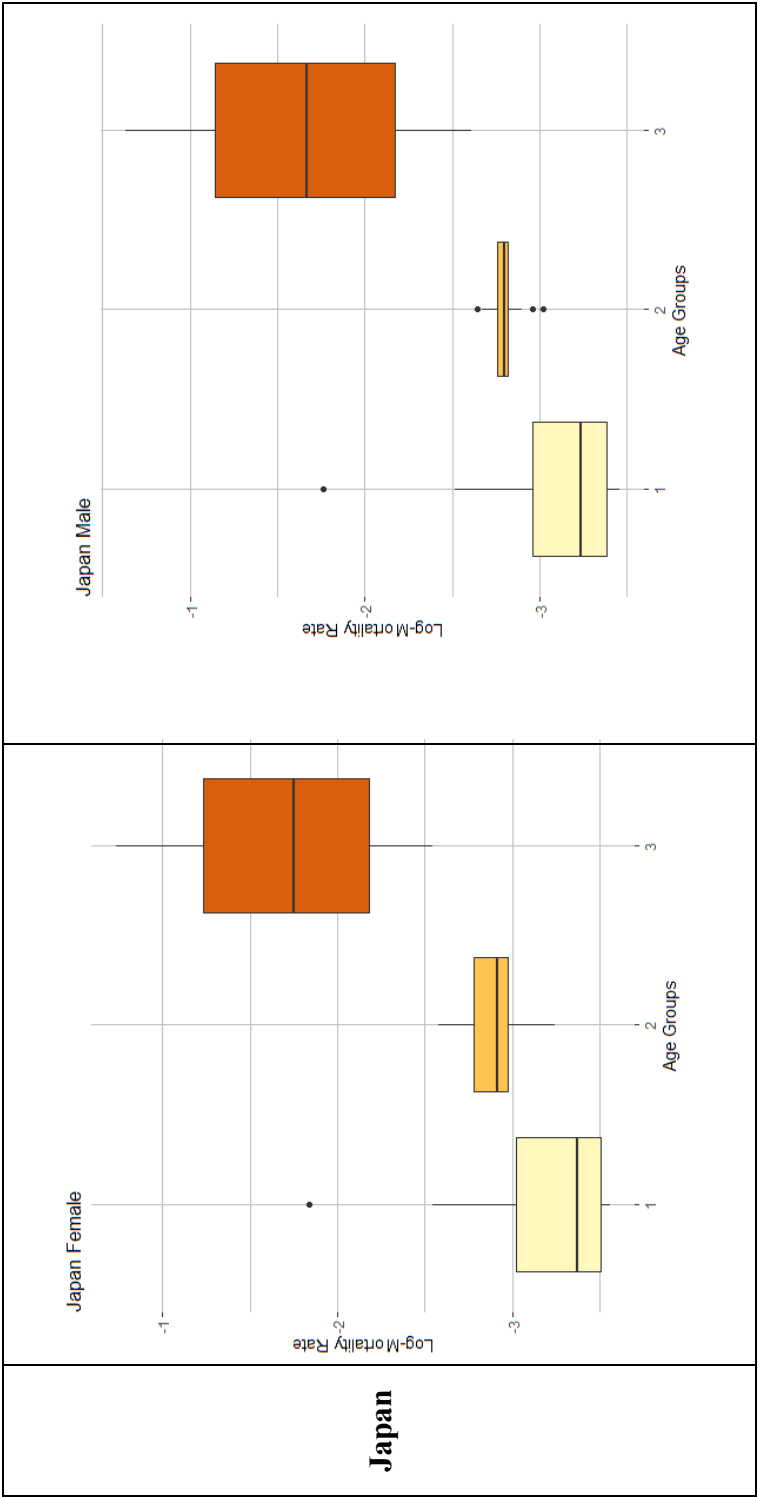
Figure 4.8. Boxplots of mortality rates for segmented age groups over Japanese subpopulations

45
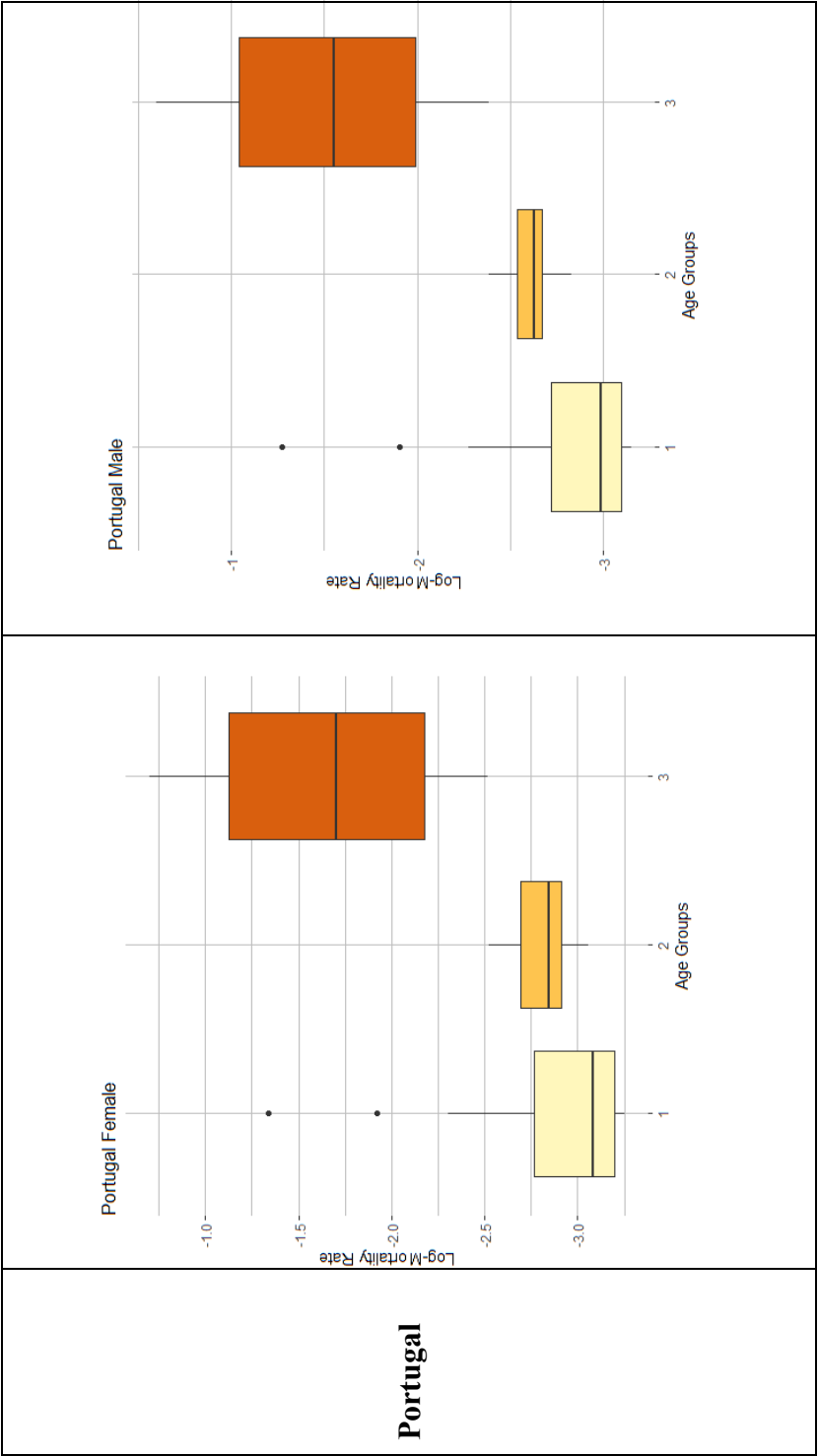
Figure 4.9. Boxplots of mortality rates for segmented age groups over Portuguese subpopulations

4.5. Forecasting Mortality Models

This section aims to evaluate the performance of distinguished forecasting methods for selected countries; Australia, Japan and Portugal. Since the age span of countries divided into three different age intervals for both female and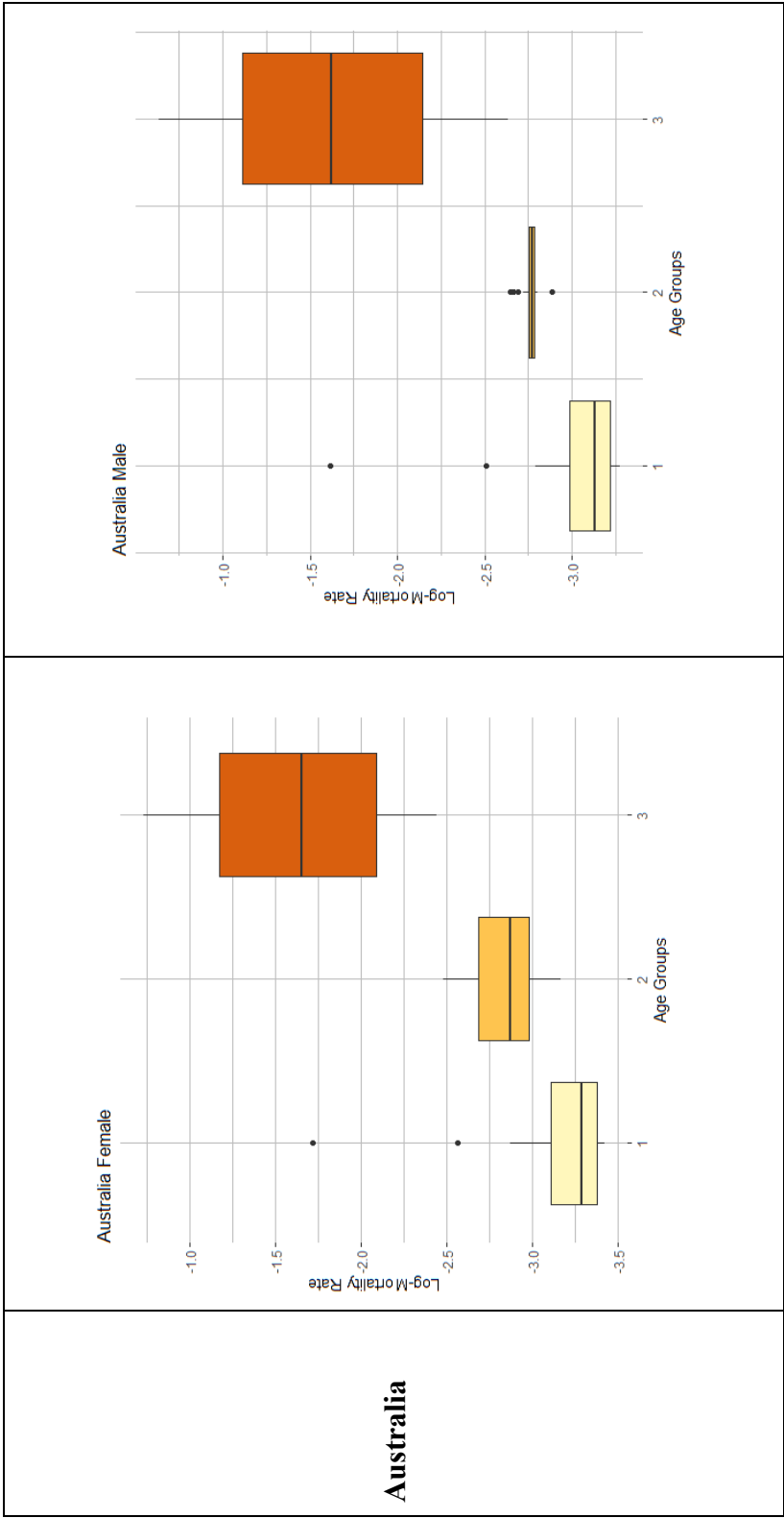 male populations, in total 18 series are obtained. The selected countries have data before 1950 to keep consistent comparisons between models (Shang et al., 2010). It was crucial to have a long period of data, with regards to get consistent estimators (Box, Jenkins & Reinsel, 2008). Note that the ages are from 0 to 89 with regards to avoid erratic rates at older ages (Shang, 2016).

The models used in the study consists of Lee-Carter variants by followed Shang et al. (2011) which are Lee & Carter (1992), Lee & Miller (2001), Booth et al. (2002) and nonparametric variants of Hyndman-Ullah (2007) methods as below:

      i.   Lee-Carter method (LC)
     ii.   Lee-Carter method without adjustment (LCnone)
    iii.   Lee-Miller method (LM)
    iv.   Booth-Maindonald-Smith method (BMS)
     v.   Hyndman-Ullah method (HU)
    vi.   Hyndman-Ullah method with data from 1950 (HU50)
   vii.   Weighted Hyndman-Ullah method (HUw)

In Table 4.4, the starting year for fitting period of each country are presented. Since BMS method chooses the optimal fitting period independently; commencing years of subpopulations show some differences in this method. Accordingly, these years are different for each age interval. Also, due to its theoretical using, the starting period Lee-Miller method is set for the year of 1950.

The point forecasts of each method, evaluation and comparison of their forecast accuracies and all other analyses conducted in this thesis are implemented by "demography" package of R (R Development Core Team, 2019).

Table 4.4. *Starting years of fitting period for each country and methods*

Country	LC	LCnone	LM	BMS[f]*
Australia	1921	1921	1950	1922, 1921, 1940
Japan	1947	1947	1950	1947, 1952, 1948
Portugal	1940	1940	1950	1955, 1953, 1940
Country	BMS[m]*	HU	HU50	HUw
Australia	1922, 1924, 1922	1921	1950	1921
Japan	1951, 1951, 1948	1947	1950	1947
Portugal	1940, 1952, 1943	1940	1950	1940

4.5.1. Forecast Evaluation

For estimating mortality for future projections, the out-of-sample validation technique is used. According to this technique, comparison observed mortality rates and the point forecast of log age-specific death rates over all age intervals and years are conducted.

The procedure for splitting the observed data and then evaluating forecasts is widespread in forecasting literature (Chatfield, 2000). In applications, it is agreed that particular methods should be assessed for accuracy by utilising point out-of-sample tests rather than in-sample tests (Tashman, 2000). This procedure commences with splitting the historical data series into (i) fitting period as a training set and (ii) forecasting period as a test set. The fitting period is employed to determine and estimate models; on the other hand, the forecasting period is used to assess the accuracy of forecasting models via forecast horizon which is the number of time periods between these two sets (Tashman, 2000).

For this study, as mentioned above, the data of each country is divided into a fitting period and forecasting period. The length of the fitting period varies across the countries as the starting period differs among countries. On the other hand, the forecasting period is set the last 30 years (i.e., 1987 – 2016) for all countries; hence the rest of the data is reserved as fitting period.

By using the data from the fitting period, one-step to 30-step-ahead point forecasts are computed and forecast errors are determined by comparing the forecasts with actual out-of-sample data for each country and each subpopulation across age intervals. For each country, seven out-of-sample validation are conducted for each age interval and sex. In total, a-hundred and twenty-six out-of-sample exercises are performed.

4.5.2. Assessing the Performance of Mortality Forecasting Models

Suppose a data set involves observations $m_1, m_2, ..., m_T$, and the data is divided into two parts: (i) training data $(m_1, m_2, ..., m_N)$ and (ii) test data $(m_{N+1}, m_{N+2}, ..., m_T)$. To assess the accuracy of related forecasting method, the parameters are estimated by using training data, accordingly the next $T - N$ observations are forecasted. Then forecasted values are compared with test data which consist of actual observations.

The h-step-ahead forecast is denoted by $\hat{m}_{N+h|N}$ which is an estimation of actual observation. Therefore, the forecast errors are defined as the difference between the actual values in test set and forecasts by training set as (Hyndman, 2015):

$$e_t = m_t - \hat{m}_{t|N}$$

for $t = N+1, N+2, ..., T$

In this study, to assess and compare the performance of mortality methods, the mean absolute forecast error (MAFE) is used as a measure of accuracy for the point forecasts for log-mortality rates. The MAFE is an average of absolute errors across different age intervals, forecast horizons and forecast periods (Shang et al., 2010). It measures the precision of forecasts, regardless of sign (Shang, 2015):

$$MAFE = \frac{1}{30 \times I} \sum_{k=1}^{30} \sum_{x=i}^{p} \left| m_{x,k} - \hat{m}_{x,k} \right| \tag{4.2}$$

where $m_{x,k}$ represents the actual age specific log-death rates in age group x in the kth year of forecasting period and $\hat{m}_{x,k}$ is the related point forecast. Note that, since the age span of country data is split into three intervals, *I* describes the total number of single ages in the corresponding age interval.

4.5.3. Out-of-Sample Validation

By using the above mentioned error measure, the point forecast accuracies are investigated of the seven methods by fitting periods. MAFEs are calculated for one-step-ahead to 30-step-ahead forecasts averaged over countries, female and male subpopulations and age intervals.

Table 4.5 and Table 4.6 provide summaries for the point forecast accuracy of female and male subpopulations based on the MAFE averaged over countries and corresponding age intervals. The highest forecast accuracy, which has the lowest error is remarked by drawing bold symbols in these tables.

In general, MAFE measure for male groups tend to show higher values compared to female groups. The result of the analyses shows that nonparametric variants of Hyndman-Ullah methods are more accurate than parametric Lee-Carter variants. Specifically, Weighted Hyndman-Ullah method (HUw) forecasts are more accurate than the other six methods at total by minimum MAFE measure. According to this measure, out of 18 indicators, HUw method is the best performer with 5 indicators which is followed by HU50 and LM methods (4 indicators for each), HU method (3 indicators) and the original LC method and LCnone method (1 indicator for each).

On the other hand, the analyses show notable results for female and male populations across different age intervals. If the general pattern of age intervals is classified as (i) young, (ii) adult and (ii) advanced ages for each subpopulation; especially for female populations at adult ages LC methods and its variants seem to have lower MAFEs; while at advances ages HU methods and its variants have more accurate forecasts in terms of MAFEs.

Additionally, the examinations across countries reveal some specific features for each country uniquely. The methods which gave the most accurate forecasts for each country over different age intervals are specified. The results show that each country has a particular model on its behalf for age intervals approximately for female and male subpopulations.

Table 4.5. *Accuracy of point forecast of log-mortality rates for female subpopulations measured by MAFE of piecewise mortality modelling approach*

Australia	LC	LCnone	LM	BMS	HU	HU50	HUw
Age 0-16	0.0004421917	0.0001750068	**0.00000929568**	0.0001171726	0.0001197628	0.00010703	0.000104938
Age 17-46	0.0005512675	0.000270154	**0.000130429**	0.0002281018	0.000182197	0.000156612	0.000143512
Age 47-89	0.00375938	0.003994833	0.0002166487	0.002794897	0.004393973	**0.0020089485**	0.003562463
Japan							
Age 0-16	0.000291741	0.00034393	0.000280285	0.000298038	**0.0002226059**	0.000234115	0.000231599
Age 17-46	0.000433697	0.000320031	0.000395241	0.000355396	**0.00028211**	0.000345749	0.000338765
Age 47-89	0.00398776	0.005318617	0.00387632	0.00476665	0.003339783	**0.002181222**	0.002408127
Portugal							
Age 0-16	0.0004405309	0.000368389	0.0002242953	0.0001192063	0.000336905	0.000309474	**0.00014944**
Age 17-45	0.000296812	**0.0001166867**	0.000382263	0.00019035	0.000190661	0.000198897	0.000202869
Age 46-89	0.00831739	0.009873963	0.005744253	0.00917593	0.008327933	0.004790917	**0.004765823**

Table 4.6. *Accuracy of point forecast of log-mortality rates for male subpopulations measured by MAFE of piecewise mortality modelling approach*

Australia	LC	LCnone	LM	BMS	HU	HU50	HUw
Age 0-16	0.0005669573	0.0002913200	**0.00007846590**	0.0002430023	0.0003403110	0.0001538151	0.0001242826
Age 17-36	0.0011561500	0.0005835790	0.00003997063	0.0005995820	0.0006153337	0.0004937523	**0.0003191189**
Age 37-89	0.0085215033	0.012127537	0.006139363	0.009455283	0.005283373	**0.0027530697**	0.010341117
Japan							
Age 0-16	0.000309905	0.000381666	0.00028646	0.000290256	**0.000232037**	0.000249542	0.000259843
Age 17-38	0.0006302	0.000456919	0.000524537	0.000480173	0.000343628	**0.000235863**	0.00029974
Age 39-89	0.002587783	0.003038185	**0.002259638**	0.003058419	0.002402663	0.002649482	0.003009737
Portugal							
Age 0-16	0.000574369	0.000559551	0.001940187	0.00042654	0.000549895	0.000566528	**0.0000171853**
Age 17-40	**0.00041806**	0.000429916	0.000647236	0.00069499	0.000471052	0.000569725	0.0005244
Age 41-89	0.008729617	0.010628763	0.00627344	0.009510457	0.010072393	0.007059287	**0.00451035**

4.5.4. Optimistic Methods for Age-Segmented Mortality

The forecast accuracy of all 7 models (four LC variants; three HU variants) across countries and corresponding age intervals for female and male populations in terms of mean absolute forecast error (MAFE) for mortality rates are given in Table 4.5 and Table 4.6.

When the forecast accuracies of different methods are investigated specific to countries, it can be seen that for different age intervals, different methods have the highest forecast accuracy. On the other hand, in general, specifically to countries both genders show the same method in terms of higher accuracy at age groups which are classified as young ages. This condition is only valid for young ages. As age is getting older, consistency between the methods for female and male groups in terms of adult and advances age intervals become dissimilar.

Australia female and male populations have different methods in terms of MAFE values for each age interval. However, when the results are compared regarding age intervals describing as (i) young, (ii) adult and (iii) advanced age, with regards to accurate method, only for adult age group, mortality method differentiates by gender giving other two methods for remaining age intervals the same.

 In terms of MAFE, for Australia female subpopulation, Lee-Miller method is most accurate model for both age interval (i) 0 – 16 years and (ii) 17 – 46 years while Hyndman-Ullah method starting from fitting period of 1950 gives the most accurate results in terms of forecasting mortality rates. Observed log-mortality rates of years 1987 to 2016 and corresponding 30 years ahead forecast of log-mortality rates are presented in Figure 4.10 which displays different mortality models for each three age intervals with regards to the lowest errors.

For male subpopulation, in terms of MAFE, the lowest errors are found for (i) Lee-Miller method for age interval 0 – 16; (ii) weighted Hyndman-Ullah method for age interval 17 – 36 and (iii) Hyndman-Ullah method starting from fitting period of 1950 for age interval 37 – 89 (Figure 4.10).

Figure 4.10. Observed and forecasted log-mortality rates for Australian subpopulations between years 1987 to 2016 according to optimistic mortality methods

Figure 4.11. Observed and forecasted log-mortality rates for Japanese subpopulations between years 1988 to 2017 according to optimistic mortality methods
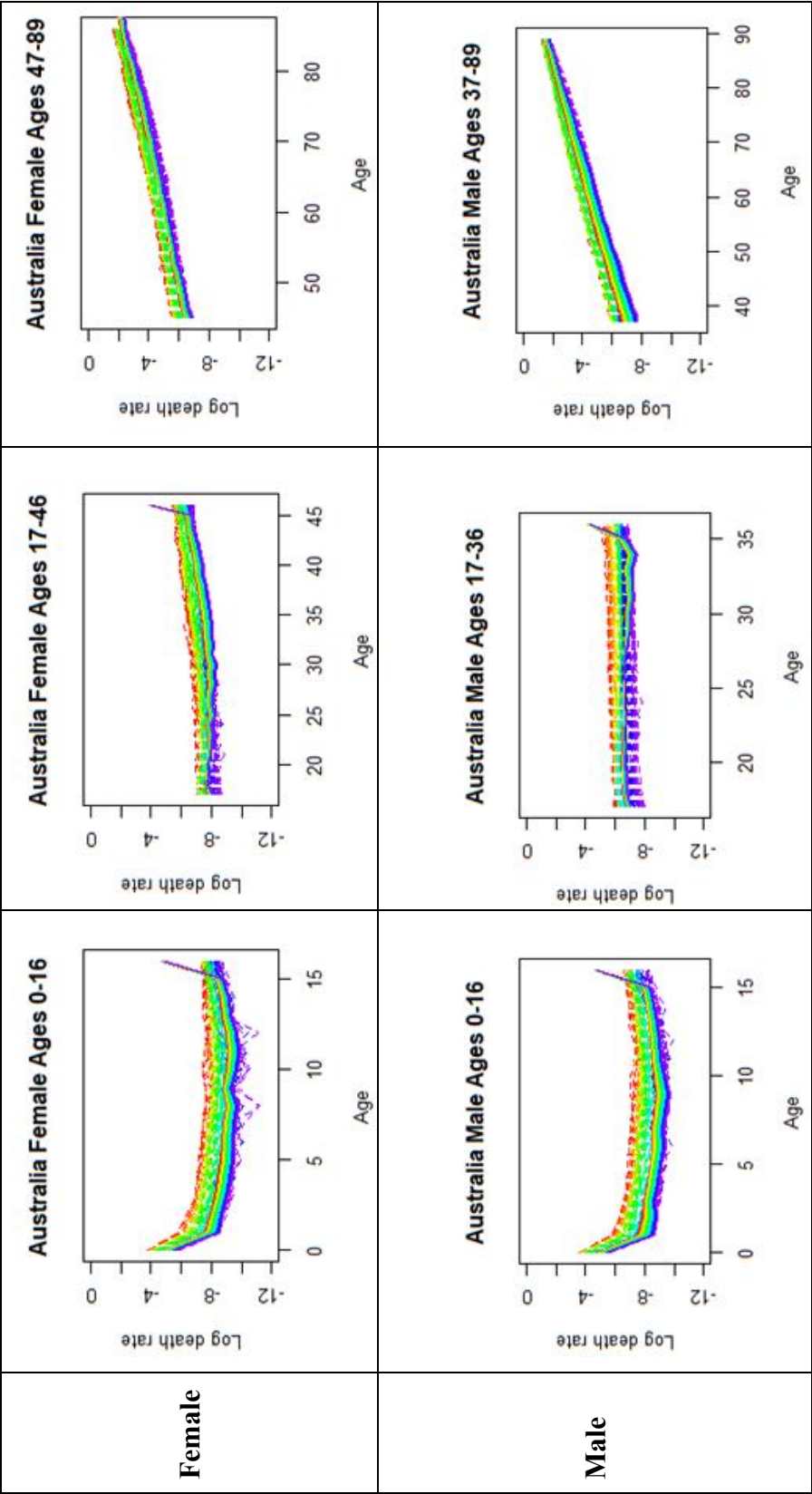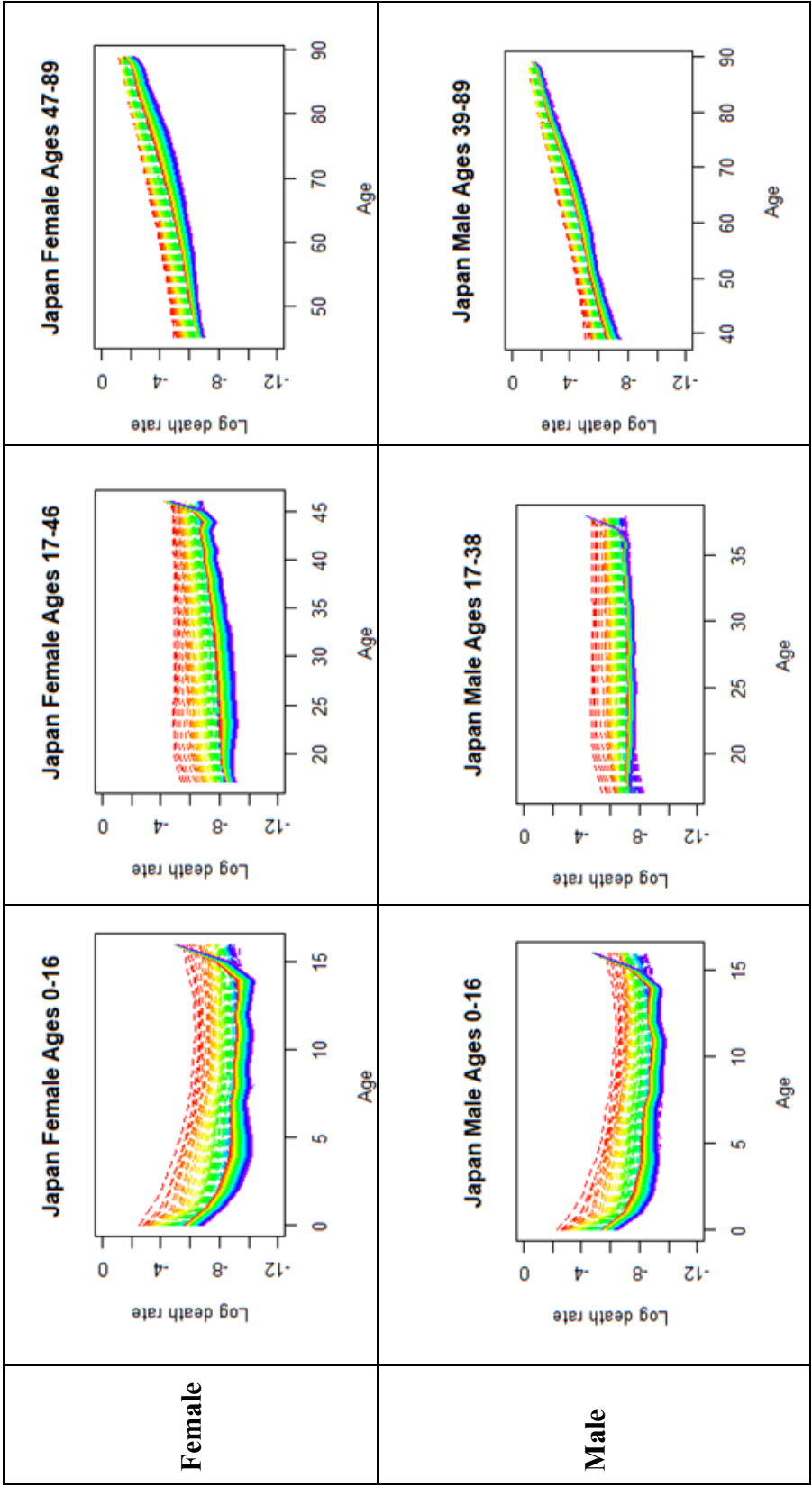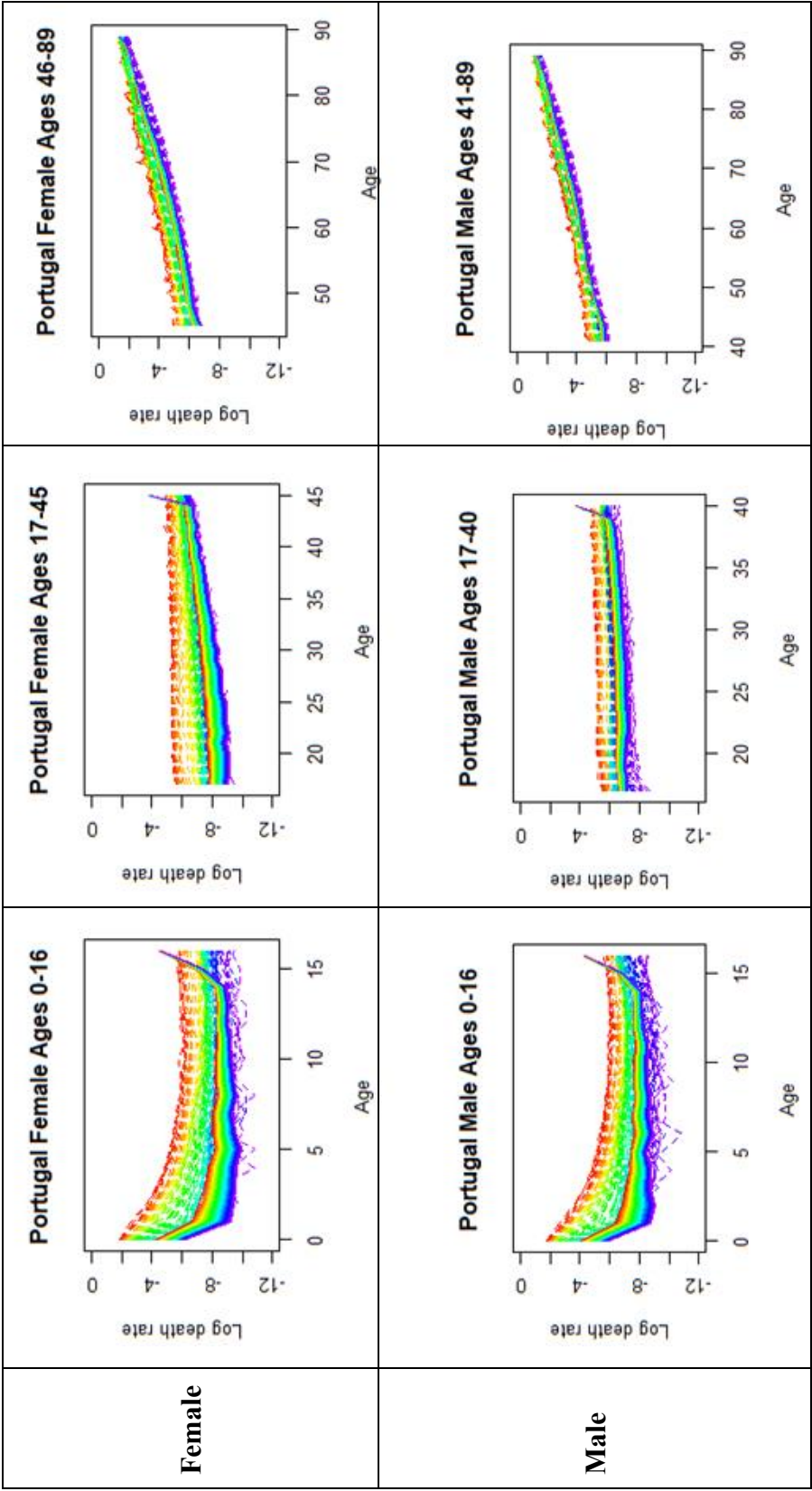
Figure 4.12. Observed and forecasted log-mortality rates for Portuguese subpopulations between years 1986 to 2015 according to optimistic mortality methods

57

In terms of the lowest MAFE values for first age interval corresponding young ages (0 – 16 years), for both Japan female and male, Hyndman-Ullah method produce optimistic forecasts.

For Japan female subpopulation, Hyndman-Ullah method is most accurate model for both age intervals 17 – 46 years. On the other hand, for the age interval containing 47 – 89 years, Hyndman-Ullah method starting from the fitting period of 1950 gives the most accurate results in terms of forecasting mortality rates (Figure 4.11).

For male subpopulation, in terms of MAFE, the lowest errors are found for Hyndman-Ullah method starting from the fitting period of 1950 for age interval of 17 – 38 years and Lee-Miller method for age interval 39 – 89 of years (Figure 4.11).

On the other hand, analyses for Portugal display almost the same results for female and male subpopulations. For both subpopulations, the age interval of years 0 – 16 years, weighted Hyndman-Ullah method produce optimistic forecasts. For female subpopulation, Lee-Carter method provides the lowest error for age interval of years 17 – 45; while for male subpopulation, Lee-Carter method without adjustment produces the lowest error with regard to MAFE for age interval of years 41 – 89 years (Figure 4.12).

If the results are examined within the general perspective, it is inferred that each country has different appropriate methods with regards to different age intervals which is classified as (i) young, (ii) adult and (iii) advances ages. In general, for Australia LM method seems to give optimistic results at total; while for Japan and Portugal, HU and HUw methods seem to generate optimistic forecasts respectively.

4.5.5. Comparison of Piecewise Mortality Modelling Approach with Overall Modelling

To validate effectiveness of piecewise mortality modelling approach; general way of mortality modelling process for the data of selected countries over female and male subpopulations is performed as an additional step.

At this procedure, mortality data of each country and subpopulations are utilised for forecasting mortality over the whole age span; instead of divided age span with breakpoints of age and comparison of these two approaches is performed in terms of accuracy of point forecast for mortality rate at specified age.

At this stage, mortality rates of selected countries for female and male subpopulations are examined for the whole age span which contains the ages 0 – 89 overall. As done in previous sections, the forecast accuracy of all 7 models (four LC variants; three HU variants) across countries for female and male populations in terms of mean absolute forecast error (MAFE) for log-mortality rates are calculated. MAFEs are obtained for one-step-ahead to 30-step-ahead forecasts averaged over countries, female and male subpopulations according to out-of-sampling framework.

Table 4.7 presents summaries for the point forecast accuracy of female and male subpopulations based on the MAFE averaged over countries for overall age span. The highest forecast accuracy, which has the lowest error is remarked by drawing bold symbol in the table.

Table 4.7. *Accuracy of point forecast of log-mortality rates for female and male subpopulations measured by MAFE of overall mortality modelling*

Australia	LC	LCnone	LM	BMS	HU	HU50	HUw
Female	0.002175703	0.004294883	0.001108628	0.002778548	**0.00110314**	0.001603078	0.001278675
Male	0.006242874	0.009670903	0.003875019	0.008337981	0.007391802	**0.002672606**	0.004776318
Japan							
Female	0.002428908	0.004201201	0.002468378	0.001966682	**0.001351229**	0.00158445	0.001384185
Male	0.002015928	0.003086938	0.001755444	**0.001493869**	0.002221661	0.002499534	0.003146136
Portugal							
Female	0.005116703	0.006294393	0.004005916	0.005699608	0.003149944	0.004644673	**0.00181640**
Male	0.00588561	0.007030764	0.004895947	0.006413962	0.006089977	0.00591237	**0.002189487**

When the forecast accuracies of different methods are investigated specific to countries, different methods have the highest forecast accuracy in terms of lowest error term. The result of the analyses showed that nonparametric variants of Hyndman-Ullah methods (5 indicators) are more accurate than parametric Lee-Carter variants (1 indicators). Specific to countries (i) HU and HU50 methods are most accurate models for Australian female and male subpopulations respectively, (ii) HU and BMS methods are most accurate models for Japanese female and male subpopulations respectively and (iii) HUw method is the most accurate model for both female and male subpopulations.

To perform the comparison between piecewise mortality modelling approach with overall mortality modelling; age 15 is chosen as a pattern. Accordingly, absolute forecast errors of age 15 for the last year of forecasting period (i.e. 2016 for Australia, 2017 for Japan and 2015 for Portugal) are considered for each country and subpopulations. For piecewise mortality modelling approach, the optimistic method for corresponding age interval is selected and point forecast error for age 15 is calculated. On the other hand, for overall mortality modelling, point forecast error of age 15 is calculated by optimistic method for each subpopulation. As a final step, corresponding values are compared with regards to minimum of absolute forecast error:

$$e_t = m_t - \hat{m}_{t|N} \ \text{ for } t = N+1, N+2, ..., T$$

The names of optimistic methods for proposed piecewise mortality modeling approach and general approach along with the corresponding forecast errors of point forecast for age 15 are presented in Table 4.8. Here, the highest forecast accuracy, which has the lowest error is remarked by drawing bold symbol.

Table 4.8. *Comparison of forecast errors for age 15 by proposed approach and general approach*

Country	Subpopulation	Piecewise Mortality Modelling		Overall Mortality Modelling	
		Method	|Forecast Error|	Method	|Forecast Error|
Australia	Female	LM	**0.000361101**	HU	0.000815218
	Male	LM	**0.0002110182**	HU50	0.0003531971
Japan	Female	HU	-	HU	-
	Male	HU	**0.00058479**	BMS	0.000743755
Portugal	Female	HUw	-	HUw	-
	Male	HUw	-	HUw	-

The results show notable information about these two approaches. As seen in Table 4.8, piecewise mortality modelling approach gives the minimum errors with regards to forecast of log-mortality rate of age 15 across female and male populations and different countries. The proposed method presents more accurate forecast in terms of mortality rates. This provides the principle that the division of the whole age span into age intervals proposing more realistic forecasts of mortality rate. The results also implicate that age breaks comprise critical information in terms of forecast accuracy.

CHAPTER 5

CONCLUSION

Mortality which can also be described as the risk for occurring of death, has vigorous structure and changes over the years, ages and genders continuously. As the significant and utter changes have taken into place in human lives in the last century, the term of mortality has become much more critical. The importance lying behind is to understand the corresponding population for both present mortality dynamics and future demographical projections. Accordingly, the mortality structure has been affecting many areas; social policies, health care, pension finances, education etc. both public and private levels.

Standing at central place, the first attempts of mortality analyses and modeling have begun centuries ago and as of now, various methods have been developed. One of these methods and probably change the mortality forecasting concept is Lee-Carter method (Lee & Carter, 1992). The principle of the Lee-Carter models is based on extrapolation of the past rends of mortality in age and time by forecasting mortality risk, and hedging longevity risk (Liu et al., 2019; Booth et al, 2006). It is still the most used method for mortality projections of official data of several countries due to its simplicity and reliability of the results. On the other hand, several variants and extensions of Lee-Carter model have been proposed with regards to improve different aspects of mortality trend fitting and projection of this method (Booth & Tickle, 2008). Accordingly, some of the variants of Lee-Carter model which were proposed by Lee & Miller (2001), Booth et al., (2002) and Hyndman & Ullah (2007) have been accepted and used widely. Especially, Hyndman-Ullah model which is also called as functional demographic (data) model, is among the most accurate models (D'Amato et al., 2011).

Although the performance of mortality models in the case of structural change in time has been studied in the literature (van Berkum et al., 2014; Coelho & Nunes, 2011), there are no concrete examples for mortality modeling in terms of structural breakpoints for age. From this point of view, in this thesis, Lee-Carter model and its extensions and functional demographic models are utilized within the perspective of age breakpoints. We take into consideration demographic data for Australia, Japan and Portugal which is available in the Human Mortality Database (2019) that ensures the number of deaths and the exposure to risk for individuals age x in the calendar year t specifically. Using this data of countries, the purpose of this study is to model and forecast log-mortality rates for female and male populations independently with regards to breakpoints of age. The pattern of age structure for each population on the mean of log-mortality rates is investigated within the framework proposed by Bai & Perrron (2003). With the results of the sequential testing framework; two breakpoints for age are found significant for each country and female and male populations. By two age breakpoints, log-mortality rate of each country over female and male subpopulations are divided into three parts accordingly; resulting in three different age intervals. After obtaining the specific age intervals particular to each subpopulation, series of Kruskal-Wallis tests are performed to determine whether there are statistically significant difference between the age intervals. These three age intervals for every six subpopulations are found significantly differentiating from one to another ($p < 0.05$).

After breakpoints of age for each three countries and two genders determined; a total of 18 subpopulations are obtained according to corresponding age intervals. Specific to these subpopulations uniquely, different mortality forecasting methods which are Lee-Carter variants and Hyndman-Ullah variants are performed. These methods are evaluated in terms of mean absolute forecast error (MAFE) for mortality rates. Results display consistent outcomes with studies previously conducted in related literature (Rabbi & Mazzuco, 2018; Shang, 2015; Shang et al. 2011).

However, in addition to the literature, the results of this study show that different methods give the most accurate and optimistic forecasts even for the same gender group but different age intervals. Additionally, to validate the effectiveness of the proposed approach, comparison procedure is conducted. As expected, the proposed piecewise mortality modelling approach gives more accurate forecasts rather than overall mortality modelling approach.

In conclusion, this study presents evidence based information to perform mortality modelling for age intervals which is segmented differently by their common statistical features. This approach puts forward a new methodology for mortality modelling field by constructing models to specific age intervals which are chosen with regards to divided breakpoints of age. Consequently, the division of the whole age span into characteristic age intervals enables us to give more realistic forecasts of mortality rate by piecewise mortality modelling. The results implicate that age breaks comprise crucial information in terms of forecast accuracy of different methods and thus for a more realistic mortality forecasting process, this approach should be taken into consideration. Specifically, the different age breaks for female and male subpopulations uniquely should be considered and best suited alternative methods for these age intervals should be preferred when it comes to mortality forecasting for the future. Additional new methods should be applied for this problem to enable to assess the best fitting model for a given of age intervals instead of the whole age span. At this point, joint distribution of mortality rates within the different age intervals should be investigated and alternative nonparametric approaches should be employed.

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