OPTION PRICING IN INTEREST RATE DERIVATIVES

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ABSTRACT

OPTION PRICING IN INTEREST RATE DERIVATIVES

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The valuation of interest rate derivatives and embedded options in fixed-income securities is crucial for market practitioners. Although there have been many models to price interest rate derivatives, the inconsistency across the assumptions of the models creates difficulty in aggregating interest rate exposures. Besides, the models tend to be applicable to specific cases. In this regard, adaptation of a general methodology to price all interest rate derivatives without making additional assumptions has critical importance. This study is expected to contribute to the literature by providing a general approach that can be applied to any fixed-income security with regular or irregular cash flows using the Vasicek model. The methodology involves four main steps: (i) deriving the closed-form solution for the interest rate derivatives traded in the market, (ii) estimating the Vasicek model parameters, (iii) deriving the exhibit solution for the interest rate derivatives and (iv) plugging the estimated Vasicek model parameters to price the security. This methodology provides a general solution that is applicable to all interest rate derivatives with regular or irregular cash flows. Additionally, it allows aggregation of exposures to different interest rate derivatives and allows the derivation of sensitivities of the option values to the changes in model parameters. Although the study provides empirical evidence for European type of options, it also can be applied to price American or Bermudan type of options as well. Besides, the methodology can be implemented using other interest rate models with desirable properties.

Keywords: Interest rate derivatives, Vasicek model, swaption, prepayment option

ÖZ

FAİZE DAYALI TÜREV ÜRÜNLERDE OPSİYON FİYATLAMASI

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Sabit getirili menkul kıymetlerdeki faize dayalı türev ürünlerin ve gömülü opsiyonların değerlemesi piyasa oyuncuları açısından önem arz etmektedir. Faize dayalı türev ürünlerini fiyatlandırmak için akademik yazında çok sayıda model geliştirilmiş olmasına rağmen, söz konusu modellerin varsayımları arasındaki tutarsızlık, özellikle faiz oranındaki değişimlere karşı risklerin birleştirilmesi kapsamında zorluk oluşturmaktadır. Ayrıca, modellerin belli durumlara yönelik olarak geliştirildiği ve genelleştirme kapsamında uygun olmadığı gözlenmektedir. Bu bağlamda, ürün bazlı olarak varsayım yapmadan genel bir metodolojinin kullanılarak faize dayalı türev ürünlerin fiyatlandırılması kritik önem taşımaktadır. Bu çalışmanın, Vasicek modeli altında düzenli veya düzensiz nakit akışları olan sabit getirili menkul kıymetlere uygulanabilecek genel bir yaklaşım sağlayarak literatüre katkı sağlaması beklenmektedir. Uyarlanan yöntem temel olarak dört adımdan oluşmaktadır: (i) piyasada işlem gören faize dayalı türev ürünler için Vasicek modeli ile ilgili formülün türetilmesi, (ii)Vasicek model parametrelerinin tahmin edilmesi, (iii) fiyatlandırılması amaçlanan faize dayalı türev ürün için de Vasicek modeli altında formülasyon gerçekleştirilmesi ve (iv) tahmin edilmiş olan Vasicek model parametre tahminlerinin fiyatlama kapsamında kullanılması. Bu metodoloji, düzenli veya düzensiz nakit akışlarına sahip tüm faize dayalı türev ürünlerine uygulanabilecek genel bir çözüm sunmaktadır. Ayrıca, faiz kaynaklı maruziyetlerin tutarlı bir şekilde hesaplanabilmesi ve faize dayalı opsiyon değerlerinin model parametrelerine olan duyarlılıklarının türetilmesi de sağlanmak-

tadır. Çalışmada yer alan sonuçlar Avrupa tipi opsiyonlar için bulunmuş olmakla beraber çalışmanın sonuçları Amerikan ve Bermudan tarzı opsiyon sözleşmelerine de uyarlanabilir. Ek olarak, metodolojinin gerekli özellikleri sağlayan diğer faiz modelleri ile de uygulanması mümkündür.

Anahtar Kelimeler: Faize dayalı türev ürünler, Vasicek modeli, swaption, erken ödeme opsiyonu

To My Wife Canan Küçüksaraç and My Newborn Son Uygar Küçüksaraç

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TABLE OF CONTENTS

LIST OF FIGURES

LIST OF ABBREVIATIONS

$M(t)$	The value of money market account at time t
$D(t, T)$	The stochastic discount factor from time t to T
$P(t, T$	Price of a bond with maturity T at time t
$\mathbb{E}^{\mathbb{Q}}$	Expectation under the risk-neutral measure
$\mathbb{E}^{\mathbb{T}}$	Expectation under the T-forward measure
$A(t, t_m, t_n)$	The value of annuity at time t which starts at time t_m and matures at time t_n
$V_{t_0, fixed\, leg}$	The value of fixed-rate leg in cross-currency swap
$V_{t_0, floating\, leg}$	The value of fixed-rate leg in cross-currency swap
S_{t_0}	The value of one unit of foreign currency in terms of domestic currency at time t_0
$L(t_{i-1}, t_i)$	The floating interest rate for the period between t_{i-1} and t_i
$fs(t, t_m, t_n)$	The forward swap rate at time t which starts at time t_m and matures at time t_n
$V_{swaption}(t, t_m, t_n)$	The value of swaption at time t where the underlying swap starts at time t_m and matures at time t_n
$r(t)$	The instantenous short rate
α	The mean-reversion rate in the Vasicek model
β	The long-run mean of short rate in the Vasicek model
σ	The volatility of short rate in the Vasicek model
W^Q	The Brownian motion under the risk-neutral measure
W^{Q_T}	The Brownian motion under the T-forward measure
$\sigma_P(t, T)$	The volatility of bond price $P(t, T)$
$\mathbb{E}^{Q_T}(r(t_i))$	The expected value of short rate at time t_i under the T-forward measure
$Var^{Q_T}(r(t_i))$	The variance of short rate at time t_i under the T-forward measure
$ZBC(t, t_m, t_i, K)$	The value of a call option at time t with maturity date t_m on a zero-coupon bond maturing at t_i with strike price K
$ZBP(t, t_m, t_i, K)$	The value of a put option at time t with maturity date t_m on a zero-coupon bond maturing at t_i with strike price K

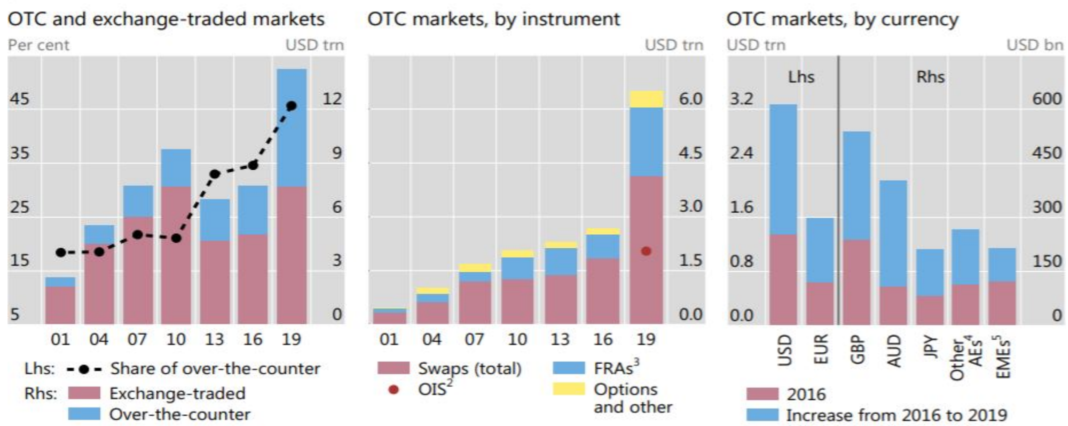
CHAPTER 1

SWAPTION IMPLIED DENSITY FUNCTIONS

1.1 INTRODUCTION

The interest rate derivatives market has expanded enormously during the last decade. In fact, the most commonly traded financial instruments in the global financial markets are interest rate derivatives. As of April 2019, the notional amount outstanding for interest rate contracts is approximately 12 trillion USD where the most traded securities are interest rate swaps and forward rate agreements according to the estimates of the Bank of International Settlements (BIS). The BIS study also indicates that the trading of interest rate derivatives has increased in the last three years mainly due to expectations about short-term interest rates of developed markets as well as the increase in non-market facing trades and compression trades. Besides the trade volumes, the instruments traded tend to be more complicated, which increases the need for sophisticated and practical models for pricing these instruments. Additionally, the popularity of the fixed-income securities with embedded options such as callable and puttable bonds or prepayment options on mortgage loans fuel the demand for hedging against changes in interest rates.

Given the size of interest rate derivatives and embedded options in the global financial markets, it is crucial to extract the interest rate expectations embedded in these contracts. Measuring interest rate expectations is especially crucial for portfolio managers, investors, risk managers and policymakers in terms of formation of trade recommendations, valuation of fixed-income securities including financial derivatives, inferring market assessments and estimating potential risk of portfolios. Besides, extraction of interest rate expectations of the market participants provides significant information for policymakers and regulators. Central banks place emphasis on obtaining market expectations mainly in order to control or direct these expectations with the aim of strengthening the transmission mechanism of monetary policy. Besides, since interest rate expectations also provide a signal about the state of the economy, recession concerns and inflation expectations priced in the financial securities, central banks put significant effort to interpreting the signals in the fixed income products. Additionally, since the interest rate derivatives and other fixed income securities constitute the largest share in global financial markets, their trading has crucial implications on financial stability.



[1] Adjusted for local and cross-border inter-dealer double-counting, ie "net-net" basis. [2] OIS are included in total swap turnover. Data available only from 2019. [3] Forward rate agreements. [4] CAD, CHF, DKK, NOK, NZD and SEK. [5] AED, ARS, BGN, BHD, BRL, CLP, CNY, COP, CZK, HKD, HUF, IDR, ILS, INR, KRW, MXN, MYR, PEN, PHP, PLN, RON, RUB, SAR, SGD, THB, TRY, TWD and ZAR.

Sources: BIS derivatives statistics and Triennial Central Bank Survey.

Figure 1.1: Turnover and Composition in Interest Rate Derivatives Markets

Given the importance of interest rate expectations priced in the market, the questions of how to extract the expectations and which fixed-income securities to use for this purpose arises. There are a variety of fixed income securities that can be used: Treasury bonds, swaps, bond futures, bond forward contracts, caps, floors or swaptions. In this regard, fixed income products with option characteristics such as interest rate caps, floors or swaptions can be used to obtain density functions for interest rates, allowing us to have an idea about the probability attached to different interest rate levels at a future point in time whereas the other fixed income securities without option characteristics would give an idea about the expected value rather than a density function. Therefore, it is possible to interpret the market's assessment of the degree of uncertainty or the direction of expected changes in interest rates. Besides, it is possible to obtain the likelihood of extreme changes in interest rate levels, which is invaluable for portfolio managers and risk managers as well as market regulators.

A key challenge is that the pricing of and hedging with interest rate derivatives are more difficult than those for equity and foreign currency derivatives due to several factors. First, the behavior of interest rates tends to be more complicated, which generally exhibits a mean-reverting process. Secondly, the pricing of most of the interest rate products requires modeling of the entire yield curve. Additionally, the fact that interest rates are used both for discounting and defining the payoff of interest rate derivatives makes pricing more complicated, which generally requires changing measures to make computations more plausible. In this regard, it is crucial to obtain robust and simple pricing mechanisms for interest rate derivatives, especially for the ones with option characteristics. Given the fact that density functions can be used to price similar fixed-income securities, it is a useful pricing tool for market practitioners since it provides an opportunity to compute prices that are consistent with market prices. Additionally, since the fixed-income securities are traded in the market, it allows using these securities to hedge against interest rate changes.

Given the usefulness of the density functions for policymakers and portfolio managers, obtaining density functions for the underlying interest rates implied by the market prices has drawn attention in the literature. However, most of the studies have focused on estimating risk-neutral density functions mostly for equity and FX markets and the studies regarding interest rates are relatively limited in number. For instance, density functions for European FX options are widely estimated by using data from emerging markets. Models used in previous studies have different advantages and disadvantages. Typically, the methodologies adopted in the literature require a great deal of data processing and use complex estimation techniques. As a result, option-based density functions are less widely applied by the market practitioners. Instead, nonparametric models have become popular and are widely used by many central banks due to the fact that these models have a good fit to data and offer flexibility in application.

This thesis is expected to contribute to the literature in terms of extracting interest rate expectations for different maturities that can be used for pricing purposes and as a hedging tool for other fixed-income positions. The density functions also are expected to be useful for policymakers in their efforts to form interest rate expectations and to analyze the systemic risks for financial stability. Additionally, this is the first study that provides density functions for Turkish lira interest rates.

The next section reviews the literature regarding the methods used to extract the density functions in the previous studies, covering the methodologies implemented for FX and equity as well. Next, preliminary mathematical properties and characteristics of fixed-income securities used are introduced. The empirical findings section presents the implied volatility curve that can be used to price similar products with different strike rates, as well as density functions obtained between January 2013 and November 2019 along with the moments of the density functions.

1.2 LITERATURE REVIEW

Density functions obtained through option contracts reflect the variations in true probability as well as risk premia that market participants attribute to market prices of various assets such as exchange rates, stock prices and interest rates. These density functions enable investors and policymakers to examine the impact of economic events and policy changes on market prices from a forward-looking perspective. In the literature, there are numerous methods/approaches for the extraction of expectations from a sound option pricing mechanism, and it is possible to categorize them as structural and non-structural approaches.

Within these approaches, there exist subgroups that resemble each other in terms of the characterization of the underlying asset. The approaches adopted in the literature for the derivation of density functions are illustrated in the figure below. Full characterization of the pricing mechanism of the underlying asset is presented under structural approaches whereas nonstructural models aim to approximate density functions via parametric or nonparametric methods without figuring out the complete pricing process of the underlying asset. Option pricing based on the Black-Scholes-Merton formula or the Black volatility model assumes that the price of underlying asset follows a lognormal distribution and implied volatility is constant over all strike prices.

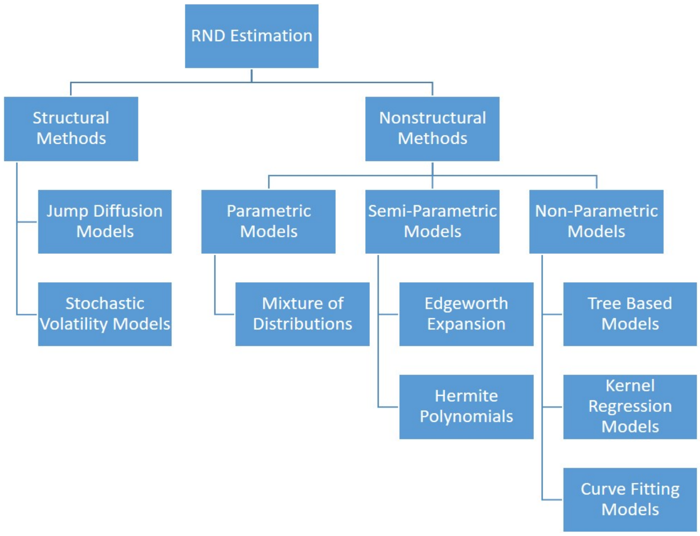


Figure 1.2: Density Estimation Methodologies

Although these assumptions do not hold in practice, the Black-Scholes-Merton formulation is used extensively in the market. In the case of swaptions, the Black volatility is used in practice as a volatility quote.

Structural models aim to make up for the inability of the Black-Scholes-Merton model by defining the stochastic process of the underlying asset such that the empirical properties of asset returns would be in line with the process defined. Jump diffusion and stochastic volatility models are two mainstream approaches under this category in the literature. Merton (1976) was the first one to use jump diffusion models allowing the price of the underlying asset to exhibit sudden jumps. In this case, the likelihood of tail events is not negligible, satisfying the leptokurtic feature of the empirical asset return distributions. Kou (2002) and Zhang et al (2012) are examples of studies adopting jump diffusion models for characterizing the asset price movements. The main advantage of these models is that the risk-neutral densities obtained have heavier tails that are consistent with the empirical features of realized asset prices. The drawback of these models is the oversimplifying assumptions regarding the jump process in an attempt to come up with a closed-form solution. Kou (2002), and Kou et al. (2004) assume an exponential distribution whereas Merton assumes lognormal distribution for the price jumps with reference to their tractability properties.

Another stream of structural models is the stochastic volatility models which account for the time-varying property of implied option volatility. Heston (1993) is the pioneer in this strand of the risk-neutral density literature. Heston assumes a mean-reverting process for the volatility and with this assumption, he demonstrates that spot returns over long periods have asymptotically normal distributions and that the positive correlation between volatility and spot returns produces a fat right tail and thin left tail. Overparametrization and calibration considerations are the problems preventing these models from being used extensively in the literature although the progress over the Black-Scholes-Merton model provided by these models cannot be denied. For instance, Rosenberg and Engle (2002) fit a stochastic volatility model to S&P 500 index returns as an application of stochastic volatility models for incorporating the risk prevailing in the market and decompose true densities from risk-neutral densities.

On the other end of the literature, models do not aim to characterize the process of the underlying asset price completely. These non-structural models define a structure for the terminal distribution directly so that the parameters of the assumed structure, if any, are chosen to approximate the underlying asset price data as much as possible. Models in the non-structural group can be further classified as parametric, semi-parametric and non-parametric, depending on the degree of the characterization of the risk-neutral distribution. In this regard, risk-neutral densities are defined completely by parametric models, as compared to a partial characterization proposed by semi-parametric and non-parametric models.

In the context of parametric models, Melick and Thomas (1997) use a linear combination of lognormal distributions for estimating the distribution of the price of underlying asset. This mixture of distributions approach has been proposed to overcome the problem of underestimation of tail events in the case of the Black-Scholes-Merton model. This stream of models reflects the market participants' expectations regarding extreme events more accurately. These models have a degrees of freedom problem due to an inadequate number of strike price against a high number of parameters to be estimated when several distributions are taken into account. Therefore, the power of this approach is diminished by data limitations. Furthermore, there is no objective way or a consensus for the determination of the number of distributions to include in the mixture of distributions. Lastly, this approach is not useful for the construction of dynamic hedging as it does not characterize the process of the prices of underlying assets explicitly.

The second branch of nonstructural models is the semi-parametric models, where the risk-neutral densities are aimed to be approximated at maturity. Edgeworth expansion and Hermite polynomials are examples of approximation techniques to distributions of options in this category. The Hermite polynomial approximation method involves approximation of risk-neutral density by an expansion around a lognormal distribution using Hermite polynomials. The theoretical foundations of this approach are shown by Madan and Milne (1994) and an application is given in Abken, Madan, and Ramamurtie (1996) and Coutant, Jondeau, and Rockinger (2001). Another kind of approximation of option valuation is the Edgeworth expansion proposed by Jarrow and Rudd (1982).

The final category to be reviewed in this section for the estimation of risk-neutral densities obtained from options is nonparametric methods which do not assume any form of a parametric model for the distribution and therefore allow greater flexibility in fitting risk- neutral distribution to observed data. Kernel regressions, tree-based models and curve-fitting models are the three most widely used approaches in this category. For instance, Ait-Sahalia and Lo (1998) extracts the risk-neutral densities from S&P 500 index option prices, consider both time series and cross-sectional variation in option prices and assume that option pricing formulation is a nonlinear function of option characteristics. As in the case of the mixture of distributions approach, the use of Kernel regression methods is problematic as a result of data limitations.

Rubinstein (1994) first introduces tree-based models for the pricing of options where state-contingent prices are obtained through observed European option prices. These prices are then used to characterize the tree by minimizing the gap between the tree implied probabilities and probabilities obtained from the tree of Cox et al (1979). Rubinstein's model is generalized by Jackwerth (1997) for obtaining a better fit. At the same time as Rubinstein, Dupire (1994) fit an implied trinomial tree, Derman and Kani (1994) use a binomial tree as Rubinstein did with one difference such that their tree models combine multiple maturities matching RNDs for different maturity dates.

Curve fitting models aim to characterize the implied volatility structure in the space of strike prices or option deltas based on market observations obtained for transactions or quotations. Shimko (1993) proposes a method for fitting the implied volatilities with respect to strike prices through a quadratic polynomial function so that option prices can be obtained for a continuum of strike prices using the Black-Scholes-Merton option price formulation. Breeden and Litzenberger (1978) results are then utilized to transform option prices into risk-neutral densities. As an alternative approach to the method proposed by Shimko, Malz (1996) estimates implied volatility by using option deltas in order to avoid consistency problems that arise with the use of strike prices. In addition, the smoothness of the volatility smile is improved as the degree of smoothness for the delta space is higher as compared to the one obtained with strike prices.

Several studies exist in the literature comparing different approaches for the derivation of risk-neutral densities from option prices, however, there is not a single best method which dominates others in terms of its applicability in practice. For instance, Campa et al. (1998) compare cubic splines, an implied binomial tree, and a mixture of lognormal distributions methods, in addition to the studies by Cooper (1999) and Jondeau et al. (2000) comparing different methods. The results of these studies are inconclusive in terms of the superiority of a single method. Additionally, the number of studies on the extraction of density functions for interest rates is relatively limited. Among the limited number of studies, Malz (2014) is the one that uses swaption market to find the density function of swap rates. The Malz study also provides evidence that the use of nonparametric models are preferable over other models in terms of simplicity and goodness of fit to data. This thesis uses the Malz methodology and analyzes the cross-currency swaption market for estimating the density function for Turkish interest rates.

1.3 PRELIMINARY

This section provides the basic mathematical properties used in the following sections. Besides the mathematical properties, this section also introduces the financial securities, cross-currency swaps, interest rate swaps and interest rate swaptions with their basic characteristics and payoff structures. Finally, since swaption pricing is performed under swap measure, first the measure change theorem is introduced together with the numeraire applications widely used in the literature.

1.3.1 Measure Change

Given that $\mathbb{P}$ and $\mathbb{Q}$ are two probability measures, there exists a unique nonnegative $\mathbb{F}$-measurable function f such that

$$\mathbb{P}(A) = \int_A f d\mathbb{Q} \tag{1.1}$$

The measurable function f is said to be the Radon-Nikodym derivative. It is also called as the density of $\mathbb{P}$ with respect to $\mathbb{Q}$ and is denoted by $\frac{d\mathbb{P}}{d\mathbb{Q}}$. It can be stated that for any random variable X for which $\mathbb{E}^{\mathbb{Q}}[X\frac{d\mathbb{P}}{d\mathbb{Q}}] < \infty$

$$\mathbb{E}^{\mathbb{P}}[X] = \mathbb{E}^{\mathbb{Q}}\left[X\frac{d\mathbb{P}}{d\mathbb{Q}}\right] \tag{1.2}$$

If the numeraires of N and M with measures $\mathbb{Q}^{\mathbb{N}}$ and $\mathbb{Q}^{\mathbb{M}}$ respectively, then the price of any asset V relative to N or M is martingale under $\mathbb{Q}^{\mathbb{N}}$ or $\mathbb{Q}^{\mathbb{M}}$ respectively, and the following holds:

$$N(t)\mathbb{E}^{\mathbb{Q}^{\mathbb{N}}}\left[\frac{V(T)}{N(T)}|\mathcal{F}_t\right] = M(t)\mathbb{E}^{\mathbb{Q}^{\mathbb{M}}}\left[\frac{V(T)}{M(T)}|\mathcal{F}_t\right] \tag{1.3}$$

This can be expressed as follows:

$$\mathbb{E}^{\mathbb{Q}^{\mathbb{N}}}\left[\frac{V(T)}{N(T)}|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}^{\mathbb{M}}}\left[\frac{V(T)}{N(T)}\frac{N(T)/N(t)}{M(T)/M(t)}|\mathcal{F}_t\right] \tag{1.4}$$

The Radon-Nikodym derivative is given by

$$\frac{d\mathbb{Q}^{\mathbb{N}}}{d\mathbb{Q}^{\mathbb{M}}} = \frac{N(T)/N(t)}{M(T)/M(t)} \tag{1.5}$$

1.3.2 Numeraire Applications

This section describes money market account, zero-coupon bond and annuity as examples of numeraires that are commonly used to price interest rate derivatives.

1.3.2.1 Money Market Account as a Numeraire

Money market account can be defined as a deposit that continuously earns the instantenous short rate, which can be deterministic or stochastic. Let $M(0)$ be the initial value of the money market account. The process for the money market account at time t, $M(t)$, is given by the following differential equation:

$$dM(t) = r(t)M(t)dt \tag{1.6}$$

Since a money market account earns the instantaneous risk-free rate, the volatility of a money market account is equal to zero. The value of $M(t)$ is equal to the accumulated value of the money market account across time.

$$M(t) = e^{\int_0^t r(s)ds} M(0) \tag{1.7}$$

The money market account is also associated with the stochastic discount factor. If $D(t, T)$ stands for the stochastic discount factor from time t to T, it can be expressed as follows:

$$D(t, T) = \frac{M(t)}{M(T)} = e^{-\int_t^T r(s)ds} \tag{1.8}$$

In this regard, the value of any asset $V(t)$, can be found using the following equation:

$$V(t) = \mathbb{E}^{\mathbb{Q}}\left[V(T)D(t, T)\right] = \mathbb{E}^{\mathbb{Q}}\left[V(T)e^{-\int_t^T r(s)ds}\right] \tag{1.9}$$

The money market account is used frequently to price derivatives that have an underlying asset of equity or foreign exchange. Such pricing typically assumes that interest rate is deterministic. Since the effect of the interest rate on the value of derivatives with the underlying asset of equity or foreign exchange is small, the assumption of deterministic interest rate is considered to be reasonable. If interest rates are assumed to be deterministic, then the factor $D(t, T)$ can be taken outside of the expectation operator. However, the use of money market accounts in the valuation of interest rate derivatives is not realistic given that the payoff structure is dependent on interest rates. Since both variables are stochastic, it requires the joint distribution between $V(t)$ and $D(t, T)$ to be estimated in order to compute the expectation. Therefore, the risk-neutral measure that is related to money market accounts is not the most practical measure to price interest rate derivatives.

1.3.2.2 Bond Price as a Numeraire

A zero-coupon bond denoted by $P(t,T)$ is a financial security that pays a certain amount of money at a predetermined time T in the future. The credit risk for a zero-coupon security is assumed to be negligible. In the case of deterministic interest rates, both money market accounts and bond prices are equilavent in terms of their use in the pricing of derivatives. If interest rates are stochastic, then both differ from each other in the sense that the value of bond price at maturity T is known at time t whereas the value of money market account at time T can only be known at time T. The measure associated with the zero-coupon bond, $P(t,T)$, is called as T-forward measure. In the environment of stochastic interest rates, the bond price and stochastic discount factor are related to each other as follows:

$$P(t,T) = \mathbb{E}^{\mathbb{Q}}\left[P(T,T)D(t,T)|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[D(t,T)|\mathcal{F}_t\right] \tag{1.10}$$

In other words, the zero-coupon bond price is equal to the expected value of the stochastic discount factor under the risk-neutral measure. However, the risk-neutral measure is not comfortable to price securities dependent on interest rates. The relation between the risk-neutral measure and T-forward measure can be expressed as follows:

$$V(t) = \mathbb{E}^{\mathbb{Q}}\left[V(T)D(t,T)|\mathcal{F}_t\right] \tag{1.11}$$

$$V(t) = \mathbb{E}^{\mathbb{T}}\left[V(T)D(t,T)\frac{D(T,T)P(t,T)}{D(t,T)P(T,T)}|\mathcal{F}_t\right] = P(t,T)\mathbb{E}^{\mathbb{T}}\left[V(T)|\mathcal{F}_t\right] \tag{1.12}$$

Since the discount factor to price the security $V(t)$ is outside the expectations operator under the T-forward measure, this approach requires the calculation of the expected value of $V(T)$ under the T-forward measure, which avoids the calculation of joint probability between stochastic variables. In this regard, the zero-coupon bond maturing at time T is especially useful to price derivatives with the same maturity T.

1.3.2.3 Annuity as a Numeraire

One of the most commonly used numeraires in the valuation of fixed-income securities is annuity measure. Annuity is a series of payments made at equal predetermined intervals, which is, in fact, a linear combination of bond prices. In this respect, the value of annuity at time t which starts at time t_m and matures at time t_n is expressed as follows:

$$A(t,t_m,t_n) = \sum_{j=m+1}^{n} P(t,t_j)(t_j - t_{j-1}) \tag{1.13}$$

The measure associated with annuity numeraire is called as the swap measure. This measure is especially useful for the securities whose value dependent on the entire yield curve. Consider the case that the value of security $V(t_m)$ can be expressed as follows:

$$V(t_m) = Spread A(t_m,t_m,t_n) \tag{1.14}$$

where $Spread$ is a stochastic variable that will be determined at time t_m. The valuation of the security under the risk-neutral measure as stated below requires the calculation of joint distribution, which complicates the situation.

$$V(t) = \mathbb{E}^{\mathbb{Q}}\left[V(t_m)D(t,t_m)|\mathcal{F}_t\right] \tag{1.15}$$

However, the price of the security under the swap measure can be written as follows:

$$V(t) = \mathbb{E}^{\mathbb{A}}\left[V(t_m)D(t,t_m)\frac{D(t_m,t_m)A(t,t_m,t_n)}{D(t,t_m)A(t_m,t_m,t_n)}|\mathcal{F}_t\right] \tag{1.16}$$

Since the value of $V(t_m)$ is dependent on $A(t_m,t_m,t_n)$, $V(t)$ can be expressed as follows.

$$V(t) = A(t,t_m,t_n)\mathbb{E}^{\mathbb{A}}\left[Spread|\mathcal{F}_t\right] \tag{1.17}$$

The use of swap measure requires the expected value of the $Spread$ under the swap measure rather than calculating the joint distribution. In this regard, the swap measure is useful for the pricing of swaption and constant maturity swaps whose value is dependent on the entire yield curve.

1.3.3 Cross-Currency Swaps

A cross-currency swap is an agreement between two counterparties for the exchange of interest payments based on a notional principal amount at predetermined periods of time. Consider two counterparties A and B in a cross-currency swap. Counterparty A makes floating payments to Counterparty B in foreign currency at specific time intervals whereas Counterparty B makes fixed payments to Counterparty A in domestic currency during the life of swap contract. At the initiation of the swap and at maturity, both parties exchange notional principal amounts in respective currencies. In this regard, from the point of view of a fixed rate receiver in domestic currency, a cross-currency swap can be considered a short position in a floating rate bond in foreign currency and a long position in a fixed rate bond in domestic currency.

$$V_{t_0} = V_{t_0,fixed\,leg} - V_{t_0,floating\,leg} S_{t_0} \tag{1.18}$$

where $V_{t_0,fixed\,leg}$ and $V_{t_0,floating\,leg}$ stand for the value of fixed-leg in domestic currency and the value of floating-leg in foreign currency, respectively. S_{t_0} stands for the value of one unit of foreign currency in exchange for domestic currency at time t_0. The value of fixed leg is the discounted value of fixed cash flows; coupon payments and notional amount.

$$V_{t_0,fixed\,leg} = \sum_{i=1}^{m} S_{t_0} N c \tau P(t_0, t_i) + S_{t_0} N P(t_0, t_m) \tag{1.19}$$

where c is the fixed rate in domestic currency, N stands for the notional amount and τ denotes the payment frequency. The value of the floating leg in foreign currency can be expressed as follows:

$$V_{t_0,floating\,leg} = \sum_{i=1}^{m} N L(t_{i-1}, t_i) \tau P^f(t_0, t_i) + N P^f(t_0, t_m) \tag{1.20}$$

where $L(t_{i-1}, t_i)$ stand for the floating interest rate for the period between t_{i-1} and t_i. Although the floating cash flows are not known at the initiation of the swap contract, the value of floating leg can be found out. In this regard, the value of swap can be written as follows:

$$\begin{aligned} V_{t_0} &= \sum_{i=1}^{m} S_{t_0} N \tau c P(t_0, t_i) + S_{t_0} N P(t_0, t_m) \\ &- \left(\sum_{i=1}^{m} S_{t_0} N \left\{ P^f(t_0, t_{i-1}) - P^f(t_0, t_i) \right\} + S_{t_0} N P^f(t_0, t_m) \right) \end{aligned} \tag{1.21}$$

$$V_{t_0,floating\,leg} = \sum_{i=1}^{m} N \left\{ P^f(t_0, t_{i-1}) - P^f(t_0, t_i) \right\} + N P^f(t_0, t_m) = N \tag{1.22}$$

$$V_{t_0} = \sum_{i=1}^{m} N \tau c S_{t_0} P(t_0, t_i) + S_{t_0} N P(t_0, t_m) - S_{t_0} N = 0 \tag{1.23}$$

since the discounted value of $L(t_{i-1}, t_i)\tau$ is equal to the $(P^f(t_0, t_{i-1}) - P^f(t_0, t_i))$.

Cross-currency swap pricing is said to determine the fixed rate in the swap contract such that the value of swap is equal to zero at the initiation of the contract.

$$c = \left(\frac{1}{\tau}\right)\left(\frac{1 - P(t_0, t_m)}{\sum_{i=1}^{m} P(t_0, t_i)}\right) \tag{1.24}$$

It can be seen that the cross-currency swap rate is in fact a function of entire yield curve. The pricing of interest rate swaps is also quite similar to the pricing of cross-currency swaps.

1.3.4 Interest Rate Swaps

An interest rate swap is a contract between two counterparties to exchange fixed interest rate payments against floating interest rate payments on a predetermined notional amount for a given period of time. In this regard, the interest rate swaps can be considered as the exchange of fixed-coupon bond with floating-coupon bond with the same maturity. The floating rates used in the interest rate swap contracts are generally Libor rates of the domestic currency. Since the payments of interest rate swap contracts are in the same currency for both fixed and floating rate payers, the notional amounts are not exchanged at the initiation and at the maturity of the swap contract. The fixed rate in an interest rate swap is determined such that the value of the swap at the initiation of the contract is equal to zero. The value of an interest rate swap from the perspective of fixed rate payer can be considered as a long position in a floating rate bond and a short position in a fixed rate bond. Since the value of floating rate bond is equal to its par value at the initiation, the fixed rate is equal to the coupon rate of fixed coupon bond traded at par value. In this respect, the fixed rate in an interest rate swap contract at time t_0 which matures at time t_m is the same as with the cross-currency swap rate contract, which is shown below:

$$c = \left(\frac{1}{\tau}\right)\left(\frac{1 - P(t_0, t_m)}{\sum_{i=1}^{m} P(t_0, t_i)}\right) \tag{1.25}$$

The pricing mechanism of interest rate swaps and cross-currency swaps is exactly the same, which leads to the same fixed rate. In theory, fixed rates in both interest rate swaps and cross-currency swaps are expected to be the same and in practice, it is observed that they tend to move quite closely where there are minor differences mainly due to liquidity conditions and counterparty risk.

1.3.5 Swaptions

Interest rate swaption is an option on an interest-rate swap which gives the holder to enter into the swap at a predetermined date and predetermined rate. Swaptions are used to benefit from favorable interest rate fluctuations, which is a kind of insurance against changes in interest rates. In return for the insurance or protection, the option holder pays a premium at the initiation of the swaption contract. The swaptions are mainly classified based on the types of legs where the option holder has the right to pay or receive the fixed rate; receiver swaption or payer swaption. A receiver (payer) swaption gives the holder the right to enter into a swap contract where the holder receives (pays) the fixed rate. In this respect, the option will be exercised for a receiver (payer) swaption if the swap rates are below (above) the predetermined strike rates. In this respect, a receiver (payer) swaption provides a kind of protection against declining (increasing) swap rates. This can be seen in the following payoff equations for swaptions. The payoff of a European payer swaption at the maturity t_m can be expressed as:

$$Payoff_{Payer\,Swaption} = A(t_m, t_m, t_n) max\left\{fs(t_m, t_m, t_n) - K; 0\right\} \tag{1.26}$$

and the payoff of receiver swaption at the maturity t_m is;

$$Payoff_{Receiver\,Swaption} = A(t_m, t_m, t_n) max\left\{K - fs(t_m, t_m, t_n); 0\right\} \tag{1.27}$$

where K is strike rate, $fs(t_m, t_m, t_n)$ is forward swap rate and $A(t_m, t_m, t_n)$ is the value of annuity at time t_m as

$$A(t_m, t_m, t_n) = \sum_{i=m+1}^{n} P(t_m, t_i)(t_i - t_{i-1}) \tag{1.28}$$

Swaptions must specify not only the maturity of the option but also the tenor of the underlying swap. Therefore, swaption prices or implied volatilities are quoted on a matrix. Since the interest rate swap and cross-currency swap rates are expected to be the same in theory, the methodology for interest rate swaption is also applicable for the cross-currency swaption as well.

tullett prebon TRYUSD Log Vols IBOR 66) MSG Contributor 05:57:15

Zoom 100%

Tullett Prebon -> Int Rate Volatility -> Swaption Skews -> IBOR L Vol -> TRYUSD -> Digital

					Straddle	Strike
Term	-200	-100	-50	-25	ATM	ATM
1) 1M1Y	31.11	34.40	35.88	36.59	37.27	10.094
2) 1M2Y	22.86	26.26	27.80	28.52	29.22	10.729
3) 1M5Y	21.60	25.35	26.98	27.74	28.47	10.818
4) 1M10Y	27.51	28.43	29.12	29.49	29.88	10.543
5) 1Y1Y	44.37	46.38	47.27	47.69	48.10	11.239
6) 1Y2Y					47.75	11.099
7) 3M1Y	27.70	32.09	33.91	34.75	35.55	9.838
8) 3M2Y	23.46	27.63	29.37	30.17	30.94	10.666
9) 3M5Y	25.24		30.96	31.75	32.50	10.771
10) 6M1Y	36.09	39.70	41.24	41.95	42.64	9.937
11) 6M2Y	31.47	34.94	36.43	37.13	37.79	10.678
12) 6M5Y	26.58	30.23	31.77	32.49	33.17	10.761
13) 6M10Y	31.50	32.12	32.62	32.90	33.19	10.473
Term	-200	-100	-50	-25	ATM	ATM

SwapMarker plus

Figure 1.3: Bloomberg Data for Swaptions

1.4 DATA AND METHODOLOGY

This section presents the details about the cross-currency swaption and cross-currency swap data used in the density function estimation. In this regard, this part also provides a summary about the Nelson-Siegel model which is used to obtain the zero-coupon swap rates together with the Black model used in the pricing of cross-currency swaption.

1.4.1 Data

The value of the swaption is a function of several variables including time-to-maturity, strike price, implied volatility and forward swap rate. The data for swaption contracts on USDTRY cross-currency swaps is obtained from Bloomberg. These products are traded in over the counter (OTC) markets and the data is quotation data rather than transaction data. The data set covers the daily observations for the period between January 2013 to November 2019. Since swaption requires the specification about not only about the maturity of the option but also the tenor of the underlying swap, the quotations for the maturity of the swaptions include the range from one-month to 10-years whereas the tenor of the underlying swaps is in the range of three months to 10-years. However, the anecdotal evidence indicates that most of the traded contracts are concentrated on certain maturities such as 1-year and 5-year specifications.

Hence, the study uses swaption contracts in these specifications. The daily data is obtained from Bloomberg, and the biggest price/quotation provider is Tullett Prebon. At-the-money (ATM) convention in the swaption market is quoting ATM forward (ATMF) strike level. The swaption market quotes are in terms of implied volatilities on Black-Model. Lastly, the implied volatility is available for nine different strike levels as shown in the figure. As seen in the figure, ATM forward level for 1Y1Y swaption is 11.24 percent. Implied Black volatility for 1Y1Y swaption is 44.37 percent with strike rate 200 bps less than ATM Forward level which corresponds to 9.24 percent. Besides the implied volatility in swaption, the term structure of forward swap rates is needed for the computation for the annuity factor in swaption pricing. In this regard, the term structure of currency swap rates is obtained through Nelson-Siegel methodology which the details are presented in the following parts.

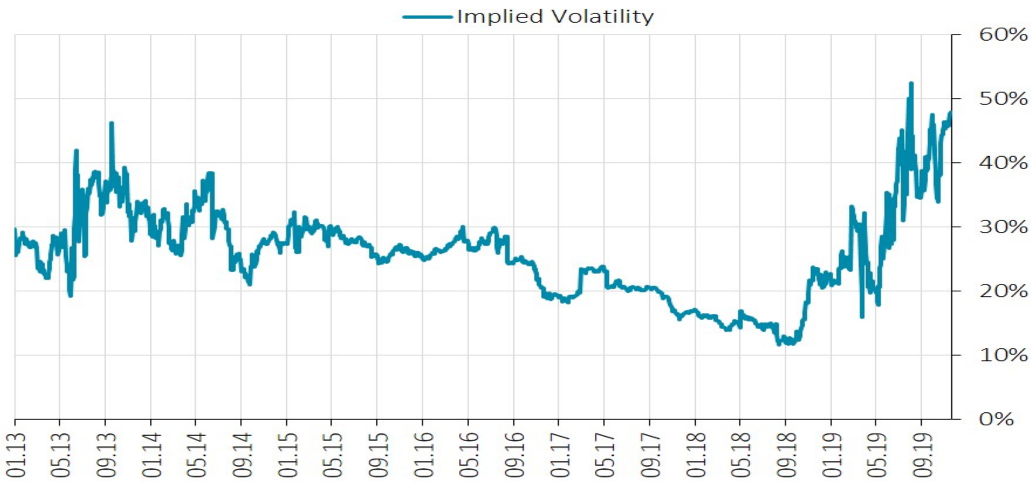


Figure 1.4: Black Implied Volatility for 1Y1Y Swaptions

The study aims to extract the density function for swap rates for different specifications, including 1y1y, 1y10y and 5y5y swaptions. To simplify, 1y10y swaption allows extracting the 10-year swap rates in 1-year forward time. The methodology can be adapted to other swaption methodologies easily, but the study uses the most active part of the swaption curve. The following figures show the Black implied volatility for 1y1y, 5y5y and 1y10y swaption specifications for the period between January 2013 to November 2019. It can be observed that implied volatility levels tend to show fluctuations during the last two years whereas it was relatively stable for the period between 2013 and 2017. Another interesting observation is that implied volatilities for 5y5y specification, 5-year swap rate for 5-year forward tends to be more stable compared to 1y1y and 1y10y swaption specifications, which means that the market expectations tend to be more volatile in the short term.

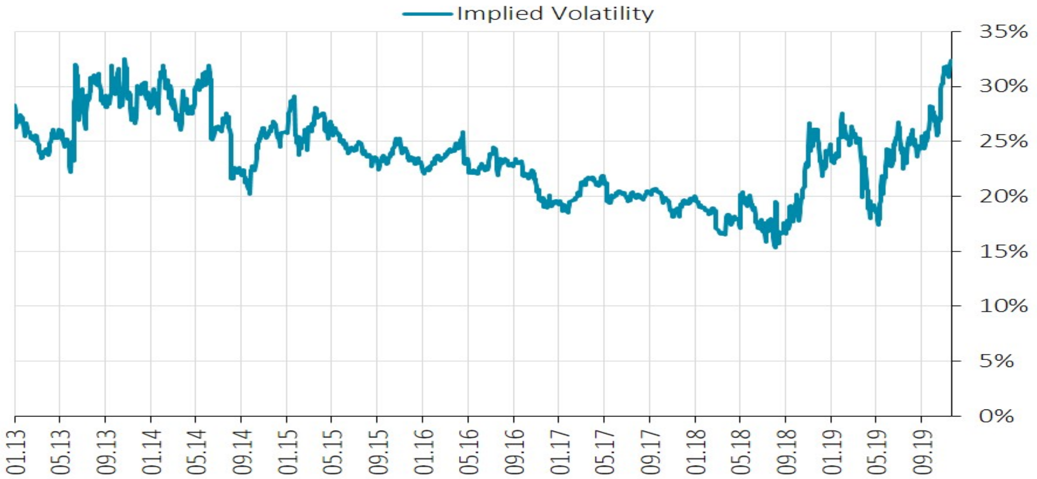


Figure 1.5: Black Implied Volatility for 1Y10Y Swaptions

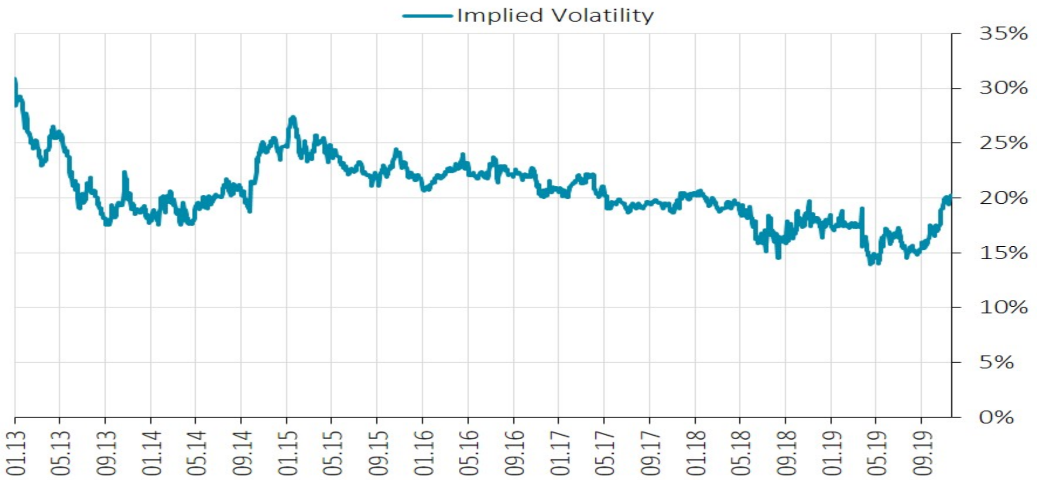


Figure 1.6: Black Implied Volatility for 5Y5Y Swaptions

1.4.2 Methodology

The section firstly presents the Nelson-Siegel model to derive the zero-coupon swap yield curve. Then it introduces the Black model for swaptions and provides the mathematical properties of swaption pricing. Lastly, the transition from swaption prices to density function is summarized.

1.4.2.1 Nelson-Siegel Model

One of the most popular and most commonly used yield curve methodologies is the Nelson-Siegel method. The Nelson-Siegel methodology, which has been extensively utilized by many central banks and market practitioners, assumes that the zero rates can be described explicitly by the following functional form.

$$r(m) = \beta_0 + \beta_1 \left(\frac{1 - e^{-\frac{m}{\tau}}}{-\frac{m}{\tau}} \right) + \beta_2 \left(\frac{1 - e^{-\frac{m}{\tau}}}{-\frac{m}{\tau}} - e^{-\frac{m}{\tau}} \right) \tag{1.29}$$

where m denotes the time to maturity, β, τ are the parameter set to be estimated. The model is quite flexible in terms of fitting negatively/positively sloped and humped yield curves. Additionally, the model-in-fit tends to provide robust and accurate results so that it can also be used to find out the fair price for the securities. Besides, the parameters of the model characterize the yield curve in such a way that β_1 and β_2 stand for the level and slope of the yield curve, respectively whereas β_3 is interpreted as the curvature of the yield curve.

Due to the flexibility and good of fit data, the Nelson-Siegel methodology is implemented to find out the zero-coupon swap rates for the value of annuity in the Black swaption pricing formula. In this regard, the common methodology to find out the optimal parameter set is to minimize the squared difference between weighted price errors, which are obtained by the difference between actual and fitted prices.

In this regard, the Bloomberg swap quotation data at maturities from 1 month to 120 months are used to find out zero-coupon currency swap rates for each day. As mentioned above, currency swap rates are quoted as par-rates rather than zero-coupon rates. In order to find out the zero-coupon swap rate, the fixed leg of the cross-currency swap is treated as a fixed coupon bond with coupon rate equal to the currency swap rate and the value of the fixed leg of the cross-currency swap rate is equal to the par value. This comes from the fact that the value of a floating bond is equal to par value. In order to find out Nelson-Siegel parameters for cross-currency swap rates, the difference between the actual prices and fitted prices weighted by the inverse of Macaulay duration of the fixed leg of the swap is minimized. The objective function can be expressed as follows:

$$min_\beta \sum_{i=1}^{L} \left(\frac{P_{t_0}^{i} - P_{t_0}^{i,fitted}}{D_{t_0}^{i}} \right)^2 \tag{1.30}$$

where $D_{t_0}^{i}$ stands for the Macaulay duration of fixed-leg of the cross-currency swap. $P_{t_0}^{i}$ and $P_{t_0}^{i,fitted}$ stand for the actual and fitted prices for bond i, respectively. L denotes the number of bonds used in the estimation.

1.4.2.2 Black Model

The Black model is one of the convenient models to value financial derivatives whose value depends on the level of interest rates but not the shape of the entire yield curve. In this regard, it is widely implemented for financial derivatives such as European interest rate options such as caplets, floorlets, bond options, futures options and swaptions. Due to the simplicity and practical use of the model, most of the traded products in the fixed income securities are quoted in terms of Black model implied volatilities. In this regard, the traders use it a lot to quote their prices directly using the Black model and hedge their interest rate exposure using Black delta.

The main property of the Black model is that it does not assume that the underlying variable satisfies any particular stochastic dynamics, it only states that the final distribution of the underlying is known. In the case of swaptions, the underlying interest rate, forward swap rate, is assumed to be lognormally distributed. In this regard, the model requires only the estimation of the volatility of forward swap rates. Since the model assumes that the forward swap rate is lognormally distributed, it allows an explicit formula for pricing European options.

Under the Black model, the forward swap rate is assumed to have a lognormal distribution under the swap measure. Then forward swap rate at $t_m$can be expressed as

$$fs(t_m, t_m, t_n) = fs(t, t_m, t_n) e^{\sigma\epsilon\sqrt{t_m - t} - \frac{1}{2}\sigma^2(t_m - t)} \tag{1.31}$$

where ϵ and σ stand for standard normal variable and the Black volatility, respectively. The volatility parameter does not necessarily say anything about the standard deviation of the logarithm of underlying at times other than T. The volatility parameter is expressed as the lognormal annualized volatility of the underlying rate. The Black model for swaptions treats the par rate since it is the rate at which the present value of the floating legs is equal to the present value of the fixed legs. In this regard, the valuation of swaptions under the Black model considers only the excess of the par rate over the strike rate, which is a kind of call option on the rate.

1.4.2.3 Swaption Pricing

This section provides the methodology behind the swaption pricing under the Black model using the swap measure. In this regard, the swaption pricing formula for payer swap is illustrated here. The payer-receiver swaption parity can be used to find out the price of receiver swaption. Fixed rate payer position in an interest rate swap can be replicated by a long position on floating rate bond and short position on fixed rate bond. In this regard, the value of a fixed rate payer swap at time t_m , $V_{swap}(t_m, t_n)$, can be expressed as follows:

$$V_{swap}(t_m, t_n) = V_{floating}(t_m, t_n) - V_{fixed}(t_m, t_n) \tag{1.32}$$

where t_m is starting time of swap and t_n is the maturity of swap. Now consider the value of the swaption at maturity, t_m, with the underlying swap expiring at t_n.

$$V_{swaption}(t_m, t_m, t_n) = [V_{swap}(t_m, t_n, fs(t_m, t_m, t_n)) - V_{swap}(t_m, t_n, K)]^+ \tag{1.33}$$

where $V_{swap}(t_m, t_n, fs)$ is the value of swap starting at t_m, ending at $t_n$with swap rate fs and $V_{swap}(t_m, t_n, K)$ is the value of swap starting at t_m, ending at t_n with swap rate K.

$$V_{swaption}(t_m, t_m, t_n) = max \left\{V_{swap}(t_m, t_n, fs(t_m, t_m, t_n)) - V_{swap}(t_m, t_n, K), 0\right\} \tag{1.34}$$

Let A denote the annuity and $P(t_m, t_i)$ is price of a bond at $t_m$with maturity t_i.

$$A(t_m, t_m, t_n) = \sum_{i=m+1}^{n} P(t_m, t_i)(t_i - t_{i-1}) \tag{1.35}$$

Then the value of swaption at the maturity can be written as factoring out annuity.

$$V_{swaption}(t_m, t_m, t_n) = max \left\{fs(t_m, t_m, t_n) - K, 0\right\} A(t_m, t_m, t_n) \tag{1.36}$$

One of the approaches to find out the value of the swaption at time t is to compute the expectation of V under Q-measure where $t < t_m$., which is described as follows:

$$V_{swaption}(t, t_m, t_n) = \mathbb{E}_t^Q \left[A(t_m, t_m, t_n) e^{-\int_t^{tm} r(s)ds} max \left\{ fs(t_m, t_m, t_n) - K; 0 \right\} \right] \tag{1.37}$$

or in integral form

$$V_{swaption}(t, t_m, t_n) = \int_{-\infty}^{\infty} A(t_m, t_m, t_n) e^{-\int_t^{tm} r(s)ds} max \left\{ fs(t_m, t_m, t_n) - K, 0 \right\} dP^Q \tag{1.38}$$

Finding out the price of swaption under the risk-neutral measure requires computing the joint distribution between the stochastic discount factor, forward swap rate and annuity, which makes calculations complicated. However, it is possible to price the swaption using numeraire change. In this regard, right side of the equation is multiplied and divided by $A(t, t_m, t_n)$ to change probability measure.

$$\int_{-\infty}^{\infty} \frac{A(t_m, t_m, t_n)}{A(t, t_m, t_n)} e^{-\int_t^{tm} r(s)ds} dP^Q = \int_{-\infty}^{\infty} \frac{\sum_{i=m+1}^{n} P(t_m, t_i) e^{-\int_t^{tm} r(s)ds}}{A(t, t_m, t_n)} dP^Q \tag{1.39}$$

It is known that under the risk-neutral measure the price of a zero-coupon bond price can be found as follows:

$$\int_{-\infty}^{\infty} P(t_m, t_i) e^{-\int_t^{tm} r(s)ds} dP^Q = P(t, t_i) \tag{1.40}$$

Since the following holds under the risk-neutral measure,

$$\frac{\sum_{i=m+1}^{n} P(t, t_i)}{A(t, t_m, t_n)} = \int_{-\infty}^{\infty} dP^A = 1 \tag{1.41}$$

it is possible to define the new probability measure, which is called as swap measure.

$$dP^A = dP^Q \frac{A(t_m, t_m, t_n)}{A(t, t_m, t_n)} \frac{M(t)}{M(t_m)} \tag{1.42}$$

Therefore, the value of swaption under the probability measure A can be stated as follows:

$$V_{swaption}(t, t_m, t_n) = \int_{-\infty}^{\infty} A(t, t_m, t_n) max \left\{ fs(t_m, t_m, t_n) - K, 0 \right\} dP^A \tag{1.43}$$

Since $A(t, t_m, t_n)$ is t-measurable it can be factored out the expectation operator.

$$\begin{aligned} V_{swaption}(t, t_m, t_n) &= A(t, t_m, t_n) \int_{-\infty}^{\infty} max \left\{ fs(t_m, t_m, t_n) - K; 0 \right\} dP^A \\ &= A(t, t_m, t_n) \mathbb{E}_t^A \left[fs(t_m, t_m, t_n) - K \right]^+ \end{aligned} \tag{1.44}$$

As can be seen from the equation, the only stochastic variable under the swap measure is the forward swap rate. The Black model assumes that forward swap rate is lognormally distributed under swap measure. Then forward swap rate at t_m can be expressed as follows:

$$fs(t_m, t_m, t_n) = fs(t, t_m, t_n)e^{\sigma\epsilon\sqrt{t_m - t} - \frac{1}{2}\sigma^2(t_m - t)} \tag{1.45}$$

where σ is the Black volatility. The good side of the Black model is that it only requires the volatility of forward swap rate.

$V_{swaption}(t, t_m, t_n)$ is defined as follows:

$$A(t, t_m, t_n) \int_{-\infty}^{\infty} \left[\left(fs(t, t_m, t_n)e^{\sigma\epsilon\sqrt{t_m - t} - \frac{1}{2}\sigma^2(t_m - t)} - K \right)^+ e^{-\frac{1}{2}\epsilon^2} \frac{1}{\sqrt{2\pi}} d\epsilon \right] \tag{1.46}$$

For $fs(t, t_m, t_n)e^{\sigma\epsilon\sqrt{t_m - t} - \frac{1}{2}\sigma^2(t_m - t)} > K$;

$$\epsilon > \frac{\ln(\frac{K}{fs(t, t_m, t_n)}) + \frac{1}{2}\sigma^2(t_m - t)}{\sigma\sqrt{(t_m - t)}} = -\left[\frac{\ln(\frac{fs(t, t_m, t_n)}{K}) - \frac{1}{2}\sigma^2(t_m - t)}{\sigma\sqrt{(t_m - t)}} \right] \tag{1.47}$$

The term between brackets on the right hand side of equation called as d_2. Hence $\epsilon > -d_2$, equation becomes

$$\begin{aligned} V_{swaption}(t, t_m, t_n) = A(t, t_m, t_n) &\int_{-d_2}^{\infty} fs(t, t_m, t_n)e^{\sigma\epsilon\sqrt{t_m - t} - \frac{1}{2}\sigma^2(t_m - t) - \frac{1}{2}\epsilon^2} \frac{1}{\sqrt{2\pi}} d\epsilon \\ &- A(t, t_m, t_n)K \int_{-d_2}^{\infty} e^{-\frac{1}{2}\epsilon^2} \frac{1}{\sqrt{2\pi}} d\epsilon \end{aligned} \tag{1.48}$$

$\int_{-d_2}^{\infty} e^{-\frac{1}{2}\epsilon^2} \frac{1}{\sqrt{2\pi}} d\epsilon$ is equal to $N(d_2)$. Define $\epsilon' = \epsilon - \sigma\sqrt{t_m - t}$.

Then, $V_{swaption}(t, t_m, t_n)$ is defined as follows:

$$A(t, t_m, t_n) \left(\int_{-d_1}^{\infty} fs(t, t_m, t_n)e^{-\frac{1}{2}\epsilon'^2} \frac{1}{\sqrt{2\pi}} d\epsilon' - K \int_{-d_2}^{\infty} e^{-\frac{1}{2}\epsilon^2} \frac{1}{\sqrt{2\pi}} d\epsilon \right) \tag{1.49}$$

where $d_1 = -\frac{\ln(\frac{fs(t, t_m, t_n)}{K}) + \frac{1}{2}\sigma^2(t_m - t)}{\sigma\sqrt{(t_m - t)}}$.

So the value of the swaption by the Black model at time t is

$$V_{swaption}(t, t_m, t_n) = A(t, t_m, t_n) \left[fs(t, t_m, t_n)N(d_1) - KN(d_2) \right] \tag{1.50}$$

$$d_1 = -\left[\frac{\ln(\frac{fs(t,t_m,t_n)}{K}) + \frac{1}{2}\sigma^2(t_m - t)}{\sigma\sqrt{(t_m - t)}}\right] \tag{1.51}$$

$$d_2 = -\left[\frac{\ln(\frac{fs(t,t_m,t_n)}{K}) - \frac{1}{2}\sigma^2(t_m - t)}{\sigma\sqrt{(t_m - t)}}\right] \tag{1.52}$$

1.4.2.4 Breeden-Litzenberger Method

The extraction of density function from the option prices traded in the market is based on the Breeden-Litzenberger study. They state that the second mathematical derivative of the option price with respect to strike price leads to the formation of density function under the relevant measure. In other words, if the pricing scheme is under the risk-neutral measure, the density obtained from option prices is risk-neutral measures. For the case of swaptions, the value of payer swaption in a swap measure is expressed as follows:

$$\begin{aligned} PS(t, t_m, t_n, K) &= A(t, t_m, t_n) \int_{-\infty}^{\infty} max\left\{fs(t_m, t_m, t_n) - K; 0\right\} g(fs) dfs \\ PS(t, t_m, t_n, K) &= A(t, t_m, t_n) \int_{K}^{\infty} (fs(t_m, t_m, t_n) - K) g(fs) dfs \end{aligned} \tag{1.53}$$

where $g(fs)$ is the density function under annuity measure, K is the strike price and $fs(t_m, t_m, t_n)$ is the forward swap rate at time t_m starting at t_m and maturing at t_n. The first derivative of payer swaption with respect to strike price leads to the following equation.

$$\frac{\partial PS(t, t_m, t_n, K)}{\partial K} = -A(t, t_m, t_n)\left[\int_{K}^{\infty} g(fs) dfs\right] \tag{1.54}$$

The second derivative with respect to K gives the relation between density function and option prices.

$$\frac{\partial^2 PS(t, t_m, t_n, K)}{\partial K^2} = A(t, t_m, t_n) g(fs)|_{K}^{\infty} = A(t, t_m, t_n) g(K) \tag{1.55}$$

Then; the density function can be written as follows. In this respect, using the option prices traded in the market it is plausible to extract densities.

$$g(K) = \frac{\partial^2 PS(t, t_m, t_n, K)}{\partial K^2} \frac{1}{A(t, t_m, t_n)} \tag{1.56}$$

1.5 EMPIRICAL FINDINGS

This section presents the empirical findings for swaption implied density function. Firstly, the following figure summarizes the steps for the estimation of the density function of swap rates.

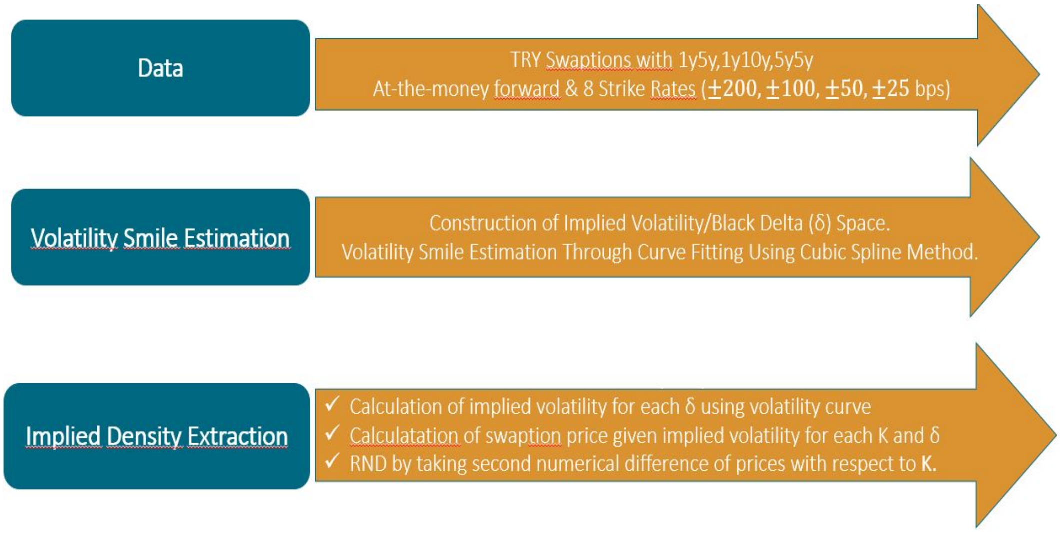


Figure 1.7: Roadmap for Swaption Implied Density Function Estimation

Swaption implied density function estimation requires the implied Black implied volatilities of plain-vanilla swaptions with different deltas. However, as mentioned above, the strike prices are quoted based on the difference with respect to the forward swap rate. The Bloomberg provides quotations 200, 100, 50, 25 basis points around the forward swap rate, which means that there are 9 quotations for swaptions. In this regard, the first step in the density function estimation is to translate strike rates into deltas. In this respect, Black model is used to find out implied deltas. It should be noted that although the strike rates are always the same around the forward swap rates, the Black model implied deltas differ each day, which makes swaption quotations different from FX option implied quotations where they are quoted based on deltas. To transform strike rates into Black model implied deltas, the following equation is solved numerically.

$$\delta = N\left(-\left[\frac{ln(\frac{fs(t,t_m,t_n)}{K}) + \frac{1}{2}\sigma^2(t_m - t)}{\sigma\sqrt{(t_m - t)}}\right]\right) A(t, t_m, t_n) \tag{1.57}$$

The daily evolution of the Black delta values of 1 year swaptions with 1 and 10 year swap maturity and 5 year swaption with 5 year swap maturity are calculated for the given strike prices. The results for the observation period from January 2013 to November 2019 identify the evolution of market interest rates as well as markets' pricing behavior in the respective maturities. For instance, the rise in spot interest rates in 2018 led to increase in the delta values for swaptions with strike levels, which reflect the level shift in the expectations.

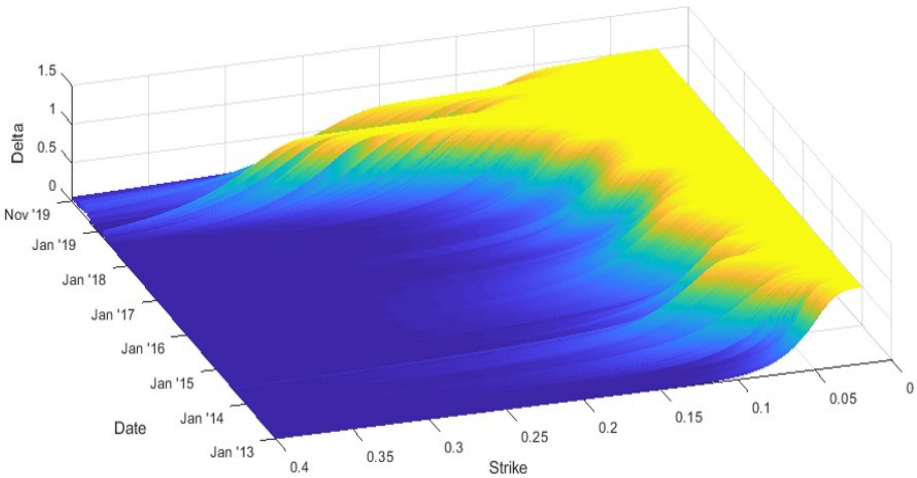


Figure 1.8: Black Delta for Strike Rates in 1Y1Y Payer Swaptions

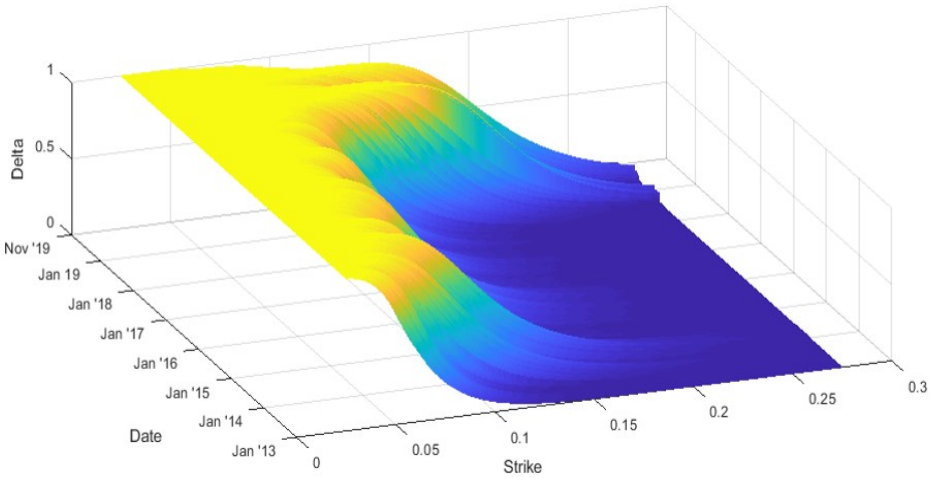


Figure 1.9: Black Delta for Strike Rates in 1Y10Y Payer Swaptions

Using the implied volatilities for swaptions it is possible to obtain swaption prices using Black model. Since the market quotations are in Black implied volatility, the calculation shows the swaption premium demanded in the market. In this regard, the following figures clearly demonstrate the periods of market stress and deteriorations in market expectations about the future interest rates. For instance, following the Fed's tapering of its quantitative easing monetary policy announcement in May 2013, there had been substantial amount of capital outflows from the emerging market countries. During this period capital flows to Turkey also weakened and volatility in the financial markets heightened. When negative global financing conditions were combined with the relative deterioration in the inflation of the Turkish economy both the CBRT policy interest rates and market interest rates increased. And as it can be clearly seen from the figures markets' pricing of higher interest rate levels also elevated. Although it was relatively milder than the post Taper Tantrum period another episode of unfavorable global financial conditions and deterioration in Turkish macroeconomic indicators led to significant amount of capital outflows and turmoil in domestic financial markets in the third quarter of 2018. The rise in the level of swaption premiums after August 2018 is striking where the premiums increased substantially during that period and subside in the following months reflecting the limited normalization of market sentiment.

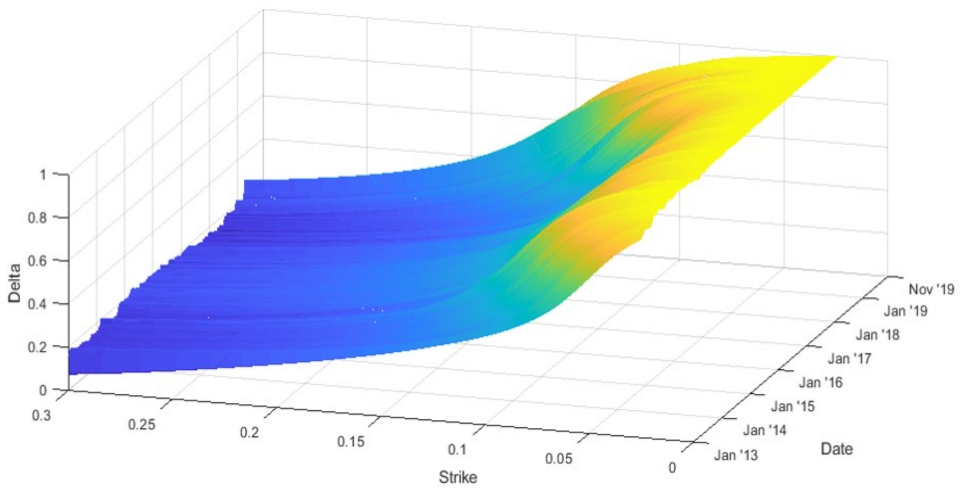


Figure 1.10: Black Delta for Strike Rates in 5Y5Y Payer Swaptions

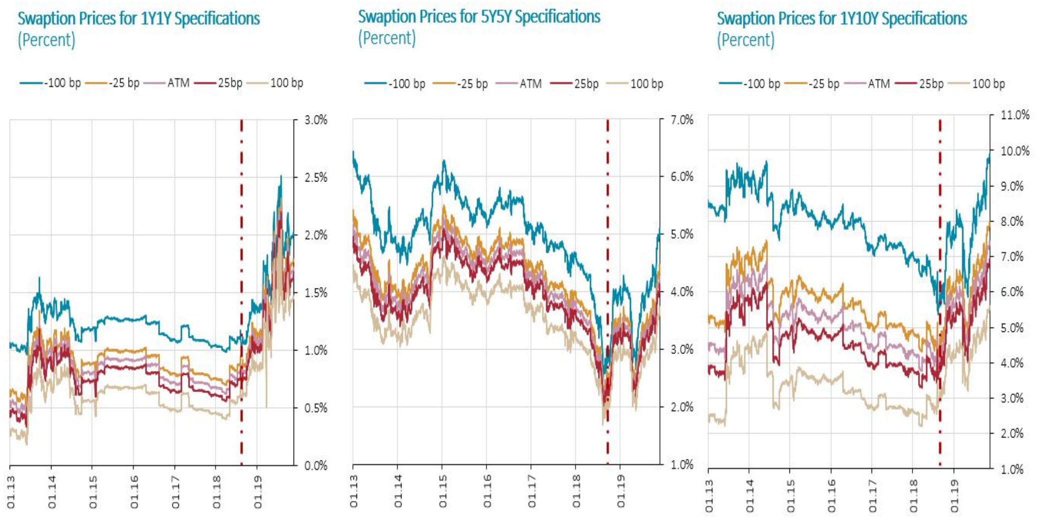


Figure 1.11: Swaption Prices for Different Specifications

1.5.1 Volatility Smile

Through the translation of strike rates into deltas, the characterization of implied Black volatility/Black delta space is completed, which is an important step since the volatility-delta pairs are used for density function estimation. The next step is to estimate the implied volatility curve. In this regard, the implied volatility curve is estimated through clamped cubic spline methodology based on the following equation. Using implied volatilities for all nine deltas available for each day, the clamped-cubic spline approach produces a fitted implied volatility that matches the data almost perfectly where the clamped property ensures that extrapolated implied volatilities for deep out-of-the-money or deep-in-the-money options are equalized to the implied volatilities of the options with the closest deltas in the input data. In this regard, the implied volatility is a function of Black delta in order to capture the volatility smile.

$$\sigma_t^{malz}(\delta) = \gamma_1 + \gamma_2(\delta - \delta(atmf_t)) + \gamma_3(\delta - \delta(atmf_t))^2 + \gamma_4(\delta - \delta(atmf_t))^3 \quad (1.58)$$

Using the estimated coefficients of the cubic polynomial function, the implied volatility levels are calculated for a range of strike prices. The relationship between the strike rates and implied volatility is illustrated in the following figures for the period between January 2013 and November 2019. The parameter optimization results show that the difference between the fitted implied volatilities and the actual implied volatilities is quite negligible. In order to work with the largest possible strike rate range, on the upside the implied volatility curve is extrapolated up to 50 percent for some of the cases. On the other hand, as a natural constraint of the applied Black Model, zero percent constitutes a natural lower bound for the strike price range.

For all of the swaption contracts, there exists a persistent positive difference between the implied volatilities of out-of-the-money or in-the-money options and at-the-money options and this observation is more significant for the 5 year swaptions with 5 year swap maturity. These positive differences indicate that there exists volatility smile, especially for longer maturity options. Inspection of the movements of the volatility curves throughout the time provides information regarding risk perception of market participants.

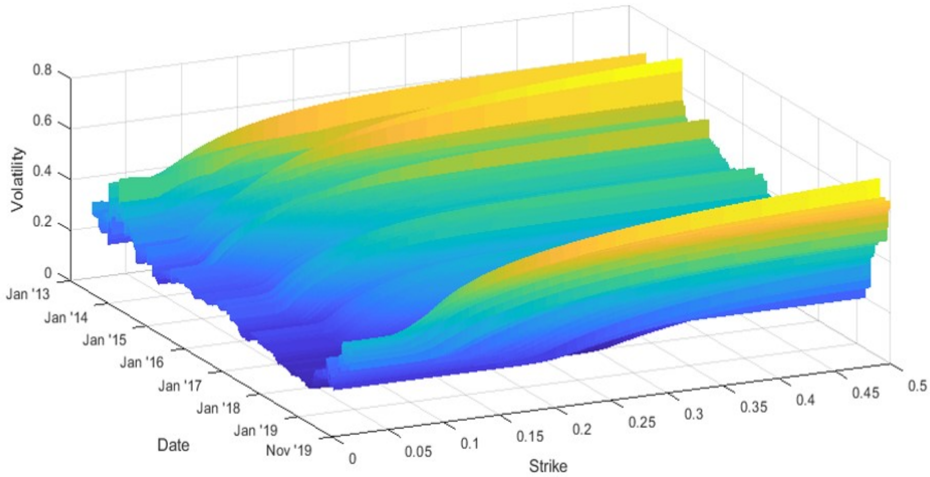


Figure 1.12: Implied Volatility Surface in 1Y1Y Swaptions

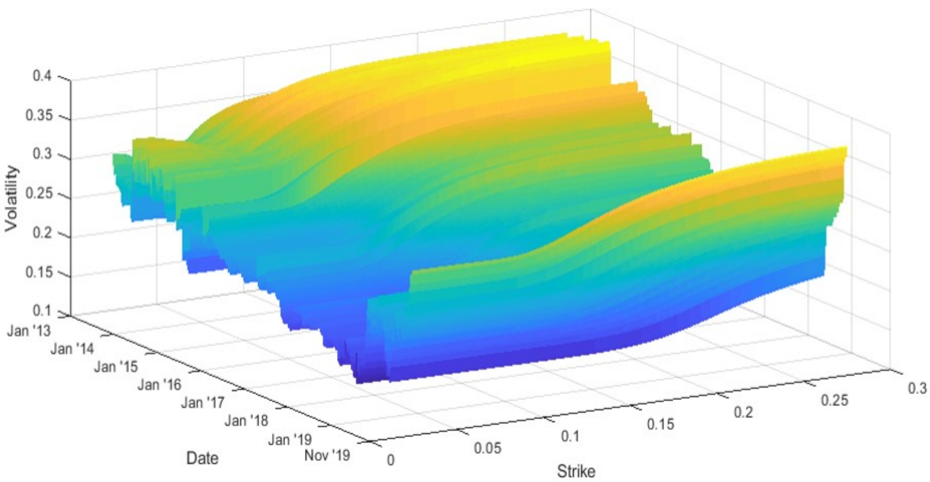


Figure 1.13: Implied Volatility Surface in 1Y10Y Swaptions

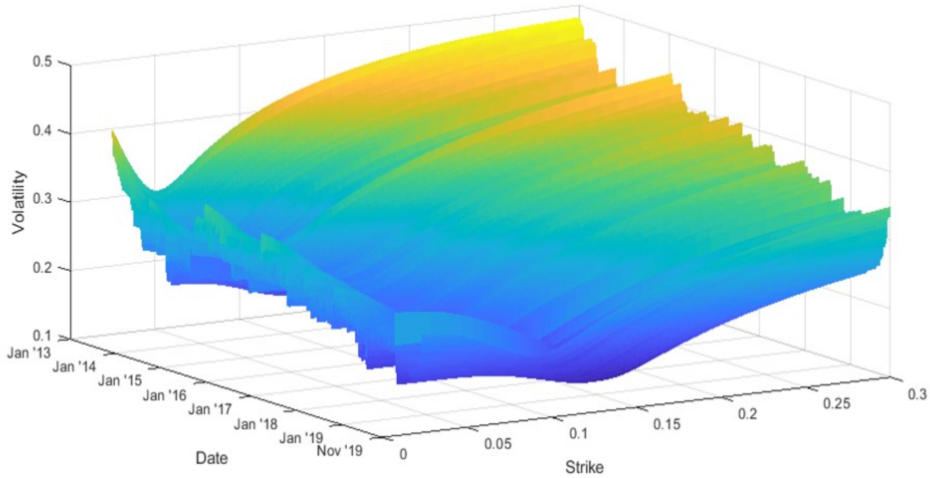


Figure 1.14: Implied Volatility Surface in 5Y5Y Swaptions

1.5.2 Swaption Implied Density Functions

In this regard, for each strike rate level which the methodology aims to assign a probability, strike rates are converted into relevant deltas and implied volatility is calculated using the estimated the volatility-delta pair. After the estimation of implied volatility, the next step is to calculate the implied payer swaption prices using Black option model. In this regard, swaptions with 1 and 5 year maturity on 1, 10 and 5 year swaps are valued using the implied volatilities estimated with cubic polynomial functions. The density function estimation requires the second derivative of payer swaption price with respect to strike price. In this regard, the incremental changes in strike rates should be small so that the numerical differentiation gives precise results. Using this methodology, swaption implied density functions are obtained for 1y1y,1y10y and 5y5y swaption specifications.

The density function estimated for three representative dates in 2019 are displayed in the following figures. Firstly looking to the developments in 1Y1Y swaption density it can concluded that shorter term expectations are more sensitive to current market interest rates. Since the interest rates were relatively high at the beginning of 2019 and have declined gradually throughout the year, market expected annual rates had a higher mean and median level in January and also the probability assigned to realization of higher interest rate in one year time was also high. Following the downward trend in the market rates the density functions also shifted to left during the year and likelihood of lower future interest rates increased. However, for 1Y10 and 5Y5Y swaptions the patterns are a little different. Especially, for 5 year maturity swaption it seems that long term interest rate pricings seem more dependent on macroeconomic indicators, since a significant leftward shift in densities have observed in late 2019 when the macroeconomic back drop have become more supportive.

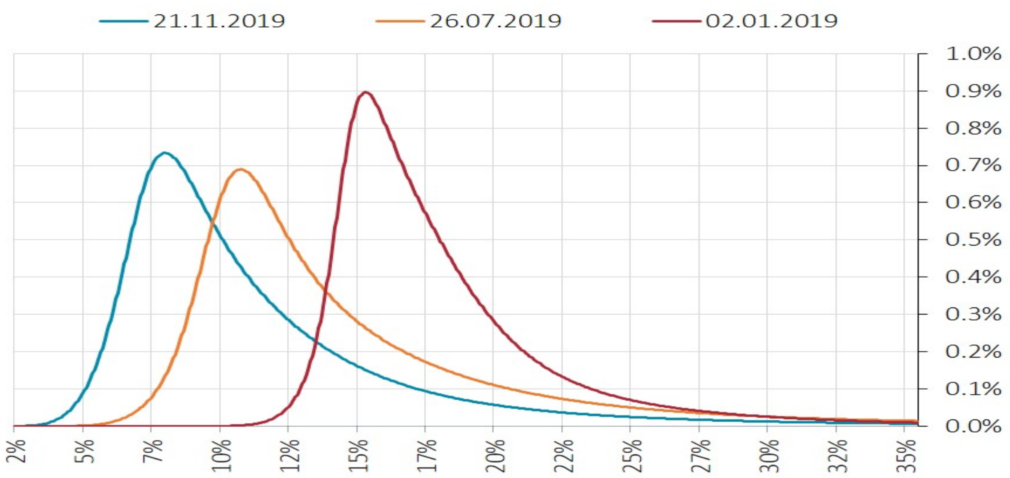


Figure 1.15: Comparison of Swaption Implied Density Functions for 1Y1Y Swaptions

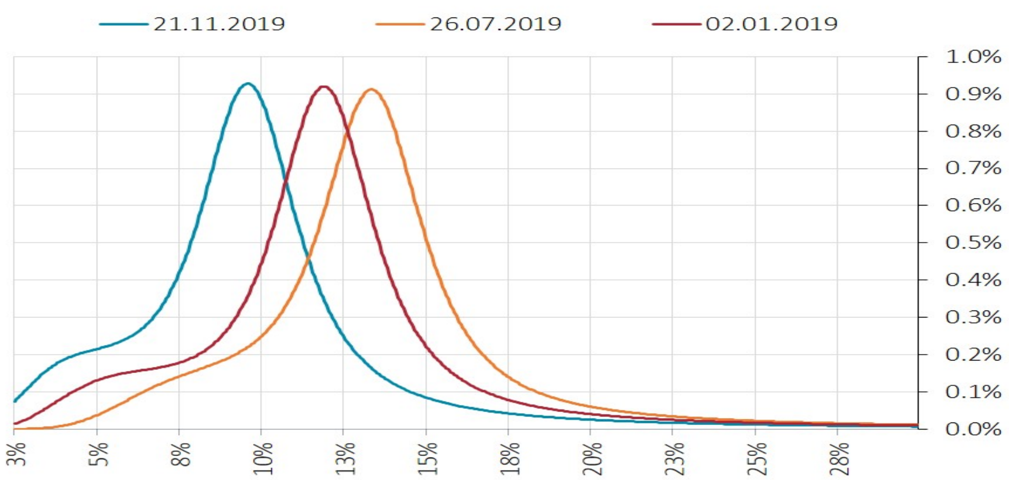


Figure 1.16: Comparison of Swaption Implied Density Functions for 1Y10Y Swaptions

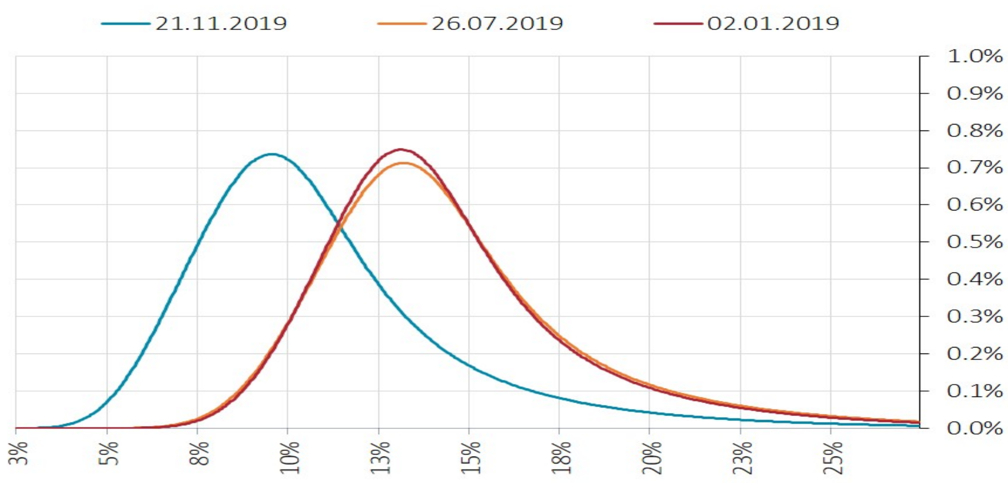


Figure 1.17: Comparison of Swaption Implied Density Functions for 5Y5Y Swaptions

The historical development of the density functions are presented in the following figures. The first observation is there had been a persistent rightward trend until the last quarter of 2018 which indicates that the market participants pricing of higher future interest rate levels both in the short and in the medium term. However, from thereafter swaption implied densities for all of the swaption contracts have started to shift to the left, reflecting an improvement in the expectations for the funding costs. Besides, the dispersion of the distributions widens sharply after 2018, which reflects the dramatic rise in the uncertainty regarding the market expectations in this period. Accordingly, the probability of extreme values for future interest rates increase during the aforementioned risk-off periods, which is visible through the extended tails of density functions.

The changes in the density functions might also be related to the changing behavior of liquidity for the underlying asset cross-currency swap and swaption as well. Since there is a regulation which limits the FX swap and cross-currency swap positions with offshore counterparties, which creates decline in the liquidity for a temporary period, the shifts in the density functions might be related to the changing conditions in liquidity rather than shift in expectations. In this regard, the density functions implicitly covers the liquidity conditions and changes in risk-premia as well as liquidity conditions.

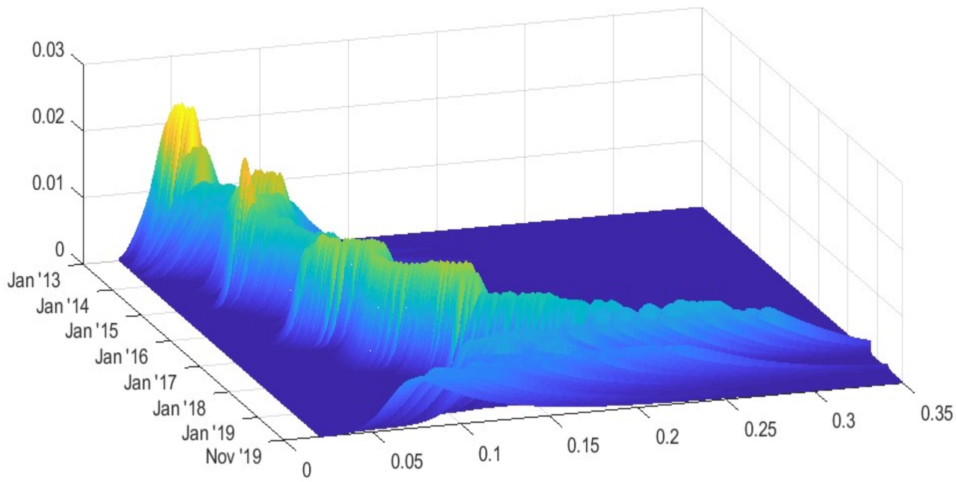


Figure 1.18: Swaption Implied Density Function in 1Y1Y Swaptions

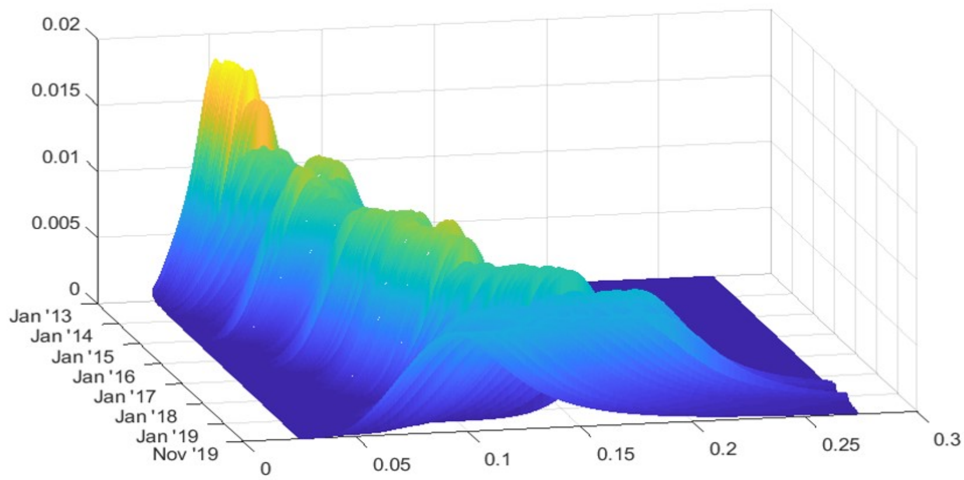


Figure 1.19: Swaption Implied Density Function in 1Y10Y Swaptions

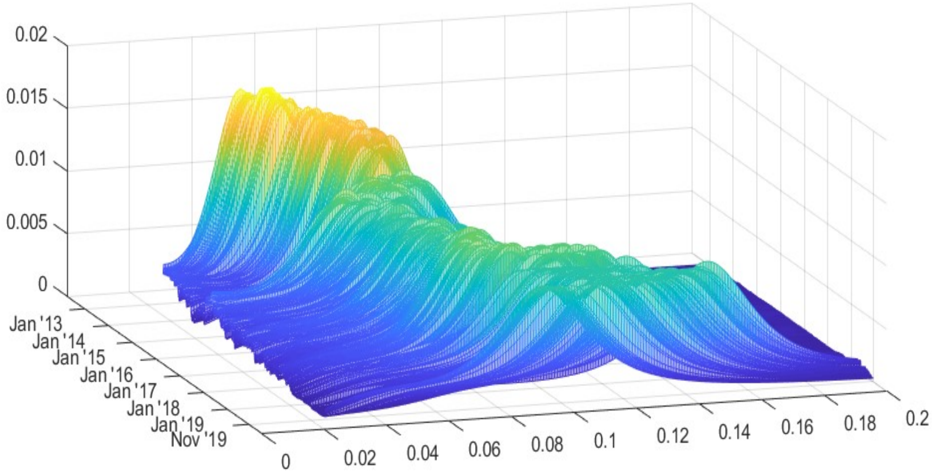
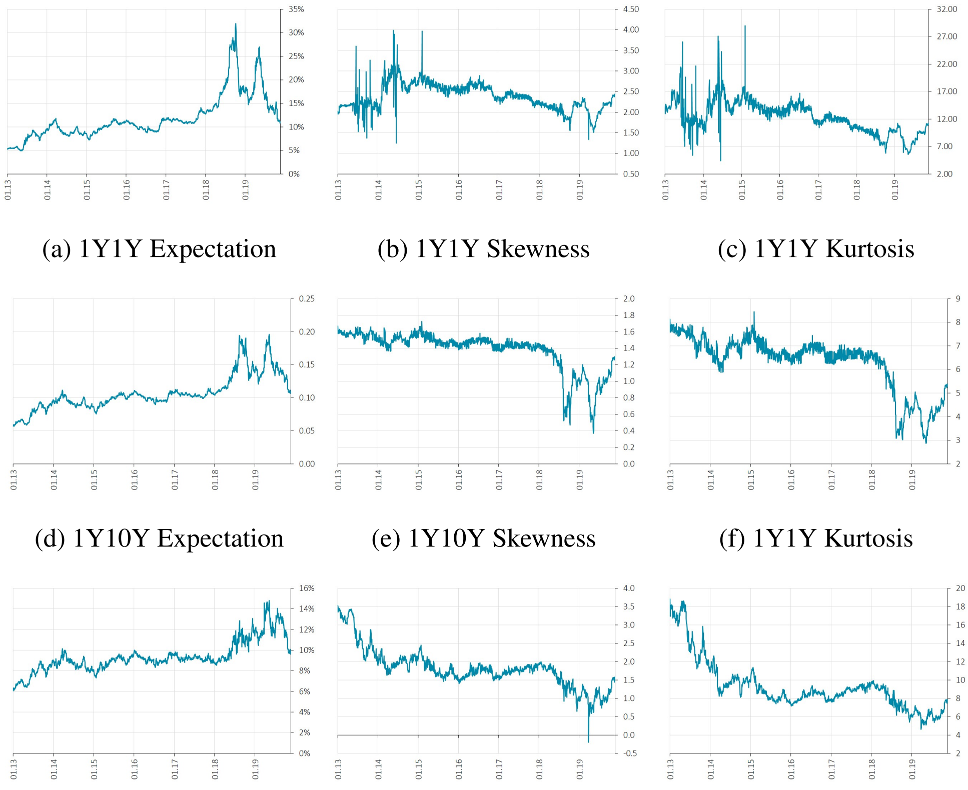


Figure 1.20: Swaption Implied Density Function in 5Y5Y Swaptions

These inferences are also supported by the evolution of first four moments of the the swaption implied density functions. The upward trend in the expected values of the density functions from the second half of 2013 throughout the following year was followed by a relatively stable period around 10 percent for the following four years. However, as it is clearly represented that the market expectations deteriorated in the following period, volatility have heightened and the level of uncertainty have elevated.

The density function can also be used to price swaptions with different characteristics. Consider the case that an investor has an interest in a 1y10y swaption with a strike price not quoted in the market. In this regard, swaption implied density estimation might provide a useful solution due to the fact that it is possible to obtain probabilities for forward swap rates under annuity measure. Since the swaption price is equal to the multiplication of annuity (a kind of discounting for multiple cash flows) with the swaption payoff under the annuity measure, swaption implied density estimation provides a solution consistent with market prices.

Another equilavent solution is to use implied volatility/delta relation to find out the relevant delta to find out the implied volatility. Then the next step is to obtain implied volatility relevant to the delta level and calculate the swaption price. Similarly, the methodology can be extended to the cases where the swaption maturity is not quoted. In this regard, one of the most plausible ways is to bootstrap the swaption implied volatility across the time-to-maturity. In other words, suppose that investor is interested in a 1,5y10y swaption. In this regard, since the market quotes for 1y10y,2y10y,3y10y,5y10y and 10y10y are available, it is possible to bootstrap the 1,5y10y swaption implied volatility fitting a cubic polynomial curve. Then it is possible to price swaptions with different strike rates. This methodology in this regard is quite useful and extendable to many specifications.



(a) 1Y1Y Expectation (b) 1Y1Y Skewness (c) 1Y1Y Kurtosis

(d) 1Y10Y Expectation (e) 1Y10Y Skewness (f) 1Y1Y Kurtosis

(g) 5Y5Y Expectation (h) 5Y5Y Skewness (i) 5Y5Y Kurtosis

Figure 1.21: Density Moments Swaption Specifications

1.6 CONCLUSION

Interest rate derivatives are the most commonly traded securities in the global financial markets. Given the increasing size of fixed-income securities and complexity in the securities issued, extraction of interest rate expectations from the traded securities provides crucial information for portfolio managers, investors, risk managers and policymakers in terms of formation of trade recommendations, valuation of fixed-income securities including financial derivatives, inferring market assessment, estimating potential risk of the portfolios. In this regard, the fixed income products with option characteristics are used to obtain density functions for interest rates so that it is possible to interpret the market's assessment of the degree of uncertainty or the direction of expected changes in interest rates. Given the wide range of areas applicable for policymakers and portfolio managers, extraction of density functions for the interest rates consistent with the market prices is crucial. In this regard, this study implements one of the most popular and widely used methodologies, Malz methodology, to extract the swaption implied density functions for Turkish lira swap rates. This is the first study that extracts the the density functions for interest rates in Turkey, which uses cross-currency swaptions data from the period between January 2013 to November 2019.

The methodology implemented in the study involves many steps including from translation of strike rates into Black deltas, estimating an implied volatility curve and pricing swaptions for a wide range of strike rates and obtaining density function. The empirical findings for Turkish lira swap rates for 1y1y, 1y10y and 5y5y specifications provide critical observations about the changing market expectations about the interest rates, the probability of extreme increases or decreases priced in the cross-currency swaptions. The density functions for the period between January 2013 and November 2019 show that the investor's expectations tend to deteriorate after 2018 where the density functions shifted to the higher levels and tails of the density functions become flattered. The shifts in the density functions might also be attributed to the changing liquidity conditions in cross-currency swap and cross-currency swaption markets as well. Additionally, the swaption implied density functions provide important clues for policymakers and fixed-income portfolio managers in terms of extracting interest rate expectations and forming trade strategies. Given the size of interest rate dependent securities it is crucial to figure out the possible effect of deteriorating outlook in the interest rates on the balance sheets of banking sector. Although the swaptions might not be used by the financial institutions directly, the information about the probabilities of relevant interest rates provide useful tool for risk managers. Besides that, swaption implied density functions can be used to price similar products since it allows to obtain state prices for each possible interest rate level, which can be used to price other fixed-income securities with relevant adjustments. Additionally, the information about the possibilities about different states also provide signals for policymakers about the recession fears or inflation concerns.

In this regards, the swaption implied density function is expected to contribute to the literature in terms of extracting interest rate expectations for different maturities and pricing and hedging tool for other fixed-income positions. Since the density functions have a wide range of areas that can be used, it is believed to be useful for policymakers in terms of extracting the interest rate expectations and analyzing the systemic risks for the financial stability. Additionally, since the density functions include liquidity conditions of the relevant interest rate markets, it might be useful tool to measure the liquidity premium priced into the financial securities.

CHAPTER 2

OPTION PRICING IN INTEREST RATE DERIVATIVES

2.1 INTRODUCTION

The trading of interest rate derivatives and fixed-income securities has been substantially increased in the global financial markets. The changing expectations about the interest rates in developed and developing countries have increased the demand for hedging and speculative trading for interest rate derivatives. Besides the growing issuance of new fixed-income securities by corporates and sovereigns has been another factor for the increase in trading of interest rate derivatives. Although most of the traded interest rate derivatives are concentrated on plain-vanilla products such as interest rate swaps, forward rate agreements or cross-currency swaps, the trading volume of interest rate derivatives with option characteristics has been growing as well. One of the reasons for the growing size of interest rate derivatives with option characteristics might be attributed to the low level of interest rates, which tend to increase speculative positions. Additionally, the zero-interest rate levels in many developed countries put a downward floor on the market rates, which might also lead to an increase in demand for interest rate options. Another important factor might be growing options embedded in fixed-income securities such as callable or putable bonds or embedded options in loans such as prepayment options on mortgage loans or mortgage-backed securities. In this regard, developing a methodology to price fixed-income securities and hedge against changes in interest rates are crucial for portfolio managers, strategists and risk managers as well as policymakers.

Considering the growing size and complexity of interest rate derivatives, the valuation of these securities is crucial for both market practitioners and academics as wells. However, market practice is to use different models for different interest rate derivatives. For instance, the forward swap rate is assumed to be lognormally distributed under the Black model whereas the cap rate is assumed to be in another application, which creates inconsistency among the assumptions used in these models. In this regard, from the perspective of portfolio managers, this inconsistency creates difficulty in terms of aggregating the exposures among interest rate derivatives. Additionally, this practice does not allow generalization for the pricing of interest rate derivatives, which provides partial solutions for each interest rate derivative.

Although there have been many models used to price out interest rate derivatives, these models tend to have drawbacks in terms of complexity, overparameterization, the allowability of a closed-form solution, consistency with the market prices and applicability. Additionally, some of the models tend to be only applicable to specific cases, which might not be generalized. In this regard, it is important to develop simple models consistent with market prices. Additionally, allowability of a closed-form solution is also useful for market practitioners in terms of pricing and hedging. In this regard, this study employs the Vasicek model which can be used to price all interest rate derivatives due to the analytical tractability of the model. In this regard, the study aims to contribute to the literature in terms of applying a model which can be generalized to many areas in interest rate derivatives such as caps, floorlets, call or put options on bonds and swaptions. Besides the methodology also allows pricing embedded options on mortgage loans, callable bonds and puttable bonds with any cash flow structure, including regular and irregular cash flow structures. Lastly, the methodology can be adapted to the pricing of American or Bermudan type of option pricing.

The methodology relies on the fact that it is possible to obtain a closed-form solution for European call or put options on zero-coupon bonds under the Vasicek model. Then using the traded securities in the market, it is possible to obtain the Vasicek model parameters, which can be used to price out fixed-income securities with a similar maturity structure. Then the next step is to derive the closed-form solution for the interest rate derivatives that're demanded to be priced out under the Vasicek model. Since the bond prices are assumed to be lognormally distributed under the Vasicek model and the Vasicek model captures the main properties of short rate including mean-reversion and existence of long-run mean of short rate, the methodology can be adapted to many interest rate derivatives such as caps, floors, swaptions, bond options including zero-coupon or coupon-bearing bonds. Besides these derivatives, it can be applicable to embedded options such as prepayment options on mortgage loans or callable bonds. Another advantage of the methodology is that it can be used for fixed-income securities with irregular cash flow structures. Since the methodology allows a closed-form solution for many interest rate derivatives or embedded options, it is also possible to obtain the sensitivities of the relevant interest rate derivatives with respect to the model parameters; mean-reversion, short rate, long-run mean of short rate and volatility of short rate.

In this regard, this study uses cross-currency swaptions data which is used to find out the Vasicek parameters. Firstly, the closed-form solution for European swaptions under the Vasicek model is obtained and the parameters are estimated using nonlinear least squares. The results indicate that the Vasicek model parameters tend to have a good model-in-fit to the market data. Then the study provides time-series of European call and put option values for callable and puttable bonds for the period between January 2013 and November 2019 using the estimated Vasicek model parameters. Additionally, the methodology is adapted to the European prepayment option on a mortgage loan. Then the sensitivities of the option values to the model parameters are presented, which provide important implications for hedging exposures in interest rate derivatives. Besides pricing European options, the methodology can be implemented for American or Bermudan type of options. In this regard, the contribution of the study is to provide an empirical framework that can be enhanced using other interest-rate models with desirable properties and allows aggregation of interest rate exposure in a consistent way.

The next section provides the preliminary mathematical properties that will be used in the following parts including Vasicek model, measure change from risk-neutral measure to forward measure and pricing a call option on zero-coupon bond under the Vasicek model. Then the following section derives the swaption premium using the Vasicek model and provides details about the estimation of Vasicek model parameters using the swaption data. Besides, the section also provides the explicit formula for a call option on callable bond, put option on puttable bond and prepayment option on a mortgage loan. Then the last section provides empirical findings, which the Vasicek model is implemented to price the aforementioned interest rate derivatives from January 2013 to November 2019. The section also presents the comparative statistics, the sensitivities of the option values with respect to the changes in the model parameters.

2.2 PRELIMINARY

This section presents the preliminary models and the mathematical properties of the relevant models used in the following sections. In this regard, the Vasicek model is introduced together with the closed-form solution for bond prices. Then the methodology of measure change from risk-neutral measure to forward measure is presented. The new dynamics for short rate and bond prices are shown under the forward measure. Then, the properties of moments for short rate are derived. Lastly, the pricing of a call option on a zero-coupon bond under the Vasicek model is derived, which will be used to price options in fixed-income securities, where can be expressed as a linear function of bond options.

2.2.1 Vasicek Model

Vasicek term structure model is one of the most extensively used interest rate models for the pricing of fixed income derivatives. The main property of the model is that the underlying short rate assumes to revert back to its long-run mean at a rate which is called as the mean reversion rate. Under the risk-neutral measure, the instantaneous short rate, $r(t)$ follows the stochastic differential equation:

$$dr(t) = \alpha(\beta - r(t))dt + \sigma dW^Q(t) \tag{2.1}$$

where α and β stand for the mean-reversion rate and long-run mean of the instantaneous short rate, respectively. $W^Q(t)$ denotes the Brownian motion under the risk-neutral measure and the volatility term governs the size of the fluctuations in the short rate. One of the important properties of the Vasicek model is that it allows a positive probability of getting negative interest rates. Under the model, the bond prices are assumed to be lognormally distributed so that a closed-form solution for zero-coupon bond prices can be derived. In this regard, the price of a zero-coupon bond is expressed as follows:

$$P(t,T) = A(t,T)e^{-C(t,T)r(t)} \tag{2.2}$$

where

$$A(t,T) = exp\left(\frac{\{C(t,T)-(T-t)\}\left\{\alpha^2\beta - \frac{\sigma^2}{2}\right\}}{2} - \frac{\sigma^2 C(t,T)^2}{4\alpha}\right) \tag{2.3}$$

$$C(t,T) = \frac{1-e^{-\alpha(T-t)}}{\alpha} \tag{2.4}$$

Besides the zero-coupon bond prices, an explicit form for European option on a zero-coupon bond or coupon-bearing bond together with swaptions, caplets and floorlets can be obtained under the Vasicek model. However, the option pricing under the Vasicek model requires changing measure from risk-neutral measure to relevant measure, which will be covered in the next part.

2.2.2 Measure Change

The money market account grows at instantaneous short rate, $r(t)$. In this regard, the incremental change in the value of money market account $M(t)$ is equal to the following:

$$dM(t) = M(t)r(t)dt \tag{2.5}$$

As can be seen from the equation, the incremental change in the value of money market account is deterministic and there is no volatility in the value of money market account.

Under the risk-neutral measure, zero-coupon bond price, $P(t,T)$ has the following dynamics

$$dP(t,T) = P(t,T)r(t)dt + \sigma_P(t,T)P(t,T)dW^Q(t) \tag{2.6}$$

where $\sigma_P(t,T)$ stands for volatility of bond price at time t that's maturing at time T.

Consider a replicating porfolio, $V(t)$, consisting of θ units in money market account and γ units in zero-coupon bond.

$$V(t) = \theta(t)M(t) + \gamma(t)P(t,T) \tag{2.7}$$

The incremental change in value of the replicating portfolio is equal to the following:

$$\begin{aligned} dV(t) &= \theta(t)dM(t) + \gamma(t)dP(t,T) \\ &= \left[\frac{V(t) - \gamma(t)P(t,T)}{M(t)}\right] M(t)r(t)dt + \gamma(t)dP(t,T) \end{aligned} \tag{2.8}$$

The next step is to plug the process of zero-coupon bond price into the incremental change in the value of the replicating portfolio.

$$\begin{aligned} dV(t) &= V(t)r(t)dt - \gamma(t)P(t,T)r(t)dt + \gamma(t)P(t,T)r(t)dt \\ &\quad + \gamma(t)\sigma_P(t,T)P(t,T)dW^Q(t) \end{aligned} \tag{2.9}$$

Then the change in the replicating portfolio can be expressed as follows under the risk-neutral measure.

$$dV(t) = V(t)r(t)dt + \gamma(t)\sigma_P(t,T)P(t,T)dW^Q(t) \tag{2.10}$$

$\sigma_V(t,T)$ is defined as:

$$\sigma_V(t,T)V(t) = \gamma(t)\sigma_P(t,T)P(t,T) \tag{2.11}$$

Then, the replicating portfolio satisfies:

$$dV(t) = V(t)r(t)dt + \sigma_V(t,T)V(t)dW^Q(t) \tag{2.12}$$

The next step is to find out the dynamics of normalized price dynamics. By the product rule, the normalized derivative price dynamics is obtained.

$$\begin{aligned} d\left[\frac{V(t)}{P(t,T)}\right] &= \frac{dV(t)}{P(t,T)} + V(t)d\left[\frac{1}{P(t,T)}\right] + dV(t)d\left[\frac{1}{P(t,T)}\right] \\ &= \frac{V(t)r(t)dt + \sigma_V(t,T)V(t)dW^Q(t)}{P(t,T)} \\ &\quad + V(t)\left[\frac{-1}{P(t,T)^2}dP(t,T) + \frac{dP(t,T)^2}{P(t,T)^3}\right] \\ &\quad + \left(V(t)r(t)dt + \sigma_V V(t)dW^Q(t)\right)\left[\frac{-dP(t,T)}{P(t,T)^2} + \frac{dP(t,T)^2}{P(t,T)^3}\right] \end{aligned} \tag{2.13}$$

$$
\begin{aligned}
d\left[\frac{1}{P(t,T)}\right] &= \frac{-1}{P(t,T)^2}dP(t,T) + \frac{1}{P(t,T)^3}dP(t,T)^2 \\
&= -\frac{r(t)P(t,T)dt + \sigma_p(t,T)P(t,T)dW^Q(t)}{P(t,T)^2} \\
&+ \frac{1}{P(t,T)^3}\sigma_p^2(t,T)P(t,T)^2dt \\
&= \frac{-r(t)dt - \sigma_p(t,T)dW^Q(t)}{P(t,T)} + \frac{\sigma_p^2(t,T)}{P(t,T)}dt \\
&= \frac{\left[\sigma_p^2(t,T) - r(t)\right]}{P(t,T)}dt - \frac{\sigma_p(t,T)}{P(t,T)}dW^Q(t)
\end{aligned} \tag{2.14}
$$

$d\left[\frac{1}{P(t,T)}\right]$ is plugged in the dynamics of normalized derivative price.

$$
\begin{aligned}
d\left[\frac{V(t)}{P(t,T)}\right] &= \frac{V(t)r(t)dt + \sigma_V(t,T)V(t)dW^Q(t)}{P(t,T)} \\
&+ \frac{V(t)}{P(t,T)}\left[\sigma_p^2(t,T) - r(t)\right]dt \\
&- \frac{V(t)}{P(t,T)}\sigma_p(t,T)dW^Q(t) \\
&- \frac{\sigma_V(t,T)V(t)\sigma_P(t,T)}{P(t,T)}dt
\end{aligned} \tag{2.15}
$$

The last step provides the steps for measure change.

$$
\begin{aligned}
d\left[\frac{V(t)}{P(t,T)}\right] &= \frac{V(t)}{P(t,T)}\sigma_V(t,T)dW^Q(t) \\
&+ \frac{V(t)}{P(t,T)}\sigma_p^2(t,T)dt - \frac{V(t)}{P(t,T)}\sigma_P(t,T)dW^Q(t) \\
&- \frac{V(t)}{P(t,T)}\sigma_V(t,T)\sigma_P(t,T)dt \\
&= \frac{V(t)}{P(t,T)}\left[\sigma_p^2(t,T) - \sigma_V(t,T)\sigma_P(t,T)\right]dt \\
&+ \frac{V(t)}{P(t,T)}\left[\sigma_V(t,T) - \sigma_P(t,T)\right]dW^Q(t) \\
&= \frac{V(t)}{P(t,T)}\sigma_P(t,T)\left[\sigma_P(t,T) - \sigma_V(t,T)\right]dt \\
&+ \frac{V(t)}{P(t,T)}\left[\sigma_V(t,T) - \sigma_P(t,T)\right]dW^Q(t) \\
&= \frac{V(t)}{P(t,T)}\left[\sigma_V(t,T) - \sigma_P(t,T)\right]\left[dW^Q(t) - \sigma_P(t,T)dt\right]
\end{aligned} \tag{2.16}
$$

The new process, dW^T, is defined as;

$$dW^T(t) = dW^Q(t) - \sigma_P(t,T)dt \tag{2.17}$$

where W^T is a Brownian motion under a new measure by Girsanov's theorem. It can be observed that the normalized derivative price dynamics $\frac{V(t)}{P(t,T)}$ is martingale with respect to the new forward measure.

$$d\left[\frac{V(t)}{P(t,T)}\right] = \frac{V(t)}{P(t,T)}\left[\sigma_V(t,T) - \sigma_P(t,T)\right] dW^T(t) \tag{2.18}$$

Since the relation between Brownian motions under two different measures is known, the dynamics of short rate can be derived.

$$dr(t) = \alpha\left[\beta - r(t)\right] dt + \sigma dW^Q(t) \tag{2.19}$$

Under the new measure, the short rate satisfies the following dynamics.

$$dr(t) = \alpha\left[\beta - r(t)\right] dt + \sigma\left[dW^T(t) + \sigma_P(t,T)dt\right] \tag{2.20}$$

$$dr(t) = \alpha\left[\beta - r(t)\right] dt + \sigma dW^T(t) + \sigma\sigma_P(t,T)dt \tag{2.21}$$

It can be seen that the drift term changes under the new measure.

$$dr(t) = \alpha\left[\tilde{\beta} - r(t)\right] dt + \sigma dW^T(t) \tag{2.22}$$

where $\tilde{\beta} = \beta + \frac{\sigma\sigma_P(t,T)}{\alpha}$

The last step is to find out the dynamics of zero-coupon bond price dynamics under the new T-forward measure.

$$\begin{aligned} dP(t,T) &= r(t)P(t,T)dt + \sigma_P(t,T)P(t,T)dW^Q(t) \\ &= r(t)P(t,T)dt + \sigma_P(t,T)P(t,T)\left[dW^T(t) + \sigma_P(t,T)dt\right] \\ &= \left[r(t)P(t,T) + \sigma_P^2(t,T)P(t,T)\right] dt + \sigma_P(t,T)P(t,T)dW^T(t) \\ &= P(t,T)\left[r(t) + \sigma_P^2(t,T)\right] dt + \sigma_P(t,T)P(t,T)dW^T(t) \end{aligned} \tag{2.23}$$

which shows that the drift term of bond price dynamics differs from the short rate under the risk-neutral measure. The next part describes the characteristics of the short rate under the T-forward measure.

2.2.3 Moments of Short Rate on T-Forward Measure

In the previous section, it is shown that the short rate has the following dynamics under the T-forward measure.

$$dr(t) = [\alpha\beta - \alpha r(t) + \sigma\sigma_P(t,T)]\, dt + \sigma dW^{Q_T}(t) \tag{2.24}$$

Since there is a closed-form solution of the bond price under the Vasicek model, which is shown below, it is possible to obtain the parametric form of the volatility of bond prices as:

$$P(t,T) = e^{-C(t,T)r(t)}A(t,T) \tag{2.25}$$

Taking the logarithm of both sides gives

$$lnP(t,T) = -C(t,T)r(t) + lnA(t,T) \tag{2.26}$$

Under the new measure, the variance of the bond price can be expressed as follows:

$$\begin{aligned} Var^{Q_T}\left[lnP(t,T)\right] &= \sigma_p(t,T)^2 \\ &= C(t,T)^2 Var^{Q_T}\left[r(t)\right] \\ &= C(t,T)^2\sigma^2 \end{aligned} \tag{2.27}$$

The following equation comes from the bond pricing formula under Vasicek model.

$$\sigma_p(t,T) = -C(t,T)\sigma \tag{2.28}$$

where

$$C(t,T) = \frac{1 - e^{-\alpha(T-t)}}{\alpha} \tag{2.29}$$

Then,

$$dr(t) = \left[\alpha\beta - \alpha r(t) - \sigma^2 C(t,T)\right] dt + \sigma dW^{Q_T}(t) \tag{2.30}$$

$$\begin{aligned} d\left[e^{\alpha t}r(t)\right] &= \alpha e^{\alpha t}r(t)dt + e^{\alpha t}dr(t) \\ &= \alpha e^{\alpha t}r(t)dt + e^{\alpha t}\left[\alpha\beta - \alpha r(t) - \sigma^2 C(t,T)\right] dt + e^{\alpha t}\sigma dW^{Q_T}(t) \\ &= e^{\alpha t}\left[\alpha\beta - \sigma^2 C(t,T)\right] dt + e^{\alpha t}\sigma dW^{Q_T}(t) \end{aligned} \tag{2.31}$$

Substitute $C(t,T)$ into equation.

$$C(t,T) = \frac{1-e^{-\alpha(T-t)}}{\alpha} \tag{2.32}$$

$$\begin{aligned} d\left[e^{\alpha t}r(t)\right] &= e^{\alpha t}\left[\alpha\beta - \sigma^2\frac{1-e^{-\alpha(T-t)}}{\alpha}\right]dt + e^{\alpha t}\sigma dW^{Q_T}(t) \\ &= e^{\alpha t}\alpha\beta dt - e^{\alpha t}\left(\frac{\sigma^2}{\alpha}\right)\left(1-e^{-\alpha(T-t)}\right)dt + e^{\alpha t}\sigma dW^{Q_T}(t) \\ &= e^{\alpha t}\alpha\beta dt - e^{\alpha t}\left(\frac{\sigma^2}{\alpha}\right)dt + e^{-\alpha(T-2t)}\frac{\sigma^2}{\alpha}dt + \sigma e^{\alpha t}dW^{Q_T}(t) \end{aligned} \tag{2.33}$$

Integrate over (t_m, t_i)

$$\begin{aligned} e^{\alpha t_i}r(t_i) - e^{\alpha t_m}r(t_m) = \beta\left[e^{\alpha t_i} - e^{\alpha t_m}\right] - \frac{\sigma^2}{\alpha^2}\left[e^{\alpha t_i} - e^{\alpha t_m}\right] \\ + \frac{\sigma^2}{2\alpha^2}\left[e^{-\alpha(T-2t_i)} - e^{-\alpha(T-2t_m)}\right] + \sigma\int_{t_m}^{t_i} e^{\alpha\mu}dW^{Q_T}(u) \end{aligned} \tag{2.34}$$

$$\begin{aligned} r(t_i) = e^{-\alpha(t_i-t_m)}r(t_m) + \left[\beta - \frac{\sigma^2}{\alpha^2}\right]\left[1-e^{-\alpha(t_i-t_m)}\right] + \frac{\sigma^2}{2\alpha^2}\left[e^{-\alpha(T-t_i)} - e^{-\alpha(T+t_i-2t_m)}\right] \\ + \sigma\int_{t_m}^{t_i} e^{-\alpha(t_i-u)}dW^{Q_T}(u) \end{aligned} \tag{2.35}$$

$$r(t_i) = r(t_m)e^{-\alpha(t_i-t_m)} + \gamma(t_m,t_i) + \sigma\int_{t_m}^{t_i} e^{-\alpha(t_i-u)}dW^T(u) \tag{2.36}$$

where

$\gamma(t_m,t_i) = \left[\beta - \frac{\sigma^2}{\alpha^2}\right]\left[1-e^{-\alpha(t_i-t_m)}\right] + \frac{\sigma^2}{2\alpha^2}\left[e^{-\alpha(T-t_i)} - e^{-\alpha(T+t_i-2t_m)}\right]$.

The expected value and variance of the short rate under T-forward measure are given by

$$\mathbb{E}^{Q_T}\left[r(t_i)|\mathcal{F}_{t_m}\right] = r(t_m)e^{-\alpha(t_i-t_m)} + \gamma(t_m,t_i) \tag{2.37}$$

and

$$
\begin{aligned}
Var^{Q_T}\left[r(t_i)|\mathcal{F}_{t_m}\right] &= \sigma^2\mathbb{E}^{Q_T}\left[\int_{t_m}^{t_i} e^{-2\alpha(t_i-u)}du\right] \\
&= \sigma^2\frac{1-e^{-2\alpha(t_i-t_m)}}{2\alpha}
\end{aligned}
\tag{2.38}
$$

by Ito Isometry.

2.2.4 Pricing Call Option on a Zero-Coupon Bond

This section presents the closed-form solution for the value of a call option on a zero-coupon bond with maturity date t_i at time t. The swaption pricing or any embedded option on a coupon-bearing bond can be considered as a series of European call or put options on zero-coupon bonds with different time-to-maturity and different strike prices. Therefore, it is crucial to understand the dynamics of option pricing on zero-coupon bonds.

The first step is to model the dynamics of bond price under the risk-neutral measure, which is described below.

$$
dP(t,t_i) = P(t,t_i)r(t)dt + P(t,t_i)\sigma_p(t,t_i)dW^Q(t) \tag{2.39}
$$

Assuming that the call option expires at time t_m, the pricing of a call option on a coupon-bearing bond requires the use of Q_{t_m} measure. The next step shows the necessary adjustment from the risk-neutral measure to Q_{t_m} measure using the methodology described in previous sections.

$$
dW^{Q_{t_m}}(t) = dW^Q(t) - \sigma_p(t,t_m)dt \tag{2.40}
$$

The next step is to describe the dynamics of bond price, $P(t,t_i)$ under the new measure.

$$
dP(t,t_i) = P(t,t_i)\left[r(t) + \sigma_p(t,t_m)\sigma_p(t,t_i)\right]dt + P(t,t_i)\sigma_p(t,t_i)dW^{Q_{t_m}}(t) \tag{2.41}
$$

$$
dlnP(t,t_i) = \frac{\partial lnP(t,t_i)}{\partial t}dt + \frac{\partial lnP(t,t_i)}{\partial P(t,t_i)}dP(t,t_i) + \frac{1}{2}\frac{\partial^2 lnP(t,t_i)}{\partial P(t,t_i)^2}dP(t,t_i)^2 \tag{2.42}
$$

$$dlnP(t,t_i) = \left[r(t) + \sigma_p(t,t_m)\sigma_p(t,t_i) - \frac{1}{2}\sigma_p(t,t_i)^2\right]dt + \sigma_p(t,t_i)dW^{Q_{t_m}}(t) \tag{2.43}$$

The next step provides the functional form of the bond price that matures at time t_i.

$$\begin{aligned} lnP(t,t_i) = lnP(t_0,t_i) &+ \int_{t_0}^{t}\left[r(s) + \sigma_p(s,t_m)\sigma_p(s,t_i) - \frac{1}{2}\sigma_p(s,t_i)^2\right]ds \\ &+ \int_{t_0}^{t}\sigma_p(s,t_i)dW^{Q_{t_m}}(s) \end{aligned} \tag{2.44}$$

$$\begin{aligned} lnP(t_m,t_i) = lnP(t,t_i) &+ \int_{t}^{t_m}\left[r(s) + \sigma_p(s,t_m)\sigma_p(s,t_i) - \frac{1}{2}\sigma_p(s,t_i)^2\right]ds \\ &+ \int_{t}^{t_m}\sigma_p(s,t_i)dW^{Q_{t_m}}(s) \end{aligned} \tag{2.45}$$

$$P(t_m,t_i) = P(t,t_i)e^{\int_t^{tm}\mu_p(s)ds + \int_t^{tm}\sigma_p(s,t_i)dW^{Q_{tm}}(s)} \tag{2.46}$$

where $\mu_p(s) = r(s) + \sigma_p(s,t_m)\sigma_p(s,t_i) - \frac{1}{2}\sigma_p(s,t_i)^2$.

The mean and variance of the logreturn of bond prices are expressed under the new measure, which will be used in the following steps.

$$\begin{aligned} \mathbb{E}^{Q_{t_m}}\left[lnP(t_m,t_i)|\mathcal{F}_t\right] = lnP(t,t_i) &+ \left[\int_t^{t_m}(\sigma_p(s,t_m)\sigma_p(s,t_i) - \frac{1}{2}\sigma_p(s,t_i)^2)ds\right] \\ &+ \mathbb{E}^{Q_{t_m}}\left[\int_t^{t_m} r(s)ds\right] = \mu_m \end{aligned} \tag{2.47}$$

$$Var^{Q_{t_m}}\left[lnP(t_m,t_i)|\mathcal{F}_t\right] = Var^{Q_{t_m}}\left[\int_t^{t_m} r(s)ds\right] + \int_t^{t_m}\sigma_p^2(s,t_i)ds = \sigma_m^2 \tag{2.48}$$

The next step is to price the call option on a zero-coupon bond with maturity date t_i under the new Q_{t_m}measure.

$$\frac{V(t)}{P(t,t_m)} = \mathbb{E}^{Q_{t_m}}\left[\frac{V(t_m)}{P(t_m,t_m)}|\mathcal{F}_t\right] = \mathbb{E}^{Q_{t_m}}\left[V(t_m)|\mathcal{F}_t\right] \tag{2.49}$$

$$V(t) = P(t,t_m)\mathbb{E}^{Q_{t_m}}\left[(P(t_m,t_i) - K)^+ | \mathcal{F}_t\right] \tag{2.50}$$

where $V(t_m)$ is the value of call option on a zero-coupon bond. In integral form, the term can be expressed as follows:

$$V(t) = P(t,t_m)\int_{-\infty}^{\infty}(P(t_m,t_i) - K)^+ \frac{1}{\sqrt{2\pi}\sigma_m} e^{-\frac{1}{2}\left[\frac{lnP(t_m,t_i)-\mu_m}{\sigma_m}\right]^2} dlnP(t_m,t_i) \tag{2.51}$$

Define $e^y = P(t_m,t_i)$ so $lnP(t_m,t_i) = y$

$$V(t) = P(t,t_m)\int_{-\infty}^{\infty}(e^y - K)^+ \frac{1}{\sqrt{2\pi}\sigma_m} e^{-\frac{1}{2}\left[\frac{y-\mu_m}{\sigma_m}\right]^2} dy \tag{2.52}$$

and for the values of $e^y > K$, or $y > lnK$,

$$V(t) = P(t,t_m)\int_{lnK}^{\infty}(e^y - K) \frac{1}{\sqrt{2\pi}\sigma_m} e^{-\frac{1}{2}\left[\frac{y-\mu_m}{\sigma_m}\right]^2} dy \tag{2.53}$$

Define $z = \frac{y-\mu_m}{\sigma_m}$,therefore $dy = \sigma_m dz$

$$\begin{aligned} V(t) &= P(t,t_m)\int_{\frac{lnK-\mu_m}{\sigma_m}}^{\infty}(e^{z\sigma_m+\mu_m} - K)\frac{1}{\sqrt{2\pi}\sigma_m} e^{-\frac{1}{2}z^2}\sigma_m dz \\ &= P(t,t_m)\int_{\frac{lnK-\mu_m}{\sigma_m}}^{\infty} e^{z\sigma_m+\mu_m} e^{-\frac{1}{2}z^2}\frac{1}{\sqrt{2\pi}} dz - KP(t,t_m)\int_{\frac{lnK-\mu_m}{\sigma_m}}^{\infty} e^{-\frac{1}{2}z^2}\frac{1}{\sqrt{2\pi}} dz \end{aligned} \tag{2.54}$$

Define $e^{\mu_m+\sigma_m z-\frac{1}{2}z^2} = e^{(\mu_m+\frac{1}{2}\sigma_m^2)}e^{-\frac{1}{2}(z-\sigma_m)^2}$ and $z' = z - \sigma_m$

$$\begin{aligned} \mathbb{E}^{Q_{t_m}}[(P(t_m,t_i) - K]^+ = &\int_{\frac{lnK-\mu_m-\sigma_m^2}{\sigma_m}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{(\mu_m+\frac{1}{2}\sigma_m^2)-\frac{1}{2}(z')^2} dz' \\ &- K\int_{\frac{lnK-\mu_m}{\sigma_m}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \end{aligned} \tag{2.55}$$

$$\mathbb{E}^{Q_{t_m}}[(P(t_m,t_i) - K]^+ = e^{(\mu_m+\frac{1}{2}\sigma_m^2)}\phi\left[\frac{-lnK+\mu_m+\sigma_m^2}{\sigma_m}\right] - K\phi\left[\frac{-lnK+\mu_m}{\sigma_m}\right] \tag{2.56}$$

The next step shows the call option price on a zero-coupon bond.

$$V(t) = P(t,t_m)e^{(\mu_m+\frac{1}{2}\sigma_m^2)}\phi\left[\frac{-lnK+\mu_m+\sigma_m^2}{\sigma_m}\right] - P(t,t_m)K\phi\left[\frac{-lnK+\mu_m}{\sigma_m}\right] \tag{2.57}$$

However, μ_m and σ_m should be plugged into the formula using the following equations.

$$\begin{aligned}\mu_m &= \mathbb{E}^{Q_{t_m}}\left[lnP(t_m, t_i)|\mathcal{F}_t\right] \\ &= \mathbb{E}^{Q_{t_m}}\left[ln\left(e^{-C(t_m,t_i)r(t_m)}A(t_m, t_i)|\mathcal{F}_t\right)\right] \\ &= \mathbb{E}^{Q_{t_m}}\left[-C(t_m, t_i)r(t_m) + ln\left(A(t_m, t_i)|\mathcal{F}_t\right)\right] \\ &= ln\left(A(t_m, t_i)\right) - C(t_m, t_i)\mathbb{E}^{Q_{t_m}}\left[r(t_m|\mathcal{F}_t)\right]\end{aligned} \tag{2.58}$$

$$\begin{aligned}\sigma_m^2 &= Var^{Q_{t_m}}\left[lnP(t_m, t_i)|\mathcal{F}_t\right] \\ &= Var^{Q_{t_m}}\left[lnA(t_m, t_i) - C(t_m, t_i)r(t_m)|\mathcal{F}_t\right] \\ &= C(t_m, t_i)^2 Var^{Q_{t_m}}\left[r(t_m)|\mathcal{F}_t\right]\end{aligned} \tag{2.59}$$

The last step in the pricing is to plug the expected value and variance of short rate under the new measure, which is covered in the previous sections. Then the price of the call option on a zero-coupon bond can be found out as follows:

$$ZBC\left[t, t_m, t_i, K\right] = P(t, t_i)\phi(h) - KP(t, t_m)\phi(h - \tilde{\sigma}) \tag{2.60}$$

where $ZBC\left[t, t_m, t_i, K\right]$ stands for the value of the call option on a zero-coupon bond at time t, where the option expires at time t_m and bond expires at t_i.

$$\tilde{\sigma} = \sigma\sqrt{\frac{1 - e^{-2\alpha(t_m - t)}}{2\alpha}}C(t_m, t_i) \tag{2.61}$$

$$C(t_m, t_i) = \frac{1 - e^{-\alpha(t_i - t_m)}}{\alpha} \tag{2.62}$$

$$h = \frac{1}{\tilde{\sigma}}ln\left[\frac{P(t, t_i)}{P(t, t_m)K}\right] + \frac{\tilde{\sigma}}{2} \tag{2.63}$$

The price of a put option on a zero-coupon bond can be found out using the put-call parity for bond options.

$$ZBC(t, t_m, t_i, K) + KP(t, t_m) = ZBP(t, t_m, t_i, K) + P(t, t_i) \tag{2.64}$$

where $ZBP(t, t_m, t_i, K)$ denotes put option on a zero-coupon bond with time-to-maturity $t_m - t$ and strike price K. $P(t, t_i)$ and $P(t, t_m)$ reflect the prices of bonds with maturity t_i and t_m, respectively. At the maturity of the option, both sides have the same payoff structure, which allows to obtain put option prices on bonds using the parity condition.

	$P(t_m, t_i) > K$	$P(t_m, t_i) < K$
$Call(t, t_m, t_i, K)$	$P(t_m, t_i) - K$	0
$K\,P(t, t_m)$	K	K
Total	$P(t_m, t_i)$	K
$Put(t, t_m, t_i, K)$	0	$K - P(t_m, t_i)$
$P(t, t_i)$	$P(t_m, t_i)$	$P(t_m, t_i)$
$Total$	$P(t_m, t_i)$	K

Figure 2.1: Put-Call Parity for Bond Options

2.3 DATA AND METHODOLOGY

This section presents the data and methodology for the pricing of European call option and put option values on interest rate derivatives and embedded options on fixed income securities. In this regard, the methodology will not be only applicable to fixed income securities with regular cash flows such as coupon-bearing bonds or swaps but also fixed income securities with different cash flow structures. In this regard, the roadmap for the valuation of call or put options on fixed income securities can be shown in the following diagram.

The main idea behind the methodology is to find out European call or put option values on fixed income securities consistent with the prices of traded securities. In this regard, the methodology uses the fact that it is possible to express options on fixed income securities as a series of call or put options on zero-coupon bonds. If the call or put options on fixed income securities can be expressed as a function of call or put options on zero-coupon bonds, then it is possible to find out prices consistent with the market prices. In other words, since a closed-form solution for call or put option on a zero-coupon bond is derived under the Vasicek model, then the model can be extended to find out closed-form solutions for call or put options on fixed income securities other than zero-coupon bonds.

However, finding out prices for call or put options consistent with the market prices

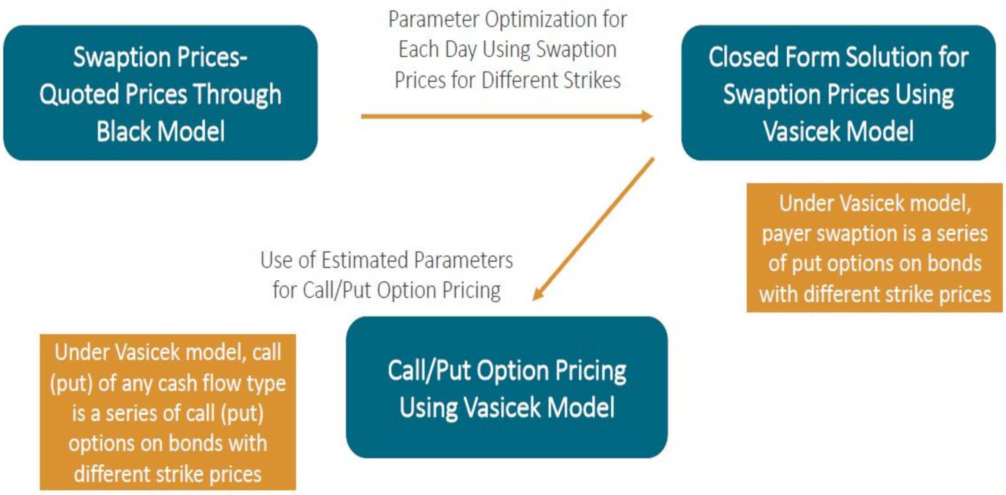


Figure 2.2: Methodology for Pricing Embedded Options

requires the parameter estimation of the Vasicek model. In this regard, the methodology firstly derives the closed-form solution for European swaptions under the Vasicek model. Then using the daily swaption quotes for Turkish lira rates from the period between January 2013 and November 2019, market prices of swaptions are obtained through the Black model since the quotes reflect the Black volatility. Then using the market prices of European swaptions and closed-form solution for swaption under the Vasicek model, the model parameters are found out using nine different strike rates for swaptions. The roadmap for the methodology is also presented in the next figure. In this respect, the next part provides the methodology for the pricing of swaption under the Vasicek model.

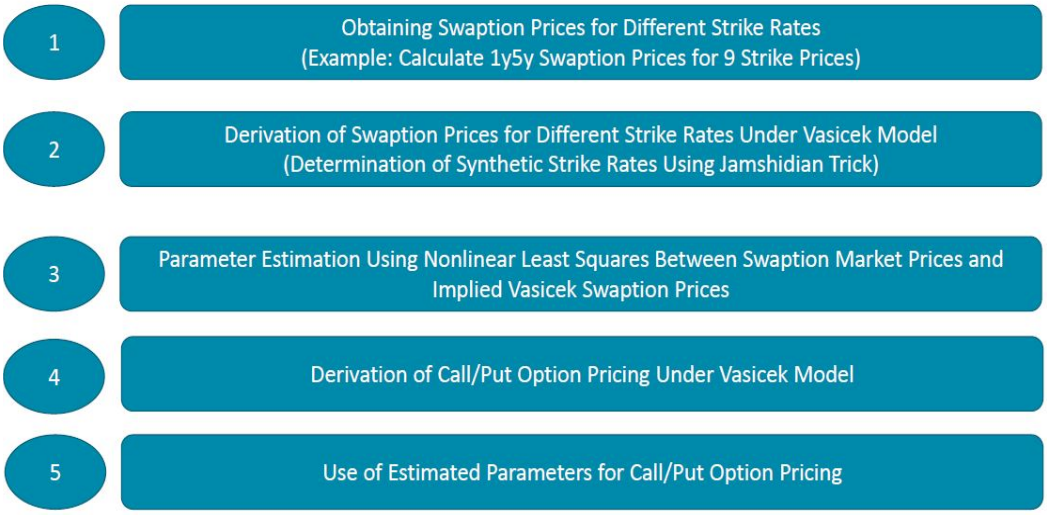


Figure 2.3: Roadmap for Pricing Embedded Options

2.3.1 Swaption Pricing Under Vasicek Model

An explicit formula for swaptions can be derived under the Vasicek model. The payoff of a European payer swaption at maturity is equal to the maximum of the difference between the value of swap at market swap rate and value of swap at the predetermined strike rate and zero. In other words, if the market swap rate at the maturity of the swaption is greater than the agreed strike rate, then the payer swaption is exercised since the option holder is able to pay at a lower rate than the market rate. However, when the market swap rate is lower than the agreed strike rate, the payer swaption is not exercised since it will be optimal to pay at a lower rate in the market. In this regard, as described in the previous sections, the value of swaption at the maturity can be expressed as follows:

$$
\begin{aligned}
V_p(t_m, t_m, t_n) &= max\left\{P(t_m, t_m, r) - P(t_m, t_n, r) - \sum_{i=m+1}^{n} CP(t_m, t_i, r); 0\right\} \\
V_p(t_m, t_m, t_n) &= max\left\{P(t_m, t_m, r) - \sum_{i=m+1}^{n} CF_i P(t_m, t_i, r); 0\right\}
\end{aligned}
\tag{2.65}
$$

where C stands for the predetermined strike price of the swaption and CF_i denotes the cash flows stemming from the fixed payments of the swap together with the one at the maturity.

The next step is to decompose the cash flows such that the swaption can be priced out in an easier way. In this regard, the trick is to consider $P(t_m, t_m)$ as the strike price since it is deterministic and to find out the critical short rate which equates the value of future payments after the maturity of swaption with the strike price. In other words, the yield curve is constructed with the critical short rate and future payments are discounted with this artificial yield curve. The fact that bond price is a monotonic function of short rate allows using this approach, which is known as the Jamshidian approach which states that the prices of options on coupon-bearing bonds can be obtained using options on zero-coupon bonds in a one-factor interest rate model.

$$
V_p(t_m, t_m, t_n) = max\left\{\sum_{i=m+1}^{n} CF_i P(t_m, t_i, r^*) - \sum_{i=m+1}^{n} CF_i P(t_m, t_i, r); 0\right\} \tag{2.66}
$$

where $P(t_m,t_m,r) = \sum_{i=m+1}^{n} CF_i P(t_m,t_i,r^*)$ and r^* is the artificial short rate which equates the future cash flows of the swap with the par amount, $P(t_m,t_m)$.

Then the next step is to apply the Jamshidian approach, which relies on the fact that bond prices are monotonic functions of short rate.

$$\begin{aligned} V_p(t_m,t_m,t_n) &= max\left\{\sum_{i=m+1}^{n} CF_i(P(t_m,t_i,r^*) - P(t_m,t_i,r));0\right\} \\ V_p(t_m,t_m,t_n) &= \sum_{i=m+1}^{n} CF_i max\left\{P(t_m,t_i,r^*) - P(t_m,t_i,r);0\right\} \end{aligned} \tag{2.67}$$

Since the option on a zero-coupon bond can be found out using the forward measure, it can be expressed as follows:

$$\begin{aligned} V_p(t,t_m,t_n) &= P(t,t_m)\mathbb{E}^{Q_{t_m}}\left[\sum_{i=m+1}^{n} CF_i[max\left\{P(t_m,t_i,r^*) - P(t_m,t_i,r);0\right]|\mathcal{F}_t\right] \\ V_p(t,t_m,t_n) &= P(t,t_m)\sum_{i=m+1}^{n} CF_i \mathbb{E}^{Q_{t_m}}\left[max\left\{P(t_m,t_i,r^*) - P(t_m,t_i,r);0\right\}|\mathcal{F}_t\right] \end{aligned} \tag{2.68}$$

The term in brackets $max\{P(t_m,t_i,r^*) - P(t_m,t_i,r);0\}$ is equal to the payoff of a put option on a bond worth $P(t_m,t_i,r)$ with strike price $P(t_m,t_i,r^*)$ at time t_m. Since a closed-form solution exists for European put option on a zero-coupon bond under the Vasicek Model, European swaption price can be obtained under the Vasicek model.

$$\begin{aligned} ZBP(t_m,t_m,t_i) &= max\left\{P(t_m,t_i,r^*) - P(t_m,t_i,r);0\right\} \\ ZBP(t,t_m,t_i) &= P(t,t_m)\mathbb{E}^{Q_{t_m}}\left[max\left\{P(t_m,t_i,r^*) - P(t_m,t_i,r);0\right\}|\mathcal{F}_t\right] \end{aligned} \tag{2.69}$$

where $ZBP(t,t_m,t_i)$ reflects the price of European put option on a zero-coupon bond at time t. Using the closed-form solutions for the European put options on bonds under Vasicek model, it is possible to express the swaption price as follows:

$$V(t,t_m,t_n) = \sum_{i=m+1}^{n} CF_i ZBP(t,t_m,t_i,K_i) \tag{2.70}$$

After the derivation of the closed-form solution for the swaption under the Vasicek model, the next step is to present the methodology that will be used to obtain the

Vasicek model parameters using swaption quotes. In this regard, since the swaption prices are quoted in the Black model, firstly swaption prices are obtained using the Black model. Then the Vasicek model parameters (short rate, mean-reversion rate, long-run mean of short rates and volatility of short rate) are estimated through nonlinear least squares. In this regard, the objective function is the square of the difference between the market prices and the Vasicek model estimated prices.

The steps for the parameter estimation can be summarized as follows:

For each of the specifications for 5y5y, 1y1y and 1y10y swaptions, the Vasicek model parameters are estimated seperately for each day from January 2013 to November 2019. The estimation methodology can be examplied for the specification of 5y5y swaptions.

- Calculate the option premium of payer swaption for different strike rates using Black implied volatility
- Derive the swaption premium for payer swaption using Vasicek model
- Estimate the Vasicek model parameters using nonlinear least squares through minimization of the squared differences between the market prices and Vasicek implied model prices
- Obtain the parameter estimates for Vasicek model

The next section presents the methodology of European call and put option pricing on fixed-income securities.

2.3.2 Call Option Pricing

This part provides the methodology of how to price call options on fixed income securities. In this regard, call option on fixed income securities such as coupon-bearing bond, mortgage loan or any security with irregular cash flow is exercised only if the discounted value of the remaining cash flows at the maturity of the option is greater than the strike price. Therefore, the value of a European call option at time

t_m on fixed income security with cash flows CF_i at time t_i is expressed as follows:

$$V_c(t_m, t_m, t_n) = max\left\{\sum_{i=m+1}^{n} CF_i P(t_m, t_i, r) - K; 0\right\} \tag{2.71}$$

Using the Jamshidian approach, there exists a critical short rate such that the discounted value of the remaining cash flows with the critical short rate is equal to strike price.

$$V_c(t_m, t_m, t_n) = max\left\{\sum_{i=m+1}^{n} CF_i P(t_m, t_i, r) - \sum_{i=m+1}^{n} CF_i P(t_m, t_i, r^*); 0\right\} \tag{2.72}$$

such that $K = \sum_{i=m+1}^{n} CF_i P(t_m, t_i, r^*)$. Then, maximum operator can be factored out so that the value of swaption at maturity becomes equal to the series of European call options on zero-coupon bonds with different time-to-maturities and different strike prices.

$$\begin{aligned} V_c(t_m, t_m, t_n) &= max\left\{\sum_{i=m+1}^{n} CF_i(P(t_m, t_i, r) - P(t_m, t_i, r^*)); 0\right\} \\ V_c(t_m, t_m, t_n) &= \sum_{i=m+1}^{n} CF_i max\left\{P(t_m, t_i, r) - P(t_m, t_i, r^*); 0\right\} \end{aligned} \tag{2.73}$$

Then the last step is to compute the expected value under the forward measure so that the closed-form solution can be found out for call options on fixed income securities with irregular cash flows.

$$\begin{aligned} V_c(t, t_m, t_n) &= P(t, t_m)\mathbb{E}^{Q_{t_m}}\left[\sum_{i=m+1}^{n} CF_i[max\left\{P(t_m, t_i, r) - P(t_m, t_i, r^*); 0\right\}]|\mathcal{F}_t\right] \\ V_c(t, t_m, t_n) &= P(t, t_m)\sum_{i=m+1}^{n} CF_i \mathbb{E}^{Q_{t_m}}\left[max\left\{P(t_m, t_i, r) - P(t_m, t_i, r^*); 0\right\}\right]|\mathcal{F}_t] \end{aligned} \tag{2.74}$$

The term in brackets $max\left\{P(t_m, t_i, r) - P(t_m, t_i, r^*); 0\right\}$ is equal to the payoff of a call option on a bond worth $P(t_m, t_i, r)$ with strike price $P(t_m, t_i, r^*)$. Since a closed-form solution exists for call options on zero-coupon bonds under the Vasicek Model, the European prepayment option price or European call option value on callable bonds can be considered as a series of European call options on bonds with different time-to-maturities and different strike prices.

$$\begin{aligned} ZBC(t_m, t_m, t_i) &= max\left\{P(t_m, t_i, r) - P(t_m, t_i, r^*); 0\right\} \\ ZBC(t, t_m, t_i) &= P(t, t_m)\mathbb{E}^{Q_{t_m}}\left[max\left\{P(t_m, t_i, r) - P(t_m, t_i, r^*); 0\right\} |\mathcal{F}_t\right] \end{aligned} \tag{2.75}$$

where $ZBC(t, t_m, t_i)$ reflects the price of call option on a zero-coupon bond at time t. Using the closed-form solutions for the call options on bonds under Vasicek model, it is possible to express as follows:

$$V_c(t, t_m, t_n) = \sum_{i=m+1}^{n} CF_i\, ZBC(t, t_m, t_i, K_i) \tag{2.76}$$

$$\begin{aligned} V_c(t, t_m, t_n) &= \sum_{i=m+1}^{n} CF_i\, ZBC(t, t_m, t_i, K_i) \\ &= \sum_{i=m+1}^{n} CF_i \left\{P(t, t_i)\phi(h_i) - K_i P(t, t_m)\phi(h_i - \tilde{\sigma_i})\right\} \end{aligned} \tag{2.77}$$

where

$$h_i = \frac{1}{\tilde{\sigma_i}} ln\left[\frac{P(t, t_i)}{P(t, t_m)K_i}\right] + \frac{\tilde{\sigma_i}}{2} \tag{2.78}$$

$$\tilde{\sigma_i} = \sigma_i \sqrt{\frac{1 - e^{-2\alpha(t_m - t)}}{2\alpha}} C(t_m, t_i) \tag{2.79}$$

$$C(t_m, t_i) = \frac{1 - e^{-\alpha(t_i - t_m)}}{\alpha} \tag{2.80}$$

Then the next part applies the same methodology for put options on fixed income securities.

2.3.3 Put Option Pricing on Bonds

In the previous section, it is shown that the call options on fixed income securities with any cash flow structure can be expressed as a series of European call options on zero-coupon bonds with different time-to-maturities and different strike prices. Similar to the previous approach, it is possible to express a put option on security with irregular or regular cash flow as a series of European put options on zero-coupon bonds.

The value of a put option on security with irregular cash flow (which generalizes the methodology) at the maturity is equal to the following:

$$V_p(t_m, t_m, t_n) = max\left\{K - \sum_{i=m+1}^{n} CF_i P(t_m, t_i, r); 0\right\} \tag{2.81}$$

Similarly, there exists a critical short rate that equates the discounted value of remaining cash flows with the critical rate to the strike price.

$$V_p(t_m, t_m, t_n) = max\left\{\sum_{i=m+1}^{n} CF_i P(t_m, t_i, r^*) - \sum_{i=m+1}^{n} CF_i P(t_m, t_i, r); 0\right\} \tag{2.82}$$

Factoring out maximum operator leads to composition that the put option on a fixed income security is a series of put options on zero-coupon bonds.

$$\begin{aligned} V_p(t_m, t_m, t_n) &= max\left\{\sum_{i=m+1}^{n} CF_i (P(t_m, t_i, r^*) - P(t_m, t_i, r)); 0\right\} \\ V_p(t_m, t_m, t_n) &= \sum_{i=m+1}^{n} CF_i max\left\{P(t_m, t_i, r^*) - P(t_m, t_i, r); 0\right\} \end{aligned} \tag{2.83}$$

Then using the closed-form solution for put options on zero-coupon bonds under the Vasicek model, an explicit formula for put option on any cash flow structure can be obtained.

$$V(t, t_m, t_n) = \sum_{i=m+1}^{n} CF_i ZBP(t, t_m, t_i, K_i) \tag{2.84}$$

where the put option price can be found out using Put-Call Parity for bond options.

$$ZBP(t, t_m, t_i, K_i) = ZBC(t, t_m, t_i, K_i) + K_i P(t, t_m) - P(t, t_i) \tag{2.85}$$

2.4 EMPIRICAL FINDINGS

This section firstly presents the estimated Vasicek model parameters using the swaption quotes. Then call option values on a coupon-bearing bond and prepayment option on a mortgage loan are priced out using the estimated Vasicek model parameters from

January 2013 to November 2019. Besides, put option values on a puttable bond are computed to compare the behaviour of put and call option values during the same period. Lastly, the section also provides comparative statistics to analyze the effect of model parameters on the value of call and put options, which is crucial for hedging and speculation against the interest rate changes.

2.4.1 The Estimation of Vasicek Model Parameters

Since the main motivation is to price call and put options consistent with the market prices of similarly traded securities, firstly the results of the Vasicek model parameter estimation are presented. The figures below show the estimation for short rate, mean-reversion rate, long-run mean of short rate and volatility of short rate obtained from the 1y10y swaption prices with different strikes. Besides the parameter estimations, it is crucial to understand the dynamics leading to changes in the model parameters and correlation between each other, which can be used for the formation of trade ideas, measuring illiquidity in the market or forecasting.

The figure shows the estimation of short rate from January 2013 to November 2019. It can be seen that the short rate seems to be quite in line with the market interest rates which is observed to rise after 2013 when the central banks of developed countries have announced the signals for exit from quantitative easing. Additionally, spikes in short rate in 2018 and 2019 are observed, which coincides with the local market developments in Turkey. In this regard, the short rate estimations seem to be stable across time and consistent with the market developments.

To figure out whether the estimated parameters provide a good model-in-fit for European swaption prices, the study also compares the quoted Black implied volatility and model-implied Black implied volatility. In this regard, the swaption prices under the estimated Vasicek models are calculated and converted into Black implied volatility to calculate the mean-absolute errors. The results indicate that the mean absolute errors for 5y5y swaption specification are around 1.16 percent compared to quoted Black implied volatility whereas the mean absolute errors for 1y10y and 1y1y swaption specifications are 2.60 percent and 2.59 percent, respectively. Given the levels for implied volatilities for swaptions, the model-in-fit of the model seems to be quite reasonable.

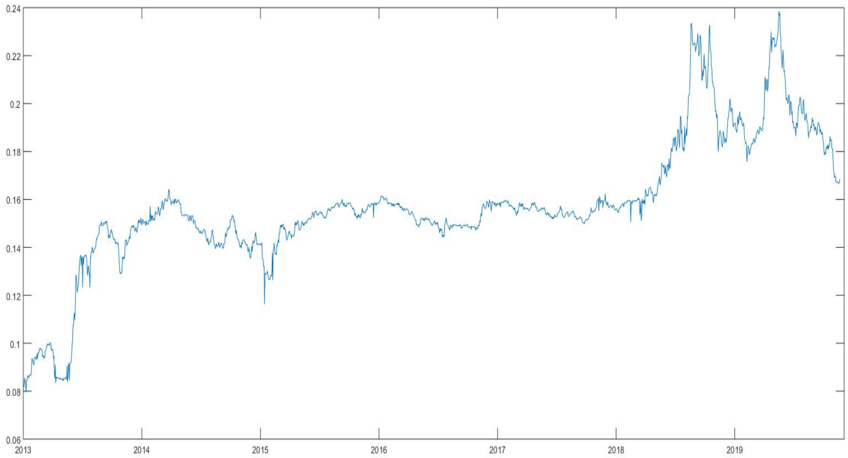


Figure 2.4: Estimated Short Rate from 1y10y Swaption

Mean-reversion rate seems to be declining after the period of 2013. However, during the period between 2014 and 2018 the mean-reversion rates are quite stable across time. Additionally, it is observed that the mean-reversion rate tends to decline in the last two years. It can be stated that the mean-reversion rate tends to decline during periods of rises in short rates. It is also observed that the mean-reversion rate tends to be at the lowest levels during the last periods, which might be associated with the liquidity of the market and changes in the expectations of the market participants. If there is uncertainty about the fair value of short rates, it is likely to cause a decline in mean-reversion rates. In this regard, the parameter estimations from mean-reversion rates seem to be robust as well.

Long-run mean of short rate estimations tends to be quite stable similar to short rate estimations. It is observed that the expectations for short rate tend to rise for Turkish rates during the periods after 2013 and peak at the periods of early 2019. However, it is observed that the model parameters tend to decline in the last ten months to the historical levels. The parameter estimation, volatility of short rate, seems to be fluctuating more compared to other parameters. However, it is observed that the estimations seem to be concentrated around the levels of 9 and 11 percent. Although there is a sharp decline in the volatility of short rate in 2018, it is observed that it tends to rise to historical levels quickly.

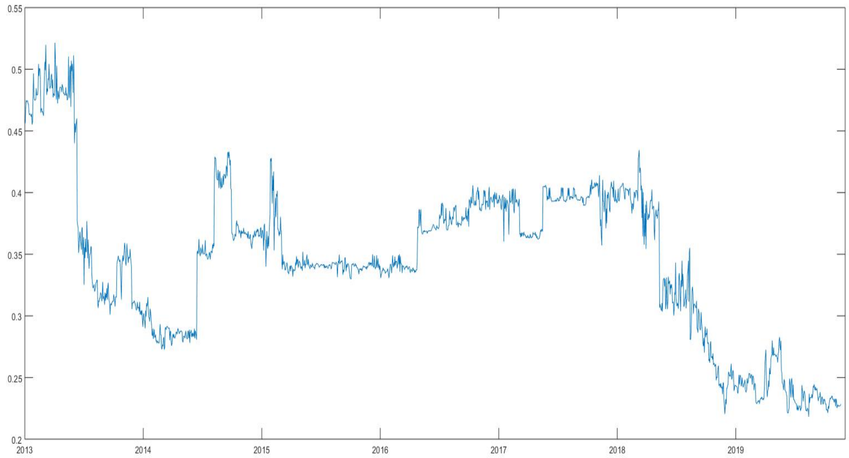


Figure 2.5: Estimated Mean-Reversion from 1y10y Swaption

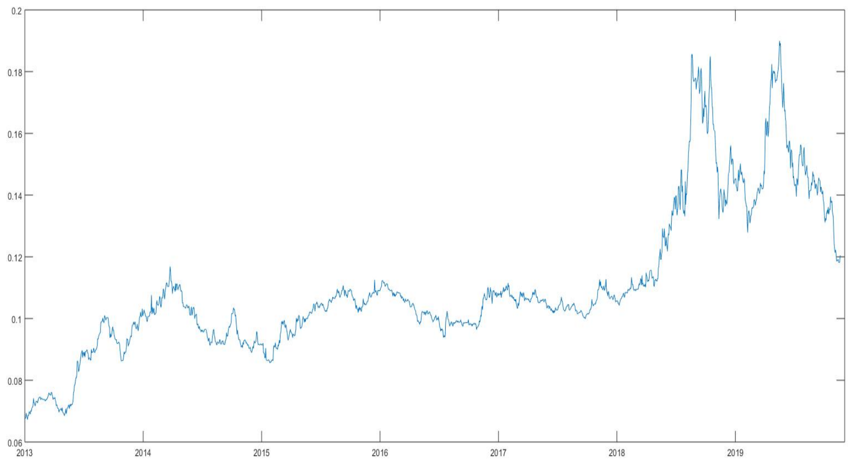


Figure 2.6: Estimated Long-Run Mean of Short Rate from 1y10y Swaption

The parameter estimations for the swaption specifications of 5y5y and 1y1y are also obtained. These parameter values are used to price out the call and option values on fixed income securities. The estimates of the parameter values obtained from different swaption specifications seem to be in line with the ones presented here.

2.4.2 Pricing Call and Put Options on Fixed Income Securities

Since the Vasicek model parameters are estimated for all swaption specifications, the next step is to use the parameter values to find out the prices of call and put options on fixed income securities. In this regard, the following charts provide option values for

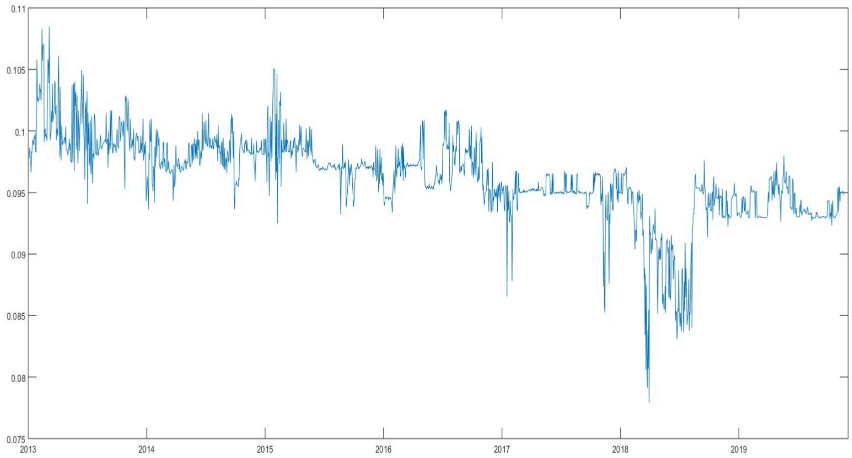


Figure 2.7: Estimated Volatility of Short Rate from 1y10y Swaption

different underlying assets; callable bonds, mortgage loan and puttable bonds from January 2013 to November 2019. Each point in the figures indicates the option value that's priced using the optimized Vasicek parameters.

Firstly, the prices of call options on callable bonds with 1y1y,1y10y and 5y5y specifications are obtained. Consider 1y10y specification, meaning that call option with the maturity of 1 year and callable bond has remaining maturity 11 years but callable at par at the end of year 1. The call option values with different specifications are also computed. The figures show that call option values tend to be more volatile in the last years. During the period of 2018, the sharp rise in short rates leads to lower call option values since the expected value of the call option declines significantly although the decline in mean-reversion rates leads to an increase in the value of call options. In other words, the rise in short rates cancels out the effect of the mean reversion rate. A similar situation is also observed for 5-year call option values on 10-year callable bonds and 1-year call option values on 2-year callable bonds. The rise in short rates tends to be a dominating factor for this period. Additionally, it is observed that call option values tend to rise in 2019 where the short rates tend to have a declining trend during this period where the other parameters such as volatility and mean-reversion rate remain relatively stable during this period.

Comparing the call option values across 1y1y, 5y5y and 1y10y specifications, it is observed that the call option value is the smallest for 1-year call option value on 2-

year callable bond since the maturity of the underlying is the shortest. Besides, call option value is greater for 1y10y specification than 1y1y specification.

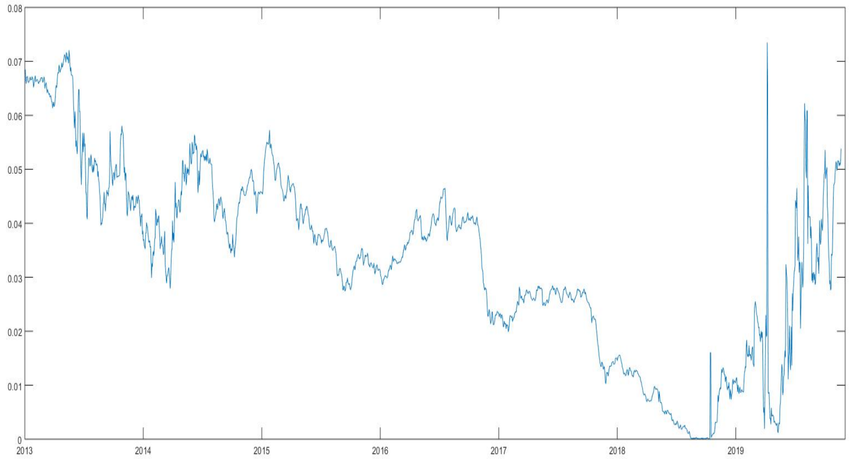


Figure 2.8: 1 Year Option Value on 2 Year Callable Bond

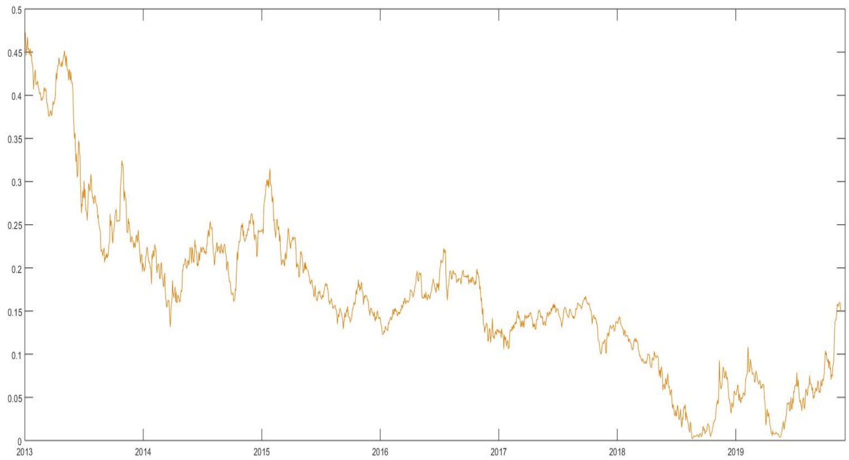


Figure 2.9: 1 Year Option Value on 11 Year Callable Bond

In addition to the call option pricing on callable bonds, the next step is to price a prepayment option on a mortgage loan with equal installments. A prepayment option on a mortgage loan can be considered as a series of call options on zero-coupon bonds with different time-to-maturities and different strike prices. In this regard, the call option values on zero-coupon bonds are weighted by the cash flow structure in mortgage loans. Since the mortgage loan is paid at equal payments, it can be stated that not only interest payments but also some portion of the notional amounts are paid during the life of mortgage loan, which makes a difference from the callable bonds.

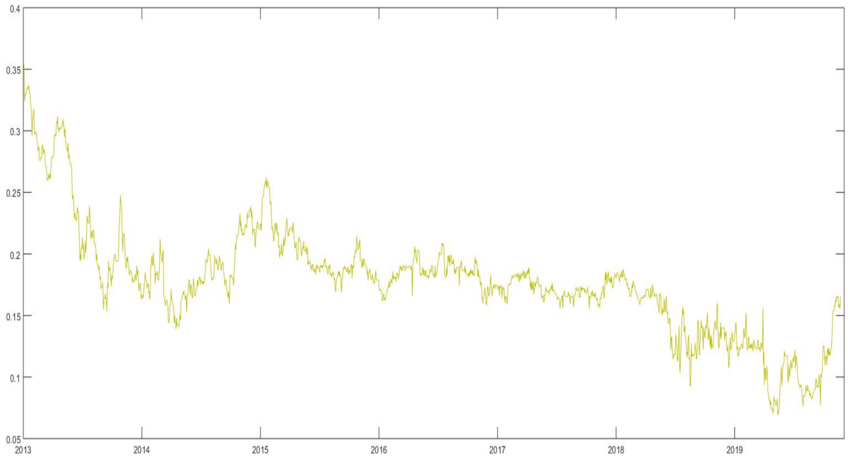


Figure 2.10: 5 Year Option Value on 10 Year Callable Bond

Additionally, the strike prices in mortgage loans are equal to the discounted value of the remaining cash flows with the initial yield-to-maturity of the loan, namely outstanding balance in a mortgage loan. Therefore, the strike prices for the prepayment option on the mortgage loan differ.

When the prepayment option costs are analyzed over time, it is observed that they are in line with the tendency of the call option values on callable bonds. However, the prepayment option values tend to be different due to the differences in strike prices and cash flow structure. Additionally, the prepayment option prices tend to increase during periods of higher short rates due to the fact that short rates are expected to revert back to its long-run mean. In this regard, the periods of spikes in short rates coincide with higher prepayment option prices for the mortgage loans initiated during those periods.

In addition to call and prepayment option values, put option values on fixed income securities can be obtained using the same methodology. In this regard, put option on fixed-income securities can be considered as a series of put options on zero-coupon bonds weighted by the cash flow structure of the underlying security. The put option values can be obtained from using the put-call option parity for bond options, which is described in the previous sections. Since the increases in short rates lead to a rise in the expected value from put options on puttable bonds, the values of put options tend to increase during the periods of spikes in short rates during 2018. However,

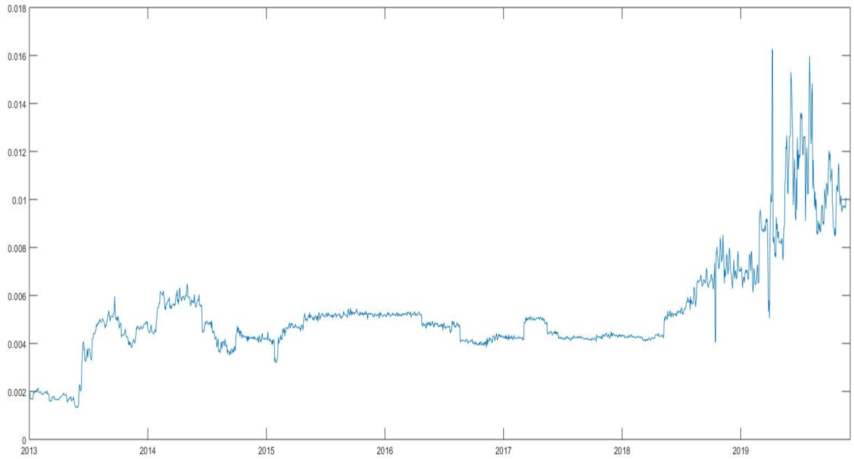


Figure 2.11: 1-Year Prepayment Option Value on 11-Year Mortgage Loan

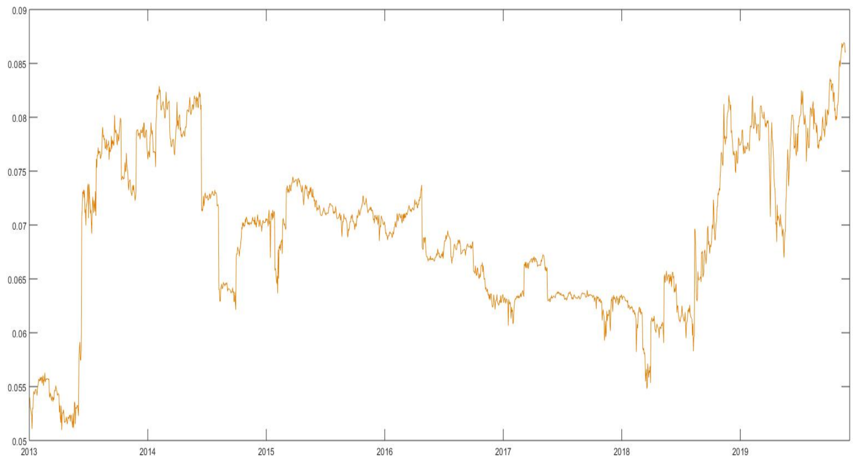


Figure 2.12: 1-Year Prepayment Option Value on 11-Year Mortgage Loan

as the rates tend to decline in 2019, the put option values on puttable bonds decline significantly. It is observed that put option values tend to be quite small before the period of 2018 whereas put option values after 2018 tend to be quite volatile. This is also observed in call option values on callable bonds and prepayment option values. It can be stated that the quick changes in the short rates across the curve tend to result in volatility in the option values, which also necessitates the need for hedging against interest rate risks. In this regard, the next section provides comparative statistics for the values of call and put options depending on the changes in the parameters; short rate, mean-reversion rate, long-run mean of short rates and volatility of short rates.

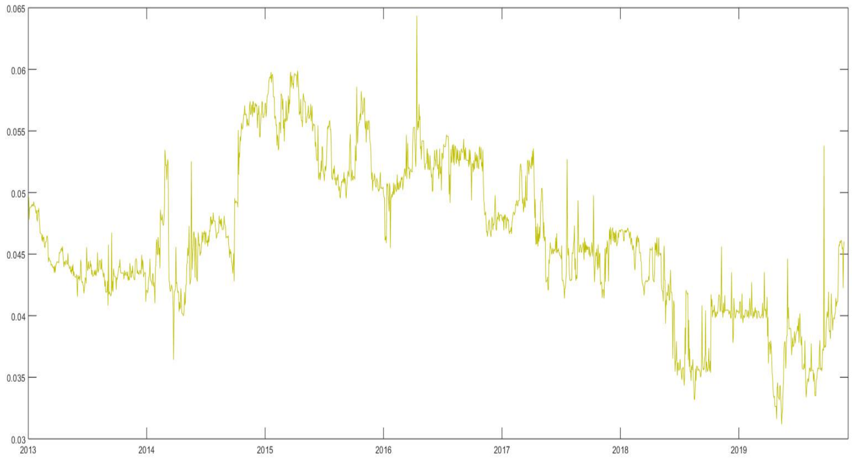


Figure 2.13: 5-Year Prepayment Option Value on 10-Year Mortgage Loan

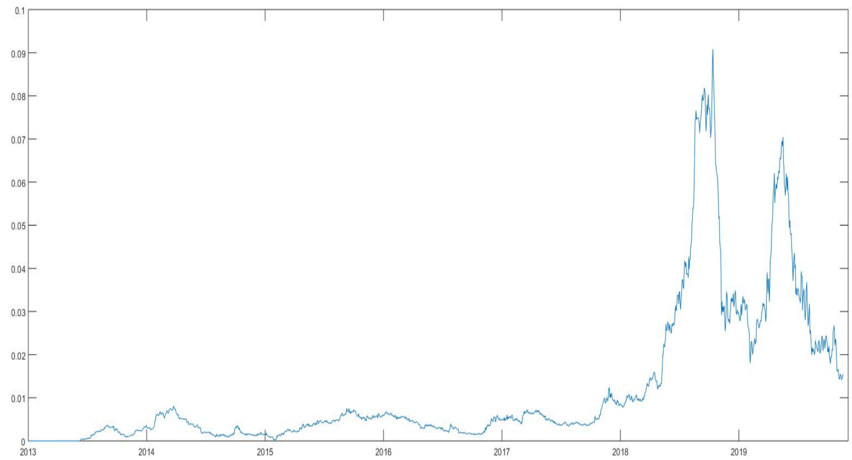


Figure 2.14: 1-Year Option Value on 2-Year Putable Bond

2.4.3 Comparative Statistics

The closed-form solution for the option values on fixed-income securities allows examining the comparative properties directly without relying on numerical methods. In this regard, the functional form regarding the option values show that the value of options on fixed-income securities depends on the short rate, mean-reversion rate, long-run mean of short rate and volatility of short rate, time to maturity of the underlying asset together with the strike price. In this regard, the sensitivities of the option values with respect to these parameters are exemplified in this section.

In this regard, the following figures are formed using the example of a 5-year call

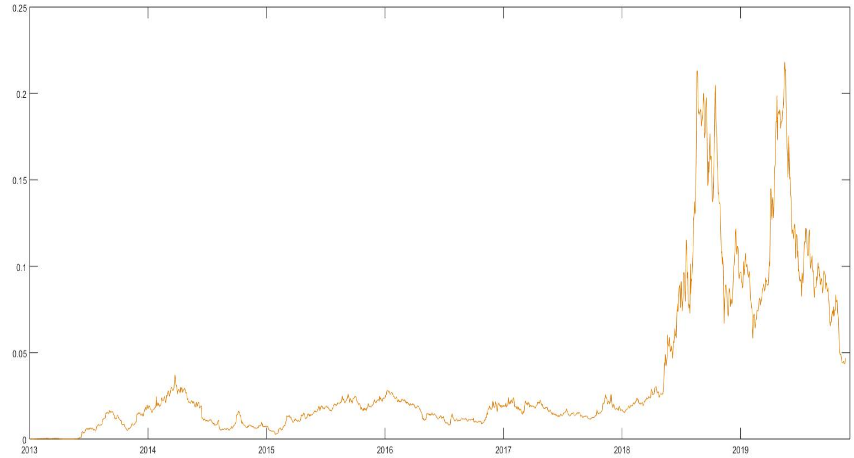


Figure 2.15: 1-Year Option Value on 10-Year Puttable Bond

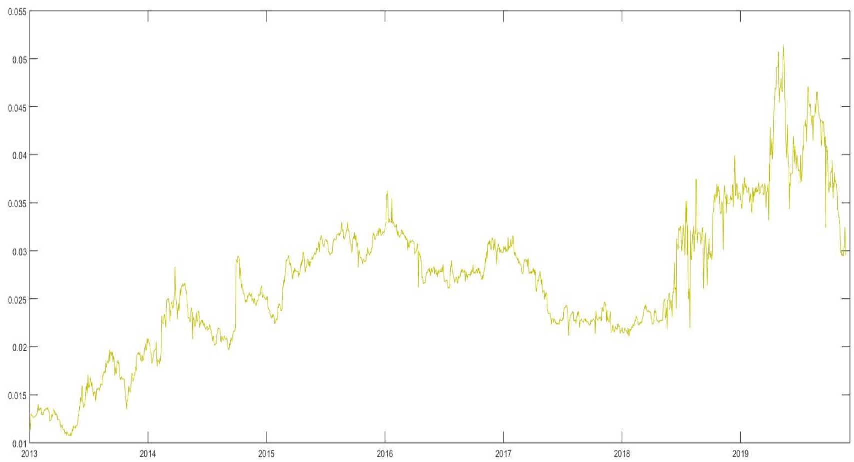


Figure 2.16: 5-Year Option Value on 10-Year Puttable Bond

option on a 10-year coupon-bearing bond with strike price at par, in other words, 10-year coupon-bearing bond callable at the end of 5^{th} year. In this regard, the bond is assumed to have monthly constant coupon payments during the life of the bond, with coupon rate equal to 12 percent annual. Besides the callable bond, the sensitivities of a 5-year put option value on 10-year coupon-bearing bond with respect to the variables are also analyzed. The differences between call option and put option values with respect to the parameters are also compared.

Firstly, the sensitivities of the option values to the changes in short rate are analyzed in the following figures. For the case of call option on a bond, it can be observed that keeping all others constant the call option value is a decreasing function of short rate.

The changes in short rate affect call option value in two ways. Firstly, the increases in short rate reduce the value of the remaining cash flows so that the expected value of the call option becomes smaller. Additionally, the increases in short rates also reduce the present value of the strike prices, which leads to an increase in the value of call option. However, the example shows that the first effect dominates the other one for the case of call option. Besides, it can be observed that the effect of short rate becomes smaller as the short rate increases.

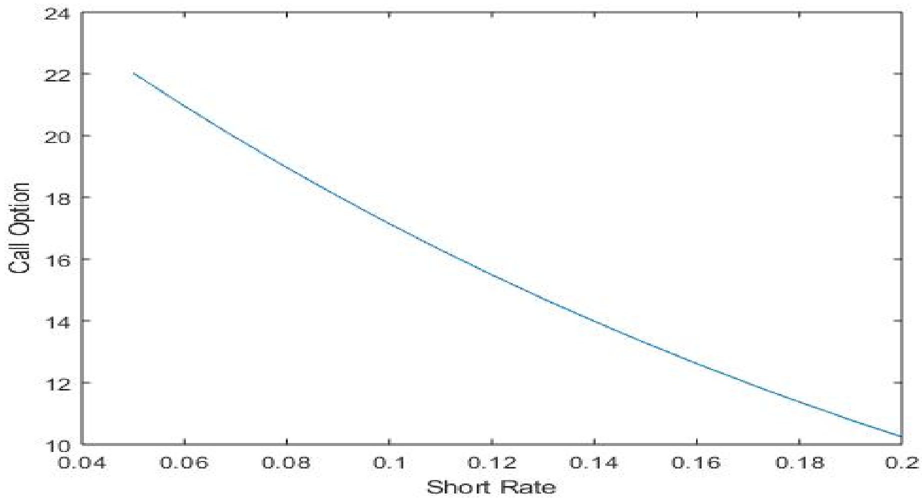


Figure 2.17: Sensitivity of Option Value on Callable Bond With Respect to Short Rate

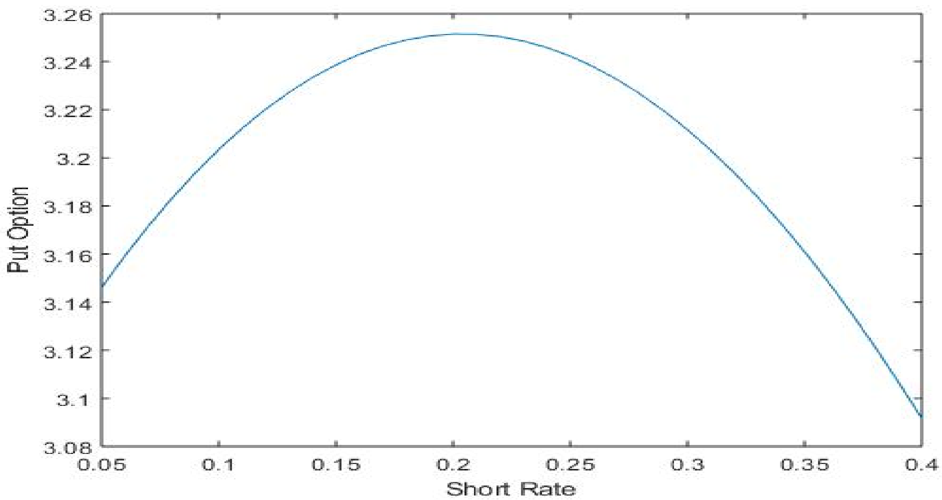


Figure 2.18: Sensitivity of Option Value on Putable Bond With Respect to Short Rate

Besides call option, the effect of short rate on put option is ambiguous due to the offsetting effects of short rate on option values. Figures show that the increases in short rate firstly lead to an increase in put option values, but after some level of short rate, the effect of short rate increases leads to a decline in put option values. Firstly, the increases in short rate initially lead to increases the value of put option due to the fact that the rises in short rate increase the expected value of the put option. However, after some level of short rate the second channel, decline in the value of strike prices, dominates the first effect. Therefore, the effect of short rate increases on the put option values is indeterminate, which is crucial for the hedging behavior for put options written on bonds or interest rate derivatives.

The effect of changes in strike prices on the value of call and put options on coupon-bearing bonds is illustrated in the following figures. It can be observed that the value of call options is negatively related to the strike prices. In other words, the value of call option on callable bonds tends to decline with the increase in strike prices. This stems from the fact that higher strike price leads to a decline in the expected payoff from the call option on coupon-bearing bond since higher strike price reduces the likelihood that call option is exercised. Additionally, it can be observed that the effect of strike prices becomes less pronounced with the increases in strike prices.

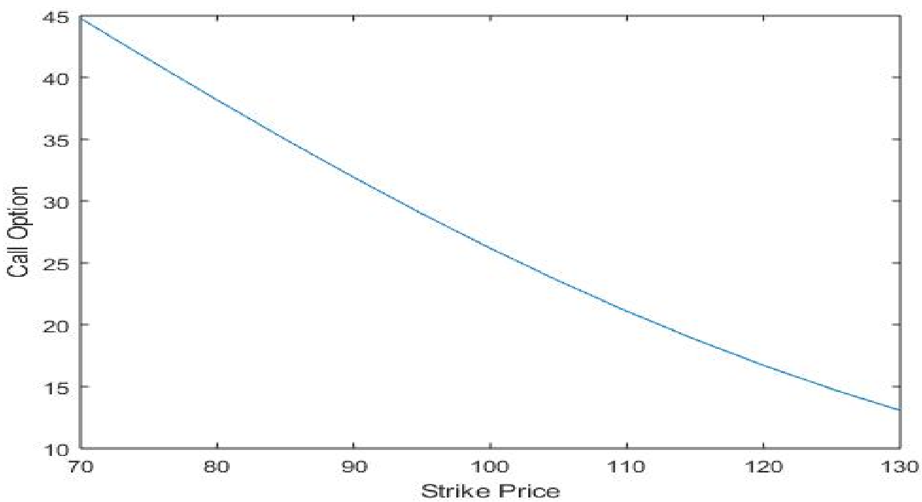


Figure 2.19: Sensitivity of Option Value on Callable Bond With Respect to Strike Price

In the case of put options on bonds or puttable bonds, the higher strike price is associated with higher put option values on puttable bonds since higher strike prices increase the expected payoff from the put option on coupon-bearing bond or puttable bonds. Besides, the relation between the strike prices and put option values tends to be nonlinear in the sense that the increase in put option values becomes more pronounced with the increase in strike prices.

One of the key determinants of the option values in coupon-bearing bonds is the volatility of short rate. Under the Black-Scholes model, option values are increasing functions of volatility regardless of call or put options. However, in the case of interest rate derivatives, the effect of volatility on call and put option values might differ depending on the level of volatility. The changes in volatility level influence the value of call and put options in two ways. Firstly, volatility has a convexity effect, which leads to an increase in the value of call and put options since increases in volatility result in higher expected payoff from the option. The second channel stems from the effect of volatility on the underlying bond prices. However, in the case of call options on bond, convexity effect dominates the second channel and call option prices are increasing functions of the volatility of short rate. However, the effect of volatility on the put option values is ambiguous. Up to some level of volatility of short rate, put option values tend to increase, which stems from the dominating effect of convexity channel. However, the effect of volatility on the value of underlying bond becomes more pronounced, which leads to a decline in the value of put options. In other words, after a threshold level of volatility, put option prices tend to be a decreasing function of volatility.

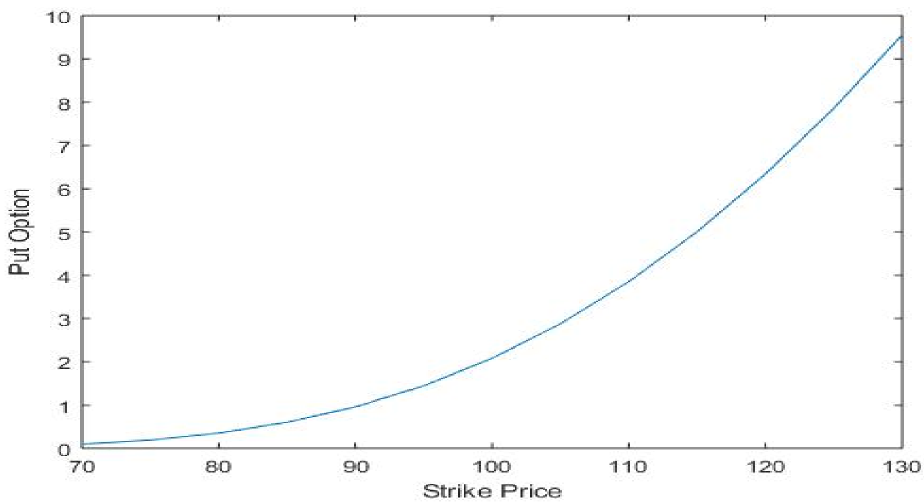


Figure 2.20: Sensitivity of Option Value on Putable Bond With Respect to Strike Price

The relationship between time-to-maturity of call and put options on coupon-bearing

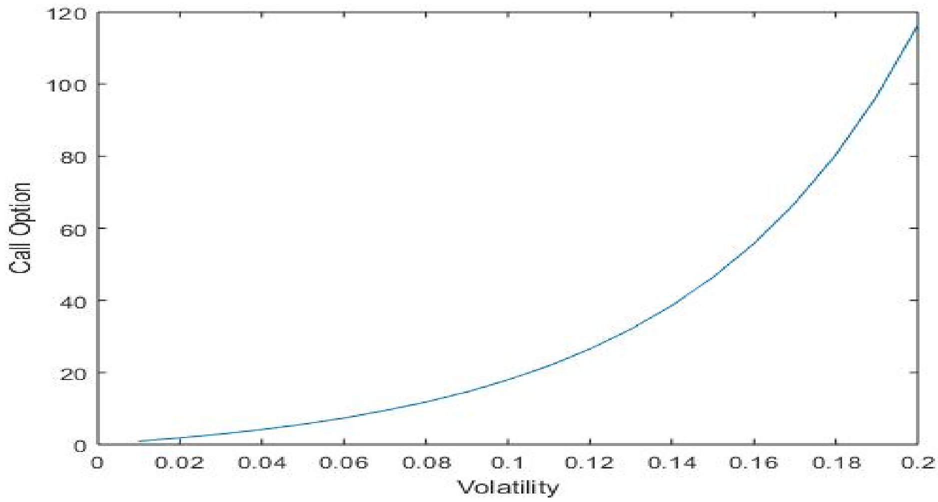


Figure 2.21: Sensitivity of Option Value on Callable Bond With Respect to Volatility

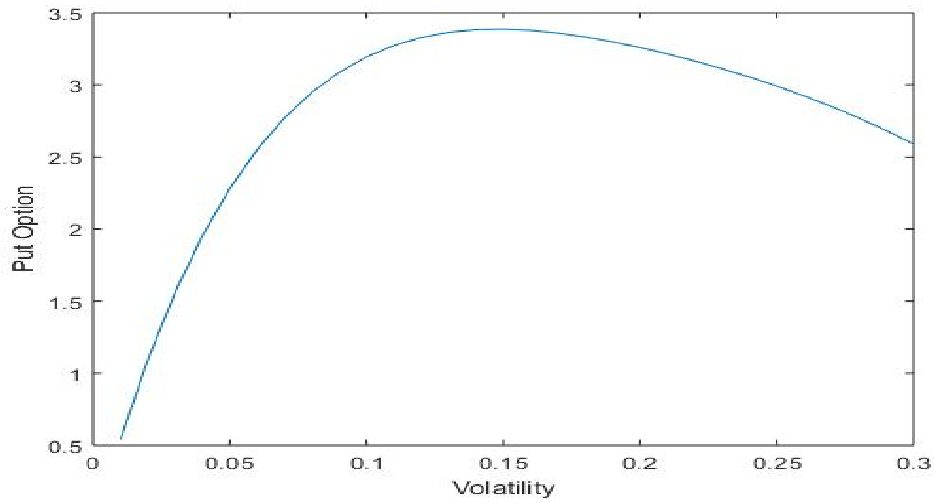


Figure 2.22: Sensitivity of Option Value on Putable Bond With Respect to Volatility

bonds is complicated. Although the increases in time-to-maturity of options lead to higher option values on the Black-Scholes model, the relationship becomes slightly different in call and put options on coupon-bearing bonds. In the case of call options on bonds, the increase in the time-to-maturity of options is generally associated with decline in call option prices. This stems from the property of the mean-reverting behavior of short rates. For small values of time-to-maturity, the option value is quite close to the intrinsic value. However, as the time-to-maturity of the option increases, the future distribution of underlying asset converges to a stationary distribution. In this regard, it can be stated that the expected payoff of call options does not increase with time-to-maturity sufficiently to offset the effect of an increase in the period which the expected payoffs are discounted. Therefore, for large values of time-to-maturity, the value of call options tends to be a decreasing function of time-to-maturity. In the case of put options on coupon-bearing bonds, the increase in the period which the expected payoffs are discounted leads to higher put option values until some level. However, the effect of time-to-maturity tends to reduce put option value. The following figures also provide an example of complicating relation between time-to-maturity and option values.

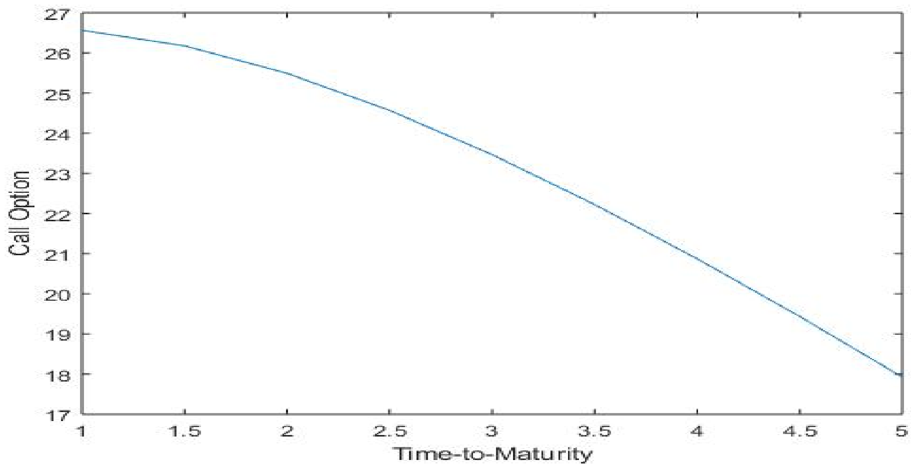


Figure 2.23: Sensitivity of Option Value on Callable Bond With Respect to Time-to-Maturity

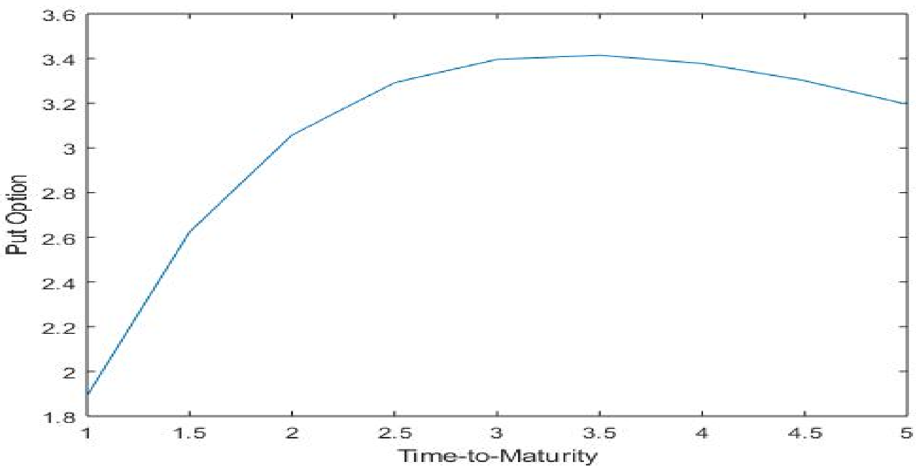


Figure 2.24: Sensitivity of Option Value on Putable Bond With Respect to Time-to-Maturity

Another important parameter for the value of call and put options on coupon-bearing bonds is the long-run mean of short rate. Under the Vasicek model, short rate reverts back to its long-run mean at a mean-reversion rate. Keeping the short rate and other parameters constant, the increases in the long-run mean of short rate tend to have a decreasing effect on the value of call option values since the short rates tend to converge to higher long-run mean. Therefore, the increases in the long-run mean of short rates lead to a decline in the expected payoff from call option. However, the increases in the long-run mean tend to have an increasing effect on the put option values due to the fact that it leads to higher expected payoffs. In this regard, the long-run mean and put option values are positively associated with each other.

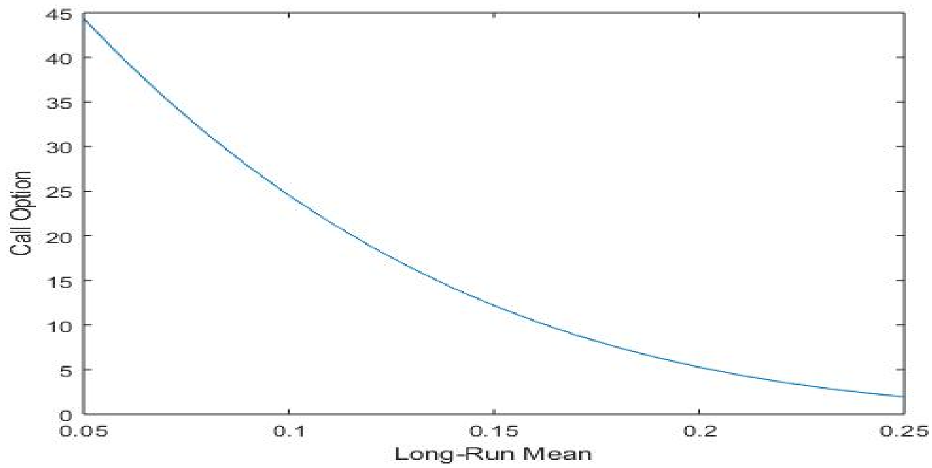


Figure 2.25: Sensitivity of Option Value on Callable Bond With Respect to Long-Run Mean

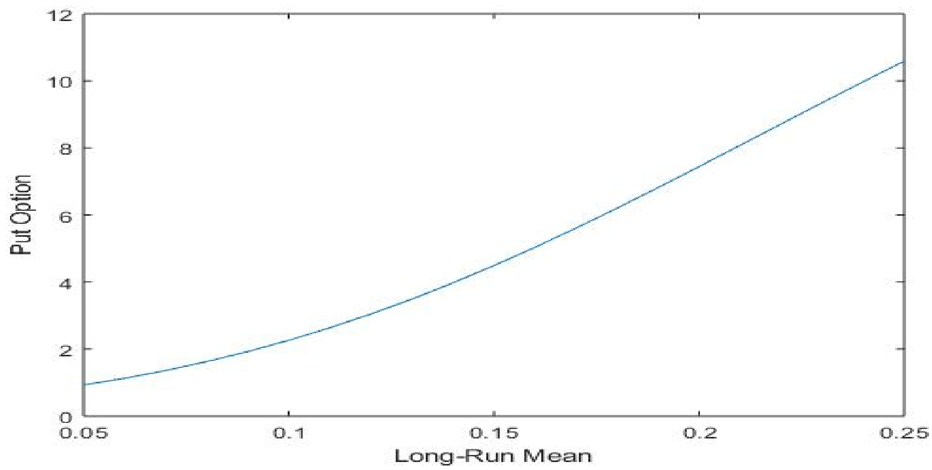


Figure 2.26: Sensitivity of Option Value on Putable Bond With Respect to Long-Run Mean

Mean-reversion rate is also another determinant of option values on coupon-bearing bonds. The increases in mean-reversion rates tend to influence negatively both the call and put option values since the higher mean-reversion rate means quicker adjustment which tends to reduce the option value on bonds. The following figures also show the negative relation between the mean-reversion rate and option values.

To sum, the sensitivities of model parameters on the value of call and put options tend to differ compared to the Black-Scholes model. The differences in the sensitivities are especially crucial for the hedging behaviour of these securities. Since changes in short rates affect both the price of underlying assets and discount factors, these effects tend to reinforce each other. Additionally, the effect of parameters such as volatility, time-to-maturity of options, mean-reversion rate and long-run mean tend to have slightly different results for call and put options.

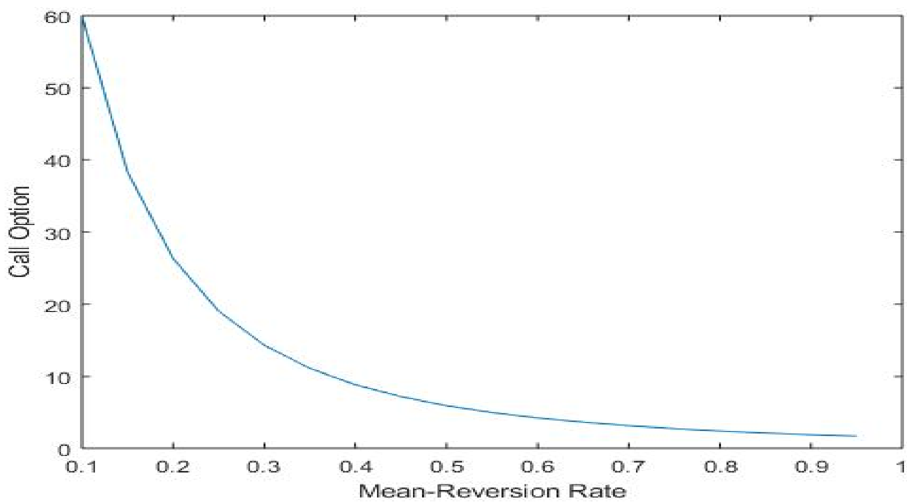


Figure 2.27: Sensitivity of Option Value on Callable Bond With Respect to Mean Reversion

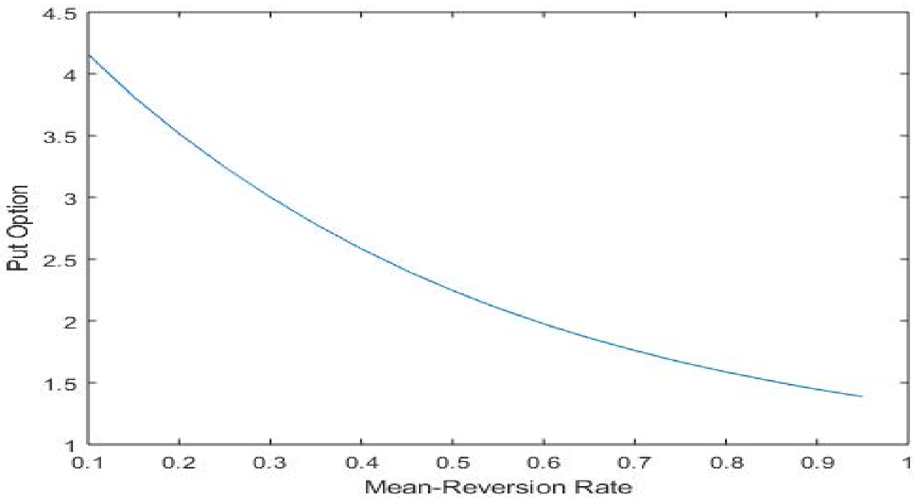


Figure 2.28: Sensitivity of Option Value on Putable Bond With Respect to Mean Reversion

2.5 CONCLUSION

The valuation of interest rate derivatives or embedded options in fixed-income securities is crucial for market practitioners and risk managers. Although there have been many models to price interest rate derivatives based on different assumptions, the inconsistency of the assumptions across different interest rate derivatives creates difficulty in aggregating interest rate exposures or risk management for fixed-income securities. Besides, the models tend to be specific to interest rate derivatives rather than providing a general picture of the interest rate derivatives. In this regard, the adaptation of a general methodology to price all interest rate derivatives without making additional assumptions has critical importance. This study is expected to contribute to the literature by providing a general approach that can be applied to any fixed-income security with regular or irregular cash flows using the Vasicek model.

The option pricing for interest rate derivatives involves four steps. The first step is to derive the explicit formula for an interest rate derivative traded in the market in order to estimate the Vasicek model parameters. In the context of the study, an explicit formula for the cross-currency swaption is derived. Then the next step is to estimate the Vasicek model parameters through nonlinear least squares for the period between January 2013 and November 2019 using daily quotes. The next step is to derive the closed-form solution for the interest rate derivative which is demanded to price out. The last step is to plug the estimated Vasicek model parameters into the closed-form solution and price the interest rate derivative. Since the methodology uses interest rate derivatives traded in the market for parameter optimization, it can be said that the option pricing on the fixed-income security is consistent with the market prices. Additionally, since there is a closed-form solution, it is much easier and useful to derive the sensitivities of the option value to the changes in the model parameters, which is crucial for hedging behaviour.

In this regard, the study estimates the Vasicek model parameters using cross-currency swap quotes traded in the market. The parameter estimates for 1y1y, 5y5y and 1y10y swaption specifications tend to be quite stable over time and consistent with each other, which provides a good signal about the reliability of the model. Additionally, model-in-fit errors tend to be quite small. Then the call option on a callable bond, put option on a puttable bond together with prepayment option on a mortgage loan are priced using the estimated Vasicek parameters. The option values tend to fluctuate during the last 2 years, where there is a significant amount of volatility in spot rates. During this period, call option and prepayment option values tend to decline as the short rate rises then increase to normal levels together with the decline in short rates. However, the behavior of put option values tends to increase at first together with the rise in short rate. The time-series observations for the option values also reflect the market sentiment about the interest rates.

Then the last step is to obtain the comparative statistics of the option values for the changes in the short rate, mean-reversion rate, volatility of short rate and long-run mean of short rate, which provides important details in terms of hedging against the interest rate risks. The sensitivities of the option values tend to behave differently than the Black-Scholes model due to the main fact that interest rates not only affect the payoff structure of the derivatives but also discount terms. Therefore, the effect of parameters tends to behave differently after threshold levels. The comparative statistics are especially important for the aggregation of interest rate exposures, which requires a more detailed examination due to the nonlinear relationship between the model parameters and option values.

The results of the study show that interest rate derivatives and embedded options in fixed-income securities can be priced out consistent with the market prices. Additionally, the use of a simple and unique model for the valuation of all interest rate contingent assets and liabilities allows portfolio-managers to aggregate the interest rate exposure without relying on different models. Besides, this also creates the opportunity to figure out the sensitivities of the interest rate dependent securities with respect to the model parameters, where it cannot be reached using different models with their specific model parameters. Although the study presents the applications for European type of options on interest rate derivatives or European type of embedded options on fixed-income securities it can be applied to Bermudan or American type of interest rate derivatives through numerical solutions. Although the study uses the Vasicek model, which provides a good model-in-fit and allows generalization to all types of interest rate derivatives, other interest rate models wit suitable properties that allow generalization can be applied as well.

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