MATHEMATICAL THINKING IN THE CLASSROOM ON DERIVATIVE: FOSTERING UNIVERSITY STUDENTS' MATHEMATICAL THINKING

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ABSTRACT

MATHEMATICAL THINKING IN THE CLASSROOM ON DERIVATIVE: FOSTERING UNIVERSITY STUDENTS' MATHEMATICAL THINKING

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The purpose of this study was to investigate the opportunities provided mathematics students to engage in mathematical thinking during the instruction of derivative concepts and categorize the students' mathematical thinking on derivative concept through test results and interview. Multiple methods of collecting data were used. A calculus class at a public university in Ankara was observed in spring semester of 2014-2015 academic year. At the end of the semester, the Thinking-about-Derivative test was applied and interviews were conducted with 5 participants. The data revealed that considering the instructions of derivative, the students were provided a platform for their formal, axiomatic, algebraic, iconic, algorithmic and enactive thinking. In addition, Thinking-about-Derivative Test results and interviews indicated that the students activated their formal, axiomatic, algebraic, iconic, algorithmic and enactive thinking while they answered the test items, as well. The findings also showed that the different mathematical thinking aspects worked collaboratively.

Keywords: Mathematical Thinking, Derivative, Calculus, Classroom Observation

ÖZ

SINIF ORTAMINDA TÜREV KONUSUNDA MATEMATİKSEL DÜŞÜNME: ÜNİVERSİTE ÖĞRENCİLERİNİN MATEMATİKSEL DÜŞÜNMELERİNİN TEŞVİK EDİLMESİ

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Bu çalışmanın amacı kalkülüs sınıfında matematik öğrencilerinin türev konusundaki matematiksel düşünmelerini sağlayan olanakları belirlemek ve öğrencilerin türev konusundaki matematiksel düşünme boyutlarını test sonuçları ve görüşmeler aracılığıyla sınıflandırmaktır. Bu amaçla birden çok veri toplama yöntemi kullanılmıştır. 2014-2015 eğitim öğretim yılı ilkbahar döneminde Ankara'da bir devlet üniversitesinde kalkülüs sınıfında gözlem yapılmıştır. Dönem sonunda Türev Test'i uygulanmış, 5 katılımcı ile görüşme yapılmıştır. Veriler 1 kalkülüs öğretmeni ve 20 test katılımcısından elde edilmiştir. Bu çalışmanın sonuçları türev konusunun öğretiminde öğrencilere biçimsel, belitsel, cebirsel, görüntüsel, algoritmik ve eylemsel düşünmeleri için olanaklar sağlandığını ortaya çıkarmıştır. Ek olarak, test ve görüşmeler öğrencilerin test maddelerini cevaplarken aynı düzeyde biçimsel, belitsel, cebirsel, görüntüsel, algoritmik ve eylemsel düşünme tiplerini kullandıklarını göstermiştir. Bulgular aynı zamanda matematiksel düşünme boyutlarının iş birliği içerisinde işlediğini ortaya çıkarmıştır.

Anahtar Kelimeler: Matematiksel Düşünme, Türev, Kalkülüs, Sınıf Gözlemi

To my parents, Arife Kartal and Haydar Kartal

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LIST OF ABBREVIATIONS

FORMTHK: Formal Thinking

AXIOTHK: Axiomatic Thinking

ALGETHK: Algebraic Thinking

ICONTHK: Iconic Thinking

ALGOTHK: Algorithmic Thinking

ENACTHK: Enactive Thinking

TDT: Thinking-about-Derivative Test

CHAPTER 1

INTRODUCTION

1.1 Mathematical Thinking and Derivative

Mathematical thinking is an important goal of schooling and important way of learning and teaching mathematics (Stacey, 2006). Because of that, many curriculum frameworks (e.g., National Council of Teachers of Mathematics, (NCTM, 2000, 2014); Turkish national elementary school curriculum (TTKB, 2018); Turkish national high school mathematics curriculums (TTKB, 2018)) emphasize that one of the main objectives school mathematics education program is to get students to gain mathematical thinking skill. Mathematical thinking helps us to solve problems in any domain since every problem-solving activity requires mathematical thinking (Henderson et al., 2001). Besides, Devlin (2012) reported that “mathematicians, scientists, and engineers need to “do math”. But for life in the twenty-first century, everyone benefits from being able to think mathematically to some extent”. Therefore, mathematical thinking is important not only for people who use mathematics directly but also for everyone.

Henningsen and Stein (1997) emphasized the crucial role of classroom-based factors in development high-level mathematical thinking. They suggested that classrooms must become environments in which students are able to engage actively in mathematical activity in order to develop students' capacities to "do mathematics".

Herrera et al. (2007) state that all students can learn to think mathematically. Besides, Henningsen and Stein (1997) claim that how academic work gets done and by whom, the attributes of tasks, and dispositions of teacher and student as the features of pedagogical and learning behaviors influence the ways of students' thinking skills.

Therefore, Bergqvist and Lithner (2012) underlines the importance of the teacher since how the teacher solves the problem is a model or example of good reasoning. Moreover, Isoda (2012) asserts that by teaching mathematical thinking consistently, the teachers can prepare children to think by and for themselves and to nurture children who think and learn mathematics by/for themselves. However, internalization of mathematical thinking requires time and great effort to develop effectively. Manouchehri, et al. (2018) states that "advancing the mathematical thinking of students is not a responsibility that one teacher should shoulder alone; neither is it a goal that can be achieved in one grade level. Its development demands overtime exposure as well as continuity in expectations and across a variety of contexts."

Derivative is one of the fundamental and most important subjects of high school mathematics and Calculus courses. However, students experienced a lot of difficulties when they learn and apply derivative. For example, students may have difficulty in defining the derivative (e.g., Orton, 1983; Habre & Abboud, 2006; Viholainen, 2007), finding the slope of the tangent line of a function at a given point, differentiating functions (e.g., Tall, 1992), some theorems on derivative and their proofs (e.g., Viholainen, 2007) sketching the graph of a function (e.g., Asiala, Cottrill, Dubinsky, & Schwingendorf, 1997; Orhun, 2012; Berry & Nyman, 2003; Ubuz, 2007; Haciomerolu et al., 2010) or solving an optimization problem (e.g., Tall, 1992; Tarmizi, 2010, Swanagan, 2012; White & Mitchelmore, 1996; Villegas, Castro, & Gutiérrez, 2009). In order to overcome such difficulties, we need to better understand the mathematical thinking aspects of the students in derivative concept and provide a platform for students' mathematical thinking. Taken together, the purpose of this study was to investigate the opportunities provided mathematics students to engage in mathematical thinking during the instruction of derivative concepts and categorize the students' mathematical thinking on derivative concept through test results and interview.

1.2 Research Questions

The present study aims to answer the following research questions:

- What kind of opportunities in Calculus class provided mathematics students to engage in mathematical thinking during the instruction of derivative concepts?

- In which mathematical thinking dimension do students answer the derivative concept questions?

1.3 Significance of the Study

The results of this study would contribute to distinguish the opportunities provided students to engage mathematical thinking in considering that classroom based factors influence the mathematical thinking on derivative and that mathematical thinking plays an important role in mathematics education. The results of the study also answer in which mathematical thinking dimensions do students answer the derivative concept questions. Therefore, another major contribution of the present study is that its findings would help instructors, researchers and, students themselves to define and categorize students' mathematical thinking aspects. Regarding the tasks and mathematical thinking skills of the students, the instructors can select appropriate instructional techniques to enhance each thinking dimensions and to provide a platform for students' mathematical thinking on derivative concept problems.

Derivative is one of the most important and comprehensive subject of mathematics. Previous researchers have examined students' understanding of derivative and difficulties regarding the derivative concept. However, there has not been much research approaching students' ways of thinking about derivative (Firouzian, 2014). Therefore, the study would contribute to the area of research on mathematical thinking on derivative and the result of this study would be beneficial for both the instructors teaching derivative concept and the researchers who are interested in students' mathematical thinking way on derivative concept.

CHAPTER 2

LITERATURE REVIEW

In this chapter, the review of related literature concerning the mathematical thinking and mathematical thinking on the derivative concept are presented.

2.1 Mathematical Thinking

The theoretical framework of mathematical thinking is used in a number of the studies (e.g., Bruner, 1966; Burton, 1984; Henderson et al., 2002; Mason, Burton & Stacey, 2010; Schoenfeld, 1992; Stewart & Thomas, 2009; Tall, 2004). Mathematical thinking is defined by Burton (1984) as a style of thinking that is a function of particular operations, processes, and dynamics recognizably mathematical and defined by Mason, Burton, and Stacey (2010) as a process by which we increase our understanding of the world and extend our choices. Henderson et al. (2002) states mathematical thinking is “applying mathematical techniques, concepts and processes, either explicitly or implicitly, in the solution of problems”. Learning to think mathematically, as Schoenfeld (1992) notes, means developing a mathematical point of view-valuing the processes of mathematization and abstraction and abstraction and having the predilection to apply them, and developing competence with the tools in the service of the goal of understanding structure-mathematical sense making. Therefore, mathematical thinking comprises “mental operations that are facilitated by the learners' mathematics knowledge and their positive disposition towards mathematical problem solving” as Beng and Yunus (2015) asserts. Taken together, the definition of mathematical thinking used in the present study as the definition of Lim and Hwa (2006) which states that mathematical thinking is “a mental operation supported by mathematical knowledge and certain kind of predisposition, toward the attainment of solution to problem.”

In order to better understand mathematical thinking, the aspects of it should also be defined clearly. Concerning mathematical thinking, Tall (2004) introduced the three worlds of mathematics. The first world is originated from our perceptions of the world called conceptual embodiment. He emphasizes here that it includes “not only our mental perceptions of real-world objects, but also our internal conceptions that involve visuospatial imagery.” The second world is proceptual-symbolic world and it is the place of symbols. These world begin with “actions that are encapsulated as concepts by using symbol that allow us to switch effortlessly from processes to do mathematics to concepts to think about.” The third world is formal-axiomatic world which is based on properties. Tall asserts that the formal world come out with the embodied conceptions and symbolic manipulation and claims that the reverse is also valid. Although Tall (2004) defines the world of mathematics by cognitive development, he says that mathematical growth of individuals in these world is different.

In the cognitive development of mental representations, Bruner (1966) suggested three ways to gain the skill of translating experience into a model within three forms of representations, namely; enactive, iconic and symbolic. Enactive representation is through action and based upon “a learning from responses and forms of habituation”. Iconic representation “depends upon visual or other sensory organization and upon the use of summarizing images”. Symbolic representation, on the other hand, is “in words or languages”.

Concerning the mathematical thinking as a process, algorithmic, intuitive and formal dimensions of knowledge, formulated by Fischbein (1994), should cooperate in the processes of concept acquisition, understanding and problem solving (Tirosh, Fischbein, Graeber, & Wilson, 1998).

Aydın and Ubuz (2015) point that mathematical thinking evolves through enactive, iconic, algorithmic, algebraic, formal, and axiomatic thinking. They state that each of these six distinct mathematical thinking aspects describe different processes with its own way, although they are interrelated. Aydın and Ubuz generated a multidimensional test to assess undergraduate students' mathematical thinking about derivative (2015). The Thinking-about-Derivative Test (TDT) including 30-item multiple-choice TDT, which comprises 6 mathematical thinking aspects, enactive, iconic, algorithmic, algebraic, formal, and axiomatic thinking, is a useful tool for mathematics education researchers and mathematicians. Results of the their study also showed that although six mathematical thinking aspects are distinct, they are interrelated. These mathematical thinking aspects should cooperate in any mathematical task and each of them stay active when students build appropriate links among them (2015).

Activation of one of the different thinking aspects can affect the others. The study conducted by Alghtani and Abdulhamied (2010) showed that using the geometric representation helps the students to read the algebraic word problem and understand the algebraic concepts through building correct image. Moreover, it enables students to think algebraically through the use of correct symbols, build geometric representations for the concepts, find out the relationship in the patterns, build the new patterns, or solve problems.

Stewart and Thomas (2009) used a framework both three world of mathematics and APOS theory to assess the feasibility of the framework as a research tool for describing the level of students' conceptual and procedural understanding of the linear algebra concepts, their difficulties with these concepts, and the effects of using embodied ideas in the teaching of linear algebra. According to their results, most of the students need to symbolize the embodiment and embody the symbolism first, and only after fully integrating them they will reach the formal world.

In their study of Manouchehri, et al. (2018) compared two teaching approaches, which are solving mathematical problems and seeking mathematical structures in order elaborated on how a focus on the development of mathematical thinking centered around repeated reasoning and mathematical structure might be nurtured by relying on problem solving and inquiry and they aimed to specific actions while helping students develop proving and reasoning skills. As a result of by two teachers differed in the type of thinking they motivated, it was claimed that both types of mathematical work discussed are crucial to the advancement of learners' mathematical thinking.

2.1.1 Formal Thinking

Tirosh et al. (1998) assert that the formal thinking is related by definitions of concepts and structures relevant to a specific content domain. Therefore, they related formal thinking with the capabilities of learners to recall and administer definitions and theorems in the problem solving situations. As cited in Aydın and Ubuz (2015), formal thinking involves using, connecting, and interpreting various conceptual representations; and recalling, distinguishing, and integrating definitions, principles, facts, and symbols in a mathematical setting (Martin, 2000).

2.1.2 Axiomatic Thinking

Özdil (2012) asserts that axiomatic thinking contains proofs and proving which are supported by verification, justification, and refutation. Therefore, axioms, theorems, and proofs are linked to axiomatic thinking. Moreover, Ko and Knuth (2009) assert that in advanced mathematical thinking, proving and refuting play a fundamental role in deciding whether and why a proposition is true or false. In addition, Tall (1985) claims that an appropriate cognitive development should eventually lead to proofs that are both intuitive and rigorous. Moreover, as Devlin (2012) states, proofs are constructed to either establish truth or communicate to others and for both purposes, proof of a statement must explain why the statement is true.

The results of the study conducted by Ko and Knuth (2009) showed that completing mathematical proof was difficult for the majority of undergraduate students. As a result, they claimed that students need both qualified knowledge of definitions, axioms, and facts, and an understanding of what counts as coherent proof.

2.1.3 Algebraic Thinking

“Algebraic symbols and procedures for working with them are a towering mathematical accomplishment in the history of mathematics and are critical in mathematical work. Algebra is best learned as a set of concepts and techniques tied to the representation of quantitative relations and as a style of mathematical thinking for formalizing patterns, functions, and generalizations” (NCTM, 2000).

Radford (2010) claims that algebraic thinking as a particular form of reflecting mathematically and introduces the three elements which make thinking algebraic. One of these interrelated elements deals with a sense of indeterminacy that is proper to basic algebraic objects such as unknowns, variables and parameters. The second element is related to the indeterminate objects which are handled analytically. The third element is the typical symbolic mode that it has to designate its objects.

Vennebush, Marquez and Larsen (2005) claim that algebra is often described as the study of generalized arithmetic; however, it focuses on operations and processes. Therefore, they state that algebraic thinking is used in any problem solving activity that combines a mathematical process with algebraic structures.

Alghtani and Abdulhamied (2010) emphasized that algebra is a way of thinking and it is not just solving for x and y. They also added that algebra helps students think about mathematics at an abstract level and it provides students with a way to reason about real-life problems. The authors stated three components of algebraic thinking as making generalizations, conceptions about the equals sign and reasoning about unknown quantities. In addition, Radfold (2010) suggested a typology of forms of algebraic thinking factual, contextual, and symbolic. Therefore, algebraic thinking is based on a coordination of gestures, words, and symbols.

To develop algebraic way of thinking, Kieran (2004) suggests some adjustments including a focus on relations and not merely on the calculation of a numerical answer; a focus on operations as well as their inverses, and on the related idea of doing / undoing; a focus on both representing and solving a problem rather than on merely solving it; a focus on both numbers and letters, rather than on numbers alone and refocusing of the meaning of the equal sign.

2.1.4 Iconic Thinking

Aydın and Ubuz (2015) state that graph reading, graph interpretation, graph construction, and graph evaluation as the four main processes of iconic thinking. Haciomeroglu et al. (2010) also consider students' solutions to be visual when the solutions are based on graphic representations. The researchers (2010) state that visual solutions are image-based and students using image-based solutions can operate on their images without feeling the necessity of another thinking process. Presmeg (1997) defines visualizer as the person who prefers to use such visual methods when there is a choice; and defines visual image as a mental construct depicting visual information. In addition, Habre and Abboud (2006) emphasize that visualizers prefer visual methods even a mathematical problem may also be solved by non-visual means.

Viholainen (2007) claims that iconic thinking is probably the most usual type of informal thinking in mathematics and visualization has a crucial role in inventing ideas in problem solving situations. By this way, Zimmermann and Cunningham (1991) define mathematical visualization as the process of forming images either mentally, or with pencil and paper, or with the aid of technology; and using such images effectively for mathematical discovery and understanding. They also emphasize to visualize a problem means to understand the problem in terms of a diagram or visual image. Therefore, as also Arcavi (2003) stated, visualization is "the ability, the process and the product of creation, interpretation, and use of and reflection upon pictures, images and diagrams".

2.1.5 Algorithmic Thinking

Curcio and Schwartz (1998) define algorithm as a procedure, an efficient method, or a rule for computation. In her study, Özdil (2012) explains the requirements of effective progress in algorithmic thinking as "identifying the situation to which procedure applies, the correct order of algorithms, the correct completion of steps, and finally recognizing the correctly completed procedure". Besides, Tirosh et al. (1998) assert that the algorithmic dimension is basically procedural in nature and involves students' capability to explain the successive steps included in various, standard procedural operations.

Fischbein (1994) states that "a widespread misconception according to which if you understand a system of concepts, you spontaneously become able to use them in solving the corresponding class of problems" and emphasizes the necessity of the skills can be acquired only by practical and systematic training. He also claims that mathematical reasoning cannot be reduced to a system of solving procedures. The formal justification of the respective procedures has to be provided and the students are taught not only the algorithms themselves but also why they do what they do.

2.1.6 Enactive Thinking

According to Turner (2016), the enactive aspect of cognition is the most interesting part of thinking and, how we enact the world underlie enactive thinking. Concerning enactive thinking, the embodied world includes not only our mental perceptions of real-world objects, but also our internal conceptions that involve visuospatial imagery (Tall, 2014). Aylwin (1988) also claims that enactive imagery is temporally and affectively organised, using transitive action and emotive structure.

Aydın and Ubuz (2015) assert that examining various attributes of a particular mathematical, physical, or social context, embodying key aspects of these attributes into a mathematical model, relating a subset of those key aspects through operations, equations, or functions and using resulting internal and external representations to solve problems activate enactive thinking.

Besides, Garderen and Montague (2003) claim that mathematical problems are challenging problems set in realistic contexts that require understanding, analysis, and interpretation. They are not simply computational tasks embedded in words; instead, they require appropriate selection of strategies and decisions that lead to logical solutions.

2.2 Mathematical Thinking on Derivative

Students encountered by the derivative concept at the first time when they were at 12th grade. According to the Turkish national high school mathematics curriculums (TTKB, 2018), students are taught the definition and meaning of the derivative, calculation the derivative of a function, and some application of derivative concept. However, the majority of students have difficulties when they learn derivative concept and apply it. Numerous studies have examined the difficulties in derivative concept (Hashemia et al., 2014; Orton, 1983; Tall,1992; Teuscher & Reys, 2010; Viholainen 2006, 2007; Zandieh, 1997), some theorems on derivative (Viholainen 2006), graph interpretations of the derivative and curve sketching (Asiala, Cottrill, Dubinsky, & Schwingendorf, 1997; Berry & Nyman, 2003; Haciomerolu et al., 2010; Orhun, 2012; Ubuz, 2007) and optimization problems (Swanagan, 2012; Tall, 1992; Tarmizi, 2010; Villegas, Castro, & Gutiérrez, 2009; White & Mitchelmore, 1996), specifically.

Zandieh (1997) explains that the functions have many representational environments which include analytic or symbolic, graphic, numeric, verbal, and physical representations and the derivative of a function may be described in terms of the such representations. Concerning the representations of derivative of a function, she clarifies that “the concept of derivative can be seen graphically as the slope of the tangent line to a curve at a point or as the slope of the line a curve seems to approach under magnification; verbally as the instantaneous rate of change; physically as speed or velocity; and symbolically as the limit of the difference quotient.”

In his study which the students’ difficulties in calculus is discussed, Tall (1992) asserts that “mental images of functions, the Leibniz notation – a ‘useful fiction’ or a genuine meaning, translating real-world problems into calculus formulation, selecting and using appropriate representations, algebraic manipulation – or lack of it, difficulties in absorbing complex new ideas in a limited time, difficulties in handling quantifiers in multiply-quantified definitions, consequent student preference for procedural methods rather than conceptual understanding” are some of the difficulties students encounter in the calculus.

In his research, Orton (1983) investigated the errors on differentiation by categorizing them as structural, arbitrary and executive and explained some misconceptions on the concept. As a result of the study, he showed students’ errors and misconceptions about rate, average rate and instantaneous rate, average rate of change from curve, carrying out differentiation, symbol of derivative, etc. The items which were concerned with understanding differentiation and graphical approaches to rate of change were the most difficult items for both high school and college students. In order to cope with such errors and misconceptions, he made suggestions. According to him, the rate of change should be understood together with the ratio and graphical representations, real-life situations should be used as the data for graphs before more algebraic approaches are used and non-linear as well as linear graphs should also be introduced. It shows that activating different mathematical thinking aspects together can decrease the errors and misconception.

The research conducted by Viholainen (2006), who claims that mathematical thinking is based on different kinds of internal representations of mathematical concepts, examined some features of two students' thinking concerning differentiability. The results showed that it was difficult to consult definitions for the students. One of the students answered questions by using an algebraic method and the other student answered the same questions in visual method. The results of the research also indicated that students had difficulty in relating continuity and differentiability. Although both students had known that differentiability implies continuity, their weak interpretation of differentiability caused them to think that a function which is not continuous can be differentiable.

In addition, Hashemia et al. (2014) stated that undergraduate students have serious difficulties in understanding conceptually of derivation. They explained that the difficulties in the conceptual understanding of derivation resulted from focusing on the symbolic aspect more than the embodied, lack of making logical connection between these aspects, and weakness of dealing with generalized question. Also, the study conducted by Habre and Aboud (2006) revealed that for most students, the algebraic representation of a function dominated their thinking.

The results of the study conducted by Haciomerolu et al (2010) to gain understanding of three calculus students' mental processes and different preferences for mathematical thinking used to create meaning for derivative graphs showed that the students constructed different representations leading to different understandings of derivative graphs. When the students sketched antiderivative graphs when presented with derivative graphs, their thinking processes were examined and two students whose cognitive preferences were strongly visual or analytic and who did not synthesize visual and analytic thinking had different difficulties associated with their preferred modes for mathematical representation and thinking.

The results of the study, which was conducted by Orhun (2012) to investigate an specific difficulties based on the graph of derived function, indicated that students had difficulty in relating the graph of derived function and the original function. According to the findings of the study, when students describe the graph of derived function, they did not use the mathematical language. Therefore, Orhun (2012) suggested that calculus course should be informal, intuitive and conceptually based on graphs and functions for an effective teaching and improvement of quality of learning and develop the understanding of calculus concepts. In addition, Rivera-Figueroa and Ponce-Campuzano (2013) state that the concept of the derivative has a geometric origin. They suggest that a highly graphical context in undergraduate calculus courses is necessary to further deeper understanding of various concepts and results about the derivative. Moreover, the study to show how students think about the links between the graph of a derived function and the original function from which it was formed which is conducted by Berry and Nyman (2003) showed the difficulty of students to make connections between the graphs of a derived function and the original function. In their study, the researchers used technological tools to conduct the experiment and the findings showed that these tools helped to externalize abstract thinking procedures.

Another difficulty that students encounter is related to problem solving including application of the derivative. Stewart (2008) states that "finding extreme values have practical applications in many areas of life and in solving such practical problems the greatest challenge is often to convert the word problem into a mathematical optimization problem by setting up the function that is maximized or minimized."

According to the results of study conducted by Swanagan (2012) showed that students failed to use their knowledge of calculus to solve the problems. In other words, although the students knew the procedure to solve the problem, they did not use it. The results showed that the students focused on learning the procedures and not on understanding why they work and how to apply them.

The findings of the study conducted by White and Mitchelmore (1996) indicated that most students either could not identify and symbolize an appropriate variable by translating quantities in the item to an appropriate symbolic form; therefore, they could not define a useable function. Therefore, Tarmizi (2010) suggests that students need correct misconceptions and monitor their problem solving steps. Moreover, he suggests much attention should be directed to fostering students' ability to follow for problem solving procedures.

CHAPTER 3

METHODOLOGY

3.1 Participants

The participants of the study were the instructor of the Calculus course (MATH 153) offered 2014-2015 academic year at METU and his 20 students majoring on mathematics. The instructor is male and has been teaching Calculus course for 37 years.

The students were the ones who failed from the course in the previous semesters and were taking this course for the second or more times. 11 of these 20 students are female and the rest are male. These 20 students were the ones who attended to the course regularly.

3.2 Data Collection

The data of the present study were collected through conducting classroom observation, administering Thinking-about-Derivative Test (TDT) and conducting interviews with some selected students. Prior to the data collection, students were informed about the purpose and the significance of the study.

3.2.1 Observation

The course were 4 class hours separated into 2 days in a week. The classroom observations were conducted in 30 class hours during 10 weeks. The observations were recorded by an audio recorder which was borrowed from METU Faculty of Education Technological Resources Center. The observer (researcher) attended to the all class sessions as a complete observer and sat at the back corner of the classroom in order not to disturb students. In addition, the lecture notes were taken by the observer. In these 30 class hours, differentiation and application of differentiation chapters were thought.

3.2.2 Thinking-about-Derivative Test

The “Thinking-about-Derivative Test (TDT)” developed by Aydın and Ubuz (2015) and proven to be valid and reliable test were administered to the students upon the completion of observation. The time allocated for administering the test was 90 minutes. The 30 multiple choice items have already been classified according to mathematical thinking dimensions, namely; enactive, iconic, algorithmic, algebraic, formal, and axiomatic thinking.

Table 3.1 Table of Specification of TDT Items

Content	*Mathematical Thinking Aspects and Item Numbers*					
	FORMTHK	*AXIOTHK*	*ALGETHK*	*ICONTHK*	*ALGOTHK*	*ENACTHK*
Derivative and Rate of Change	Item 1 Item 6 Item 16				Item 21 Item 23 Item 26	
Differentiation Rules	Item 5				Item 20 Item 22	
Some Theorems on Derivative		Item 7 Item 8 Item 9 Item 10 Item 11	Item 12 Item 13 Item 14 Item 15			
Curve Sketching	Item 2 Item 3 Item 4			Item 16 Item 17 Item 18 Item 19 Item 20	Item 22 Item 25	
Optimization Problems						Item 24 Item 27 Item 28 Item 29 Item 30

3.2.3 Interviews

After application of TDT, all students were invited to participate in the interviews in order to examine their mathematical thinking aspects further. Five students who were volunteered to participate in the interview sessions were interviewed. The students who accepted to be interviewed were assured that any data collected about them will be held in confidence. The interviews were held at Mathematics department and scheduled regarding the interviewees' free times. Each interview lasted about 1 hour in average.

At the beginning of each interview, a copy of the student own TDT paper was given to him/her and asked to explain how he/she answered the items. The interviews were recorded by an audio recorder. According to TDT results, the number of correct answers of each interviewee in each mathematical thinking aspect is shown in the table below. Number of items which measures the specific thinking aspect is given in parenthesis.

Table 3.2 The number of correct answers of interviewees on TDT

Interviewees	*FROMTHK Items (1,2,3, 4,5,6)*	*AXIOTHK Items (7,8,9, 10,11)*	*ALGETHK Items (12,13, 14,15)*	*ICONTHK Items (16,17,18, 19,20)*	*ALGOTHK Items (21,22,23, 25,26)*	*ENACTHK Items (24,27,28, 29,30)*	*Total*
Student 1	4	3	0	5	4	1	17
Student 2	5	4	2	5	5	0	21
Student 3	5	3	1	5	5	3	22
Student 4	6	3	1	5	5	3	23
Student 5	5	5	4	5	4	3	26

Students were given a number from 1 to 5, regarding their TDT results. Student 1 and Student 5 are male; the other interviewees are female.

3.3 Analysis of Data

The indicators for six different mathematical thinking aspects were adopted from Aydın and Ubuz (2015). They interpreted students' mathematical thinking and related embodied objects to construct mental perceptions of real world objects with the enactive thinking; embodied objects to develop internal conceptions with regard to visuo-spatial imagery with the iconic thinking; symbolic objects to execute procedural computations with the algorithmic thinking; symbolic objects to construct algebraic manipulations with the algebraic thinking; defined objects to recognize the properties of concepts with the formal thinking and defined objects to deduce these properties from theoretical systems with the axiomatic thinking. According to these distinctions among mathematical thinking dimensions, the examples compiled. The six main indicators which were used as units of analysis, were used to analyze the opportunities in Calculus class provided students to engage in mathematical thinking during the instruction of derivative concepts. The specific indicators for each mathematical thinking aspects, which has been adopted from Özdil (2012), are at the table below.

Table 3.3 The Indicators for six Mathematical Thinking Dimensions

Mathematical Thinking Dimensions	Indicators
FORMTHK	Recalling the definitions, notations, and conventions relevant to differentiation concepts. Recognizing the differentiation facts, rules, and terminology.
AXIOTHK	Analyzing differentiation theorems to determine, describe, and use relationships between mathematical facts and situations. Generalizing the justifications to which the inferences are applicable by restating them in more widely terms of differentiation.
ALGETHK	Integrating linkages between a differentiation theorem and its hypotheses. Synthesizing the differentiation theorem to apply its hypotheses. Justifying the truth or falsity of a differentiation theorem statement by reference to its hypotheses.

ICONTHK	Retrieving information from the graph of a function and/or derivative function to interpret a set of graphical information. Constructing graphical representations for a given differentiation relationship.
ALGOTHK	Computing derivatives via an appropriate differentiation method. Evaluating derivatives to carry out algorithmic differentiation procedures. Solving routine differentiation problems.
ENACTHK	Modelling the differentiation problem in the context of a real-life phenomenon. Generating an appropriate model such as an equation to translate the differentiation problem into a function to be maximized or minimized.

3.3.1 Analysis of Observation

In order to explain platforms in Calculus class provided mathematics students to engage in mathematical thinking during the instruction of derivative concepts, the class observations which were audio recorded were converted to the text. By developers of the TDT, the mathematical thinking dimensions were connected with specific items of the test. Therefore, the instructions related with the content of the test items of each mathematical thinking aspect were categorized (see Table 3.1) and the results were presented under related mathematical thinking aspect. To determine the potentialities activating students' different mathematical thinking aspects, specific examples were sought for indicators of six different mathematical thinking aspects during the instructions.

3.3.2 Analysis of the TDT Results

Answer key of TDT was provided by the developers of TDT. The response for each item was categorized as correct or incorrect; therefore, each correct response, "1" and for each incorrect or no response “0” point was given. Therefore, the scores on TDT consisting 30 multiple choice items ranged between 0 and 30. The items were analyzed also with respect to their difficulty index. The difficulty index was provided via ITEMAN program. In addition of the correct responses of items, other responses which were preferred by some participants were examined.

3.3.3 Analysis of the Interviews

In this part of result section, responses of 5 interviewees who were participants of TDT were analyzed according their TDT results. Since the items have already been grouped of different mathematical thinking aspects with respect to their nature, it was accepted that an interviewee has the mathematical thinking aspects which the item measures, if he/she had correct answer of the item in TDT. Then the audio recorded interview sessions were transcribed and all of the written solutions at the interview papers were scanned. Then the interviews were examined regarding the indicators of different mathematical thinking aspects(see Table 3.3).

3.4 Trustworthiness

In order to ensure trustworthiness of this study, different and multiple methods of collecting data were used. By this way, observation, Thinking-About-Derivative test and interviews were used to provide credibility and confirmability of the study as suggested by Guba (1981). In addition, in-depth methodological description allowing the study to be repeated (Shenton, 2014) was made to lead dependability and the indicators which were used for making inferences about different mathematical aspects were explained clearly. As Guba (1981) suggested that in order to provide transferability purposive sampling was done and thick descriptions were developed. Detailed description of data collection methods and data analysis were explained. Moreover, the interview sessions were audio recorded in order to have correct data, and avoid a lack of information. The exam papers of test participants were saved. When reporting the data and making inferences, solutions, designations or verbal explanations of students reported directly, as much as possible. Furthermore, the length of the observation supported the trustworthiness of the study.

3.5 Ethical Issues

The classes were observed and, an audio recorded during class sessions by the permission of the instructor. The students in Calculus were also informed about the study, and the researcher observed the class as a non-participant observer in order not to disturb both the instructor and students. The interviewees were selected regarding their willingness. Although all participants of the test were asked to participate in the interview sessions to increase the number of data sources, the interviews were conducted with only 5 participants of TDT who want to take part in the study voluntarily. They were informed that their data would be held in confidence. Besides, the names of interviewees were not used explicitly in anywhere. Instead, they were numbered regarding their TDT results.

CHAPTER 4 .

RESULTS

In this chapter, observations under mathematical thinking aspects and test and interview results within the frame of mathematical thinking aspects and contents in the test are presented. In this regard, six sections are given, and under each section, the observation, test and interview results are discussed.

4.1 Formal Thinking

In this section, observations of instructions under formal thinking aspects and students' performance on formal thinking items are discussed.

4.1.1 Opportunities in the Classroom

As the main aim of the instructions was to engage students in formal thinking, the instructions observed related to the definition of the derivative and rate of change, the product and the quotient rule, and basic terminology of curve sketching are presented here.

The formal thinking was also activated when the instructor reminded definitions when he gave examples and solved problems. However, in this section, formal thinking just during the development of the content is handled.

The Derivative and the Rate of Change

The definition of derivative was constructed through using the definitions and formulas of known objects. As seen in the following excerpt, the instructor first defined secant line (Line 1) and difference quotient (Line 2) by using the graph (Figure 1) and used the same graph for limiting process (Line 3) to gain definition of the derivative. He defined the limit of the difference quotient and slope of the tangent line as the formal definition of the derivative (Line 6). Besides, as seen in (Line 4), the instructor emphasized that differentiability implies continuity by using graphical aspects of a continuous function (Line 5). To sum up, the definitions of the tangent line (Line 7) and slope of the tangent line (Line 8) were restated.

Excerpt 1 (Observation)

Instructor:

(1) … This is the graph of f(x).

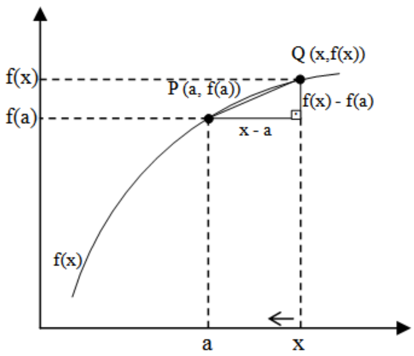


Figure 4.1. The graph used by the instructor to introduce the tangent line

This value here (refers to $f(a)$), is the value of the function at a. This point has coordinates $(a, f(a))$ as we all know. Suppose you are close to P. So the point that would have coordinates in $(a, f(a))$ and suppose you have another point Q. Suppose this is the point Q whose coordinates are x and $f(x)$, this point is $(x, f(x))$. And you consider this line that passes through P and Q. So I consider the line joining the points P and Q. This line is called the secant line. It is very important; it is called just the secant line.

(2) But let's consider this angle and let us consider the tangent of this angle. It is a right triangle so it is this line which is $f(x) - f(a)$ divided into length of this side $x - a$. This is called a difference quotient. Because we have a difference of $f(x) - f(a)$ and then quotient because you divided into $(x - a)$. And we all know in this right triangle tangent of the angle (teta) is the opposite right hand side divided into adjoint side.

(3) But as a fact that suppose the point Q moves along the graph of f and suppose Q be to nearer and nearer point to the point P. So, how can I write this? And consider the slope of the line adjoining PQ, let's called this m_{PQ}, also which is $f(x)$ minus $f(a)$ divided into x minus a. As you bring the point Q nearer and nearer to so you will be having this, one has Q moves on the graph of f towards P. Of course, we should be able to move along the curve f because if f has gaps or vertical asymptotes or something like this, then we would not be possible to move along the curve to being Q to the point near.

(4) But as it happens, differentiability implies continuity. In order to be talking about such a thing, our function has to be continuous anyway.

(5) It is logical because as you move along the curve, we must not fall it to holes or jumps. You should be able to bring Q towards P. P is fixed. Then you consider slopes of the secant lines. Let x approaches a, Q is coming towards P at the same that the point with coordinate x comes nearer and nearer to a. So as x approaches to a, the point Q moves on the curve towards P. I said you bring Q to near to P on the curve, so I said that if you bring Q like this, you could take this point and move along. You bring Q towards P along the curve whose graph is given by $f(x)$. This is logical. Another way of expressing saying that let x approach to the point a, then the corresponding point images of the point $(x, f(x))$ will move towards to the point $(a, f(a))$.

(6) The limit of this difference quotient, let's say m_{PQ} as Q moves towards P, is called the slope of the tangent. Let me first define it, on the curve towards P, then the limit of the difference quotient as Q towards P exists. You know that limit sometimes does not exist, as we all know. This limit exists $f(x)$ minus

$f(a)$ divided into x minus a. If this limit exists, then this is called the slope of the tangent line. It is the definition, formal definition, slope of the tangent line.

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

The limit of the secant line

So let's see what actually happens. This is the secant line. As you bring Q towards P, let see what happens. Just consider the top line please, top edge. This is the secant line. As Q moves towards P, we will have this, this, this, secant lines. So the point Q is here. Bring towards P more, and it will be moving like this. When you actually Q becomes P, this will be the shape of secant line.

(7) As Q moves towards P, this line is called the tangent line through to the curve of defined by f at the point $x = a$. It proves everything because it is a local property, and I am talking about a certain line is an infinite line, you see. This passes from this point. So if the limit exists it is a certain line goes this way, goes that way. But only touches … it should always touch the curve at a point $(a, f(a))$. So it is the slope of a certain infinite line which is the limit of this, whatever they are, they are tangents of these angles that may be horizontal.

(8) The line joining PQ defined by m_{PQ} this right angle triangle and it is the limit exists at Q goes to P whatever that is, actually it is slope it is tangent of this line, then it is called the slope of the tangent line. It is not a trivial thing. You may learn it like this. It is limit exist we know how to judge when limit exists…. if this limit exists and the slope of the certain tangent line, but actually to grasp this idea that this limit exists, it is not obviously to do; you really have to do consider. Formally, if this limit exists, then it is called the limit of the tangent line at the point a. As I said the idea, it is not the mechanics of it. We can solve the following problem (see the Excerpt 1

(Observation) in the algorithmic thinking section), why mechanically without any thinking what slope is.

As seen in Excerpt 1, the instructor activated FORMTHK in order to introduce the derivative concept. Introduction to the derivative concept and the definitions which are activating FORMTHK was handled with the contribution of graphical representation activating ICONTHK. Also, using "variables, symbols, expressions, and equations as structures of general representation" energized ALGETHK.

The Product Rule

The instructor reminded that "real numbers is a group under multiplication" (Line 2) and introduced the product rule (Lines 1 and 3). He emphasized the condition that both of the factors are differentiable at x (Line 4) and stated the rule also verbally (Line 5), as seen in the following excerpt.

Excerpt 2 (Observation):

Instructor:

(1) Theorem: If f and g are differentiable at x then so is $f \cdot g$ at x then

$$(f \cdot g)(x) = f(x) \cdot g(x)$$

(2) But you see this is well defined because real numbers is a group under multiplication. So $(f \cdot g)(x)$ does make sense. This product $(f \cdot g)(x)$ is defined to be $f(x) \cdot g(x)$. This is the only way to define product of two functions.

(3) So, $(f \cdot g)'(x) = f'(x) \cdot g(x) + g'(x) \cdot f(x)$ the expression does make sense again because real numbers is a group under multiplication which one do I write first, which one do I write, these are numbers. It says this is the product rule for differentiation.

(4) Derivative of a product if both of the factors are differentiable at x, so is the product defined is on the right and the derivative is:

(5) Derivative of the first one times the second one plus derivative of the second one times the first one.

Therefore, the instructor stated the product rule (FORMTHK) with both symbols and verbal expression (ALGETHK).

The Quotient Rule

When the instructor introduced the quotient rule, he stated that the quotient of two differentiable functions is also differentiable and the rule was given after the product rule (Line 1). He emphasized that the denominator cannot be 0 to the quotient be defined (Line 2) and stated the rule algebraically (Line 3).

Excerpt 3 (Observation)

Instructor:

(1) Corollary: If f and g are $R \to R$ differentiable at x and $g(x) \neq 0$ then $\frac{f}{g}$ is differentiable at x.

(2) Of course I have to put the condition $g(x)$ is not 0 because I want $\frac{1}{g}$ to be defined. And the derivative of $\frac{f}{g}$ at x is the following:

(3)

$$\mathrm{D}\left(\frac{\mathrm{f}}{\mathrm{g}}\right)(\mathrm{x}) = \frac{\mathrm{f}'(\mathrm{x}) \cdot \mathrm{g}(\mathrm{x}) - \mathrm{g}'(\mathrm{x}) \cdot \mathrm{f}(\mathrm{x})}{\mathrm{g}(\mathrm{x})^2}$$

As in the excerpt above, the instructor stated the quotient rule algebraically (ALGETHK) by stating it as the corollary. Therefore, he activated AXIOTHK as well as FORMTHK.

Increasing and Decreasing Functions (in Theorems)

As seen in the excerpt below, the instructor first defined (Line 1) increasing, decreasing, nondecreasing, nonincreasing functions on an interval and he told that the sign of the first derivative is the tool for determining whether the function satisfies any of these properties. The instructor emphasized the interval (Line 3) and the theorem (Line 4) stating the relation between the sign of the first derivative of the function and either the function increasing, decreasing, nonincreasing, and nondecreasing on an interval was given.

Excerpt 4 (Observation)

Instructor:

(1) Definition: We have an interval I, interval (a, b) and x_1, x_2 in the interval I. So we say that if $x_1$is less than x_2 implies $f(x_1)$ is less than $f(x_2)$ then f is increasing on I. Let me tell, for every x_1, x_2 whenever x_1 is less than x_2 implies $f(x_1)$ is less than $f(x_2)$ then f is called increasing. If same condition implies that $f(x_1)$ is greater than $f(x_2)$, f is called decreasing on I. If x_1 is less than or equal to x_2 implies $f(x_1)$ is less than or equal to $f(x_2)$, f is called nondecreasing on I. And the same condition implies $f(x_1)$ is greater than or equal to $f(x_2)$, f is called nonincreasing on I.

(2) Next theorem answers do we know the criteria for a function to be increasing, decreasing, nondecreasing, nonincreasing, etc? Yes. And the tool used to decide whether the function satisfies anyone of these properties. It is determined by the sign of the first derivative and here is the theorem. Again let J be an interval, I is another interval containing J, I may include one or both of the endpoints of J.

(3) Theorem: J be an interval J $\in$ I, may include one or both of the endpoints of J. So what I am saying that J is (a, b), I is may be $[a, b]$ or $(a, b]$ or $[a, b)$ like this, one of the endpoints is included in the larger interval I.

(4) If f is continuous on I and differentiable on J. Now the conditions are: a) if derivative of f is positive for every x in J, then f is increasing on I. b) if

derivative of f is negative for every x in J, then f is decreasing on I. c) if $f'(x)$ is greater or equal to 0 for every x in J, then f is nondecreasing on I. Finally if $f'(x)$ is less than or equal to 0 for every x in J, then f is nonincreasing on I.

The excerpt shows that the instructor used definitions and stated theorem. Therefore, FORMTHK was activated during the introduction.

Critical Points

The instructor first used the term “critical point” when he explained Fermat’s Theorem (see Excerpt 6 (Observation) in the axiomatic thinking section). Below, there is an excerpt when curve sketching was introduced.

Excerpt 5 (Observation)

Instructor: (the instructor orders the steps of curve sketching) …. the fifth step is find out

(1) $f'(0) = 0$, these are called critical points.

(2) When the derivative is 0, it has a horizontal tangent such as in my picture.

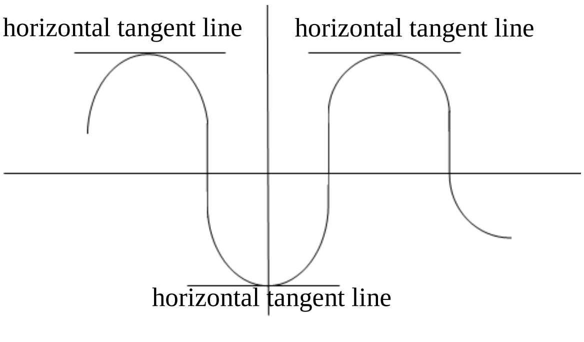


Figure 4.2. The graph used by instructor to show horizontal tangent lines

The picture is important. These are exactly when we have horizontal tangent line. So I would like to add one more thing.

(3) When f' does not exist… So when may be this function:

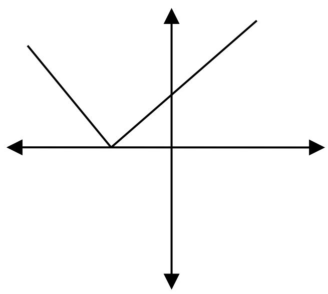


Figure 4.3. The graph used by the instructor to exemplify cusp point

(4) It looks very ordinary elsewhere but at this point, it has cusp. The derivative does not exist. So find out when the derivative is 0 or and when derivative does not exist, such as absolute value. Some authors add critical points, when the derivative does not exist to the critical points. I don't, but as far as the graphs are concern, you must also do this find out when the derivative does not exist. When the derivative is 0 is important,

The excerpt above shows that the definition of the critical point (Line 1) and horizontal tangent line (Line 2) by using graphical representation. Therefore, FORMTHK was activated by definition and via ICONTHK. In Line 3, the instructor emphasized the points where the derivative does not exist, and (Line 4) they are important for graphs as well as critical points.

Concavity and Inflection Points

When the introduction of concavity, the graphs were mainly used to show the difference between concave up and concave down. After the definitions of concave upwards and concave down were given, the inflection point was defined.

Excerpt 6 (Observation)

Instructor:

(1) This is $y = x^2$ and this is where $y = -x^2$. You see one of them has shape like this, and the other one has shape like this.

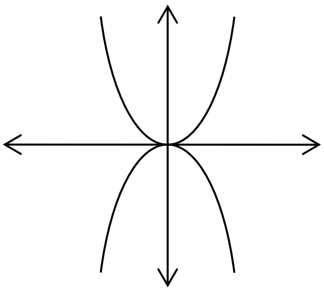


Figure 4.4. The graph used by the instructor to introduce concavity

(2) Shapes like these (refers to the graph of x^2) are called concave up, and shapes like these (refers to the graph of $-x^2$) are called concave down.

(3)…Can you describe the difference between a parabola like this, parabola like this using mathematics?

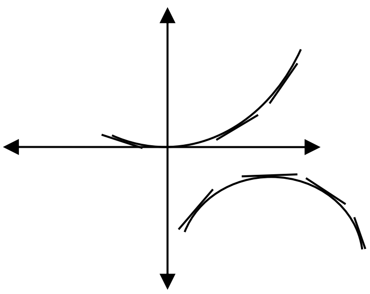


Figure 4.5. The graph used by the instructor regarding convavity and tangent lines

(4) The one above has a property that you see here if you draw the tangent at any point the curve is always above the tangent lines. And the second one has the property that if you draw the tangents, the curve is always below the

tangent lines. So the difference between the two. And lots of time our functions are differentiable.

(5) If it is differentiable, then it is continuous. So what must I say, concave up concave down. What must I say to this? Lots of time, our functions are differentiable and, therefore, continuous functions, so what must I add to this? Concave up, concave down, especially our function is continuous. What did we learn about continuous functions? A continuous function does not make violent changes. A continuous function is a reasonable function, does not make violent changes. It is a stable function. It is a smooth function.

(6) Definition: A graph of f is called concave upwards. If f is differentiable on the interval and the tangent line is always below the graph like $y = x^2$.

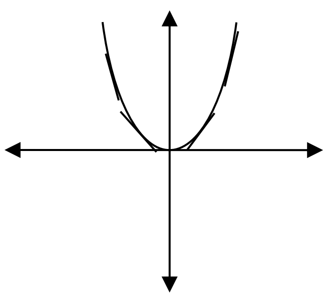


Figure 4.6. The graph used by the instructor to introduce the "concave up"

(7) Similarly, a graph of f is called concave down if f is differentiable on the interval and the tangent line is always above the curve, like $y = -x^2$.

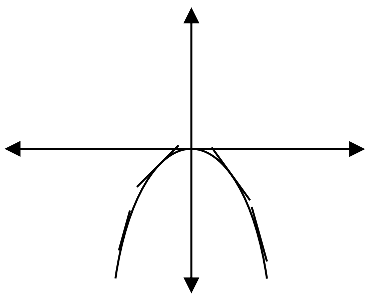


Figure 4.7. The graph used by the instructor to introduce "concave down"

(8) One more definition. You have a function like this:

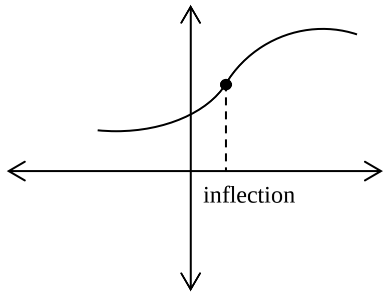


Figure 4.8. The graph used by the instructor to introduce inflection point

(9) And from here to here it is concave up and all the tangents are below the graph but if you pass from this point on the tangents always above. So, concavity changes at this point. This is called an inflection point. So;

(10) Definition: A point is called an inflection point if concavity changes at that point. Say, from concave up to concave down or from concave down to concave up.

The excerpt shows that (Line 1) the instructor was benefited from the graphs of $y = x^2$ and $y = -x^2$ (ICONTHK) to (Line 2) introduce the terms concave up and concave down (FORMTHK). In Line 3, he used different graphs to emphasize (Line 4) the relation between whether the curves are below or above the tangent lines. In Line 6, he reminded that if a function is differentiable, then it is continuous (FORMTHK). Then, the definition of concave upwards (FORMTHK) was stated, and the graph of $y = x^2$ (ICONTHK) was given as an example (Line 6). Next, the definition of concave down (FORMTHK) was stated and as an example, the graph of $y = -x^2$ (ICONTHK) was drawn (Line 7). In Line 8, the graph (ICONTHK) was used to show (Line 9) the point where the concavity changes and (Line 10) the inflection point was defined (FORMTHK) as the point where the concavity changes.

The excerpt below shows how the teacher related concavity and the sign of the second derivative.

Excerpt 7 (Observation)

Instructor:

(1) Concavity can be decided by the sign of second derivative.

(2) Corollary: a) If $f''(x)$ is greater than 0 for all x in an interval I, then the graph of f is concave upwards.

(3) Think $y = x^2$. Because the second derivative is 2, positive.

(4) b) If $f''(x)$ is negative for all x in an interval I, then the graph is concave downwards on I.

The instructor highlighted that the concavity can be decided by the sign of second derivative (Line 1) (FORMTHK) and stated the relation by corollary (AXIOTHK) in Line 2 and Line 4. To support the first assertion, he related (Line 3) the sign of the second derivative of the function $y = x^2$, which is positive (ALGOTHK), and the graph of the well-known function (ICONTHK).

Increasing and Decreasing Functions (in Curve Sketching)

The increasing and decreasing functions were defined after the Mean Value Theorem and Rolle's Theorem. However, when the curve sketching was introduced, the functions redefined, and the proof was made.

Excerpt 8 (Observation)

Instructor: Proposition: If $f'(x)$ is positive on an interval I, you must always think about the hypotheses,

(1) the first derivative must exist to be positive, and

(2) it is differentiable therefore continuous, so it is positive on an interval then f is increasing. These are simple question completely because I said if what does f' say about f.

(3) If f' is positive on interval then f is increasing and on the other hand if f' is negative then f is decreasing.

(4) I will prove 1, the first case. The other is very similar. Suppose x_1, x_2 are two elements of interval I and suppose x_1 is less x_2, we need to show that $f(x_1)$ is less than $f(x_2)$ if $f'(x)$ is greater than 0 for every x in I. This is what we have to show. To say that a function is increasing is to say that whenever x_1 is less than x_2, $f(x_1)$ is less than $f(x_2)$. Ok so use the Mean Value Theorem on the interval $[x_1, x_2]$. $[x_1, x_2]$are differentiable on the interval that contains. $[x_1, x_2]$ are differentiable on the closed interval which is interval I given. So there exists a point c in x_1 and x_2 such that

(5) $$f'(c)\ (x_2 - x_1) = f(x_2) - f(x_1)$$

(6) Now look at right hand side $x_2 > x_1$ therefore this is positive. $f'(c)$ is positive for each in the interval. This is also positive. So, the difference $f(x_2) - f(x_1)$ is positive. That means $f(x_1)$ is less than $f(x_2)$. And the other one is proved similarly.

As seen in the excerpt, hypotheses (Lines 1 and 2) were emphasized, and the proposition was introduced (Line 3). One of the cases was proved (Line 4) by using the Mean Value Theorem (AXIOTHK), and the algebraic expression (ALGETHK) (Line 5) was interpreted concerning signs of the $f'(c)$ and $f(x_2) - f(x_1)$.

The First Derivative Test

By the First Derivative Test, local maximum and local minimum of a function at the critical point were defined, and it was emphasized when the sign of the first derivative changes at the critical point, the critical point is the local maximum or local minimum of the function. Moreover, the instructor exemplified the cases in which the sign of the first derivative of the function may not change at the critical point; therefore, the function has no maximum or minimum at the critical point.

Excerpt 9 (Observation)

Instructor: Suppose c is a critical point of a function f. The first assertion is

(1) If the sign of f' changes from positive to negative at c, then f has a local maximum at c. So the picture is like this:

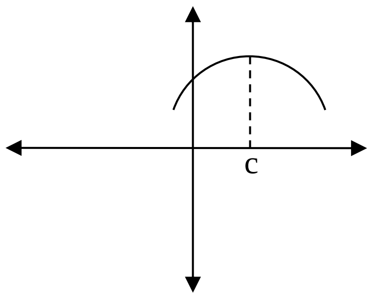


Figure 4.9. The graph used by the instructor to introduce the local maximum at c

….. Everything depends on f'. If f increasing, increasing and then start decreasing so we must have, so increasing, increasing and then decreasing.

(2) You see this is differentiable function, continuous, so it does not make jump.

(3) It starts increasing, increasing, positive, positive then become 0, and one point it has to be 0. So there has to be a point where derivative must be 0. That means that derivative is changing sign. And that is in a lot of information about the graph.

(4) Secondly, second part of the assertion is the following. If the sign of f' changes from negative to positive at c, the f has a local minimum at c. So it is decreasing, decreasing, decreasing and then starts increasing. So we have a local minimum.

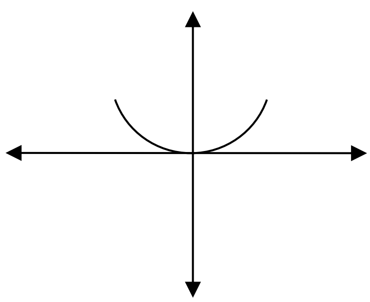


Figure 4.10. The graph used by the instructor to introduce the local minimum at c

(5) But I would like to have 3, this is important, suppose we have a critical point but if f' does not change sign at c, if f' is positive or negative on both sides of c, then there is no local maximum or minimum at c. So we have a picture like this:

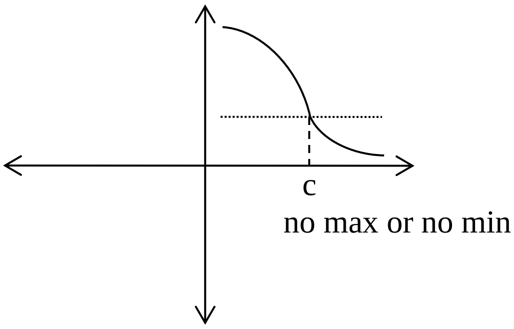


Figure 4.11. The graph used by the instructor to introduce the point c where the function has no maximum or minimum

At this point, c this is the third case. It is decreasing in both sides of c, as you approach c from left hand side, it is negative. If you approach c from the right hand side the sign of the first derivative is still negative. Therefore, no max or no min. So you see, if the derivative does not change sign, completely another word.

(6) You may have a function like this:

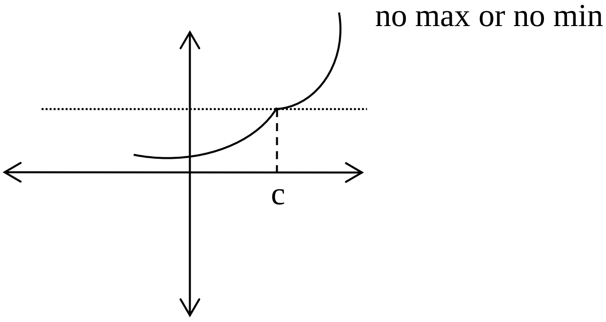


Figure 4.12. The second graph used by the instructor to introduce the point c where the function has no maximum or minimum

(8) You see the tangent is 0 here; the first derivative is 0 here, at this point. But again this point is no maximum or no minimum. In other words, having a critical point at any point does not show that the function has maximum or minimum at that point.

The excerpt shows that the instructor explained (FORMTHK) the assertions (Lines 1, 4, and 5) of the First Derivative Test and visualized each case (ICONTHK). He reminded (Line 2) differentiable functions are continuous (FORMTHK) and do not make jump (ICONTHK). It was emphasized that there has to be a point where derivative must be 0 since, at this point, the first derivative is changing sign either from negative to positive or positive to negative (Line 3), (FORMTHK). The instructor used one more graph (Line 6) (ICONTHK) to exemplify the third assertion and emphasized (Line 7) having a critical point at any point does not guarantee that the function has maximum or minimum at that point (FORMTHK).

The Second Derivative Test

By the Second Derivative Test, the instructor used the sign of the second derivative of a continuous function at a critical point to determine local maximum and local minimum.

Excerpt 10 (Observation)

Instructor:

(1) If the second derivative of f is continuous at $x = a$, and

(2) if the first derivative is 0 at a, the value of the first derivative is greater than 0 then f has a local minimum at $x = a$. And secondly again,

(3) if a is a critical point and value of second derivative is less than 0 then f has a local maximum at $x = a$.

The instructor first stated the primary condition that the second derivative of f is continuous at $x = a$ (Line 1) and then assertions (Lines 2 and 3) were given

(FORMTHK). When the assertions were stated, the expressions "the first derivative is 0 at a" (Line 2) and "a is a critical point" (Line 3) were used. Therefore, the instructor defined implicitly that the point where the first derivative is 0 is called critical point (FORMTHK).

4.1.2 Students' Mathematical Thinking Performance

Six items regarding the definitions of the derivative (item 1), the inflection point (item 2), increasing and decreasing function (item 3), and local minimum and local maximum points (item 4), the product rule (item 5) and derivative symbol (item 6) are formal thinking items in TDT.

Table 4.1. *TDT Results of FORMTHK Items*

	Item 1	*Item 2*	*Item 3*	*Item 4*	*Item 5*	*Item 6*
Difficulty index	.5	.65	.9	.9	1.00	.55

The investigation of the item difficulties reveals that almost all participants responded to items 3, 4, and 5 correctly. Regarding item 3, the interviewees were asked to draw a graph representing decreasing and increasing function. All interviewees drew correctly increasing and decreasing function on a linear function.

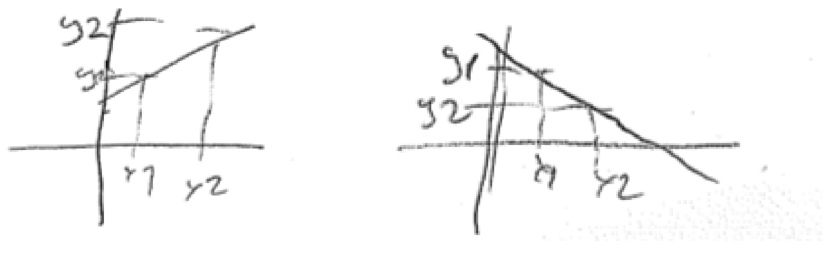


Figure 4.13. The graphs drawn by Student 1 regarding item 3

Student 2, Student 3 and Student 4 also reasoned functions' decreasing or increasing interval by the sign of the slope of the function on an interval. The example from Student 4 is below.

Excerpt 1 (Interview)

> Student 4: I considered that all statements are true. I have already known this.
> Researcher: What do the statements say?
> Student 4: For example, let there is a function, $f(x) = y = x$. When x is 1, it is 1. When x is 2, it is 2. It shows that it is increasing. In this way, all are true.
> Researcher: Can you give an example for the second statement?
> Student 4: When we consider the function $y = -x$, it is -1 at $x = -1$, it is -2 at $x = -2$. It is converse of the first statement. In the third statement, as I said before, if the first derivative is greater than 0, it is increasing.
> Researcher: If the function is increasing, the first derivative is greater than 0. Why?
> Student 4: The first derivative is greater than 0... I could not....If we take x^3, the first derivative is $3x^2$, greater than 0. Let me show this. Let me take x^3. Its first derivative is always greater than 0.
> Researcher: What is the sign of the first derivative?
> Student 4: Positive. So it is increasing. By drawing the graph.

Regarding item 4, all interviewees explained that where the sign of the first derivative changes, there is a local maximum or a local minimum point. As an example, the excerpt from Student 2 is below.

Excerpt 2 (Interview)

> Student 2: When the sign of the first derivative changes from positive to negative at the point c, the function changes from increasing to decreasing.

Therefore, at point c, there is a local maximum. When the function changes from decreasing to increasing at the point, there is a local minimum.

Students 1, 2 and 3 also explained that they did not choose 3rd and 4th statement since the statements were not true according to the "The Second Derivative Test" which says if $f'(c) = 0$ and $f''(c) > 0$, then has a local minimum at c and if $f'(c) = 0$ and $f''(c) < 0$, then has a local maximum at c, assuming f'' continuous near c. However, Student 4 and 5 claimed that the second derivative of the functions used for concavity does not tell about the local maximum or local minimum. The interviewees activated FORMTHK when they explained their answers.

Regarding Item 5 about differentiation rule for $[(f \cdot g)(x)]$, interviewees who provided the correct answer in TDT were asked to prove the product rule. It was expected that they could prove the rule by using the formal definition of the derivative as it was proved in class by the instructor. Student 1 told that the product rule is proved by using the formal definition of the derivative, but he did not know how to prove it. Students 2 and 3 explained that they did not remember how to prove it, and they did not try to do it. On the other hand, Students 4 and 5 started to proof by using the formal definition of the derivative, but they could not complete. The proof of the product rule made by Student 4 is below.

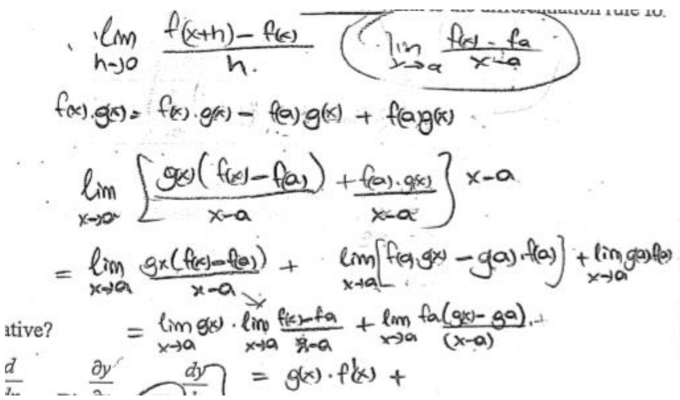


Figure 4.14. Student 4's study to prove the product rule.

Since the item is related to “knowing and recalling”, all interviewees showed their FORMTHK skills. However, considering the proof, the interviewees failed in AXIOTHK.

Three items (items 1, 2, and 6) were responded correctly by half of the participants. Regarding item 1, four interviewees, who had given the correct answer in TDT, explained correctly with self confidence that “the instantaneous rate of change of f with respect to x” and “the function that represents the slope of the tangent lines to f at each point” are the definition of the derivative function. 3 participants selected option D which is partially correct. These participants considered the definition of derivative as “the instantaneous rate of change of f with respect to x” and “the function that represents all the tangent lines to f at each point”. The interview with one of these students who selected option D is given below.

Excerpt 3 (Interview)

> Student 1: First, I had thought derivative is the instantaneous rate of change with respect to x as said in 3rd statement. I have thought this is true.
> Researcher: Why? What is the instantaneous rate of change?
> Student 1: Instantaneous change. It is the change with respect to x. Then I chose option D. It is the function that represents of tangent lines of f at each point. However, I realized later that this does not represent all.
> Researcher: Which statement can be true?
> Student 1: At 6th statement, it says each point. That represents the slope of the tangent lines to f at each point.

As seen in Excerpt 3, Student 1 provided the correct definition of derivative during the interview.

In addition, 3 participants selected option C which is also partially correct. The participant defined derivative as “change in a function f with respect to x” and “the function that represents the slopes of the tangent lines to f at each point". 4 participants selected option B, which is totally incorrect. They considered the definition of derivative as "function of a function f" and "tangent line to f at each point". Considering the nature of item 1, all interviewees have shown the ability of FORMTHK. Since the interviewees used just definition in their explanations, there is no clue about another aspect of mathematical thinking here.

Regarding item 2 about the definition of inflection point, 13 participants selected the correct option. Four interviewees, who had given the correct answer in TDT, provided similar explanations during the interviews. The following excerpt from Student 5 is an example of the explanations of the interviewees who had answered the question correctly in TDT.

Excerpt 4 (Interview):

> Student 5: Inflection point is the point where the second derivative is equal to 0. Therefore, the second statement is true. Since the inflection point is related to concavity, the point where concavity changes there is an inflection point. So this statement is also true.

However, 4 participants selected either option C or D which are partially correct. Two of these participants selecting option C considered that the inflection point is "the point where the function $f: [a, b] \to R$ changes from increasing to decreasing or from decreasing to increasing" and "the point where the function $f: [a, b] \to R$ changes from convex to concave or from concave to convex". The other 2 participants, on the other hand, selected option D, which includes "only" the statement "the point where the function $f: [a, b] \to R$ changes from convex to concave or from concave to convex" as the definition of the inflection point. The answers of two participants who selected option A were totally incorrect, and they defined the inflection point as "the point where the first derivative of the function $f: [a, b] \to R$ equals to zero" and "the point where the function $f: [a, b] \to R$ changes from increasing to decreasing or from decreasing to increasing" . Student 1, who did not select an option for item 2 in TDT, explained that he did not remember what inflection point is. Therefore, the definitions of interviewees reflected their FORMTHK.

Regarding item 6 about the derivative symbol, 11 participants responded correctly selecting option E, "$\frac{dy}{dx}$". Student 1 explained that he had difficulty in deciding whether $\frac{d}{dx}$ or $\frac{dy}{dx}$ was the symbol of the derivative, but then he has chosen $\frac{dy}{dx}$ since there is a function "y" is given. In addition, Student 4 selected "$\frac{dy}{dx}$" and explained that this symbol contains "d" and there is a function which is differentiated with respect to a variable. On the other hand, 8 participants selected option C which is "$\frac{d}{dx}$". The interview with Student 5, who selected option C, is given below.

Excerpt 5 (Interview)

> Student 5: The symbols which are not "d" are not symbols of the derivative. Therefore, I selected option C. Because, we can put any function whatever we want, it can be y, it can be f. Why is this (E) ($\frac{dy}{dx}$) not? It is the derivative of y. If there was y here, how could we show derivative of another function?"

Student 3, who also selected $\frac{d}{dx}$ as the symbols of the derivative in TDT, realized that she had mistaken, and decided on the correct answer. The excerpt is below.

Excerpt 6 (Interview)

> Student 3: I had thought in general, but I had to think regarding a function. Therefore, it should be $\frac{dy}{dx}$, the symbol of the derivative. In addition, the symbols in option A, B and D are not the symbols of the derivative."

Finally, Student 2, who selected option D which is "$\frac{\partial y}{\partial x}$", explained that she had oscillated between options E ($\frac{dy}{dx}$) and D ($\frac{\partial y}{\partial x}$), but she has chosen option D. The excerpt is below.

Excerpt 7 (Interview)

> Researcher: You selected option D for item 6.
> Student 2: I first could not decide whether option E or option D is correct, then I selected option D since it is one of the symbols which I used when differentiating.
> Researcher: Why did you eliminate the symbols in options A, B and C?
> Student 2: I have never seen the symbols given in options A and B when I differentiate, and I did not select C because it means differentiation with respect to x, but it does not state a differentiation of a function.

The item measures FORMTHK regarding its nature, interviewees activated FROMTHK by describing the derivative symbol.

4.2 Axiomatic Thinking

In this section, observations of instructions under axiomatic thinking aspects and students' performance on axiomatic thinking items are discussed.

4.2.1 Opportunities in the Classroom

As the main aim of the instructions was to engage students in axiomatic thinking, the instructions observed related with proof of the product and quotients rules which were stated in the previous section, the Mean Value Theorem, Rolle's Theorem, the Intermediate Value Theorem, Fermat's Theorem and the theorem which states that if "f is differentiable at a, then f is continuous at a" and their proofs are presented in this part of study.

Proof of the Product Rule

The proof of the product rule was made by using known facts such that real numbers is a commutative group under multiplication, the limit of a sum is the sum of the limits, the limit of a product is the product of the limits and differentiability at any point implies continuity at that point.

Excerpt 1 (Observation)

Instructor: I prove $(f \cdot g)'$(x), this is by definition,

$$(f \cdot g)'(x) = \lim_{h \to 0} \frac{f(x+h)g(x+h) - g(x)f(x)}{h}$$

I have to show this. I have to find what this limit is. The trick here:

(1) If you add or subtract the same number and the thing to add here $-f(x)g(x+h)$ and $+f(x)g(x+h)$ so when I do this, I group them as follows:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \, . \, g(x+h) + f\ (x) . \frac{g(x+h) - g(x)}{h}$$

Nothing changes because real numbers is a commutative group under multiplication. So

(2) now what will I use. As always, limit of a sum is the sum of the limits. So I will write this as this limit:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} . g(x+h) + \lim_{h \to 0} f\ (x) . \frac{g(x+h) - g(x)}{h}$$

Then I will use the product so this is equal to:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} . \lim_{h \to 0} g(x+h) + \lim_{h \to 0} f\ (x) . \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

(3) as your familiar theorem. Here I used the theorem that the limit of a product is the product of the limits. This is the value of derivative f' at x it exists by hypothesis; it is the derivative of f.

(4) g is a continuous function. Limit always goes inside. You first bring g limit inside. $(x + h)$ goes to x when x goes to 0. So the second one is $g(x)$. Why? Because limit goes inside as g is a continuous function. Differentiability at x implies continuity at x. And you apply the same trick here. What is this one? This is not to do with h, this is just $f(x)$. But the second one is $g'(x)$.

$$f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

(5) Therefore you have the product rule. Two functions differentiable at x, then the derivative exists and given by the product rule, $f'(x) \cdot g(x) + f(x) \cdot g'(x)$."

The proof was made (Line 1) by the formal definition of the derivative. Even though knowing the product rule is an indicator of FORMTHK, AXIOTHK was also employed since the instructor proved the theorem by using the formal definition of the derivative. The instructor followed limiting procedure (ALGOTHK) in Line 2 and Line 3 and activated FORMTHK by reminding background information (Lines 2 and 3) and the relation between differentiability and continuity (Line 4), and then by summarizing the product rule (Line 5).

Proof of the Quotient Rule

The proof of the quotient rule was made by using the product rule. The excerpt is below.

Excerpt 2 (Observation)

Instructor: Proof.

(1) Since it is corollary, it will be derived from previous theorem (refers to product rule). I will give you a hint. You can think of a function $\frac{f}{g}$ at x as $f(x) \cdot \frac{1}{g(x)}$ as since we know how to take the derivative of a product and this we can write the. I have to use two results. I can think of the function $\frac{f}{g}$ as $f \cdot \frac{1}{g}$ and then use the product rule of differentiation.

(2) Proof: $\frac{f}{g}(x) = f(x) \cdot \frac{1}{g(x)}$

$\left(\frac{f}{g}\right)'(x) = D(f) \cdot \frac{1}{g(x)} + f(x) \cdot D\left(\frac{1}{g}\right)$ This is the product rule:

$=f'(x)g(x) + f(x).g'(x)$

$= f'(x).g(x) + f(x) \cdot -\frac{g'(x)}{g(x)^2}$ Then if you do the algebra here we get the formula:

$$\frac{f'(x) \cdot g(x) - g'(x) \cdot f(x)}{g(x)^2}$$

So we know how to take derivative of the products, we know how to take the derivative of the reciprocal function "$\frac{1}{f}$", then we can take the derivative of the quotients.

Since the quotient rule was given as corollary (Line 1), its proof was made by using product rule (Line 2), which has already taught. In this way, the instructor activated AXIOTHK as well as FORMTHK.

The Relation between Differentiability and Continuity

The excerpt below shows how the instructor introduced the theorem which states the relation between differentiability and continuity and how a student proved the theorem.

Excerpt 3 (Observation)

Instructor: Now I come to an important theorem. What is the relation between continuity and differentiability?

(1) We have two properties; both are local properties. You can talk about differentiability of a function at a point; you can talk about continuity of a function at a point. So we have something in common. So what I am asking is the relation between continuity and differentiability. Does this question make sense to you?

(2) Student: If a function is differentiable, then it is continuous.

(3) Instructor: OK. This is the relation. We have set of differentiable functions and then these are continuous functions. This set of functions, which have derivative, say x_0, is contained functions that are continuous at x_0. This is correct. Every differentiable, well is a local property so I must correct putting it would be, every function which is differentiable at x_0. is continuous at x_0.

(4) But as we observed the converse is not true because we already have an example (see Excerpt 1 (Observation) in the algorithmic thinking section). The function $|x|$ is a continuous function at $x = 0$ but it is not differentiable, derivative does not exist at 0. So this, in fact, is a straight containment. There are continuous functions which do not smaller set of differentiable functions. Obviously, as every mathematical statement, this needs a proof. But anybody knows the proof?

(5) Student: Suppose f is differentiable.

Instructor: What do you show? If it is differentiable, it is continuous.

(6) Student: Yes. At the point c.

$$f(c+h\,) \ = \ f(h+c) - f(c) \ + \ f(c)$$

$$f(c+h) = \frac{[f(c+h) - f(c)]h}{h} + f(c)$$

$$\lim_{h\to 0} f(c+h) = \lim_{h\to 0} \frac{[f(c+h)-f(c)]h}{h} + \lim_{h\to 0} f(c)$$

(7) Instructor: Which theorem did you use here?

(8) Student: Limit of a sum is sum of the limits.

(9) $$\lim_{h\to 0} f(c+h) = \lim_{h\to 0} \frac{[f(c+h)-f(c)]}{h} . \lim_{h\to 0} h + \lim_{h\to 0} f(c)$$

$$\lim_{h\to 0} f(c+h) = f'(c).0 + f(c)$$

$$\lim_{h\to 0} f(c+h) = f(c)$$

Instructor: If $f(x)$ goes to value of the function at c, this means it is continuous at c. OK. So what have you done? She proved that $\lim_{h\to 0} f(c+h) = f(c)$.

(10) Student: We showed it is continuous.

(11) Instructor: They are both numbers.

$\lim_{(c+h)\to c} f(c+h) = f(c)$.

For every h, as $c+h$ goes to 0, and therefore this limit is obvious $f(c+h)$ goes to $f(c)$. This is exactly continuous. Every differentiable function is a continuous function because she adjusts from that if the derivative exists then if you take x is going to c, the $f(x)$ goes to $f(c)$, value of the function at the limiting point. OK. This proves that every differentiable function is a continuous function.

(12) But as a mathematician, you would ask whenever something is true P implies Q, you should ask yourself, is the converse true? Is it say that Q implies P? No. Because there are continuous functions as shown by absolute value functions, which is definitely continuous but not differentiable. OK. Now, so we this process. Another thing that we have done about continuous functions that they have some algebraic properties.

As seen in the excerpt, the instructor first emphasized that both continuity and differentiability are local properties (Line 1) (FORMTHK) and stated the theorem by using sets of continuous and differentiable functions (Line 3) (FORMTHK). Also, the instructor reminded the example which was given to show a function is not differentiable and noted that the converse of the theorem is not true (Line 4) (FORMTHK). The proof of the theorem (AXIOTHK), on the other hand, was made by a student who stated the theorem (Line 2) when the instructor asked the relation between continuity and differentiability. The student made proof (Lines 6 and 9) and explained he used the theorem “limit of a sum is the sum of the limits” (AXIOTHK). Then, the instructor summarized the proof (Line 11) (AXIOTHK) and restated the converse of the theorem statement is not true (Line 12) (FORMTHK).

The Mean Value Theorem

The Mean Value Theorem was introduced by two different graphs, then the theorem and its hypotheses were stated.

Excerpt 4 (Observation):

Instructor: Today, I will talk about lots of theorems. The first theorem is this:

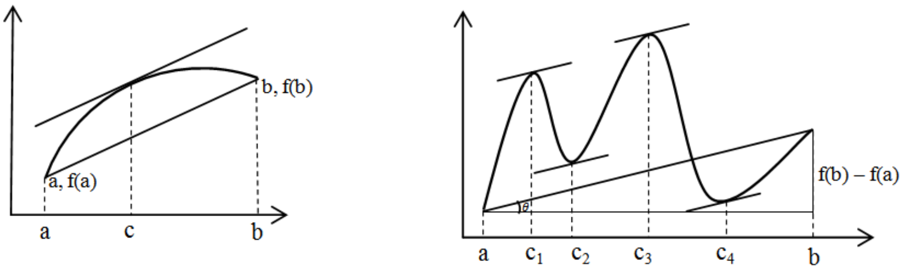


Figure 4.15. The graphs used by the instructor regarding the Mean Value Theorem

(1) The common aspect of these two pictures is this. Here is $(a, f(a))$ this point $(b, f(b))$ there is a point c such that the slope of the tangent line at c is parallel to line joining $(a, f(a))$ and $(b, f(b))$. Similarly, there are lots of points, here, here, where the let’s call the c_1, c_2, c_3, c_4 my answer I will looking for, first of all, the functions represented by these graphs are differentiable functions and continuous functions.

(2) But if it is differentiable, it is certainly continuous. And these points there could be one, in the first graph there is only one point, but in the second there are 4 points where the slope of the tangent line is parallel to the cord joining $(a, f(a))$ to $(b, f(b))$. This is what is known as Mean Value Theorem.

(3) There is a distinction between the interval (refers to (a, b)) and (refers to $[a, b]$). This is called the closed interval where endpoints a and b are included, in the first interval (a, b) it is called an open interval where the points a and b are not included. So here is the statement of Mean Value Theorem, which I will write as MVT. It is called Mean Value Theorem.

(4) f is continuous on $[a, b]$ and f is differentiable on (a, b).

(5) Then there exist c in a,b such that $f(b)- f(a)$ divided into $b- a$ is equal to value of the derivative of f at c and c is between a and b not end points. This is so called Mean Value Theorem. First of all, how are we going to remember this theorem?

(6) If the function to say that f is differentiable, it means that f is continuous everywhere in the open interval. So what about at the endpoints? At the endpoints, f may not be differentiable, but still it is continuous there. At the endpoints, f may not be differentiable because we know differentiability at left hand point and we know differentiability about right hand point. But f is continuous on the closed interval, and f is differentiable on the open interval. But this is a typical theorem in mathematics, called existence theorem. It says there exists such a point…. But does it say how many? There is a graph there is one c, here is a graph there is four c's. It does not tell me how many. It does not also tell me how to find this point c. However, this is the slope of the line joining $(a, f(a))$ the left hand side is the slope of the line, this is the tangent of this, this is here, this is $f(b)- f(a)$ divided into $b- a$ you get the tangent of teta this is slope of this line joining $(a, f(a))$ to $(b, f(b))$. Then there is a point between a and b, not endpoints. So please be careful. Such that a is between not equality c is between a, b such that slope of the tangent at c, is equal to slope of the line joining $(a, f(a))$ to $(b, f(b))$. It is an existence theorem. It does not tell you how to find it. But it says there exists such a point. This is called Mean Value Theorem.

The Mean Value Theorem was introduced by the contribution of 2 graphs (see Figure) (ICONTHK), and common aspects of the graphs which show the slope of the tangent lines at c's are parallel to line joining endpoints were emphasized (Line 1). In Line 2 and Line 6, it is restated that differentiability implies continuity (FORMTHK), and the aspects of the open and closed interval was reminded (Line 3) (FORMTHK). Then, the hypotheses of the theorem (Line 4) and the theorem was given (Line 5) (FORMTHK). In addition, it was emphasized that since the theorem does not tell how to find the points where the slope of the tangent line equals the slope of the line joining endpoints, it is an existence theorem.

The proof of the Mean Value Theorem was made after all theorem was introduced.

Excerpt 5 (Observation)

Instructor: Now the proof.

(1) Remember the conditions. f was continuous on $[a, b]$, f was differentiable on (a, b). First of all, the line joining $(a, f(a))$ to $(b, f(b))$. We can write the equation of this line. And the equation is:

$$y = f(a) + \frac{f(b) - f(a)}{b - a} \cdot (x - a)$$

We know how to write the equation of a line that passes from these two points. These points are $(a, f(a))$, $(b, f(b))$. Now I am going to introduce another function.

(3) This function is the function which majors the distance between the graph of f and the graph of this line and I will tell you what this function is. Let us introduce another function g. $g(x)$ majors the vertical distance between line joining $(a, f(a))$ to $(b, f(b))$ and $f(x)$. The function is the following function:

$$g(x) = f(x) - \left(f(a) + \frac{f(b) - f(a)}{b - a} \cdot (x - c)\right)$$

Now this was the line joining $(a, f(a))$ to $(b, f(b))$. This is $g(x)$. This is the vertical distance between this line and $f(x)$. Now, I can make certain observations about g. g is a continuous function, why? Because f is continuous, a line is continuous. Difference of two continuous functions is continuous. g is also differentiable, why?

(4) Because, difference of two differentiable functions is differentiable. Because differentiable functions and continuous functions are back to spaces.

(5) You can do algebra with continuous and differentiable functions. Difference of two continuous functions is continuous, difference of two differentiable functions is differentiable. That is to say, g satisfies the conditions of the Mean Value Theorem.

(6) So observe that, $g(a)$ is equal to $g(b)$ equal 0. So if you put here a, $f(a)$ will be 0. So by Rolle's Theorem, there exits c in open interval (a, b), such that $g'(c) = 0$. What is it mean to say that $g'(c) = 0$. So let's write this:

$$g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}$$

So this is the Mean Value Theorem.

The Mean Value Theorem was proved (AXIOTHK) by using algebraic rules of continuous and differentiable functions (Lines 4 and 5) (ALGOTHK) and Rolle's Theorem (Line 6) (AXIOTHK).

Rolle's Theorem

Rolle's Theorem was introduced immediately after Mean Value Theorem.

Excerpt 6 (Observation)

Instructor: There is a close Rolle's Theorem.

(1) It says that if, under the same condition, f is continuous on $[a, b]$, differentiable on (a, b) such that $f(a)$ is equal to $f(b)$ so $f(b)$–$f(a)$ is 0.

Then there exists a point c in a, b, such that $f'(c)$ is equal to 0. So it is really easy to get this theorem from this, you see.

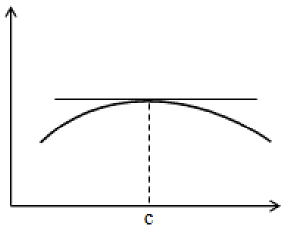


Figure 4.16. The graph used by the instructor to introduce the Rolle's Theorem

(2) This refers to Rolle's Theorem. $f(a)$ equal to $f(b)$ so there is a point c such that the tangent at the point c is parallel is 0.

(3) It is the same theorem except this was proved earlier to the point, it is a line it is parallel to the line joining $(a, f(a))$ to $(b, f(b))$.

(4) However, since $f(b) - f(a)$ is 0, then tangent to the chord or line joining $(a, f(a))$ to $(b, f(b))$ is 0.

As seen in the excerpt above, the hypotheses were given first (Line 1), and then the theorem was stated (FORMTHK). By using the graphical representation (ICONTHK) (2), the tangent at the point c is 0 and parallel to the x-axis was showed (Lines 3 and 4) (FORMTHK).

Fermat's Theorem

Fermat's Theorem was stated after Min-Max Theorem was introduced and then it was proved by the Mean Value Theorem.

Excerpt 7 (Observation)

Instructor: Min-Max Theorem. It says the following. A continuous function on a closed interval which is certainly compact by a…achieves maximum and minimum. It is very simple theorem.

(1) If f is continuous on $[a, b]$ then for every x in a, b $f(x)$ is less than or equal to $f(p)$ greater than or equal to $f(q)$, for some p and q in a, b. If f is continuous function achieves a minimum, all the values are greater or equal to some $f(q)$ or .. and achieves a maximum. This is called Min-Max Theorem. A continuous function, but it is very important that this function there is point p and there is a point q where has a minimum at q, and f has a maximum at p. This is a property of continuous function, as well as the set of which it is defined. Let's remember another definition. A function f from reals to reals is called bounded if there exists k such that $f(x)$ less than or equal to k for every x in the domain of f. Certainly a continuous function, on a closed finite interval like this, then it is certainly bounded because you see then $f(x)$ lies between $f(p)$ and $f(q)$.

(2) What about these points, p and q whose existence is guaranteed by this theorem? These are important points because at these points p and q, are functions continuous function will take its minimum and will take its maximum. Now I will tell you how to find them. This is another theorem.

(3) Theorem: f is defined in a finite interval $[a, b]$ and achieves a max, maximum I mean, or minimum at c in a, b then $f'(c)$ must be 0. If f is continuous, this theorem guarantees that there exist such a point p and q where our continuous function achieves its minimum or maximum. This theorem tells me what these point p and q must be. They must be critical points because the point where the first derivative is 0 which is called critical point. This says if a continuous function achieves maximum minimum like p and q, then the first derivative mathematically must be 0 there. I would like to prove this theorem. Because the proof is an easy application of the Mean Value Theorem.

(4) Without loss of generality, either have a maxima or minima, suppose f has a maximum point. f has maximum at c, then $f(x) - f(c)$ is clearly less than or equal to 0 because $f(x)$ is less than or equal to $f(c)$ for every x in a, b because we assume that f has a maximum there. So $f(x) - f(c)$ is negative.

So choosing such a point if x is between a point where the maximum achieves and less than or equal to right hand point b, then we have

$$\frac{\mathrm{f(x)} - \mathrm{f(c)}}{x - c} \leq 0$$

Because $x - c$ is positive but $f(x) - f(c)$ is negative. Something positive divided into something negative is always negative. So value of derivative at c which is

$$\lim_{x \to c^+} \frac{\mathrm{f(x)} - \mathrm{f(c)}}{x - c} \geq 0$$

Similarly, if I take, again if I look at

$$\lim_{x \to c^-} \frac{\mathrm{f(x)} - \mathrm{f(c)}}{x - c} \leq 0$$

Therefore, if I take the limit now, when x goes to c from the left hand side, by this theorem,

The same limit value of the derivative at c positive, negative together imply that the value that limit at c must be 0. Because the only number, real number, which is positive and negative must be 0. Right hand limit is positive and left hand limit is negative. Therefore, the common value of the limit must be 0. So, conclusion is, if a point p or q is where a max or min is achieved than the first derivative at p or the first derivative at q vanishes, $f'(p) = 0$, $f'(q) = 0$. I would like say statement in a different way, I would like the following statement. Points where f form interval $[a, b]$ into real numbers achieves a maximum or minimum is a among the points where $f'(x) = 0$.

(5) So a point of a continuous function on a closed finite interval where a maximum or minimum exists is guaranteed by this theorem, is among the points where the slope of the tangent line is 0.

(6) Among the critical points because the points where the first derivative are 0 are called critical points.

The excerpt above shows that the instructor stated the Min-Max Theorem (Line 1) and Fermat's Theorem (Line 3) to find the points where the function achieves maximum or minimum. The proof of Fermat's Theorem (Line 4) was made (AXIOTHK), and it emphasized that the point where maximum or minimum of a function exists (Line 5) and the points is among the critical points (Line 6) where the first derivative are 0 (FORMTHK).

The Intermediate Value Theorem

Excerpt 8 (Observation)

Instructor: A continuous function f from the interval (a, b) into R has the following property $f(x_1) < c < f(x_2)$ then there exist a such that $x_1 < a < x_2$ such that $f(a) = c$.

(1) It also says that a continuous function maps intervals to intervals, range of a continuous function is an interval. If you take any c in the interval $f(x_1)$, $f(x_2)$; then must be one a in the interval a,b such that $f(a)$ equal to. This is called Intermediate Value Theorem. It says a continuous function in each of a connected set under a continuous function is a connected set. If you take value the real numbers, it says f of an interval like this is an interval. That is another way of saying. Now, a continuous function defined in an interval. This is what is says. If you take c there exists a in the domain such that $f(a)$ is equal to c. Image of an interval is an interval.

The Intermediate Value Theorem was stated (FORMTHK). However, the proof was not made.

4.2.2 Students' Mathematical Thinking Performance

In TDT, five items regarding the Rolle's Theorem (item 7), the Mean Value Theorem (item 8), Fermat's Theorem (item 9), the Intermediate Value Theorem (item 10), and the theorem which states that if f is differentiable at a, then f is continuous at a (item 11) are axiomatic thinking items.

Table 4.2. *TDT Results of AXIOTHK Items*

	Item 7	*Item 8*	*Item 9*	*Item 10*	*Item 11*
Difficulty index	.75	.4	.9	.3	.85

Based on the difficulties of item 9 and item 11, the items were answered correctly by almost all participants. Regarding item 9, all interviewees stated Fermat's theorem and reasoned their answers. Student 5 explained the theorem, but he did not emphasize that if the derivative of the function at c exists then it is 0. Then he discussed the statements and gave the correct answer by irrelevant explanations. However, he selected the correct statement by eliminating the invalid statements regarding the relations among the options. Student 3 also stated the theorem correctly and emphasized that there exists a relationship between the local minimum and local maximum, and the critical points of the function. Students 1, 2, and 4 supported their answers emphasizing that "if there is a local minimum or maximum at a point, the function may not be differentiable at the point". In addition, Student 1 stated that "at sharp turn, the function is not differentiable and it is an obvious fact". On the other hand, Student 2 reasoned her answer by giving the absolute value function $|x|$ as an example, and the pointed that "there is a local minimum at $x = 0$, but the function is not differentiable there". Therefore, it indicated that Student 1 and 2 showed ICONTHK and FORMTHK as well as AXIOTHK.

Regarding item 11, the interviewees' explanations were almost the same. Here is an example.

Excerpt 1 (Interview)

> Student 1: "Every differentiable function is a continuous function, but every continuous function is not differentiable."

The explanations of interviewees showed their AXIOTHK, since they made valid inferences with respect to the theorem.

Item 7 was answered correctly by 15 participants. Interviewees were asked to explain the theorem and later if they were able to visualize an example for the given information in the theorem. All interviewees explained the theorem verbally and geometrically, examined the statements in options of the item accurately, and found the true inference with respect to Rolle's Theorem both in TDT and during the interview. However, explanations of Rolle's Theorem by Student 5 are worth deeper analysis. The excerpt from his explanation is here.

Excerpt 2 (Interview)

> Student 5: Rolle's Theorem is a different form of The Mean Value Theorem. If they are equal, $f(b) - f(a)$ becomes 0. In this case, the derivative at that point must be 0. In other words, if we think as the Mean Value Theorem, $f(a)$ and $f(b)$ is different at Mean Value Theorem. However in this case, when their values are equal, $f(b) - f(a)$ will be $f'(c)$. Therefore, here becomes 0. It shows that $f'(c)$ is equal to 0, automatically.

The student related two theorems and interpreted Rolle's Theorem by using The Mean Value Theorem. Building robust connections between these two theorems showed very active AXIOTHK of Student 5.

Moreover, 1 participant considered the function has more than one root on the open interval (a, b) and 1 participant considered the first derivative of the function is always positive or always negative on that interval. 3 participants, on the other hand, considered that "the tangent drawn to the function f at the point $(c, f(c))$ is perpendicular to the x-axis" can be inferred with respect to Rolle's Theorem. Student 1, who had considered that the tangent drawn to a function at a point is perpendicular to the x-axis in TDT, explained the theorem correctly during interview, and drew a correct example graph. Then, he found the correct answer when discussing the statements. The excerpt from the interview of Student 1 is below.

Excerpt 1 (Interview):

Student 1: (Reading the theorem) It means there is a local minimum or maximum.

Researcher: Can you draw?

Student 1: I do. Let it be such a thing. $f(b)$ is equal to $f(a)$ there. There are points, a and b. Here, at c, there is an extreme value.

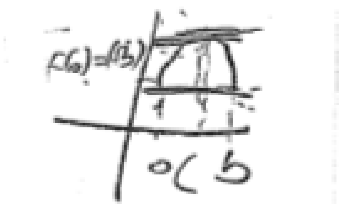


Figure 4.17. The graph drawn by Student 1 to show Rolle's Theorem

Let me say, it is parallel passing through here.

Researcher: What is the derivative of the function at that point, c?

Student 1: It is 0.

Researcher: You selected option C. You said "parallel," but in TDT you had considered "perpendicular."

Student 1: I think that I am confused by it. I thought it as parallel.

Student 1 related his incorrect answer because of the error in his language. To get more profound analysis of the thinking of Student 1 and to allow him to find the correct answer, the other options were also discussed. The interviewee stated that it could not decide whether the function has more than one root on the open interval (a, b) concerning the theorem. When he read the statement given in option B, he realized the correct answer, and the function had at least one critical point on the interval. The interviewee was asked whether there can be more than one critical point. He cited constant functions as example and stated all values on the interval are equal by drawing graph of a constant function. Regarding the other options, he stated that since there is a critical point, the function is positive on one side and negative on the other side. In addition, he explained that $f(a)$ is equal to $f(b)$; therefore, the slope of the secant line is equal to the slope of the tangent line at c.

When we look for mathematical thinking aspects of interviewees at this item, they first showed their AXIOTHK by analyzing the theorem and making correct inferences with respect to it. When they discussed the theorem and examined the statements, they used their ICONTHK aspects. They also showed their FORMTHK by using the definition of the terms in the statements given in the options.

The investigation of the item difficulties reveals that less than half of the participants responded to items 8 and 10 correctly. Regarding Item 8, interviewees were asked to explain the theorem verbally and geometrically. They all stated the theorem correctly and visualized it by drawing an example graph. Student 5, having the correct answer in TDT, explained that the average rate of change of the function on the open interval (a, b) can be determined with respect to the Mean Value Theorem. 5 participants of TDT considered that the instantaneous rate of change of the function on the open interval (a, b) can be determined; 3 participants considered the function has at least one root on the open interval (a, b); 2 participants considered the number of roots that the function has on the closed interval $[a, b]$ can be determined. Student 1, who had no answer in TDT, explained that he had not understood the theorem at first step. When he drew the graph according to the theorem and evaluated the statements, he stated that “average rate of change is $\frac{f(b)-f(a)}{b-a}$, therefore, that average rate of change of function can be determined seemed like the correct answer”. Student 2, 3 and 4 gave the same incorrect answer in TDT. They explained $\frac{f(b)-f(a)}{b-a}$ as the slope of the secant line and stated it is equal to the slope of the tangent line at c. They defined the derivative as the instantaneous rate of change. Therefore, they thought that the instantaneous rate of change of the function f on (a, b) can be determined as the valid inference with respect to given theorem although the statement does not state the point c.

According to explanations of the interviewees, they activated AXIOTHK making inferences with respect to the theorem. Their lack of definition of the average rate of change and misperceive the statement caused their failure. However, they used also FROMTHK.

Item 10 was answered correctly by 6 participants who considered the function f takes on every value between the function values $f(a)$ and $f(b)$ at least one, and other 6 participants considered that the intermediate value that the function takes between the function value $f(a)$ and $f(b)$ can be found, with respect the Intermediate Value Theorem. Students 1, 2, and 5 stated the theorem correctly and emphasized that the function takes on every between the function value $f(a)$ and $f(b)$ at least once. While Student 1 and 5 found the true statement by emphasizing that the function can take each value more than one, Student 2 reasoned her answer with continuity. 6 participants considered that the intermediate value that the function takes between the function value $f(a)$ and $f(b)$ can be found, with respect to the Intermediate Value Theorem. Student 3 claimed that the intermediate value could be found. Conversely, Student 4 realized her mistakes during interview, corrected her answer by valid explanations. As seen in the explanations of the interviewees, they activated their AXOTHK.

4.3 Algebraic Thinking

In this section, observations of instructions under algebraic thinking aspects and students' performance on algebraic thinking items are discussed.

4.3.1 Opportunities in the Classroom

As the main aim of the instructions was to engage students in algebraic thinking, in addition to the instructions presented previous sections, the instructions observed related with the examples of verifying whether a function satisfies the equality of right-hand side and left-hand side limit at a point and conditions when some theorems on derivative are not used regarding their hypotheses.

As seen in the excerpts given previous observation sections, the definitions, terms, rules, theorems and their proofs were presented algebraically. For example, the formal definition of the derivative (see Excerpt 1 (Observation) in the formal thinking section) is introduced as means variables, x and a.

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

The use of variables in instructions employs a deep ALGETHK aspect.

In the example below, the instructor analyzed the function to verify whether it satisfies the equality of right-hand side and left-hand side limit at a point.

Excerpt 1 (Observation):

> Instructor: Consider y = $|x|$ at $x = 0$. This is the following function whose graph is:

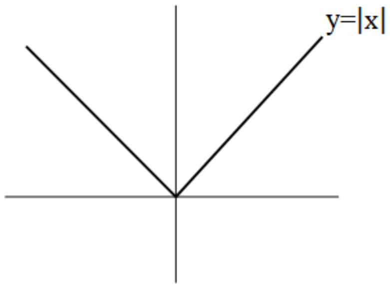


Figure 4.18. The graph of y = $|x|$

(1) It is defined x if x positive and x is 0, and defined $-x$ if x is less than 0. A function likes this. Let's look at the following. Take a is 0. But I will write a here. I am approaching to 0 from right hand side so my function is x, this is equal to $(0 + h)$ but h is also positive. So it is h. Minus value of an absolute value of a function at 0, this is 0, divided into h. But you see h is positive so right-hand limit of this absolute value of function at 0 because of the definition of absolute value right-hand limit this difference quotient is 1. Let's now look at the left-hand limit. But a is 0, do not forget. This is limit h goes to 0 from the negative side so it is h, but h is negative because we are only near so it is absolute of negative number is $(-h - 0)$ divided into h. This limit is -1.

(2) So you see the limit does not exist and it is clear from here you see when you differentiate tangent lines which are here they all have slope, the angle is 45, when here it is -45. So right-hand side is 1 and left-hand side limit is -1, so $\frac{|0+h|-|0|}{h}$ does not exist. So f is not differentiable."

In the excerpt above, the instructor analyzed a given function to verify whether it satisfies the hypotheses of a differentiation theorem (2), which is an indicator for ALGETHK. Besides, he followed procedural techniques to solve the problem and remind the rule (1); therefore, he ALGOTHK and FORMTHK were also employed.

The graphs drawn after the Mean Value Theorem and Rolle's Theorem were introduced to show the conditions when Mean Value Theorem and Rolle's Theorem are not used regarding their hypotheses.

Excerpt 2 (Observation)

Instructor: …Let me first show you what they going on. As every theorem, the hypotheses are very important.

(1) These theorems (refers to the Mean Value Theorem and Rolle's Theorem) are through under these circumstances, f to be continuous on the closed interval; f is differentiable on the open interval. Now let me draw some graphs what they will be wrong.

(2) Here is that graph of a function,

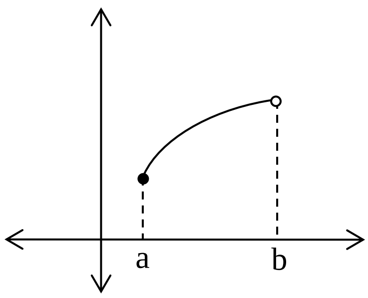


Figure 4.19. The graph of f, which is not continuous at b

it is not defined b, so here, it jumps from here to here. So this is the line f joining a and b, so there is no point parallel, f is not continuous at b. So you see there is no point on the graph which is parallel to the line joining $(a, f(a))$ to $(b, f(b))$. Because f is not continuous at b, right-hand points.

(3) In this case you have there is one, then the function jumps here and something like this.

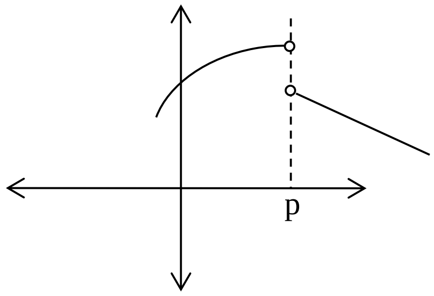


Figure 4.20. The graph of f, which is not continuous at p

f is not continuous at p. Therefore, there is no point as I say in the theorem and

(4) here is another graph, such a point it is not a differentiable.

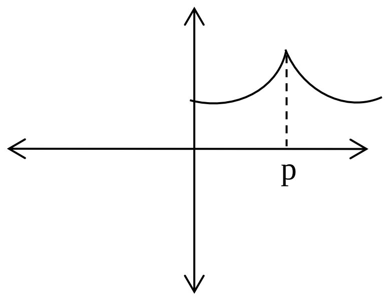


Figure 4.21. The graph of f, which is not differentiable at p

(4) These points are called cusp points. f is not differentiable at p. And all these three examples are where we don't have such a point. No such c in the theorem. Because the hypothesis is not met.

As seen in the excerpt above, the instructor stated that the hypotheses are fundamental and restated the hypotheses of Mean Value Theorem and Rolle's Theorem: "f to be continuous on the closed interval" and "f is differentiable on the open interval" (Line 1) (FORMTHK). Then he drew graphs of the functions (Lines 2 and 3) (ICONTHK), which are not continuous and drew a graph of the function

(Line 4) (ICONTHK), which is not differentiable at a point. The cusp points were redefined (Line 5) (FORMTHK). All three graphs were used to show the conditions when the Mean Value Theorem and Rolle's Theorem are not applied (ALGETHK).

The following excerpt indicates that the instructor used examples to show the cases in which the hypotheses of Fermat's Theorem were not met.

Excerpt 3 (Observation)

Instructor: Now I will give you examples where maximum may not exist. Conditions are very important.

(1) Every theorem has a hypotheses and the theorem is true whenever the hypotheses are met. You have to accept this. Now, here are examples what way go wrong. So here is what may go wrong. If the conditions are met, consider this function

(2) f(x) = $\frac{1}{x}$ on the interval (0,1). The graph is like this.

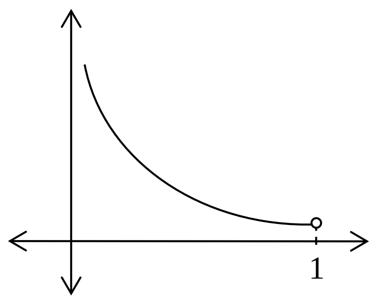


Figure 4.22. The graph of $f(x) = \frac{1}{x}$ on the interval (0,1)

This function has no maximum. Because $\frac{1}{x}$ goes to infinity when x goes to infinity. So it has no maximum. This is not surprising because the interval is not closed interval, it has no max on (0,1). The function f, has also no minimum on 0. Because f has discontinuity at $x = 0$.

(3) Second example. $f(x) = x$ on $[0,1)$

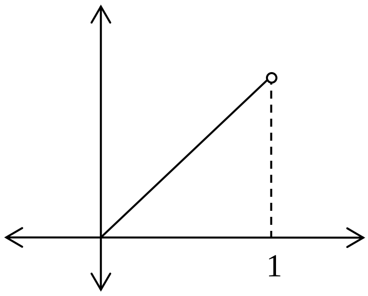


Figure 4.23. The graph of $f(x) = x$ on $[0,1)$

This time although it has minimum, 0 but no maximum. Because the interval is open because 1 is another ….0 although the function is continuous.

(4) So if I take both of them, but this time has no max has no min are not on the interval (0,1)

(5) The last one is, x equal p. This function also has no maximum discontinuous at $x = p$. Therefore, no max no min.

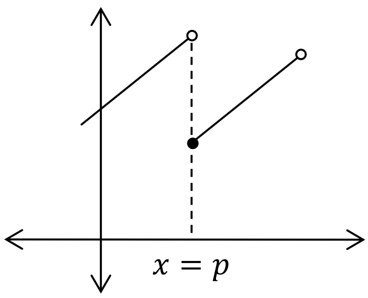


Figure 4.24. The graph of a function, which has max or min at $x = p$

(6) So intervals and functions are important. So if any of these conditions are not met, where close interval open interval for our theorem to be true, it must be closed. Endpoints must be in. Then I am sure that there exist a maxima and minima. f has to be continuous and the interval has to be closed, meaning that the endpoints, left hand point, right hand point must be in the set.

The excerpt shows that the instructor restated that the hypotheses are very important (Line 1) (FORMTHK). Then he drew graphs of the functions (Lines 2 and 5) (ICONTHK) which have no maximum or minimum and drew a graph of the function (Line 3) (ICONTHK) which has minimum but no maximum. The instructor reemphasized that the interval must be closed to apply the Fermat's Theorem (Line 6) (FORMTHK). All three graphs were used to show the conditions when Fermat's Theorem is not applied (ALGETHK).

4.3.2 Students' Mathematical Thinking Performance

In TDT, four items regarding hypotheses of the Mean Value Theorem (item 12), the Rolle's Theorem, Fermat's Theorem and the Intermediate Value Theorem (item 13), the Rolle's Theorem and Fermat's Theorem (item 14) and the Mean Value Theorem (item 15) are algebraic thinking items.

Table 4.3. *TDT Results of ALGETHK Items*

	Item 12	*Item 13*	*Item 14*	*Item 15*
Difficulty index	.3	.4	.35	.5

6 participants of TDT answered Item 12 correctly. 6 participants, on the other hand, selected option C, which including the one correct expression and also incorrect one. They also considered the function f takes on the value of 45 at least one on the closed interval [6,15]. 3 participants also considered the average value of the function f on the closed interval [6,15] is 45. Besides, the item was not responded to by 5 participants.

Item 12 was answered correctly in TDT by only Student 5 among interviewees. His explanation is in the excerpt below.

Excerpt 1 (Interview):

> Student 3: The function is continuous between 6 and 15. The minimum value is -2, and the maximum value is 88. Therefore, I thought that it takes every value between these values.

Student 1, who had no answer in TDT, explained that he did not know how to apply the theorem. The other interviews calculated the maximum value of $f(15)$ and found 88 given in the first expression. However, it was seen that the students had no idea about the average value of the function, Student 3 and 5 claimed that average value of the function could be found by adding the smallest and the greatest values and then dividing the sum in to two. Student 5 also explained that in order the average value to be 45 maximum value of $f(15)$ must be 90 and so accepted the expression is not true.

The item showed how the interviewees activated ALGETHK. According to their explanations, it can be inferred that except from Student 1 the all interviewees also activated their ALGOTHK since they found the maximum value of a function at a given point.

Item 13 covering Rolle’s Theorem, Fermat’s Theorem, and Intermediate Value Theorem together was answered correctly by 8 test participants. 4 participants selected option D, which also the function satisfies the hypotheses of Fermat’s Theorem. The other 4 participants selected option E, containing all expressions. The item was analyzed in terms of the explanations of interviewees on each theorem. Student 3, 4 and 5 told that the function does not satisfy Rolle's Theorem since $f(a)$ and $f(b)$ are not equal on the closed interval. The students 1 and 2, on the other hand, stated the condition $f(a) = f(b)$, but when they found the values of the function at end points, they realized they are not equal, and it does not satisfy the theorem and so their answers were incorrect. None of the students had a reasonable answer about whether the function satisfies Fermat's Theorem. Most of them stated there could be a maximum or minimum point.

Item 14 was answered by 7 test participants correctly. The interviewees, apart from Student 1, verified whether the function satisfies the condition that $f(a) = f(b)$, when a is 0 and b is 1. Since they saw the condition was satisfied, they found the appropriate number c. It was assumed that since the given function is polynomial, they had already known it is continuous on the closed interval and differentiable on the open interval. Therefore, they did not need to discuss it. Student 2,3 and 5 canceled the statements, including the Intermediate Value Theorem, since the function does not satisfy the Intermediate Value Theorem. The explanation of Student 4 was different from the other interviewees' explanations. The excerpt is below.

Excerpt 2 (Interview):

> Student 4: $f(0)$ is 0, $f(1)$ is 0. So it cannot be.
>
> $f(0) < f(c) < f(1),\ 0 < f(c) < 0.$
>
> Therefore, the probability of that $f(c)$ is $\frac{1}{2}$ is 0.

Her reasoning was quite interesting. Her explanation was seemed valid at first glance, but she missed out the specific hypotheses of the Intermediate Value Theorem which is $f(a) \neq f(b)$.

Moreover, regarding item 14, 5 participants had no answer, and 5 participants were mistaken. They could not find the appropriate number c with respect to Rolle's Theorem and found a value M with respect to the Intermediate Value Theorem, although the function does not satisfy the hypotheses of the theorem. Student 1 had no answer and emphasized that he had difficulty at the items including application.

Therefore, interviewees activated ALGETHK and AXIOTHK. Besides, except for Student 1, all interviewees found the c by following the routine procedure, so they revealed their ALGOTHK.

Item 15 was answered by 10 test participants correctly. Student 2 and 3 stated the theorem but could not apply it. The students, had the correct answer in TDT, were asked to explain the procedure which he followed. Student 5 expressed that he tried each option. He continued to express his procedure through the function which he selected in TDT.

Excerpt 3 (Interview):

> Student 5: Let me differentiate first the function, $3x^2 - 1$. At the same time, I am concerned Mean Value Theorem and finding the difference of the values. It must be 0. When it is equal to 0, I obtain $\frac{1}{\sqrt{3}}$ since I know that it must be 0. Since it is a value between these, it satisfies the theorem."

Furthermore, Student 2 had the correct answer, and rationally explained her answer. The episode is below.

Excerpt 4 (Interview):

> Researcher: You selected option C in TDT, and I saw that you eliminated options B and E. But you did not eliminate the others.
>
> Student 2: I eliminated options B and D, since the function should be continuous. Nevertheless, in option B, the function is undefined, so I eliminated it. At the point 0, $|x|$ it is not differentiable. Therefore, I eliminated it too. I tried the other options. I set -1 and 1 and the answer became option C.
>
> Researcher: How did you find?
>
> Student 2: I wrote $f(1)$ then subtracted $f(-1)$, it became $1 - (-1)$, and here become 2. Then I differentiated this $(x^3 - x)$. We know $f'(c)$ is 0. We said

there is such a point. We can write $f'(c)$ as $\frac{f(1)-f(-1)}{1-(-1)}$. From $\frac{f(1)-f(-1)}{1-(-1)}$, it became 2, here will be 0. Then I calculated $f'(c)$ and I found the result.

The item was not responded correctly by 6 participants and 4 participants had no answer. Student 1, 3, and 4 explained that they could not answer the question. During the interview, they were asked to try to solve it. Student 1 applied the theorem properly and verified the function $x^3 - x$ satisfies the hypotheses of the Mean Value Theorem on $[-1,1]$. According to the results of the interviewees, the Student 1, 2 and 5 activated their ALGETHK, and by taking the derivative of the function and verifying the hypotheses of the theorem, they showed ALGOTHK.

4.4 Iconic Thinking

In this section, observations of instructions under iconic thinking aspects and students' performance on iconic thinking items are discussed.

4.4.1 Opportunities in the Classroom

As the opportunity in the class to engage students in iconic thinking, the instructions observed related to the graphical connection between a function and its derivative and how derivative affects the graph of a function are presented here as the main aim of the instructions was to engage students in iconic thinking.

As seen in the excerpts given previous observation sections, almost all the definitions, theorems, and examples were presented by the contribution of graphical representation of functions. For example, the formal definition of the derivative (see Figure 4.1) is introduced as a means of the graph.

Excerpt 1 (Observation):

> Instructor: We know that the first derivative is very much related to the slope of the tangent line at $x_0 = a$ is the value of the first derivative at $x_0 = a$.

As seen in the excerpt above, the instructor emphasized that the derivative is highly graphical-based content (ICONTHK).

Besides, when basic terminology of the curve sketching was given (as seen in Excerpts 5,6,7,8,9,10 (Observation) in the formal thinking section), the visual examples were used intensively. It means that the information related to how the derivative affects the graph of a function (FORMTHK) was given by helping of ICONTHK. Besides, the content which canalizes ICONTHK was introduced by using definitions via FORMTHK.

Moreover, the examples which are discussed in the algorithmic thinking section include graphs and the optimization problems, which are discussed enactive thinking section includes images and diagrams which activate ICONTHK.

4.4.2 Students' Mathematical Thinking Performance

In TDT, five items regarding retrieving information from the graph to decide the derivative of a function at a point (item 16), inflection points (item 17), construction of a derivative graph of a function with respect to graph of the original function (item 18), construction of a graph of a function with respect to graph of the derivative function (item 19), and retrieving information from the graph of a function to find the derivative of another function at a point (item 20) are iconic thinking items.

Table 4.4. *TDT Results of ICONTHK Items*

	Item 16	*Item 17*	*Item 18*	*Item 19*	*Item 20*
Difficulty index	.85	.95	.85	.7	.8

Considering the item difficulties, items 16, 17, and 18 responded correctly by almost all participants. Item 16, which involves the application of Product rule, requires retrieving information from the graph in order to find the slope of the tangent line and values of a function at a point. Additionall, it requires considering the relation between derivative of a function at a point and slope of the tangent line at that point. All interviewees, who gave the correct answer in TDT, also solved the question correctly during the interviews. They differentiated both sides of the given equation, $h(x) = x \cdot f(x)$, in order to find the derivative of h at a given point. They used the product rule to differentiate $x \cdot f(x)$ and attained $h'(x) = f(x) + f'(x) \cdot x$. So they got the expression: $h'(-3) = f(-3) + f'(-3) \cdot (-3)$. To find corresponding y coordinate to x and define the sign of the slope of the tangent line, the graph interpretation was must. In order to find the derivative of h at the point $x = -3$, they needed $f(-3)$ and $f'(-3)$. Therefore, they obtained the information from the graph. Since it was given that d is tangent to the graph of a function f at the point A at the verbal part of the question and showed on the graph, they calculated the slope of the d to find the derivative of f at $x = -3$. In addition, they decided the sign of the derivative of the function at $x = -3$ regarding the slope of d, which is the tangent line of the function at the point. The excerpt from Student 4 is below.

Excerpt 1 (Interview):

> Student 4: …It is asked to find $f'(-3)$. This is actually equal to its slope at that point.

Then she found the slope of the tangent line that touches the curve at $x = -3$. The explanation of Student 4 reasoned her answer relating derivative of the function at the point A and slope of the tangent line to the graph of the function at that point. Student 4, in this way, activated FORMTHK more than the other interviewees.

The solutions of the interviewees showed their ICONTHK. Since the solution of the question also required the definition of the Product rule and the relation of derivative of a function at a point and slope of the tangent line at that point, the item help students to activate also FORMTHK. Moreover, the solution required the application of the Product rule; ALGOTHK was also activated.

Item 17, which is related to the interpretation of the graph of the second derivative of a function f, asks to find inflection points of f. In order to find the correct answer of this item, the relation between inflection point and the second derivative of a function should be known, and the given graph should be interpreted correctly. The item was responded correctly by all the interviewees in TDT. Since the item required the definition of an inflection point and Student 1 has already stated that he did not know what inflection point is, his correct response was unexpected. However, the explanation of Student 1 showed that he selected the correct answer by an irrelevant solution.

Excerpt 2 (Interview):

> Student 1: When I examined the options, the correct option should be either A or E. Then, instead of maximum points, I decided to choose the points where the graph intercepts x-axis, and I found 2,6 and 10.

The correct answer was given from other interviewees who had correct explanations for 2nd item and by correct interpretation of the graph. Therefore, the item activated interviewees' ICONTHK and FORMTHK.

Item 18, which all interviewees had correct answers in TDT, asks students to choose the graph of the derivative of the function given at the beginning of the question. All interviewees made explanations in the same way: Here is a sample of answers.

Excerpt 3 (Interview):

> Student 2: Here, a function was given. We have this information: When the first derivative of f is positive, greater than 0, f is increasing. When the first derivative of f is negative, less than 0, f is decreasing. If f is increasing on $(-2, -1)$, the sign of the first derivative has to be positive. Then I continue. Since it is decreasing $(-1, 0)$, it should be negative. Then it has to be positive, and then it has to be negative again. These are provided in option B. So, I selected option B.

Item 19, on the other hand, was answered by 14 participants correctly and, 5 participants selected option C, which including the graph, which is symmetric of the graph of the derivative of the function with respect to the origin. The interviewees all had correct answers in TDT, and they reasoned their answer by relating properly the sign of the first derivative and the intervals in which the function is decreasing or increasing.

Item 20, which was responded correctly by 16 participants, was not responded by 2 participants. In addition, the other 2 participants selected the same incorrect option C. In order to reach the correct solution of item 20, the quotient rule is applied correctly, and then the first derivative of a function at a local minimum point is discussed, and the value of a function at any point is determined by reading the given graph. At this item, all interviewees had correct answers both in TDT and in the interview. They applied quotient rule properly and attained the equation, $h'(x) = \frac{f'(x)\cdot x-1\cdot f(x)}{x^2}$.

All interviewees reached the information about the first derivative of $f(x)$ at $x = 3$, by using the key point in the question. Since $A(-3,1)$ is the local minimum point of f, the interviewees were expected to explain that first derivative of $f(x)$ at $x = 3$ is 0, if it exists, regarding Fermat's Theorem.

Excerpt 4 (Interview):

Researcher: Why the derivative is 0 at $x = 3$?

Student 4: Since it is a local minimum, it is a critical point.

During the interview, the interviewees read the graph and found the value of the function at $x = 3$. The interviewees used the information given the verbal part of the question, A(-3,1) is the local minimum point of f. In addition, it was assumed that interviewees saw $f'(x)$ exists at $x = 3$ by interpreting the graph. The question, with its nature, measures ICONTHK. Nevertheless, it is valuable to measure also FORMTHK, AXIOTHK, and ALGOTHK.

4.5 Algorithmic Thinking

In this section, observations of instructions under algorithmic thinking aspects and students' performance on algorithmic thinking items are discussed.

4.5.1 Opportunities in the Classroom

As the main aim of the instructions was to engage students in algorithmic thinking, the instructions observed related with computing derivative, computing derivative of a function by using Chain Rule, evaluating the values on an interval where the given function is decreasing/increasing, computing the product of the unknowns in a function via using its local extremum point and inflection point, and evaluating whether a given partial function is differentiable at a given point are presented here.

4.5.1.1 Derivative as the Slope of the Tangent Line

Excerpt 1 (Observation)

Instructor: Find the equation of the tangent line at $P(1,1)$ of the curve $y = x^2$.

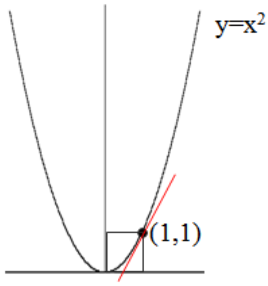


Figure 4.25. The graph of $y = x^2$ and tangent line passing through $P(1,1)$

(1) You have a curve which is a parabola passing through the point $P(1,1)$. What you really want is to find the line which touches the parabola, so you want to find the equation of this red line, tangent line to the parabola $y = x^2$ at $(1,1)$. It is tangent line. If you know one

(2) point on it and the slope because the equation of a line is $y = ax + b$ or the slope is $\frac{y-y_0}{x-x_0}$. If you know one point (x_0, y_0) on it, then we calculate the slope $\frac{y-1}{x-1}$ this could be m. So if you know one point from your analytic geometry courses if you know slope and if you know one point on it, you can always write a line. But as it happens in this example, the line is a very special line. It is given tangent line it touches to parabola

(3) $y = x^2$ only once at the points (1,1). And the slope is given by this difference quotient in this case is equal to $\frac{x^2-1}{x-1}$. If you compute the limit of this thing as x goes to 1 as x_0 is 1 of $\frac{(x+1)(x-1)}{(x-1)}$ this limit could be 2. Hence you are looking for a curve if you use this equation $2 = \frac{y-1}{x-1}$ then the curve $y = 2x - 1$. This is the curve. This is the line $2x - 1$ which touches the parabola at $x = 1, y =$

1 and has slope 2. You can calculate the slope you already given one point on it so you can write the equation of the line. But actually this is not an ordinary line. It is a line whose slope is given by mechanically structured.

(4) These are limit of secant lines you care to choose parabola as you go from say a near by point to the point (1,1). You consider each of these secant lines and take the limit of this difference quotient as you go towards as x goes to (1,1). That corresponds to the idea you move along the curve from x come to the point $x = 1$ either left or right because you said the limit exist with the right hand limit is the same as left hand limit. You should not make any difference whether which point you will approach to x either from left or from right. Any case you should be able to take this limit, to calculate the slope then the second step is to find a line whose slope is given by this limiting process and the point on the curve that passes through (1,1). As I said this could be basic concern.

(5) To this formal expression here, you can approach another way. A second approach to the limit in $m_{PQ} = \lim_{Q \to P} \frac{\text{f(x) - f(a)}}{x-a}$ where $Q = (x, f(x))$, $P = (a, f(a))$. What you do you bring Q towards P or you go you take x towards a, and naturally $f(x)$ would becoming towards $f(a)$. A second approach to the problem is the following. Instead of taking Q as $(x, f(x))$ we consider the following. Let $h = x - a$ the distance is h, which is sometimes go step by step. So h, because P has coordinates $(x, f(x))$, h to be $(x - a)$. And consider the difference quotient defining $m_{PQ} = \frac{\text{f(x) - f(a)}}{x-a}$ as in $\lim_{h \to 0} \frac{\text{f(a+h) - f(a)}}{h}$. Because now if h is $x - a$, point x has coordinate $x + h$. So you have $f(a + h)$ instead of $f(x)$, and $f(a)$ this time $(x - a)$ is h. And how would you describe x is moving towards a by simply limit h is going to 0 because as h goes to 0, the distance between x and a, h the point x goes to, but we will be discussing.

(6) Don't forget in order to decide whether this limit exists or not we have to take the right hand limit left hand limit and see they are equal. So another way

second approach to the limit of this difference quotient exists is to say that this limit exists. This limit exists. But please do not forget that the limit above is equality of when $h \to 0^+$ and $h \to 0^-$. We say that this differences quotient, slope of secant line. Secant line exist and here we discuss this way instead of letting h, $x - a$, replacing everything $f(a + h)$– $f(a)$. Existence of this limit is actually existence of right hand and left hand limit. Then they exist and equal, then the limit exists. This is the limiting process

(7) that we should already know. So I think I can define $f: R \to R$ is differentiable if the following limit exists.

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

If this limit, if this quotient exists as h goes to 0, then we say f is differentiable. In order to appreciate the definition maybe we should look at the following. In example of function which is not differentiable.

In order to visualize the problem, the graph of the function was used, so ICONTHK was employed. As seen in the graph and the excerpt, the FORMTHK was activated by stating the line passing through a point of a curve is the tangent line (1). In addition, the FORMTHK was employed in order to remind the equation of a line and how the slope of the lines can be found (2). The solution of the task given as an example, required to follow limiting techniques (3), so ALGOTHK.

By using the data from the example (4) and to introduce another approach to the limit of the difference quotient (6), the instructor emphasized that the limit exists if the right-hand limit is the same as left-hand limit. Therefore, FORMTHK was activated, also introducing equal formula of the definition of the derivative (5, 6, 7).

Excerpt 2 (Observation)

Instructor: Example: Find the points on $y = x^4 – 6x^2 + 4$ for which the tangent is horizontal?

(1) It is a polynomial, it is a differentiable function and the derivative exists and the point is on this curve.

(2) OK. The graph is something like this:

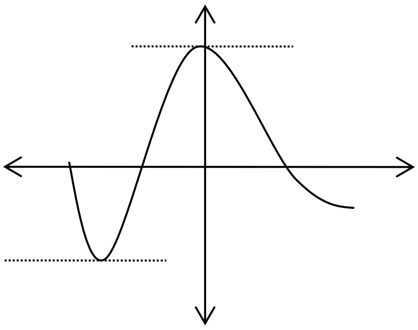


Figure 4.26. The graph drawn by the instructor for $y = x^4 - 6x^2 + 4$

(3) So, we ask to find where the tangent is horizontal that means find the points on $y = f(x)$ where $\frac{dy}{dx}$ = 0. Because at this point slope is parallel to the x-axis and the slope is 0. So, question is find the point on this curve, so we take derivative:

(4) $$\frac{dy}{dx} = 4x^3 - 12x = 4x(x^2 - 3) = 0$$

(5) So, we should equate that to 0 because we want to find the points where the tangent is horizontal that means it has slope 0. And there are 3 points: at $x = 0$, $x = \mp\sqrt{3} \Longrightarrow \frac{dy}{dx} = 0$

(6) And finding the second coordinate, y coordinate of such points we see that at (0,4) because if you put $x = 0$, you get y.

(7) at (0,4), ($\sqrt{3}$, -5)(-$\sqrt{3}$,-5) are points where the tangent is parallel to x-axis or $\frac{dy}{dx}$ = 0.

As seen in the excerpt above, the instructor reminded (Line 1) that a polynomial is a differentiable function, so the derivative exists, and the point is on this curve (FORMTHK). He drew the graph (Line 2) (ICONTHK) and explained that (Line 3) tangent is horizontal at the points where the derivative is 0 at that point, and the slope, which 0, is parallel to the x-axis (FORMTHK). Then (Line 4) he took the derivative of the function and equated the algebraic expression (ALGETHK) to 0 and then found the roots (Line 5). By using the roots (Line 6), he found the y coordinate of the points where the tangent is parallel (Line 7). Therefore, by following the procedure, from (Line 4) to (Line 7), he employed ALGOTHK.

Excerpt 3 (Observation)

Instructor: Example: Consider the function given as the product

$$f(x) = (2x + 1)^5(x^3 - x + 1)^4$$

(1) This is the differentiable function at every point because it is a product of two functions, $(2x + 1)^5$ which is actually itself a composition of two functions x goes to $2x + 1$, then you applied the fifth power, this is a composition of two functions differentiable, the second one is a polynomial $(x^3 - x + 1)^4$ is a differentiable functions.

(2) Polynomials are differentiable everywhere and then raise to the power 4.

(3) This is the composition of polynomial and then power function.

(4) It is differentiable by Chain Rule.

(5) And then it is a product of two differentiable functions and the product is differentiable and the derivative is taken by the product rule.
So let's take it.

(6) $\frac{df}{dx}$, please notice that I am using different notation sometimes I use prime, sometimes I use $\frac{df}{dx}$, $\frac{dy}{dx}$, etc. You must get used to both of these notations. So by the chain rule;

(7) This is 5 times $(2x + 1)^4$ it is time we go from inside the derivative is 2, times this function, $(x^3 - x + 1)^4$ plus I now apply the product rule and chain rule to this one, $4 \cdot (x^3 - x + 1)^3$ times derivative of inside which is $(3x^2 - 1)$ times by the product rule $(2x + 1)^5$.

$$\frac{df}{dx} = 5(2x+1)^4 \cdot 2 \cdot \ (x^3 - x + 1)^4 + 4 \cdot (x^3 - x + 1)^3 \cdot (3x^2 - 1) \cdot (2x+1)^5$$

According to the excerpt above, the instructor noted (Lines 1 and 4) the function is differentiable (FORMTHK). He also restated that polynomials are differentiable everywhere (Line 2), and the product of two differentiable functions is differentiable and the derivative is taken by the product rule (Line 5) (FORMTHK). He also addressed the symbols of the derivative (Line 6) (FORMTHK). Finally, he took the derivative of the function by using both the product rule and chain rule (Line 7) (ALGOTHK).

Excerpt 4 (Observation)

Instructor: Example: Find the derivative of y = $\frac{3x^3-4}{x}$ at $x = -2$.

(1) If f is in these cases $3x^3$ – 4, it is a polynomial. It is a differentiable function. And the derivative is $9x^2$. The function x is a polynomial of degree 1, it a polynomial.

(2) So when x is not 0, and it is given $x = -2$ here, it is quotient of two differential functions.

(3) So let's take the derivative. The derivative of this function is: It is differential function for the reasons we have explained and therefore this is equal to:

$$D\left(\frac{3x^3-4}{x}\right) = \frac{9x^2 \cdot x - (3x^3 - 4)}{x^2}$$

$$= \frac{9x^3 - 3x^3 + 4}{x^2}$$

and the value of the derivative is;

At $x = -2$ the value of the derivative is -11. Substitute $x = -2$

The equation of the tangent line to this curve $\frac{3x^3-4}{x}$ is (-2). How will I find the rest? We will substitute $x = -2$ in the equation. You cannot solve the problem if the point is not on the curve. But actually it is because when you put -2 you will get $y = 14$, by putting x equals -2 we calculate the y coordinate and is:

$\frac{y-14}{x-(-2)} = -11$ You can solve.

$$\frac{y-14}{x+2} = -11$$

As seen in the excerpt above, the instructor reminded that polynomials are differentiable (Line 1) and the quotient of two differential functions is differential (Line 2) (FORMTHK). Then, quotient rule was used to differentiate the function and product rule was applied step by step in (Line 3) (ALGOTHK).

Excerpt 5 (Observation)

Instructor:

Example: Sketch the graph of f such that f has the following properties.

(1) 1) $f'(x)$ should be positive for all x in the interval $(-\infty, 1)$ and $f'(x)$ is less than 0 on $(1, \infty)$. It tells f' says lots of things about f. Between $-\infty$ and 1, the function is increasing, increasing and since the sign of the first derivative is positive. And then decreasing.

2) $f''(x)$ is greater than 0 on $(-\infty, -2)$ and $(2, \infty)$ but is less than 0 that $f''(x)$< 0 on $(-2,2)$ and

(2) 3) $\lim_{x\to-\infty} f(x) = -2$ and $\lim_{x\to\infty} f(x) = 0$

What does the third line say?

(3) At -2, the function has an asymptote. When x goes minus infinity, our function goes to the line $y = -2$. Same thing, when it goes to plus infinity, it is 0.

(4) Our function is increasing … So at 1, there seems to be a local maximum because it is increasing, increasing, increasing and starts decreasing so there is a local maximum. This uses the first line.

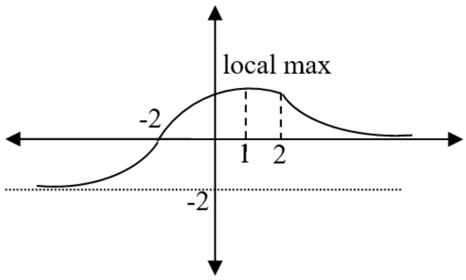


Figure 4.27. The graph drawn as the solution of the example given in Excerpt 5.

(5) Concavity can be decided by using the sign of second derivative.

(6) Because the second derivative is the derivative of the first derivative. Using when it is positive or negative we can decide you see here before -2 we are told there are second derivative is positive between 2 and -2 it is negative and the it is again positive says something about the concavity.

The excerpt shows how the shape of a graph (ICONTHK) was decided by using the first (Lines 1 and 4) and second (Line 5 and 6) derivative of the function and the asymptotes (Line 2 and 3) (FORMTHK).

Excerpt 6 (Observation)

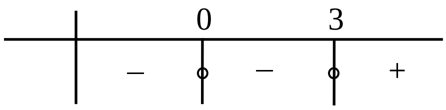
Instructor: Example: Find local maximum and local minimum for $f(x) = 3x^4 - 12x^3 + 5$

What is given here is a polynomial of degree 4.

(1) How many zeros do we have? It means, how many times does it intercept x-axis? Why does it have 4 zeros? By fundamental theorem of algebra. Any polynomial. this is fundamental theorem of algebra. This algebraic equation has 4 roots. So well-known is called fundamental theorem of algebra. Any algebraic equation, like this, to have as many as the highest degree which is 4. Any polynomial is continuous so is a continuous function without .. any continuous function polynomial algebraic equation cuts x-axis 4 times and put x equals to 0 and find it cuts the y-axis. So there are roots, 1,2,3,4 and 5

it cuts the y-axis. As far as the domain of this function is whole real numbers. Because domain of definition of polynomials is whole real numbers. For rational functions, a polynomial divided into a polynomial we just call the rational function which are also very important. When denominator is 0, then we have a problem. Then we will have vertical asymptotes. So I would like to investigate the derivative.

(2) Derivative of the function is a polynomial of third degree which is $12x^3 - 36x^2 = 0$. 0 and 3 are critical points. These are the roots of $x^2(12x - 36) = 0$. So value put before 0, it "−", here



increasing up to 3, then the sign of the first derivative changes and it becomes +.

(3) At $x = 3$ there is a local minimum. You just look at the sign of the first derivative. 0 is a critical point but nothing happens at 0. Decreasing, decreasing up to 3 and after 3 it is starting increase so by the First Derivative Test. This is what the first derivative test tells us.

(4) If the sign of the first derivative is negative, negative, negative and then stop becoming positive, then there is a local minimum by the first derivative test. OK.

As seen in the excerpt, the instructor first reminded the basic information concerning polynomials (Line 1) (FORMTHK). Then by using the roots of the derivative function, he decided the sign of the first derivative (Line 2) (ALGOTHK). The point $x = 3$, where the sign of the first derivative changes from negative to positive, was called the local minimum (Line 3) (FORMTHK) according to the First Derivative Test (Line 4) (FORMTHK).

Excerpt 7 (Observation)

Instructor: Example: Discuss the concavity of the function $y = x^4 - 4x^3$, a polynomial with inflection points and local maxima and minima. This question can also be asked in the following question, local maxima minima (extremum points).

So we are going to discuss inflection points. So we can only answer this question if we know what is asked.

(1) Inflection points are the points where the concavity changes.

(2) Inflection point means you have to find the set of points where the second derivative is 0 and discuss the sign of the second derivative.

(3) If the second derivative positive then it is concave up, if the sign of the second derivative is negative there is concave down.

(4) And a local maxima and minima, collection of first derivative is 0 together with singular points that is to say where the first derivative does not exist.

(5) So we have to find critical points, these are the points where $f'(x) = 0$ and the tangent line is horizontal, and singular points where derivative does not exist. What else do we have to decide?

(6) And find the second derivative is 0 these are inflection points, then we have to discuss sign of the second derivative for concavity.

(7) After we find where the first derivative is 0 and derivative does not exist then we discuss the sign of the first derivative so if it changes from increasing to decreasing we have a local maximum, if it goes from decreasing to increasing we have a local minimum.

(8) So we are given a polynomial, which is differentiable everywhere.

(9) Given $x^4 - 4x^3$, a polynomial of degree 4, differentiable and continuous everywhere.

(10) Such functions in mathematics are called entire functions.

(11) The first derivative of this function:

$$f'(x) = 4x^3 - 12x^2$$

(12) And the second derivative is

$$f''(x) = 12x^2 - 24x$$

(13) This says no singular points no points where because first and second derivative exist everywhere. There is no cusp points like this. It is differentiable everywhere. So we found first and second derivative.

(14) So let us find where the critical points are. $4x^2(x - 3) = 0$. So the critical points are 0 and 3. So if you look at the sign of the first derivative it is 0 at 0 and also 0 at 3. It is minus before 0. $4x^2$ is always positive so everything depends on the sign of the first derivative. If you put less than 0 it negative, negative, plus.

x		0		3	
	–	o	–	o	+

(15) So the conclusion from the sign of the first derivative we see that from f', f decreases on the interval $(-\infty, 3)$ and increasing in the interval $(3, \infty)$.

(16) So the sign of the first derivative is very simple there is no change so 0 is a false critical point because derivative does not change sign at 0. This expected because x^2 is always positive so everything depends on 3.

(17) So, from the first derivative test, we conclude that there exists, it is decreasing and then increasing, so there exist a local minimum at $x = 3$.

(18) Let's look at the sign of the second derivative which is $12x^2 - 24x$. So this is $12x(x - 2)$. So if you want to discuss the graph of the sign of the first derivative, it is 0 at 0 and 2. So it has 2 roots.

x		0		2	
	+		−		+

Inflection point seems to be 0 at 2 and sign of the second derivative.

The graph of f is up in $(-\infty, 0)$ and $(2, \infty)$ because sign of the second derivative is positive in this interval and it is concave down on the interval (0, 2).

So we already know that it has a minimum at 3 so what is that value, substituting $x - 3$ in the function $x^4 - 4x^3$, so $(0,0)$ is an inflection point. Why? Because the sign of the first derivative changes from + to 0 as well as it is an inflection point and $x = 2$ is also an inflection point. It is also clear that putting $x = 0$ on $f(x) = x^4 - 4x^3$, it is 0. So x intercept is 0. And putting $x = 4$ is also an x intercept. So putting on these together, it is continuous function, and there is a minimum at $x = 3$, it cuts x-axis at 4. After this example I would like to add few more about second derivative test.

(19) So you may wonder what happens if the second derivative is 0. So it is inconclusive if the second derivative at a is equal to 0 that is to say such a point there may be a maximum or maximum or neither of these. The second derivative test may also fail if second derivative does not exist. So in this case, we use the first derivative test. So it is advised that to utilize the first derivative test. In my opinion, the first derivative test is easier to use.

As seen in the excerpt above, the instructor activated FORMTHK by restating (Line 1) the definition of inflection point; (Lines 2 and 6) the relation between inflection point and the sign of the second derivative; definitions of (Line 3) concave up and concave down, (Lines 4 and 7) local maximum, local minimum, and (Line 4) singular points, (Line 5) critical points; (Line 5) that the tangent line is horizontal at the critical point where $f'(x) = 0$, (Line 6) the relation between concavity and the sign of the second derivative; and (Lines 8 and 9) that polynomials are differentiable everywhere. Then, he found (Line 11) the first derivative of this function, (Line 12) the second derivative of this function, (Line 13) critical points, (Line 14) the sign of the first derivative and (Line 15) he decided in which interval where the function increasing and decreasing. The instructor found, then, (Line 17) local minimum, (Line 18) inflection point, and decided where the function concave up and concave down. The step by step procedures from (Line 11) to (Line 18) shows how the instructor activated ALGOTHK. In addition, it was emphasized that the second derivative test is inconclusive when the second derivative at a point is equal to 0 or does not exist.

Excerpt 8 (Observation)

Instructor:

(1) Example: A farmer has 1200 meters of fencing and wants to fence off a rectangular field. No fencing on the river side.

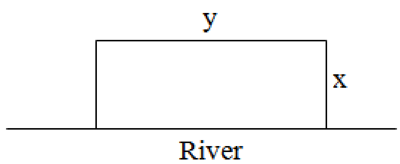


Figure 4.28. The fencing of the rectangular area

Find the dimensions of the field with maximum area. Let this side be x, let this side be y.

(2) So we know that $2x + y$ should be the length of the fencing 1200 m and we want to maximize the area. Area of the rectangle here is x times y.

(3) But as usual there are too many variables x and y. So we have to eliminate at from here so we y equal we have $1200 - 2x$ for y. Okay, they have y because the river side needs no fencing. We only fence the length side. Okay, so substitute here, we have x times $(1200 - 2x)$.

(4) So the function to be maximized is $1200x - x^2$. This is our area, the function to be maximized. x is between 0 and 600. Why? Because, anything greater 600 would give you negative area. And area cannot be negative and cannot be 0. So x changes is here, so the function to maximize is the function $A = 1200x - 2x^2$.

(5) Taking the derivative, this is $1200 - 4x$ and for critical points we have to need this derivative is $4x$. We have critical point at x equals 300. Okay. Is it maximum in this problem? It is maximum. They cannot be minimum because the minimum area it is not to clever to asked the minimum, because minimum area is the 0. So the 300 can only be, so the maximum area that is fenced with 1200 m of fencing is A(300) equal to $1200(300) - 2(300)^2$ and it turns out that it is 180 000 m^2. So, 1200 long fencing enable to enclose a maximum area of 180 000 m^2.

(6) And I know that in this case, this function has no singularity. Observe that the function has no singularity because area function in this case $1200x - x^2$ is a polynomial. Singularity means where the derivative does not exist. and it is differentiable everywhere. Okay. And at end points the area that 0 is 0 so we have compare with 180 000 m^2, $A(0)$ is 0 and at the end point $A(600)$, (writing 1200 (600) – 2 (600)^2), I think this is less than (writing 180 000). So, we also have to check the endpoint, nevertheless the maximum cannot be at the endpoints. It can only be at critical point and the area maximum. If you are not sure, but let me make a remark, there may be more than one critical point.

(7) Suppose there are two critical points or may be three, so we have to calculate $A(x_1)$, $A(x_2)$, $A(x_3)$, I choose the maximum among these. They could be three of points where the maximum may appear I like this problem because when you take the derivative of the area function it is polynomial of degree 1, so it has only one 0, one critical point. But in some problems, we have a polynomial again, and the maximum may happen at three critical points. Then what do we do? We choose the maximum; we choose the one which yields to greatest value so that would be the absolute maximum.

(8) So let us try to reach some conclusions from this example. First of all, you must understand the problem, understand what the problem is. Second case, identify the variables, dependent and independent, like in this case. Then, try to reduce variables to one variable. Like in this case, the problem is maximization problem where you have to maximize the area. So the area of a rectangle obviously x times y, when x is the short side and y is the long side. Fair enough, but if you write this, then we have too many variables and the other information that it is a rectangular field gives you that $2x + y$ some of the sides, but we don't fence the river side, so $2x + y$ is the length of the fencing to be used so $2x + y$ is 1200 that would give us the function to be maximized. Then you realize that the maximum can happen, you see first of all, the variable changes between 0 and 600, then you have to think about this, because you see if x is more than 600, the area would be negative, which is not very clever. Because area cannot be. It is a length, so it is always greater or equal to 0. So we have this, we know that this is a polynomial, the function to be maximized is polynomial. Polynomials are differentiable functions everywhere. Therefore, they are continuous everywhere. And in a closed finite interval, there would have a maximum. However, this maximum may not be the absolute maximum because among the maximums we want to biggest one. So in any case, we have to find the critical points, we have only one critical point, but we also have to check what happens at the endpoints. There is no singularity, sometimes the maximum happens at

a point where the function is not differentiable. But in this case, since we have polynomial this function has no singularity. So, you forget about the singularities but you check what happens at the endpoints. okay. For instance, look, you have 0 1 you have a function like this, you see the minimum appears at the endpoint and maximum appears at the endpoint. So you have to check at what happens at the endpoints for absolute maxima and absolute minima.

(9) Try to reduce the variables to one and then among the values of the function at critical points, singular points and end points, choose the one which yields the absolute maximum. What I called absolute maximum is the biggest value among the maximums. You may have three maximums, you may have one singular point and you may have two endpoints. So you have all together six points to be checked. So, you look at the function as I have done, area of x_1, area of x_2, are of x_3 and you choose the one, if you are asked for maximum you choose the one which yields the biggest value that is maximum and then you choose the one which yields the smallest value that is the absolute minimum.

4.5.2 Students' Mathematical Thinking Performance

In TDT, five items regarding computing derivative of a function by using Chain Rule (item 21), evaluating the values on an interval where the given function is always decreasing (item 22), evaluate the derivative of a given function by using the formal definition of the derivative (item 23), computing the product of the unknowns in a function via using its local extremum point and inflection point (item 25), and evaluating whether a given partial function is differentiable at a given point (item 26) are algorithmic thinking items.

Table 4.5. *TDT Results of ALGOTHK Items*

	Item21	*Item 22*	*Item 23*	*Item 25*	*Item 26*
Difficulty index	.85	.7	.7	.45	.55

Based on the difficulties of the items, Item 21 was answered correctly by almost all participants. Item 21 ragarding the Chain Rule, which is used to differentiate composite functions. All participants, except Student 5, had the correct answers in TDT. Student 5, who has the highest score in TDT, was asked to find the derivative of the function at the given point in the interview. Although he has known that he should apply Chain Rule and followed the correct procedure at the first steps, in other words, he worked from outside to the inside, he failed when taking derivative of inner functions. Therefore, as the other interviewees who applied the rule correctly, he also showed the ability of FORMTHK as well as ALGOTHK.

Three-quarters of the participants answered correctly item 22 and item 23, and most of the other participants did not select any option for both items. The item is a question that requires both the application of quotient rule and the relation of the sign of the first derivative with decreasing interval of a function in order to solve it. The question was answered all interviewees apart from Student 1. Student 1 explained that "$f'(y)$ must be less than 0, but I had difficulty when I have taken the derivative of the function". The other interviewees solved the question again and found the interval by solving the inequality $\frac{k \cdot (x+k) - 1 \cdot (kx+1)}{(x+k)^2} < 0$. Although Student 1 has known the product rule, as it is seen at his solution of item 16, he failed when he differentiated the function given at item 22. From this, it can be inferred that he failed when he followed the routine procedure in ALGOTHK. However, he was still in activating FORMTHK.

Item 22 is a question that requires both quotient rule and sign of the first derivative affecting decreasing interval of a function in order to solve it. The question was answered all interviewees apart from Student 1. Student 1 explained that "$f'(y)$ must be less than 0, but I had difficulty when I have taken the derivative of the function" It means that the interviewee did not solve question correct. However, it is not because of the lack of information about the relation between decreasing of a function on an interval and negative sign of the derivative of the function at any point on that interval. Therefore, any interviewees did not fail in FORMTHK.

On the other hand, interviewees, apart from Student 1, solved the question again during the interview and found the interval by solving the inequality $\frac{k \cdot (x+k)-1 \cdot (kx+1)}{(x+k)^2}$ < 0. Student 2, 3 and 4 followed the routine procedure by the table and discussed the interval. Student 5 followed the routine procedure and explained the decreasing interval without using a table. After he reached the simpler form of y', namely $\frac{k^2-1}{(x+k)^2}$, explained that "We can use -1 and $+1$. Because here will always be positive (by pointing the denominator). The values make here negative (by pointing the nominator). Because, any numbers on the interval makes the upper part negative. Then our derivative becomes automatically less than 0, therefore this interval."

At this item, Student 2, 3, 4 and 5 showed the ALGOTHK by implementing technical procedures to find the derivative of a quotient.

Regarding item 23, all interviewees explained that since the limit given in the item is the definition of the derivative, they have found the derivative of the function, which is given with $f(x) = e^{tanx}$ when x is $\pi/4$. By application of correct differentiation algorithms, which is an indicator of ALGOTHK, they found the derivative of e^{tanx} when x is $\pi/4$. In addition, their description of the limit of the difference quotient as the derivative showed their FORMTHK aspects.

Item 25 was answered by half of the participants. A quarter of the participants had no answer, and a quarter of the participants had an incorrect answer.
Item 25 was again answered correctly by all interviewees, and the same explanations came by all of them. Since he did not select the item 2 and explained that he did not remember what inflection point is, the explanation of Student 1 was interesting.

Excerpt 1 (Interview):

Researcher: What is inflection point at x?

Student 1: It means that the second derivative is 0.

F: So inflection point is?

Student 1: When it is asked in question, it is okay. However, it does not happen when it says what inflection point is.

Student 1 failed FORMTHK since he could not define the inflection point. Nevertheless, he found the correct answer of the item requiring knowing what inflection point is and how it can be determined when it is asked in ALGOTHK based item.

Item 26 was responded by half of the participants correctly, and more than a quarter of the participants had no answer. Item 26 is related to the differentiability of a function at a given point. It requires the application of both differentiability and continuity algorithms for the partial function to be differentiable at the given point. All interviewees had the correct answer. The interviewees solved the question quickly, and some of them highlighted the relation between differentiability and continuity. Here is an example.

Excerpt 2 (Interview)

Student 2: We know that if it is differentiable, then it is continuous. If it is continuous, when it is equal to 1, when it is smaller and greater than 1, all values will be equal. Therefore, when we set to 1, we know that $x^2 + 2$ is equal to 3. We know $ax + b$ will be equal to 3 when we set to 1. So we know directly what $a + b$ is. We know it is 3. Then we will differentiate them. Their derivatives will also be equal. Then we will find a and b.

Since the item requires differentiability and continuity algorithms, it is evidence to see students' ALGOTHK. Therefore, the interviewees revealed their ALGOTHK. In addition, although all did not repeat the theorem, "If f is differentiable at a, then f is continuous at a." during the interview, they found the values in order the partial function make continuous as well as make differentiable. Since the interviewees used the theorem properly, it reflected their FORMTHK aspects.

4.6 Enactive Thinking

In this section, observations of instructions under enactive thinking aspects and students' performance on enactive thinking items are discussed.

4.6.1 Opportunities in the Classroom

As the opportunity in the class to engage students in enactive thinking, the instructions observed related with optimization problems are presented here as the main aim of the instructions was to engage students in enactive thinking.

Excerpt 1 (Observation)

Instructor: Suppose an object moves along a straight line. Then the equation of the motion $S = f(t)$ where t is typed usually S is the distance traveled in time t. S in physics is often called displacement.

(1) Suppose we move from $t = a$ to $t = a + h$ in time. This corresponds to we are at a, and then we are at $(a + h)$. But this time the variable is in time. Then the change in position is $f(a + h) - f(a)$. So if the equation of motion with respect to time is $f(t)$ and then, you are moving in time from a to $(a +$

h), displacement function will change from $f(a+h)$ to $f(a)$. This is the displacement corresponding to the time change h. Then average velocity is distance travelled divided into time. So this is displacement/time which is $\frac{f(a+h)-f(a)}{h}$. So you see things are very familiar look at the definition of differentiability at a and if you look at the formula of average velocity as displacement/time, you see they are in fact the same.

(2) If we compute the average velocities over shorter and shorter time intervals, that is, when we take the limit of average velocity as shorter and shorter time intervals mean over time h is going to 0 shorter and shorter time intervals, say over $[a, a+h]$ when h is small, the result is velocity.

$$V(t)=\frac{f(a+h)-f(a)}{h}$$

The excerpt above shows that although the instructor modeled real-life phenomena by differentiation symbolism in (1), and activated ENACTHK. He also activated FORMTHK by using definition of the derivative and ALGETHK (2) by relating derivative with instantaneous velocity.

Excerpt 2 (Observation)

Instructor: Here is an application (of Fermat's Theorem)

(1) Example: What is the largest rectangular area which is enclosed by a fence of length 200 m.

So, we have a fence, this side is x, this side is y.

(2)

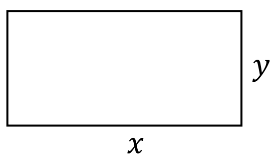


Figure 4.29. The rectangular area which is enclosed by a fence

(5) So, $2x + 2y = 200$, this is given and you want to find the maximum area of a rectangle which can be enclosed by fence of length 200 m.

(6) So the area is clearly x times y. But from here, we have $x + y = 100$.

(7) But it is too many variables $y = 100 - x$.

(8) So here, we have $A = x \cdot (100 - x)$.

(9) We want to maximize this function. We want to find the maximum area, maximum of the function $100x - x^2$. Does the maximum exist? Of course, x changes between (writes $0 \leq x \leq 100$). Does the maximum exist? Can I solve this problem? The answer is yes. Because I have a polynomial.

(10) Polynomials are continuous everywhere and the polynomial is $100x - x^2$. So polynomial of degree 2, polynomials are continuous everywhere. In particular, in this finite interval 0,100 as well. So, I know by previous theorem, the maximum can happen whenever $A'(x)$ equal 0. And $A'(x)$ is $100 - 2x = 0$ from each I contain I solve for x is 50. If x is 50 then y equals to 50. So the maximum area is a square, because x equals y, square of side 50. We know that the area must be a rectangle; rectangle must be square of length 50.

The excerpt above shows that the instructor solved a real life problem (1) to support the instruction of Fermat's Theorem. He used a figure (2) to visualize the problem and solved (ALGOTHK) the problem by algebraic manipulation (3)(4)(5) (ALGETHK) and obtained the function to be maximized (6)(7). In addition, the instructor reminded that polynomials are continuous everywhere (8) (FORMTHK).

Excerpt 3 (Observation)

Instructor: Optimization Problems. Mathematical background is the following:

(1) First of all, your function has to be continuous. Nice function takes its maximum on the nice set. Nice set is the full geometry is a compact set.

(2) So a continuous function from f to R has an absolute maximum, there exist a point x_1 in $[a, b]$ such that $f(x_1)$ is maximum of $f(t)$'s when t belongs to $[a, b]$ and has an absolute minimum, there exist a point x_2 in $[a, b]$ such that $f(x_1)$ is minimum of the value of $f(t)$'s in [a,b]. Among the local maxima, this is the biggest one but it is not only that.

(3) It is very economic problem. If you want to minimize the cost, it is a minimum question, if you want to maximize your profit, there is a maximum question.

(4) But as long as your variable changes in a closed interval, including the endpoints, this always happens. It could be more than one.

(5) These points x_1 and x_2 may be among critical points as well as end points on the interval a and b or where f' does not exist. So what we will do is the following:

(6) First of all we will find critical points of, zeros of f', singular points where the derivative does not exist and third is endpoints. Then I am going to take; $f(x_1)$, ……., $f(x_n)$, $f(y_1)$, ……., $f(y_k)$ and $f(a)$, $f(b)$. I am going to find critical points, zeros of the first derivative, then I am going to find singular points where the derivative does not exist and

(7) calculate value of f at critical points, value at singular points, value at a and b. And these values are real numbers, finite.

(8) If you are going to find maxima which is the maximum of these values of f is absolute maximum. Minimum of these is absolute minimum.

The excerpt shows that the instructor first reminded the condition (ALGETHK) to a function that has a maximum (Line 1) and then defined (FORMTHK) absolute maximum and absolute minimum (Lines 2 and 8). In Line 3, he activated ENACTHK by referring to real-life phenomena. He emphasized that a maximum or minimum point on a closed interval may be more than one (Line 4), and these points may be among critical points, singular points, and endpoints (Line 5). He defined critical points as zeros of f' and singular points as where the derivative does not exist (Line 6). Therefore, he activated FORMTHK by definitions. He added that after calculation of the values of the function at critical points, singular points, and endpoints, the absolute maximum and absolute minimum could be decided (Line 7).

4.6.2 Students' Mathematical Thinking Performance

In TDT, five items (item 24, 27, 28, 29, and 30) regarding the optimization problems are enactive thinking items.

Table 4.6. *TDT Results of ENACTHK Items*

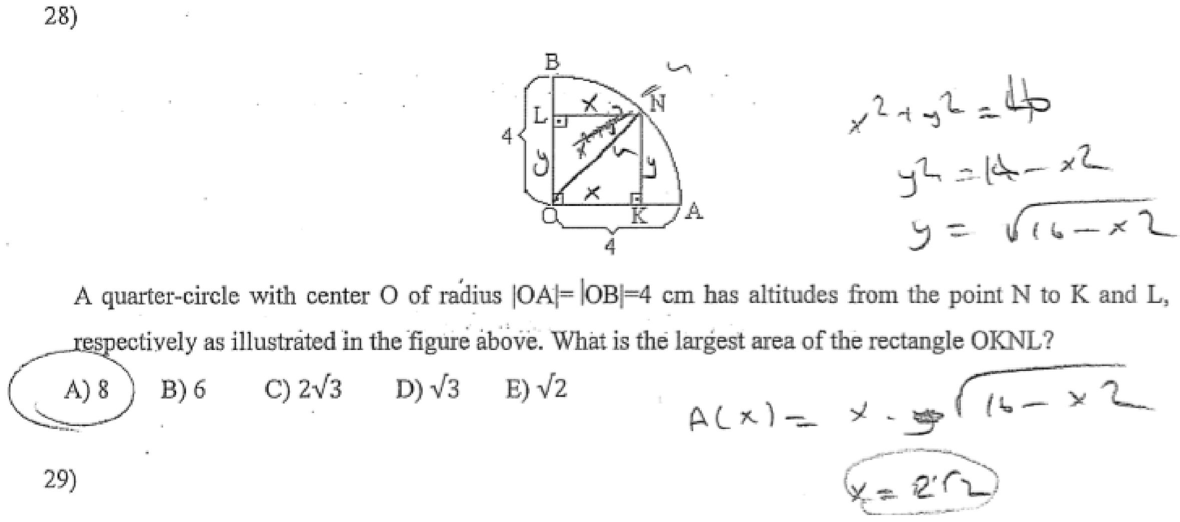
	Item 24	*Item 27*	*Item 28*	*Item 29*	*Item 30*
Difficulty index	.35	.35	.35	.25	.25

Before the analysis of interviewees' reasoning on the problems, it should be noted that Student 2, who had not solved any optimization problems in TDT, did not solve the problems also in the interview. Despite the directions of the researcher, she did not try to solve the problems, and she stated that she could not formulate the problems. Then, she added that even if she formulates, she does not know how she continues. In this part, therefore, the explanations of the other interviewees are handled.

Concerning the difficulties of the item, 7 participants responded correctly, and 8 participants did not respond for each of the Items 24, Item 27, and Item 28. Student 1 and Student 5 solved the item 24 by representing their mathematical ideas during interview. They followed the problem solving step and reached the correct solution. Besides, Student 4, who had an operational mistake in TDT, solved the question in the interview and reached the correct answer.

Item 27 was solved Student 3, 4, and 5 correctly in TDT. During the interview, it was seen that although Student 1 understood the problem, identified variables, and wrote the equation of the function, he failed while differentiating the function.

Item 28 was solved Student 3, 4 and 5 correctly in TDT. Student 1 showed that he understood the problem, identified variables, and wrote the equation of the function as $x^2 + y^2 = 16$. However, he could not reduce the variables to one variable; therefore, he could not differentiate the function.


28)

$x^2+y^2=16$
$y^2=16-x^2$
$y=\sqrt{16-x^2}$

A quarter-circle with center O of radius |OA|= |OB|=4 cm has altitudes from the point N to K and L, respectively as illustrated in the figure above. What is the largest area of the rectangle OKNL?

A) 8 B) 6 C) $2\sqrt{3}$ D) $\sqrt{3}$ E) $\sqrt{2}$

$A(x)= x \cdot \sqrt{16-x^2}$

$x=2\sqrt{2}$

29)

Figure 4.30. The solution made by Student 5 regarding item 28

Item 29 and Item 30 was responded correctly by a quarter of the participants, and the items were nor responded almost half the participants. Item 29 was solved Student 3 and 4 correctly in TDT. The excerpt is below:

Excerpt 1 (Interview)

> Student 5: It is given the volume. The volume is $\pi r^2 h$. In this case, it is 1000 cm^3. We need to area because we want it is being minimum. And the area is $2\pi rh + 2\pi r^2$. When I differentiate this equation, it is $2\pi - 2000/r^2 + 4\pi r^2$.
>
> Researcher: What happened to h?
>
> Student 5: I wrote h as $1000/\pi r^2$, from formula of the volume. If I equate this to 0, I found that r^2 is 5, but I could not continue. After I found that r^2 is 5, I find r in π. Options are in r, I could not find it in r.

However, none of the interviewees solved correctly the item 30 in TDT. Student 1 stated the reason that he could not decide how to apply to volume, how to apply it in 3D. Furthermore, Student 5 expressed that he did not understand what kind of shape would be obtained. The student 3 explained the correct procedure, but she did not solve question. The student 4, on the other hand, solved the problem and reasoned all steps accurately. The excerpt from Student 4 is below.

Excerpt 2 (Interview):

> Student 4: When I cut x unit from every corner, the rest becomes $6 - 2x$ and the height is x. The volume of this is $(6-x)^2$ multiplied by x. Then I differentiated it. I found $x = 1$ and $x = 3$. Since it asks me maximum value, when I set 3 it becomes 0, when I set 1 it becomes 16. Therefore, it is 16.

Since the students constructed mental perceptions of real world objects activating ENACTHK, their solution showed whether they activate their ENACTHK or not. In addition, knowing why maximum/minimum points should be found to solve the problem and how they are determined are indicators for FORMTHK, differentiation the function properly is ALGOTHK, identification of the variables and trying to reduce the variables to one variable is ALGETHK, and visualizations are ICONTHK.

CHAPTER 5

DISCUSSION, CONCLUSION AND IMPLICATIONS

The purpose of this study was to investigate the opportunities provided mathematics students to engage in mathematical thinking during the instruction of derivative concepts and categorize the students' mathematical thinking on derivative concept through test results and interview.

Considering the opportunities in Calculus class provided mathematics students to engage in mathematical thinking during the instruction of derivative concepts, the results indicated that all of the enactive, iconic, algorithmic, algebraic, formal, and axiomatic thinking aspects were activated throughout the instruction of the derivative concept. This result was in line with Aydın and Ubuz (2015) stating that each of these six distinct mathematical thinking aspects describe different processes.

Formal thinking was activated when the instructor introduced the derivative concept by using definitions, rules and theorems and, reminded these essential elements activating formal thinking throughout the class times. Besides, axiomatic thinking was mainly acivated while proving some theorems on derivative.

Since the nature of derivative concept is related to algebraic structures like formal definition of derivative, equations, variables, algebraic thinking was highly activated throughout the instructions. It confirms that algebraic thinking is used in any problem solving activity that combines a mathematical process with algebraic structures (Vennebush, Marquez & Larsen, 2005). However, the items those concerned with algebraic thinking were not responded correctly by most of the participants.

Iconic thinking was activated while introducing the definitions and theorems, and solving problems. Particularly graphical representations were used while introducing the definitions and theorem to explain them graphically. They were used to visualize the problems graphically prior to solve them. As graphical context is necessary to deepen understanding of derivative concept (Rivera-Figueroa & Ponce-Campuzano, 2013), students were succesfful while responding to items concerned with iconic thinking. Item regarding construction of a graph of a function with respect to graph of the derivative function item was not responded correctly by a quarter of the participants. The finding contributes a clear understanding of the findings of the study conducted by Berry and Nyman (2003) showed the students'difficulties in construction of a graph of a function with respect to graph of the derivative function.

Algortihmic thinking was activated while differentiating functions using differentiation rules and solving problems. Instructions was supported by examples and related problems by explaining algorithmic procedures step by step. Since algorithmic dimension involves students' capability to explain the successive steps included in procedural operations (Tirosh et al., 1998), student were quite succesful while solving the items regarding algorithmic thinking.

Enactive thinking was mainly activated while solving five optimization problems in two class hours. Therefore, the items concerned with enactive thinking were the most difficult items for the students. As optimation problems requires to formulate the problem mathematically, students experienced difficuly particularly while formulazing the problem statement. This result was in line with Stewart (2008), stating that converting the word problem into a mathematical form by generating the function, which is maximized or minimized, is the most difficult part of the solution. If they were able to formulate the problem mathematically, the experienced difficulty while visualizing the context (ICONTHK), implementing the differentiation procedure (ALGOTHK) and reducing the variables to one variable (ALGETHK).

In conclusion, in this study it was shown that the relation between mathematical thinking dimensions cannot be ignored. The instructor used lots thinking aspects as much as possible concerning the content, and the students reflected the instruction. Although the items were categorized according to the distinct mathematical thinking aspects, the students also activated another aspect of mathematical thinking. The collaboration of mathematical thinking aspects provides correct answer to students and the success in TDT.

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