AN ANALYSIS OF MIDDLE SCHOOL STUDENTS' GENERALIZATION OF LINEAR PATTERNS

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Prof. Dr. Yaşar KONDAKÇI
Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

Assoc. Prof. Dr. Elvan ŞAHİN
Head of Department

This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

Assoc. Prof. Dr. Zelha TUNÇ PEKKAN
Co-Supervisor

Prof. Dr. Mine IŞIKSAL BOSTAN
Supervisor

Examining Committee Members

Prof. Dr. Ayhan Kürşat ERBAŞ	(METU, MSE)	____________
Prof. Dr. Mine IŞIKSAL BOSTAN	(METU, MSE)	____________
Assoc. Prof. Dr. Zelha TUNÇ PEKKAN	(MEF Uni., MFBEB)	____________
Prof. Dr. Yezdan BOZ	(METU, MSE)	____________
Assist. Prof. Dr. Reyhan TEKİN SİTRAVA	(Kırıkkale Uni., MATÖ)	____________

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Name, Last name: Zeycan KAMA

Signature :

ABSTRACT

THE ANALYSIS OF MIDDLE SCHOOL STUDENTS' GENERALIZATION OF LINEAR PATTERNS

Kama, Zeycan

M.S., Department of Elementary Science and Mathematics Education

Supervisor: Prof. Dr. Mine IŞIKSAL-BOSTAN

Co-Supervisor: Assoc. Prof. Dr. Zelha TUNÇ PEKKAN

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The purpose of the present study was to explore sixth, seventh, and eighth grade students' generalizations of patterns using arithmetical generalization, algebraic generalization, and naïve induction. In addition to studying their generalization process, the study also focuses on how this process of generalization differs according to their grade level. The study employed a qualitative case study design. The data were collected from five sixth grade, four seventh grade, and five eighth grade students during the spring semester of the 2015-2016 academic year. Data were collected through the Pattern Test and individual interviews.

The findings revealed the use of four generalization approaches: (i) algebraic generalization strategies only, (ii) a combination of arithmetical generalization and algebraic generalization strategies, (iii) a combination of arithmetical generalization and naïve induction strategies, and (iv) a combination of arithmetical generalization, algebraic generalization, and naïve induction strategies. It was found that the combination of arithmetical generalization and algebraic generalization was the most frequent generalization approach, while the combination of arithmetical generalization, algebraic generalization, and naïve induction was the least frequent ones used by the students in all grade levels in this study. Moreover, the use of algebraic generalization strategies only was observed by the sixth graders only. It was also seen that sixth, seventh, and eighth-grade students used arithmetical generalization strategies in order to find near terms of the pattern. In order to find the far terms or the general term, they either used algebraic generalization strategies or naïve induction strategy.

Keywords: Pattern Generalization, Middle School Students, Arithmetical Generalization, Algebraic Generalization, Naïve Induction

ÖZ

ORTAOKUL ÖĞRENCİLERİNİN DOĞRUSAL ÖRÜNTÜLERİ GENELLEMELERİNİN İNCELENMESİ

Kama, Zeycan

Yüksek Lisans, İlköğretim Fen ve Matematik Eğitimi

Tez Yöneticisi: Prof. Dr. Mine IŞIKSAL-BOSTAN

Ortak Tez Yöneticisi: Doç. Dr. Zelha TUNÇ PEKKAN

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Bu çalışmanın amacı altıncı, yedinci ve sekizinci sınıf öğrencilerinin doğrusal örüntüleri aritmetik genelleme, cebirsel genelleme ve naif tümevarım kullanarak genelleme süreçlerini incelemektir. Genelleme süreçlerini incelemeye ek olarak, bu çalışma aynı zamanda genelleme süreçlerinin sınıf seviyelerine göre gösterdiği farklılıklara da odaklanmaktadır. Bu çalışmada nitel durum çalışması deseni kullanılmıştır. Veriler 2015-2016 akademik yılının bahar döneminde beş altıncı sınıf,

dört yedinci sınıf ve beş sekizinci sınıf öğrencisinden toplanmıştır. Veri kaynaklarını Örüntü Testi ve bireysel görüşmeler oluşturmaktadır.

Bulgular, (i) sadece cebirsel genelleme stratejileri, (ii) aritmetik genelleme ve cebirsel genelleme stratejilerinin kombinasyonu, (iii) aritmetik genelleme ve naif tümevarım stratejilerinin kombinasyonu ve (iv) aritmetik genelleme, cebirsel genelleme ve naif tümevarım stratejilerinin kombinasyonunu içeren dört tür genelleme sürecini ortaya çıkarmıştır. Bulgular, öğrencilerin sınıf düzeyine göre incelendiğinde, aritmetik genelleme ve cebirsel genelleme kombinasyonunun tüm sınıf seviyelerinde en sık yapılan genelleme süreci türü olduğunu, aritmetik genelleme, cebirsel genelleme ve naif tümevarım kombinasyonunun ise en az yapılan genelleme süreci olduğunu göstermiştir. Ayrıca, sadece cebirsel genelleme stratejilerini içeren genelleme süreci türü yalnızca altıncı sınıf düzeyinde görülmüştür. Ek olarak, altıncı, yedinci ve sekizinci sınıf seviyesindeki öğrencilerin örüntünün yakın terimlerini bulmak için aritmetik genelleme stratejisini kullandıkları, uzak terimleri veya genel terimi bulmak için ise ya cebirsel genelleme stratejileri yada naif tümevarım stratejilerini kullandıkları görülmüştür.

Anahtar Kelimeler: Örüntü Genelleme, Ortaokul Öğrencileri, Aritmetik Genelleme, Cebirsel Genelleme, Naif Tümevarım

This thesis is dedicated to my dearest husband,

Batuhan KAMA

and to the miracles of my life,

Doğa and Mavi KAMA.

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LIST OF ABBREVIATIONS

NCTM	National Council of Teachers of Mathematics
MONE	Ministry of National Education
AG	Arithmetical Generalization
ALG	Algebraic Generalization
I	Naïve Induction
P	Participant

CHAPTER 1

INTRODUCTION

Algebraic reasoning is an essential landmark for students to progress in mathematics at school and their career in science, engineering, and economics (Greenes, Cavanagh, Dacey, Findell, & Small, 2001; Moses & Cobb, 2001). It is the fundamental element of mathematical thinking (Windsor, 2010), which enables students to go into a "deeper underlying structure of mathematics" (Cai & Knuth, 2005, p. 1). Students with algebraic reasoning skills can develop advanced ways of thinking, including solving problems, noticing structures between quantities, generalizing, modeling, and justifying (Cai & Knuth, 2011). Due to the importance of algebraic reasoning, an increasing number of researchers, policy-makers, and mathematics educators emphasized that algebra instruction should become a part of the elementary mathematics curriculum (Carraher, Schliemann, Brizuela, & Earnest, 2006). Furthermore, the National Council of Teachers of Mathematics [NCTM] (2000, 2006, 2010) suggested that algebra is an integral strand in K-12 curriculum and it is important to develop algebraic reasoning starting from kindergarten.

By integrating objectives related to algebraic reasoning into all grade levels in the K-12 strand, it was aimed to start the development of algebraic reasoning among young students and to provide the necessary ground for higher-level abstract mathematics (NCTM, 2000). Nevertheless, there is a severe barrier for students to develop algebraic reasoning skills and gain the above objectives. The obstacle in front of students is the dominant focus on arithmetic at primary grades (Warren, 2003). State differently, traditionally, it was believed that algebra should be taught after arithmetic for students to be cognitively ready (Patton & De Los Santos, 2012). Therefore, until recently, the primary school curriculum focused on arithmetic and middle school curriculum focused on algebra (Kamol & Ban Har, 2010). Because algebra follows arithmetic in most elementary school curricula, students could not get used to the thinking ways required for algebra (Warren, 2003). Lee (1996, p. 87) used the phrase “cultural shock” to portray students’ reactions while entering into the algebraic culture from that of arithmetic. The cultural-shock, which was experienced by elementary students, was articulated as ‘transition from arithmetic to algebra’ in mathematics education literature.

To case the transition from arithmetic to algebra, various researchers recommended the necessity of meaningful experiences in arithmetic, which would help students develop algebraic thinking (Mcrae-Childs, 1995). There are some big ideas, which are the foundation of both arithmetic and algebra (Carpenter, Franke, & Levi, 2003). Having these ideas develops students’ arithmetic knowledge and forms the basis for algebraic knowledge. Generalization of patterns is one of the big ideas that students should acquire in elementary grades. In the related literature, the root of algebraic reasoning is considered as generalization (Carraher et al., 2006; Mason, Graham, &

Johnston-Wilder, 2005) because generalization enables students to have powerful mathematical ideas by developing the knowledge structure of the mind (Carpenter & Levi, 1999; Dreyfus, 1991). It exists “both within and outside of mathematics” (Kaput, 2000, p. 3). Kieran (2004) expressed that much of the meaning-making process occurs during generalization activities.

As a way of introducing elementary students to generalization, literature has recommended linear patterning tasks (NCTM, 2000, 2010; Van de Walle, Karp, & Bay-Williams, 2007). Among the objectives of the Common Core Standards, NCTM recommended

> …numerical and geometric patterns and express them mathematically in words or symbol…analyze the structure of the pattern and how it grows or changes...and use their analysis to develop generalizations about the mathematical relationships in the pattern (2000, p. 159).

This recommendation suggests that the generalization process centers strongly around patterns. Patterning tasks are not only the start of the way through algebraic reasoning (Kieran, 1989) but also a helpful way of introducing students with formal algebraic thinking (Lannin, 2005).

According to Dienes (1961), algebra can be understood when students generalize a pattern to any term after generalizing it to some near and far terms. Similarly, Radford (2006) defined algebraic generalization as a process of searching for a common point that can be generalized to all near and far terms of the pattern and which can be used to express any term. In parallel with related literature, in middle school mathematics curricula, patterns generally include terms in an ordered sequence, i.e., from near terms to far terms in ascending order (Ministry of National Education [MONE], 2013). While students expand the pattern from near terms to far terms, they experience both near and far generalization processes. The movement from near terms to far terms necessitates the transition from arithmetic thinking to algebraic thinking. For example, a student can generalize a pattern to the 5th term by adding the constant difference to the previous terms through arithmetic thinking. On the other hand, a student's generalization of a pattern to the 100th term necessitates the algebraic relationship between the term and the term number through algebraic thinking. NCTM (1997) recommended students having experience with patterning activities since the flow from near terms to far terms helps to make a connection from the numeric-elementary level to a more general-algebraic level.

1.1. Statement of the Problem and Research Questions of the Study

Despite the considerable importance of the concept of generalization in terms of the transition from arithmetic to algebra, literature indicated some problematic areas regarding the concept of pattern-generalization, first of which was the emphasis on procedural skills of students during the instruction. It was reported that mathematics instruction dominantly centered on the procedures of forming the general rule of the pattern (Lannin, Barker, & Townsend, 2006). Therefore, students could not develop a conceptual understanding of the nature of the generalization; instead, they developed their techniques of how to generate the rule of the pattern (Maudy, Didi, & Endang, 2018). For example, in a study conducted by Girit and Akyüz (2016), it was reported that students "got used to multiply something and add something for getting a rule" (p. 261). Therefore, the first problematic area was generalizing patterns procedurally as a result of rule-based instruction. As a result of this problem, students might not be able to develop algebraic generalization skills, since algebraic generalization can be understood when students generalize a pattern to any term after generalizing it to some near and far terms (Dienes, 1961). Indeed, they might be more tended to use short-cut strategies such as the trial and error strategy.

The trial and error strategy was named differently by different researchers. For example, Lannin (2005, p. 234) named it as "guess and check" strategy and Radford (2008, p. 85) called it "naïve induction" strategy. Students' tendency to use trial and error/naïve induction strategy has been reported many times in the literature (Becker & Rivera, 2005; Lannin, 2005; Radford, 2008, 2010a; Vale & Pimentel, 2009). Through naïve induction, students do not look for a generality in the pattern. They just make guesses with the purpose of finding the rule of the pattern. Therefore, naïve induction strategy does not belong to the nature of generalization (Radford, 2010a). In order to overcome this problem, students' algebraic generalization skills will be investigated focusing on not only the general rule of the pattern, but also the near and far terms in detail in this study. In other words, in this study, students' detailed process of generalizing near terms and far terms of the pattern will be investigated in terms of arithmetical generalization, algebraic generalization, and trial and error/naïve induction. The detailed process of near and far generalizations will highlight the subtle steps behind students' generalization process. By starting from first few terms and progressing through distant terms, students will be able to show their development of algebraic generalizations in a progressive way. It will also reveal whether students connect the process of near and far generalization to the process of finding the general rule in a conceptual way. All in all, the present study would provide valuable information about how students conduct the pattern generalization process in a progressive way.

Another problematic area in terms of the concept of pattern-generalization was the existence of "zone of emergence of algebraic thinking" (Radford, 2010b, p. 36). Zone of emergence of algebraic thinking referred to the gap between students' beginning to think algebraically and their capability to use symbolic algebra (Radford, 2010b). Algebraic generalization has traditionally been recognized as symbolic generalization. Nevertheless, this is a limited perspective, which leaves many children, who are unable to use symbolic algebra, behind. Students do not reach the level of symbolic algebra all at once. They pass through a progressive process (Aké, Godino, Gonzato, & Wilhelmi, 2013; Garcia-Cruz & Martinón, 1998; Godino et al., 2014; Maudy et al., 2018; Radford, 2010a). This process starts with realizing a common point in given first few terms of a pattern (Aké et al., 2013; Godino et al., 2014; Radford, 2010a). According to the literature, students typically tend to realize the additive relationship between consecutive terms of the pattern as a first step. This type of generality is called as *arithmetic.* Then, they notice a "factual" generality with *algebraic* nature, which enables to make a relation between the positions of the given terms and their numerical value (Radford, 2003, p. 46). By using this common point, students can find the numerical value of particular terms. Lee (1996, p. 95) called it as “algebraically useful pattern”. One step ahead, students no longer deal with particular terms. Instead, they directly express how to find the numerical value of any term in the form of a general rule by using natural language through a “contextual generalization” (Radford, 2003, p. 50). Eventually, they express their general rule by using symbols through symbolic algebra (Maudy et al., 2018; Radford, 2010a). All in all, as literature showed, there are subtle shifts from arithmetical generalization to symbolic generalization. In other words, lack of algebraic symbols does not necessarily show inability to think algebraically (Zazkis & Liljedahl, 2002). Thus, it is of great importance to investigate students’ generalization skills by including the subtle shifts of algebraic generalization. In the present study, students’ generalization process will be investigated within the scope of Radford’s generalization layers, which are factual, contextual, and symbolic generalizations. Thus, an analysis of students’ reasoning would give precious information about how students reach symbolic algebra by passing through factual and contextual generalizations.

Considering the problematic areas mentioned above, which are the emphasis on procedural skills of students during the instruction and the existence of zone of emergency of algebraic thinking, the purpose of the study is to explore sixth, seventh, and eighth grade students’ generalizations of patterns using arithmetical generalization, algebraic generalization, and naïve induction. Based on curricular restrictions, linear patterns were used in the present study. In addition to studying their generalization process, the study focuses also on the ways in which this process of generalization differs according to their grade level. Research questions of the study were given below:

- How do sixth, seventh, and eighth grade students generalize linear patterns using arithmetical generalization, algebraic generalization, and naïve induction?
- To what extent do these generalizations differ in terms of their grade level?

1.2. Definition of Important Terms

Pattern is defined as structural or numerical regularity (Papic & Mulligan, 2005). Patterns can be classified according to their structure (Van de Walle et al., 2007) or according to the expression of the general rule (Stacey, 1989). Structurally, a pattern is called as *numeric* if their terms include numbers. It is called as *figural* if their terms include geometric figures (Chua & Hoyles, 2014). On the other hand, a pattern is called as *linear* or *quadratic* since their general terms can be expressed as *an+b [a refers to the common difference of the pattern; n refers to the position of the term; b refers to the constant of the pattern]* or *an2+bn+c [a refers to the half of the constant amount between the differences of successive terms of a quadratic pattern; n refers to the position of the term; b refers to the '2nd term-1st term-3a'; c refers to the '1st term-b-a'],* respectively (Chua & Hoyles, 2014). Since Turkish middle school mathematics curriculum (MONE, 2013, 2018) included linear patterns in numeric and figural form, only linear-numeric and linear-figural patterns are referred in the present study. More specifically;

Linear-figural pattern refers to the patterns whose terms are in the form of geometric figures (Van de Walle et al., 2007) and whose general term can be expressed as an+b, n refers to the position of the terms (Stacey, 1989). For example, consider the linear-figural pattern in Figure 1 whose terms are in the form of circles and general term can be expressed as 2n+3, where n is the position of the terms.

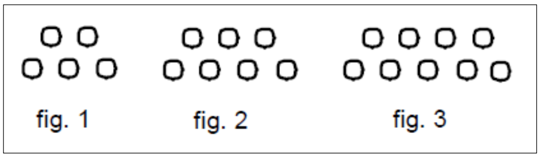


Figure 1. A linear-figural pattern (Radford, Bardini, & Sabena, 2006, p. 395)

Linear-numeric pattern refers to the patterns whose terms are in the form of numbers (Van de Walle et al., 2007) and general term can be expressed as an+b (Stacey, 1989). For example, in Figure 2, terms are in the form of numbers and general term can be expressed as 6n-2, where n is the position of the terms.

4, 10, 16, 22, __, __, __

Figure 2. The linear-numeric pattern example from Stacey (1989, p. 149)

Pattern generalization is defined as perceiving the common ground on some terms of the pattern, being able to apply this commonality to all terms of the pattern and providing a direct expression about general term of the pattern (Radford, 2008). In this study, pattern generalization refers to a process in which students notice a commonality on given terms, expand the pattern to near terms and generate a rule to reach the far terms and express the general rule either with natural language or with letters.

Near terms of the pattern refer to the terms of the pattern, which can be reached by counting step by step onto previous term in a practical way (Stacey, 1989). In this study, fourth, fifth, and tenth terms of the pattern are accepted as near term.

Far terms of the pattern refer to the terms of the pattern, which can not be practically reached by counting step by step onto previous term (Stacey, 1989). As an instance, 100th term, 1000th term, and so on.

General rule/term of the pattern is defined as a general formula that can be applied to any term of the pattern (Van de Walle et al., 2007). For example, general rule/term of given pattern '2, 4, 6, …, …, 12, …' can be expressed as *2n* or *twice the term number*.

Arithmetical generalization is defined as a process in which students notice a commonality based on the additive /recursive relationship between the consecutive terms of a linear pattern (Gutiérrez, 2013; Radford, 2012).

Algebraic generalization is defined as a process of searching for a common point based on the relationship between the position of the terms and their numerical value that can be generalized to all near and far terms of the linear pattern and which can be used to express any term (Radford, 2008).

Naïve induction is defined as trial and error strategy, through which students make guesses with the purpose of finding the general rule without looking for a generality in the linear pattern (Radford, 2010a).

1.3. Significance of the Study

One of the reasons behind the high significance of pattern-generalization tasks is that they enable students experience both recursive and explicit reasoning during generalizing the pattern to near and far terms. To be clearer, a typical pattern generalization task enables students express two types of rules, which are recursive rule and explicit rule. A recursive rule includes a step-by-step approach, i.e. a recursive rule includes the relationship from output to output; while an explicit rule includes the relationship from input to output, in which the value of any output can be found by using inputs, where input refers to the term numbers and output refers to corresponding terms (Lannin et al., 2006; Rubenstein, 2002). For instance, in following the number sequence, 2, 4, 6, 8, ..., 'the expression 'add two to find the next term' is a recursive expression; while the expressions 'multiplying term number with 2 gives the related term' or '2n' are examples of explicit expressions. It can be said that recursive rules have an arithmetical nature, and explicit rule have an algebraic nature. Thus, the connection between recursive and explicit rules helps students to overcome the transition from arithmetic to algebra (Lannin et al., 2006; NCTM, 2000). Furthermore, it helps to construct meaningful algebraic generalizations and to make sense of symbol use in algebra (Moss, Beatty, Shillolo, & Barkin, 2008; Stacey & MacGregor, 2001). Nevertheless, several studies reported students' difficulty not only in forming explicit rules more than recursive rules (Chua & Hoyles, 2014), but also in connecting recursive and explicit rules to each other (Arzarello, 1992; Lannin et al., 2006; Swafford & Langrall, 2000). According to Lee (1996), the problem for most of the students was not "seeing the pattern", but it was "seeing an algebraically useful pattern" (p. 95). Therefore, investigating how middle school students form recursive and explicit rules toward pattern generalization tasks is essential. The results of the present study would give precious information related to the abilities and difficulties of students in terms of constructing recursive and explicit rules with arithmetical and algebraic generalizations, respectively. Furthermore, considering the limited number of studies focusing on the relationship between recursive and explicit rules increases the significance of the current study.

In accordance with the general trend, almost entire mathematics curricula expect beginner algebra students to be able to use symbolic algebra. This expectation brings into existence "the zone of emergence of algebraic thinking", which is between students' beginning to think algebraically and their capability to use letters as algebraic symbols (Radford, 2010b, p. 36). In order to fill the zone, literature indicates pre-symbolic type of generalizations (Redden, 1996; Stacey & MacGregor, 1995) It is because algebraic generalization has a progressive nature, which develops from pre-symbolic type of generalizations into symbolic type of generalizations (Rivera, 2013). Pre-symbolic type of generalizations include generalizations conducted with presymbolic forms such as gestures, pictures, words, numbers, and combinations of forms (Rivera, 2013). Yet, as reported in the literature, many teachers do not recognize presymbolic type of generalizations (Demonty, Vlassis, & Fagnant, 2018). Therefore, they design their instructions by focusing on practicing techniques to form symbolic generalizations (Lannin et al., 2006). Nevertheless, this approach prevents students from understanding the progressive nature of pattern generalization (Lannin et al., 2006). Additionally, it leads the mathematics educators and researchers to view students' generalization as dependent on accurate usage of symbolic generalization rather than as a process from pre-symbolic generalization to symbolic generalization. Radford called presymbolic type of generalizations as factual generalization and contextual generalization and symbolic type of generalization as symbolic generalization. According to Radford (2003), factual, contextual, and symbolic generalizations follow each other so as to produce symbolic algebra. One of the focuses of the present study is to analyze students' generalization processes depending on presymbolic and symbolic type of generalizations, i.e. factual, contextual, and symbolic generalizations. Thus, the results of present study might provide valuable information for mathematics educators and policy makers in revealing how students use the pre-symbolic type of generalizations before symbolic generalization. It might contribute to the related literature by offering valuable educational implications, which eliminate the *zone of emergence of algebraic thinking* in designing the algebraic course contents of elementary students.

The other significance of the study is that findings of the study has the potential to provide information on progressive development of students' algebraic reasoning based on the schooling level. Most of the elementary mathematics curricula around the world, including Turkish mathematics curriculum, introduce students with algebra at the middle school (grades 5-8). Thus, students at the fifth or sixth grade level are called as *beginning algebra students*. According to the literature, the transition from arithmetic to algebra problem was mostly seen with *beginning algebra students* (Baroody & Ginsburg, 1983; Falkner, Levi, & Carpenter, 1999; Knuth, Alibali, Weinberg, McNeil, & Stephens, 2005; Knuth, Stephens, McNeil, & Alibali, 2006; Sfard, 1995; Sfard & Linchevski, 1994). Actually, while students progress in middle school grades, they are expected to connect arithmetic and algebraic reasoning in the first sense and to develop increasingly complex abstract algebraic reasoning afterwards (Knuth et al., 2005). In this sense, some researchers investigated algebraic reasoning levels of students from different grade levels and reached supportive results. According to the literature, students' algebraic reasoning levels increased as their grade level increased (Kama & Işıksal-Bostan, 2016; Ley, 2005). Through this study, in light of the literature, sixth, seventh, and eighth grade students' algebraic reasoning skills were investigated through pattern-generalization activities in order to see whether they show more complex algebraic reasoning skills and a variety of generalization ways across increasing grade levels. Therefore, the results of this study would reveal the existing trends of algebraic reasoning in terms of different grade levels.

CHAPTER 2

LITERATURE REVIEW

The purpose of this study is to explore the sixth, seventh, and eighth grade students' generalizations of patterns using arithmetical generalization, algebraic generalization, and naïve induction. In addition to studying their generalization process, the study also focuses on the ways in which this process of generalization differs according to their grade level. Accordingly, the literature review section was organized in a way that in the first part, various definitions and components of algebraic thinking were reviewed in the light of historical development of algebra. In the second part, pattern generalization was explained in detail and Radford's pattern generalization framework was explored as the theoretical framework of the current study. In the third part, related studies on generalization strategies of students and difficulties students experience during pattern generalization were reviewed. Finally, the summary of literature review was given.

2.1. Historical Development of Algebraic Thinking

To understand the nature of algebraic thinking, it is necessary to know the emergence of algebra throughout history. Algebra has emerged centuries after arithmetic, almost a millennium-time (Carraher, Schliemann, Brizuela, & Earnest, 2014; Reves, 1951).

About 4000 years ago, the first traces of algebra were seen in Mesopotamia (Katz & Barton, 2007). According to the literature, history of algebra is divided into three stages: the rhetorical stage, the syncopated stage, and the symbolic stage (Nesselmann, 1842). This distinction was made based on the development of language in algebra. In other words, the rhetorical stage was purely verbal, while both words and symbols were used in the syncopated stage. On the other hand, the symbolic algebra stage was only symbolic and grounded modern algebra with symbolism, which is still used today (Heefer, 2009).

The rhetorical stage lasted from the beginning of the algebra until nearly 250 AD. During this stage, mathematical problems, solutions and calculations were completely expressed by using words and everyday language (Puig & Rojano, 2004). Unknown was expressed as 'heap' by Egyptians, as 'length' or 'area' by Babylonians and Greeks, and as 'thing' or 'root' by Arabics (Van Amerom, 2002). For example, in order to solve the quadratic equation of the type 'squares and numbers equal to roots', which can be expressed as x2+c=bx in modern algebraic language, Al-Khwarizmi offered a completely verbal solution (Katz, 2007). It included directions such as taking half the number of 'things', squaring it, subtracting the constant, finding the square root and then adding it to the roots that were found (Katz, 2007). No symbols or abbreviations were used.

Around 250 AD, Diophantus presented abbreviations as the shortened forms of words (Van Amerom, 2002). This was the beginning of the syncopated algebra stage (Van Amerom, 2002). This stage lasted until the middle of the 17th century (Nesselmann, 1842). During this stage, mathematical expressions were presented by using both natural language and abbreviations (Spagnolo, 2000). For instance, Arabics used the first letter of the words to express the powers of the unknown in the 9th century (Van Amerom, 2002). Western Europeans used r and s by shortening res and cosa in the 13th century (Van Amerom, 2002).

In the 17th century, French mathematician François Viete used capital letters to represent numerical quantitites (Sfard, 1995). He made a distinction between coefficients and parameters (Sfard, 1995). By doing so, he made it possible to use letters to represent more than one quantity as well as to introduce the concept of variable (Sfard, 1995). Viete's work was the beginning of the symbolic algebra stage. After that time, all mathematical calculations and relations were represented by using sign systems. Furthermore, the symbolic algebra stage brought a new dimension to algebra beyond equation-solving. Signs and symbols represented general quantities, not a single unknown (Harper, 1987).

In conclusion, as the history of algebra reviewed above shows, algebra was built based on arithmetical techniques over long years (Van Amerom, 2002), especially in rhetorical and syncopated stages. In both these stages, the aim behind algebraic algorithms was to find the solutions of the equations and unknowns (Katz, 1997). Unknown was a specific number as maintained by Al-Khwarizmi: "What people generally want in calculating... is a number" (as cited in Katz, 1997, p. 31). Even though letters were used in the syncopated stage, they did not have the function of expressing generality (Erbas, 2005). They were used to represent unknowns;

therefore, the use of abbreviations could not develop algebra one step further, which is generality (Van Amerom, 2002). Till the introduction of modern algebraic symbolism by Viete in the 17th century, the aim behind algebra was to find the solutions of the equations and unknowns in an arithmetic way (Katz, 2007). The work of Viete enabled to generalize and abstract arithmetic.

All in all, historical development of algebraic thinking was grounded on arithmetical needs such as searching short cuts for solving equations or finding unknowns arithmetically. However, over the years, algebraic thinking has emerged with the generalization of arithmetical facts. In the following part, various definitions and characteristics of algebraic thinking are reviewed.

2.1.1. Algebraic thinking.

Algebra and algebraic thinking have been defined by many researchers in the mathematics education literature. According to the widely accepted definition of algebra in the literature, algebra is generalized arithmetic (Booth, 1988; Carraher, Schliemann, & Schwartz, 2007; Gavin & Sheffield, 2015; Mason, 1996; Philipp & Schappelle, 1999; Samo, 2009; Subramaniam & Banerjee, 2004; Usiskin, 1988). According to the literature, viewing algebra as generalized arithmetic is a milestone through the development of algebraic thinking (Usiskin, 1988). Vygotsky (1986) supports this notion by describing arithmetic concepts as preconcepts and algebraic concepts as real concepts. According to Vygotsky, the progress from preconcepts to real concepts can be achieved by abstracting and generalizing arithmetical facts. In other words, algebra is the generalization of given arithmetical rules, operations and statements (Wang, 2015). Based on the related literature, it can be stated that arithmetic thinking is a crucial part of algebraic thinking (Ralston, 2013). For example, Peck and Jencks (1988) consider it as a necessity in a way that algebraic thinking should come naturally as the result of students' observations of the way arithmetic works. By referring to algebraic thinking, Kaput (1999) states that it is a process of generalizing mathematical structures from particular examples, justification and expression of generalizations. In addition, NCTM (2000) set four standards related to algebraic thinking which are "understand patterns, relations, and functions; represent and analyze mathematical situations and structures using algebraic symbols; use mathematical models to represent and understand quantitative relationships; analyze change in various contexts" (p. 37). Among these standards, understanding patterns, relations and functions is a continuous standard for school mathematics for all grade levels (NCTM, 2000), since it provides meaningful experiences during the transiton from arithmetic thinking to algebraic thinking (Orton & Orton, 1999). As parallel to the standarts of NCTM (2000), Van de Walle et al. (2007) defined algebraic thinking as generalizing numbers and operations, formalizing them with a meaningful sign system, and exploring the patterns and functions.

All in all, it is obvious that generalization is one of the common points in various definitions of algebraic thinking, and it is accepted as the key element of algebraic thinking (Mason, 1996). In the next section, the concept of generalization is reviewed in the light of the literature.

2.2. Generalization

Generalization is a reasoning process, which goes beyond particular instances and reaches relationships of those instances (Kaput, 1999). Thus, it is extremely important in terms of transition from arithmetic to algebra. Carraher, Martinez, and Schliemann (2008) define it as “some property or technique holds for a large set of mathematical objects or conditions” (p. 3). Making mathematical generalization is crucial since it enables to construct mathematical knowledge and experience (Mason, Burton, & Stacey, 2011). According to the literature, generalization has three components, which are (i) “grasping a commonality noticed on some elements of a sequence S”, (ii) “being aware that this commonality applies the all the terms of S”, and (iii) “being able to use it to provide a direct expression of whatever term of S” (Radford, 2010a, p. 42). As the components of generalization indicate, it is fundamental to develop a generalization based on some concrete examples, and then to show its currency for abstract examples or any number, and finally to express it algebraically. Therefore, it can be deduced that the components of generalization represent the transition from arithmetic to algebra due to the flow from concrete to abstract terms.

The literature has also indicated a “cognitive gap” which beginning algebra students experience during the progression from arithmetic to algebra (Herscovics & Linchevski, 1994, p. 63). It is argued that the progress from arithmetic to algebra is possible when students learn to operate with unknowns instead of specific numbers (Warren, 2003). In order to fill the gap, the recent mathematics education literature has suggested some pedagogical approaches to introduce algebra. One of these approaches is the pattern generalization approach (Carraher et al., 2008; Jones, 1993; Kieran, 1989; Orton & Orton, 1999; Witzel, 2015).

Pattern generalization approach has a progressive nature, which provides a proper base to develop algebraic thinking considering algebra as generalized arithmetic (Tall, 1992). In order to generalize a pattern, students are required to experience a gradual process, which involves three important steps related to near terms, far terms, and the general term (Radford, 2008). In other words, to generalize a pattern algebraically, it is required (i) to notice the common structure in the given terms and to find near terms by using commonality, (ii) to expand the commonality to far terms, and (iii) to establish a general description to find any term in the sequence (Radford, 2008).

Studies conducted so far have shown that when students are asked about near terms of the pattern, they tend to use additive relationship in an arithmetical sense (Stacey, 1989). In other words, they mostly focus on the relationship between consecutive terms and use expressions to find a term by using previous terms (Van de Walle et al., 2007). On the other hand, when far terms of the pattern are asked, additive relationship is not sufficient (Van de Walle et al., 2007). There is a need for an algebraic rule to find the far term. For example, in the given number sequence '2, 4, 6, …, …, 12, …', students need to notice the twice relationship between terms and the term number to be able to calculate the hundredth term since it is not practical to expand the given pattern by adding 2 till the hundredth term. In this regard, it can be said that arithmetic thinking can be adequate to reach near terms, while algebraic thinking is necessary to find far terms and that the movement from near terms to far terms necessitates transition from arithmetic thinking to algebraic thinking. NCTM (1997) recommended that students have experience with patterning activities since the flow from near terms to far terms helps to make a connection from the numeric-elementary level to a more general-algebraic level. As students work with near terms through arithmetic thinking, they start to notice the limitations of the arithmetic processes and tend more to use algebraic thinking (Lannin et al., 2006). Thus, by working on far terms, students develop their algebraic thinking skills and explore the general term of the pattern (Radford, 2014). Far generalization helps students overcome the difficulties about expressing generality with formal algebraic language (Zazkis, Liljedahl, & Chernoff, 2008). In this regard, as Ontario Ministry of Education (2013) stated, algebraic generalization can be constructed by moving from near terms to far terms.

The following section describes the pattern generalization framework used in the present study. After describing the theoretical framework of the present study, related studies on the generalization process of students are summarized.

2.2.1. Theoretical framework of the present study.

In the present study, Radford's pattern-generalization framework was used. Radford (2000, 2001) developed and applied the Theory of Knowledge Objectification (TKO) in the field of algebraic generalization. The reasons behind this choice were his longitudinal research since the 1990s in this field and the fact that generalization is both universal and learnable (Radford, 2008). In the TKO, the meaning of 'objectification' is as important as the meaning of 'knowledge'. Object refers to anything that can be referred, directed at or indicated (Sabena, Radford, & Bardini, 2005). Mathematical object is defined as anything which can be indicated or labeled during mathematical constructions or communications (Godino, 2002) such as mathematical language (terms, expressions, notations, etc.), mathematical situations (problems, exercises, etc.), mathematical actions (operations, algorithms, procedures, etc.), mathematical concepts (line, point, function, etc.), and so on (D'Amore, 2007). Thus, objectification is a process of showing a [mathematical] object to someone (Sabena et al., 2005).

On the other hand, Radford defined knowledge as culturally-historically encoded actions in people's memory. Therefore, objectification of knowledge refers to the process in which students participate in an activity in order to notice and make meaning of knowledge (Radford, 2010b). The main principle behind objectification of knowledge is its progressive manner. According to the TKO, individuals obtain knowledge in a progressive manner (Radford, 2003). For example, in patterning activities, students first perceive the common point in the given terms of the sequence, then generalize it beyond the given terms to apply it to other elements, and finally reach an expression of generality for any term. All these steps point to different levels of algebraic generality, some of which are more complex than the others (Radford, 2010a).

In his theory, Radford focused on the main difference between arithmetical and algebraic generalization, which is the fact that algebraic generalization allows to calculate indeterminate objects (Radford, 2008). In other words, algebraic generalization gives results beyond arithmetical generalization. In line with this, Radford (2010a) defined generalization of patterns as “the capability of grasping a commonality noticed on some elements of a sequence S, being aware that this commonality applies to all the terms of S and being able to use it to provide a direct expression of whatever term of S.” (p. 42). As the definition indicates, algebraic generalization has different levels, some of which are more complex than the others (Radford et al., 2006). Through the Theory of Knowledge Objectification, Radford formed a generalization framework based on the levels.

Before explaining the generalization framework, there is an important point to mention. Radford made a distinction between generalization and non-generalization, which he called as naïve induction (Radford, 2010a). Through naïve induction, individuals form some rules based on their predictions and then check whether they are valid or not on a few cases (Radford, 2010a). For example, in a linear-figural patterning activity, in which there are 4, 6, and 8 points in the first three terms of the sequence, students tried the rule ‘the number plus 3’ or ‘n+3’; however, it worked only for the first term. Then, they used ‘4n’ and then ‘2n+1’, and finally, they tried the ‘2n+2’ rule and reached the given terms of the pattern. Since this process is conducted based on probability and includes trial and error, naïve induction is different from the generalization framework in the Theory of Knowledge Objectification.

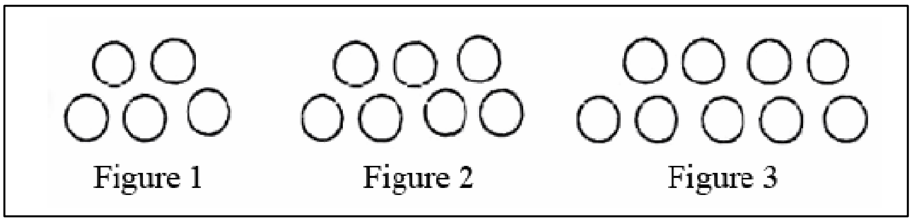


Figure 3. Linear-figural pattern from Radford (2006, p. 4)

The generalization framework distinguishes between arithmetic generalization and algebraic generalization. In arithmetic generalization, students notice the commonality in the given figures at an arithmetic level and provide a recursive expression in the form of ‘Ux+1=Ux+common difference’ (Gutiérrez, 2013). However, they do not provide a general expression in an algebraic way. For instance, when students are asked about the 100th term of the sequence in Figure 3, their solution way of expanding the sequence term by term till the term 100 is an example of arithmetical generalization. The most important characteristic of arithmetic generalization is lack of analyticity (Radford, 2012), i.e. dealing with indeterminate objects as if they were known, which is the main characteristic of algebra.

Algebraic generalization, as different from arithmetic generalization, is the pathway towards explicit expressions. Explicit expression is defined as functional expression, which allows for calculating dependent variables based on the independent variables (Barbosa, Vale, & Palhares, 2009). In pattern generalization tasks, independent variable is the position of the terms, i.e. term number, while the dependent variable is the value of the term itself. When students reach algebraic generalization and form an explicit expression, they can calculate any term based on the relationship between term and term number. In addition to this, Radford (2001) defined three hierarchical levels of algebraic generalization, which are factual, contextual, and symbolic generalizations. The first level of algebraic generalization defined by Radford (2001) is factual generalization. Below is an example of factual generalization of the pattern given in Figure 3:

> Student: It [the first figure] is one, one, plus three; [the second figure is] two, two, plus three; [the third figure is] three, three, plus three. For this reason, the 25th figure becomes twenty-five, twenty-five, plus three.

Factual generalization is not only the lowest level of algebraic generalization, but also the most concrete form among the others. It is limited with specific terms of the pattern. To express factual generalization, particular terms of the pattern are used within a concrete formula (Radford, 2001). In factual generalization, students explore the mathematical structure of the pattern and notice another type of commonality that they can apply to any particular term just as students' application of the factual rule to the 25th figure in the above example.

The second level of generalization is contextual generalization (Radford, 2001). It is more abstract than factual generalization. It is beyond the specific terms of the pattern. It is the first step that students give a name to "indeterminacy" (Radford, 2010a, p. 51). To deal with indeterminacy, students need to use abstract terms such as 'number of the term' or 'the figure'. A general rule, which is expressed through contextual generalization, includes descriptive terms such as "the general rule is 2 times the number of the figure and plus 1". Through contextual generalization, any term can be found within an abstract formula. They conduct mathematical operations on those abstract concepts such as the expression 'doubling the number of the figure and plus three'. An example of contextual generalization of a given pattern in Figure 3 is illustrated below:

> Student: If it is 1 [the first figure], I do one, one, plus three. If it is five [the fifth figure], I do five, five plus three. Right? I always do this. How can I say that?... I add the figure plus the figure, and plus three. I mean, it is always doubling the number of the figure and plus three.

Symbolic generalization is the most abstract level of algebraic generalization. To express symbolic generalization, alphanumeric symbols and letters are used such as "the general rule is "2n+1" (Radford, Bardini, & Sabena, 2007). It is one-step further than contextual generalization. In this type of generalization, the letter 'n' is the replacement of 'the number of the figure' (Radford, 2010a). In both contextual and symbolic generalizations, students reach an explicit expression, which is about the relationship between the term and position of the term [number of the term]. However, as contextual generalization includes abstract natural language terms such as 'the figure' or 'the number of the figure', symbolic generalization reduces the form of contextual generalization through alphanumeric symbols just as the example given (Radford et al., 2007). Below is an example of symbolic generalization of the pattern represented in Figure 3:

> Student: So, it would be n plus n and plus 3… It is n+n+3! (The student writes (n+n)+3). There are two n's. I think, I can write it as 2.n +3.

In sum, Radford's framework covers pattern generalization as a process from arithmetic to algebra. In this process, there are soft and subtle shifts, which are factual, contextual, and symbolic generalizations. The aim of the present study is to explore the pattern generalization process of students through transition from arithmetic to algebra, which is related to Radford's generalization framework. For this reason, Radford's generalization framework was used in this study. The uses of Radford's pattern generalization framework in mathematics education research were reviewed in the section that follows.

2.2.1.1. Uses of Radford's generalization framework in mathematics education research.

Hunter and Miller (2018) conducted an early algebra study using Radford's generalization framework. The aim of their study was to reveal how patterning tasks can develop students' understanding of growing patterns. With this purpose, they selected 27 second grade (6 year old) students. Through the study, the students developed the concept of linear growing patterns in 30 minute lessons. In each lesson, the students did pair work and engaged in group discussions through teacher facilitation. At the end of the study, the students reached factual and contextual generalizations. Similarly, Miller (2014) conducted an early algebra study whose purpose was to explore young Australian Indigenous students' generalization process. The students were from second and third grade level. The researcher asked the students to generalize linear figural patterns. The result of the study showed that the students were capable of using only contextual generalization. They did not engage in factual generalization or use letters through symbolic generalization.

On the other hand, Cooper and Warren (2011) conducted a study with third, fourth, and fifth grade students on how they generalize patterns in terms of Radford's generalization layers. They also used linear growing patterns during the study. As a result of the study, the researchers reported that the students moved through factual to contextual and symbolic generalizations while generalizing linear patterns. In parallel with Cooper and Warren (2011), Miller and Warren (2012) also reported students' movement from factual to contextual and symbolic generalizations while generalizing linear patterns.

In addition to Radford's hierarchical generalization strategies, the literature represents a wide range of generalization strategies. In the next part, related studies on generalization strategies of students are reviewed.

2.3. Related Studies on Students' Generalization Strategies

As explained above, Radford's generalization strategies are mainly distinguished from each other based on their arithmetic or algebraic nature. In parallel with Radford's sense, there are a variety of strategies in the related literature, which have either arithmetic or algebraic nature. Nevertheless, they were named differently by different researchers in spite of having similar meanings. For example, Radford's arithmetical generalization strategy was called as counting strategy in Stacey (1989), recursive in Ley (2005), looking for difference in Orton and Orton (1999), procedural activity in Garcia-Cruz and Martinón (1998), and so on. Some detailed explanations were given below.

In the related literature, while most of the studies focused on the generalization of linear patterns, some studies examined the generalization of both linear and quadratic patterns. Since the current study investigated the generalization of linear patterns, detailed information on the results of linear patterns are specifically reviewed in this section. As examining the generalization skills of students, related patterning tasks included items related to near terms, far terms, and general term. As students were conducting near generalization, far generalization, and global generalization processes, past studies dominantly focused on which strategies students used during those processes and what kind of difficulties students experienced meanwhile. According to the review of the literature, past studies related to the pattern-generalization processes are conducted with students from primary school level (7-11 years old), middle school level (12-14 years old) or high school level (15-18 years old).

In one of the studies conducted with primary grade students, Hargreaves, Threllfal, Frobisher, and Shorrocks-Taylor (1999) examined primary students' methods of generalizing number sequences. 487 students whose ages varied from 7 to 11 participated in the study. Students were asked about continuing/completing the linear and quadratic patterns to/with near terms and explaining, describing or providing a general rule about the pattern. Students were not necessarily expected an algebraic general rule due to their early ages. Results of the study revealed three methods of generalization: *looking for difference*, *looking at the nature of the numbers*, and *looking for multiplication tables*. According to the researchers, the first two strategies, i.e. *looking for difference and looking at the nature of the numbers*, have low complexity, while *looking for multiplication tables* has high complexity. The strategy of *looking for difference* focuses on the constant difference between successive terms of the pattern. It includes the recursive relationship of term-to-term. It corresponds to the *arithmetical generalization* strategy of Radford (2001). The second strategy of *looking at the nature of the numbers* includes noticing a common property related to the nature of the numbers that is valid for all numbers in the pattern such as oddness and evenness of the terms. For example, a student classifies the given number sequence, 3-8-13-18-23, as an odd number and an even number right after each other. However, this strategy does not allow students to reach any kind of generalization. The last strategy of *looking for multiplication tables* involves forming a relationship between the pattern and another sequence from the multiplication table. For example, a student generalizes the given number sequence, 2-5-8-11-14, in a way that ‘it goes on in 3s, yet, it is always 1 less than 3 times table, 3-6-9-12-15’. The researchers viewed the last strategy among the others as the closest one to algebraic generalization, since it may lead students to extend the pattern to other near and far terms by using the relationship ‘3 times table minus 1’. According to the results of the study, almost all students could find the near terms with looking for difference strategy at continuing/completing patterns to/with near terms. Yet, few students could answer the general rule question. Additionally, researchers reported two types of generalization process, which are single-type (i.e. using one strategy at answering questions) and mixed-type (i.e. using more than one strategy at answering questions). They resulted that there were some students who used mixed strategies. However, students who used single type of strategy outnumbered students who used mixed strategies at this level. In addition, as students’ grade level increased, the frequency of mixed strategies increased.

In another study, Bourke and Stacey (1988) worked with 371 primary students from fourth, fifth, and sixth grade levels (9-11 years old) in order to examine their problem-solving skills. They formed a problem-solving test, which included several problems from different mathematical domains. One of the problems was about the generalization of a linear pattern, which included a Ladder figure (see Figure 4).

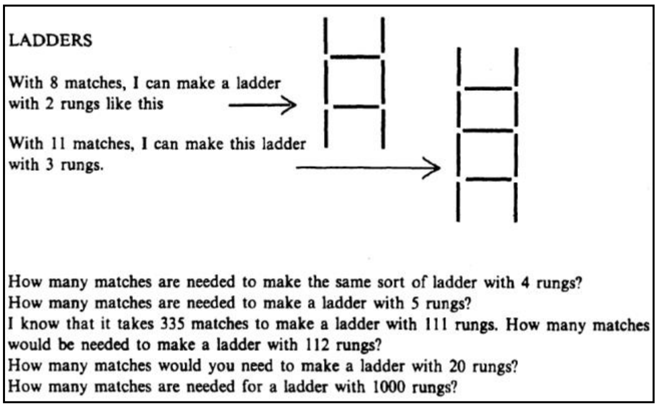


Figure 4. Ladder question from Stacey (1989, p. 148)

In Ladders, students were asked about generalizing given pattern to the 4th and 5th terms as near generalization and to the 20th and 1000th terms as far generalization. Upon Bourke and Stacey (1988), Stacey (1989) represented the analysis results of students' responses to the Ladders by naming four main generalization strategies that fourth, fifth, and sixth grade students used in Bourke and Stacey (1988). Those strategies were *counting* strategy, *difference* strategy, *whole-object* strategy, and *linear* strategy. The *counting* strategy included the repetitive and recursive counting process till the asked term such as counting by 3s to the 100th term with a calculator. The *difference* strategy involves multiplying the common difference, which is 3 matches in the question of Ladders, with the asked term such as multiplying 1000 with 3 to find the 1000th term. The *whole-object* strategy includes the assumption that a multiple of a smaller term gives the value of a bigger term (If the 3rd term includes 11 matches, the 30th term includes 110 matches). Lastly, the *linear* strategy includes explicit reasoning. For instance, in Stacey's study, the students recognized the structure of the pattern and formed the general rule accordingly such as finding the number of matches in the 1000th term as 3002 since there are 1000 matches on each side of the ladder (students ignore the 2 matches at the top) and 1000 matches in the middle and 2 at the top of the ladder, the sum of which is 3002. According to the results, fifth and sixth grade students showed similar performances with each other and better performance than fourth grade students did in Ladders pattern.

In another study conducted by Ley (2005), primary students' generalization strategies were investigated through five different types of linear patterns: figural, geometric, table, numeric, and word problem. In Ley's study (2005), 97 students from second grade to fifth grade were asked to generalize linear patterns to 5th term, 9th term, and 41th term. Throughout the study, three generalization strategies were observed, which are *recursive*, *whole-object*, and *explicit*. Ley's *whole-object* strategy carries the same meaning as Stacey's (1989). The *recursive* strategy, on the other hand, refers to adding repetitively upon previous term to find the further terms, which corresponds to the *counting* strategy in Stacey's (1989) study. The *explicit* strategy corresponds to the *linear* strategy of Stacey (1989). It includes understanding the structure of the pattern and reaching an algebraic rule related to any term of the pattern. Researcher also defined *ineffective* strategy, when students' responses included guessing or another random answer. According to the results of the study, *recursive* reasoning was the mostly seen strategy on near generalization tasks (5th and 9th terms) with a percentage of 61.4 in average. On the other hand, *explicit* strategy was seen far less than *recursive* reasoning with a percentage of 20.2 in average on far generalization task (41th term). Similar with Hargreaves et al. (1999), Ley (2005) observed participants using more than one strategy in their generalization process; yet, she did not analyze students' use of mixed strategies and reported the highest strategy among mixed strategies of students. From grade level perspective, there was no developmental trend in the use of *recursive* strategy. However, as students' grade level increased, the use of explicit strategy increased and the use of *ineffective* strategies decreased.

In addition to these studies conducted with primary school students, literature is wealth on middle school students pattern-generalization processes (Amit & Neria, 2008; Barbosa, 2011; Lannin, 2005; Lin & Yang, 2004; Orton & Orton, 1999; Stacey, 1989). In one of them, seventh and eighth grade Taiwanese students' reasoning ways of pattern generalization activities were investigated by Lin and Yang (2004). It is worth mentioning that elementary mathematics curricula in Taiwan does not include pattern generalization topic. Therefore, participants did not have any knowledge related to the pattern generalization. 1181 seventh graders and 1105 eighth graders participated in the study. Students' responses to the survey were coded under six categories: (i) seeing an improper pattern, (ii) seeing some useful but incomplete pattern or only with correct result, (iii) seeing a complete pattern only with correct arithmetic or photo-picture (manipulation), (iv) seeing a complete pattern with correct result and verbal explanation, (v) seeing a complete pattern towards correct algebraic strategies, and (vi) not showing to see any pattern. According to the results of the study, 34.9% of seventh grade students and 45% of eighth grade students had arithmetic reasoning, while 0.1% of seventh grade students and 0.6% of eighth grade students showed algebraic reasoning. In addition to that, almost half of seventh and eighth graders either did not see any pattern or saw an improper pattern at linear patterns.

In another study conducted with pre-algebra middle school students, Amit and Neria (2008) investigated the generalization strategies of 50 students who are at the beginning of sixth and seventh grade level. They asked students to generalize given patterns (figural-linear, figural non-linear and verbal non-linear) to the next term as near generalization, to the 10th term as far generalization, and to write any term by using n. To classify students' answers, they defined two strategies: *recursive/operational/local* strategy and *functional/conceptual/global* strategy. *Recursive/operational/local* strategy refers to the *counting* strategy, while *functional/conceptual/global* strategy refers to the *linear* strategy in Stacey's (1989) study. According to the results of the linear patterning task, almost all students first generalized the pattern to the next term by using the *recursive* strategy. After finding the next term, some students continued to generalize the pattern to 10th term with recursive relationship. Yet, they failed to find the any term, since they insisted on *recursive* strategy. On the other hand, some of them jumped to finding the any term and found the general rule of the pattern with functional relationship. When they found the general rule, they used it to calculate the far terms. In other words, those students did not follow the gradual generalization format which includes the order of near generalization, far generalization, and global generalization. Instead, they created their generalization process the order of which is near, global, and far. Furthermore, researchers observed that students, who could successfully generalize the linear pattern, generally used more than one strategy within the same question.

One of the major studies on pattern generalization was conducted by Stacey (1989). Stacey (1989) investigated middle school students' generalization strategies of linear patterns including Ladders pattern (see Figure 4) in order to compare primary and middle students' responses to the same pattern. The participants were 140 students from seventh and eighth grade level (12-13 years old). Results of the study showed that students mostly used the *counting* strategy for near generalization regardless of their grade level. When they were asked about generalizing the pattern to far terms, the *counting* strategy was inadequate. In such a situation, they employed either the *difference* or *whole-object* strategy. Very few students used the linear strategy for far generalization. Furthermore, there was no change in the strategies of students across different grade levels. Additionally, the students were more successful at generalizing the pattern to near terms than far terms. From grade level perspective, seventh and eighth grade students showed similar performances at near and far generalization tasks. Another important finding of Stacey's (1989) study was that 64% of the students used more than one strategy while generalizing linear patterns. Stacey (1989,

p. 147) called this situation as “inconsistency of choice of model”. In detail, 22% of the students used a combination of the *whole-object* and *difference* strategies, while 21% of the students used a combination of the *whole-object* and *linear* strategies, 15% of the students used a combination of the *difference* and *linear* strategies, and 6% of the students used a combination of the *whole-object*, *difference,* and *linear* strategies.

Orton and Orton (1999) extended the study of Stacey (1989). They studied with 10-13 year old students about their pattern generalization methods when there are linear and quadratic patterns. They reached the same results with Stacey (1989) that many students changed methods in near and far generalization tasks within the same question. They expressed that participants could get the most success when they combined *recursive* method in near generalization and *explicit/linear* method in far generalization. They also observed the combination of *whole-object* and *difference* methods during changing of methods. Yet, they did not present detailed analysis of results related to changing of methods. Else than changing of methods, Orton and Orton (1999) presented their results based on the age of the students. During presenting their results, they grouped students based on their ages as 10-11 year old students, 11-12 year old students, and 12-13 year old students. According to the results, at finding 20_{th} and 100_{th} terms, 10-11 year olds and 11-12 year olds performed similarly with fifty percent approximately; while 12-13 year olds performed better than them (over 60 percent). On the other hand, few students could generalize the pattern to nth term regardless of their grade level. 0.7% of 10-11 year old students, 7.0% of 11-12 year old students, and 19.4% of 12-13 year olds could find nth term through linear method. In other words, as students aged, they performed better. According to the researchers, the reasons behind students' having difficulty at generalizing patterns are their being incompetent at arithmetic, their persistency at recursive method, and the use of ineffective methods such as whole-object and difference.

Else than cross-sectional studies on pattern generalization such as Stacey (1989), Orton and Orton (1999) or Amit and Neria (2008), Barbosa (2011) investigated middle school students' pattern generalization development in time with a longitudinal study. She studied with 54 Portuguese students from sixth grade level (11-12 years old). Through the study, near and far generalization tasks of increasing linear patterns were asked to students in clinical interviews over 6 months. Students were grouped in 27 pairs. The results of the study revealed that the students used five generalization strategies, which are *counting*, *whole-object*, *difference*, *explicit,* and *guess and check*. According to the results of the first task, more than half of students used *counting* strategy at near generalization and almost a quarter of students used *whole-object* strategy. Furthermore, there was only 1 pair of students who used *recursive* strategy. After 4 months of experience with patterning activities, another increasing linear pattern was asked to the participants of the study. According to the result of the second task, 22 pairs used *counting* strategy, 4 pairs used *difference* strategy, and 2 pairs used *explicit* strategy at near generalization. Besides, 22 pairs used *explicit* strategy, 4 pairs used *difference* strategy, and 1 pair used *counting* strategy at far generalization. In other words, the number of students who used *explicit* strategy increased from 12 pairs to 22 pairs at far generalization. The number of students who used *difference* strategy increased from 16 pairs to 21 pairs at near generalization. The number of students who did not answer the question or incorrectly used the *whole-object* strategy decreased from 6 pairs to 0 pairs at far generalization. Overall, students were more successful at near generalization than far generalization.

Just as Barbosa (2011), Lannin (2005) also examined sixth grade students' generalization processes. As different than Barbosa (2011), Lannin (2005) used spreadsheets throughout the study and formed a framework involving *explicit* and *non-explicit* strategies. During the study, 25 sixth grade students were asked to generalize and justify their generalizations through computer spreadsheets. While he called *counting* and *recursive* strategies as *non-explicit*, he classified the *whole-object*, *guess-and-check*, and *contextual* strategies as *explicit* strategies. Since the first three strategies were explained before, the last two strategies are explored here. The *guess-and-check* strategy involves trying many rules on the given pattern. The *contextual* strategy involves figuring out a rule based on the given structure in the pattern. Lannin's *contextual* strategy (2005) might correspond to Stacey's *linear* strategy (1989) or Ley's *explicit* strategy (2005). According to Lannin (2005), since doing operations on spreadsheets is easier, in his study, the students were more likely to use the *guess-and-check* strategy. Additionally, he described the *whole-object* and *guess-and-check* strategies as distractor strategies since they focus on empirical-particular results more than the general structure of the pattern. Lannin (2005) also expressed that sixth grade students' inadequate operational skills in mathematics prevents them developing algebraic generalizations.

Else than studies conducted with primary and middle school students, studies conducted with high school students also present important results. In a study conducted by Mason, Graham, Pimm, and Gowar (1985), the pattern-generalization process of students were categorized into four stages, which are seeing, saying, recording, and testing. To identify the pattern, to describe it with words, to record the findings and to test the formula are the corresponding explanations of the four stages. Lee and Wheeler (1987) analyzed high school students' generalization processes using Mason et al. (1985)'s framework. They asked students a 'dot rectangle problem' as shown in Figure 5. According to the results of the study, 163 out of 176 students could find the number of dots in the fifth rectangle, while only 26 students could find the 100th term and the general term of the pattern. Lee and Wheeler (1987) found that "seeing the pattern" was not a problem for students. What was difficult for students was "seeing an algebraically useful pattern" (p. 95). In other words, students were able to notice the arithmetical structure of the pattern, but they could not relate it to algebra. The researchers also concluded that as far as the saying phase is concerned, students did not describe the pattern verbally nor did they test their findings in the testing phase.

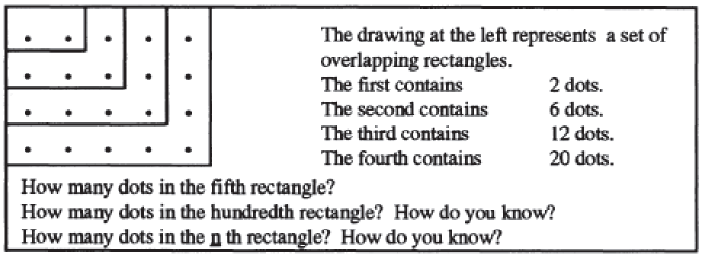


Figure 5. The dot rectangle problem (Lee, 1996, p. 94)

In another study by Becker and Rivera (2005), ninth grade students' generalization skills were examined through a qualitative study. Two of the purposes of the study were to examine the generalization strategies of successful students and to understand the obstacles behind students' having successful generalizations. They asked students linear patterning tasks including items asking for the next few terms, 10th term, and general term. They worked with 22 students from a beginning algebra course. The results of the study revealed 23 different generalization strategies that students used. These strategies were either numerical or figural. Successful students mostly used at least two strategies while solving different items of the question. The results of the study also showed that students had difficulties in answering all the items of the generalization questions. Five of the 22 students were able to generalize all parts, while four students were able to make generalization partially. The remaining 13 students were not able to make generalizations at all. According to the researchers, students, who had difficulties at generalizing, mostly used numerical strategies at the beginning of the question and insisted on using only one strategy at all items of the question. They also could not connect different generalization strategies with each other.

Garcia-Cruz and Martinon (1998) also conducted a study with high school students who are 15-16 years old. They firstly conducted interviews with 11 students and then performed a teaching experiment including small group and whole class discussions with 18 students. Researchers asked students about how to generalize a linear pattern to 4th, 5th, 10th, 20th, and nth term. At the end of the study, they defined three hierarchical levels of generalization, which are *procedural activity*, *local generalization*, and *global generalization*. The procedural activity involves the recursive relationship between successive terms of the pattern. It mostly focused on the constant difference of the linear pattern. At this stage, students find the required term of the pattern by adding the constant difference onto known terms repetitively. The second stage, local generalization, involves the use of a specific rule to apply on specific terms. For example, in a given number sequence '6, 11, 16, 21, …..', students identify the relationship of the pattern with multiples of 5 in a way that 'the first term is 5 plus 1, the second term is 2 times 5 plus 1, the third term is 3 times 5 plus 1. It is always 1 more than the multiples of 5. Thus, the 100th term is 100 times 5 plus 1. As the example shows, the student formed a specific rule by adding 1 to the required multiple of 5. The final stage is global generalization. In global generalization, students adapt the rule, which is formed during local generalization, for new patterns. In other words, the specific rule, which is formed before, is used for new situations. In addition to defining three hierarchical levels of generalization, researchers resulted that students dominantly shifted from procedural activity to local generalization within the same question.

2.4. Summary of the Literature Review

In conclusion, the related literature showed that pattern generalization activities increase students' development of algebraic thinking skills and provide experience with numerical relationships between inputs and outputs of a pattern (NCTM, 2000; Orton & Orton, 1999). It helps beginning algebra students to make the transition from arithmetic to algebra due to its progressive nature from near terms to far terms and the general term. Yet, related studies showed that students have difficulties in seeing the algebraic structure during pattern generalization (Becker & Rivera, 2005; Lee & Wheeler, 1987; Lin & Yang, 2004; Orton & Frobisher, 2004; Rivera & Becker, 2006). The difficulties experienced by students generally arise from the persistence on the recursive approach, the insufficient understanding of arithmetical operations, the misusage of some strategies such as the usage of whole-object strategy without adjusting to the pattern, and the search for short-cut strategies such as guess and check strategy. To be able to overcome these difficulties, it is necessary to analyze students' strategy use during pattern generalization process. In spite of its important role in terms of the development of algebraic thinking, the studies on patterning strategies at elementary grade levels are inadequate in the literature (Vale, 2009; Waters, 2004). As reported in the literature, generalization strategies are basically distinguished from each other based on their arithmetic or algebraic nature. For that reason, results of the present study would be a significant contribution to the literature in terms of revealing the abilities and difficulties of students in constructing and connecting arithmetic and algebraic generalizations.

The review of the literature showed that past studies which examined students' pattern generalization processes dominantly applied a gradual generalization format which follows a path from near generalization to far generalization and general term/nth term (Barbosa, 2011; Becker & Rivera, 2005; Garcia-Cruz & Martinon, 1998; Lee & Wheeler, 1987; Ley, 2005; Orton & Orton, 1999; Stacey, 1989). In these studies, students were not offered any flexibility of creating their own generalization processes. Yet, some studies showed that when students were free to choose, they could create different generalization processes such as starting off with near generalization, jumping to finding general term/nth term, and returning to far generalization through calculation of general term (Amit & Neria, 2008; Rivera & Becker, 2006). Considering the gains of each generalization process, the order of the generalization processes students applied could provide important clues about the specific points that they had difficulties at the whole process. In other words, when a student chose to conduct near generalization at first, to find nth term as second, and to conduct far generalization at last, it can be inferred that the student can apply near generalization easily while s/he can apply far generalization difficultly. Yet, there is a gap in the literature in this area. In the current study, researcher did not apply a gradual generalization format during the data collection process. Even if the questions in the Pattern Test included items consisting of the order of near, far, and global generalizations, students were free to create their own generalization process. Thus, results of this study would provide valuable information about general trends in students' generalization process sequences.

Related literature revealed two types of generalization process: single type generalization, mixed-type generalization. Single type generalization included only one generalization strategy at whole-generalization process, while mixed-type generalization included at least two generalization strategies. As indicated in the literature, there is evidence that students use mixed generalization strategies within the generalization process more than single type generalizations (Barbosa et al., 2009; Noss, Healy, & Hoyles, 1997; Rivera, 2010; Stacey, 1989). However, many studies in the literature have so far investigated single type of generalizations, which included only one generalization strategy (Barbosa, 2011; Cai & Knuth, 2011; Hargreaves et al., 1999; Lannin, 2005; Ley, 2005; Rivera & Becker, 2007). In other words, when students used more than one strategy at answering one generalization question, past studies did not give detailed information about which strategies were combined, how the frequency of each strategy was or how the order of the strategies was. Instead, they just expressed that they observed multiple strategies by giving shallow information (Becker & Rivera, 2005; Garcia-Cruz & Martinon, 1998; Hargreaves et al., 1999; Orton & Orton, 1999; Stacey, 1989). In addition to this, some studies ignored the situation of students' using multiple strategies. When a student used more than one strategy at answering one generalization question, the student's answer was coded either by the more general strategy (Stacey, 1989) or by the highest-ranking strategy (Ley, 2005). In other words, past studies did not provide sufficient information on the use of multiple strategies during the whole-generalization process. There is a gap in the literature on how students use mixed generalization strategies.

All in all, the purpose of the present study is to investigate students' generalization process in detail including both single-type strategies and mixed-type strategies.

CHAPTER 3

METHOD

The aim of this study is to reveal sixth, seventh, and eighth grade students' generalizations of patterns using arithmetical generalization, algebraic generalization, and naïve induction as well as whether their generalization process differs in terms of their grade level. In line with these purposes, research design, the procedure, the pilot study, the participants, the data collection procedure, the data collection tool, data analysis, validity, reliability, and limitations of the study are explained in this chapter.

3.1. Research Design

The purpose of the study is to understand elementary students' algebraic thinking skills deeply through pattern tasks. Based on the purpose of the study, qualitative research methods were used in this study, since qualitative research strategies enable researchers to have a detailed understanding of the issue and gain an insight into the deeper thoughts and behaviors of participants (Creswell, 2007).

Creswell (2007) defined five approaches of qualitative inquiry design, which are narrative, phenomenology, grounded theory, ethnography, and case study. Each approach was built on one another in a way that they all share common points as well as basic differences. Narrative design involves experiences and stories told by participants, while phenomenology design focuses on the commonality in stories and experiences told by participants (Creswell, 2013). Grounded theory design, on the other hand, aims at exploring a theory and forms a framework based on the data gathered from the participants who had the same experiences (Creswell, 2013). Ethnography design shows similar characteristics with grounded theory in terms of analyzing many participants who have gone through same processes. Yet, it differs from grounded theory in terms of shared locations of the participants where they form common behaviors, beliefs, and languages (Creswell, 2013). The aim of ethnography is to find out the mechanism behind the culture, not to develop an in-depth understanding of the issue through the case. Lastly, case study aims at understanding an issue thoroughly using the case/s.

Case study has been described in many ways in the literature. It involves “an in-depth description and analysis of a bounded system” (Merriam, 2009, p. 40). Yin (2009) defined it as an inquiry method, which examines a real-life phenomenon deeply whose boundaries with the context are not apparent. Creswell (2013), on the other hand, mostly focused on the procedure by defining it as

> A qualitative approach in which the investigator explores a real-life, contemporary bounded system (a case) or multiple bounded systems (cases) over time, through detailed, in-depth data collection involving multiple sources of information (e.g., observations interviews, audiovisual material, and documents and reports), and reports a case description and case themes (p. 459).

Definitions of the case study indicate that cases are identified as bounded systems within a case study. In a case study, a case can only be identified as a unit of analysis in a bounded system (Merriam, 2009). The literature emphasizes the importance of defining the bounded system in detail while describing the cases (Stake, 2005; Yin, 2009). In a typical case study, a unit of analysis might be a person (Yin, 2009). It can also be a program, an organization, or a small group of people. Yin (2009) categorized case study designs in a 2x2 framework based on the unit of analysis (see Figure 6). Categories of the framework are determined according to two criteria: the number of cases (single or multiple) and the number of unit of analysis (holistic or embedded). Single case designs including single unit of analysis are characterized as holistic, whereas single case designs including multiple unit of analysis are characterized as embedded (Yin, 2009). Multiple case designs including single unit of analysis are characterized as holistic while multiple case designs including multiple unit of analysis are characterized as embedded (Yin, 2009).

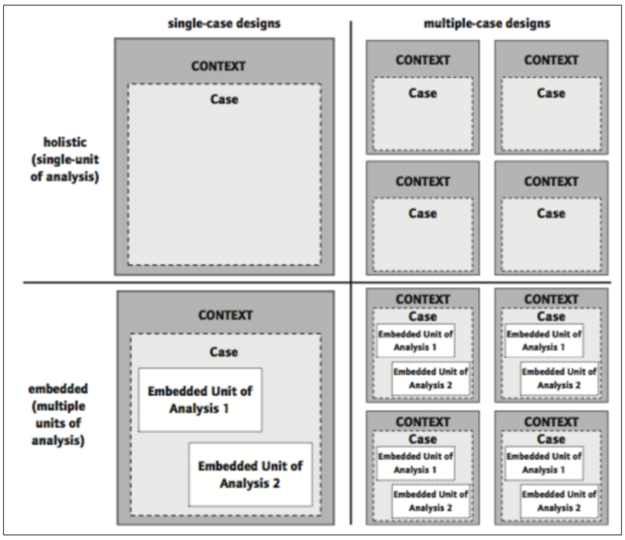


Figure 6. Case study categorization developed by Yin (2009, p. 46)

The current study is a multiple case study. There are three cases, which are the small groups of students from sixth, seventh, and eighth grade levels, as the present study addresses middle school students' generalizations of patterns through linear pattern tasks. The unit of analysis in this study is the pattern generalization structures of middle school students. The context of the study is bounded with a public middle school in Ankara. Therefore, the design of the study is holistic-multiple case study as seen in Figure 7.

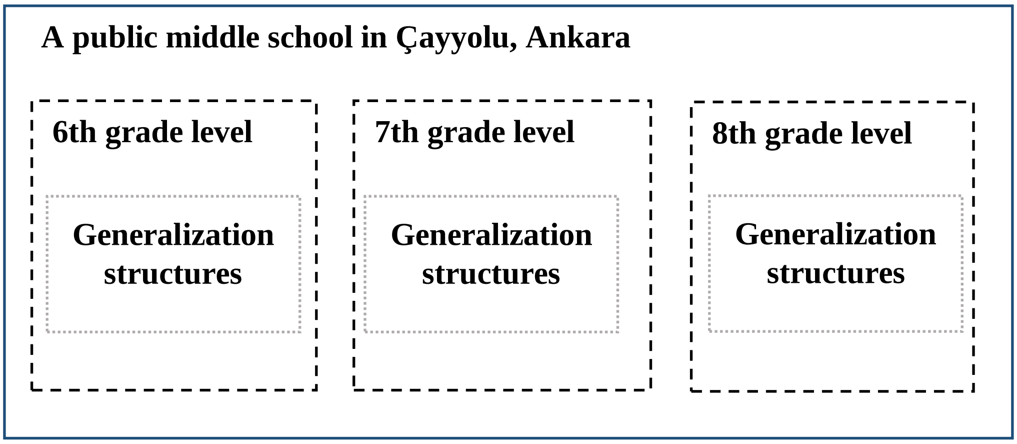


Figure 7. Design of the study: holistic multiple case study

3.2. Sampling and Participants

Main sampling strategies are probability and nonprobability sampling (Merriam, 2009). Probability sampling enables researcher to generalize from sample to the population (Merriam, 2009). However, the main concern of qualitative research is not statistical generalization. Thus, nonprobability-sampling strategies have been accepted as the basic sampling strategy in qualitative designs (Merriam, 2009). The present study uses purposeful sampling among nonprobability sampling strategies.

Purposeful sampling allows choosing informative cases and reach in-depth understanding of the study (Patton, 2002). Due to the convenient location and time, the participants of the study were selected from the public school in which the researcher was working as a mathematics teacher for eight months by the time data collection started.

The school was a state middle school located in Çankaya district of Ankara. It was located in a rural area. Yet, with the construction projects around the village, the rural area was turning into urban area. In this neighbourhood, there were four state middle schools at all. This school was not a crowded school since it was opened in 2013-2014 academic year. In the public school during 2015-2016 academic year, there were about 120 middle school students in all grade levels (i.e., 5th, 6th, 7th, and 8th). There were two classes for each grade level in the school. The size of the classes was about 15 students. There were two mathematics teachers in the school. While one of the teachers, i.e. the researcher, was assigned to teach mathematics lesson in one sixth grade, one seventh grade, and two eighth grade classes, the other teacher was assigned to teach in two fifth grade, one sixth grade, and one seventh grade classes. The researcher was also assigned to teach math application lesson in all sixth, seventh, and eighth grade classes.

Students' age ranged from 11 to 14. Socioeconomic statuses of students were generally moderate. The parents of the students were usually graduated from high school. Their mothers were mostly housewives, while their fathers were either civil servant or worker in the private sector. Students dominantly had one or two brothers/sisters, with whom they share the same room at their houses. Furthermore, students generally had a moderate academic profile in terms of mathematics lesson. They were neither highly successful nor unsuccessful in mathematics lesson. They were also willing to participate in lessons and to take responsibilities academically.

During the selection of the participants, all students were clearly informed about the purpose of the study. Since the researcher knew all the students personally, she selected the most suitable participants who could provide rich information for the study. There were three criteria during the selection process of the participants. The first criteria was the grade level of the students. Since the topic 'generalization of linear patterns' belongs to 6th, 7th, and 8th grade level in the National Elementary Mathematics Education Curriculum, it was necessary for the participants to be at sixth, seventh or eighth grade level (MONE, 2013). The second criterion was students' enthusiasm about the mathematics lesson. The researcher determined the enthusiastic students based on three ways, which are (i) observing students personally in mathematics and math application lessons, (ii) taking the opinion of the second mathematics teachers about each student, and (iii) observing students' activities in Math Club. This criterion arised from my view that students who are enthusiastic about the mathematics lesson would be more open to give answers to the questions in the interview protocol and provide rich information. The last criterion was students' talkativeness. The researcher operationalized this criterion based on students' participation in class discussions. In the current study, the individual interviews were conducted with one participant at one time; therefore, it was important that the participants be willing to talk. In order to overcome possible biases during the selection of the participants was reviewed by the second mathematics teacher in the school. Then, 14 participants (five students from 6th grade level, four students from 7th grade level, and five students from 8th grade level) were selected among the volunteer students.

As mentioned above, the present study includes three cases, which are sixth grade students, seventh grade students, and eighth grade students. As a result of selection process, five sixth grade students among 35 students, four seventh grade students among 38 students, and five eighth grade students among 30 students were selected as participants. They were from the both classrooms that the researcher and the other teacher taught. There was one female, four male students from eighth grade; one female, three male students from seventh grade; and two female, three male students from sixth grade.

3.3. Data collection

3.3.1. Setting.

The context of the present study was bounded with a public school located in Çayyolu, Ankara. It had eight classrooms, one conference room, and one library. The study took place in the conference room of the school. The conference room was 7 meter by 8 meter in size. It had windows on east side. During the study, all chairs were placed at the back side of the room in order to increase available space (see Figure 8). The room had dark blue curtains with a blackboard at one end. To set up best video lighting during recording, curtains were all closed and lightings were on during the study.

In the course of interview, a table of 1 meter by 2 meter in size was placed in the center of the room with two chairs at each side, one for the participant, one for the researcher. The location of the researcher's chair was suitable to see the test paper of the participant. There were two cameras with two tripods and one voice-recorder. The voice recorder was placed in front of the participant next to the test paper. One of the cameras took close-up record on the test paper of the participant with hand movements of the participant. The other one took mid-shot record focusing on the face of the participant.

These physical settings were fixed for all participants.

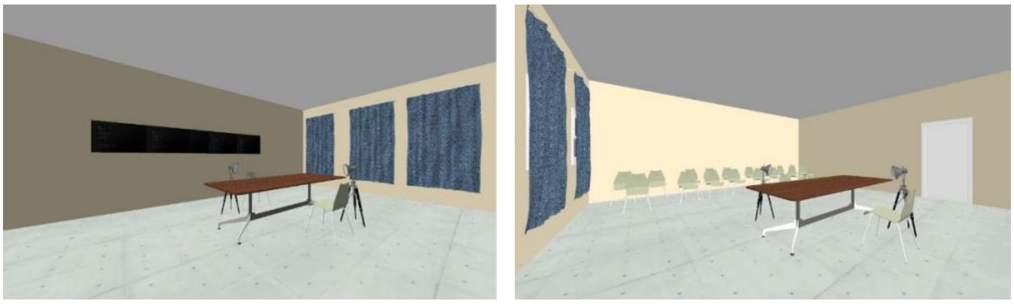


Figure 8. Physical setting of the conference room

3.3.2. Data collection instruments.

In this study, data were collected through a Pattern Test and interviews.

3.3.2.1. The pattern test.

The Pattern Test included six open-ended pattern tasks adapted from the literature. While selecting the questions, a table of specification (see Table 1) was prepared considering the related objectives from the National Elementary Mathematics Education Curriculum (MONE, 2013, 2018).

Table 1. Table of Specification for the Pattern Tasks based on the objectives of National Mathematics Education Curriculum (MONE, 2013, 2018)

Objectives	Grade Level	Related Tasks
Students should be able to recognize the number patterns with fixed difference between consecutive terms of the patterns, find the rule of the pattern, and complete the pattern by identifying the missing item. [Aralarındaki fark sabit olan sayı örüntülerini tanır, örüntünün kuralını bulur ve eksik bırakılan ögeyi belirleyerek örüntüyü tamamlar.] (MONE, 2018)	2	T2, T4, T6
Students should be able to expand and form the number pattern with fixed difference between consecutive terms of the pattern. [Aralarındaki fark sabit olan sayı örüntüsünü genişletir ve oluşturur.] (MONE, 2018)	3	T2, T4, T6
Students should be able to form increasing or decreasing number patterns according to a certain rule and explain the rule. [Belli bir kurala göre artan veya azalan sayı örüntüleri oluşturur ve kuralını açıklar.] (MONE, 2018)	4	T2, T4, T6
Students should be able to construct the required steps of the given number and shape patterns whose rule are given. [Kuralı verilen sayı ve şekil örüntülerinin istenen adımlarını oluşturur.] (MONE, 2013, 2018)	5	T1, T2, T3, T4, T5, T6

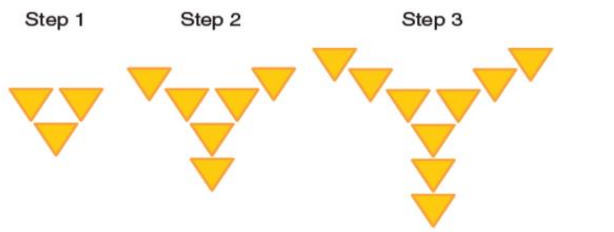
Table 1 (continued)

Students should be able to express the rule of the arithmetic sequences with letter and find the required step of the sequence whose rule is expressed with letter. [Aritmetik dizilerin kuralını harfle ifade eder; kuralı harfle ifade edilen dizinin istenilen terimini bulur.] (MONE, 2013)	6	T1, T2, T3, T4, T5, T6
Students should be able to express the rule of the number patterns with letter and find the required step of the pattern whose rule is expressed with letter. [Sayı örüntülerinin kuralını harfle ifade eder, kuralı harfle ifade edilen örüntünün istenilen terimini bulur.] (MONE, 2018)	7	T1, T2, T3, T4, T5, T6

Based on the curricular restrictions, the questions in the Pattern Test were classified as linear-numeric and linear-figural questions. Additionally, the purpose of the instrument was to collect data about participants' algebraic reasoning skills. Basic algebraic reasoning skills are to identify the pattern, to extend the pattern to near and far terms, to find out the general term, and to generate a rule for the pattern (Threlfall, 1999). Therefore, while determining the questions for the Pattern Test, the researcher aimed to select items which would enable the participants to identify a pattern, to predict near and far terms, and to find the general term. Students were expected to apply anyone of the generalization strategies among arithmetical generalization, algebraic generalization, and naïve induction during solving each queston in the Pattern Test. Since each question was adapted from the literature, the validity and reliability of the questions were provided in the related studies. Accordingly, each question included four or five items related to near generalization, far generalization, and the general rule.

The Pattern Test included three linear-figural pattern questions. The difference between the consecutive steps of the figural patterns in the questions was the same. These questions are first, third, and fifth questions.

The first question in the Pattern Test was adapted from Van de Walle et al. (2007). In the original version of the task (see Figure 9), students were asked to fill in the blanks in the given *table* for near and far terms. Additionally, it included a direction about writing the general rule with *words* and *symbols.*



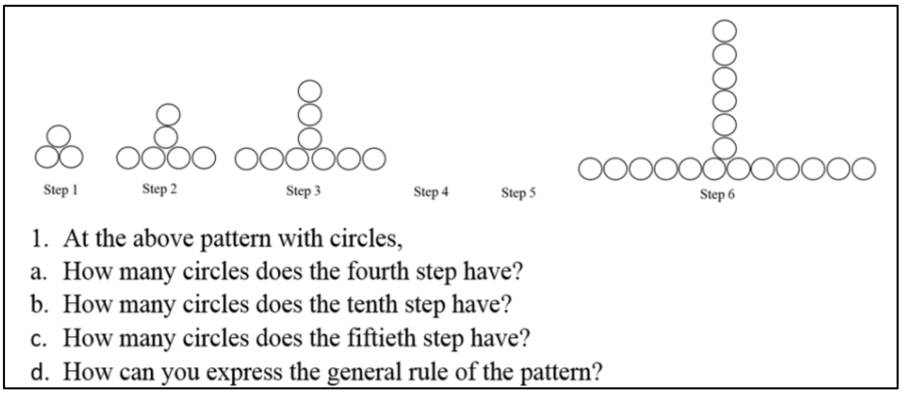
- **Complete a table that shows number of triangles for each step.**

Step Number	1	2	3	4	5 . . .	10	20
Number of Triangles							

- **How many triangles are needed for step 10? Step 20? Step 100? Explain your reasoning.**
- **Write a rule (in words and/or symbols) that gives the total number of pieces to build any step number (*n*).**

Figure 9. The original version of first question (Van de Walle et al., 2007, p. 269)

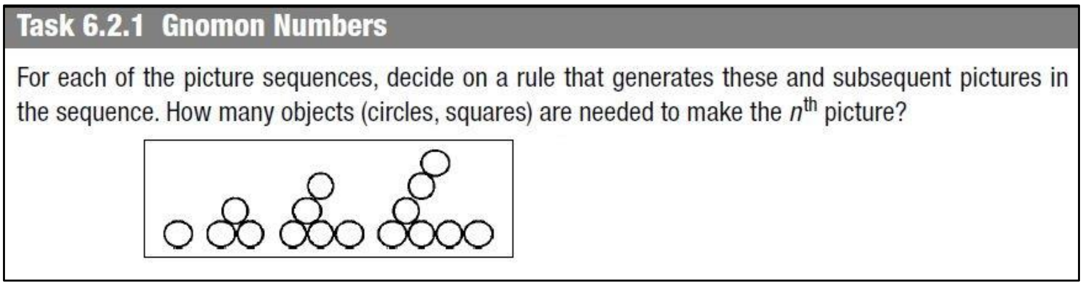
In order not to lead students in a particular solution direction, the researcher removed the table part from the question. In addition, the general rule was asked without directing students to use words or symbols. The reason behind is not to limit students' answers with the use of table, words, and symbols. Instead, near, far, and general terms were asked in regular items. Additionally, rather than triangles, circles were used due to the practicality of drawing as in Figure 10.



1. At the above pattern with circles,
a. How many circles does the fourth step have?
b. How many circles does the tenth step have?
c. How many circles does the fiftieth step have?
d. How can you express the general rule of the pattern?

Figure 10. The adapted version of first question from the instrument

The third question in the Pattern Test was adapted from Mason et al. (2005). In its original version, only general rule of the pattern was asked in terms of nth term as seen in Figure 11. There was no item related to near or far terms.



Task 6.2.1 Gnomon Numbers

For each of the picture sequences, decide on a rule that generates these and subsequent pictures in the sequence. How many objects (circles, squares) are needed to make the n^{th} picture?

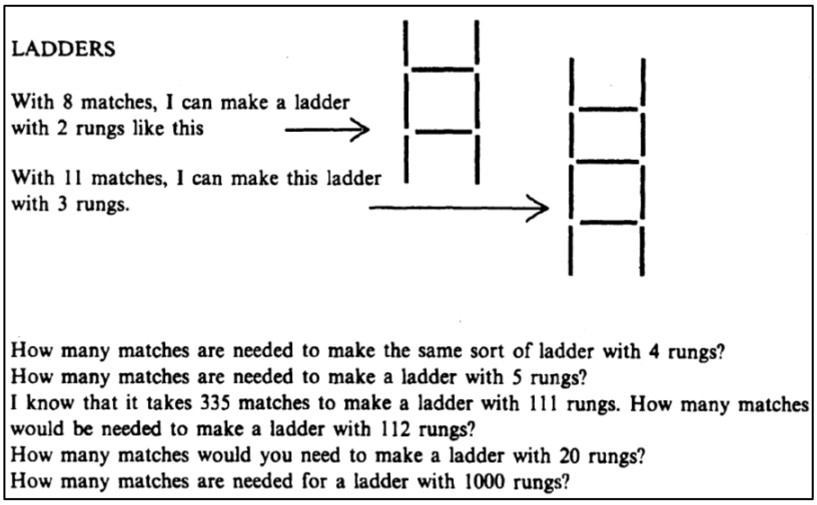
Figure 11. The original version of third question (Mason et al., 2005, p. 117)

In order to use the question, the researcher added new items to the question which asks for near and far steps such as fourth, fifth, tenth, and fiftieth steps. These items were added in order to reach the objectives of the study related to students' generalization processes of near and far terms. Moreover, 'the general rule' expression was used instead of the expression 'nth term' not to direct students to *symbol* use. The adapted question was as follows:

3. At the above pattern with circles,
a. How many circles does the fourth step have?
b. How many circles does the fifth step have?
c. How many circles does the tenth step have?
d. How many circles does the fiftieth step have?
e. How can you express the general rule of the pattern?

Figure 12. The adapted version of third question from the instrument

The fifth question in the Pattern Test was adapted from Stacey (1989). It included the construction of a ladder using matches as seen in Figure 13. Stacey (1989) presented the pattern with two images rather than a sequence image. In addition, she contextualized the step number as ‘the number of rungs’. Moreover, although near and far terms of the pattern were asked in original version, the question did not ask about general term of the pattern as follows:


LADDERS

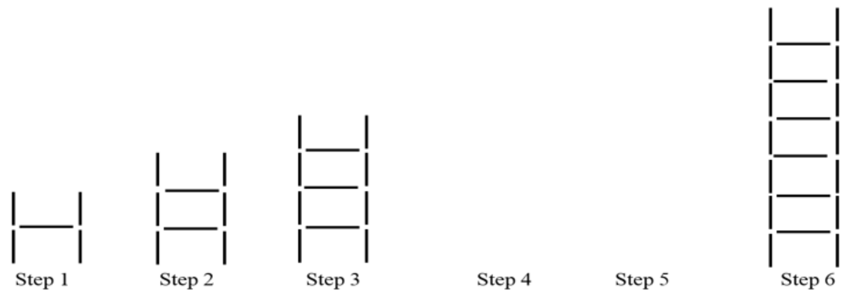
With 8 matches, I can make a ladder with 2 rungs like this

With 11 matches, I can make this ladder with 3 rungs.

How many matches are needed to make the same sort of ladder with 4 rungs?
How many matches are needed to make a ladder with 5 rungs?
I know that it takes 335 matches to make a ladder with 111 rungs. How many matches would be needed to make a ladder with 112 rungs?
How many matches would you need to make a ladder with 20 rungs?
How many matches are needed for a ladder with 1000 rungs?

Figure 13. The original version of fifth question (Stacey, 1989, p. 148)

In order to use the question in the present study, staging of the question was modified in a way that the pattern was broken into growing stages. Also, the expression of ‘step number’ was used instead of the expression of ‘the number of rungs’. Lastly, an item related to general term of the pattern was added in order to see students’ processes of generalizing a pattern to any term in accordance with the objectives of the study. The adapted question was as given:



5. At given pattern above, the ladder is constructed by using matches.
a. How many matches does the fourth step have?
b. How many matches does the fifth step have?
c. How many matches does the tenth step have?
d. How many matches does the hundredth step have?
e. How can you express the general rule of the pattern?

Figure 14. The adapted version of fifth question from the instrument

The Pattern Test also included three linear-numeric pattern questions. The difference between the consecutive steps of the numeric patterns was the same. These questions are second, fourth, and sixth questions. Each question was adapted from related literature. The original versions and the changes are given below.

The second question in the study was adapted from Graham (2005). It included a numeric pattern whose first term is 5 and which increases by 3 at each step. In the question, only near terms were asked. There was no item related to far terms or the general rule as given in Figure 15:

Find the difference between terms for each sequence and hence write down the next two terms of the sequence. 5, 8, 11, 14, 17, . . .

Figure 15. The original version of second question (Graham, 2005, p. 263)

In order to use in the present study, new items, which ask for far terms and the general term, were added in order to reach the objectives of the study. No further modification was made as below:

5, 8, 11, …, …, 20, …, … a. What would be the fourth term in the sequence? b. What would be the fifth term in the sequence? c. What would be the tenth term in the sequence? d. What would be the hundredth term in the sequence? e. How can you express general term in the sequence?

Figure 16. The adapted version of second question

The fourth question in the Pattern Test was adapted from Sacey (1989). It included a numeric pattern whose first term is 4 and which increases by 6 at each step. In its original version, Stacey (1989) asked about near terms of the pattern in ‘fill in the blank’ type question and far term. She did not mention the general rule as given below:

4, 10, 16, 22, __, __, __,

a. Fill in the blanks.
b. Find the hundrenth term in the sequence.

Figure 17. The original version of fourth question (Stacey, 1989, p. 149)

In order to use in the present study, the first term of the sequence was modified as 12 and the constant difference remained the same. The reason behind was to observe students' generalization processes in a numeric sequence whose terms are multiples of 6 when the difference between the consecutive terms is 6. In addition, a new item was added related to the general rule of the pattern as the objectives of the study required. Thus, the adapted version of the question was as follows:

12, 18, 24, …, …, 42, …, …

a. What would be the fourth term in the sequence?
b. What would be the tenth term in the sequence?
c. What would be the fiftieth term in the sequence?
d. How can you express general term in the sequence?

Figure 18. The adapted version of fourth question

The sixth question in the Pattern Test was adapted from Rivera and Becker (2011). It included the construction of a square using smaller blue squares as seen in Figure 19. In its original version, the question was asked as a figural pattern whose sequence is 4, 8, 12, and 16. Furthermore, students were asked to find the general formula in two different ways. There was no item related to near or far terms as given below:

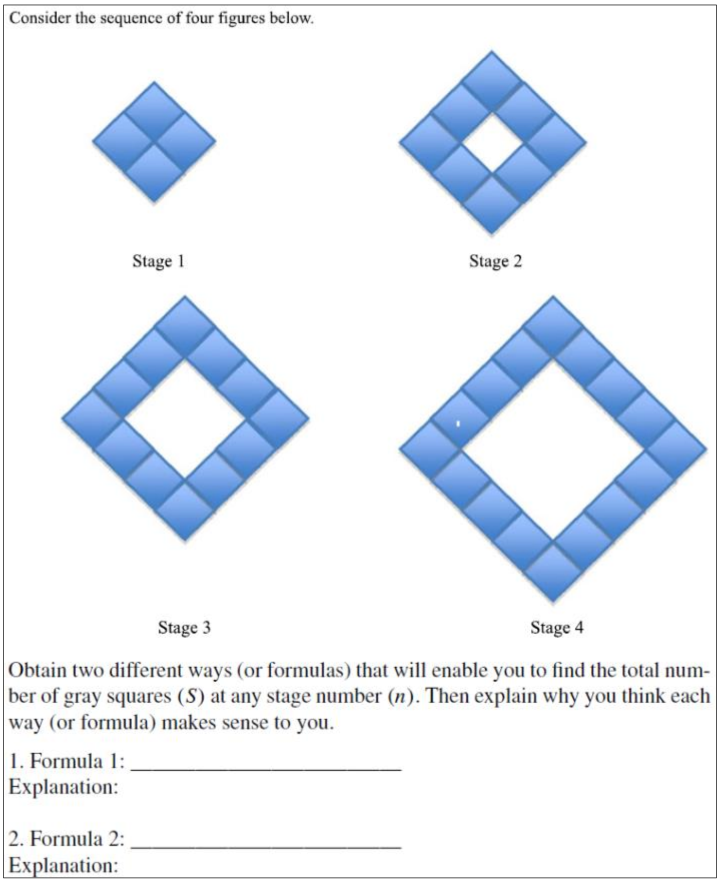


Figure 19. The original version of sixth question (Rivera & Becker, 2011, p. 339)

In order to use the question in the present study, figural pattern was transformed into numeric pattern in order to see students' generalization processes at linear-numeric patterns. Then, new items were added related to near and far terms to observe students' different generalization processes in both near and far terms. Lastly, the expression *at any stage number (n)* was removed not to limit students with *letter* use. The adapted version of the question was as follows:

4, 8, 12, …, …, 24, …, …

a. What would be the fourth term in the sequence?

b. What would be the fifth term in the sequence?

c. What would be the tenth term in the sequence?

d. What would be the hundredth term in the sequence?

e. How can you express general term in the sequence?

Figure 20. The adapted version of sixth question

3.3.2.2. The interviews.

According to Patton (2002), qualitative interviews can be conducted in three ways that are the standardized open-ended interview, the general interview guide approach, and the informal conversational interview. In the standardized open-ended interview approach, the predetermined questions are asked to the interviewee in the same manner by protecting the question order (Patton, 2002). The general interview guide approach is less structured than the standardized open-ended interview (Turner, 2010). In this approach, the interviewer can change the order of the questions according to the interviewee's responses (Turner, 2010). There is a list of topics to be asked and the interviewer is free to explore those topics in a limited time based on the information taken from the interviewee. The informal conversational interview, i.e. unstructured interview, contains no predetermined questions. Interview questions are generated according to the interviewee and flow of the talk (McNamara, 2009). While its advantages include flexibility and deeper communication, necessity of a lot of time and an experienced interviewer is the disadvantage of the informal conversational method. Patton (2002) mentioned that it is possible to combine these three approaches as interviewing in a qualitative study. In this study, I selected the general interview guide approach (see Figure 21 for the General Interview Guide) since it enables to change the order of the questions depending on the interviewee's responses.

The General Interview Guide for Patterning Test

How did the interviewee describe the pattern?

- (for figural) shape
- (for all) numeric relationship

In what ways did the interviewee generalize the pattern?

- Near generalization (4th term, 5th term, 10th term, etc.)
- Far generalization (100th term, 1000th term, etc.)

(If the interviewee used trial-and-error strategy) What did the interviewee consider during the trial and error?

How did the interviewee reach the general rule of the pattern?

- Examples
- Counter-examples

How did the interviewee express the general rule of the pattern?

- Letters
- Natural language (Speech, writing language)

Figure 21. The General Interview Guide for the Patterning Test

3.3.3. Pilot study.

The pilot study of the present study was conducted during the spring semester of 2015-2016 academic year. The purposes of the pilot study were to determine the suitableness of the pattern questions in the Pattern Test, to review the interview protocol, and to predict the necessary time for completing the Pattern Test.

The first version of the Pattern Test included twelve pattern questions in three categories: linear-numeric, linear-figural, and linear-verbal (word problem). There were four numeric linear questions, four figural linear questions, and four verbal linear questions. Each question had three items for near terms, two items for far terms, and one item for the general term of the pattern. The first version of the instrument was piloted in a public school in Çankaya, Ankara with 25 seventh grade students. The participants of the pilot study were selected according to the convenient time and location. They were asked to answer all the questions in the Pattern Test. Additionally, they were asked to put a question mark in case they did not understand something in the Pattern Test. After completing the Pattern Test, three volunteer students out of 25 students were selected for the pilot study of the interview protocol. Predetermined interview questions were asked to them.

Students completed the Pattern Test in approximately 90 minutes. A few students left the Pattern Test unsolved. Some students mentioned that too many items (items (a), (b),(c), (d), (e), and (f)) related to each question as well as plenty of the same-kind of questions bored them. For instance, here is a student's reaction to the Pattern Test:

> It asks for a hundredth step of the pattern…Again…I had already explained in the previous item…I really got bored…As I said before, the general rule of the pattern is three times the term number and plus two. Thus, the number on a hundredth step is 302.

After the pilot study of the first version of the instrument, the results of the pilot study were discussed with two experts from the field. In the light of their recommendations, some changes were made on the instrument such as (i) the number of questions were decreased in order to reduce the necessary time to complete the Pattern Test, (ii) linear-verbal pattern questions were removed from the Pattern Test since they gave similar results with linear-numeric patterns, (iii) six items were decreased to four or five items not to discourage participants, and (iv) 'Explain your answer' part which was at the end of each item was deleted for near term items. Thus, the predetermined version was changed as seen in Table 2.

Table 2. Predetermined and revised version of the pattern tasks after the first pilot study.

Predetermined Version	**Revised Version**
four linear-numeric questions	three linear-numeric questions
four linear-figural questions	three linear-figural questions
four linear-verbal questions	

The revised version of the instrument was piloted again with 15 volunteer sixth grade students from a public school in Çankaya, Ankara. It included six questions that were three linear-numeric and three linear-figural. Students completed the revised instrument in 40-50 minutes. After the second pilot study, the final version of the Patterning Test was constructed. Then a further interview was conducted with one selected student among 15 sixth grade students. During the pilot study of the interview protocol, the researcher aimed to determine whether there were any unnecessary, unreasonable or confusing questions. The final version of the interview protocol was prepared after the feedbacks from experts and from the pilot study. After the pilot study, one question was removed from the interview protocol since students mentioned the question as confusing and gave irrelevant answers. The removed question was '(If the interviewee use trial-and-error strategy) What did the interviewee consider during trial and error?' Furthermore, the *counter-example* expression was removed from the interview protocol since it leaded students to unclear answers.

3.3.4. Data collection process.

The data were collected during the spring semester of 2015-2016 academic year, after official permissions were obtained from Middle East Technical University Human Subjects Ethics Committee (see Appendix A) and Ministry of Education (see Appendix B). After informing the administration of the school, the researcher selected the participants of the main study among the volunteer students. Each participant was asked to sign Informed Consent Form and Parental Approval Form.

The data of the study were collected through task-based interviews. The researcher interviewed with one participant at one time while s/he was answering the questions in the Pattern Test. The data were collected during the school hours. For example, while a participant was solving the first question in the Pattern Test, the researcher asked about how she could describe the given pattern. Then, when s/he found the near and far terms of the given pattern, the researcher asked about how the participant generalized the pattern to those terms. All in all, the researcher asked the interviewing questions from the Interview protocol for each of the questions in the Pattern Test, as the participant was solving them. The implementation of the Pattern Test and the interview took almost one hour. Interviews were conducted one by one at a private room in the school. The physical setting of the room was arranged accordingly in order to prevent any interruption. Additionally, to make the participant comfortable, the researcher reminded the confidentiality of the identities and the voluntary participation at the beginning of each interview session. All the interviews were audiotaped and videotaped.

3.4. Data Analysis

Qualitative studies include huge amounts of data and to make sense of the data is the most difficult part of the process, which is data analysis. Through qualitative data analysis, the data are transformed into findings by "preparing and organizing the data for analysis, reducing the data into themes through a process of coding and condensing the codes, and finally representing the data in figures, tables or discussion" (Creswell, 2007, p. 148). According to Patton (2002), there is no formula or recipe, but just guidance.

The data in qualitative research can be analyzed inductively or deductively. The difference between them lies at existing of a framework. In inductive analysis, patterns, themes, and codes were explored through qualitative data without an existing framework which is called as open-coding (Patton, 2002). In contrast, deductive analysis includes an existing framework and hypothesis related to relationship between concepts. In this study, data were analysed with a deductive approach.

In this study, data analysis was conducted in order to examine middle school students' pattern-generalization process with regard to arithmetical generalization, algebraic generalization, and naïve induction. In order to analyze data, deductive approach was employed. Main categories were determined and defined up-front which were naïve induction, arithmetic generalization, and algebraic generalization, taken from Theory of Knowledge Objectification proposed by Radford (2000). Pre-determined categories of data analysis process were presented in Figure 22 (see in detail in Theoretical Framework part). Coding system for the three categories were explained, respectively.

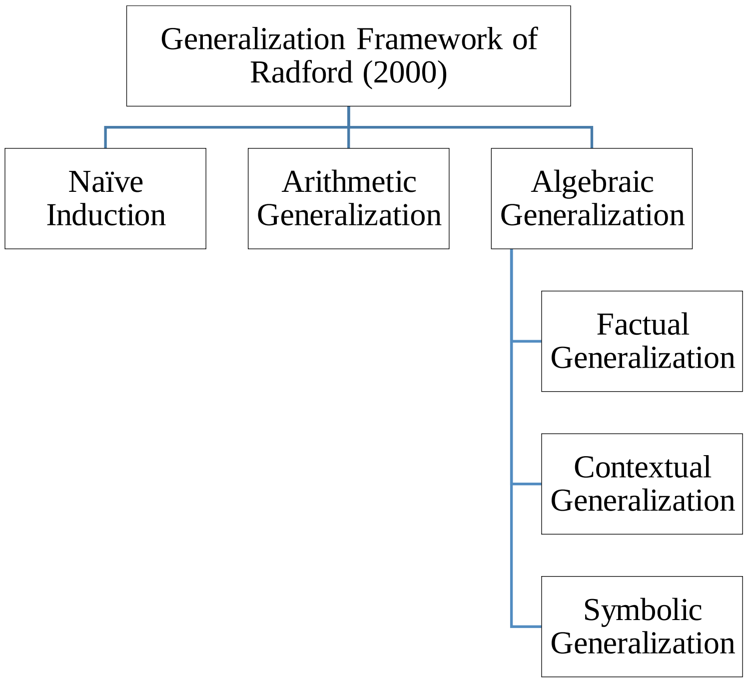


Figure 22. Generalization Framework of Radford

The coding system for the three categories of Radford's generalization framework were explained in detail based on the pattern example given in Figure 23.

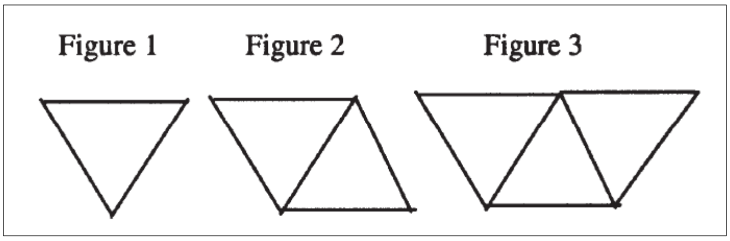


Figure 23. The toothpick pattern from Radford (2003, p. 45)

Students' answers were coded as *naïve induction* if trial and error strategy was employed during pattern generalization. Radford (2010a), as mentioned in Chapter 2, made a distinction between generalization and naïve induction. Naïve induction includes trial and error strategy. Students try simple rules related to given pattern such as *'step number times 3 plus 2' or 'step number times 3'* and check it on particular steps of the pattern. According to Radford (2010a), they just make inductions through hypothesis, not a generalization. Thus, responses of students were coded as *naïve induction* if there is trial and error. Below is an example of students' answers, which could be categorized under naïve induction for the toothpick pattern in Figure 23:

> How can I find the general rule of this pattern? If I multiply term number with 3, it works for the first term (1 times 3 equals 3 toothpicks), but it does not work for the second term (2 times 3 equals 6 toothpicks). Second term has 5 toothpicks. Then, I will try to multiply term number with 2 and add 1. It works for the first term (1 times 2 plus 1 is 3 toothpicks); it also works for second and third terms (2 times 2 plus 1 is 5 toothpicks and 3 times 2 plus 1 is 7 toothpicks). Yes, that is the general rule. Multiply with 2 and add 1.

Students' answers were coded as *arithmetical generalization* if students explored the recursive relationship between consecutive terms of the pattern by focusing on the common difference. For example, a student's answer '*this pattern always continues* *by adding two to the previous term.'* for the pattern in Figure 23 could be categorized under arithmetical generalization.

Algebraic generalization is distinguished from arithmetic generalization includes three sub-categories that are factual generalization, contextual generalization, and symbolic generalization.

Students' answers were coded as *factual generalization* if students found the particular terms of the pattern within a concrete formula. Factual generalization, as explained in Chapter 2, is the most concrete form of algebraic generalization. A rule, which is formed with factual generalization, does not include any abstract terms. Furthermore, it is applied to only particular terms. In the following, there are some examples of factual generalization for the pattern in Figure 23:

> Look! First term has 3 toothpicks, I can say it is 1 plus 2. Second term has 5 toothpicks, which is 2 plus 3. Third term has 7 toothpicks, which is 3 plus 4. Then, 10th term will have 10 plus 11 toothpicks. 100th term will have 100 plus 101 toothpicks.

As seen from the example, student noticed a concrete rule on the first three terms of the pattern as 1 plus 2 for the first term, 2 plus 3 for the second term, and 3 plus 4 for the third term. Then, she found 10th and 100th terms with this rule. As seen from the example, student did not mention any abstract term during the factual generalization. Only particular terms were included in the process of factual generalization.

Students' answers were coded as c*ontextual generalization* if students used abstract cooncepts such as 'number of the term' or 'the figure' and conducted mathematical operations on those abstract concepts such as the expression '*doubling* the number of the figure and *plus* three'. As an example of contextual generalization of the pattern given in Figure 23:

> Therefore, it can be said that the general rule of this pattern is the number of the term plus the number of the next term.

Students' answers were coded as *symbolic generalization* if they expressed the generalization with alphanumeric symbols just as the expression of '2n+3' (Radford et al., 2007). It is one-step further than *contextual generalization.* In this type of generalization, the letter 'n' is the replacement of 'the number of the figure' (Radford, 2010b). In both *contextual* and *symbolic* generalizations, students reach an explicit expression, which is about the relationship between term and position of the term [number of the term]. Symbolic generalization allows reducing the form of contextual generalization through alphanumeric symbols (Radford et al., 2007). As an example of symbolic generalization of the pattern given in Figure 23:

> I can write the general rule of the pattern as: N+N+1.

First step of the analysis was to determine a manageable classification system or codebook as seen above. Secondly, all the audio-taped and video-taped interviews were transcribed in order to prepare for content analysis. Content analysis is a technique to analyze texts, especially verbatim transcripts which includes identifying, coding, categorizing, classifying, and labeling the primary patterns in the data (Patton, 2002, p. 463). In this study, content analysis began by reading all field notes and verbatim transcripts over and over again. Then, with the help of colored markers and post-its, primary coding of generalization types was conducted with regard to predetermined codebook. If a new category was observed in the data more than a few times, sensitizing concept was formed to orient new data.

The transcribed data is first coded by the researcher. In order to ensure dependability, transcription was coded by another coder, a mathematics teacher, with given coding sheet. After both the researcher and the teacher coded the same sample (the transcribed data of 5 students' answers to all questions among 14 students), the results were compared. The interrater reliability was 83% for the initial coding. After discussion, the researcher and the second coder had an overall inter-rater reliability of 90%. As a last step of data analysis, the transcript of data is analyzed in terms of categories of Radford's generalization framework, which are arithmetical generalization, algebraic generalization (including sub-levels: factual generalization, contextual generalization, and symbolic generalization), and naïve induction.

3.5. Trustworthiness

3.5.1. Credibility.

Lincoln and Guba (1985) used the term *credibility* to describe internal validity. Credibility in qualitative research is about the correspondence of research results with reality (Merriam, 2009). To ensure credibility, Merriam (2009) mentioned five strategies which are triangulation, member checks, adequate engagement in data collection, researcher's role/reflexivity, and peer review. In this study, credibility was provided by applying *researcher's role/reflexivity* and *peer review*.

The first strategy was *researcher's role/reflexivity* which means that "the process of reflecting critically on the self as researcher, the 'human as instrument' " (Lincoln & Guba, 2000, p. 183). The researcher of the present study explained her role clearly with all biases and assumptions about the study. The detailed information about the researcher's role is explained below. The second strategy was *peer review* which is a way of the external check of the research process as the same as inter-rater reliability in quantitative research (Creswell, 2007). The role of peer reviewer is described as an individual that keeps the researcher honest, asks difficult questions about methodology and interpreting the data (Lincoln & Guba, 1985). To ensure peer review, one of my colleagues from mathematics education, who is a doctoral student in the Mathematics and Science Education department, participated during the study. She reviewed the study through the preparation of interview protocol, collection of data, and analysis of data. Additionally, the feedbacks of my advisor and co-advisor were regarded carefully.

3.5.2. Dependability/Consistency.

The second concern of the trustworthiness in qualitative research is to ensure reliability. Lincoln and Guba (1985) used the term *dependability/consistency* to describe the reliability in qualitative research. Reliability generally refers to stability of responses to multiple coders of data sets in qualitative research which was called inter-rater reliability (Creswell, 2009). In this study, reliability was established through inter-rater reliability.

After transcribing interviews, the researcher coded the transcription and formed a coding sheet which explains the codes. Then, transcription was coded by another coder, a mathematics teacher, with the given coding sheet. After both the researcher and the teacher coded the same sample, the results were compared. The researcher and the second coder had an overall inter-rater reliability of 90%. Lastly, the final version of coded transcripts and agreement on codes and categories were established.

3.5.3. Transferability.

The third concern of the trustworthiness in qualitative research is to ensure external validity. External validity is about the generalizability of the results from sample to the population; yet, qualitative studies do not have concerns related to statistical generalizability (Merriam, 2009). Therefore, Lincoln and Guba (1985) used an equivalent term ‘transferability’ in qualitative study rather than external validity in quantitative study. To ensure transferability, the researcher should give detailed description of the study (Merriam, 2009). Lincoln and Guba (1985) called this strategy thick description. Since the researcher describes participants and settings of the study in detail, researcher enables readers whether to transfer information to other settings due to the shared characteristics (Creswell, 2007). In this study, the researcher tries to provide rich and thick descriptions of the cases and the findings in order to communicate the findings effectively.

3.6. Role of the Researcher and Biases

During the study, the researcher was working as a mathematics teacher at the public school in which data was collected. There were only two mathematics teachers in the school. Thus, all students in the school knew the researcher from mathematics lessons. Moreover, the researcher had a good communication with each student in the school. This situation was for the benefit of the researcher in several ways. Firstly, when the researcher mentioned the purpose of the study, most of the students volunteered to participate in the study. Secondly, since the researcher knew students in person, she could select the information-rich participants among volunteer students. Also, due to the good communication with participants, participants felt comfortable expressing themselves during data collection. Nevertheless, there was a risky situation related to the effects of teacher-student relationship. They might either think their participation compulsory, or fear low grades in the class. To prevent it, the researcher underlined the voluntary participation repeatedly after expressing the data collection process in detail. Additionally, at the beginning of each interview session, the participants were explained that they could express any of their ideas without fear of wrong answers. Researcher also gave the guarantee of not grading participants according to the interviews.

As mentioned before, the data was collected in a way that researcher interviewed one participant at one time while s/he was answering the questions on the data collection tool. At the beginning of each session, the participants were asked to express their answers and thoughts loudly all the time. Additionally, the duration of each interview was identified according to the time students needed to complete the test. Lastly, orienting the interviewee through welcomed answers by thanking, by shaking the head or by confirmative sounds is among general mistakes that researcher makes during the interview (Patton, 2002). To overcome it, the interviewer paid great attention not to use words of thanks or confirmation, or bodily gestures which give clue in the Pattern Test. The interviewer said phrases like *'I see'* or *'I understand'* after interviewee's responses. If interviewer observes interviewee's working hard for any question or activity, supportive sentences were used such as *'I am aware that was a challenging question and you worked really hard'*. When the interviewer needed more in-depth responses, phrases like '*Could you explain what you meant in detail?'* or *'Can you explain your work here?'* were used. In addition, if the interviewee is in struggle for a long time, interviewee says '*I see you struggling around. Do you think there can be an alternative way of this question?'*. Lastly, when the interviewee keeps silent for a long time, interviewer breaks the silence with phrases like *'What do you think?'* or *'Do you have anything in your mind?'*.

3.7. Limitations

There were some limitations and possible biases in this study. Firstly, inexperience of researcher was one of the limitations of this study. Patton (2002) states that while experienced, well-trained observer increases credibility of the inquiry, there are doubts in the report of inexperienced observers. I had no experience of interviewing or qualitative study, thus to reduce this limitation, I have worked with my advisor and co-advisor at all stages of data collection and a second observer watched all the video-records and checked transcriptions of the video-records. Additionally, this study explored students' algebraic reasoning only with researcher's pattern questions. Asking different questions could give different results and enable to make different interpretations of algebraic reasoning.

3.8. Ethics

In order to ensure ethical issues, firstly, all participants' identities in this study are protected and extra care is given to ensure that none of the information collected would embarrass or harm them. Secondly, participants are treated with respect during the research. They are not lied or audio/video taped without their permission. They are informed about the research and interview process. Furthermore, participants are ensured that they can quit any time they wanted. Lastly, it is ensured that no physical or psychological harm will come to participants.

CHAPTER 4

FINDINGS

The aim of this study is to explore the sixth, seventh, and eighth grade students' generalizations of patterns using arithmetical generalization, algebraic generalization, and naïve induction. In addition to studying their generalization process, the study focuses also on the ways in which this process of generalization differs according to their grade level. Research question of the study was given in the following:

- How do sixth, seventh, and eighth grade students generalize linear patterns using arithmetical generalization, algebraic generalization, and naïve induction?
- To what extent do these generalizations differ in terms of their grade level?

In the guidance of the research question, which was given above, this chapter was organized based on the sixth, seventh, and eighth grade students' generalization of linear patterns arithmetically, algebraically or in terms of naïve induction.

4.1. Sixth, Seventh, and Eighth Grade Students' Generalization of Linear Patterns

As expressed above, the present study aimed to reveal sixth, seventh, and eighth grade students' generalization approaches, which included one or more than one generalization strategies such as arithmetical generalization, algebraic generalization, and naïve induction. In this study, 6 linear patterning tasks were analyzed for 5 sixth grade students, 4 seventh grade students, and 5 eighth grade students. The analysis of the students' answers revealed that sixth, seventh, and eighth grade students mostly used at least two generalization strategies (arithmetical generalization, algebraic generalization or naïve induction) on the 6 patterning tasks. These students were grouped according to the sets of strategies they used within the process of generalization. All in all, the analysis of the students' answers revealed four generalization approaches which are (i) algebraic generalization strategies only, (ii) the combination of arithmetical generalization and algebraic generalization strategies, (iii) the combination of arithmetical generalization and naïve induction strategies, (iv) the combination of arithmetical generalization, algebraic generalization, and naïve induction strategies.

4.1.1. Sixth grade students' generalization of linear patterns.

Table 3 represents the generalization approaches the sixth grade students used during the generalization of patterns. According to Table 3, the combination of arithmetical generalization and algebraic generalization was the most frequent generalization approach used by the sixth graders. The second and third mostly used generalization approaches were algebraic generalization only and the combination of arithmetical generalization and naïve induction, respectively.

Table 3. Sixth grade students' generalization approaches including the variation of strategies used

	ALG only	AG and ALG	AG and I	AG, ALG, and I
Q1	P1 P3	P2 P4 P5		
Q2	P1	P4 P5	P2 P3	
Q3		P2 P4 P5	P1 P3	
Q4	P1 P3	P4 P5	P2	
Q5	P3	P1 P5	P2	P4
Q6	P1 P3	P4 P5	P2	

Notation: P, participant; ALG, algebraic generalization; AG, arithmetical generalization; I, naïve induction

4.1.1.1. Generalization approach including algebraic generalization only.

The generalization approach including only algebraic generalization strategies was the second most frequently used one among the sixth grade students. It emerged eight times in the present study. The detailed generalization processes were represented in Table 4.

Table 4. Sixth grade students' generalization approach including algebraic generalization only

# of Questions	Algebraic Generalization Strategies
Q1	P1 contextual generalization [general rule] *(calculates near and far terms by applying the general rule)* symbolic generalization [general rule]
	P3 contextual generalization [general rule] *(calculates near and far terms by applying the general rule)* symbolic generalization [general rule]
Q2	P1 contextual generalization [general rule] *(calculates near and far terms by applying the general rule)* symbolic generalization [general rule]
Q3	-
Q4	P1 contextual generalizaton [general rule] *(calculates near and far terms by applying the general rule)* symbolic generalization [general rule]
	P3 contextual generalization [general rule] *(calculates near and far terms by applying the general rule)* symbolic generalization [general term]
Q5	P3 symbolic generalization [general rule] *(calculates near and far terms by applying the general rule)*
Q6	P1 factual generalization contextual generalization [general rule] symbolic generalization [general rule] *(calculates near and far terms by applying the general rule)*
	P3 factual generalization [near & far term] symbolic generalization [general rule]

Notation: P, participant

As indicated in Table 4, two students in first, fourth, and sixth questions and one student in second and fifth questions generalized linear patterns using only the algebraic generalization strategies. As an example for the fourth question,

> Q4/P3: The general rule of this pattern is the number of the next step times 6.. As an instance, in order to find the number in the first step, I multiply 2 by 6, which is 12. In order to find the number in the second step, I multiply 3 by 6, which is 18. 4 times 6 is 24, which is the number in the third step. (Student writes 36 for the fifth step, 66 for the tenth step. Then, he multiplies 51 by 6 and writes 306 for the fiftieth step. At last, he writes n+1.6 for the general rule.)

As seen from the example, P3 started the generalization process by forming the general rule of the pattern by mentioning an abstract word ‘the step number’, which was coded as *contextual generalization*. After expression of the general rule, he exemplified the general rule for the calculation of the numbers in the second, third, and fifth steps. Then, he calculated the numbers in 10th and 50th steps with the general rule he found. At last, he wrote the symbolic form of the general rule, which was coded as *symbolic generalization.* Below is another example of the generalization process from sixth question including only algebraic generalization strategies:

> Q6/P1: (Student writes the number of the terms below the terms.) In this question, 1 times 4 is 4, which is the number in the first step. 2 times 4 is 8, which is the number in the second step. 3 times 4 is 12, which is the number in the third step. Therefore, the rule is the step number times 4. So, I write n times 4 (student writes n4).. Then, the number in the 10th step will be 40. The number in the 20th step will be 80. The number in the 100th step will be 400.

As the above example showed, P1 firstly generalized the pattern to first, second, and third steps by forming a numeric rule of ‘multiplying with 4’, which was coded as *factual generalization*. Then, he expressed the general rule by mentioning ‘step number’, which was coded as *contextual generalization.* At the end, he wrote the general rule with *symbolic generalization.*

4.1.1.2. Generalization approach including the combination of arithmetical generalization and algebraic generalization.

The sixth grade students' generalization processes most frequently included both arithmetical generalization and algebraic generalization. The combination of arithmetical generalization and algebraic generalization emerged 14 times in the present study. The detailed generalization processes were represented in Table 5.

Table 5. Sixth grade students' generalization approach including the combination of arithmetical generalization and algebraic generalization

# of Questions	The Combination of Arithmetical Generalization and Algebraic Generalization Strategies
Q1	P2 arithmetical generalization contextual generalization [general rule] *(calculates near and far terms by applying the general rule)*
	P4 arithmetical generalization [near term] factual generalization [near & far term] symbolic generalization [general rule]
	P5 arithmetical generalization [near term] symbolic generalization [general rule] *(calculates near and far terms by applying the general rule)*
Q2	P4 arithmetical generalization [near term] symbolic generalization [general rule] *(calculates near and far terms by applying the general rule)*
	P5 arithmetical generalization symbolic generalization [general rule] *(calculates near and far terms by applying the general rule)*

Table 5 (continued)

# of Questions	The Combination of Arithmetical Generalization and Algebraic Generalization Strategies
Q3	P2 arithmetical generalization [near term] contextual generalization [general term] *(calculates far terms by applying the general rule)*
	P4 arithmetical generalization [near term] symbolic generalization [general term] *(calculates near and far terms by applying the general rule)*
	P5 arithmetical generalization symbolic generalization [general term] *(calculates near and far terms by applying the general rule)*
Q4	P4 arithmetical generalization symbolic generalization [general term] *(calculates near and far terms by applying the general rule)*
	P5 arithmetical generalization symbolic generalization [general term] *(calculates near and far terms by applying the general rule)*
Q5	P1 arithmetical generalization contextual generalization [general rule] symbolic generalization [general rule]] *(calculates near and far terms by applying the general rule)*
	P5 arithmetical generalization symbolic generalization [general term] *(calculates near and far terms by applying the general rule)*
Q6	P4 arithmetical generalization [near term] symbolic generalization [general term] *(calculates near and far terms by applying the general rule)*
	P5 arithmetical generalization factual generalization symbolic generalization [general term] *(calculates near and far terms by applying the general rule)*

Notation: P, participant

In detail, three students in the first and third questions, two students in the second, fourth, fifth, and sixth questions generalized linear patterns using the arithmetic and algebraic generalization models. Below is a related example from P1 in the fifth question:

> Q5/P1: There are 5 [matches] in the first step. There are 8 [matches in the second step] and 11 [matches in the third step]. Since it is increased by 3, it [the general rule] is the step number times 3 and plus 2 (student writes n3+2). When I multiplied 1 by 3 and added 2, it is 5 here [in the first step]. When I multiplied 2 by 3 and added 2, it is 8 [in the second step]. When I multiplied 3 by 3 and added 2, it is 11 [in the third step]. In the fourth step, 4 times 3 is 12 and 12 plus 2 is 14. In the fifth step, 5 times 3 is 15 and 15 plus 2 is 17.. So, [in order to calculate the tenth step] 10 times 3 is 30. 30 plus 2 is 32. In the 100th step, 100 times 3 is 300 and 300 plus 2 is 302.

Another example could be given from P4's answer to question 1:

> Q1/P4: In this question, it is 3, 6, and 9. It is increased by 3 in each step. So, I will add 3. So, this [fourth step] is 12. In the fifth step, it will be increased by 3 again, it [the fifth step] will be 15.
>
> Q1/R: How many circles are needed to reach the tenth step?
>
> Q1/P4: To reach the tenth step, I should multiply 10 by 3.
>
> Q1/R: Why?
>
> Q1/P4: Because it always increases by 3. Therefore, (student writes 10x3=30); 30 circles are needed. To reach the 50th step, I should multiply 50 by 3. Because, all of it increases by 3. It [the fiftieth step] is 150.
>
> Q1/R: How can you express the general rule of the pattern?
>
> Q1/P4: The general rule of the pattern can be expressed as 3n, because it is increased by 3 in each step.
>
> Q1/R: What is n?
>
> Q1/P4: n ... Rule ...
>
> Q1/R: For example, when you asked the tenth step, what did you put in place of n?
>
> Q1/P4: I put 10.

Q1/R: Can you give us an example of this general rule?

Q1/P4: For example, we have to do 2 times 3 to find the second step. n should be replaced by 2. So we find that the second step is 6 and if we want to find the third step, we should replace n with 3 and we can see that it's 9.

The above dialogue represents the whole generalization process of P4 at the first question. As seen from the example, P4 reached the number of circles in the fourth and fifth terms by adding the common difference, which is 3, to the previous terms. This was coded as *arithmetic generalization.* In order to reach the number of circles in the 10th term, she multiplied 10 by the common difference, just as she multiplied 50 by the common difference to calculate the 50th term. Since she formed a rule to find the number of circles in the 10th and 50th terms at the numerical level, this part of the generalization process was coded as *factual generalization,* which is the first sub-model of *algebraic generalization.* When she was asked about the general rule, she expressed it as 3n. Thus, she moved to *symbolic generalization*, which is the third sub-model of *algebraic generalization.*

4.1.1.3. Generalization approach including the combination of arithmetical generalization and naïve induction.

In the present study, the combination of arithmetical generalization and naïve induction was used 7 times within a generalization process. Two students in second and third questions and one student in fourth, fifth, and sixth questions used both the arithmetical generalization and naïve induction during pattern generalization. Table 6 shows the detailed structures.

Table 6. Sixth grade students' generalization approach including the combination of arithmetical generalization and naïve induction strategies

# of Questions	**The Combination of Arithmetical Generalization and Naïve Induction Strategies**
Q1	-
Q2	P2 arithmetical generalization [near & far term] naïve induction [general rule]
	P3 arithmetical generalization [near term] naïve induction [general rule] *(calculates near and far terms by applying the general rule)*
Q3	P1 arithmetical generalization [near term] naïve induction [general rule] *(calculates near and far terms by applying the general rule)*
	P3 arithmetical generalization [near term] naïve induction [general rule] *(calculates near and far terms by applying the general rule)*
Q4	P2 arithmetical generalization [near term] naïve induction [general rule] *(calculates far terms by applying the general rule)*
Q5	P2 arithmetical generalization [near term] naïve induction [general rule] *(calculates near and far terms by applying the general rule)*
Q6	P2 Arithmetical generalization [near & far term] Naïve induction [general rule]

Notation: P, participant

As seen in Table 6, each student firstly used arithmetical generalization to generalize the linear patterns to near and/or far terms. Then, they employed the naïve induction strategy to find the general rule. Below is an example from the fifth question;

> Q5/P2: There are 5 in the first step, 8 in the second step, and 11 in the third step. So it has increased by 3. The fourth step becomes 14 and the fifth step becomes 17.. Term number times 2? Not suitable for the second step. Term number times 3..plus 2? (For the first term) 1 times 3 plus 2 is 5. It worked. (For the second term) 2 times 3 plus 2 is 8. Yes, the general rule is term number times 3 plus 2'.

P2 reached the number of toothpicks in the fourth and fifth terms by adding 3 to the previous terms that was coded as arithmetical generalization. Then, he used the naïve induction strategy while trying some rules for finding the general rule as it can be seen in the example.

4.1.1.4. Generalization approach including the combination of arithmetical generalization, algebraic generalization, and naïve induction.

In the current study, the combination of arithmetical generalization, algebraic generalization, and naïve induction strategies was the least frequent generalization approach that the sixth grade students used. This combination emerged once. One student in the fifth question used all three strategies during the pattern generalization process. The detailed structure was represented in Table 7.

Table 7. Sixth grade students' generalization approach including the combination of arithmetical generalization, algebraic generalization, and naïve induction strategies

# of Questions	**The Combination of Arithmetical Generalization, Algebraic Generalization, and Naïve Induction Strategies**
Q1	-
Q2	-
Q3	-
Q4	-

Table 7 (continued)

Q5	P4 arithmetical generalization [near term] naïve induction symbolic generalization [general rule] *(calculates near and far terms by applying the general rule)*
Q6	-

Notation: P, participant

Here is an example from question 5,

> Q5/P4: There are 5 toothpicks (shows the first step). There are 8 toothpicks (shows the second step) is 8. There are 11 toothpicks (shows the third step). The difference between them is 3. They increase by 3. Then, it's supposed to be 14 in the fourth step. Let's show it by drawing (draws toothpicks). Already, I have found the answer as 14. Let's check the answer now (counts the toothpicks he draws). There are 14 toothpicks here, so I found it right. In order to build the fifth step, we need to draw 6 rows of toothpicks (draws right below) because they build more than one toothpick per step. In the fifth step (counts the toothpicks he drew), 17 toothpicks were needed. We need to check our rule again. 11 plus 3 was 14. 14 plus 3 is 17. We found it right. In the sixth step, there are 20 toothpicks. 17 plus 3, 20 is correct.
>
> To build the tenth step, we need to do 11 times 11 as it builds one more toothpick from step number in each step. Therefore, we need to multiply 11 by 11 and find 111. To find the hundredth step, we need to multiply 101 by 101. 1101.
>
> R: Have you reached any general rule?
>
> P4: Yeah. I'll write the rule now. N+1 times N+1. For example, we want to find the seventh step. We replace N with 7. Then we should calculate 7 plus 1 is 8. 8 times 8, 64. It didn't work.. However, I know these are true (shows part a, b, c, and d)… 3N plus 1? (writes the general rule as 3N+1 times 3N+1). I will calculate the sixth step again. 6 times 3, 18 plus 1, 19. 19 times 19 does not work again.
>
> R: How many toothpicks do you think are in the tenth step?
>
> P4: 111.. Let me figure out how many toothpicks are in the seventh step (draws the seventh step). 23 toothpicks need to be in the seventh step. As we found, it continues to increase by 3. Then our rule should include 3N. Yes, our rule should be 3N+2. Let me give an example. The first step 3 times 1 plus 2, 5. Correct. In the second step, 3 times 2 plus 2, 8. Correct. In the third step 3

> times 3, 9 plus 2, 11. Correct. In the fourth step, 4 times 3, 12 plus 2, 14 is Correct. Therefore, that is our rule. The tenth step must be 3 times 10 plus 2 according to the rule 3N+2 so it is 32. Then to find 100th step, you have to have 3 times 100 plus 2. Then it is 302.

As seen from the above example, P4 found the number of toothpicks in the fourth, fifth, and sixth steps by counting the toothpicks in her drawing and by adding the constant difference onto the previous terms, which was coded as arithmetical generalization. Then, she erroneously calculated the number of toothpicks in the 10th and 100th steps by forming a rule '(n+1).(n+1)'. By trial and error strategy, she tried two rules, which are (n+1) times (n+1) and (3n+1) times (3n+1); yet, she noticed that they were not working. So, this part of the generalization process was coded as naïve induction. After all, she noticed that the general rule should include 3n since the common difference is 3 and expressed the general rule as 3n+2. She ended her generalization process with symbolic generalization.

4.1.2. Seventh grade students' generalization of linear patterns.

Table 8 represents the generalization approaches the seventh grade students used during the generalization of patterns. According to the Table 8, the combination of arithmetical generalization and algebraic generalization was the most frequent approach the seventh grade students used. The second and third mostly used generalization approaches were the combination of arithmetical generalization, algebraic generalization, and naïve induction and the combination of arithmetical generalization and naïve induction, respectively. On the other hand, only algebraic generalization strategies were never used by the seventh grade students.

Table 8. Seventh grade students' generalization approaches including the variation of strategies used

	ALG only	**AG and ALG**	**AG and I**	**AG, ALG, and I**
Q1		P6 P7 P8 P9		
Q2		P7 P9		P6 P8
Q3		P7 P8		P6 P9
Q4		P6 P8 P9		P7
Q5		P7 P8		P6 P9
Q6		P7 P8 P9	P6	

Notation: P, participant; ALG, algebraic generalization; AG, arithmetical generalization; I, naïve induction

4.1.2.1. Generalization approach including the combination of arithmetical generalization and algebraic generalization.

Since the use of only algebraic generalization was not seen among the seventh grade students, the results about the combination of arithmetical generalization and algebraic generalization were represented directly. The seventh grade students' generalization processes most frequently included both arithmetical generalization and algebraic generalization. This combination emerged 16 times in the present study. The detailed generalization processes were represented in Table 9.

Table 9. Seventh grade students' generalization approach including the combination of arithmetical generaliation and algebraic generalization

# of Questions	The Combination of Arithmetical Generalization and Algebraic Generalization Strategies
Q1	P6 arithmetical generalization factual generalization [near & far term] *(student passed the question for some time)* arithmetical generalization [near term] symbolic generalization [general rule]
	P7 arithmetical generalization [near term] factual generalization [near & far term] *(student passed the question for some time)* symbolic generalization [general rule]
	P8 arithmetical generalization [near term] factual generalization [near & far term] contextual generalization [general rule]
	P9 arithmetical generalization [near term] factual generalization [far term] contextual generalization [general rule]
Q2	P7 arithmetical generalization [near term] factual generalization [far term] contextual generalization [general rule] *(student passed the question for some time)* symbolic generalization [general rule]
	P9 arithmetical generalization factual generalization [far term] contextual generalization [general rule]
Q3	P7 arithmetical generalization contextual generalization arithmetical generalization [near term] factual generalization [near term] contextual generalization [general rule] *(calculates far terms by applying the general rule)* symbolic generalization [general rule]

Table 9 (continued)

# of Questions	**The Combination of Arithmetical Generalization and Algebraic Generalization Strategies**
Q3	P8 arithmetical generalization [near term] contextual generalization [general rule] *(calculates far terms by applying the general rule)*
Q4	P6 arithmetical generalization [near term] contextual generalization [general rule] symbolic generalization [general rule] *(calculates near and far terms by applying the general rule)*
	P8 arithmetical generalization [near term] factual generalization [near term] contextual generalization [general rule] *(calculates far terms by applying the general rule)*
	P9 arithmetical generalization [near term] factual generalization [near term] contextual generalization [general rule] *(calculates far terms by applying the general rule)*
Q5	P7 arithmetical generalization [near term] factual generalization [near & far term] symbolic generalization [general rule]
	P8 arithmetical generalization [near term] contextual generalization [general rule] *(calculates far terms by applying the general rule)*
Q6	P7 arithmetical generalization [near term] symbolic generalization [general term] *(calculates near and far terms by applying the general rule)*
	P8 arithmetical generalization [near term] contextual generalization [general rule] *(calculates near and far terms by applying the general rule)*
	P9 arithmetical generalization [near term] factual generalization [far term] contextual generalization [general rule]

Notation: P, participant

In detail, one student in the first and fourth questions, one student in all questions except fourth question, another student in all questions except second one, and another student in the first, second, fourth, and sixth questions generalized linear patterns using the arithmetic and algebraic generalization models. Below is a related example from question 3:

> Q3/P8: There are 3 circles here (at the first step). There are 5 circles here (at the second step). There are 7 circles here (at the third step) . When 2 is added to 3, it is 5. When 2 is added to 5, it is 7. Now, I have to reach the sixth step. I can add 2 to 7 to find the fourth step, which is 9. When I add 2 to 9, it is 11 which is the fifth step. When I add 2 to 11, it is 13 and it is the sixth step. This pattern increases by 2 at each step.
>
> R: How can we find the number of circles in the tenth step?
>
> P8: We can multiply the step number with the remaining number. 2 times 4 is 8. 8 plus 1 is 9. For the fifth step 2x5=10. When 1 is added to 10, it is 11. In this sixth step, I multiply 6 with 2, 12. Then, I add 1, 13. Always, we should multiply by 2 and add 1.
>
> R: Can you explain the general rule you found to find the number in any step?
>
> P8: Since this pattern always increases by 2, I multiply 2 with the desired number of steps. Then I add 1. Because I saw that, I had to add 1 in the first step. I multiply 10 by 2 in the tenth step, 20, I add 1, 21. When we multiply 50 by 2 for the fiftieth step, 100. When we add 1 to 100, it is 101. The general rule in this question is to multiply the incrementing number by the step number and add 1.

P8's generalization process exemplified the movement from arithmetical genealization to algebraic generalization strategies. Firstly, he repetitively added 2 to the previous terms and found the number of circles in the fifth and sixth steps, which was coded as arithmetical generalization. Then, in order to reach the number of circles in the tenth step, he expressed the general rule of the pattern as 'multiplying step number with 2 and adding 1’. So, this part of the generalization process was coded as contextual generalization. There is another example from question 5 in the following:

> Q5/P7: Drawing: Each step increases by 3. 5 in the first step and 8 in the second step. Can I draw?
>
> R: Whatever you want.
>
> P7: (Draws an arrow towards fourth step) I draw an arrow, so I would not draw it again. I added three on it. There are 11 in the third step. There are 14 in the fourth step. Fifth step is 17.
>
> R: How did you find 17?
>
> P7: Adding 3 at each time.
>
> R: What number is there in the tenth step?
>
> P7: Tenth step is 32.
>
> R: How did you find 32?
>
> P7: If the first step was 3, then it'd be 30. However, the first step is 5 which is 2 more than 3. If I started at 3, it was 30; I added 2, because the first step is 5. Therefore, it is 32.
>
> R: Which number is included in the hundredth step?
>
> P7: 302.
>
> R: How can the general rule be expressed?
>
> P7: n times 3 plus 2.
>
> R: How did you get this rule?
>
> P7: In the second step, the step number is 2, 2 times 3, 6; plus 2, 8. In the second step there are 8 bars. (Student writes (n.3) +2) In the third step 3 times 3, 9; plus 2, 11. Also, there is 11 bars really.

As illustrated in the example, P7 found the number of bars in the fifth step by adding 3 to the number of bars in the fourth step with arithmetical generalization. When he was asked about the tenth term, he developed a rule by assuming that the pattern began with 3; and he also calculated the hundredth term in this way, which was an example of the factual generalization. At last, he expressed the general rule of the pattern with letters as n times 3 plus 2. P6's generalization process of the pattern in question 4 is also an example of this trend:

> Q4/P6: First, let me write term numbers on top of the terms. Now, how much it is increased in each step? It is increased by six. It is always six. 36 comes after 30. It is always increased by six.
>
> I probably found the general rule. If I say say n to step number, then it is nx6 + 6.
>
> R: How did you find?
>
> P6: Firstly, I thought about that. For example, I multiplied by 6 and then I added 6.
>
> R: Have you tried for the others?
>
> P6: Yes, I tried. For example, the fifth step is 36. I will try in the tenth step; it is 10x6 + 6 = 66. I think it is true. 50x6 + 6 = 306 in the fiftieth step. This is the general rule.

P6 used arithmetical generalization and found the number in the fifth step as 36. Then, she expressed the general rule of the pattern as n times 6 plus 6 with symbolic generalization and calculated the numbers in the 50_{th} and 100_{th} steps by applying this general rule.

4.1.2.2. Generalization approach including the combination of arithmetical generalization and naïve induction.

In the current study, the combination of arithmetical generalization and naïve induction strategies was the least frequent generalization approach that the seventh grade students used. This combination emerged twice. One student in the fifth question and one student in the sixth question used this combination of strategies during the pattern generalization process. The detailed structure was represented in Table 10.

Table 10. Seventh grade students' generalization approach including the combination of arithmetical generalization and naïve induction strategies

# of Questions	**The Combination of Arithmetical Generalization and Naïve Induction Strategies**
Q1	-
Q2	-
Q3	-
Q4	-
Q5	P9 arithmetical generalization [near term] naïve induction
Q6	P6 arithmetical generalization [near term] naïve induction [general rule] *(calculates near and far terms by applying the general rule)*

Notation: P, participant

As seen in Table 10, the student firstly used arithmetical generalization to generalize the linear patterns to near terms. Then, she employed the naïve induction strategy to find the general rule. As an example;

> Q6/P6: I'm writing the steps again. 1, 2, ..., 8. Now, it increases by 4 in each step. This is 16 (points the fourth step). This is 20 (points the fifth step), This is 28 (points the seventh step). This is 32 (points the eighth step). Fifth step is 20. Fourth step is 16. Now, I'm going to find the rule. Again, I say n to the step number.. I think, it was nxn + 3.
>
> R: How did you get that rule?
>
> P6: In the first step. It is 1x1 + 3 = 4…. I'm thinking of another rule right now. Oh, I found it. It is n times 4. That is so simple.
>
> R: Why is that? Can you explain?

> P6: When I write the differences between each step, it can be found immediately. I did so. I tried, for example. When I made n times 4 in the first step, it turned out right. I thought I would try it because it is always 4. It went right up to the eighth step. It is probably true. There are 10x4 = 40 in the tenth step. There are 20x4 = 80 in the twentieth step. There are 400 in the hundredth step.

As exemplified in above example, P6 reached the numbers in the fifth, sixth, seventh, and eighth terms by adding 3 to the previous terms that was coded as arithmetical generalization. Then, she used the naïve induction strategy and tried some rules for finding the general rule such as n times n plus 3 and n times 4. When she noticed that the rule ‘n times 4’ works, she used it to calculate the numbers in the tenth, twentieth and hundredth steps.

4.1.2.3. Generalization approach including the combination of arithmetical generalization, algebraic generalization, and naïve induction.

In the present study, the combination of arithmetical generalization, algebraic generalization, and naïve induction was used 6 times within a generalization process. One student in the second, third, and fifth questions, another student in the third and fourth questions, and one student in the second question used all three strategies during pattern generalization. Table 11 shows the detailed structures.

Table 11. Seventh grade students' generalization approach including the combination of arithmetical generalization, algebraic generalization, and naïve induction strategies

# of Questions	The Combination of Arithmetical Generalization, Algebraic Generalization, and Naïve Induction Strategies
Q1	-

Table 11 (continued)

# of Questions	The Combination of Arithmetical Generalization, Algebraic Generalization, and Naïve Induction Strategies
Q2	P6 arithmetical generalization [near term] naïve induction *(student passed the question for some time)* symbolic generalization [general rule] *(calculates far terms by applying the general rule)*
	P8 arithmetical generalization [near term] naïve induction factual generalization contextual generalization [general rule] *(calculates far terms by applying the general rule)*
Q3	P6 naïve induction arithmetical generalization contextual generalization [general rule] symbolic generalization [general rule] *(calculates near and far terms by applying the general rule)*
	P9 arithmetical generalization [near term] factual generalization [near term] naïve induction [general rule] *(calculates far terms by applying the general rule)*
Q4	P9 arithmetical generalization [near term] factual generalization [near and far term] naïve induction [general rule]
Q5	P6 contextual generalization [general rule] naïve induction arithmetical generalization [near term] naïve induction [general rule] *(calculates near and far terms by applying the general rule)*
Q6	-

Notation: P, participant

Here is a related example from question 3:

> Q3/P6: Just like we solved in the previous question. So, (n + 1) times (n + 1). n is the step number again. I will try again. It did not work in this question. I guess, I found it; (N + 1) x2.
>
> R: How did you get it?
>
> P6: I counted both here and there, therefore it is 4. I mean, I calculated as 8, but it was 7. (N + 1) x 2 does not work, either. Let me find the fourth step. Here it is increasing only horizontally. There will be 5 horizontally. In addition, it is also increasing by 1 upwards. Oops, (step number) it is here. The bottom increases by 1 and it becomes equal to the above step. N + 1 + N. I found the rule now. The fourth step is 4 + 1 + 4 = 9. It works. In the fifth step it is 11. Yes, it is also true. 11 + 10 = 21 in the tenth step. In the fiftieth step, it is 101.

P6's answer starts with the naïve induction process. He tried two rules, but none of them gave the terms of the pattern. Suddenly, he saw the arithmetical increase between terms of the pattern in a figural way and then noticed the relationship of the number of circles with the term number, which leaded him to contextual and symbolic generalizations.

4.1.3. Eighth grade students' generalization of linear patterns.

Table 12 represents the generalization approaches the eighth grade students used during the generalization of patterns. According to Table 12, the combination of arithmetical generalization and algebraic generalization was the most frequent set the eighth grade students used. The second and third mostly used generalization approaches were the combination of arithmetical generalization and naïve induction and the combination of arithmetical generalization, algebraic generalization, and naïve induction, respectively. On the other hand, only algebraic generalization was the least among the others.

Table 12. Eighth grade students' generalization approaches including the variation of strategies used

	ALG only	AG and ALG	AG and I	AG, ALG, and I
Q1	P11	P12 P13 P14		P15
Q2		P13 P14	P11 P12	P15
Q3		P11 P14	P12 P13	P15
Q4		P11 P12 P13 P14 P15		
Q5		P12 P14	P11 P13 P15	
Q6	P11	P12 P13 P14 P15		

Notation: P, participant; ALG, algebraic generalization; AG, arithmetical generalization; I, naïve induction

4.1.3.1. Generalization approach including only algebraic generalization strategies.

The generalization process including only algebraic generalization was the least frequently used generalization process among the eighth grade students. It emerged twice in the present study. The detailed generalization processes were represented in Table 13.

Table 13. Eighth grade students' generalization approach including algebraic generalization only

# of Questions	Algebraic Generalization Strategies
Q1	P11 factual generalization symbolic generalization [general rule] *(calculates near and far terms by applying the general rule)*
Q2	-
Q3	-
Q4	-
Q5	-
Q6	P11 symbolic generalization [general rule] *(calculates near and far terms by applying the general rule)*

Notation: P, participant

As indicated in Table 13, in the first and sixth questions, one student generalized linear patterns using only the algebraic generalization strategies. Below is an example from question 1:

> Q1/P11: In the first step, there are three circles. There are six circles in the second step. When I multiply 1 with 3, it gives first term. When I multiply 2 with 3, it gives second term. Therefore, the rule is n times 3.
>
> (He writes 10 x 3 = 30 as the tenth term; he writes 50 x 3 = 150 as the fiftieth term)
>
> (He writes general rule as n x 3)
>
> R: What does n represent?
>
> P11: Term number.

P11 firstly expressed the first and second terms by using a rule, multiplying with 3 rule that was coded as factual generalization and then stated the general rule by using letters as n times 3.

4.1.3.2. Generalization approach including the combination of arithmetical generalization and algebraic generalization.

The eighth grade students' generalization processes most frequently included both arithmetical generalization and algebraic generalization. The combination of arithmetical generalization and algebraic generalization emerged 18 times in the present study. The detailed generalization processes were represented in Table 14.

Table 14. Eighth grade students' generalization approach including the combination of arithmetical generaliation and algebraic generalization

# of Questions	**The Combination of Arithmetical Generalization and Algebraic Generalization Strategies**
Q1	P12 arithmetical generalization [near rule] factual generalization [far rule] contextual generalization [general rule]
	P13 arithmetical generalization [near rule] factual generalization [near rule] contextual generalization [general rule] *(calculates near and far terms by applying the general rule)*
	P14 arithmetical generalization factual generalization [near & far rule] contextual generalization [general rule] symbolic generalization [general rule]
Q2	P13 arithmetical generalization [near rule] factual generalization [near rule] (student passed the question for some time) factual generalization contextual generalization [general rule] *(calculates far terms by applying the general rule)*
	P14 arithmetical generalization [near term] factual generalization [near & far term] symbolic generalization [general term]

Table 14 (continued)

# of Questions	The Combination of Arithmetical Generalization and Algebraic Generalization Strategies
Q3	P11 arithmetical generalization [near rule] (student passed the question for some time) contextual generalization [general rule] symbolic generalization [general rule] *(calculates far terms by applying the general rule)*
	P14 arithmetical generalization factual generalization contextual generalization [general rule] symbolic generalization [general rule] *(calculates far terms by applying the general rule)*
Q4	P11 arithmetical generalization [near rule] factual generalization [far rule] symbolic generalization [general rule]
	P12 arithmetical generalization [near rule] symbolic generalization [general rule] *(calculates near and far terms by applying the general rule)*
	P13 arithmetical generalization [near rule] factual generalization [far rule] contextual generalization [general rule]
	P14 arithmetical generalization [near rule] symbolic generalization [general rule] *(calculates far terms by applying the general rule)*
	P15 arithmetical generalization factual generalization symbolic generalization [general rule] *(calculates near and far terms by applying the general rule)*
Q5	P12 arithmetical generalization [near rule] symbolic generalization [general rule] *(calculates near and far terms by applying the general rule)*
	P14 arithmetical generalization [near rule] symbolic generalization [general rule] *(calculates near and far terms by applying the general rule)*

Table 14 (continued)

# of Questions	The Combination of Arithmetical Generalization and Algebraic Generalization Strategies
Q6	P12 arithmetical generalization [near rule] factual generalization [near rule] symbolic generalization [general rule] *(calculates near and far terms by applying the general rule)*
	P13 arithmetical generalization [near rule] contextual generalization [general rule] *(calculates near and far terms by applying the general rule)*
	P14 arithmetical generalization [near rule] symbolic generalization [general rule] *(calculates near and far terms by applying the general rule)*
	P15 arithmetical generalization [near rule] symbolic generalization [general rule] *(calculates near and far terms by applying the general rule)*

Notation: P, participant

In detail, all five students in the fourth question, four students in the sixth question, three students in the first question, and two students in the second, third, and fifth questions generalized linear patterns using the arithmetical and algebraic generalization strategies. Below is a related example from question 4:

> Q4/P13: Here (in the second step) 6 is added. Here, 6 is added also (in the third step). When 6 is added to 24, it is 30. When 6 is added to 30, it is 36. When 6 is added to 36, it is 42. When 6 is added to 42, it is 54. When 6 is added to 54, it is 60. There's 36 in fifth step. In the tenth step ..... How can I find the tenth step? I'll keep adding. 66 is the ninth step. When 6 is added to 66, it is 72 which is the tenth step. My fiftieth step .... How can we find my fiftieth step?

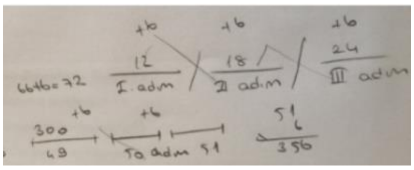


Figure 24. The schema drawn by P13

> (Through the schema he drew) There is something like this. For example, when the second step is multiplied by 6, then it is 12 (it refers to the first term). Multiplying 3 by 6 is 18 (refers to the second term). It asks for the fiftieth step and the previous step is the forty-ninth step. It should also be at the fifty-first step. We do not know the number in the fiftieth step. However, 6 is added. That is 50 times 6, 300 (writes for the forty-ninth step). It is 51 times 6, 356 (writes for the fiftieth step). How can we explain the general rule here? I multiply the step number of the next step by increasing the number. For example, multiply 51 and 6 for the fiftieth step.

P13's generalization process started with arithmetical generalization. She found the number in the tenth term by constantly adding 6 to the previous terms. Then, she explored that the product of 2 and 6 gives the first term and the product of 3 and 6 gives the second term as the factual generalization rule which she expands to the fiftieth term as the product of 51 and 6. She expresses the general rule without using letter with contextual generalization. Another example could be given from P15's generalization process from question 6:

> Q6/P15: Let me find the increasing number. It increased by 4. Then, I increased by 4 again. I mean, it increased by 4. Why did I try so hard? (Deletes the table) It increased by 4. It increased by 4 and then it became 16; It increased by 4 and then it became 20. n times 4 is the general rule (writes 4n)
>
> R: How did you find it?
>
> P15: I looked at the fifth step again. 5 times 4, 20. 6 times 4 is 24 for the sixth step. The tenth step is 40, while the twentieth step is 80 and the hundredth step is 400.

As shown in P15's generalization process, she firstly mentioned the constant increase between successive terms of the pattern as 4 with arithmetical generalization. Then, she symbolizes the general rule as 4n with symbolic generalization. After symbolic generalization, she calculated the numbers in the tenth, twentieth, and hundredth step by applying the rule. P14's generalization process also included the combination of arithmetical and algebraic generalization strategies.

4.1.3.3. Generalization approach including the combination of arithmetical generalization and naïve induction.

In the present study, the combination of arithmetical generalization and naïve induction was used 7 times within a generalization process. One student in the second and fifth questions, one student in the second and third questions, and three students in the fifth question used both arithmetical generalization and naïve induction during pattern generalization. Table 15 shows the detailed structures.

Table 15. Eighth grade students' generalization approach including the combination of arithmetcal generalization and naïve induction strategies

# of Questions	The Combination of Arithmetical Generalization and Naïve Induction Strategies
Q1	-
Q2	P11 arithmetical generalization [near rule] naïve induction [general rule] *(calculates far terms by applying the general rule)*
	P12 arithmetical generalization [near rule] naïve induction [general rule] *(calculates far terms by applying the general rule)*
Q3	P12 arithmetical generalization [near rule] naïve induction [general rule] *(calculates far terms by applying the general rule)*
	P13 arithmetical generalization naïve induction [general rule] *(calculates near and far terms by applying the general rule)*
Q4	-

Table 15 (continued)

# of Questions	**The Combination of Arithmetical Generalization and Naïve Induction Strategies**
Q5	P11 arithmetical generalization [near rule] naïve induction [general rule] *(calculates far terms by applying the general rule)*
	P13 arithmetical generalization [near rule] naïve induction [general rule] *(calculates far terms by applying the general rule)*
	P15 arithmetical generalization [near rule] naïve induction [general rule] *(calculates near and far terms by applying the general rule)*
Q6	-

Notation: P, participant

As seen in Table 15, each student firstly used arithmetical generalization to generalize the linear patterns to near terms. Then, they employed the naïve induction strategy to find the general rule. Below is an example from question 3;

> Q3/P12: 3, 5, 7. It increases by two. 7, 9, 11, 13. Okay, it increased by two. The fifth step is 11. 13, 15, 17. 7 and 15 are not related. I was going to find by using them. 17, 19, 21. The difference is seven times.
>
> R: What do you mean by saying the difference is seven times?
>
> P12: The number in the tenth step [which is 21] is seven times the number in the first step [which is 3]. Then, the number in the twentieth step must be seven times the number in the tenth step. Doesn't it? Does n + 1 work? It works. But does it work for three (does it work for the third step)? N + 4-1 Excuse me. But it works for the second step. I'll see something... If we multiply 1 by 3, it does not work.
>
> R: Where does it work, where not?
>
> P12: Aaa. I found it. nX2 + 1.
>
> R: Where did you find? How did you find?
>
> P12: It came to my mind.

R: Why did you choose to multiply by 2?

P12: To reach 3 (which is the first step).

R: So, you're trying.

P12: Yeah. Then I got 50. 101. I found it right.

R: Does this rule work for all steps?

P12: Yeah.

R: Can you give an example?

P12: It holds for 3 (the third step). Also, it holds for 2 (the second step). Then it works for others, too. For example, does it hold for the fourth step? Multiply 4 by 2. Yeah, it works. Done.

P12's generalization process first included arithmetical generalization. He found the number of circles in the tenth step using arithmetical generalization. Then, he tried many rules such as n plus 1, n plus 4 minus 1 or multiplying by 3. Yet, none of them worked for all the terms of the pattern. Then, he found a working rule, which is n times 2 plus 1, which was coded as naïve induction. Another example from question 5 is given as follows:

Q5/P11: Toothpick sticks are increased by 3. If we increase 3, the fourth step is 5, 8, 3, 11, 3, and 14. The fifth step is 17. I'm trying to find the general rule. I found it. 10 times 3 plus 2.

R: How did you get this answer?

P11: According to the numbers between each step. I tried one by one. Firstly, I did n times 5; it did not hold. Then I multiplied with 4 (multiplied the step number). Then I tried it with 3. It worked when I added the required number. (He writes nx3 + 2 to general rule) The tenth step is 32nd. In addition, the hundredth step is 302.

The dialogues of P11 started with arithmetical generalization. He found the fifth term by adding 3 to the fourth term. Then, P11 tried some rules such as multiplying term number with 5, with 4, and with 3 in order with the naïve induction process.

4.1.3.4. Generalization approach including the combination of arithmetical generalization, algebraic generalization, and naïve induction.

In the current study, the combination of arithmetical generalization, algebraic generalization, and naïve induction strategies was the third most frequent generalization approach that the eighth grade students used. This combination emerged three times. In detail, one student used this generalization approach in the first, second, and third questions. The detailed structure was represented in Table 16.

Table 16. Eighth grade students' generalization approach including the combination of arithmetical generalization, algebraic generalization, and naïve induction strategies

# of Questions	**The Combination of Arithmetical Generalization, Algebraic Generalization, and Naïve Induction Strategies**
Q1	P15 arithmetic generalization [near rule] naïve induction arithmetical generalization [near rule] factual generalization [far rule] symbolic generalization [general rule]
Q2	P15 arithmetical generalization [near rule] naïve induction (student passed the question for some time) factual generalization symbolic generalization [general rule] *(calculates far terms by applying the general rule)*
Q3	P15 arithmetical generalization [near rule] naïve induction arithmetical generalization [near rule] factual generalization [near rule] symbolic generalization [general rule] *(calculates far terms by applying the general rule)*
Q4	-

Table 16 (continued)

# of Questions	The Combination of Arithmetical Generalization, Algebraic Generalization, and Naïve Induction Strategies
Q5	-
Q6	-

Notation: P, participant

An example is given from question 1 as follows:

> Q1/P15: This pattern consists of circles; there are three (circles in the first step), then six (circles in the second step), then nine (circles in the third step). Then, I guess it will be 12 (fourth step). How many circles are required? Probably 12. In fact, exactly 12. How can we do it now? If we give n, it becomes (n + 2) (In the first step, he put 1 in the place of n and added 2 to reach 3). However, n + 2 does not work, because in the second step, when I replace n with 2, I have to find 4. Can it be the exponential of 3? (He calculates over the second step) No, that would be ridiculous.
> Then, let me just say as 3, 6, 9; it is easier. (On the figure, he writes 3 to the first step, 6 to the second step, 9 to the third step, 12 to the fourth step, 15 to the space in the fifth step, 18 to the sixth step.)
> Let me think of the rest as a table (he writes the number of steps on the bottom line and the terms on the top line as seen in Figure 25).

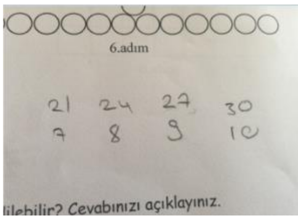


Figure 25. The terms and term numbers *w*ritten by P15

> To find the number of circles in the fiftieth step ..... If the first step is 3, then what is the fiftieth step? Can there be a crossover here? No way. To find the fiftieth step, we can multiply 50 with 3; therefore, there are 150 circles.
> If it is increased by 3 in the first step, then it becomes 150 in the fiftieth step. For example, let me try in others. In the fourth step, I multiply 4 by 3 and it is 12, which is similar to what I found by counting. In the fifth step, I multiply 5 by 3 and the result is 15. Yes, the same answer. Hence, the rule is correct.
> The general rule of the pattern is n times 3 (he writes nx3). n, the step number. For example, n = 2 for the second step. 2 times 3 is 6. It is also true for the third step.

As illustrated in the above example, the student finds the number of circles at the fourth step with arithmetical generalization. Afterwards, he tried some rules through naïve induction; but they did not work. After the failure of the induction process, he moved back to arithmetical generalization and expanded the pattern to the sixth step. After arithmetical generalization, he formed a rule, which is multiplying by 3 and expanding the pattern to fourth and fifth terms by using this rule. This part of the generalization process exemplified factual generalization. Finally, he wrote the general rule with letters through factual generalization.

4.1.4. Differences/Similarities between the generalization approaches of sixth, seventh, eighth graders.

Sixth, seventh, and eighth grade students' generalization approaches have some similarities and differences between them. Table 17 represents the distribution of the generalization approaches of sixth, seventh, and eighth grade students.

Table 17. The distribution of the generalization approaches of sixth, seventh, and eighth graders

		6th grade level	**7th grade level**	**8th grade level**
the combination of AG and ALG	*arithmetical g. factual g. contextual and/or symbolic g.*	*2 times*	*6 times*	*9 times*
	arithmetical g. contextual and/or symbolic g.	*12 times*	*6 times*	*7 times*
total number of the combination of AG and ALG		14 times	12 times	16 times

Table 17 (continued)

		6th grade level	**7th grade level**	**8th grade level**
ALG only	*factual g.* *contextual and/or symbolic g.*	*2 times*	-	*1 time*
	contextual and/or symbolic g.	*6 times*	-	*1 time*
total number of ALG only		8 times	-	2 times
the combination of AG and I		7 times	2 times	7 times
the combination of AG, ALG, and I		1 time	6 times	2 times
not classified		-	4 times	3 times

Notation: AG, arithmetical generalization; ALG, algebraic generalization; I, naïve induction

As seen from Table 17, the first similarity was that the most frequent generalization approach at all grade levels was the combination of arithmetical generalization and algebraic generalization. On the other hand, considering the sub-levels of algebraic generalization (factual, contextual, and symbolic generalizations), sixth graders differ from seventh and eighth graders in terms of the dominance of the movement from arithmetical generalization to contextual and/or symbolic generalizations over the movement from arithmetical generalization to factual generalization and contextual and/or symbolic generalizations.

There was a difference between the sixth, seventh, and eighth grade students' generalization approaches, which is about the algebraic generalization strategies only. It can be summarized that the generalization approach including algebraic generalization strategies only was the second mostly used approach at sixth grade level, while the seventh and eighth grade students either did not use it at all or used it rarely. Furthermore, considering the sub-levels of algebraic generalization, sixth grade students mostly used contextual and/or symbolic generalizations more than the movement from factual generalization to contextual and/or symbolic generalizations.

Thirdly, the sixth and eighth graders used the generalization approach including the arithmetical generalization and naïve induction more than the seventh graders. In other words, the movement from arithmetical generalization to naïve induction was the third and second mostly used approach at the sixth and eighth grade level, respectively. Nevertheless, it was used twice by the seventh graders.

Lastly, the combination of arithmetical generalization, algebraic generalization, and naïve induction was observed at the seventh grade level more than the sixth and eighth grade level.

4.2. Summary of the Findings

In sum, middle school students dominantly used at least two generalization strategies while generalizing linear patterns. The analysis of students' answers revealed four categories of generalization approaches, which included (i) only algebraic generalization strategies, (ii) the combination of arithmetical generalization and algebraic generalization strategies, (iii) the combination of arithmetical generalization and naïve induction strategies, and (iv) the combination of arithmetical generalization, algebraic generalization, and naïve induction strategies. It was found that the combination of arithmetical generalization and algebraic generalization was the most frequent generalization approach at all grade levels, while the combination of arithmetical generalization, algebraic generalization, and naïve induction was the least frequent one used by the students in all grade levels. Furthermore, the use of algebraic generalization strategies only was observed by the sixth graders only.

CHAPTER 5

DISCUSSION AND CONCLUSION

The purpose of this study is to explore the sixth, seventh, and eighth grade students' generalizations of patterns using arithmetical generalization, algebraic generalization, and naïve induction. In addition to studying their generalization process, the study also focuses on the ways in which this process of generalization differs according to their grade level.

Accordingly, in the first part of the chapter, the findings related to the students' usage of arithmetical generalization, algebraic generalization, and naïve induction during pattern generalization are discussed in the light of the previous studies. Then, in the second part of the chapter, the implications of the study are discussed and some recommendations for future studies are given in the third part.

5.1. Discussion of the Generalization Process of Linear Patterns

The research question of the current study is about the sixth, seventh, and eighth grade students' generalization process of linear patterns. In order to answer it, Radford's generalization strategies were considered during the data analysis, which are arithmetical generalization, algebraic generalization, and naïve induction. The analysis of students' answers revealed that the students used a variety of generalization strategies while generalizing patterns to near and far terms and while finding the general rule. In other words, they used at least two generalization strategies while generalizing linear patterns to near, far, and nth terms. Therefore, the data analysis revealed four categories of generalization types based on the sets of strategies the students used, which are the combination of algebraic generalization strategies, the combination of arithmetical generalization and algebraic generalization strategies, the combination of arithmetical generalization and naïve induction strategies, and the combination of arithmetical generalization, algebraic generalization, and naïve induction strategies.

After identifying students' generalization types based on the sets of strategies, the frequency of the emergence of each generalization type was compared for different grade levels. When the generalization types of sixth, seventh, and eighth grade students were compared based on their grade level, it was seen that there were some similarities and differences between them. The first similarity was that the generalization including the combination of arithmetical generalization and algebraic generalization was the most frequent generalization approach regardless of students' grade level. Specifically, more than half of the students from each grade level generalized linear patterns using both arithmetical generalization and algebraic generalization strategies. Indeed, the findings also showed that they used arithmetical generalization strategy in order to generalize the pattern to near terms and algebraic generalization strategies in order to generalize the pattern to far terms or to find the general rule. In other words, the students followed a path from arithmetical generalization to algebraic generalization.

This result is consistent with the related literature (Amit & Neria, 2007; Lannin, 2004; Lannin et al., 2006; Orton & Orton, 1999; Stacey & MacGregor, 2001). Past studies indicated that it is natural to start generalizing patterns with recursive reasoning through arithmetical generalization (Lannin, 2004); since it is easy to add the constant difference onto the previous term while expanding the pattern to near terms (Garcia-Cruz & Martinón, 2002). In addition, the literature also showed that the nature of the patterning tasks might help students determine the generalization strategies (Barbosa & Vale, 2015; Lannin et al., 2006). According to Lannin et al. (2006), the patterning tasks might lead students to recursive thinking if the pattern is represented sequentially, i.e. step by step. In the present study, all six patterns in the Patterning Test were in the form of a sequential pattern. On the other hand, generalizing a pattern to far terms also leads students to have more tendency to understand the algebraic structure of the pattern (Stacey, 1989). Indeed, near and far generalization are of great importance to students as they make students feel the need for a general rule to reach far terms easily (Chua & Hoyles, 2014). Therefore, the reason behind the flow from arithmetical generalization to algebraic generalization might be either the sequential nature of the patterning tasks or the near and far generalization questions, which might direct students to use arithmetical generalization strategies first and to use algebraic generalization strategies afterwards.

Another similarity between different grade level students' generalization process is the naïve induction strategy. As indicated in the literature, naïve induction is just about predicting the general rule of the pattern by trying out rules and checking whether the rules work or not. The data analysis of the present study revealed two types of generalization processes including naïve induction strategy, which are the combination of arithmetical generalization and naïve induction and the combination of arithmetical generalization, algebraic generalization, and naïve induction. According to the frequencies, the combination of arithmetical generalization and naïve induction was hardly observed at the seventh grade level, whereas almost one-fifth of the sixth and eighth grade students generalized linear patterns using both arithmetical generalization and naïve induction. On the other hand, the combination of arithmetical generalization, algebraic generalization, and naïve induction was seldom observed at the sixth and eighth grade level, while one-quarter of the seventh grade students used this type of generalization. All in all, it can be deduced that the naïve induction strategy existed at all grade levels although its frequency is low. This finding showed consistencies with the related literature, which reported students' tendency to use naïve induction in patterning tasks (Barbosa, 2011; Lannin et al., 2006; Ozdemir, Dikici, & Kultur, 2015; Rivera & Becker, 2005). As the literature indicated, mathematics instruction mostly focused on the procedures of constructing the general rule of the pattern (Lannin et al., 2006). Then, students could not understand the algebraic structure of the pattern conceptually (Noss et al., 1997). Instead, they could just practice their procedural skills about how to construct a general rule. Therefore, the lack of conceptual instruction on pattern generalization might have had an impact on students' usage of the naïve induction strategy.

On the other hand, there was also a difference between the sixth, seventh, and eighth grade students. The difference was about the generalization type including only the algebraic generalization strategies. According to the findings, at the sixth grade level, the generalization type including only the algebraic generalization strategies was used 8 times by two students in the first, fourth, and sixth questions, by one student in the second question and another student in the fifth question. At the seventh grade level, this type of generalization was not observed. At the eighth grade level, it was used 2 times by one student in the first and sixth questions. It can be concluded that the generalization type including only the algebraic generalization strategies was the second mostly used type for the sixth grade students, whereas the seventh and eighth grade students either did not use it at all or used it rarely. This finding revealed that the sixth grade students showed more complex algebraic generalization skills in terms of algebraic generalization strategies compared to the seventh and eighth grade students. This result was inconsistent with the related literature, which showed progressive development of students' algebraic generalization skills across increasing grade levels (El Mouhayar, 2018). However, some past studies reported younger students' tendency to use algebraic strategies more than older students (Rivera, 2013). When the Turkish middle school mathematics curriculum was reviewed, it was observed that the curricular objectives related to pattern generalization are only seen in the fifth and sixth grade mathematics curriculum (MONE, 2013). Students in the fifth grade are expected to construct the required steps when the rule of the number and shape patterns is given, and the students in the sixth grade are expected to express the general rule of the linear patterns with letters and to find the required steps when the rule is expressed with letters (MONE, 2013). Thus, while students in the fifth and sixth grade levels could engage in patterning activities, the seventh or eighth grade students did not have any patterning activities. Thus, the high frequency of the generalization process including only algebraic generalization strategies at the sixth grade level and absence of it at the seventh and eighth grade levels might be attributed to the differences in mathematics curricula of different grade levels.

In addition to the similarities and differences between different grade level students, the current study revealed another important finding when students' generalization process was examined in terms of near, far, and nth terms. Almost all the students extended the patterns to near terms with arithmetical generalization as the first step of their generalization process. Then, they were asked to generalize the pattern to far terms. However, they were reluctant to look for generalization of far terms. They mostly skipped that question and directly tried to find the general term in a procedural way. In other words, the majority of the students from each grade level were engaged in a generalization process in which they gave priority to find the general rule over finding particular terms especially far terms. When they found the general rule, they used it to calculate the far terms, which were asked as sub-questions of the patterning task. Put differently, since the students did not generalize the pattern to far terms with factual generalization, they lacked the connection between arithmetical generalization and algebraic generalization.

While this result showed consistencies with some of the past studies (Ozdemir et al., 2015), it was inconsistent with some of them (Cooper & Warren, 2011; Miller & Warren, 2012; Radford, 2003). According to Radford (2003), factual, contextual, and symbolic generalizations should follow each other so that students could move from numerical level generalization to algebraic level generalization in a meaningful manner. There were also many studies, which focused on the importance of far term generalization. Being able to generalize the pattern to far terms shows students' conceptual understanding of the nature of the generalization (Lannin et al., 2006). Thus, it is highly important to enable students to engage with factual generalization to generalize the pattern to far terms. This result might stem from two reasons. First of all, as expressed before, there are two objectives related to pattern generalization in the Turkish middle school mathematics curriculum; however, those objectives do not include the process of generalizing the patterns to near or far terms. Instead, they focus on finding the general rule (MONE, 2013). In addition, the pattern generalization tasks in Turkish mathematics textbooks might not enable students to explore the nature of the generalization, since they include patterning tasks, which encourage students to find the rule of the pattern before extending the pattern to far terms (Ayber, 2017). Furthermore, they do not include sufficient tasks related to 'patterns' topics (Ayber, 2017). All in all, the reason behind the finding of the present study, which is students' giving priority to finding the general rule over generalizing the pattern to far terms, might stem either from the rule-based objectives about pattern-generalization in the Turkish middle school mathematics curriculum or from the rule-based pattern generalization tasks included in Turkish mathematics textbooks.

5.2. Implications

To ensure conceptual understanding of algebraic generalization, educational environments should be designed in a way that students could explore the nature of generalization tasks and conduct near and far generalizations progressively, instead of practicing the techniques of finding the general term of the pattern in a rule-based way. In order to provide such educational environments, first, the Turkish mathematics education curriculum should cover pattern-generalization more conceptually. In other words, there are two objectives related to pattern-generalization in the elementary mathematics curriculum, the first of which is "Students should form the desired steps in numeric and figural linear patterns whose rule is given" at the fifth grade level and "Students express the general rule of linear patterns with letters and finds the desired terms of the patterns when the rule is expressed with letters" at the sixth grade level (MONE, 2013). In both of these objectives, the focus is on the rule of the pattern. Thus, objectives related to the structure of the patterns and near and far generalization of the patterns could be added to the middle school mathematics curriculum before emphasizing the general rule of the pattern.

Furthermore, as reported by Ayber (2017), Turkish mathematics textbooks include insufficient number of patterning tasks. Furthermore, the patterning tasks in the books lead students to find the general rule before near and far generalizations. Thus, the content of the textbooks could be revised accordingly. The number of patterning tasks could be increased and the content of the tasks could include near and far generalizations before general rule, progressively.

Moreover, as the literature revealed, teachers do not have sufficient knowledge about the relationship between arithmetic and algebra (Demonty et al., 2018) and about the generalization of patterns (Girit, 2016). They introduce generalization of patterns to students in a rule-based way (Lannin et al., 2006). Thus, teachers could develop themselves in terms of arithmetical and algebraic generalizations. In order to enhance teachers' knowledge on arithmetical and algebraic generalizations, seminars or workshops could be organized.

5.3. Recommendations for Further Research

The present study focused on the generalization process of the sixth, seventh, and eighth grade students using arithmetical generalization, algebraic generalization, and naïve induction. Based on the results, some recommendations for further studies could be made.

First, the results were limited with the sample of the present study from a public school in Çankaya district of Ankara. Thus, it would be helpful to select students from different type of schools such as private schools. Additionally, the participants of the present study were selected based on the predetermined criteria, which were students' grade level, enthusiasm about mathematics lesson, and talkativeness. By including students’ success among the criteria, existing trends between successful students and/or unsuccessful students could be revealed.

Secondly, the results of this study were limited with the sixth, seventh, and eighth grade students’ trends of patten-generalization approaches. Nevertheless, while some reserchers reported the development of students’ algebraic reasoning as grade level increased, some studies resulted similar algebraic reasoning structures regardless of grade level. For this reason, by expanding the age range from the fifth grade to twelfth grade, existing trends across increasing grade levels could be revealed.

In addition, there were six pattern tasks in the present study, all of which were represented in a sequential manner. According to the literature, the sequential patterns might encourage students to use recursive relations and discourage them from using algebraic relations. Therefore, in further studies, the nature of the pattern tasks might not be limited with sequential patterns. Lastly, the pattern tasks were numeric or figural pattern tasks in this study. Therefore, a further study could be conducted with the purpose of finding out the differences between the students’ approaches of generalizing numerical patterns and figural patterns.

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APPENDICES

A: APPROVAL OF THE METU HUMAN SUBJECTS ETHICS COMMITTEE

UYGULAMALI ETİK ARAŞTIRMA MERKEZİ
APPLIED ETHICS RESEARCH CENTER

ORTA DOĞU TEKNİK ÜNİVERSİTESİ
MIDDLE EAST TECHNICAL UNIVERSITY

DUMLUPINAR BULVARI 06800
ÇANKAYA ANKARA/TURKEY
T: +90 312 210 22 91
F: +90 312 210 79 59
ueam@metu.edu.tr
www.ueam.metu.edu.tr

Sayı: 28620816 /119

22 MART 2016

Gönderilen: Doç.Dr. Mine Işıksal BOSTAN

İlköğretim Bölümü

Gönderen: Prof. Dr. Canan SÜMER

İnsan Araştırmaları Komisyonu Başkanı

İlgi: Etik Onayı

Sayın Doç.Dr. Mine Işıksal BOSTAN'ın danışmanlığını yaptığı yüksek lisans öğrencisi Zeycan KAMA'nın "Öğrencilerin Cebirsel Düşünme Becerilerinin Genelleme Aktiviteleri Yoluyla İncelenmesi" başlıklı araştırması İnsan Araştırmaları Komisyonu tarafından uygun görülerek gerekli onay **2016-EGT-038** protokol numarası ile **31.03.2016-31.03.2017** tarihleri arasında geçerli olmak üzere verilmiştir.

Bilgilerinize saygılarımla sunarım.

Prof. Dr. Canan SÜMER

Uygulamalı Etik Araştırma Merkezi

İnsan Araştırmaları Komisyonu Başkanı

Prof. Dr. Meliha ALTUNIŞIK

İnsan Araştırmaları Komisyonu

Üyesi

Prof. Dr. Ayhan SOL

İnsan Araştırmaları Komisyonu

Üyesi

Prof. Dr. Mehmet UTKU

İnsan Araştırmaları Komisyonu

Üyesi

Yrd.Doç.Dr. Pınar KAYGAN

İnsan Araştırmaları Komisyonu

Üyesi

B: APPROVAL OF THE MINISTRY OF NATIONAL EDUCATION

T.C.
ANKARA VALİLİĞİ
Milli Eğitim Müdürlüğü

ÖİDB

Sayı : 14588481-605.99-E.6000624 31.05.2016
Konu : Araştırma İzni

ORTA DOĞU TEKNİK ÜNİVERSİTESİNE
(Öğrenci İşleri Daire Başkanlığı)

İlgi: a) MEB Yenilik ve Eğitim Teknolojileri Genel Müdürlüğünün 2012/13 nolu Genelgesi.
b) 26/04/2016 tarihli ve 1788 sayılı yazınız.

Üniversiteniz İlköğretim Anabilim Dalı yüksek lisans öğrencisi Zeycan KAMA'nın **"Öğrencilerin Cebirsel Düşünme Becerilerinin Genelleme Aktiviteleri Yoluyla İncelenmesi"** konulu tez kapsamında uygulama talebi Müdürlüğümüzce uygun görülmüş ve uygulamanın yapılacağı İlçe Milli Eğitim Müdürlüğüne bilgi verilmiştir.

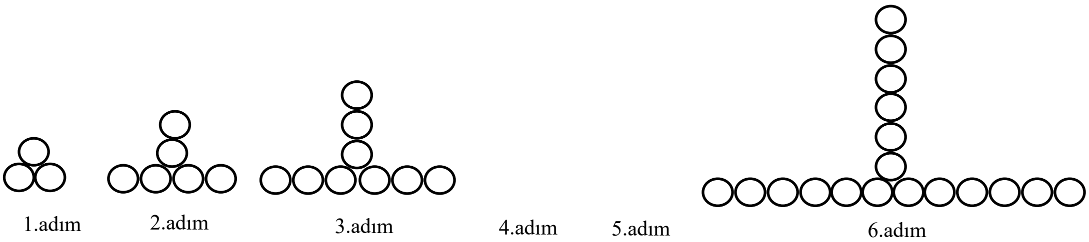
Görüşme formunun (9 sayfa) araştırmacı tarafından uygulama yapılacak sayıda çoğaltılması ve çalışmanın bitiminde bir örneğinin (cd ortamında) Müdürlüğümüz Strateji Geliştirme (1) Şubesine gönderilmesini arz ederim.

Müberra OĞUZ
Müdür a.
Şube Müdürü

Güvenli Elektronik İmzalı
Aslı İle Aynıdır.
01/06/2016

03-06-2016-9034

C: PATTERN TEST/ÖRÜNTÜ TESTİ

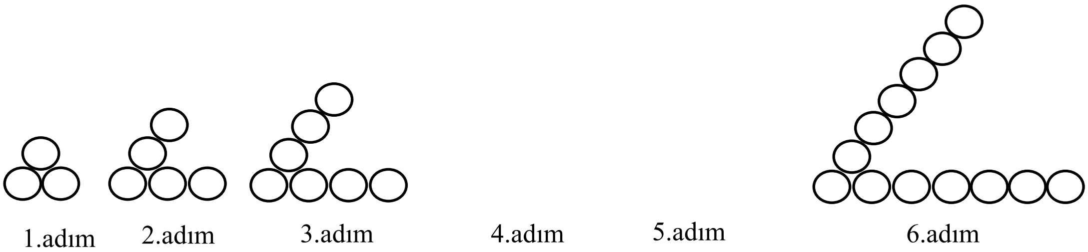


1. Çemberler kullanılarak oluşturulan yukarıdaki örüntüde:

a. Dördüncü adımı oluşturmak için kaç çember gereklidir?
b. Beşinci adımı oluşturmak için kaç çember gereklidir?
c. Onuncu adımı oluşturmak için kaç çember gereklidir? Cevabınızı açıklayınız.
d. Yüzüncü adımı oluşturmak için kaç çember gereklidir? Cevabınızı açıklayınız.
e. Herhangi bir adımdaki çember sayısını bulmak için örüntünün genel kuralı nasıl ifade edilebilir? Cevabınızı açıklayınız.

2. 5, 8, 11, ..., ..., 20, ..., ... olarak verilen örüntünün:

a. Dördüncü adımında hangi sayı vardır?
b. Beşinci adımında hangi sayı vardır?
c. Onuncu adımında hangi sayı vardır? Cevabınızı açıklayınız.
d. Yüzüncü adımında hangi sayı vardır? Cevabınızı açıklayınız.
e. Herhangi bir adımdaki sayıyı bulmak için örüntünün genel kuralı nasıl ifade edilebilir? Cevabınızı açıklayınız.

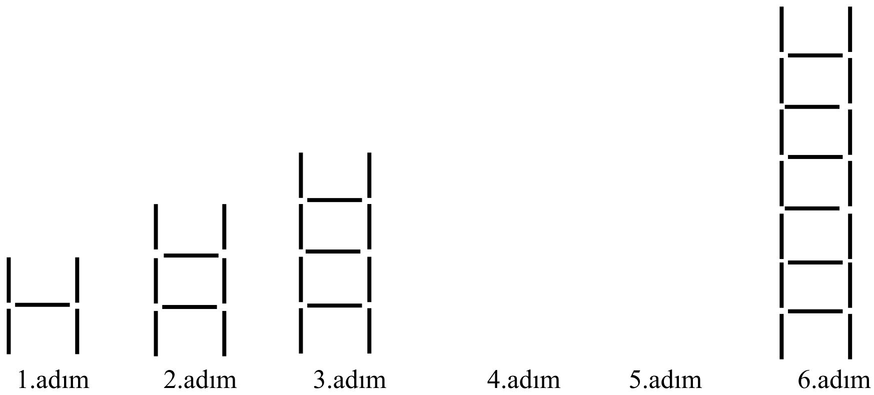


3. Çemberler kullanılarak oluşturulan yukarıdaki örüntüde:

a. Dördüncü adımı oluşturmak için kaç çember gereklidir?
b. Beşinci adımı oluşturmak için kaç çember gereklidir?
c. Onuncu adımı oluşturmak için kaç çember gereklidir? Cevabınızı açıklayınız.
d. Yüzüncü adımı oluşturmak için kaç çember gereklidir? Cevabınızı açıklayınız.
e. Herhangi bir adımdaki çember sayısını bulmak için örüntünün genel kuralı nasıl ifade edilebilir? Cevabınızı açıklayınız.

4. 12, 18, 24, ..., ..., 42, ..., ... olarak verilen örüntünün;

a. Dördüncü adımında hangi sayı vardır?
b. Beşinci adımında hangi sayı vardır?
c. Onuncu adımında hangi sayı vardır? Cevabınızı açıklayınız.
d. Yüzüncü adımında hangi sayı vardır? Cevabınızı açıklayınız.
e. Herhangi bir adımdaki sayıyı bulmak için örüntünün genel kuralı nasıl ifade edilebilir? Cevabınızı açıklayınız.



5. Yukarıda verilen örüntüde kürdanlardan merdiven inşa edilmektedir. Buna göre:

a. Dördüncü adımı inşa etmek için kaç kürdan gereklidir?
b. Beşinci adımı inşa etmek için kaç kürdan gereklidir?
c. Onuncu adımı inşa etmek için kaç kürdan gereklidir? Cevabınızı açıklayınız.
d. Yüzüncü adımı inşa etmek için kaç kürdan gereklidir? Cevabınızı açıklayınız.
e. Herhangi bir adımdaki kürdan sayısını bulmak için örüntünün genel kuralı nasıl ifade edilebilir? Cevabınızı açıklayınız.

6. 4, 8, 12, ..., ..., 24, ..., ... olarak verilen örüntünün:

a. Dördüncü adımında hangi sayı vardır?
b. Beşinci adımında hangi sayı vardır?
c. Onuncu adımında hangi sayı vardır? Cevabınızı açıklayınız.
d. Yüzüncü adımında hangi sayı vardır? Cevabınızı açıklayınız.
e. Herhangi bir adımdaki sayıyı bulmak için örüntünün genel kuralı nasıl ifade edilebilir? Cevabınızı açıklayınız.

D: TURKISH SUMMARY/TÜRKÇE ÖZET

ORTAOKUL ÖĞRENCİLERİNİN DOĞRUSAL ÖRÜNTÜLERİ
GENELLEME SÜREÇLERİNİN İNCELENMESİ

Giriş

Kültür, bir topluma özgü düşünce ve eserlerin bütünü olarak tanımlanırken; kültür şoku, kültür bakımından büyük değişmeler karşısında şaşırma, olaylara akıl erdirememe olarak tanımlanmaktadır. Lee (1996) öğrencilerin aritmetik kültürden cebirsel kültüre geçiş sürecindeki durumlarını kültür şoku söz öbeğiyle ifade eder. Lee (1996)'ye göre, cebirsel kültüre sahip kişiler ortak kuralları, ortak iletişim yollarını ve ortak dili paylaşmaktadır.

Bilindiği üzere, ilkokul matematik eğitimi ağırlıklı olarak aritmetiksel kazanımlara odaklı iken, ortaokul matematik eğitimi cebirsel kazanımlara odaklı sürdürülmektedir (Kamol ve Ban Har, 2010). Tarihsel süreçte, cebirin aritmetikten asırlar sonra ortaya çıkmış olmasının bu geleneğe kaynaklık ettiği düşünülmektedir (Patton ve De Los Santos, 2012). Tarihsel sürece benzer olarak, öğrenciler önce aritmetikte uzmanlaşmakta, ardından cebirle tanışmaktadırlar. Bu durum, öğrencilerin aritmetikten cebire geçiş sürecinde uyum problemleri yaşamalarına ve bilişsel eksikliklere sebep olmaktadır (Booth, 1984, 1988; Kieran, 1991, 1992; Linchevski ve Herscovics, 1996; Sfard ve Linchevski, 1994). Mason (1996), aritmetikten cebire geçiş sürecinde yaşanan zorlukların üstesinden gelmek için örüntü/genelleme aktivitelerinin en etkili yol olduğunu ifade etmektedir.

Örüntülerin genellenmesi yaklaşımı, erken cebir eğitimi ve cebirsel düşünme becerilerinin gelişimi açısından özel bir yere sahiptir. Zazkis ve Liljedahl (2002), cebirde her şeyin örüntülerin bir genellemesi olduğundan, örüntülerin matematiğin kalbi ve özü olduğunu ifade etmişlerdir. Bundan başka, cebirin literatürde en çok kabul gören tanımı cebirin aritmetiğin genellemesi olduğudur (Booth, 1988; Carraher, Schliemann ve Schwartz, 2007; Gavin ve Sheffield, 2015; Mason, 1996; Philipp ve Schappelle, 1999; Samo, 2009; Subramaniam ve Banerjee, 2004; Usiskin, 1988). Buna göre, genellemenin cebirin ve cebirsel düşünmenin doğasında olduğu çıkarımına ulaşılabilir. Aynı çıkarıma, cebirsel düşünmenin çeşitli tanımlarına bakıldığında da ulaşılmaktadır. Van de Walle, Karp ve Bay-Williams (2007)'a göre, cebirsel düşünme, örüntüleri ve fonksiyonları keşfetme, sayılar ve şekiller arasındaki ilişkilere dayanarak genellemelere ulaşma ve bu genellemeleri sembollerle ifade etmedir. National Council of Teachers of Mathematics [NCTM] (2000), erken cebirsel düşünme becerilerini sayı ve şekil örüntülerinin yapısını analiz edebilme, arasındaki ilişkileri keşfedebilme ve bulguları kelimelerle yada sembollerle belirtebilme olarak ifade etmiştir.

Mason, Graham ve Johnston-Wilder (2005), genellemenin doğal bir içgüdü olduğunu, okula başlayan her öğrencinin genelleme ve soyutlama içgüdüsü olduğunu ifade etmiştir. NCTM (2000)'e göre, 'örüntüleri ve ilişkileri anlamak' anaokulundan ortaöğretime bütün sınıf seviyelerinde sürekli bir konudur. Örüntüler, hem cebirsel düşünme becerilerinin gelişimine katkıda bulunurken (Lee, 1996; Mason, 1996), hem de cebirsel sembolleri kullanma üzerine temel inşa etmektedir (Zazkis ve Liljedahl, 2002).

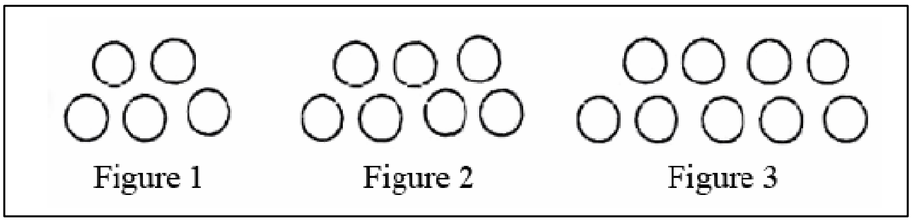
Araştırma Soruları

Bu çalışmada ele alınan iki araştırma sorusu aşağıda verilmiştir.

- Altıncı, yedinci ve sekizinci sınıf öğrencileri, aritmetik genelleme, cebirsel genelleme ve naif tümevarım kullanarak doğrusal örüntüleri nasıl geneller?
- Bu genellemeler sınıf seviyelerine göre ne ölçüde farklılık gösterir?

Kuramsal Çerçeve

Bu çalışmada benimsenen kuramsal çerçeve Radford (2006)'un örüntü genelleme kuramıdır. Radford (2003) 1990 yıllarında 120 adet sekizinci sınıf öğrencisinin katıldığı, 3 yıl süren bir çalışma yürütmüştür. Bu çalışmanın amacı öğrencilerin cebirsel düşünme becerilerinin oluşumunu ve gelişimini derinlemesine anlayabilmektir. Radford (2010a)'a göre, öğrencilerin cebirsel düşünmeye başlamaları ve formal cebirsel dili doğru kullanabilmeleri arasında bilişsel bir boşluk bulunmaktadır. Radford'un teorisi, bu bilişsel boşluğu doldurmaya yönelik ihtiyaca dayanmaktadır.



Şekil 1. Doğrusal Şekil Örüntüsü (Radford, 2006, s. 4)

Radford, aritmetik genelleme ve cebirsel genelleme yöntemlerini genelleme yöntemleri olarak; naif tümevarım yöntemini ise genelleme olmayan yöntem olarak sınıflandırmıştır. Naif tümevarım, kuralı deneme yanılma yoluyla tahmin etmeye yönelik bir yöntem olarak tanımlanmıştır (Radford, 2010a). Örneğin, Şekil 1'de verilen örüntünün genel kuralını bulmaya yönelik olarak şu öğrencinin cevabı 'Bu örüntünün kuralı 5n olsa, 5 kere 1, 5 eder. Birinci terimde tutuyor. 5 kere 2, 10. İkinci terimde tutmadı. 3 ile çarpsam 2 eklesem, ikinci adımda gene tutmadı. 2 ile çarpsam 3 eklesem, birinci adımda tutuyor. İkinci adımda da tutuyor. Üçüncü adımda da tutuyor. O zaman örüntünün kuralı 2 ile çarpıp 3 eklemek.' Naif tümevarım sürecine örnek olarak verilebilir.

Bunun dışında, Radford'a göre, öğrencilerin genelleme biçimleri aritmetik genelleme ve cebirsel genelleme olmak üzere ikiye ayrılmaktadır. Cebirsel genelleme ise kendi içinde olgusal genelleme, kavramsal genelleme ve sembolik genelleme olarak üçe ayrılmaktadır. Aritmetik genelleme, öğrencilerin örüntünün ardışık terimleri arasındaki ortak farkı farkedip, 'Ux+1=Ux+ortak fark' formunda yinelemeli bir ifade oluşturmasıdır (Gutiérrez, 2013). Örneğin, Şekil 1'de verilen şekil örüntüsünde, öğrencilerin örüntünün her adımında ikişer ikişer arttığını söylemesi, aritmetik genellemeye ait bir ifadedir.

Olgusal genelleme, temel seviyedeki cebirsel genellemedir. Olgusal genelleme kapsamındaki ifadeler örüntünün verilmiş adımları için geçerlidir (Radford, 2010). Örneğin, öğrencilerin Şekil 1'de verilen örüntünün ilk terimini "bir, bir, artı üç.....", ikinci terimini "iki, iki, artı üç..." olarak ortaya koyması ve 25. Terimi '25, 25, artı 3' olarak ifade etmesi, olgusal genelleme olarak adlandırılır. Ancak, herhangi bir adımdaki çember sayısını bulmak için, aritmetik genelleme yada olgusal genelleme yeterli olmamaktadır. Bu yüzden, Radford, kavramsal genellemeyi tanımlamıştır. Kavramsal genelleme, aritmetik ve olgusal genellemelerden daha derin olup, örüntünün verilmiş adımlarının dışında herhangi başka bir adıma örüntünün kuralını soyutlayabildikleri durumu ifade eder. Kavramsal genellemeye örnek olarak, öğrencinin Şekil 1'deki soruya şu cevabı verilebilir: 'birinci adımda bir, bir, artı üç; ikinci adımda iki, iki, artı üç; dolayısıyla onuncu adımda on, on, artı üç olur... O zaman genel kural adım sayısını iki ile çarpıp üç eklemektir.' (Radford, 2010a). Son olarak, sembolik genelleme, örüntünün kuralını alfanümerik sembollerle ifade etmeye denmektedir. Şekil 1'de verilen örüntünün kuralı 'Eğer genel kural adım sayısını 2 ile çarpıp 3 eklemek ise, formül de 2.n+3 olur.' olarak ifade edildiğinde, bu sembolik genellemeye örnek oluşturur.

Mevcut çalışma kapsamında, öğrencilerin cebirsel düşünme becerileri, Radford'un örüntü genelleme kuramına dayanılarak araştırılmıştır. Öğrencilerin örüntüleri genelleme biçimleri, Radford'un tanımladığı dört biçim (aritmetik, olgusal, kavramsal, sembolik) altında incelenmiştir.

Yöntem

Çalışma Deseni

Ortaokul öğrencilerinin aritmetik genelleme, cebirsel genelleme ve naif tümevarım yöntemlerini kullanarak yaptığı genelleme süreçlerini keşfetmeyi amaçlayan bu çalışmada, nitel araştırma yöntemlerinden durum çalışması deseni kullanılmıştır. İlgili literatüre göre, dört farklı türde durum çalışması vardır: bütüncül tek durum deseni, bütüncül çoklu durum deseni, iç içe geçmiş tek durum deseni ve iç içe geçmiş çoklu durum deseni (Yin, 2009). Mevcut çalışmada bu türler arasından bütüncül çoklu durum deseni kullanılmıştır.

Katılımcılar

Katılımcılar, araştırmanın amacına yönelik olarak Ankara'daki bir devlet ortaokulunun, altıncı, yedinci ve sekizinci sınıflarında eğitim gören öğrenciler arasından seçilen 5 altıncı sınıf, 4 yedinci sınıf, 5 sekizinci sınıf öğrencisidir. Bu çalışmada toplanan verinin zenginliği önemli olduğu için amaçlı örneklem kullanılmıştır. Uygun yer ve zaman nedeniyle, araştırmaya katılanlar araştırmacının sekiz ay boyunca matematik öğretmeni olarak çalıştığı devlet okulundan seçilmiştir. Katılımcıların seçiminden önce, öğrencilere çalışmanın amacı hakkında açıkça bilgi verilmiştir. Araştırmacı, tüm öğrencileri kişisel olarak tanıdığından, çalışma için zengin bilgi sağlayabilecek en uygun katılımcıları seçmiştir. Öğrencilerin yaşları 11 ile 14 arasında değişmektedir. Öğrencilerin sosyoekonomik durumları genellikle ortalamadır. Öğrencilerin aileleri kamu ve özel sektör çalışanlarıdır.

Veri Toplama Araçları

Bu çalışmada veriler Örüntü Testi ve bireysel görüşmeler vasıtasıyla toplanmıştır. Veri toplama araçları ile ilgili detaylı bilgi aşağıda verilmiştir.

Örüntü Testi

Bu çalışmanın verileri Örüntü Testi yoluyla toplanmıştır. Örüntü Testi'nde 6 adet açık uçlu doğrusal örüntü sorusu vardır. Sorular geçmiş çalışmalarda kullanılan sorulardan uyarlanmıştır. Soruları seçerken, Ulusal İlköğretim Matematik Eğitim Müfredatındaki ilgili hedefleri dikkate alarak bir belirtke tablosu hazırlanmıştır. Müfredat doğrusal sayı ve şekil örüntüleri ile sınırlıdır (Milli Eğitim Bakanlığı [MEB], 2013, 2018). Müfredat kısıtlamalarına dayanarak, testteki sorular doğrusal-sayısal ve doğrusal-şekilsel sorular olarak sınıflandırılmıştır. Ek olarak, bu testin amacı katılımcıların cebirsel akıl yürütme becerileri hakkında veri toplamaktır. Temel cebirsel akıl yürütme becerileri, örüntüyü tanımlamak, örüntüyü yakın ve uzak terimlere genişletmek, genel terimi bulmak ve örüntü için genel bir kural oluşturmaktır (Threlfall, 1999). Bu nedenle, teste ilişkin soruları belirlerken araştırmacı, katılımcıların bir örüntü tanımlamasını, örüntüyü yakın ve uzak terimlere genellemesini ve genel terimi bulmasını sağlayacak maddeleri seçmeyi amaçlamıştır. Buna göre, her soru yakın genelleme, uzak genelleme ve genel kural ile ilgili dört veya beş maddeyi içermiştir. 25 öğrenci ile yapılan pilot çalışma sonucu Örüntü Testi tekrar düzenlenmiştir.

Bireysel Görüşmeler

Bu çalışmada, görüşülen kişinin yanıtlarına bağlı olarak soruların sıralamasını değiştirmeye ve gerekirse ek sorular sormaya olanak verdiği için, yarı-yapılandırılmış görüşme yaklaşımı kullanılmıştır. Yarı-yapılandırılmış görüşme rehberi, bir öğrenci ile yapılan pilot çalışma sonucunda tekrar düzenlenmiştir.

Veri Toplama Süreci

Veri toplama sürecinde öğrenciler ile teker teker görüşme yapılmıştır. Bu görüşmeler sırasında öğrenciler Örüntü Testi'ndeki soruları cevaplamışlar, araştırmacı ise bu sırada gerekli durumlarda öğrenciye soruların cevaplarına dair nasıl ve neden soruları sormuştur. Örneğin 'Örüntünün onuncu adımındaki çember sayısını nasıl buldun?'. Son olarak, bütün görüşme, ses ve görüntü kaydına tabi tutulmuştur.

Veri Analizi

Nitel araştırmalarda veri analizi, verinin analiz için hazırlanması ve düzenlenmesi, verinin kodlama süreci sonucunda belirli bir düzene indirgenmesi ve nihayetinde verinin şekiller, tablolar veya tartışma şekilinde ifade edilmesi süreçlerini kapsar (Creswell, 2007). Bu yüzden, ilk olarak bütün ses ve görüntü kayıtları yazıya dökülmüştür. İkinci adım kodlama olmuştur. Kodlar (i) geçmişte konuyla ilgili yapılan çalışmalar ve (ii) ana çalışma süresince toplanan veriler göz önüne alınarak türetilmiştir.

Yazıya dökülen veriler ilk önce araştırmacı tarafından kodlanmıştır. Güvenirliği sağlamak için, bir başka kodlayıcı kodlama protokolünü araştırmacının hazırladığı kodları açıklayan bir kodlama anahtarını kullanarak tekrar etmiştir. Veri analizinin son adımı olarak, yazıya dökülmüş veri kodları, kategoriler açısından analiz edilmiş, figürler, tablolar ve tartışma şeklinde temsil edilmiştir.

BULGULAR

Bu bölüm altıncı, yedinci ve sekizinci sınıf öğrencilerinin aritmetik genelleme, cebirsel genelleme ve naif tümevarım yöntemlerini kullanarak oluşturduğu genelleme süreçlerine dayanarak düzenlenmiştir.

Altıncı, Yedinci ve Sekizinci Sınıf Seviyesindeki Öğrencilerin Doğrusal Örüntüleri Genelleme Süreçleri

Öğrencilerin genelleme süreçlerinin analizi, altıncı, yedinci ve sekizinci sınıf öğrencilerinin örüntü genelleme sürecinde en az iki genelleme stratejisi (aritmetik genelleme, cebirsel genelleme, naif tümevarım) kullandıklarını ortaya koymuştur. Bulgulara göre, (i) sadece cebirsel genelleme stratejilerini, (ii) aritmetik genelleme ve cebirsel genelleme stratejilerinin kombinasyonunu, (iii) aritmetik genelleme ve naif tümevarım stratejilerinin kombinasyonunu ve (iv) aritmetik genelleme, cebirsel genelleme ve naif tümevarım stratejilerinin kombinasyonunu içeren dört tür genelleme süreci ortaya çıkmıştır.

Sadece Cebirsel Genelleme Stratejilerini İçeren Genelleme Süreci

Mevcut çalışmada sadece cebirsel genelleme stratejilerini içeren genelleme süreci altıncı sınıf seviyesinde 8 kez, sekizinci sınıf seviyesinde 2 kez görülmüştür. Yedinci sınıf seviyesinde ise görülmemiştir.

Detaylı olarak, altıncı sınıf seviyesinde, birinci, dördüncü ve altıncı sorularda iki öğrenci ve ikinci ve beşinci sorularda bir öğrenci; sekizinci sınıf seviyesinde, birinci ve altıncı sorularda başka bir öğrenci doğrusal örüntüleri sadece cebirsel genelleme stratejileri kullanarak genellemiştir. Aşağıda altıncı sınıf seviyesindeki bir öğrencinin altıncı sorudaki örüntüyü genelleme sürecinden bir örnek verilmiştir:

> S6/P1: (İlk olarak terimlerin altına terim sayılarını yazar) Bunda 1 ile 4'ü çarptım, 4 etti. 2 ile 4'ü çarptım, 8 etti. 3 ile 4'ü çarptım, 12 etti. Bunun kuralı ise adım sayısı çarpı 4. Bununla (ikinci adımda 2'yi gösterir) 4'ü çarptığımda 8 ediyor. O zaman n çarpı 4 yapıyorum (n4 yazar).
>
> R: Kuralını örnekleyebilir misin?
>
> P1: Dördüncü adımda da 16 ediyor. Beşinci adımda da 20 ediyor. Onuncu adımda da 40 ediyor. Yirminci adımda da 80 ediyor. Yüzüncü adımda da 400 ediyor.

Örnekten de anlaşılacağı üzere, P1 ilk önce, olgusal genelleme olarak kodlanan '4 ile çarpma' sayısal kuralını oluşturarak birinci, ikinci ve üçüncü adımlara genellemiştir. Ardından genel kuralı, kavramsal genelleme olarak kodlanan 'adım numarası'nı söyleyerek ifade etmiş, sonunda ise genel kuralı sembolik genelleme ile yazmıştır.

Aritmetik Genelleme ve Cebirsel Genelleme Kombinasyonunu İçeren Genelleme Süreci

Altıncı, yedinci ve sekizinci sınıf seviyesindeki öğrencilerde en çok görülen genelleme süreci aritmetik genelleme ve cebirsel genelleme kombinasyonunu içermiştir. Mevcut çalışmada bu genelleme süreci altıncı sınıf seviyesinde 14 kez, yedinci sınıf seviyesinde 16 kez, sekizinci sınıf seviyesinde ise 18 kez görülmüştür.

Detaylı olarak, altıncı sınıf seviyesinde, birinci ve üçüncü sorularda üç öğrenci, ikinci, dördüncü, beşinci ve altıncı sorularda iki öğrenci; yedinci sınıf seviyesinde, birinci ve dördüncü sorularda bir öğrenci, dördüncü soru hariç diğer tüm sorularda bir öğrenci, ikinci soru hariç diğer tüm sorularda başka bir öğrenci ve birinci, ikinci, dördüncü ve altıncı sorularda bir öğrenci; sekizinci sınıf seviyesinde, birinci soruda üç öğrenci, ikinci, üçüncü, beşinci sorularda iki öğrenci, dördüncü soruda beş öğrenci ve altıncı soruda dört öğrenci doğrusal örüntüleri hem aritmetik hem cebirsel genelleme stratejilerini kullanarak genellemiştir. Aşağıda altıncı sınıf seviyesindeki bir öğrencinin beşinci soruyu genelleme sürecinden bir örnek verilmiştir:

> S5/P1: İlk adımda 5 tane var, (ikinci adımda) 8 tane var, (üçüncü adımda) 11 tane var. Burda 3'er 3'er arttığı için adım sayısı çarpı 3 artı 2 (n3+2 yazar). Burda (birinci adımda) 3'ü çarptığımızda 3 ediyor, 2 eklediğimizde 5 ediyor. (İkinci adımda) 3 ile 2'yi çarptığımızda 6 ediyor, 2 eklediğimizde 8 ediyor. Üçte, 3 ile çarptığımızda 9, 2 eklediğimizde 11 ediyor. Dördüncü adımda da 4 ile 3'ü çarpacağız, 12 ediyor, 2 eklediğimizde 14 ediyor. Beşinci adımda 5 ile 3'ü çarptığımızda 15, 2 daha, 17. Onuncu adımda da 10 ile 3'ün çarpımı 30 ediyor, 2 ekleyeceğiz, 32. Yüzüncü adımda da 100 ile 3'ü çarpacağız, 300 edecek, 2 ekleyeceğiz, 302.

P1 genelleme sürecine ardışık adımlar arasındaki sabit farkı vurgulayarak aritmetik genelleme ile başlamıştır. Ardından genel kuralı 'adım sayısı çarpı 3 artı 2' diye ifade ederek kavramsal genellemeye geçmiştir. Hemen ardından kavramsal genelleme ile bulduğu genel kuralı harfler ile ifade etmiş, bu da sembolik genelleme olarak kodlanmıştır. Son olarak birinci, ikinci, üçüncü, dördüncü, beşinci, onuncu ve yüzüncü adımdaki sayıları bulduğu genel kuralı uygularak hesaplamıştır.

Aritmetik Genelleme ve Naif Tümevarım Kombinasyonunu İçeren Genelleme Süreci

Mevcut çalışmada aritmetik genelleme ve naif tümevarım kombinasyonunu içeren genelleme süreci altıncı sınıf seviyesinde 7 kez, yedinci sınıf seviyesinde 1 kez, sekizinci sınıf seviyesinde ise 7 kez görülmüştür.

Detaylı olarak, altıncı sınıf seviyesinde, ikinci ve üçüncü sorularda iki öğrenci, dördüncü, beşinci ve altıncı sorularda ise bir öğrenci; yedinci sınıf seviyesinde, altıncı soruda bir öğrenci; sekizinci sınıf seviyesinde ise birinci, dördüncü ve altıncı sorularda iki öğrenci ve ikinci ve beşinci sorularda bir öğrenci; sekizinci sınıf seviyesinde ikinci ve beşinci sorularda bir öğrenci, ikinci ve üçüncü sorularda bir öğrenci, beşinci soruda üç öğrenci doğrusal örüntüleri aritmetik genelleme ve naif tümevarım stratejilerini kullanarak genellemiştir. Aşağıda altıncı sınıf seviyesindeki bir öğrencinin beşinci sorudaki örüntüyü genelleme sürecinden bir örnek verilmiştir:

> S5/P2: Birinci adımda 5 tane, ikinci adım 8, üçüncü adımda 11 tane olduğu için 3'er 3'er artmış. 11'e 3 eklersem 14 eder. Demek ki dördüncü adımı inşa etmek için 14 tane kürdan gerekli. Beşinci adım için 14'e 3 eklersek 17 eder, 17 tane kürdan lazım.. Adım sayısı çarpı 2 desek.. İkinci adımda tutmuyor. Adım sayısı çarpı 3 artı 2 desek? (Birinci adımda) 1 kere 3 artı 2, 5 eder. Oldu. (İkinci adımda) 2 kere 3 artı 2, 8 eder. Evet, genel kural adım sayısı çarpı 3 artı 2.

P2, aritmetik genelleme olarak kodlanan, önceki terimlere 3 ekleyerek dördüncü ve beşinci terimlerdeki kürdan sayısına ulaşmıştır. Daha sonra genel kuralı bulmak için bazı kurallar denereyerek naif tümevarım stratejisini kullanmıştır.

Aritmetik Genelleme, Cebirsel Genelleme ve Naif Tümevarım Kombinasyonunu İçeren Genelleme Süreci

Mevcut çalışmada, aritmetik genelleme, cebirsel genelleme ve naif tümevarım kombinasyonunu içeren genelleme süreci altıncı sınıf seviyesinde 1 kez, yedinci sınıf seviyesinde 7 kez, sekizinci sınıf seviyesinde ise 3 kez görülmüştür.

Detaylı olarak, altıncı sınıf seviyesinde beşinci soruda bir öğrenci; yedinci sınıf seviyesinde, ikinci, üçüncü ve beşinci sorularda bir öğrenci, üçüncü, dördüncü ve beşinci sorularda bir öğrenci ve dördüncü soruda başka bir öğrenci doğrusal örüntüleri aritmetik genelleme, cebirsel genelleme ve naif tümevarım stratejilerini kullanarak genellemiştir. P6'nın üçüncü sorudaki genelleme sürecine ilişkin örnek aşağıda verilmiştir:

> S3/P6: Aynı arkada çözdüğümüz gibi. Yani (n+1) çarpı (n+1). n yine adım sayısı. Tekrar deneyeceğim. Bunda tutmadı. Aaa buldum sanki. (n+1)x2.
>
> R: Nasıl ulaştın buna?
>
> P6: Şöyle hem böyle yana hem burayı saydım 4 oldu. Yani ben 8 yaptım ama 7'ymiş. (N+1)x2... Bu da tutmuyor. Önce dördüncü adımı bulayım. Burada sadece yatay artıyor. 5 tane yatay olacak. Yukarıya doğru da yine 1 artıyor. Aaa (adımın) kendisi varmış burada. Alttaki 1 artıp yukarıdaki adımın kendisi oluyor. N+1+N. Kuralı buldum şimdi. Dördüncü adım 4+1+4=9. Bu tuttu. Beşinci adımda da 11. Evet bu da doğru. Onuncu adımda 11+10=21. Ellinci adımda da 101.

P6 genelleme sürecine naif tümevarım ile başlamıştır. İki kural denemiş, fakat bunların hiçbiri örüntünün terimlerini sağlamamıştır. Birden örüntünün ardışık figürleri arasındaki aritmetik artışı görmüş ve terimlerdeki çember sayısı ve terim sayısı arasındaki ilişkiyi farketmiştir. Bu sebeple, öğrencinin bu süreci kavramsal genelleme ve sembolik genelleme olarak kodlanmıştır.

Tartışma

Bu çalışmada öğrencilerin aritmetik genelleme, cebirsel genelleme ve naif tümevarım stratejilerini kullanarak yaptığı, doğrusal örüntüleri genelleme süreçleri incelenmiştir. Öğrencilerin genelleme süreçlerinin analizi altıncı, yedinci ve sekizinci sınıf öğrencilerinin örüntü genelleme sürecinde en az iki genelleme stratejisi (aritmetik genelleme, cebirsel genelleme, naif tümevarım) kullandıklarını ortaya koymuştur. Bulgulara göre, (i) sadece cebirsel genelleme stratejilerini, (ii) aritmetik genelleme ve cebirsel genelleme stratejilerinin kombinasyonunu, (iii) aritmetik genelleme ve naif tümevarım stratejilerinin kombinasyonunu ve (iv) aritmetik genelleme, cebirsel genelleme ve naif tümevarım stratejilerinin kombinasyonunu içeren dört tür genelleme süreci ortaya çıkmıştır.

Öğrencilerin, kullandıkları genelleme stratejilerini temel alan genelleme türleri belirlendikten sonra, altıncı, yedinci ve sekizinci sınıf seviyelerine göre kıyaslandığında aralarında bazı benzerlikler ve farklılıklar olduğu görülmüştür. İlk benzerlik, aritmetik genelleme ve cebirsel genelleme kombinasyonunu içeren genellemenin, öğrencilerin sınıf seviyesine bakılmaksızın en sık kullanılan genelleme süreci olduğudur. Bulgular aynı zamanda, örüntüyü yakın terimlere genellemek için aritmetik genelleme stratejisini, ve uzak terimlere genellemek veya örüntünün genel kuralını bulmak içinse cebirsel genelleme stratejilerini kullandıklarını göstermiştir. Başka bir deyişle, öğrenciler aritmetik genellemeden cebirsel genellemeye doğru bir yol izlemiştir. Bu bulgular, geçmiş çalışmaların sonuçları ile tutarlılık göstermektedir (Amit ve Neria, 2007; Lannin, 2004; Lannin, Barker ve Townsend, 2006; Orton ve Orton, 1999; Stacey ve MacGregor, 2001). Geçmişte yapılan çalışmalar, örüntüleri genellemeye aritmetik genelleme ile başlamanın doğal olduğunu göstermiştir (Garcia-Cruz ve Martinón, 2002; Lannin, 2004). Buna ek olarak, literatür aynı zamanda genelleme stratejilerini belirlerken örüntü sorularının doğasının öğrenciler için belirleyici olabileceğini göstermiştir (Barbosa & Vale, 2015; Lannin vd., 2006). Örneğin, örüntü sorularının sıralı dizi şeklinde adım adım gösterilmesi, öğrencileri yinelemeli akıl yürütmeye, yani aritmetik genellemeye, yönlendirebilir. Mevcut çalışmada da, Örüntü Testi'ndeki tüm örüntüler sıralı dizi şeklinde adım adım gösterilmiştir. Öte yandan, literatürde, bir örüntüyü uzak bir terime genellemenin örüntünün cebirsel altyapısının anlaşılmasına yardım ettiği ifade edilmiştir. Bu yüzden, bu çalışmada aritmetik genellemeden cebirsel genellemeye akışın yüksek oranda olmasının sebebi ya örüntünün terimlerinin sıralı dizi şeklinde verilmesi yada yakın ve uzak genelleme sorularının sorulması olabilir.

Bütün sınıf seviyelerinde ortak olan başka bir bulgu da her sınıf seviyesinde düşük oranda naif tümevarımın yani deneme yanılma stratejisinin kullanılmış olmasıdır. Benzer sonuçlar, geçmiş çalışmalarda da görülmüştür (Barbosa, 2011; Lannin vd., 2006; Özdemir, Dikici ve Kültür, 2015; Rivera ve Becker, 2005). Literatürde belirtildiği gibi, matematik eğitimi çoğunlukla örüntünün genel kuralını oluşturmaya yönelik prosedürel bilgiye odaklanmıştır (Lannin vd., 2006). Bu yüzden öğrenciler örüntülerin cebirsel yapısını kavramsal olarak anlayamamaktadır (Noss, Healy ve

Hoyles, 1997). Bunun yerine, genel bir kuralın nasıl oluşturulacağıyla ilgili prosedürlere yönelik becerilerini pratik etmektedirler. Bu nedenle, örüntü genelleme ile ilgili kavramsal matematik öğretiminin bulunmayışı, öğrencilerin naif tümevarım stratejisini kullanmalarını etkilemiş olabilir.

Bu iki benzerliğin yanında, sınıf seviyeleri arasında görülen bir farklılık da, sadece cebirsel genelleme stratejilerinin yalnızca altıncı sınıf seviyesinde kullanılmış, yedinci ve sekizinci sınıf seviyelerindeyse neredeyse hiç kullanılmamış olmasıdır. Bu bulgu, sınıf seviyesi arttıkça öğrencilerin cebirsel genelleme becerilerinin arttığını gösteren geçmiş çalışmalar ile tutarsızlık göstermektedir (El Mouhayar, 2018). Bu bulgunun altında yatan sebep, Ulusal Ortaokul Matematik Müfredatı’nda örüntü genelleme ile ilgili kazanımların sadece beşinci ve altıncı sınıf seviyelerinde bulunması olabilir (MEB, 2013).

Bahsedilen benzerlik ve farklılıklarla beraber, çalışmanın önemli bir bulgusu olarak, neredeyse bütün öğrenciler genelleme sürecinin ilk adımı olarak aritmetik genelleme ile örüntünün yakın terimlerini bulmuşlardır. Ardından, örüntüyü uzak terimlere genellemeleri istendiğinde, bu terimleri hesaplamak yerine genel terimi bulmaya yönelmişlerdir. Ve buldukları genel terimle uzak terimleri hesaplamayı tercih etmişlerdir. Bu sonuç, bazı geçmiş çalışmalar ile tutarlılık gösterirken (Özdemir vd., 2015), bazıları ile tutarsızlık göstermiştir (Cooper ve Warren, 2011; Miller ve Warren, 2012; Radford, 2003). Bu sonuç şu iki sebepten kaynaklanmış olabilir. Öncelikle, Ulusal Ortaokul Matematik Müfredatı’nda bulunan örüntü genellemeye yönelik kazanımlar, yakın veya uzak genelleme süreçlerini kapsamamaktadır. Bunun yerine, bu kazanımlar örüntünün genel kuralını bulmaya yada genel kuralı verilmiş bir örüntüde istenilen terimi hesaplamaya odaklanmıştır (MEB, 2013). İkinci olarak Ulusal Matematik ders kitaplarındaki örüntü genelleme soruları müfredatımızdaki kazanımlara paralel olarak öncelikle öğrencinin genel terimi bulmasını istemektedir. Bulduğu genel terimle yakın ve uzak terimleri hesaplattırmaktadır. Özetle, mevcut çalışmada, Ulusal Ortaokul Matematik Eğitim Müfredatı'nda yakın veya uzak genellemeden ziyade genel kuralı bulmaya öncelik veren kazanımların varlığı veya Ulusal Matematik ders kitaplarında bulunan örüntü genelleme sorularının da genel kural bazlı sorular içermesi, öğrencilerin önceliği yakın yada uzak genellemeye değil de, genel kuralı bulmaya vermesinin sebebi olabilir.

Doğurgalar

Bu çalışmanın bulguları, ortaokul matematik öğretmenleri ve program geliştiriciler için önemli bilgiler sunmaktadır.

Cebirsel genellemenin kavramsal olarak anlaşılmasını sağlamak için, eğitim ortamları, örüntünün genel terimini kural tabanlı bir şekilde bulma tekniklerini uygulamak yerine, öğrencilerin genellemelerin doğasını keşfedebilecekleri ve gelişimsel olarak yakın ve uzak genellemeler yapabilecekleri şekilde tasarlanabilir. Bu tür eğitim ortamlarını sağlamak için, ilk olarak, matematik eğitimi müfredatının örüntü genellemeyi daha kavramsal olarak kapsaması önerilmektedir. Başka bir deyişle, ilköğretim matematik dersi öğretim programında örüntü genellenmesi ile ilgili iki kazanım vardır. Bunlardan ilki, beşinci sınıf düzeyindeki "Kuralı verilen sayı ve şekil örüntülerinin istenen adımları oluşturur." kazanımı (MEB, 2013, s. 2) ve ikincisi, altıncı sınıf düzeyindeki "Aritmetik dizilerin kuralını harfle ifade eder; kuralı harfle ifade edilen dizinin istenilen terimlerini bulur." (MEB, 2013, s. 18) kazanımlarıdır. Bu amaçların her ikisinde de, odak, örüntünün genel kuralını bulmak ve bulunan genel kuralı uygulamak üzerinedir. Öte yandan, örüntülerin cebirsel yapısını anlamlandırmak için yakın ve uzak genelleme süreçlerini yürütmek büyük önem taşımaktadır. Bu nedenle, örüntülerin cebirsel altyapısını anlamlandırmaya ve yakın ve uzak genelleme süreçlerini yürütmeye dayanan kazanımlar, örüntünün genel kuralını vurgulamadan önce ortaokul matematik müfredatına eklenebilir.

Ayrıca, Ulusal Matematik ders kitaplarında yetersiz sayıda örüntü genelleme soruları olduğu ve bu sorularda da öğrencilere yakın ve uzak genellemeden önce genel kuralı buldurmaya yönelik alt sorular olduğu bildirilmiştir (Ayber, 2017). Dolayısıyla, ders kitaplarının içeriği buna göre revize edilebilir. Örüntü genelleme sorularının sayısı arttırılabilir ve aşamalı olarak genel kuraldan önce yakın ve uzak genellemeleri içerebilir.

Bundan başka, literatürde, öğretmenlerin aritmetik ve cebir ilişkisi (Demonty, Vlassis ve Fagnant, 2018) ve örüntü genelleme ile ilgili yeterli bilgiye sahip olmadıkları belirtilmiştir (Girit, 2016). Öğretmenler, örüntüleri öğrencilere kurala dayalı bir şekilde ortaya koymaktadırlar (Lannin vd., 2006). Dolayısıyla, öğretmenler aritmetik ve cebirsel genellemeler açısından kendilerini geliştirebilirler. Öğretmenlerin aritmetik ve cebirsel genellemeler hakkındaki bilgilerini arttırmak için seminerler veya hizmet içi çalışmalar düzenlenebilir.

Gelecek Çalışmalar için Öneriler

Bu çalışma, altıncı, yedinci ve sekizinci sınıf öğrencilerinin aritmetik genelleme, cebirsel genelleme ve naif tümevarım kullanarak yürüttüğü doğrusal örüntüleri genelleme sürecine odaklanmıştır. Sonuçlara göre, bu kısımda ileri çalışmalar için bazı önerilerde bulunulmuştur.

Bu çalışmaya, altıncı, yedinci ve sekizinci sınıf seviyelerinden öğrenciler katılmıştır; dolayısıyla çalışmanın sonuçları, altıncı, yedinci ve sekizinci sınıf öğrencileri ile sınırlıdır. Gelecek çalışmalarda yaş aralığı beşinci sınıftan on ikinci sınıfa genişletilerek, artan sınıf seviyeleri arasındaki mevcut eğilimler incelenebilir.

Bundan başka, bu çalışmada öğrenciler önceden belirlenmiş 3 kritere göre seçilmişlerdir, bu kriterler öğrencilerin sınıf seviyesi, matematik dersine karşı istekli olmaları ve konuşkan olmalarıdır. Gelecek çalışmalarda, öğrencilerin başarısını da kriterler arasına dahil ederek, yüksek/düşük başarı bazında öğrencilerin genelleme yaklaşımları incelenebilir.

Ek olarak, bu çalışmada, tümü sıralı dizi şeklinde temsil edilen altı örüntü genelleme sorusu yer almıştır. Literatüre göre, sıralı dizi şeklinde sunulan örüntüler, öğrencileri yinelemeli ilişkileri kullanmaya ve cebirsel ilişkilerden caydırmaya teşvik edebilir. Bu nedenle, ilerideki çalışmalarda, örüntü genelleme soruları sıralı dizi şeklinde sunulma ile sınırlı olmayabilir.

Son olarak, mevcut çalışmada, örüntü soruları sayısal ve şekilsel örüntüler içermektedir. İleri çalışmalarda, sayısal ve şekilsel örüntüleri genelleme yaklaşımları arasındaki farkı bulma maksatlı bir çalışma yürütülebilir.

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YAZARIN / AUTHOR

Soyadı / Surname : KAMA
Adı / Name : Zeycan
Bölümü / Department:İlköğretim Fen ve Matematik Eğitimi / Elementary Science and Mathematics Education

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