HOUSING MARKET DYNAMICS AND ADVANCES IN MORTGAGES: OPTION BASED MODELING AND HEDGING

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ABSTRACT

HOUSING MARKET DYNAMICS AND ADVANCES IN MORTGAGES:
OPTION BASED MODELING AND HEDGING

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In the last two decades, academicians and professionals intending to study in any area of real estate and finance not only must master advanced financial mathematics concepts and mathematical/econometric models but also should be able to implement those concepts computationally to improve real estate markets' efficiency. This comprehensive thesis mainly aims to combine the theory of financial mathematics with an emphasis on real-life applications in keeping with the way, both investors and policymakers, in today's real estate markets. Unlike most studies on real estate markets, housing markets and mortgages, the thesis covers both non-parametric statistical modeling methods (Multivariate Adaptive Regression Splines (MARS) and Generalized Linear Models (GLM)) and stochastic calculus (Stochastic Differential Equations (SDE), Malliavin calculus theory) with Monte Carlo simulations, Capital Asset Pricing Model (CAPM) and Fama French three-factor model with its extensions. The thesis offers thorough models in the subject of housing markets and provides hedging strategies of default and prepayment options embedded into mortgage contracts. Along with with the theoretical aspects, the thesis presents numerous applications for pricing, investment decision, risk management via hedging strategies, and portfolio management. The numerical illustrations are on determining the housing market price drivers and the effect of large investors, house price forecasting by using the US housing market data, determining hedging strategies for both mortgage default and prepayment options by computing the hedging coefficients via using Monte Carlo simulations and analyzing the T-REITs returns performance in various aspects.

Keywords: Housing Markets, Large Investors, Portfolio Optimization, MARS, GLM, SDE, Malliavin Calculus, Mortgage, Hedging, CAPM, T-REIT, Fama-French Model

ÖZ

KONUT PİYASASI DİNAMİKLERİ VE MORTGAGELARDA İLERİ TEKNİKLER: OPSİYONA DAYALI MODELLEME VE HEDGİNG

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Son yirmi yılda, gayrimenkulün ve finansın herhangi bir alanında çalışma yapmak isteyen akademisyenler ve uzmanlar sadece ileri finansal matematik kavramlarında ve matematiksel/ekonometri modellerinde uzmanlaşmanın yanı sıra gayrimenkul piyasalarının etkinliğini arttırmak için bu kavramları aynı zamanda uygulayabilmelidirler. Bu kapsamlı tez, finansal matematik teorisinin gerçek hayat uygulamalarını göz önüne alarak, hem yatırımcılar hem de hükümetler için bu günün gayrimenkul piyasaları uygulamaları ile birleştirmeyi hedeflemektedir. Gayrimenkul piyasaları, konut piyasaları ve mortgage üzerine yapılan çoğu araştırmanın aksine bu tez, hem parametrik olmayan istatistiksel modelleme yöntemlerini (Çok Değişkenli Adaptif Regresyon Splineları (MARS) ve Genelleştirilmiş Lineer Modeller (GLM)) hem de stokastik analizi (Stochastik Diferansiyel Denklemlerini (SDE), Malliavin analiz teorisini), Monte Carlo simülasyonları ile beraber sermaye varlıkları fiyatlandırma modelini (CAPM) ve genişletilmiş halleriyle beraber Fama-French üçlü faktör modellerini kapsamaktadır. Bu tez, konut piyasaları konusunda kapsamlı modeller sunmakta ve mortgage içerisinde bulunan erken ödeme ve ödemeyi durdurma opsiyonları için hedge stratejileri sağlamaktadır. Teorik bakış açılarıyla birlikte bu tez, fiyatlandırma, yatırım kararları, hedge yoluyla risk yönetimi ve portföy yönetimi ile ilgili sayısız uygulama sunmaktadır. Sayısal uygulamalar konut piyasalarını etkileyen faktörleri ve büyük yatırımcıların etkilerini belirlemekte, ABD konut piyasası verilerini kulla-

narak konut fiyat tahminleri yapmakta, Monte Carlo simülasyonu ile hem mortgage erken ödeme hem de mortgage ödemeyi bırakma opsiyonlarının hedge katsayılarını hesaplanmakta ve hedge stratejileri belirlemekte ve GYO'larının getirilerinin performanslarını çeşitli yönlerden analiz etmektedir.

Anahtar Kelimeler: Konut Piyasaları, Büyük Yatırımcılar, Portföy Optimizasyonu, MARS, GLM, SDE, Malliavin Analiz, Mortgage, Hedging, CAPM, GYO, Fama-French Modeli

To My Family
and
Dear Wife

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her.

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LIST OF ABBREVIATIONS

ADF	Augmented Dickey-Fuller
BIST	Borsa Istanbul Stock Exchange
C^2	The Class of Twice Differentiable Functions
C_P^∞	The Class of Infinitely Differentiable Functions with Polynomial Growth
CAPM	Capital Asset Pricing Model
CIR	Cox Ingersoll Ross short rate process
CME	Chicago Mercantile Exchange
CPI	Consumer Price Index
DSM	Debt Securities Market
$\mathbb{D}^{1,2}$	Ones Differentiable and Twice Integrable Stochastic Sobolev Space
EFFR	Effective Federal Funds Rate
ER	US / Euro Foreign Exchange Rate, US Dollars to One Euro
EREIT	Equity Real Estate Investment Trust
EPRA	The European Public Real Estate Association
FHA	Federal Housing Administration
FRM	30-Year Fixed Mortgage Rate
FTSE	Financial Times Stock Exchange
GCV	Generalized Cross-Validation
GDP	Gross Domestic Product
GLM	Generalized Linear Regression Models
GSP	Henry Hub Natural Gas Spot Price
HML	High Minus Low
HPI	S&P Case Shiller Home Price Index
IMI	Investable Market Index
IPO	Initial Public Offering
JB	Jarque-Bera Test
L^2	The Square Integrable Function Space

L-GLM	Linear-GLM
LOF	Lack-of-Fit
LSM	Least Square Method
MAE	Mean Absolute Error
MARS	Multivariate Adaptive Regression Splines
MBS	Mortgage-Backed Securities
MSCI	Morgan Stanley Capital International
NAREIT	National Association of Real Estate Investment Trusts
OU	Ornstein–Uhlenbeck
P/B	Price-to-Book ratio
PDE	Partial Differential Equations
PWI	Proportion of Residuals within Three Sigma
R^2	Coefficient of Determination
RC	Recession Cycle in the Economy
REIT	Real Estate Investment Trust
RDI	Real Disposable Personal Income: Per Capita
RMSE	Root Mean Square Error
RSS	Residual Sum of Squares
$\mathcal{S}$	The Class of Smooth Random Variables
SC	Small Caps
SDE	Stochastic Differential Equation
SMB	Small Minus Big
TBill	10-Year Treasury Constant Maturity Rate
TÜBİTAK	The Scientific and Technological Research Council of Turkey
TREIT	Tayland REIT
T-REIT	Turkish REIT
TCU	Capacity Utilization: Total Industry, Percent of Capacity
UER	Civilian Unemployment Rate
Q-GLM	Quadratic-GLM
VA	Department of Veterans Affairs
WAPR	Working-Age Population Aged

CHAPTER 1

INTRODUCTION

1.1 Motivation

The real estate market represents all transactions, including ownership rights, use of properties, and property-based products. It is one of the leading and locomotive markets in national economies due to its high dependence on domestic capital, the creation of high added value, the magnitude of its employment potential, and strong association with other markets, such as financial and commodity markets. Starting from the early nineties, the development in the acceleration of capital flows across countries thanks to the economic globalization, increasing liquidity and addition of property based investment tools to the field in which capital flows are interested has magnified the influence of the real estate market on national economies. Especially housing markets, which are the largest sub-markets of real estate markets, commence to influence national economies remarkably following the early nineties. Hence, the real estate market, particularly housing markets, has unavoidable implications on national economies.

In all industrialized countries, houses are one of the most substantial household expenditures. Hence, they are the most valuable assets in households' total wealth portfolio. As a consequence, housing investments are one of the most significant ingredients of individuals' total wealth. Furthermore, in recent years, the home-ownership also recognized as one the leading investment field to increase households' wealth. Householders do not hesitate, adding houses to their wealth portfolio since a home does not only allow the owner a place to live but also they symbolize owners' living standards. Therefore, housing is a plentiful portion of total wealth in many developed countries. As a result, investments in housing markets have a dominant impact on both householders' consumption and saving habits. From this standpoint, housing investments impact socio-economic statuses and have a tremendous influence on national economies and house price movements, therefore, have been of concern to both investors such as individual and real estate investment trusts and policymakers.

Even though purchasing a house is one of the most significant expenditures of householders, the home-ownership rate in many countries, including the European region, has increased significantly in the last two decades [185]. The growth in the home-ownership rate partially is a result of the government policy of favoring home-ownership as a portion of an asset-based welfare scheme. The supported welfare system relies on the notion that home-ownership is going to increase householders' wealth by the growth of housing equity. Generally, price changes in housing markets play a meaningful role in the generation of housing equity and households' wealth inherent to home-ownership. Usually, price changes in housing markets occur in cycles of downward and upward trends. These cycles might be driven from various sets of fundamental determinants, such as economic circumstances.

In the long run, households accumulate a significant amount of housing equity, yielding welfare benefits to households, even in periods that house price decline may consume the housing equity value that assessed across several years. Therefore, the home-ownership carries significant financial risks that may unfavorably affect householders' balance sheet. Even in some cases, it may strongly affect national and global economies. Such risks require proper measures and better management. In this respect, it is essential to identify the price dynamics of housing markets. A thorough identification of price dynamics of housing markets is necessary to distinguish innovative approaches of insuring against the risks connected with home-ownership. Therefore, there is a requirement to investigate housing market dynamics and progress of return performance of investments in housing markets with the help of the sophisticated instruments developed in mathematical finance. Monitoring the dynamics of housing markets and their price determinants are necessary. Furthermore, it may provide extensive information to both investors and policymakers. Therefore, the focus on housing markets is increasing, and as a result, the number of studies that are highlighting housing market models is rising in the last three decades.

In the last third decades, the association between real estate markets and the macro & micro-economy has received accelerating attention in the property management and property investment literature, partly as a consequence of the increasing significance of properties as a notable asset type in both direct and indirect investments. Various examples of econometric applications of statistical models exist in the literature that focuses on favorable externalities of investments in Real Estate Investment Trust (REIT) shares. It may be critical to incorporate both macro-economic indicators for economic growth in the short-term and high added value of real estate markets to the socio-economic significance. Accordingly, although real estate markets natural risks, and the latest collapse throughout the global economic crisis that we faced in 2008, it has become a key research field of center on supply and demand, from practitioners such as developers, contractors, investors, banks, etc., to policymakers and individual householders who are seeking a house to live in or invest.

Beginning from early 2000, the real estate market has also become a significant priority for policymakers and investors in Turkey. As a consequence of compact politicians and private investor actions, Turkey's real estate market has made delightful benefactions to Turkey's national economic indicators such as Gross Domestic Product (GDP), employment rate, and especially in the volume of mortgage credit, notwithstanding the weak relationship within the real estate market and the financial industry of Turkey. From this point of view, it has been detectable that compared to its worldwide banking equivalents, Turkey's housing credit volume to its GDP ratio is comparatively weak in Turkey's banking system, at approximately 6.1% as of year 2015 [105]. Furthermore, in Turkey, the secondary markets of mortgage and its associated insurance products are both significantly underdeveloped compared to developed countries. This fact strengthens concerns over real estate markets' efficiency, their sustainability, and their benefits. In such a perspective, the performance of REITs is crucial in order to improve the market of real estate efficiency and national economies.

Turkey's Real Estate Investments Trusts (T-REITs) industry is a remarkable illustration of an immediately expanding market along with some unique governing provision within some significant advantages on taxes and distribution payout privilege for the previous two decades (see Appendix A for further details). Besides, REITs might be acknowledged as a liquid sub-real estate market. Therefore, investigating companies whose shares are traded in exchange markets and analyzing their advantages is essential to all market participants.

1.2 Gap in the Literature

Housing markets are defined by the allocation mechanism of supply and demand as in perfectly competitive markets. Like any other market, the price of transactions in housing markets is settled, in the first place, by the interaction between supply and demand. However, many natural peculiarities of housing markets distinguish them from perfectly competitive markets. For instance, the most notable differences are; in perfectly competitive markets, there exists a large number of purchasers and dealers, and all associates are price takers. Moreover, participants may enter perfectly competitive markets and retreat from those markets at any time they request. Assets in such markets are generally homogeneous, and the market-place is clearly defined.

Furthermore, in fully competitive markets, information about assets is observed by the market, and the price is known by all market participants spontaneously. However, in housing markets, the price of houses is kept as a mystery between buyers and sellers, which limits the market from observing house prices. More importantly, housing markets are characterized by severe heterogeneity due to their location dependency and physical characteristics, which make houses unique, of a property. Additionally, houses are distinguished to varying degrees in housing markets, and information on the quantity and the quality of amenities that constitute house prices are challenging and costly to obtain. Hence, in the literature of real estate finance and economics, the real estate market is often referred to as entirely asset-differentiated.

Unlike perfectly competitive markets, such as exchange and currency markets, transactions in housing markets generally involve only one buyer and one seller who bargain over an unknown price. Hence, housing markets are not fruitful to observe house prices. As a result of this intuition, these markets can not inform market participants as intelligent as other markets. Moreover, entering into housing markets and retreat from them is severely constrained by a large capital sum involved as well as by relatively high transaction costs [124]. For instance, the durability, stability, and heterogeneity nature of houses imply that transaction costs have a significant effect on housing markets. Another essential characteristic of housing markets that differentiate them from perfectly competitive markets is the in-elasticity of housing supply and demand in the short-run. An increase in demand causes a simultaneous rise in the housing supply to adjust the market price. However, in many countries, the construction of a house or property requires more than six months. Thus, the housing supply cannot cover the housing demand in the short term [122]. As a consequence, this result causes an increase in house prices, considering the housing supply cannot adjust in the short-run.

Some historical events show that fluctuations in house prices, whether they are in a balloon or burst mode, hold the potential to lead national economies to collapse or exalt. As an illustration, consider the large volume of capital that invested into Korea property market throughout the period 1980s, which caused a tense raise in their house prices, and ended with dramatic house price fall during the Asian currency crisis occurred in 1997. As another illustration, consider the latest global financial crisis of 2008, triggered by the United States (US) housing market demand side, all countries faced. During the newest crisis period, house prices and their related product prices stagnated or decreased slightly almost in all US states. Most of the nations, such as, the US, Spain, Ireland, the United Kingdom (UK), the Scandinavian countries as well as most of the Eastern European and Asian countries announced brisk capital extension throughout the 1990s and price drops at 2008, which triggered or exacerbated the economic declines [109]. Hence, housing markets are highly related to financial markets, and house prices, therefore, are of great interest to both housing markets and financial markets participants [122].

The trend of house prices, paths of house price fluctuations, or house price volatility may be used as an indicator to understand housing market dynamics in a city/submarkets or even in a country. The house price volatility may be managed by adjusting factors that influence prices. Hence, determinants of price or volatility of price change should be clarified. Housing markets affected by macro-economic indicators, spatial diversity (such as the intensity of house), features of the community, and environmental conveniences of the area [122]. In addition to these factors, the expectation of capital gains of investors from investments in housing affects house prices by expanding the housing demand, which in turn would provoke an increase in volatility of house prices [171].

A satisfactory theoretical price of houses might be used to analyze housing markets more efficiently. An empirical model used to estimate house prices has various practical utilization in fields such as housing market research, tax assessment, housing investment, and land-use planning. In the real estate finance and economic literature, most of the studies that are addressing house price models are conducted with hedonic modeling and multiple regression analysis. These approaches are generally more appropriate to a straightforward evaluation of the association between house prices and various characteristics of houses used as independent variables. However, these approaches might have drawbacks if the schedule of the valuation is widened to involve features such as outliers, non-linearity, spatial, and other varieties of dependence among observations, discontinuity, and fuzziness [121]. There are, however, some sophisticated alternative modeling approaches that rely on machine learning, namely neural networks and spatial modeling approaches, which are far better to deal with these features. These modeling approaches have in common that they introduced to obtain a better fit to observed data. However, they do not demonstrate how house prices are generated endogenously by the actions of market participants. On the other hand, unlike preferably liquid and fully competitive markets, the studies rely on mathematical tools in financial mathematics are still limited in the context of real estate finance and economics literature. One of the potential reasons for that is the price behavior of the housing market still puzzles researchers and practitioners.

There is a large number of study that investigates the change in house price at national and local levels. However, the number of studies exploring the effect of large investors in property markets is limited. The impact of a large investor on a local housing market depends on supply and demand behaviors as in fully competitive markets. On the supply side, developers construct new houses increasing the number of housing units, types, and quality of houses in the local market. Whereas, the demand side sets the price of new houses to a level that is competitive within the current prices and attractive to both investors and consumers who are seeking an investment opportunity or a domicile.

Traditionally, theoretical studies emphasize that constructing new houses is generally likely to generate both positive and adversarial externalities for landlords in local housing markets [191, 198]. In extension to the fiscal, social, and ecological advantages, new buildings might produce advantages for joining individual house owners also. New houses may produce certain spillover influences on current blocks by generating a more lively district as empty lots become populated [198]. If abandoned properties perform external dis-amenities through attracting dumping, enabling illegal use, or producing a deformity, then constructing new houses may reduce the external dis-economy, increase the local residents, promote the aesthetics of the local housing market and increase neighboring real estate values [60].

New constructions might be more aesthetically charming than disorderly properties or neglected apartments which improve the views. Nevertheless, such infill improvement can also cause an adverse effect on surrounding real estates due to the rise in traffic flow and reduction of green areas on the district [139]. New residences may also contend instantly with present residences in the same housing market or indirectly into associated sub-markets, potentially diminishing the prices of nearby current apartments by extending the supply while the demand stays constant [178].

Housing markets, generally, affect economies indirectly. These markets show their effect on economies through Mortgage-Backed Securities (MBS). Therefore, miss-pricing of MBSs have been widely blamed for triggering the latest global financial crisis that we faced in the middle of the previous decade. The strong dependence among housing markets and national economies leads many researchers to determine the price movements in housing markets.

A mortgage is a type of commercial contract which is considered as a fixed-income product. This product is a legal record that pledges a house as collateral for repayment of a borrowed amount, and the agreement is invalidated if the loan fully repaid. Thus, the mortgage interpreted as a kind of security derivative. The value of a mortgage is derived from the growth of the economy, the underlying dwelling price that is written on the mortgage, and the term structure of spot rates, as underlying factors.

Mortgages are prime examples of financial merchandise that may be appraised using the celebrated option-pricing theory in the context of financial mathematics. The standard option based mortgage valuation method assumes that there are three embedded options in a mortgage. The first option is a financial option to prepay that may happen at any time throughout the survival period of the loan, which is considered as an American option. Second, a financial option to default that may occur only at payment dates, which is recognized as a European put option and, finally, a non-callable bond that relies on monthly payments. Even though there exist significant studies on the option-based mortgage valuation method, still studies focusing on hedging of mortgages is missing in the literature.

The use of sophisticated mathematical tools, which are famous in analyzing financial markets, in housing markets ranging from Partial Differential Equations (PDE) to stochastic calculus and numerical methods have grown steadily during the past few decades. On the one hand, the mathematical tools and their results have shaped the way of housing markets. They are used to model housing markets to understand how housing risk is assessed and managed. However, the structure of housing markets presents several essential mathematical and computational challenges to practitioners and academic researchers in the industry. One of which is the hedging of mortgages. Therefore, studies that investigate the hedging of mortgages are missing in the real estate economics literature.

Housing markets are illiquid markets due to their nature. However, REITs satisfy liquidity to housing markets since they have shares tied to the real estate market that traded in exchange markets. REITs have a range of advantages that distinguish their assets from other assets. As an alternative to investing in properties directly, REITs tender at least two unique advantages. Its first advantage is the liquidity benefit of their share. The REITs shares allow investors to take positions in real estate markets without cumbersome transaction costs and lengthy delays in execution due to its securitized nature. The second advantage of REITs is the diversification benefit. Their relatively low-cost helps provide investors an opportunity to the allocation of funds across the real estate resulting in diverse portfolio holdings. Moreover, as REITs share are traded on exchange markets, they operate in well established regulatory environments that providing a level of governance that is typically not offered in the direct real estate market. However, the studies that are investigating the investment benefits of REITs shares, especially studies examining REITs in emerging markets, are again limited in the literature.

1.3 Aim and Scope of the Thesis

Starting from the early seventies, the importance of housing markets has led to an accelerating increase in the research interest of both practitioners and academicians on house price tendency. The fact that the majority of consumers lack adequate savings to purchase a house. It increases the use of mortgages, and the relative ease of converting the house into capital owing to a variety of property based investment tools have made it mandatory to an accurate estimate of the price trend in housing markets. With a well-specified statistical or econometric model, the price pattern may be determined for housing markets [13]. Furthermore, the model also might determine the economic indicators force on significant house prices changes.

Moreover, having a model in hand that may help to clarify the driving indicators of housing markets, and revealing the trend is of great importance for both housing market participants and policymakers. However, our view is that the number of studies that examine the relationship between economic indicators and housing markets is enormous. Hence, the researchers are aware of the importance of specifying underlying drivers of prices in housing markets and assessing their financial implications to housing markets.

Assets in housing markets constitute a substantial proportion of households' welfare and nations' GDP. Hence, housing markets naturally shape the long-term development of countries. Housing market bubbles crash generally destabilizes the economy. More importantly, it causes significant economic recessions, and even in some cases, it may become a global financial crisis as we experienced in 2008. In this sense, housing markets perform a critical task in economic activities. Therefore, the state of the current economy and the recent global financial crisis lead researchers to pay more attention to the role of housing markets on economies and long term investments as a whole [189].

This thesis is about to systematically analyze and explain housing markets price trends against macro-economic indicators on housing markets, the effect of large investors, and hedging of the embedded options, which are mortgage default and prepayment options, into the mortgage under the assumption of the standard two-state stochastic model in housing markets. Furthermore, it analyzes the performance of value and growth shares of REITs in an emerging country, namely Turkey. The calculations are associated with the US housing market and Monte Carlo (MC) simulations, while explanations are couched in the mathematical finance literature on methods such as Stochastic Differential Equations (SDEs), Generalized Linear Model (GLM), Multivariate Adaptive Regression Splines (MARS), Malliavin calculus, Capital Asset Pricing Model (CAPM), Fama-French three-factor model along with some its extensions.

The primary aims of the thesis are; to provide models that rely on SDEs, GLM, and MARS for housing markets, to determine the effect of large investors on housing markets, to apply hedging strategies to mortgage default and prepayment options, and analyze investing benefits in REITs shares. To achieve our first goal, we propose a two-state stochastic process under continuous trading assumptions for the US housing market index and apply an optimization algorithm to estimate the model parameters. Again, to illustrate housing market determinants, we use GLM and MARS methods to the US housing market. Second, for the sake of simplicity, we consider a one-state stochastic process for housing markets that contains the effect of large investors. Then, we optimize the portfolio of large investors who have a log and power utility function. Third, we consider a two-state stochastic process, which is standard in the real estate finance and economics literature, to compute the hedging parameters for both default and prepayment options. In the computation of the hedging coefficients first, we use the Malliavin calculus and obtain formulas for the hedging parameters. Then, we apply a crude MC algorithm for the computation of hedging coefficients. The offered models for housing markets, financial analysis of these markets, and such hedging parameters for both default and prepayment options embedded into mortgages may conduct banks, investors, and policymakers in optimal investment decisions. Since practitioners in this field need to be aware of the evaluation of housing market prices based on historical events, they need to consider the uncertainty inherited by the economy. Finally, we investigate the benefits of investing in REITs by using T-REITs data and the celebrated CAPM, Fama-French three-factor model, and its extensions.

Overall, modeling housing prices, determining economic drivers of housing markets, analyzing the effect of large investors and, more importantly, hedging mortgage default and prepayment options are essential concepts for real estate economics and finance literature. There is a large amount of study, which deals with econometric models or statistical models and performance of the investment in REITs' shares, exist in the literature. However, studies dealing with stochastic models, analyzing large investors' effect, and hedging of mortgages default and prepayment options are missing. Hence, we consider a one-state and a two-state stochastic model. Then we analyze large investors' effects on housing markets and offer hedging strategies for both options embedded into mortgage under the standard two-state stochastic market assumption.

1.4 Contributions of the Thesis

The contributions of this thesis intend to make are apparent. The specific contributions of the thesis are fivefold. First, we illustrate the importance of macro-economic indicators on housing markets by establishing models based on nonparametric modeling methods. However, the number of explanatory variables may change due to country specifics, and other economic indicators may be added to the housing market analysis. Second, to sharpen the focus, we extend the modeling of housing markets on the potential application of SDEs on housing markets. We analyze S&P/Case-Shiller US National Home Price Index (HPI) and 30-Years Fixed Mortgage rate (FRM) together and explain the house price variability by using stochastic processes. Thus, we offer a model that estimates HPI values by using stochastic processes. Third, we analyze the effect of large investors on housing markets. We consider a one-state stochastic process that carries the impact of large investors along with the economic state. Then, we optimize their portfolios under the assumption of both log and power utility functions and observe the effect of large investors on housing markets. Forth, we motivate how the computation of Greeks or hedging coefficients can be consistent in the hedging of both default and prepayment options embedded into the mortgage. Hence, we compute the hedging parameters of both default and prepayment options by applying the finite-dimensional Malliavin calculus. Furthermore, we discuss how these coefficients may signify the hedging and offer hedging strategies for mortgages. Fifth, we examine the benefits of investing in REIT shares according to their risk diversification effects. As an empirical analysis, we analyze REIT operated in Borsa İstanbul (BIST100) stock exchange by employing both the CAPM and the Fama-French three-factor modeling approaches along with its extensions. Our empirical analysis indicates that T-REITs are more effective than Turkey's banks, but worse than T-TRUSTs to diversify the risk. The thesis also reveals that the T-REITs generally have a defensive management structure; they are small and financially distressed.

The major contributions of the thesis to the real estate finance and economics literature are the followings:

- The GLM and MARS methods adopted to construct non-parametric statistical models that identify the potential effects of macro-economic indicators on housing markets. As an application, we study the US housing market. Starting from the basic linear model, we include only three models among many possible models, two for GLM and one for MARS. These three models highlight the potential effects of macro-economic indicators on the US housing market. Further, they serve potential users to predict the house price trend in the US housing market. The thesis also reveals that the model based on MARS is better than GLM models for the validation period according to performance measures in the context of the thesis. However, the efficiency of the MARS may change with respect to the validation period length.

- With the MARS and GLM models, we show that there is a large number of explanatory variables for housing markets. One may extend the number of explanatory variables, even for the US housing market. Hence, to avoid the high number of explanatory variables and decrease their numbers sufficiently, we adopt a two-state stochastic process to the US housing market. As an empirical analysis of the offered modeling strategy and an application, we assume that HPI evolves from a geometric Brownian motion, and FRM grows from the

Ornstein–Uhlenbeck (OU) process. The superiority of the two-state stochastic process to the models relies on GLM and MARS methods are the less number of explanatory variables and its forecasting power. The empirical analysis shows that the SDE that we offer is successful for estimating the HPI values and forecasting future values.

- Following the stochastic modeling idea, we define housing markets with a one-state stochastic model for the sake of simplicity. Then, we apply a portfolio optimization procedure to the model based on the different states of the economy, maintenance cost, rental income, and the effect of large investors. The results indicate that the investment strategies of large investors depend on the balance among economic state, maintenance cost, rental income, interest rate, and investment willingness of large investors to housing. Furthermore, their investment decisions have positive externalities on housing markets.

- We follow the modeling studies of housing markets by offering hedging strategies of mortgages. In accordance with this purpose, we analyze the mortgage default and prepayment options independently. Here, we adopted one of the bases of the computation of Greeks, namely Malliavin calculus, to obtain the hedging parameters of both default and prepayment options. This work assumes that the underlying house price evolves from the standard two-state stochastic process. In the literature, there are two inherent practices of these coefficients. First, they enable users to discover the consequences of the spot rate, the price of the underlying house, and the house price volatility change on both options. Second, both borrowers and lenders might replicate and thus hedge their leading portfolio by utilizing the balance within the underlying asset and options with the help of these coefficients.

- In the analysis of mortgage default and prepayment options hedging, we suggest using REIT companies' share to use in hedging instead of the underlying house. It is because houses cannot be short-sell and not divisible. Hence, we analyze the benefits of investing in REITs shares. As an empirical analysis, we investigate the return performance of REITs companies, operated in BIST. This study shows that while a portfolio contains only T-REITs' shares contributes to a lower level of risk diversification advantage than a collection, and it only consists of investment trusts' shares. It also has a higher level of risk diversification advantage than a portfolio that only includes banks' shares. The results also indicate that T-REITs show a degree of variety according to their property focus, and they have essentially defensive, small, and financially distressed management characteristics. This work also provides the readers' suggestions for the capacity of T-REITs, and it develops their return enhancement capacity on efficient portfolio management.

1.5 Outline of the Thesis

The content of the thesis is three dimensional, modeling and simulation of housing markets, analyzing the large investors' effect on housing markets, and the computation of hedging coefficients for mortgages. To achieve our goals, we comprise the thesis five chapters and two appendices. The organization of the remaining body of this thesis carried out is summarized as follows:

In Chapter 2, we analyze the dynamic effect of some macro-economic indicators on housing markets. However, this chapter does not only focuses on the impacts of macro-economic indicators but also it proposes a variety of models that rely on the GLM and MARS methods. By using models, we identify the macro-economic drivers of the US housing market, and further, we forecast house prices in the US. Our empirical analysis within this chapter concentrates on the US national housing market since the US has adequate and long-term retention of regular housing market data. Specifically, the illustrations of the suggested models are done through the actual achievements of HPI and the US macro-economic indicators for the period 2000 to 2018.

In Chapter 3, we aim to illustrate a two-state stochastic process, defined in the form of SDEs, for estimating the price trend in housing markets. First, to determine an active stochastic process, a generalization of the price structure concerning house price indices and mortgage rates is proposed. Then, a calibration procedure on monthly HPI values and US FRM values is employed to estimate the initial parameters of the differentiable functions specified in given SDEs. The estimation and forecast capability of the two-state stochastic model is confirmed by an MC algorithm for one-year ahead of monthly forecasting of the HPI value.

The purpose of Chapter 4 is to fill the gap in the literature by SDEs and stochastic control methods to investigate the impact of large investors purchasing single-family houses for the rental business. Besides, we also consider the effects of economic states on housing markets using the Markov switching model. Therefore, the key ingredient and contribution of this chapter are to examine the presence of large investors jointly with the effect of economic state on housing markets to maximize investors' wealth.

Chapter 5 aims to explore the hedging parameters of the mortgage financial option to default and mortgage financial option to prepay, which are inserted options in mortgages. Hence, we start the chapter by a literature review and the definition of the economic environment. Then, we give a brief summary of the Malliavin calculus since we used this calculus in the computation of mortgage default and prepayment hedging coefficients. We compute the hedging coefficients for the change in interest rate, the change in underlying house price, and the volatility of underlying house prices by applying the finite-dimensional Malliavin calculus. From the Malliavin calculus, we obtain all of the hedging coefficients as a product of mortgage default and prepayment option's payoff and an independent weight function. And, finally, using the product, we run an MC algorithm and compute the hedging coefficients.

Chapter 6 is dedicated to analyzing REITs and the investment benefits in their shares in an emerging country. In this chapter, we first illustrate the REITs properties and T-REITs market. For our analyze purposes, we remind both the CAPM and its extension Fama-French three-factor model. Then, we examine REITs operating in Borsa İstanbul (BIST). Hence, this chapter investigates the return performance of T-REITs from various perspectives throughout July 2008-March 2015 by employing these models and the extension of Fama-French three-factor model. Furthermore, in this chapter, we also classify the T-REITs according to their operating topics.

Finally, the thesis concludes in Chapter 7 with a discussion of our empirical findings and a future research agenda.

CHAPTER 2

UNDERSTANDING HOUSING MARKETS PRICE DYNAMICS: THE US HOUSING MARKET CASE

Beginning in the early seventies, the importance of housing markets in national economic activities has led to an increase in the research interest on housing market tendency. The fact that the majority of the community who are seeking a housing lack adequate savings to purchase a house. It increases the use of mortgages, and the relative ease of converting the house into capital owing to a variety of property based investment tools have made it mandatory to an accurate estimate of the price trend in housing markets. A well-specified statistical or econometric model may be useful to determine the price pattern in housing markets [13]. Furthermore, the model also might determine the economic indicators force on significant house price changes.

Moreover, having a model in hand that may help to clarify the driving indicators of housing markets, and revealing the trend is of great importance for both housing market participants and policymakers. However, our view is that the number of studies that examine the relationship between economic indicators and housing markets is enormous. As commonly believed, it is vital to specify underlying drivers of prices in housing markets and to assess their financial implications to housing markets.

Even though there exist studies that investigate housing markets, analyzing these markets is more complicated and cumbersome than analyzing fully competitive markets since house prices do not respond to economic fluctuations as fast as fully competitive markets. Generally, prices in housing markets show a steady downward price movement since house-owners resist selling their houses under a specific price barrier during recession periods. As a result, house prices have a decreasing trend through high inflationary periods rather than through formal price reduction [2]. Besides, the price inertia also affects the house price behavior during economic booms. It is because high expectations of householders, generally, promote housing balloons.

Besides, along with these problems, housing markets have specific characteristics distinguished from fully competitive markets. First of all, housing markets are highly illiquid markets due to high transaction costs and the time spent on the decision of a house to purchase. Generally, real house prices have kept a mystery, and they are known only by the buyer and seller, which prevents markets from observing house prices. More importantly, enrolling and retreating housing markets are relatively burdensome since they require a notable amount of cost. Moreover, housing markets are highly heterogeneous since houses are unmovable and attach to a specific location.

Despite the challenges in analyzing housing markets we mentioned above, it is essential to specify underlying indicators of prices of houses and their effects on housing markets. Therefore, the extreme dependence on housing markets and national economies leads us to construct statistical models, which capture the behavior of housing markets as well as its underlying factors by using the celebrated Generalized Linear Models (GLM) and Multivariate Adaptive Regression Splines (MARS) methods and determine the impact of macro-economic indicators on housing markets. Hence, unlike many other studies, this chapter examines the effects of macro-economic indicators on house prices by using these two non-parametric regression models for the first time in this area.

In the proposed models as its rigorous mathematical descriptions include the influence of the historical prices as well as the impact of macro-economic indicators. Even though there exist studies that investigate the relationship between economic indicators and housing markets, this chapter is distinct from these studies as methods in these studies do not include GLM and MARS, while this chapter focuses exclusively on these methods. Moreover, we would like to address the effect of Capacity Utilization: Total Industry, Percent of Capacity is a critical difference in this thesis.

Concerning the central role that housing markets play as a catalyst to the financial crisis, it is critical to understand the determinants of the house price dynamics adequately. Hence, the primary purpose of the current chapter is to identify macro-economic drivers of housing markets. Accordingly, this chapter examines the influence of macro-economic indicators of the US by using monthly US market data over the period 1999-January to 2018-June as empirical analysis. More specifically, S&P/Case-Shiller National Home Price Index is analyzed in this chapter by concerning Consumer Price Index, Civilian Unemployment Rate, 10-Year Treasury Constant Maturity Rate, 30-Year Fixed Rate Mortgage Average in the US, US/Euro Foreign Exchange Rate, Effective Federal Funds Rate, Working-Age Population: Aged 15-64: All Persons, Crude Oil Prices: West Texas Intermediate, Real Disposable Personal Income: Per Capita, and Recession Cycles in the US economy within the period that we investigate. The comparison of models resembling a better fit is made through their accuracy with method free error measures.

The organization of the chapter is as follows. The following section positions a literature survey on the association between macro-economic indicators and housing markets. Section 2.2 exhibits a short demonstration of the US housing market and macro-economic indicators that are used in our empirical analysis. In Section 2.3, we summarize both statistical methods, GLM and MARS. Section 2.3 is supported by our empirical findings of the recommended statistical models.

2.1 A Literature Survey on the Relationship of Housing Markets and Economic Indicators

In the last three decades, there is accelerating interest in the number of research about the association among housing markets and macro-economic indicators, especially after the sub-prime mortgage crisis and the subsequent financial crisis in the mid-2000s, which is fundamentally due to the central collateral role of house prices. For a while, central banks have successfully held inflation in check through their targeting. However, then, they failed to prevent house prices from bursting and having adverse effects on the economy. Hence, price changes in housing markets might be a significant origin of macro-economic fluctuations for inflation targeting that central banks may want to respond to [28]. As a consequence, there is a vast amount of study in the real estate and finance literature that emphasizes the relation between housing markets and macro-economic indicators, such as [3, 14, 106, 128] and references within these studies.

The majority of studies in real estate finance and economics research area identify that interest rate, mortgage rate, inflation, unemployment rate, and financial indicators, such as exchange rate, industrial production as the most important explanatory variables. These studies also show that the explanatory variables inevitably influence householders in consumption behaviors and investment decisions. In turn, macro-economic indicators affect housing markets with a lag depended on propagation mechanism speed. The circulation speeds profoundly affected by the effectiveness of the organizational frame, such as land availability, administration arrangements, and the rate of regulatory processes [2]. Alongside these indicators, it is known that credit supply, transaction expenses, and innovations in mortgage products also have significant roles in housing markets. To exemplify, if the fluctuations in spot rates affect mortgage rates instantly, an expansion in the capital supply influences housing markets quicker than the fixed mortgage rate case.

The sensitivity of householders' behaviors to spot rate movements depends on whether the spot rate on the debt is predominantly fixed or variable over the survival time of the loan. For instance, Poterba et al. (1991, [166]) find that variations in borrower costs connected with the spot rate fluctuations and country tax policy are one of the crucial determinants of movements in real house prices. It determines whether householders, financial intermediaries, or pension funds are mostly exposed to fluctuations in the spot rate based on the location. In turn, this case will influence the short-term impact of movements on the spot rate.

Johnes and Hyclak (1999, [108]) examine the association between labor income and house price. They find some evidence that changes in unemployment affect house prices significantly. Moreover, in similar empirical studies to Johnes and Hyclak [108], such as [68, 140, 166], it is proved that income is one of the most significant drivers of house prices. Generally, most of these studies depend on average income measures, such as per capita disposable income. Such average price measures of housing markets capture the fact that wealthier households demand more consumption good and, thus, more house to purchase than poorer householders [160].

Some studies identify house prices, displaying a feedback reaction to national economic activities. For instance, Adams and Füss (2010, [2]) prove that an increase in house prices makes house-owners feel wealthier because of their collateral size and the value of their house. Furthermore, if a house-owner has a liquidity constraint, the rise in her house price may be her only chance to get a loan. Such kind of wealth shocks causes an increase in households' consumption. In addition to this, if house prices decline, the change leads to an adverse impact on householders' expenditures. It is because a decrease in house prices generally increases the number of mortgage defaults, which reduces the lenders' credit supply as lenders lose part of their capital [85].

Even though there exist studies that investigate housing markets, analyzing housing markets is much more complicated and cumbersome than analyzing fully competitive markets since house prices do not respond to economic fluctuations as fast as other markets. Generally, house prices show a steady downward price movement since house-owners resist selling their houses under a specific price barrier during recessions. As a result, house prices have a decreasing trend through high inflationary periods rather than through formal price reduction [2]. Besides, the price inertia also affects the house price behavior throughout economic blasts since high expectations of householders promote ballons in housing markets. Moreover, along with these problems, housing markets have specific characteristics distinguished from fully competitive markets. First of all, housing markets are highly illiquid due to high transaction costs and the time spent on the decision of a house to purchase. Generally, real prices are known only by the buyer and seller, which prevents the market from observing house prices. Entering and retreating housing markets are relatively tricky since they require a significant amount of cost. Moreover, housing markets are highly heterogeneous since houses are unmovable and stick to a location.

Mortgages also perform a vital task in the propagation of real house prices to macroeconomy [2]. At this point, it is worth to emphasize that the more substantial mortgage debt implies greater leverage, through variations in the spot rate can affect householder consumption [2]. For instance, Case et al. (2000, [36]) find that the influence of house prices on expenses is substantial in the US, where two-thirds of all residents are also owner-occupants. In their following study, Case et al. (2005, [37]) point that fluctuations in house prices might even affect householder expenses more heavily than variations in prices in exchange markets. It might be because homeownership more fairly allocated across householders than exchange markets' wealth.

Rising house prices have underpinned growth in measures of householders' total wealth in many countries over the past two decades. There exist empirical studies, such as Case et al. (2005, [37]) and the references therein, which establish a positive relationship between increasing aggregate householders' wealth and their consumption, although these studies based on micro-economic data that are less conclusive. However, while empirically a positive relationship may be observed between rising house prices and householders consumption, in theory, the expected issue is much less clear. For example, Bajari et al. (2005, [17]) present a model where house price rises have a small adverse effect on householders' consumption.

Mankiw and Weil (1989, [143]) investigate the impact of demographic variations on price fluctuation in housing markets. Mankiw and Weil's analysis suggests that the decay in the portion of the US residents in the prime-house purchasing age bracket through the 1990s caused a plentiful fall in actual house prices of the US housing market. In another study, Cutler and Poterba (1991, [57]) discover that variations in user costs connected with spot rate changes and tax policy are essential determinants of campaigns in real house prices. Johness and Hyclak (1999, [108]) also find some evidence that the changes in unemployment affect house prices.

It is known that immigration affects house prices through different and opposing mechanisms, which makes the overall effect ambiguous and difficult to understand. Migrants not only carry their skills but also carry their traditions, customs, and attitudes to the country of destination themselves. These issues differentiate immigrates from the hosts in many aspects. Immigrant's cultural differences affect the consuming behavior of settled public or hosts. The consumption of immigrants is particularly effective in the supply side of housing markets, and the shifts in housing demand since immigrants need a place to live in. Hence, as an expected result, there is a strong correlation between net immigration and house price in the short run [110, 145].

2.2 The US Housing Market

In this section, we first take the liberty of briefly summarizing some of the critical points of the US housing market. Then, we introduce the characteristics of economic variables that we use as explanatory variables for the US housing market. The evidence of housing market models we present within the current chapter is based on the association between the US housing market and its economic indicators. The data set that we analyzed is gathered from publicly available data-set of Federal Reserve Bank of ST. Louis. This section contains sufficient materials for the empirical examination within the current chapter that concentrates on the US housing market.

For decades after the Great Depression period, the US endured a healthy housing market, which is believed to increase the wealth of middle-class US citizens. However, the US housing market has experienced a high degree of volatility cycle relative to macro-economic indicators, such as consumer price index and real income levels, during the period 1994-2009 due to significant structural changes and fluctuations in its economy [92]. While affordable house prices cause difficulties for lower-income communities and homelessness continues a pervasive puzzle for a small portion of the community, the overall quality of housing, even for the most impoverished families, improved through the past few decades.

The purpose of Figure 2.1 is to illustrate the US homeownership rate evolution. This figure presents the homeownership rate in the US for the period 1985-2016. The graph clearly reveals that the increment in homeownership precedes at a relatively constant rate over the period 1994-2004. During this period, the homeownership expanded rapidly in the US and achieved record highs from 1985 to 2004. However, homeownership has started to decline from 2004. At the end of the 1994-2004 period, 69% of the US population owned homes compared with 63.5% in 1985. However, the homeownership rate decreased each year following 2004, and it reached a level of 63.7% in the year 2016. Within this period, the US housing market and its finance policy interacted to produce a bubble from 2005 to 2007, which is finally burst in 2008. Note that, 2008 is also the year that the latest global financial crisis we faced. During the period, the decrease in the homeownership rate has been most dramatic among younger adults. From 2004 to 2013, rates for 25–34 years old are down nearly 8%

and for 35–44-year-old 9%. But the homeownership rate for middle-aged households has also fallen by at least 4%. Looking over a more extended period, the Current Population Survey illustrates that the homeownership rate for all 10-year age groups between 25 and 54 are at their lowest level in the US.

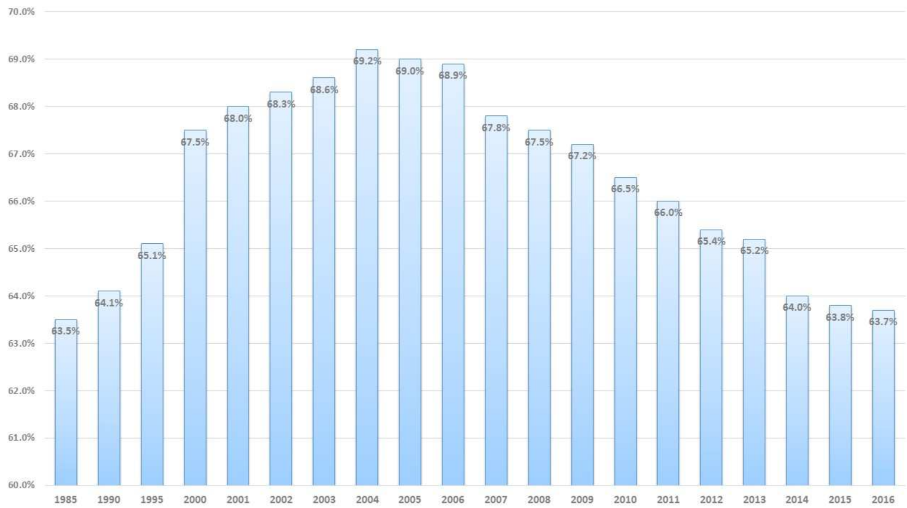


Figure 2.1: Home-ownership rate in the US in 1985-2016

In housing markets, external funding is mostly done through mortgages (see Figure 2.3), having a high association with spot and mortgage rate movements. Considering a 30-year fixed mortgage rate (FRM), 6- Month Treasury Bill, Effective Federal Funds Rate in the US, and S&P/Case-Shiller National Home Price Index (HPI) series between 1975 and 2016, we observe that spot rates and mortgage rate follow a similar pattern at which mortgage rate yield is higher than the spot rates. It is an expected result as the treasury notes are the safest investment instruments, since the US government issues guarantees on them, whereas the mortgage rate is not. Furthermore, the duration of the mortgage rate has to be longer due to the nature of the house financing business. We also see that both mortgage and spot rates show a reverse pattern and negative association to the house price index. We observe two striking dynamic structures from Figure 2.2:

- The periodic patterns, and opposite trend components are consistent features of the housing market.

- A sharp rise is find, even during the 2001 recession, reaching a remarkable increase in 2006 compared to the position of mortgage rate which is commonly perceived as a bubble [62].

Here, it is essential to note that an increase in the US house prices, even resembling a bubble, is triggered by the preceding prices in time. Also, the mortgage and spot rates of the US appear less vulnerable to financial crises compared to house prices. It is logical to infer that the decreasing mortgage rate increases the demand in the US housing market.

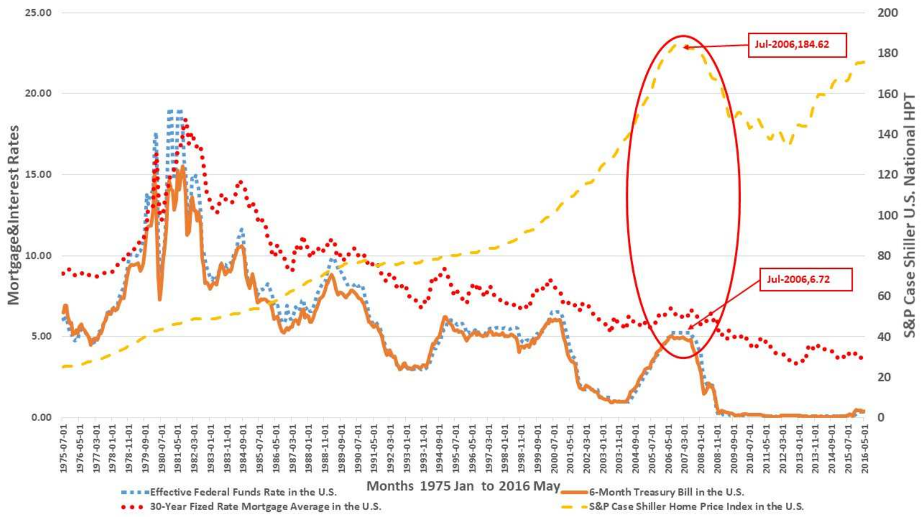


Figure 2.2: The development of the US HPI, FRM, and interest rates between years 1975 and 2016

The FRM dominated the US housing market within the period 1994-2004 (see Figure 2.3). As a result, the mortgage rate is one of the essential features in explaining the US housing market during the period. However, after the latest financial global crisis that we faced in 2008, which is triggered by the demand side of the US housing market, the nontraditional mortgage products that allow homeowners easy access to credit challenged the dominance of the FRM. The challenged cause a steep decline in the number of mortgage usage from 2006 to 2010.

In the period of 2006-2010, the number of mortgage usage decreased to 58.3% from 90.4% percent. Figure 2.3 reveals that there is an increase in the use of Federal Housing Administration (FHA) insured and the Department of Veterans Affairs (VA) guaranteed to finance in the period 2006-2010. However, as the figure highlights, the use of mortgages in the US housing market is still more popular than the remaining financing types. Hence, even though it caused a financial crisis, the US banking system still supports the use of mortgages. Moreover, householders are willing to use the mortgage since its' rate is lower than other financing system rates. The studies on mortgage relating to its direction in which the US mortgage market has been moving since 2008 have been mixed and continue recovery in the US housing market.

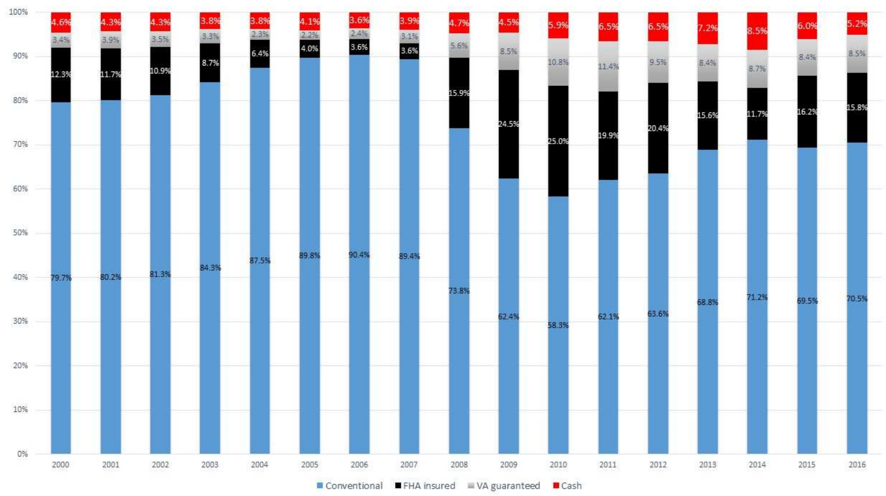


Figure 2.3: House financing types in the US from 2000 to 2016

In many countries, consumers, who are seeking a house to purchase, use nontraditional mortgage products to purchase more expensive houses than they could afford with the expectation of a rise in house prices in the future. However, such usage of alternative mortgage products in purchasing a house may cause negative externalities in housing markets through speculations [81]. As we witnessed, during the period 2004-2009, housing markets and financial crisis elevated mortgage delinquencies, and defaults dampen house prices. Furthermore, it increases the pessimism among community and investors. Then, eventually, the mortgage usage ruined the US financial markets and spread to the financial markets worldwide [45, 168]. Almost all countries experienced the global financial crisis over the period 2004-2009 that is originated from the demand side of the US housing market and spread throughout the other countries.

In Figure 2.4, we aim to reveal the importance of the US housing market for the US economy. This figure illustrates the number of privately owned new housing units in the US starting each year (solid line), and recession periods as determined by the National Bureau of Economic Research (shaded areas). Figure 2.4 confirms that it is common for a substantial decline in the demand for housing preceded by a recession. In other word, the housing demand is quite delicate to variations in the US economic state. Hence, in this respect, the decrease in the housing demand may be adopted as an early warning to recessions in the economy. From this figure, it is also clear that we may use the housing demand trend to discover the state of the economy.

Figure 2.4: The recession (shaded vertical lines) and the number of privately owned new houses (solid line)

At this point, it is logical to conclude that both Figure 2.3 and Figure 2.4 show the US housing market is about to finish its recovery period. We may include four severe aspects behind the recovery of the US housing market that needs to be considered as the most important ones. Which are expressed as follows,

1. Unlike the speculative increase in house prices before the crisis, current house prices rising due to the fundamental strength of the US economy.

2. The US population growth and the increase in the housing demand.

3. The rise in US citizens' wealth, namely, the increase in gross income per house-

hold after the financial crisis.

4. The labor market new jobs opportunities during the recovery period.

2.2.1 Data Description and Preliminary Analyzes

The evidence of housing market models presented in the current chapter is based on the relation among the US housing market, recessions in the US economy, and its economic indicators. The data is retrieved from the commercially available data source of the Federal Reserve Bank of St. Louis. For the empirical analysis of this chapter, we use the US housing market since it has educated data and easy accessibility of its data, which are very important in constructing models, especially constructing non-parametric models. In addition, data on the US housing market is arguably more reliable than data on the housing market in many countries.

We use the US data with monthly frequency to determine the predictive efficiency of our models and the direction of statistically significant indicators of the US housing market. In this chapter, based on the guiding literature on the determinants of housing markets, we selected 11 economic variables. We define these variables as substantial factors for the US housing market price variability. More specifically, monthly observation of Civilian Unemployment Rate, Consumer Price Index, 30-Year Fixed Rate Mortgage Average, 10-Year Treasury Constant Maturity Rate, US/Euro Foreign Exchange Rate, Effective Federal Funds Rate, Crude Oil Price: West Texas Intermediate, Henry Hub Natural Gas Spot Price, Capacity Utilization: Total Industry, Working-Age Population: Aged 15-64 and Real Disposable Personal Income: Per Capita are taken into account as significant factors that can be used in determining the US housing market. Further data available from the Federal Reserve Bank of St. Louis data source, which may be relevant to the context of this chapter. However, we used the most commonly used indicators to avoid the over dimensionality problem.

Here, it is worth emphasizing that among these variables, we are the first to analyze the effect of Capacity Utilization: Total Industry on housing markets. In addition to these explanatory variables, another variable of our interest that has generated in several studies related to economy and finance is recession cycles in economies. Hence, we also examine the impact of recession cycles of the US economy on the US housing market.

In this and following chapter, we represent the US housing market by S&P/Case-Shiller US National Home Price Index (HPI). It is one of the leading measures of prices in the US housing market, to trail variations in the value of houses both nationally as well as in metropolitan regions. Although the variables that we discussed above could be the variables that may affect housing markets, still it is very cumbersome to detect and analyze which of the variables have a significant effect on housing markets. More importantly, these variables may vary according to country specifics characteristics, such as local, political, and cultural differences. This data set expected to provide information on the US housing market price variability and the price trend in the US housing market.

We provide a summary of the dependent variable that represents the US national housing market, and primary explanatory variables contributing to the price variation in the US national housing market, along with the data notations and their abbreviations in Table 2.1.

Table 2.1: Outline of the variables having influence on the US national house price

Variable	**Abbreviation**
S&P/Case-Shiller U.S. National Home Price	HPI
10-Year Treasury Constant Maturity Rate	$TBill$
30-Year Fixed Rate Mortgage Average in the United States	FRM
Consumer Price Index for All Urban Consumers: All Items	CPI
US/Euro Foreign Exchange Rate, U.S. Dollars to One Euro	ER
Effective Federal Funds Rate	$EFFR$
Crude Oil Prices: West Texas Intermediate (WTI)	COP
Henry Hub Natural Gas Spot Price	GSP
Working-Age Population: Aged 15-64: All Persons	WAP
Real Disposable Personal Income: Per Capita	RDI
Capacity Utilization: Total Industry, Percent of Capacity	TCU
Civilian Unemployment Rate	UER
Recession Cycle in the Economy	RC

Concerning other explanatory variables, in addition to those listed in Table 2.1, one may continue to add explanatory variables by taking into account the country-specific variables for analyzing housing market price variability. It is clear that although the listed variables that we introduce in Table 2.1 could be the ones that may affect the US housing market, it is still cumbersome to detect and analyze which of those variables have a significant effect on the US housing market. It is because of the complexity of housing markets and the specific characteristics that distinguish them from fully competitive markets.

In Table 2.2, we report the descriptive statistics of the dependent and explanatory variables that we introduce in Table 2.1. Table 2.2 does not contain the recession periods since they are represented with dummy variables. From Table 2.2, we perceive that while the means of indicators are varying from 3.62 to 194632738.11, the standard deviation ranges from 1.30 to 9265241.76. From this respect, using the variables as in the given form may cause externalities in modeling the HPI. Further, the table illustrates the following logical inferences.

- Approximately, half of the variables have right-skewed or right tail of distributions ($TBill$, FRM, $EFFR$, COP, GSP, RDI, UER).
- Average values of the spot rates range between 1.92% to 3.62% while FRM yields 5.38% over the period January 1999-June 2018.
- Within the period under investigation, the average of HPI and UER is relatively high due to their data structure.
- The variables CPI, COP, WAP and RDI show the highest variability.
- The standard normality test, Jargue-Bera (JB) test, shows any of the variables listed above satisfy the normality assumption.

Table 2.3 shows that there exists not only a strong association within the dependent and independent variables but also, there exists a considerable amount of association among some of the explanatory variables (Table 2.3). According to Pearson correlation, the response variable, HPI has the highest association with RDI (79%) and WAP (78%) followed by CPI (74%) and FRM (-56%). Interestingly, although many studies have noted a high and positive relation between house price and unemployment (for example, [90, 91, 161] and references therein), the data yields meager

Table 2.2: Descriptive statistics of the selected housing market indicators

	Mean	Median	Standard Deviation	Kurtosis	Skewness	JB
HPI	151.91	151.52	28.03	2.44	-0.32	1
$TBill$	3.62	3.69	1.30	2.02	0.21	1
FRM	5.38	5.40	1.35	2.05	0.30	1
CPI	210.02	213.45	25.36	1.72	-0.19	1
ER	1.21	1.23	0.17	2.42	-0.29	1
$EFFR$	1.92	1.15	2.06	2.25	0.86	1
COP	59.80	57.31	28.01	5.11	0.34	1
GSP	4.58	4.01	2.16	5.63	1.50	1
WAP	194632738.11	196967646.90	9265241.76	2.17	-0.59	1
RDI	11582.54	11548.30	1443.08	2.11	0.04	1
TCU	77.25	77.33	3.16	4.17	-0.90	1
UER	5.95	5.40	1.77	2.58	0.90	1

and negative association (-13%). It is also clear that interest rates ($TBill$ and $EFFR$) and FRM are all negatively correlated with HPI. Note that Figure 3.1 also supports the results that $TBill$, $EFFR$, and FRM are negatively correlated with HPI and positively correlated with each other. Unfortunately, the linear correlation between $TBill$ and FRM (98%), $TBill$ and $EFFR$ (83%), FRM and $EFFR$ (83%) among the explanatory variables appears to be the highest (ranging from 83% to 98%). The correlation between CPI and RDI (98%) and WAP and CPI (98%) are also relatively high. Also, the correlation between CPI and RDI is also very high (95%). The relation between these two variables is investigated in many studies. An example of high-income increasing CPI is the Lawson boom of the late 1980s, which is followed by the recession of 1981, observed in the UK. The CPI and spot rates and FRM are also linked. Generally, lower spot rates increase the number of people that borrow from banks. The result is that householders have more money to spend; causes a growing impact on the economy and strengthen the inflation. On the contrary, if spot rates increase, the householders tend to save money, and with less RDI to spend, the economy slows, and so inflation decreases.

The relationship between the variables introduced in Table 2.1 is investigated in many empirical studies. An example of high-income increasing CPI is the Lawson boom of the late 1980s, which is followed by the recession of 1981, observed in the United

Table 2.3: The association among house price index and explanatory variables

	HPI	$TBill$	FRM	CPI	ER	$EFFR$	COP	GSP	WAP	RDI	TCU	UER
HPI	1											
$TBill$	-0.52	1.00										
FRM	-0.56	0.98	1.00									
CPI	0.74	-0.88	-0.89	1.00								
ER	0.45	-0.39	-0.40	0.48	1.00							
$EFFR$	-0.27	0.83	0.83	-0.67	-0.36	1.00						
COP	0.41	-0.49	-0.49	0.62	0.82	-0.39	1.00					
GSP	0.23	0.30	0.31	-0.21	0.35	0.32	0.25	1.00				
WAP	0.78	-0.88	-0.90	0.98	0.54	-0.69	0.64	-0.13	1.00			
RDI	0.79	-0.86	-0.87	0.98	0.38	-0.63	0.50	-0.23	0.97	1.00		
TCU	-0.01	0.40	0.39	-0.27	-0.11	0.65	-0.02	0.28	-0.31	-0.24	1.00	
UER	-0.13	-0.43	-0.44	0.28	0.54	-0.66	0.52	-0.12	0.34	0.16	-0.65	1.00

Kingdom (UK). The CPI is significantly linked with the $TBill$, $EFFR$ and FRM. Generally, lower spot rates increase the number of consumers that borrow and increase consumption in the economy. The consequence is that households have more money to spend, resulting in the growth of the economy and accelerates the CPI. On the contrary, if spot rates increase, the households tend to saving, and with less disposable income to spend, the economy slows, and so inflation decreases.

Green and Hendershott (2001, [90]) emphasize that adult cohorts have both larger homeownership rates and lower unemployment rates than the younger groups. Hence, as the population gets older in a country, it is more likely to have both greater homeownership and unemployment rates. From this point of view, we may anticipate that the aging communities in a country would generate a negative correlation between homeownership and unemployment rate. The correlation coefficient between HPI and WAP is also negatively correlated in this study, as it is seen from Table 2.3. However, the correlation coefficient is relatively small compared to similar studies.

On the other hand, conversely, the correlation between HPI and the WAP is positively correlated and relatively high. It is also worth mentioning at this point that the GLM result illustrated in this chapter also supports that unemployment has a negative and has the smallest association with the housing market. The reason behind the negative correlation between unemployment and the housing market may lie the length of the observed period since the results also have a link with the observation period.

2.3 Modeling Methodologies

The classical linear regression models have been studied for many years, and they are still the most commonly used methods to address scientific inquiries within the finance literature. Notably, before the sub-prime mortgage market crisis and financial crisis in 2008, which is triggered by the US housing market, researchers used those linear models to investigate the link within the macro-economy and also the dynamics of housing markets. However, non-parametric models become more attractive recently. Especially in the modeling of the non-linear relationship of the data. Therefore, this part of the thesis uses the GLM and MARS methods to determine the effect of macro-economic indicators on housing markets.

In this section, we avoid conveying a detailed clarification of both GLM and MARS methods since finding the impact of the macro-economic indicators on housing markets is our primary aim and not teaching the methods. Hence, we give explanations at an introductory level for both of the methods. Still, the readers who are interested in in-depth explanations can find the details of both approaches within the references that we used in this section.

2.3.1 The Generalized Linear Models (GLM)

Nelder and Wedderburn (1972, [155]) unify many of the regression models under the framework of Generalized Linear Models (GLM). Hence, GLM is an enlargement of the classical regression models. This extension removes the scaling problem in conventional regression models. Moreover, within these models, normality and constant variance assumptions are no longer a requirement for the error component. The GLM provides users a unified framework for multi-factor regression analysis by allowing the use of multiple regression, logistic regression, variance, and covariance analysis [144]. The GLM consists of three components: a link function that specifies the conversion of the response indicator to be modeled with a linear relation of explanatory variables, an error distribution that appropriates for each type of response, and a variance function that specifies the linkage among the mean and variance of the error distribution.

Given a random variable, Y, with mean μ, that observed with n realizations, y, and a matrix, X, of order $n \times p$, and a p-dimensional parameter vector β that will be estimated, the primary objective of a GLM model is to investigate the link between expected value $\mu = \mathbb{E}[Y]$ and the matrix X [144]. In this representation, the vector μ represents the systematic part of the model. It can be written utilizing the existence of co-variates $x_1, x_2, \ldots, x_p$ and estimated parameters β as follows

$$\mu = \sum_{j=1}^{p} \beta_j x_j. \tag{2.1}$$

The systematic part of the model given in Equation (2.1) can be represented as in the following finite sum

$$\mathbb{E}[Y] = \mu_i = \sum_{j=1}^{p} \beta_j x_{ij}, \quad i = 1, \ldots, N,$$

where x_{ij} denotes the value of observation i's j^{th} co-variate. Here, it should be emphasized that $Y \sim N(\mu, \sigma^2)$ and co-variates $x_1, x_2, \ldots, x_p$ produce a linear map, which can be given as

$$\eta = \sum_{j=1}^{p} \beta_j x_j.$$

We may introduce the relationship between the systematic and the random components as

$$\eta = \mu, \tag{2.2}$$

where the parameters η and μ are identical.

Here, if Equation (2.2) is written as

$$\eta_i = g(\mu_i),$$

where the functional g is the link function, it is clear that GLM allows two extensions; the first one is that the distribution may come from an exponential family, and the second one is that the link function, g, may be chosen any monotonic and differentiable function [134].

2.3.2 Multivariate Adaptive Regression Splines (MARS)

For uncovering and complex data patterns, Multivariate Adaptive Regression Splines (MARS) is a popular non-parametric regression method used for estimation of general functions of large dimensional arguments. The most crucial advantage of MARS is that this method performs no specific hypothesis concerning the underlying functional relationship among the response variable and explanatory variables [20, 78, 96]. As a result, it has been widely used in a variety of numerical applications in fields such as finance, medicine, and engineering, and it gives promising results for estimation and forecasting [79, 97, 133].

There are many other advantages of MARS models over classical regression-type models. Its primary advantage is its ability to handle a large number of explanatory or predictor variables, which generally have non-linearity form with response variables. The efficiency of MARS is due to an adaptive model selection/fitting procedure that uses regression splines as its basis functions in the framework of the least-square. Moreover, the choice of its basis functions is data-related and explicit to the problem that users are dealing with. Another distinctive advantage of MARS is that the model constructed with MARS can efficiently predict the benefactions of basis functions so that both of the additive and the multi-way interactions of the estimators.

In modeling with MARS, the relationship between explanatory variables and the response variable is given with a general form defined as

$$Y = f(\boldsymbol{\beta}, \boldsymbol{x}) + \epsilon, \tag{2.3}$$

where Y represents the response variable, $\boldsymbol{x} = (x_1, x_2, \ldots, x_p)^T$ is a vector of estimators and $\epsilon \sim \mathcal{N}(0, \sigma)$ is an error term with constant σ.

In modeling with MARS the main aim is to form reflected pairs for inputs x_j $(j = 1, 2, \ldots, p)$ by using p-dimensional knots $\boldsymbol{\tau}_i = (\tau_{i,1}, \tau_{i,2}, \ldots, \tau_{i,p})^T$ at each input data vector or explanatory variable $\boldsymbol{x}_i = (x_{i,1}, x_{i,2}, \ldots, x_{i,p})^T$, for $\{i = 1, 2, \ldots, N\}$.

MARS uses piecewise linear basis functions of the form

$$h_+(x_j, \tau) = [(x_j - \tau)]_+ = \begin{cases} (x_j - \tau) & \text{if} \quad x_j > \tau \\ \\ 0 & \text{otherwise,} \end{cases}$$

and

$$h_{-}(x_j, \tau) = [-(x_j - \tau)]_+ = \begin{cases} 0 & \text{if } x_j > \tau \\ \\ (\tau - x_j) & \text{otherwise}. \end{cases}$$

In this representation, $\tau \in \{x_{1,j}, x_{2,j}, \ldots, x_{N,j}\}$, where $j \in \{1, 2, \ldots, p\}$. Here, each function is piecewise linear with an uni-variate knot at the value τ, which allows for non-linear changes in the predictor variables. These functions are also called hinge functions or hockey-stick. A key advantage of these functions is that the smoothing parameter does not need to be estimated. As a result of this advantage, large amounts of computational cost and time is saved.

The functional form, $f(\boldsymbol{\beta}, \boldsymbol{x})$, introduced in Equation (2.3) can be illustrated by a linear combination of the basis functions and intercepts as follows

$$Y = \beta_0 + \sum_{m=1}^{M} \beta_m H_m(\boldsymbol{x}^m) + \epsilon.$$

Here, β_ms are the unknown constant parameters for the m^{th} basis function $(m = 1, 2, \ldots, M)$ and for the constant 1 $(m = 0)$. Here, H_m $(m = 1, 2, \ldots, M)$ are basis functions taken from a set of M linearly independent basis elements. The basis functions can be in a form of main or interaction form.

Interacted basis functions are constructed by product of existing basis functions with a truncated linear function involving a new variable. Both the existing basis function and the newly constructed basis functions are used in the MARS approximation. For the given data $(\boldsymbol{x}_i, y_i)$ $(i = 1, 2, \ldots, N)$, the form of the m^{th} basis function for the multi-predictor case is given as

$$H_m(\boldsymbol{x}^m) := \prod_{j=1}^{K_m} [s_{\kappa_j^m} \cdot (x_{\kappa_j^m} - \tau_{\kappa_j^m})]_+, \tag{2.4}$$

where K_m denotes the number of truncated linear functions multiplied in the m^{th} basis function, $x_{\kappa_j^m}$ denotes the input variable corresponding to the j^{th} truncated linear function in the m^{th} basis function, $\tau_{\kappa_j^m}$ is the knot value corresponding to the variable $x_{\kappa_j^m}$, and $s_{\kappa_j^m}$ is the selected sign $+1$ or -1.

The method of model-fitting in the MARS algorithm relies on the lack-of-fit (LOF) criterion, which is used in the comparison of candidate basis functions. The investigation of new basis functions can be restricted to interactions of maximum order. For instance, if we allow only up to two-factor interactions, then, $K_m \leq 2$ would be restricted in Equation (2.4). The MARS algorithm for estimating the model function $f(\boldsymbol{\beta}, \boldsymbol{x})$ consists of two main steps [78]:

i. **The forward stepwise algorithm:** Here, forward stepwise search for the basis function and at each step, the split that minimized some LOF criterion from all the possible splits on each basis function is chosen. The process stops when a user-specified value M_{max} is reached.

ii. **The backward stepwise algorithm:** The purpose of this step is to prevent from over-fitting by decreasing the complexity of the model without degrading the fit to the data. Therefore, the backward stepwise algorithm involves removing from the model basis functions that contribute to the smallest increase in the residual sum of squares (RSS) at each stage.

In both steps we define above, MARS algorithm uses Generalized Cross-Validation (GCV) as a variable selection criterion [78] for the model. The GCV value is computed as follows

$$GCV = \frac{1}{N} \frac{\sum_{i=1}^{N} \left(Y_i - \hat{f}_M(\boldsymbol{\beta}, \boldsymbol{x}) \right)^2}{\left(1 - Q(M)/N\right)^2}.$$

In the GCV equation, the numerator is the usual RSS, the function $Q(M)$ in denominator represents the cost penalty measure of MARS models with M basis functions. In the literature, it is assumed that the MARS model is constructed when we reach the minimum GCV value.

2.4 Empirical Results

In this section, we aim to propose GLMs and MARS models to analyze the US housing market. In our modeling applications in this chapter, we construct the MARS model by using the R package “Earth” (Milborrow, 2009) and form the GLMs by using MATLAB. Here, it is worth emphasizing that before the construction of our models, all variables are all normalized to make the variables comparable since the descriptive statistics of the variables are too different from each other (See Table 2.2).

As none of the variables follow the normal distribution and to reduce the influence of other hidden factors such as auto-correlation and multi-collinearity, we normalized the variable time series using the usual normalization formula,

$$X_N = \frac{X - \min(X)}{\max(X) - \min(X)}.$$

Further, we employ the Augmented Dickey-Fuller (ADF) test to detect the unit root properties of all series, which indicates that all series are non-stationary at the level. Thus, we use the first difference of the series in our modeling analyzes.

Case 1: Linear-GLM (L-GLM)

The linear link function yields the estimated model to be

$$\begin{aligned} \hat{Y}_{HPI} = & - 0.004 + 0.0788 \cdot CPI + 0.0411 \cdot EFFR - 0.004 \cdot GSP \\ & + 0.070 \cdot WAP + 0.0386 \cdot TCU - 0.0243 \cdot UER \\ & + 0.0112 \cdot RC. \end{aligned} \tag{2.5}$$

Here, Equation (2.5) contains only statistically significant explanatory variables since we have use the stepwise method in modeling. The statistically significant parameter estimates conclude that if an increase occurs in CPI, $EFFR$, WAP, and RC, there will be an increase in US house prices. On the other hand, a rise in GSP and UER will lead to a decrease in the prices in the US housing market. Here, we analyze the Granger casuality test for the variables and observe that while $EFFR$ causing HPI, FRM and $TBill$ is not causing HPI. Hence, these two explanatory variables are insignificant in this model. More specifically, under the assumption of ceteris paribus, we conclude the following:

(i) The inflation hedging ability of housing markets is a well-known phenomenon in the real estate and finance literature. As a result, in the high inflationary periods, house prices increase due to the rise in the levels of housing service in response to community demand [180]. On the other hand, inflation affects housing markets over the long term. Although the majority of society considers the price increase in housing markets as improvement of housing markets, generally, the reason behind the scene is inflation. It is because, when we consider inflation while evaluating the price increase of a house, it will be observed that the real growth will be smaller than we observe. Besides, when the inflation rate increases, so do the cost of construction is increased, which causes an increase in house prices. Andrews (2010, [11]) gives proof of the positive effect of inflation on housing markets. However, there exist studies that claim that inflation is having an adverse consequence on housing markets. For instance, Follain (1982, [75]) investigates the connection between inflation and housing markets, and he reports that inflation hurts house prices. In the current chapter, inflation has a positive impact on the US housing market, which conflicts with the results of [75] but coincides with the findings of [11].

(ii) Residential investments tended to turn prior to house prices in business cycles. In the recent decade, investments in housing markets have shown high growth in many countries due to governmental supports. Low-spot rates have been one of the driving factors as they stimulate the housing demand. Therefore, the lower spot rates cause an increase in house prices and, in turn, stimulate residential investments [13]. However, according to the linear GLM, as $EFFR$ increases, house prices increase. Although the model result seems to be a contradiction to [13], it is economically significant according to two essential aspects. First (lending standards), higher spot rates may provide lenders with more of an incentive to make loans and a little bit of a cushion against risk. Second (households psychology), the expectation of an increase in spot rates causes a housing demand increase since householders who are willing to purchase a house will purchase a house before rates go up. Moreover, this situation may increase the quality of mortgage contracts since it pushes costumers to purchase houses only they may afford monthly payments.

(iii) Prices in housing markets are increased as a result of better employment opportunities and higher incomes enjoyed by residents in an expanding economy since the demand for housing is dependent upon household income. Indeed, the higher economic growth and a rise in the household's income will lead families to spend more on houses. On the other hand, employment and household income are highly dependent on each other [132]. As a result, housing markets have a healthy relationship with the employment level of countries. Especially, booms in housing investments increase the employment rate, as the construction sector covers more than 20% of employment gains since early 2000 in the US. GSP is highly related to the energy market and labor markets. For instance, some of the communities benefited from the construction of energy-producing facilities, which built to allow export of GSP. The construction sub-sector has led to job opportunities for households. Thus, job growth has led to an increase in demand for housing. Therefore, GSP increases house prices. The linear model (Equation (2.5)) shows that GSP has a positive effect on the US housing market.

(iv) Mankiw and Weil (1989, [143]) study on the link between demographics and housing markets, and they conclude that an increase in the newborn will affect housing markets twenty years later. It means that the working population has a strong relation with housing markets.

(v) TCU measures the efficiency of the resources by corporations and factories to produce goods in manufacturing, mining, electricity, and gas utilities located in the US. Therefore, capacity utilization highly depends on demand and scheduling production for the most efficient use of facilities in a county. From this point of view, it will affect the cost of new houses and the house values in markets. The linear model finds that when efficiency has an upward trend, the house values will increase. It is an expected result since the increase in TCU increased the income of the consumers and triggered the demand in housing.

(vi) The unemployment causes a recession in housing markets, which is also declared in [170]. However, Oswald (1999, [161]) and Blanchflower and Oswald (2013, [30]) propose that home-ownership increases the unemployment rate as it affects the labor mobility. Contrary to [161] and [30], our model shows that a rise in unemployment will cause a decrease in the house prices as in the study [32].

(vii) The model also proofs that the US housing market effected by recessions periods in the US economy.

Case 2: Quadratic-GLM (Q-GLM)

The previous model introduced in Case 1 gives us the results under the linear relation assumption. The linear relationship moderated by explanatory variables is a simplistic way to explain statistically significant dependent variables, but they have several serious drawbacks. For instance, linear models may not capture certain nonlinear relationships, and they may make no sense for some of the specific parameters. More importantly, generally the real life is more complicated than linear relationships. For accurate modeling, the inclusion of the two-way interactions is crucial since mutual influence can be observed in the parameters of the interaction terms.

On the other hand, understanding a significant interaction is slightly less straightforward than the parameters in linear models. Therefore, like Case 2, we construct a model for HPI using a quadratic link function that allows interactions among explanatory variables in GLM. Mathematically, the quadratic link function yields a polynomial equation that illustrates the influence of statistically significant variables with interactions on the US housing market.

Under the quadratic link function assumption, the housing market model becomes,

$$\begin{aligned}\hat{Y}_{HPI} = \quad & - \quad 0.006 + 0.009 \cdot TBill + 0.089 \cdot EFFR + 0.085 \cdot TCU \\ & + \quad 0.016 \cdot RC + 0.815 \cdot EFFR \times TCU - 0.0419 \cdot EFFR \times RC \\ & - \quad 0.0620 \cdot TCU \times RC - 0.003 \cdot RC^2. \end{aligned} \tag{2.6}$$

Here, the model illustrated by Equation (2.6) also contains only statistically significant terms since the stepwise method drop out the non-significant terms.

The quadratic model given by Equation (2.6) involves the main effect of four explanatory variables namely; $TBill$, $EFFR$, TCU, and RC, and four interaction terms between variables: $EFFR$ has interactions with TCU and RC; TCU has interactions with $EFFR$ and RC; RC has interactions with TCU and itself which are selected according to their relative association to each other. The positive coefficients of the interaction terms suggest that the house prices become more favorable as the variables increase. However, the size and precise nature of these effects are not easy to divine from the examination of interaction coefficients alone.

The significance of quadratic terms signal that the relationship between HPI and explanatory variables may be non-linear. Therefore, it is cumbersome to interpret the individual coefficients in Case 2 since variables tied to each other. However, intuitively, we may interpret the following from Equation (2.6).

(i) $TBill$ is tied to any of the variables. Thus, its effect may be explained as in the L-GLM case.

(ii) However, notice that there are three terms which essentially contain $EFFR$. So if we combine these terms, the aggregate effect of $EFFR$ is being $(0.089 + 0.815 \cdot TCU - 0.0419 \cdot RC)$. Thus, on the contrary to Case 1, for some values of TCU and RC the effect of $EFFR$ is negative. The quadratic model shows that the effect of $EFFR$ depends on the levels of TCU and RC. So, sort of a way the coefficients of TCU and RC adjusting the effective price of $EFFR$.

(iii) The aggregate effect of TCU is determined by $(0.085+0.815\cdot EFFR-0.0419\cdot RC)$. The aggregate effect shows that if the variables $EFFR = RC = 0$, the rate of change will be 0.085. The coefficient -0.0419 tells both the direction and steepness of the curvature. Thus, it indicates that RC has a concave down effect on HPI.

(iv) Similarly, the aggregate effect of RC has a concave down impact on HPI.

Case 3: MARS

In the construction of the model using the MARS method, the maximum number of basis functions (M_{max}) and the highest degree of interactions (K_m) are determined by trial and error, which are summarized in Table 2.4.

The final decision for the MARS model is decided according to minimum GCV value (MARS models having the same GCV value may also be compared according to RSS, and Coefficient of Determination (R^2) values). Therefore, among many alternative MARS models, the eighth model in Table 2.4 is chosen as the best model to fit the US housing market for the explanatory variables that we considered in our analysis. Here, it is essential to clarify that in our modeling with the MARS method, we assign the parameters M_{max} and K_m as 100 and 2, respectively.

Table 2.4: Alternative MARS models with comparison measures

Case	M_{max}	K_m	GCV	RSS	R^2
1	50	1	2.4151e-05	0.0044	0.7679
2	50	2	2.1075e-05	0.0034	0.8239
3	50	3	2.1075e-05	0.0034	0.8239
4	50	4	2.1075e-05	0.0034	0.8239
5	50	5	2.1075e-05	0.0034	0.8239
6	50	6	2.1075e-05	0.0034	0.8239
7	100	1	2.3893e-05	0.0042	0.7791
8*	100	2	2.0129e-05	0.0023	0.8800
9	100	3	2.0690e-05	0.0021	0.8878
10	100	4	2.0531e-05	0.0019	0.9025
11	100	5	2.0531e-05	0.0019	0.9025
12	100	6	2.0531e-05	0.0019	0.9025
13	150	1	2.3893e-05	0.0042	0.7791
14	150	2	2.0331e-05	0.0033	0.8257
15	150	3	2.1848e-05	0.0036	0.8127
16	150	4	2.1102e-05	0.0023	0.8781
17	150	5	2.1102e-05	0.0023	0.8781
18	150	6	2.1102e-05	0.0023	0.8781

* Indicates the best configuration according to GCV value

The estimated HPI ($\hat{Y}_{HPI}$) becomes,

$$\begin{aligned}\hat{Y}_{HPI} = \quad & 0.0069 - 0.2387 \cdot H_1 - 0.2369 \cdot H_2 - 0.1202 \cdot H_3 \\ & +0.3872 \cdot H_4 + 0.0640 \cdot H_5 - 0.1546 \cdot H_6 - 0.0161 \cdot H_7 \\ & +0.0083 \cdot H_8 - 0.0749 \cdot H_9 + 3.9605 \cdot H_{10} - 2.3779 \cdot H_{11} \\ & -1.2907 \cdot H_{12} + 2.0095 \cdot H_{13} - 2.0001 \cdot H_{14} + 2.3246 \cdot H_{15} \\ & -19.5806 \cdot H_{16} + 7.3737 \cdot H_{17} - 0.3601 \cdot H_{18} + 0.3043 \cdot H_{19} \\ & -0.9922 \cdot H_{20} + 0.0625 \cdot H_{21} + 0.1166 \cdot H_{22} + 0.4982 \cdot H_{23} \\ & -9.3394 \cdot H_{24} + 20.0889 \cdot H_{25} - 10.6264 \cdot H_{26} + 0.0320 \cdot H_{27} \\ & -0.1394 \cdot H_{28}. \end{aligned} \tag{2.7}$$

where $\hat{Y}_{HPI}$ is the HPI prediction of MARS model. The corresponding basis functions in Equation (2.7), H_m, $m = 1, 2, \cdots, 28$, are determined as follows

$$H_1 = \max\{0, ER - 0.06\}, \; H_2 = \max\{0, EFFR + 0.02\},$$
$$H_3 = \max\{0, 0.01 - EFFR\}, \; H_4 = \max\{0, EFFR - 0.01\},$$
$$H_5 = \max\{0, COP + 0.07\}, \; H_6 = \max\{0, 0.03 - GSP\},$$
$$H_7 = \max\{0, 1 - RC\}, \; H_8 = \max\{0, RC - 1\},$$
$$H_9 = ER \cdot \max\{0, RC - 1\},$$
$$H_{10} = \max\{0, CPI - 0\} \cdot \max\{0, 0.04 - COP\},$$
$$H_{11} = \max\{0, 0.06 - ER\} \cdot \max\{0, COP - 0.04\},$$
$$H_{12} = \max\{0, ER + 0.02\} \cdot \max\{0, GSP - 0.03\},$$
$$H_{13} = \max\{0, ER + 0.01\} \cdot \max\{0, GSP - 0.03\},$$
$$H_{14} = \max\{0, ER - 0.02\} \cdot \max\{0, GSP - 0.03\},$$
$$H_{15} = \max\{0, ER - 0.04\} \cdot \max\{0, GSP - 0.03\},$$
$$H_{16} = \max\{0, 0.01 - EFFR\} \cdot \max\{0, COP - 0.03\},$$
$$H_{17} = \max\{0, 0.01 - EFFR\} \cdot \max\{0, COP - 0.01\},$$
$$H_{18} = \max\{0, 0.01 - EFFR\} \cdot \max\{0, RC - 1\},$$
$$H_{19} = \max\{0, 0.04 - COP\} \cdot \max\{0, 0.35 - GSP\},$$
$$H_{20} = \max\{0, 0.04 - COP\} \cdot \max\{0, UER - 0.02\},$$

$$H_{21} = \max\{0, 0 - COP\} \cdot \max\{0, 1 - RC\},$$
$$H_{22} = \max\{0, COP - 0\} \cdot \max\{0, 1 - RC\},$$
$$H_{23} = \max\{0, GSP - 0.03\} \cdot \max\{0, RDI - 0\},$$
$$H_{24} = \max\{0, GSP - 0.09\} \cdot \max\{0, 0.03 - TCU\},$$
$$H_{25} = \max\{0, GSP - 0.1\} \cdot \max\{0, 0.03 - TCU\},$$
$$H_{26} = \max\{0, GSP - 0.11\} \cdot \max\{0, 0.03 - TCU\},$$
$$H_{27} = \max\{0, 0.27 - GSP\} \cdot \max\{0, 1 - RC\},$$
$$H_{28} = \max\{0, 0 - TCU\} \cdot \max\{0, 1 - RC\}.$$

MARS model (Equation (2.7)) runs an algorithm that starts with the inclusion of all variables and then eliminates insignificant explanatory variables. Equation (2.7) includes a total of 28 basis functions to explain the inherently complex nature of the US housing market. It is because MARS models are developed automatically and adaptively, requiring less applicant expertise. For the prediction of such complex data, the MARS algorithm explores the inherent structure of the data set easily. MARS produces a robust prediction by building models over all possible combinations of explanatory variables and all values of each variable as candidates of knots automatically. Therefore, there is a large group of knot points in Case 3.

Besides its complexity, the MARS model is capable of exploring both linear and nonlinear links among variables through the additive and interaction basis functions determined as above. The most frequently used variables in the MARS model are GSP, COP, RC, ER, and $EFFR$. In order to assess the relative importance of each independent variable, the complete MARS model is evaluated in detail concerning both additive basis functions such as the first eight basis functions (H_1 to H_8) and interaction basis functions (interaction between only two independent variables) such as H_9 to H_{28}. It should be noted that the knot values of basis functions are the first difference of the series.

The results of the MARS model indicate that independent variables ER, COP, GSP, RC, and $EFFR$ involved in both types of basis functions have the highest effect on the dependent variable (HPI) when compared with the other independent variables.

The knot point for basis function H_3 is 0.01. The interpretation of this basis function is that as $EFFR$ values get smaller values than 0.01, HPI decreases. On the other hand, the basis function H_{16} contains the basis function H_3 to express the interaction between the independent variables $EFFR$ and COP. Similarly, the basis function H_{18} represents the interaction between the independent variables $EFFR$ and RC.

Among basis functions $H_{16}, H_{17}, H_{24}, H_{25}$, and H_{26} have the largest effect on the US housing market (HPI). In the model in Case 3, only basis functions $H_{17} = \max\{0, 0.01 - EFFR\} \cdot \max\{0, COP - 0.01\}$ and $H_{25} = \max\{0, GSP - 0.1\} \cdot \max\{0, 0.03 - TCU\}$ have a positive impact on the US housing market. While the basis function H_{17} contains the interaction between $EFFR$ and COP, the basis function H_{25} has the interaction between GSP and TCU.

The MARS model includes 28 terms, while Q-GLM and L-GLM include much fewer terms. However, there exist some limitations on using basis functions due to their interaction forms. For instance, basis functions H_{12}, H_{13}, H_{14} and H_{15} contain the main function which is $\max\{0, GSP - 0.03\}$. Hence, some of them do not affect HPI since they get zero values related to ER's value. On the other hand, basis functions H_{24}, H_{25} and H_{26} includes the main function which is $\max\{0, 0.03 - TCU\}$. Therefore, their values depend on the change in GSP values. For instance, while GSP value is less than 0.09, they have no effect on HPI. They all have an effect on HPI when GSP value greater than 0.11. On the other hand, if GSP value is between 0.1 and 0.11 only H_{25} and H_{24} have effect on HPI. Similar cases observed for basis functions H_{16}, H_{17} and H_{21}, H_{22}. In this respect, some of the basis functions do not affect HPI at the same time.

Moreover, Figure 2.5 is introduced to visualize the monthly observed and predicted HPI values for the period 1999-2018. The data contains the latest global financial crisis. The MARS, Q-GLM and L-GLM captures/detects and quantifies the crisis since they include Recession Cycles (RC). We observe that there is compliance with the observed and predicted HPI values even during the global financial crisis for all models. The evolution of the anticipated prices indicates that all models predictions are relatively significant, and they can be used to determine the direction of the prices for the US housing market.

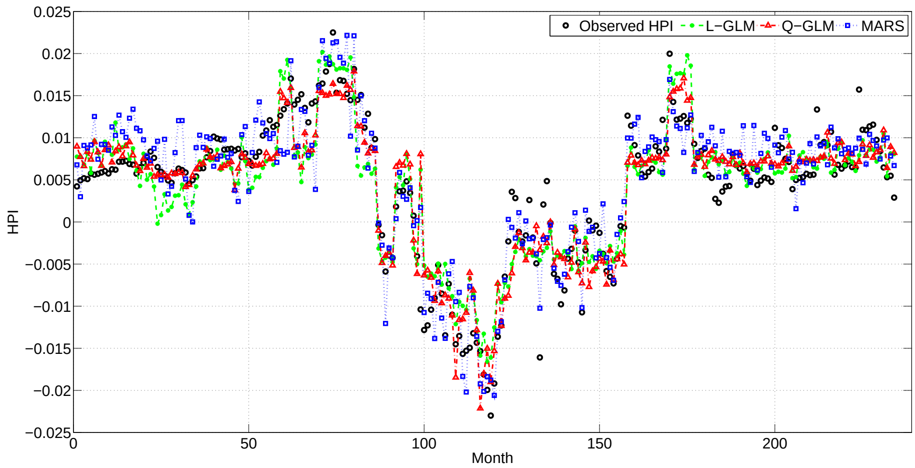


Figure 2.5: MARS and GLM model fits on real data (1999-2018)

2.4.1 Performance of the Models

To evaluate and compare the performances of models (MARS, L-GLM, Q-GLM), we divide the data set into two parts: we use 175 observations as a training sample and 59 observations as a validation sample. The first part of these samples is used for the estimation of model parameters. The second part is employed in the validation of models. Prediction results from MARS and GLMs are further evaluated concerning well-known performance measures such as Mean Absolute Error (MAE), Root Mean Square Error (RMSE), Coefficient of Determination (R^2), and Proportion of Residuals within Three Sigma(PWI). The smaller values for MAE and RMSE indicate a reasonable estimation of model parameters. There are no well-defined bounds on these performance measures values. On the one hand, other measures (R^2 and PWI) state a better performance if their values are close to 1.

We summarize performance measures of models that we constructed and their forecasting power for comparing their efficiency in Table 2.5. The table reveals that for the training data-set, the Q-GLM shows the best performance according to most of the measures (MAE, RMSE, and R^2). However, there is no significant difference between the performance of the models for training data. We also see that the MARS model performs much better than Q-GLM and L-GLM, according to almost all measures for the validation data set. From these results, we may conclude that, in both the training sample and the validation sample, the MARS model has an excellent prediction capability and discovers the main structure of the data very well. It is because the MARS model uses the power of piecewise functions in capturing the data structure. As a result of this, we can apply MARS successfully in the validation sample after the model-building procedure.

Table 2.5: Performances of MARS and GLMs for on train and test data sets

	Train Data			Validation Data		
Performance measure	MARS	Q-GLM	L-GLM	MARS	Q-GLM	L-GLM
MAE	0.0036	0.0031	0.0035	0.0038	0.0036	0.0042
RMSE	0.0044	0.0042	0.0047	0.0051	0.0053	0.0057
R^2	0.76	0.79	0.74	0.66	0.63	0.57
PWI	0.97	0.96	0.96	0.95	0.93	0.97

This chapter provides an insight into macro-economic factors connected with housing markets. It has two main contributions to the real estate finance literature. The first contribution is to highlight the relationship between macro-economic indicators and housing markets with their direction of the interaction. The second contribution is to shed new light on the mechanism of housing markets and to employ non-parametric statistical models that can be used to predict housing markets' price trend. The empirical models provide a coherent set of practical and prediction results. The models also confirm the importance of changes in housing markets.

However, the models we provide in this chapter require so much information about the economy since they use so many explanatory variables. Furthermore, the number of explanatory variables can be increased. The number of explanatory variables may cause a problem while forecasting HPI. Therefore, in the following section, we offer a two-state stochastic process to decrease the number of explanatory variables.

CHAPTER 3

A STOCHASTIC APPROACH TO MODEL HOUSING MARKETS: THE US HOUSING MARKET CASE

The pattern of house prices, trends in their price changes, or their price volatility can be used as an indicator to understand the housing market dynamics. Given the implications of such changes on welfare, there has been considerable interest in identifying driving factors of house price volatility and its further implications [63]. The volatility of house prices can be managed by understanding its influencing factors. Hence, it is vital to reveal determinants behind house prices or reasons for experiencing volatility in housing markets. Although many fundamental elements are well-known, especially after the latest global financial crisis that we faced in 2008, studies on the influence of macro-economic indicators on housing markets have received considerable attention in recent years; see, in particular, studies [12, 88, 106, 107, 188, 195] and the references in these studies.

Along with the impact of many macro-economic indicators on the housing market behavior, mostly, the mortgage rate takes the first place as an external financing source to purchase a house. The demand in housing markets is known to be highly related to the cost of borrowing funds to purchase a house. Therefore, an increase or a decrease in the mortgage rate certainly causes variability to prices in housing markets.

In housing markets, the external funding is generally done through mortgages, whose rate has a high positive correlation with spot or interest rates. Taking into account some essential macro-economic and financial indicators of the US such as 30-year Fixed Mortgage Rate (FRM), 3-Month Treasury Bill ($TBill$), 6-Month $TBill$, 3-Month Eurodollar deposit rate (London), and housing market price index (S&P/Case-

Shiller National Home Price Index (HPI)) series between 1975 and 2016, we observe that interest rates and the mortgage rate follow a similar pattern at which mortgage rate yields a higher rate than the chosen interest rates (Figure 3.1). Even though the duration of mortgage longer than other loans, due to the nature of financing a house purchase, it is an expected result as the treasury notes are the safest investment instruments since the US government issues guarantees on Bonds. In contrast, mortgages do not have such a warranty. Both mortgage rate and interest rates show a reverse pattern and adverse correlation with house price index HPI as emphasized by Andrews (2010, [11]).

Figure 3.1, reveals the following dynamics.

i) The periodic patterns and reverse trend components are consistent features of the housing market.

ii) A strong rise is consistent, even during the 2001 recession, reaching to a remarkable point in 2006 compared to the position of mortgage rates.

The second structure is commonly perceived as a bubble, which is indicated by a circle in Figure 3.1 quoting to the global financial crises in 2008 [59, 62]. At this point, it is worth to emphasize that an increase in house prices, even resembling a bubble, is triggered by preceding prices in time. Besides, the mortgage rate appears to be less vulnerable to financial crises compared to interest rates and HPI.

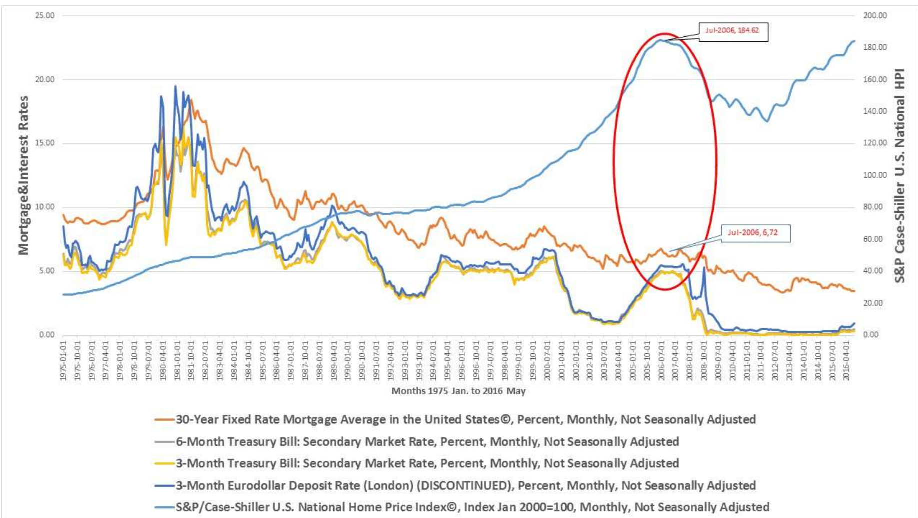


Figure 3.1: US National Home Price Index and financial market indicators

In housing markets, traditional price prediction methods rely on comparisons of the cost and selling house prices without any internationally pronounced standards and certification process. A vast number of studies describing housing market dynamics utilize many approaches, mostly in the frame of econometric models. Those models usually reproduce and capture a sub-sample of characteristics in housing markets.

Common house valuation methods such as hedonic and multiple regression enable researchers to display the importance and impact of significant housing characteristics and recommend a wide range of variables, such as market regulation, physical conditions, spatial and national economic indicators, on the evolution of house prices [34, 65, 82, 140]. Chapter 2 is a genuine example of these modeling approaches. However, especially, the rigid assumptions underlying in conventional models such as normality, independence among explanatory variables, and linearity become an obstacle for assuring the prediction accuracy of such models. Therefore, a model that has a high forecasting power and requiring a minimum number of contributing variables enables housing markets participants to portray housing markets tendency conveniently.

Without dealing with assembling all related explanatory variables in the time frame used in the analysis, one can explain housing markets' price behavior and capture the stochastic response concerning a major indicator. Besides, the association (multicollinearity) among contributing variables may distort the accuracy of the prediction power in conventional methods. Nevertheless, selecting the most contributing variable as a significant explanatory indicator of house price change reduces the cumbersome search for all relevant information, simplifies the housing market modeling, and it may increase the accuracy of the house price prediction.

The strong dependence on housing markets and the mortgage rate leads us to define a stochastic process that mirrors the behavior of house prices as well as its underlying factors using the Stochastic Differential Equations (SDE) approach. Our concern in SDE models stems from their use in modeling stochastic behavior of financial assets, their financial applications such as option pricing, and current widespread availability of data. Therefore, unlike many other studies in the literature, this chapter is based on one indicator that governing house price dynamics. More importantly, this assumption reduces the complexity of dealing with many explanatory variables affecting house price dynamics. Hence, our main aim is to provide an SDE representation of housing markets, while many original studies on housing markets use statistical and econometric models.

In the proposed process, an SDE is utilized, as its rigorous mathematical descriptions include the influence of the historical HPI and FRM values as well as the impact of other explanatory variables. If the present HPI can be reasonably modeled in terms of only the past HPI and FRM values, we have the enticing prospect that forecasting HPI is possible. The SDE for identifying a plausible stochastic model for a national housing market is given in this chapter, along with the parameter estimation techniques and forecasting for the introduced model.

From the modeling perspective and for the empirical analysis, to emphasize the uncertainty component of house price developments, we study on the US house price index and fixed mortgage rate whose dynamics are observable over long years. We consider FRM as the fundamental indicator due to its essential role in increasing (decreasing) housing demand and its high association with interest rates. Based on proposed SDEs, we analytically acquire theoretical house prices associated with mortgage rates whose pattern also follows a certain SDE. We operated a calibration procedure to the given SDE and required parameters, which are inherited into the model to predict the US housing market price behavior. The parameter set is estimated based on US historical HPI and FRM data. In this chapter, to achieve our goal, we adopt an optimization algorithm to the estimation process, which considers underlying probabilistic assumptions of random terms in the given SDEs.

The critical and cumbersome part of the house pricing approach that we use in this chapter is the precise description of stochastic processes governing the behavior of house prices and the mortgage rate. The main characteristics of these processes are to capture the exact nature of both variables. The current chapter contributes to the existing literature by deriving HPI in terms of mortgage rates using SDEs as an alternative method to the econometric methods.

The organization of the chapter is as follows. Section 3.1 introduces the literature on traditional housing market forecasting. Section 3.2 gives an outline of the background of stochastic models, which aid in analytical solutions of the proposed SDEs. The system of SDE equations in modeling house price index in terms of mortgage rates and its theoretical derivation is presented in Section 3.3. Model calibration and forecasting the house prices based on real-life data are illustrated in Section 3.4.

3.1 A Review of Structural Housing Market Modeling Literature

In recent decades, many studies that forecasting house prices can be found due to its growing impact on consumption, commodity, and financial markets, see, for instance, [126, 141, 168, 181]. In the literature, many authors employ econometric models to determine the effect of particular housing characteristics on house prices [53, 74, 121, 122, 183]. However, these models require numerous explanatory variables as we highlight one of its examples in Chapter 2. As a result, these models increase models cost due to the characteristics of explanatory variables. More importantly, the econometric models have a high potential in generating multi-collinearity.

Furthermore, the performance of house price prediction based on the suggested models may lose their accuracy over time. As indicated by Zietz et al. (2008, [200]), outcomes of such models differ not only concerning the size of dependent and explanatory variables and their statistical significance but also in some cases, the sign of coefficients of explanatory variables. Therefore, such models reflect only outcomes for a specific time and location.

Moreover, Rapach and Strauss (2009, [168]) emphasize that no single variable can be considered as the most contributing variable. Therefore, it is challenging to identify apriori and particular variables or a small set of variables. In this regard, the utilization of econometric models in forecasting house prices may not always be as precise as expected. However, the existing literature indicates that relatively few studies use alternative modeling techniques for house price forecasting, such as multivariate time series, which requires a strong underlying theoretical relationship [19, 92, 199]. There exist studies that employing a univariate time series approach with a particular focus on nonlinear price dynamics [56, 126, 149]. These studies explore a variety of predictors of housing markets, which are beyond simple auto-regressive models. For instance, Rapach and Strauss (2009, [168]) focus on the forecastability of house prices in states of the US, and they show that auto-regressive models perform relatively well for interior states. In contrast, they do not perform well for coastal states. They interpret their results as evidence of disconnection among prices and fundamentals of housing markets.

There are remarkable studies that represent a micro-economic derivation of diffusion type processes for house prices, which includes one-state and two-state geometric Brownian motions [115, 119, 120, 153, 175]. Notably, in the pricing of mortgage contracts, the context of the literature uses the arbitrage pricing theory to rationally price a mortgage contract by assuming the underlying house price evolves from a geometric Brownian motion.

As summarized above, the housing market modeling is mostly fulfilled using linear and nonlinear time series techniques and some other econometric techniques. However, to our knowledge, the implementation of a fully stochastic model in forecasting prices in housing markets does not exist in the literature. In a similar direction to Kau and Keenan (1995, [115]) as guiding literature, we implement an SDE structure to understand housing markets in terms of the HPI and FRM. The model in this chapter differs from [115] in the following aspects: i) The SDE structure is composed of different contributing functions, and their analytical solutions under these assumptions are derived. ii) We employ the HPI and FRM, whereas the guiding literature considers only the prices and interest rates. iii) The Ornstein–Uhlenbeck process is deemed to be model the FRM, and the model is calibrated using the observed US data, whereas the guiding literature illustrates findings with simulations.

3.2 Preliminaries for Stochastic Modeling

An SDE explains continuous paths of a variable incorporating both with random and deterministic components at which the random component is presented generally by a Wiener process. It is, therefore, a mathematical construction that models an experiment consisting of states occurring randomly in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where each component in the triple determines the set of all possible outcomes, set of events, and the probability of each event, respectively.

A stochastic process, X_t, on a given probability space is a sequence of $\mathbb{R}$-valued random variables that are measurable concerning the filtration $\mathcal{F}_t$ defined in the probability space and it carries all relevant information along with the time (t). More specifically, if a path of a stochastic process up to a time $t \geq 0$ is observed, it is possible to decide whether an event $A \in \sigma\left(X_s, s \leq t\right)$ has occurred or not occurred based on the relevant information carried by the process itself where $\sigma\left(X_s, s \leq t\right)$ denotes a filtration. More formally, we may define filtration as in the following definition.

Definition 3.1 (Filtration)**.** A filtration $\mathcal{F}_t$ is a non-decreasing family of sub-σ-algebras of $\mathcal{F}$ in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

If $\mathcal{F}_t = \sigma\left(X_s, s \leq t\right)$, then it is called the natural filtration of the given process X_t on $(\Omega, \mathcal{F}, \mathbb{P})$ and X_t is an adapted process to the filtration $\mathcal{F}_t$, if it is $\mathcal{F}_t$ measurable for every $t \geq 0$. Hence, based on Definition 3.1, a stochastic process is always adapted with respect to its natural filtration. This enables us to employ stochastic integrals and stochastic differential equations based on martingales, M_t, justifying the certain requirements given in Definition 3.2 [21, 123].

Definition 3.2 (Martingale)**.** Let $\left(\mathcal{F}_t\right)_{t\geq 0}$ be a filtration on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then, a process M_t is a continuous-time martingale with respect to the filtration $\left(\mathcal{F}_t\right)_{t\geq 0}$ if it satisfies;

i. $\mathbb{E}\left[|M_t|\right] < \infty$,

ii. M_t is $\left(\mathcal{F}_t\right)_{t\geq 0}$ measurable for all t,

iii. $\mathbb{E}\left[M_t|\mathcal{F}_s\right] = M_s$ almost surely, if $s \leq t$.

In the literature, the random component in stochastic processes generally assumed to be a Brownian motion [21], which is given in the following definition.

Definition 3.3 (Brownian motion)**.** A process $W(t,\omega) : [0,\infty) \times \Omega \mapsto \mathbb{R}$ is a Brownian motion if the following properties holds;

i. $W(0) = 0, \quad \mathbb{P}\left(\omega; W\left(0,\omega\right) = 0\right) = 1$,

ii. $W(t)$ is a continuous function of t,

iii. If $0 = t_0 \leq t_1 \leq t_2 \leq \ldots \leq t_n$, then the consecutive differences

$$\{W(t_1) - W(t_0)\}, \ldots, \{W(t_n) - W(t_{n-1})\}$$

are independent, normally distributed with

$$\mathbb{E}\left[\{W(t_{t+k}) - W(t_k)\}\right] = 0, \quad \mathbb{E}\left[\{W(t_{t+k}) - W(t_k)\}^2\right] = t_{k+1} - t_k.$$

The properties given in Definition 3.3 enable us to employ a multi-dimensional Gaussian process. However, a Brownian motion, W_t, does not attain always absolutely continuous paths. To tackle this, Young Theorem [197] is used to evaluate the stochastic integration of the form $\int_0^T X_s \, dW_s$ together with employing quadratic variation property defined as

$$\langle X \rangle_t = \lim_n \sum_{t_i \in \tau_n} \left(X_{t_{i+1}} - X_{t_i}\right)^2, \quad \forall t \geq 0.$$

Here, it should be noted that the quadratic variation of a Brownian motion equals to t. To evaluate stochastic integrals in terms of quadratic variation approach, the Itô's formula needs to be employed.

To introduce Itô integral we first define the space of the set of square integrable processes, u_t, to be progressively measurable with respect to the filtration $\mathcal{F}_t$, on $L^2(\Omega, \mathcal{F}_t, \mathbb{P})$ expressed as

$$u_t = \sum_{i=1}^{n-1} F_i \times \mathbb{1}_{(t_i, t_{i+1}]}(t),$$

where $0 = t_0 \leq \ldots \leq t_n = T$ is a partition of the interval $[0, T]$ and F_i is an $\mathcal{F}_{t_i}$ measurable random variable with $\mathbb{E}[F_i^2] < \infty$. A set of simple processes, $\mathcal{E}$, which consists of the set of u_t's satisfy $\mathcal{E} \subset L^2(\Omega, \mathcal{F}_t, \mathcal{F}, \mathbb{P})$.

Theorem 3.4 (Itô integral)**.** *Given a linear map* $\mathcal{I} : L^2(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}) \to L^2(\Omega, \mathcal{F}, \mathbb{P})$ *having the properties*

i. For $u = \sum_{i=0}^{n-1} F_i \times \mathbb{1}_{(t_i, t_{i+1}]} \in \mathcal{E}$, $\mathcal{I}(u) = \sum_{i=0}^{n-1} F_i(B_{t_{i+1}} - B_{t_i})$,

ii. For $u \in L^2(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, $\mathbb{E}\left(\mathcal{I}(u)^2\right) = \mathbb{E}\left(\int_0^{+\infty} u_s^2 ds\right)$, *the map $\mathcal{I}$ is unique and called the Itô integral defined as*

$$\mathcal{I}(u) = \int_0^{+\infty} u_s dB_s.$$

Before we proceed let us introduce also Itô formula. Itô formula relates differentiation and integration, and also provides a practical method for computation of stochastic integrals.

Theorem 3.5 (Itô's formula)**.** *Let X_t be a continuous and adapted semi-martingale process and let $f : \mathbb{R} \to \mathbb{R}$ be a function which is at least twice continuously differentiable ($f \in \mathcal{C}^2$). Then, for almost every $\omega \in \Omega$, the process $(f(X_t))_{t \geq 0}$ is a semi-martingale, and the following change of variable formula holds:*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) dX_s + \frac{1}{2} \int_0^t f''(X_s) d\langle X, X \rangle_s.$$

If we consider the process, X_t, satisfies the following SDE

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dB_t,$$

then, the existence and uniqueness of solutions to such SDE's can be guaranteed only if the certain conditions given in Theorem 3.6 holds.

Theorem 3.6. *Let $\mu : \mathbb{R} \to \mathbb{R}$, and $\sigma : \mathbb{R} \to \mathbb{R}$ be two real valued functions. Then, there exists a constant $C > 0$ such that*

$$\|\mu(x) - \mu(y)\| + \|\sigma(x) - \sigma(y)\| \leq C\|x - y\|, \; x, y \in \mathbb{R}.$$

Hence, for every $x_0 \in \mathbb{R}$, we can find a unique continuous and adapted process, $(X_t^{x_0})_{t \geq 0}$, such that for $t \geq 0$

$$X_t^{x_0} = x_0 + \int_0^t b(X_s^{x_0}) ds + \int_0^t \sigma(X_s^{x_0}) dB_s.$$

Moreover, for every $T \geq 0$,

$$\mathbb{E}\left(\sup_{0 \leq s \leq T} \mid X_s \mid^2 \right) < +\infty.$$

3.3 The Economic Implications and the Model

Stochastic models, especially in the form of SDEs, are commonly used in financial derivatives pricing and real options analysis under the assumption of a fully competitive market [33, 84, 99, 146, 167, 172]. Among those Gaussian or Geometric Brownian motion structures are the most commonly implemented ones due to their excellent properties [29, 55, 148].

In this chapter, the proposed housing market pricing model is an arbitrage model in which the stochastic behavior of house price indices and a co-factoring variable, the mortgage rate is exogenously given. The SDE approach in developing a price structure for housing markets is defined according to assumptions on its specific characteristics for its analytical simplicity. The price of any house on housing markets can then be retrieved as a function of the given variables, which grand the condition of no-arbitrage for housing markets as in perfectly competitive markets.

In an efficient market, where asset prices instantly change to mirror newly arrived information to market, asset prices would be an accurate reflection of all currently known factors of supply and demand. Moreover, they also contain an adjustment based on expected future changes in the supply and demand relationship. Although the efficient market hypothesis in housing markets was first conducted in the mid-eighties by Linneman (1986, [135]) and some other studies following his study on the efficiency of housing markets, support the weak and semi-strong forms of the efficient market hypothesis [94]. There is still a lack of unanimous conclusion in the effectiveness of housing markets. Under the assumption that the actions of the ignorant and irrational investors are uncorrelated and random, their efforts may cancel out, and housing markets can agree on the same prices [26]. Hence, in this chapter, we aim to present a continuous-time model relying on stationary diffusion processes for the price behavior of housing markets.

Most buyers finance their housing primarily through a debt, which causes a high correlation between housing price and the mortgage rate, as illustrated earlier (Figure 3.1). Such a close association and high statistical dependence between house prices and mortgage rates are expected to define a stochastically interacting system of equations. Therefore, the model of the house price is assumed to follow an evaluation process similar to the Black-Scholes-Merton (BSM) described as follows.

Proposition 3.7. *General Model Let $\left(\Omega, \mathcal{F}, (\mathcal{F})_{t\in[0,T]}, \mathbb{P}\right)$ be a filtered probability space with required conditions. Consider an economy with two correlated state variables which are two non-negative adapted processes, the house price, h_t, and mortgage rate, r_t, defining a joint stochastic process,*

$$\begin{aligned} dh_t &= f_1\left(r_t, h_t\right) dt + f_2\left(h_t, \sigma_h\right) dZ_t, & (3.1)\\ dr_t &= f_3\left(r_t\right) dt + f_4\left(r_t, \sigma_r\right) dW_t, & (3.2)\\ dZ_t dW_t &= \rho dt, \end{aligned}$$

where Z_t and W_t are two correlated Gaussian random variables, with corresponding means zero and variances t, defined under the same natural probability measure $\mathbb{P}$ and the correlation coefficient is given with ρ. Here, at least twice differentiable functions, $f_i, i = 1, 2, 3, 4$, determine the random components at which $f_i, i = 1, 3$ correspond to the drift rates, whereas $f_i, i = 2, 4$ describe the instantaneous part of the unanticipated price and mortgage rate due to the volatiles related to each variable.

It should be rephrased in Equation (3.2) that the mortgage rate is expected to change at any time at a rate $f_3\left(r_t\right)$. However, the actual change stems from the unbiased component $f_4\left(r_t, \sigma_r\right)$, which is serially uncorrelated and follow normally distributed disturbances in the economy. It should be noted that the mortgage rate diffusion processes can be either identified with observed historical data or in particular, with any of the well-known short rate interest rate models [54, 104, 190], which may increase the complexity of the model. Hence, the SDE system is given in Proposition 3.7 sets up a generalization of this feature to construct a more robust pricing structure. Another point to be mentioned is that the model can be easily tractable and flexible concerning the choice of the functions $f_i, i = 1, 2, 3, 4$. In this frame, the selection of the most appropriate model is an important issue and may not always be trivial. Based on the guiding literature [29, 55, 99, 148, 167], selection of f_i's should be made. Specifically, selecting a linear relation between the house price based on [172] resembles econometric models commonly utilized in the literature for estimating the house price.

Additionally, it should be emphasized that Equation (3.1) is a standard stochastic process structure for the underlying asset price allowing for a convenience yield, which evolves with a stochastic process structure introduced in Equation (3.2). In this setting, if one selects a fixed rate, r, the model will become a one-factor model, and the effect of the mortgage rate on the house price is not adequately observed.

Proposition 3.7 can be extended by adding a jump-diffusion to insert sudden changes in the economy, such as financial crises and large catastrophic events that affect the whole economy. The extension of the proposition is given in the following proposition.

3.3.1 Model Selection

The SDE approach is assumed to explain well the characteristics of prices in terms of the underlying distribution of housing returns (log-prices) interpreted intuitively from observed house prices. One-factor models such as Geometric Brownian Motion for the log return of the index price process and an Ornstein-Uhlenbeck (OU) process to represent the mortgage rate are proposed with pre-specified parameters. Analogously, the selection of $f_i, i = 1, 4$, in Proposition 3.8, yields the proposed model for HPI. In index return modeling (Equation (3.3)), the parameter μ_h represents the total expected return on the HPI return, σ_h is the constant volatility and λ is the rate at which r_t reverts to μ_h.

Similarly, the mortgage rate model contains the parameters μ_r representing the mean level at which the short rate will evolve around in the long run; κ denoting the rate of reversion that characterizes the speed at which future trajectories will revert back to μ_r, and σ_r standing for the volatility of mortgage rate (Equation (3.4)).

This system of modeling in equity markets is firstly introduced by Gibson and Schwartz (1990, [84]), which is taken in this chapter as the guiding model to adopt the process of house price index log-returns h_t, as a geometric Brownian motion where the growth rate is adjusted by mortgage rate r_t. Since the house price index is represented as a function of the mortgage rate, this modest and flexible model embeds the properties in both dimensions.

Proposition 3.8. *Given the parameters* $\lambda > 0, \mu_h \in \mathbb{R}, \sigma_h > 0, \kappa > 0, \mu_r \in \mathbb{R}$ *and* $\sigma_r > 0$*, we define*

$$\begin{aligned}
\frac{dh_t}{h_t} &= \lambda\left(\mu_h - r_t\right)dt + \sigma_h dZ_t, & (3.3)\\
dr_t &= \kappa\left(\mu_r - r_t\right)dt + \sigma_r dW_t, & (3.4)\\
dZ_t dW_t &= d\rho,
\end{aligned}$$

whose solutions are derived as

$$\begin{aligned}
h_T &= h_t e^{\left(\lambda\mu_h - \frac{\sigma_h^2}{2}\right)(T-t) + \sigma_h(Z_T - Z_t) - \lambda\int_t^T r_s\, ds},\\
r_T &= r_t e^{-\kappa(T-t)} + \kappa\mu_r\left(1 - e^{-\kappa(T-t)}\right) + \sigma_r\int_t^T e^{-\kappa(T-s)} dWs.
\end{aligned}$$

Proof. Using Itô's Theorem (Theorem 3.5) and taking the logarithm we find

$$\begin{aligned}
\ln(h_T) &= \ln(h_t) + \int_t^T \frac{1}{h_t} dh_s - \frac{1}{2}\int_t^T \frac{1}{h_s^2} d[h,h]_s\\
&= \ln(h_t) + \int_t^T \lambda\left(\mu_h - r_s\right)ds + \int_t^T \sigma_h dZ_s\\
&- \frac{1}{2}\int_t^T \sigma_h^2\left(\rho^2 + 1 - \rho^2\right)ds. \qquad (3.5)
\end{aligned}$$

By rearranging Equation (3.5) we obtain,

$$\ln\left(\frac{h_T}{h_t}\right) = \int_t^T \left(\lambda\left(\mu_h - r_s\right) - \frac{\sigma_h^2}{2}\right)ds + \int_t^T \sigma_h dZ_s. \qquad (3.6)$$

which yields the house price index as

$$h_T = h_t e^{\left(\lambda\mu_h - \frac{\sigma_h^2}{2}\right)(T-t) + \sigma_h(Z_T - Z_t) - \lambda\int_t^T r_s\, ds}.$$

To derive the expression for mortgage rate, r_t, we solve the OU process by defining a new process as in the following form

$$X_t = r_t e^{\kappa t},$$

which results in

$$\begin{aligned}
dX_t &= e^{\kappa t}\left(\kappa r_t dt + dr_t\right)\\
&= e^{\kappa t}\left(\kappa\mu_r dt + \sigma_r dW_t\right). \qquad (3.7)
\end{aligned}$$

By integrating Equation (3.7), we obtain

$$X_T = x_t + \kappa\mu_r\left(e^{\kappa T} - e^{\kappa t}\right) + \sigma_r\int_t^T e^{\kappa s} dW_s. \qquad (3.8)$$

Rearrangement of Equation (3.8) ends up with

$$r_T = r_t e^{-\kappa(T-t)} + \kappa\mu_r \left(1 - e^{-\kappa(T-t)}\right) + \sigma_r \int_t^T e^{-\kappa(T-s)} dWs.$$

□

Note that the solution of r_t is composed of a sum of a deterministic term and an integral of a deterministic function concerning the Wiener process. Having this property in r_t, we assume, under the normality assumption, the mean and the variance of mortgage rate can be derived using the martingale and Itô isometry of Brownian motion.

Corollary 3.9. *Given the mortgage rate, r_t, is normally distributed, the expected value, $\mathbb{E}[r_T]$ and the variance, Var (r_T), of mortgage rate are derived as follows:*

$$\begin{aligned} \mathbb{E}[r_T] &= r_t e^{-\kappa(T-t)} + \kappa\mu_r \left(1 - e^{-\kappa(T-t)}\right) \\ \mathit{Var}(r_T) &= \frac{\sigma_r^2}{2\kappa} \left(1 - e^{-2\kappa(T-t)}\right). \end{aligned}$$

3.4 Model Justification and Forecasting: S&P Case-Shiller Case

The main obstacle in the practical implementation of house pricing with SDEs arises from the determination of the parameters associated with the proposed processes. The reason is: the underlying assets in housing markets are physically produced immobile products, and the market liquidation is not as easy as in the fully competitive markets. Moreover, since housing markets are characterized by experimentation and bargaining among the potential buyers and sellers, both parties naturally use their experience, which may manipulate house prices in the market [65]. Also, houses are not traded frequently, and the house prices that both parties agree on are not directly observable by the market. The validity of the model is performed in a market whose historical data in a long-time frame is available and is known to be robust against the extreme shocks in financial markets. For this reason, we implement our model to monthly collected S&P Case-Shiller Home Price index, h, and the monthly mortgage rate, r, obtained from Federal Reserve Bank of St. Louis for the period in between 1975 and 2016.

The descriptive statistics of the variables (Table 3.1) illustrate that house price index (log-returns) exposes a heavy tail and high kurtosis, which can be a sign of the frequent occurrence of extreme events. On the other hand, the mortgage rate follows an approximately normal distribution ($kurtosis \approx 3.0$) with a mean rate of 8.38% and a standard deviation of 3.24%.

Table 3.1: Descriptive statistics of HPI and mortgage rate (1975-2015)

	Min	Max	Mean	Std	Skewness	Kurtosis
h	25.2	184.62	96.26	47.55	0.36	1.84
$\log$-h	-0.023	0.02	0.004	0.006	-0.76	4.84
r(%)	3.32	18.44	8.38	3.24	0.79	3.33

The model we introduce in Proposition 3.8 contains the potential to represent the behavior of the real-world market. However, before any practical application, the coefficients in the model-specific parameters have to be observed and estimated using the real market data. Therefore, the calibration of the model to determine the market reflecting the model parameters is vital. Indeed, the calibration is a procedure that minimizes the sum of the differences between the market data and the estimated values based on the prespecified model.

In order to implement the prescribed model, we first discretize the time into a finite set of intervals $\{t\}_{i=1}^{N}$, where $t_i < t_{i+1}$ for all $i \in [0, N]$, with $t_0 = 0$, and $t_N = T$. Using a sufficiently large N and an evenly spaced time-lattice $t_i = \frac{iT}{N}$, we approximate the house price and mortgage rate. Starting from initial values observed from the real data, the house price and mortgage rate are determined as follows:

$$h_{t+\Delta t} = h_t + \lambda \left(\mu_h - r_t\right) h_t \Delta t + \sigma_h h_t (Z_{t+\Delta t} - Z_t), \tag{3.9}$$

$$r_{t+\Delta t} = r_t + \kappa \left(\mu_r - r_t\right) \Delta t + \sigma_r (W_{t+\Delta t} - W_t), \tag{3.10}$$

where $\Delta_t = t_{i+1} - t_i$.

By rephrasing Equations (3.9) and (3.10) in terms of their parameters, we obtain

$$\left(\hat{\lambda}, \hat{\mu}_h\right) = \underset{\lambda,\mu_h}{\operatorname{argmin}} \sum_{i=1}^{N-1} \left(\left(\frac{h_{i+1} - h_i}{h_i} \right) - \left(\lambda \mu_h \Delta t + \lambda r_i \Delta_t \right) \right)^2, \tag{3.11}$$

$$\left(\hat{\kappa}, \hat{\mu}_r\right) = \underset{\kappa,\mu_r}{\operatorname{argmin}} \sum_{i=1}^{N-1} \left(r_{i+1} - r_i - \kappa \mu_r \Delta t + \lambda r_i \Delta_t \right)^2. \tag{3.12}$$

Using Least Square Method (LSM) the parameter pairs, κ, μ_r and λ, μ_h, which minimizes Equations (3.11) and (3.12) are estimated. Afterward, to determine the dispersion in the fitted model based on these calibrated estimates, the standard deviations of residuals between actual and estimated values in both SDEs are calculated.

Table 3.2 contains the calibration results of the SDE model that we introduce above. The table shows that the log-return of house prices mortgage rate reverts to the mean value 5.23 with a rate of 16.30 with a volatility rate 6.26%, which are much higher than the ones in the log-transformed index returns. Given the turbulent period under our consideration, it is perhaps unsurprising that the S&P Case-Shiller Home Price Indices have a relatively high mean-reverting parameter.

Similarly, the mortgage rate yields a high κ referring to a fast rate of reversion that future observations reverting to the mean value of -0.01% and much higher volatility, 0.31% compared to the original mortgage rate values, which are also justified by higher standard deviation value in the mortgage rate. The estimation result illustrates that the proposed model in this chapter also includes the hidden volatility in the time series under investigation.

Table 3.2: Estimates of the parameters using calibration

$\hat{\lambda}$	$\hat{\mu}_h(\%)$	$\hat{\sigma}_h(\%)$	$\hat{\kappa}$	$\hat{\mu}_r$ (%)	$\hat{\sigma}_r$ (%)	ρ
16.30	5.23	6.23	7.74	-0.01	0.31	-0.77

As next, we demonstrate the accuracy of the joint calibration (parameter estimation) by performing Monte Carlo simulations of monthly S&P Case-Shiller Home Price Index values based on the estimated SDE model parameters given in Table 3.2 and observed values collected between 1975-2015. Figure 3.2 shows that the model having the estimated parameters generates plausible variability in index returns. The accuracy of the fitted model is justified by quantifying the Root Mean Square Error (RMS) (0.45%) and Mean Absolute Error (MAE) (0.31%).

The estimated stochastic house price index model is empowered to predict the monthly future realizations with Monte Carlo simulations for a duration of a year, which are shown in Figure 3.3. The gray lines in the graph represent the path space trajectory forecasts of the coming twelve months with a hundred simulation, and the red line

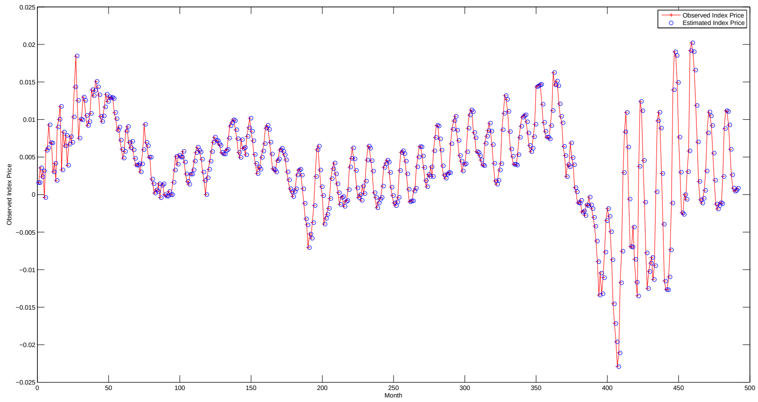


Figure 3.2: Simulated and observed S&P Case-Shiller Home Price Indices

represents the Monte Carlo result of these forecasts. Needless to say, the forecasting values using SDE as future realizations capture the pattern in the index accurately. These results verify the grasping ability of the proposed SDE's. Here, the accuracy of the model can be increased by applying other calibration producers. Besides, the forecast power also might be increased by increasing the number of simulations.

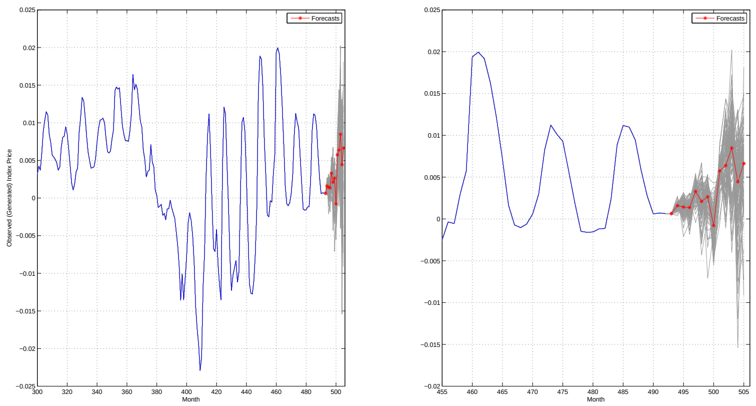


Figure 3.3: Observed and predicted S&P Case-Shiller Home Price Indices

The outcomes of this chapter enable researchers to understand the house price behavior in terms of the random components. We implement the SDE approach through a real-life case, yields a good prediction in future house prices as well it captures the real volatility which is not foreseen accurately in the original series. Measuring price fluctuations and imitating the market evolution together with the mortgage rate using the proposed approach gains importance, certainly for housing markets whose historical observations in terms of all contributing factors are either scarce or not fully available.

CHAPTER 4

LARGE INVESTORS IMPACT ON THE PORTFOLIO OPTIMIZATION OF SINGLE-FAMILY HOUSES IN HOUSING MARKETS

As a consequence of the crash in real estate markets in 2008, some of the large investors came on to the stage and invested in a significant amount of wealth into single-family houses to construct a portfolio of rental dwellings, whose income are securitized in the capital. In some local housing markets, these investors own remarkable numbers of single-family housing units. Furthermore, the trading activities of these investors have resulted in a new investment/business strategy, which exacerbates property wealth concentration and polarization. This new investment strategy and its portfolio optimization inspire curiosity on its influence on housing markets. Hence, this chapter is dedicated to analyzing the effect of large investors on housing markets by taking advantage of the classical portfolio optimization theory.

4.1 A Brief on Large Investors in Housing Markets and Motivation

The most recent global financial crisis triggered by the US housing market collapse in 2008 diminished the home-ownership rate almost in all countries. As a result, following the 2008 crisis, housing markets attracted some single-family real estate investors (such as large private equity firms, institutional investors, or real estate investment trusts), who are called large investors, entered the stage with a new business model as an opportunity to construct a portfolio of single-family housing units in rental housing markets [7]. The large investors are, especially, the biggest buyers in struggling local housing markets where house prices increase quicker compared to healthy housing markets. Hence, contrary to the common belief in the market, after the sub-prime mortgage crisis, large investors dominate local housing markets in return to benefiting from the leading edge of rising house prices and the rental income in their portfolio.

This new business form differentiates substantially from the usual business models in housing markets since the large investors purchase vast numbers of single-family housing units and, therefore, own a sizable extensive rental house portfolio. This strategy conduces a new asset class that a broader group of investors may hold single-family rental houses in their portfolio without purchasing and managing houses by themselves. Contrary to the conventional belief on the inefficiency of large pools of single-family rental housing units due to their managing scattered-site housing units and cost as in the early eighties [192], the large-scale buy-to-rent investment strategy in housing markets has emerged naturally in the last two decades [150].

There are three major reasons behind the notoriety of the new business model. (i) The vast house inventory on housing markets following the 2008 crisis made it easier for the large investors to construct geographically concentrated pools of almost identical properties as the unusually high number of single-family houses in the market creates an opportunity to summon a rental house portfolio. (ii) The tight mortgage financing policy of policymakers gives the large investors an advantage over small and local investors, and further spurring demand for rental housing units due to their less dependence on mortgage financing. (iii) The recent technological developments (such as cloud computing, personal mobile devices, and mobile internet connectivity and their integrity with the banking system) have allowed scattered-site property renovation, maintenance, and management in a more flexible and efficient manner [150].

Regarding the first reason, the unusually high number of single-family houses on markets creates an opportunity for large investors to construct a portfolio that consists of rental houses close to each other and almost identical characteristics. More importantly, large investors are also sophisticated and intelligent consumers, generally cash purchasers, and they have superior negotiation skills and more experience than small and local consumers. This opportunity allows large investors to take advantage of constructing a portfolio with less initial wealth since, at this period, generally houses offered with a discount.

Concerning the second reason, large investors are less dependent on mortgage financing than local and small investors. More importantly, sources of mortgage financing that are available to large investors (for instance, private equity and bank lines of credit) tend to be less costly than mortgage financing that is available for small investors. Consequently, large investors pay less interest than local and small consumers for the loan used for financing the house purchasing. The tight mortgage financing policy of countries due to the crisis also causes an increase in demand for rental houses. Another issue in this item is that large investors generally make their investments in areas where the people with lower credit scores live and most likely prevented from purchasing houses by the tight mortgage financing policy.

On the subject of the third reason is the construction of proprietary software for property acquisition and management. More importantly, such software reduces the cost of assembling and maintaining a sizable rental house portfolio.

By depending on the efficient market hypothesis in the context of the modern portfolio theory, we may claim the price of a house mirrors all available information in the housing market. Therefore, it is likely the price of the house is indifferent to the type of purchaser, whether the buyer is a large or an individual investor [7]. However, the price diverts from its fundamental value. It deviates across investor size because of factors such as high cost of information, lack of adequate competition, high transaction costs, agency costs, and frictions in financing types. For instance, some investors may enjoy the liquidity of their portfolio and transactional efficiencies of the market (advanced targeting of potential acquisition houses, cash acquisitions, superior negotiation talents, and expertise, etc.), and operational efficiencies (real estate and portfolio management expertise of investors) in local housing markets whose purchasers may not have [7].

On top of that, some investors might also enjoy the monopsony power during distressing periods and might be qualified to utilize their purchasing capability and negotiation abilities to purchase houses at a discount to their market value. On the other hand, acquisitions by large investors may increase the total demand in local housing markets, consume house inventory of distressed local housing markets, and, more importantly, cause an increase in house prices. Furthermore, large investors are primarily non-local at which they suffer from high research expenses and informational disadvantage concerning local housing markets [7].

The impact of large investors on housing markets and their investment performance is significant for both local and small investors and inhabitants. It also creates potential interest to accelerate the magnitude of recovery in local housing markets, mainly, with a substantial portion of distressed housing markets [7]. Furthermore, whether acquisition activities by large investors promote or suppress fundamental house prices, it is also crucial for the overall economy since the recovery in housing markets is one of the leading indicators of national economic growth [37, 83, 89, 127, 129].

A vast number of studies investigate the change in house prices at national and local levels. However, the number of studies analyzing the effect of large investors on housing markets is limited. The impact of a large investor on a local housing market typically depends on supply and demand behaviors as in fully competitive markets. On the supply side, developers construct new houses, causing an increase in the number of housing units, types, and quality of houses in the local market. Whereas, the demand side sets the price of new houses to a competitive level within the current house prices and attractive to both investors and potential consumers who are eagerly seeking an attractive investment opportunity to housing. Traditionally, theoretical studies emphasize that constructing new houses is generally likely to generate both positive and adversarial externalities for house owners in local housing markets [191, 198]. In addition to the fiscal, social, and ecological benefits, new buildings can also generate advantages for joining private house owners. The newly constructed houses may have positive spillover influences on present neighborhoods by producing a more lively district as empty lots become populated [198]. If abandoned lots form external dis-amenities through attracting dumping, providing illegal use, or causing a deformity, then building new houses will progressively reduce the external dis-economy, increase the local population, develop the aesthetics of the local housing market and increase surrounding property values [60]. New constructions might be more aesthetically delightful than unkempt lots or dilapidated houses which improve the views. However, such infill development may also cause an adverse effect on surrounding properties due to increased traffic flow, noise pollution, and reduction of green areas on the district [139]. New houses may also compete directly with current houses in the same housing market segment or indirectly through associated sub-markets, potentially decreasing the values of nearby current housing units by expanding the housing supply while the housing demand remains constant [178].

While most of the attention focuses on the effect of state expenditures and grants for city center redevelopment impact surrounding house values and the impact of private residential development on local housing markets, the effect of large investors on housing markets in this aspect remains an unresolved question. This chapter aims to address this gap in the literature by employing Stochastic Differential Equations (SDEs) and stochastic control methods to investigate the impact of large investors purchasing single-family houses for the rental business. On top of that, the effect of economic states on housing markets is taken into account using the Markov switching model. Therefore, the key ingredient and contribution of this chapter are to examine the presence of large investors jointly with the effect of economic state on housing markets to maximize investors' wealth.

The organization of the chapter is as follows: Section 4.2 introduces a theoretical framework for the economic environment of the investigated housing market. Furthermore, it includes preliminaries, proposed theorems along with their proofs to model the housing market with the effect of large investors. Section 4.3 illustrates the numerical results of the proposed model.

4.2 The Economic Environment

In this chapter, the large investors as purchasers are assumed to utilize housing units in their portfolio only for business purposes. The context of large investors contains "corporate" investors. Corporate investors are the purchasers that intend to lease and/or resell the property without leasing or occupying it. On the other hand, individual investors are the ones where the purchaser is not an organization, and his legal posting address obliges for at least three transactions [150].

Definition 4.1. A large investor is a buyer who has corporate structure and does not intend to use the housing units for the personal or company use.

In the context of the chapter, we assume that the objective of the large investors is to find self-financing strategies in the long-term period that maximize their expected utility concerning their terminal wealth. To express house price dynamics we define the triplet $(\Omega, \mathcal{F}, \mathbb{P})$ to be a filtered probability space with filtration, $\mathbb{F} = \left(\mathcal{F}_{t\in[0,T]}\right)$, satisfying the usual conditions for some fixed but arbitrary time horizon $T \in (0, \infty)$. Consider a continuous-time finite-state Markov chain, Y, which represents the uncertainty of the economic state. Let $\epsilon = \{e_1, \ldots, e_K\}$ denote the state space where e_k is the basis of $\mathbb{R}^K$ and assume that the Markov chain, Y has a generator $Q = (Q^{ij})$ with a predetermined initial distribution denoted by $\pi = \left(\pi^1, \ldots, \pi^K\right)$.

Suppose there exists a large investor with a predetermined initial wealth $x \in \mathbb{R}^+$ in the market, and her objective is to attain an investment strategy that maximizes her expected utility from her terminal wealth. Moreover, suppose that there are two available instruments in the market: a bank account (or risk-free bond as a risk-free asset) and houses as risky assets. Therefore, the investor can invest in houses and a risk-free bond as the only available investment instruments in the market.

Assume the risk-free bond price process dynamics is given as

$$dB_t = rB_t dt, \quad r > 0, \quad and \quad t \geq 0. \tag{4.1}$$

Suppose $\alpha_t \in \mathbb{R}$ represents the fraction of wealth invested into the houses at $t \in [0, T]$ whereas, $1 - \alpha_t$, corresponds to the wealth that invested in the risk-free bond. Here, it is worth emphasizing that due to its nature, the short selling of a house is not possible, which may create analytical difficulties. To avoid such obstacle, Assumption 4.2 is stated.

Assumption 4.2. Given $\alpha_t \in [0, L]$, $L \in \mathbb{R}^+$ for all $t \in [0, T]$ and L always can be chosen large enough to ensure that the optimal solution of the corresponding portfolio problems is an interior point in the given interval if a finite optimal solution exists.

In the standard approach, a house price is assumed to evolve from a diffusion process whose mean, and variance are determined by a two-state diffusion process [119, 120]. However, in this chapter, for the sake of simplicity, we employ a diffusion process specified by one state variable, which contains a Markov process representing the state of the economy. A two-state diffusion process to define the evolution of house prices is more complicated than the process we propose in this chapter. To present a model with the simultaneous effect of a large investor and the state of the economy, we consider a diffusion process in the lead of the studies [22, 66]. Extending the result of [66] from exchange markets to housing markets, we drive an SDE model for the effect of large traders in housing markets.

Let H_t denote the original price of the house at time t, which is assumed to obey a diffusion process whose drift is a function of the current state of the economy and the fraction of wealth invested by the large investor. We propose house price dynamics to be as follows.

$$\begin{aligned}\frac{dH_t}{H_t} &= (\mu(Y_t) + g(\alpha_t, m))\, dt + \sigma dW_t, \\ H_0 &= h_0.\end{aligned} \tag{4.2}$$

Here, the parameter $\sigma > 0$ represents the volatility, m is the maintenance cost, W_t is a Brownian motion which is independent of the Markov chain Y_t, and $g(\cdot, \cdot)$ denotes the influence of the large investor on the drift of the house price process. It should be noted that $\mu(Y_t) = MY_t$ with $M^k = \mu(e_k)$, $1 \leq k \leq K$ which is a consequence of the finite-state property of the Markov chain.

The collective impact of the large investor and maintenance are inserted into the model since the value of a house generally increases with regular maintenance. More importantly, the provision of more and better houses accommodates more households, develops the appearance of the region, and hence, brings in new infrastructure, additional spending, and investments in local shops and services. Contrary to other commodities, these effects in the local housing market affect house prices gradually. For this purpose, the impact of the large investor is added to the drift of the process given with Equation (4.2). Here, it is worth emphasizing that if the large investor purchases a house from an underdeveloped zone, it requires him to spend a fraction of her wealth for the maintenance of the house. Therefore, the effect of maintenance costs on the house is added to the model. Clarifying the role of investment effect and maintenance is especially crucial in the current social and political environment [95].

On the other hand, the case of volatility, σ, under partial information, leads us to an exciting setting where activities of the large investors generate a trade-off among the increase in the controlled part of the drift and reduction in the correctness of estimates of the unobserved portion of the process. Nevertheless, such a model setting is even technically acceptable since one investor's movements will not make a significant change in volatility.

The dynamics in Equation (4.2) indicate that the portfolio selection of the large investor in the housing market might be perceived as a signal by the small and local investors. That is, in a local area, the portfolio selection of the large investor serves as a determinant ruling out the drift term of the house price process. Note that we focus on the case with no influence on the diffusion part of the house price process, which allows a unique solution granted that the functional $g(\cdot,\cdot)$ is adequately regular.

As an immediate consequence of Equation (4.2), we make the following remark.

Remark 4.3. If the house price evolves from the stochastic process defined in Equation (4.2), the following holds

i) If $\mu(Y_t) \gg 0$, depending on its form the value of the impact functional $g(\cdot,\cdot)$ need not to be too large for yielding a decent return from houses.

ii) If $\mu(Y_t) \approx 0$, the impact of the large investor may cause an increase in the house price. For instance, by depending on the form of $g(\cdot,\cdot)$, a substantial α_t and high maintenance can result in a decent return from houses.

To construct a portfolio within $[0, T]$, we assume a market where householders may sell without a cost, a fractional interest in their home. Here, the housing choice decision does not contain individual asset allocation. Further, we classify any buyer who has no intention of personal use of the housing unit as an investor. Thus, the term investors here include corporate investors [150]. Under these assumptions, the portfolio dynamics of the wealth of the large investor satisfies,

$$\frac{dX_t^\alpha}{X_t^\alpha} = \alpha_t \frac{dH_t}{H_t} + (1-\alpha_t)\frac{dB_t}{B_t} + \delta\alpha_t dt - m\alpha_t dt,$$

where δ and m denote the rental income from the housing units and maintenance cost of the investor, respectively.

The investor benefits from the rental income of houses in her portfolio, and thus she has a monthly income proportional to her investment amount to the housing. On the other hand, the investor has a cost due to the maintenance, which is also proportional to her housing investment. While the rental income has a positive effect on her wealth, the maintenance yields an adverse impact as a cost.

Equivalently, we can write the wealth process as

$$\begin{aligned}\frac{dX_t^\alpha}{X_t^\alpha} = & \quad \left[\alpha_t\left(\mu_t(Y_t) + g(\alpha_t, m)\right) + (1-\alpha_t)\, r + \delta\alpha_t - m\alpha_t\right] dt \\ & + \quad \alpha_t \sigma dW_t, \quad X_0^\alpha = x > 0. \end{aligned} \tag{4.3}$$

To guarantee that the wealth process given in Equation (4.3) is well defined, it is necessary to consider only investment strategies satisfying certain conditions given in Assumption 4.4.

Assumption 4.4. Let X_t^α be the wealth process defined in Equation (4.3). Then, we call an investment strategy α_t admissible if X_t^α satisfies

$$\int_0^T \left(\alpha_s X_s^\alpha\right)^2 ds < \infty, \quad a.s.$$

In modeling fully competitive markets, generally, it is assumed that all investors are price takers. However, in the proposed model given with Equation (4.2), this assumption is violated since we allow the large investor to influence the house price. Therefore, in such a case, we can not depend on the no-arbitrage condition given for the classical models defined for financial markets. However, under these considerations, we can propose the no-arbitrage condition on functional g as in the following theorem.

Theorem 4.5. *Let H_t be the house price process introduced in Equation (4.2). Given Assumptions 4.2 and 4.4 hold, and the function g satisfies the condition $|g\left(\alpha_t, m\right)| \leq C|1+\alpha_t|$ for constant $C \in \mathbb{R}^+$, then, the market becomes arbitrage free.*

Proof. For an admissible strategy α_t, suppose

$$\theta(t) = \frac{\mu(Y_t) - r}{\sigma}, \quad 0 \leq t \leq T.$$

It is clear that $\theta(t)$ is adapted to $\mathcal{F}_t$ since the Markov chain Y_t is adapted and the parameter σ is constant. Using Girsanov's theorem, there exists an equivalent probability measure $\tilde{\mathbb{P}}$ under which

$$\tilde{W}_t = W_t + \int_0^t \theta(s) ds$$

is a Brownian motion. Here, $|\theta(t)| \leq \frac{|\mu(Y_t)+r|}{\sigma}$ and the Novikov condition is satisfied. The Radon-Nikodym derivative is given by

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \exp\left\{-\int_0^T \theta(t)dW_t - \frac{1}{2}\int_0^T \theta^2(t)dt\right\}.$$

Now, define

$$L_t = \exp\left\{-\xi R_t\right\}, \quad \xi \in \mathbb{R}_+,$$

where

$$R_t = \int_0^t \alpha_s[\mu(Y_s) + g(\alpha_s, m) - r]ds + \int_0^t \alpha_s \sigma dW_s.$$

Then, L_t can be written as

$$L_t = \exp\left\{-\xi\left(\int_0^t \alpha_s g(\alpha_s, m)ds + \int_0^t \alpha_s \sigma d\tilde{W}_s\right)\right\}.$$

By applying Itô's formula we obtain

$$dL_t = L_t\left[\left(-\xi\alpha_t g(\alpha_t, m) + \frac{1}{2}\xi^2\alpha_t^2\sigma^2\right)dt - \xi\alpha_t\sigma d\tilde{W}_t\right].$$

Here, if $\xi > \frac{2C}{\sigma^2}$, then the drift becomes negative ($L_t > 0$). By considering the integrability condition on L_t, we obtain L_t as a super martingale on $\tilde{\mathbb{P}}$, hence

$$\tilde{\mathbb{E}}\left[L_T\right] \leq \tilde{\mathbb{E}}\left[L_0\right] = 1, \tag{4.4}$$

where $\tilde{\mathbb{E}}$ denotes the expectation under $\tilde{\mathbb{P}}$ measure.

Now, suppose α_t to be an admissible strategy that satisfies

$$\mathbb{P}\left(e^{-rT}X_T^{(\alpha)} \geq X_0^{(\alpha)}\right) = 1,$$

which corresponds to $\mathbb{P}\left(R_T \geq 0\right) = 1$. From the equivalent property, $\tilde{\mathbb{P}}\left(R_T \geq 0\right) = 1$. By Equation (4.4) we have $\tilde{\mathbb{P}}\left(R_T = 0\right) = 1$, implying $\tilde{\mathbb{P}}\left(e^{-rT}X_T^{\alpha} = X_0^{\alpha}\right) = 1$, which means α_t is not an admissible strategy. This final result contradicts with our assumption that α_t is an admissible strategy. $\square$

Assume we are provided a utility function, which is concave, increasing, and at least twice continuously differentiable, defined as $U : \mathbb{R}^+ \longmapsto \mathbb{R}$. Then, the optimization problem of the large investor's value function, $V(x, i)$, in investing housing market becomes

$$V(x, i) = \max_{\alpha} \mathbb{E}^{x,i}\left[U\left(X_T^{(\alpha)}\right)\right]$$

subject to

$$X_t^{(\alpha)} = x, \quad Y_t = i.$$

Throughout the current chapter, we consider the case that the large investor is assumed to recognize the true state of the economy. Subsequently, a portfolio strategy is admissible if $\alpha_t \in \mathcal{F}_t$ for all $t \in [0,T]$ and both Assumptions 4.2 and 4.4 are hold.

Let $\mathcal{H}$ define the set of all admissible portfolio strategies. Then, under the logarithmic utility function, it is possible to solve the optimization problem for a general impact function, g, which is regular enough.

Proposition 4.6. *Suppose that the impact function g is continuously differentiable and the large investor has the utility function $U(x) = \log(x)$. Then, the optimal investment strategy is*

$$\alpha^*(t,i) = \arg\max_{\alpha\in\mathcal{H}} \mathbb{E}^{x,i}\left[U\left(X_T^{(\alpha)}\right)\right],$$

and furthermore, for all $(t,i) \in [0,T] \times \epsilon$, $\alpha^*(t,i) \in H_i^{\log}$ where

$$H_i^{\log} = \{0, L\} \cup \left\{ l : M^i - r + g(l,m) + \delta l - ml + l\left(\frac{\partial g(l,m)}{\partial l} - \sigma^2\right) = 0\right\}.$$

Proof. Given the wealth process dynamics in Equation (4.3), we apply Itô's formula for the utility function $U(x) = \log(x)$ and obtain

$$\begin{aligned} U(X_T^{(\alpha)}) = & \quad \log(x) \\ & + \int_t^T \left(\alpha_s\left(\mu(Y_s) + g(\alpha_s, m)\right) + (1-\alpha_s)r + \delta\alpha_s - m\alpha_s - \frac{1}{2}\alpha_s^2\sigma^2\right) ds \\ & + \int_t^T \alpha_s \sigma dW_s. \end{aligned}$$

For any admissible strategy $\alpha^* \in \mathcal{H}$, $\int_t^T \alpha_s^* \sigma dW_s$ is well defined and $\mathbb{E}\left[\int_t^T \alpha_s^*\sigma dW_s\right] = 0$ since it is a martingale. Hence, we have

$$\begin{aligned} \mathbb{E}^{x,i}\left[U(X_T^{(\alpha)})\right] = \log(x) + \mathbb{E}\Bigg[\int_t^T \Bigg(& \alpha_s\left(\mu(Y_s) + g(\alpha_s,m)\right) + (1-\alpha_s)r \\ + & \delta l - m\alpha_s - \frac{1}{2}\alpha_s^2\sigma^2\Bigg) ds\Bigg]. \end{aligned} \tag{4.5}$$

Now, let us denote the integrand in Equation (4.5) as

$$f(s,l) = l\left(\mu(Y_s) + g(l,m)\right) + (1-l)r + \delta l - ml - \frac{1}{2}l^2\sigma^2.$$

Then, from the continuity of the functional $g(\cdot,\cdot)$ for any $s \in [t,T]$, $f(s,\cdot)$ is a continuous function defined on the compact set $[0,L]$. Hence, there exists a number in

$[0, L]$ that maximizes $f(s, \cdot)$. More specifically, the maximizer satisfies $l : \frac{\partial f(s,l)}{\partial l} = 0$ or $\{0, L\}$, which is $\alpha_s^* \in H_i^{\log}$. $\square$

Remark 4.7. One can extend Proposition 4.6 to the case where the impact function $g(\cdot, \cdot)$ is differentiable except for finitely many points in the domain $[0, L]$. Let H^0 denotes the set of the points where $g(\cdot, \cdot)$ is not differentiable. Then, Proposition 4.6 holds with the optimal solution $\alpha^*(t, i) = \left(H_i^{\log} \cup H^0\right)$.

To start with an easy and specific case application, consider the optimization problem under complete information with a linear impact function and logarithmic utility. Namely, for simplicity set the impact function as $g(\alpha, m) = \beta(\alpha + m)$ with $\beta > 0$. Then, we give the following corollary as an immediate result of Proposition 4.6.

Corollary 4.8. *Suppose an investor with a utility function $U(x) = \log(x)$ and an impact function $g(\alpha, m) = \beta(\alpha + m)$ where $\beta > 0$. Then, we have*

$$H_i^{\log} = \left\{0, L, \left(\frac{M^i + \delta + (\beta - 1)m - r}{\sigma^2 - 2\beta}\right)^+\right\}.$$

In particular, depending on the given set of model parameters we have the following cases:

i) if $2\beta - \sigma^2 < 0$, then, for all $(t, i) \in [0, T] \times \epsilon$, the optimal strategy is given by

$$\alpha^*(t, i) = \left(\frac{M^i + \delta + (\beta - 1)m - r}{\sigma^2 - 2\beta}\right)^+,$$

and the value function has the following stochastic representation:

$$V(t, x, i) = \log(x) + r(T - t) + \mathbb{E}^{x,i}\left[\int_t^T \frac{[(\mu(Y_s) + \delta + (\beta - 1)m - r)^+]^2}{2(\sigma^2 - 2\beta)} ds\right].$$

ii) if $2\beta - \sigma^2 \geq 0$, then, for all $(t, i) \in [0, T] \times \epsilon$, the optimal strategy is given by

$$\alpha^*(t, i) = L \mathbb{1}_{M^i + \delta + (\beta - 1)m - r \geq -L\left(\beta - \frac{1}{2}\sigma^2\right)}$$

and the value function in this case has the following stochastic representation:

$$\begin{aligned} V(t, x, i) = \log(x) \quad &+ \quad (T - t)\left(\left(\beta - \frac{\sigma^2}{2}\right) L^2 + r\right) \\ &+ \quad \mathbb{E}^{x,i}\left[L \int_t^T (\mu(Y_s) + \delta + (\beta - 1)m - r)\, ds\right]. \end{aligned}$$

Remark 4.9. As a consequence of Corollary 4.8, we infer:

i) If parameters satisfy $M^i + \delta + (\beta - 1)m - r < 0$, the solution is not an optimal solution for the housing market. This case occurs if the economy is in the bad state and the interest rate is high which an indication of an unfavorable housing market. The optimal strategy leads investors to have a positive bank account. Further, if the case in item $i)$ occurs, the effect of large investors on housing markets are going to be significantly small as it is expected since $\frac{\sigma^2}{2} > \beta$.

ii) If $\beta > \frac{\sigma^2}{2}$ the influence of the large investor on the house price is too high.

iii) Corollary 4.8 also implies that there has to be a balance between rental income and maintenance. Hence, by using the balance, large investors may have a favorable investment environment.

iv) On the other hand, if the parameter condition in item $ii)$ holds and if $M^i + \delta + (\beta - 1)m - r \geq 0$, then the optimal action is to borrow as much as possible from the bank to purchase houses. This case clarifies that if the economy is in the good state, large investors may invest in the housing by borrowing loans from the bank as much as possible.

In the following proposition we consider the power utility function, $U(x) = \frac{1}{\theta}x^\theta,\ 0 < \theta < 1$, with the case with a linear impact function and obtain explicit results. This utility function gives Constant Relative Risk Aversion (CRRA) type preferences with risk aversion $(1-\theta)/x$. In this case, we address this problem by the dynamic programing approach. To this, for any function $v \in C^{1,2}$ and $(t, x, i) \in [0, T] \times \mathbb{R}_+ \times \epsilon, \alpha \in \mathcal{H}$, we define the differential operator

$$\begin{aligned} \mathcal{A}^\alpha v(t,x,i) = \frac{\partial v(t,x,i)}{\partial t} &+ \frac{\partial v(t,x,i)}{\partial x} x \left(\alpha(M^i + g(\alpha, m)) + (1-\alpha)r + \alpha\delta - \alpha m\right) \\ &+ \frac{1}{2}\frac{\partial^2 v(t,x,i)}{\partial x^2} x^2 \alpha^2 \sigma^2 + \sum_j \left(v(t,x,j) - v(t,x,i)\right) Q^{ij}. \end{aligned}$$

Here, from the standard verification result, we need to solve the following Hamilton-Jacobi-Belman (HJB) equation.

$$\begin{aligned} &\sup_\alpha \mathcal{A}^\alpha v(t,x,i) = 0 \\ &v(T,x,i) = \frac{1}{\theta}x^\theta \quad for\ all\ (x,i) \in \mathbb{R}_+ \times \epsilon. \end{aligned} \tag{4.6}$$

We solve the HJB equation as in the following proposition.

Proposition 4.10. *Suppose* $U(x) = \frac{1}{\theta}x^\theta$ *and* $g(\alpha, m) = \beta(\alpha + m),\ \beta > 0$ *then,*

i) If $2\beta - (1-\theta)\sigma^2 < 0$*, the optimal strategy* α^* *is given by*

$$\alpha^*(t,i) = \left(\frac{M^i + \delta + (\beta - 1)m - r}{(1-\theta)\sigma^2 - 2\beta}\right)^+, \tag{4.7}$$

and $V(t,x,i) = \frac{1}{\theta}x^\theta u(t,i)$*, for all* $(t,x,i) \in [0,T] \times \mathbb{R}^+ \times \epsilon$*, where* $u(t,i) > 0$*, with* $U(T,i) = 1, i \in \epsilon$*, is the unique solution of the following system of linear differential equations*

$$\frac{\partial u(t,i)}{\partial t} + a(i)u(t,i) + \sum_j (u(t,j) - u(t,i))Q^{ij} = 0, \tag{4.8}$$

with $a(i) = \theta r + \frac{\theta(M^i+\delta+(\beta-1)m-r)^2}{2((1-\theta)\sigma^2-2\beta)}$*. Moreover, the value function has the following stochastic representation*

$$\begin{aligned} V(t,x,i) = \quad & \frac{x^\theta}{\theta}\exp(r\theta(T-t)) \times \\ & \mathbb{E}^{x,i}\left[\exp\left(\int_t^T \frac{[\theta(\mu(Y_s) + \delta + (\beta-1)m - r)^+]^2}{2((1-\theta)\sigma^2 - 2\beta)} ds\right)\right]. \end{aligned}$$

ii) If $2\beta - (1-\theta)\sigma^2 \geq 0$*, the optimal strategy* α^* *is given by*

$$\alpha^*(t,i) = L\mathbb{1}_{M^i+\delta+(\beta-1)m-r \geq -L\left(\beta - \frac{1}{2}(\theta-1)\sigma^2\right)},$$

and $V(t,x,i) = \frac{1}{\theta}x^\theta u(t,i)$*, for all* $(t,x,i) \in [0,T] \times \mathbb{R}^+ \times \epsilon$*, where* $u(t,i) > 0$*, with* $U(T,i) = 1, i \in \epsilon$*, is the unique solution of the following system of linear differential equations*

$$\frac{\partial u(t,i)}{\partial t} + a(i)u(t,i) + \sum_j (u(t,j) - u(t,i))Q^{ij} = 0, \tag{4.9}$$

with $a(i) = \theta r + \theta L\left(M^i + \delta + (\beta-1)m - r\right) + \theta L^2\left(\beta + \frac{(\theta-1)\sigma^2}{2}\right)$*. Moreover, the value function has the following stochastic representation*

$$\begin{aligned} V(t,x,i) = \frac{x^\theta}{\theta}\mathbb{E}^{x,i} \quad \Big[\quad & \exp\left(\theta(T-t)\left(L^2\left(\beta - \frac{(1-\theta)\sigma^2}{2}\right) + r\right)\right) \\ + \quad & \theta L \int_t^T \mu(Y_s) + \delta + (\beta-1)m - r\ ds\Big]. \end{aligned}$$

Proof. It follows from the utility function form and the linearity of the wealth process for all $i \in \{1, \cdots, K\}$ that the value function can be written as $v(t,x,i) = \frac{1}{\theta}x^\theta u(t,i)$, for some $u \geq 0$ with $u(T,i) = 1$. This yields

$$\begin{aligned}
\frac{\partial v(t,x,i)}{\partial t} &= \frac{1}{\theta}x^\theta \frac{\partial u(t,i)}{\partial t}, \\
\frac{\partial v(t,x,i)}{\partial x} &= x^{\theta-1}u(t,i), \\
\frac{\partial^2 v(t,x,i)}{\partial x^2} &= (\theta-1)x^{\theta-2}u(t,i).
\end{aligned}$$

Substituting these and $g(\alpha,m) = \beta(\alpha+m)$ in Equation (4.6), we have

$$\begin{aligned}
-ru(t,i) = & \sup_{\alpha\in[0,L]} \Big\{ \alpha \left(M^i + \delta + (\beta-1)m - r \right) u(t,i) \\
&+ \alpha^2 \left(\beta + \frac{\theta-1}{2}\sigma^2 \right) u(t,i) \Big\} \\
&+ \frac{1}{\theta}\frac{\partial u(t,i)}{\partial t} + \frac{1}{\theta}\sum_j Q^{ij}\left(u(t,j) - u(t,i)\right), \qquad (4.10) \\
u(T,i) &= 1 \quad for\ all\ i \in \{1, \cdots, K\}.
\end{aligned}$$

A necessary condition for the maximizer

$$2\alpha \left(\beta + \frac{\sigma^2(\theta-1)}{2} \right) u(t,i) + \left(M^i + \delta + (\beta-1)m - r \right) u(t,i) = 0$$

is defined. Suppose $2\beta < (1-\theta)\sigma^2$. These conditions together with $u(t,i) > 0$ imply that the necessary conditions are also sufficient, i.e. the maximizer is given with Equation (4.7). The positivity of $u(t,i)$ is also explained in Remark 4.11. After inserting the maximizer we obtain Equation (4.8). The differential equation given with Equation (4.8) has a unique solution u. The Feyman-Kac representation of $u(t,i)$ is found as

$$v(t,x,i) = \exp\left\{r\theta\left(T-t\right)\right\} \mathbb{E}^{x,i}\left[\exp\left\{ \int_t^T \frac{\theta(\mu(Y_s) + \delta + (\beta-1)m - r)^2}{2((1-\theta)\sigma^2 - 2\beta)} ds \right\} \right].$$

Infact, $v(t,x,i) = \frac{1}{\theta}x^\theta u(t,i)$ is a solution of the HJB Equation (4.6), $v \in C^{1,2}$, and satisfies $|v(t,x,i)| \leq K(1+|x|)$ for an appropriate $K \in \mathbb{R}$. By applying a verification theorem, we obtain $v(t,x,i)$ as the optimal value function $V(t,x,i)$.

Next, suppose $2\beta \geq (1-\theta)\sigma^2$. In this case the maximum is attained in one of the boundary points of the interval $[0,L]$. It is clear that our maximizer depends on the value of $(M^i + \delta + (\beta-1)m - r)$. Namely, $\alpha^*(t,i) = L$ for $M^i + \delta + (\beta-1)m - r >$

$-L\left(\beta - \frac{1}{2}\sigma^2\right)$ and $\alpha^*(t,i) = 0$ for $M^i + \delta + (\beta - 1)m - r < -L\left(\beta - \frac{1}{2}\sigma^2\right)$. Hence, by inserting these into Equation (4.10) we obtain Equation (4.9). $\square$

Remark 4.11. The representation of u above implies that $u(t,i)$ is always positive provided that the given parameter restrictions are satisfied.

Remark 4.12. Proposition 4.10 suggests that for any parameter condition the current value function dominates the value function given in [22]. This means that the investor benefits from the presence of the price impact also in the case of power utility preferences.

4.3 Numerical Application

To illustrate the house price evaluation in terms of large investor and economic state proposed in Equation (4.2), we employ a Monte Carlo simulation procedure. Under the assumption that there exists an operating large investor in the housing market whose utility function is the logarithmic function, we compute the optimal weight of the housing investment in the large investor portfolio. Then, we run a Monte Carlo simulation process to find expected house prices for trading days within a year. While doing this computation, we compute expected house prices by considering different economic states, investment willingness of the large investor, and house price volatility levels to observe their effect on the house prices.

Visual representation of the expected house prices (Figures 4.1, 4.2, and 4.3) and illustration of the house price paths over a one year horizon (Figures 4.4 and 4.5) depict the similarity in the behavior of simulated and expected house prices in terms of the considered factors: the state of economy, investment willingness of the large investor, and house price volatility. Here, it is also worth to mention that we measure the investment willingness of the large investor via the slope, β, of the impact function g.

In simulations, without loss of generality, we use parameters given in Table 4.1. To compute expected house prices, we used 10.000 simulations for trading days in one year. Here, note that we purposely consider three interest rates since the interest rate changes according to the state of the economy: Bad (0), Neutral (1), and Good (2).

For instance, if the economy is in a bad state, the interest rate is going to be high due to its relationship with inflation. Besides, we also avoid analyzing the effect of rent amount and interest rate since they are both determined by the market, not by investors.

Table 4.1: The model parameters

H_0	δ	m	M	σ	r	β	Y_i
100,000	0.003	0.001	0.06	0.3	0.03	0.002	0
				0.6	0.05	0.005	1
				0.9	0.07	0.008	2

Figure 4.1, is a chart of the expected house price for the economic states and the bank account, where each time step corresponds to one of the trading days within the year. Figure 4.1 illustrates that if the economy is in the bad state, the house price does not increase as in the other two economic states, which is a consequence of demand decrease due to high-interest rate. Hence, investment in housing is not favorable during the bad economic state. The figure also reveals that investing in a bank account is favorable to housing during the bad economic state period. On the other hand, if the economy is in neutral or good states, housing investment is favorable since, during both periods, the interest rate is low compared to the bad economic state. Moreover, it is found that the large investor should borrow and invest in housing if the economy is in the good state since the return on the house price is higher than the return on the bank account. Hence, one of the most significant results is that the house price reacts to the economic state.

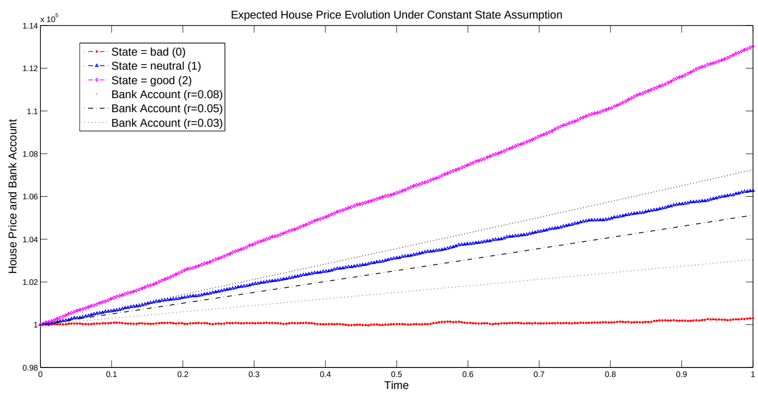


Figure 4.1: The effect of the economic state on the house price

Figure 4.2 shows the effect of the large investor's investment willingness to the housing when the economy is in the neutral state. The graphs for the other two states of the economy are not given for two main reasons: (i) In a good state, the investor should invest as much as possible to housing, (ii) In the bad economic state housing investment is not favorable. As in Figure 4.1, we see that if the economy is in the neutral state, the large investor should prefer investing in housing as much as possible because the willingness of the large investor leads to rising house prices. Expected house prices in terms of the large investor's investment willingness can be seen exerting a similar level of variability between 105.000 and 107.000. The figure shows a significant deviation among the house price paths, particularly after the mid of the year.

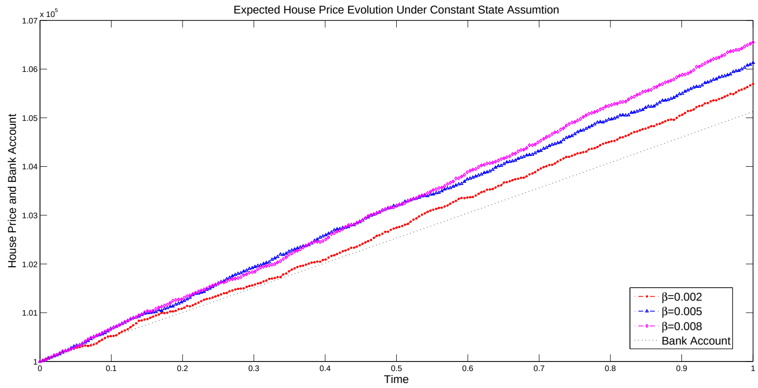


Figure 4.2: The effect of β on the house price under neutral economy state

The sensitivity of the house price to its volatility is shown in Figure 4.3. As in the geometric Brownian motion, the value of the house is increasing when the volatility increased.

Contrary to previous figures, Figure 4.4 and Figure 4.5 expose the paths for the house price for three volatility levels. The change in the states of the economy is taken into account while all other assumptions remain the same. Figure 4.4 explains the house price evolution for the volatility levels 0.3, 0.6, and 0.9 with assumptions that the state of the economy is known at the beginning of the year and it remains constant within the year. However, Figure 4.5 illustrates the price evolution of the house under the varying economic state. To compute the house price, here, we allow that the economic state may have changed quarterly. In this case, since the economic state is changed,

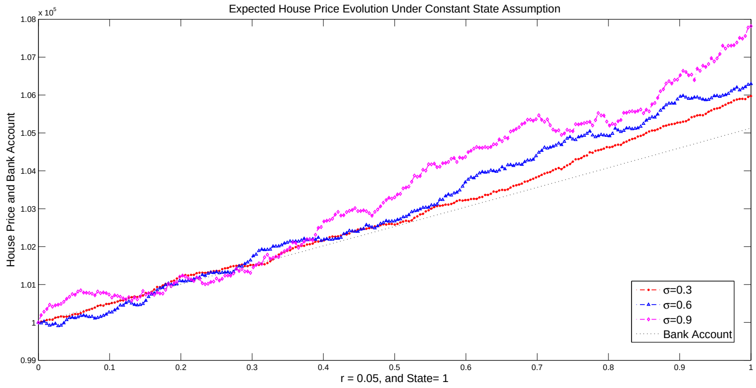


Figure 4.3: The effect of the house price volatility on its price

the investment amount to the housing is also changed. Therefore, the large investor has to change his investment strategy when the economic state changes. Due to the reallocation, the deviation among the paths that represent house prices is higher than in Figure 4.4.

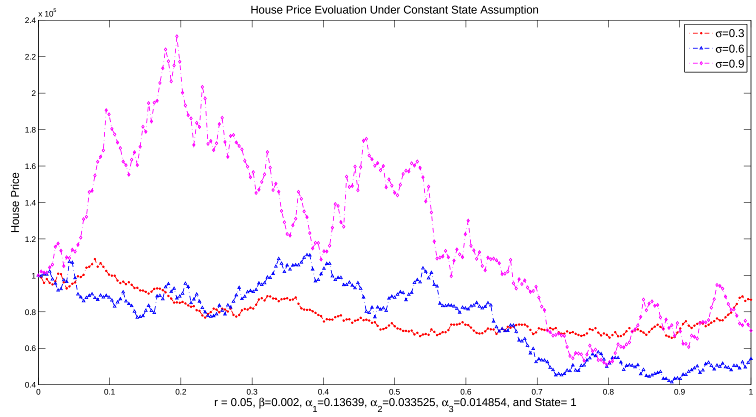


Figure 4.4: House price evolution for all economic states under constant economic state assumption

Figure 4.5: The house price evolution for three volatility level under the non-constant economic state assumption

CHAPTER 5

COMPUTATION OF MORTGAGE HEDGING COEFFICIENTS: THE MALLIAVIN CALCULUS APPROACH

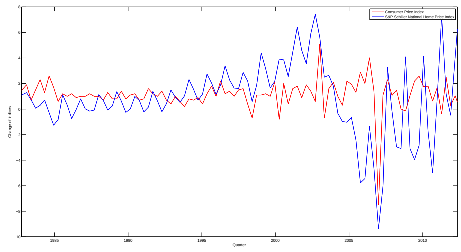
Pricing an option is the primary objective of managing trades in financial markets. However, when options are settled, their price does not remain constant. Instead, their price follows dynamic paths during options' survival time. Hence, investors should protect their portfolio against unexpected price changes by managing variations in the price of options. Consequently, on the contrary to previous two chapters, this chapter is dedicated to explore the hedging parameters of the financial option to default and to prepay, which are embedded into mortgages that rely on the variation in spot rate, price of the underlying house, and underlying house price volatility. The balance within these coefficients and such options can be used by buyers and sellers to protect their main portfolios against the related risks. From this point of view, the outcomes of this chapter are the first attempt to compute hedging strategies for mortgage hedging in real estate finance and economic literature.

It is well known that both parties, namely lenders and borrowers, expose to the risk of variation in the spot rate, price of the underlying house, and its volatility. Lenders' risk is associated with falling in the spot rate, and as a result of such a fall, less income than they anticipated is going to be produced by the lent capital. On the other hand, borrowers have to concern the risk of a rise in the spot rate and its anticipation of spending more interest on the loan than they anticipate. Therefore, the borrowers are the most important bearers of the spot rate risk in housing markets. In this point of view, borrowers are the most reliable candidates to benefit from such hedging activities in housing markets [39].

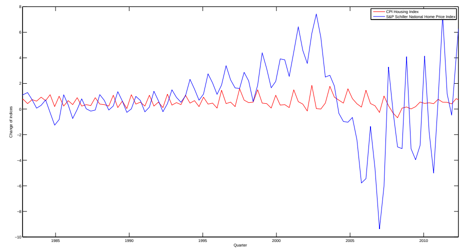
However, the standard hedging activities, which are commonly used and benefit in financial markets, are problematic for derivatives traded in housing markets. As an example, underlying securities in housing markets, more precisely houses, are indivisible. As a result, the underlying assets in housing markets cannot be traded in small fractions. Furthermore, due to a considerable amount of transaction costs and expenses of taxes, housing markets have very low liquidity compared to fully competitive markets. More significantly, the market price of houses can not be perceived by the housing market. On the other hand, despite these difficulties, both parties, namely borrowers and lenders, may hedge their wealth portfolios against variations in the price of the underlying house, the spot rate, and change in underlying house price volatility. To hedge their portfolios, they can use strategies as investing in derivatives, such as futures and options contracts, which are attached to S&P Case-Shiller Home Price Index (HPI) submitted by Chicago Mercantile Exchange (CME) and other related indexes and some of the company shares traded in exchange markets [38, 101].

On the one hand, although the futures and options that are tied to HPI offered by CME appeared on the stage during late 1980 in an effort to improve the liquidity of real estate markets, the derivatives market is still in its first steps [162]. As a result, investors in real estate markets concentrate on alternative substitution vehicles like investing in REITs, which are investment companies that maintain commercial properties, or mortgage-backed securities and futures addressed on inflation and the spot rate to reduce or eliminate the risks that we discuss above. Moreover, these investors tend toward futures addressed on $TBill$s to reduce the risk inherited to the spot rate. Figure 5.1 illustrates a variety of illustrations of the possible indexes that might be utilized in the hedging of derivatives rely on real estate assets, including inflation (CPI) and $TBill$.

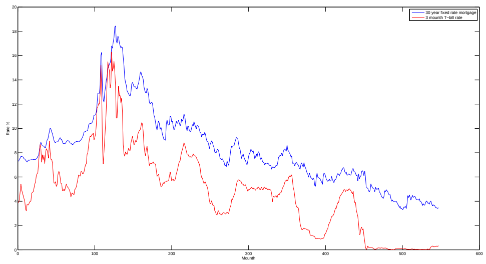
More specifically, Figure 5.1c reveals that 30-Year Fixed Mortgage Rate (FRM) and 3-month $TBill$ rate are nearly indistinguishable. Besides, Figure 5.1a and 5.1b demonstrate the relationship of HPI with CPI and CPI Housing index, respectively. Despite the disagreements in their computational methods, both figures exhibit that HPI progress concomitantly with CPI and CPI Housing index, although the correlations between the indices are not precise. Besides, these three figures also reveal that fluctuations in recognized values of both spot rates and indexes are positively associated with FRM and HPI, respectively.



(a)



(b)



(c)

Figure 5.1: Realization of HPI and other related indices over year: (a) HPI and CPI (1975-2016), (b) HPI and CPI housing (1975-2016), (c) Mortgage rate (30-year fixed) and $TBill$ (1975-2016)

The distinctive natural features of housing markets increase the risk of mortgages and the options of default and prepay, which are set in mortgages. Therefore, these options have to be replicated to compensate for the surprising variation in spot rates, price of the underlying house, and house price volatility to prevent investors from unanticipated losses related to these risk sources. However, the studies investigating hedging activities such options are limited in the real estate finance and economics literature. Therefore, this chapter aims to present a contribution to the literature with finding the corresponding hedging parameters of such options applying the option-based mortgage valuation method under the standard economic environment hypothesis that is introduced in the pioneering studies [16, 114, 117, 118, 153, 175]. Hence, for this chapter, the stochastic processes which we will deal with are the ones introduced in these studies.

The hedging coefficients that we compute in the current chapter allow both parties to replicate their wealth portfolios, related to mortgage, against the risk inherited to the change in the price and volatility of the underlying house and the spot rate. To address the problem of calculation of these coefficients, we used the finite-dimensional Malliavin calculus by pointing to the pioneering researches [76, 77] to accomplish our task. To yield useful information, we should emphasize that the Malliavin calculus named after Paul Malliavin, who extends the calculus of variations from deterministic functions to stochastic processes. This calculus is also called the calculus of stochastic variation. In particular, the Malliavin calculus allows users to compute derivatives concerning to chance parameter.

Paul Malliavin's suggestions led to a proof that Hölmender's condition implies the presence and regularity of a density for the solution of SDEs. Hölmender's original proof is based on the Partial Differential Equation's (PDE) theory. The Malliavin calculus also has been employed for the stochastic PDEs. The Malliavin calculus contains an integration by parts formula with random variables that allow users to compute the hedging coefficients of financial derivatives. Among many other researchers that used the Malliavin calculus, Bismut (1984, [27]) further developed the calculus and extended the integration by parts formula to multi-dimensional SDEs.

The genuineness of the Malliavin calculus approach, which is assumed to be one of the cornerstones in the calculation of Greeks, is that hedging parameter arises as a product of options' payoff and an independent weight function. In mathematical finance literature, the weight function is called Malliavin weight. This feature of the Malliavin calculus allows users to obtain predicting for options hedging coefficients by operating a Monte Carlo (MC) algorithm. More importantly, this method does not demand any solution to PDEs. In this respect, the Malliavin calculus is an encouraging vehicle, particularly for the calculation of hedging parameters or options' Greeks.

Here, it is worth emphasizing that throughout the current chapter, our analysis proceeds on the premise that the prepayment and default options are separate judgments, and hence they are independent actions. Such an assumption is particularly useful for us in the design of both options' payoff functions and for the calculation of their hedging parameters.

In this chapter, our simulations exhibit the financial option to default and to prepay is both more sensitive to a variation in spot rate than a variation in the price and volatility of the underlying house. There are two possible use areas of such coefficients. First, the coefficients enable users to determine the effects of the spot rate, the price, and volatility of underlying house change on options related to mortgage. Second, both parties can replicate and hedge their portfolio by managing the balance among the coefficients and the mortgage-related options.

The rest of the current chapter progresses as follows. In Section 5.1, a literature survey on the option-based mortgage valuation is presented in a relatively detailed manner. Section 5.2 introduces the economic environment of the housing market that we work on and the payoffs for both default and prepayment options embedded into the mortgage. We illustrate a brief review of the Malliavin Calculus in Section 5.3. Sections 5.4 and 5.5 are the chapters where we present the calculation of hedging parameters and the crude MC simulation illustrations, respectively.

5.1 Literature survey on the Option-Based Mortgage Valuation

In the recent two decades, the option-based mortgage valuation method, which is very popular in financial markets, has driven attention from many researchers as an advanced mortgage valuation method. During this period, a substantial number of study has been conducted on option-based mortgage valuation topics, see, for instance, the studies [15, 115, 117, 118, 175, 177] and references in these studies. In this research field, the studies in the early years usually concentrate on the right to prepay the loan, controlling the opportunity of a mortgage default and prepay. In the subsequent reviews, see, for instance, the studies [115, 117], it is pronounced that such a method is incomplete since a mortgage need to recognize the probability of immediate termination, utilizing default and prepayment.

The logic behind the mortgage default and prepay are quite complicated for researchers and market players. Many researchers attempt to incorporate the complexity of the mortgage default and prepay [115, 131, 184]. Generally, these researches emphasize that the mortgage default and prepay might happen for financial and non-financial speculations. We may consider unemployment, indicators touching upon affordability, or the diminished affordability to sustain a mortgage, such as a tense fall in borrowers' income, householders divorcing and moving into a new city or a house as some of the reasons. However, there is a consensus that mortgage prepayments generally occur when the refinancing rate is sufficiently low in the market.

On the other hand, especially mortgage defaults occur when borrowers are in a negative equity situation. To illustrate, Azevedo et al. (2000, [15]) state that the early close of mortgage is no longer an exogenous proceeding. Alternately, this decision is installed in the composition of the mortgage model. After that, some supplements have been developed within the mortgage valuation framework. We can give the study of Hilliard et al. (1998, [100] as a good example that provides a less unfavorable mathematical procedure operating with a bi-variate binomial lattice model and the paper of Ambrose et al. (2000, [9]) that splits the mortgage default option into two parts. First, a right to prevent doing monthly payments temporarily and second, a right to leave the house itself.

To arrive at an adequately accurate and more manageable procedure for the numerical valuation of a mortgage and its natural ingredients, namely prepayment and default options of mortgage, the SDEs are utilized in the real estate finance and economics literature. Some of these studies contain the major significant contributions introduced by Dunn and McConnell (1981, [64]). Dunn and McConnell (1981, [64]) offer a one-state stochastic process. More specifically, they suggest a CIR process, whereas Schwartz and Torous (1989, [173]) proposed a two-state stochastic process for spot rates. In their experimental study, Chatterjee et al. (1998, [43]) indicate that concerning pricing efficiency, the two-state stochastic process is the most effective of all the alternative option-based mortgage valuation methods that are introduced to the real estate finance and economics literature.

In both one and two-state processes, models give birth to a PDE that has to be solved numerically since their complexity prevents users from finding a closed-form solution. An alternative method is to use reduced-form models where the terminus is displayed as a function of a set of supplementary variables. For instance, in studies [173, 174], a two-state stochastic process with a hazard rate is used to obtain a corresponding PDE for the mortgage. The study of Pliska (2006, [165]) includes a valuable survey, including the current applications of credit risk methods on a mortgage valuation. From a practitioners' perspective, Kalotay et al. (2004, [111]) focus on the refinancing action of buyers in a model based on an optimal exercise strategy, and Longstaff (2005, [136]) concentrates on a multi-factor term structure procedure combining premium rate refinancing.

Although in some researches it is argued that the inherent assuming made for the framework of the option-based mortgage valuation method are violated [177], many studies have been introduced following the pioneering studies, consequently utilizing such a powerful technique, which is produced originally for ordinary financial markets, to real estate market and especially for mortgage valuation [115, 117, 153, 175]. Even though there are some severe difficulties with its inherent hypotheses, the option-based mortgage valuation method still contributes useful penetrations into the values and behavior of mortgages. Therefore, as a result of its insights, this method is widely studied in real estate finance literature for mortgage valuation. Furthermore, in some recent researches, authors favor applying the celebrated arbitrage-free pricing theory with two-state processes, one for spot rate, and one for the price of the underlying house in the valuation of mortgage contracts. The option-based mortgage valuation structure includes a non-callable bond (loan), financial options to prepay, and default. This arrangement gives growth to a substantial amount of investigations on mortgage valuation due to associated risks in mortgages, namely the risks of prepay and default [177].

In the usual option-based mortgage valuation method, the mortgage value is broken into three components: A non-callable bond, a default option, and a prepayment option. The standard approach acknowledges the discounted value of contracted monthly repayments to the lenders as a non-callable bond, acknowledges giving up purchasing the house, and trading it back to the seller as a default option and acknowledges the early payment of mortgage loans to the lender as prepayment. Usually, while the standard approach views mortgage default option as an exercise of a European put option, it views mortgage prepayment option as an American call option [10, 115, 153]. Although the standard approach considers that prepayment occurs instantly after the spot rate becomes sufficiently small, however in actual markets, mortgage loans might not be paid instantly when spot rates fall. In practice, mortgage agreement holders usually choose the option of prepaying their mortgage loans, succeeding what the standard mortgage valuation method requires.

More importantly, some borrowers avoid using the prepayment option even when the FRM is above the current spot rate [182]. Besides, borrowers who have an expectation of mortgage rate may decrease further soon may choose to delay the prepayment option instead of terminating the mortgage by early paying the reaming mortgage loan immediately [136]. Also, the standard option-based mortgage valuation fails due to its weakness to model borrowers who must terminate the mortgage by prepaying but avoid early paying and instead prefer to wait [111].

As a result of the suspension we mention above, the American type call option provides mortgage values that are lower than the perceived values. From this point of view, the American type call option is not a realistic characterization of the mortgage prepayment option. As a result of this judgment, it will be more beneficial to displace the ordinarily used American type call option in the standard option based mortgage valuation with a modification of the Parisian type call option. The Parisian type call option allows the mortgage prepayment option to be a kind of time-dependent barrier option that depends on the mortgage value [175]. Consequently, to withdraw the mispricing and do a reasonable valuation on the mortgage prepayment option, the puzzle addressed in this chapter uses a two-state stochastic process model, which is defined initially in the studies [115, 117]. Furthermore, it concentrates on a modification of the Parisian type call option or time-dependent barrier option presented to the literature for prepayment option for the first time by Sharp et al. (2009, [175]).

The scope of the next section is to introduce the details of the economic environment for our specific housing market studied in the current chapter. Here, we present a spot rate process and a house price process. Also, we point out the necessary financial hypotheses to establish the options of a mortgage default and prepayment as clearly as possible. Further, the hedging strategies of these options will be analyzed within the Section 5.4, which relies on the Malliavin calculus approach used in the computation of Greeks.

5.2 Economic Environment of the Housing Market

The option-based mortgage valuation method becomes a standard method in the literature, and it resembles almost to pioneering studies of Titman et al. (1989, [186]) and Kau et al. (1992, [117]. Generally, this method considers an arbitrage theory with two-state stochastic variables; one for the spot rate and one for the underlying house price to simultaneously compute the mortgage value and the embedded options default and prepayment. For empirical fidelity of our study, the choice of both processes for given variables, which denotes the economic environment, is consistent with the standard method in the recent literature [16, 114, 117, 153, 175]. In this setting, the model provides the probabilities of the mortgage default and prepay, and they are regarded as independent operations from each other. When enabling the likelihood of mortgage default and prepay, first the underlying house price and spot rate are modeled with SDEs, since we assume that the choice whether to default or prepay relies on the price of the underlying house and the current spot rate level.

We argue that the price of the house exchanged in the market reflects all accessible information about the fundamental house price by relying on the celebrated efficient market hypothesis. Moreover, it also mirrors the discounted illustrate the value of the expected future cash flows related to the house under consideration. As a result, the economic environment builds on the assumption that the house price and spot rate both evolve from the stochastic processes that we introduced in the previous literature.

Our exact assumptions on the regularity of housing market economic environment can be favorably defined by considering the filtered probability space, $\left(\Omega, \mathcal{F}, \mathcal{F}_{t\in[0,T]}, \mathbb{Q}\right)$ with a finite interval $[0, T]$. Here, the filtration, $\mathcal{F}_{t\in[0,T]}$, is assumed to be rich enough to compile with a two-dimensional Brownian motion. In this setting, $(\mathcal{F}_t)_{t\in[0,T]}$ represents the filtration produced by two independent standard Brownian motions represented by $(W_t^i)_{t\in[0,T]}$, $i = 1, 2$. Here, it is also worth to emphasize that, we suppose that the given filtration fulfils the right continuity and completeness with respect to $\mathbb{Q}$.

Now, suppose there is a purchaser in the market that bought a house. She financed the home that she purchased with FRM at $t \in [0, T]$. Moreover, suppose the economic environment of the housing market evolves from a two-state stochastic variable, which is a pricing characterization consistent with earlier studies [153, 175]. Let us denote the state variables or processes that correspond to the house value (h_t) and the spot rate (r_t). To be more precise in the goal of the current chapter, we present the numerical notations below for the processes of house value and spot rate.

In this chapter, h_t, is thought to evolve from a log-normal diffusion process under risk neutrality arguments as Merton (1973, [147]) granted for the ordinary markets and later employed by many researchers. In our setting, a parameter that we should include while forming a model for the house value is the service flow that corresponds to benefits from the house. Generally, the service flow considered to act as a dividend. Now, under these assumptions, the SDE that the house price process satisfies can be given as

$$dh_t = (r_t - \delta)h_t\, dt + \sigma_h h_t dW_t^h, \quad h_0 = x > 0, \tag{5.1}$$

where the parameters r_t, δ, σ_h and W_t^h denotes spot rate, service flow produced by the underlying house, the variance related to returns on the house, and a regular Wiener process on the filtered probability space that we introduced above, respectively. Here, we suggest that δ is proportionate to the house value since the holder of the option tied to the mortgage has no claim to the service flow, and it has an adverse effect since the owner gains advantages from rent.

As we observe from Equation (5.1), the variation in the house value relies on the current house value, h_t, and the current spot rate, r_t, plus an unknown ingredient, which is called the diffusion term of the model. Furthermore, the house value process, h_t, is assumed to be a continuous-time Markov process. The house value depends only recently observe house values. Remark that in this market, the house value process has an engaging barrier like $h_t = 0$. More precisely, if the underlying house lost its value and its price reaches zero at any $t \in [0, T]$, its price remains zero after that specific t.

The second process and source of market indecision is the instant spot rate represented by r_t. In this thesis, we assume that it incorporates all information about the expected spot rate, and hence r_t forces the entire rates. The structure of r_t dynamics are assumed to evolve from the classical CIR [54] short rate model that is described as follows

$$dr_t = \kappa(\theta - r_t)dt + \sigma_r\sqrt{r_t}dW_t^r, \quad r_0 = r > 0. \tag{5.2}$$

The CIR model provided by Equation (5.2) is a kind of mean-reverting process that contains a square root. In this equation, θ describes the steady-state rate of r_t, κ denotes the adjustment parameter, which regulates its speed, σ_r expresses its variance, and W_t^r is a regular Wiener process correlated with W_t^h. The correlation coefficient among the Brownian motions is assumed to be ρ. The shiny part of such a model is the negative values are prevented if $2\kappa\theta \geq \sigma_r^2$ and $r_0 \geq 0$ satisfied, and in the long run the short rate is going to converge to mean value.

The spot rate process given by Equation (5.2) indicates that it is expected to variate at any $t \in [0, T]$ with a rate of $\kappa(\theta - r_t)$. But, the exact variation differs in an unbiased form as a result of the normally distributed and serially uncorrelated variations in the economy, which is a role of W_t^r. We capture the volatility of the fluctuations with $\sigma_r\sqrt{r_t}$.

The house value process in Equation (5.1) might be described in a alike method as in the interpretation of Equation (5.2). In this modeling structure, the fluctuations in h_t are associated with the variations in r_t through the correlation coefficient ρ. Here, W_t^h and W_t^r may be denoted by two independent Brownian motions $W_t^i, i = 1, 2$, with Cholesky decomposition as

$$dW_t^h = dW_t^1$$

and

$$dW_t^r = \rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2.$$

For the computation purposes, now, suppose that we can merge h_t and r_t into a two-dimensional process. More specifically, the underlying asset is considered to evolve from $(X_t)_{t\in[0,T]}$, which is a two dimensional Markov process that has values in $\mathbb{R}^2$.

The dynamics of this new two-dimensional process are given by the following SDE

$$dX_t = \beta(X_t)dt + \sigma(X_t)d\mathbb{W}_t, \; X_0 = x, \tag{5.3}$$

where $x = [h_0 \; r_0]^\top$, and

$$X_t = \begin{bmatrix} h_t \\ r_t \end{bmatrix}, \; \beta(X_t) = \begin{bmatrix} (r_t - \delta)h_t \\ \kappa(\theta - r_t) \end{bmatrix}, \; \sigma(X_t) = \begin{bmatrix} \sigma_h h_t & 0 \\ \rho\sigma_r\sqrt{r_t} & \mu\sigma_r\sqrt{r_t} \end{bmatrix}, \; \mathbb{W}_t = \begin{bmatrix} W_t^1 \\ W_t^2 \end{bmatrix}.$$

Note that, $(\mathbb{W}_t)_{t\in[0,T]}$ is a two-dimensional Brownian motion defined on $\left(\Omega, \mathcal{F}, \mathcal{F}_{t\in[0,T]}, \mathbb{Q}\right)$ with $\mu = \sqrt{1-\rho^2}$. Here, the functions β, and σ are both assumed to be at least twice continuously differentiable with bounded derivatives. Also, they satisfy the Lipschitz condition.

Now, the default and prepayment options can be provided using the economic environment that we state above. In the following section, we provide rigorous formulations of these options.

5.2.1 Financial Options to Default and Prepay

The first step of our computations consists of definitions of the payoffs of the options under investigation. We consider the general payoff function, European put option payoff, for the mortgage default, which is commonly used in the standard method. However, we use a time-variant Parisian option for the mortgage prepayment instead of the American call option widely used in the conventional method. This section aims to introduce these options to the readers in a detailed manner.

A default option is a contract embedded into mortgage whose owner has a claim to terminate her down payments for the loan and surrender owner rights of the underlying house that she purchased by financing with a mortgage. From this point of view, this option is a contract that relies on t, r_t, and h_t which are evolving from processes given with Equations (5.1) and (5.2), respectively. In the standard approach, this option recognized as a European put with the purchased house as the underlying asset, one-month maturity, and strike is the present value of the remaining payments.

Now, let us define the mathematical representation of the mortgage default option. Hence, let X_t indicates the joint process of h_t and r_t, which evolves from the diffusion process defined in Equation (5.3) even though the underlying assets are not regular assets, they are consumption goods whose payoffs are service flow [115]. In this case, h_t, might be used instead of X_t since the default relies on h_t. Then, its payoff is

$$
\begin{aligned}
D(t,r,X) &= \max\left(B(t,r) - X_t, 0\right), \\
D(t,r,h) &= \max\left(B(t,r) - h_t, 0\right),
\end{aligned}
$$

where the functional $B(t, r)$ represents the current value of the remaining constant monthly payments.

In a fully competitave market, the default option value is nothing but the discounted value of the expected present value of its future benefits. This option, generally characterized by a function of h_t. Therefore,

$$\begin{aligned} P_D(x) &= \mathbb{E}\left[e^{-\int_0^T r_s\, ds} D\left(T, r, X\right) |\mathcal{F}_0\right] \\ &= \mathbb{E}\left[e^{-\int_0^T r_s\, ds} D\left(T, r, h\right) |\mathcal{F}_0\right], \end{aligned} \tag{5.4}$$

where $D\left(T, r, h\right)$ or $D\left(T, r, X\right)$ stand for the terminal value of the mortgage default option that expiring at maturity T.

In the standard approach, prepayment is recognized as an American call option, see, for instance, the studies [117, 116, 153]). However, Kalotay et al. (2004, [111]) suggest that such an option fails to represent buyers who have to make an early termination by paying all remaining loan but instead avoid terminating the mortgage and favor to remain for r_t decrease further. So, the American option results in mispricing. Consequently, on the contrary to conventional theory, Sharp et al. (2009, [175]) introduce the following Parisian option for mortgage prepayment option that is originally intended to have a knock-in stipulation.

To evade the mispricing that related to the American call option, we also used in this thesis that the mortgage prepayment performs like a Parisian option like introduced in [175]. The concepts of such an option are specified as the Parisian option, and they are introduced below.

Let T_m and t_m denote the complete time and the elapsed time in month m, respectively. Next, the maturity in month m is going to be $\tau_m = T_m - t_m$. Hence, the face value of mortgage balance $FV(\tau_m)$ equals to the following

$$FV(\tau_m) = [1 + c(T_m - \tau_m)]OB(i),$$

where $OB(i)$ and c represents the outstanding balance of the mortgage after the i^{th} regular payment and FRM, respectively.

Describe the face value of the remaining loan as a barrier level at t and define $\bar{T}_m$ as the predetermined time interval, which is named as the excursion interval. The barrier option is stimulated when the PV constantly waits beyond the barrier level longer than $\bar{T}_m$. But, if PV goes back across the barrier, the option clock is reset, and it becomes zero. Since prepayment considered as an up-and-in option, the time consumed above the barrier $FV(\tau_m)$ is watched.

To formulate the option we defined above; it is beneficial to present the following functional form

$$g_t^{FV}(PV) = \sup\{s \leq t | PV(s,r) = FV\},$$

which denotes the lates t before PV o reaches the barrier $FV(\tau_m)$.

Now, let us denote the first t that PV remains longer than $\bar{T}_m$, above the barrier with τ^c. The mathematical representation of τ^c is as in the following form

$$\tau^c = \inf\left\{t > 0 | \left(t - g_t^{FV}(h)\right) \mathbb{1}_{PV_t \geq FV} \geq \bar{T}_m\right\},$$

where c denotes continuity.

Next, the mortgage prepayment option' payoff might be represented as a function of PV and face value as in the following functional form

$$P(t,r,X) = \max\left(PV(t,r) - (1+\xi)FV(t), 0\right). \tag{5.5}$$

Here, the function $PV(t,r)$ denotes the present value of the mortgage, and the parameter ξ denotes the penalty parameter for early paying, which satisfying $0 < \xi < 1$. Generally, the penalty parameter used by lenders to avoid early termination of mortgages.

Using the payoff function that we introduce in Equation (5.5), we can define "no-arbitrage price" of an up-and-in Parisian option with a maturity T as in the following form

$$P_P(x) = \mathbb{E}\left[e^{-\int_0^T r_s\, ds} P(T,r,X)\right].$$

At this point, it is clear from the nature of the Parisian option that buyers are not prepared to remain until maturity. Instead, they prefer to exercise the prepayment option immediately if the necessary conditions of the Parisian option are satisfied. As a result, an adjustment of the Parisian option value, which may be of concern, is presented below.

The Parisian option terminated instantly if the requirement is met instead of waiting till to the maturity. More clearly, the termination occurs at time τ^c in the current definition of the Parisian option. As a result, the price of the mortgage prepayment option becomes

$$P_P(x) = \mathbb{E}\left[e^{-\int_0^{\tau^c} r_s \, ds} P(\tau^c, r, X)\right]. \tag{5.6}$$

Now, observe that the pricing formulas in Equations (5.4) and (5.6) include two origins of uncertainty; (r_t), and (h_t). More importantly, both options distributions are unknown. Therefore, using the Malliavin calculus is the best choice to calculate the hedging parameters for both options with an underlying variable evolve from the two-dimensional stochastic process defined by X_t, which we introduce in Equation (5.3).

The following section aims to present an intuition to the Malliavin calculus to remind the ones who are familiar and give an introduction to the ones who are not familiar with the Malliavin calculus.

5.3 A Brief Review on Malliavin Calculus

The purpose of this section is to familiarize readers with Malliavin calculus and introduce some primary results from its context, which are especially related to the computation of hedging coefficients of options. We refer to readers who are interested in the Malliavin calculus concept to the books [61, 157].

The Malliavin calculus is a robust instrument to handle the anticipating processes. It becomes a significant vehicle for the computation of hedging coefficients starting with the pioneering studies Fournie et al. (1999, 2001 [76, 77]) since the theory executes potentially to differentiate concerning the random component. To understand our computations within the current chapter, readers have to be familiar with the primary tools of the Malliavin calculus that we used in the calculations. Therefore, the present section is devoted to giving a short introduction to the theory of Malliavin Calculus. Mainly, we review the celebrated integration by parts formula, the chain rule, and the Bismut-Elworthy-Li formula, without any proof, which are the cornerstones of the computation of the hedging parameters of options. However, the interested readers can find the documentation and further details in [61, 157].

Now, let us start with the definition of smooth random variables, which is defined in Hilbert spaces.

Definition 5.1. Let H be a real separable Hilbert space with a scalar product represented as in the form $\langle \cdot, \cdot \rangle_H$. The set of smooth random variables denoted by $\mathcal{S}$ and it includes random variables F of the structure,

$$F = f\left(W(h_1), \ldots, W(h_n)\right),$$

where $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ denoted the class of all infinitely continuously differentiable functions with its derivatives satisfying the polynomial growth condition $(f \in C_P^\infty(\mathbb{R}^n))$, and $h_i \in H, i = 1, 2, \ldots, n$ for $n \geq 1$ and $W = \{W(h), h \in H\}$ is an isonormal Gaussian process described on the complete probability space given with $(\Omega, \mathcal{F}, P)$.
In this setting, W is a centered Gaussian class of random variables and defer $\mathbb{E}\left[W(h)\, W(g)\right] = \langle h, g \rangle_H$, $\forall h, g \in H$ [157].

Now, since we introduce the definition of smooth random variables, we may present the definition of the Malliavin derivative.

Definition 5.2. Let $F \in \mathcal{S}$ and $H = L^2([0,T], \mathcal{B}, \mu)$. In that case, the Malliavin derivative $D : \mathcal{S} \mapsto L^2\left([0,T], \beta, \mu\right)$ of F is introduced as

$$DF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\left(W(h_1), \ldots, W(h_n)\right) h_i,$$

where $\frac{\partial f}{\partial x_i}$ is the derivative of functional $f \in C_P^\infty(\mathbb{R}^n)$ concerning its i^{th} component.

The derivative operator, D, is closable from $L^2(\Omega)$ to $L^2\left(\Omega \times [0,T]\right)$ [157]. Therefore, the domain of the derivative operator may be extended to the stochastic Sobolev space $\mathbb{D}^{1,2}$, which is a closure of $\mathcal{S}$ concerning the following norm

$$||F||_{1,2}^2 = \mathbb{E}[F^2] + \mathbb{E}[||DF||_H^2].$$

Moreover, the domain of the Malliavin derivative, $\mathbb{D}^{1,2}$, is also a Hilbert space with the scalar product given by,

$$< F, G >_{1,2} = \mathbb{E}[FG] + \mathbb{E}[< DF, DG >_H],$$

where $F, G \in \mathbb{D}^{1,2}$.

The Malliavin calculus also has a chain rule as in the classical calculus. It is given as in the following proposition.

Proposition 5.3 (Chain Rule). *Let $F = (F_1, \ldots, F_n)$ be a vector of random components where the components are from $\mathbb{D}^{1,2}$ and let $\varphi : \mathbb{R}^n \mapsto \mathbb{R}$ be a functional from $f \in C_P^\infty(\mathbb{R}^n)$. At that rate, $\varphi(F) \in \mathbb{D}^{1,2}$ and*

$$D_t\varphi(F) = \sum_{i=1}^{n} \frac{\partial\varphi}{\partial x_i}(F)\ D_tF_i,\ a.s. \quad t \in [0,T].$$

Proof. If φ is a smooth function the proof is easy to obtained by the chain rule in the classical analysis. Otherwise, the function has to be mollified. To mollify φ, we may use the mollifier $\rho_\epsilon(x) = \epsilon^n\rho(x)$, where $\rho(x) = ce^{\frac{1}{x^2-1}}$ and c is an arbitrary chosen coefficient that makes the integral $\int_{\mathbb{R}^n} \rho(x)dx = 1$, to obtain a smooth approximation $\varphi * \rho_\epsilon$. By considering the smooth approximations F_n of F, we obtain $\varphi * \rho_\epsilon(F_n) \mapsto \varphi(F)$ for $\min \epsilon, n \mapsto \infty$ in the space L^p. Then, by the closeness of the derivative operator D, we have

$$\begin{aligned}
& \left\| D\varphi(F) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\varphi(F)D_tF_{ni} \right\|_p \\
\leq \quad & \left\| D\varphi(F) - D\varphi * \rho_\epsilon(F) \right\|_p \\
+ \quad & \left\| D\varphi * \rho_\epsilon(F) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\varphi * \rho_\epsilon(F_n)D_tF_{ni} \right\|_p \\
+ \quad & \left\| \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\varphi * \rho_\epsilon(F_n)D_tF_{ni} - \sum_{i_1}^{n} \frac{\partial}{\partial}\varphi(F)D_tF_i \right\|_p \mapsto 0.
\end{aligned}$$

□

As an instantaneous outcome of Proposition 5.3, we can give the following lemma.

Lemma 5.4. *Let the sequence $F_n \in \mathbb{D}^{1,2}$ converging to F in the space $L^2(\Omega, \mathcal{F}, \mathbb{P})$ satisfying $\sup_n \mathbb{E}\left[\|DF\|_H^2\right] < \infty$. Then, $F \in \mathbb{D}^{1,2}$ and DF_n weakly converges to DF in $L^2(\Omega \times [0,T])$.*

Proof. A proof of this lemma can be proved directly from the Banach-Alaoglu theorem on the weak compactness of the unit ball of a dual space, every bounded sequence in the dual of a separable topological vector space has a converging sub-sequence. □

Furthermore, we may generalize Proposition 5.3 to the non-differentiable functions.

Proposition 5.5. *Consider a function φ, which satisfies the following condition for $K \in \mathbb{R}^+$,*

$$|\varphi(x) - \varphi(y)| \leq K\,|x-y|\,, \quad x, y \in \mathbb{R}^n$$

and $F \in \mathbb{D}^{1,2}$. Then, $\varphi(F) \in \mathbb{D}^{1,2}$ and we can find an n-dimensional vector $G \in \mathbb{R}^n$, $|G| < K$ with random components such that

$$D_t(\varphi(F)) = \sum_{i=1}^{n} G_i D_t F_i, \quad t \in [0,T].$$

Proof. Utilizing the same mollifier ρ_ϵ, defined in the proof of Proposition 5.3, we can obtain $\varphi * \rho_\epsilon$ that converges to φ. The sequence $D_t(\varphi * \rho_\epsilon)(F)$ is bounded in $L^2\left(\Omega \times [0,T]\right)$ space. This is because, $|\nabla(\varphi * \rho_\epsilon)| \leq K$ for some large ϵ. From Lemma 5.4, $\varphi(F) \in \mathbb{D}^{1,2}$ and the Malliavin derivative $D_t(\varphi * \rho_\epsilon)(F) \mapsto D_t(\varphi(F))$ in the weak sense. Besides, $\nabla(\varphi * \rho_\epsilon)(F)$ weakly converges to a vector $G \in \mathbb{R}^n$, $|G| < K$. So, we can take the the weak limit in

$$D(\varphi * \rho_\epsilon)(F) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\varphi(F) D_t F_i$$

to lead us to the result. $\square$

The Malliavin derivative $D_t : \mathbb{D}^{1,2} \longmapsto L^2\left(\Omega \times [0,T]\right)$ is a closed linear, unbounded mathematical operator with a dense domain in $L^2(\Omega)$. Then, an adjoint operator denoted by δ of the derivative operator can be introduced such that $\delta : L^2\left(\Omega \times [0,T]\right) \mapsto L^2\left(\Omega\right)$ and $Dom(\delta)$ represents the domain of δ.

Definition 5.6. Let $u \in L^2\left([0,T]\,, \beta, \mu\right)$ and it satisfies $u \in Dom(\delta)$. At that rate, $\forall F \in \mathbb{D}^{1,2}$ and

$$\left| \mathbb{E}\left[\int_0^T D_t F u_t dt\right] \right| \leq c\,\|F\|_{L^2(\Omega)}\,,$$

where c is a constant depending on u,

$$\delta(u) = \int_0^T u_t \delta W_t$$

is in $L^2(\Omega)$ and the duality formula holds,

$$\mathbb{E}\left[\int_0^T D_t F u_t dt\right] = \mathbb{E}\left[F\delta(u)\right].$$

Until now, we present all theoretical backgrounds to give the integration by parts formula. Based on the conceptual framework above, we can define the integration by parts formula as in the following proposition.

Proposition 5.7 (Integration by Parts Formula)**.** *Let* $F \in \mathbb{D}^{1,2}$ *and* $Fh \in Dom(\delta)$ *for* h *in the Hilbert space* $\in H$. *In that case, the following is hold*

$$\delta\left(Fh\right) = FW(h) - \langle DF, h\rangle_H .$$

Further, when $F = 1$ *a.s.,*

$$\delta(h) = W(h).$$

Proof. A detailed proof might be found in [157]. □

We may give the following two remarks without any proof since their proofs are already existed in [157].

Remark 5.8. If $H = L^2([0,T], \mathcal{B}, \mu)$, where μ is a σ-finite atomless measure on the measurable Borel space $([0,T], \mathcal{B})$, Proposition 5.7 transform

$$\int_0^T Fh_t \delta W_t = F\int_0^T h_t \delta W_t - \int_0^T D_t F h_t dt.$$

Remark 5.9. The domain of the Skorohod integral includes the adapted stochastic processes in $L^2\left(\Omega \times [0,T]\right)$. If the integrand is altered, the Skorohod integral sycronize with the classical Itô integral [157], i.e.

$$\delta(h) = \int_0^T h_t dW_t,$$

and

$$\int_0^T Fh_t dW_t = F\int_0^T h_t dW_t - \int_0^T D_t F h_t dt,$$

where $F \in \mathbb{D}^{1,2}$ and $\mathbb{E}\left[\int_0^T (Fh_t)^2 dt\right] < \infty$.

For our computation purposes, it is essential to introduce the following Remark.

Remark 5.10. If $F \in \mathcal{S}$ and it is $d - dimensional$ and h_t is a $d \times d$ dimensional matrix process, Remark 5.9 becomes

$$\delta(hF) = \int_0^T Fh_t dW_t = F\int_0^T h_t DW_t - \int_0^T Tr((D_t F)u_t)dt,$$

with the precept that the classical Itô integral for a matrix process is now a vector of random variables. In this setting, "Tr" denotes the trace of the random matrix.

The first variation process of X_t is the derivative of X_t concerning its x for the time $t \in [0, T]$. The first variation process is given by following definition.

Definition 5.11. Suppose $(X_t)_{t\in[0,T]}$ is an Itô process in $\mathbb{R}^2$ and its dynamics are obtained from Equation (5.3). At this rate,

$$dY_t = \beta'(X_t)\, Y_t dt + \sum_{i=1}^{2} D^x \sigma_i'(X_t)\, Y_t\, dW_t^i, \quad Y_0 = \mathbb{1}_2, \tag{5.7}$$

is recognized as its first variation process. Herein, $'$ represents the derivative and σ_i is the i^{th} column of diffusion matrix σ. Further, $\mathbb{1}_2$ account for a identity matrix with a dimension of 2×2 and we define $Y_t = D^x X_t$. Here, the parameters β and σ are assumed to be functions that are twice continuously differentiable, and their derivatives are bounded. Moreover, we assume that the process X_t has continuous trajectories.

We may write Equation (5.7) as in the following form.

$$\begin{aligned}
\begin{bmatrix} dY_t^{11} & dY_t^{12} \\ dY_t^{21} & dY_t^{22} \end{bmatrix} = & \begin{bmatrix} (r_t - \delta) & h_t \\ 0 & -\kappa \end{bmatrix} \cdot \begin{bmatrix} Y_t^{11} & Y_t^{12} \\ Y_t^{21} & Y_t^{22} \end{bmatrix} dt \\
& + \begin{bmatrix} \sigma_h & 0 \\ 0 & \frac{\rho\sigma_r}{2\sqrt{r_t}} \end{bmatrix} \cdot \begin{bmatrix} Y_t^{11} & Y_t^{12} \\ Y_t^{21} & Y_t^{22} \end{bmatrix} dW_t^1 \\
& + \begin{bmatrix} 0 & 0 \\ 0 & \frac{\mu\sigma_r}{2\sqrt{r_t}} \end{bmatrix} \cdot \begin{bmatrix} Y_t^{11} & Y_t^{12} \\ Y_t^{21} & Y_t^{22} \end{bmatrix} dW_t^2.
\end{aligned}$$

Corollary 5.12. *From the model that represents the economic environment given by Equations (5.1) and (5.2), we may conclude that the spot rate process r_t is independent of underlying house price process h_t. Thus $Y_t^{21} = 0$ and the renaming elements satisfy*

$$\begin{aligned}
dY_t^{11} &= (r_t - \delta)\, Y_t^{11} dt + \sigma_h Y_t^{11} dW_t^1, \quad Y_0^{11} = 1, \\
dY_t^{12} &= \left[(r_t - \delta) Y_t^{12} + h_t Y_t^{22}\right] dt + \sigma_h Y_t^{12} dW_t^1, \quad Y_0^{12} = 0, \\
dY_t^{22} &= -\kappa Y_t^{22} dt + \frac{\sigma_r}{2\sqrt{r_t}} dW_t^r, \quad Y_0^{22} = 0.
\end{aligned}$$

It is true that we may represent Y_t as $Y_t = \nabla_x X_t$, and further, due to continuously differentiablty of β and σ, their bounded derivatives and X_t is continuous, we can give the Malliavin derivative of X_t as (See [77] for further details)

$$D_s X_t = Y_t Y_s^{-1} \sigma(X_s) \mathbb{1}_{s \leq t}. \tag{5.8}$$

As an immediate result, we can also introduce the following remark.

Remark 5.13. Using Itô lemma we may express Y_t^{11} in terms of the house price as in the following form,

$$Y_t^{11} = \frac{1}{h_0} h_t, \quad a.s.$$

Proof. By applying Itô lemma to Equation (5.1) for the $\log$ function, first we obtain

$$\begin{aligned} \log(h_t) &= \log(h_0) + \int_0^T \frac{1}{h_t} dh_t - \frac{1}{2} \int_0^T \frac{1}{h_t^2} \sigma_h^2 h_t^2 dt \\ \log(\frac{h_t}{h_0}) &= \int_0^T (r_t - \delta - \frac{\sigma_h^2}{2}) dt + \int_0^T \sigma_h dW_t^h. \end{aligned} \tag{5.9}$$

Then, Equation (5.9) yields that

$$h_t = h_0 e^{\int_0^T (r_t - \delta - \frac{\sigma_h^2}{2}) dt + \int_0^T \sigma_h dW_t^h}. \tag{5.10}$$

Now, applying Itô lemma to Y_t^{11} again with $\log$ function we have

$$\begin{aligned} \log(Y_t^{11}) &= \log(Y_0^{11}) + \int_0^T \frac{1}{Y_0^{11}} dY_t^{11} - \frac{1}{2} \int_0^T \frac{1}{(Y_t^{11})^2} \sigma_h^2 (Y_t^{11})^2 dt \\ \log(\frac{Y_t^{11}}{Y_0^{11}}) &= \int_0^T \left(r_t - \delta - \frac{\sigma_h^2}{2} \right) dt + \int_0^T \sigma_h dW_t^1. \end{aligned} \tag{5.11}$$

Then, it is clear from Equation (5.11)

$$Y_t^{11} = e^{\int_0^T (r_t - \delta - \frac{\sigma_h^2}{2}) dt + \int_0^T \sigma_h dW_t^h}. \tag{5.12}$$

We also know that $W_t^h = W_t^1$. Thus, from Equation (5.10) and Equation (5.12) we end up with the desired result,

$$Y_t^{11} = \frac{1}{h_0} h_t, \quad a.s.$$

□

Now, let us give the price process of a contingent claim.

Definition 5.14. Consider a square integrable option payoff $\Phi = \Phi(X_{t_1}, \ldots, X_{t_n})$, which is evaluated at time $t_1, t_2, \ldots, t_n$. Suddenly, the price of any contingent claim is nothing but the discounted value, and it is displayed as

$$v(x) = \mathbb{E}\left[e^{-\int_0^T r_t\, dt}\Phi(X_{t_1}, \ldots, X_{t_n})|\mathcal{F}_0\right]. \tag{5.13}$$

The purpose of the computation of hedging coefficients is to take derivatives of the option concerning the parameters inherited into the model. To perform our calculations, it is inevitable to consider σ to be a uniformly elliptic matrix. Thus,

$$\exists\, \eta > 0,\ \text{such that} \quad \xi^\top \sigma(x)^\top \sigma(x)\xi \geq \eta\, |\xi|^2, \quad \text{for any} \quad \xi, x \in \mathbb{R}^n. \tag{5.14}$$

5.3.1 Variations in the Initial Condition

To obtain a logical calculation issue, it is undeniable to ensure that the Malliavin weights do not corrupt with probability one. Consequently, Fournié et.al. (1999, [77]) demonstrate the following class of square-integrable functions

$$\Gamma = \left\{\alpha \in L^2\left([0,T]\right); \int_0^{t_i} \alpha(t)dt = 1, \forall i = 1, \cdots, n\right\},$$

in $\mathbb{R}$ to withdraw degeneration. In this case, t_i's denotes a separation of finite time horizon $[0, T]$.

To compute the sensitivity concerning the initial underlying asset price, we require the celebrated Bismut-Elworthy-Li formula proposed in [27, 67].

Proposition 5.15 (Bismut-Elworthy formula)**.** *Suppose σ is the uniformly elliptic matrix, and Φ is a function that is continuously differentiable with bounded gradient. And so, for any functional $\alpha(t) \in \Gamma$ we have*

$$(\nabla v(x))^\top = \mathbb{E}\left[e^{-\int_0^T r_t\, dt}\Phi(X_{t_1}, \ldots, X_{t_n}) \int_0^T \alpha(t)(\sigma^{-1}(X_t)Y_t)^\top d\mathbb{W}_t\right].$$

Proof. Since Φ is continuously differentiable with bounded derivative, it is possible to interchange the differentiation and expectation. Applying chain rule (Proposition 5.3), we obtain

$$\begin{aligned}\nabla p\left(x\right)^{\top} &= \mathbb{E}\left[e^{-\int_0^T r_t dt}\sum_{i=1}^{n}\nabla_i\Phi\left(X_{t_1},\ldots,X_{t_n}\right)D^x X_{t_i}\right]\\ &= \mathbb{E}\left[e^{-\int_0^T r_t dt}\sum_{i=1}^{n}\nabla_i\Phi\left(X_{t_1},\ldots,X_{t_n}\right)Y_{t_i}\right],\end{aligned} \tag{5.15}$$

where ∇ denotes the gradient and Y is the first variation process introduced in Definition 5.7. By Equation (5.8), one can write the Malliavin derivative of X_t in terms of first variation process

$$D_t X_{t_i} = Y_{t_i} Y_t^{-1} a(X_t) \mathbb{1}_{t \le t_i}.$$

Then,

$$Y_{t_i} = \int_0^T \left(\alpha(t) D_t X_{t_i}\right) \sigma^{-1}(X_{t_i}) Y_t \, dt, \tag{5.16}$$

where $\alpha \in \Gamma_n$. Substituting Equation (5.16) into Equation (5.15) and applying the chain rule, we obtain

$$\begin{aligned}\nabla p(x) &= \mathbb{E}\left[e^{-\int_0^T r_t dt}\sum_{i=1}^{n}\nabla_i\Phi\left(X_{t_1},\ldots,X_{t_n}\right)\int_0^T \alpha(t)\left(D_t X_{t_i}\right)\sigma^{-1}(X_{t_i})Y_t dt\right]\\ &= \mathbb{E}\left[e^{-\int_0^T r_t dt}\int_0^T \alpha(t) D_t(\Phi\left(X_{t_1},\ldots,X_{t_n}\right))\sigma^{-1}\left(X_t\right)Y_t dt\right].\end{aligned}$$

Note that the diffusion matrix satisfy the uniform ellipticity condition. Hence, $\alpha(t)a^{-1}(X_t)Y_t$ is a square integrable random variable for all fixed time in the interval $[0, T]$, which enables us to utilize the integration by parts formula. Applying the integration by parts formula (Proposition 5.7), the gradient of the price process may be rewritten as in the following form

$$\left(\nabla p\left(x\right)\right)^{\top} = \mathbb{E}\left[e^{-\int_0^T r_t dt}\Phi\left(X_{t_1},\ldots,X_{t_n}\right)\int_0^T \alpha(t)\left(\sigma^{-1}(X_t)Y_t\right)^{\top} d\mathbb{W}_t\right].$$

Since the continuously differentiable functions class with a bounded gradient is dense in L^2 space, it can be used, precisely as in Fournié et al. (1999, [77]) to generalize the results. □

5.3.2 Variations in the Drift Coefficient

In this subsection, we define a perturbed process to assess the sensitivity of the price of the option to variations in the drift as

$$dX_t^\epsilon = \left[\beta(X_t^\epsilon) + \epsilon\gamma(X_t^\epsilon)\right] dt + \sigma(X_t^\epsilon)\, d\mathbb{W}_t, \quad X_0^\epsilon = x,$$

where $\epsilon \in \mathbb{R}$ and γ is a bounded.

The following Proposition explains to the user how sensitive the price of an option to the perturbed price process to ϵ for $\epsilon = 0$.

Proposition 5.16. *Let σ satisfies the uniformly ellipticity condition, and Φ is a square-integrable continuous payoff function. For our perturbed underlying asset price process*

$$v^{\epsilon}(x) = \mathbb{E}\left[e^{-\int_0^T r(t)dt}\Phi\left(X^{\epsilon}_{t_1}, \ldots, X^{\epsilon}_{t_n}\right)\right],$$

the following is obtained

$$\frac{\partial}{\partial \epsilon} v^{\epsilon}(x)|_{\epsilon=0} = \mathbb{E}\left[e^{-\int_0^T r(t)dt}\Phi(X_{t_1}, \ldots, X_{t_n}) \int_0^T \left(\sigma^{-1}(X_t)\gamma(X_t)\right)^{\top} d\mathbb{W}_t\right].$$

5.3.3 Variations in the Diffusion Coefficient

In the computation of the contingent claim, v, sensitivity concerning the volatility, we use the perturbed price process defined below.

$$dX^{\epsilon}_t = \beta\left(X^{\epsilon}_t\right) + \left[\sigma\left(X^{\epsilon}_t\right) + \epsilon\gamma(X^{\epsilon}_t)\right] d\mathbb{W}, \quad X^{\epsilon}_0 = x,$$

where $\epsilon \in \mathbb{R}$ and γ is a function that is continuously differentiable and has bounded derivatives. Moreover, the uniform ellipticity condition holds for$\sigma + \epsilon\gamma$. To proceed with our calculations, we also need to define the variation process concerning ϵ as follows

$$dZ^{\epsilon}_t = \beta'\left(X^{\epsilon}_t\right) Z^{\epsilon}_t dt + \sum_{i=1}^{2}\left(\sigma'_i\left(X^{\epsilon}_t\right) + \epsilon\gamma(X^{\epsilon}_t)\right) Z^{\epsilon}_t dW^i_t + \gamma(X^{\epsilon}_t) d\mathbb{W}_t$$

with $Z^{\epsilon}_0 = 0_n$. Here, one may notice that $\frac{\partial X^{\epsilon}_t}{\partial \epsilon} = Z^{\epsilon}_t$. Also, $Z^{\epsilon}_t|_{\epsilon=0}$ is defined as Z_t. Moreover, we define the class of square-integrable functions on $\mathbb{R}$ as

$$\Gamma_n = \left\{\alpha \in L^2\left([0,T)\right) : \int_{t_{i-1}}^{t_i} \alpha(t)dt = 1, \quad \forall i = 1, \ldots, n, t_0 = 0\right\}. \tag{5.17}$$

Immediately, based on the recent perturbed process given with Equation (5.34) the following proposition may be given.

Proposition 5.17. *Let σ be a matrix that satisfies uniformly ellipticity condition (5.14) and for $B_{t_i} = Y^{-1}_{t_i} Z_{t_i}$, $i = 1, \ldots, n$ the value $\sigma^{-1}(X_t)Y_t B_t \in Dom(\delta)$ available. In* *that case,*

$$v^{\epsilon}(x) = \mathbb{E}\left[e^{-\int_0^T r(t)dt}\Phi\left(X^{\epsilon}_{t_1},\ldots,X^{\epsilon}_{t_n}\right)\right],$$

and for any $\alpha \in \Gamma_n$ the following is obtained

$$\frac{\partial}{\partial\epsilon}v^{\epsilon}(x)|_{\epsilon=0} = \mathbb{E}\left[e^{-\int_0^T r(t)dt}\Phi\left(X_{t_1},\ldots,X_{t_n}\right)\delta\left(\sigma^{-1}(X_{\cdot})Y_{\cdot}\tilde{B}_{\cdot}\right)\right],$$

where

$$\tilde{B}_t = \sum_{i=1}^{n}\alpha(t)\left(B_{t_i} - B_{t_{i-1}}\mathbb{1}_{t\in[t_{i-1},t_i)}\right).$$

Proof. The payoff function Φ is at least twice continuously differentiable with bounded gradient. Thus, we have right to write

$$\frac{\partial}{\partial\epsilon}v^{\epsilon}(x) = \mathbb{E}\left[\sum_{i=1}^{n}\nabla_i\Phi\left(X^{\epsilon}_{t_1},\ldots,X^{\epsilon}_{t_n}\right)Z^{\epsilon}_{t_i}\right].$$

Next define $B_t = Y_t^{-1}Z_t = Y_t^{-1}Z_t^{\epsilon}|_{\epsilon=0}$. Since $\alpha \in \Gamma_n$ we have

$$\begin{aligned}
\int_0^T (D_tX_t)\left(\sigma^{-1}(X_t)Y_t\tilde{B}_t\right)dt &= \int_0^{t_i} Y_{t_i}\tilde{B}_t dt \\
&= Y_{t_i}\sum_{j=1}^{i}\int_{t_{j-1}}^{t_j}\alpha(t)(B_{t_j} - B_{t_{j-1}})dt \\
&= Y_tB_{t_i} \\
&= Z_i.
\end{aligned}$$

Then,

$$\begin{aligned}
\frac{\partial}{\partial\epsilon}v^{\epsilon}(x)|_{\epsilon=0} &= \mathbb{E}\left[\int_0^T\sum_{i=1}^{n}\nabla_i\Phi\left(X^{\epsilon}_{t_1},\ldots,X^{\epsilon}_{t_n}\right)(DX_t)\sigma^{-1}(X_t)Y_t\tilde{B}_t dt\right] \\
&= \mathbb{E}\left[\int_0^T D_t\Phi(X^{\epsilon}_{t_1},\ldots,X^{\epsilon}_{t_n})\sigma^{-1}(X_t)Y_t\tilde{B}_t dt\right].
\end{aligned}$$

By our assumption and linearity property of Skorohod integral, $\sigma^{-1}(X)Y\tilde{B} \in Dom(\delta)$. Thus,

$$\frac{\partial}{\partial\epsilon}v^{\epsilon}(x)|_{\epsilon=0} = \mathbb{E}\left[\Phi(X^{\epsilon}_{t_1},\ldots,X^{\epsilon}_{t_n})\sigma^{-1}(X_t)Y_t\tilde{B}_t dt\right]$$

$\square$

Remark 5.18. If $B_t \in \mathbb{D}$, we can calculate $\delta\left(\sigma^{-1}(X_.)Y_.\tilde{B}_.\right)$ as follows

$$\begin{aligned}\delta\left(\sigma^{-1}(X_.)Y_.\tilde{B}_.\right) = \ & \sum_{i=1}^{n} \ \Big\{B_{t_i}^{\top}\int_{t_{i-1}}^{t_i}\alpha(t)\left(\sigma^{-1}(X_t)Y_t\right)^{\top}d\mathbb{W}_t \\ - & \int_{t_{i-1}}^{t_i}\alpha(t)Tr\left((D_tB_{t_i})\sigma^{-1}(X_t)Y_t\right)dt \\ - & \int_{t_{i-1}}^{t_i}\alpha(t)\left(\sigma^{-1}(X_t)Y_tB_{t_{i-1}}\right)^{\top}d\mathbb{W}_t\Big\}.\end{aligned}$$

5.4 The Hedging Coefficients of Mortgage Default and Prepayment Options

This section develops theoretical formulas for hedging parameters of options of mortgages, namely default and prepayment options, based on the finite-dimensional Malliavin calculus. As the housing market defined in the economic environment that we explained above to examine our research problem, first, we find the mathematical representations of the hedging coefficients in this chapter.

Mortgage users are representatives of two brands of options: A Parisian call option that defines prepaying the loan [175] and a European put option that specifies defaulting and moving out from the house [38]. There is a common faith that prepayment is not influenced by the change in the underlying house price as much as default. As the underlying house price decreases, the probability of default raise, and the expense to moneylenders likewise rise. While default probabilities and losses grow as a result of the decreasing house price, the damages inherited to default increase quicker than house price collapse without any linear association. In other respects, if the spot rate reduces sufficiently, buyers favor refinancing their mortgages with the current rate since it is cheaper. In another word, mortgage prepayment raises a satisfactory reduction in the conventional interest rates. Consequently, both parties have to utilize a progressive hedging procedure for their mortgages.

Commonly, in hedging parameter computations, we can utilize the methods of likelihood, pathwise, and finite-difference. However, the calculation of these parameters under the standard two-state variable hypothesis is cumbersome. It is because the recommended model does not produce a precise distribution, or it is because the payoffs are not differentiable. Consequently, both likelihood and pathwise methods are not proper in our circumstances. But, we can utilize the finite difference method since this method based on the resemblance of derivatives as variations in a dependent variable over a small interval.

Besides, the finite difference method can be represented by utilizing a tiny class of differential operators. However, it is computationally costly and questionable for discontinuous functions. Consequently, the use of it is circumvented, and the Malliavin calculus is practiced in the computation of the hedging parameters by following the Fournié et al. (1999, 2001, [77, 76]) for two central deductions. 1) it is computationally less costly contrasted to the finite difference, and the payoffs of both options are discontinuous. 2) the method introduced by Fournié et al. (1999, 2001 [77, 76]) allows quicker convergence for problems that contains Markov diffusions.

In this chapter, X_t is in $\mathbb{R}^2$ (see Equation (5.3)). Here, we suppose that X_t is a Markov diffusion process with an initial value of $X_0 = x$. Suddenly, the value of options written on this asset is

$$p(x) = \mathbb{E}\left[e^{-\int_0^T r_t\, dt}\Phi(X_{t_1}, \ldots, X_{t_n})\right]. \tag{5.18}$$

We know that the hedging parameters are the derivatives of the price function provided with Equation (5.18) concerning the corresponding variables. Consequently, as a critical consequence of Proposition 5.15, we can give the following proposition that classifies the hedging parameters for the variations in underlying house prices.

Proposition 5.19. *Let β and σ be functions both defined as in Equation (5.3). Both of them continuously differentiable functions with bounded derivatives. Further, the uniform ellipticity condition holds for σ. Now, think a payoff function denoted by Φ. It is at least twice continuously differentiable and also has bounded derivatives. In this regard, the sensitivity concerning h_0 is*

$$\frac{\partial p}{\partial h_0} = -\mathbb{E}\left[\frac{e^{-\int_0^T r_t dt}}{h_0\sigma_h T}\Phi\left(h_{t_1}, \ldots, h_{t_n}\right)\left(W_T^1 - \frac{\rho}{\mu}W_T^2\right)\right]. \tag{5.19}$$

Proof. The inverse of the diffusion matrix σ is

$$a^{-1}\left(X_t\right) = \begin{bmatrix} \frac{1}{\sigma_h x_1} & 0 \\ \frac{-\rho}{\sigma_h x_1 \mu} & \frac{1}{\mu\sigma_r\sqrt{x_2}} \end{bmatrix}.$$

Then,

$$\left(\sigma^{-1}\left(X_t\right)Y_t\right)^\top = \begin{bmatrix} \frac{1}{\sigma_h x_1}Y_t^{11} & \frac{-\rho}{\sigma_h x_1 \mu}Y_t^{11} + \frac{\rho}{\sigma_r \mu \sqrt{x_2}}Y_t^{21} \\ \frac{1}{\sigma_h x_1}Y_t^{12} & \frac{-\rho}{\mu \sigma_h x_1}Y_t^{12} + \frac{1}{\mu \sigma_r \sqrt{x_2}}Y_t^{22} \end{bmatrix}. \tag{5.20}$$

By Proposition 5.15 we have,

$$\frac{\partial p}{\partial h_0} = \mathbb{E}\left[\frac{e^{-\int_0^T r_t dt}}{h_0 \sigma_h T}\Phi\left(X_{t_1},\ldots,X_{t_n}\right)\left(\int_0^T \frac{Y_t^{11}}{x_1}dW_t^1 + \int_0^T \frac{-\rho Y_t^{11}}{\mu x_1}dW_t^2\right)\right]. \tag{5.21}$$

Here, $Y_t^{21} = 0$ since the process h_t is not affect r_t. Moreover, in previous section and $\alpha(t)$ is chosen as $\frac{1}{T}$. Then, we come to final result easily by Remark 5.13. $\square$

In the present chapter, the spot rate assumed to be either constant or deterministic. Preferably, it evolves from Equation (5.3). Accordingly, the hedging coefficient concerning the spot rate is not straightforward. So, rather than immediately differentiating the option price, a perturbation parameter ϵ is included. Then, the effect of ϵ on options is recognized. At this point, it is undeniable to clarify what we meant by the perturbed process.

We define the perturbed process as X_t^ϵ and give its dynamics as

$$\begin{aligned} dX_t^\epsilon &= \left[\beta\left(X_t^\epsilon\right) + \epsilon\gamma\left(X_t^\epsilon\right)\right]dt + \sigma\left(X_t^\epsilon\right)d\mathbb{W}_t, \\ X_0^\epsilon &= x, \end{aligned} \tag{5.22}$$

where $\epsilon \in \mathbb{R}$, and γ is a function that is continuously differentiable. Here, $\beta + \epsilon\gamma$ and σ satisfies all regularity conditions that we explained above. To describe the influence of a structural variation in the drift, we require a perturbed price process.

Definition 5.20. Let X_t^ϵ be a solution (5.22) for $t \in [0, T]$ and Φ is a continuously differentiable with bounded derivatives payoff function. Consequently, the perturbed option price $p^\epsilon(x)$ equals to

$$v^\epsilon(x) = \mathbb{E}\left[e^{-\int_0^T r_t^\epsilon\, dt}\Phi\left(X_{t_1}^\epsilon,\cdots,X_{t_n}^\epsilon\right)\right]. \tag{5.23}$$

By relying on Definition 5.23, the hedging parameter that corresponds to the change in spot rate is of the form as in Proposition 5.16.

Proposition 5.21. *Consider β and σ as we discussed above. Then, for any payoff that is continuously differentiable and have bounded derivatives as* $\Phi : \mathbb{R}^2 \times \mathbb{R}^2 \times \ldots \times$

$\mathbb{R}^2 \longmapsto \mathbb{R}$, $\epsilon \longmapsto p^\epsilon(x)$ *is differentiable concerning all* $x \in \mathbb{R}^2$*. Consequently, the hedging parameters of any option that corresponds to the spot rate fluctuations is*

$$\begin{aligned}\frac{\partial p}{\partial r} &= \mathbb{E}\left[e^{-\int_0^T r_t dt}\Phi\left(X_{t_1},\cdots,X_{t_n}\right)\frac{1}{T}\left(\frac{W_T^1}{\sigma_h}-\frac{\rho W_T^2}{\sigma_h\mu}\right)\right]\\ &- \mathbb{E}\left[Te^{-\int_0^T r_t dt}\Phi\left(X_{t_1},\cdots,X_{t_n}\right)\right].\end{aligned}$$

Proof. We may represent Equation (5.22) by and integral form as follows

$$X_t^\epsilon = X_0^\epsilon + \int_0^t \left[\beta(X_s^\epsilon)+\epsilon\gamma(X_s^\epsilon)\right]ds + \int_0^t \sigma(X_s^\epsilon)d\mathbb{W}_s. \tag{5.24}$$

Then subtracting the original process X_t from Equation (5.24) we have

$$X_t^\epsilon - X_t = \int_0^t \left[\beta(X_s^\epsilon)+\epsilon\gamma(X_s^\epsilon)-\beta(X_s)\right]ds + \int_0^t \left[\sigma(X_s^\epsilon)-\sigma(X_s)\right]d\mathbb{W}_s. \tag{5.25}$$

By dividing both sides of Equation (5.25) with ϵ we obtain

$$\frac{X_t^\epsilon - X_t}{\epsilon} = \int_0^t \left[\gamma\left(X_0^\epsilon\right)+\frac{\beta(X_s^\epsilon)-\beta(X_s)}{\epsilon}\right]ds + \int_0^t \frac{\sigma(X_s^\epsilon)-X_s}{\epsilon}d\mathbb{W}_s. \tag{5.26}$$

In the limiting case, Equation (5.26) becomes the derivative of X_t with respect to ϵ, i.e. $Z_t^\epsilon = \frac{\partial X_t^\epsilon}{\partial \epsilon}$. Then, using the Gâteaux derivative property we obtain

$$Z_t^\epsilon = \int_0^t \left[\gamma(X_s) - D^x\beta(X_s)Z_s^\epsilon\right]ds + \int_0^t \sum_{i=1}^2 D^x\sigma_i(X_s)Z_s^\epsilon dW_s^i. \tag{5.27}$$

The process Z_t^ϵ satisfies the SDE system

$$dZ_t = \left[\gamma(X_t)+D^x\beta(X_t)Z_t\right]dt + \sum_{i=1}^2 D^x\sigma_i(X_t)Z_t dW_t^i, \quad Z_0 = 0_2. \tag{5.28}$$

The equivalent representation of Z_t^ϵ is

$$Z_t^\epsilon = \int_0^t Y_t Y_s^{-1}\gamma(X_s)ds, \quad t\in[0,T]. \tag{5.29}$$

Now let us differentiate the perturbed price process defined by Equation (5.23) with respect to ϵ.

$$\begin{aligned}\frac{\partial p^\epsilon(x)}{\partial \epsilon} &= \mathbb{E}\left[\Phi\left(X_{t_1}^\epsilon,\ldots,X_{t_n}^\epsilon\right)\frac{d}{d\epsilon}e^{-\int_0^T r_t^\epsilon\,dt}\right]\\ &+ \mathbb{E}\left[e^{-\int_0^T r_t^\epsilon\,dt}\sum_{i=1}^n \nabla_i\Phi\left(X_{t_1}^\epsilon,\ldots,X_{t_n}^\epsilon\right)\frac{\partial X_{t_i}^\epsilon}{\partial\epsilon}\right].\end{aligned} \tag{5.30}$$

Let us define the last component of the right-hand side of the equation as I_2. From Equations (5.8) and (5.29) we have

$$Z_{t_i}^{\epsilon} = \int_0^T D_t X_{t_i} \sigma^{-1}(X_t)\gamma(X_t) \mathbb{1}_{t \le t_i} dt. \tag{5.31}$$

Then, by the chain rule we can write following for I_2.

$$\begin{aligned} I_2 &= \mathbb{E}\left[e^{-\int_0^T r_t^{\epsilon}\, dt} \sum_{i=1}^{n} \nabla_i \Phi\left(X_{t_1}^{\epsilon}, \ldots, X_{t_n}^{\epsilon}\right) \int_0^T D_t X_{t_i} \alpha(t) \sigma^{-1}(X_{t_i})\gamma(X_{t_i}) dt \right] \\ &= \mathbb{E}\left[e^{-\int_0^T r_t^{\epsilon}\, dt} \int_0^T D_t \left(\Phi\left(X_{t_1}^{\epsilon}, \ldots, X_{t_n}^{\epsilon}\right)\right) \alpha(t) \left(\sigma^{-1}(X_t)\gamma(X_t)\right)^{\top} dt \right]. \end{aligned}$$

Then, by integration by parts formula (Proposition 5.7), we get

$$I_2 = \mathbb{E}\left[e^{-\int_0^T r_t^{\epsilon}\, dt} \Phi\left(X_{t_1}, \ldots, X_{t_n}\right) \int_0^T \alpha(t) \left(\sigma^{-1}(X_t)\gamma(X_t)\right)^{\top} d\mathbb{W}_t \right].$$

Inserting this equation into Equation (5.30), finally we have

$$\begin{aligned} \frac{\partial p^{\epsilon}(x)}{\partial \epsilon}\Big|_{\epsilon=0} &= \mathbb{E}\left[e^{-\int_0^T r_t(X_t^{\epsilon}) dt} \Phi\left(X_{t_1}, \ldots, X_{t_n}\right) \int_0^T \alpha(t) \left(\sigma^{-1}(X_t)\gamma(X_t)\right)^{\top} d\mathbb{W}_t\big|_{\epsilon=0} \right] \\ &- \mathbb{E}\left[\Phi\left(X_{t_1}^{\epsilon}, \ldots, X_{t_n}^{\epsilon}\right) \frac{\partial}{\partial \epsilon} e^{-\int_0^T r_t^{\epsilon}\, dt}\big|_{\epsilon=0} \right] \end{aligned} \tag{5.32}$$

Now choose $\gamma(X_t) = [h_t, 0]^{\top}$. Then,

$$\left(\sigma^{-1}(X_t)\gamma(X_t)\right)^{\top} = \left[\frac{1}{\sigma_h}, \frac{-\rho}{\mu \sigma_h} \right]. \tag{5.33}$$

Thus, inserting $\alpha(t) = \frac{1}{T}$ and Equation (5.33) into Equation (5.32) for $\epsilon \mapsto 0$ we deduce the desired result. $\square$

As in the calculation of the hedging parameters corresponding to the spot rate, we need a perturbed process to calculate the hedging coefficient that corresponds to the volatility. But, now, we define the perturbed process X_t^{ϵ} as

$$dX_t^{\epsilon} = \beta\left(X_t^{\epsilon}\right) + \left[\sigma\left(X_t^{\epsilon}\right) + \epsilon\gamma(X_t^{\epsilon})\right] d\mathbb{W}, \quad X_0^{\epsilon} = x, \tag{5.34}$$

where again $\epsilon \in \mathbb{R}$ and $\gamma \in \Gamma_n$ is an at least twice continuously differentiable function that having bounded derivatives. Also, the uniform ellipticity condition is satisfied by $(\sigma + \epsilon\gamma)$. Next, as an instant issue, we can give the following proposition by using Equation 5.34, Proposition 5.17, and Remark 5.18.

Proposition 5.22. *Suppose both β and σ are at least twice continuously differentiable functions. Also, σ satisfies the usual condition that is necessary for our computations. Next, for any continuously differentiable payoff having bounded derivatives $\Phi : \mathbb{R}^2 \times \mathbb{R}^2 \times \ldots \times \mathbb{R}^2 \longmapsto \mathbb{R}$, $\epsilon \longmapsto p^\epsilon(x)$ is differentiable with $x \in \mathbb{R}^2$. Suddenly, the hedging parameter of any option that is corresponding to the volatility is*

$$\frac{\partial p}{\partial \sigma_h} = \mathbb{E}\left[e^{-\int_0^T r_t dt}\Phi\left(X_{t_1}, \cdots, X_{t_n}\right)\left(\frac{W_T^1}{T\sigma_h}\left(W_T^1 - \frac{\rho W_T^2}{\mu}\right) - \frac{1}{\sigma_h}\right)\right].$$

Proof. To proceed our computations we need the followings.

$$\beta' = \begin{bmatrix} (r_t - \delta) & h_t \\ 0 & -\kappa \end{bmatrix}, \quad \gamma\left(X_t\right) = \begin{bmatrix} h_t & 0 \\ 0 & 0 \end{bmatrix},$$
$$\sigma_1'\left(X_t\right) = \begin{bmatrix} \sigma_h & 0 \\ 0 & \frac{\rho\sigma_r}{2\sqrt{r_t}} \end{bmatrix}, \quad \sigma_2'\left(X_t\right) = \begin{bmatrix} 0 & 0 \\ 0 & \frac{\mu\sigma_r}{2\sqrt{r_t}} \end{bmatrix}.$$

Now, we may define Z_t explicitly as

$$\begin{aligned} d\begin{bmatrix} Z_t^1 \\ Z_t^2 \end{bmatrix} &= \begin{bmatrix} (r_t - \delta) & h_t \\ 0 & -\kappa \end{bmatrix} \cdot \begin{bmatrix} Z_t^1 \\ Z_t^2 \end{bmatrix} dt + \begin{bmatrix} \sigma_h & 0 \\ 0 & \frac{\rho\sigma_r}{2\sqrt{r_t}} \end{bmatrix} \cdot \begin{bmatrix} Z_t^1 \\ Z_t^2 \end{bmatrix} dW_t^1 \\ &+ \begin{bmatrix} 0 & 0 \\ 0 & \frac{\mu\sigma_r}{2\sqrt{r_t}} \end{bmatrix} \cdot \begin{bmatrix} Z_t^1 \\ Z_t^2 \end{bmatrix} dW_t^2 + \begin{bmatrix} h_t & 0 \\ 0 & 0 \end{bmatrix} d\mathbb{W}_t. \end{aligned} \tag{5.35}$$

Here, using the fact that the spot rate process (r_t) does not depend on the house price process (h_t), we may deduce that $Z_t^0 = 0$. Thus, from Equation (5.35) we have

$$dZ_t^1 = \left(r_t - \delta\right) Z_t^1 dt + \sigma_h Z_t^1 dW_t^1 + h_t dW_t^2.$$

Furthermore, from Itô lemma the solution of this SDE is $Z_t^1 = h_t W_t^1$. Then we have

$$B_t = \begin{bmatrix} \frac{1}{Y_t^{11}} & -\frac{Y_t^{12}}{Y_t^{11}Y_t^{22}} \\ 0 & \frac{1}{Y_t^{22}} \end{bmatrix} \cdot \begin{bmatrix} Z_t^1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{Z_t^1}{Y_t^{11}} \\ 0 \end{bmatrix} = \begin{bmatrix} h_0 W_t^1 \\ 0 \end{bmatrix}.$$

So that

$$\tilde{B}_t = \frac{h_0}{T} \cdot \begin{bmatrix} W_t^1 \mathbb{1}_{[0,T]} \\ 0 \end{bmatrix}$$

for $\alpha(t) = \frac{1}{T} \in \Gamma_n$.

Now, we can compute the following

$$\sigma^{-1}(X_t)Y_t = \begin{bmatrix} \frac{1}{\sigma_h h_t} & 0 \\ -\frac{\rho}{\mu\sigma_h h_t} & \frac{1}{\mu\sigma_r\sqrt{r_t}} \end{bmatrix} \cdot \begin{bmatrix} Y_t^{11} & Y_t^{12} \\ 0 & Y_t^{22} \end{bmatrix} = \begin{bmatrix} \frac{Y_t^{11}}{\sigma_h h_t} & \frac{Y_t^1 2}{\sigma_h h_t} \\ -\frac{\rho Y_t^{11}}{\mu\sigma_h h_t} & -\frac{\rho Y_t 12}{\mu\sigma_h h_t} + \frac{Y_t^{22}}{\mu\sigma_r\sqrt{r_t}} \end{bmatrix}.$$

We also know that

$$D_t B_T = \begin{bmatrix} h_0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus,

$$(D_t B_T)\left(\sigma^{-1}(X_t)Y_t\right) = \begin{bmatrix} \frac{h_0 Y_t^{11}}{\sigma_h h_t} & \frac{h_0 Y_t^{12}}{\sigma_h h_t} \\ 0 & 0 \end{bmatrix},$$

and

$$Tr\,(D_t B_T)\left(\sigma^{-1}(X_t)Y_t\right) = \frac{h_0 Y_t^{11}}{\sigma_h h_t} = \frac{1}{\sigma_h}$$

since $Y_t^{11} = \frac{h_t}{h_0}$.

Now, we can compute the following integral easily as follows

$$\int_0^T \alpha(t) Tr\,(D_t B_T)\left(\sigma^{-1}(X_t)Y_t\right) dt = \frac{1}{T}\int_0^T \frac{1}{\sigma_h} dt = \frac{1}{\sigma_h} \tag{5.36}$$

since $\alpha(t) = \frac{1}{T}$.

Before proceeding the second integral first let us find the following

$$\left(\sigma^{-1}(X_t)Y_t\right)^\top = \begin{bmatrix} \frac{Y_t^{11}}{\sigma_h h_t} & -\frac{\rho Y_t^{11}}{\mu\sigma_h h_t} \\ \frac{Y_t^{12}}{\sigma_h h_t} & -\frac{\rho Y_t^{12}}{\mu\sigma_h h_t} + \frac{Y_t^{22}}{\mu\sigma_r\sqrt{r_t}} \end{bmatrix}.$$

Then, we have

$$\int_0^T \alpha(t)\left(\sigma^{-1}(X_t)Y_t\right)^\top d\mathbb{W}_t = \begin{bmatrix} \int_0^T \frac{Y_t^{11}}{\sigma_h h_t} dW_t^1 - \int_0^T \frac{\rho Y_t^{11}}{\mu\sigma_h h_t} dW_t^2 \\ \int_0^T \frac{Y_t^{12}}{\sigma_h h_t} dW_t^1 - \int_0^T \left(\frac{\rho Y_t^{12}}{\mu\sigma_h h_t} + \frac{Y_t^{22}}{\mu\sigma_r\sqrt{r_t}}\right) dW_t^2 \end{bmatrix}$$

Now, we have

$$\begin{aligned} B_T^\top \int_0^T \alpha(t)\left(\sigma^{-1}(X_t)Y_t\right)^\top d\mathbb{W}_t &= \frac{h_0 W_T^1}{T}\left(\int_0^T \frac{Y_t^{11}}{\sigma_h h_t} dW_t^1 - \int_0^T \frac{\rho Y_t^{11}}{\mu\sigma_h h_t} dW_t^2\right) \\ &= \frac{h_0 W_T^1}{T}\left(\int_0^T \frac{1}{\sigma_h h_0} dW_t^1 - \int_0^T \frac{\rho}{\mu\sigma_h h_0} dW_t^2\right) \\ &= \frac{W_T^1}{T}\left(\frac{W_T^1}{\sigma_h} - \frac{\rho W_T^2}{\mu\sigma_h}\right). \end{aligned} \tag{5.37}$$

The desired result can be obtained from Equations (5.36) and (5.37). □

The following section presents some numerical illustrations of the results for the hedging parameters that we introduced in this section.

5.5 Numerical Illustrations of Hedging Coefficients

The finite-dimensional Malliavin calculus in the context of mathematical finance, more precisely its application to the hedging briefly discussed in Section 5.3. It is relatively generic, in the sense that this method does not require any solution to PDE of the option. Contrarily, this calculus based on the utilization of MC algorithms. Hence, it is relevant to use a crude MC algorithm for our numerical illustrations for some hedging coefficient. Consequently, this section examines the dilemma of state variables with some predetermined model parameters as a fundamental example. In this framework, we calculate the hedging parameters by appropriating an MC scheme. In the computations, first, we generate representation paths of house price and spot rate, and then we calculate the hedging parameters for this simulated paths. Later, we generate the corresponding paths repeatedly, and we calculate the resulting hedging parameters. The parameters converge to the hedging parameter of the options as we increase the simulation number.

The corresponding parameters of the mortgage default and to prepayment against the changes in the initial house price, the spot rate, and the volatility of house prices are accomplished by replacing the analogous phrases in Propositions 5.19, 5.21 and 5.22. To visualize the application of those propositions, we illustrate our computations in Figures5.2, 5.3 and 5.4 to realize penetrations that stimulate our computations. For a meaningful empirical study, we chose the parameters arbitrarily and introduced them in Table 5.1.

Typically, in our calculations, instead of practicing a relatively poor number of simulations, we perform a crude MC algorithm having 10,000 simulations. In the setting of this thesis, we chose the corresponding parameters within the model to reproduce a flexible variety of functional employment and in such a way that the Novikov condition holds for the spot rate process, which we consider as the CIR process [54] two avoid negative spot rate.

The hedging coefficients play a central role in hedging activities — these coefficients, also recognized as delta, rho, and vega are the ones that we compute. The arbitrage valuation models rely on the notion that any option can be perfectly hedged against the

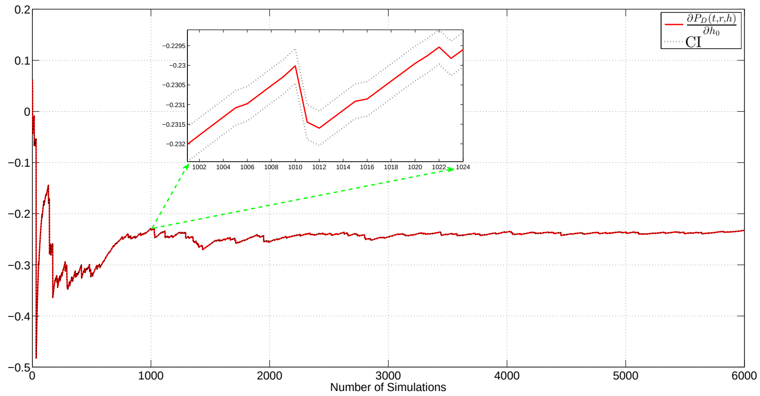
Table 5.1: The parameters for our numerical illustrations

Parameter	Abraviation	Value
Initial House Price	H_0	100000
Service or Rental Flow	δ	0.06
House Price Volatility	σ_h	0.6
Correlation	ρ	-0.8
Initial Interest Rate	r_0	0.08
Adjustment Factor	κ	0.5
Steady State Parameter	θ	0.6
Interest Rate Volatility	σ_r	0.5
Fixed Mortgage Rate	c	0.15
Maturity	T	30 Years
Excursion Period	$\bar{T}_m$	5 Month
Month	i	5
Number of Simulation	N_{sim}	10000

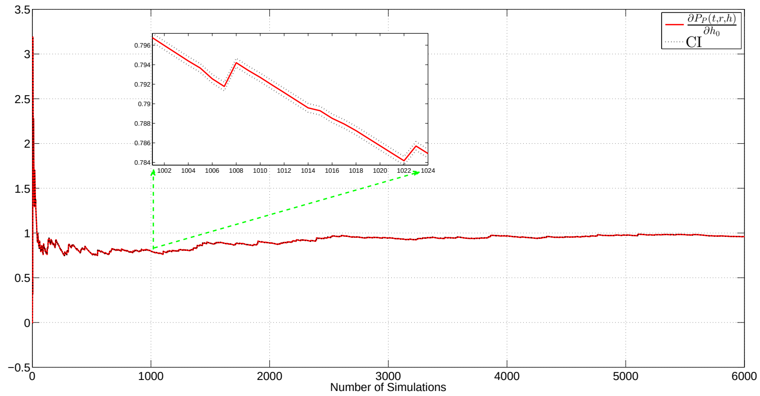
associated risk using the balance between underlying assets and the options. Namely, the hedging activities have a central roll in mathematical finance concerning the theory of option pricing. These coefficients enable investors to assemble a replicating portfolio of their options. The so-called hedging tactics that rely on these coefficients broadly used to overcome the risks associated with the corresponding components. As an example, the first hedging parameter is called delta in mathematical finance (Figure 5.2). It is adopted to decrease the risk of change in the house price. The second parameter is called rho (Figure 5.3). It is utilized to diminish the risk of change in the spot rate. The last parameter is named as vega. This one is engaged in reducing the risk of volatility.

Figure 5.2 represents the MC illustrations of the delta for both options against the number of MC simulations. Figure 5.2 (a) demonstrates that the delta of default option is around -0.23 with the parameter assumption above from Proposition 5.19. Figure 5.2 (b) reveals that delta of prepayment option is roughly 0.84 with the same parameters and Proposition 5.21. Both graphs reveal that practicing 3000 simulations is sufficient to expose the delta of both options. Consequently, the delta can be achieved even with fewer simulations, and the cost can be decreased adequately. Figure 5.2 also reveals that a trader who owns a default option must consider a short position to replicate his option. However, the trader must consider a long position to replicate his mortgage prepayment option. Users should recognize that when a trader holds a short position in the option, he has a poor potential to gain a profit while there is a possibility of infinite losses.

The boxes in both figures and the following figures illustrate zoom-in of a small simulation interval to show upper and lower confidence intervals at 0.5 level. These boxes clearly indicate that even though confidence intervals are all very tight, the estimation of hedging coefficients are all stays within confidence intervals. Hence, the evaluations are all statistically consistent.



(a)

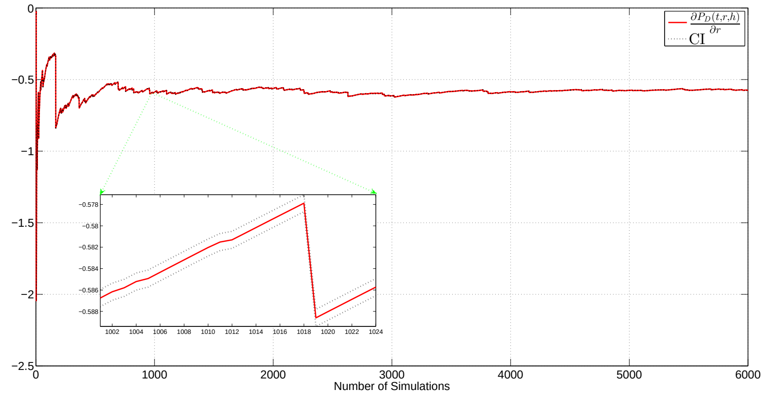


(b)

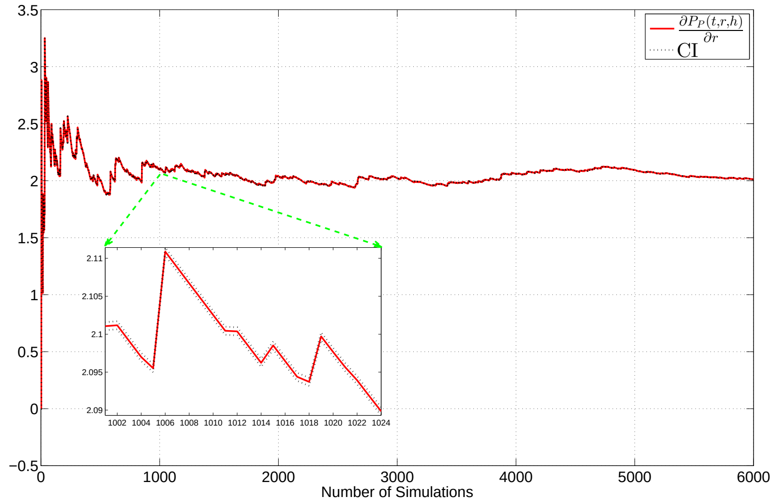
Figure 5.2: Simulations of the options delta coefficients: (a) Delta of default option, (b) Delta of prepayment option

Figure 5.3 portrays the MC illustrations of the rho for both options against the simulation number. In Figure 5.3 (a), it is shown that the rho for the default option is roughly

0.66. In Figure 5.3 (b), it is shown that the rho for the mortgage prepayment option is almost 2. More importantly, since the hedging coefficients may be accomplished with fewer simulations, the computation cost may be reduced sufficiently. Figures 5.3 (a) and (b) both reveal that both options are more sensitive to fluctuations in the spot rate.



(a)

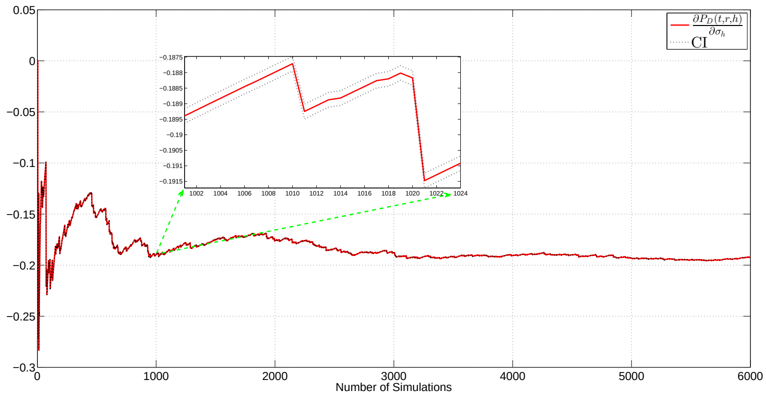


(b)

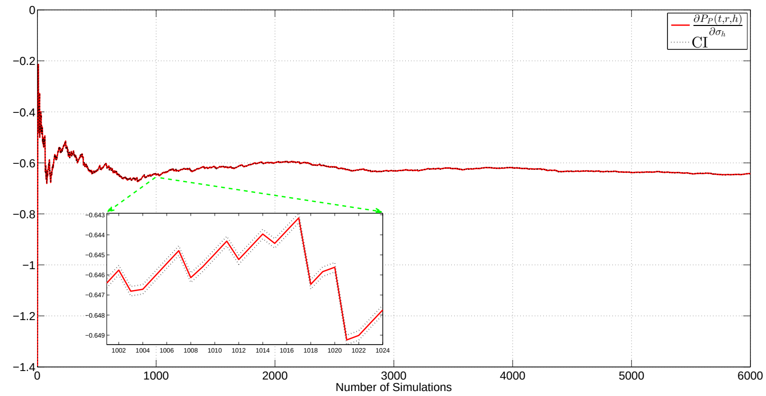
Figure 5.3: Simulations of the options rho coefficients: (a) Rho of default option, (b) Rho of prepayment option

Figure 5.4 plots the MC simulations of the vega of both options against the simulation number. In Figure 5.4 (a)it is shown that the vega of the default option is about -0.19. In Figure 5.4 (b), it is depicted that the vega of the prepayment option is around

-0.63 . For these computations, the hedging coefficients may be obtained as well with a small number of simulations. Therefore, the computation cost may be reduced sufficiently.



(a)



(b)

Figure 5.4: Simulations of the options vega coefficients: (a) Vega of default option, (b) Vega of prepayment option

It is essential to note that this chapter aims to recognize that the Malliavin calculus, yielding the hedging equations in the previous section, produces the Malliavin weights. These weights involve powers of Brownian motions. Such "global" weights can increase the cost and decelerate our computations. However, such a problem might be resolved by localization of the corresponding integration-by-parts around the singularity points [77]. However, it is out of the range of the thesis. Additionally, this thesis only intends to compute the hedging parameters for the default and prepayment options. Even though the current forms are questionable, the computation cost of our illustrations is relatively quick.

To summarize, we may emphasize the results deducted for hedging parameters that we compute above as follows:

1. Hedging with Delta: By relying on the delta of the options, to decrease the risk related to house price investors should have the following strategies:
   Case 1 (For the mortgage default option): Consider a house whose current price is $\$100$. Also, suppose the default option on this house has a price of $\$10$, and its delta is -0.23. Then, a financial institution sells out ten options to its customers so that the customers have a right to buy 1000 shares at maturity. To compose a delta neutral position, the investor must short sell $0.23 \times 1000 = 230$ shares of the underlying house. If the price of the house increases $\$1$, the mortgage default option decreases by $\$0.23$. Hence, the financial institution has a $\$230$ $(\$1 \times 230)$ loss from the house and a $\$230$ $(\$0.23 \times 1000)$ gain from the default option.
   Case 2 (For the mortgage prepayment option): Suppose that there is a house in the market with a current market value value $\$100$, there is a prepayment option whose current price is $\$10$, and the prepayment option has a delta of 0.84. A financial institution sells out ten options to its customers so that the customers have the right to purchase 1000 shares at maturity. To compose a delta neutral position, the institution must purchase $0.84 \times 1000 = 840$ shares of the underlying house. If the house price rises $\$1$, the mortgage prepayment option rises $\$0.84$. In such a case, the institution has a $\$840$ $(\$1 \times 840)$ profit from the house and a $\$840$ $(\$0.84 \times 1000)$ waste from the option.
   Notice that in both cases, the total loss of the institution is zero.

2. Hedging with Rho: Using the rho, one may reduce the risk related to the spot rate. He can use the following neutral portfolio strategies to reduce its risk.
   Consider an investor having a delta neutral-portfolio. The portfolio includes both default and prepayment options along with the underlying house. Its' portfolio's rho is 3. Its' mortgage default option's delta is -0.23, and rho is -0.58. The mortgage prepayment option's delta is 0.84, and rho is 2. It may compose a replicating portfolio, which is both delta and rho neutral, just by

including ≈ -92.65 of the mortgage default and ≈ -25.37 of the mortgage prepayment options.

3. Hedging with vega: The vega can suggest the following strategies to reduce the risk associated with house price volatility by assembling a delta and vega neutral portfolio.
   Think of a delta-neutral portfolio that includes both the mortgage default, prepayment option, and also the house that the options are written. Suppose that the vega of this portfolio is given as -2. Furthermore, suppose that the mortgage default option's delta is -0.23, and its vega is given as -0.19. Suppose that the mortgage prepayment option's delta is given as 0.84, and its vega is given as -0.63. To assemble a replicating portfolio that is both delta and vega neutral at the same time, the investor must include $w_D \cong -5.52$ of the default and $w_p \cong -1.51$ of prepayment option into its main portfolio.

By considering the investment strategies that we introduce as an example above, financial institution and individual investors may construct portfolios, which satisfies delta, delta&rho, and delta&vega neutral investment positions and benefit these hedging strategies. Based on these investment strategies, hedgers obliged to short or long a tiny proportion of the underlying house. However, due to their nature, houses are unified, and they are indivisible. As a result, we cannot trade them in small units. As a consequence, hedgers have to use alternative investment instruments such as CPI or contingent claims tied to CPI, REITs shares traded in exchange markets. Or, they can invest in $TBill$ to replicate the options against the spot rate fluctuations. However, the mortgage default and prepayment options are both attached to the mortgage and, thus, they are not traded independently in markets. This point resides that the hedgers must adopt alternative investment instruments such as options written on REITs and options attached to HPI, which is administered by the Chicago Mercantile Exchange (CME).

A hedging activity is a full or partial reduction of an asset's price risk by using contracts, which offsets the risk. The existence of market participants who are willing to construct a hedging portfolio is one of the fundamental conditions to the establishment and success of derivative markets. In the absence of this will, it is difficult to close contracts between counter-parties, since the demand and supply of risk and return are likely to be out of balance. Thus, in this chapter, we derive Malliavin calculus-based expressions for the hedging coefficients of the mortgage default and prepayment options. The specialty of this method is that the illustrations that developed are a product of option's payoff and an independent weight, which is called the Malliavin weight. In this respect, it permits the user to incorporate formulas resulting from the Malliavin calculus and to derive estimations for the hedging coefficients by running an MC algorithm.

These expressions to the mortgage default and prepayment options are of particular interest since they have no closed-form solutions for these options. The performance of the illustrations has been illustrated, and numerical results have been presented using a crude MC algorithm. The numerical results reveal that both default and prepayment options are more sensitive to a change in the underlying house price volatility than a change in the underlying house price and interest rate. There are many potential usages of these computations in the genuine markets. Although market participants face risks related to increases in spot rate, prepayment, default, and foreclosure in housing markets, they are not well informed about the use of hedging strategies to reduce the risk. Thus, the housing market participants would benefit from the ability to hedge in such markets by using the outcomes of this chapter. Furthermore, the use of such results would be likely to evolve as insurance and as development in consumer products of financial service companies to take advantage of the housing markets.

CHAPTER 6

DIVERSIFICATION BENEFIT AND RETURN PERFORMANCE OF REITS: AN EMERGING MARKET CASE

This chapter reviews returns of Real Estate Investment Trusts (REITs) and diversification advantage of these companies. REITs have been the subject of extensive interest in the field of real estate finance from both practitioner and an academic point of view. The primary concern of them is about consistency about REITs share returns with core holdings of the REITs and trend in overall exchange markets. Consequently, as empirical analysis, this chapter reviews the return performance of REITs operating in BIST, which are denoted as T-REITs, of several viewpoints for the specific interval July 2008-March 2015.

The relationship between REIT's share returns and core properties of these companies portfolios is essential for both investors and academicians, considering allocating funds to real estate. Investors are attracted to real estate as an investment tool due to their potential diversification benefits of the asset class is believed to offer [18]. Besides, specific benefits of REITs, direct investment in real estate, for instance, purchasing a property, maybe infeasible for many reasons. High transaction costs, liquidity, lot size, inability to construct a sufficiently diversified portfolio of properties, uncertainty about prices and valuations, and limited experience in managing real estate assets are some of the main reasons. Even investing in common investment tools, such as open-ended funds, may not completely solve these obstacles to invest in the real estate market straight. However, REITs offer a form of investment strategy to real estate markets, especially in the case of traded REITs' shares in stock exchange markets, which may overcome many of these obstacles.

The methodology within this chapter relies on both the celebrated CAPM in the derivatives market and its extension Fama-French three-factor model with some expansion of Fama-French to observe political and currency risk. The theoretical background of these methods generally investigated, and thus so many studies exist in the literature. Consequently, these methods are both well-established approaches. Hence, in this chapter, we avoid giving detailed explanations of both methods not to bore potential readers.

As it is well known, the CAPM is a model that relies on a single factor, which interprets the expected return of assets traded in stock exchange markets relative to the market risk. On the other hand, the Fama-French three-factor model expands the celebrated CAPM by distinguishing risk sources by attaching size and some value variables to the model. Along with the traditional FAMA-French factors, further influential variables might be included to estimate variations in returns of assets shares. Two of the most significant reliable risk sources are the currency risk that has a great influence on prices of asset [4, 138], and the risk associated with political activities that are essentially started by numerous circumstances like elections and developments in the market regulations [1, 40, 93, 196]. As a consequence, these two risk sources included in the Fama-French model as an additional risk source that represents an innovation within the current chapter. Since real estate markets highly depend on financial crises and political regulations, in addition to these two risk sources, we attach the influence of the global crisis and election periods into the Fama-French three-factor model to include the political and global risk sources.

The main goals of the current chapter are to investigate the profitability and return variability of REITs from many aspects. To provide background and further control of the sample size, we narrow our analysis throughout July 2008-March 2015 that spans real estate market recovery in Turkey. Investigations within the current episode give birth to four new research fields in the real estate finance literature. First, as one of our essential investigation purposes, the diversification advantage of T-REIT shares is analyzed by rivaling banks' shares and trust companies' shares in Borsa Istanbul Stock Exchange (BIST). Second, the primary objective of this chapter includes the investigation of REITs returns variability by using both the celebrated models, followed by a measurement of corresponding models to demonstrate their relevant performance. Third, the study endeavors to extend the Fama-French three-factor model by attaching additional risk sources, namely, some foreign exchange rates, the general and local elections, and the global crisis, in particular, significant risk components in emerging economies. Therefore, this chapter presents additional risk sources exceeding the standard three-factor Fama-French three-factor model to mirror the consequences of both local and international risk sources. Final and fourth, this chapter classifies the specialty (namely, property focus), management structure (defensive or aggressive), size (big or small), and financial states (distressed or not distressed) of T-REITs. Hence, this thesis is a pioneering study in this respect.

Furthermore, this chapter also examines the association between the specialties and risk yielding enhancement willingness of T-REITs. In that respect, it allows potential readers a better understanding of T-REITs from various innovative aspects. Moreover, it affords a meaningful contribution to asset allocation decisions for investors who are operating in BIST and willing to invest in T-REITs' shares. The implications of Turkey's real estate market experience extend beyond the local market analysis frame. They provide REITs analysis with globally practical approaches since the experimental study on investigated topics is still in a nutshell, particularly in emerging economies. In this sense, our review also has international dimensions related to the vast amount of international investments in Turkey.

We organize the chapter as follows. In Section 6.1 we summarize REITs. We document the stylized realities on the REITs in the chosen emerging economy, and especially Turkey's market, contribute further examination for the association within submarkets of real estate and REITs in Section 6.2. Section 6.3 critiques the literature. Section 6.4 gives a preliminary of the well-known models that we employed. The utilization of our recommended study on T-REITs is illustrated in Section 6.5. Appendix A also displays corresponding benefits and crucial features of REITs regulations in Turkey's economy.

6.1 Real Estate Investment Trusts

REITs are companies operating on a closed form-end fund that consists of real properties and mortgage-related assets or both. These companies are organized by the US Congress in 1960 for the first time as an investment vehicle to grant investors a potential opportunity to invest in real assets and benefit from their activities. Since sophisticated investors consider the resulting investment environment for traditional investment methods as low return, many of them turned to alternative investment vehicles as a way of satisfying their return expectations and probably to a lesser extent as a risk controlling tool. Indeed, alternative investment opportunities provide an opportunity to earn an acceptable return with manageable risk.

As an alternative investment vehicle, REITs were unpopular among investors until the end of the 1960s. During the period 1968-1970, the number of REITs increased from approximately 61 to 161 in the US. After this point, these companies succeeded a widespread acceptance from investors in US stock exchange markets and spread internationally. Further, they start operating in developed countries such as Japan, Australia and have increasingly been presented in the Europe region. The extension in the number of REITs and their total assets (parallel to the increase in their number) is a result of several factors. For instance, to make REITs shares more charming to investors, the US Congress abandoned the corporate-level income tax on REITs if these companies satisfy conditions offered by tax laws. For example, the REITs obligated to issue most of their earnings as dividends to their shareholders [41].

Before the establishment of REITs, the only investment opportunity of investors into real properties is purchasing real properties. However, with the establishment of REITs, investors are allowed to trade real properties in exchange markets since REITs are in the form of a corporation, and their shares are traded in stock exchange markets. As a consequence, investors who are willing to invest in real assets or property-based investment tools can purchase REITs shares from exchange markets with the intention of purchasing real properties. From this point of view, REITs give an investment opportunity to small investors in real assets and benefit from exchange markets return opportunities at the same time. REITs offer to participate in various markets and use investment strategies, which are unavailable to the general investing for the public.

In the last two decades, there have been produced some crucial changes in the regularity law for REITs. Such as, changes in regularity have allowed REITs to manage their properties and provide benefits from related services of properties in their asset portfolio. From this viewpoint, some of the improved REITs might also be viewed as operating companies. However, even these modern REITs, real properties are still the most significant component of their asset portfolios. Investors who are prepared to invest in REITs shares should understand both real estate markets and exchange market dynamics to compose a successful investment since REITs viewed as pools of properties that traded in exchange markets. Thus, investors and portfolio managers who possess real estate and exchange markets have a substantial advantage over investors who do not have any perception of these markets.

6.2 REITs Performances in Emerging Markets: Turkey Case

6.2.1 REIT Indices Relies on Emerging Markets and T-REITs

The weight of REITs in global finance is demonstrated by utilizing global REIT indices. For instance, MSCI Emerging Markets IMI Core REIT Index includes large, mid, and small-cap shares. In early 2016 and the following periods, the index incorporates 28 REITs with a total market cap of 28 billion dollars. Developed countries concerning their weights in the index are South Africa (52%), Mexico (33%), Malaysia (8.3%), China (3.8%) and Turkey (1.8%). Based on the industry classification, central REIT classes in the index of MSCI are included as diversified (62.2%), retail (24.54%), industrial (9.95%), hotel&resort (2.14%) and office (1.17%) (MSCI, 2016).

While analyzing FTSE EPRA/NAREIT Emerging Index, starting from 2015, we observe that the number of driving REITs and the net market cap of them are 153, 165.5 billion dollars, respectively. FTSE EPRA/NAREIT Emerging Index, published in 2016 in March, provides a variety of interesting facts to readers. For instance, REITs in emerging countries have recorded a downswing, and REITs number along with their net market capitalization have decreased to 149 and 142 billion dollars, respectively. In this index, Chinas market seems to be the leading market, with 54 REITs where they have a 65.7 billion dollar net market capitalization, and China's weight in the index is 46.3%. South Africa, the Philippines, and Mexico follow China with the weights $11.3\%, 8.1\%$, and 6.6%, respectively.

In the FTSE EPRA/NAREIT Emerging Index, Turkey weight is 1.7%, along with with with a 2.4 billion dollars net market cap in March 2016 (Table 6.1). The real estate market collapse illustrated the diversified REITs have an 83.9 billion dollar net market capitalization along with the industry weight of 59.1%. The residential and retail REITs listed as the second and third highest REIT types in this index, with 35 billion dollars (24.7%), and 13.7 billion dollars (9.7%) net market capitalizations, respectively (FTSE, 2016).

Table 6.1: Overview of FTSE EPRA/NAREIT emerging index (May 2015 – March 2016)

Country	Number of REITs		Net market Cap (USD mn)		Weight (%)	
	May 2015	Mar. 2016	May 2015	Mar. 2016	May 2015	Mar. 2016
Brazil	20	17	8254	7183	4.99	5.07
Chile	1	1	998	1029	0.60	0.73
China	50	54	75,219	65,715	45.44	46.34
Egypt	1	2	405	511	0.24	0.36
Greece	-	1	-	326	-	0.23
India	5	5	2262	1929	1.37	1.36
Indonesia	12	11	8625	7727	5.21	5.45
Malaysia	14	11	6249	6083	3.77	4.29
Mexico	6	6	9519	9332	5.75	6.58
Philippines	6	7	11,908	11,505	7.19	8.11
Poland	1	-	390	-	0.24	-
Russia	1	1	2511	1953	1.52	1.38
South Africa	12	11	18,461	16,028	11.15	11.30
Taiwan	1	1	177	179	0.11	0.13
Thailand	14	14	5990	5788	3.62	4.08
Turkey	4	4	2564	2372	1.55	1.67
UAE	5	3	12,012	4140	7.26	2.92
Total	153	149	165,544	14,1801	100.00	100.00

The MSCI Emerging Markets IMI Core REIT Index presents 8.59 percent total annualized return through the period of November 1994 - March 2016 (MSCI, 2016). The yearly achievement of the FTSE indexes implies that even though the FTSE EPRA/NAREIT Emerging Index does not present a more desirable total return, it might exhibit substantial return volatility comparable to other market indexes throughout 2006-2017 period (Table 6.2).

Table 6.2: Yearly performance (overall return %) of the selected FTSE indexes

Index % (USD)	2006	2007	2008	2009	2010	2011	2012	2013	2014	2015	2016	2017
FTSE EPRA-NAREIT Emerging	63.7	42.9	-63.5	91.2	25.2	-29.2	42.4	-14	5.2	-4.8	1.1	52.4
FTSE Emerging	33.1	39.7	-52.9	82.6	19.8	-19.0	17.9	-3.5	1.6	-15.2	13.5	32.5
FTSE EPRA-NAREIT Developed	42.4	-7.0	-47.7	38.3	20.4	-5.8	28.7	4.4	15.9	0.1	5.0	11.4

The facts that we highlight above successfully shows the importance of REITs operating in emerging markets and economies, including Turkey, on the international economy, and the potential to afford diversification advantages of these companies to investors for their portfolio management.

6.2.2 Turkey's Real Estate Economy and T-REITs

The real estate market, along with its related markets, such as the construction market, has been evolving with major developments in Turkey's economy. These developments have played a significant role in changing real estate demand and supply dynamics in the real estate market of Turkey. Hence, the real estate market has grown significantly in Turkey. The market has displayed a remarkable accomplishment in the last three decades. In correspondence to the development in demand for property and high-quality office and retail properties, the lately proposed mortgage financing scheme and the falling spot rate trend have been the primary incentives for the remarkable recovery of the real estate market of Turkey (EPRA, 2014).

Across the last three decades, international agents have given accelerating relevance in Turkey's market of real estate. Their attention produces a total worth of M&A transaction capacity increase that arrived at a level of 17.5 billion dollars, along with 217 transactions (Deloitte, 2014). The modern hotel and office market, as a real estate sub-market perspective, are quite young in İstanbul [125]. Especially, investments in mall projects come to the stage after 2005 due to high global liquidity, residential, office, and commercial real estate have made solidly in the following years of 2008 in Turkey (Deloitte, 2014).

Following developments in the real estate market of Turkey, rental income and property prices of both retail and residential real estate have proceeded to increase across the two last decades. An analysis of Turkey's real estate market illustrates notable development in recent times. Still, the opportunity is unclear for the second half of 2015 concerning the demand, supply, and rental income, and density of retail(JLL, 2015). Moreover, the office market of the A-Class vacancy rate in İstanbul has increased 25.88%, and rental incomes declined in some local markets, (Colliers International, 2015). Analyses on Retail and residential markets also show the power of both malls and housing in the economy of Turkey, notwithstanding some significant risks [194]. These significant real estate market developments are the fundamental reasons for the latest market balloon in T-REITs that are legally established in 1995 and started to trade in BIST in 1997. Since then, T-REITs have become essential investment vehicles for both local and foreign investors, and thus, these companies are encouraged to grow by favorable regulations.

Although there are an increasing political and economic risks in Turkey, T-REITs have shown significant development in the last three decades, by the effect of the favorable economic conditions in the real estate market, and also the similar benefits of investors on T-REITs (observe Appendix A). A structural change has occurred with legal regulations and improvements in Turkey's real estate market. More importantly, the real estate market of Turkey has managed to adapt itself by increasing specialization of activity areas from the general-purpose mall and office buildings to special-purpose flats.

The evolution of T-REITs as a subcategory within Turkey's real estate market has arisen primarily as a result of the changing process of the needs of the community in the last three decades. Therefore, Figure 6.1 shows that both T-REITs total asset value and the market cap have expanded concerning dollar by 518 percent and 281 percent, respectively. More importantly, the number of T-REITs has risen sharply 14 to 31 between the years 2008-2015 (CMB, 2015). Here, it is worth to emphasize that Figure 6.1 also shows that expanding the number of T-REITs through the recent Initial Public Offering (IPO) boom produced favorable benefactions to the industry performance measures. The growing importance of assets in Turkey's real estate market might also result in the increasing amount of investments in Turkey's real estate market and T-REITs shares at BIST. From the worldwide investment standpoint, the

T-REITs market has a weight 1.7% in the FTSE EPRA/NAREIT Emerging Index and weights 1.8% in MSCI Emerging Markets IMI Core REIT Index, in 2016 (FTSE, 2016; MSCI, 2016). In this respect, the significance of T-REITs role in international investment opportunities is increasing, which may increase the number of foreign investors in BIST. It is important to note that Figure 6.1 illustrates that even though there is a significant increase in the number of T-REITs and total assets, the market cap is not increasing. The reason behind this result might be due to the management structure of these companies. This result also can be interpreted as these companies are not operating well. Hence, their management operating methods should be improved.

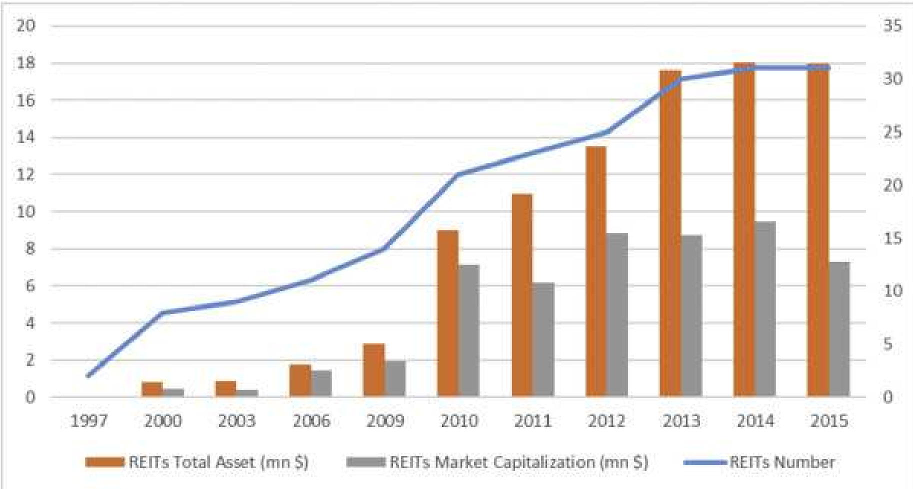


Figure 6.1: T-REITs industry facts for the period 1997-2015

6.3 A Literature Survey on REITs

The increasing popularity of real estate markets and REITs in the last three decades has canalized many researchers to analyze the behavior of REITs return performance from various angles, with a diversified data composition. Thus, there exist many studies in the real estate finance and economics literature, which examine characteristics and components of real estate markets and REITs shares. Most of these studies assume to have a similar character to real estate markets, and they are commonly inspected as a portfolio diversifier or a tool to reduce portfolio risk. More importantly, some of these studies consider REITs shares as a hedging vehicle against losses related to inflation.

A large amount of study in the literature regards the advantages of REITs in the portfolio management context within the asset allocation, risk minimization, and risk diversification features [42, 103]. For instance, Chun et al. (2004, [51]) illustrate the advantage of the property diversification benefit compared to riskless and small-cap assets in periods of weak expenditure extension possibilities. In an investigation of diversification advantages utilizing the classical mean-variance tests, Chen et al. (2005, [46]) prove that REITs undoubtedly expand the mean-variance frontier and expand the investment opportunity set. Furthermore, Chen et al. (2005, [46]) also confirm the economic consequence of investments in REITs from the asset allocation attitude. The diversification and risk-reducing advantages of REITs are proved to be vital, mainly, for particular classes of shares, like mixed and mortgage class REITs, but not for the equity REITs [80, 102, 137].

Chaudhry et al. (2010, [44]) show two co-integrating vectors within equity REITs and energy-related investment instruments. From an international investors' viewpoint, there is a more comprehensive aim for risk diversification in struggling economies compared to developed economies, as the latter is previously thoroughly combined into the international money markets. Notwithstanding its high potential, the risk diversification advantages of REITs in emerging economies have not recognized the interest they deserve and rarely studied in emerging markets [154, 158].

Concerning specialization of REITs and their performance measured by risk and return, Capozza and Seguin (1999, [35]) show diversification over real estate classes, such as office, warehouse, retail or residence, unfavorably influences REITs market value. Capozza and Seguin (1999, [35]) also detect no indication of fluctuations in cash flows accessible to stockholders. Similar to Capozza and Seguin (1999, [35]), Benefield et al. (2009, [23]) also prove diversified REITs considerably better compared to specialized REITs throughout the years1995-2000. In similar research to Benefield et al. (2009, [23]), for the period 1997-2006 Ro and Ziobrowski (2011, [169]) compared the performance of the specialized REIT versus the diversified REIT shares utilizing the CAPM and the Fama- French three-factor model. They ended up with no proof of better performance connected with REITs specializing in a single property class. Consistent by the existing studies, Ro and Ziobrowski (2011, [169]) attain a more significant market risk on the specialized REITs than the diversified REITs.

The return performance of REITs, precisely through the CAPM and Fama-French three-factor model, have widely investigated in the real estate finance and economics literature. For instance, Karolyi and Senders (1998, [113]) show REITs returns contain an essential economic risk premium where the regular multiple-beta asset pricing models neglect to identify. Chiang et al. (2004, [47]) notify investors about the observation might deceive interested investors who use the CAPM to evaluate the risk of REITs and sensitivities of variable that are mostly symmetric in the Fama-French three-factor model case. Parallel to [47], Chiang et al. (2008, [48]) emphasize the limited scope of both models to represent the performance of REITs.

Despite their apparent disadvantages in the various researches, the Fama-French three-factor and CAPM models have continued to apply in analyzes of the risk and return performance of REITs' shares. In this respect, the application of the Fama-French three-factor model on returns of REITs initially allowed researchers to verify the robustness of the current results and the practical benefit of the methodology [98, 130]. By examining REITs' shares risk and return performance utilizing the Fama-French five-factor model [72], Peterson and Hsieh (1997, [164]) show that mortgage REITs' risk premiums are considerably connected to three exchange and two bond markets factors. Chiang et al. (2005, [49]) also show that the Fama-Frenc three-factor model is preferred to the CAPM in illustrating the return variety of the Equity Real Estate Investment Trust (EREIT) shares, and in presenting steady estimations of market betas.

Xiao et al. (2012, [193]) use the traditional CAPM, and multi-factor models developed from the four-factor model in [52], as a frame to highlight the association between REITs with other asset categories. The authors demonstrate the return of REITs displays the highest sensitivity to market return, supported by the large-and small-cap stock, bond, and real estate indexes. Presumably, the effect of size is one of the inferences on portfolios that consist of REITs' shares performing better than the portfolios consists of popular stock shares. Consequently, the investigations using both CAPM and Fama-French model have generally concentrated on REITs in developed economies, which recommends a gap in the literature for the examination of emerging economies.

Studies that are emphasizing the management structure of REITs as in the form of defensive or aggressive management are also rare in both advanced and emerging economies. Sing et al. (2016, [179]) estimate the time-varying US REITs betas over the period 1972-2013. In this study, they illustrate that a primary difference observed in time-varying beta aspects of US REITs in the 2000s. Moreover, within the same study, they discover that while mortgage REIT betas proceeded to decay, equity REIT betas reveal a dramatic reversal of the descending trend.

Glascock et al. (2004, [86]) remark REITs generally considered as low risk and return companies, which present defensive management features. But, by using a dynamic conditional correlations bi-variate threshold GARCH model, Wu et al. (2010, [151]) observe that more than seven registered REITs have defensive management characteristics in Taiwan. In the study of Chiang et al. (2013, [50]), it is suggested that currently, REITs have less defensive management structure than stable market periods. Therefore, REITs can not be proper security for the financial chaos in Taiwan, Hong Kong, Singapore, and Japan. However, contrary to Chiang et al. (2013, [50]), Newell and Osmadi (2009, [156]) discover conflicting results reported as the İslamic REITs generally demonstrate defensive features in Malaysia. In another study, Wu et al. (2012, [152]) notice ten property-type REITs in the US have a defensive management structure in the period 2007-2010.

There is a variety of CAPM and Fama-French model application on BIST. Within these studies, we can find proof for the significance of factors producing by company size and the book to equity market ratio. For instance, Eraslan (2013, [69]) states that the Fama-French three-factor model represents the variety in general stock returns, and reveals that yields are favorably influenced by BE/ME and adversely correlated to the size of the company. Gokgoz (2007, [87]) also indicates that CAPM and Fama French three-factor models are relevant to BIST. Eraslan (2013, [69]) and Gokgoz (2007, [87]) also confirm that the Fama French three-factor model has a superior performance according to pricing error measures. As an application, Bereket (2014, [24]) examines CAPM, Fama-French three-factor, and four-factor models. He also shows the supremacy of the Fama-French three-factor model to remaining models. However, on the contrary to Eraslan (2013, [69]) and Gokgoz (2007, [87]), Dalgin et al. (2012, [58]) find CAPM neglects to estimate adequately the excess returns of assets traded in BIST.

Studies investigating risk and return features of T-REITs remain comparatively limited in the real estate finance and economics literature. In this limited literature, Erol and Tirtiroglu (2008, [71]), attain that T-REITs present a more reliable hedge versus both real and predicted inflation compared to BIST ordinary asset indices. On the other hand, Altinsoy et al. (2010, [8]) notice no sign of the asymmetric time-varying behavior for the betas of T-REITs. In the second study of Erol and Ileri (2013, [70]), they examine economic sources of time-varying risk premia for T-REITs returns. Mandaci et al. (2014, [142]) prove the absence of co-movement between T-REITs and US REITs indices for the period 2003-2009. Akinsomi et al. (2016, [5]) find some herding responses, the appearance of directional asymmetry, and the linear relation among volatility and returns in T-REITs between the years 2007-2016.

Aktan and Ozturk (2009, [6]) illustrate CAPM fails to reflect the market information successfully. On the other hand, they prove that the Fama-French three-factor model demonstrates that a small caps ratio indicating the company size has a more notable influence on returns than the standard price-to-book ratio measure. Lu et al. (2013, [137]) also discover that emerging economies, such as economies of South Korea and Turkey, displayed a more prominent downside risk t international REITs under the conventional market conditions.

The research survey in this section highlights a common absence of examination on time and market-dependent features of REITs in emerging economies. Therefore, there exist apparent literature gaps in areas of emerging REITs and return fluctuations of REITs by matching and developing CAPM and Fama-French three-factor models, diversification advantages, management structure, property focus and return enhancement of risk-yielding relation. The chapter tries to charge such gaps for T-REITs in particular.

6.4 Methodological Background

The preliminary analysis considers T-REITs returns and evaluates their performance by using CAPM and Fama-French model and their persistence over time. Therefore, this section dedicated to these models.

6.4.1 Capital Asset Pricing Model

The celebrated Capital Asset Pricing Model (CAPM) emphasizes that only the non-diversifiable risk is significant. Besides, the idiosyncratic risk, which is also referred to as unsystematic risk is irrelevant. It is because, according to modern portfolio theory, the opposite of the systematic risk, the idiosyncratic risk may be mitigated through diversification in an investment portfolio [159].

In an ideal exchange market, risky assets indicate that all tradeable assets accessible to the market agents. Moreover, there exists a riskles asset with a rate of r_f, which serves for lending and borrowing objects in infinite quantities. In this chapter, we assume that all knowledge is open to all market agents. Then, we may compute the systematic and unsystematic risk of assets by using CAPM [176].

Now, consider a market which contains a set of returns, $r_i, i = 1, 2 \ldots, n$ of a risky asset and a risk-free asset, r_f. Then, the CAPM is given as in the given formula

$$R_i = \alpha_i + \beta_i R_M + \epsilon_i, \tag{6.1}$$

where $R_i = (r_i - r_f)$ represents the excess return of the asset, $R_M = (r_M - r_f)$ denotes the excess return of the market under investigation, α_i is the non-market return component, β_i is the beta of the risky asset, and the parameter ϵ_i is a random error term that comes from the normal distribution family having zero mean and constant variance.

It is worth to note that, in the no abnormal return case or an intercept no intercept, $\alpha_i = 0$, this model matches with the classical CAPM.

Indistingushably, the excess return of asset portfolios, which is denoted by $R_P = (r_P - r_f)$, under the assumption of equally-weighted assets, CAPM model return to the form given in Equation (6.1),

$$R_P = \alpha_P + \beta_P R_M + \epsilon_P. \tag{6.2}$$

In Equation (6.2), components corresponds to $\alpha_P = \frac{1}{n}\sum_{i=1}^{n}\alpha_i$, $\beta_p = \frac{1}{n}\sum_{i=1}^{n}\beta_i$, and $\epsilon_P = \frac{1}{n}\sum_{i=1}^{n}\epsilon_i$. According to Equation (6.2), the portfolio has a sensitivity to the return on the market given by the mean value of assets β_is including the mean of the components related to company operations. Thus, variance of the portfolio is equal to

$$\sigma_P^2 = \beta_P^2\sigma_M^2 + \sigma^2(\epsilon_P). \tag{6.3}$$

The first term, $\beta_P^2\sigma_M^2$, in Equation (6.3) denotes the non-diversifiable risk ingredient of the variance of portfolio. The variance relies on the sensitivity parameters of assets in the portfolio. The second term, $\sigma^2(\epsilon_P)$, represents the non-systematic risk ingredient of the variance of the portfolio. It is considered as the risk of firm operations. Here, ϵ_i's are independent. Thus, the non-systematic risk can be decomposed as the form

$$\sigma^2(\epsilon_P) = \sum_{i=1}^{n}(\frac{1}{n})^2\sigma^2(\epsilon_i) = \frac{1}{n}\bar{\sigma}^2(\epsilon), \tag{6.4}$$

where the component $\bar{\sigma}^2(\epsilon)$ denotes the average firm-specific variances.

It is clear from Equation (6.4) that the firm-specific variances depend on the number of assets in the main portfolio. Therefore, the portfolio variance can be ignored for large values of n. It means as more assets included in the portfolio, the portfolio risk attributable to firm-specific actions gets progressively smaller, and the risk occurring related to the firm operations is theoretically diversified. But, the systematic risk resides within the portfolio, regardless of the number of assets included in the main portfolio. Besides, the risk of assets specified by companies management structure as a part of the CAPM. In this point of view, an asset has an aggressive management structure if $|\beta| > 1$ [31, 187].

6.4.2 The Fama-French Three Factor Model

In the CAPM, calculations are oversimplified by the hypothesis that only a single systematic source influences the return of assets. However, it ignores the other determinants that are changing asset returns, such as the influence of business cycles, fluctuations in the spot rate, inflation, exchange rate, and global oil prices. The vulnerability to such determinants is expected to influence asset' risk and return significantly. To highlight this statement, Fama and French (1996, [73]) propose a multi-factor asset-pricing modeling approach, which consolidates the influence of supplementary risk sources to illustrate the behavior of asset return.

Fama and French (1996, [73]) added two classes, which are reflecting market behavior that relies on order statistics, to the standard CAPM to discover the influence of Small Caps (SC) and Price-to-Book ratio (P/B) on the performance of the main portfolio. In the Fama-French three-factor modeling, the impact of economic indicators is defined by relying on the company features. Hence, such a modeling approach appears to be a proxy for exposure to non-diversifiable risk when the empirical grounds on.

The Fama-French model factors consider the size of portfolios as Big (B), Small (S), and three quartiles as Low (L), Medium (M), and High (H) within a prespecified time. SC is the difference among the average returns of three small and three big portfolios, whereas P/B attains the difference among the average returns of two value portfolios and two growth portfolios, which are illustrated as Small Minus Big (SMB) and High Minus Low (HML). SMB is the quantity by which the return of a small stocks portfolio is in excess return on a large stocks portfolio, and HML is the quantity that returns of a portfolio with a high book-to-market ratio that is more than the return on a portfolio with a low book-to-market ratio.

Therefore, describing the return, R_{it}, of i^{th} asset at time t with the Fama-French three-factor model is expressed as

$$R_{it} = \alpha_i + \beta_{i,M} R_{Mt} + \beta_{i,SMB} SMB_t + \beta_{i,HML} HML_t + \epsilon_{it}.$$

Comparable to β coefficients of the CAPM, the model consolidates two supplementary β coefficients demonstrating the size (market capitalization) and financially distressed status of the assets in the market and $\epsilon_{it} \sim \mathcal{N}(0, \sigma)$, where σ is constant. In this model, if $\beta_{i,SMB}$, and $\beta_{i,HML}$ are interpreted according to their magnitude and sign, the companies are referred as Small (S) if $\beta_{i,SMB} > 0$ and financially Distressed (D) if $\beta_{i,HML} > 0$, respectively [31].

6.4.3 Some Additional Risk Factors

The CAPM and Fama-French model has the insufficient capability to analyze asset returns [48]. In this regard, regarding the potential economic and industry circumstances to obtain positive or negative influences on REIT returns, extending the Fama-French three-factor model can increase the explanatory skill of the Fama-French three-factor. Hence, we apply three unusual supplementary risk factors; political risk, currency risk, and global crises. Considering the political risk, the most effective measure is the election period if the economy generally passes into the standby state. Consequently, first, we embedded dummy variables into the model that represent both pre and post-election periods. Second, by considering the foreign exchange rate, we may have devastating outcomes on fluctuations of assets return, as perceived recently in countries like Brazil, Russia, Turkey, and South Africa. Also, we engage in exchange rate (USD) risk as an additional risk source. Lastly, we further performed the influence of the global financial crises. There are studies that examine the influence of such risk sources on assets [1, 25, 40, 93, 112]. However, the analysis in this chapter is original research to perform these three risk sources into the Fama-French model to measure variations of REITs' returns.

During the investigation period, 2008-2015, we observe six elections in Turkey. Namely, we faced three general, two local, and one presidential election. These elections caused various impacts on the economy and Turkey's real estate market. Thus, to discover how the return of T-REITs reacts to the political risk, the dates of elections are incorporated into the Fama-French model. In the following equation, D_{1t} identifies the influence of a specific month where the election takes place. In contrast, D_{2t} identifies the effect of 5 months pre and post-election months.

$$D_{1t} = \begin{cases} 1, & \text{election month} \\ 0, & \text{otherwise} \end{cases}$$

$$D_{2t} = \begin{cases} 1, & \text{election month } \pm 5 \\ 0, & \text{otherwise.} \end{cases}$$

Notable variations in foreign exchange rates, mainly in EURO and USD exchange rates, are observed in Turkey's economy during our investigation period. To demonstrate their impact on the market, we add the USD exchange rate into the model as an additional risk source. In our analysis, the structural break test that is measuring the influence of the global crisis is used to identify whether adding it to the model is required or not.

6.5 Empirical Analysis

6.5.1 Data, Property Focus and Summary Statistics of T-REITs

Our primary preference measures for selecting T-REITs are the accessibility of data, and their market share reached over the highest attainable period. As Figure 6.2 illustrates, the chosen 11 T-REITs amongst 17, in Jully 2008, formed 79% of T-REITs market value. The price data obtain from BIST and Google Finance, and we utilize both CAPM and Fama-French three-factor model to explain diversification advantages of T-REITs starting from July 2008 and ending March 2015. We detect their excess returns by subtracting the returns of the portfolio performance repo index (DSM) as the risk-free rate from the market returns.

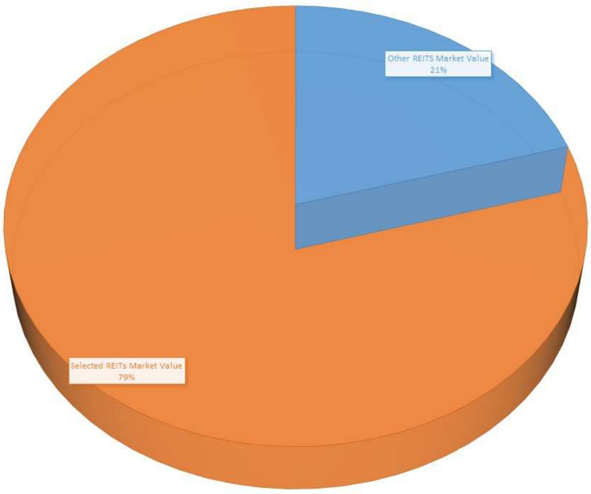


Figure 6.2: The chosen T-REITs dominance in BIST (2008 July)

This chapter makes a significant contribution as rendering the first complete classification effort of T-REITs based on their focus on their property portfolio. In this sense, the thesis allows for more nuanced evidence about T-REITs. In this regard, we animate that T-REITs display equity REIT properties since the overall REIT industry has not historically associated with mortgage commitments. Second, in terms of substitutability, the T-REITs examined in this chapter are listed as publicly-traded organizations in BIST. Third, as an innovative classification effort, we additionally classify T-REIT types based on their property weight in their portfolio. Mainly, we concentrated on a governing weight of properties in T-REITs balance sheets by employing portfolio weights displayed in their audit and financial reports. Also, we use corporate expert views in some cases. To describe specializations, we use various classification procedures as in the paper of Clayton and MacKinnon (2003, [52]).

In the classification decisions, we fundamentally utilize the appraised values of T-REITs' property inventories and ongoing projects on real estate presented in their audit reports to determine the main real asset portfolio during the period 2008-2014. In this period, we observe that the specified properties that T-REITs focused on display no notable variation. Our investigation implies that T-REITs display a level of diversity in their focus on the property. They show four residential, six diversified, six retails, and one specialty REITs characteristics. Furthermore, the retail and residential T-REITs mirror the principal trends in Turkey's national real estate market.

As a summary, Table 6.3 reflects T-REITs that we analyze in this thesis along with their abbreviations and preliminary statistics. Also, the JB test proves that most of the data is not coming from a normal distribution.

Table 6.3: The property focus and descriptive statistics of T-REITs

T-REIT	Code	Specialization	Mean	Median	St. Dev.	Skewness	Kurtosis	JB^a
Alarko	AL	Diversified (resort, industrial, retail)	1.577	1.948	12.39	-0.020	1.001	0
Avrasya	AV	Specialty (lease)	4.259	0.000	24.33	0.592	4.447	1
Dogus	DG	Diversified (retail, office)	3.596	1.266	18.01	2.912	13.643	1
Is	IS	Diversified (office, retail-mall)	1.988	2.299	10.73	-0.014	1.728	1
Nurol	NU	Residential	3.895	-1.081	18.67	1.714	4.761	1
Ozderici	OZ	Residential	1.039	-1.282	18.00	2.680	14.947	1
Pera	PE	Retail (mall)	-1.382	-1.613	13.16	0.061	2.191	1
Saf	SAF	Retail (mall)	1.859	0.741	15.36	1.492	5.441	0
Vakıf	VK	Diversified (land investment/retail)	3.834	0.707	18.76	0.851	2.648	1
Yapı Kredi	YK	Residential	1.075	1.869	14.76	-0.963	3.826	1
Yesil	Y	Residential	0.479	-1.493	18.47	0.937	2.938	1
Sinpaş	SN	Diversified (residential/land investment)	-0.012	-0.007	0.168	0.084	2.633	1
Akmerkez	AK	Retail (mall)	0.007	0	0.158	-0.220	7.097	1
ATA	AT	Diversified (retail/land investment)	0.009	0.009	0.082	-0.734	2.906	1
Deniz	DZ	Retail (mall)	0.013	0.009	0.138	0.414	4.870	1
EDIP	EP	Retail (mall)	-0.012	-0.019	0.158	0.552	1.936	1
Atakule	A	Retail (mall)	0.003	-0.008	0.118	0.618	0.845	1
Market	MT	–	0.945	0.287	8.623	-0.057	0.434	0

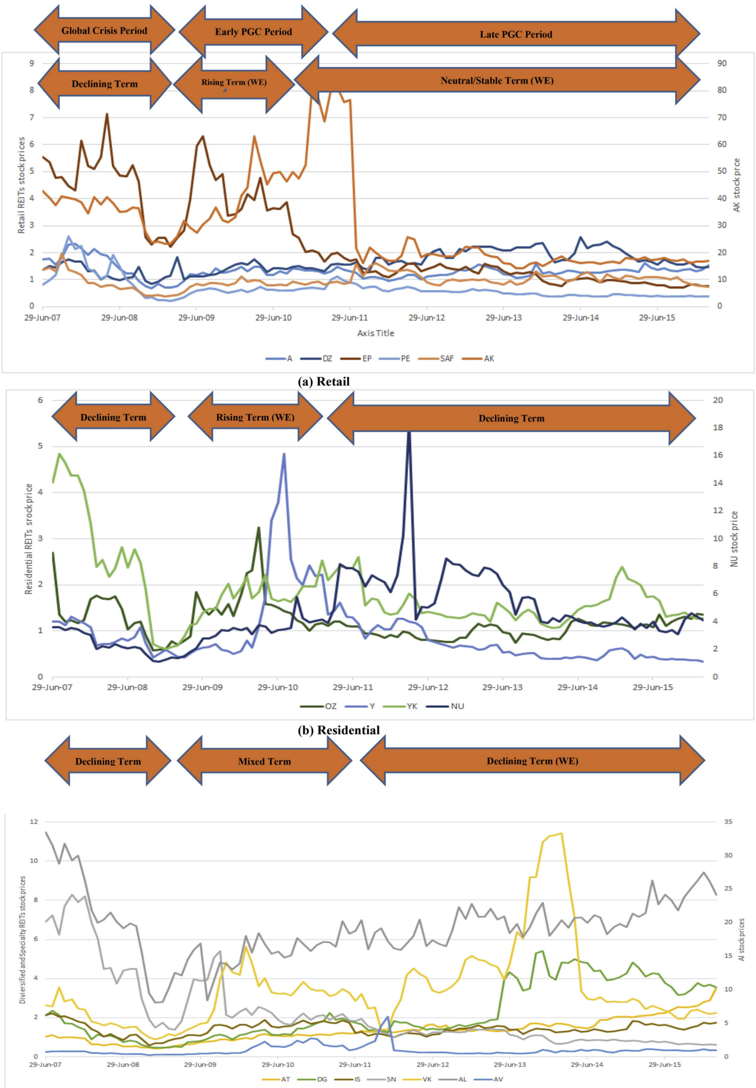
[a] Jarque-Bera test: Normally distributed if its value is zero.

We separate data series of T-REITs into two: the financial crisis (2007 Q2-2009 Q1) period and the post-financial crisis (after the 2009 Q1) period. We can also divide the second period into two, like the early period (about 2009 Q2-2011 Q2), and the post-period (approximately speaking 2011 Q3-2015 Q2). Our primary conclusion is all T-REITs are in the decay cycle through the period of the latest financial crisis. Still, throughout the period of post-crisis, the real estate trends are modified, with reasonable exemptions of T-REITs. Second, most of the retail T-REITs display stable economic features, exclusively noticeable for all retail T-REITs during the recent post-crisis years. Third, there exist solid, growing, and decaying return cycles in this period.

In the setting of T-REITs asset portfolios, generally, the T-REITs in the diversified class have partly growing drifts. However, the T-REITs in the residential type are commonly in decay, especially in the second half of 2011 (Figure 6.3). Hence, we can hypothesize that the T-REITs return dynamics expose no clear devotion among property focus and their return fluctuations in periods that we consider.

The descriptive statistics of T-TREITs, illustrated in Table 6.3, display three outstanding features. First, most of the T-REITs' seem to right-skewed distribution characteristics. Second, the performance of T-REITs average returns during the period under investigation varies in -1.4 and 4.3 that producing AV as the largest and PE is the lowest. From the focus of the property aspect, retail T-REITs present the weakest return, 0.49, where both diversified and residential T-REITs present 3.14 and 2.00, respectively. Hence, the highest performance of diversified T-REITs can indicate an association within property focus and yielding growth in T-REITs. But, this is a very suggestive remark that should have proved more robust analysis. In the end, it appears some of the T-REITs exhibit greater volatility than other T-REITs, and their volatility level can be associated with the specialty of T-REITs.

The return volatility of T-REITs can be connected to factors that rely on the performance of BIST, such as growing market values and asset portfolios, expectations on BIST performance in the near future, and conditions related to company activities. In this regard, while AV, VK, and NU have comparatively larger volatility, AT, A, SN, AK, EP, and DZ have lower volatility. Unexpectedly, retail T-REITs illustrate the



(c1)Diversified (c2) Specialty (AV)

Abbreviations: PGC; post-global crisis. WE; with exception(s).

Figure 6.3: The historical price evolution of T-REITs

weakest volatility except PE and SAF (see Table 6.3 and Figure 6.4). Consequently, we can remark that most of the retail T-REITs present a low return-low risk character compared to diversified and residential T-REITs as they illustrate high risk-high return character.

Correlation coefficients determine the collective behavior of T-REITs shares and the BIST100 index, summarized in Table 6.4. The table shows some exciting features. First, T-REITs generall have low correlations that imply diversification over T-REITs can grant privileges. Second, all T-REITs have a positive association with the BIST100 index except SN, varying 10% to 71%. Third, T-REITs positively correlated since they are operating in the same business.

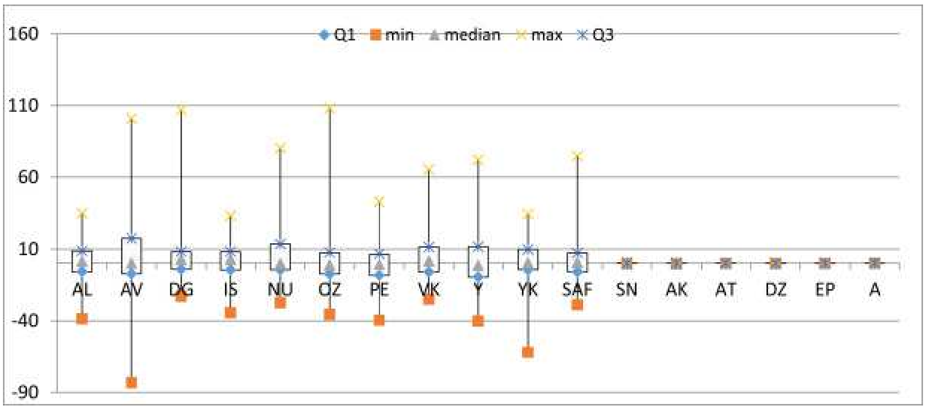


Figure 6.4: The distributional properties of excess returns

The three features that we highlight above suggest that T-REIT share price changes are parallel with the BIST100 to a certain amount. Furthermore, association with the BIST100 from the focus of property can accommodate exciting knowledge on their return properties. For instance, IS, which has the largest market value and a diversified REIT, is more reliant on the market compared to others. Considering less profit-generating assets in their main portfolios, and having less property in their portfolio, DZ, EP, VK, AV, and A have the most profound association with BIST100. Such a connection to the BIST100 index and the diversified T-REITs is 0.326, the residential T-REITs have an association of 0.507, and the retail T-REITs have an association of 0.263. As a result, considering the higher dependence of the T-REITs in the residential class to the BIST100 index, we can demonstrate there might be an attachment among property focus and yield growth potential.

Table 6.4: The associations of T-REITs and BIST100

	AL	AV	DG	IS	NU	OZ	PE	VK	Y	YK	SAF	SN	AK	AT	DZ	EP	A
AV	18	100															
DG	37	0.1	100														
IS	60	15	33	100													
NU	47	12	11	36	100												
OZ	38	9	16	35	22	100											
PE	50	24	30	46	41	53	100										
VK	35	-13	16	27	26	8	17	100									
Y	35	25	16	45	25	34	56	12	100								
YK	56	5	44	61	40	28	58	29	46	100							
SAF	56	24	20	36	29	37	42	24	38	45	100						
SN	-10	1	-2	-5	5	-7	-1	-13	20	-12	-2	100					
AK	45	8	20	30	23	21	26	14	33	43	29	3	100				
AT	1	1	23	10	1	11	8	-12	20	13	10	36	14	100			
DZ	17	1	11	-2	0	17	-6	-1	5	-8	9	15	28	30	100		
EP	9	-8	5	1	5	1	1	-3	7	-6	0	35	21	22	24	100	
A	-11	-10	-6	-11	-8	-11	-14	-15	5	-18	14	61	1	36	20	29	100
MT	56	14	32	71	46	44	60	16	51	62	40	-1	25	22	13	10	-10

6.5.2 The Diversification Benefit of T-REITs

REITs' shares are used as the primary investment opportunity for the real estate market, and institutional investors that use them benefit from a higher return and a more significant set of diversified investment opportunities, which consider both essences and not essence industry segment [163]. In a sense, we may claim that REITs are the liquid side of real estate markets. Therefore, these companies are mostly used in the hedging of inflation. As a result, adding their shares into our portfolios may be beneficial to protect ourselves from inflation.

The diversification benefit of REITs is comparatively examined in T-REIT, bank, and trust shares in BIST throughout the investigation period. To discover the diversification benefit of T-REITs, we listed them according to their risk level within each sector independently. We measure the risk as in the standard method with standard deviation (σ) over time, and we add them into the portfolio. The chosen REITs having the most imminent risk, we indicate their risk diversification level isn't influenced if we include other T-REITs. Such an examination expresses although the risk of portfolio decrease as T-REITs increase in our portfolio, the systematic risk limits the reduction in the risk

Figure 6.5 illustrates that the analysis of T-REITs' returns achieved, the risk, and the asset allocation demonstrate that the diversification strategy implies T-REITs are performing better than banks to diversify the risk. Our empirical evidence shows the equally weighted portfolio composed of T-REITs better in diversifying risk than banks. However, its diversification advantage resides lower compared to trusts. Therefore, T-REITs' shares might be accepted as an alternative asset class due to their diversification benefits in asset portfolios. Here, it is necessary to emphasize that we construct portfolios according to an equally weighted assumption scheme. Mainly, we assume one may invest only T-REIT' shares, bank' shares, or trust company shares. However, if there are two different T-REITs, then half of the wealth is going to invest in the first asset, and the remaining half is going to invest in the second asset. Moreover, Figure 6.5 clearly shows trust shares diversifying almost all non-systematic risk if an investor invests in trusts' shares. Bank shares diversify less risk compared to the other two company shares. It is because the banks are highly related to BIST and the economy of Turkey. Here, the most important finding is, even though their return performances are not sufficient, T-REITs have successful to diversify the unsystematic risk. Hence, the return improvement and diversification advantages of T-REIT over banks indicate a difficulty to local and foreign investors, that generally attend to invest in banks.

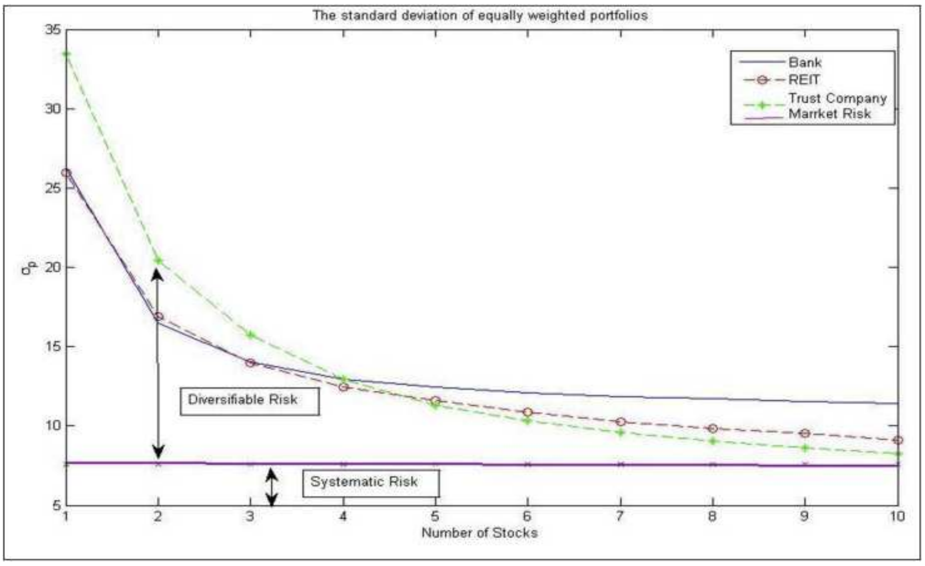


Figure 6.5: The diversification advantages of REIT

6.5.3 The Management Style of T-REITs

In the current section, we summarize the findings that we observe from the CAPM application. Table 6.5 displays estimated parameters of CAPM for all T-REITs. Here, it is necessary to note that despite the low R-squared (R^2) values, the CAPM is significant for nine T-REITs. The similar pattern is recognized in the t-statistics.

Parallel to the literature, such as [86, 152], we find that T-REITs generally have managed in the defensive structure. AV and VK have the lowest association with the BIST100 index. Also, they are defined as "Defensive" management since $|\beta| < 1$. Such outcomes might be a result of companies' real asset management policies. In this setting, their records review of AV throughout the investigation period exposes its income-producing property portfolio produced moderately limited income and revealed a small difference in its portfolio composition, by reason of the absence of efficient acquisitions. In VK case, the poor profit-generating potential of VK's property portfolio is also observable in the period 2008-2014. Therefore, we can hypothesize the assumed less active property acquisition strategy, and the short profit-generating potential of T-REITs might have adverse influences on association with the market that follows defensive management. Three residential T-REITs, namely NU, YK, and Y, generate aggressive management, where $|\beta| > 1$. It implies two main features. First, there may be a relationship among risk and their property focus due to the larger risk-return feature of the residential T-REITs. Second, if we manage a T-REIT aggressively, its return can reveal investors' expectations on profits. The expansion in the return variability, likely to lead a surprisingly bigger spread that reflecting β.

Table 6.5: The estimates of CAPM

T-REIT	α	β	$R^2(\%)$	p-value	M. Structure
AL	0.424 (0.3682)	0.852 (6.083*)	32	0.000*	Defensive
AV	3.420 (12.631*)	0.420 (1.277)	2	0.206	Defensive
DG	2.549 (13.291*)	0.705 (3.023*)	10	0.003*	Defensive
IS	0.786 (0.9208)	0.919 (8.853*)	50	0.000*	Defensive
NU	2.614 (14.02)	1.029 (4.541*)	21	0.000*	Aggressive
OZ	-0.194 (-0.1068)	0.962 (4.356*)	19	0.000*	Defensive
PE	-2.623 (-22.128)	0.974 (6.755*)	37	0.000*	Defensive
VK	3.029 (14.583*)	0.372 (1.473)	3	0.145	Defensive
Y	-0.888 (-0.4979)	1.148 (5.290*)	26	0.000*	Aggressive
YK	-0.284 (-0.2198)	1.136 (7.217*)	40	0.000*	Aggressive
SAF	0.780 (0.4926)	0.750 (3.896*)	16	0.000*	Defensive
SN	-0.557 (-12.0845)	0 (0.748)	0.7	0.456	NA
AK	-0.542 (-11.705*)	0 (1.648)	3.32	0.100	NA
AT	-0.538 (-12.144*)	0 (1.290)	1.9	0.218	NA
DZ	-0.534 (-11.865*)	0 (1.199)	1.78	0.234	NA
ED	-0.558 (-12.296*)	0 (1.132)	1.59	0.261	NA
A	-0.541 (-12.125*)	0 (0.546)	3.8	0.586	NA

6.5.4 The Fama-French Three-Factor Model

We employ SMB and HML over the investigation period to evaluate the Fama-French three-factor model. The examination outcomes, we summarize in Table 6.6, demonstrate T-REITs allow a good fit and Fama-French three-factor model acceptably increasing the explanatory power (R^2). However, DG is exceptional, and it's R^2 is not increasing.

Table 6.6: The parameters of Fama-French Model

T-REIT	α	β_M	β_{SMB}	β_{HML}	$R^2(\%)$	p-value FF	Size	State
AL	0.237 (0.168)	0.909 (5.548*)	0.604 (1.918)	-0.011 (-0.033)	40	0.000*	S	ND
AV	2.952 (0.877)	0.288 (0.740)	1.045 (1.400)	0.259 (0.342)	5	0.385	S	D
DG	2.954 (1.217)	0.686 (2.440*)	-0.168 (-0.312)	0.055 (0.100)	10	0.0100*	B	D
IS	0.510 (0.502)	0.941 (7.981*)	0.272 (1.201)	0.343 (1.494)	55	0.000*	S	D
NU	3.852 (1.627)	0.970 (3.540*)	1.097 (2.083*)	0.698 (1.308)	24	0.0023*	S	D
OZ	-1.110 (-0.490)	0.938 (3.571*)	1.538 (3.046*)	0.784 (1.533)	29	0.000*	S	D
PE	-3.680 (-2.51*)	1.015 (5.990*)	1.011 (3.103*)	0.714 (2.162*)	45	0.000*	S	D
VK	3.727 (1.626)	0.576 (2.171*)	0.250 (0.490)	-0.434 (-0.840)	11	0.085*	S	ND
Y	-1.790 (-0.847)	1.156 (4.698*)	1.706 (3.609*)	0.660 (1.379)	40	0.000*	S	D
YK	-1.490 (-0.979)	1.216 (6.838*)	1.172 (2.977*)	0.405 (1.169)	50	0.000*	S	D
SAF	0.878 (-0.049)	0.792 (3.749*)	0.939 (2.314*)	0.361 (-0.089)	25	0.0005*	S	D

Table 6.7 illustrates three aspects that especially exciting. 1) while all residential and two retail, PE and SAF, T-REITs positively respond to SMB, only PE positively responds to HML. Consequently, we can discuss that SMB is more substantial than HML in demonstrating T-REITs return variability. With the sign of such evidence, it is reasonable that the return performance of the residential and retail T-REITs are more sensitive to variations in SMB (and so changes in the market capitalization and profitability), in addition to the BIST100 index ingredient. This consequence might also refer that portfolio management strategies of T-REITs may have some impacts on SMB. 2) all T-REITs present small company specialties, excluding DG. Besides AL and VK, all T-REITs illustrate a financially distressed state. Such main results are a sign of lower profitability obstacles for the T-REITs under investigation. T-REITs' net profit at the end of the year before tax might also verify our judgment. With this regard, for the period that we consider, T-REITs and the years they have a loss are listed as: AV (2008; 2010), DG (2008), NU (2013; 2014); OZ (2012); PE (2008; 2011; 2012; 2013; 2014), SAF (2008; 2012; 2013), YK (2008; 2009; 2010; 2011; 2012; 2013), Y (2008; 2009; 2010). AL, IS, and VK, has announced a net profit in our investigation period. Accordingly, their net profits and financially Not-Distressed (ND) state of IS, AL, and VK, and also the historical price properties of the diversified T-REITs (see Figure 6.3 3c1) empirically indicate better achievement in precise periods compared to specialized REITs. Recognizing that the residential T-REITs have higher β values than remaining T-REITs, one might claim that there are mixed outcomes on property focus and risk improvement of T-REITs. Such sign indicates the requirement for market agents to promote their performance by precisely examining companies' asset values, their management styles, financial status, and

their speculative ingredients of share prices, and thus, T-REITs' returns.

We summarize the parameters that we estimate for the CAPM and Fama-French models in Table 6.7 to compare the models. Here, we can note that generally, the Fama-French model improves the influence of the BIST100 index (β_M) except for the REITs AV, DG, NU, and OZ. It also increases the explanatory power of the model, except for the REIT DG. The increase in the explanatory power (R^2) of the models supports the recent improvement. Besides, standard errors also seem to be alike and display poor dispersion. Our results imply that the three-factor Fama-French model develops the participation of the market compared to the CAPM. Also, the addition of extra variables improves the estimating efficiency of returns.

Table 6.7: Parameters of the models

	CAPM			FAMA-FRENCH					
T-REIT	α	β	$\sigma(\epsilon_p)$	α	β_M	β_{SMB}	β_{HML}	$\sigma(\epsilon_P)$	$\Delta R^2(\%)$
AL	0.424	0.852	10.32	0.237	0.909	0.604	-0.011	10.81	25
AV	3.420	0.420	24.27	2.952	0.288	1.045	0.259	25.73	150
DG	2.549	0.705	17.19	2.954	0.686	-0.168	0.055	18.55	0
IS	0.786	0.919	7.65	0.510	0.941	0.272	0.343	7.77	10
NU	2.614	1.029	16.71	3.852	0.970	1.097	0.698	18.08	14
OZ	-0.194	0.962	16.28	-1.110	0.938	1.538	0.784	17.34	53
PE	-2.623	0.974	10.63	-3.680	1.015	1.011	0.714	11.20	22
VK	3.029	0.372	18.62	3.727	0.576	0.250	-0.434	17.51	267
Y	-0.888	1.148	15.99	-1.790	1.156	1.705	0.660	16.25	54
YK	-0.284	1.136	11.60	-1.490	1.216	1.017	0.405	11.74	25
SAF	0.780	0.750	14.19	0.878	0.792	0.939	0.361	13.94	56

The evidence we find above recommends the following. Agents can not only use the information retrieved from the CAPM but also use the information recovered from the Fama-French because of its' partial development in getting the variety in returns. But, we may remark the advantage of the Fama-French over the CAPM is comparable, due to its refined but yet poor explanatory power for the return variability.

6.5.5 Additional Risk Sources Effect

By regarding the possible influences of various economic and industry-specific risk sources on T-REITs, an extended Fama-French can develop the R^2. Consequently, we used pre- and post-election periods (as the political factor) and the financial crisis.

Additionally, the impact of the foreign exchange rate is considered for the univariate model.

We add a factor to the Fama-French three-factor model that represents the influence of the financial crisis presents no statistical significance. To estimate the effectiveness of the political influence, we insert D_{1t} and D_{2t} into the model. Although Turkey's financial market is comprehended to respond to the political changes, our examination reveals that T-REITs are generally not affected by the elections (Table 6.8 and 6.9).

Here, to demonstrate the influence of USD/TL currency on T-REITs that we defined above, we stimulate a simple linear regression to reveal the association among T-REITs and USD/TL currency. But, the influence of the exchange rate on T-REITs shows no significance when it involved in the Fama-French model. Thus, we fit a simple linear regression to measure its affect on T-REIT separately. We apply it as in the following equation:

$$R_i = \alpha_{i,USD} + \beta_{i,USD} R_{USD} + \epsilon_i, \quad i = 1, \dots, n. \tag{6.5}$$

Table 6.8: The effect of political risk on T-REIT at the election month

T-REIT	α	β_M	β_{SMB}	β_{HML}	β_{D_1}	$R^2(\%)$	p-value (FF)
AL	0.532 (0.364)	0.916 (5.566*)	0.646 (2.021*)	0.002 (0.007)	-4.636 (-0.811)	40	0.000*
AV	2.752 (0.787)	0.281 (0.714)	1.015 (1.327)	0.253 (0.330)	3.420 (0.250)	5	0.546
DG	3.523 (1.407)	0.701 (2.486*)	-0.086 (-0.158)	0.078 (0.142)	-9.108 (-0.930)	12	0.140
IS	0.908 (0.878)	0.951 (8.162*)	0.329 (1.455)	0.360 (1.586)	-6.313 (-1.561)	57	0.000*
NU	3.764 (1.531)	0.967 (3.489*)	1.084 (2.015*)	0.696 (1.292)	1.579 (0.164)	24	0.005*
OZ	-0.39 (-0.168)	0.958 (3.660*)	1.642 (3.229*)	0.814 (1.598)	-11.583 (-1.275)	31	0.000*
PE	-4.011 (-2.653*)	1.005 (5.902*)	0.963 (2.911*)	0.702 (2.120*)	5.481 (0.927)	46	0.000*
VK	4.024 (1.694)	0.584 (2.181*)	0.292 (0.562)	-0.421 (-0.810)	-4.676 (-0.503)	12	0.146
Y	-1.471 (-0.668)	1.164 (4.690*)	1.753 (3.636*)	0.674 (1.397)	-5.170 (-0.600)	40	0.000*
YK	-1.821 (-1.149)	1.205 (6.748*)	0.969 (2.795*)	0.393 (1.131)	5.461 0.881	51	0.000*
SAF	-0.190 (-0.100)	0.789 (3.697*)	0.925 (2.230*)	0.357 (0.860)	1.634 (0.221)	27	0.002*

Table 6.9: The effect of political risk on T-REIT (pre&post-election)

T-REIT	α	β_M	β_{SMB}	β_{HML}	β_{D_2}	$R^2(\%)$	p-value (FF)
AL	0.247 (0.172)	0.908 (5.302*)	0.605 (1.859)	-0.009 (-0.028)	-0.003 (-0.001)	40	0.000*
AV	2.067 (0.172)	0.064 (0.164)	1.399 (1.882)	0.301 (0.409)	9.827 (2.182*)	13	0.105
DG	3.108 (1.261)	0.721 (2.457*)	-0.224 (-0.403)	0.050 (0.090)	-1.587 (-0.470)	11	0.179
IS	0.739 (0.734)	0.994 (8.294*)	0.187 (0.820)	0.335 (1.485)	-2.401 (-1.739)	24	0.000*
NU	3.950 (1.643)	0.992 (3.463*)	1.063 (1.955)	0.696 (1.293)	-0.977 (-0.296)	24	0.004*
OZ	-0.936 (-0.407)	0.979 (3.574*)	1.474 (2.832*)	0.778 (1.512)	-1.821 (-0.577)	29	0.001*
PE	-3.949 (-2.704*)	0.946 (5.439*)	1.120 (3.391*)	0.727 (2.226*)	3.023 (1.510)	48	0.000*
VK	3.300 (1.446)	0.466 (1.717)	0.423 (0.819)	-0.413 (-0.809)	4.790 (1.531)	15	0.064
Y	-1.637 (-0.759)	1.192 (4.644*)	1.647 (3.377*)	0.655 (1.357)	-1.658 (-0.561)	40	0.000*
YK	-1.641 (-1.057)	1.176 (6.360*)	1.079 (3.073*)	0.413 (1.188)	1.711 (0.803)	51	0.000*
SAF	-0.571 (-0.321)	0.672 (3.169*)	1.129 (2.803*)	0.382 (0.960)	5.285 (2.165*)	32	0.000*

We eventually discover that the influence of USD on T-REITs is meaningful (Table 6.10). Hence, we can suggest that T-REITs and the market index have an opposite tendency concerning the USD currency. Therefore, investors who are willing to invest in REITs' shares should consider the currency rate while investing in Turkey's exchange market to increase their portfolio performance. The evidence that we present here implies three useful hints. First, it can be useful to employ additional variables to illustrate the variability of T-REITs' returns beyond the CAPM and Fama-French models. Second, the general counter association defined among USD and T-REITs indicates fund managers may be more willing to invest in T-REITs when dollar currency is in decreasing trend. Third, the insignificance of T-REITs' to the political shocks implies TREITs present stability in negative shocks.

Table 6.10: The impact of USD on returns

T-REIT	α_{USD}	β_{USD}	p-value	T-REIT	α_{USD}	β_{USD}
AL	1.325 (-3.26*)	-1.032 (1.310)	0.001*	PE	-1.218 (-3.07*)	-1.045 (-0.857)
AV	4.3151 (-1.352)	-0.883 (1.581)	0.180	VK	3.893 (-1.782)	-0.889 (1.860)
DG	3.468 (-1.259)	-0.610 (1.713)	0.211	Y	1.013 (-3.41*)	-1.598 (0.518)
IS	2.277 (-4.82*)	-1.233 (2.13*)	0.000*	YK	1.771 (-5.34*)	-1.842 (1.228)
NU	4.390 (-3.24*)	-1.541 (2.21*)	0.001*	SAF	1.810 (-1.779)	-0.728 (1.059)
OZ	1.105 (-1.882)	-0.899 (0.553)	0.063	BIST 100	1.506 (-6.46*)	-1.167 (1.995)

t-values are in the parenthesis, * at 5% significance level.

CHAPTER 7

CONCLUSION

As housing markets are becoming an essential part of the economy in many developed and emerging countries, in this thesis, various models with statistical and stochastic structures are introduced to the real estate finance and economics literature. The first emphasis of the thesis reported here is to develop statistical models and analysis of a potential influence of particular national economic indicators on housing markets, which will enable the use of models in the estimation of the trend in housing markets. Housing markets can be represented by using house price indexes such as S&P Case Shiller Home Price Index (HPI). Hence, we model the US housing market with non-parametric statistical methods, and contributions of its macro-economic indicators on the US housing markets are determined. However, statistical methods require many initial assumptions that are difficult to satisfy. Furthermore, in our case, there exist so many explanatory variables, which generally may cause a multi-dimensionality problem. To avoid the multi-dimensionality problem of statistical methods, we model housing markets as a two-state stochastic process; one for HPI and one for FRM.

Along with the macro-economic indicators, it is also essential to clarify the effect of large investors on housing markets. Hence, to analyze the impact of large investors on housing markets, we consider a simple one-dimensional stochastic process, which represents the house price. As a result, we obtain explicit optimization results for the large investors' portfolios and compute the effect of large investors on housing markets.

Determining the price in housing markets is not enough for investors since house prices are highly related to financial markets through mortgages. Hence, mortgage values should be fairly determined. There are studies computing mortgage value by solving the mortgage PDE using finite difference and finite element methods. Even though determining the fair price of mortgages is the primary point of managing trades in housing markets, when mortgages are settled, their price does not remain constant. Instead, their price follows dynamic paths during mortgages' survival time. Hence, investors in housing markets should protect their main portfolio against unexpected house price changes by managing variations in the price of mortgages. Thus, by considering the standard two state-variable process, we compute hedging coefficients for mortgage default and prepayment options by applying the finite-dimensional Malliavin calculus.

In this thesis, with detailed analysis, the effectiveness of our models are discussed, and their comparisons are made. The effect of large investors on housing markets are analyzed comprehensively. Furthermore, hedging of mortgages default and prepayment options are analyzed and quantified through sensitivity analysis. In this respect, this thesis has theoretical and practical contributions to the real estate finance and economics literature, as we summarize below.

- The models are driven by the application to work with GLMs and MARS methods with linear and interaction terms in the model structure. As the empirical analysis shows, the explanatory variable selection and modeling methods are the keys to construct statistical models. Thus, concerning other explanatory variables, in addition to those listed in Table 2.1, one may continue to add explanatory variables for the explanation of housing market variability by looking at the country-specific variables. The modeling methods presented here makes the use of non-linear models with interaction terms for the first time in housing markets. The methods suggest significant advances in nonparametric modeling with economic indicators for the US housing market. The US housing market participants may benefit from the computationally driven models and modeling interactions by an explicit estimation of the appropriate relative coefficients of variables.

- The thesis provides an insight into factors connected with housing markets. It highlights the link between the critical macro-economic indicators and housing markets with their direction of the interaction to explain the price fluctuations in housing markets. Housing markets have a strong relationship with other markets, and the potential volatility in housing markets can have a dramatic impact on the overall economies, even in some cases, it can cause a global crisis. It also sheds new light on the mechanism of housing markets and offers statistical models that can be used to predict housing market price behavior. The empirical models provide a coherent set of empirical and prediction results. The models also confirm the importance of changes in national economic conditions, employment, inflation, income, interest rates, crude oil prices, and capacity utilization that affect the housing market. In addition to these contributions, it is worth emphasizing that it is possible to forecast the direction of changes in house prices by using models within the thesis.

- The preliminary analysis in Chapter 2 exemplifies that the US housing market exhibit a number of puzzling features, including a strong positive and negative correlation between house prices and its macro-economic indicators that we analyze.

- The past two decades have seen a proliferation of financial instruments that are linked to the house prices, such as futures, options, and mortgage pool linked bonds. Besides, we witnessed a substantial rise in house prices in early 2000, followed by a severe global financial crisis related to the real estate market between late 2006 and early 2009. These breakdowns urge more sophisticated and, at the same time, flexible models to explain the behavior of housing markets and their related assets. Although there have been many econometric and statistical models studied in the real estate literature that aims to resemble the behavior of housing markets, we propose a fully stochastic, flexible model whose outcomes allow better efficiency in prediction with allowing a reduced number of exogenous variables. In particular, we offer a two-factor stochastic differential equation for housing markets. Unlike statistical and econometric models, our approach incorporates the uncertainties of the market via a significant factor, mortgage rate. We develop an estimation and calibration procedure for the model based on a real data set, the S&P Case-Shiller Home Price index, and a 30-year fixed mortgage rate, and then implement a Monte Carlo algorithm to estimate S&P Case-Shiller Home Price index values.

- The outcomes of this thesis enable researchers to understand the house price behavior in terms of the random pattern in mortgage rates. The stochastic model implemented through a real-life case yields a good prediction in HPI values as well. It captures the real volatility, which is not foreseen accurately in the original series. Measuring the price fluctuations and imitating the market evolution together with the mortgage rate using the proposed approach gains importance, certainly for housing markets whose historical observations in terms of all contributing factors are either scarce or not fully available.

- We investigate the effect of large investors on housing markets by maximizing the expected utility from the terminal wealth of these investors. The optimal investment problem of large investors is solved explicitly under the linear impact function, complete information, and log and power utility functions assumptions. We show that the optimal investment decisions depend on the balance among economic state, maintenance, rental income, interest rate, and willingness to invest of the large investor.

- The outcomes also show that investors should invest in the bond instead of investing in housing if the economy is in the bad state. However, if the economy is in the neutral and good state, investing in housing is a favorable investment compared to the bond. Besides, investors can improve their wealth by adjusting the maintenance cost. Moreover, the model clearly shows that in the good economic state, an investor should borrow as much as possible and invest in the housing market.

- These outcomes also identify the effect of large investor activities on housing markets and guide investors in maximizing their wealth. We observe that large investors operating with log and power utility function assumptions may gain benefits from the price impact by choosing optimal investment strategies.

- An essential key ingredient and contribution of the thesis is to identify the mortgage default and prepayment options hedging coefficients. Hedging is a sufficient or partial decrease in an asset's risk by joining a contingent claim, which balances the risk. The presence of agents in need of a replicating portfolio is a primary requirement to establish and increase the success of derivatives markets. In the lack of such a requirement, it is troublesome to conclude settlements among counter-parties. It is because the demand and the supply risk and asset return are anticipated to be out of equilibrium. Hence, in the context of this thesis, we derive Malliavin calculus-based phrases for the hedging parameters for the options that inherited to mortgages. The masterpiece of this approach is that the phrases that we introduced are a product of the option's payoff the Malliaivin weights. In this regard, this method authorizes the possible users to consolidate the specifications resulting from this calculus and to obtain calculations for the corresponding hedging parameters by performing MC simulations.

- The phrases to the options embedded into mortgages are our particular interest since these options do not have closed-form resolutions. The achievement of our formulas is illustrated, and our numerical results are displayed by utilizing a crude MC algorithm and performing simulations. Our empirical outcomes exhibit that both mortgage options are more perceptive to changes in the volatility compared to changes in the house price and spot rate. There are many possible practices and applications of our results in the actual markets. Even though housing market agents bear risks associated with an increase in the spot rate, mortgage prepayment, mortgage default, and foreclosure in housing markets, the investors generally not notified properly concerning the application of hedging strategies to reduce their risk. Therefore, the agents in housing markets are potential pragmatists by the capability to hedge their positions in such housing markets by applying the outcomes of our thesis. Moreover, the application of such findings may be anticipated to emerge as insurance and improvement in investment instruments of the companies that are providing financial services to receive benefits from the investments in housing markets.

- The thesis investigates five previously never considered fundamental research subjects about T-REITs throughout the period of July 2008–March 2015. Our first attention is the diversification advantage of the T-REITs compared to banks and trusts operating in BIST. Our second problem compares to describing the return fluctuations of T-REITs by comparing the CAPM and Fama-French three-factor model. Our third interest is linked to whether the expansions of the classical Fama-French model may develop its expressive power or not. In this regard, we operate for the first time currency risk, global crisis, and political risk as supplementary sources in the standard Fama-French model to mirror the consequences of global economic/political factors on T-REITs. Fourth, we also represent property focus and examine management structure and size, the financial state of T-REITs by appropriating CAPM and Fama-French models. Finally, the thesis also seeks to explore whether there is an association between property focus and risk-taking/yield enhancement in T-REITs.

- In the context of diversification benefits, defining return enhancement and risk reduction benefits potential of T-REIT shares over the bank shares at BIST, presents a challenge for domestic/foreign fund managers, who tend to choose major bank stocks for their asset allocation. Investors in BIST should consider including T-REITs in their portfolios to achieve diversification benefits, and hence, improve their investment opportunity sets.

- Portfolio managers and investors can, in addition to utilizing the knowledge deriving from the CAPM, also incorporate the information retrieved from the Fama-French model, due to its relatively improved capacity to capture the variation in T-REITs returns. However, the superiority of the Fama-French three-factor model over the CAPM is relative due to its still limited explanatory power for explaining the return variability of T-REITs.

- Additionally, we define that the inclusion of the new independent variables to the Fama-French model may increase the explanatory power of the model. Based on the expanded Fama-French model outcomes, we find that T-REIT shares are generally insensitive to the election periods and T-REITs returns, and market index has reverse movement concerning the USD change.

- By utilizing portfolio weights presented in the audit/financial reports and also corporate expert views, we identify that T-REITs show a degree of diversity in property focus with four residential, six diversified, six retails, and one specialty REITs. Moreover, the high number of retail and residential T-REITs available also shows the main investment trends in Turkey's real estate industry. All except one T-REITs show small status, and T-REITs management structures are found to be mainly defensive. Except for two diversified T-REITs, all T-REITs show a financially distressed state. The latter point is the evidence of a lower earnings (profitability) problem for most of the studied T-REITs. Based on the multiple observations, it is possible to argue that there would be a linkage between property focus and yield improvement/risk-taking structure of T-REITs. In this respect, we find that diversified T-REITs show the highest excess return performance, and some show a relatively better return performance in certain periods during the late post-global crisis period compared to other specialties. On the other hand, the majority of retail T-REITs generally show a low return-low risk profile compared to high risk-high return profiles of the majority of diversified and residential T-REITs. We also identify that residential T-REITs are more sensitive to the market index, and have higher betas than other specialties.

- In the context of company-specific investment strategy, the above evidence implies that improvement of portfolio performance requires careful analysis of asset values, management strategies, financial information and speculative/realistic components of share prices, and hence returns of T-REITs.

- Future research concentrates on investigating the growth patterns in total assets and market caps in the T-REITs industry, expanding the Fama-French model with additional variables, and exploring the connection between firm-specific idiosyncrasies and return variability in T-REITs. For the latter, some further variables may be added, such as management quality, portfolio management strategies, the share of institutional/foreign investors, and short/long term expectations of investors.

The results presented in the thesis will serve as a benchmark for housing market modeling, analysis and mortgage default, and prepayment hedging. Using the results of our study on housing markets, one can make a productive investment in housing markets and hedge mortgages. Moreover, the stochastic model can be calibrated for the jump model structure successfully mimic possible the large price deviations. Besides, as future work, we aim to investigate the effect of large investors having credit limitations and different interest rates for borrowing and lending assumptions. Application of the modeling methodologies in the thesis to the Turkish housing market may determine its deriving factors. It can find how accurate the stochastic model captures the price pattern and growth of the Turkey's housing market within an emerging economy.

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APPENDIX A

ADVANTAGES OF T-REITS

A REIT is a company that established to invest in income-producing real estate assets as a public company. Even though T-REITs history began in the mid-1990s, they have recently achieved recognition as an investment tool in Turkey. As a financial intermediary, T-REITs have several financial instruments and some regulatory limitations as in the other countries.

T-REITs enjoy the positive effects of gaining leverage through the capacity to issue debt instruments, real estate certificates, and asset-covered securities, and the right to borrow the amount of up to five times of their shareholders' equity. As prohibited activities, T-REITs are not authorized to perform construction works, operate on real estates, offer project development and supervision, lend credit, or to have permanent involvement with the short-term trading of real estate. These restrictions are designed to ensure that T-REITs are limited to manage portfolios consists of real estate.

Regulations on the T-REITs are first issued in 1995, and the latest update on major regulation is enacted in 2013. The first T-REIT publicly listed in BIST in 1997. Recently, the number of T-REITs in BIST has increased dramatically, which may be related to the following leverage opportunities and policies:

- Although REITs transactions are subject to value-added tax and other taxes, their profits are exempt from corporate tax (20%), and their dividend withholding tax rate is 0%. At an investor level, the transaction of shares is subject to a 0% of withholding tax for domestic and foreign investors. This favorable tax provision is scarce in the Turkish public finance tradition, highlighting strong state support for the development of the real estate industry through REITs.

- In all countries, REITs require a certain level of dividend to be paid as a minimum dividend requirement. For instance, REITs are required to distribute 90% of their taxable income as dividends to their shareholders in the US. Since policies of Capital Markets Board of Turkey (CMB) do not require dividend payout requirements, T-REITs also have permission to define their own dividend policy. Despite this exemption, industry practices reveal that some of the T-REITs may prefer to distribute a dividend. Hence, we may argue that optional dividend payout policy of T-REITs represents an essential source of support for T-REITs liquidity management at the expense of shareholders' short-term benefits. In this respect, Turkey's far less restrictive regime facilitates the development of REITs.

- In addition to the tax and dividend payout exemptions, the minimum ratio of issued capital for T-REITs declined from 49% to 25% in December 2009, causing, the number of T-REITs to increase to 31 as of 2015, from 14 as of 2009, due to the favorable market environment and above mentioned supportive regulatory framework.

In light of the factors that we listed above the existing regulatory structure and policies can be said to provide an industry-friendly environment for T-REITs evolution. The apparent reasons behind this explicit support are to improve transparency in Turkish real estate industry, to increase tax revenues from real estate sales, and more importantly, to enhance contributions of the Turkish real estate economy, via T-REITs, to the economic growth.

APPENDIX B

MONTHLY PAYMENT AND PRINCIPAL BALANCE

In the fix rate mortgage, the mortgage loan is repaid by a series of equal Monthly Payments (MP) on pre-determined payment dates. The monthly payments of a fix mortgage rate for a householder and the Outstanding Balance (OB) following each payment are calculated using standard annuity formula, which is given as

$$\begin{aligned} MP &= OB(0)\frac{(\frac{c}{12})(1+\frac{c}{12})^m}{(1+\frac{c}{12})^m-1}, \\ OB(t) &= OB(0)\frac{(1+\frac{c}{12})^m-(1+\frac{c}{12})^t}{(1+\frac{c}{12})^m-1}, \end{aligned}$$

where the parameter c is the fixed yearly mortgage rate, OB(0) is the initial loan amount and m represents the life of the mortgage loan in months.

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EDUCATION

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M.S.	Institute of Applied Mathematics, METU	2014
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PROFESSIONAL EXPERIENCE

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Jan 2008 - Jan 2009	IPM International Project Management	Project Manager

PUBLICATIONS

Journal Publications

- **Bilgi Yilmaz**, A. Sevtap Selcuk-Kestek. Forecasting House Prices in Turkey: A Time Series Approach. International Journal of Computational Intelligence and Applications, 2019 (Under Review).
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- **Bilgi Yilmaz**, Fatma Yerlikaya-Özkurt, A. Sevtap Selcuk-Kestel. Analyzing Housing Market Dynamics using Linear and non-Parametric Models. 11.International Statistics Days Conference, Muğla, Turkey, October 3-7, 2018.

- **Bilgi Yilmaz**, A. Sevtap Selcuk-Kestel. Default and Prepayment Risk Management Using Option Based Mortgage Contract Pricing Method. European Actuarial Journal Conference 2018, Leuven, Belgium, September 9-11, 2018.

- **Bilgi Yilmaz**, A. Sevtap Selcuk-Kestel. Determination of Sensitivities for Mortgage Default and Prepayment Options. 8th General AMaMeF Conference, Amsterdam, Holland, June 19-23, 2017.

- **Bilgi Yilmaz**, A. Sevtap Selcuk-Kestel. The Effect of Macro-Economic Factors on Housing Markets: US Case. IRSYSC 2017-3rd International Researchers, Statisticians and Young Statisticians Congress, Konya, Turkey, May 24-26, 2017.

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- **Bilgi Yilmaz**. Comparison of Different Methods to Compute the Greeks. 8th International Statistic Congress, Antalya, Turkey, October 27-30, 2013.

- **Bilgi Yilmaz**, B. Alper Inkaya, Yeliz Yolcu Okur. Application of the Malliavin Calculus for Computation of Greeks in Black-Sholes and Stochastic Volatility Models. 26th European Conference on Operational Research, Rome, Italy, July 1-4, 2013.

National Conference Presentations

- **Bilgi Yilmaz**. Computation of Greeks via Malliavin Calculus. 2nd Ankara-İstanbul Workshop on Stochastic Processes, İstanbul, Turkey, June 18-19, 2015.

Workshops

- Advanced Modelling in Mathematical Finance Workshop in Honor of Azize Hayfavi's 75th Birthday, Financial Modeling, Monte Carlo Simulations, Institute of Applied Mathematics, Middle East Technical University, Ankara, Turkey, October 23, 2017 (Organizer/National).

- VCMF Educational Workshop, Limit Order Book / High Frequency Trading, Credit Risk / Systemic Risk, Computational Methods and Calibration, New Financial Markets, Stochastic Volatility Models, Risk Measures and Optimization, WU Wien-Vienna University of Economics and Business, Vienna, Australia, September 15-16, 2016 (International)

- 3rd Ankara-Istanbul Workshop on Stochastic Processes, Bayesian and hidden Markov models, Diffusion and jump-diffusion models in finance, Levy and renewal processes, Malliavin calculus and its applications to finance, Probabilistic approach to operator theory, Processes in random media Random walks on graphs, Stochastic flows, Middle East Technical University, Ankara, Turkey, June 16-17, 2016 (National).

- 2nd Ankara-Istanbul Workshop on Stochastic Process, Bayesian and hidden Markov models, Diffusion and jump-diffusion models in finance, Levy and renewal processes, Malliavin calculus and its applications to finance, Probabilistic approach to operator theory, Processes in random media, Random walks on graphs, Stochastic flows, Koç University, Istanbul, Turkey, June 18-19, 2015 (National).

- 1st Ankara-Istanbul stochastic days, Bayesian and hidden Markov models, Diffusion and jump diffusion models in finance, Levy and renewal processes, Malliavin calculus and its applications to finance, Probabilistic approach to operator theory, Processes in random media, Random walks on graphs, Stochastic flows, Boğaziçi University, Istanbul, Turkey, June 12-13, 2014 (National).