A MATHEMATICAL PROGRAMMING EVALUATION APPROACH FOR MULTIPLE CRITERIA SORTING PROBLEMS

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submitted by **MERVE CIVELEK** in partial fulfillment of the requirements for the degree of **Master of Science in Operational Research Department, Middle East Technical University** by,

Prof. Dr. Halil Kalıpçılar
Dean, Graduate School of **Natural and Applied Sciences**

Prof. Dr. Yasemin Serin
Head of Department, **Industrial Engineering**

Prof. Dr. Esra Karasakal
Supervisor, **Industrial Engineering, METU**

Examining Committee Members:

Assoc. Prof. Dr. Serhan Duran
Industrial Engineering, METU

Prof. Dr. Esra Karasakal
Industrial Engineering, METU

Assist. Prof. Dr. Sakine Batun
Industrial Engineering, METU

Assist. Prof. Dr. Gülşah Karakaya
Business Administration, METU

Assist. Prof. Dr. Özlem Karsu
Industrial Engineering, Bilkent University

Date: 28.08.2019

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Surname: Merve Civelek

Signature:

ABSTRACT

A MATHEMATICAL PROGRAMMING EVALUATION APPROACH FOR MULTIPLE CRITERIA SORTING PROBLEMS

Civelek, Merve
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Supervisor: Prof. Dr. Esra Karasakal

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Multiple criteria sorting problem is to assign alternatives, evaluated according to multiple criteria, into predefined preference ordered classes. In this study, a new distance metric based sorting method is developed to solve multiple criteria sorting problems without specifying class thresholds between preference-ordered classes. The aim of the proposed method is to assign each alternative to one class or a set of possible adjacent classes considering the distance to class centroids. In the proposed method, two cases are considered. In the first case, centroids of the classes are estimated using the whole data set. In the second case, class centroids are estimated using only the training data set. Distance of alternatives to the centroids are used as criteria aggregation function. A mathematical model is formulated to determine the weights of the criteria. Assignment is performed according to the weighted distance of each alternative to each class centroids. The proposed method is applied to five data sets with four different distance norms and several performance measures are calculated. The results show that centroid information is not so important to obtain better performances. The performance of the proposed method is compared with PDIS method and UTADIS method. The computational studies show that with relatively large data sets the proposed method performed better than the other methods.

Keywords: Multi Criteria Sorting, Distance Function, Distance Based Sorting, Sorting without Class Threshold

ÖZ

ÇOK KRİTERLİ SIRALI SINIFLANDIRMA PROBLEMLERİNİN DEĞERLENDİRİLMESİNDE MATEMATİKSEL PROGRAMLAMA TABANLI BİR YÖNTEM

Civelek, Merve
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Tez Danışmanı: Prof. Dr. Esra Karasakal

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Çok kriterli sıralı sınıflandırma problemi, çok kriterlere değerlendirilmiş alternatiflerin ön tanımlı, sıralı sınıflara atamasıdır. Bu çalışmada, çok kriterli sıralı sınıflandırma problemlerini tercihe göre sıralanmış sınıflar arasında sınıf eşik değerleri tanımlamadan çözmeye yarayan uzaklık ölçütü tabanlı bir sıralama metodu geliştirilmiştir. Önerilen metodun amacı, her alternatifi sınıf merkezlerine olan uzaklığına göre bir sınıfa veya olası komşu sınıflara atamaktır. Önerilen bu metotta iki farklı durum ele alınmıştır. İlk durumda sınıf merkezleri tüm veri setinin kullanılmasıyla yaklaşık olarak hesaplanmaktadır. İkinci durumda, sınıfların merkezleri eğitim veri seti kullanılarak yaklaşık olarak hesaplanmaktadır. Alternatiflerin sınıf merkezlerine olan uzaklığı kriter birleştirme fonksiyonu olarak kullanılmaktadır. Kriterlerin ağırlıklarının belirlemesi için matematiksel bir model formüle edilmiştir. Atama işlemi her alternatifin her sınıf merkezine ağırlıklı mesafesine göre yapılmaktadır. Önerilen metot, dört farklı uzaklık ölçüsü kullanılarak beş farklı veri setine uygulamış ve birçok performans ölçüsü hesaplanmıştır. Alınan sonuçlar merkez bilgisinin daha iyi performans elde etmek için çok da önemli olmadığını göstermiştir. Önerilen metodun performansı literatürdeki PDIS ve UTADIS metotları ile karşılaştırılmıştır. Farklı veri setleri ile yapılan deneysel çalışmalar, önerilen bu metodun göreceli olarak daha büyük veri setlerinde diğer metotlardan daha iyi sonuç verdiğini göstermektedir.

Anahtar Kelimeler: Çok Kriterli Sınıflandırma, Uzaklık Fonksiyonu, Uzaklık Fonksiyonuna Bağlı Sınıflandırma, Eşik Değersiz Sınıflandırma

To My Mother…

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CHAPTER 1

INTRODUCTION

Multi criteria decision aid (MCDA) is a very popular area among researchers from various disciplines. Multi criteria decision making approaches can be applied to real life problems. Therefore, MCDA has numerous application fields (Doumpos and Zopounidis, 2002b).

Obtaining preference information from the decision maker (DM) creates a bottleneck for researchers. When a set of alternatives are considered with multiple objectives, three types of methods can be used to provide information to the DM (Roy, 1996). These are:

(1) Choice methods: for choosing the best alternative from a set of alternatives

(2) Ranking methods: for ranking the alternatives form best to worst

(3) Classification methods: for classifying the alternatives into predefined classes.

Multi criteria sorting problem is a type of classification problem where alternatives are assigned into predefined ordered classes according to their criterion values. Like general multi criteria problems, sorting is a commonly encountered problem in everyday life situations. Some of the application fields are medicine, human resource management, pattern recognition, marketing, financial and risk management, inventory management and energy policies (Doumpos and Zopounidis, 2002b).

Many studies are performed in order to classify alternatives into predefined ordered classes by using different parameters or criteria aggregation functions. Yet to our knowledge there are a few multi criteria sorting methods that do not require class threshold values for each class.

The goal of this study is to develop a multi criteria sorting method that helps DM to assign alternatives to the predefined classes based on distance metrics without estimating class threshold values. Preference information for a small portion of alternatives, which is called training data set, can be obtained from the DM. Centroids can be either estimated for each class or directly obtained from the DM. In the proposed mathematical model only the preference of DM on the training data set and estimated centroid information are used. Weight information for each criterion is the output of the mathematical model. Objective function of the proposed method is to minimize the classification error for training data set. Distance between the centroid of each class and rest of the data set, which is called as test data set, are calculated by using the weight information. According to the distance information, alternatives are assigned to the closest class. In this approach, different distance norms are used to find the most suitable one to assign alternatives to the closest classes. Different performance measures are used to evaluate the performance of the proposed method.

The performance of the proposed method is compared with PDIS (Celik et al., 2015). and UTADIS (Devaud et al., 1980) methods. The computational studies show that with relatively large data sets the proposed method performs better than the other methods.

The remainder of the thesis is organized as follows: literature on sorting problems are reviewed in Chapter 2; theoretical background about proposed method and terminology used in the thesis are given in Chapter 3; solution approach, proposed mathematical models are described in Chapter 4; the details of used data sets, interpretation of mathematical models, computational results are presented in Chapter 5; finally some concluding remarks and future research directions are given in Chapter 6.

CHAPTER 2

LITERATURE REVIEW

Multi criteria decision aid (MCDA) is a well-known branch of decision making and Operations Research. It deals with decision problems in the presence of conflicting objectives. MCDA can be applied to real life problems at the cost of complexity of the model.

Earliest roots of MCDA depends on empirical approaches. Benjamin Franklin's pros and cons analysis in 1770's is an example of these approaches. Decision making problems with multiple criteria are studied and efficiency term is defined by Pareto (1896). During 1940's, utility theory is introduced as one of the most important approaches for MCDA (Von Neumann and Morgenstern, 1944). Efficient set and non-dominated alternative are defined by Koopmans (1951). In 1960's, goal programming is introduced by Charnes and Cooper (1961). Goal programming is an extension of traditional mathematical programming approach. Several different extensions of utility theory for multi criteria decision making were studied by Fishburn (1965). Outranking relation approach was introduced by Roy (1968). During 1970's and 1990's MCDA evolved through theoretical and practical aspects with contribution of information technology and computer science.

DM may design decision problem in different ways. Roy (1968) proposed three different MCDA problems for a decision problem with an alternative set. These problems are choice, ranking and classification/sorting problems. The last problem is divided into classification and sorting. In main aspects, classification and sorting problems are equivalent to each other. The only difference is that classes are defined in a nominal way in classification problems while they are defined in an ordinal way for sorting problems. MCDA methods, developed to address sorting problems, can be divided into two main categories. First category is techniques based on direct interrogation of DM. Second category is preference disaggregation analysis (PDA) methods.

2.1. Techniques Based on Direct Interrogation of Decision Maker

In these methods, DM directly gives the preferential information in order to develop the model. This part of the study is discussed in three subsections: “Methods Based on Outranking Relations”, “Methods Based on Data Envelopment Analysis” and “Methods Based on Utility Functions”.

2.1.1. Methods Based on Outranking Relations

In recent studies, sorting methods based on outranking relations are developed. One of the most important and commonly used method with outranking relations is ELimination Et Choix Traduisant la REalité - ELimination and Choice Expressing the Reality (ELECTRE). This method is initially introduced by Roy (1968). ELECTRE method takes special attention of researchers in the presence of complex real life problems. The method employs a multi-criteria aggregation procedure and uses both weights and possible veto-threshold values (Figueira et al., 2005, chap. 4). ELECTRE method handles different preference modelling cases such as indifference, strict preference, weak preference and incomparability with the concept of concordance and discordance. These cases are also called as pseudo-criterion (Figueira et. al, 2013). Other developed ELECTRE based methods are ELECTRE TRI, ELECTRE TRI-C and ELECTRE TRI-nC. All ELECTRE methods satisfy the conditions which are conformity, homogeneity, monotonicity and stability.

ELECTRE TRI is a member of ELECTRE family which is developed for sorting problems. ELECTRE TRI is first studied by Yu (1992) and it is based on general framework of ELECTRE-III method (Roy, 1991). In this method, limiting profiles are used to define classes. Sorting is performed in two stages. In the first stage, outranking relations are defined according to comparison of alternatives with reference profiles. In the second stage, final classification is made (Doumpos and Zopounidis, 2002a). ELECTRE TRI method finds weights, threshold values, category bounds and cutting level (Figueira et. al, 2013). ELECTRE TRI method employs two different assignment procedures which are optimistic and pessimistic. In this method, both pessimistic and optimistic version of assignment procedures are modelled. ELECTRE TRI method has been applied to a number of real life problems in literature such as financial management, performance evaluations, business and energy management (Govindan and Jepsen, 2016).

Another member of ELECTRE family is ELECTRE TRI-C (Almeida-Dias et al, 2010). In this method, the categories are defined by a single characteristic central profile instead of limiting profiles. Central profile information is obtained from DM (Figueira et. al, 2011). The rest of ELECTRE TRI-C method is the same with ELECTRE TRI method.

ELECTRE TRI-C method is generalized to ELECTRE TRI-nC (Almeida-Dias et al., 2012). In this method, each category is defined by several reference characteristic actions instead of one reference characteristic action which is also called as central profile. This enriches the definition of each category and allows to obtain better ranges for the possible categories.

ELECTRE-SORT method is developed by Ishizaka and Nemery (2014). In this method, classes can be partially ordered. Partially ordered means that central profiles are naturally ordered but this order can change from one criterion to another. Therefore, classes become incomparable. In ELECTRE-SORT method, concordance index, discordance index and degree of credibility are calculated in the same way with the other ELECTRE family methods. In this method, classes are shown by levels and incomparable classes are represented in same level. The assignment is applied by two different rules. These are descending (optimistic) and ascending (pessimistic) assignment rules. Both rules assign alternatives to the levels, not directly to classes. One or more classes can be belong to a level. If there is only one class in that level, alternative is assigned to that class. If there are more than one class, then alternative is indifferent to these classes. Thus, the alternative can be assigned to any class in that level.

FlowSort method is based on the relative position of an alternative according to the reference profiles (Nemery and Lamboray, 2008). This method integrates both limiting profiles and central profiles. The assignment rule of this method depends on global comparison of all profiles. In this method, sorting problem is addressed by means of ranking method which is PROMETHEE method. PROMETHEE method ranks the alternatives using positive, negative or net flow values (Brans and Mareschal, 2002 and Brans and Mareschal, 2005). In FlowSort method, each class is defined by one central profile or two limiting profiles. The method evaluates “preferred to” and “being-preferred to” character of each alternative. To use this characterization, net flow values of alternatives are calculated. According to the class profiles, alternatives are assigned to the classes with respect to their net flow values. This is called as net-flow assignment rule of FlowSort method.

2.1.2. Methods Based on Data Envelopment Analysis

Data Envelopment Analysis (DEA) is used to measure performances of decision making units (DMUs). The method calculates each DMU’s efficiency score and obtain efficiency frontier (Charnes et. al., 1978). In the literature, two-stage DEA method is also formulated. This method evaluates the performance of DMUs considering their performance of internal resource utilization (An et. al., 2016). Although DEA is mainly used for ranking problem, there are few studies that use DEA for sorting purposes as well. Johnson and Zhu (2003) demonstrate how DEA can be evaluated as a fair screening and sorting tool for decision making process.

Aker and Karasakal (2015) developed a multi criteria sorting method based on DEA. The method is applied to the problem of choosing R&D projects. In this method, weight intervals of each criterion are obtained from Interval Analytic Hierarchy Process (Interval AHP). Interval judgments are more rational to the human judgment, easier to use for group decision making and leads less information loss. Therefore, this method applies interval AHP and constructs interval comparison matrices. Interval AHP handles inconsistent comparison matrices. In the method, two threshold estimation and five assignment models are developed. First threshold estimation method, PM1, satisfies the assurance region constraints. Second threshold estimation method, PM2, keeps criterion weights close to each other. First assignment model, APM1, is used for first estimation method PM1. Assignment models, APM2, APM3, APM4 and APM5 are used for second threshold estimation method PM2. APM1 is assurance region used form of basic DEA model. APM2 and APM4 evaluate the unevaluated alternatives individually. APM3 and APM5 evaluate the same alternatives in a single model. In APM3 and APM5, the effects of each alternative can be observed. Therefore the results of APM3 and APM5 can be considered as more realistic and stable than result of APM2 and APM4 methods.

2.1.3. Methods Based on Utility Functions

Analytic Hierarchy Process (AHP) developed by Saaty (1980) to evaluate multiple conflicting criteria combines qualitative and quantitative factors. AHP is a well-defined decision making method for choice and ranking problems. AHPSort is a new variant of AHP for sorting of alternatives to the predefined ordered classes (Ishizaka et al., 2012). AHPSort is based on eight steps. In the first step, goals, criteria and alternatives of the problem are defined. In the second step, classes are defined and ordered. In the third step, local limiting profiles and local central profiles are defined. In the fourth step, importance of criteria are evaluated as pairwise and weights of each criterion are obtained with eigenvalue method of AHP. In the fifth step, a pairwise comparison matrix of each alternative with limiting profiles and central profiles are constructed. In the sixth step, local priority for each alternative and local priority of limiting profiles are derived. In the seventh step, global priority for alternatives and global priority of limiting profiles are derived by aggregating the weighted local priorities. In the final step, all alternatives are classified. With the help of this procedure, AHPSort removes high number of comparison and makes an effective sorting.

2.2. Preference Disaggregation Classification Methods

The main difficulty of MCDA problems is obtaining DM's preference information in the form of values or weights. To obtain preference information from the DM, preference disaggregation classification methods use case-based reasoning (i.e., illustrative cases). These cases may contain:

- Past decisions made by same DM
- Decisions made for limited but realistic set of alternatives
- Decisions made from a representative subset of alternatives (Chen et. al., 2008a)

The main advantage of case-based reasoning tool is explained in Doumpos and Zopounidis (2002b) as "decision makers may prefer to make exemplary decisions than to explain them in terms of specific functional model parameters". Case-based reasoning tool utilizes the DM's point of view and generates better sorting.

Methodological framework of preference disaggregation analysis (PDA) and case-based reasoning depend on minimizing effort required to obtain information from DM and also time required to perform the decision making process (Doumpos and Zopounidis, 2002a). The Preference Disaggregation Classification methods are described in the following three subsections.

2.2.1. Methods Based on Utility Functions

Preference disaggregation method is developed by Jacquet-Lagreze and Siskos (1978). UTilites Additives DIScriminantes (UTADIS) method is introduced in order to solve sorting problems (Devaud et al., 1980). It is developed as a variant of UTilites Additives (UTA) method. UTA method is proposed to ensure optimality with linear programming techniques and this method is mainly used for ranking problems (Jacquet-Lagreze and Siskos, 2001).

The aim of UTADIS method is to develop a criteria aggregation model to determine the sorting of the alternatives. This aggregation method evaluates overall performance of each alternative along all criteria. Criteria aggregation function is expressed as an additive utility function, which is also piecewise linear function for this method. The utility function gives a value to represent overall performance of each alternative and estimates the preference information of DM as well. The method assigns the alternative with the highest score into the best class. Alternatives with lower scores are assigned to worse classes gradually. The main structural parameters of UTADIS method are criteria weights, marginal utility functions and utility thresholds. These parameters are defined by regression based philosophy of preference disaggregation analysis (Zopounidis and Doumpos, 2002a). The objective function of the model is to minimize average classification error value. The error value represents the difference between estimated class and classification of preference set. UTADIS uses single utility function estimation in the beginning of the procedure.

Zopounidis and Doumpos (2000a) proposed a different method which is called the Multi-group Hierarchical Discrimination Method (MHDIS). MHDIS uses utility–based framework like UTADIS. In MHDIS method, two different objective functions are considered. First one is to minimize the total number of misclassification. Second one is to maximize the clearness of the classification (Zopounidis and Doumpos, 2002a). Clearness means maximization of variance among classes. The first and second objectives are optimized in a sequential manner. Hierarchical discrimination is applied in this method. Hierarchical discrimination is a progressive classification process. The process includes several stages and each stage is considered as a two-group classification problem. According to hierarchical discrimination procedure, a number of utility function is developed. In MHDIS method, model development procedure is implemented by two linear programs and one mixed integer program to calibrate the developed model.

2.2.2. Methods Based on Distance Functions

Distance function is another approach to obtain a preference estimation from DM. Methods based on distance functions use distance of each alternative to ideal point or reference point.

Chen et. al. (2007) develop two different case-based sorting models with distance functions. In the first model, they used *right distance concept*. This model depends on cardinal criteria. It uses a reference point, a *centroid* of sorting classes, and distance threshold value, R. If the distance between an alternative and the centroid is less than defined distance threshold, this alternative belongs to that class. However, if the distance to the centroid is greater than distance threshold, the alternative belong to a worse class. In this model, alternative set is defined to construct ellipses. When distance threshold is defined as a constant, R, these ellipses are closer to circles. This model uses Weighted *Euclidean Distance* function since *Euclidean Distance* can be easily interpreted geometrically and understood by DM. In the model, *Squared Euclidean Distance* (i.e., the Euclidean Distance function without square root operation) instead of original Euclidean Distance function is used due to its computational advantage. Objective function of the method is to minimize error terms. Model gives the weight of each criteria, distance threshold vectors, an upper bound and a lower bound for error terms.

In addition to these, the second model considers ordinal criteria. Basic and most efficient way to handle ordinal criteria is directly asking preferences of DM and connecting ordinal data to cardinal data. In the second model, criteria are divided into two parts as cardinal and ordinal. Distance function values are calculated accordingly.

Chen et al. (2008a) integrated the previously explained method with ABC analysis and it is applied to the inventory planning and controlling problem. Post-optimality analysis is also performed for the new method. Chen et al. (2008b) applied screening procedure via case based distance method for different MCDA problems. Screening procedure selects candidates of best class and eliminates the other alternatives. This method decreases the effort spent to evaluate alternatives

Soylu (2011) developed a multi criteria sorting procedure based on Tchebycheff function as utility function. Tchebycheff function can find efficient alternatives even they are located in non-convex part of the efficient frontier. This method uses a strategy to provide the most optimistic condition for the alternatives and assigns favorable weights to alternatives. To be a candidate member of a class, the alternative should outperform some of the reference alternatives with its own weights. The study considers both known and unknown criteria weights cases.

Another distance based sorting approach is developed by Celik et al. (2015). The proposed solution has two phases. These phases are called Model-1 and Model-2. In Model-1, maximum and minimum threshold values of each class and weight set for each criterion, which depends on training data set, are obtained. The objective function of Model-1 is to minimize classification error. In Model-2, maximum and minimum of class thresholds determined in Model-1 and weights of each criterion are taken as parameters. Total classification error is taken as upper bound for new model. In Model-2, test data set is also included with training data set. Model-2 result gives maximum and minimum values for each alternative in the test data set. According to the Model-1 and Model-2 results, alternatives are assigned to the possible classes.

2.2.3. Interactive Methods

Koksalan and Ulu (2003) developed an interactive procedure for placing alternatives in preference classes. The method is based on the assumption that DM has a linear utility function. The developed algorithm calculates importance weight of criteria according to DM's utility function for each iteration. If utility value of an alternative is between the lower and the upper bound values of the class, the alternative is assigned to that class. If utility value of an alternative is greater than the upper bound of the class, the alternative is assigned to a higher class. And if the utility value of an alternative is less than the lower bound of the class, the alternative is assigned to a lower class. In the algorithm, a mathematical model is evaluated to check the convex dominance between the alternatives in each iteration.

Koksalan and Ozpeynirci (2009) proposed a new interactive sorting method for additive utility functions to improve their previously explained study. This method guarantees to place all alternatives to the correct classes considering DM's preferences as additive utility function. In this method, one alternative is assigned to a class in each iteration using linear models. For each alternative the worst and the best possible classes are determined.

In this thesis, a new distance based sorting without class threshold method is proposed. The method utilizes distance function as criteria aggregation function. The proposed method is unique in terms of both using the centroid estimation for each class and assigning alternatives without using class thresholds. And also 35 % of the whole data set which is randomly selected is used to estimate the weights of each criterion. The rest of alternatives in the data set are assigned to the closest classes by using these weight information.

CHAPTER 3

THEORETICAL BACKGROUND

In this chapter, theoretical background about proposed method and terminology used in the thesis are given. In Section 3.1, distance norms and their formulations are defined. In Section 3.2, PDIS Method that employs weighted distance norms as criteria aggregation function is described (Celik et al., 2015). In Section 3.3, details of UTADIS Method is explained.

3.1. Distance Norms

The formulation of weighted distance between two points, X and Y, in p-norm, $D^p(X,Y)$, is given in Equation (3-1):

$$D^p(X,Y) = \left[\sum_{q=1}^{Q}(w_q\left|X_q - Y_q\right|)^p\right]^{1/p} \qquad (3\text{-}1)$$

where X$= \{X_1, X_2, \dots, X_q, \dots, X_Q\}$, $Y = \{Y_1, Y_2, \dots, Y_q, \dots, Y_Q\}$ are the coordinates of points X and Y in Q dimensional vector space.

When distance norm, p, is equal to one, the distance equation transforms into *Rectilinear Distance* form. Rectilinear distance is also called as *Manhattan Distance*. The formal formulation of Rectilinear Distance function, $D^1(X,Y)$, is given in Equation (3-2):

$$D^1(X,Y) = \sum_{q=1}^{Q} w_q\left|X_q - Y_q\right| \qquad (3\text{-}2)$$

When p is equal to two, the distance equation becomes the well-known *Euclidean Distance* given in Equation (3-3).

$$D^2(X,Y) = \sqrt{\sum_{q=1}^{Q}(w_q\left|X_q - Y_q\right|)^2} \qquad (3\text{-}3)$$

When p gets higher values, the larger valued weighted difference becomes dominant. Therefore, when p goes to infinity, the distance is equal to the greatest of the difference of vectors along any coordinate dimension. This distance is called as *Tchebycheff distance*. The formal formulation of Tchebycheff Distance function, $D^{\infty}(X,Y)$, is given in Equation (3-4):

$$D^{\infty}(X,Y) = max(w_q|X_q - Y_q|) \qquad (3\text{-}4)$$

Rectilinear distance gives an upper bound for the distance between two different points while Tchebycheff distance provides a lower bound.

3.2. A Probabilistic Distance Based Sorting (PDIS) Method

Celik et al. (2015) developed a preference disaggregation analysis method (PDIS) to help DM to sort alternatives. In this method, weighted distance to the ideal point which is the optimum point for all criteria is used as the criteria aggregation function. Weight of each criterion and class threshold values are estimated by using case-based reasoning. The solution approach of PDIS method has two phases which are called Model-1 and Model-2. Model-1 uses only training data set and Model-2 uses both training data set and test data set. The classification of the alternatives are performed with minimization of the classification error rule. In Model-1, total classification error value, maximum and minimum threshold values of each class are obtained. Model-1 outputs are used as input to Model-2. This model gives maximum and minimum distances to the ideal point for each alternative in test data set. Alternatives are assigned to the classes according to both Model-1 and Model-2 results.

The method gives a probability for each alternative to belong to a class. Probabilistic approach is also incorporated into UTADIS method in order to handle alternative optimal solutions. Details of Model-1 and Model-2 are explained in following subsections.

Model-1

In Model-1, class thresholds and target classification error values are determined. Different weight vectors are used to find maximum and minimum threshold values of each class. The objective function of Model-1 is to minimize total classification error of each alternative in training data set.

Indices, parameters and decision variables used in this model are defined below.

Indices:

n= Number of classes

q= Number of criteria

K= Number of alternatives in preference set

$t = \begin{cases} 1 \text{ for maximum threshold values} \\ 2 \text{ for minimum threshold values} \end{cases}$

$i \in \{1,2, ..., n\text{-}1\}$ for thresholds

$j \in \{1,2, ..., q\}$ for criteria

$k \in \{1,2, ..., K\}$ for alternatives in preference set

Parameters:

a_{kj}= Value of alternative k on criterion j

I_j^*= Value of ideal point on criterion j

∂ = A small constant (0.005)

C_i= Set of alternatives in preference set which belongs to class i

Decision Variables:

w_{tj} = *Weight* of criterion j

e_{kt}^{+} = Error of assignment of alternative k to a lower class

e_{kt}^{-} = Error of assignment of alternative k to a higher class

$Tmax_i$ = Maximum value of threshold i separating class i and i+1

$Tmin_i$ = Minimum value of threshold i separating class i and i+1

The formulation of Model-1 is given below.

Model-1:

Min $\sum_{t=1}^{2}\sum_{k=1}^{K}(e_{kt}^{+} + e_{kt}^{-}) + \partial \sum_{i=1}^{n-1}(Tmin_i - Tmax_i)$

Subject to:

$$\left[\sum_{j=1}^{q}(w_{1j}\left|a_{kj} - I_j^{*}\right|)^{p}\right]^{1/p} - e_{k1}^{+} \leq Tmin_1 \quad \forall k \in C_1 \tag{3-6}$$

$$\left[\sum_{j=1}^{q}(w_{1j}\left|a_{kj} - I_j^{*}\right|)^{p}\right]^{1/p} + e_{k1}^{-} \geq Tmin_{i-1} \quad \forall k \in C_i \ (i = 2,3,\dots,n) \tag{3-7}$$

$$\left[\sum_{j=1}^{q}(w_{1j}\left|a_{kj} - I_j^{*}\right|)^{p}\right]^{1/p} - e_{k1}^{+} \leq Tmin_i \quad \forall k \in C_i \ (i = 2,3,\dots,n-1) \tag{3-8}$$

$$\left[\sum_{j=1}^{q}(w_{2j}\left|a_{kj} - I_j^{*}\right|)^{p}\right]^{1/p} - e_{k2}^{+} \leq Tmax_1 \quad \forall k \in C_1 \tag{3-9}$$

$$\left[\sum_{j=1}^{q}(w_{2j}\left|a_{kj} - I_j^{*}\right|)^{p}\right]^{1/p} + e_{k2}^{-} \geq Tmax_{i-1} \quad \forall k \in C_i \ (i = 2,3,\dots,n) \tag{3-10}$$

$$\left[\sum_{j=1}^{q}(w_{2j}\left|a_{kj} - I_j^{*}\right|)^{p}\right]^{1/p} - e_{k2}^{+} \leq Tmax_i \quad \forall k \in C_i \ (i = 2,3,\dots,n-1) \tag{3-11}$$

$$Tmax_i \geq Tmax_{i-1} \quad \forall i \in \{1,2,\dots,n-1\} \tag{3-12}$$

$$Tmin_i \geq Tmin_{i-1} \quad \forall i \in \{1,2,\dots,n-1\} \tag{3-13}$$

$$Tmax_i \geq Tmin_i \quad \forall i \in \{1,2,\dots,n-1\} \tag{3-14}$$

$$\sum_{j=1}^{q} w_{tj} = 1 \quad \forall t \in \{1,2\} \tag{3-15}$$

$$w_{tj} \geq 0 \quad \forall t \in (1,2); j \in \{1,2,\dots,q\} \tag{3-16}$$

$$e_{kt}^{+} \geq 0 \quad \forall t \in (1,2); k \in \{1,2,\dots,K\} \tag{3-17}$$

$$e_{kt}^{-} \geq 0 \quad \forall t \in (1,2); k \in \{1,2,\dots,K\} \tag{3-18}$$

In Model-1, two different criteria aggregation functions are used. To explore alternative solutions for minimum and maximum threshold values, an index, *t*, is used in the model. Index t is defined with 1 and 2 values. One of them gives the minimum threshold values for classes and the other gives the maximum threshold values of each class. In this model, primary objective is to minimize total classification error. The secondary objective is to maximize total range of threshold values that is to minimize the minimum threshold value and to maximize the maximum threshold values. Constraint sets (3-6) and (3-9) are defined for the alternatives which are assigned to the Class 1 by the DM. They determine the minimum threshold value and the maximum threshold values for Class 1, respectively. Constraint sets (3-7), (3-8), (3-10) and (3-11) are defined for the alternatives which are assigned to the Class 2 to Class n by DM. They determine the minimum and the maximum threshold values for Class 2 to Class n. Constraint sets (3-12) and (3-13) ensure an order of threshold values. Constraint set (3.14) ensures that the maximum threshold values must be greater than corresponding minimum threshold values. Equation (3-15) determines that sum of the weights must be equal to 1. Constraint set (3-16), (3-17) and (3-18) are non-negativity constraints.

The value of total classification error in the optimal solution of Model-1 is used as an upper bound in the Model-2 which is given in Equation (3-19).

$$\sum_{t=1}^{2}\sum_{k=1}^{K}(e_{kt}^{+} + e_{kt}^{-}) = E^{*} \quad (3\text{-}19)$$

<u>Model-2</u>

Objective function of Model-2 is to find the maximum and minimum values of alternatives in the test data set.

<u>Additional Indices:</u>

L= number of alternatives in test set

$l \in \{1,2, ..., L\}$ for alternatives in test set

<u>Additional Parameters:</u>

a_{lj} = Score of alternative l on criterion j

E^{*} = Total classification error

$Tmax_i$ = Maximum values of threshold i separating class i and i+1

$Tmin_i$ = Minimum values of threshold i separating class i and i+1

Decision Variables:

w_{ltj} = *Weight* of criterion j

e_{klt}^{+} = Error of assignment of alternative k to a lower class

e_{klt}^{-} = Error of assignment of alternative k to a higher class

T_{lti} = Value of threshold i

$Vmax_i$ = Maximum value of alternative i in test set

$Vmin_i$ = Minimum value of alternative i in test set

The formulation of Model-2 is given below.

Model-2:

Min $\sum_{l=1}^{L}(Vmin_l - Vmax_l)$

Subject to:

$$\left[\sum_{j=1}^{q}\left(w_{ltj}\left|a_{kj} - I_j^*\right|\right)^p\right]^{1/p} - e_{klt}^{+} \leq T_{lt1}$$

$\forall\ k \in C_1; \forall\ t \in \{1,2\};\ \forall\ l \in \{1,2,\dots,L\}$

$$\left[\sum_{j=1}^{q}\left(w_{ltj}\left|a_{kj} - I_j^*\right|\right)^p\right]^{1/p} + e_{klt}^{-} \geq T_{lti-1}$$

$\forall\ k \in C_i\ (i = 2,3,\dots,n); \forall\ t \in \{1,2\}; \forall\ l \in \{1,2,\dots,L\}$

$$\left[\sum_{j=1}^{q}\left(w_{ltj}\left|a_{kj} - I_j^*\right|\right)^p\right]^{1/p} - e_{klt}^{+} \leq T_{lti}$$

$\forall\ k \in C_i\ (i = 2,3,\dots,n-1);\ \forall\ t \in \{1,2\}; \forall\ l \in \{1,2,\dots,L\}$

$$\left[\sum_{j=1}^{q}\left(w_{l1j}\left|a_{lj} - I_j^*\right|\right)^p\right]^{1/p} \geq Vmax_l \quad \forall\ l \in \{1,2,\dots,L\}$$

$$\left[\sum_{j=1}^{q}\left(w_{l2j}\left|a_{lj} - I_j^*\right|\right)^p\right]^{1/p} \leq Vmin_l \quad \forall\ l \in \{1,2,\dots,L\}$$

$T_{lti} \geq T_{lti-1} \quad \forall\, t \in \{1,2\}, \forall\, l \in \{1,2,\dots,L\}, \forall\, i \in \{2,3,\dots,n-1\}$

$\sum_{j=1}^{C} w_{ltj} = 1 \quad \forall\, t \in \{1,2\}, \forall\, l \in \{1,2,\dots,L\}$

$\sum_{l=1}^{L} \sum_{t=1}^{2} \sum_{k=1}^{K} (e_{kt}^{+} + e_{kt}^{-}) \leq LE^{*}$

$Tmax_i \geq T_{lti} \qquad \forall\, i \in \{1,2,\dots,n-1\};\ \forall\, t \in \{1,2\};\ \forall\, l \in \{1,2,\dots,L\}$

$Tmin_i \leq T_{lti} \qquad \forall\, i \in \{1,2,\dots,n-1\};\ \forall\, t \in \{1,2\};\ \forall\, l \in \{1,2,\dots,L\}$

$w_{ltj} \geq 0 \qquad \forall\, t \in (1,2); j \in \{1,2,\dots,q\};\ \forall\, l \in \{1,2,\dots,L\}$

$e_{klt}^{+} \geq 0 \qquad \forall\, t \in (1,2);\ \forall\, l \in \{1,2,\dots,L\};\ k \in \{1,2,\dots,K\}$

$e_{klt}^{-} \geq 0 \qquad \forall\, t \in (1,2);\ \forall\, l \in \{1,2,\dots,L\};\ k \in \{1,2,\dots,K\}$

After solving Model-1 and Model-2, optimum values of the maximum and the minimum values of each alternative and the maximum threshold value and the minimum threshold value of each class are found. These values are used to assign each alternative in the test data set to the classes. Model-1 and Model-2 are valid for all *p* norms except for $p=\infty$. When distance norm becomes infinity, distance function becomes Tchebycheff distance function (which is equation (3-4)). Therefore, the formulations of Model-1 and Model-2 are modified in order to use Tchebycheff distance as follows. Some variables and parameters are added to the Tchebycheff model and the objective function is modified as explained below.

Model-1

Additional Decision Variables:

Dp_{kt} = Weighted Tchebycheff distance of alternative k to ideal point

Additional Parameters:

∂ = A small constant (0.005)

β = A small positive constant greater than ∂ (0.01)

Model-1*:*

Min $\beta \sum_{t=1}^{2} \sum_{k=1}^{K} (e_{kt}^{+} + e_{kt}^{-}) + \partial \sum_{i=1}^{n-1} (Tmin_i - Tmax_i) + \sum_{k=1}^{K} \sum_{t=1}^{2} Dp_{kt}$

Subject to:

$Dp_{kt} \geq w_{tj} \left| a_{kj} - I_j^* \right| \quad \forall k \in \{1,2, \dots, K\}; \forall t \in (1,2); j \in \{1,2, \dots, q\}$

$Dp_{k1} - e_{k1}^{+} \leq Tmin_1 \quad \forall k \in C_1$

$Dp_{k1} + e_{k1}^{-} \geq Tmin_{i-1} \quad \forall k \in C_i \ (i = 2,3, \dots, n)$

$Dp_{k1} - e_{k1}^{+} \leq Tmin_i \quad \forall k \in C_i \ (i = 2,3, \dots, n-1)$

$Dp_{k2} - e_{k2}^{+} \leq Tmax_1 \quad \forall k \in C_1$

$Dp_{k2} + e_{k2}^{-} \geq Tmax_{i-1} \quad \forall k \in C_i \ (i = 2,3, \dots, n)$

$Dp_{k2} - e_{k2}^{+} \leq Tmax_i \quad \forall k \in C_i \ (i = 2,3, \dots, n-1)$

$Tmax_i \geq Tmax_{i-1} \quad \forall i \in \{1,2, \dots, n-1\}$

$Tmin_i \geq Tmin_{i-1} \quad \forall i \in \{1,2, \dots, n-1\}$

$Tmax_i \geq Tmin_i \quad \forall i \in \{1,2, \dots, n-1\}$

$\sum_{j=1}^{q} w_{tj} = 1 \quad \forall t \in \{1,2\}$

$w_{tj} \geq 0 \quad \forall t \in (1,2); \ \forall j \in \{1,2, \dots, q\}$

$e_{kt}^{+} \geq 0 \quad \forall t \in (1,2); \forall k \in \{1,2, \dots, K\}$

$e_{kt}^{-} \geq 0 \quad \forall t \in (1,2); \forall k \in \{1,2, \dots, K\}$

In this model criteria aggregation function is referred as Dp_{kt}.

Model-2

Additional Decision Variables:

Dp_{klt} = Weighted Tchebycheff distance of alternative k to ideal point

Dt_{lt} = Weighted Tchebycheff distance of alternative l to ideal point

Additional Parameters:

∂ = A small constant (0.005)

Model-2:

Min $\partial \sum_{l=1}^{L}(Vmin_l - Vmax_l) + \sum_{l=1}^{L}\sum_{k=1}^{K}\sum_{t=1}^{2} Dp_{klt} + \sum_{l=1}^{L}\sum_{t=1}^{2} Dt_{lt}$

Subject to:

$Dp_{klt} \geq w_{ltj}|a_{kt} - I_j^*|$

$\forall k \in \{1,2,\dots,K\}; \forall t \in (1,2); j \in \{1,2,\dots,q\}; \forall l \in \{1,2,\dots,L\}$

$Dt_{lt} \geq w_{ltj}|a_{lt} - I_j^*| \quad \forall t \in (1,2); j \in \{1,2,\dots,q\}; \forall l \in \{1,2,\dots,L\}$

$Dp_{klt} - e_{klt}^{+} \leq T_{lt1} \quad \forall k \in C_1; \forall t \in \{1,2\}; \forall l \in \{1,2,\dots,L\}$

$Dp_{klt} + e_{klt}^{-} \geq T_{lti-1} \quad \forall k \in C_i \ (i = 2,3,\dots,n); \forall t \in \{1,2\}; \forall l \in \{1,2,\dots,L\}$

$Dp_{klt} - e_{klt}^{+} \leq T_{lti} \quad \forall k \in C_i \ (i = 2,3,\dots,n-1); \forall t \in \{1,2\}; \forall l \in$
$\{1,2,\dots,L\}$

$Dt_{l1} \geq Vmax_l \quad \forall l \in \{1,2,\dots,L\}$

$Dt_{l2} \leq Vmin_l \quad \forall l \in \{1,2,\dots,L\}$

$\sum_{j=1}^{C} w_{ltj} = 1 \quad \forall t \in \{1,2\}; \forall l \in \{1,2,\dots,L\}$

$\sum_{l=1}^{L}\sum_{t=1}^{2}\sum_{k=1}^{K}(e_{klt}^{+} + e_{klt}^{-}) \leq LE^*$

$Tmax_i \geq T_{lti} \quad \forall i \in \{1,2,\dots,n-1\}; \forall t \in \{1,2\}; \forall l \in \{1,2,\dots,L\}$

$Tmin_i \leq T_{lti} \quad \forall i \in \{1,2,\dots,n-1\}; \forall t \in \{1,2\}; \forall l \in \{1,2,\dots,L\}$

$w_{ltj} \geq 0 \quad \forall t \in (1,2); j \in \{1,2,\dots,q\}; \forall l \in \{1,2,\dots,L\}$

$e_{klt}^{+} \geq 0 \quad \forall t \in (1,2); \forall l \in \{1,2,\dots,L\}; \forall k \in \{1,2,\dots,K\}$

$e_{klt}^{-} \geq 0 \quad \forall t \in (1,2); \forall l \in \{1,2,\dots,L\}; \forall k \in \{1,2,\dots,K\}$

3.3. UTilities Additives DIScriminates (UTADIS) Method

The UTADIS method was introduced for sorting problems by Devaud et al. (1980). This method is a variation of well-known UTA (UTilities Additives) method which is used for ranking problem. In this method, C_1 class is the best class and C_q is the worst class. Within the sorting framework, the UTADIS method develops a criteria aggregation function to represent overall performance of each alternatives. The criteria aggregation function is in the form of additive utility function. The global utility function is given in Equation (3-20).

$$U(a) = \sum_{j=1}^{q} w_j u_j(a_j) \qquad \text{(3-20)}$$

where $a = \{a_1, a_2, \ldots, a_q\}$ represents the vector for evaluation criteria, w_j represents weights of each criterion and $u_j(a_j)$ represent the marginal utility function. Sum of the weights is equal to 1. Marginal utility function shows the value score of a_j over criterion j according to the DM's preferences. UTADIS Method aims to estimate marginal utility function scores with minimum classification error.

In this method, marginal utility functions are defined as monotone functions. The following two conditions, Equation (3-21), must be satisfied:

$$u_j(a_{j*}) = 0\ ;\ u_j\left(a_j^*\right) = 1 \qquad \text{(3-21)}$$

a_{j*} represents the least preferred criterion value and a_j^* represents the most preferred criterion value.

To avoid the nonlinearity, the global utility function is transformed into following Equation (3-22):

$$U(a) = \sum_{j=1}^{q} u_j'(a_j) \qquad \text{(3-22)}$$

In the equation, $u_j'(a_j)$ is equal to $w_j u_j(a_j)$. According to the Equation (3-21), $u_j'(a_j^*)$ becomes w_j and $u_j'(a_{j*})$ becomes 0. In this form of global utility function, the overall values are in the range of *(0, 1)* and marginal utility values are in the range of $(0, w_j)$ for criterion *j*. To obtain best estimation for marginal utility function, it is modelled as a piece-wise linear function. Each criterion is divided into subintervals. p_j represents the number of breakpoints for piece-wise linear function. And number of subintervals calculated by number of breakpoints minus 1, p_j-1. The marginal utilities are calculated by using breakpoint estimation according to Equations (3-23) and (3-24):

$$w_{js} = u_j'(a_j^s) - u_j'(a_j^{s-1}) \qquad (3\text{-}23)$$

$$u_j'(a_j^h) = \sum_{s=1}^{h-1} w_{js} \qquad (3\text{-}24)$$

w_{js} is the utility value for criterion j in interval s. The score of an alternative k for criterion j is represented by a_{kj} and its marginal utility function can be found using the linear interpolation in Equation (3-25)

$$u_j'(a_{kj}) = \sum_{s=1}^{r_{kj}} w_{js} + w_{j,r_{kj}} \frac{a_{jk} - a_j^{r_{kj}}}{a_j^{r_{kj+1}} - a_j^{r_{kj}}} \qquad (3\text{-}25)$$

where r_{kj} represents the subinterval which alternative k belongs the subinterval for criterion j. Global utility function is given in Equation (3-26)

$$U(a_k) = \sum_{j=1}^{q} (\sum_{s=1}^{r_{kj}} w_{js} + w_{j,r_{kj}} \frac{a_{jk} - a_j^{r_{kj}}}{a_j^{r_{kj+1}} - a_j^{r_{kj}}}) \qquad (3\text{-}26)$$

The objective function of the method is to minimize classification error of alternatives in the training data set. To calculate these errors, two different error terms are defined for each alternative.

$$\varepsilon_k^+ = \max\{0, u_i - U(a_k)\} \qquad \forall\, k \in C_i, i = 1, \ldots, n-1$$

$$\varepsilon_k^- = \max\{0, U(a_k) - u_{i-1}\} \qquad \forall\, k \in C_i, i = 2, \ldots, n$$

ε_k^+ indicates that to assign a misclassified alternative k correctly, its global utility function value should be increased by $u_i - U(a_k)$ units. On the other hand, ε_k^- indicates that to assign a misclassified alternative k correctly, its global utility function value should be decreased by $U(a_k) - u_{i-1}$units.

The linear programming model which finds the optimal utility values of each criterion breakpoints and class thresholds while minimizing total classification error is given below.

Indices:

n= Number of classes

q= Number of criteria

K= Number of alternatives in training set

$i \in \{1,2, ..., n\}$ for classes
$j \in \{1,2, ..., q\}$ for criteria

$k \in \{1,2, ..., K\}$ for alternatives in training set

$s \in \{1,2, ..., p_j\text{-}1\}$ for intervals on each criterion

Parameters:

r_{jk} = subinterval number that alternative k belong on criterion j

C_i = set of alternatives in training data set that belongs to class i

x_k = alternative k in training data set

m_i = number of alternatives that belong to class i

a_{jk} = score of an alternative k criterion j

a_j^t = breakpoint t on criterion j

p = small positive constant (0.001)

δ_1 = small positive constant (0.0001)

δ_2 = small positive constant (0.001)

Decision Variables:

w_{js} = utility value of interval s and criterion j

u_i = threshold value between class i and class i+1

ε_k^+ = classification error of alternative k to a lower class

ε_k^- = classification error of alternative k to a higher class

UTADIS Model:

$$\min \ \sum_{i=1}^{n} \left[\frac{\sum_{\forall x_k \in C_i} (\varepsilon_k^+ + \varepsilon_k^-)}{m_i} \right]$$

Subject to:

$$\sum_{j=1}^{q} \left[\sum_{s=1}^{r_{kj}} w_{js} + w_{j,r_{kj}} \frac{a_{jk} - a_j^{r_{kj}}}{a_j^{r_{kj+1}} - a_j^{r_{kj}}} \right] - u_1 + \varepsilon_k^+ \ \geq \ \delta_1 \quad \forall \, x_k \in C_1$$

$$\sum_{j=1}^{q} \left[\sum_{s=1}^{r_{kj}} w_{js} + w_{j,r_{kj}} \frac{a_{jk} - a_j^{r_{kj}}}{a_j^{r_{kj+1}} - a_j^{r_{kj}}} \right] - u_i + \varepsilon_k^+ \ \geq \ \delta_1 \quad \forall \, x_k \in C_i (2, \dots, n-1)$$

$$\sum_{j=1}^{q} \left[\sum_{s=1}^{r_{kj}} w_{js} + w_{j,r_{kj}} \frac{a_{jk} - a_j^{r_{kj}}}{a_j^{r_{kj+1}} - a_j^{r_{kj}}} \right] - u_{i-1} - \varepsilon_k^- \ \leq -\, \delta_2 \quad \forall \, x_k \in C_i (2, \dots, n-1)$$

$$\sum_{j=1}^{q} \left[\sum_{s=1}^{r_{kj}} w_{js} + w_{j,r_{kj}} \frac{a_{jk} - a_j^{r_{kj}}}{a_j^{r_{kj+1}} - a_j^{r_{kj}}} \right] - u_{n-1} - \varepsilon_k^- \ \leq -\, \delta_2 \quad \forall \, x_k \in C_n$$

$$\sum_{j=1}^{q} \sum_{s=1}^{p_j - 1} w_{js} = 1$$

$$u_i - u_{i+1} \geq p \quad \forall \, i = 1, \dots, n-2$$

$$w_{js} \ \geq 0 \quad \forall \, j = 1, \dots, q, s = 1, \dots, p_j - 1$$

$$\varepsilon_k^- \ \geq 0$$

$$\varepsilon_k^+ \ \geq 0$$

The model gives the optimal values of w_{js} and u_i. By using these values, utility values of each alternative in the test data set is calculated and assignment to the classes are made.

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CHAPTER 4

SOLUTION APPROACH

In this study, a distance based sorting method is proposed. The method uses distance functions as the criteria aggregation function. A mathematical model is formulated to determine the DM's criteria weights. Without specifying class thresholds, the alternatives are assigned to the closest classes by using these obtained weights.

Solution approach of the proposed method and terminology used in the thesis are given in this chapter. In Section 4.1, overview of the distance based sorting method is given. In Section 4.2, notation used in the proposed method is defined. In Section 4.3, details of mathematical model for Lp norm is explained. In Section 4.4, details of mathematical model for L∞ norm, the differences between Lp norm model and L∞ norm model are explained.

4.1. Overview of Distance Based Sorting Without Class Threshold Method (DISWOTH)

In sorting models, preference information is obtained from the DM by using training data set and/or evaluating her/his past decisions. Next, the DM's preference information is used to estimate parameters of the method. Some methods such as UTADIS use class thresholds, some methods such as ELECTRE methods use limiting profiles to assign alternatives to the classes. A few sorting methods use centroids to represent each class. Centroid is a symbolic alternative which is used to represent the alternatives in all classes. Distance function based methods which are developed by Chen et al. (2007, 2008a and 2008b) use one centroid which represents all classes to assign alternatives.

This study proposes a new multi criteria sorting method that uses class centroids and distance functions and minimize average classification error. Classification error is defined as difference between weighted distance of an alternative to the centroid of actual class and weighted distance of that alternative to the centroid of predicted class if the alternative is incorrectly classified. Distance Based Sorting without Class Threshold Method (DISWOTH) does not estimate any class threshold value. Alternatives are assigned to one class or a set of possible adjacent classes considering their distances to class centroids.

In the DISWOTH method, DM is asked to classify the alternatives in training data set into the classes. Training data set is a part of the data set that preference information of DM is obtained. The rest of data set is called test data set. Class centroids are estimated by using training data set. Using the training data set and estimation of class centroids, the weights of the criteria are determined. A mathematical model is formulated to estimate the criterion weights. Objective function of mathematical model minimizes average classification error of alternatives. Weighted distance between alternatives in the test data set and the class centroids are calculated by using obtained weight information. Assignment is performed according to the weighted distance of each alternative to the class centroids. This solution approach is summarized in Figure 4.1. Steps of the solution approach are explained in the following subsections.

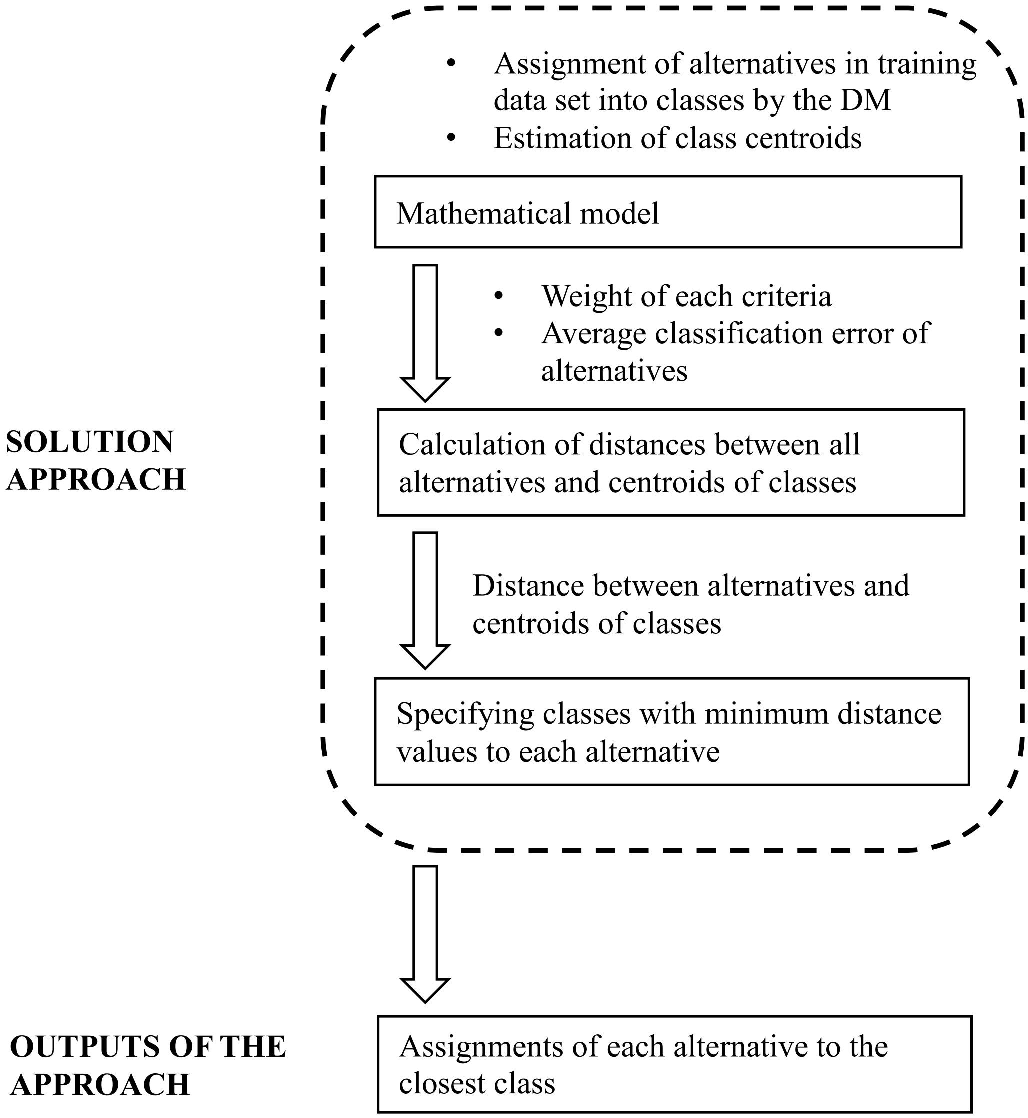

Figure 4.1. Steps of the solution approach

Overview of the solution approach that assigns alternatives to their closest classes is given in Figure 4.1. Figure 4.2 shows how the solution approach is applied to obtain the final assignment.

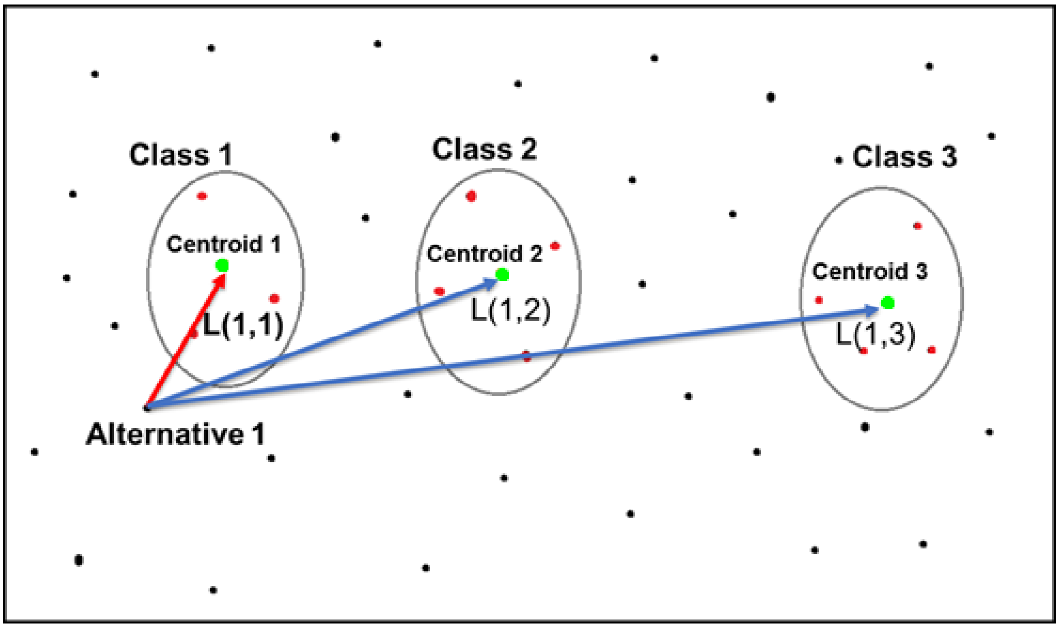

Figure 4.2 Sorting of alternatives to the classes

In the beginning, alternatives in the training data set are assigned to the classes by DM. Next, class centroids are estimated by using this assignment information and mathematical model is used to determine criterion weights. Distances between alternatives in the test data set and class centroids are calculated by using criterion weights. The closest class centroid is determined for each alternative in the test data set. Finally, alternatives are assigned to the closest class.

4.2. Notation

The DISWOTH approach sorts alternatives in the training set $\{ X_1, X_2, \dots, X_K\}$ into N predefined classes $\{ C_1, C_2, \dots, C_N\}$ with minimum classification error. Classes are ordered for sorting problem such as C_1 represents the most preferred class and C_N represents the least preferred class or C_N represent the best class and C_1 represents the worst class. Class centroids are denoted as $\{ y_1, y_2, \dots, y_n, \dots, y_N\}$. In the DISWOTH method, centroids are calculated by averaging of criteria values of alternatives belonging to the classes. Each alternative is defined with a set of criteria; A$=\{ a_1, a_2, \dots, a_q\}$ such as $X_k = \{ a_{k1}, a_{k2}, \dots, a_{kq}\}$ where a_{kj} shows the score of alternative k on criterion j. Criterion weights are defined as $\{ w_1, w_2, \dots, w_q\}$. In the DISWOTH method, distance function is used as preference aggregation function. By using Equation (3-1), Distance, $D^p(X_k, y_n)$, between an alternative X_k and class centroid y_n in Lp norm is calculated as

$$D^p(X_k, y_n) = \left[\sum_{j=1}^{q} (w_j |a_{kj} - y_n|)^p\right]^{1/p} \quad (3\text{-}1)$$

4.3. Mathematical Model for Lp Norm

The objective function of mathematical model is to minimize average classification error. Model uses assignment information of alternatives in the training data set and estimation of class centroids as inputs. Mathematical model determines criterion weights by using these inputs. Distance between each alternative in the test data set and class centroids are determined by using this weight information.

Mathematical model considers distance function as preference disaggregation function. When distance norm is equal to one or infinity, mathematical model becomes a linear programming model. For the other distance norms, two or three, the model becomes nonlinear. The general form of the mathematical model, which is called *Lp Model*, is as follows.

Definition of the decision variables and parameters of the Lp Model is given below:

Indices:

K = Number of alternatives in training data set

q = Number of criteria

N= Number of class

$k \in \{1,2, \dots ,K\}$ for alternatives in training data set

$j \in \{1,2, \dots ,q\}$ for criteria

$n \in \{1,2, \dots ,N\}$ for classes

Parameters:

p = Distance function norm

r = Small positive constant to avoid equality case (0.001)

a_{kj} = jth criterion value of alternative k in training data set

C_n = Number of alternatives assigned to class n in training data set

y_n = Centroid of class n

Calculated by averaging of a_{kj} for each criterion.

$D^p(X_k, y_n)$= Distance function value between alternative k and class centroid n according to distance norm p.

Decision Variables:

$\mathcal{E}_k$ = classification error of alternative k

w_j = weight of criterion j

z= objective function value

Lp Model

Min $z = \frac{1}{N}\sum_{n=1}^{N}\frac{\sum_{\forall k \in C_n} \varepsilon_k}{C_n}$

Subject to:

$D^p(X_k, y_n) - \varepsilon_k \leq D^p(X_k, y_t) - r \quad k \in \{1,2, \dots, K\};\ \forall\, t \neq n \in \{1,2, \dots, N\}$ (4-1)

$\sum_{j=1}^{q} w_j = 1$ (4-2)

$\alpha_k \geq 0 \quad \forall\, k \in \{1,2, \dots, K\}$ (4-3)

$w_j \geq 0 \quad \forall\, j \in \{1,2, \dots, q\}$ (4-4)

Objective function of the Lp Model is to minimize the average classification error. To obtain average classification error, first all classification errors of alternatives in a class is summed and this summation is divided by the number of alternatives in the class. Then, obtained values for each class is also summed and divided by the number of classes to calculate average classification error. Equation (4-1) ensures that the distance between an alternative in the training data set and the class centroid which it is assigned is less than the distances between the alternative and the other class centroids. Equation (4-2) ensures sum of criterion weights is equal to one. Equation (4-3) and (4-4) are the non-negativity constraints of Lp Model.

Lp Model determines a weight vector that minimizes the average classification error. The criterion weights are used to calculate the distances between alternatives in the test data set and all class centroids. According to the distance values, alternatives are assigned to the classes with minimum distances.

4.4. Mathematical Model for L∞ Norm

Lp Model, which is explained in Section 4.3, is valid for all L_p distance norms. However, when distance norm is equal to infinity, Lp Model is modified to use Tchebycheff distance function as preference aggregation function. The formulation of the Tchebycheff distance is given in Equation (3-4)

Tchebycheff model, which is called *L∞ Model*, is a linear programming model. Definition of additional decision variables and parameters of the L∞ Model is given below:

Parameters:

b = small positive constant greater than *r* (0.005)

Decision Variables:

L_{kn} = Tchebycheff distance of alternative *k* to class centroid *n*

L∞ Model

Min $z = b\sum_{n=1}^{N}\frac{\sum_{\forall k\in C_n}\varepsilon_k}{C_n} + \sum_{k=1}^{K}\sum_{n=1}^{N}L_{kn}$

Subject to:

$L_{kn} \geq w_j * |a_{kj} - y_{nj}|$

$\forall\, k \in \{1,2,\dots,K\}; \forall\, n \in \{1,2,\dots,N\}; \; \forall\, j \in \{1,2,\dots,q\}$ (4-5)

$L_{kn} - \varepsilon_k \leq L_{kt} - r \qquad \forall\, k \in \{1,2,\dots,K\}; \; \forall\, t \neq n \in \{1,2,\dots,N\}$ (4-6)

$\sum_{j=1}^{q} w_j = 1$ (4-7)

$\alpha_k \geq 0 \qquad \forall\, k \in \{1,2,\dots,K\}$ (4-8)

$w_j \geq 0 \qquad \forall\, j \in \{1,2,\dots,q\}$ (4-9)

The objective function of the $L\infty$ Model has secondary objective additional to the objective given in Lp Model. Primary objective sums all Tchebycheff distances. And secondary objective takes average of the classification errors. The objective of $L\infty$ Model is to minimize the sum of primary objective and secondary objective which is multiplied by small constant b. Equation (4-5) calculates the Tchebycheff distance function value for each alternative in the training data set and each class centroid. Equation (4-6) is similar with Equation (4-1) in Model-1. Only difference is that Equation (4-6) uses Tchebycheff distance values. Equation (4-7) is the same with Equation (4-2) which ensures sum of criteria weights is equal to one. Equation (4-8) and (4-9) are non-negativity constraint for decision variables.

CHAPTER 5

COMPUTATIONAL EXPERIMENTS

The General Algebraic Modeling System (GAMS) v. 23.9 with CPLEX and BARON solver is used to solve the developed mathematical models. CPLEX solver is employed to solve the linear mathematical models with L_1 and L_∞ distance norms, whereas BARON solver is used for the non-linear mathematical models with L_2 and L_3 distance norms. BARON solver is preferred for the non-linear models due to the nature of its algorithm which guarantees optimality for nonlinear models since the models for different distance norms are convex. In all computational experiments, ASUS ET2311NKHB024M Intel ® Core ™ i7 4770S CPU @ 3.10 GHz Processor, 16GB, 64bit OS Windows 10 1703 Operating System is used.

In GAMS default optimality gap is 10%, whereas in the mathematical models it is set as zero in order to obtain global optimum solutions.

Section 5.1 describes all the data sets used in the computational experiments. Performance measures are explained in Section 5.2, followed by the results of the distance based sorting method without class threshold in Section 5.3. Results of PDIS and UTADIS methods are discussed in Sections 5.4 and 5.5, respectively, and the chapter is concluded with Section 5.6 by comparing the results of the proposed method with the results of PDIS and UTADIS methods.

5.1. Data Sets

The proposed method is applied on five data sets. Four of them are taken from UCI Machine Learning Repository (UCI, n.d.) and one of them is taken from the study of Fernandez et al. (2009). Some of the criteria in the data sets are categorical, which are transformed into quantitative form by assigning numbers to each category, and others are continuous which are normalized and evaluated in (0, 1) range. Data sets are divided into two parts as training data set and test data set. Training data is 35% of the whole data set and test data is 65% of the whole data set. Summary of the data sets is given below Table 5.1.

Table 5.1. Summary of Data Sets Used in the Computational Experiments

Data Set Name	**Number of Alternative**	**Number of Criteria**	**Number of Classes**
LENS	24	4	3
R&D Projects	81	4	8
Teaching Assistant	66	3	3
Credit	1000	20	2
Car	1728	6	4

5.1.1. Lens Data Set

Lens data set is taken from UCI Machine Learning Repository. It is the smallest data set with 24 alternatives considered in this study.

The alternatives are sorted into 3 classes which are:

- Class 1: The patient should be fitted with hard contact lenses.
  4 alternatives are classified into Class 1.
- Class 2: The patient should be fitted with soft contact lenses.
  5 alternatives are classified into Class 2.
- Class 3: The patient should not be fitted with contact lenses.
  15 alternatives are classified into Class 3.

Data set includes 4 criteria. Details of criteria are given in Table 5.2

Table 5.2. Details of Criteria for the Lens Data Set

Criteria	Criterion Values
Age of the patient	Young (1) Pre-presbyobic (2) Presbyobic (3)
Spectacle prescription	Myope (1) Hypermetrope (2)
Astigmatic	No (1) Yes (2)
Tear production rate	Reduced (1) Normal (2)

All criteria are categorical in the data set. Therefore, normalization is not required for criteria values.

5.1.2. R&D Projects Data Set

R&D Projects data set is taken from the study of Fernandez et al. (2009). The data set includes 81 different alternatives.

The projects are sorted into 8 classes which are:

- Class 1: Exceptional.

  6 alternatives are classified into Class 1.
- Class 2: Very high.

  28 alternatives are classified into Class 2.
- Class 3: High.

  25 alternatives are classified into Class 3.
- Class 4: Above average.

  6 alternatives are classified into Class 4.
- Class 5: Average.

10 alternatives are classified into Class 5.

- Class 6: Below average.

  2 alternatives are classified into Class 6.

- Class 7: Low.

  2 alternative are classified into Class 7.

- Class 8: Very low.

  2 alternative are classified into Class 8.

Data set involves 4 criteria. Details of each criterion is given in Table 5.3.

Table 5.3. Details of Criteria for the R&D Projects Data Set

Criteria	Criterion Values
Economic outcomes	1-7
Social outcomes	1-7
Scientific outcomes	1-7
Improvement of research competence	1-7

All criteria are categorical and they are defined with the same ranges in the R&D Projects data set. Therefore, standardization is not required for criteria values.

5.1.3. Teaching Assistant Data Set

Teaching Assistant data set is taken from UCI Machine Learning Repository. The original data set includes 151 alternatives, however, some of the data is problematic since there exists repetitive alternatives in the data set and several alternatives that have the same scores for each criterion are assigned to different classes which can cause biased results for the proposed method. Therefore these problematic alternatives are eliminated from the data set in the computational experiment of the proposed method and only unique alternatives are included in the data set. The number of unique alternatives in the data set is reduced to 66 after applying this approach.

These unique alternatives are sorted into 3 classes which are:

- Class 1: Low performance of assistant.

  20 alternatives are classified into Class 1.
- Class 2: Medium performance of assistant.

  22 alternatives are classified into Class 2.
- Class 3: High performance of assistant.

  24 alternatives are classified into Class 3.

The data set includes 2 categorical criteria and 1 continuous criterion. Details of each criterion are given in Table 5.4.

Table 5.4. Details of criteria for the Teaching Assistant data set

Criteria	**Criterion Value**
Native English Speaker	Native (1) Non-Native (2)
Course Semester	Regular (1) Summer (2)
Class size	Number of students registered (Continuous)

For the continuous criterion "Class size", criterion value and criterion ranges are not compatible with those of the other criteria. Therefore this continuous criterion range is normalized to (0, 1) range as shown in Equation (5-1) in order to prevent one criterion dominance over the others.

$$a'_{k,j} = \frac{a_{k,j} - min_k\,(a_{k,j})}{max_k\,(a_{k,j}) - min_k\,(a_{k,j})} \qquad (5\text{-}1)$$

Where $a_{k,j}$ is original value of alternative k on criterion j and $a'_{k,j}$ is normalized value of alternative k on criterion j. $min_k\,(a_{k,j})$ represents the minimum value of alternative k on criterion j. $max_k\,(a_{k,j})$ represents the maximum value of alternative k on criterion j.

5.1.4. Credit Data Set

Credit data set is taken from UCI Machine Learning Repository. The data set includes 1000 alternatives.

Credit applicants are sorted into 2 classes which are:

- Class 1: Not approved applicants.

  700 alternatives are classified into Class 1.

- Class 2: Approved applicants.

  300 alternatives are classified into Class 2.

The data set includes 14 categorical criteria and 6 continuous criterion. Details of each criterion are given in Table 5.5.

Table 5.5 *Details of Criteria for the Credit Data Set*

Criteria	Criterion Values
Status of existing checking account	No check account (1) Account with no money (2) Account with less than $200 money (3) Account with more than $200 money (4)
Duration of credit	Months (4 - 60)
Credit history of applicant	Critical account (1) Delay in paying off in the past (2) Existing credits paid back duly till now (3) All credits at this bank paid back duly (4) No credits taken / all credits paid back duly (5)
Purpose of the credit application	New Car (1) Used Car (2) Furniture / Equipment (3) Radio / Television (4) Domestic appliances (5) Repair (6) Education (7) Vacation (8) Retraining (9) Business (10)
Credit amount	Dollar (Continuous)
Applicants savings account/bonds	No info / No account (1) Account is less than $ 100 (2) Account is between $100 and $500 (3) Account is between $500 and $1000 (4) Account is greater than $1000 (5)
Employment status of applicant	Unemployed (1) Employed less than 1 year (2) Employed between 1 and 4 years (3) Employed between 4 and 7 years (4) Employed more than 7 years (5)
Installment rate in percentage of disposable income	Percentage (0-100)

Table 5.5 Continued

Personal status and gender of the applicant	Male and divorced (1) Female and divorced / married (2) Male and single (3) Male and married / widowed (4) Female and single (5)
Whether other debtors and guarantors exist or not	None (1) Co-applicant (2) Guarantor (3)
Duration of the residence of the applicant	Years (Continuous)
Property owned by applicant	No info / No property (1) Car / Other (2) Building society savings agreement / life insurance (3) Real estate (4)
Age of applicant	Years (Continuous)
Whether applicant have other installment plans or not	Plan to bank (1) Plan to stores (2) No plan (3)
Housing information of the applicant	Rent (1) Owns the house (2) House for free (3)
Number of existing credits at this bank	Number (Continuous)
Job of the applicant	Unemployed/ Unskilled - Non-resident (1) Unskilled resident (2) Skilled employee / Official (3) Manager / Self-employed / Highly qualified employee / Officer (4)
Number of people being liable to provide maintenance for the applicant	1 person (1) 2 people (2)
Whether applicant has telephone or not	None (1) Yes, registered under applicant's name (2)
Whether applicant is foreign worker or not	Yes (1) No (2)

Continuous criteria are normalized to (0, 1) range using the Normalization Equation (5-1).

5.1.5. Car Data Set

Car data set is taken from UCI Machine Learning Repository. The data set includes 1728 alternatives.

The credit applicants are sorted into 4 classes which are:

- Class 1: Unacceptable.

  1209 alternatives are classified into Class 1.
- Class 2: Acceptable.

  384 alternatives are classified into Class 2.
- Class 3: Good.

  69 alternatives are classified into Class 3.
- Class 4: Very good.

  65 alternatives are classified into Class 4

The data set includes 6 criteria and all of them are categorical. Therefore, normalization is not required for criteria scores. Details of each criteria is given in Table 5.6.

Table 5.6. Details of criteria for the Car data set

Criteria	Criterion Values
Price	Low (1) Medium (2) High (3) Very high (4)
Maintenance cost	Low (1) Medium (2) High (3) Very high (4)
Number of doors	2 doors (1) 3 doors (2) 4 doors (3) 5 and more doors (4)
Number of people can be carried	2 people (1) 4 people (2) 6 and more people (3)
Luggage - Boot capacity	Small (1) Medium (2) Big (3)
Safety level	Low (1) Medium (2) High (3)

5.2. Performance Measures

Two different cases are studied in order to evaluate the performance measures. In Case-1, centroids of the classes are estimated by taking the averages of the data in the given class of the training data set, whereas in Case-2 centroids are estimated using the whole data set (i.e., taking averages of all data in the given class.) in order to analyze the effect of estimation of centroids in the experiments. Therefore, Case-2 represents the perfect information situation. In both cases, models are run for four different distance norms which are one, two, three and infinity. The diagram related with this categorization is shown in Figure 5.1.

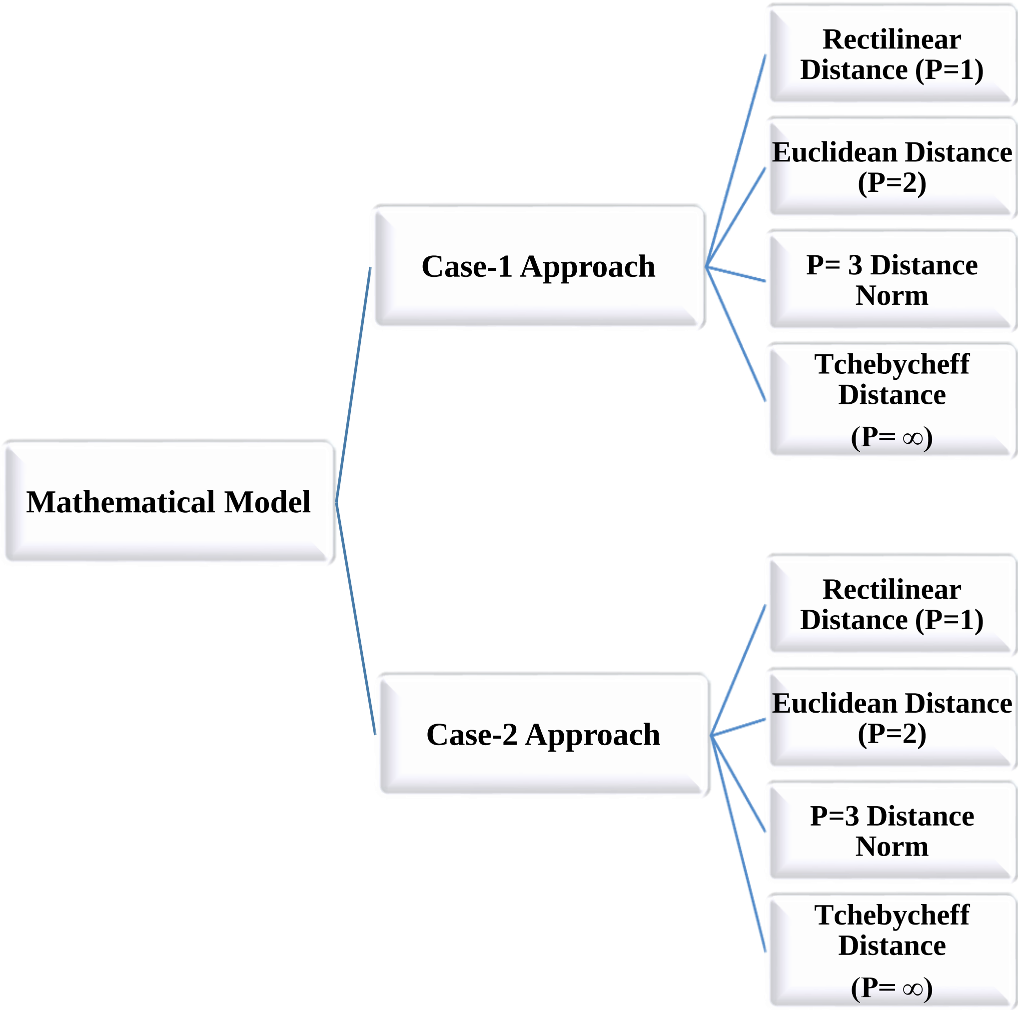

Figure 5.1. Diagram of the Developed Mathematical Models

Mathematical models determine weight sets using the training data set. Distance values between the test data set and each class centroids are calculated by using these weight sets and alternatives in the test data set are assigned to the closest classes according to their distance values.

Mainly two different performance measures are used to validate the model. One of the performance measure used to evaluate the model is *success rate*. Success rate is the proportion of correctly classified alternatives to the whole data set. Table 5.7 shows the number of correctly and incorrectly classified alternatives for a given data set.

Table 5.7. Number of correct and incorrect assigned alternatives for a data set

	Training Data	**Test Data**	**Whole Data**
Number of Alternatives in Data Set	A	B	A+B
Number of Correctly Classified Alternatives	C	D	C+D
Number of Incorrectly Classified Alternatives	A-C	B-D	(A+B)-(C+D)

The success rates for the data set are calculated as follows:

$$\text{Success rate for training data set} = \frac{C}{A}$$

$$\text{Success rate for test data set} = \frac{D}{B}$$

$$\text{Success rate for whole data set} = \frac{C+D}{A+B}$$

Accuracy is defined as the success rate for test data set of the method. Accuracy is one of the most commonly used performance measure for sorting problems. The success rates for training data set and whole data set and accuracy level are calculated for each distance norm models. The best accuracy level is selected as the performance of the method for that data set. The details of this evaluation for each data set is explained in the Section 5.3.

Another performance measure that is used to validate the model is *range length*. The range length shows how accurately the model assigns alternatives to the classes. Range length is calculated with two different ways. The first one is called *Full Range* and the second one is called *Partial Range.* The steps of calculating range length is shown in Figure 5.2.

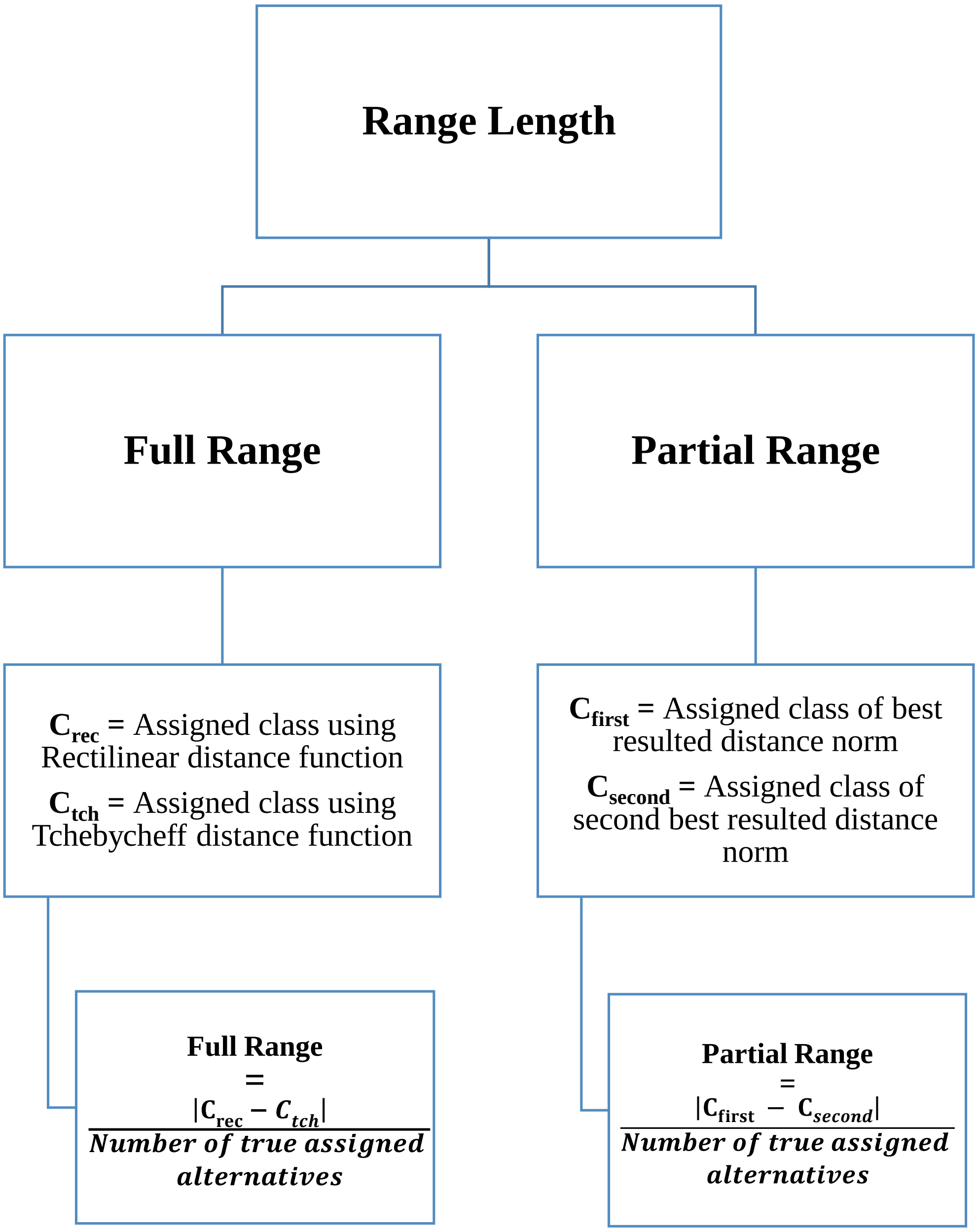

Figure 5.2. Steps of Range Calculation

In the full range, Rectilinear and Tchebycheff distance norms are used and their assignment results are evaluated. These two distance norms give the whole range for possible class assignments, representing a lower bound and an upper bound. Alternatives, which are located between the lower and upper bound are labelled as correctly classified and the rest of the alternatives are labelled as incorrectly classified.

The *full range* of an alternative is calculated as:

$$\text{Full Range} = \frac{\sum|C_{rec} - C_{tch}|}{Number\ of\ correctly\ classified\ alternatives} \quad (5\text{-}2)$$

C_{rec} represents the assigned class number by using Rectilinear distance norm in Equation (5-2). Similarly, C_{tch} represents the assigned class number by using Tchebycheff Distance norm in Equation (5-2). Rectilinear distance result gives a lower value for some of the alternatives in the data set and Tchebycheff distance result gives a lower value for the rest. Therefore absolute value is used to calculate the range. Ranges are calculated for all of the correctly assigned alternatives in a data set.

In partial range, test data set assignment results for four distance norms are compared and two of the distance norms with the highest accuracies are selected. These two results give the best possible class range for an alternative. The *partial range* of an alternative is calculated as:

$$\text{Partial Range} = \frac{\sum|C_{first} - C_{second}|}{Number\ of\ correctly\ classified\ alternatives} \quad (5\text{-}3)$$

C_{first} represents the assigned class number of the distance norm with the best accuracy level and C_{second} represents the assigned class number of the distance norm with the second best accuracy level, in Equation (5-3). The alternatives belonging to the class which is between the best two results are labelled as correctly classified and the rest of the alternatives are labelled as incorrectly classified. The assigned class number for best accuracy distance norm might be a smaller value compared to that of the second best accuracy distance norm, therefore absolute value is used in the formulation. Range measures are calculated for all correctly assigned alternatives and their average gives the *partial range length*.

The calculation of range lengths are explained in the following example. Assume that test data set includes 4 alternatives and their assignment results are given below Table 5.8.

Table 5.8. Example Results for Range Calculation

Alternatives	DM's Assignment Information	Distance Norms				$\lvert C_{rec} - C_{tch} \rvert$ (Nominator of Full Range Equation)	$\lvert C_{first} - C_{second} \rvert$ (Nominator of Partial Range Equation)
		L_1	L_2	L_3	L_∞		
1	2	2	2	1	2	0	1
2	1	1	1	1	2	1	0
3	2	3	1	1	3	FALSE	FALSE
4	3	3	3	2	1	2	1

In Table 5.8, C_{rec} represents the assigned class number by using Rectilinear distance; L_1 and C_{tch} represents the assigned class number by using Tchebycheff distance; L_∞. C_{first} represents the assigned class number of the distance norm with the best accuracy level which is L_2 for this example. C_{second} represents the assigned class number of the distance norm with the second best accuracy level which is L_3 for this example. Also, *FALSE* represents that the alternative is incorrectly classified for both distance norms within the related range calculation. In this example, alternative 3 is labeled as FALSE. Because it is incorrectly classified for full range with L_1 and L_∞ distance norms. It is also the same for partial range with L_2 and L_3 distance norms. Therefore, the number of correctly classified alternatives are 3 for the both range calculations. So ranges are calculated according to the Equations (5-2) and (5-3) respectively:

Full range = $\frac{\sum\lvert C_{rec} - C_{tch}\rvert}{Number\ of\ correctly\ classified\ alternatives} = \frac{0+1+2}{3} = 1$

Partial Range = $\frac{\sum\lvert C_{first} - C_{second}\rvert}{Number\ of\ correctly\ classified\ alternatives} = \frac{1+0+1}{3} = 0.667$

5.3. Results of Distance Based Sorting Without Class Threshold Method (DISWOTH)

In this method, four different distance norms are evaluated for each case and their results are compared to identify the best distance norm of the data set.

The results of the DISWOTH method for different performance measures such as accuracy, success rate of training data set, success rate of whole data set, computation times and range lengths are given in following subsections for each data set separately.

5.3.1. Result of Lens Data Set

In this subsection, performance measures are given for Lens data set. Case-1 results are given in Table 5.9.

Table 5.9. Case-1 Results for Lens Data Set

Distance Norm	Accuracy	Success Rate for Training Data	Success Rate for Whole Data	Computation Time (Sec.)
L_1	0.667	1.000	0.792	0.000
L_2	0.533	1.000	0.708	0.050
L_3	0.667	1.000	0.792	0.080
L_∞	0.933	0.889	0.917	0.030

For Case-1, L_∞ distance norm has the highest accuracy level of 0.933, and the success rate for the whole data set is better than the other distance norms. Computation time of 0.03 second for L_∞ distance norm is also one of the shortest, therefore L_∞ distance norm is selected as the best performing distance norm for Case-1 approach. Assignment of test data set for best performed model, L_∞ distance norm, is given in Appendix A.

Table 5.10. Case-2 Results for Lens Data Set

Distance Norm	Accuracy	Success Rate for Training Data	Success Rate for Whole Data	Computation Time (Sec.)
L_1	0.800	1.000	0.875	0.000
L_2	0.867	1.000	0.917	0.100
L_3	0.733	1.000	0.833	0.060
L_∞	0.867	0.667	0.792	0.000

Case-2 results are presented in Table 5.10. L_2 and L_∞ distance norms give the best accuracy levels with 0.867. In terms of success rate of training data and whole data measures, L_2 distance norm gives higher levels than L_∞ distance norm. Therefore, L_2 distance norm is accepted as the best for Case-2 of Lens data set. L_2 distance norm has 0.1 second computation time. Assignment of test data set for best performed model, L_2 distance norm, is reported in Appendix B.

Range length is calculated depending on accuracy level. Solutions of the test data set are compared and two of the highest accuracy level distance norms are selected for partial range calculation. For Case-1 approach, the best accuracy level is obtained from L_∞ distance norm. Second best accuracy level is obtained from both L_1 and L_3. To be able to differentiate the partial range from full range, L_3 distance norm is selected as the second best accuracy level model. For Case-2, L_2 and L_∞ distance norms give the highest accuracy level of 0.867. Therefore, these two best distance norms are used to calculate partial range. Table 5.11 shows the range length of the data set.

Table 5.11. Range Length Values for Lens Data Set

	Case-1		**Case-2**	
	FULL RANGE (L_1 - L_∞)	**PARTIAL RANGE (L_3 - L_∞)**	**FULL RANGE (L_1 - L_∞)**	**PARTIAL RANGE (L_2 - L_∞)**
Range for Whole Data	0.347	0.318	0.500	0.416
Range for Test Data	0.500	0.461	0.533	0.400
Range for Training Data	0.111	0.111	0.444	0.444

The Lens data set includes 3 classes, therefore the whole range is equal to 2. The range lengths of the test data for full and partial ranges are 0.500 and 0.461 respectively, for Case-1 approach, whereas the full range result is 0.533 and partial range result is 0.400 for Case-2. For both cases partial range is smaller than full range as expected in the Lens data set. All of the range lengths, presented in Table 5.10, are smaller than 1. This shows how accurate the DISWOTH method assigns alternatives to the correct classes.

5.3.2. Result of R&D Projects Data Set

In this subsection, performance measures are evaluated for R&D Projects data set. Case-1 approach results are given in Table 5.12.

Table 5.12. Case-1 Results for R&D Projects Data Set

Distance Norm	Accuracy	Success Rate for Training Data	Success Rate for Whole Data	Computation Time (Sec.)
L_1	0.404	0.793	0.543	0.000
L_2	0.423	0.690	0.519	55.430
L_3	0.385	0.759	0.519	123.560
L_∞	0.288	0.552	0.383	0.030

For Case-1, L_2 distance norm has the best accuracy rate with 0.423, and the computation time is found as 55.430 seconds. The assigned classes by Case-1 approach for the test data set and the classes given by data itself is provided in APPENDIX C.

Table 5.13. Case-2 Results for R&D Projects Data Set

Distance Norm	Accuracy	Success Rate for Training Data	Success Rate for Whole Data	Computation Time (Sec.)
L_1	0.462	0.517	0.481	0.000
L_2	0.596	0.621	0.605	68.480
L_3	0.538	0.621	0.568	57.470
L_∞	0.577	0.586	0.580	0.000

Case-2 approach results are presented in Table 5.13. The best performance is L_2 distance norm with 0.596 accuracy level. L_2 distance norm has 68.48 seconds computation time which is longer than other distance norms since L_2 distance norm model is a nonlinear model. The assigned classes by Case-2 approach for the test data set and the classes given by data itself is provided in APPENDIX D.

For partial range calculation, the best two accuracy distance norm results from the test data set are selected. For Case-1 approach, L_1 and L_2 distance norms give the highest accuracy rates, which are 0.404 and 0.423 respectively. Therefore, partial range is calculated with L_1 and L_2 distance norms. For Case-2 approach, L_2 and $L\infty$ distance norms give the highest rate for test data set with 0.596 and 0.577 respectively. Thus, L_2 and $L\infty$ distance norms are used for partial range calculation. Range length calculation results of Case-1 approach and Case-2 approach are presented in Table 5.14.

Table 5.14. Range Length Values for R&D Projects Data Set

	Case-1		**Case -2**	
	FULL RANGE (L_1 - $L\infty$)	**PARTIAL RANGE (L_1 – L_2)**	**FULL RANGE (L_1 - $L\infty$)**	**PARTIAL RANGE (L_2 - $L\infty$)**
Range for Whole Data	0.760	0.906	0.825	0.754
Range for Test Data	1.214	1.000	0.811	0.692
Range for Training Data	0.500	0.583	0.850	0.836

The R&D Projects data set includes 8 classes, therefore the whole range is 7. The range lengths of test data for full and partial ranges are 1.000 and 1.214 respectively, for Case-1 approach. For Case-2 approach, the full range result is 0.811 and the partial range result is 0.692. Both range values for Case-2 approach are smaller than Case-1 approach results. When all results are compared with the whole range, which is 7, all results are very small. This shows how accurate the DISWOTH method assigns alternatives.

5.3.3. Results of Teaching Assistant Data Set

Performance measures of data set are evaluated in this subsection. Case-1 approach results are given in Table 5.15.

Table 5.15. Case-1 Results for Teaching Assistant Data Set

Distance Norm	Accuracy	Success Rate for Training Data	Success Rate for Whole Data	Computation Time (Sec.)
L_1	0.349	0.304	0.333	0.201
L_2	0.419	0.348	0.394	3.260
L_3	0.395	0.391	0.394	0.790
$L\infty$	0.419	0.435	0.424	0.373

For Case-1, L_2 and $L\infty$ distance norms have the highest accuracy level of 0.419. For the training and whole data set, $L\infty$ distance norm gives higher success rates than L_2 distance norm. $L\infty$ distance norm has also very small computation time with 0.373 second. Therefore, the best performance of Case-1 is selected as $L\infty$ distance norm. Assignment of test data set for best performed model, $L\infty$ distance norm, is reported in Appendix E.

The result for L_3 distance norm given in Table 5.15 is obtained with 1 % optimality gap within 0.790 seconds of computation time. The same model is also solved without optimality gap and even after 72271.17 seconds of execution time the model could not find a global optimal solution. The results of two models which are with gap and without gap also found to yield the same objective function and weight set but since their computation times are very different the results of the model with gap is given here.

Table 5.16. Case-2 Results for Teaching Assistant Data Set

Distance Norm	Accuracy	Success Rate for Training Data	Success Rate for Whole Data	Computation Time (Sec.)
L_1	0.512	0.565	0.530	0.242
L_2	0.535	0.609	0.561	4.310
L_3	0.488	0.652	0.545	0.580
L_∞	0.535	0.652	0.576	0.240

The Case-2 approach results are reported in Table 5.16. L_2 and L_∞ distance norms have the highest accuracy level of 0.535. For training data set and whole data set, L_∞ distance norm gives higher success rates than L_2 distance norm. And L_∞ distance norm has very short computation time as 0.24 second. Therefore, the best performance is L_∞ distance norm. Assignment of test data set for the best performed model, L_∞ distance norm, is reported in Appendix F.

The best two distance norms are selected for partial range calculation by comparing test data set results. For Case-1 approach, L_2 and L_∞ distance norms give the highest accuracy levels for test data, which is 0.419 for both of them. For Case-2 approach, again L_2 and L_∞ distance norms give the highest accuracy levels, which is 0.535 for both distance norms. Results of range calculation for both cases of Teaching Assistant data set are given in Table 5.17.

Table 5.17. Range Length Values for Teaching Assistant Data Set

	Case-1		**Case-2**	
	FULL RANGE (L_1 - L_∞)	**PARTIAL RANGE (L_2 - L_∞)**	**FULL RANGE (L_1 - L_∞)**	**PARTIAL RANGE (L_2 - L_∞)**
Range for Whole Data	0.500	0.190	0.522	0.519
Range for Test Data	0.333	0.129	0.607	0.536
Range for Training Data	0.800	0.363	0.375	0.187

The Teaching Assistant data set includes 3 classes, therefore the whole range is 2. The range length of the test data for full and partial ranges are 0.333 and 0.129 respectively for Case-1 approach, whereas for the full range result is 0.607 and partial range result is 0.536 for Case-2. Partial range lengths are smaller than full range lengths as expected. Because the partial range limits are selected from the best results of different distance norms. When all results are compared with the whole range, all of them are very small. This shows that the DISWOTH method performs well in assigning alternatives to the correct classes.

5.3.4. Result of Credit Data Set

In this subsection, performance measures are evaluated for Credit data set. Case-1 approach results are presented in Table 5.18.

Table 5.18. Case-1 Results for Credit Data Set

Distance Norm	Accuracy	Success Rate for Training Data	Success Rate for Whole Data	Computation Time
L_1	0.598	0.583	0.593	0.036
L_2	0.674	0.720	0.690	0.890
L_3	0.675	0.746	0.700	1.660
L_∞	0.623	0.591	0.612	5.550

For Case-1, L_3 distance norm has the highest accuracy level of 0.675. Computation time of the best distance norm is 1.66 seconds. The assigned classes by Case-1 approach for the test data set and the classes given by data itself is provided in APPENDIX G.

The model result for L_3 distance norm given in Table 5.18 is obtained with 2 % optimality gap within 1.660 seconds of computation time. Same model is also solved without optimality gap and even after 72021.38 seconds of execution time the model could not find a global optimal solution. The model result for L_2 distance norm presented in Table 5.18 is obtained with 2 % optimality gap within 0.890 second of computation time. The same model is also solved without optimality gap and even after 72028.72 seconds of execution time the model could not find a global optimal solution. The results of two models which are with gap and without gap also found to yield the same objective function and weight set but since their computation times are very different the results of the model with gap is given here.

Table 5.19. Case-2 Results for Credit Data Set

Distance Norm	Accuracy	Success Rate for Training Data	Success Rate for Whole Data	Computation Time
L_1	0.649	0.686	0.662	0.000
L_2	0.662	0.723	0.683	0.540
L_3	0.686	0.760	0.712	0.730
L_∞	0.645	0.586	0.624	4.727

Case-2 approach results are presented in Table 5.19. The best performance is L_3 distance norm with 0.686 accuracy level. L_3 distance norm has 0.73 second computation time, which is very short. The assigned classes by Case-2 approach for the test data set and the classes given by data itself is provided in APPENDIX H.

The model result for L_3 distance norm is executed for 72024.84 seconds and could not find optimal solution. The model is also solved with 2% optimality gap. L_2 distance norm model is executed for 72021.54 seconds and could not find optimal solution. The model is also solved with 2% gap. The results of two models which are with gap and without gap with different distance norms give the same objective function and weight set but their computation times are very different. Therefore the results of models with gap are given in Table 5.19.

Range length is calculated for Credit data set depending on accuracy level. Test data set results are compared and the best two distance norms are selected for partial range analysis. For Case-1 approach, L_2 and L_3 distance norms give the highest accuracy level of 0.674 and 0.675 respectively. So, these distance norms are used for partial range length calculation. For Case-2 approach, L_2 and L_3 distance norms give the highest accuracy levels for test data as 0.662 and 0.686 respectively. Therefore, partial range is calculated with L_2 and L_3 distance norms. Results of range length for Case-1 and Case-2 are given in Table 5.20.

Table 5.20. Range Length Values for Credit Data Set

	Case-1		**Case-2**	
	FULL RANGE (L_1 - L_∞)	**PARTIAL RANGE (L_2 - L_3)**	**FULL RANGE (L_1 - L_∞)**	**PARTIAL RANGE (L_2 - L_3)**
Range for Whole Data	0.313	0.252	0.324	0.117
Range for Test Data	0.323	0.259	0.337	0.124
Range for Training Data	0.296	0.240	0.302	0.106

The Credit data set includes 2 classes, therefore the whole range is 1. The range lengths of the test data for full and partial ranges are 0.323 and 0.259 respectively for Case-1 approach. For Case-2 approach, the full range result is 0.337 and the partial range result is 0.124. For both cases, the partial range results are smaller than the full range results as expected.

5.3.5. Result of Car Data Set

Performance measures of data set are evaluated in this subsection. Case-1 approach results are presented in Table 5.21.

Table 5.21. Case-1 Results for Car Data Set

Distance Norm	Accuracy	Success Rate for Training Data	Success Rate for Whole Data	Computation Time (Sec.)
L_1	0.602	0.615	0.599	0.011
L_2	0.733	0.702	0.722	2.890
L_3	0.773	0.759	0.768	5.780
L_∞	0.734	0.754	0.741	2.940

For Case-1, L_3 distance norm has the highest accuracy level of 0.773 and the success rates are better than the other distance norms. For the best distance norm model, the computation time is 5.78 seconds. Assignment of test data set for best performed model, L_3 distance norm, is reported in Appendix I.

The model result for L_3 distance norm given in Table 5.21 is performed with 2 % optimality gap within 5.780 seconds of computation time. The same model is also solved without gap and even after 72024.11 seconds of execution time the model could not find a global optimal solution. The model result for L_2 distance norm given in Table 5.21 is performed with 3 % optimality gap within 2.890 second of computation time. The same model is also solved without gap and even after 72005.61 seconds of execution time the model could not find a global optimal solution. The results of two models which are with gap and without gap also found to yield the same objective function and weight set but since their computation times are very different. The results of the model with gap is presented here.

Table 5.22. Case-2 Results for Car Data Set

Distance Norm	Accuracy	Success Rate for Training Data	Success Rate for Whole Data	Computation Time (Sec.)
L_1	0.631	0.615	0.626	0.000
L_2	0.762	0.717	0.750	12.260
L_3	0.806	0.790	0.800	7.240
L_∞	0.768	0.792	0.773	3.170

Case-2 approach results are presented in Table 5.22 The best performance of the approach is L_3 distance norm with 0.806 accuracy level. L_3 distance norm has 7.24 seconds computation time. Assignment of test data set for best performed model, L_3 distance norm, is given in Appendix J.

The model result for L_3 distance norm is executed for 72024.19 seconds and could not find optimal solution. Therefore, the model is also solved with 2% optimality gap. L_2 distance norm model is executed for 72042.75 seconds and could not find optimal solution. Therefore, the model is also solved with 3% optimality gap. The results of two models give the same objective function and weight set but their computation times are very different. Therefore the results of the models with gap are given in Table 5.22.

Car data set has 1728 alternatives, 605 of them are in the training data and 1123 of them are in the test data set, which can be accepted as a large data set. In this large data set, the accuracy level of 0.806 is better than the accuracy level of previous smaller data sets, indicating the performance of the method increases as the data set gets larger within these five data sets.

Test data set results are compared and the best two distance norms are selected for partial range calculation. For Case-1, L_3 and L_∞ distance norms give the highest rates for test data set, which are 0.773 and 0.734 respectively. Therefore, L_3 and L_∞ distance norms are used to calculate partial range length. For Case-2, L_3 and L_∞ distance norms give the highest accuracy levels for test data, which are 0.762 and 0.768, respectively.

Therefore L_3 and $L\infty$ distance norms are used for partial range calculation. Range length of Case-1 approach and Case-2 approach are given in Table 5.23.

Table 5.23. Range Length Values for Car Data Set

	Case-1		**Case-2**	
	FULL RANGE (L_1 - $L\infty$)	**PARTIAL RANGE (L_3 and $L\infty$)**	**FULL RANGE (L_1 - $L\infty$)**	**PARTIAL RANGE (L_3 and $L\infty$)**
Range for Whole Data	0.459	0.069	0.443	0.029
Range for Test Data	0.439	0.063	0.425	0.025
Range for Training Data	0.494	0.081	0.477	0.037

The Car data set includes 4 classes, therefore the whole range is 3. The full and partial range lengths for the test data are 0.439 and 0.063 respectively for Case-1 approach. For Case-2 approach, the full range result is 0.425 and the partial range result is 0.025. For both cases, the partial range results are smaller than the full range results as expected. When all results are compared with the whole range, all of them are too small. This shows how accurate the DISWOTH method assigns alternatives to the correct classes.

5.4. Results of Probabilistic Distance Based Sorting (PDIS) Method

PDIS method (Celik et al, 2015) is applied to the same data sets explained in Section 5.1. To run the mathematical model of PDIS method, the same solvers, CPLEX and BARON, are used. Performance measures, namely accuracy, success rate of training data set, success rate of whole data set and computation time, are calculated for PDIS method. The PDIS method gives threshold values for each class. But in some cases, the method cannot distinguish all class thresholds and may result in missing classes. As a performance measure, number of missing classes are also evaluated.

According to the PDIS model results, class threshold values are obtained from Model-1. Maximum and minimum values of alternatives in test data set are found by Model-2. If a class contains both the maximum and the minimum values of an alternative, then this alternative is assigned to the class. Alternatives are labelled as correctly classified when the model assignment results give the same class with data itself. Accuracy level is calculated for the correctly classified alternatives in the test data set. The performance measure results of PDIS method are evaluated for each data set separately.

Table 5.24 shows performance measures of the PDIS method for Lens data set.

Table 5.24. Performance Measures of PDIS Method for Lens Data Set

Distance Norm	Accuracy	Success Rate for Training Data	Success Rate for Whole Data	Computation Time (Sec.)	Number of Missing Class
L_1	0.267	0.555	0.375	0.3	0
L_2	0.267	0.555	0.375	45.25	0
L_3	0.333	0.444	0.375	8.82	0
L_∞	0.533	0.778	0.625	0.24	1

For Lens data set, L_∞ model gives the best accuracy level as 0.533. Its computation time is 0.24 second. But, this model could not distinguish 2nd and 3rd classes and skips 2nd class. L_3 distance norm model gives the second best accuracy level with 0.333 for the PDIS method. And also L_3 distance norm distinguishes all classes. So, Tchebycheff model can be the best performed model however L_3 can be also the best performed among the ones distinguishing all classes.

For R&D Projects data set, PDIS method models are run for all distance norms. However, L_2 and L_3 distance norm models could not find any feasible solution at the end of the 86400 seconds for both. Therefore, performances of L_2 and L_3 distance norms could not be measured. Table 5.25 shows the result of L_1 and L_∞ distance norms.

Table 5.25. Performance Measures of PDIS Method for R&D Projects Data Set

Distance Norm	Accuracy	Success Rate for Training Data	Success Rate for Whole Data	Computation Time (Sec.)	Number of Missing Class
L_1	0.692	0.827	0.740	0.65	0
L_∞	0.269	0.379	0.308	3.75	2

L_1 model gives the best accuracy level with 0.692 for R&D Projects data set. The model is solved in 0.65 seconds without any missing class. Tchebycheff model gives 0.269 accuracy level. This model could not distinguish all classes. The number of missing classes is found as 2. The method skips 6^{th} and 7^{th} classes. For the PDIS method, L_1 model is selected as the best performed distance norm.

For Teaching Assistant data set, PDIS method is executed for all distance norms. But, the method gives the same accuracy level without any missing class. The only difference is computation times for different distance norms. Table 5.26 shows the result of PDIS method for Teaching Assistant data set.

Table 5.26. Performance Measures of PDIS Method for Teaching Assistant Data Set

Distance Norm	Accuracy	Success Rate for Training Data	Success Rate for Whole Data	Computation Time (Sec.)	Number of Missing Class
L_1	0.326	0.434	0.363	0.37	0
L_2	0.326	0.434	0.363	307.54	0
L_3	0.326	0.434	0.363	86421.8	0
L_∞	0.326	0.434	0.363	2.33	0

When all model solutions are compared, L_1 is selected as best performed model with the shortest computation time, 0.37 second.

PDIS method models are run with all distance norms for Credit data set. However, L_2, L_3 and $L\infty$ distance norms could not find any feasible solutions after 88108.28, 88291.41 and 88662.73 seconds computation times, respectively. For the L_1 distance norm, the model results with 0.092 as class threshold value. And also maximum and minimum values for all alternatives in the test data set are the same, 0.092. Therefore, any alternative in the test data set could not be distinguished between 1st and 2nd classes. Performance measures could not be evaluated for Credit data set.

Car data set models with different distance norms, L_1, L_2, L_3 and L_∞ are executed for PDIS method. However, any of them could not find feasible solution at the end of the 24 hours computation time. Therefore, performance measures could not be evaluated for Car data set.

For Credit and Car data sets, the DISWOTH method give high performances. However, PDIS method and DISWOTH method could not be compared for these data sets. Therefore, UTADIS method is also applied to the same data sets.

5.5. Results of Utilities Additives DIScriminates (UTADIS) Method

UTADIS method (Devaud et al., 1980) is applied to all five data sets explained in Section 5.1 like PDIS method. To run the mathematical model of UTADIS method, CPLEX is used.

In the UTADIS method, breakpoint and interval information is needed. If the criteria are categorical, the breakpoints of criteria are determined as these categories. But, if the data set includes continuous criteria values, determining the subintervals can be a challenge. Doumpos and Zopounidis (2002) propose a heuristic, HEUR1, to solve this problem. HEUR1 states that:

> "*Define pj-1 equal subintervals, such that there is at least one alternative belonging in each interval*."

In mathematical model of UTADIS method, breakpoint and interval information is obtained by using HEUR1. According to the mathematical model results, class threshold values and utility values for criteria and intervals are obtained. These utility values for criteria and intervals are used to calculate global utility function of each alternative in the test data set. If the utility function value of an alternative is between the class threshold values, the alternative is assigned to the corresponding class. If alternatives are assigned to the correct classes given in the data, they are labelled as correctly classified. Accuracy level is calculated for the correctly classified alternatives in the test data set.

Performance measures are calculated for UTADIS method such as accuracy, success rate for training data set, success rate for whole data set and computation time. The UTADIS method gives all threshold values for each class and no missing class is occurred. Therefore, number of missing class is not evaluated as a performance measure.

Table 5.27 shows performance measures of the UTADIS method for all data sets.

Table 5.27. Performance measures of UTADIS Method for all Data Set

Distance Norm	Accuracy	Success Rate for Training Data	Success Rate for Whole Data	Computation Time (Sec.)
Lens	0.400	0.778	0.514	1.422
R&D Projects	0.519	0.758	0.605	0.808
Teaching Assistant	0.279	0.305	0.288	0.191
Credit	0.318	0.342	0.327	0.209
Car	0.396	0.557	0.452	0.334

It is seen that computation time of all models are very short. And the model gives the best accuracy level which is 0.519 for R&D Projects data set. The comparison of the UTADIS method and the DISWOTH method is provided in Section 5.6.

5.6. Comparison with PDIS Method and UTADIS Method

In this section, results of PDIS method (Celik et al, 2015), UTADIS method (Devaud et al., 1980) and the DISWOTH method are compared. The best results of Case-1 approach and Case-2 approach for DISWOTH method is selected to compare with PDIS and UTADIS methods.

The best results of the DISWOTH method, PDIS method and UTADIS method are compared in Table 5.28 for Lens data set.

Table 5.28. Comparison of Methods for Lens Data Set

Model Name	Accuracy	Success Rate for Training Data	Success Rate for Whole Data	Computation Time (Sec.)
Case-1 of DISWOTH Method (L_∞)	0.933	0.889	0.917	0.030
Case-2 of DISWOTH Method (L_2)	0.867	1.000	0.917	0.100
PDIS Method (L_∞)	0.533	0.778	0.625	0.240
UTADIS	0.400	0.778	0.514	1.422

For the Lens data set, Case-1 approach of DISWOTH method gives the best performance with 0.933 accuracy level. The accuracy level of PDIS method and UTADIS method are lower than both cases of DISWOTH method. The computation time of the methods are very short but Case-1 has the shortest computation time.

The best performed Case-1 and Case-2 of DISWOTH method, PDIS method and UTADIS method results for R&D Projects data set are compared in Table 5.29.

Table 5.29. Comparison of Methods for R&D Projects Data Set

Model Name	Accuracy	Success Rate for Training Data	Success Rate for Whole Data	Computation Time (Sec.)
Case-1 of DISWOTH Method (L_2)	0.423	0.690	0.519	55.430
Case-2 of DISWOTH Method (L_2)	0.596	0.621	0.605	68.480
PDIS Method (L_1)	0.692	0.827	0.740	0.65
UTADIS	0.519	0.758	0.605	0.808

For the R&D Projects data set, PDIS method gives the best accuracy level with 0.692. Also the accuracy level of Case-2 approach of DISWOTH method is better than the accuracy level of UTADIS method.

For the Teaching Assistant data set, best results of the three methods are compared in Table 5.30.

Table 5.30. Comparison of Methods for Teaching Assistant Data Set

Model Name	Accuracy	Success Rate for Training Data	Success Rate for Whole Data	Computation Time (Sec.)
Case-1 of DISWOTH Method (L_∞)	0.419	0.435	0.424	0.373
Case-2 of DISWOTH Method (L_∞)	0.535	0.652	0.576	0.240
PDIS Method (L_1)	0.326	0.434	0.363	0.37
UTADIS	0.279	0.305	0.288	0.191

For this data set, Case-2 approach of DISWOTH method gives the best performance with 0.535 accuracy level. The accuracy level of PDIS method is lower than both cases of DISWOTH method. And UTADIS method gives the lowest accuracy level for Teaching Assistant data set. The computation times are very short for all methods.
Credit and Car data sets are accepted as large data set with 1000 and 1728 alternatives respectively. PDIS method could not solve these data sets with any distance norms in 24 hours. Therefore, comparison of the performance measures is not possible for Credit and Car data sets with PDIS method. The comparison is performed between the results of DISWOTH method and UTADIS method.

Table 5.31 shows the comparison results for the Credit data set.

Table 5.31. Comparison of Methods for Credit Data Set

Model Name	Accuracy	Success Rate for Training Data	Success Rate for Whole Data	Computation Time (Sec.)
Case-1 of DISWOTH Method (L_3)	0.675	0.746	0.700	1.660
Case-2 of DISWOTH Method (L_3)	0.686	0.760	0.712	0.730
UTADIS	0.318	0.342	0.327	0.209

For this data set, Case-2 approach of DISWOTH method gives the best performance with 0.686 accuracy level. And Case-1 approach result gives slightly lower accuracy level, 0.675. The accuracy level of UTADIS method is significantly lower than both cases of DISWOTH method, 0.318. The computation times are short for all methods. Table 5.32 shows the comparison results for Car data set.

Table 5.32. Comparison of Methods for Car Data Set

Model Name	Accuracy	Success Rate for Training Data	Success Rate for Whole Data	Computation Time (Sec.)
Case-1 of DISWOTH Method (L_3)	0.773	0.759	0.768	5.780
Case-2 of DISWOTH Method (L_3)	0.806	0.790	0.800	7.240
UTADIS	0.396	0.557	0.452	0.334

For this data set, Case-2 approach of DISWOTH method gives the best performance with 0.806 accuracy level. And Case-1 approach gives 0.773 accuracy level. The accuracy level of UTADIS method is lower than both cases of DISWOTH method, 0.396.

Both Case-1 and Case-2 approaches of DISWOTH method perform better than PDIS method and UTADIS method in four of the five data sets, Lens, Teaching Assistant, Credit and Car, for the defined performance measures. PDIS method performs better than the DISWOTH method in only one data set and UTADIS method is not as good as the DISWOTH method in any data set. Computation time of UTADIS method on the other hand, is shorter than the DISWOTH method for most of the data sets. For the DISWOTH method, even though the nonlinear models (L_2 and L_3 distance norms) give longer computation times than the linear models, the computation times for almost all of the cases are shorter than 1 minute, which can be considered acceptable. Finally, it is important to mention that the accuracy levels of both Case-1 and Case-2 approaches increase for larger data sets in these five data sets.

CHAPTER 6

CONCLUSION

In this thesis, a new distance based sorting without class threshold method (DISWOTH) is developed. This method uses preference disaggregation analysis approach with distance functions. The distance function is used as the criteria aggregation function in the proposed method. The mathematical model of DISWOTH method determines criteria weights that represent DM's preferences using training data set. The alternatives in the test data set is assigned to the closest classes by using these weight information.

To the best of our knowledge, in the literature there is no sorting method that assigns alternatives without estimating class thresholds or preference profiles. There are clustering methods such as k-means algorithm that do not consider class thresholds but estimate class centroids instead. We incorporate this idea into sorting approach. We did not develop an iterative method that update class centroid estimates since DISWOTH method gives high quality results.

Class centroids used in the mathematical model are estimated in two ways, Case-1 and Case-2. In Case-1, class centroids are estimated by taking the averages of the data in the given class of the training data set. In Case-2, class centroids are estimated by the use of whole data set. For both cases, different distance norms which are one, two, three and infinity are evaluated with five data sets. The highest accuracy levels are obtained for large data sets used in the computational experiments. Computation time of nonlinear programming models with L_2 and L_3 distance norms are longer than linear programming models with L_1 and L_∞ distance norms, as expected. All mathematical models provide results less than 1 minute with high accuracy rates. Also, in most of the sorting methods, more than half of the whole data set is used as training data set

to obtain preference information from DM. The DISWOTH method, performs well when 35% of the whole data set is used as training data set.
PDIS method and UTADIS method are also evaluated with the five data sets. The results of these methods are compared with the DISWOTH method. The experiments show that the DISWOTH performs better than the other methods in different data sets. For Lens data set, the proposed mathematical models with different distance norms give the accuracy levels between 0.533 and 0.933. However, accuracy levels of the other methods are less than the lowest accuracy level of the DISWOTH method. For TA data set, the best accuracy level of the DISWOTH method is 0.596 and the worst accuracy level is 0.349. PDIS method give accuracy level of 0.326 and UTADIS method give accuracy level of 0.279. Again, accuracy levels of both methods are less than the worst accuracy level of the DISWOTH method. For Credit data set, the DISWOTH method's models give the accuracy levels between 0.686 and 0.598. However, UTADIS method gives accuracy level of 0.318 which is far less than the worst accuracy level of the DISWOTH method. And PDIS method could not find any feasible solution. For Car data set, the best accuracy level is 0.806 and the worst accuracy level is 0.602. However, UTADIS method could obtain only 0.396 accuracy level while PDIS method could not find any feasible solution.

While applying the DISWOTH method to the data sets, no alternative solution is obtained. However, different data sets can give alternative optimal solutions for training data sets. Improving the DISWOTH method in order to handle alternative optimal solutions can be a future research direction. Applying the DISWOTH method on a real life problem can be another future research direction.

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APPENDICES

A. L∞ DISTANCE NORM RESULT OF CASE-1 FOR LENS DATA SET

Alternatives	Assigned classes by data itself	Assigned classes by Case-1 model
1	3	3
2	3	3
3	3	3
4	2	1
5	3	3
6	2	2
7	3	3
8	3	3
9	2	2
10	3	3
11	3	3
12	3	3
13	3	3
14	3	3
15	3	3

B. L_2 DISTANCE NORM RESULT OF CASE-2 FOR LENS DATA SET

Alternatives	Assigned classes by data itself	Assigned classes by Case-1 model
1	3	3
2	1	3
3	3	3
4	1	1
5	3	3
6	2	2
7	3	3
8	3	3
9	2	2
10	2	3
11	3	3
12	3	3
13	3	3
14	3	3
15	3	3

C. L_2 DISTANCE NORM RESULT OF CASE-1 FOR R&D PROJECTS DATA SET

Alternatives	Assigned classes by data itself	Assigned classes by Case-1 model
1	1	1
2	1	1
3	2	2
4	4	3
5	8	5
6	8	7
7	8	8
8	4	3
9	2	2
10	3	3
11	2	2
12	2	2
13	8	3
14	2	2
15	2	2
16	2	2
17	2	2
18	8	5
19	8	3
20	4	5
21	1	1
22	2	1
23	8	3
24	2	2
25	4	5
26	8	5

Alternatives	Assigned classes by data itself	Assigned classes by Case-1 model
27	8	3
28	4	4
29	7	6
30	2	2
31	2	2
32	8	4
33	8	2
34	8	3
35	3	3
36	3	3
37	8	5
38	8	2
39	8	3
40	4	4
41	8	3
42	2	3
43	3	2
44	3	2
45	1	2
46	8	5
47	8	2
48	2	2
49	8	3
50	8	3
51	3	3
52	8	4

D. L_2 **DISTANCE NORM RESULT OF CASE-2 FOR R&D PROJECTS DATA SET**

Alternatives	Assigned classes by data itself	Assigned classes by Case-2 model
1	1	1
2	1	1
3	2	2
4	3	3
5	8	5
6	7	7
7	8	8
8	2	3
9	2	2
10	3	3
11	2	2
12	1	2
13	3	3
14	2	2
15	2	2
16	2	2
17	2	2
18	8	5
19	3	3
20	7	5
21	1	1
22	1	1
23	3	3
24	2	2
25	4	5
26	8	5

Alternatives	Assigned classes by data itself	Assigned classes by Case-2 model
27	8	3
28	8	4
29	8	6
30	2	2
31	2	2
32	8	4
33	2	2
34	3	3
35	3	3
36	3	3
37	8	5
38	2	2
39	3	3
40	4	4
41	3	3
42	2	3
43	3	2
44	3	2
45	3	2
46	8	5
47	3	2
48	2	2
49	2	3
50	3	3
51	4	3
52	8	4

E. **L∞ DISTANCE NORM RESULT OF CASE-1 FOR TEACHING ASSISTANT DATA SET**

Alternatives	Assigned classes by data itself	Assigned classes by Case-1 model
1	2	3
2	3	3
3	1	3
4	3	3
5	3	3
6	3	3
7	1	1
8	2	1
9	1	1
10	2	1
11	2	1
12	1	1
13	2	1
14	1	2
15	3	3
16	2	3
17	3	3
18	1	3
19	1	3
20	2	1
21	2	2
22	3	3

Alternatives	Assigned classes by data itself	Assigned classes by Case-1 model
23	3	3
24	2	3
25	3	3
26	1	3
27	3	3
28	2	3
29	3	3
30	3	1
31	2	1
32	1	1
33	3	1
34	2	1
35	1	2
36	1	2
37	2	3
38	1	3
39	2	3
40	1	3
41	3	3
42	3	3
43	2	3

F. L∞ DISTANCE NORM RESULT OF CASE-2 FOR TEACHING ASSISTANT DATA SET

Alternatives	Assigned classes by data itself	Assigned classes by Case-2 model
1	2	3
2	3	3
3	1	3
4	3	3
5	3	3
6	3	3
7	1	2
8	2	2
9	1	2
10	2	2
11	2	2
12	1	2
13	2	2
14	1	1
15	3	1
16	2	3
17	3	3
18	1	3
19	1	3
20	2	3
21	2	2
22	3	3

Alternatives	Assigned classes by data itself	Assigned classes by Case-2 model
23	3	3
24	2	3
25	3	3
26	1	3
27	3	3
28	2	3
29	3	3
30	3	2
31	2	2
32	1	2
33	3	2
34	2	2
35	1	1
36	1	1
37	2	1
38	1	1
39	2	3
40	1	3
41	3	3
42	3	3
43	2	3

G. L_3 DISTANCE NORM RESULT OF CASE-1 FOR CREDIT DATA SET

Alternatives	Assigned classes by data itself	Assigned classes by Case-1 model
1	1	1
2	1	2
3	2	2
4	1	2
5	1	2
6	1	1
7	2	2
8	2	2
9	2	2
10	1	2
11	2	2
12	2	2
13	2	1
14	1	2
15	1	1
16	1	2
17	2	2
18	1	2
19	2	1
20	1	1
21	2	2
22	2	2
23	2	2
24	1	2
25	1	2
26	1	2
27	1	1
28	1	2
29	2	2
30	1	1
31	1	1
32	1	1
33	2	2
34	2	2
35	2	2
36	1	1
37	1	2
38	1	2
39	1	1
40	1	2
41	1	2
42	1	1
43	1	1
44	1	1
45	1	1
46	1	2
47	2	1
48	1	2
49	1	2
50	2	1
51	1	1
52	2	1
53	1	1
54	1	1
55	1	1
56	1	2
57	1	2
58	1	1
59	2	2
60	1	2
61	1	2
62	1	2
63	1	1
64	1	1
65	2	2
66	2	2
67	1	1
68	1	2
69	1	1

Appendix G Continued

70	2	2
71	1	2
72	1	1
73	2	1
74	1	1
75	1	1
76	1	2
77	1	2
78	2	2
79	2	1
80	2	2
81	1	1
82	1	1
83	2	2
84	1	2
85	2	2
86	1	1
87	1	2
88	1	1
89	1	2
90	1	2
91	1	1
92	1	1
93	2	1
94	2	2
95	1	1
96	1	2
97	2	2
98	1	1
99	1	1
100	1	2
101	1	1
102	2	2
103	2	1
104	2	1
105	1	2

106	2	1
107	1	1
108	1	2
109	2	2
110	2	2
111	1	2
112	1	2
113	1	1
114	1	1
115	1	1
116	1	1
117	2	2
118	2	2
119	2	2
120	1	1
121	1	2
122	1	2
123	2	2
124	2	1
125	2	2
126	1	2
127	2	2
128	1	1
129	2	2
130	2	1
131	2	1
132	2	2
133	1	2
134	1	1
135	2	2
136	1	2
137	1	1
138	1	1
139	2	1
140	1	2
141	1	2

Appendix G Continued

142	1	1
143	1	1
144	1	2
145	1	2
146	1	1
147	1	1
148	1	1
149	1	1
150	1	1
151	1	1
152	2	2
153	1	1
154	1	2
155	1	1
156	1	1
157	1	1
158	2	1
159	1	1
160	1	1
161	2	2
162	2	2
163	2	2
164	2	1
165	1	1
166	1	1
167	1	1
168	2	1
169	2	2
170	2	2
171	1	1
172	1	2
173	1	2
174	1	1
175	1	1
176	1	2
177	1	1

178	2	2
179	1	1
180	1	1
181	1	1
182	1	2
183	1	2
184	2	1
185	1	2
186	1	1
187	1	2
188	1	1
189	1	1
190	2	2
191	1	1
192	2	1
193	1	2
194	1	1
195	2	2
196	2	1
197	1	1
198	2	1
199	1	1
200	1	2
201	2	1
202	1	2
203	1	1
204	1	1
205	1	2
206	2	2
207	2	2
208	2	1
209	1	2
210	1	1
211	1	1
212	1	1
213	1	1

Appendix G Continued

214	2	2
215	2	1
216	1	1
217	2	1
218	1	1
219	1	1
220	1	1
221	1	1
222	1	1
223	2	1
224	2	1
225	1	2
226	1	1
227	1	1
228	1	1
229	1	1
230	1	1
231	1	1
232	2	1
233	1	1
234	1	1
235	1	1
236	1	1
237	1	1
238	1	1
239	1	1
240	1	1
241	1	1
242	1	2
243	2	2
244	1	1
245	2	1
246	1	2
247	2	1
248	1	1
249	1	2

250	2	1
251	1	1
252	2	2
253	1	2
254	2	2
255	1	1
256	1	1
257	1	1
258	2	2
259	1	1
260	2	2
261	1	1
262	1	1
263	2	1
264	1	1
265	1	1
266	1	1
267	1	1
268	1	1
269	1	1
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271	2	2
272	1	1
273	1	1
274	1	1
275	1	1
276	1	1
277	1	2
278	1	2
279	2	1
280	2	1
281	2	2
282	2	2
283	1	1
284	1	1
285	1	1

Appendix G Continued

286	1	1
287	1	1
288	1	1
289	1	1
290	1	1
291	1	1
292	1	1
293	1	1
294	1	1
295	1	1
296	1	1
297	1	1
298	1	1
299	1	1
300	1	1
301	2	2
302	1	2
303	2	2
304	2	2
305	2	2
306	2	2
307	2	2
308	2	2
309	1	2
310	2	2
311	2	1
312	1	2
313	1	2
314	2	2
315	2	2
316	1	1
317	1	1
318	1	1
319	2	2
320	1	2
321	2	2

322	1	2
323	2	2
324	2	2
325	2	1
326	1	2
327	1	1
328	1	2
329	2	2
330	1	1
331	1	1
332	1	2
333	1	1
334	2	2
335	1	1
336	1	2
337	1	2
338	1	2
339	1	1
340	1	2
341	1	1
342	2	1
343	2	2
344	1	2
345	2	2
346	1	2
347	1	2
348	1	2
349	2	1
350	1	1
351	1	2
352	1	1
353	1	1
354	2	2
355	1	1
356	2	2
357	1	2

Appendix G Continued

358	1	1
359	1	1
360	2	1
361	2	1
362	1	1
363	1	2
364	1	1
365	1	1
366	1	1
367	1	2
368	1	2
369	1	2
370	1	1
371	2	2
372	2	2
373	1	1
374	1	1
375	1	1
376	1	1
377	1	2
378	1	1
379	2	1
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381	1	2
382	1	2
383	2	2
384	1	2
385	1	2
386	1	1
387	1	2
388	2	2
389	2	2
390	1	1
391	1	1
392	1	2
393	1	1

394	2	2
395	1	1
396	1	1
397	1	1
398	1	1
399	1	1
400	1	2
401	1	2
402	1	1
403	1	2
404	2	2
405	1	1
406	1	1
407	2	2
408	1	2
409	1	1
410	1	2
411	2	1
412	1	1
413	1	1
414	1	2
415	1	2
416	1	1
417	1	2
418	2	2
419	1	1
420	1	1
421	1	2
422	2	2
423	1	1
424	2	1
425	1	1
426	2	2
427	1	2
428	2	1
429	1	2

Appendix G Continued

430	1	1
431	2	2
432	1	1
433	1	1
434	1	1
435	1	1
436	2	2
437	1	2
438	2	1
439	2	2
440	1	2
441	2	2
442	1	2
443	1	1
444	2	2
445	1	1
446	1	2
447	1	2
448	1	1
449	2	2
450	2	1
451	2	2
452	2	2
453	2	2
454	1	1
455	2	2
456	1	2
457	1	1
458	1	2
459	1	1
460	1	2
461	1	1
462	1	1
463	1	1
464	2	1
465	1	1

466	1	1
467	2	2
468	2	2
469	1	2
470	1	1
471	1	1
472	2	2
473	1	1
474	1	2
475	2	2
476	1	1
477	1	2
478	2	2
479	1	2
480	2	1
481	1	2
482	1	1
483	2	2
484	1	2
485	1	2
486	1	1
487	1	2
488	2	2
489	1	1
490	1	1
491	1	1
492	1	1
493	1	1
494	1	2
495	1	1
496	2	2
497	2	2
498	1	2
499	2	2
500	1	1
501	1	1

Appendix G Continued

502	1	1
503	2	2
504	2	2
505	1	1
506	1	1
507	2	2
508	1	2
509	1	1
510	2	1
511	2	2
512	1	2
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514	1	2
515	1	2
516	2	2
517	1	2
518	1	1
519	1	2
520	1	2
521	1	2
522	1	1
523	2	2
524	1	2
525	1	1
526	1	1
527	2	2
528	2	2
529	2	2
530	1	1
531	2	2
532	1	1
533	1	1
534	1	1
535	1	1
536	2	1
537	1	1

538	1	1
539	1	2
540	1	1
541	1	2
542	1	2
543	1	1
544	1	2
545	2	1
546	1	1
547	1	1
548	1	2
549	1	1
550	2	1
551	2	2
552	1	1
553	1	1
554	1	1
555	1	1
556	2	2
557	1	1
558	1	1
559	1	2
560	1	2
561	1	2
562	2	1
563	1	1
564	1	1
565	1	1
566	1	1
567	1	1
568	2	1
569	2	1
570	1	1
571	1	1
572	1	1
573	1	1

Appendix G Continued

574	1	1
575	1	1
576	1	1
577	2	2
578	1	1
579	1	1
580	2	2
581	2	1
582	1	1
583	1	1
584	1	1
585	1	1
586	1	1
587	1	1
588	1	1
589	1	1
590	2	1
591	1	1
592	1	1
593	1	1
594	1	1
595	1	1
596	1	1
597	2	1
598	1	1
599	1	1
600	1	1
601	1	1
602	1	1
603	2	1
604	1	1
605	1	1
606	1	1
607	1	2
608	2	1
609	1	1

610	1	1
611	2	2
612	1	1
613	1	1
614	1	1
615	1	1
616	1	2
617	1	1
618	2	1
619	1	1
620	1	2
621	1	1
622	1	1
623	1	1
624	1	1
625	1	1
626	1	1
627	1	1
628	1	1
629	1	1
630	1	1
631	1	1
632	1	2
633	1	1
634	1	1
635	2	1
636	1	1
637	1	1
638	1	1
639	1	1
640	1	1
641	1	1
642	2	2
643	2	2
644	1	1
645	1	1

Appendix G Continued

646	1	1
647	2	2
648	1	1
649	1	1
650	1	2

H. L_3 **DISTANCE NORM RESULT OF CASE-2 FOR CREDIT DATA SET**

Alternatives	**Assigned classes by data itself**	**Assigned classes by Case-2 model**
1	1	2
2	1	1
3	2	2
4	1	2
5	1	2
6	1	2
7	2	2
8	2	2
9	2	1
10	1	2
11	2	2
12	2	2
13	2	1
14	1	2
15	1	1
16	1	2
17	2	2
18	1	2
19	2	1
20	1	1
21	2	2
22	2	2
23	2	2
24	1	2
25	1	2
26	1	2
27	1	2
28	1	1
29	2	2
30	1	1
31	1	1
32	1	1
33	2	2
34	2	2
35	2	2
36	1	1
37	1	1
38	1	2
39	1	1
40	1	2
41	1	1
42	1	1
43	1	1
44	1	1
45	1	1
46	1	1
47	2	1
48	1	2
49	1	2
50	2	1
51	1	1
52	2	1
53	1	2
54	1	2
55	1	1
56	1	2
57	1	1
58	1	1
59	2	2
60	1	2
61	1	2
62	1	2
63	1	1
64	1	1
65	2	2
66	2	2
67	1	1
68	1	2
69	1	1

Appendix H Continued

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Appendix H Continued

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Appendix H Continued

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Appendix H Continued

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Appendix H Continued

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Appendix H Continued

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Appendix H Continued

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Appendix H Continued

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I. L_3 DISTANCE NORM RESULT OF CASE-1 FOR CAR DATA SET

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Appendix I Continued

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Appendix I Continued

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Appendix I Continued

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Appendix I Continued

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Appendix I Continued

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Appendix I Continued

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Appendix I Continued

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Appendix I Continued

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Appendix I Continued

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Appendix I Continued

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Appendix I Continued

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J. L_3 DISTANCE NORM RESULT OF CASE-2 FOR CAR DATA SET

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Appendix J Continued

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Appendix J Continued

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Appendix J Continued

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Appendix J Continued

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Appendix J Continued

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Appendix J Continued

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