POINCARÉ’S PHILOSOPHY OF MATHEMATICS AND THE IMPOSSIBILITY OF BUILDING A NEW ARITHMETIC

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ABSTRACT

POINCARÉ'S PHILOSOPHY OF MATHEMATICS AND THE IMPOSSIBILTY OF BUILDING A NEW ARITHMETIC

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This thesis examines Poincaré's philosophy of mathematics, in particular, his rejection of the possibility of building a new arithmetic. The invention of non-Euclidean geometries forced Kant's philosophy of mathematics to change, leading thinkers to doubt the idea that Euclidean postulates are synthetic *a priori* judgments. Logicism and formalism have risen during this period, and these schools aimed to ground mathematics on a basis other than the one that was laid down by Kant. With regards to the foundations of mathematics, Poincaré adopted Kant's philosophy and remained an intuitionist, though naturally, he had to make significant changes in Kant's thought. Poincaré argued that the branch of mathematics that contains synthetic *a priori* judgments is arithmetic, which is completely independent of experience and therefore pure. What gives arithmetic its object of knowledge and justifies the use of its fundamental principles is not experience, but a pure intuition. On the other hand, Poincaré claimed that our ideas about space and the geometric postulates are not imposed upon us, that they are not known *a priori* but are rather conventions – "definitions in disguise". The role experience plays in the foundations of geometry has given us the possibility of building non-Euclidean geometries. However, since arithmetic is completely independent of experience, it is not possible for a change similar to that in geometry to take place in arithmetic, which would alter its basic concepts or principles that we consider to be true. It is argued in this thesis that it is possible to develop the intuition which lies at the basis of arithmetic and this may become the starting point of a new arithmetic. It will be shown that this is what Cantor has actually achieved when establishing transfinite ordinal arithmetic.

Keywords: Intuitionism, conventionalism, synthetic *a priori*, non-Euclidean geometries, transfinite arithmetic.

ÖZ

POINCARÉ'NİN MATEMATİK FELSEFESİ VE YENİ BİR ARİTMETİK İNŞA ETMENİN OLANAKSIZLIĞI

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Bu tez Poincaré'nin matematik felsefesini, özel olarak da kendisinin yeni bir aritmetik kurmanın imkanını reddedişini incelemektedir. Öklid-dışı geometrilerin icadı Kant'ın matematik felsefesini değişime zorlamış, düşünürleri Öklid postulatlarının sentetik *a priori* yargılar olduğu fikrinden şüphe duymaya itmiştir. Mantıkçılık ve biçimcilik okulları bu dönemde yükselmiş ve matematiği Kant'ın öne sürdüğü temellerden başka temellere oturtmayı amaçlamıştır. Poincaré ise matematiğin temellerine dair Kant'ın felsefesini benimsemiş ve sezgici kalmıştır; fakat elbette Kant'ın düşüncesinde köklü değişiklikler yapması gerekmiştir. Poincaré matematiğin sentetik *a priori* yargılar barındıran alanının, deneyimden tümüyle bağımsız ve dolayısıyla saf olan aritmetik olduğunu öne sürmüştür. Aritmetiğe bilgi nesnesini veren ve temel ilkelerinin kullanımını meşru kılan şey deneyim değil, saf bir sezgidir. Buna karşın Poincaré, uzaya dair fikirlerimizin ve geometrik postulatların ise bize dayatılmadığını, bunların *a priori* bilinmediğini ve aslında birtakım uzlaşımlar, "kılık değiştirmiş tanımlar" olduğunu söylemiştir. Deneyimin geometrinin temellerindeki payı bize Öklid-dışı geometriler kurmanın imkanını vermiştir; fakat aritmetik tümüyle deneyimden bağımsız olduğundan, geometridekine benzer bir değişimin aritmetikte yaşanması ve buradaki temel kavramların veya doğru kabul edilen ilkelerin değişmesi mümkün değildir. Bu tez, aritmetiğin temelinde yatan sezginin geliştirilebileceğini ve bunun da yeni bir aritmetiğin başlangıç noktası olabileceğini öne sürmektedir. Cantor'un sonluötesi ordinal aritmetiği kurarken esasında bunu başardığı gösterilecektir.

Anahtar Kelimeler: Sezgicilik, uzlaşımcılık, sentetik *a priori*, Öklid-dışı geometriler, sonluötesi aritmetik.

To My Parents

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TABLE OF CONTENTS

LIST OF FIGURES

The Great book of nature can be read only by those who know the language in which it is written, and this language is mathematics.

- Galileo

In fine, it is our mind that furnishes a category for nature. But this category is not a bed of Procrustes into which we violently force nature, mutilating her as our needs require. We offer to nature a choice of beds among which we choose the couch best suited to her stature.

- Poincaré

CHAPTER 1

INTRODUCTION

Almost every philosopher who has questioned the value and certainty of the knowledge produced by humankind has written on mathematics. Unlike the physical sciences, the exactness found in mathematics and the fact that its truths are unassailable have always astonished philosophers. Many theories have been put forward to explain what the source of the fundamental difference between the two sciences is, and to what mathematics owes its certainty. A great number of thinkers have portrayed the contrast between the object of mathematics and physics. Mathematics was said to be about *ideal* objects (such as numbers, points, and lines), whose properties are immediately conceived by the mind; whereas physics is about *sensible* objects (such as molecules, rocks, and planets), whose properties are perceived by the senses. The two kinds of objects have brought with them an idea of ‘two worlds’: a world of forms, which is invisible and perfect; and a world of matter, which is visible and ephemeral. It is possible to find a similar terminology being used in almost every period in the history of philosophy, for since Plato, there has never been a time when this theory had difficulty in finding proponents. Though thinking about the nature of mathematics does not necessarily lead to such a duality, the idea has given rise to certain problems that still remain to be solved, and even today these questions continue to motivate thinkers to propose new theories. Some of these questions can be formulated as follows: How did the thought of ideal objects first emerged? What does it mean for such objects to exist? And more importantly, what is the nature of the agreement between the ideal and sensible, such that it makes the former essential in explaining and predicting the behavior of the latter? Every philosopher of mathematics must face these questions, and as we can see, these are linked to some fundamental problems in epistemology and ontology. This is unavoidable, because ultimately, what is questioned in philosophy of mathematics is the long sought relationship between mind and matter, thought and world.

French mathematician and philosopher of science Jules Henri Poincaré (1854-1912) has dealt with these and other similar questions, and he has found reasonable explanations for almost all of them. He is a substantial figure in philosophy of mathematics, because different attitudes philosophers have today towards the nature of mathematical entities and mathematical knowledge[1] were first rigorously outlined and separated from each other in his era, through debates between his contemporaries and of course himself. Poincaré based the existence of mathematical entities and the necessary character of their relations on the *pure intuitions of the human mind* and on the common nature of these minds which were trained in a similar environment. He is therefore considered a naturalist (see Folina, 2014; Stump, 1989) and an intuitionist regarding the foundations of mathematics, though his intuitionism differs from Brouwerian intuitionism, which is what many contemporary thinkers consider when it comes to intuitionism in mathematics. Poincaré was certainly not a platonist, and he has also defended his position against formalists such as David Hilbert (1862-1943) and logicists such as Bertrand Russell (1872-1970). Not only mathematics but philosophy of science in general owes a lot to him. Some see Poincaré as the father of *conventionalism,* and many scientists today – those who question the essence of their practice at least – hold some form of conventionalism concerning the nature of scientific truth.

The subject of this thesis is Poincaré's philosophy of mathematics, more specifically, his rejection of the possibility of building a new arithmetic as in the case of non-Euclidean geometries. Poincaré's intuitionism, which he built to a significant extent on Kant's philosophy, has allowed him to give an account of how it was possible to build non-Euclidean geometries, but led him to deny the same possibility for arithmetic. In order to have a full grasp of Poincaré's rejection, the thesis begins by presenting the key concepts in Kant's philosophy of mathematics, namely, synthetic *a priori* and *pure intuition*. Then, in Chapter 3, Poincaré's interpretation of these concepts is given and the main aspects of his idiosyncratic intuitionism are explained. This chapter closes with a clear demonstration of the difference between the foundations of arithmetic and geometry and Poincaré's reasons for rejecting the possibility of a new arithmetic. Finally, in Chapter 4, Georg Cantor's (1845-1918)

[1] The most prominent of these schools are logicism, intuitionism, and formalism.

transfinite ordinal arithmetic is presented, and it is discussed from which aspects can this theory be considered a new arithmetic.

Although the thesis largely adheres to Poincaré's intuitionism, it adds something to it: the intuition which lies at the basis of arithmetic can be developed. According to Poincaré, the possibility of rejecting Euclidean postulates lies in the fact that geometry and experience are not completely independent. The principles of arithmetic – mainly, the principle of mathematical induction – however, are completely independent of experience; they originate from the affirmation of a power of the mind itself, and we have a direct intuition of this power. This power is basically the ability to conceive the idea of indefinite repetition. It is argued in this thesis that this power, and hence the intuition that corresponds to it, can be subject to improvement. It is claimed that this is what Cantor has achieved, and it became the starting point of a new arithmetic.

CHAPTER 2

KANT'S PHILOSOPHY OF MATHEMATICS

After Plato and Aristotle the third famous figure in Western philosophy is perhaps German philosopher Immanuel Kant (1724-1804). His influence in shaping the history of thought is certainly comparable to the two important thinkers of antiquity. Kant has synthesized the two prevalent schools of his day: Continental Rationalism and British Empiricism, among whose most celebrated proponents we can cite René Descartes (1596-1650) and David Hume (1711-1776) respectively. The name Kant gave to his doctrine was *Transcendental Idealism*. This doctrine was mainly put forward to explain the nature and origin of our knowledge, and it was an attempt to secure the objective knowledge of the empirical world, without adhering to the existence of a mind-independent intelligible world.

Kant's genius lies in his reformulation of *a priori* philosophy[2]. He observed that all knowledge begins with experience, but not all knowledge arises out of experience (1929/1781, p. 41, A1/B1). Since experience is the way it is, that is, organized and conceptualisable, Kant thought that there must be a certain framework that we impose upon experience in order to make it intelligible, and that this framework itself can be a source of knowledge. Such a framework, which is epistemologically *prior* to any conceptualisable experience and constitutive of it, is the condition of possibility of having an objective knowledge of the world, in other words, of doing *science.* The 'lawful' aspect of reality should be ascribed to the faculties of the human being which is experiencing it. Kant thought that if something is held as being valid universally and necessarily, then it must be known *a priori*, meaning that it is related to the active contribution of our minds: "We can know *a priori* of things only what we ourselves have put into them" (1929, p. 23, Bxviii).

[2] The term *a priori* is of Latin origin and means "from the earlier", which is opposed to *a posteriori*, meaning "from the later". Before Kant, *a priroi* was used to indicate a reasoning that proceeds "from causes to the effect", and *a posteriori* "from effects to causes". Kant used *a priori* in order to specify knowledge that is independent of experience – knowledge that is universal and necessary – as opposed to *a posteriori*, i.e. knowledge that is derived from experience.

This shift of attention from the object of experience to the experiencing subject is the Copernican revolution in the history of philosophy. Even though some tenets of Kant's philosophy are abandoned today, the idea that the subject must have a framework independent of and prior to experience in order to make experience intelligible, still finds many proponents.

Kant's epistemology, ethics, and aesthetics are all centered on his theory of the *a priori*, and at the heart of his theory we find arguments concerning the nature of mathematical knowledge. In his first *Critique*, Kant takes arithmetic and geometry as proper examples of *a priori* knowledge, and he often appeals to the epistemic status of their propositions to explain what it means for something to be known independently of experience and belong to the subjective constitution of our mind. According to Kant, mathematical propositions are *synthetic a priori* judgments, and what gives mathematics its object of knowledge is not experience, but *pure intuitions*.

These two concepts, i.e. synthetic *a priori* and intuition, are essential in understanding Poincaré's philosophy of mathematics, because Poincaré followed Kant and classified the propositions of arithmetic as synthetic judgments, whose truth is known *a priori* on the basis of a pure intuition. This, though, was a quite different intuition than the one Kant described. What really separates Poincaré from Kant, however, is Poincaré's ideas on the nature of space and geometry. This does not change the fact that Poincaré remained Kantian to a significant degree, for he has certainly built his theory by adapting the Kantian perspective, and by using the concepts he introduced. These concepts will be examined in detail as they were first formulated by Kant, then in Chapter 3 we will see how they were revised by Poincaré.

2.1 The Distinction between Analytic and Synthetic Judgments

Kant argued that we possess two faculties, one passive and the other active; he referred to them as *sensibility* and *understanding* respectively[3]. He claimed that,

[3] There is also a third faculty, i.e. imagination, but it will not be discussed in this thesis. Nevertheless, it is worth noting that for Kant, in addition to sensibility and understanding, imagination is essential in acquiring knowledge: "Synthesis in general [...] is the result of the power of imagination, a blind but

in the first place, we must have a capacity for being affected by objects, and sensibility is the faculty that grants us this capacity; it is the faculty through which objects are given to us (1929, p. 65, A19/B33). Our sensibility has a certain structure, a certain form, to which objects conform, and Kant referred to space and time as the forms of our sensibility (see. 2.4), which was a revolutionary idea. Through these forms, objects affect us, and we obtain *intuitions* of them. Kant described intuition as a means of cognition which relates to its object immediately (p. 65, A19/B33). He focused on intuition and not on perception of objects, because he thought that even perception involves a certain act of cognition: it is always found in a particular time and space, which means that it is received through the forms of our sensibility. All perceptions are therefore grounded on pure intuition (p. 141, A116).

Kant argued that intuition is something that is not conceptualized, and so it cannot be an object of thought, because according to him, "thought is knowledge by means of concepts" (p. 106, A69/B94). By itself, intuition cannot yield knowledge: "Intuitions without concepts are blind". In making something an object of thought, and therefore of knowledge, the mind must play an active role. For Kant, this role is played by the understanding; it conceptualizes the manifold given by sensibility.

There are some pure concepts in our understanding, i.e. *categories*, which are not derived from experience, but which, nevertheless, organize the manifold given by sensibility. We derive further concepts through the employment of these pure concepts (or rather these rules) accompanied by what is given through sensibility. Kant wrote, "The only use which the understanding can make of these concepts is to judge by means of them" (p. 105, A68/B93). The mind knows only through judgments, and in fact, "we can reduce all acts of understanding to judgments" (p. 106, A69/B94). A judgment is the combining of two concepts in the subject-predicate form, e.g. snow is white, some bodies are heavy, or in its general form, *A is B*. Kant observed that there are two kinds of judgments, two different ways of combining concepts: a judgment is either *analytic*, meaning that the concept of the predicate is already contained in the concept of the subject and logical analysis of the subject is sufficient for showing that the predicate is implied in it, e.g. all bachelors are unmarried; or a judgment is *synthetic*, meaning that the concept of the predicate

indispensable function of the soul, without which we should have no knowledge whatsoever, but of which we are scarcely ever conscious" (1929, p. 112, A78/B103).

is not already contained in the concept of the subject and so the former cannot be deduced logically from the latter but must be added to it, which means that a synthesis between them is required, e.g. all bachelors are unhappy.

Kant has provided two criteria for distinguishing analytic and synthetic judgments. First, as we have mentioned, in analytical judgments the concept of the predicate is already thought in the concept of the subject. Kant's example is "all bodies are extended". He asserted that 'to be extended' is already thought in the concept 'body'[4], though obscurely. Thus, in making this judgment, "I do not require to go beyond the concept which I connect with 'body'" (p. 49, A7/B11); what I do is rather to become conscious of what other concepts are already thought while I am thinking the concept of the subject. If through this method I find that the concept of the predicate is already thought in the concept of the subject, then the judgment is analytical. If not, then I must add the predicate synthetically to the subject[5]. Kant's example is "some bodies are heavy". He observed that he does not include the predicate of weight in the concept of a body at all[6], but experience nevertheless teaches that the two concepts belong together. Heaviness can therefore be added to the concept of a body synthetically on the basis of what experience furnishes.

The second criterion to test a judgment and see whether it is analytical is to deny its truth and derive a contradiction. Kant argued that if denying the truth of a judgment, e.g. all bodies are extended, leads to a contradiction, then the judgment must be analytical, because "all analytic judgments rest entirely on the principle of contradiction" (2004/1783, p. 17, 4:267). Kant believed that we cannot judge that there is a body, perhaps a very strange one, which is not extended, because extension is found in the definition of body; if we judge otherwise, we would be contradicting

[4] Together with impenetrability, shape, figure, etc. (1929, p. 49, A8/B12)

[5] Even though we can add synthetically any two concepts as long as they are not contradictory, not every synthetic judgment will have an object; there has to be a basis on which we can perform this synthesis. According to Kant, this basis is either experience or pure intuition. See 2.4.

[6] "I do not include in the concept of a body in general the predicate 'weight', nonetheless this concept indicates an object of experience through one of its parts, and I can add to that part other parts of this same experience, as in this way belonging together with the concept […] [L]ooking back on the experience from which I have derived this concept of body, and finding weight to be invariably connected with the above characters, I attach it as a predicate to the concept; and in doing so I attach it synthetically, and am therefore extending my knowledge" (1929, pp. 49-50, A8/B12)

ourselves. Similarly, we cannot judge that there is a bachelor who has a wife, because bachelor is defined as unmarried man. In both cases, the concept of the predicate can be deduced logically from the concept of the subject, and so denying their relation must inevitably lead to a contradiction. Notable that neither situation requires us to consult to experience in order to see whether we are judging right, whether there is a bachelor who has a wife, for instance, because logic already teaches us *a priori* that these concepts are contradictory.

Kant himself never actually referred to these two as two different criteria, he took them to be equivalent. It seems that for him, if a concept is implied in the definition of another, and so deducible from it by logic, this guaranteed that in thinking the former we also think the latter, and vice versa. But later philosophers, especially logicists like Gottlob Frege[7] (1848-1925) and Alfred Jules Ayer[8] (1910-1989) criticized Kant's view of analyticity and argued that these two criteria were not actually equivalent. Ayer (1964) wrote that a concept being already thought in another is a *psychological* criterion; whereas being deducible from another using the principle of contradiction is a *logical* one. If the concept of the predicate is deducible from the definition of the subject by simple rule following, then it is auxiliary whether our thinking accompanies this procedure or not. Therefore the two criteria cannot be used interchangeably as Kant did. These philosophers held that the psychological criterion was inadequate for showing whether a judgment was analytical; only the fact that the judgment cannot be denied without self-contradiction – that is, resting on the principle of contradiction – is the proper criterion of analyticity. This debate will be relevant in Chapter 3.4 where Poincaré's views about the epistemic status of arithmetical propositions are discussed.

[7] See Frege (1960), §17; §88. "Kant obviously – as a result, no doubt, of defining them too narrowly – underestimated the value of analytic judgments" (p. 99).

[8] See Ayer (1964). "Kant does not give one straightforward criterion for distinguishing between analytic and synthetic propositions; he gives two distinct criteria, which are by no means equivalent" (p. 294).

2.2 Mathematical Propositions are Synthetic Judgments

Kant claimed that unlike logical propositions, which are analytical and "can be entitled explicative" (1929, p. 48, A7/B11) for they add nothing new to our knowledge, mathematical propositions are synthetic judgments; we cannot expect to arrive at a mathematical truth simply by analyzing the concepts involved. This is true for propositions of both arithmetic and geometry.

Kant appealed to the psychological criterion in order to demonstrate this result. He wrote:

> One might well at first think: that the proposition 7 + 5 = 12 is a purely analytic proposition that follows from the concept of a sum of seven and five according to the principle of contradiction. However, upon closer inspection, one finds that the concept of the sum of 7 and 5 contains nothing further than the unification of the two numbers into one, through which by no means is thought what this single number may be that combines the two. (2004, p. 18, 4:268)

And this "can be seen all the more plainly in the case of somewhat larger numbers" (p. 19, 4:269). Kant is drawing attention to the fact that when we are calculating a sum, the result is not known by us beforehand, and he is taking this as an indication that 12 is not derivable from 7 + 5 using the principle of contradiction alone. He never proved this though; he believed that having shown the predicate is not already thought in the subject implied that the judgment was synthetic. Kant concluded that propositions of arithmetic are not explicative but *ampliative* (p. 16, 4:266), the mere analysis of the subject does not give us the predicate, so we 'go beyond' the concept and perform a synthesis (p. 18, 4:269).

The same is true for geometric propositions. Propositions such as "in a triangle two sides together are greater than the third" or "the sum of interior angles of a triangle is equal to two right angles" are synthetic judgments according to Kant. He argued that neither conclusion could be derived simply by analyzing the concept of a figure enclosed by three straight lines; however long we meditate on this concept, we cannot discover what relation the sum of its angles bear to a right angle, or its sides to each other. Kant concluded that geometric propositions are synthetic. In the first *Critique*, Kant explains how the second proposition is proved by the geometer by constructing a triangle. His geometer draws a triangle using straightedge and compass and derives the desired result with the help of Euclid's postulates:

> He at once begins by constructing a triangle. Since he knows that the sum of two right angles is exactly equal to the sum of all the adjacent angles which can be constructed from a single point on a straight line, he prolongs one side of his triangle and obtains two adjacent angles, which together are equal to two right angles. He then divides the external angle by drawing a line parallel to the opposite side of the triangle, and observes that he has thus obtained an external adjacent angle which is equal to an internal angle – and so on. In this fashion, through a chain of inferences guided throughout by intuition, he arrives at a fully evident and universally valid solution of the problem. (1929, p. 579, A716/B744)

The method of the geometer involves something else other than purely logical principles, and therefore it is synthetic. As in the case of arithmetic, in geometry we also go beyond the given concept to see what other concepts are in relation with it, and make a judgment accordingly.

2.3 The Truth of Mathematical Propositions is Known A Priori

Now in 'going beyond' a concept we usually make use of what is given in *experience.* Experience teaches us whether the sky is blue, or whether Earth is bigger than Venus. What guides us in amplifying our cognition regarding these concepts is the actual perception of objects, and judgments about them are therefore *a posteriori*, i.e. known through experience. But perception of objects can never be the basis of a mathematical judgment, because as Kant observed, "[Mathematical judgments] carry with them necessity, which cannot be derived from experience" (1929, p. 52, A10/B15). We can conceive the possibility of the sky being red for example (as in a sunset); the concepts sky and blue are not necessarily related, because what is taught by experience can someday be corrected by it. However, it seems that we cannot conceive 7 + 5 to be something different than 12; these concepts are related necessarily and experience cannot be the source of such necessity. According to Kant, what is taken as necessary must be known *a priori*, i.e. prior to the experience of objects, and as such, it must pertain to the subjective constitution of our minds. He concluded that the synthesis of mathematical concepts was performed *a priori*.

Like many other philosophers, Kant also pointed out that the proper object of mathematics is not the objects given in experience. Counting either pebbles or sheep plays no role in determining the truth of a proposition in arithmetic. Similarly, a three-sided figure built out of wood or plastic can never be the basis of a geometric truth. Experiments done with pebbles or wood can only yield knowledge about the material with which the experiment is performed, and this always *a posteriori*. These experiments can never teach a purely mathematical relation, because mathematics is concerned with ideal objects (such as numbers and lines), which are not given in experience, and whose relations we conceive to be necessarily true. Even though sensible objects may exemplify mathematical objects, the former can never be the basis of the necessity and universality in mathematical sciences. Experience can therefore only serve as the ground of synthesis of judgments *a posteriori*; the synthesis of mathematical judgments, which express necessity, must be carried out *a priori*, i.e. independently of experience.

The most important question now manifests itself: If mathematical judgments are synthetic, and it is shown that the objects we encounter in experience cannot be the basis of their synthesis, then what is the ground on which we make mathematical judgments? What guides us in arithmetic and geometry if not the familiar objects around us? Kant answered that rather than the objects given *in* experience, what guides us in these sciences are the *forms* to which every object of experience must conform. Space and time are these forms, and they are the forms of our sensibility.

2.4 Forms of Sensibility: Space and Time

Kant described sensibility as the faculty through which objects are given to us. As being forms of our sensibility, space and time are indispensable, and in fact, constitutive elements of experience. Every object (or to be more accurate, every appearance), without exception, is always found in a particular space and time. Kant argued that this was not because space and time had an *objective reality* independent of our cognition. There is not an absolute space and time in which objects reside when they are not perceived by us; rather, space and time pertain to the subjective constitution of our minds. These are frameworks which lie ready in the mind prior to the experience of objects; they constitute and limit our experience. Space is the form of all outer experience, and time is the form of all inner experience (1929, §3; §6). Kant wrote that Space and Time are *empirically real,* meaning that their reality is necessarily recognized by every thinking, judging, experiencing human being. But at the same time they are *transcendentally ideal*, meaning that they have their seats in our subjective constitution, and in this respect they cannot be called objectively real (1929, p. 72, A28/B44).

In synthesizing concepts and making a judgment we usually make use of what is given in experience, and what is given in experience are sensible objects. These objects affect us, and we obtain intuitions of them. According to Kant, every concept that is not empty has a corresponding intuition, in other words a *content*, and this intuition is the ground on which a concept is added synthetically to another. What determines the truth of judgments *a posteriori,* such as 'the sky is blue', is the corresponding intuitions we have of the relevant concepts, and since these are obtained from sensible objects, Kant referred all such intuitions as *sensible intuition.* Sensible intuition cannot teach us something necessary, and so cannot serve as a basis for mathematical judgments. But Kant asserted that even without any sensible object to stir intuitions, *we nevertheless possess an intuition of the framework to which every sensible object must conform*; when everything sensible and intelligible is abstracted from the representation of an object[9], the form of sensibility still remains, and what is intuited then is this form. In contrast to sensible intuition, Kant called the intuition of the form of our sensibility pure, and referred to space and time as *pure intuitions*[10]. According to him, it is these pure intuitions that serve as a basis for mathematical judgments; they allow us to construct (or represent to ourselves) mathematical objects. Since these forms lie ready in the mind prior to the experience of objects, they are a legitimate source of *a priori* knowledge, and hence the necessity and universality in mathematics can be explained.

[9] "If, then, I take away from the representation of a body that which the understanding thinks in regard to it, substance, force, divisibility, etc., and likewise what belongs to sensation, impenetrability, hardness, colour, etc., something still remains over from this empirical intuition […]. These belong to pure intuition, which, even without any actual object of the senses or of sensations, exist in the mind *a priori* as a mere form of sensibility" (1929, p. 66, A21/B35)

[10] "This pure form of sensibility may also itself be called *pure intuition*" (1929, p. 66, B35/A21)

In mathematics we go beyond the concepts by making use of what is given in pure intuition. According to Kant, in order to determine the truth of a proposition in arithmetic we call the aid of intuition of time, and in geometry the intuition of space. Mathematical objects are not given in experience, but they are not empty symbols devoid of content either. We are able to construct mathematical objects on the basis of pure intuition and become conscious of the relations these objects bear to each other. A number is constructed by a successive addition of units (1929, p. 134, A103)[11]. We can conceive the idea of succession, in other words we can *count*, because Kant argues that for us, time is a pure intuition. Similarly, we can represent to ourselves a line in space and go on to construct more complex figures in pure intuition, and thereby become conscious of yet unknown geometric relations. According to Kant, in both cases, that which guides us is pure intuition.

2.4.1 Arithmetic and Time

Kant thought that when we are asked to do the addition 7 + 5, "[We] use the intuition that corresponds to one of the two, such as one's five fingers, or five points, and in that manner adding the units of the five given in intuition step by step to the concept of seven" (2004, p. 19, 4:269). Even though we may use fingers to represent the concept five, as sensible objects our fingers do not have a mathematical character. The intuition that corresponds to the number five is actually a successive addition of units: 1 + 1 + 1 + 1 + 1, and fingers are used only because they resemble the pure units whose addition we originally represent to ourselves in time. By adding the units of the five to the number seven (which is also represented as a successive addition of units), that is by counting, we "see the number 12 come into being" (1929, p. 53, B16). Kant argued that although it may seem as if by this method nothing new is said in the predicate (=12) which is not already thought in the subject (7 + 5), this is actually not the case. "That 7 should be added to 5, I have indeed already thought in the concept of a sum = 7 + 5, but not that this sum is equal to the

[11] This is true for integers and rational numbers. But the construction of irrationals poses a serious problem. An irrational number cannot be constructed by a successive addition of units, because a 'unit' by whose operations we can calculate an irrational number cannot be found. For a discussion about Kant and irrational numbers, see Van Atten (2012).

number 12" (p. 53, B16). In order to judge that 7 + 5 is 12, we must represent to ourselves the concept of the subject in pure intuition, and this is done simply by counting; only then can we decide whether the proposition 7 + 5 = 12 is true or not. For Kant, this was an indication that propositions of arithmetic were synthetic. He thought that they differed from logical propositions whose truth we can show only with the help of the principle of contradiction. There is something else in arithmetic other than logic, and Kant found this in the pure intuition of time.

2.4.2 Geometry and Space

Just like arithmetic, geometry also presupposes something else other than logic. Surely we make use of some fundamental logical propositions in geometry, such as $a = a$, the whole is equal to itself; or $a + b > a$, the whole is greater than its parts. Kant held that these propositions are analytic *a priori* and they rest on the principle of contradiction. But in geometry they "serve only as links in the chain of method and not as principles" (1929, p. 54, B17). We cannot constitute geometry using these propositions alone, because contrary to logic, which is devoid of content, geometry has a subject matter, i.e. space. We need principles other than purely logical ones that would describe space. In geometry we study lines, surfaces, figures, etc. It is evident that we have an intuition corresponding to each of these concepts. We know what a line or a point means; we can represent these to ourselves, or in Kantian terms, we can give to ourselves an object in intuition (p. 86, A48/B65). According to Kant, when we represent to ourselves two points, we immediately see that there can only be one straight line between the two; or similarly, that through any line and a point not on the line, only one parallel can be drawn passing through this point. Now for Kant, these results are not attained by logic alone, and they are not derived from experience either. Geometry is not the study of the relations between sensible objects, because if it were, then "that there should be one straight line between two points would not be necessary, but only what experience always teaches" (p. 69, A24) (we will see in Chapter 3.5 that for Poincaré this was more or less the case). Kant concluded that geometry is the study of the framework to which sensible objects must conform, and that this framework is given *a priori*, because it is the form of our sensibility. Geometry owes its necessity and universality to this *a priori* framework. We construct lines and figures in space, and when we represent to ourselves a line, Kant believed that we immediately see only one parallel can be drawn to it from a given point. In this respect the parallel postulate and all the results that are derived thereof are intuitive results; they are grounded upon the pure intuition of space.

2.5 Summary

The key concepts in Kant's philosophy of mathematics are outlined above. Kant classified mathematical propositions as synthetic *a priori* judgments. He called them synthetic, because he believed that in a mathematical proposition the result is not just a mere rephrasing of what is given (as in the case of analytical judgments), but it covers something more. Kant expressed this by saying that contrary to logical judgments, in mathematical judgments the concept of the predicate is not already thought in the concept of the subject.

Moreover, the truth of mathematical propositions is known *a priori,* that is, prior to any particular experience and without consulting to them. Rather than investigating the relations between sensible objects, which, for Kant, are incapable of exhibiting the necessary character of mathematical relations, in mathematics we investigate ideal objects which we give to ourselves, or construct, in pure intuition. Mathematics is the study of the frameworks to which every object of experience must conform. These frameworks are called space and time and we have a pure intuition of them, because they pertain to our subjective constitution, i.e. they are the forms of our sensibility.

As we can see from the preceding lines, Kant established mathematics as a science, even though he thought that its subject matter was not the objects given in experience. By determining the truth of a proposition in mathematics we learn something new; mathematics expands our knowledge. Thus it would be wrong to claim that mathematics is a branch of logic, which is no doubt an indispensable aid for mathematics, but actually a body of tautologies and therefore not a science. Besides, the truths of mathematics are conceived as *laws*; we hold them to be inviolable and valid for every mind. Hence mathematics meets the requirements of being a science; it teaches us previously unknown laws, not of sensible objects but of our mode of perceiving them. Since our mode of perceiving objects is prior to their perception and a determining factor in it, all other sciences that aim to describe the perceived relations between these objects should be founded upon mathematics.

Everything that has been said so far is essential in understanding Poincaré's philosophy of mathematics, for he has taken a Kantian standpoint in the face of numerous problems concerning the foundations of mathematics. Poincaré thought, for example, that the propositions of arithmetic (but not all of them) are synthetic *a priori* judgments, and he found the justification for their truth in a pure intuition. He distinguished mathematics from logic and treated the former as a science. Furthermore, he agreed with Kant that space and time were not independent realities but frameworks that we impose upon nature to make it intelligible and suitable for experimenting. However, he disagreed with Kant on the idea that principles of geometry – or more precisely, postulates of Euclidean geometry – were synthetic *a priori*. The reason, as might be expected, is that Poincaré was exceedingly familiar with non-Euclidean geometries, where the truths Kant held as necessary and universal turn out to be false. Lalande (1954) wrote, “Nobody has done more than [Poincaré] in France to bring home to educated men the idea that Euclid's system of axioms is not endowed with metaphysical truth, and that on this point it does not differ in any way from Riemann's or Lobachevski's axiomatic systems” (p. 598). Poincaré also disagreed with Kant on the nature of the intuition which lies at the basis of arithmetic, and exactly which propositions in arithmetic are synthetic.

The next chapter is devoted to show how Poincaré transformed the Kantian outlook and established his original theory, which, I believe, was more accurate and in tune with the scientific discoveries of his day. Our principal aim will be to provide a firm basis on which we can make sense of the following claim asserted by Poincaré: Building a new arithmetic is not possible as in the case of non-Euclidean geometries. This is the central claim that this thesis aims to disprove. In Chapter 4, Georg Cantor's theory of transfinite ordinal numbers is presented, which can be said to contain a new arithmetic. But let us first identify Poincaré's position and understand what he thinks the difference between geometry and arithmetic is; later, a detailed analysis of transfinite ordinal arithmetic will be given.

CHAPTER 3

POINCARÉ'S PHILOSOPHY OF MATHEMATICS

3.1 The Intellectual Climate after Kant

Kant's critical method and his ideas concerning almost every branch of philosophy have attracted much attention from subsequent philosophers. Naturally, his 'intuitionistic' philosophy of mathematics had its share of this attention. After all, Kant's ideas about mathematics occupy a central place in his overall philosophy.

However, over time the belief in the Kantian idea that there exist in our minds frameworks which are independent of experience was weakened in the light of new scientific evidence and the criticisms that followed. One of the principal reasons behind this weakening was the invention of non-Euclidean geometries. The first person to take the possibility of these geometries seriously and to work on them was Carl Freidrich Gauss (1777-1855). In 1792, when he was only 15 years old, Gauss started to work on the fifth postulate of Euclid[12], and in 1817 he was convinced that this postulate was independent of the other four postulates, i.e. that it could not be derived from them. He then started to work on alternative geometries where this postulate was rejected, yet he never published these works. The first publication concerning non-Euclidean geometries came from Hungarian mathematician Janos Bolyai (1802-1860) in 1825, in the form of an appendix to his father Farkas Bolyai's book, who was a close friend of Gauss. Russian mathematician Nikolai Lobachevsky (1792-1856), unaware of Bolyai's work, published his own studies in Russian in 1829, in a local university publication the *Kazan Messenger*. The works of these two mathematicians were discussed in a rather small circle, but this changed in 1854 when Bernhard Riemann (1826-1866) gave an inaugural lecture about his works on

[12] "If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines will intersect each other on that side if extended far enough". This postulate is equivalent to what is known as the parallel postulate, which simply states that given a line and a point not on it, there exists only one line that passes through this point and never intersects the given line.

non-Euclidean geometries where he completely reformulated the notion of space. Finally in 1868, Eugenio Beltrami (1835-1900) set these non-Euclidean geometries on a firm basis and reduced the problem of their consistency to the problem of the consistency of Euclidean geometry.

These mathematicians showed that Euclid's postulates were not the only candidates for constituting a consistent system of geometry, and that it was possible to have different geometries that describe space, even though these may seem unintuitive. In these geometries the parallel postulate was replaced by other postulates[13], and so the results derived from the parallel postulate were false. For instance, in non-Euclidean geometries, the sum of interior angles of a triangle adds up to something greater or less than two right angles. Furthermore, two triangles having the same interior angles but different side lengths cannot be drawn – in other words, there are no similar triangles in these geometries – yet they are as consistent and as rich as Euclidean geometry. Schiller (1896) wrote, "If it is a universal and necessary truth that the angles of a triangle are equal to two right angles, it cannot be an equally universal and necessary truth that they are greater" (p. 179). Philosophers were thus led into questioning the existence of an *a priori* framework that our minds imposed upon experience inexorably, giving rise to Euclid's postulates, for non-Euclidean geometries clearly showed that there were other logically possible frameworks for describing space. Naturally, the following questions were raised: What is it that has led us into treating Euclidean geometry as intuitive, that is, why *this* form of sensibility rather than another? And among these geometries, which one is the *true* geometry?

Another factor that undermined Kant's idea of a pure form of sensibility was that the theory of evolution of biological species by natural selection was becoming increasingly dominant[14]. The path that led to this theory was opened chiefly by

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[13] Lobachevsky's and Riemann's postulates. Lobachevsky's postulate states that there exist two lines parallel to a given line through a given point not on the line. Riemann's postulate states that there are no parallel lines. Riemann also had to reject the second postulate of Euclid, which states that a line segment can be extended indefinitely. Riemann assumed that it cannot, and on these suppositions he has laid the foundations of spherical geometry.

[14] In fact, in *Critique of the Power of Judgment*, Kant argues for something like a proto-Darwinian theory of evolution, although not yet for natural selection, which nevertheless shows how farsighted and exceptionally brilliant he was. He suggests that merely mechanical means can account for the

French biologist Jean-Baptiste Lamarck (1744-1829) and his theory of inheritance of acquired characteristics, which first appeared in 1801. But the theory of evolution by natural selection was first formulated after almost 60 years by Charles Darwin (1809-1882) in his book *On the Origin of Species* (1859). With the help of the technological advancements of 19th and 20th century, but more importantly, with the establishment of modern biology and genetics, the number of observations confirming the idea that the different traits in biological species were the result of mutation, adaptation, and selection continued to grow. Philosophers then naturally questioned whether the frameworks we impose upon experience or the 'pure' intuitions we possess were shaped by such an evolution, as in the case of physical and behavioral traits, meaning that they were not that pure after all[15].

Seeing the problems with founding mathematics on our forms of sensibility and the pure intuitions of these forms which are supposedly independent of experience, thinkers sought another basis on which they can ground mathematics. This led some philosophers like Frege to reject Kant's position and attempt to reduce mathematical principles to principles of logic. Others such as Hilbert held that mathematics was simply the study of formal systems whose principles are like the rules of a game which are otherwise meaningless. And there were still others who committed to intuitionism but sought to revise Kant's original position. Poincaré was a member of the last group; he remained a Kantian and an intuitionist. He accepted that there are propositions in mathematics which are synthetic and known *a priori* on

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variation in biological species: "The agreement of so many genera of animals in a certain common schema [...] strengthens the suspicion of a real kinship among them in their generation from a common proto-mother, through the gradual approach of one animal genus to the other, from that in which the principle of ends seems best confirmed, namely human beings, down to polyps, and from this even further to mosses and lichens, and finally to the lowest level of nature that we can observe, that of raw matter: from which, and from its forces governed by mechanical laws [...] the entire technique of nature [...] seems to derive" (2000, p.287, 5:419). But apparently Kant did not consider the possibility that the form of our sensibility or understanding could have such an origin; the hypothesis can be used to explain only the physical constitution of living beings, and even this was regarded by Kant as a "daring adventure of reason".

[15] A similar view was expressed by Poincaré in *Science and Method*, where he mentioned the role of adaptation and natural selection in acquiring the idea of space. According to him, the distinctive movements which allow us to parry incoming threats or reach desired objects are constitutive of space: "Certain hunters learn to shoot fish under the water, although the image of these fish is raised by refraction; and, moreover, they do it instinctively. Accordingly they have learnt to modify their ancient instinct of direction or, if you will, to substitute for the association A1, B1, another association A1, B2, because experience has shown them that the former does not succeed." (2008, p. 116).

the basis of a pure intuition, but as Janet Folina (1986) writes, compared to Kant, "Poincaré's theory of the synthetic *a priori* is much more minimal" (p. 30).

3.2 Poincaré against Logicism and Formalism

Poincaré presented his views in three books: *Science and Hypothesis* (1903), *The Value of Science* (1905), and *Science and Method* (1908). In these books he formulated and often defended his idiosyncratic intuitionism against Russell and Hilbert, the champions of logicism and formalism of his era. After Kant came the logicist attempts, mainly by Frege and Russell, to rid mathematics of any need of intuition and to reduce it to logic, thereby show that mathematical propositions are analytic *a priori*. These philosophers thought that something could be known *a priori* only in virtue of its lack of factual content; there were no *a priori* intuitions that could serve as a basis to synthetic propositions. In truth, mathematical reasoning was not different than logical reasoning and it had nothing to do with forms of sensibility or pure intuitions. A. J. Ayer (1910-1989), another important defender of logicism in the 20th century wrote, "To say that a proposition is true *a priori* is to say that it is a tautology" (1964, p. 301) and mathematics is only a "special class of analytic propositions, containing special terms" (p. 297).

Poincaré was one of the fiercest opponents of this tradition. He believed that contrary to logic, mathematics was not a gigantic tautology but a *science*, and it had a "creative virtue" (2011, p. 3). In *The Value of Science*, Poincaré likens a logicist – whose only tool is *analysis*[16] – to a person who checks whether each move is made in accordance with the rules of the game in order to understand a game of chess. He argues that the person must rather recognize the strategy and the plan behind every move in order to truly understand the game. "We need a faculty which makes us see the end from afar, and intuition is this faculty" (p. 22). There is a parallelism between chess and mathematics, in the sense that both are rule following procedures yet analysis alone is not sufficient for understanding either of them. Just like ascertaining the correctness of every move in a game of chess is not sufficient, so is ascertaining every step of a mathematical proof.

[16] Referred to as "*division and dissection"* (1907, p. 23).

> When we have examined these operations one after the other and ascertained that each is correct, are we to think we have grasped the real meaning of the demonstration? Shall we have understood it even when, by an effort of memory, we have become able to repeat this proof by reproducing all these elementary operations in just the order in which the inventor had arranged them? Evidently not; we shall not yet possess the entire reality; that I know not what which makes the unity of the demonstration will completely elude us. (1907, p. 22)

The quote certainly ends in an obscure manner regarding what intuition is. But Poincaré clarified his views in the following chapters, and so will we. Here it is necessary to add that there is obviously a limit to the analogy between chess and mathematics: the former can "never become a science, for the different moves of the same piece are limited and do not resemble each other" (2011, p. 21).

Poincaré raised a similar criticism against Hilbert and his formalist program. Hilbert wished to reduce the number of the fundamental assumptions of geometry to a minimum. Some of these assumptions might be understood intuitively, but Hilbert held that in essence they were simply formal rules from which theorems could be deduced by purely analytic procedures. There were others such as Giuseppe Peano (1858-1932) who have tried to accomplish what Hilbert did in geometry for arithmetic and analysis. In the very beginning of his *Foundations of Geometry*, Hilbert (1950) wrote:

> Let us consider three distinct systems of things. The things composing the first system we will call *points* […] those of the second we will call *straight lines* […] and those of the third system we will call *planes* […] We think of these points, straight lines, and planes as having certain mutual relations, which we indicate by means of such words as "are situated", "between", "parallel", "congruent", "continuous", etc. The complete and exact description of these relations follows as a consequence of the *axioms of geometry*.

Even though Poincaré admitted that he thought very highly of Hilbert's book, he still condemned Hilbert's approach. Regarding the 'things' Hilbert considered at the beginning of his book, Poincaré writes: "What these 'things' are we do not know, and we do not need to know – it would even be unfortunate that we should seek to know; all that we have the right to know about them is that we should learn their axioms" (2008, p. 122). And again:

> [In Hilbert's Formalism] in order to demonstrate a theorem, it is neither necessary nor even advantageous to know what it means. The geometer might be replaced by the *logic piano* imagined by Stanley Jevons; or, if you choose, a machine might be imagined where the assumptions were put in at one end, while the theorems came out at the other, like the legendary Chicago machine where the pigs go in alive and come out transformed into hams and sausages. No more than these machines need the mathematician know what he does. (2014, Book II, Ch. 3)

We can see from the above lines that Poincaré's emphasis was principally on *understanding*. If we completely neglect our intuitions that play a role in mathematics and adopt a strong formalist standpoint, then we would sacrifice an integral part of mathematics: we would "not divine by what caprice all these [theorems] were erected in this fashion one upon another" (1907, p. 22), and we would not see why among countless possible assumptions these particular ones were judged preferable to others (2008, p. 148). It would therefore be very difficult, if not impossible, to learn and understand mathematics if it is presented to us as a purely formal practice, and this is why Poincaré wrote that he would not recommend Hilbert's book to a schoolboy (2008, p. 122).

There is another, perhaps an even more serious criticism that Poincaré raised against Hilbert's program, and consequently against every other program that aims to prove the consistency of mathematics within a formal system, such as Peano's or Zermelo-Fraenkel's Axiomatic Systems. Poincaré argued that the principle of mathematical induction is an indispensable tool for all branches of mathematics, which states that if a theorem is true for $\gamma = 1$ and if it is shown to be true for $\gamma + 1$ when it is true for an arbitrary γ, then the theorem is true for all natural numbers. Now where does this principle come from? If it is a purely logical principle, then its negation must be capable of being reduced to the principle of contradiction. But how can we be sure that this principle never leads to a contradiction? Poincaré maintained that every attempt to show mathematics is consistent needs to prove, at some point, that the principle of mathematical induction is exempt from contradiction. Since this principle states something about an infinite number of cases, a direct verification showing the principle is true for a finite number of cases would not suffice. "We must then have recourse to processes of demonstration, in which we shall generally be forced to invoke that very principle of complete induction that we are attempting to verify" (2008, p. 153). Thus, every attempt to prove the consistency of the principle of induction will make use of the principle itself, "for that is the only instrument which enables us to pass from the finite to the infinite" (2011, p. 14). This poses a problem for formalists and logicists, yet it is not a problem for an intuitionist like Poincaré, because just like Kant, Poincaré thought that this principle is an *a priori* synthetic judgment and is grounded upon a pure intuition; we immediately become conscious of its validity because it pertains to the subjective constitution of our minds and we have a direct intuition of it (see Chapter 3.4).

After examining the quotations above, it might seem as if Poincaré used the term intuition in several different ways. In fact, he admitted that the meaning of this term was quite vague and he tried to elucidate it. In *The Value of Science*, he writes: "To make any science, something else than pure logic is necessary. To designate this something else we have no word other than *intuition*. But how many different ideas are hidden under this same word?" (1907, p. 19). We saw that he used the term intuition to designate the faculty "that which makes us see the end from afar". For Poincaré, this faculty is an integral part of understanding. A person who has not developed this kind of intuition in a particular field will lack something very crucial in comparison to a person who has, even though both are bound by the same rules and doing the same operations. Seen under this light, intuition appears to be something psychological. This led some thinkers such as Warren Goldfarb (1988) to claim that Poincaré's concern in invoking intuition in mathematics was to explain the psychology of mathematical thinking. However, though an accurate observation, this is only one-half of Poincaré's intuitionism. Intuition understood this way is not exclusive to the mathematician; a chess player, a composer, and even a logicist requires its aid. What is intuited in all these practices is a certain strategy peculiar to that field. It is developed through many experiences and it allows the practitioner to immediately see beforehand what steps she should take. But Poincaré mentions another kind of intuition, one which is pure and reminiscent of Kant's, which gives rise to mathematical reasoning. What is intuited here is not the strategy or the plan in this or that practice, and unlike Kant, Poincaré did not relate this to the intuition of the pure forms of our sensibility. This is rather the intuition of a certain power of the mind, which consists in "conceiving the indefinite repetition of the same act, when the act is once possible. The mind has a direct intuition of this power, and experiment can only be for it an opportunity of using it" (2011, p. 16). This intuition is pure, and unlike the intuition a chess player has, it is given prior to all experience. This is actually the intuition we have mentioned in the previous paragraph, i.e. the one to which we owe the principle of mathematical induction. Let us now specify the two kinds of intuition as Poincaré formulated them, and distinguish the one that serves as a basis to synthetic judgments in arithmetic, giving rise to the 'science of number'.

3.3 Poincaré's Intuitionism

It is true that one of Poincaré's intentions was to explain the psychology of the mathematician and that he used the term intuition in order to do this. There are many passages in which he described what goes on in the "soul" of a mathematician. The most detailed of these passages is found in *Science and Method*. In the third chapter of the first book, Poincaré explains the intuition that guides the mathematician in a mathematical invention by describing in detail the mental processes after which he managed to establish the existence of different classes of Fuchsian functions. Reflecting on his experiences, Poincaré concludes that both conscious and unconscious mental procedures play a role in mathematical invention. The conscious procedures consist of many trials with different combinations, carried out gropingly. These are indispensable for a mathematical proof, since through them the mathematician gains an acquaintance with his problem and becomes more competent. But more importantly, the conscious work of the mathematician "set[s] the unconscious machine in motion" (2008, p. 56). The unconscious procedures are reasonings that go on in the mind of the mathematician even after his focus is turned away from the problem. Among countless possible combinations, only a few can be constructed consciously by the mathematician, which, in most cases, will appear to be barren and useless in the beginning. But Poincaré observes that the unconscious ego, or the "subliminal ego", once stirred by a conscious work, would keep reasoning about the problem by eliminating the useless and encumbering combinations among the extremely numerous possibilities, selecting the most fruitful ones. What guides the mathematician in this unconscious work is an intuition that is certainly not operating arbitrarily, but following extremely subtle and delicate rules, which are practically impossible to be stated in precise language; “They must be felt rather than formulated” (2008, p. 57). What is felt here is the “mathematical beauty; of the harmony of numbers and forms and of geometric elegance. It is a real aesthetic feeling that all true mathematicians recognize” (p. 59). Poincaré states that the results of this unconscious work present themselves to the mind in moments of sudden illumination. He cites experiencing several of such instances; one when he was walking on the cliffs of Caen and another when he was serving his time in the army in Mont-Valérien.

What we need to notice is that the intuition Poincaré described in these passages is not exclusive to the mathematician. In fact, there are other thinkers who have noted that this intuition is not limited to mathematics. Folina (1986) regards it as a faculty that glosses over the incomplete character of both mathematical and empirical experience (p. 86). Gerhard Heinzmann (1988) likens it to “the awareness of the mastery of a schema” (p. 48). We can easily assume the mathematician in the above paragraph to be replaced with a composer and the proof with a musical piece the composer is working on. Or similarly we can imagine it to be a chess player trying to devise a new and unheard strategy. Analogous to Poincaré’s mathematician, the composer would most likely say what guides her in her process of creation is a certain kind of intuition. We can imagine her saying that through this intuition, which is very difficult to elucidate, she appreciates harmony and able to recognize the patterns that arouse in her a feeling of beauty, and what she is looking for very often comes to her in moments of inspiration.

This kind of intuition is also what distinguishes a professor from a student of mathematics. Surely what causes the difference between them is primarily experience, but the abundance of experiences would be useless if it did not help to develop in the mathematician a certain kind of intuition, i.e. the feeling towards mathematical beauty. Because this feeling is highly improved and polished in a professor, he would not waver in the face of a problem that would easily overwhelm a student.

If intuition in mathematics was confined solely to what is described here, then it would truly be something entirely psychological, not something over and above the intuition developed in a specialist concerning his or her particular field. However, even though the student of mathematics – or an amateur in any particular field – lacks this kind of intuition compared to a specialist, both the student and the professor of mathematics share something in common: both possess a mind that is capable of conceiving the idea of indefinite repetition, and therefore able to count. They both have a direct intuition of this capacity and as Poincaré wrote, experience is only an opportunity for them of using it. Because the student also has this capacity, he is able to construct numbers just like the professor, and the reasoning behind the method of proof by mathematical induction is not going to be a mystery for him. Through countless experiences the student would eventually develop the kind of intuition we described in the previous paragraphs. But he would never have learned mathematics and understood mathematical reasoning if he did not have an intuition of this distinctive mental capacity in the first place, which amounts basically to the ability to count indefinitely; without it, the concept of 'number' would be completely meaningless.

Poincaré offered a reliable criterion for distinguishing these different kinds of intuition. He wrote that there are intuitions that may deceive us, and then there are intuitions that may never do so. For instance, the intuition a chess player has developed may sometimes deceive him. We can imagine the game he planned being outmaneuvered by a more brilliant strategy coming from his opponent. Also the mathematician guided by his feeling towards mathematical beauty "need[s] to work out the results of the inspiration" (2008, p. 56). A more striking example Poincaré gives of a deceptive intuition is *geometric intuition*. He writes: "If we try to imagine a line […] our representation must have a certain breadth. Two lines will therefore appear to us under the form of two narrow bands". The geometer "[imagines] a line as the limit towards which tends a band that is growing thinner and thinner, and the point as the limit towards which is tending an area that is growing smaller and smaller" (2011, p. 31). On the basis of our representations we intuitively conclude that whenever we imagine a curve, there are going to be an infinite number of lines intersecting this curve at only one point, i.e. that the function this curve represents will be everywhere differentiable. By establishing the existence of functions which are continuous everywhere but differentiable nowhere, Karl Weierstrass (1815-1897) has showed once again that we were mistaken in trusting our intuitions in geometry. What the chess player and the geometer have in common is the fact that the basis of their intuition is *experience*: the player's intuition depends upon the countless games he played, and the geometer's intuition depends upon the countless observations he had of the motion of the most notable objects around him, in our case, solid bodies. The role experience plays in geometry and how the motion of solid bodies relates to it are subjects of Chapter 3.5. At the moment it is sufficient to know that Poincaré asserted that geometry is not completely independent of experience and this is why geometric intuition sometimes deceives us.

On the other hand, the intuition we have of our capacity to iterate indefinitely – consequently of the principle of mathematical induction – can never deceive us, because "it is only the affirmation of a property of the mind itself" (2011, p. 17). Poincaré called this "the intuition of pure number" (1907, p. 20). Nothing empirical plays a role in formulating the principle of mathematical induction; its truth is known *a priori* on the basis of a pure intuition and it is "imposed upon us with such an irresistible weight of evidence" (2011, p. 16), for what is intuited here is simply a mental capacity. The term intuition is used both for that which makes us conscious of a distinctive mental capacity and also that which makes us conscious of the strategy behind any practice, giving us competence, be it in chess, mathematics, or even logic. This is because in both cases we become conscious of the object of our inquiry *immediately*, without surveying all the intermediary steps. Our intuition is "an incomplete summary of a piece of true reasoning" (2011, p. 216), or in Folina's words, a glossing-over faculty. The difference is that in pure intuition what is summarized is an *a priori* reasoning; whereas in the intuition a practitioner (or a geometer) develops what is summarized is a reasoning that is borrowed from experience. This is precisely the reason why we can never be deceived by pure intuition but may be misled by what we will call *sensible intuition.* However, when we use the term sensible intuition in the context of Poincaré's philosophy, it should be noted that this is not perfectly in line with Kant's conception. Both thinkers took this to mean an intuition which is derived from experience (though in Kant's case, this already presupposes a pure, *a priori* form of sensibility) and as such, incapable of being a basis for necessary and universal truths. But unlike Kant, Poincaré thought that the intuition which lies at the basis of geometry is not pure; it is to a certain degree sensible. He believed that the intuition we have of our forms of sensibility is not pure, because our forms of sensibility – the mental frameworks to which objects conform – are not given *a priori*; these are convenient frameworks invented by us under the guidance of experience. This is perhaps the fundamental difference that separates Poincaré from Kant and it is the origin of Poincaré's stronger empiricist tendencies. How experience guides us in adopting the most convenient framework among other possible options is discussed in Chapter 3.5.

One final remark on the difference between the intuition developed in a specialist and the intuition of pure number: it may be argued that this distinction is not really necessary, for these two intuitions can be reduced to one. In fact, there is no doubt that these intuitions are related to a certain degree. A very important element of the intuition that helps a professional – be it a chess player or a mathematician – in his or her particular field, is the recognition of patterns that are recurring. Recognition of these repeating patterns is actually constitutive of the 'feeling of beauty' that guides a specialist. Thus, it would not be wrong to say that the idea of repetition is an integral part of the intuition developed in a specialist. How these two intuitions are related and whether they can be reduced to one is out of the scope of this paper. However, it is worth noting that we cannot fully explain the intuition in mathematics in terms of the intuition developed in a specialist, because as we have said, the latter is sometimes deceptive. Contrary to this, according to Poincaré, the intuition of pure number on which arithmetic is grounded can never deceive us. What this intuition amounts to is the capacity to conceive the idea of indefinite repetition. From repetition, number arises, and the principle of mathematical induction is simply the affirmation of this mental capacity.

3.4 Mathematical Induction and the Intuition of Pure Number

As we have mentioned above several times, Poincaré disagreed with Kant as to which propositions in arithmetic should be considered synthetic. According to Poincaré, propositions in the form 7 + 5 = 12 are not synthetic but analytic judgments. He wrote that the method for showing the truth of these propositions is not a proof properly so called but *verification.* "Verification differs from proof precisely because it is analytical, and because it leads to nothing" (2011, p. 5). For Poincaré, these are rather uninteresting for a mathematician. What is truly interesting and fruitful in mathematics are *theorems,* for example the proposition which states that every natural number is either even or odd. It is these propositions that needs to be proved, and in which "the conclusion is in a sense more general than the premises" (2011, p. 5).

7 + 5 = 12 can be verified when the operation $x + 1$ (adding 1 to any given number) is defined. Poincaré wrote, "Whatever may be said of this definition [i.e. $x + 1$], it does not enter into the subsequent reasoning" (2011, p. 7). 7 + 5 = 12 is simply a rule following procedure. Poincaré considered 7 + 5 to be an instance of the general formula $x + a$. He defined $x + a$ as

$$x + a = [x + (a - 1)] + 1 \qquad (1)$$

"We can know what $x+a$ is when we know what $x+(a-1)$ is, and since I have assumed that to start with we know what $x+1$ is, we can define successively and 'by recurrence' the operations $x+2$, $x+3$, etc." (2011, p. 8). Poincaré notes that (1) is not a purely logical definition, "It contains an infinite number of distinct definitions, each having only one meaning when we know the meaning of its predecessor" (p. 8). The general formula $x + a$ is of a different character than 7 + 5, which is a particular instance of this formula. On the basis of the definitions above, we can reach 7 + 5 = 12 after a finite number of syllogisms. We will know what 7 + 5 is when we know what 7 + 4 is. We know what 7 + 1 is from the definition of $x + 1$, it is 8. We know what 8 + 1 is, it is 9, and so on. 7 + 5 is only a logical step in this reasoning.

As we can see that for Poincaré, the fact that the result of the sum 7 + 5 is not immediately recognized, or to express this in Kantian terms, the fact that the concept of the predicate is not already thought in the concept of the subject, does not necessarily make 7 + 5 = 12 a synthetic judgment. It is therefore accurate to say that Poincaré has sided with the logicists on this issue, that is, the psychological criterion not being satisfied does not indicate that the judgment is synthetic. Even though we may not immediately see the result of a sum, this does not mean that we cannot also reach it by a purely analytical reasoning. Once $x + 1$ is defined, we can always *verify* in a finite number of syllogisms the truth of the sum of two numbers, however big the numbers may be. In order to calculate this sum we do not need to use intuition and find a correspondence for these concepts in terms of fingers or points added to each other in temporal succession. We can do that, but that is not what Poincaré thought the role of intuition in arithmetic was.

"If mathematics could be reduced to a series of such verifications it would not be a science" (2011, p. 5). As we have said, what is truly valuable in mathematics for Poincaré are theorems. Theorems must be proved rather than verified. $7 + 5 = 12$ is a verification, but the proposition, for instance, that every natural number is either even or odd is a theorem. The truth of any particular instance of this theorem such as $(7 + 5)$ can always be verified. By following the procedure described above, we can show by way of analysis that $7 + 5$ is 12 and that this is an even number (assuming the definition of even and odd is given). We can do the same for any natural number say 4, $(11 + 8)$, 3^4, etc. We can easily verify that all of these numbers are either even or odd. But we can never reach our theorem by conjoining countless of such verifications. The list of our verifications will always be finite, so how to prove this theorem that states something about an infinite number of cases? Now for Poincaré this is where synthesis in mathematics comes in. Logical reasoning can only verify the particular instances of this theorem, but in order to reach the general theorem where an infinite number of verifications are contained in only a few lines, a synthesis is required, and this is carried out on the basis of the principle of mathematical induction, which, according to Poincaré, is "mathematical reasoning *par excellence*" (2011, p. 12). The principle of mathematical induction is an instrument which the logicist does not possess; it is the only instrument that enables us to pass from the finite to the infinite. "This instrument is always useful, for it enables us to leap over as many stages as we wish; it frees us from the necessity of long, tedious, and monotonous verifications which would rapidly become impracticable" (2011, p. 14).

In order to formulate our theorem we first need to define what even and odd means. We suppose $x + 1$ is defined and properties of addition and multiplication are given[17]. A natural number x is called *even* if it is a multiple of 2, in other words, if there is a natural number n where $x = 2n$.

$$x = 2n \qquad (2)$$

A natural number x is called *odd* if it is not a multiple of 2, in other words, if there is a natural number n where $x = 2n + 1$.

$$x = 2n + 1 \qquad (3)$$

Together with (1) where the addition of any two numbers is defined, we now have three definitions. The theorem we wish to prove states that for every natural number x there is a natural number n such that either $x = 2n$ or $x = 2n + 1$. It is easy to verify this theorem for a particular number and increase the number of our verifications, but it is clear that none of these verifications will be considered a *proof* of the theorem. Rather, we must arrive at our result by reasoning *inductively*. First, we need to show that the theorem holds for the base case, $x = 1$. Next we need to assume that it holds for an arbitrary x and show that it holds for $x + 1$. If both conditions are satisfied, we can conclude that *every* natural number is either even or odd. The principle itself requires no proof, pure intuition of number guarantees its truth *a priori*. We assert without a doubt that the theorem will be true if it is true for $x + 1$ when it is true for x, because we immediately recognize that there will be nothing that can disturb this reasoning at a later step. For Poincaré, this is neither a logical nor an empirical reasoning, but a mathematical reasoning, and as a matter of fact, mathematical reasoning *par excellence*. Let us see the proof in order to appreciate the value of this principle.

<u>Theorem:</u> $\forall x, x \in N \; [\exists n, n \in N : x = 2n \text{ or } x = 2n + 1]$

[17] In fact, in *Science and Hypothesis*, Poincaré (2011) treats each of these properties as a theorem and proves them using the definition of addition (1) and the principle of mathematical induction. For instance, with regards to associativity of addition, he considers $a + (b + c) = (a + b) + c$ as a theorem and shows that this is true for $c = 1$, and then shows that when it is true for $c = \gamma$ it is true for $c = \gamma + 1$ (pp. 8-11). He states that this indicates even at such an early stage of arithmetic where the properties of the most basic operations are defined, we must appeal to the intuition of pure number.

Proof by induction

Base Case: $x = 1$

$$1 = (2 \cdot 0) + 1, \text{ therefore odd (3)}$$

Successor Case: Assume x is either even or odd. Show $(x + 1)$ is either even or odd.

If x is odd, then $(x + 1)$ is the sum of two odd numbers. The sum of two odd numbers is always even, because for every natural number a and b where $a = 2n + 1$ and $b = 2m + 1$

$$a + b = (2n + 1) + (2m + 1)$$
$$a + b = 2n + 2m + 2$$
$$a + b = 2(n + m + 1), \text{ therefore even (2)}$$

Hence if x is odd then $x + 1$ is even.

If x is even, then $(x + 1)$ is the sum of an even number and an odd number. The sum of an even number and an odd number is always odd, because for every natural number a and b where $a = 2n$ and $b = 2m + 1$

$$a + b = 2n + (2m + 1)$$
$$a+b = 2n + 2m + 1$$
$$a + b = 2(m + n) + 1, \text{ therefore odd (3)}$$

Hence if x is even then $x + 1$ is odd.

Either way, $x + 1$ is either even or odd, which is what we needed to prove. We may thus conclude that every natural number x is either even or odd.

Each step of this proof follows directly from a previous step (or from the definitions we have given) according to the principle of contradiction. For instance, while showing the sum of two odd numbers is always even we actually compared two definitions, one of which was only a little more complex than the other, and demonstrated that they were in fact identical. But in asserting that the theorem would hold for every natural number if it is shown to hold for an arbitrary number and its successor, we did not rely upon the principle of contradiction but on the principle of mathematical induction, and therefore on the intuition of pure number. This simple proof then illustrates the role of intuition in establishing the truth of a mathematical proposition where a generalization is made over infinitely many elements. According to Poincaré it is these propositions where the conclusion is more general than the premises, and these are grasped intuitively.

3.4.1 The Intuition of Pure Number and Time

In the first pages of *Science and Hypothesis*, Poincaré gives (or rather proves) the rules of algebraic calculus and the definitions required to do elementary arithmetic on the supposition that $x + 1$ is given, in other words, that we already know what it means to add the number one to any given number. But can we not say that there is already a difficulty in defining $x + 1$? Looking from a Kantian point of view, we may argue that the concepts ‘addition’ and ‘number’ must have corresponding intuitions. Besides the principle of mathematical induction, and perhaps even prior to it, the basic concepts of arithmetic must be understood intuitively instead of as mere logical symbols without any significance. Poincaré would have certainly agreed. When he is arguing against logicists, Poincaré writes: “You give a subtle definition of numbers; then, once this definition given, you think no more of it; because, in reality, it is not it which has taught you what number is; you long ago knew that” (2014, Book II, Chapter 3). What has taught us what numbers are is the intuition of pure number, i.e. the ability to conceive the idea of indefinite repetition. But Poincaré separated himself from Kant by claiming that what is intuited in arithmetic is not the pure form of our sensibility. For Poincaré, space and time are mental frameworks which are imposed by us upon nature, yet they are not given *a priori*, because experience plays a certain role in their foundations. These frameworks are invented by us in order to accommodate our particular field of experience, and it is experience that guides us in choosing the most convenient framework for representing natural phenomena. If such a framework were to be a basis for the basic concepts and principles of arithmetic, then arithmetic would not be an *a priori* science, for experience would then have a determinant role in its foundations. Thus Poincaré was careful not to relate the intuition of pure number to the intuition of time as a form of sensibility. Although he did not say directly that the ability to conceive the idea of indefinite repetition pertains to the form of our *understanding*, he did say that the idea of *group*[18], which is given *a priori* and which "preexists in our minds, at least potentially […] is imposed on us not as a form of our sensitiveness, but as a form of our understanding" (2011, p. 82). We can infer that the ability to iterate indefinitely and hence the principle of mathematical induction is also imposed upon us by the form of our understanding. The intuition of pure number pertains not to sensibility but to understanding, because it seems that it is the form of the latter which Poincaré considered as being independent of experience. In this sense there is a significant deviance from the Kantian thought on Poincaré's side. Poincaré has completely separated arithmetic from sensibility; both from sensible objects and from the forms of our sensibility.

It may sound odd to argue that there is absolutely no relationship between the intuition of pure number and time, since 'succession' and 'repetition' seem to be temporal concepts. In fact, in *The Value of Science*, Poincaré writes that it is repetition which gives space its quantitative character, and that "repetition supposes time; this is enough to tell that time is logically anterior to space" (1907, p. 72). But he did not recourse to the idea of time while explaining the foundations of arithmetic and the intuition of pure number. Especially in his paper "The measure of time"[19], Poincaré treated time as a convenient framework just like space. He wrote, "We have not a direct intuition of simultaneity, nor of the equality of two durations" (1907, p. 35). In order to measure time, we need to make use of certain means; these may include pendulums, the revolution of the Earth around itself, or the speed of light. In making one of these methods the standard way of measuring time, we need to adopt some rules whose truth we cannot know *a priori* but must choose in terms of

[18] A group in mathematics is a set with a binary operation, which combines any two elements to create a third in such a way that group axioms are satisfied, namely, closure, associativity, identity, and invertability. Poincaré argued that not any particular group but the idea of group in general pre-exists in our minds as a form of our understanding. The concept of group and its role in the genesis of geometry is discussed in Chapter 3.5.1 (A).

[19] Poincaré (1898), *Revue de Métaphysique et de Morale*, *6*, pp. 1-13. Also appears in *The Value of Science* (2008), Chapter II.

convenience and simplicity[20]. Nevertheless, Poincaré mentioned a *psychological time*, which indeed appears as a form pre-existing in our minds. He described this as the feeling by which we distinguish "present sensation from the remembrance of past sensations or the anticipation of future sensations" (1907, p. 26). It is also the feeling which informs us that between the memories we can recall, there always remain other instances of which we may not have a memory. Poincaré did not take psychological time into account in the foundations of arithmetic, nor did he refer to it when he is explaining the intuition of pure number. Moreover, he argued that psychological time alone is not sufficient for constructing the temporal framework in which we wish to put *everything*, including the phenomena of our consciousness, of other's consciousness, and also the physical facts, "which no consciousness sees directly" (1907, p. 27). The construction of this framework requires certain assumptions whose truth we cannot know *a priori.* Contrary to this, the truth of the principle of mathematical induction is known *a priori*, because unlike our ideas about time, it involves nothing empirical; it is imposed on us directly by the form of our understanding and we have a pure intuition of this form.

3.4.2 The Difference in the Foundations of Arithmetic and Geometry

But can we not raise to Poincaré the same criticism he raised against Kant? That is, could experience play a role in the form of our understanding as it does for our sensibility? Poincaré answers this in the negative and compares the principles of arithmetic with that of geometry to endorse his view. As we have said, Poincaré argued that experience cannot be the source of the principle of mathematical induction, but only an opportunity of using it; the principle is a perfect example of a synthetic *a priori* judgment and it is imposed on us with an irresistible weight of evidence by the very nature of our minds. This is why Poincaré believed that we can never conceive the rejection of this principle and construct a "false arithmetic" that

[20] With respect to pendulums, and therefore to all sorts of clocks, it is assumed that all the beats of the pendulum are of equal duration. With respect to the revolution of the Earth, it is assumed that two complete rotations of the Earth about its axis have the same duration. And with respect to the speed of light, Poincaré claimed that it is *assumed* that light has a constant velocity and that it is the same in all directions: "This postulate could never be verified directly by experiment; it might be contradicted by it if the results of different measurements were not concordant" (1907, p. 34).

would follow from a contrary proposition (2011, p. 57). But there is something fundamentally different in geometry. According to Poincaré, principles of geometry are not given *a priori* but chosen conventionally under the guidance of experience, and this is exactly why it was possible to construct non-Euclidean geometries. Poincaré thought it was evident, for instance, that the fifth postulate of Euclid is not forced upon our minds, for we are able to reject it and construct alternative geometries using Riemann's or Lobachevsky's postulates (see footnote 14). For Poincaré this showed that the Euclidean framework, which we find intuitive and with the help of which we reason about physical phenomena, is not unique; there are different possible frameworks, equally conceivable and coherent. The reason we find the Euclidean framework intuitive is not because it is a form pre-existing in our minds prior to all experience, but because it is the simplest and the most convenient one that describes our particular field of experience. If we were taken to another world in which our impressions would change radically and where the motion of objects would follow altogether different laws from the ones we are accustomed, then, given enough time, our sensibility would adapt its form. Our straightedge and compass would no longer describe the lines we are familiar with, and so we would be forced to choose different principles describing different frameworks. In other words, we would be led to adopt different conventions, and it is going to be these conventions that we will be teaching in the geometry classes of our hypothetical secondary schools. "So that beings like ourselves, educated in such a world, will not have the same geometry as ours" (2011, p. 79). On the other hand, Poincaré believed that the same is not true for arithmetic. He held that no experience can render the principle of mathematical induction useless and compel us to abandon it; contrary to Euclid's postulates, this principle is not a convention, but an affirmation of a property of the mind itself. Poincaré writes: "Let us next try to […] reject [mathematical induction] and let us construct a false arithmetic analogous to non-Euclidean geometry. We shall not be able to do it" (2011, p. 57). Here for the first time we see clearly the reason behind Poincaré's rejection of the possibility of constructing an alternative arithmetic. The reason is simply that arithmetic is a pure, *a priori* science, whose basic concepts and principles are known independently of experience by a pure intuition; whereas the postulates and the meaning of the basic concepts of geometry cannot be known *a priori,* these are determined by convention: "It is impossible to discover in geometric empiricism a rational meaning" (2011, p. 115). So far we have explained Poincaré's views on the nature of arithmetic and specified briefly how it is different from geometry. But we have not yet elaborated on his ideas about how we have constructed geometry and what exactly is the role experience plays in its foundations. This will be our next subject.

3.5 Conventionalism in Geometry

Poincaré argued that it is impossible to decide *a priori* whether the concept of straight line, whose definition is 'the shortest distance between two points', corresponds to a Euclidean or a non-Euclidean line, or how many 'lines' can pass through two points. There is not a rational decision to be made concerning these matters. Like Euclid, we may assume that lines can be extended indefinitely, that there is only one line passing through two points, and that there is only one line parallel to a given line. We would then be giving the name straight to the sides of Euclidean triangles and by accepting his other postulates we can derive the rest of Euclidean geometry. But we may equally assume, as Riemann did, that lines cannot be extended indefinitely, that there are more than one line passing through two points, and that there are no parallels. We would then be giving the name straight to the sides of spherical triangles and derive a body of geometry which is equally coherent and interesting as Euclidean geometry. Poincaré's reasoning on this matter is straightforward: If Euclid's postulates were *a priori* intuitions as Kant affirmed, "They would then be imposed upon us with such a force that we could not conceive of the contrary proposition, nor could we build upon it a theoretical edifice. There would be no non-Euclidean geometry" (2011, p. 57). Since there is, then the truth of Euclid's postulates is not the result of an *a priori* reasoning, and this becomes even clearer when we compare them with what Poincaré called a true synthetic *a priori* judgment, i.e. the principle of mathematical induction.

What, then, is the source of these postulates, and why is it that we feel unwilling to oppose them? Is it because they are actually experimental facts? But as Kant has shown and later Poincaré confirmed, experience cannot be the basis of a geometric truth: “We do not make experiments on ideal lines or ideal circles; we can only make them on material objects” (2011, p. 58). Experiments done with material objects will always remain approximate, whereas in geometry there is certainty. According to Poincaré, if someday we observe objects which are moving differently from what we are used to and whose movements more or less resemble non-Euclidean motions, we would not conclude that this experiment refutes Euclidean geometry, but rather prefer to draw conclusions about these particular objects[21]; Euclidean geometry and all other geometries would remain unaffected by such an experiment. “No experiment will ever be in contradiction with Euclid’s postulate; but, on the other hand, no experiment will ever be in contradiction with Lobachevsky’s postulate” (2011, p. 86). Experience can neither directly verify nor refute geometric postulates.

The situation in geometry is rather peculiar. There are propositions in geometry whose truth we hold as self-evident and therefore consider as axioms, but the truth of these propositions cannot be shown by reasoning *a priori* (since we can equally conceive contrary propositions), neither by way of experience, and yet still, there is no doubt that geometry is in some connection with experience, for it lies in the foundations of almost all physical sciences. “It then follows for Poincaré […] that we have here – in this very special situation – a conventional choice or free stipulation” (Friedman, 1995, p. 312). The axioms of geometry belong to a new epistemic category: they are *conventions*.

Conventions are “intermediary principles found in scientific disciplines that lie on the border between pure mathematics (the synthetic *a priori*) and the natural sciences (the synthetic *a posteriori*)” (Folina, 2014, p. 26). These are not forced upon us by the nature of our minds, nor are they experimental facts. Even though experience cannot directly refute a convention, it does guide us in choosing the most useful one among other possible hypotheses. This is exactly the situation in Euclidean geometry[22]. Contrary to spherical or hyperbolic geometries, Euclid’s

[21] See *Science and Hypothesis* (2011), Chapter V.

[22] Conventions are not limited to geometry. According to Poincaré, they are found in physical sciences as well, e.g. the principle of least action, the principle of inertia, etc. For the discussion about the conventions in physical sciences, see *Science and Hypothesis*, Chapter VI.

geometry, in other words plane geometry, is the geometry that seems most intuitive to us, and it is this geometry that we consider as standard. Poincaré thought that this is because the framework built using Euclid's axioms is the most convenient one that describes the movements of the most significant objects around us, including our own bodies, which are called *solid objects*. Poincaré defined them as "[the] objects whose displacements may be corrected by a correlative movement of our body"[23] (2011, p. 70). For us, these objects are such that they do not change their shape or size as they are moving. Euclid's postulates provide us with the most simple and convenient framework for describing the movements of objects which are displaced without being deformed, and since this is characteristic of the most significant objects around us, it is not a mystery that we have intuitively adopted the Euclidean framework. However, this movement – a displacement without deformation – is never perfectly realized. We know that due to changes in pressure or heat, very small variations which are imperceptible by us occur in the shape and size of these objects. "But in laying the foundations of geometry we neglect these variations; for besides being but small they are irregular, and consequently appear to us to be accidental" (2011, p. 77).

In Poincaré's philosophy, experience is an integral part of geometry. According to him, we construct a framework of spatial associations "by studying the laws by which [sensation of objects] succeed one another" (2011, p. 67). For Poincaré, these associations are what we call the evidence for geometric truths – the repugnance we feel towards breaking very old habits (2008, p. 104). "It is just because this association is useful for the defense of the organism, that it is so old in the history of the species and that it seems to us indestructible" (1907, p. 71). But what is associated can be dissociated, though in the case of spatial associations this is very difficult, because "we have to overcome a multitude of associations of ideas which are the fruit of a long personal experience and of the still longer experience of

[23] Correcting a displacement means to perform the movements that would establish the initial impressions of the object before its movement. If I see a car moving a meter to the left in front of me, I can trace its movement with my eye and by walking a meter to the left I can reestablish my impressions of the car. A detailed explanation of what a displacement is and what it means to correct it is given in 3.5.1. (B).

the race" (1907, p. 70). However, since the laws of succession of the sensation of objects is not something we can discover by reasoning *a priori* but by experience, Poincaré maintained that there is nothing preventing us from imagining these objects to be succeeding one another according to laws which differ from the ones we are accustomed. According to Poincaré, this is exactly what non-Euclidean geometries amount to – frameworks that describe objects which are moving according to unconventional laws.

3.5.1 Conditions for Constituting Geometry

Poincaré cites both *a priori* and (unlike in the case of arithmetic) *empirical* conditions that play a role in the genesis of geometry, and according to him, these empirical conditions *define* the subject matter of geometry (Folina, 2014, p. 13). When I say *a priori* conditions, I mean that which is given prior to all experience and which pertains to the constitution of the mind; and when I say empirical conditions, I mean that which is given in experience and which a mind having the described constitution encounters.

3.5.1.1 *A priori* Conditions

First of all, in order to build the framework we call space an organism needs to have a mind that is capable of conceiving indefinite repetition. As we have said, Poincaré thought that only then can number be a part of geometry and space may have a quantitative character. The intuition of pure number is therefore necessary for constituting geometry. Besides, Poincaré argued that this intuition plays a very important role in the invention of the *mathematical continuum*, and the space of the geometer is actually a three dimensional mathematical continuum. The invention of the mathematical continuum deserves a detailed investigation on its own; here we can only deal with it briefly[24]. We can summarize Poincaré's position in the following way: The rough data of our senses sometimes give us contradictory results. It may happen, for instance, that our sense of weight cannot distinguish two objects *A*

[24] It will be explained how a continuum of the *first-order* – as Poincaré called it – is created. This is the continuum composed of integers and fractions. But the real continuum of the mathematician is a continuum of the *second-order*, which is composed of real numbers. For the creation of a second-order continuum, see *Science and Hypothesis* (2011), Chapter II.

and *B*, weighing 10 and 11 grams respectively, while it can distinguish *A* from *C*, which is weighing 12 grams, but cannot distinguish *B* from *C*. The results of our experience may then be expressed as “A = B, B = C, A < C, which may be regarded as the formula of the physical continuum. But here is an intolerable disagreement with the law of contradiction, and the necessity of banishing this disagreement has compelled us to invent the mathematical continuum” (2011, p. 28). Even though we may develop better and more delicate instruments, according to Poincaré we will never be able to escape from the inherent contradiction of the physical continuum by following such a method. There will come a time when we encounter again with a new, indistinguishable term. “We only escape from [this contradiction] by incessantly intercalating new terms between the terms already distinguished, and this operation must be pursued indefinitely” (2011, p. 29). Now since we can immediately become conscious of the possibility of such an operation on the basis of the intuition of pure number, we can create a mathematical continuum where every element is completely distinguishable from one another and therefore escape from the inherent contradiction of the physical continuum when experience gives us contradictory results.

Another *a priori* condition for constituting geometry is for the mind to have the notion of group. A mathematical group is a set with a binary operation that combines any two elements to create a third in such a way that group axioms are satisfied (see footnote 19). Poincaré writes, “What mathematicians call a group is the *ensemble* of a certain number of operations and of all the combinations which can be made of them” (Poincaré, 1898, p. 13). For instance, the set of all integers together with the addition operation forms a group. According to Poincaré, the general notion of group pre-exists in our minds, at least potentially, as a form of understanding, and “the object of geometry is the study of a particular ‘group’ […] only, from among all possible groups, we must choose one that will be the standard, so to speak, to which we shall refer natural phenomena” (2011, p. 82). Here ‘natural phenomena’ designates the observable motion of objects. Poincaré argues that we consider the *displacements* of the objects around ourselves as forming a group, and the object of geometry is the group of these displacements.

Displacements, which we shall soon discuss as an empirical condition for geometry, are movements performed by objects that can be corrected by a correlative movement of our body, and the objects that perform such a movement are called solid objects. For us, these objects move continuously without changing their shape or size; they do not go through sudden changes, shrink or extend. According to Poincaré, the possibility of such a motion is admitted implicitly in Euclid's postulates, and the properties of the group describing this particular motion are used in his demonstrations[25]. But "the possibility of the motion of an invariable figure is not a self-evident truth" (2011, p. 53). Euclid adhered to the properties of a particular group intuitively, as we all do, not because it was the 'pure' form of his sensibility, but because it just happens that the objects whose movements we can correct with a corresponding movement of our body are those that do not change their shape as they are moving, and the displacements of these objects obey *approximately* to the properties of this particular group. This group already exists in our minds potentially, along with countless other possible groups, yet experience gives us the opportunity to reach it (2011, p. 82) and to employ it in explaining natural phenomena.

Not all these groups are suitable for describing displacements. Poincaré believed that those that are suitable are determined by a theorem of Norwegian mathematician Sophus Lie (1842-1899), which characterizes all manifolds where free mobility of figures is possible[26]. Poincaré argued that Lie's group-theoretic solution to Helmholtz's problem of space[27] shows that the number of geometries in

[25] This point is mentioned in almost all of Poincaré's writings on geometry, but the most detailed explanation is given in "On the foundations of geometry". Poincaré (1898) writes, "When I pronounce the word 'length', a word which we frequently do not think necessary to define, I implicitly assume that the figure formed by two points is not always superposable upon that which is formed by two other points; for otherwise any two lengths whatever would be equal to each other. Now this is an important property of our group". And again, "How do we proceed in our reasonings? By displacing our figures and causing them to execute certain movements. I wish to show that at a given point in a straight line a perpendicular can always be erected, and to accomplish this I conceive a movable straight line turning about the point in question. But I presuppose here that the movement of this straight line is possible, that it is continuous, and that in so turning it can pass from the position in which it is lying on the given straight line, to the opposite position in which it is lying on its prolongation. Here again is a hypothesis touching the properties of the group" (p. 33).

[26] In modern terms, free mobility amounts to constant Riemannian curvature. For Poincaré's interpretation of Lie, see Poincaré (1898, p.37; 2011, p.55).

which free mobility is possible is limited, and for Poincaré, this amounts to saying that there are only a limited number of groups suitable for describing the displacements of objects. In all these geometries (1) space has *n* dimensions, (2) the movement of an invariable figure is possible, and (3) *p* conditions are necessary to determine the position of this figure in space. "*The number of geometries compatible with these premises will be limited*. I may even add that if *n* is given, a superior limit can be assigned to *p*" (2011, p. 55). When *n* is 3, only three systems of geometry can be established: flat, hyperbolic, and spherical; in other words, zero curvature, constant negative curvature, and constant positive curvature. A particular group of displacements corresponds to each of these geometries, and the choice as to which of these groups is going to refer to natural phenomena – the observable motion of objects – remains free. But the choice is not made arbitrarily; here it is experience that guides us. We have chosen the geometry of zero curvature, i.e. the Euclidean group, because, first of all, it is simpler[28]; and second, the most significant objects that approximate to what we call an invariable figure are such that they do not change their shape as they are moving, and this is the group that best describes such movements. But certain movements that may be described by other groups are not ruled out *a priori*. In *Science and Hypothesis* Poincaré imagines a hypothetical world whose inhabitants would most likely adopt a non-Euclidean group to refer to the phenomena of their world.

> Suppose, for example, a world enclosed in a large sphere and subject to the following laws:—The temperature is not uniform; it is greatest at the center, and gradually decreases as we move towards the circumference of the sphere, where it is absolute zero. The law of this temperature is as follows:—If *R* be the radius of the sphere, and *r* the distance of the point considered from the center, the absolute temperature will be proportional to$R^2 - r^2$. Further, I shall suppose that in this world all bodies have the same co-efficient of dilatation, so that the linear dilatation of any body is proportional to its absolute temperature. Finally, I shall assume that

[27] In *Philosophy of geometry from Riemann to Poincaré*, Roberto Torretti (1984) formulates Helmholtz's problem of space as follows: "Which among the infinitely many geometries whose mathematical viability has been shown by Riemann's theory of manifolds are compatible with the general conditions of possibility of physical measurement" (p. 154). See Torretti (1984), Chapter 3.1 for further discussion of Helmholtz's problem of space and Lie's solution of it.

[28] "It is the simplest in itself, just as a polynomial of the first degree is simpler than a polynomial of the second degree" (2011, p. 59).

> a body transported from one point to another of different temperature is instantaneously in thermal equilibrium with its new environment. There is nothing in these hypotheses either contradictory or unimaginable. A moving object will become smaller and smaller as it approaches the circumference of the sphere[29].

To us who are living outside the sphere, the objects moving inside the sphere would no longer appear as invariable figures, for they would undergo deformations which we would be unable to correct by a corresponding movement of our body. An object *O* inside the sphere moving from point *A* to point *B* would shrink or expand, and no matter where we move around this sphere we will never be able to have the same impression of *O* when it was in *A*. But the creatures who are living *inside* the sphere and therefore subject to the same deformation *O* undergoes can reestablish their impressions by performing a corresponding movement with their bodies. They would therefore consider *O* as an invariable figure, and the movement it performs as a displacement. The object of their geometry would be the group of these particular displacements, i.e. “The laws of motion of solids *deformed by the differences of temperature*” (2011, p. 76). “Beings educated there would no doubt find it more convenient to create a geometry different from ours, and better adapted to their impressions” (p. 82), they would adopt a non-Euclidean group and this will be the one that they find intuitive. Poincaré treated these groups as *models* that lie ready in the mind: “We have within us, in a potential form, a certain number of models of groups, and experience merely assists us in discovering which of these models departs least from reality” (1898, p. 13).

The list of *a priori* conditions for establishing geometry can no doubt be extended. For example among these conditions we can cite the logical rules of inference, and perhaps self-awareness, for in order to establish a system of spatial associations the organism needs to have a certain degree of awareness of its own movements – these should be performed voluntarily. It also has to be able to recall these movements as well as the movements of other objects. These and a few other conditions were mentioned by Poincaré as well, but the two that are discussed above were the ones that interested him the most.

[29] In the following paragraphs, Poincaré supplements this picture with hypotheses concerning the relationship between the laws of heat and the refraction of light, thereby altering the impressions of the inhabitants of this world even further. See (2011, pp. 75-79).

3.5.1.2 Empirical Conditions

Like the list of *a priori* conditions, the list of empirical conditions is certainly much longer than what is written here. Only two of these conditions will be discussed, though these are the ones that were emphasized the most in Poincaré's writings.

The first empirical condition that needs to be satisfied for an organism to establish geometry is for the organism to have *mobility*. According to Poincaré, a motionless being could have never acquired the idea of space (2011, p. 69). Space does not necessarily have to be a pure intuition, and in fact it is not, because there are other conceivable spaces than the one we find intuitive; rather, we are able to conceive a certain group of spatial relations simply because we are creatures who can perform certain movements. Mobility should be granted if an organism is to have an idea of space, otherwise the organism would have no means of discovering spatial relations; this is something it cannot do by reasoning alone. If the organism is motionless, then there is no way for it to correct the movements of objects by performing a corresponding movement with its own body. It will therefore be unable to distinguish *changes of state* from *changes of position*. Poincaré considered these as the two main changes that objects may undergo. An object can either change its state or it can change its position. If an object undergoes a change of position, we say that the object has been *displaced.* The possibility of making this distinction lies at the basis of geometry, and Poincaré argued that it is impossible for an immobile creature to make it.

But it is not enough for the creature to be mobile. In order to distinguish changes of state from changes of position the motion of objects must also meet certain criteria. Objects should move in such a way that we can correct them with a correlative movement of our body and restore our initial impressions of the objects; only then can we say that objects are displaced. This, then, is the second empirical condition for establishing geometry – the possibility of displacement. So far we have mentioned this concept several times, especially when explaining the concept of group, and there we gave its definition: Displacements are movements performed by objects that can be corrected by a correlative movement of our body. We have also said that objects capable of performing such a movement are called solid objects.

That there are such objects is not a self-evident truth, but an experimental fact, and Poincaré argued that from this experimental fact we are led to distinguish the two main classes of changes that objects may undergo:

> We see at first that our impressions are subject to change; but among the changes that we ascertain, we are very soon led to make a distinction. Sometimes we say that the objects, the causes of these impressions, have changed their state, sometimes that they have changed their position, that they have only been displaced. (2011, p. 68)

We are able to make this distinction, because it just happens that there are objects whose changes are such that we can "restore the primitive aggregate of impressions by making movements which would confront us with the object in the same *relative* situation" (2011, p. 68). We classify the changes that can be corrected by this means as changes of position, and those that cannot be corrected are called changes of state. Objects which frequently experience displacements that may be thus corrected are called solid bodies. If "there were no solid bodies in nature there would be no geometry" (2011, p. 71), because we would not be able to distinguish changes of position from changes of state and so could not obtain the idea of space.

A quick digression is necessary. We should notice that in claiming that there are 'solid' bodies in nature Poincaré is not actually affirming a property of the things in themselves[30]. Here solid body only refers to a relation being ensured between objects and ourselves, that a certain compensation is possible. Our classification of objects as solid does not directly concern what they are as they are in themselves; what distinguishes these objects is determined in relation to us. And in fact, an object that is considered solid by an observer and whose movements are recognized as displacements may not be considered solid by another observer. Remember the

[30] Poincaré agreed with Kant on the idea that nothing concerning the *things in themselves* can be an object of knowledge; all that can be known is our specific relationship with objects. "[The physical theories] teach us now, as they did then, that there is such and such a relation between this thing and that thing; only, the something which we then called *motion*, we now call *electric current*. But these are merely names of the images we substituted for the real objects which Nature will hide for ever from our eyes. The true relations between these real objects are the only reality we can attain" (2011, p. 179) In fact, there are passages where Poincaré went even further and argued that the true reality is nothing but these relations; an investigation into the properties of things as they are in themselves actually has no object. "External objects, for instance, for which the word object was invented, are really *objects* and not fleeting and fugitive appearances, because they are not only groups of sensations, but groups cemented by a constant bond. It is this bond, and this bond alone, which is the object in itself, and this bond is a relation" (1907, p. 138).

hypothetical creatures living inside the sphere. The objects these creatures would consider solid and whose movements they would classify as displacements would not be considered solid by us, for there is no way for us to "restore the primitive aggregate of impressions by making movements which would confront us with the object in the same relative situation". Thus, in saying an object is solid Poincaré is not pointing at an objective reality, but only stating that a certain relationship can be ensured between the observer and the object in question. Let us now return to our main subject.

Assume that two objects, first O_1 and then O_2 (let these be a red sphere and a blue cube), move from point *A* to point *B*. For the sake of the argument let us also assume that we do not yet know geometry, and so cannot relate the changes these objects are undergoing to points or to space in general. We can only ascertain that O_1, which was causing the aggregate of impressions α, is now causing the impressions α'; and O_2, which was causing the aggregate of impressions β, is now causing the impressions β'. Poincaré argues that in terms of sense data there is nothing in common between α and β, and so between α' and β'; these are two completely different groups of sensations. But still, even without knowing geometry and having an idea of space, it is possible for us to assert that both O_1 and O_2 merely changed their position and performed the same displacement. This is possible if, by the same correlative movement of our body, we can reestablish the initial impressions α and β. If we can thus correct our impressions then we call both changes a displacement, and indeed, the same displacement. This possibility is granted to us by experience. We then consider all these displacements – that particular class of movements which can be corrected by a correlative movement of our body – as forming a group, and the true object of geometry is this group. According to Poincaré, when we represent an object in space, we are actually thinking about a certain group of displacements; when we say an object is at some point in space, "it simply means that we represent to ourselves the movements that must take place to reach that object" (2011, p. 67). So we may argue that what Kant thought was a pure form of sensibility – that which remains after everything sensible is abstracted from the representation of an object – was, in fact, a representation of a certain group of movements, which are performed frequently by the most significant objects around us, including our bodies.

According to Poincaré, the group that we have chosen which best describes displacements pre-exists in our minds potentially along with other groups, but we have chosen the one that helps us accommodate our particular field of experience. The choice cannot be made *a priori* but it is not arbitrary either; it is nature that shows us the couch best suited to her stature (1898, p. 43). The displacements of material objects only approximately obey to the properties of the group we have in mind. But Poincaré writes that this is enough for us to consider this displacement, by an artificial convention, as a change resulting from two other component changes: one that is obeying the properties of the group rigorously; and the other, which is small, is regarded as a qualitative alteration (1898, p. 11).

We cannot disregard the role experience plays in the genesis of geometry; we cannot separate space from the distinctive movements of our bodies and of other objects. The primitive concepts of geometry, such as distance, point, and line are understood on the basis of these particular movements, to which geometric postulates are also fundamentally related. This is the empirical part of geometry, for the possibility of these movements is given to us by experience. Precisely for this reason, Poincaré stated that geometry is not as pure as arithmetic[31]. There is nothing preventing us from imagining the impressions of objects to be succeeding each other according to laws which differ from the ones we are accustomed, and in fact, as we have seen above, Poincaré imagined a world where such impressions would be actual when the given physical hypotheses are granted. The geometry of the inhabitants of such a world will be different from ours, and neither our, nor their geometry will be the *true* geometry. One geometry can only be more *convenient* than another. The principles which these creatures would find intuitive are going to be different from the ones we find intuitive, and these may even be contradicting with each other, yet neither group of principles would be imposed by the nature of the minds of those who follow them; these will be merely convenient hypotheses, or “definitions in disguise” (2011, p. 59). It is possible for experience to render one of these principles useless one day (though not by directly contradicting it) and lead its followers to

[31] “We must seek mathematical thought where it has remained pure – *i.e.*, in Arithmetic” (2011, p. 6)

adopt another convention. However, a similar situation cannot happen in arithmetic, for according to Poincaré, the principles of arithmetic are imposed directly by the nature of the mind; this is a pure, *a priori* science.

3.6 Summary

The distinction between arithmetic and geometry in Poincaré's philosophy of mathematics is now clear. In order to give a detailed explanation of this distinction we first presented Poincaré's criticisms to logicism and formalism that emerged after Kant. According to Poincaré, mathematics is neither a branch of logic, nor merely a formal game whose rules have no intrinsic meaning; mathematics is a science and it is based on the intuitions of the human mind. But contrary to what Kant believed, Poincaré held that the intuition underlying mathematics is not the intuition of the pure forms of our sensibility, in other words, of space and time. The idea that there are frameworks in our minds which are given prior to all experience but which, nevertheless, determine the spatial and temporal relations between objects was unacceptable for Poincaré. He maintained that these frameworks were invented by us under the guidance of experience. It is true that nature does not impose them upon us, we impose them upon nature, yet we do this under her guidance and counsel. Poincaré argued that we do not have a pure intuition of space and time, "the persons who believe they possess this intuition are dupes of an illusion" (1907, p. 27); there are no 'pure' frameworks in our minds which are completely independent of experience. The invention of non-Euclidean geometries showed that there are other conceivable, yet unintuitive spaces, and for Poincaré this was proof that our framework is not unique. He believed that the reason for choosing this particular framework among other possible options cannot be given entirely on rational grounds; experience must be playing a determinant role in making this decision. He argued that it does this by showing us the most convenient framework among the possible ones.

According to Poincaré, the pure intuition underlying mathematics pertains not to sensibility but to the form of our understanding, and this means for him an intuition of a certain capacity, a power of the mind. This is not a mysterious power, but simply the capacity to conceive indefinite repetition. This is the intuition that gives meaning to the concept of number, and the principle of mathematical induction is the formulation of this intuition. The truth of synthetic judgments in arithmetic is shown with the help of this principle, and the truth of the principle itself is known *a priori*, because it is simply the affirmation of a property of the mind itself. For Poincaré, nothing empirical plays a role in the formulation of this principle; experience is only an opportunity of using it. This is why he considered arithmetic pure and restricted solely by the mind itself. But in geometry there are other principles. Geometric proofs are based on certain axioms whose truth is considered self-evident, but which, according to Poincaré, are not imposed upon us by the nature of our minds; they are convenient hypotheses. Euclid's axioms are not self-evident truths of reason. This point is almost indisputable, for it is possible to replace, for instance, the fifth postulate with a contrary proposition and still build a new, though 'unintuitive' geometry. The choice as to which propositions should be considered as axioms of geometry remains – to a certain degree – free, yet our choice is guided by experience. We are confronted with certain objects and their motion (or more precisely, aggregates of impressions), and experience shows us which of the potential frameworks in our minds is best suited to describe their motion. The most remarkable objects around us are solid bodies – objects whose displacements we can correct with a corresponding movement of our body, and which do not change their shape or size (at least perceptibly) as they are moving. Euclid's axioms seem intuitive to all of us, and for a long time no one has doubted the truth of these axioms, because the framework built on the basis of these axioms describes the motion of figures that undergo no deformation as they are being translated, which is what the solid bodies in our field of experience approximates to. If we accept that lines can be extended indefinitely, and from a point only one parallel can be drawn to a given line, then a figure drawn between these lines can be translated without being deformed, and this figure can also be superimposed with another figure so that their sizes can be easily compared. But as Poincaré said, the motion of an invariable figure is not a self-evident truth. This possibility is shown to us by experience, for it has confronted us with certain objects whose movements approximately agree with the movement of a perfectly invariable figure. But since the familiar movements of objects is something taught by experience, there is nothing preventing us from imagining different kinds of movements. We can imagine the impressions of objects to be succeeding each other according to different laws, and for Poincaré this is the starting point of a non-Euclidean geometry. But he held that something similar cannot be accomplished for arithmetic, because the principle of mathematical induction involves nothing empirical; its truth is directly imposed upon us by the nature of our minds. Trying to reject this principle and build an alternative arithmetic based on an opposite proposition would be equivalent to trying to cease thinking mathematically, for what we are trying to reject is the basis of mathematical reasoning. Poincaré concluded that it is impossible to build a new arithmetic as in the case of non-Euclidean geometries.

CHAPTER 4

CANTOR'S TRANSFINITE ORDINAL ARITHMETIC

Is rejecting the principle of mathematical induction the only way to build a new arithmetic? It seems reasonable that unless we want mathematical thinking to cease, we should not ignore the pure intuition that gives meaning to the concept of number. But could not this intuition itself give rise to new principles? Could it be possible to establish a new arithmetic by developing, not by rejecting this intuition, and refining the primitive notions of arithmetic? I maintain that this is possible, and that Georg Cantor's transfinite ordinal arithmetic provides an adequate example.

In his famous paper *Grundlagen einer allgemeinen Mannigfaltigkeitslehre*[32] (1976/1883), Georg Cantor introduced to the world an entirely new type of number called *transfinite numbers*. These numbers were neither absolutely infinite nor finite; they designated a 'many' which is infinite, but which, at the same time, could be thought of as 'one' – as a determinate, *completed* unit. Transfinite numbers had a peculiar arithmetic where, for example, the commutative law failed to be generally valid in the case of addition and multiplication. In order to build the theory of these numbers German mathematician adhered to a principle whose author was no one but himself: the *second principle of generation*[33]. On the basis of this principle he defined the first transfinite ordinal number[34] ω (sometimes called the smallest limit

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[32] "Foundations of a General Theory of Manifolds".

[33] This principle will be examined in detail in Chapter 4.2. Cantor's formulation is as follows: "If any definite succession of defined whole real numbers is given of which there is no greatest, then on the basis of this second principle of generation a new number is created, which can be thought of as a limit of those numbers, i.e. can be defined as the next greater number to all of them" (1976, p. 87). For instance, when the sequence of natural numbers is given, i.e. 1, 2, 3, 4, …, Cantor asserts that a new number ω is created by which we can count all the elements of the sequence and which, therefore, is greater than all of them. See 4.2.

[34] The two primary types of number Cantor discusses in his theory of manifolds are cardinals and ordinals. Cardinal numbers are not in the scope of this paper, though for some mathematicians they are the true subject, and arguably constitute the more interesting part, of set theory. In simplest terms,

ordinal), and then by using the first and the second principles of generation (see 4.2) he defined $\omega + 1$, $\omega + 2, \ldots, \omega \cdot 2, \omega \cdot 3, \ldots, \omega^2, \ldots, \omega^\omega, \ldots$ etc. Furthermore, unlike what Poincaré proposed, instead of rejecting the principle of mathematical induction and replacing it with a contrary proposition, Cantor extended this principle so that it would be applicable to his new, transfinite numbers. This new principle is called *the principle of transfinite induction.* Underlying all of Cantor's mathematical innovations was his unusual but groundbreaking conception of infinity. Cantor brought a new meaning to the concept of infinity, but more importantly, he managed to make this a subject of mathematical investigation. I contend that the basis of Cantor's new principles and novelties is the same as that which grounds the principle of mathematical induction – it is what Poincaré called the intuition of pure number. On this view, what justifies the truth of Cantor's principles is the intuition of a distinctive capacity of the mind, only that Cantor has improved this capacity and refined the primitive mathematical notions accordingly, which became the starting point of a new arithmetic.

Poincaré praised Cantor for the services he rendered to science; nevertheless, he openly contested Cantor's new conception of infinity, more precisely, his acceptance of the *actual infinite*. The distinction between potential and actual infinity was first put forward by Aristotle to designate the difference between an uncontainable, never-ending progression; and a containable, yet infinite progression (see 4.1.1). In *Science and Method*, Poincaré (2008) wrote: "*There is no actual infinity.* The Cantorians forgot this, and so fell into contradiction" (p. 195). Poincaré believed that Cantor's treatment of 'all' real numbers as a completed totality was inherently flawed. The idea of actual infinity has given rise to *impredicative definitions* (see 4.3), which in turn resulted in what Poincaré called *Cantorian Antinomies*[35]. But a lot of his criticisms are mainly directed to transfinite *cardinal* numbers, not ordinals, and more generally to non-predicative definitions. With

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a cardinal number is how many of something there is, whereas an ordinal number is what the order of something is. If we have a set of five objects, e.g. {a, b, c, d, e}, then the cardinal number that corresponds to this set is 5, for the set has that many elements. But if we are discussing about the element 'c' in this set, this will have the ordinal number 3, for it is the third element in the set.

[35] Burali-Forti's antinomy, The Zermelo-König antinomy, Richard's antinomy. See Poincaré (2011), p.185.

regards to transfinite ordinals, Poincaré was content only with stating the Burali-Forti Paradox and asserting that it was wrong to think 'all ordinal numbers' as forming a set. We will clarify Poincaré's reasons for rejecting the idea of actual infinity, though these reasons have not prevented Cantor's theory from gaining even more popularity and becoming one of the most interesting fields of today's mathematics. Today, students of mathematics in higher education are introduced with these unusual whole numbers. Students are expected to give up their intuitive idea of infinity, or rather to see this concept under a new light, which was handed down to us by Cantor. What is astonishing is that when this is achieved and the theory of transfinites is understood, it allows us to solve some problems that we can formulate in the old theory but cannot solve without accepting the new one[36]. Like the invention of non-Euclidean geometries, which once demanded interpretation, we now have a 'non-standard', seemingly unintuitive arithmetic that we need to make sense of. The origin and the justification of the principles of this new arithmetic requires clarification. Surely the emergence of transfinites and their peculiar arithmetic did not take place exactly in the same manner as that of non-Euclidean geometries. First of all, unlike in the case of Euclid's postulates, Cantor did not replace the principle of mathematical induction with a contrary proposition, but he rather improved the principle itself. Thus, the results of transfinite arithmetic are not in contradiction with the results of our standard, finite arithmetic; the new arithmetic is more of an extension of our previous system. Furthermore, the possibility of rejecting Euclid's postulates stems from the role experience plays in their foundations. We have arrived at the postulates of non-Euclidean geometries by assuming the empirical conditions underlying Euclidean geometry to be different, in other words, by assuming the impressions of objects to be succeeding each other according to laws which differ from the ones we are accustomed. But a similar method cannot be adopted in arithmetic. If we believe that arithmetic is a pure, *a priori* science, and as such, independent of experience, then we must hold that changes in empirical conditions – either actual or hypothetical – cannot result in a refinement in the basic concepts and principles of arithmetic. But such a refinement should nevertheless be possible, for there is no doubt that Cantor has succeeded in making it. I maintain that this is possible, not by rejecting the

[36] Defeating the Mathematical Hydra and Goodstein's Theorem are examples of such problems. These are discussed in Chapter 4.3.

intuitive results but by developing intuition itself, and this is in fact what Cantor did. In spite of the fact that I find Poincaré's intuitionistic philosophy of mathematics accurate and Cantor's realism a bit too overwhelmed with metaphysics[37], unlike Poincaré I believe Cantor in no way contradicted the previously given definitions, or the pure intuition on which primitive notions are based; instead, he has *naturally extended* them. Poincaré was wrong only in thinking that the intuition of pure number is limited and that it compels us to work only with finite whole numbers. Cantor paved the way for the idea of 'infinite whole numbers' to gain recognition, and in order to do this he had to change the commonly held, intuitive understanding of mathematical infinity. Let us first see how Cantor had managed to do this; later his theory will be presented in more detail.

4.1 Theory of the Actual Mathematical Infinite

In a paper published in 1899, Poincaré wrote: "Before, one began with a large number of concepts regarded as primitive, irreducible and intuitive; such were the concepts of whole number, fraction, continuous magnitude, space, point, line, surface, etc. Today only one remains, that of whole number" (as cited in McLarty, 1997). Unfortunately, a few years before this was written, Cantor had published his *Grundlagen* in which he was presenting a reformulation of the concept of whole number – the last remaining frontier. The first section begins with the following lines:

> […] my investigations in the theory of manifolds[38] has reached a point where its continuation becomes dependent upon an extension of the concept of a real whole number beyond the present boundaries; in particular, this extension goes in a direction in which, to my knowledge, no one has so far looked for it. (1976, p.70)

[37] Cantor's metaphysical and even religious views play a determinant role in his mathematical theory of the infinite. The arguments to which Cantor appeals in order to explain a detail in his theory are sometimes metaphysical in nature. For example, according to Cantor (1976), "The true infinite or absolute, which is in God, admits of no kind of determination" (p. 76). He also believed that a clearly conceived mathematical entity will always have a counterpart in the external world because of the principle of "the unity of the all, to which we ourselves belong as well" (p. 79).

[38] Also used as "set" or "aggregate".

Cantor argued that the traditional understanding of 'whole number' admits only *finite* numbers. He held that this concept should be made more comprehensive so that it allows *infinite* numbers as well. According to Cantor, the reason no one had considered the possibility of an infinite whole number lies in the widely accepted but rather limited understanding of infinity. He cited many philosophers, such as Descartes, Locke, and Leibniz, whom he thought had this understanding, while at the same time tracing the idea back to Aristotle. Cantor claimed that with regards to mathematics, the majority of philosophers held the following as an incontestable proposition taken from Aristotle: *infinitum actu non datur*[39] . They believed that infinity in mathematics is meaningful only when it is used to designate a procedure that continues without ever terminating – "a variable magnitude, either growing beyond all limits or diminishing to an arbitrary smallness, always, however remaining *finite*"[40] (1976, p.70). The obvious example is the generation of natural numbers by constantly adding 1 to itself, in other words, *counting*. It is easy to see that this procedure is never complete; there is no last element and thus it must be called *potentially infinite*. Clearly none of the steps of this procedure is the 'infinite' step; the procedure is never *actually infinite*. If we follow this line of reasoning then we must accept that infinity cannot be thought of as a determinate number, but only as an indefinitely varying magnitude or a non-terminating, endless progression. In order to argue for the existence of his transfinite numbers which were actually infinite, Cantor felt compelled to defeat Aristotle's arguments. He claimed that the misconception about the state of affairs concerning the finite and the infinite, which was rooted in Aristotle's writings, led philosophers to assume that no modifications can exist between the absolute infinite and the finite. Cantor proved otherwise:

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[39] "Actual infinity does not exist". This was used almost as a motto by scholastic philosophers.

[40] In truth, Cantor said that infinity in mathematics has occurred so far under not one but two different forms. He called the one we have cited the *non-genuine* infinite, and this was the prevailing understanding for a long time. Cantor argued that the other one emerged in modern times, in particular in function theory: "It has become necessary and in fact common practice to imagine in the plane representing the complex variable a single point at infinity, i.e. an infinitely distant but determinate point" (1976, p.70). Cantor called this the *genuine* infinite. In contrast to non-genuine infinite, which he characterized as "variable finitude", the genuine infinite was not variable but determinate. However, Cantor claimed that his transfinite numbers can be captured by neither of these two. For more on this see Cantor (1976), Section 1.

> What I maintain and believe I have proved in this paper as well as in my earlier endeavors is that after the finite there is a *transfinitum* (which also could be called *suprafinitum*), i.e. an unlimited gradation of determinate modes which in their nature are not finite but infinite, yet which, much as the finite, can be determined by determinate, well-defined and distinguishable *numbers*. I am convinced, therefore, that the domain of definable magnitudes is *not* limited to the finite magnitudes; accordingly, the limits of our cognition may be extended further without it being necessary to do any kind of violence to our nature. (1976, p.76)

4.1.1 Cantor's Response to Aristotle's Rejection of Actual Infinity

In *Grundlagen*, Cantor discussed two arguments which were given by Aristotle against the existence of an infinite number. First, in *Metaphysics* Book XI, Aristotle argues that a number is that which is arrived at by counting, and (1) only finite numbers can be counted; therefore, only finite numbers exist. The second argument is that if an infinite number were to exist, then (2) it would annul the finite numbers; since finite numbers do exist, then an infinite number does not. We take (1) and (2) as the two main premises which were used by Aristotle to prove the non-existence of actual mathematical infinity. Cantor believed that his theory of transfinites, in which he gave a proper definition of an infinite number and made it subject to mathematical investigation, is proof that both (1) and (2) are false.

Cantor claimed that (1) is an undemonstrated proposition; in taking this as a true premise and concluding that only finite numbers exist, Aristotle has committed a *petitio principii*. He thought that only finite numbers can be counted, "because only counting procedures with respect to finite aggregates were known to him" (1976, p.75). Cantor believed that there are counting procedures with respect to infinite aggregates as well, and he designed such a procedure on the basis of his second principle of generation (see 4.2). This principle allowed him to define first the smallest and then the rest of the transfinite ordinal numbers, which were required to count infinite aggregates. ω is defined by the second principle of generation as the next greater ordinal number to all finite ordinals[41]; this is a number that has no

[41] More precisely, ω denotes the order type of a set in which there is a first element and after every element comes a next element in accordance with a certain rule. The obvious example is {0, 1, 2, 3,

immediate predecessor and is used to count all the elements of the set of natural numbers. Every ordinal number has an immediate successor; this is something we hold intuitively and it is explained in Chapter 3.4 how this is rooted in the intuition of pure number. But in Cantor's theory we find that not all ordinals have an immediate *predecessor*, and ω is the smallest number having this property[42]. It is impossible to reach ω with a successive addition of units, in other words, by finite counting. For Cantor, however, this does not make transfinite ordinals less real than finite ordinals, for on the basis of his new generation principle he has succeeded in defining the former as determinate and precise as the latter, and extended counting beyond finite numbers, showing that number formation does not end with finite numbers. But how was Cantor able to arrive at this principle which was unnoticed by Aristotle? Everyone who has read Cantor and knows a little about his life should notice the theological motives behind his mathematical research. Cantor thought that his ambitious investigation into the concept of the infinite was inseparable from the study of God, i.e. the absolutely infinite being. In fact, throughout his investigations, Cantor believed that time to time, God was with him. In his book *Georg Cantor: His Mathematics and Philosophy of the Infinite* (1990) J. W. Dauben writes that Cantor believed the fundamentals of set theory were divinely inspired to him by God (p. 298). I maintain that it is possible to free Cantor's theory from these theological connotations and argue that what he has actually accomplished was to develop a pure intuition that was already present in him. On this view, it is the intuition of a mental capacity that underlies the second principle of generation. This capacity consists in conceiving an indefinitely repeating process as a single event and assigning to it a determinate number when a certain order can be found among its members. In the language of set theory this means *well-ordering* of a set that has infinitely many elements. This point will be elaborated in Chapter 4.2.

With respect to (2), Cantor did not deny that an infinite number annuls a finite number, but he held that this happens only under certain circumstances, and since

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…}, i.e. the set of all natural numbers. The set of all natural numbers is said have the order type ω, or similarly, ω is the ordinal number that corresponds to the set of all natural numbers. See 4.3.

[42] $\nexists x\, [x + 1 = \omega],\ x \in N$

Aristotle did not properly define an infinite whole number and gave it a determinate meaning, his intuitive reasoning cannot be accepted without criticism. In truth, Aristotle did not use (2) directly in order to show the non-existence of an infinite number, but rather to show that a body – "that which is bounded by planes" – or likewise any of the elements that make up a body – fire, water, earth, air – cannot be infinite. He wrote, "If one of the two bodies falls at all short of the other in potency, the finite will be destroyed by the infinite"; and similarly, if an element like fire or water were to be infinite, then "it would be the destruction of the contrary elements" (Aristotle, trans. 2015, p. 114). Cantor saw this argument as directly applicable to mathematics, and held that it is concerned not with the magnitude of a body or an element, but with numbers in general. This is quite understandable, for when we are first introduced with the idea of mathematical infinity in high school – which is shown with the symbol ∞[43] – most of us are naturally led to think like Aristotle: we consider this in terms of potential infinity and believe that any finite number that goes into operation with it would be destroyed. ∞ is not a determinate number, it is rather thought of as a non-terminating process. This is why instead of writing $x = \infty$, we write $x \to \infty$. Any finite number, however big it may be, would be utterly insignificant when it is standing next to ∞, and when such a number is added to or multiplied by ∞, it is unreasonable to think that the result would be something determinate: the finite number will always be annulled by the infinite. However, there is an error in this reasoning. ∞ cannot be used freely in arithmetical operations, because arithmetical operations for it are not defined. Surely there is an idea behind this term which we are all familiar with, but this is not enough to conceive it as a determinate number and make it subject to further mathematical investigation. Arithmetical operations with ∞ must therefore be left undefined, as is usually the case in calculus. On the other hand, contrary to ∞, Cantor defined ω as a determinate number and showed its relation to our traditional numbers, it is therefore possible to write $x = \omega$. ω belongs to a class of a new type of numbers, namely, *limit ordinals*, which are defined by the second principle of generation. When Cantor defined arithmetical operations for limit ordinals, he saw that if ω is added to a finite number,

[43] This symbol was first used by English mathematician John Wallis (1616-1703) in 1650s.

the result is ω: the finite number is destroyed by the infinite; but when a finite number is added to ω, the result is a new number. He expressed this as follows:

$$1 + \omega = \omega$$

On the other hand,

$$\omega + 1 \neq \omega$$

$\omega + 1$ is a whole new number, which is greater than ω, more precisely, its successor. Therefore, “We can very well adjoin a finite number to an infinite number (if the latter is thought of as determinate and complete) and unite the finite number with the infinite number *without* bringing about the annulment of the former” (1976, p. 75). This result will be explained in detail in Chapter 4.2.1, at the moment it is sufficient for appreciating Cantor’s reasons for rejecting Aristotle’s argument.

4.1.2 The Intuition of Pure Number and Potential Infinity

According to the theory suggested in this thesis, the affinity we feel for the idea of potential infinity is caused by the intuition of pure number. What has led Aristotle, and is leading us even today, to resist the idea of actual infinity in mathematics is the particular meaning that the concept of infinity acquires through the intuition of pure number.

It is quite easy to relate this intuition to the idea of potential infinity. The intuition of pure number amounts simply to becoming conscious of mind’s ability to conceive indefinite repetition. When we want to express that a certain quantity is infinite (i.e. the number of elements in the set of natural numbers), or similarly a procedure is infinite (i.e. counting), we actually make use of this intuition, for what we mean is nothing other than that there is no end to what we are measuring; it can increase, or diminish, indefinitely. “Understanding indefinite iterability is what enables us to understand the potential infinity of a set like the natural numbers” (Folina, 1986, p. 35).

Poincaré stated that even though we may think ourselves very far from the idea of infinity while discussing the foundations of arithmetic, this idea is already playing a preponderating part, and in fact, what makes mathematics a science is for the most part its relationship with infinity (2011, p. 14). A similar view was expressed by Hilbert (1964) with the famous phrase: "Mathematical analysis is a symphony of the infinite" (p. 138). I believe that the concept of infinity, like that of number, is principally understood in terms of the intuition of pure number. This intuition, taken simply as ordinary counting, gives us the natural numbers and warrants proofs by mathematical induction. But in doing so, it also attaches a meaning to the concept of infinity. By means of it, infinity is conceived as an indefinite progression, or similarly, as an ever growing quantity. This intuitive conception, however, does not let us treat infinity as a determinate number, for when we wish to determine it, we feel that we must stop the progression; and when we stop it, what we get is always a finite number. The progression in its entirety, which is what we really mean when we use the word infinite, remains as something indeterminate. Our intuitive ability of ordinary counting leads us into adopting the idea of potential infinity, and so like Aristotle, we naturally think that infinity is not a number.

But is the possibility of accepting the proposition "infinity *is* a number" ruled out *a priori*? We have said that in arguing against the synthetic *a priori* status of Euclid's postulates, Poincaré wrote: "Are they synthetic *a priori* intuitions, as Kant affirmed? They would then be imposed upon us with such a force that we could not conceive the contrary propositions, nor could we build upon it a theoretical edifice. There would be no non-Euclidean geometry" (2011, p. 57). By the same token, if the proposition "infinity is not a number" was known *a priori*, then Cantor could not have conceived the opposite proposition and built upon it a new arithmetic. But he did, and he was praised by many great mathematicians such as Hilbert and Gödel for doing so. This indicates that the meaning of the concept of infinity is not given *a priori*; it acquires a certain meaning through the intuition of pure number, but it is open for interpretation, because the intuition of pure number itself can be developed.

4.2 Fundamentals of Transfinite Ordinal Arithmetic

Cantor, just like Poincaré, thought that the sequence of natural numbers 1, 2, 3, … has its origin in the repeated addition of units (Dauben, 1990, p.97). In

Grundlagen, Cantor called the process of defining numbers by a successive addition of units the *first principle of generation* (1976, p .87). This refers to ordinary counting and it is the most intuitive and straightforward way of generating new numbers. However, Cantor believed that number formation does not end here. There is a second, and even a third principle of generation[44], by which infinite whole numbers are defined. After showing that Aristotle's arguments are not strong enough to banish infinite whole numbers from mathematics, Cantor presented the principles and the conceptual framework required to establish the theory of these numbers. This conceptual framework was his theory of manifolds, sometimes called the *naive* set theory.

Cantor did not define what a set is in a rigorous way, though it is not clear whether such a definition can be given at all. He was content with stating that a set is "a multiplicity that allows itself to be thought of as one"[45] (1976, p. 93). We know, however, that there are some multiplicities that cannot be thought of as one, such as *Ω*, i.e. the 'set' of all ordinal numbers; or Russell's famous example, "the set of all sets that are not members of themselves". It has been shown that when we try to think of these multiplicities as one, we fall into contradiction. Cantor's intuitive conception did not specify which multiplicities can be thought of as one, and this presented serious problems. This is why his theory is usually referred to as a naive set theory.

Although Cantor's definition of set was problematic, his definition of a *well-ordered* set was rigorous and proved itself to be very useful. This concept played an

[44] Cantor (1976) called the third principle of generation the *principle of limitation,* or *inhibiting principle* (p. 71). This principle allowed him to "produce natural breaks in the sequence of transfinite numbers" (Dauben, 1990, p. 98). With the use of the first and second principles of generation it is possible to create an unlimited sequence of transfinite ordinals. Cantor used the inhibiting principle to impose limits on the endless formation process and thus obtain distinct *number classes*. The first number class is simply the finite ordinals generated by the first principle of generation, and the second number class is the entire sequence of transfinite ordinals created by the first and second principles of generation. However, this principle was rather overlooked, because it was closely related with the continuum hypothesis (see footnote 51) and higher cardinalities. Cantor thought that the second number class was actually an infinity of ordinal numbers which belong to one and the same cardinal number, $\aleph_0$ (1915, p. 159), and higher cardinalities were in turn related to ordinals from higher number classes.

[45] Cantor also writes the following in *Contributions to the Founding of the Theory of Transfinite Numbers*: "By an 'aggregate' (*Menge*) we are to understand any collection into a whole (*Zusammenfassung zu einen Ganzen*) M of definite and separate objects *m* of our intuition or our thought" (1915, p. 85).

essential role in the generation of infinite whole numbers, for the second principle of generation creates a number that is used to count the elements of a set with infinitely many elements only when this set can be well-ordered. A well-ordered set is one in which "the elements are bound together by a specifically pre-assigned law of succession, according to which there exists both a *first* element […] and there follows after every single element another specific element" (Cantor, 1976, p.72). If we put aside the problems concerning the definition of set and assume that a whole composed of elements can be given, we can say that this is a well-ordered set when it is possible to arrange its elements in a specific order. Let us assume a set *S* is given which is composed of four elements, e.g. {4, 1, 3, 2}. We can arrange the elements of this set as {1, 2, 3, 4}, therefore give it an ascending order. We can also give it a descending order by arranging the elements as {4, 3, 2, 1}. Even {4, 1, 3, 2}, which appears to have no specific order, is a well-ordering of *S*. Here the order is simply the following: first element is 4, second element is 1, third element is 3, and last element is 2. Well-ordering of a set with finitely many elements is always possible, because even if there is not a practical way of arranging the elements we can always order them, however numerous they may be, by selecting one element at a time, and this will be our law of succession. The different orderings we have given of *S* does not change the *order type* of *S*. This is characteristic of finite sets: different orderings of the elements of a finite set does not change the order type of the set. In the case of *S*, the sets of all different orderings of *S* will be isomorphic to each other; they will all have a first element and the last element in each set will be the 4th element. While discussing transfinite ordinals we are going to see that different orderings of the elements of an infinite set changes the order type.

Cantor thought that a set with infinitely many elements can also always be well-ordered, and he considered this as "a law of thought which seems to be basic and consequential" (1976, p.72). He wrote in *Grundlagen* that he will return to this law in a later treatise. However, he did not, and it was seen later that this should not be taken as a law but as a theorem that requires proof, which is today referred to as the *well-ordering theorem*[46]. There are some sets, like the set of all real numbers or

[46] The well-ordering theorem states that every set can be well-ordered. This should not be confused with the well-ordering principle. The well-ordering principle states that every non-empty set of non-negative integers contains a least element, and thus has a 'natural' order, which makes it a well-

the set of points in space, in which it seems impossible to determine a first element and find a rule of succession that will order the rest of the elements. In 1904, Ernst Zermelo introduced the *axiom of choice*[47] as an "unobjectionable logical principle" to prove the well-ordering theorem. This axiom basically states that there is always a function that selects one element at a time from a nonempty set even if the set has infinitely many elements, making it a well-ordered set. But since the axiom only states that such a function is possible and does not specify what this function is, it was not accepted by all mathematicians, for it was a paradigm of non-constructive mathematics. This axiom is still the subject of great controversy, and it plays the role of a catalyst in determining the different attitudes mathematicians have towards the nature of their art.

If we put aside the problematic sets, there are a lot of sets with infinitely many elements which are fairly easy to be put in a well-ordered form. The obvious example is *N*, the set of all natural numbers. We simply start with 1 and arrange the rest of the elements in ascending order: {1, 2, 3, 4, …}. Similarly *Z*, the set of all integers can also be well-ordered in the following way: {0, -1, 1, -2, 2, -3, 3, …}. Notice that {…, -3, -2, -1, 0, 1, 2, 3, …} is not a well ordering of *Z*, for there is not a first element in this arrangement. Cantor proved with an ingenious method that *Q*, the set of all rational numbers, can also be well-ordered. This method is called *diagonalization*[48]. Cantor visualized a table that is supposed to contain all the rational numbers where the first row is composed of integers, the second row is composed of fractions with 2 as a denominator, the third row is composed of fractions with 3 as a denominator, etc. Starting with 0 which is in the left uppermost corner, Cantor

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ordered set. It has been proved that the well-ordering theorem is logically equivalent to the axiom of choice. In fact, there are numerous other mathematical statements, from both set theory and other branches of mathematics, which are logically equivalent to the well-ordering theorem, e.g. Zorn's lemma. See Moore (2013), pp. 330-334.

[47] For every indexed family $(S_i)_{i\in I}$ of nonempty sets there exists an indexed family$(x_i)_{i\in I}$ of elements such that $x_i \in S_i$ for every $i \in I$.

[48] This should not be confused with the diagonalization used in proving the nonlistability of the set of real numbers. Cantor is the inventor of both proofs and he used diagonals in both of them. However, the only thing that is common for both is the visual method of diagonalization, what they demonstrate is completely different. For the diagonal proof of the nonlistability of real numbers, see Stillwell (2010), pp. 6-10.

moved on to 1, and then made a diagonal down to ½, and by following this 'diagonal movement' he has ordered all the elements of the set of rational numbers.

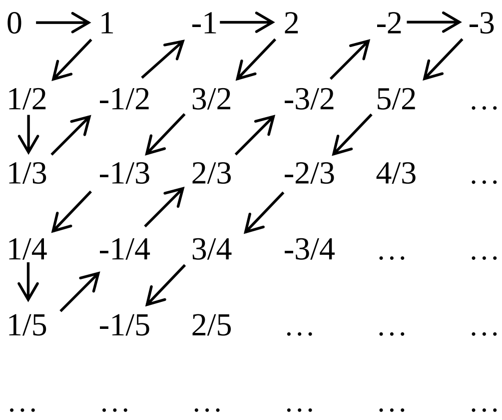

Figure 1. Well-ordering of *Q*

The diagonal method gives us the following set: {0, 1, 1/2, 1/3, -1/2, -1, 2, 3/2, …}. This is a well-ordering of *Q*. It should now be clear that many sets other than *N, Z,* and *Q,* such as the set of all even numbers, powers of 2, rationals between 1 and 2, etc. (mainly the subsets of *N, Z,* and *Q*) can also be well-ordered.

When Poincaré described the 'mathematical spirit', he wrote that it is this spirit "which has taught us to give name to things differing only in material, to call it by the same name" (1907, p. 77). As a mathematician in whom this spirit was highly matured, this was exactly what Cantor did with regards to sets similar to ones we mentioned above. He saw that all these sets could be well-ordered and that they had the same order type: (1) they all had a first element, (2) a rule with which the rest of the elements are ordered, (3) and no last element. "Cantor believed there was nothing improper in thinking of a new number *ω,* which expressed the natural, regular order of the *entire set"* (Dauben, 1990, p. 97). Cantor conceived a potentially infinite series as a *completed* totality, thus as actually infinite, under a new number that expressed the natural order of the series. He saw this as "an extension, or actually a continuation, of the sequence of real whole numbers beyond the infinite […] this extension will come to be regarded as a thoroughly simple, appropriate, and natural one" (1976, p. 70). Not only Cantor but also Hilbert regarded it as a natural extension. He wrote: "When we have counted 1, 2, 3, … we can regard the objects thus enumerated as an infinite set existing all at once in a particular order. If, following Cantor, we call the type of this order ω, then counting continues naturally with $\omega+1$, $\omega+2$ …". And again, "We arrive at [transfinite numbers] simply by counting beyond the ordinarily enumerably infinite, i.e. by a natural and uniquely determined consistent continuation of ordinary finite counting" (1964, p. 140).

Counting past the finite whole numbers requires admitting the second principle of generation. This principle is not a violation of our intuition but an extension of it. The second principle of generation suggests that

> If any definite succession of defined whole real numbers is given of which there is no greatest, then on the basis of this second principle of generation a new number is created, which can be thought of as a *limit* of those numbers, i.e. can be defined as the next greater number to all of them. (1976, p.87)

Cantor tells us that it is possible to conceive the ever growing series of naturals, as long as they are well-ordered, as a distinct *number*, and simply go on counting with transfinite numbers. In Chapter 3, we have discussed Poincaré's idea suggesting that we have a mind capable of conceiving indefinite repetition. Now Cantor is showing us that it can also conceive an indefinite repetition as a single event once it has found a way of arranging its steps, and can go on repeating this new event indefinitely. Once this is admitted, it then becomes possible to arrange the elements of an infinite set in infinitely many new ways, and thereby give it new order types. These new order types are what the unlimited gradation of transfinite ordinals amount to. For instance, it becomes possible to arrange the elements of N in the following way: {2, 3, 4, …; 1}. This set would seem completely meaningless if the second principle of generation is not admitted, in other words, if it is rejected that our minds can conceive some form of actual infinity. We would then be unable to count past infinity, fall back to Aristotle's reasoning and be compelled to oppose to the forming of such a set. But thanks to Cantor, this set now has a definite meaning, because we now know that the number 1 in this set can said to be in the ω^{th} place. The sets {1, 2, 3, 4, …} and {2, 3, 4, …; 1} have different order types. The former, as having a first and no last element, has the order type ω. The latter, however, has a last element, and this element is one that has no immediate predecessor; it is in the ω^{th} place in this set. The order type of this set is $\omega + 1$.

$$\{1, 2, 3, 4, \ldots\} = \omega$$

$$\{2, 3, 4, \ldots; 1\} = \omega + 1$$

We can also form a set in which the natural numbers are arranged as odd and even numbers, i.e. $\{1, 3, 5, 7, \ldots; 2, 4, 6, 8, \ldots\}$, whose order type will be $\omega \cdot 2$. Even a set can be formed where the powers of each prime number are arranged in ascending order, i.e. $\{2, 4, 8, \ldots; 3, 9, 27, \ldots; 5, 25, 125, \ldots; 7, 49, 343, \ldots; \ldots\}$, and the order type of this set will be ω^2.

4.2.1 Formal Notation[49]

The conceptual framework required to establish the theory of transfinite ordinals is set theory. There are, however, two main approaches to set theory, namely, naive set theory and axiomatic set theory. Naive set theory is primarily due to Cantor, and it has been discussed briefly in the previous chapter. The main difference between the two approaches is that naive set theory does not include axioms or definitions which are given using formal logic, but is rather defined informally using natural language. Axiomatic set theories, on the other hand, begin by stating their axioms formally, and build the rest of the theory based upon the given axioms. The first axiomatization of set theory was given by Zermelo in 1908, though there are several other axiomatic systems used in set theory, e.g. NBG (Neumann-Bernays-Gödel set theory), MK (Morse-Kelley set theory), etc. The axioms of ZFC (Zermelo-Fraenkel set theory with the axiom of choice) will be used throughout this section.

The building up of set theory is based first upon the *Axiom of the Empty Set*. This axiom states that there exists a set with no members:

$$\exists A[\forall x, x \notin A]$$

[49] The notation adopted in this section is from James Clark's (2017) *Transfinite Ordinal Arithmetic*.

The empty set is represented symbolically as $\emptyset$. In order to create the rest of the ordinals we require another axiom, the *Axiom of Infinity*. We can treat this axiom as the refined and modernized version of Cantor's second principle of generation. The Axiom of Infinity guarantees the existence of an infinite set *I*. Essentially, ordinals are specific combinations of members of the infinite set. *I* is a set that contains the empty set as a member, and for every *x* that is a member of *I*, the set formed by the union of *x* with its *singleton* $\{x\}$ – a set that only contains *x* – is also a member of *I*.

$$\exists I[\emptyset \in I \wedge (\forall x \in I)[x \cup \{x\} \in I]]$$

This gives us the following set:

$$I = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset,\{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset,\{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \ldots\}.$$

Thanks to John von Neumann we have a simple way of relating the natural numbers with the members of *I*. Von Neumann (1976) defined an ordinal as "the set of all ordinals that precede it" (p. 347) and gave each ordinal a label from non-negative integers. With this method it becomes possible to relate $\emptyset$ to 0, $\{\emptyset\}$ to 1, $\{\emptyset, \{\emptyset\}\}$ to 2, etc. To denote this labeling Clark (2017) used the delta-equal-to symbol ($\triangleq$).

$$0 \triangleq \emptyset$$

$$1 \triangleq \{\emptyset\} = \{0\}$$

$$2 \triangleq \{\emptyset, \{\emptyset\}\} = \{0,1\}$$

$$3 \triangleq \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\} = \{0, 1, 2\}$$

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This notation is compatible with our intuitive results. Here it is easy to show, for instance, that the successor of 2 is 3. The successor relationship for ordinals is defined as follows:

$$a + 1 \triangleq a \cup \{a\}$$

This means having the union of a with its singleton $\{a\}$. Hence,

$$2 + 1 = 2 \cup \{2\} = \{0, 1\} \cup \{2\} = \{0, 1, 2\} = 3$$

Every ordinal has an immediate successor, but we have said that in Cantor's theory there are ordinals that have no immediate predecessor. The latter are called transfinite ordinals or limit ordinals. Clark formulates this by saying that every ordinal a falls into one of two *classes*, K_I or K_{II}, which designate the non-limit ordinals and limit ordinals respectively. Symbolically this is represented as,

$$\alpha = 0 \lor \exists\beta[\beta + 1 = \alpha], \forall\alpha \in K_I$$

$$\alpha \neq 0 \land \nexists\beta[\beta + 1 = \alpha], \forall\alpha \in K_{II}$$

Note that K_I and K_{II} are not sets but classes.This is in order to escape from paradoxes like that of Russell's or Burali-Forti's, though this approach has problems of its own. These will be discussed in the next chapter.

The existence of the smallest limit ordinal ω is guaranteed by the Axiom of Infinity. ω is defined as the set of all finite ordinals as well as their supremum, $\omega \triangleq \{0, 1, 2, 3, \ldots\}$. This limit ordinal also has an immediate successor, which is greater than it:

$$\omega + 1 = \omega \cup \{\omega\} = \{0, 1, 2, 3, \ldots\} \cup \{\omega\} = \{0, 1, 2, 3, \ldots, \omega\}$$

We can now demonstrate why the commutative law fails in the addition of transfinite ordinals. In general, the addition of two ordinals $\alpha + \beta$ means that we count α

number of times and then count β number of times afterwards. The most appropriate way of formalizing this is by using a Cartesian product.

$$\alpha + \beta \triangleq a \times \{0\} \cup \beta \times \{1\}$$[50]

This formula tells us to count a times and then β times. Therefore $\omega + 1$ means that we count ω times and then 1 more.

$$\omega + 1 = \omega \times \{0\} \cup 1 \times \{1\} = \{0, 1, 2, 3, \ldots\} \times \{0\} \cup \{0\} \times \{1\}$$

This gives us,

$$= \{(0,0), (1,0), (2,0), (3,0), \ldots\} \cup (0,1)$$

$$= \{(0,0), (1,0), (2,0), (3,0), \ldots, (0,1)\}$$

Notice that this set is isomorphic to $\{0, 1, 2, 3, \ldots, \omega\}$, where there is a last element with no immediate predecessor. However, adding 1 to ω means that we count to one and then to ω. This gives a set where there are no limit ordinals, it is therefore of a different order type than the previous set.

$$1 + \omega = 1 \times \{0\} \cup \omega \times \{1\} = \{0\} \times \{0\} \cup \{0, 1, 2, 3, \ldots\} \times \{1\}$$

$$= (0,0) \cup \{(0,1), (1,1), (2,1), (3,1), \ldots\}$$

$$= \{(0,0), (0,1), (1,1), (2,1), (3,1), \ldots\}$$

This set is isomorphic to $\{0, 1, 2, 3, \ldots\}$, hence to ω. Now we see why $1 + \omega \neq \omega + 1$. Therefore we write,

[50] Using a and b or any other symbol instead of 0 and 1 is unimportant. 0 and 1 are selected only for convenience. If we were to add 3 numbers we would then require 3 symbols: 0, 1, and 2; or a, b, and c.

$$1 + \omega = \omega$$

$$1 + \omega \neq \omega + 1 > \omega$$

Similar results are achieved in the case of multiplication. Multiplication with transfinite ordinals, like that of finite whole numbers, is understood in terms of repetitive addition. The formal definition of ordinal multiplication is the following:

$$\alpha \cdot \beta \triangleq \sum_{\gamma<\beta} \alpha$$

Therefore if we take, for instance, the product of $2 \cdot 3$, this means to add 2 three times.

$$2 \cdot 3 = 2 + 2 + 2$$

If we adopt the previous formalism we get the following:

$$2 + 2 + 2 = 2 \times \{0\} \cup 2 \times \{1\} \cup 2 \times \{2\}$$

$$= \{(0,0), (1,0), (0,1), (1,1), (0,2), (1,2)\}$$

This set is isomorphic to $\{0, 1, 2, 3, 4, 5\} = 6$. Thus $2 \cdot 3 = 6$.

If we take the product of $2 \cdot \omega$, which means to add 2 indefinitely, this will be represented as,

$$2 + 2 + 2 + ... = 2 \times \{0\} \cup 2 \times \{1\} \cup 2 \times \{2\} \cup \ ...$$

$$= \{(0,0), (1,0), (0,1), (1,1), (0,2), (1,2), ...\}$$

This set is isomorphic to $\{0, 1, 2, 3, ...\} = \omega$. Thus $2 \cdot \omega = \omega$. However, if we take the product of $\omega \cdot 2$, which means to add ω two times, we get a different result:

$$\omega \cdot 2 = \omega + \omega = \omega \times \{0\} \cup \omega \times \{1\}$$

$$= \{(0,0), (1,0), (2,0), (3,0), ...\} \cup \{(0,1), (1,1), (2,1), (3,1), ...\}$$

$$= \{(0,0), (1,0), (2,0), (3,0), ..., (0,1), (1,1), (2,1), (3,1), ...\}$$

This set is isomorphic to $\{0, 1, 2, 3, ..., \omega, \omega+1, \omega+2, \ldots\} = \omega \cdot 2$. That $2 \cdot \omega \neq \omega \cdot 2$ is now clear. Therefore we write,

$$2 \cdot \omega = \omega$$

$$2 \cdot \omega \neq \omega \cdot 2 > \omega$$

It is possible to derive further and equally interesting results. For instance, from what has been demonstrated so far it is possible to show that the product of a transfinite ordinal with a finite ordinal will always be greater than a finite ordinal added to a transfinite ordinal. Thus, for instance, $\omega \cdot 2$ is greater than all of the following: $\omega + 2$, $\omega + 2^{10}$, $\omega + 2^{100}$, $\omega + 2^{1000}$, … Similarly, a finite power of a transfinite ordinal will always be greater than a product of a transfinite ordinal with a finite ordinal. For instance if we take ω^2, this is greater than all of the following: $\omega \cdot 2$, $\omega \cdot 2^{10}$, $\omega \cdot 2^{100}$, $\omega \cdot 2^{1000}$, … etc. (see Stillwell (2010) for details).

With respect to the principle of induction, since we now have new numbers, we need to extend this principle so that it becomes applicable to limit ordinals. We do this by adding a third step to standard induction, turning it into *transfinite induction*. There are two steps in standard induction. First, we verify that the theorem is true for the base case, $\gamma = 0$. Then we assume that the theorem is true for an arbitrary γ and show that it is also true for $\gamma + 1$. In addition to these two steps, in transfinite induction we assume that γ is a limit ordinal, i.e. $\gamma \in K_{II}$. We assume that the theorem is true for all the elements of γ and show that it is also true for γ. As we have explained in the previous chapter, besides the capacity to conceive indefinite repitition, we have the capacity to conceive this entire repetition as a single event – a determinate number – once we find an order in it, and continue with our repetition using this new number. We owe the concept of a limit ordinal to the intuition of this mental capacity. Since every ordinal is either a limit ordinal or a non-limit ordinal, a theorem holds for all ordinals if it holds for all three steps.

4.3 Objections to Transfinite Arithmetic

There are three possible questions that can be raised against the use of transfinite numbers. First, is the theory of these numbers coherent, can we be sure that its results do not lead to a contradiction? Second, is it intuitive? Is it reasonable to expect for people to understand it with sufficient effort? Third, are these numbers useful? Do they help us in broadening our mathematical knowledge and present solutions to previously known problems? Poincaré raised the first two of these questions; here we are going to address all three of them.

Cantor's theory stirred many controversies after it was made public. These were mainly about the transfinite cardinal numbers, the continuum hypothesis[51], and most importantly, the definition of *set*. Cantor's definition allowed for sets to be formed unrestrictedly, that is, a set could be formed of objects that satisfy any given property. Ironically, it was Cantor who first saw the problem with the unrestricted formation of sets. When he considered the entire set of ordinals Ω, he saw that this was a well-ordered set, so it must have had an order type δ greater than all ordinals. But since δ was an ordinal, it must have been contained in Ω, which results in $\delta < \delta$. Cantor concluded that Ω was not actually a set but an *inconsistent multiplicity*[52]. This contradiction is the main reason that in the previous chapter we treated K_I and K_{II} as

[51] The continuum hypothesis states that there is no set whose cardinality is between the cardinality of the set of real numbers ($\aleph_1$) and the cardinality of the set of integers ($\aleph_0$), which means that $\aleph_1$ is the next cardinal number to $\aleph_0$. For Gödel's formulation of the hypothesis, see Gödel (1964b). In 1963 it was proved by Paul Cohen, who was complementing Gödel's work of 1940, that the truth of the continuum hypothesis is independent of – or rather undecidable in – ZFC, thus either the hypothesis or its negation can be added as an axiom to ZFC.

[52] According to Cantor, an inconsistent multiplicity is still a multiplicity, a collection-like object, though in which it is impossible to "think without contradiction all its elements as being together, and consequently, of the set itself as a unified thing in itself" (as cited in Jané, 1995). In addition to the sequence of all ordinals, another clear example of an inconsistent multiplicity is the sequence of all cardinal numbers, i.e. the set of all alephs. Cantor claimed that neither could be an object of further mathematical consideration. For a more detailed account of inconsistent multiplicities, see Jané (1995).

classes of ordinals and not as sets. Because when they are taken as sets, they become subject to set-theoretical operations, which means we can take their union $K_I \cup K_{II}$ and derive the set of all ordinals Ω – a set that is shown to be contradictory. By claiming that K_I and K_{II} are not sets but classes, we escape from this contradiction. There is, however, a strong disposition to treat this solution as a mere play of words. What are these 'classes' if not sets? It seems they are playing an integral role in our theory, yet we cannot make them subject to further mathematical investigation. This is still an ongoing debate and there is an extensive literature written on it[53].

Even though this contradiction was first noticed by Cantor, it has gained recognition with Russell and his famous paradox. Russell considered "the set of all sets which are not members of themselves" and showed that the idea was paradoxical; it exhibitied a similar contradiction with that of Ω. Russell concluded that the problem was with the naive set theory in which the definition of set was not given rigorously, where it was possible to form for any property a set of objects that satisfy it. He asserted that not every property defines a set; those definitions that define a set he called *predicative*, and those that do not *impredicative*. An impredicative definition is one where there is a generalization over the totality to which the entity being defined belongs. These definitions create a certain kind of circularity, for they define a totality "whose existence would entail the existence of certain new elements of the same totality, namely elements definable only in terms of the whole totality" (Gödel, 1964a, p. 217). Russell held that contradictions mainly occur in set theory when a set is defined impredicatively. In order to amend these problems he proposed three solutions: the zigzag theory, theory of limitation of size, and the no-class theory[54].

Poincaré also considered impredicative definitions as a major problem with Cantor's theory. He wrote, "It is the belief in the existence of the actual infinite which has given birth to those non-predicative definitions" (2008, p. 194). Poincaré saw the problem in the fact that an impredicatively defined set cannot be studied mathematically, because the classification would then be *mutable*, "The appearance

[53] See Jané (1995) and Welch & Horsten (2016).

[54] See Russell (1906). Also see Gödel (1964a) for Gödel's evaluation of Russell's solutions.

of a new object [would] oblige us to modify the classification" (p. 195). In fact, Poincaré held that in his diagonal proof[55] where Cantor showed that the set of real numbers have a higher cardinality than the set of natural numbers, he actually used an impredicative definition and mistook the fact that he failed to establish a one-to-one correspondence between these sets for an indication that *R* has a higher cardinality than *N*. According to Poincaré, the reason that no one-to-one correspondence was found between these sets should be ascribed to Cantor's way of defining *R* impredicatively: "To represent points in space by the sentences which serves to define them […] is to construct a classification which is not predicative, one which entails all the inconveniences" (1963, p. 61). Poincaré offered his own solution: "The important thing is never to introduce entities not completely definable in a finite number of words" (2008, p. 45). He thus rejected the use of infinite numbers and remained a *finitist*, because for him,

> [E]very mathematical theorem must be capable of verification […] and the verifications apply only to finite numbers, it follows that every theorem concerning infinite numbers or particularly what are called infinite sets, or transfinite cardinals, or transfinite ordinals, etc., can only be a concise manner of stating propositions about finite numbers. (1963, p. 62)

Although concepts like Ω or $\aleph_1$ (the cardinal number that corresponds to the set of real numbers) were problematic, no one has seriously doubted the set of all natural numbers being well-ordered and having a certain order type, ω. Poincaré, for instance, did not devote a chapter to ordinals as he did to cardinals. The second principle of generation works perfectly when the sequence of natural numbers is considered. It is not an impredicative definition of ω, for even though it generalizes over all finite ordinals, the principle is intended to define an *infinite* ordinal: "It [is] a uniquely determined extension of the concept of 'number' to the infinite sets" (Gödel, 1964b, p. 258). This extension is legitimate, but only when it is restricted. The need for such a restriction definitely needs explanation; it must be answered why is it that we are unable to use the principle in the case of Ω while it is possible to use it in the case of natural numbers. However, this alone should not be a reason for rejecting the theory of transfinite numbers completely, which, as we will see shortly,

[55] This proof is the one that we have *not* mentioned; it is different than the diagonal proof that shows the well-ordering of *Q*. For details, see Stillwell (2010), pp. 6-10.

is very useful, even necessary for solving certain problems. In its restricted form, the theory of transfinite ordinals and their arithmetic remains intact.

But is defining something without contradiction necessarily implies its existence in mathematics? Should not the thing being defined have an intuitive aspect to it? Poincaré's answer is quite complicated. When describing how Kronecker defined irrationals as a "particular method of division of commensurable numbers[56]" (2011, p. 26), Poincaré wrote: "Mathematicians do not study objects, but the relations between objects […] If we did not remember it, we could hardly understand that Kronecker gives the name of incommensurable number to a simple symbol" (p. 25). What we should expect from a symbol is therefore to express a relation and not an object. If we remember that "'possible' in the language of geometers simply means exempt from contradiction" (p. 24), and if we have shown that our definition is non-contradictory, then we can say that the relation being defined 'exists'. But how do we show that it is non-contradictory? For Poincaré, if the consequences of the definition are finite, we do this directly by giving (or constructing) a concrete example for which the consequences hold. However, if the consequences are infinite, then we have to recourse to mathematical induction, which, as we have explained, is based on a pure intuition. Thus, definitions that are intended to specify an infinite number of relations are always given in intuition. In this sense, Kronecker's definition is not problematic. However, "in Poincaré's view, set theory does 'banish' intuition, for it *contradicts* it" (Folina, 1986, p. 118). Thanks to our ability to iterate indefinitely, we have an intuition of a never ending sequence – a potential infinity – but not of an actually infinite set. Theorems can only be verified for finite numbers, even when we are using induction to prove for infinitely many of them, but never for an infinite number. Transfinite numbers, then, is "a *violation* of our prior conception of mathematical objects […] our glossing over faculties does not *address* such 'objects'" (Folina, 1986, p. 119). Furthermore, against the set theorists who wanted to make the theory of finite numbers depend upon transfinites, Poincaré wrote, "This method is evidently contrary to all sane psychology; it is certainly not in this way that the human mind proceeded in constructing mathematics" (2008, p. 145). I grant that this is true, but there is nothing

[56] Poincaré used the term commensurable for rational numbers and incommensurable for irrational numbers.

wrong in accepting that the mind is now advancing in a different way. Poincaré rejected the idea that we have an intuition of an infinite number. He argued that the intuition in mathematics – the intuition of pure number – gives only finite numbers and warrants proofs by mathematical induction. I have tried to show how it is possible to treat transfinite ordinals as being not in contradiction with the intuition of pure number, but rather as extensions of it. By examining our primitive conception of infinity, and with the help of the concept of a well-ordered set, it is possible to make our 'glossing over' faculties address infinite sets by their order types. Contrary to what some platonists believe, the fact that we can address these sets is not because they exist in a mysterious, set theoretical realm, and we can somehow interact with them; we rather *construct* these sets – or in Poincaré's terms, these relations – with the help of a pure intuition, now adjusted to allow counting with infinite numbers.

As for the usefulness of transfinite arithmetic, I am going to choose a rather meaningful example. When he was criticizing the problems of set theory, Poincaré wrote: "We are sure to see [these problems] resurrected with insignificant alterations, and some of them have already risen several times from their ashes. Such long ago was the Lernaean hydra[57] with its famous heads which always grew again" (2008, p. 145). Today, there is a proof for slaying a mathematical hydra which is growing heads a lot more rapidly than the one in the myth, and funny that this proof uses transfinite ordinals. In 1982, mathematicians Laurie Kirby and Jeff Paris proposed a new variation of the story of the hydra where the creature is a mathematical beast. In their version the hydra is shaped like a tree:

[57] The Lernaean Hydra is a monster in Greek mythology. The hydra has many heads, and when one of them is cut off, the creature grows two more heads. Killing the hydra was the second of the twelve labours of Hercules, and he managed to kill it with the help of his nephew Iolaus, who cauterized the wound every time Hercules cut off a head.

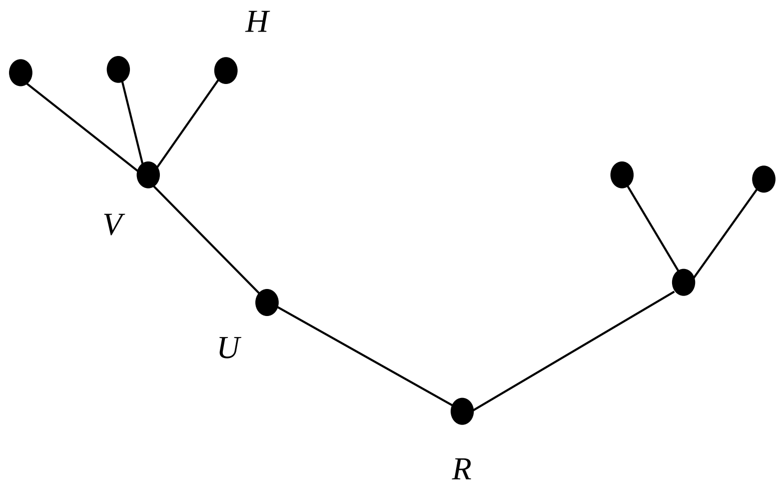

Figure 2. A mathematical hydra

R is taken as the body of the hydra and all the dots above it – such as *U*, *V*, and *H* – are its heads. Our mathematical hydra grows heads out of its heads, and we can only chop off heads from which no heads are growing, for instance in the above case we can remove *H* but not *V*. Like the one in the myth, our hydra also grows new heads when one of its heads is removed, but it has a special rule for doing this, and this rule can be chosen arbitrarily as long as new heads are growing out from the head that is one level below the removed head (for instance the rule can be, "grow a copy of the decapitated portion of the hydra that remains below the removed head". In this case, when *H* is removed, a copy of all the heads from *V* to *R* will grow out of *U*). The creature dies when there remains no heads that grow out of *R*. Now the question is: Is it possible for Hercules to kill a significantly tall mathematical hydra? Is there a way, a strategy for chopping off the heads of this creature so that he will eventually reach *R*, or is the beast destined to grow to infinity? Kirby and Paris proved that given any hydra with an initial composition and a rule for growing heads, *we can always defeat it in a finite number of steps no matter the order we chop off the heads*. This may sound counterintuitive. It is easy to see that after only a few decapitations the hydra will have a lot more heads compared to the original, and apparently with each decapitation it will continue to grow new heads. But Kirby and Paris showed that eventually the hydra will be defeated, and this is true regardless of the order of decapitations. They used transfinite ordinal arithmetic in their proof, and their method is based on a special labeling of the complexity of the hydra in terms of limit ordinals. We will not go into the details of the proof, for a summary see Stillwell (2010), pp. 51-54. We can see that the theory of transfinite ordinals is not a useless occupation; it has helped us in solving a problem which is fairly easy to formulate in the language of ordinary mathematics. Besides, perhaps it might have given Hercules relief and motivation if he had known the theory of these numbers, then he wouldn't need to call his nephew Iolaus for help to cauterize the wound each time a head was removed.

'Killing the Hydra' is actually a special case derived from a more general theorem called the *Goodstein's theorem*. This is a theorem about the natural numbers which was first put forward and proved by Reuben Goodstein in 1944. It states that every Goodstein sequence converges to zero. A Goodstein sequence is a certain class of integer sequences that grows very quickly. In order to create one, we take any positive integer and represent it in hereditary base 2 notation (which means basically to write it in base 2 where 2 is the largest number). For instance if we take 13, this will be represented as

$$13 = 2^{2+1} + 2^2 + 1$$

We then replace all 2s with 3s and subtract one to create the next term in the sequence.

$$3^{3+1} + 3^3 + 1 - 1 = 108$$

We write this new number in hereditary base 3 notation, and then replace all 3s with 4s and subtract one to create the third term in the sequence, etc.

$$108 = 3^{3+1} + 3^3$$

$$4^{4+1} + 4^4 - 1 = 1279$$

The first three terms of our Goodstein sequence are 13, 108, and 1279. The fourth term will be strictly larger. At first glance, the sequence seems to be growing very rapidly. However, Goodstein proved that this sequence, and *every* Goodstein sequence, eventually terminates. This is a stunning result. The base bumping operation gives us a much larger number, and it is highly suspicious that subtracting one from it will ever lead to the termination of the sequence. Nevertheless, this is indeed the case, and Goodstein proved it using transfinite ordinals. With a special labeling of the terms of this sequence with transfinite ordinals and exploiting the rules of their arithmetic, it is possible to show that just like in the case of the hydra, even though the terms seem to be growing very rapidly, subtracting one at each step gradually *decomposes the entire structure* (see Stillwell (2010), pp.47-51). What is even more astonishing is that Kirby and Paris have proved that Goodstein's theorem is unprovable in Peano Arithmetic. This is a fairly simple theorem of number theory and it is easy even for a student to formulate it, yet it is unprovable in the basic language in which it is formulated[58]. This is because the terms become unmanageably big in no time when they are handled using only finite arithmetic. The Goodstein sequence that starts with 3 terminates in five steps. According to Kirby and Paris, the one that starts with 4, however, terminates in $3 \cdot 2^{402,653,211} - 1$ steps. Transfinite ordinals are not only useful but also necessary for showing that Goodstein's theorem, a simple theorem found in number theory, is true.

Although the benefits we have cited so far have only been valuable for pure mathematics, I do not see any serious obstacle to the use of transfinite numbers in the explanation of some phenomena that will be observed in the future. As in the case of imaginary numbers, finding an application for a subject of pure mathematics to observable phenomena sometimes takes centuries.

[58] After Gödel's incompleteness theorem and Gerhard Gentzen's proof of the unprovability of ε_0-induction in Peano arithmetic, Goodstein's theorem is the third example of a true statement that is unprovable in Peano arithmetic.

CHAPTER 5

CONCLUSION

The examination of the theory of transfinite ordinals shows that it is possible to consider what Cantor achieved in arithmetic to be similar to what Riemann and Lobachevsky did in geometry, and thus we may argue that Cantor has laid the foundations of a new arithmetic. It follows that Poincaré was mistaken in rejecting the possibility of building a new arithmetic. Of course, we cannot treat Cantor as the Riemann of arithmetic; the building up of transfinite ordinals did not take place exactly in the same manner as that of non-Euclidean geometries, because there is a fundamental difference between both in the foundations and in the subject matter of arithmetic and geometry.

In order to emphasize the difference between the two main branches of mathematics I have adopted Poincaré's philosophy, and in most cases I adhered to his intuitionism, which is built to a significant extent on Kant's philosophy. As we have seen in Chapter 3, for Poincaré, the difference between arithmetic and geometry lies in the fact that geometry and experience are inseparably bound; the observable motion of objects and the distinctive motion of our bodies play a constitutive role in the establishment of the framework we call space. However, experience has no such role in the foundations of arithmetic; the concept of 'number', and the fundamental rules that these objects obey, pertain nothing but to the subjective constitution of our minds, and these are given by a pure intuition, in other words, by a direct awareness of a mental capacity. Poincaré held that the affinity we feel for the postulates of Euclidean geometry is caused by the observation of solid objects, and assuming laws which are different from the ones we are accustomed has given us the postulates of non-Euclidean geometries. Neither group of postulates is imposed upon us directly by the nature of our minds; according to Poincaré, they are *conventions*, and not synthetic *a priori* judgments.

If it is admitted that arithmetic is independent of experience in the sense Poincaré described, then it is easy to see that something similar to what happened in geometry cannot take place in arithmetic. However, it is undeniable that today we have a new and unusual, yet a functioning arithmetic, whose details we have given in Chapter 4. Once it is admitted that the intuition underlying arithmetic can be developed, that our mental capacities can be subject to improvement, then it becomes possible to provide an intuitionistic basis to the theory of transfinite ordinals. On this view, transfinite ordinals and the principle of transfinite induction are grounded upon the same intuition that gives us natural numbers and justifies the use of standard induction – the intuition of the mind's capacity to conceive indefinite repetition. Cantor has shown that our minds, which are capable of conceiving indefinite repetition, can also conceive such a repetition as a definite number once it has found an order among its steps. By carefully examining the common Aristotelian conception of infinity, and by supplementing this with his concept of 'well-ordering', Cantor showed that a mental capacity that already exists in all of us can be improved. He was able to establish a new type of number and the theory of its unordinary arithmetic, which is surprisingly useful, and appears quite meaningful to the mathematicians who are willing to follow his steps.

What is presented in this thesis can be considered as a defense of a slightly different version of Poincaré's intuitionism. According to this view, empirical observation plays no role in the foundations of mathematics; this science is based upon the intuitions of the human being. Contrary to what some philosophers believed, mathematical intuition is not a mysterious faculty (Ayer, 1964); it is basically mind's becoming conscious of its own constitution and capabilities. In the case of Poincaré, this amounts to the capacity of conceiving indefinite repetition – a quite ordinary and natural ability. What is suggested in this thesis is that this capacity can be developed, and this is the starting point of a new arithmetic. We have realized that geometric concepts were necessarily understood under the guidance of experience, and that the evidence for geometric postulates lies partly in observation. By imagining different laws for the motion of objects, in other words, by adopting new conventions, we were able to derive non-Euclidean geometries. Since experience plays no role in the foundations of arithmetic, we cannot expect for a new observation, or a modification in the ordinary empirical conditions, to result in a change in the meaning of the basic concepts or principles of arithmetic. Here the thesis departs from the general Poincaréan picture and speculates that the possibility of such a change lies in the development of intuition. It suggests that Cantor had managed to do this, and thus he was able to establish the theory of transfinite ordinal arithmetic.

In this picture, intuition in mathematics is not – as platonists such as Gödel like to think – something like a perception of physical objects. Mathematical objects such as transfinite numbers do not exist in a mind-independent mathematical realm, and our intuition is not a way of interacting with these mysterious objects; rather, our mind *constructs* these objects. In fact, these are not ‘objects’ in the ordinary sense but rather *relations*, which the mind is able to conceive within the limits of its capacities. The rules of transfinite ordinal arithmetic are not some arbitrary rules for manipulating meaningless symbols. The justification for these rules comes from the intuition of a mental capacity, just as in the case of natural numbers and finite arithmetic. It is therefore possible to make sense of transfinite ordinal arithmetic as an intuitionist, without committing to either platonism or nominalism regarding the existence of mathematical objects.

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APPENDICES

A. TURKISH SUMMARY/ TÜRKÇE ÖZET

Bu tez Poincaré'nin matematik felsefesini konu almaktadır. Matematiğin temellerine, özellikle de uzaya ve geometrinin doğasına dair birçok problemde Poincaré'nin sezgiciliği ve uzlaşımcılığı savunulmakta, fakat Öklid-dışı geometrilere benzer yeni bir aritmetik kurulamayacağı savı eleştirilmektedir. Poincaré'nin yeni bir aritmetiği reddedişinin ardındaki temel sebep, aritmetiğin tümüyle deneyimden bağımsız olduğuna, geometrinin ise deneyim ile ayrılmaz bir biçimde bağlı olduğuna inanmasıdır. Öklid postulatlarını reddedip karşıt önermeler temelinde yeni geometrilerin inşa edilmesine imkan tanıyan, deneyimin geometrinin temellerindeki rolüdür. Buna karşın Poincaré'ye göre aritmetikte deneyimin hiçbir rolü yoktur. Aritmetik ilkelerin – ki ona göre bunların en öne çıkanı matematiksel tümevarım ilkesidir – doğruluğu bize bizzat zihnimiz tarafından dayatılır. Bunlar deneyim ile değil saf bir sezgi aracılığıyla bilinir ve belli başlı zihinsel kabiliyetlere indirgenir. Dolayısıyla aritmetikte Öklid postulatlarında olduğu gibi reddedilip yerine karşıtı konulabilecek bir ilke yoktur. Bu tezde, zihinsel kabiliyetlerimizin ve onlara dair sahip olduğumuz sezginin gelişebileceği savı öne sürülmekte, bunun da yeni bir aritmetiğin başlangıç noktası olabileceği iddia edilmektedir. Poincaré'ye eklenen tek şey, aritmetikte faydalandığımız sezginin gelişebileceği varsayımıdır. Cantor'un temellerini attığı, alışılmadık yeni tür sayıların ve ilkelerin bulunduğu sonluötesi ordinal aritmetik teorisi yeni bir aritmetiğe örnek olarak alınmış ve bu teoriyi kurarken Cantor'un esasında hepimizde olduğu gibi kendisinde de var olan zihinsel bir kabiliyeti geliştirdiği öne sürülmüştür. Böylece birebir aynı olmasa da Öklid-dışı geometrilere benzer yeni bir aritmetiğin sezgici bir temelde nasıl izah edilebileceği gösterilmiştir.

Bu sonuca varmak için en başta Kant'ın matematik felsefesi özetlenmektedir, zira Poincaré tam bir Kantçı olmasa da görüşünü büyük ölçüde Kant'ın tanıttığı kavramlar üzerine kurmuştur, ki bunların başlıcaları 'sentetik *a priori*' ve 'saf sezgi'dir. Kant matematiğin, doğanın hakiki işleyişinde değil, esasında kendi zihinlerimizin yapısında temellendiğini öne sürmüştür. Bu bakımdan matematik tam anlamıyla nesnel değildir, çünkü nihayetinde fiziksel nesneler hakkında değil, bu nesneleri duyumsayan ve onlar hakkında düşünen zihin hakkındadır. Buna karşın yine de matematiksel bilgi zorunlu ve evrenseldir, çünkü zihinlerimizin hepimizce paylaşılan yapısı ile ilgilidir; matematiğe ancak bu bakımdan nesnel denebilir.

2. Bölümde anlatıldığı gibi Kant'a göre bilişsel aktiviteyi mümkün kılan iki melekemiz vardır; bunlar duyarlılık ve anlama yetisidir. Duyarlılık, nesneleri sezmemize yarayan melekedir. Esasında bu bilişsel bir aktivitedir ve nesneler ile doğrudan ilişki kurmamıza yarar; fakat Kant'a göre sezgi kavramsallaşmamıştır ve dolayısıyla tek başına bilgi üretmek, yargıda bulunmak için yeterli değildir. Duyarlılık tarafından bize verilenlerin sınıflandırılması, organize edilmesi gerekir ve bu yolla kavramlar oluşturulur. Bunu gerçekleştiren ise anlama yetisidir. Anlama yetisinin kendine has bir yapısı, onda halihazırda bulunan birtakım saf kavramları vardır; bunlar kategorilerdir. Kategoriler bir bakıma duyarlılığın verdiklerini kavramsallaştırmanın kurallarıdır. Kavramlar yargılarımızın nihai öğeleridir ve biliş ancak yargılar aracılığıyla gerçekleşir. Bilgi üretmek için bu iki melekenin birlikte uyum içinde çalışması gerekir.

Kant duyarlılığımızın da belli bir biçime, bir yapıya sahip olduğunu öne sürmüştür. Bu yapı deneyimden bağımsızdır; dahası, deneyimden önce verili böyle bir yapı anladığımız türden deneyimin de koşuludur. Nesnelerin tecrübe edilebilmesi için bu yapıya uymaları gerekir. Kant için uzay ve zaman işte bu tür yapılardır; Poincaré'nin deyimiyle onlar zihinsel çerçevelerdir. Kant'a göre hiçbir nesneyi sezmesek, çevremizden tamamıyla soyutlanmış olsak bile, duyarlılığımızın yapısını, yani deneyimden önce zihnimizde halihazırda bulunan bu çerçeveyi sezmek mümkündür. Uzay ve zaman saf sezgiler, veya kimilerinin deyişiyle saf görülerdir. Tüm nesneler bir uzayda ve zamanda tecrübe edilir; uzayda ve zamanda olmayan bir nesne olanaklı değildir. Kant'a göre bundan çıkarılacak sonuç, uzay ve zamanın nesnelere değil, zihnimize, daha doğrusu duyarlılığımızın yapısına atfedilmesi gerektiğidir.

Duyarlılığımızın yapıları olarak uzay ve zaman Kant için matematiksel önermelere de gereken temeli sağlar. Matematiksel bir önermenin doğruluğunu araştırırken fiziksel nesnelerle ilgilenmeyiz, çünkü Kant'a göre bu nesneler matematiksel önermelerde bulunan zorunluluğun ve evrenselliğin kaynağı olamazlar. Böyle bir zorunuluğu ve evrenselliği ancak deneyimden bağımsız, ondan önce verilen, yani *a priori* bir zeminde bulabiliriz. İşte uzay ve zaman Kant için bu zemini sağlar. Aritmetik bir önermenin – Kant'ın örneğini alırsak 7 + 5 = 12'nin – doğruluğunu araştırırken zaman sezgisinden faydalanırız. En başta böyle bir önerme Kant'a göre analitik değil sentetik bir yargıdır, yani verilenlerin başka bir biçimde ifade edilişi değil, onlarla yapılan bir sentez, bilgimize katılan yeni bir şeydir. Mantıksal önermelerin tümü analitik yargılardır ve bu yargıların doğruluğu nihayetinde çelişmezlik ilkesine indirgenir, bir başka deyişle onlar totolojilerdir. Matematik ise mantık değildir; matematikte sentetik yargılar da bulunur. İşte bu tür yargıların zemini alışılmışın aksine tecrübe ettiğimiz nesneler değil, bizzat kendi duyarlılığımızın yapısı, bu yapıya dair sahip olduğumuz saf sezgidir. Kant'a göre 12, 7 + 5'ten analitik olarak çıkan bir sonuç değildir. 7 + 5'i düşündüğümüzde sonucun 12 olduğunu doğrudan söyleyemeyiz; bu toplama işlemini yapmamız, yani saymamız gerekir ki bu da Kant için bir sentezi işaret eder. Bu sentezi yapmamızı sağlayan zaman sezgimizdir. Toplama işlemi, daha da temelde sayı kavramı, yineleme fikrinden türer. Kant'a göre bir sayı soyut birimlerin art arda eklenmesidir ve bu 'art arda eklenme' fikrinin kaynağı tecrübe edilen nesneler değil, duyarlılığımızın bir yapısı olarak zamandır. Özetle 7 + 5 = 12 önermesi sentetik bir önermedir ve zaman sezgisi aracılığıyla *a priori* bilinir.

Geometrik önermeler için de benzer bir durum söz konusudur. 'İki noktadan yalnız bir doğru geçer' önermesi de Kant için sentetik bir yargıdır ve deneyime dayanmadan, *a priori* bilinir. Bu yargının doğruluğunu bize gösteren gözlemlenen nesneler değil, bu nesnelerin uymak zorunda olduğu zihinsel yapı, yani uzaydır. Noktalar, doğrular ve şekiller bizce tasarlanır; Kant'a göre bu kavramlara karşılık gelen sezgiler duyarlılığımızın yapısı tarafından sağlanır, bu sebeple de *a priori* bilinirler. Kant uzayda iki nokta tasarladığımızda aralarından bir doğru geçebileceğini doğrudan gördüğümüzü söyler. Bu doğruyu bir noktanın etrafında döndürebileceğimizin; belli kurallara uyarak çemberler, üçgenler inşa edebileceğimizin bilgisini bize veren duyumsanan nesneler değil, duyarlılığın yapısıdır. Bu bakımdan geometrik önermeler ve en başta da Öklid postulatları Kant için sentetik *a priori* yargılardır. Salt mantık bu önermeleri doğrulamaktan aciz, deneyim de onlardaki kesinliği vermek için yetersizdir; bu önermelerin doğruluğunu bize gösteren, aynı zamanda deneyimin de koşulu olan, duyarlılığımızın yapısıdır.

Bolyai, Lobachevsky ve Riemann gibi 19. yüzyıl matematikçileri Öklid postulatlarının bazılarını karşıtlarıyla değiştirerek yeni geometriler kurmayı başarmış, böylece Öklid uzayının olanaklı tek uzay olmadığını göstermiştir. Bu sonuç Kant'ın Öklid postulatlarının *a priori* sentetik yargılar olduğu fikrini oldukça şüpheli bir hâle getirmiştir ve düşünürleri geometrik postulatların mahiyetini bir kez daha sorgulamaya itmiştir. Öklid-dışı geometrilere hayli aşina olan Poincaré de Öklid postlatlarının *a priori* bilindiği fikrini reddetmiştir. Eğer duyarlılığımızın gerçekten de Kant'ın inandığı gibi bize Öklid uzayını dayatan *a priori* bir yapısı varsa nasıl olmuş da Riemann gibi matematikçiler bu dayatmadan sıyrılmış ve Öklid postulatlarının karşıtlarıyla değiştirildiği uzaylar düşünebilmiştir? Poincaré için bu, deneyimin geometride bir rolü olduğunun göstergesidir. Olanaklı uzaylar arasından Öklid uzayının seçilmesi *a priori* bir seçim değildir; bu spesifik uzayı tanımlayan önermeleri doğru kabul edişimizde deneyimin bir rolü olmalıdır.

Buna rağmen Poincaré matematikte sentetik *a priori* yargılar bulunuyor olduğu fikrini tümden dışlamamıştır; yalnızca bunların geometride değil, matematiksel düşüncenin en saf kaldığı alanda, aritmetikte olduğunu söylemiştir. Tıpkı Kant gibi Poincaré de matematikte mantıktan öte bir şey olduğu fikrindedir; ona göre mantığın aksine matematik bir bilimdir ve yaratıcı bir güce sahiptir, yani yalnız analitik yargılar barındıran dev bir totolojiye indirgenemez. Fakat burada da Kant ile bir anlaşmazlık vardır. Poincaré'ye göre $7 + 5 = 12$ gibi önermeler analitik yargılardır; hakiki sentetik *a priori* yargılar ise tümevarımın kullanıldığı, sonsuz durum için geçerli olan genel teoremlerdir. Bu teoremlere bir örnek 3.4 numaralı bölümde verilmiş ve ispatının nasıl yalnız çelişmezlik ilkesine dayanarak verilemeyeceği gösterilmiştir. Poincaré'ye göre $7 + 5 = 12$ önermesinin analitik bir yargı olmasının sebebi önermenin doğruluğunun verilen bir kuralın (bu durumda $x + 1$) sonlu kez uygulanması sonucu bulunabilmesidir. Toplama bir kez tanımlandığında yalnızca çelişmezlik ilkesine dayanarak yapacağımız sonlu tane kıyas sonucu 7 +

5’in 12 ettiğini doğrulayabiliriz; birbirine eklenen parmakları veya taşları düşünmeden, işin içine sezgi katmadan bu ifadenin doğruluğunu gösterebiliriz. Poincaré’ye göre 7 + 5 = 12 gibi tekil önermeler ispatlanmaz, bunlar doğrulanır ve bir doğrulama da Poincaré için analitik bir iştir; sonuçta verilenlerden öte bir şey bulunmaz.

‘Bütün doğal sayılar ya tek ya çifttir’ gibi bir önermeyi aldığımızda ise bunun 7 + 5 = 12 durumunda olduğu gibi doğrulanamayacağını görürüz; bu önermenin ispatlanması gerekir. Verili bir sayı için, hatta verilecek tüm sayılar için önerme doğrulanabilir; fakat bu doğrulamalardan hiçbiri tüm doğal sayılar hakkında bir yargı belirten bu önermenin ispatı olarak alınamaz. Eğer matematik yalnızca doğrulamalardan ibaret olsaydı mantıktan bir farkı kalmaz ve bir bilim olmazdı, bu tür önermelerin doğruluğu da bizim için erişilemez olurdu, zira bunları ancak sonsuz doğrulama yaparak gösterebiliriz. İşte Poincaré için matematikte sezginin rol oynadığı, bir sentezin söz konusu olduğu yerler bu teoremlerin ispatlarıdır. Bu teoremlerin ispatı için çelişmezlik ilkesinden başka bir ilkeye ihtiyaç duyarız, bu da matematiksel tümevarım ilkesidir. Bu ilke bize bir teoremin herhangi bir n doğal sayısı için doğruyken $n + 1$ için de doğru olduğu gösterilirse teoremin tüm doğal sayılar için doğru olduğunu söyler. İşte bu ilke Poincaré için mantığın alanının dışındadır; o salt matematiksel bir ilke, hatta matematiksel akıl yürütmenin temelidir. Poincaré bu noktada Kantçı davranmış ve bu ilkenin deneyimden öğrenilmediğini, tersine onun deneyime dayatıldığını ve deneyimi mümkün kıldığını söylemiştir. Matematiksel tümevarım ilkesi hakiki bir *a priori* sentetik yargı örneğidir. Fakat Poincaré bunu duyarlılığımızın yapılarıyla veya zaman ile ilişkilendirmemiştir. Ona göre ne uzaya ne de zamana ilişkin saf bir sezgimiz vardır. Doğru, bunlar birtakım zihinsel çerçevelerdir ve duyarlılığımızın yapıları olduğu kabul edilebilir; fakat onlar deneyimden tümüyle bağımsız değildir. Poincaré için deneyimden tümüyle bağımsız olan, ancak yine de bize nesneler arasındaki uzamsal ve zamansal ilişkilerin nasıl olması gerektiğini söyleyen zihinsel bir yapının varlığı kabul edilebilir bir fikir değildir. Bu yapılar veya çerçeveler *a priori* verilmez, Poincaré’ye göre onlar deneyimin rehberliğinde icat edilir. Bu bakımdan nesneleri içine oturttuğumuz uzay ve zaman denen çerçeveler sentetik yargılar için *a priori* bir temel olamazlar.

Matematiksel tümevarım duyarlılığımızın değil, anlama yetimizin yapısında temellenmiştir ve bizim bu yapıya dair doğrudan bir sezgimiz vardır. Poincaré için anlama yetisinin bu yapısı basitçe bir olayın sonsuz defa yinelenmesini kavrama gücüdür. Bizim zihnimiz öyledir ki bir kere gerçekleşen bir olayı sonsuz kez tekrar ettirmeye muktedirdir. İşte sayma kabiliyetimizin de temeli budur. Poincaré için bu saf bir sezgidir, fakat kendisi bunu zamanla ilişkilendirmemiştir. Poincaré burada Kant'tan ayrılmış ve aritmetiği hem duyulur nesnelerden, hem de duyarlılığımızın yapılarından ayırmıştır. Matematiksel tümevarım ilkesi anlama yetimizle ilişkilidir ve bu belli bir zihinsel kapasiteye işaret etmektedir. Bu ilkenin doğruluğu zihnimize bizzat kendi yapısı tarafından dayatılır. Matematikçinin gerçekten ilgisini çeken teoremlere matematiksel tümevarım ilkesiyle varılır ve bunlar sentetik *a priori* yargılardır.

Geometrik postulatlar ise matematiksel tümevarım ilkesi ile aynı mahiyeti taşımaz. Poincaré'ye göre Öklid postulatlarının reddedilebilmesi Öklid uzayının zihnimize dayatılan bir çerçeve olmadığının göstergesidir. Nesneleri içine yerleştireceğimiz uzamsal bir çerçeve için birden fazla olasılık vardır; tıpkı Öklid uzayı gibi hiperbolik ve küresel uzaylar da düşünülebilir. Bizim bu uzaylar arasından Öklid uzayını seçmemizin ardında yatan sebep ise bu çerçevenin çevremizdeki en dikkate değer nesnelerin hareketlerini en uygun ve kolay şekilde tarif etmemize yarayan çerçeve olmasıdır. Çevremizdeki en dikkate değer nesneler katı cisimlerdir. Bu cisimler hareket ederken şekil değiştirmeyen ve izlenimlerini vücudumuzun karşılıklı bir hareketiyle düzeltebildiğimiz cisimlerdir. Katı cisimlerin hareketi Öklid uzayında kolayca tarif edilir. Diğer postulatlarla beraber iki noktadan yalnızca bir doğru geçtiği ve verili bir doğruya ancak bir paralel çizilebildiği kabul edilirse, bu uzayda inşa edilecek şekiller biçim değiştirmeden, uzayıp kısalmadan bir yerden bir yere taşınabilir. Poincaré'ye göre diğer geometrilerin değil de Öklid geometrisinin standart kabul edilmesinin nedeni katı cisimlerin hareketinin gözlenmesine dayanmaktadır. Fakat katı cisimlerin hareketi – yani şekil değiştirmenin eşlik etmediği bir hareket – *a priori* vardığımız bir sonuç değildir, Poincaré için bu deneysel bir olgudur. Tam da bu sebeple deneyimin bize verdiği kuralların aksini düşünerek, yani cisimlerin alışılmadık kurallara göre hareket ettiğini varsayarak, yeni geometriler inşa etmek mümkün olmuştur. Örneğin Riemann postulatında olduğu gibi iki noktadan birden fazla, hatta kimi zaman sonsuz doğru geçtiğinin varsayılması, Poincaré için cisimlerin alışıldık hareketlerinin değişikliğe uğradığının düşünülmesidir; bu postulatların tarif ettiği uzaylarda artık bir cisim şekil değiştirmeden yer değiştiremeyecektir. Nihayetinde Poincaré'ye göre uzay yalnızca bir hareketler grubudur; uzayda bir nokta tasarlarsak bu yalnızca o noktaya varmak için yapmamız gereken hareketleri tasarlıyoruz anlamına gelir. Uzayı vücudumuzun ve gözlemlediğimiz cisimlerin alışıldık hareketlerinden koparmak mümkün değildir; fakat bu hareketlerin imkanı bize deneyim tarafından verildiğinden, farklı tür hareketler ve bu hareketleri tarif eden uzayları düşünmenin önünde bir engel de yoktur. Öklid postulatları bu yeni uzayları tarif edecek postulatlardan daha doğru değil, yalnızca daha kullanışlıdır. Bu postulatlar Poincaré için nihayetinde birtakım uzlaşımlara, kılık değiştirmiş tanımlara indirgenir.

3. Bölüm aritmetik ve geometrinin temelleri arasındaki bu farkın açıkça belirtilmesiyle kapanmaktadır. Özetle aritmetik, geometriden farklı olarak deneyimden tümüyle bağımsızdır. Öklid postulatlarının nihayetinde deneysel bir yanı olduğundan bunları karşıtlarıyla değiştirmek mümkün olmuştur; fakat matematiksel tümevarım ilkesinin deneysel hiçbir yanı yoktur. Poincaré için bu ilkenin doğruluğu bize zihnimiz tarafından dayatılır, çünkü aslında bu ilke zihinsel bir kabiliyetin doğrudan sezilmesidir – sonsuz yineleme kabiliyeti. Poincaré Öklid postulatlarında olduğu gibi bu ilkeyi reddedip yeni bir aritmetik kurmaya çalışılırsa bunun başarılamayacağını söylemiştir; bunu yapmak bizzat matematiksel düşünmeyi yok saymak anlamına gelecektir.

4. Bölüm Cantor'un sonluötesi ordinal aritmetik teorisine adanmıştır. Cantor'un teorisi yeni bir aritmetik olarak alınmakta ve bunun hangi yönlerden Öklid-dışı geometrilere benzediği soruşturulmaktadır. Sonluötesi aritmetiği yeni bir aritmetik olarak görmek yanlış olmaz, zira bu teoride tamamiyle yeni tür sayılar ve bunların tabi olduğu, ilk bakışta yabancı gelen birtakım kurallar bulunur. Cantor tümevarım ilkesini reddetmemiştir, fakat bu ilkeyi geliştirmiş ve ondan sonluötesi tümevarım ilkesini çıkarmıştır. Aynı şekilde alışılmış sayı kavramının da anlamını genişletmiş ve sonsuz sayıları matematiğin bir konusu hâline getirmiştir. Cantor sonsuz büyüklüklerin nasıl birer sayı olarak düşünülebileceğini göstermiş ve bunların alışkın olduğumuz doğal sayılar ile ilişkisini ortaya koymuştur. Sonluötesi aritmetik

bu yeni tür sayıların aritmetiğidir ve sonluötesi tümevarım ilkesi bu teorideki birtakım ispatları yapmak için kullanılır. Bu aritmetikte toplamanın ve çarpmanın değişme özelliği yoktur, üstelik sonluötesi sayıların büyüklük ve küçüklük ilişkileri de alıştığımızdan farklıdır; örneğin sonluötesi ordinal bir sayıyı iki ile çarpmak, ne kadar büyük olursa olsun sonlu bir sayıyı bu sayıya eklemekten her zaman daha büyük bir sayı verecektir ($\omega \cdot 2 > \omega + 2^{1000}$). Bu teori ve özellikle de sonluötesi sayılar bize ne kadar yabancı gelse de, bugün matematiğin en heyecan verici ve yeniliklere gebe alanlarından biridir. Dahası, birtakım problemlerin çözümünde bu sayıların yalnızca faydalı değil, aynı zamanda gerekli de olduğu gösterilmiştir ve bu mesele 4.3 numaralı bölümde detaylıca anlatılmaktadır.

Cantor sayarak ulaşılamayacak bir sayının varlığını kabul etmiştir. Bir başka deyişle bu öyle bir sayıdır ki kendisinden hemen önce gelen bir sayı bulmak imkansızdır. Yine de bu Cantor için bu sayıyı bir hayal ürünü, bir kuruntu yapmaz. Bu sayı doğal sayılar kadar gerçektir, yalnızca ona erişmek için bildiğimiz sayma prosedüründen farklı bir prosedür izlememiz gerekir ki bu da Cantor'un ikinci üretme ilkesinde tarif edilmiştir. Bu ilkede Cantor sonsuz elemana sahip fakat aralarında hiç bozulmayan bir sıralamanın olduğu kümelere bitmiş, tamamlanmış birer nesne, bir sayı olarak bakmanın mümkün olduğunu söyler. Doğal sayılar kümesi sonsuz elemana sahiptir ve asla kapanmayacaktır, çünkü 'en büyük doğal sayı' yoktur. Buna rağmen elemanlar arasında bir sıra vardır. İşte Cantor bu sıraya bir sayı atfetmiş ve buna ω, en küçük sonluötesi ordinal sayı demiştir. Böylece doğal sayılar kümesinin elemanlarını, hatta daha da ötesini saymak mümkün olmuştur. Bu ilkenin kabul edilmesi bize sonluötesi sayıları, sonluötesi tümevarım ilkesini ve bu sayıların kendilerine has aritmetiğini vermiştir. Başlarda kimi çelişkilere yol açmış olsa da bugün bu ilke kusursuzlaştırılmıştır ve sınırlandırılmış bir hâlde geçerliliği sağlanmıştır, üstelik son derece de faydalı olduğu gösterilmiştir. Peki nasıl olmuş da Cantor bu ilkeyi öne sürebilmiştir? Bu tezde bu soruya bir cevap verilmektedir: Cantor, Poincaré'nin tarif ettiği son derece anlaşılır ve doğal olan bir zihinsel kabiliyeti, yani sonsuz yineleme kabiliyetini geliştirmiştir. Cantor bize, sonsuz tekrarı kavramaya muktedir olan zihnilerimizin, yaptığı bu tekrarda bir sıra bulabildiği taktirde bütün bu süreci tek bir olay gibi kavramaya da muktedir olduğunu göstermiştir. Sonluötesi sayıların ve yeni aritmetik ilkelerin doğuşu Öklid-dışı geometrilerde olduğu gibi deneysel koşulların farazi bir şekilde değiştirilmesinden kaynaklanmamaktadır; bunlar bizzat zihinsel yetilerimizin geliştirilmesi ile ortaya çıkmıştır.

Eğer aritmetiğin Poincaré'nin tarif etiği gibi deneyimden tümüyle bağımsız olduğu kabul edilirse, geometridekine benzer bir durumun aritmetikte gerçekleşemeyeceğini görmek kolaydır. Yine de bugün yeni ve alışılmadık, üstelik son derece faydalı olan ve işleyen bir aritmeğimiz olduğu yadsınamaz. Bu teorinin saf matematiğe faydaları 4.3'te anlatılmaktadır. Eğer aritmetiğin temelinde yatan sezginin gelişebileceği, zihinsel yetilerimizin ilerlemeye tabi olabileceği kabul edilirse, o zaman sonluötesi ordinal teorisine sezgici bir temel öne sürmek mümkün olur. Bu görüşte sonluötesi ordinaller ve sonluötesi tümevarım ilkesi bize doğal sayıları veren ve standart tümevarım ilkesinin kullanımını meşru kılan aynı sezgide temellenmiştir – zihnin sonsuz yineleme kabiliyeti. Cantor bu tür bir tekrarı kavrayabilen zihnin, bir sıra bulduğu taktirde bu tekrarın bütününü belirli bir sayı olarak düşünebildiğini göstermiştir. Aristoteles'in uzun süre matematiğe hakim olmuş sonsuzluk anlayışını dikkatle analiz ederek ve bunu kendi öne sürdüğü 'iyi-sıralama' kavramı ile destekleyerek Cantor hepimizde bulunan bir zihinsel yetinin geliştirilebileceğini göstermiştir. Kendisi yeni tür sayılar ve bunların sıradışı aritmetiğini kurmayı başarmıştır, ki bunlar şaşırtıcı derecede faydalı olmuştur ve Cantor'un adımlarını takip etmeye razı olan matematikçilere oldukça anlamlı gelmektedir.

Bu tezde sunulana, Poincaré'nin sezgiciliğinin bir miktar değiştirilmiş bir versiyonunun savunulması olarak bakılabilir. Bu görüşe göre empirik gözlemin matematiğin temellerinde hiçbir rolü yoktur; bu bilim insanoğlunun sezgilerinde temellenmiştir. Bazı filozofların (Ayer, 1964) düşündüğünün aksine matematiksel sezgi gizemli bir meleke değildir, o basitçe zihnin kendi yapısı ve kabiliyetleri hakkında doğrudan bir kavrayışa sahip olmasıdır. Poincaré'nin durumunda bu sonsuz yineleme kabiliyetine tekabül eder. Bu tezde öne sürülen, bu kabiliyetin gelişebileceği ve bunun da yeni bir aritmetiğin başlangıç noktası olabileceğidir. Geometrideki kavramların deneyimin rehberliğinde anlaşıldığı ve geometrik postulatlara dair delillerin kısmen gözlemde bulunduğu görülmüştür. Cisimlerin hareketi için alışılmışın dışında kurallar varsayarak, bir başka deyişle yeni uzlaşımlar benimseyerek, Öklid-dışı geometrileri kurmak mümkün olmuştur. Deneyim aritmetiğin temellerinde hiçbir rol oynamadığından, yeni bir gözlemin veya alışıldık deneysel koşullarda meydana gelecek gerçek veya farazi bir değişikliğin aritmetiğin temel kavramlarında veya ilkelerinde bir değişikliğe sebep olması beklenemez. Bu noktada tez genel Poincaréci görüşten ayrılmakta ve böyle bir değişikliğin sezginin geliştirilmesinde bulunabileceğini iddia etmektedir. Cantor'un bunu başardığı ve böylece sonluötesi ordinal aritmetik teorisini kurduğu savunulmaktadır.

Bu görüşe göre matematikteki sezgi, Gödel gibi bazı filozofların düşündüğünün aksine fiziksel nesnelerin duyumsanmasına benzeyen bir şey değildir. Sonluötesi sayılar gibi matematiksel nesneler zihinden bağımsız matematiksel bir diyarda değildirler ve sezgimiz de bu gizemli nesnelerle etkileşime geçmek için bir yöntem değildir. Daha ziyade zihnimiz bu nesneleri inşa eder. Aslında bunlar alışıldık anlamıyla 'nesne' değil, zihnin kendi yetileri çerçevesinde kavramaya muktedir olduğu birtakım ilişkiler, bağıntılardır. Sonluötesi ordinal aritmetiğin kuralları anlamsız birtakım sembolleri işlemek için ortaya koyulmuş keyfi ve boş kurallar değildir. Bu kuralların meşruiyeti tıpkı doğal sayılar ve sonlu aritmetikte olduğu gibi belli bir zihinsel kabiliyetin sezilmesine dayanır, yalnızca bu kabiliyetin geliştirildiği kabul edilmelidir. Dolayısıyla sonluötesi ordinal aritmetiği platonizm veya nominalizme düşmeden, bir sezgici olarak anlamlandırmak mümkündür.

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