DETERMINATION OF STRESSES IN VLASOV BEAM SECTIONS

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ABSTRACT

DETERMINATION OF STRESSES IN VLASOV BEAM SECTIONS

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In this thesis, the normal and shear stresses in Vlasov beams are determined. The shape of the considered cross-sections may be arbitrary. For the computation of shear stresses, two-dimensional triangular finite element formulations are developed. The stiffness matrices and force vectors are derived for transversal forces, uniform torsion, and nonuniform torsion. The proposed finite element algorithm is validated through the analytical solutions, structural engineering books, and related articles. The numerical examples include beams with different cross-section types such as solid, thick-walled, closed thin-walled, and open thin-walled sections.

Keywords: Vlasov Beams, Finite Element Method, Normal and Shear Stresses, Cross-section properties, Warping Function
ÖZ

VLASOV KİRİŞ KESİTLERİNDE GERİLİMLERİN BELİRLENMESİ

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Anahtar Kelimeler: Vlasov Kirişleri, Sonlu Elemanlar Yöntemi, Normal ve Kayma Gerilmeleri, Kesit Alanı Özellikleri, Çarpılma Fonksiyonu
To my deceased father Hüsamettin Erdoğan and my family
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<tr>
<td>FE</td>
<td>Finite Element</td>
</tr>
<tr>
<td>FEM</td>
<td>Finite Element Method</td>
</tr>
<tr>
<td>FEA</td>
<td>Finite Element Analysis</td>
</tr>
<tr>
<td>BEM</td>
<td>Boundary Element Method</td>
</tr>
<tr>
<td>Symbol</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
</tr>
<tr>
<td>$u$</td>
<td>Translational displacement in y-direction</td>
</tr>
<tr>
<td>$v$</td>
<td>Translational displacement in z-direction</td>
</tr>
<tr>
<td>$\theta$</td>
<td>Rotational displacement in x-direction</td>
</tr>
<tr>
<td>$y_o$</td>
<td>Y coordinate of shear center w.r.t centroid</td>
</tr>
<tr>
<td>$z_o$</td>
<td>Z coordinate of shear center w.r.t centroid</td>
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CHAPTER 1

INTRODUCTION

1.1. Introduction

Beams are structural elements used in modeling various engineering structures such as aircraft wings, aircraft frames, helicopter rotor blades, automobile chassis, and edge beams in buildings. There are multiple different cross-section types of beams such as solid, thick-walled, thin-walled closed, and thin-walled open sections. The structural behavior of beams is characterized by the material they are made of and their geometry. Geometrical features play an important role in their behavior. The amount of deflection in a beam is directly related to its length while cross-sectional shape affects stresses occurring on the beam section. For this reason, the accuracy of cross-section properties is crucial for reliable stress analysis. While performing beam analyses under transversal and torsional loadings, it is necessary to obtain complicated section properties such as the shear center, torsional constant, and warping constant. For structures having a complex shape, it is difficult to calculate these properties by analytical means. To compute these parameters, numerical methods such as the boundary element method, the finite difference method, and the finite element method are needed.

Figure 1.1: Representation of Numerical Methods [1]
In solid mechanics, it is crucial to comprehend the analysis of structures subjected to twisting moments correctly. For this reason, torsion has long been the subject of theoretical and practical interest in this field. The Saint Venant torsion theory [3] is referred to as uniform torsion and the Vlasov torsion theory [4] is known as nonuniform torsion. In the historical development of these works, Agustin Cauchy [5] demonstrated experimentally that non-circular beams under a torsional loading experience an out-of-plane displacement called warping. Based upon this, Saint Venant defined an axial displacement, which is a function of the in-plane direction. In uniform torsion, the warping displacement and angle of twist rate become identical in all cross-sections. This theory is valid if the beam is free to warp at the ends under constant torque. However, in more realistic cases, beams usually have boundary conditions. In other words, the warping and twist can be restrained at one or more cross-sections. Vlasov presented the nonuniform torsion theory. In this theory, the torque varies along the beam, or it is not free to warp at any cross-section. In a parallel manner to this, the angle of twist rate and axial displacement along its axis are not constant. These effects result in some internal forces such as warping moments and warping torques. In general, the warping effects are taken into account if the section type is open thin-walled since it has low torsional rigidities. For a conservative design, beam structures are analyzed considering not only the effect of Saint-Venant torsion but also the warping effects for all types of beams.

Figure 1.2: Uniform Torsion and Non-uniform Torsion [2]
1.2. Thesis Objective

The purpose of this thesis is to determine cross-section properties and stresses in Vlasov beam sections using the finite element method. In the formulations of this study, the stiffness matrices and force vectors for transversal shear forces, uniform torsion, and non-uniform torsion are offered. For the discretization of the geometry, the numerical model uses 2D triangular mesh elements. This mesh type makes dealing with complex cross-sectional shapes possible. In the development of this study, several engineering software tools are integrated. As a pre-processor, Hypermesh provides input data with node coordinates and element connectivities of the meshed geometry. As a processor, Fortran organizes the FEM algorithm and solves the equilibrium matrix equations. As a post-processor, Tecplot visualizes output data with stress distributions, shear flow graphs, and warping characteristics. In numerical examples, beams with solid, thick-walled, open thin-walled, and closed thin-walled sections are used. Materials defined in the problems are homogeneous, isotropic, and linearly elastic. The reliability of the presented finite element algorithm is confirmed by already existing analytical solutions and the literature.

1.3. Thesis Plan

This dissertation is comprised of five chapters. It starts with Chapter 1 introducing the problem outline, thesis objective, and thesis plan. In Chapter 2, the related studies are represented in a literature review which focuses on previous works. This chapter also includes the analytical and numerical approaches to cross-sectional analyses. In Chapter 3, the formulations are separated into two subchapters. The first part deals with the internal forces on cross-sections. In the second part of Chapter 3, normal and shear stress equations are formulated. Chapter 4 verifies the proposed finite element model by comparing those found in the literature. Finally, Chapter 5 presents a conclusion as a result of the issues covered in the earlier chapters. This chapter also addresses recommendations for future researches in this area.
CHAPTER 2

LITERATURE REVIEW

This chapter is comprised of two subchapters that review analytical solutions and numerical solutions such as the boundary element method, the finite difference method, and the finite element method. In these subchapters, previous works in the literature related to the determination of cross-section properties and stresses on cross-sections are mentioned.

2.1. Analytical Solutions


W. F. Chen, T. Atsuta [9] offered an analytical solution based on the Laplace equation for torsion analysis of structures having I-beam, C-beam, and T-Beam cross-sections by dividing them into rectangular sub-domains. In this study, the authors ignored corner fillets in the formulation.

Hematiyan and Doostfatemeh [10] presented an approximate analytical method for a hollow isotropic polygonal shape under torsional loading. This method enabled the authors to estimate the variation of the shearing stress across the thickness of the beam. Although this method provided acceptable results for thin-walled and thick-walled beams, it did not address cover beams with open sections.

theory in the presence of warping shear and normal stresses for thin-walled beams is addressed in the textbooks of V. Z. Vlasov [4] and Gjelsvik [13].

M. A. Gurel, R. K. Pekgokgoz, M. Kisa [14] proposed an approximated model and mathematical formulation for the uniform torsion analysis of thin-walled, thick-walled, and solid cross-sections. In their article, the formulas for the maximum shearing stress and the angle of twist was derived. They compared the results with several cross-sections having exact or numerical solutions. According to the obtained results, the formulations gave highly accurate values for thin-walled sections. For thick-walled and solid ones, however, this accuracy decreased gradually.

J. Francu, P. Novackova, P. Janicek [15] suggested an analytical solution for the constant cross-section using the Airy stress function. The contribution dealt with triangular, rectangular, and some other profiles. In the formulations, the authors solved the rectangular profile case through Fourier series.

2.2. Numerical Solutions

Katori, H. [16] implemented formulations to determine the shear center of arbitrary cross-sections using the finite element method. The author derived equations concerning the warping, angle of twist, shear deflection and Lagrange's multipliers. In the case study, the change of the coordinates of the shear center and centroid of the circular cross-sections by circular notches of various diameters were investigated. This problem was validated by the analytical solution of W.J. Strong, and T.G. Zhang [17].

B. D. Mixon [18] developed a finite element tool for the calculation of beam cross-section properties. In this thesis, warping independent and warping dependent cross-section properties were achieved using three-node triangles and six node triangles. The results were validated with Beam Tool of ANSYS which is a commercial software program that is highly useful in determining said properties.
G. H. Holze, C. P. Pulver, and Y. G. Giorgis [19] divided cross-section properties into basic and advanced ones such that the basic properties were area and centroids, and the advanced properties were torsional constants and the shear centers. They developed a boundary element method to solve the equations of Laplace and Poisson.

F. Gruttmann, R. Sauer, and W. Wagner [20] proposed approximate calculations of shear stress in prismatic beams exposed to pure torsion and torsion-free bending using the finite element method. For numerical calculations, Dirichlet boundary conditions of torsion-free bending problems are converted into Neumann boundary conditions.

D. Banić, G. Turkalj, J. Brnić [21] developed a two-dimensional finite element formulation for the stress analysis of elastic beams with arbitrary cross-sections subjected to non-uniform torsion. The element stiffness matrix and load vectors were obtained by primary warping functions corresponding to uniform torsion and secondary warping functions to corresponding non-uniform torsion. The accuracy of the presented algorithm was validated with analytical solutions of I-Beams and C-Beams [9][22].

A. Stefan, M. Lupoae, D. Constantin, C. Baciu [23] aimed to determine the tangential stresses of rectangular sections and L-sections subjected to uniform torsion using the finite difference method. They concluded that the method displayed difficulties in some areas of use, such as when applied to curved boundary domains.

El Darwish and Johnston [24] applied an approximate method to perform torsion analyses of some structural shapes, such as T-beams, by using Prandtl's stress function via the finite difference method. They offered a formulation for the value of the torsional rigidity and shearing stress at the midpoint of the T-beam flanges. However, it is impossible to find shearing stress at the web-flange junction fillet by this method.

Lamancusa and Saravanos [25] presented the torsional analysis of hollow square tubes by a two-dimensional thermal analogy using the finite element method. They evaluated the dependence of torsional properties on wall thickness. The obtained
results were used to generate closed-form algebraic expressions for maximum shear stress and torsional stiffness.

Sapountzakis and Mokos [26] developed a boundary element method for bars with arbitrary cross-sections subjected to an arbitrarily distributed or concentrated twisting moment. The proposed procedure took into account the warping effects along the member length.

Walter Pilkey [27], in the book Analysis and Design of Elastic Beams, explained all forms of loading conditions that can be subjected to beams. Based on these explanations, two programs named ThinWall and PlotStress were developed; these programs took advantage of the finite element method.
CHAPTER 3

FORMULATIONS

3.1. Governing Equations and Internal Forces in Beams

In Figure 3.1, a beam with arbitrary cross section is shown in three-dimensional view. In this thesis, all derivations and solutions are evaluated according to the coordinate system shown in Figure 3.1.

![Figure 3.1: 3-D Representation of a Beam with Arbitrary Cross-section](image)

The governing equations of a beam are

\[ EA(u'' - \alpha \Delta T') = -f_x \]
\[ E(l_y l_z - l_{yz}^2) v^{iv} = l_y f_y + l_{yz} f_z \]
\[ E(l_y l_z - l_{yz}^2) w^{iv} = l_z f_z + l_{yz} f_y \]
\[ EI_w \theta^{iv} - GJ \theta'' = m_x \]  \hspace{1cm} (3.1)
where \( \theta \) is the angle of twist, \( u, v, w \) and \( f_x, f_y, f_z \) are the displacements and the intensities of distributed forces in \( x, y, z \) directions, respectively, \( m_x \) is the intensity of distributed torque, \( \Delta T \) is the thermal force, \( E \) is the elasticity modulus, \( G \) is the shear modulus, \( \alpha \) is thermal expansion coefficient of the beam material, \( A \) is the area, \( I_y, I_z, I_{yz} \) are the second moments of area, \( J \) is the torsional constant and \( I_w \) is the warping constant of the cross section.

Then, the stress resultants in a cross section are

\[
\begin{align*}
N &= EA(u' - \alpha \Delta T) \\
M_y &= EI_{yz}v'' - EI_y w'' \\
M_z &= EI_{x}v'' - EI_{yz} w'' \\
M_w &= EI_{w} \theta'' \\
S_y &= EI_{yz}w''' - EI_{x} v''' \\
S_z &= EI_{yz}v''' - EI_{y} w''' \\
M_x &= GJ \theta' - EI_{w} \theta'' \\
T_s &= GJ \theta' \\
T_w &= -EI_{w} \theta''' \\
T &= T_s + T_w
\end{align*}
\]

(3.2)

where \( N \) is the axial force, \( S_y, S_z \) are the shear forces, \( M_y, M_z \) are the bending moments, \( M_w \) is the warping moment, \( T_s \) is the St.Venant torque, \( T_w \) is the warping torque, and \( T \) is the total torque acting on the cross section. The resultants are calculated either by solving the governing equations or by using a computational method. One of the choices in computational methods is the finite element analysis. The resultants \( N, M_y, M_z, S_y, S_z \) are obtained by using the standard bar and Euler-Bernoulli beam finite elements. The resultants \( T_s, T_w, M_w \) can be obtained by a torsion finite element.
Torsion Finite Element

The weak form of the torsion equation is

$$\int g(E_l w \theta'' - G J \theta'' - m_x) \, dx = 0$$

where $g(x)$ is a test function. Using integration by parts, the weak form can be expressed as

$$E_l w \int g'' \theta'' \, dx + G J \int g' \theta' \, dx = (g T)_B + (g' M_w)_B + \int g m_1 \, dx$$

where the subscript $B$ indicates the boundaries. Consider a finite element of length $L$ with node-1 at $x = 0$ and node-2 at $x = L$.

The twist angle $\theta$ can be assumed as

$$\theta = \Omega \delta$$

where

$$\Omega = \frac{1}{L^3} \left[ (L + 2x)(L - x)^2 \quad Lx(L - x)^2 \quad x^2(3L - 2x) \quad Lx^2(x - L) \right]$$

$$\delta = \begin{bmatrix} \theta_1 \\ \theta_1' \\ \theta_2 \\ \theta_2' \end{bmatrix}$$

Let $G$ be the vector of test functions. Using Galerkin method in which $G = \Omega$, the weak form can be written as

$$E_l w \int \Omega''^T \Omega'' \delta \, dx + G J \int \Omega^T \Omega' \delta \, dx = (\Omega^T T)_B + (\Omega' M_w)_B + \int \Omega^T m_1 \, dx$$

Let

$$k_w = \int \Omega''^T \Omega'' \delta \, dx \quad \text{and} \quad k_s = \int \Omega^T \Omega' \delta \, dx$$
\[ f = (\Omega^T \Omega)_B + (\Omega^T M_w)_B + \int \Omega^T m_x dx \]

Then, the element level equilibrium equation can be expressed as

\[ [E l_w k_w + G j k_s] \delta = f \]

Assembling the elements and imposing the boundary conditions, the nodal values of the twist angle \( \theta \) can be determined. Then, the resultants \( T_s, T_w, M_w \) are computed as

\begin{align*}
T_s &= G j \Omega' \delta \\
T_w &= -E l_w \Omega''' \delta \\
M_w &= E l_w \Omega'' \delta
\end{align*}

In this thesis, the normal and shear stress distributions over a cross section subjected to \( N, M_y, M_z, S_y, S_z, T_s, T_w, M_w \) are determined.
3.2. Determination of Stresses

Triangular Finite Element

Consider a three-node finite element of area $\alpha$ with vertex nodes in a coordinate system $yz$.

![Figure 3.1: Representation of Triangular Element](image)

The triangular coordinates are

$$\xi_i = \frac{\alpha_i}{\alpha}, \quad \xi_1 + \xi_2 + \xi_3 = 1$$

(3.4)

The relations between $yz$ and

$$\begin{bmatrix} y \\ z \end{bmatrix} = \frac{1}{2\alpha} \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} + \frac{1}{2\alpha} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

(3.5)

where

$$a_1 = z_2 - z_3 \quad b_1 = y_3 - y_2 \quad c_1 = y_2z_3 - y_3z_2$$

$$a_2 = z_3 - z_1 \quad b_2 = y_1 - y_3 \quad c_2 = y_3z_1 - y_1z_3$$

$$a_3 = z_1 - z_2 \quad b_3 = y_2 - y_1 \quad c_3 = y_1z_2 - y_2z_1$$

(3.6)

For area of the triangular element

$$\alpha = \frac{1}{2} (a_2a_3 - a_3a_2)$$

(3.7)
Differentiation

\[ \frac{\partial \gamma}{\partial y} = \sum_{i=1}^{3} a_i \frac{\partial}{\partial \xi_i} \]

\[ \frac{\partial \gamma}{\partial z} = \sum_{i=1}^{3} b_i \frac{\partial}{\partial \xi_i} \]

Note that \( \xi_i = 1 \) at node \( i \) and zero at nodes \( j \) and \( k \). Then, a function \( f (y, z) \) can be interpolated over a triangular domain as

\[ f = \xi_1 f_1 + \xi_2 f_2 + \xi_3 f_3 \]

where \( f_n = f (y_n, z_n) \).

Line Integration

A function \( \xi_i^l \xi_j^l \) is integrated along an edge \( ij \) of length \( L_{ij} \) as

\[ \int_{ij} \xi_i^l \xi_j^l \, ds = L_{ij} \frac{l!j!}{(l+j+1)!} \] (3.8)

Area Integration

A function \( \xi_i^l \xi_j^l \xi_k^k \) is integrated over a triangle of area \( \alpha \) as

\[ \int_A \xi_i^l \xi_j^l \xi_k^k \, ds = 2\alpha \frac{l!j!k!}{(l+j+k+2)!} \] (3.9)

Centroid, Cross-Sectional Area, Second Moments of Area

Let the section be discretized by \( N \) number of triangular elements. Let's \( rs \) be an arbitrary frame
The cross-sectional area is

\[ A = \sum_{n=1}^{N} \alpha_n = \frac{1}{2} \sum_{n=1}^{N} (a_2 b_3 - a_3 b_2)_n \]  

(3.10)

Note that

\[
\int_A r \, dA = \sum_{n=1}^{N} \int \left( r_1 \xi_1 + r_2 \xi_2 + r_3 \xi_3 \right)_n dA = \frac{1}{3} \sum_{n=1}^{N} \alpha_n (r_1 + r_2 + r_3)_n 
\]

\[
\int_A s \, dA = \sum_{n=1}^{N} \int \left( s_1 \xi_1 + s_2 \xi_2 + s_3 \xi_3 \right)_n dA = \frac{1}{3} \sum_{n=1}^{N} \alpha_n (s_1 + s_2 + s_3)_n 
\]

Then, the coordinates of the centroid \( C(\bar{r}, \bar{s}) \) are

\[
\bar{r} = \frac{1}{A} \int_A r \, dA = \frac{1}{3A} \sum_{n=1}^{N} \alpha_n (r_1 + r_2 + r_3)_n 
\]

\[
\bar{s} = \frac{1}{A} \int_A s \, dA = \frac{1}{3A} \sum_{n=1}^{N} \alpha_n (s_1 + s_2 + s_3)_n 
\]
Let \( yz \) be a centroidal frame parallel to \( rs \). Then
\[
y = r - \bar{r} \quad z = s - \bar{s}
\]
And the second moments of area are defined as
\[
I_y = \int_A z^2 \, dA = \sum_{n=1}^{N} \int \left( z_1 \xi_1 + z_2 \xi_2 + z_3 \xi_3 \right)^2 \, dA
\]
\[
= \frac{1}{6} \sum_{n=1}^{N} \alpha_n \left( z_1^2 + z_2^2 + z_3^2 + z_1 z_2 + z_2 z_3 + z_3 z_1 \right)_n
\]
(3.11a)
\[
I_z = \int_A y^2 \, dA = \sum_{n=1}^{N} \int \left( y_1 \xi_1 + y_2 \xi_2 + y_3 \xi_3 \right)_n^2 \, dA
\]
\[
= \frac{1}{6} \sum_{n=1}^{N} \alpha_n \left( y_1^2 + y_2^2 + y_3^2 + y_1 y_2 + y_2 y_3 + y_3 y_1 \right)_n
\]
(3.11b)
\[
I_{yz} = -\int_A y z \, dA = -\sum_{n=1}^{N} \int \left( y_1 \xi_1 + y_2 \xi_2 + y_3 \xi_3 \right)_n \left( z_1 \xi_1 + z_2 \xi_2 + z_3 \xi_3 \right)_n \, dA
\]
\[
= -\sum_{n=1}^{N} \alpha_n \left[ \frac{1}{6} \left( y_1 z_1 + y_2 z_2 + y_3 z_3 \right)_n + \frac{1}{12} \left( y_1 z_2 + y_2 z_1 + y_2 z_3 + y_3 z_2 + y_3 z_1 + y_1 z_3 \right)_n \right]
\]
(3.11c)

**Divergence Theorem**

Consider a beam section in the \( yz \) plane. Let \( A \) be the area and \( B \) be the boundary of the cross section. The boundary \( B \) consists of the outer boundary \( B_O \) and the outer boundaries \( B_i (i = 1, \ldots, n) \). Along each boundary, a set of coordinates \( ns \) is defined such that \( s \) is tangent to boundary, \( n \) is normal to the boundary and \( xns \) is a right-handed coordinate system. \( n = n_x j + n_z k \) is the unit vector in the direction \( n \).
Figure 3.3: Normal Vectors in Arbitrary Cross Section

Let \( \mathbf{F} = F_y \mathbf{j} + F_z \mathbf{k} \) be a vector function. Then,

\[
\int_A \nabla \cdot \mathbf{F} \, dA = \int_B \mathbf{F} \cdot \mathbf{n} \, ds
\]

where

\[
\nabla = \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k}
\]

Let \( g(y, z) \) be a scalar function. Then,

\[
\int_A \nabla \cdot g\mathbf{F} \, dA = \int_B g\mathbf{F} \cdot \mathbf{n} \, ds
\]

Note that

\[
\nabla \cdot g\mathbf{F} = \left( \frac{\partial}{\partial y} g F_y + \frac{\partial}{\partial z} g F_z \right) = \frac{\partial}{\partial y} (g F_y) + \frac{\partial}{\partial z} (g F_z)
\]

\[
= F_y \frac{\partial g}{\partial y} + g \frac{\partial F_y}{\partial y} + F_z \frac{\partial g}{\partial z} + g \frac{\partial F_z}{\partial z} = g \left( \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right) + \left( F_y \frac{\partial g}{\partial y} + F_z \frac{\partial g}{\partial z} \right)
\]
\[
= g \left( \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k \right) \cdot (F_y j + F_z k) + \left( \frac{\partial g}{\partial y} j + \frac{\partial g}{\partial z} k \right) \cdot (F_y j + F_z k)
\]

\[
= g \nabla \cdot F + \nabla \cdot (g \nabla F)
\]

Then,

\[
\int_A (g \nabla \cdot F + \nabla g \cdot F) dA = \int_B g \nabla \cdot F d\sigma - \int_A \nabla g \cdot F dA
\]

Let \( F = \nabla f \) where \( f = f(y, z) \) is a scalar function. Then,

\[
\int_A g \nabla \cdot \nabla f dA = \int_B g \nabla f \cdot n d\sigma - \int_A \nabla g \cdot \nabla f dA
\]

\[
\rightarrow \int_A g \nabla^2 f dA = \int_B g \nabla f \cdot n d\sigma - \int_A \nabla g \cdot \nabla f dA
\]

Consider \( N \) number of functions \( g_n (n = 1, \ldots, N) \). Let

\[
g = \begin{bmatrix} g_1(y, z) & g_1(y, z) & \ldots & g_N(y, z) \end{bmatrix}
\]

Then,

\[
\int_A g^T \nabla^2 f dA = \int_B g^T \nabla f \cdot n d\sigma - \int_A \nabla g^T \cdot \nabla f dA
\]

(3.12)
3.2.1. Stresses due to Normal Force and Bending Moments

The normal stress due to axial force \( N \) and bending moments \( M_y \) and \( M_z \) is

\[
\sigma_{xx} = \frac{N}{A} + \frac{1}{(I_y I_z - I_{yz}^2)} \left[ M_y (I_z z + I_{yz} y) - M_z (I_y y z + I_{yz} y) \right]
\]  

(3.13)

3.2.2. Stresses due to Shear Forces

Consider a beam cross section in the \( yz \) plane. Let the shear forces \( S_y \) and \( S_z \) act through the shear center \( O \). Therefore, there is no twisting in the section. The stresses on the section are the normal stress \( \sigma_x \) and the shear stresses \( \tau_{xy} \) and \( \tau_{xz} \). The equilibrium requires that

\[
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0
\]

Noting that

\[
\sigma_{xx} = \frac{1}{(I_{yy} I_{zz} - I_{yz}^2)} \left[ M_y (I_{zz} z + I_{yz} y) - M_z (I_{yy} y z + I_{yz} y) \right]
\]

\[ M'_y = S_z \]

\[ M'_z = S_y \]
we get
\[ \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = -h(y, z) \]

where
\[ h(y, z) = \frac{1}{(I_{yy}I_{zz} - I_{yz}^2)} \left[ S_y(I_{zz}z + I_{yz}y) + S_z(I_{yy}z + I_{yz}y) \right] \]  \hspace{1cm} (3.14)

Let \( \phi(y, z) \) be a stress function such that
\[ \tau_{xy} = \frac{\partial \phi}{\partial y} \text{ and } \tau_{xz} = \frac{\partial \phi}{\partial z} \]

Then, the governing equation is obtained by substituting eq. (2) and eq. (3) into eq. (4)
\[ \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = -h(y, z) \rightarrow \nabla^2 \phi + h = 0 \]

The Cauchy’s relation can be written on the boundaries as
\[
\begin{bmatrix}
\sigma_{xx} & \tau_{xy} & \tau_{xz} \\
\tau_{xy} & 0 & 0 \\
\tau_{xz} & 0 & 0
\end{bmatrix}
\begin{bmatrix}
n_y \\
n_z
\end{bmatrix} =
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

noting that the outer and inner boundaries of the beam are stress free. The second and third rows are identically satisfied. The first row gives
\[ \tau_{xy}n_y + \tau_{xz}n_z = 0 \]

which can be written as
\[ \frac{\partial \phi}{\partial y} n_y + \frac{\partial \phi}{\partial z} n_z = 0 \rightarrow \left[ \frac{\partial \phi}{\partial y} \quad \frac{\partial \phi}{\partial z} \right] \begin{bmatrix} n_y \\ n_z \end{bmatrix} = 0 \rightarrow \nabla \phi \cdot \mathbf{n} = 0 \text{ for } B_i \ (i = 0, 1, ..., N) \]

This condition applies to both outer and inner boundaries.

The weak form is
\[ \int_A g(\nabla^2 \phi + h) dA = 0 \]

Consider \( N \) number of functions \( g_n(n = 1, \ldots, N) \). Let
\[ \mathbf{g} = [g_1(y, z) \quad g_1(y, z) \ldots \quad g_N(y, z)] \]
Then,
\[ \int_A \mathbf{g}^T (\nabla^2 \varphi + h) dA = 0 \quad \text{rearranging} \quad \int_A \mathbf{g}^T \nabla^2 \varphi dA = - \int_A \mathbf{g}^T h dA \]

Using the Green-Gauss Theorem,
\[ \int_A \mathbf{g}^T \nabla^2 \varphi dA = \int_B \mathbf{g}^T \nabla \varphi \cdot \mathbf{n} ds - \int_A \nabla \mathbf{g}^T \cdot \nabla \varphi dA \]

Then, the weak form can be written as
\[ \int_A \nabla \mathbf{g}^T \cdot \nabla \varphi dA = \int_B \mathbf{g}^T \nabla \varphi \cdot \mathbf{n} ds - \int_A \mathbf{g}^T h dA \]

The first integral on the right-hand side is zero due to the boundary condition \( \nabla \varphi \cdot \mathbf{n} = 0 \). Then, the weak form is
\[ \int_A \nabla \mathbf{g}^T \cdot \nabla \varphi dA = \int_A \mathbf{g}^T h dA \]

The solution of the equation (7) is carried out with the finite element formulations.

**Finite Element Formulation**

Assume the stress function within the \( n^{th} \) element as
\[ \varphi = [\xi_1 \quad \xi_2 \quad \xi_3] \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{bmatrix} = \mathbf{Pq} \]

Then,
\[ \nabla \varphi = \nabla \mathbf{Pq} = \begin{bmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{bmatrix} = \begin{bmatrix} \frac{\partial \xi_1}{\partial y} & \frac{\partial \xi_2}{\partial y} & \frac{\partial \xi_3}{\partial y} \\ \frac{\partial \xi_1}{\partial z} & \frac{\partial \xi_2}{\partial z} & \frac{\partial \xi_3}{\partial z} \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{bmatrix} \]

\[ \mathbf{Bq} = \frac{1}{2A} \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{bmatrix} \]

In Galerkin method, the test functions are chosen that
\[ \mathbf{g} = \frac{\partial \varphi}{\partial \mathbf{q}} \rightarrow \mathbf{g} = \mathbf{P} \rightarrow \nabla \mathbf{g} = \nabla \mathbf{P} = \mathbf{B} \]

Then, the weak form can be written for a finite element as

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\[
\int_{A} B^T B q \, dA = \int_{A} P^T h \, dA
\]

Then, the element level equilibrium equations are

\[kq = f\]

where

\[
k = \int_{A} B^T B \, dA
\]

\[
f = \int_{A} P^T h \, dA
\]

\[
= \frac{S_y}{I_{yy}I_{zz} - I_{yx}^2} \int_{A} P^T (I_{yy}y + I_{yz}z) \, dA + \frac{S_z}{I_{yy}I_{zz} - I_{yx}^2} \int_{A} P^T (I_{zz}z + I_{yx}y) \, dA
\]

\[
= \frac{A}{12(I_{yy}I_{zz} - I_{yx}^2)} \left[ \begin{array}{ccc}
2 & 1 & 1 \\
1 & 2 & 1 \\
1 & 1 & 2
\end{array} \right] \left[ \begin{array}{c}
y_1 \\
y_2 \\
y_3
\end{array} \right] + \left[ \begin{array}{c}
\frac{S_y I_{yy} + S_z I_{yz}}{I_{yy}I_{zz} - I_{yx}^2} \\
\frac{S_z I_{zz} + S_y I_{yz}}{I_{yy}I_{zz} - I_{yx}^2}
\end{array} \right] \left[ \begin{array}{c}
z_1 \\
z_2 \\
z_3
\end{array} \right]
\]

As a result, the shear stresses in the element due to shear force \( S \) are calculated as

\[
\begin{bmatrix}
\tau_{xy} \\
\tau_{xz}
\end{bmatrix} = Bq = \frac{1}{2A} \begin{bmatrix}
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3
\end{bmatrix} \begin{bmatrix}
\varphi_1 \\
\varphi_2 \\
\varphi_3
\end{bmatrix}
\]

**Shear Center**

The torque created by \( \tau_{xy} \) and \( \tau_{xz} \) is

\[
T = \int_{A} (\tau_{xz} y - \tau_{xy} z) \, dA = \sum_{n=1}^{N} A_n \left[ (\tau_{xz})_n \bar{y}_n - (\tau_{xy})_n \bar{z}_n \right]
\]

\[
= \frac{1}{3} \sum_{n=1}^{N} \alpha_n \left[ \tau_{xz} (y_1 + y_2 + y_3) \right]_n - \tau_{xy} (z_1 + z_2 + z_3)_n
\]
Nothing that, for the present three-node element, the shear stresses $\tau_{xy}$ and $\tau_{xz}$ are constant over the element. The shear center $O(y_0, z_0)$ is a point in the cross-sectional plane such that the shear forces acting through it do not cause twisting of the section. Then, $y_0$ is determined by taking $S_z \neq 0$ and $S_y = 0$, and $z_0$ is determined by taking $S_y \neq 0$ and $S_z = 0$ as

$$y_0 = \frac{T}{S_z} \text{ and } z_0 = -\frac{T}{S_y}$$

(3.19)

The shear center is also the twist center about which a section rotates under the action of a torque.

### 3.2.3. Stresses due to Saint Venant’s Torque

Consider uniform torsion such that $\theta'$ is constant and $\theta = \theta'x$. In this situation, torque and angle of twist rate are constant along the beam. In addition, the warping displacement is dependent of $y$ and $z$ coordinates, and independent of $x$ coordinate.

Then, the displacements are

$$u = \tilde{\omega}\theta'$$
$$v = -(z - z_0)\theta'$$
$$w = (y - y_0)\theta'$$

(3.20)

where $\tilde{\omega}(y, z)$ is the warping function. Then, the nonzero strains and stresses are

$$\gamma_{xy} = \theta' \left( \frac{\partial \tilde{\omega}}{\partial y} - z + z_0 \right) \quad \gamma_{xz} = \theta' \left( \frac{\partial \tilde{\omega}}{\partial z} + y - y_0 \right)$$

$$\tau_{xy} = G\gamma_{xy} = G\theta' \left( \frac{\partial \tilde{\omega}}{\partial y} - z + z_0 \right) \quad \tau_{xz} = G\gamma_{xz} = G\theta' \left( \frac{\partial \tilde{\omega}}{\partial z} + y - y_0 \right)$$

Then, St. Venant torque is defined as

$$T_s = \int_A \left( \tau_{xz}y - \tau_{xy}z \right) dA = GJ\theta'$$
where

\[
J = \int_A \left[ (y - y_0) \frac{\partial \tilde{\omega}}{\partial z} - (z - z_0) \frac{\partial \tilde{\omega}}{\partial y} + (y - y_0)^2 + (z - z_0)^2 \right] dA
\]

\[
= I_y + I_z + A(y_0^2 + z_0^2) + \int_A \left[ (y - y_0) \frac{\partial \tilde{\omega}}{\partial z} - (z - z_0) \frac{\partial \tilde{\omega}}{\partial y} \right] dA
\]

is the torsional constant of the section.

The equilibrium condition gives

\[
\frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0 \rightarrow \frac{\partial^2 \tilde{\omega}}{\partial y^2} + \frac{\partial^2 \tilde{\omega}}{\partial z^2} = 0
\]

Thus, the governing equation is obtained as \( \nabla^2 \tilde{\omega} = 0 \)

Cauchy’s relation gives the boundary conditions as

\[
\left( \frac{\partial \tilde{\omega}}{\partial y} - z + z_0 \right) n_y + \left( \frac{\partial \tilde{\omega}}{\partial z} + y - y_0 \right) n_z = 0
\]

\[
\rightarrow \left( \frac{\partial \tilde{\omega}}{\partial y} \right) n_y + \left( \frac{\partial \tilde{\omega}}{\partial z} \right) n_z = (z - z_0)n_y - (y - y_0)n_z
\]

Let \( \lambda = (z - z_0)n_y - (y - y_0)n_z \)

Then, boundary condition is \( \nabla \tilde{\omega} \cdot \mathbf{n} = \lambda \)

This condition must be satisfied along external and internal boundaries \( B_i \) (\( i = 0, 1, ..., N \))

The weak form is obtained by using the governing equation as

\[
\int_A \mathbf{g}^T \nabla^2 \tilde{\omega} dA = 0
\]

Using the divergence theorem
\[
\int_A g^T \nabla^2 \bar{\omega} dA = \int_B g^T \nabla \bar{\omega} \cdot ndS - \int_A \nabla g^T \cdot \nabla \bar{\omega} dA = \int_B g^T \lambda dS - \int_A \nabla g^T \cdot \nabla \bar{\omega} dA
\]

Then, the weak form becomes
\[
\int_A \nabla g^T \cdot \nabla \bar{\omega} dA = \int_A g^T \lambda dS
\]

**Finite Element Formulation**

Assume the warping function within an element as
\[
\bar{\omega} = Pq = \begin{bmatrix} \xi_1 & \xi_2 & \xi_3 \end{bmatrix} \begin{bmatrix} \bar{\omega}_1 \\ \bar{\omega}_2 \\ \bar{\omega}_3 \end{bmatrix}
\]

Then,

Recall that \( g = P \to \nabla g = \nabla P = B \). Then, the weak form can be written for a finite element as
\[
\int_A B^T Bq dA = \int_\beta P^T \lambda dS
\]

where \( \beta \) denotes the element boundary. Then, the element level equilibrium equations are

\[
kq = f
\]

where

\[
k = \int_A [B]^T B dA
\]

\[
= \frac{1}{4a} \begin{bmatrix} (a_1a_1 + b_1b_1) & (a_1a_2 + b_1b_2) & (a_1a_3 + b_1b_3) \\
(a_2a_1 + b_2b_1) & (a_2a_2 + b_2b_2) & (a_2a_3 + b_2b_3) \\
(a_3a_1 + b_3b_1) & (a_3a_2 + b_3b_2) & (a_3a_3 + b_3b_3) \end{bmatrix}
\]  
(3.21)

The force vector can be expressed as
\[
f = f_{12} + f_{23} + f_{31}
\]  
(3.22)
where $f_{ij}$ is the force vector related with the edge $ij$. Note that the normal vector components for the edge $ij$ are

$$n_y = \frac{(z_j - z_i)}{L_{ij}} \quad n_z = \frac{(y_j - y_i)}{L_{ij}}$$

Then, $f_{ij}$ can be evaluated as follows:

If the edge-12 is not on the boundary, then

$$f_{12} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

If the edge-12 is on the boundary, then

$$f_{12} = \frac{1}{L_{12}} \int_{12} \begin{bmatrix} \xi_1 \\ \xi_2 \\ 0 \end{bmatrix} \left[ (z_1 \xi_1 + z_2 \xi_2 - z_0)(z_2 - z_1) + (y_1 \xi_1 + y_2 \xi_2 - y_0)(y_2 - y_1) \right] ds$$

If the edge-23 is not on the boundary, then

$$f_{23} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

If the edge-23 is on the boundary, then

$$f_{23} = \frac{1}{L_{23}} \int_{23} \begin{bmatrix} 0 \\ \xi_2 \\ \xi_3 \end{bmatrix} \left[ (z_2 \xi_2 + z_3 \xi_3 - z_0)(z_3 - z_2) + (y_2 \xi_2 + y_3 \xi_3 - y_0)(y_3 - y_2) \right] ds$$

If the edge-31 is not on the boundary, then

$$f_{31} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

If the edge-31 is on the boundary, then
\[ f_{31} = \frac{1}{L_{31}} \int_{\xi_3}^{\xi_1} \left( (z_3 \xi_3 + z_1 \xi_1 - z_0)(z_1 - z_3) + (y_3 \xi_3 + y_1 \xi_1 - y_0)(y_1 - y_3) \right) ds \]

Define

\[ \tilde{\omega} = \frac{1}{A} \int_A \tilde{\omega} dA = \frac{1}{A} \sum_{n=1}^{N} \int_{\xi_1}^{\xi_2} \left[ \begin{array}{c} \tilde{\omega}_1 \\ \tilde{\omega}_2 \\ \tilde{\omega}_3 \end{array} \right] dA = \frac{1}{3A} \sum_{n=1}^{N} \alpha_n (\tilde{\omega}_1 + \tilde{\omega}_2 + \tilde{\omega}_3)_n \]

Then, the warping function is normalized as \( \omega = \bar{\omega} - \tilde{\omega} \). And, \( \tilde{\omega} \) is replaced by \( \omega \) in all equations. Note that, within a finite element

\[ \left[ \begin{array}{c} \frac{\partial \omega}{\partial y} \\ \frac{\partial \omega}{\partial z} \end{array} \right] = \nabla \omega = Bq \rightarrow \left[ \begin{array}{c} \frac{\partial \omega}{\partial y} \\ \frac{\partial \omega}{\partial z} \end{array} \right] = q^T B^T \]

Then, for a section discretized by \( N \) number of finite elements, \( J \) can be expressed as

\[ J = I_{yy} + I_{zz} + A(y_0^2 + z_0^2) + \int_A \left[ (y - y_0) \frac{\partial \omega}{\partial z} - (z - z_0) \frac{\partial \omega}{\partial y} \right] dA \]

\[ = I_{yy} + I_{zz} + A(y_0^2 + z_0^2) + \sum_{n=1}^{N} \int_A \left[ q^T B^T \left[ \begin{array}{c} z_0 - z \\ y - y_0 \end{array} \right] \right] dA \]

\[ = I_{yy} + I_{zz} + A(y_0^2 + z_0^2) \]

\[ + \frac{1}{6} \sum_{n=1}^{N} \left[ \begin{array}{c} \omega_1 \\ \omega_2 \\ \omega_3 \end{array} \right] \left[ \begin{array}{c} a_1 \\ a_2 \\ a_3 \\ b_1 \\ b_2 \\ b_3 \end{array} \right] \left[ \begin{array}{c} 3z_0 - z_1 - z_2 - z_3 \\ y_1 + y_2 + y_3 - 3y_0 \end{array} \right] \]

The shear stresses at point \((y, z)\) in an element due to St. Venant torque \( T_s \) are evaluated as
\[
\begin{bmatrix}
\tau_{xy} \\
\tau_{xz}
\end{bmatrix} = G\theta' 
\begin{bmatrix}
\frac{\partial \omega}{\partial y} - z + z_0 \\
\frac{\partial \omega}{\partial z} + y - y_0
\end{bmatrix} = G\theta' 
\left[ Bq + \begin{bmatrix}
z_0 - z \\
y - y_0
\end{bmatrix} \right] = \frac{T_s}{f} 
\left[ Bq + \begin{bmatrix}
z_0 - z \\
y - y_0
\end{bmatrix} \right]
\]

(3.24)

3.2.4. Stresses due to Warping Torque and Warping Moment

If \( \theta' = \theta'(x) \), then the torsion is non-uniform. In this situation, torque and angle of twist rate is not constant along the beam. Furthermore, the warping displacement is dependent of \( x, y, \) and \( z \) coordinates.

The normal stress in the warping is

\[
\sigma_x = E\omega\theta'''
\]

The warping moment \( M_w \) and warping torque \( T_w \) are defined as

\[
M_w = \int_A \sigma_x \omega t ds = EI_w \theta'''
\]

\[
T_w = -M_w = -EI_w \theta'''
\]

Then, the normal stress due to warping moment \( M_w \) is

\[
\sigma_x = \frac{M_w}{I_w}(\omega_n)_{ave} \quad \text{where} \quad (\omega_n)_{ave} = \frac{\omega_1 + \omega_2 + \omega_3}{3}
\]

(3.25)

The equilibrium requires that

\[
\frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + \frac{\partial \sigma_{xx}}{\partial x} = 0 \Rightarrow \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = -\frac{\partial \sigma_{xx}}{\partial x} = -E\omega\theta'''' = \frac{T_w}{I_w} \omega
\]

Let \( \phi(y,z) \) be a stress function such that

\[
\tau_{xy} = \frac{\partial \phi}{\partial y} \quad \text{and} \quad \tau_{xz} = \frac{\partial \phi}{\partial z}
\]
Then, the governing equation is obtained as

\[
\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = \frac{T_w}{I_w} \omega \rightarrow \nabla^2 \phi - \frac{T_w}{I_w} \omega = 0
\]

The Cauchy’s relation can be written on the boundaries as

\[
\tau_{xy} n_y + \tau_{xz} n_z = 0 \rightarrow \nabla \phi \cdot \mathbf{n} = 0
\]

This boundary condition applies to both outer and inner boundaries. Then, the weak form is

\[
\int_A \mathbf{g}^T (\nabla^2 \phi - \frac{T_w}{I_w} \omega) \, dA = 0 \rightarrow \int_A \mathbf{g}^T \nabla^2 \phi \, dA - \int_A \mathbf{g}^T \frac{T_w}{I_w} \omega \, dA = 0
\]

Using the Green-Gauss theorem and imposing the boundary conditions

\[
\int_A \mathbf{g}^T \nabla \phi \cdot \mathbf{n} \, dS - \int_A \nabla \mathbf{g}^T \cdot \nabla \phi \, dA = \int_A \nabla \mathbf{g}^T \cdot \nabla \phi \, dA
\]

Then, the weak form is

\[
\int_A \nabla \mathbf{g}^T \cdot \nabla \phi \, dA = -\frac{T_w}{I_w} \int_A \mathbf{g}^T \omega \, dA = 0
\]

**Finite Element Formulation**

Assume the stress function within an element as

\[
\phi = \mathbf{Pq} = [\xi_1 \ \xi_2 \ \xi_3] [\phi_1 \ \phi_2 \ \phi_3]
\]

Then,

\[
\nabla \phi = \nabla \mathbf{Pq} = \frac{1}{2A} \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} = \mathbf{Bq}
\]

Recall that \( \nabla \mathbf{g} = \nabla \mathbf{P} = \mathbf{B} \). Then, the weak form can be written for a finite element as

\[
\int_A \mathbf{B}^T \cdot \nabla \phi \, dA = -\frac{T_w}{I_w} \int_A \mathbf{g}^T \omega \, dA = 0
\]
\[
\int_{\alpha} B^T B q dA = -\frac{T_w}{I_w} \int_{\alpha} P^T \omega dA
\]

Then, the element level equilibrium equations are \( kq = f \)

\[
k = \int_{\alpha} B^T B dA = \frac{1}{4\alpha} \begin{bmatrix}
(a_1 a_1 + b_1 b_1) & (a_1 a_2 + b_1 b_2) & (a_1 a_3 + b_1 b_3) \\
(a_2 a_1 + b_2 b_1) & (a_2 a_2 + b_2 b_2) & (a_2 a_3 + b_2 b_3) \\
(a_3 a_1 + b_3 b_1) & (a_3 a_2 + b_3 b_2) & (a_3 a_3 + b_3 b_3)
\end{bmatrix}
\]

(3.26)

\[
f = -\frac{T_w}{I_w} \int_{\alpha} P^T \omega dA = \frac{\alpha T_w}{12 I_w} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}
\]

(3.27)

The shear stresses in an element due to warping torque \( T_w \) can be calculated as

\[
\begin{bmatrix} \tau_{xy} \\ \tau_{xz} \end{bmatrix} = \nabla \phi = B q = \frac{1}{2\alpha} \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}
\]

(3.28)
CHAPTER 4

NUMERICAL EXAMPLES

4.1. Solid Sections

4.1.1. Rail Section

An A100 Crane Rail is a type of crane used for railroads. It can be exposed to different loadings such as transversal and torsional forces. In this example, the rail section is subjected to the shear force in the negative z-direction. The problem is compared with the German code DIN 536 Design Specification [28], and the article “Shear Stresses in Prismatic Beams with Arbitrary Cross–Sections” [20]. The cross-section properties are matched with said German design specifications [28], and the warping function, shear stress, and shear flows are checked with the aforementioned article by F. Gruttmann, R. Sauer, and W. Wagner [20]. In Figure 4.1, the loading type and meshed geometry are shown. The cross section is discretized by 1791 elements. The geometrical data for an A100 Crane Rail is given in Appendix A.1.

![Figure 4.1: Loading Type and Meshed Geometry for the Rail Section](image)

In Table 4.1, the warping constant among cross-section properties is compared with the article [20] while other parameters are tested with the DIN 536 Design Specifications [28] since the warping constant is not available in the said article.
Table 4.1: Comparison of Cross-section Properties for the Rail Section

<table>
<thead>
<tr>
<th>Property Name</th>
<th>Present Study</th>
<th>Article and DIN 536</th>
<th>Error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Area (mm²)</td>
<td>0.95013E+04</td>
<td>0.94700E+04</td>
<td>0.33</td>
</tr>
<tr>
<td>Centroid, y_c (mm)</td>
<td>0.00000E+00</td>
<td>0.00000E+00</td>
<td>0.00</td>
</tr>
<tr>
<td>Centroid, z_c (mm)</td>
<td>0.42477E+02</td>
<td>0.42100E+02</td>
<td>0.89</td>
</tr>
<tr>
<td>Shear Center, y_s (mm)</td>
<td>0.00000E+00</td>
<td>0.00000E+00</td>
<td>0.00</td>
</tr>
<tr>
<td>Shear Center, z_s (mm)</td>
<td>-0.98812E+01</td>
<td>-0.98000E+01</td>
<td>0.82</td>
</tr>
<tr>
<td>Inertia, I_{yy} (mm²)</td>
<td>0.86852E+07</td>
<td>0.85600E+07</td>
<td>1.44</td>
</tr>
<tr>
<td>Inertia, I_{zz} (mm²)</td>
<td>0.13378E+08</td>
<td>0.13450E+08</td>
<td>0.54</td>
</tr>
<tr>
<td>Torsional Constant (mm⁴)</td>
<td>0.67555E+07</td>
<td>0.67070E+07</td>
<td>0.72</td>
</tr>
<tr>
<td>Warping Constant (mm⁶)</td>
<td>0.39299E+10</td>
<td>0.39940E+10</td>
<td>1.63</td>
</tr>
</tbody>
</table>

Having done the comparison of the cross-section parameters, the rail is analyzed under shear force. In Figure 4.2, the distribution of the shear flows is shown. They have higher intensity in the middle corner regions. According to Figure 4.3, the article [20] obtains the maximum shear stress value as -0.41 MPa. Likewise, the study conducted herein resulted in -0.39 MPa. As seen in the figure, the upper and lower zones are stress-free.

![Figure 4.2: Comparison of Shear Flows for the Rail Section](image-url)
Lastly, the warping function is plotted in Figure 4.4. At its maximum value, the article [20] obtains 1888 mm$^2$, and the study conducted for this thesis results in 1889 mm$^2$. As seen in the figure, the cross-section warps at the corner zones.

In this example, the obtained results show that cross-section properties, the shear flow diagram, shear stress values, and warping functions are consistent with those in the article [20] and the design specification document [28].
4.1.2. Elliptical Section

In Timoshenko and Goodier’s book [11], an elliptical cross-section under a torsional moment is solved analytically. In this problem, solved herein, results for torsional shear stresses and warping functions are compared to the values in this book. The analytical formulations in Timoshenko and Goodier’s book are given below. Respectively, \( a \) and \( b \) denote the long radius and short radius. \( M_t \) stands for applied torque in the problem. These parameters are shown in Figure 4.5.

The torsional shear stresses and warping function in the book [11] are formulated as:

\[
\tau_{xy} = -\frac{2M_t z}{\pi ab^3} \quad \tau_{xz} = \frac{2M_t y}{\pi a^3 b} \quad \omega = \frac{(b^2 - a^2)yz}{(b^2 + a^2)}
\]  

Following the analytical formulations, the finite element model is investigated. The cross-section is discretized by 2966 elements in Figure 4.5. To illustrate the inputs clearly, the geometrical data are given in Table 4.2. Based on these data, the shear stresses and warping function are solved by the formulations in Timoshenko’s book and the FEM Study.
Table 4.2: Input Values for the Elliptical Section

<table>
<thead>
<tr>
<th>Symbols</th>
<th>Inputs</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_t$</td>
<td>Torsional Moment (N.mm)</td>
<td>10000000</td>
</tr>
<tr>
<td>G</td>
<td>Shear Modulus (MPa)</td>
<td>28000</td>
</tr>
<tr>
<td>b</td>
<td>Short Radius (mm)</td>
<td>30</td>
</tr>
<tr>
<td>a</td>
<td>Long Radius (mm)</td>
<td>50</td>
</tr>
</tbody>
</table>

In Figure 4.6 and Table 4.3, the shear stress distribution and maximum shear stress values are illustrated, respectively.

![Figure 4.6: Shear Stress Distribution over the Elliptical Section (MPa)](image)

Table 4.3: Comparison of the Shear Stresses over the Elliptical Section

<table>
<thead>
<tr>
<th>Stress Component</th>
<th>Location (y,z)</th>
<th>The Book [11] (MPa)</th>
<th>Present Study (MPa)</th>
<th>Error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau_{xy}$</td>
<td>(0, b)</td>
<td>-141.471</td>
<td>-139.941</td>
<td>1.08</td>
</tr>
<tr>
<td>$\tau_{xy}$</td>
<td>(0, -b)</td>
<td>141.471</td>
<td>140.047</td>
<td>1.01</td>
</tr>
<tr>
<td>$\tau_{xz}$</td>
<td>(a, 0)</td>
<td>-84.882</td>
<td>-84.304</td>
<td>0.68</td>
</tr>
<tr>
<td>$\tau_{xz}$</td>
<td>(-a, 0)</td>
<td>84.882</td>
<td>84.243</td>
<td>0.75</td>
</tr>
</tbody>
</table>
In this problem, a convergence test is performed to get the optimum stress results by increasing the number of mesh elements. In Figure 4.7, the variations of the shear stresses in the y and z directions are shown, respectively.

![Figure 4.7: Variations of the Stresses in terms of the Number of Elements (MPa)](image)

In Figure 4.8, the warping functions have the absolute maximum values at the corner regions. These values range between 352.941 mm$^2$ and -352.941 mm$^2$. As shown in Table 4.4, the errors found for maximum and minimum warping values are 0.11% and 0.13%, respectively.

![Figure 4.8: Comparison for the Warping Functions over the Elliptical Section (mm$^2$)](image)

<table>
<thead>
<tr>
<th></th>
<th>The Book [11] (mm$^2$)</th>
<th>Present Study (mm$^2$)</th>
<th>Error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximum Warping Value</td>
<td>352.941</td>
<td>352.551</td>
<td>0.11</td>
</tr>
<tr>
<td>Minimum Warping Value</td>
<td>-352.941</td>
<td>-352.498</td>
<td>0.13</td>
</tr>
</tbody>
</table>

Lastly, Figure 4.8 shows that warping functions have the absolute maximum values at the corner regions. These values range between 352.941 mm$^2$ and -352.941 mm$^2$. As shown in Table 4.4, the errors found for maximum and minimum warping values are 0.11% and 0.13%, respectively.
4.1.3. Variable Section

In this example, a beam having a variable cross-section is investigated under a distributed torsional moment. The dimensions of the cross-section are presented in Figure 4.9. This example was previously solved by Sapountzakis using the Boundary Element Method described in the author’s article [26]. In this article, torsional and warping rigidities are obtained for the chosen sections. Moreover, the variations of normal and shear stresses along the beam due to Saint Venant torque, warping torque and warping moment are shown. The material properties are given in Table 4.5.

Figure 4.9: Geometrical Dimensions for the Beam of Variable Cross-section [26]
Table 4.5: Material Properties for the Beam of Variable Cross-section

<table>
<thead>
<tr>
<th>Symbols</th>
<th>Name</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E$</td>
<td>Elastic Modulus (MPa)</td>
<td>30000</td>
</tr>
<tr>
<td>$G$</td>
<td>Shear Modulus (MPa)</td>
<td>12500</td>
</tr>
<tr>
<td>$\nu$</td>
<td>Possion Ratio</td>
<td>0.20</td>
</tr>
</tbody>
</table>

Firstly, the comparisons for torsional rigidity and warping rigidity are made with Sapountzakis’s article [26]. In its formulized form, the torsional rigidity is defined as $G.J$; whereas, the warping rigidity is identified as $E.I.\nu$. While the torsional rigidities are checked in Table 4.6, the warping rigidities are matched in Table 4.7.

![Figure 4.10: Cross-sections located at Fixed End 1 (left), Middle, Fixed End 2 (right)](image)

In this example, the constraint at the larger cross-section is called fixed end 1 while the restriction at the smaller cross-section is referred to as fixed end 2. According to Table 4.6 and Table 4.7, there is a minor difference between BEM solution and FEM solution.
Then, internal forces experienced along the beam must be determined. For this purpose, the beam is divided into 10 elements and 11 nodes, as in Figure 4.11. Each element has a length of 800 mm.
In Table 4.8, loads such as the SV torque, warping torque, and warping moment are tabulated for each node. It is clear that the maximum warping loads are appeared at the fixed ends. Disparately, the SV torque does not exist at these locations. Moreover, the warping loads have relatively smaller values at middle nodes than at fixed nodes.

<table>
<thead>
<tr>
<th>Node Number</th>
<th>St. Venant Torque (N)</th>
<th>Warping Torque (N)</th>
<th>Warping Moment (N)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0000E+00</td>
<td>3.8622E+06</td>
<td>-1.2351E+09</td>
</tr>
<tr>
<td>2</td>
<td>3.3104E+06</td>
<td>2.5269E+05</td>
<td>3.7362E+06</td>
</tr>
<tr>
<td>3</td>
<td>2.7205E+06</td>
<td>9.7921E+03</td>
<td>5.8948E+07</td>
</tr>
<tr>
<td>4</td>
<td>1.9313E+06</td>
<td>-4.5165E+03</td>
<td>5.8022E+07</td>
</tr>
<tr>
<td>5</td>
<td>1.1302E+06</td>
<td>-5.3237E+03</td>
<td>5.3892E+07</td>
</tr>
<tr>
<td>6</td>
<td>3.2838E+05</td>
<td>-5.3941E+03</td>
<td>4.9559E+07</td>
</tr>
<tr>
<td>7</td>
<td>-4.7379E+05</td>
<td>-5.4000E+03</td>
<td>4.5201E+07</td>
</tr>
<tr>
<td>8</td>
<td>-1.2763E+06</td>
<td>-5.4104E+03</td>
<td>4.0834E+07</td>
</tr>
<tr>
<td>9</td>
<td>-2.0790E+06</td>
<td>-5.6644E+03</td>
<td>3.6389E+07</td>
</tr>
<tr>
<td>10</td>
<td>-2.8534E+06</td>
<td>-2.8539E+04</td>
<td>2.7270E+07</td>
</tr>
<tr>
<td>11</td>
<td>0.0000E+00</td>
<td>-2.7426E+06</td>
<td>-4.8216E+08</td>
</tr>
</tbody>
</table>

After determining the internal forces, a cross-sectional analysis is carried out for each section. First, the change of the shear stress along the beam for point A is investigated. Then, the variation of the normal stress over the beam for point B is examined. Additionally, the stress graphs are obtained for the maximum stress cases.
In Figure 4.12, the change of the shear stresses along the beam due to St. Venant torque and warping torque are illustrated for point A. The maximum stress arising from the warping torque is found as $-70.371$ kPa at fixed end 2. In Sapountzakis’s article this value is attained as $-73.774$ kPa. Moreover, the maximum value for the St. Venant torque is obtained as $-40.043$ kPa while the BEM solution results in $-43.332$ kPa. As such, the results found as a consequence of this thesis, as displayed in Figure 4.12, are significantly close to the results found by Sapountzakis. Besides, it is easily deduced from the graph in Figure 4.12 that the contribution of the shear stress due to warping torque should not be ignored.
In Figure 4.13, the absolute shear stress values for the warping torque are illustrated. At point A, the absolute value is observed as 73.774 kPa. Based on this figure, shear stress is not observed in the middle zone; this shear stress free zone is shaped like a circle.

Figure 4.13: The Absolute Resultants of the Warping Shear Stresses (kPa)

Figure 4.14: The Absolute Resultants of the St. Venant Shear Stresses (kPa)
In Figure 4.14, the absolute shear stress values for the St. Venant torque are displayed. At point A, the magnitude of the stress is stated as 43.33 kPa. Also, it is obvious that stress is concentrated at the innermost corners.

Figure 4.15: Variation of the Normal Stresses along the Beam at Point B

Lastly, the distribution of the normal stresses along the beam due to the warping moment is depicted in Figure 4.15 for point B. Based on this graph, the normal stress reaches the maximum value at fixed end 2, as it does for the warping shear stress. At this location, the normal stress is read as 154.667 kPa from the graph. In the BEM solution [26], it was found as 158.219 kPa. Furthermore, as it is seen in both Figure 4.12 and Figure 4.15, the effects of the warping decrease away from the fixed ends.
In Figure 4.16, the normal stress values due to warping moment are shown. At point B, the value is read as 154.667 kPa. Based on the figure, the maximum stress values occur at the outer corner regions.

Figure 4.16: Distribution of the Normal Stresses (kPa)
4.2. Thick-walled Sections

4.2.1. Hollow Square Section

Structural elements with hollow square cross-sections are commonly used in several engineering systems such as automotive chassis, truss and framed structures, mechanisms, and robot arms. The torsional behavior of a square having hollow tubing sections is an important design consideration in many mechanical systems. If this section has thin walls, the analytical solutions for the torsional constant and torsional shear stress can be found in various design handbooks [25]. On the other hand, in the case of thick-walled cross-section, the problem must be solved by numerical methods, such as finite element analysis. In Figure 4.17, the change of torsional constant is shown in terms of the thickness/length ratio to decide whether the cross-section is thin-walled or thick-walled.

![Figure 4.17: Hollow Square with Uniform Thickness](image)

Thin walled Theory formulations by Roark [29] are;

\[ J = t(a - t)^3 \]  
(4.3)

\[ \tau_{\text{max}} = \frac{T}{2t(a - t)^2} \]  
(4.4)
As seen in Figure 4.18, when the thickness/length ratio is higher than 0.075, the values of torsional constants found by thin walled theory and within this study differ by 5%. In addition to this, when the thickness/length ratio is 0.240, the torsional constant starts to decrease. In other words, the thin-walled theory is strictly not applicable after this ratio. Lamanusca and Saravanos [25], and this study reach the same values at each point.
Accordingly, in order to be able to evaluate a thick-walled problem the thickness/length ratio is selected as 0.250. Figure 4.19 depicts a hollow square with a 60 mm length and 15 mm thickness. The cross-section is discretized by 1211 elements. The applied load is \(5 \times 10^6\) Nmm. The torsional constant and maximum shear stress values are compared with the article “The Torsional Analysis of Bars with Hollow Square Cross-Sections” by Lamanusca and Saravanos [25]. Said authors generated an algebraic formulation for torsional stiffness and maximum shear stress.

In Figure 4.20, according to the results, the maximum shear stresses are obtained as 112.3 MPa. This value is found as 110.1 MPa in the aforementioned article [25].
4.3. Thin-walled Open Sections

4.3.1. Channel Section

Channel beams are widely used in many engineering applications such as building structures and aircraft frames. The example problem analyzed herein is compared to the thesis entitled “Finite Element Stress Analysis of Elastic Beams under Non-Uniform Torsion” [21]. The example was validated analytically by a similar example in Chen and Atsuta [9]. The geometrical data for the example problem is given in Figure 4.21.

![Channel Beam Problem under Torsional Moment](image)

Figure 4.21: Channel Beam Problem under Torsional Moment [21]

Firstly, the internal forces along the beam are calculated. These forces are the St. Venant Torque, warping torque, and warping moment. Then, the normal and shear stress values are obtained for the A-B-C-D-E-F section cuts. The comparisons of stresses are at $x = 0$, $x = l/10$, and $x = l$. 

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As can be seen from Figure 4.22, the angle of twist reaches a maximum value of 0.0302 rad. This value is the same as the aforementioned article [21].
Figure 4.24: Distribution of the Warping Torque

Figure 4.25: Distribution of the Warping Moment
As illustrated in Figure 4.2, Figure 4.2, and Figure 4.2, the maximum values of warping moment and warping torque are at the fixed end while the maximum value of Saint Venant Torque is obtained at the free end. Thus, the warping normal and shear stresses reach high values at the fixed point, but shear stresses due to Saint Venant torque have high values at the free end.

Figure 4.2: Stress Distribution for the Warping Moment at the fixed end (MPa)

Based on Figure 4.2, the magnitude of the normal stress peaks at the tip of the upper flange. The region at which the lower flange and the web intersects is stress-free. In Table 4.9, the stress values for section cuts A, C, E, and F are tabularized. These values are compatible.
Table 4.9: Comparison of the Warping Normal Stresses for the Channel Section

<table>
<thead>
<tr>
<th>Section</th>
<th>Location</th>
<th>Numerical (MPa)</th>
<th>Analytical (MPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A-A</td>
<td>at x = 0 mm (fixed end)</td>
<td>18.39</td>
<td>18.07</td>
</tr>
<tr>
<td>C-C</td>
<td></td>
<td>-12.98</td>
<td>-12.82</td>
</tr>
<tr>
<td>E-E</td>
<td></td>
<td>18.69</td>
<td>18.87</td>
</tr>
<tr>
<td>F-F</td>
<td></td>
<td>-58.99</td>
<td>-59.58</td>
</tr>
<tr>
<td>A-A</td>
<td>at x = 100 mm</td>
<td>10.41</td>
<td>9.92</td>
</tr>
<tr>
<td>C-C</td>
<td></td>
<td>-7.41</td>
<td>-7.03</td>
</tr>
<tr>
<td>E-E</td>
<td></td>
<td>10.19</td>
<td>10.36</td>
</tr>
<tr>
<td>F-F</td>
<td></td>
<td>-33.34</td>
<td>-32.70</td>
</tr>
<tr>
<td>A-A</td>
<td>at x = 1000 mm (free end)</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>C-C</td>
<td></td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>E-E</td>
<td></td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>F-F</td>
<td></td>
<td>0.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Figure 4.27: Stress Distribution for the St. Venant Torque at the free end (MPa)
Table 4.10: Comparison of the St. Venant Shear Stresses for Channel Section

<table>
<thead>
<tr>
<th>Section</th>
<th>Location</th>
<th>Numerical (MPa)</th>
<th>Analytical (MPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>B-B</td>
<td>at x = 0 mm</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>D-D</td>
<td>(fixed end)</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>B-B</td>
<td>at x = 100 mm</td>
<td>6.27</td>
<td>6.79</td>
</tr>
<tr>
<td>D-D</td>
<td></td>
<td>6.27</td>
<td>6.79</td>
</tr>
<tr>
<td>B-B</td>
<td>at x = 1000 mm</td>
<td>13.91</td>
<td>14.97</td>
</tr>
<tr>
<td>D-D</td>
<td>(free end)</td>
<td>13.91</td>
<td>14.97</td>
</tr>
</tbody>
</table>

Figure 4.28: Stress Distribution for the Warping Torque at fixed end (MPa)

Table 4.11: Comparison of the Warping Shear Stresses for the Channel Section

<table>
<thead>
<tr>
<th>Section</th>
<th>Location</th>
<th>Numerical (MPa)</th>
<th>Analytical (MPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>B-B</td>
<td>at x = 0 mm</td>
<td>1.32</td>
<td>1.27</td>
</tr>
<tr>
<td>D-D</td>
<td>(fixed end)</td>
<td>0.81</td>
<td>0.92</td>
</tr>
<tr>
<td>B-B</td>
<td>at x = 100 mm</td>
<td>0.65</td>
<td>0.70</td>
</tr>
<tr>
<td>D-D</td>
<td></td>
<td>0.59</td>
<td>0.51</td>
</tr>
<tr>
<td>B-B</td>
<td>at x = 1000 mm</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>D-D</td>
<td>(free end)</td>
<td>0.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>
The analysis results are consistent with the analytical solutions for St. Venant shear stresses, warping shear stresses, and warping normal stresses.

Figure 4.29: Shear Flows under St. Venant and Warping Torque

In Figure 4.29, the shear flows are indicated for shear stresses due to St. Venant and warping torques, respectively. In areas shown by red circles, stress concentration is observed. This undesired aspect is handled by filleting or chambering the radius.
4.4. Thin-walled Closed Sections

4.4.1. Multicell Section

Multicell sections are selected as thin-walled closed sections. This example is compared with a thin-walled analytical solution. The cross-section has outer dimensions of 60 mm x 100 mm, and a length of 300 mm. The wall thickness is 3 mm. In addition, both inner walls are centered according to the parallel walls. The material properties are given in Table 4.12.

Figure 4.30: 3D Representation of the Multicell Section under Applied Loadings

<table>
<thead>
<tr>
<th>Symbols</th>
<th>Name</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E$</td>
<td>Elastic Modulus (N.mm)</td>
<td>80000</td>
</tr>
<tr>
<td>$G$</td>
<td>Shear Modulus (MPa)</td>
<td>28000</td>
</tr>
<tr>
<td>$v$</td>
<td>Possion Ratio</td>
<td>0.25</td>
</tr>
</tbody>
</table>
In Figure 4.31, the cross section is represented as thin-walled by its midlines. The cross-section is discretized by 2314 elements.

Figure 4.31: Thin-walled Representation of the Multicell Section

In Figure 4.32, the numbering used in the thin-walled solution is shown. In this figure, $s$ denotes the distance from the starting point.

Figure 4.32: Numbering for the Thin-walled Solution
In Table 4.13, a comparison of the thin-walled solutions and those found in the present study is made. In Table 4.14, internal forces are obtained for the free end. Based on these forces, a cross-sectional analysis is performed to determine the normal and shear stresses.

### Table 4.13: Comparison of the Cross-section Properties for the Multicell Section

<table>
<thead>
<tr>
<th>Property Name</th>
<th>Thin-walled Solution</th>
<th>Present Study</th>
<th>Difference (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Centroid, ( y_c ) (mm)</td>
<td>51.344</td>
<td>51.208</td>
<td>0.26</td>
</tr>
<tr>
<td>Centroid, ( z_c ) (mm)</td>
<td>28.500</td>
<td>28.500</td>
<td>0.00</td>
</tr>
<tr>
<td>Shear Center, ( y_s ) (mm)</td>
<td>-2.844</td>
<td>-2.569</td>
<td>6.15</td>
</tr>
<tr>
<td>Shear Center, ( z_s ) (mm)</td>
<td>0.000</td>
<td>0.000</td>
<td>0.00</td>
</tr>
<tr>
<td>Area (mm²)</td>
<td>1.24100E+03</td>
<td>1.22250E+03</td>
<td>1.49</td>
</tr>
<tr>
<td>Inertia, ( I_{yy} ) (mm⁴)</td>
<td>6.11624E+05</td>
<td>6.06000E+05</td>
<td>0.92</td>
</tr>
<tr>
<td>Inertia, ( I_{yx} ) (mm⁴)</td>
<td>0.00000E+00</td>
<td>0.00000E+00</td>
<td>0.00</td>
</tr>
<tr>
<td>Inertia, ( I_{zz} ) (mm⁴)</td>
<td>1.36485E+06</td>
<td>1.35735E+06</td>
<td>0.55</td>
</tr>
<tr>
<td>Torsional Constant (mm⁶)</td>
<td>1.19103E+06</td>
<td>1.22202E+06</td>
<td>2.60</td>
</tr>
<tr>
<td>Warping Constant (mm⁶)</td>
<td>4.02331E+07</td>
<td>4.30505E+07</td>
<td>7.00</td>
</tr>
</tbody>
</table>

### Table 4.14: Internal Forces for the Multicell Section

<table>
<thead>
<tr>
<th>Internal Forces</th>
<th>Values (N)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N )</td>
<td>10000</td>
</tr>
<tr>
<td>( S_y )</td>
<td>0</td>
</tr>
<tr>
<td>( S_z )</td>
<td>-3000</td>
</tr>
<tr>
<td>( M_y )</td>
<td>285000</td>
</tr>
<tr>
<td>( M_z )</td>
<td>-456560</td>
</tr>
<tr>
<td>( T_s )</td>
<td>-506832</td>
</tr>
<tr>
<td>( T_w )</td>
<td>345226</td>
</tr>
<tr>
<td>( M_w )</td>
<td>3647112</td>
</tr>
</tbody>
</table>
In Figure 4.33, the distribution of the warping functions and warped section are visualized. In Table 4.15, the maximum and minimum warping function are compared to the analytical solutions for Point 1 and 5.

![Figure 4.33: Distribution of the Warping Functions for the Multicell Section (MPa)](image)

**Table 4.15: Comparison of the Warping Functions for the Multicell Section**

<table>
<thead>
<tr>
<th>Point</th>
<th>Thin-walled Solution (mm²)</th>
<th>Present Study (mm²)</th>
<th>Difference %</th>
</tr>
</thead>
<tbody>
<tr>
<td>Minimum Warping Function 1</td>
<td>-359.026</td>
<td>-365.704</td>
<td>1.83</td>
</tr>
<tr>
<td>Maximum Warping Function 5</td>
<td>359.026</td>
<td>365.483</td>
<td>1.77</td>
</tr>
</tbody>
</table>

In Figure 4.34, the distribution of the shear stresses due to $T_s$ and $T_w$ are illustrated, respectively. In Table 4.16, the shear stress values for $S_y$, $S_z$, $T_s$, and $T_w$ are matched with the analytical solutions for Point 2.

![Figure 4.34: Distribution of the Shear Stresses of the Multicell Section (MPa)](image)
Table 4.16: Comparison of the Shear Stresses of the Multicell Section

<table>
<thead>
<tr>
<th>Forces</th>
<th>Point</th>
<th>Thin-walled Solution (MPa)</th>
<th>Present Study (MPa)</th>
<th>Difference %</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_y, S_z$</td>
<td>2</td>
<td>-2.669</td>
<td>-2.517</td>
<td>5.70</td>
</tr>
<tr>
<td>$T_s$</td>
<td>2</td>
<td>-15.278</td>
<td>-15.312</td>
<td>0.22</td>
</tr>
<tr>
<td>$T_w$</td>
<td>2</td>
<td>-54.894</td>
<td>-53.993</td>
<td>1.64</td>
</tr>
</tbody>
</table>

In Figure 4.35, the distribution of the normal stresses due to $N$, $M_y$, $M_z$ and $M_w$ are shown, respectively. In Table 4.17, the normal stress values for $N$, $M_y$, $M_z$ and $M_w$ are checked with the analytical solutions for Point 8.

![Figure 4.35: Distribution of the Normal Stresses of the Multicell Section (MPa)](image)

Table 4.17: Comparison of the Normal Stresses of the Multicell Section

<table>
<thead>
<tr>
<th>Forces</th>
<th>Point</th>
<th>Thin-walled Solution (MPa)</th>
<th>Present Study (MPa)</th>
<th>Difference %</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N, M_y, M_z$</td>
<td>8</td>
<td>36.611</td>
<td>36.505</td>
<td>0.29</td>
</tr>
<tr>
<td>$M_w$</td>
<td>8</td>
<td>32.981</td>
<td>31.695</td>
<td>3.90</td>
</tr>
</tbody>
</table>

In this example, the cross-section properties, warping functions, normal stresses, and shear stresses are compared with the thin-walled analytical solutions. All results are appropriately similar.
CHAPTER 5

CONCLUSIONS

5.1. Summary

The proposed finite element algorithm allows for conducting stress analyses of elastic beams with arbitrary cross-sections subjected to combined loadings. This study has consisted of five chapters, namely the Introduction, Literature Review, Formulations, Numerical Examples, and this chapter entitled, Conclusions. In Chapter 1, beams as a structural element are identified. The importance of cross-section properties and the uniform and non-uniform torsion theories are clarified. Moreover, the purpose and content of the thesis are expressed. In Chapter 2, related studies are offered in the form of a literature review by focusing on previous works. This chapter also includes the analytical and numerical approaches used to perform cross-sectional analyses. In Chapter 3, the formulations are given for the determination of internal forces and stresses. In the first part, equivalent loads are found at any cross-section along the beam. In the second part, 2D analysis is performed to obtain cross-section properties, warping functions, normal stresses, and shear stresses. In Chapter 4, beams with different cross-section types are analyzed to demonstrate the range of the applications of this study.

5.2. Conclusions

The results obtained in the examples affirm the developed finite element algorithm. All results are in a good agreement. In some problems, there are slight differences between analytical solutions and numerical solutions due to the number of elements and stress concentrations. These differences can be reduced by increasing the number of elements, and/or filleting or chamfering the sharp corners. The developed
formulations can aid mechanical engineers in regards to stress analysis of cross-sections while taking into consideration the warping effects.

5.3. Future Works

The effects of large axial forces on bending and torsion have not been covered in the present study. These effects can be included in future works.
REFERENCES


APPENDICES

A.1. DIN 536 Design Specification

Crane rails
Hot rolled flat bottom crane rails (type A)
Dimensions, section parameters and steel grades


In keeping with current practice in standards published by the International Organization for Standardization (ISO), a comma has been used throughout as the decimal marker.

Dimensions in mm

1 Scope and field of application

This standard specifies requirements for hot rolled crane rails with a flat bottom (type A), with the dimensions specified in table 1 and made from steel with the properties specified in clause 4.

2 Dimensions and designation

2.1 Designation

![Diagram of crane rail]

Figure 1: Type A crane rail

Designation of a flat bottom crane rail (type A) complying with this standard, with a head width, k, of 100 mm (A 100) and made from steel with a minimum tensile strength of 800 N/mm².

Crane rail DIN 536 - A 100 - 890

Continued on pages 2 to 5.
2.2 Dimensions and tolerances

2.2.1 Crane rails shall be of the sizes and be subject to the limit deviations and geometrical tolerances specified in table 1. Any values for which no tolerance is specified shall be regarded as approximate values.

<table>
<thead>
<tr>
<th>Crane rail symbol</th>
<th>$k$</th>
<th>$b_1$</th>
<th>$b_2$</th>
<th>$b_3$</th>
<th>$f_1$</th>
<th>$f_2$</th>
<th>$f_3$</th>
<th>$h_{11}$</th>
<th>$h_2$</th>
<th>$r_1$</th>
<th>$r_2$</th>
<th>$r_3$</th>
<th>$r_4$</th>
<th>$r_5$</th>
<th>$r_6$</th>
<th>$r_1^1$</th>
<th>$r_2^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A 45</td>
<td>45</td>
<td>±0.6</td>
<td>125</td>
<td>±1.5</td>
<td>54</td>
<td>24</td>
<td>14,5</td>
<td>11</td>
<td>8</td>
<td>55</td>
<td>±1</td>
<td>24</td>
<td>20</td>
<td>4</td>
<td>400</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>A 55</td>
<td>55</td>
<td>±0.6</td>
<td>150</td>
<td>±0.5</td>
<td>66</td>
<td>31</td>
<td>17,5</td>
<td>12,5</td>
<td>9</td>
<td>65</td>
<td>±1</td>
<td>28,5</td>
<td>25</td>
<td>5</td>
<td>400</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>A 65</td>
<td>65</td>
<td>±0.8</td>
<td>175</td>
<td>±1.5</td>
<td>78</td>
<td>38</td>
<td>20</td>
<td>14</td>
<td>10</td>
<td>75</td>
<td>±1</td>
<td>34</td>
<td>30</td>
<td>6</td>
<td>400</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>A 75</td>
<td>75</td>
<td>±0.8</td>
<td>200</td>
<td>±0.5</td>
<td>90</td>
<td>45</td>
<td>22</td>
<td>15.4</td>
<td>11</td>
<td>85</td>
<td>±1</td>
<td>39.5</td>
<td>35</td>
<td>8</td>
<td>500</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>A 100</td>
<td>100</td>
<td>±1</td>
<td>200</td>
<td>±0.5</td>
<td>100</td>
<td>60</td>
<td>23</td>
<td>16.5</td>
<td>12</td>
<td>95</td>
<td>±1.5</td>
<td>46.5</td>
<td>40</td>
<td>10</td>
<td>500</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>A 120</td>
<td>120</td>
<td>±1</td>
<td>220</td>
<td>±0.5</td>
<td>120</td>
<td>72</td>
<td>30</td>
<td>20</td>
<td>14</td>
<td>105</td>
<td>±1.5</td>
<td>55.5</td>
<td>47.5</td>
<td>10</td>
<td>600</td>
<td>6</td>
<td>10</td>
</tr>
<tr>
<td>A 150</td>
<td>150</td>
<td>±1</td>
<td>220</td>
<td>±0.5</td>
<td>30</td>
<td>15</td>
<td>120</td>
<td>64.5</td>
<td>50</td>
<td>10</td>
<td>800</td>
<td>10</td>
<td>30</td>
<td>3</td>
<td>242</td>
<td>6</td>
<td>31</td>
</tr>
</tbody>
</table>

1) Cf. subclauses 2.2.2 and 2.2.4. 2) Cf. subclauses 2.2.3 and 2.2.4. 3) Cf. Explanatory notes

3.2 The mass of crane rails, as a function of the section parameters, shall be as specified in table 2.

<table>
<thead>
<tr>
<th>Crane rail symbol</th>
<th>Mass, in kg/m</th>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$A_x$ cm$^2$</th>
<th>$A_y$ cm$^2$</th>
<th>$I_x$ cm$^4$</th>
<th>$I_y$ cm$^4$</th>
<th>$J_x$ cm$^4$</th>
<th>$J_y$ cm$^4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A 45</td>
<td>22.1</td>
<td>3.33</td>
<td>4.24</td>
<td>28.2</td>
<td>17.0</td>
<td>9.6</td>
<td>39</td>
<td>90</td>
<td>170</td>
</tr>
<tr>
<td>A 55</td>
<td>31.8</td>
<td>3.90</td>
<td>4.91</td>
<td>40.5</td>
<td>24.6</td>
<td>14.6</td>
<td>88</td>
<td>178</td>
<td>337</td>
</tr>
<tr>
<td>A 65</td>
<td>43.1</td>
<td>4.47</td>
<td>5.61</td>
<td>54.9</td>
<td>33.7</td>
<td>20.2</td>
<td>173</td>
<td>319</td>
<td>606</td>
</tr>
<tr>
<td>A 75</td>
<td>56.2</td>
<td>5.04</td>
<td>6.29</td>
<td>71.6</td>
<td>44.1</td>
<td>26.9</td>
<td>311</td>
<td>531</td>
<td>1011</td>
</tr>
<tr>
<td>A 100</td>
<td>74.3</td>
<td>5.29</td>
<td>6.27</td>
<td>94.7</td>
<td>65.8</td>
<td>41.6</td>
<td>666</td>
<td>855</td>
<td>1345</td>
</tr>
<tr>
<td>A 120</td>
<td>100.0</td>
<td>5.79</td>
<td>6.53</td>
<td>127.4</td>
<td>97.1</td>
<td>58.5</td>
<td>1302</td>
<td>1361</td>
<td>2350</td>
</tr>
<tr>
<td>A 150</td>
<td>150.3</td>
<td>7.73</td>
<td>8.48</td>
<td>191,4</td>
<td>153,6</td>
<td>107.1</td>
<td>2928</td>
<td>4373</td>
<td>3605</td>
</tr>
</tbody>
</table>

1) In accordance with DIN 1080 Parts 1 and 2, the symbols have the following meaning.

$A_x$: cross-sectional area
$A_y$: surfaces subjected to shear
$I_x$: second moment of area (torsion)
$I_y$: second moments of area (flexure)
$J_x$, $J_y$: static moments of parts of cross sections delineated by and related to the principal axes

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