HEDGING PERFORMANCE OF UTILITY INDIFFERENCE PRICING OF EUROPEAN CALL OPTIONS

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ABSTRACT

HEDGING PERFORMANCE OF UTILITY INDIFFERENCE PRICING OF EUROPEAN CALL OPTIONS

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Hedging performance of the Utility Indifference Pricing model presented by Davis et. al [European option pricing with transaction costs, SIAM J. Con. & Opt., 31(2), 1993] is studied in this thesis. Their indifference pricing approach is based on utility indifference of an investor towards portfolios with and without a short position in the European call option contract. The option price is defined as a difference of the minimum amount of initial endowments that make the maximum utilities from these portfolios equal to zero. Furthermore, Davis et al. considered an incomplete market where transaction costs are included. They worked with an exponential utility function which eliminates the dependence of investments in stocks to total wealth. This framework is adopted and hedging strategy is defined as a difference of two control variables that solves the utility maximization problems for the portfolios. Thus, finding the call option price embedded in utility maximization problems is studied via Optimal Control Theory. Markov Chain Approximation is utilized to compute the problem numerically and the option price is derived. Ending wealth from the portfolio consisting of short position in the option is measured. Hedging error is defined as the losses incurred in this portfolio. Furthermore, hedging performance is measured by computing the conditional expected value of losses as a percentage of the option price. Hedging performance is evaluated against different levels of transaction costs, degree of risk aversion, volatility and option moneyness. Our findings suggest that hedging performance measure is large when volatility and risk aversion rates are low, and when transaction costs are high. We also find that moneyness of option has

a decreasing effect on the hedging performance measure.

Keywords: Stochastic Optimal Control, Indifference Pricing, Transaction Costs, Markov Chain Approximation, Hedging.

ÖZ

AVRUPA TİPİ SATIN ALMA OPSİYONU FAYDA KAYITSIZLIĞI FİYATLAMASININ KORUNMA PERFORMANSI

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Bu tezde Davis v.d. [European option pricing with transaction costs, SIAM J. Con. & Opt., 31(2), 1993] tarafından ortaya konan Fayda Kayıtsızlığı Fiyatlama modelinin korunma performansı çalışılmıştır. Yazarların kayıtsızlık fiyatlaması yaklaşımı yatırımcının Avrupa tipi alım opsiyonu olan ve olmayan portföylere karşı olan fayda kayıtsızlığına dayanmaktadır. Opsiyon fiyatı bu portföylerden elde edilen maksimum faydaları sıfıra eşitleyen minimum başlangıç donanımı miktarı farkları olarak tanımlanmıştır. Davis v.d. işlem maliyetlerini kapsayan tamam olmayan bir piyasa gözetmiştir. Yazarlar hisse senedine yapılan yatırımların toplam varlığa olan bağımlılığını kaldıran üstel fayda fonksiyonu gözetmiştir. Bu çerçeve kabul edilmiş ve korunma stratejisi portföylerin fayda maksimizasyonu problemlerini çözen iki kontrol değişkenlerinin farkları olarak tanımlanmıştır. Bu nedenle fayda maksimizasyonu problemine gömülü alım opsiyonu fiyatı Optimal Kontrol Teorisi kullanılarak çalışılmıştır. Markov Zinciri Yaklaşımı problemi sayısal olarak çözmekte yararlanılmış ve opsiyon fiyatı elde edilmiştir. Opsiyonda kısa pozisyon içeren portföyün bitiş varlık durumu ölçülmüştür. Korunma hatası bu portföydeki gerçekleşen kayıplar olarak tanımlanmıştır. Buna ek olarak korunma performansı kayıpların şartlı beklenen değerinin opsiyon fiyatı yüzdesi cinsinden hesaplanması ile ölçülmüştür. Korunma performansı farklı büyüklüklerdeki işlem maliyeti, riskten kaçınma derecesi, volatilite ve opsiyon değerliliğine karşı ölçülmüştür. Bulgularımız korunma performansı ölçümünün volatilite ve riskten kaçınma derecesinin düşük olduğu durumlarda ve işlem maliyetinin yüksek olduğu durumlarda büyük olduğunu göstermiştir. Bunun yanı sıra opsiyon değerliliğinin korunma performansı ölçümü üstünde azaltıcı bir etkisi olduğu görülmüştür.

Anahtar Kelimeler: Stokastik Optimal Kontrol, Kayıtsızlık Fiyatlaması, İşlem Maliyetleri, Markov Zinciri Yaklaşımı, Korunma.

To My Family

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$\mathbb{R}$	Set of Real Numbers
$\mathbb{R}^+$	Set of Positive Real Numbers
$\mathbb{N}$	Set of Natural Numbers
$\mathbf{C}^n$	Set of Continuous n-Differentiable Functions
$(\Omega, \mathscr{F}, \mathbb{P})$	Sample Space
$\|\|X\|\|$	$[\mathbb{E}(\|X\|^2)]^{\frac{1}{2}}$
$\mathscr{B}(U)$	Borel σ-algebra of the set U
$\mathbb{E}(X) < \infty$	Integrable random variable
$\mathbb{E}^{t,x}(X_T)$	$\mathbb{E}(X_T \| X_t = x)$ for $t < T$
$x \wedge y$	$\min(x, y)$
$x \vee y$	$\max(x, y)$
x^+	$\max[x, 0]$
x^-	$\max[-x, 0]$
HJB	Hamilton-Jacobi-Bellman

CHAPTER 1

INTRODUCTION

Option pricing is one of the central questions in research in mathematical finance. Earliest study dates back to 1900 when Bachelier [1] derived a pricing formula for options by modelling the underlying stock prices with driftless Brownian motion. Samuelson [28] introduced the idea of modelling the stock prices using Geometric Brownian Motion in 1967. Another breakthrough took place when Black and Scholes published their article [5] in 1973. They adopted the Samuelson's stock price model and proposed a European option pricing formula which involved the solution of a parabolic differential equation, which could be reduced to the solution of the standard heat equation. A more comprehensive approach has been published later by Merton [25] where he introduced a riskless portfolio consisting of options and stocks which gains the return of the risk free asset. Their work has been a major success and the authors were awarded with Nobel Prize in 1997.

Black-Scholes-Merton model has several limitations: it assumes that financial markets markets are complete. That is, investors can create a replicating portfolio in which every contingent claim is "replicated" exactly by taking appropriate positions in stocks and bonds. Therefore, taking a position in option is completely redundant. However, this is not the case in real life, one of the reasons is the presence of transaction costs and under them continuous trading, which Black-Scholes requires for hedging the option becomes prohibitively costly. Therefore, replicating portfolios cannot be constructed and trading options always bear a risk. An extensive literature has been emerged that incorporates transaction costs in option pricing [[6], [7], [9], [12], [14], [18], [23], [26], [29], [30]]. The task of option pricing by constructing a replicating portfolio under transaction costs was first studied by Leland [23]. He considered a model which preserves Black-Scholes-Merton approach and implements proportional transaction costs. He developed a hedging strategy which involves rebalancing in discrete time intervals. However, optimality of the strategy was not considered. Hodges and Neuberger [18] were first to devise a optimal strategy using dynamic programming to maximize expected utility. They were not concerned with the exact replication of the option payoff due to high probability of ruin. Taking a position in a option contract involves risk and investors' risk preferences are considered by the authors. Instead of the replication argument, they incorporated utility indifference pricing notion to their model. Davis et al. [14] also takes this approach and derive the price of a call option as $B_w - B_1$ where B_w is the minimal amount of money to get 0 maximum utility from a portfolio having a call option and B_1 is the same amount for a portfolio not containing the option. They prove that the value functions of these control problems are the viscosity solutions to the Hamilton Jacobi Bellman (HJB) equation associated with the control problem. The numerical computation of the price in [14] is based on a Markov Chain approximation of the control problem. A review of indifference utility to option pricing is given in Chapter 4. The work [14] assumes that the stock price follows Black-Scholes dynamics (i.e., constant drift and volatility driven by Brownian motion), constant proportional transaction costs and constant interest rates. It deals with a European call option written on a single stock. Many works since [14] have extended this framework in several directions; some of these are reviewed in Chapter 2.

An interesting and important problem that is little studied in [14] (which focuses on the price of the option) and in the literature that followed is the hedging performance of the hedging algorithms implied by the utility indifference pricing framework. This is an important aspect of the problem because if the optimal controls implied by these models lead to large losses then the prices they imply may not be very meaningful. The main objective of this thesis is to study hedging performance of Indifference Pricing model suggested by Davis et al. [14]. The paper doesn't directly give a hedging algorithm associated with the pricing method. The main result in [14] concerning hedging is as follows: let π_w be the optimal trading strategy maximizing the utility of the portfolio containing the call option; if transaction costs are 0 and if there is a hedging strategy π_h replicating the option payoff perfectly, then $\pi_w - \pi_h$ is an optimal trading strategy for the portfolio without the option. This result and its proof, following [14], is reviewed in Chapter 4, see Theorem 4.1.1. The work [14] doesn't treat the problem of hedging when there are transaction costs. To come up with a hedging strategy we take inspiration from Theorem 4.1.1. This theorem implies that when perfect hedging is possible when the utility maximization problems have unique optimal controls $\pi_w - \pi_1$ hedges the option perfectly. This suggests the following hedging algorithm even when perfect hedging is not possible: compute the optimal controls for the two portfolio problems, use their difference as the hedging strategy. Neither of these control problems have explicit analytic solutions. A natural approach to approximate the optimal controls is Markov Chain approximation, this is also the approach taken in [14]. Chapter 3 is a review of both the continuous time optimal control and the Markov Chain approximation frameworks. Chapter 5 presents our main results: in this chapter we give the details of the application of the Markov Chain approximation framework to the computation of the value functions and the optimal controls in the indifference pricing framework and the hedging performance of the resulting hedging algorithms. In our computations we assume that the utility function is of the form $1 - e^{-\gamma x}$; this choice of the value function ensures that the optimal control is independent of the initial bond position in the portfolio, leading to a reduction of dimension. Let us briefly comment on the resulting optimal controls. As predicted in [14] the optimal control divides the state space of the problem into three connected regions: buy, no transaction, sell. The computation of the optimal control explicitly agrees with this prediction (see Figures 5.2, 5.5, 5.7 and 5.9). We see that as increase in transaction costs causes buy and sell regions to drift away from each other. Also, no transaction region widens [shrinks] for higher [lower] transaction costs 5.5. In contrast, increase in volatility 5.7 or in risk aversion rate 5.9 causes contraction of no transaction region. In the buy [sell] region, the optimal control consists of buying [selling] enough number of shares to push the state process to the boundary between no transaction and buy [sell]. To implement our hedging strategy we compute the optimal portfolios for both of the control problems and then take their difference. The difference between the buy-no transaction boundaries of the two control problems give an idea of what the hedging portfolio looks like, this difference is shown in Figure 5.3; we notice that this looks similar to the Δ of a call option in the Black-Scholes framework except for very large values of the price (the right side of the figure) where there is a sudden drop due to the constraining boundary added to the problem to make the numerical problem finite. Section 4.2 presents are main results on the performance of the resulting algorithms. We measure the performance of the resulting hedging algorithms by expected size of loss given that there is a loss, as a percentage of the option price; see the introduction of 4.2. Subsections 4.2.1, 4.2.2, 4.2.3 and 4.2.4 study the dependence of the hedging performance on transactions costs, risk aversion, volatility and moneyness of the option. Our main observations are as follows: Hedging error increases for higher transaction fees (see Figure 5.4) and decreases for higher risk aversion (see Figure 5.6) , volatility (see Figure 5.8) and stock price (see Figure 5.10). We provide further comments on our results and possible for future work in the Conclusion 6.

CHAPTER 2

LITERATURE REVIEW

Utility Indifference Pricing approach by Davis et al. [14] has been further developed by many works. Weak convergence properties of the discrete value functions have been stuided by Davis and Panas [12]. Pricing of the American options in similar setting was studied by Zakamulin [30]. He computed indifference price for both put and call options and studied the cases leading to early exercise. Monoyios [26] find bid-ask prices and compared his results against Leland's [23] policy. Asympthotic analysis of the model [14] was carried out by Whalley and Wilmott [29]. Instead of evaluating the discretised value function, they have reduced the problem to two dimensional inhomogenous diffusion equation. More complicated incomplete market adaptations applied to this model can be found on Caflisch et al. [6], where volatility is a stochastic process, and on Cantarutti et al. [7], where underlying price dynamics follow exponential Lévy process. Particularly, Cantarutti et al. [7] worked with diffusions as well as variance gamma processes and proposed an approximating chain. Cosso et al. [9] studied the problem of indifference pricing of American Options with stochastic volatility. To best of our knowledge, studies on hedging performance evaluation of the utility indifference model are scarce. We hope to contribute this field of research in this thesis.

CHAPTER 3

PRELIMINARIES

Models in the field of financial mathematics are extensively built on probabilistic foundations. Some essential concepts from Probability Theory, Stochastic Calculus and Optimal Control Theory are used throughout this thesis. This chapter is a review of results and ideas used from these fields in our thesis and is based on the following works [[3],[15],[16],[17],[20], [22], ,[21],[27]]. Section 3.1 provides the necessary background on Probability Theory while 3.2 briefly introduces Stochastic Optimal Control Theory. Discrete Approximations to continous time problems are introduced at 3.3. A convenient numerical method for the control problems depending on time is developed at 3.4.

3.1 Probabilistic Setup

Main objective of this section is to provide a background prior to introduction of controlled diffusion processes which will be used in later chapters. Proofs of the theorems are not provided in this thesis. All concepts introduced here follows the facts from Grimmet and Stirzaker [16], Karatzas and Shreve [20], Lamberton and Lapayre [22] and Labordere [17]. Reader can refer to these works for more comprehensive analysis.

In this section, we shall work with a sequence of random variables, $\{X_n\}_{n\in\mathbb{N}}$, on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Theorem 3.1.1 (**Monotone Convergence Theorem**). *Let $\{X_n\}_{n\in\mathbb{N}}$ be a monotone sequence and X be a random variable. If $X_n \uparrow X$ and $\mathbb{E}(X_0^-) < \infty$,then,*

$$\lim_{n\to\infty} \mathbb{E}(X_n) = \mathbb{E}(X) \tag{3.1}$$

Theorem 3.1.2 (**Dominated Convergence Theorem**). *Let $X_n \uparrow X$ and U be a integrable random variable satisfying $|X_n| \leq U$ for every $n \in \mathbb{N}$, then*

$$\mathbb{E}(\lim_{n\to\infty} X_n) = \lim_{n\to\infty} \mathbb{E}(X_n) \tag{3.2}$$

Definition 3.1.1 (**State Space**). Let $E \subseteq \mathbb{R}^n$ and $\mathscr{E} = \mathscr{B}(E)$. A state space $(E, \mathscr{E})$ is defined as the Euclidean space provided with Borel σ-algebra of E. The functions mapped to this space will be defined as states.

Definition 3.1.2 (Stochastic Processes). A collection of random variables $\{X_t\}_{t\in T}$ taking values in measurable state space $(E, \mathscr{E})$ are called stochastic process.

$$X : T \times \Omega \mapsto E \tag{3.3}$$

$$X(t, \omega) = X_t(\omega) \tag{3.4}$$

If a stochastic process is indexed by the set $T = \mathbb{N}$ then it is a discrete-time process. A continuous-time process would be indexed by $T = [0, \infty)$. Functions $t \mapsto X_t(\omega)$ for every fixed $\omega \in \Omega$ are called path,or trajectory. In other words, $X_t(\omega)$ denotes the result of an experiment ω at time t .

Definition 3.1.3 (Filtration). Filtration on the probability space is an increasing family of sub σ-algebras $\{\mathscr{F}_t\}_{t\geq 0}$ satisfying,

$$\mathscr{F}_s \subset \mathscr{F}_t \subset \mathscr{F} \; \forall \; s \leq t \tag{3.5}$$

If X_t is $\mathscr{F}_t$ measurable for every $t \geq 0$, then the process $\{X_t\}_{t\in T}$ is adapted to filtration $\{\mathscr{F}_t\}_{t\geq 0}$. Therefore, for each t, elements of the adapted stochastic process $\{X_t\}_{t\in T}$ does not have more information than $\mathscr{F}_t$. In addition,the filtration generated by stochastic process is called natural filtration defined as below,

$$\mathscr{F}_t^{X_t} = \sigma(X_s : s \leq t) \tag{3.6}$$

Every stochastic process is adapted to its natural filtration. Therefore, natural filtration is the smallest family of sigma algebra where the stochastic process $\{X_t\}_{t\in T}$ is adapted.

Definition 3.1.4 (Multi-Dimensional Stochastic Processes). Throughout this section, X_t is defined as multi-dimensional unless specified otherwise. Let $X_t = (X_t^1, ..., X_t^n)$ denote a n-dimensional vector valued stochastic process defined on the filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$. In addition, X_t^i is $\mathscr{F}_t$ measurable $\forall i \in [1, n]$.

Definition 3.1.5 (Wiener Processes). A stochastic process $\{B_t\}_{t\geq 0}$ is a Wiener Process if the following conditions hold;

1. $B_0 = 0$ almost surely
2. B_t is $\mathscr{F}_t$ measurable for every $t \in T$.
3. $\hat{\mathscr{F}}_u = \sigma(B_u - B_t : t \leq u)$ is independent of $\mathscr{F}_t$.
4. $t \mapsto B_t$ are continous almost surely. That is, the sample paths of $B_t \in \mathbf{C}[0, \infty)$
5. $B_u - B_t$ have normal distribution with mean 0 and variance $u - t$.

Wiener Processes are also called Standart Brownian Motion in the literature. $B_t = (B_t^1, ..., B_t^n)$ shall denote a Wiener Process that takes values in $\mathbb{R}^n$.

Definition 3.1.6 (**Martingales**). Let the stochastic process $X = \{X_t\}_{t\in T}$ be adapted to filtration and integrable. Then,

1. X is a Supermartingale if $\mathbb{E}[X_t|\mathscr{F}_s] \leq X_s \; \forall s \leq t, \;\; s,t \in T$
2. X is a Submartingale if $\mathbb{E}[X_t|\mathscr{F}_s] \geq X_s \; \forall s \leq t, \;\; s,t \in T$
3. X is a Martingale if $\mathbb{E}[X_t|\mathscr{F}_s] = X_s \; \forall s \leq t, \;\; s,t \in T$

Definition 3.1.7 (**Stopping Time**). A $\mathscr{F}_t$ measurable random variable $\tau : \Omega \mapsto [0,\infty) \cup \{\infty\}$ with following property is called stopping time :

$$\{\tau \leq t\} \in \mathscr{F}_t, \forall t \geq 0 \tag{3.7}$$

Definition 3.1.8 (**Hitting and Escape Times**). Boundary of a set is denoted by ∂. Let $\partial E \subset E$ and the stochastic process $\{X_t\}_{t\in T}$ is taking values in the state space (E, ξ). Then, a stopping time will be called as hitting time if the following holds,

$$\tau = \inf\{t \geq 0 : X_t \in \partial E\}. \tag{3.8}$$

Hitting times denote the first time when the process has reached the target set. We can define escape times as

$$\tau = \inf\{t \geq 0 : X_t \notin E\}. \tag{3.9}$$

Definition 3.1.9 (**Markov Processes**). A stochastic process $\{X_t\}_{t\in T}$ adapted to filtration $\{\mathscr{F}_t\}_{t\geq 0}$ is a Markov process if the following propery holds ;

$$\mathbb{E}\big[f(X_t)|\mathscr{F}_s\big] = \mathbb{E}\big[f(X_t)|X_s\big] \text{ such that } s \leq t. \tag{3.10}$$

where f is a bounded Borel function. This property defined as Markov Property suggesting that the future state of a random process does not depend on its history. In fact, the future state only depends on the current value. This property is coherent with Weak Form Efficiency of Markets [24] indicating that past price movements are irrelevant in forecasting the future when markets are efficient. Present price reflects all the past knowledge.

Definition 3.1.10 (**Markov Chain**). Let $E' \subset E$ denote a finite state space and $j, i_1, ..., i_{n-1}, i$ represent the states in E' for any $n \in \mathbb{N}$. A stochastic process $\{X_n\}_{n\in\mathbb{N}}$ taking values in E' is called Markov Chain if the following is satisfied,

$$P(i,j) = \mathbb{P}(X_{n+1} = j|X_n = i) = \mathbb{P}(X_{n+1} = j|X_0 = i_0, ..., X_{n-1} = i_{n-1}, X_n = i) \tag{3.11}$$

where $P(i,j)$ is called transition probability from state "i" to "j".

Definition 3.1.11 (**Diffusion Processes**). Diffusion processes are stochastic processes consisting of a deterministic drift function μ and random diffusion function σ. Drift and diffusion parts are defined as $\mu : T \times \mathbb{R}^n \to \mathbb{R}^n, \sigma : T \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ such that

$$\int_0^t |\mu(s, X_s)| ds \quad + \quad \int_0^t |\sigma(s, X_s)|^2 ds < \infty, \text{ almost surely} \tag{3.12}$$

$\mu(s,x) = (\mu^i(s,x))_{1\leq i\leq n}$ is a n-dimensional function and $\sigma(s,x) = (\sigma^{i,j}(s,x))_{1\leq i\leq n, 1\leq j\leq m}$ is a matrix. Multidimensional Diffusion process is given as,

$$X_t = X_0 + \int_0^t \mu(s,X_s)ds + \int_0^t \sigma(s,X_s)dB_s \tag{3.13}$$

as for i^{th} component,

$$X_t^i = X_0^i + \int_0^t \mu^i(s,X_s)ds + \sum_{j=1}^m \int_0^t \sigma^{i,j}(s,X_s)dB_s^j \tag{3.14}$$

Theorem 3.1.3 (Itô Formula). *Let X_t be a diffusion process and $f(\cdot,\cdot) \in \mathbf{C}^{1,2}([0,\infty)\times\mathbb{R}^n)$. Then $f(t,X_t)_{t\in T}$ can be expressed as,*

$$f(t,X_t) = f(0,X_0) + \int_0^t \frac{\partial f}{\partial t}(s,X_s)ds + \int_0^t \frac{\partial f}{\partial x}(s,X_s)dX_s + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s,X_s)d\langle X,X\rangle_s$$

$$\begin{aligned} f(t,X_t) = f(0,X_0) &+ \int_0^t \frac{\partial f}{\partial t}(s,X_s)ds + \sum_{i=1}^n \int_0^t \frac{\partial f}{\partial x_i}(s,X_s)dX_s^i \\ &+ \frac{1}{2}\sum_{i,j=1}^n \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(s,X_s)d\langle X^i,X^j\rangle_s \end{aligned} \tag{3.15}$$

Also this can be written as,

$$\begin{aligned} f(t,X_t) = f(0,X_0) &+ \int_0^t \frac{\partial f}{\partial t}(s,X_s)ds + \sum_{i=1}^n \int_0^t \frac{\partial f}{\partial x_i}(s,X_s)\mu^i(s,X_s)ds \\ &+ \sum_{i=1}^n \sum_{j=1}^m \int_0^t \frac{\partial f}{\partial x_i}(s,X_s)\sigma^{i,j}(s,X_s)dB_s^i \\ &+ \frac{1}{2}\sum_{i,j=1}^n \sum_{k=1}^m \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(s,X_s)\sigma^{i,k}(s,X_s)\sigma^{j,k}(s,X_s)ds \end{aligned} \tag{3.16}$$

Definition 3.1.12 (Infinitesimal Generator). Let X_t be a multidimensional diffusion process and $f \in \mathbf{C}^{1,2}(\mathbb{R}^n)$. Such diffusion process can be described by its infinitesimal generator as;

$$(\mathscr{L}^t f)(t,x) = \lim_{t\to 0} \frac{\mathbb{E}[f(t,X_t)] - f(t,x)}{t}$$

if the limit exists,

$$(\mathscr{L}^t f)(t,x) = \sum_{i=1}^n \mu^i(t,x)\frac{\partial f}{\partial x_i}(t,x) + \frac{1}{2}\sum_{i,j=1}^n \sum_{k=1}^m \sigma^{i,k}(t,x)\sigma^{j,k}(t,x)\frac{\partial^2 f}{\partial x_i \partial x_j}(t,x) \tag{3.17}$$

Infinitesimal generators characterizes the movement of X_t given an infinitesimal time interval.

Definition 3.1.13 (Stochastic Differential Equations). Let $\mu(\cdot,\cdot) : T \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma(\cdot,\cdot) : T \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ be measurable functions. Following type of equations are called stochastic differential equations;

$$dX_t = \mu(t,X_t)dt + \sigma(t,X_t)dB_t \tag{3.18}$$

$$X_0 = x \in \mathbb{R}^n \tag{3.19}$$

where 3.19 is the initial condition for SDE.

Definition 3.1.14 (Strong Solution). Let $t \in T$ where T is a finite set and K be a constant. In order for 3.18 to have a non exploding unique strong solution, following conditions must be satisfied,

1. Global Lipschitz Condition $||\mu(t,x)|| - ||\mu(t,y)|| + ||\sigma(t,x)|| - ||\sigma(t,y)|| \leq K||x-y||$ for $x, y \in \mathbb{R}^n$

2. Growth Condition : $||\mu(t,x)||^2 + ||\sigma(t,x)||^2 \leq K^2(1 + ||x||^2)$

3. Square Integrable Initial Condition :$\mathbb{E}[||X_0||^2] < \infty$

Let the conditions above are satisfied, and X_t is an unique strong solution of 3.18 relative to B_t. X_t is adapted to natural filtration $\mathscr{F}_t^{B_t, X_0}$ generated by X_0 and B_t. Furthermore,such unique solution of the differential equation satisfies,

$$\mathbb{E}\left[||X_t||^2\right] \leq C(1 + E\left[||X_0||^2\right])e^{Ct}$$

where C is constant. Uniqueness is defined in strong sense therein. That is, if X^1 and X_2 on $(\Omega, \mathscr{F}, \mathbb{P})$ are strong solutions to 3.18, then $P[X_t^1 = X_t^2; \forall 0 \leq t < \infty] = 1$ indicating both solutions are pathwise unique.

Strong solutions are produced on a given probability space, filtration and Brownian Motion. In other words, if B_t is input with initial datum X_0 then the output would be X_t satisfying 3.18. Usually strong solution conditions on the drift and diffusion coefficients are too restrictive for some market models. Therefore, the concept of weak solutions are introduced.

Definition 3.1.15 (Weak Solution). Let $\hat{\mathscr{F}}_t$ be a increasing family of σ-algebras that satisfies 3.12 and 3.18. X_t is weak solution of the 3.18 if and only if it has the following properties;

1. X_t is a $\hat{\mathscr{F}}_t$ adapted process.

2. B_t is a $\hat{\mathscr{F}}_t$ martingale Brownian Motion.

It should be noted that the weak uniqueness of the solutions is defined in the sense of probability law such that $\mathbb{P}^1(X^1 \in \Gamma) = \mathbb{P}^2(X^2 \in \Gamma)$ for $X^{i=1,2}$ on $(\Omega, \hat{\mathscr{F}}^i, \mathbb{P}^i), \forall \Gamma \in \mathscr{B}(\mathbb{R}^n)$.

We are only given the drift and the diffusion functions in the weak solution concept. Finding the Brownian motion and the process that satisfies 3.18 is part of the solution. Therefore, X_t does not have to be $\mathscr{F}_t^{B_t, X_0}$ adapted process anymore. Weak solutions are quite useful in the stochastic optimal control theory which will be discussed in the next section.

3.2 Stochastic Optimal Control Theory

We shall consider systems of variables that can exist in different states in different times. These systems are called dynamic systems and Optimal Control Theory mainly concerns with control of such systems and finding optimal policies for them. This section provides necessary tools used in the Stochastic Optimal Control Theory. All Definitions and Theorems are based on the books by Kushner & Dupuis [21], Phâm [27] and Fleming & Soner [15].

Definition 3.2.1 (Time Horizon). Dynamic systems are characterized by their states at the time . Time horizon of the system can be one of the following,

1. Finite Horizon: $t \in [0, T]$ for some positive finite number T.
2. Indefinite Horizon: $t \in [0, \tau]$ where τ is a stopping time.
3. Infinite Horizon: $t \in [0, \infty)$.

Definition 3.2.2 (Control Process). Let A be a closed subset of $\mathbb{R}^n$ and $\alpha_t = \alpha(t, \omega)$ where $\alpha : [0, T] \times \Omega \to A$. Dynamics of the system is influenced by progressively measurable process α_t called as control process. Control process is a collection of the decisions made at a certain time with the information available up to that time. Therefore, α_t should be adapted to $\{\mathscr{F}_t\}_{t \geq 0}$. The set of admissible controls is given as;

$$\mathbf{A} := \{\alpha(t, \omega) : [0, T] \times \Omega \to A \mid \alpha(t, \omega) \text{ is } \mathscr{F}_t \text{ measurable } \forall t \in [0, T]\} \tag{3.20}$$

Depending on the associated control problem, the set of admissible controls that we are interested in can be subject to more strict conditions.

Definition 3.2.3 (Controlled Diffusion Process). Let the continuous function μ defined as $\mu : [0, T] \times \mathbb{R}^n \times A \mapsto \mathbb{R}^n$ and σ defined as $\sigma : [0, T] \times \mathbb{R}^n \times A \mapsto \mathbb{R}^{n \times m}$ be measurable. Also, assume that $\alpha_t \in A$ and μ and σ satisfies,

$$\int_0^T |\mu(t, X_t, \alpha_t)| dt \quad + \quad \int_0^T |\sigma(t, X_t, \alpha_t)|^2 dt < \infty, \text{ almost surely} \tag{3.21}$$

State of the system is identified by the controlled process $X = \{X_t\}_{t \in [0,T]}$ and it is governed by the stochastic differential equation as shown below;

$$dX_t = \mu(t, X_t, \alpha_t) dt + \sigma(t, X_t, \alpha_t) dB_t \tag{3.22}$$

$$X_0 = x \in \mathbb{R}^n \tag{3.23}$$

where K is constant. Since α_t driving the movement of X_t is free to choose, these equations are also called controlled stochastic differential equations. In other words, solution of the 3.22depends on X_0 and α_t. Controlled stochastic differential equation has an unique solution if the following is satisfied,

$$||\mu(t, x, a) - \mu(t, y, a)|| + ||\sigma(t, x, a) - \sigma(t, y, a)|| \leq K||x - y|| \tag{3.24}$$

$$||\mu(t, x, a)||^2 + ||\sigma(t, x, a)||^2 \leq K^2(1 + ||x||^2 + ||a||^2) \tag{3.25}$$

$$\mathbb{E}[||X_0||^2] < \infty \tag{3.26}$$

where $a \in A$ and the unique solution $\hat{X}_t$ has following property;

$$\mathbb{E}\left[||\hat{X}_t||^2\right] \leq C(1 + E\left[||X_0||^2\right])e^{Ct} \tag{3.27}$$

Definition 3.2.4 (Performance Criterion). The ultimate objective of the stochastic optimal control is to find an optimal process which minimizes (maximises) the certain cost (gain) function. This function used as a criterion by the controller in order to evaluate the performance of the system quantitatively. The performance criterion can depend on the state of the system, the control process and the time if finite time horizon is considered. Therefore, it can be defined as either $J : [0,T] \times \mathbb{R}^n \times \mathbf{A}$ or $J : \mathbb{R}^n \times \mathbf{A}$.

Firstly, the time horizon can be finite and agent who seeks optimal policy may wish to control the system up to some time. Assume that the agent operates within $[t, T]$ and will not control after time T. Accordingly, the cost or gain dynamics of the system is characterized by $J(\cdot,\cdot,\cdot)$ as,

$$J(t, x, \alpha) = \mathbb{E}\left[\int_t^T F(s, X_s, \alpha_s)ds + g(X_T)|X_t = x\right] \tag{3.28}$$

Here we consider a controlled diffusion process $\{X_s\}_{s\in[t,T]}$ with the initial condition $X_t = x$. $F(\cdot,\cdot,\cdot)$ is running cost or gain from the system and $g(\cdot)$ is boundary gain or cost. Let the continous functions $F : [0,T] \times \mathbb{R}^n \times A \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ satisfy quadratic growth conditions given some constant K,

$$\begin{aligned} F(t, x, \alpha) &\leq K(1 + |x|^2 + |\alpha|^2) && (3.29) \\ g(x) &\leq K(1 + |x|^2) && (3.30) \end{aligned}$$

Also both $F(\cdot,\cdot,\cdot)$ and $g(\cdot)$ are integrable. Secondly,we consider the scenario where the agent seeks optimal policy in infinite time horizon, the performance criterion would be

$$J(x, \alpha) = \mathbb{E}\left[\int_t^\infty F(X_s, \alpha_s)e^{-\rho s}ds|X_t = x\right] \tag{3.31}$$

where $\rho > 0$. Associated discount factor ensures that $J(\cdot,\cdot)$ has a finite value.

For the rest of this section, we consider the case 3.28 where the controller wishes to maximize his gain from the system over a finite time horizon .

Definition 3.2.5 (Value Function). The problem of finding a maximum attainable gain from the system is identified by the value function defined as,

$$V(t, x) = \sup_{\alpha\in\mathbf{A}} J(t, x, \alpha) = \sup_{\alpha\in\mathbf{A}} \mathbb{E}\left[\int_t^T F(s, X_s, \alpha_s)ds + g(X_T)|X_t = x\right] \tag{3.32}$$

with the terminal condition $V(T, x) = g(X_T)$. Value functions are used to seek for the optimal performance. Therefore, we are looking for an optimal policy $\hat{\alpha}$ such that

$$V(t, x) = \sup_{\alpha\in\mathbf{A}} J(t, x, \alpha) = J(t, x, \hat{\alpha}) \tag{3.33}$$

Theorem 3.2.1 (Dynamic Programming Principle). *Let τ be stopping time taking values in $[t, T]$. Dynamic Programming Principle can be stated as,*

$$V(t,x) = \sup_{\alpha \in \mathbf{A}} \mathbb{E}^{t,x} \left[\int_t^\tau F(s, X_s, \alpha_s) ds + V(\tau, X_\tau) \right] \tag{3.34}$$

Dynamic Programming Principle or Principle of Optimality is defined by Bellman [3]as; Regardless of the initial conditions, controller must find a decision process which preserves the optimality throughout the time horizon of the problem. If a smooth value function and corresponding optimal control exists for the problem, we shall be able to describe its local behaviour in terms of some mechanics via so-called Hamilton-Jacobi-Bellman equation.

Theorem 3.2.2 (Hamilton-Jacobi-Bellman Equation). *Let $V \in \mathbf{C}^{1,2}([0,T) \times \mathbb{R}^n)$ and an optimal control satisfying 3.33 exists. Therefore, V satisfies the following SDE,*

$$0 = \sup_{a \in A} \left[F(t,x,a) + \frac{\partial V(t,x)}{\partial t} + \mu(t,x,a) \frac{\partial V(t,x)}{\partial x} + \frac{\sigma^2(t,x,a)}{2} \frac{\partial^2 V(t,x)}{\partial x^2} \right]$$

with the boundary condition,

$$V(T,x) = g(X_T)$$

Proof. Heuristic derivation of Hamilton Jacobi Bellman equation is given below,

$$V(t,x) = \sup_{\alpha \in \mathbf{A}} \mathbb{E}^{t,x} \left[\int_t^{t+dt} F(s, X_s, \alpha_s) ds + \int_{t+dt}^T F(s, X_s, \alpha_s) ds + g(X_T) \right]$$

$$V(t,x) = \sup_{\alpha \in \mathbf{A}} \mathbb{E}^{t,x} \left[\int_t^{t+dt} F(s, X_s, \alpha_s) ds + \underbrace{\sup_{\alpha \in \mathbf{A}} \mathbb{E}^{t+dt,x+dx} \left[\int_{t+dt}^T F(s, X_s, \alpha_s) ds + g(X_T) \right]}_{\text{by the Law of Iterated Expectations}} \right]$$

$$V(t,x) = \sup_{\alpha \in \mathbf{A}} \mathbb{E}^{t,x} \left[\int_t^{t+dt} F(s, X_s, \alpha_s) ds + \underbrace{V(t+dt, x+dx)}_{\text{Applying Itô Formula}} \right]$$

$$V(t+dt, x+dx) = V(t,x) + \int_t^{t+dt} \frac{\partial V(s,x)}{\partial t} ds + \frac{\partial V(s,x)}{\partial x} dx + \frac{1}{2} \frac{\partial^2 V(s,x)}{\partial x^2} dx^2$$

where x^2 is defined as,

$$dx^2 = \mu^2 \underbrace{d\langle t,t \rangle_s}_{0} + 2\mu\sigma \underbrace{d\langle t,B \rangle_s}_{0} + \sigma^2 \underbrace{d\langle B,B \rangle_s}_{dt} = \sigma^2 dt$$

Then, we get

$$\cancel{V(t,x)} = \sup_{\alpha \in \mathbf{A}} \mathbb{E}^{t,x} \left[\int_t^{t+dt} F(s, X_s, \alpha_s) ds + \cancel{V(t,x)} + \frac{\partial V(s,x)}{\partial t} ds + \frac{\partial V(s,x)}{\partial x} \mu(s, X_s, \alpha_s) ds \right.$$
$$\left. + \underbrace{\frac{\partial V(s,x)}{\partial x} \sigma dB_s}_{\mathbb{E}[dB_s]=0} + \frac{\partial^2 V(s,x)}{\partial x^2} \frac{\sigma^2}{2} (s, X_s, \alpha_s) ds \right]$$

If we divide the equation by dt and let $dt \to 0$, the following is derived,

$$0 = \sup_{a \in A} \left[F(t,x,a) + \frac{\partial V(t,x)}{\partial t} + \mu(t,x,a) \frac{\partial V(t,x)}{\partial x} + \frac{\sigma^2(t,x,a)}{2} \frac{\partial^2 V(t,x)}{\partial x^2} \right] \tag{3.35}$$

□

Definition 3.2.6 (Markov Control). If the admissible control process α_t is $\mathscr{F}_t^{B_t, X_t}$ measurable, then it is called closed loop or feedback control. A special case for the feedback controls are Markov controls which depend on the state and time. They are in the form defined as below,

$$\alpha_t = u(t, X_t) \tag{3.36}$$

for some measurable function $u : [0,T] \times \mathbb{R}^n \to A$.

Definition 3.2.7 (Hamiltonian Function). Let S^n denote the set of symmetric matrices. $H : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times S^n$ is called Hamiltonian of the associated control problem 3.32,

$$H(t,x,V_x,V_{xx}) = \sup_{a \in A} \left[F(t,x,a) + \mu(t,x,a) \frac{\partial V(t,x)}{\partial x} + \frac{\sigma^2(t,x,a)}{2} \frac{\partial^2 V(t,x)}{\partial x^2} \right] \tag{3.37}$$

Suppose smooth function $\vartheta(\cdot,\cdot)$ is a candidate solution to the optimization problem 3.32 and we can find an admissible control $\hat{\alpha}$. Next theorem shows that this candidate solution is actually the optimal solution and $\hat{\alpha}$ is the optimal Markov control.

Theorem 3.2.3 (Verification Theorem). *Let function ϑ be a class of $\mathbf{C}^{1,2}([0,T) \times \mathbb{R}^n)$ and satisfies the condition $|\vartheta(t,x)| \le K(1 + |x|^2)$ for all $(t,x) \in [0,T] \times \mathbb{R}^n$.*

1. *Assume that*

$$\frac{\partial \vartheta(t,x)}{\partial t} + \sup_{\alpha \in \mathbf{A}} \left[F(t,x,\alpha) + \mathscr{L}^\alpha \vartheta(t,x) \right] \le 0, (t,x) \in [0,T) \times \mathbb{R}^n$$

   and $\vartheta(T,x) \ge g(X_T)$. Then, $\vartheta(t,x) \ge V(t,x)$ on $[0,T] \times \mathbb{R}^n$.

2. *Assume for all $(t,x) \in [0,T] \times \mathbb{R}^n$ there exist a measurable process $\hat{\alpha}(t,x) \in \mathbf{A}$ with*

$$\begin{aligned} \frac{\partial \vartheta(t,x)}{\partial t} + \sup_{\alpha \in \mathbf{A}} \left[F(t,x,\alpha) + \mathscr{L}^\alpha \vartheta(t,x) \right] &= \frac{\partial \vartheta(t,x)}{\partial t} + F(t,x,\hat{\alpha}) + \mathscr{L}^{\hat{\alpha}} \vartheta(t,x) \\ &= 0, \end{aligned}$$

   where differential equation $dX_t = \mu(t, X_t, \hat{\alpha}_t)dt + \sigma(t, X_t, \hat{\alpha}_t)dB_t$ has an unique solution denoted by $\hat{X}_t$. Then, $\vartheta(t,x) = V(t,x)$ on $[0,T] \times \mathbb{R}^n$. Hence, ϑ coincides with the value function and $\hat{\alpha}$ is an optimal control.

For proof of this theorem, the steps from Phâm [27] can be followed. Verification theorem justifies the optimality of the control policy if the candidate solution is smooth and actually corresponds to the value function associated to the problem. However, this is not usually the case in stochastic control problems since it is difficult to find such smooth functions. Therefore, the solution should be in weak sense when V is a non-smooth function.

Definition 3.2.8 (Viscosity Solutions). Let S^n be the set of real valued symmetric $n \times n$ matrices and $\mathcal{O}$ is open set with $\mathcal{O} \subset \mathbb{R}^n$. Let $\hat{H} : [0, T] \times \mathcal{O} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times S^n \to \mathbb{R}$ be a continuous function defined as,

$$\begin{aligned}
\hat{H}(t, x, r, q, p, M) &= \frac{\partial V(t,x)}{\partial t} + \sup_{a \in A} \left[F(t, x, a) + \mu(t, x, a) \frac{\partial V(t,x)}{\partial x} + \frac{\sigma^2(t, x, a)}{2} \frac{\partial^2 V(t,x)}{\partial x^2} \right] \\
&= q + \sup_{a \in A} \left[F(t, x, a) + \mu(t, x, a) p + \frac{\sigma^2(t, x, a)}{2} M \right] \\
&= 0
\end{aligned} \tag{3.38}$$

Also, $\hat{H}$ has parabolicity and ellipticity property,

$$\begin{aligned}
\hat{H}(t, x, r, q, p, M) &\leq \hat{H}(t, x, r, q, p, \hat{M}), \ M \geq \hat{M} \\
\hat{H}(t, x, r, q, p, M) &\leq \hat{H}(t, x, r, \hat{q}, p, M), \ q \geq \hat{q}
\end{aligned} \tag{3.39}$$

where $\mathcal{V} : [0, T] \times \mathcal{O} \to \mathbb{R}$ is locally bounded function. Let $\mathcal{V}^*$ denote an upper semicontinous function and $\mathcal{V}_*$ denote a lower semicontinous function as below,

$$\begin{aligned}
\mathcal{V}^*(t, x) &= \limsup_{(t', x') \to (t, x)} \mathcal{V}(t', x') \\
\mathcal{V}_*(t, x) &= \liminf_{(t', x') \to (t, x)} \mathcal{V}(t', x')
\end{aligned}$$

$\mathcal{V}(\cdot, \cdot)$ is viscosity subsolution on $[0, T) \times \mathcal{O}$ if the following is satisfied,

$$\begin{aligned}
\hat{H}_*(t_0, x_0, \mathcal{V}_*(t_0, x_0), \phi_t(t_0, x_0), \phi_x(t_0, x_0), \phi_{xx}(t_0, x_0)) \leq 0 \ \forall (t_0, x_0) \in [0, T) \times \mathcal{O}, \\
\forall \phi \in \mathbf{C}^{1,2}([0, T] \times \mathcal{O})
\end{aligned} \tag{3.40}$$

where (t_0, x_0) is local maximum point of $(\mathcal{V}_* - \phi)(t_0, x_0)$. Likewise, $\mathcal{V}(\cdot, \cdot)$ is viscosity supersolution if the following is satisfied,

$$\begin{aligned}
\hat{H}^*(t_0, x_0, \mathcal{V}^*(t_0, x_0), \phi_t(t_0, x_0), \phi_x(t_0, x_0), \phi_{xx}(t_0, x_0)) \geq 0 \ \forall (t_0, x_0) \in [0, T) \times \mathcal{O}, \\
\forall \phi \in \mathbf{C}^{1,2}([0, T] \times \mathcal{O})
\end{aligned} \tag{3.41}$$

where (t_0, x_0) is local minimum point of $(\mathcal{V}^* - \phi)(t_0, x_0)$. $\mathcal{V}$ is a viscosity solution if it satisfies sub and super solution conditions on $[0, T) \times \mathcal{O}$.

Equivalence of weak solutions to viscosity solutions has been shown by Ishii [19]. Following theorem states that viscosity solution is in fact unique when values of the sub and super solutions coincide at the boundary.

Theorem 3.2.4 (Comparison Theorem). *Let $\mathcal{V}$ and $\mathcal{W}$ be sub and super viscosity solution to 3.38 on $(t, x) \in [O, T) \times \mathcal{O}$ such that $\mathcal{V}(T, x) \leq \mathcal{W}(T, x)$ holds for $x \in \mathcal{O}$. Then the following inequality holds for all $(t, x) \in [0, T) \times \mathcal{O}$,*

$$\mathcal{V}(t, x) \leq \mathcal{W}(t, x) \tag{3.42}$$

Viscosity solutions enable us to find a class of solutions for discontinous non-smooth value functions. Furthermore, one of the key features of the viscosity solutions is that unique solution can be guaranteed to an optimal control problem without assuming the strong solution conditions. Reader can refer to Crandall, Ishii and Lions [11] for more information on viscosity solutions.

3.3 Markov Chain Approximation for Optimal Control Problems

HJB equations usually have only formal meaning. Most of the stochastic control problems do not admit an explicit solution as they involve solving a non-linear partial differential equation. Markov Chain Approximation provides a reliable method for solving Stochastic Optimal Control Problems numerically. In this method, optimal control problem is approximated by a discretizated process. This process is a Markov Chain and resembles more to original diffusion process as discretization steps get smaller. Also, dynamic programming equation for the approximating process coincides with HJB equation of the original problem. Therefore, we can describe the value function by constructing sequence of Markov Chains without solving the nonlinear partial differential (HJB) equation. Basic concepts regarding Markov Chain Approximation are introduced in this section and terms are adapted from the book by Kushner and Dupuis [21].

3.3.1 Interpolation of the Diffusion Process

Let G be a compact set and $G^0 = G - \partial G$ denotes its interior. We shall consider a diffusion process and a control problem as below,

$$dx(t) = \mu\big(x(t), u(x(t))\big)dt + \sigma\big(x(t)\big)dB(t) \tag{3.43}$$

$$V(x) = \inf_{u\in\mathbf{A}} W(x,u) = \inf_{u\in\mathbf{A}} \mathbb{E}_x^u\Bigg[\int_0^\tau k(x(t),u(x(t)))dt + g(x(\tau))\Bigg] \tag{3.44}$$

where $\tau = \inf\{t : x_t \notin G^0\}$. Here $\mathbb{E}_x^u$ denotes the expectation that has controlled transition probabilities $\mathbb{P}(\xi_{n+1} = y|\xi_n = x, u_n = u) = p(x,y|u)$ where $u_n = u(\xi_n)$ is a feedback control. We shall establish an approximation to 3.43 and 3.44. Firstly, the terminology used in Markov Chain Approximation will be introduced.

Definition 3.3.1 (Finite State Space). Let $E_h = \{\pm h, \pm 2h, ..., Nh\}$ be a finite discrete state space with the parameter $h > 0$ and $\{\xi^h\}_{n\in\mathbb{N}}$ be a discrete time Markov process on E_h. Components of the state space is denoted by $G_h^0 = E_h \cap G^0$.

Definition 3.3.2 (Interpolated Processes and Interval). Let $\Delta\xi_n^h = \xi_{n+1}^h - \xi_n^h$ denote the difference and length of the time interval defined as $\Delta t_n^h = \Delta t^h(\xi_n^h, u_n^h)$ satisfies the following properties,

$$\lim_{h\to 0} \sup_{x,\alpha} \Delta t^h(x,\alpha) \to 0, \Delta t^h(x,\alpha) > 0 \tag{3.45}$$

Hence, interpolation of the processes ξ and u is given as,

$$\begin{aligned}
&\xi^h(t) = \xi^h_n, \\
&u^h(\xi^h(t)) = u^h(\xi^h_n) = u^h_n, \\
&t \in [t^h_n, t^h_{n+1})
\end{aligned} \tag{3.46}$$

where $t^h_n = \sum_{i=0}^{n-1} \Delta t^h_i$.

Definition 3.3.3 (Local Consistency). In order to establish meaningful approximation to the controlled diffusion, the discretized process must have local consistency. Local consistency conditions for ξ^h are given as below,

1. $\mathbb{E}^{h,\alpha}_{x,n}\left[\Delta\xi^h_n\right] \equiv \mu_h(x,\alpha)\Delta t^h(x,\alpha) = \mu(x,\alpha)\Delta t^h(x,\alpha) + o(\Delta t^h(x,\alpha))$.

2. $\mathbb{E}^{h,\alpha}_{x,n}\left[\left[\Delta\xi^h_n - \mathbb{E}^{h,\alpha}_{x,n}[\Delta\xi^h_n]\right]\left[\Delta\xi^h_n - \mathbb{E}^{h,\alpha}_{x,n}[\Delta\xi^h_n\right]'\right] \equiv a_h(x)\Delta t^h(x,\alpha) = a(x)\Delta t^h(x,\alpha) + o(\Delta t^h(x,\alpha))$ where $a(x) = \sigma(x)\sigma^T(x)$.

3. $\sup_n |\xi^h_{n+1} - \xi^h_n| \xrightarrow{h} 0$.

$\mathbb{E}^{h,\alpha}_{x,n}$ denotes expectation with $\mathbb{P}(\xi^h_{n+1} = y|\xi^h_n = x, u^h_n = \alpha) = p^h(x,y|\alpha)$. Let N be the first time when process ξ^h_n escapes G^0_h. We can now write down the approximation of 3.43 as,

$$W^h(x,u^h) = \mathbb{E}^{u^h}_x\left[\sum_{n=0}^{N-1} k(\xi^h_n, u^h_n)\Delta t^h(\xi^h_n, u^h_n) + g(\xi^h_N)\right] \tag{3.47}$$

Furthermore, the dynamic programming principle for the performance function is given as,

$$V^h(x) = \inf_{\alpha\in A}\left[\mathbb{E}^{h,\alpha}_{x,n}\left[V^h(\xi^h_{n+1})\right] + k(x,\alpha)\Delta t^h(x,\alpha)\right] \tag{3.48}$$

$$\begin{aligned}
&V^h(x) = \inf_{\alpha\in A}\left[\sum_y p^h(x,y|\alpha)V^h(y) + k(x,\alpha)\Delta t^h(x,\alpha)\right], x \in G^0_h \\
&V^h(x) = g(x), x \notin G^0_h
\end{aligned} \tag{3.49}$$

We conclude that $V^h(x) \to V(x)$ as h approaches 0. Reader can refer to Chapter 10 of [21] for the convergence proofs.

3.3.2 Finite Difference Approximation

In order to approximate the value function (or performance function) of an optimal control problem, one must get a locally consistent chain. Thus, time interval and transition probabilities must be determined. There are many different ways to do that, as described by [21].

Finite Difference type of approximations is one way to determine such variables. Although the term "Finite Difference" is usually understood as a method for solving differential equations numerically, it is not exactly the case here. It will merely serve as a tool for getting locally consistent chains which will be used to solve 3.49 recursively.

We shall now consider one dimensional problem where $G = [0, B]$. Boundary conditions are given as $W(x, u) = g(x)$ when $x = 0, B$. Therefore, $\tau = \{t : x(t) \notin (0, B)\}$ and performance function satisfies ,

$$(\mathscr{L}^{u(x)}W)(x, u) + k(x, u(x)) = 0 \tag{3.50}$$

$$W(0, u) = g(0),\ W(B, u) = g(B) \tag{3.51}$$

Finite Difference Approximation to the differentials of $W(x, \alpha)$ are given as,

$$W_x(x, \alpha) \to \frac{W^h(x + h, \alpha) - W^h(x, \alpha)}{h},\ \text{ when } \mu \geq 0 \tag{3.52}$$

$$W_x(x, \alpha) \to \frac{W^h(x, \alpha) - W^h(x - h, \alpha)}{h},\ \text{ when } \mu < 0 \tag{3.53}$$

$$W_{xx}(x, \alpha) \to \frac{W^h(x + h, \alpha) - 2W^h(x, \alpha) + W^h(x + h, \alpha)}{h^2} \tag{3.54}$$

When these are plugged into 3.50, we can write the following instead of the infinitesimal generator of the controlled process,

$$W_x(x, \alpha)\mu(x, \alpha) + W_{xx}(x, \alpha)\frac{\sigma^2(x)}{2} + k(x, \alpha) = 0 \tag{3.55}$$

$$\begin{aligned} &\frac{W^h(x + h, \alpha) - W^h(x, \alpha)}{h}\mu^+(x, \alpha) - \frac{W^h(x, \alpha) - W^h(x - h, \alpha)}{h}\mu^-(x, \alpha) \\ &+ \frac{W^h(x + h, \alpha) - 2W^h(x, \alpha) + W^h(x + h, \alpha)}{h^2}\frac{\sigma^2(x)}{2} + k(x, \alpha) = 0 \end{aligned} \tag{3.56}$$

where control parameter is fixed at α. It can be seen that when $W^h(x, \alpha)$ terms are collected, left hand side of the equation becomes

$$\begin{aligned} W^h(x, \alpha)\frac{\mu^+(x, \alpha)}{h} + W^h(x, \alpha)\frac{\mu^-(x, \alpha)}{h} + \frac{\sigma^2(x)}{h^2}W^h(x, \alpha) &= \text{R.H.S} \\ W^h(x, \alpha)\left(\frac{|\mu(x, \alpha)|}{h} + \frac{\sigma^2(x)}{h^2}\right) &= \text{R.H.S} \\ W^h(x, \alpha)\left(\frac{|\mu(x, \alpha)|h + \sigma^2(x)}{h^2}\right) &= \text{R.H.S} \end{aligned} \tag{3.57}$$

Furthermore, the equation turns into,

$$\begin{aligned} W^h(x, \alpha) = \frac{\sigma^2(x)/2 + h\mu^+(x, \alpha)}{|\mu(x, \alpha)|h + \sigma^2(x)}W^h(x + h, \alpha) + \frac{\sigma^2(x)/2 + h\mu^-(x, \alpha)}{|\mu(x, \alpha)|h + \sigma^2(x)}W^h(x - h, \alpha) \\ + \left(\frac{h^2}{|\mu(x, \alpha)|h + \sigma^2(x)}\right)k(x, \alpha). \end{aligned} \tag{3.58}$$

Therefore, transitional probabilities and interpolation interval are constructed as,

$$p^h(x, x+h|\alpha) = \frac{\sigma^2(x)/2 + h\mu^+(x,\alpha)}{|\mu(x,\alpha)|h + \sigma^2(x)} \tag{3.59}$$

$$p^h(x, x-h|\alpha) = \frac{\sigma^2(x)/2 + h\mu^-(x,\alpha)}{|\mu(x,\alpha)|h + \sigma^2(x)} \tag{3.60}$$

$$\Delta t^h(x,\alpha) = \frac{h^2}{|\mu(x,\alpha)|h + \sigma^2(x)} \tag{3.61}$$

where $p^h(x, y|\alpha) = 0$ if $y \neq x \pm h$. In conclusion, $\mathscr{L}^{u(x)}$ is approximated by $W^h(x, u)$ satisfying

$$W^h(x,u) = \sum_y p^h(x, y|u(x))W^h(y,u) + k(x, u(x))\Delta t^h(x, u(x)) \tag{3.62}$$

In addition, if 3.62 has a solution, $W^h(x, u)$ satisfies 3.47 and associated value function is in the form of 3.49 . Finally, local consistency conditions for 3.43 are checked as below,

$$\begin{aligned}\mathbb{E}^{h,\alpha}_{x,n}\Big[\Delta\xi^h_n\Big] &= h\frac{\sigma^2(x)/2 + h\mu^+(x,\alpha)}{|\mu(x,\alpha)|h + \sigma^2(x)} - h\frac{\sigma^2(x)/2 + h\mu^-(x,\alpha)}{|\mu(x,\alpha)|h + \sigma^2(x)} \\ &= \mu(x,\alpha)\Delta t^h(x,\alpha) \end{aligned}\tag{3.63}$$

$$\begin{aligned}\mathbb{E}^{h,\alpha}_{x,n}\Big[\Delta\xi^h_n - \mathbb{E}^{h,\alpha}_{x,n}\big[\Delta\xi^h_n\big]\Big]^2 &= h^2\bigg(\frac{\sigma^2(x)/2 + h\mu^+(x,\alpha)}{|\mu(x,\alpha)|h + \sigma^2(x)} + \frac{\sigma^2(x)/2 + h\mu^-(x,\alpha)}{|\mu(x,\alpha)|h + \sigma^2(x)}\bigg) \\ &= \sigma^2(x)\Delta t^h(x,\alpha) + o(\Delta t^h(x,\alpha))\end{aligned}\tag{3.64}$$

Thus, the chain approximation has been successfully built.

3.3.3 Computation by Iteration

Constructing a locally consistent process by finding transition probabilities has been the main objective in the previous sections. Having built a consistent chain, we shall focus on solving 3.49. There are two main approaches in solving such problem. It can either be approximation in policy space or in value space. Those methods are given by [21] in detail. We shall consider approximation in value space in this thesis. One simple way to do value space approximation is Jacobi Iteration.

Theorem 3.3.1 (Jacobi Iteration). *Let $u \in \mathbf{A}$ be a feedback control and W^h_n be the sequence given as*

$$W^h_{n+1}(x,u) = \sum_y p^h(x,y|u)W^h_n(y,u) + k(x,u)\Delta t^h(x,u) \tag{3.65}$$

for any initial value W_0. Then, $W^h_n \to W$ as $n \to \infty$. Similarly, V^h_n given as

$$V^h_{n+1}(x) = \inf_{\alpha\in A}\left[\sum_y p^h(x,y|\alpha)V^h_n(y) + k(x,\alpha)\Delta t^h(x,\alpha)\right] \tag{3.66}$$

converges to the value function of the optimal control problem for any V^h_0.

3.4 Explicit Approximation Method For Bounded Time Intervals

Unbounded time intervals have been considered in the preceding section. The focus was on the first time when $x(t)$ exits the domain and the control is terminated. However, the chain $x(t)$ might have not stopped at all as the time interval is not specifically defined [1]. Therefore, value of the time was not explicitly taken into consideration. One has to make small adjustments in the tools defined in 3.3 when time dependence is considered.

There are two main different methods of approximation when time horizon is finite. First one is explicit method where the system variables evaluated at the forward time t^h_{n+1} is used to calculate the value function at t^h_n. That is, the trasitions are only happenning in x while we take Δt steps in time. Also, corresponding discrete control problem is solved by backward iterations. Second one is implicit method where time is treated as an another state variable. Its value changes for the each step and the value function is computed for trasitions both in the state and the time. We shall use explicit method which is briefly introduced in this section.

$$W(t,x,u) = \mathbb{E}^u_x\Big[\int_t^T k(x(s),u(x,s))ds + g(x(T))\Big] \tag{3.67}$$

Analogous to previous arguments $W(t,x,u)$ satisfies the equation below,

$$\begin{aligned} &W_t(t,x,u) + (\mathscr{L}^{u(x,t)}W)(t,x,u) + k(x,u(x,t)) = 0 \\ &W(T,x,u) = g(x) \end{aligned} \tag{3.68}$$

We set $\Delta t = \delta$ to simply the notation. Time interval is $t \in \{0, \delta, 2\delta, ..., N\delta\}$ where $N = \frac{T}{\delta}$. Differential operator shall be approximated as,

$$W_t(t,x,\alpha) \to \frac{W^{h,\delta}(t+\delta,x,\alpha) - W^{h,\delta}(t,x,\alpha)}{\delta} \tag{3.69}$$

$$W_x(t,x,\alpha) \to \frac{W^{h,\delta}(t+\delta,x+h,\alpha) - W^{h,\delta}(t+\delta,x,\alpha)}{h}, \text{ when } \mu \geq 0 \tag{3.70}$$

$$W_x(t,x,\alpha) \to \frac{W^{h,\delta}(t+\delta,x,\alpha) - W^{h,\delta}(t+\delta,x-h,\alpha)}{h}, \text{ when } \mu < 0 \tag{3.71}$$

$$W_{xx}(t,x,\alpha) \to \frac{W^{h,\delta}(t+\delta,x+h,\alpha) - 2W^{h,\delta}(t+\delta,x,\alpha) + W^{h,\delta}(t+\delta,x+h,\alpha)}{h^2} \tag{3.72}$$

Similar to the case without time dependence, $W^{h,\delta}(t,x,\alpha)$ satisfies,

$$\begin{aligned} W^{h,\delta}(n\delta,x,u) = &W^{h,\delta}(n\delta+\delta,x,u)p^{h,\delta}(x,x|u(x,n\delta)) \\ &+ W^{h,\delta}(n\delta+\delta,x+h,u)p^{h,\delta}(x,x+h|u(x,n\delta)) \\ &+ W^{h,\delta}(n\delta+\delta,x-h,u)p^{h,\delta}(x,x-h|u(x,n\delta)) \\ &+ k(x,u(x,n\delta))\delta \end{aligned} \tag{3.73}$$

[1] Stopping times τ can take values in $[0,\infty)$ by definition

where the transition probabilities are given as,

$$p^{h,\delta}(x, x|u(x, n\delta)) = \left[1 - \sigma^2 \frac{\delta}{h^2} - |\mu(x, u(x, n\delta))| \frac{\delta}{h}\right]$$
$$p^{h,\delta}(x, x + h|u(x, n\delta)) = \left[\sigma^2 \frac{\delta}{h^2} + \mu^+(x, u(x, n\delta)) \frac{\delta}{h}\right]$$

$$p^{h,\delta}(x, x - h|u(x, n\delta)) = \left[\sigma^2 \frac{\delta}{h^2} + \mu^-(x, u(x, n\delta)) \frac{\delta}{h}\right] \tag{3.74}$$

Therefore, the discrete counterpart of 3.67 is,

$$W^{h,\delta}(t, x, u) = \mathbb{E}^u_{x,n}\left[\sum_i^{N-1} k(\xi_i^{h,\delta}, u(\xi_i^{h,\delta}, i\delta))\delta + g(\xi_N^{h,\delta})\right]. \tag{3.75}$$

For any $t = n\delta < T$, the value function becomes

$$V^{h,\delta}(t, x) = \inf_{\alpha \in A}\left[\sum_y p^{h,\delta}(x, y|\alpha) V^{h,\delta}(y, n\delta + \delta) + k(x, u(x, n\delta))\delta\right]. \tag{3.76}$$

Solving 3.76 by backward iteration will generate an approximation to $V(t, x)$. Local consistency and convergence properties are extensively treated by [21]. We shall not asses such properties within the scope of this thesis.

CHAPTER 4

INDIFFERENCE PRICING

Call option is an contract which grants the buyer right to purchase the underlying asset on a defined price. This defined price is called strike price and will be denoted by $E \in \mathbb{R}^+$. Such contract would only be exercised if underlying asset price denoted by S_T is greater than the exercise price E. Therefore, value of the option at maturity is $C_T = (S_T - E)^+$. Black-Scholes model has treated the problem of deriving fair option value with preference free arguments. The price paid by investor to purchase option can be determined uniquely. However, this framework is only an approximation to real life case. On the other hand, utility based pricing argues that there exist a certain price where the investor is indifferent between buying the option or holding his portfolio by not going through trading. In order to illustrate this point, we shall adapt the definition from Chapter 2 of Carmona [8]. Let W_T denote the wealth of the investor at maturity and k number of shares held in the option contract. Value function is defined as a maximum attainable utility of over the wealth process ,

$$V(W, k) = \sup_{W_T \in A(W)} \mathbb{E}[U(W_T + kC_T)]. \tag{4.1}$$

It is assumed that investor has initial wealth of W and no endowment for the option. Therefore, investor will initially pay $p^b(k)$ to obtain k amounts of C_T at maturity:

$$\begin{gathered} V(W - p^b(k), k) = V(W, 0), \\ \sup_{W_T \in A(W - p^b(k))} \mathbb{E}[U(W_T + kC_T)] = \sup_{W_T \in A(W)} \mathbb{E}[U(W_T)]. \end{gathered} \tag{4.2}$$

Price of the contingent claim is implicitly determined in this framework which can be applied to incomplete markets if some risks cannot be perfectly hedged. Law Of One Price is preserved in incomplete markets but equivalent probability measure is no longer unique. Depending on the type of the market risk, probability measure corresponding to the fair option price changes. Reader can find more information about this fact on Chapter 7 of Bingham & Kiesel [4]. Black Scholes model determines same option price and risk measure for all levels of risk preference unlike utility based pricing models in incomplete markets. Therefore, utility approach is more realistic as true nature of the markets and risk preference of the investors are considered. A compilation of Indifference Pricing models can be found on [8].

In this chapter, we shall introduce and work with the key ideas from Davis et al. [14]. Firstly, utility based pricing argument is introduced when transaction costs are excluded. Secondly, transaction costs are introduced and corresponding value function is derived. Then viscosity property of the solution to the optimal control problem is investigated. Finally, numerical computation by Markov Chain Approximation is given. We demonstrate the case when stock price process is one dimensional. Although authors [14] had introduced market dynamics with multidimensional stock price processes in order to present the general idea, the option price was computed with one dimensional processes in further sections.

4.1 Pricing without Transaction Costs

Time interval is considered as $0 \leq t \leq T$ for a positive finite number T. Market dynamics are given as,

$$\begin{aligned} dS(t) &= \mu(t)S(t)dt + \sigma(t)S(t)d\mathbf{W}(t), \\ dB(t) &= rB(t)dt. \end{aligned} \tag{4.3}$$

Here $\mathbf{W}(t)$ is a one dimensional Brownian motion. Option payoff at maturity is

$$C(T) = (S(T) - E)^{+}. \tag{4.4}$$

Investor shall begin with wealth of B in cash and trades a dynamic portfolio with strategy π. Hence, $\pi \in \mathbf{A}(B)$ is a set of admissible strategies where the vector process $\pi = (B^{\pi}(t), y^{\pi}(t))$ represents holdings in cash and risky asset over a finite time horizon. Control presence is indicated on the superscripts of B^{π} and y^{π} . We shall define cash value of the position in stocks as $c(y(t), S(t)) = y(t) \times S(t)$ with $c(0, S(t)) = 0$.

We consider the writer of the call option as a person who would like to hedge his position by forming a portfolio with holdings in stocks and cash account. Also, utility function $U : \mathbb{R} \to \mathbb{R}$ is defined as an increasing and concave function. Therefore, option writer's maximum utility after the contract is exercised,

$$\begin{aligned} V_w(B) = \sup_{\pi \in \mathbf{A}(B)} \mathbb{E}\Bigg[U\Bigg(& B^{\pi}(T) + \mathbb{1}_{\{S(T) \leq E\}}\big(c(y^{\pi}(T), S(T)\big) \\ & + \mathbb{1}_{\{S(T) > E\}}\big((c(y^{\pi}(T) - 1, S(T)) + E\big)\Bigg)\Bigg]. \end{aligned} \tag{4.5}$$

Maximum utility of the portfolio without option is,

$$V_1(B) = \sup_{\pi \in \mathbf{A}(B)} \mathbb{E}\Bigg[U\Bigg(B^{\pi}(T) + c(y^{\pi}(T), S(T))\Bigg)\Bigg]. \tag{4.6}$$

$V_j(\cdot) < \infty$ and $V_j(\cdot)$ is a continuous monotone increasing function for $j = 1, w$.

Definition 4.1.1 (Indifference Price). Minimum amount of money that makes option writer indifferent between getting into market or doing nothing is defined as,

$$B_w = \inf\{B : V_w(B) \geq 0\}. \tag{4.7}$$

It should be noted writer will charge B_w to short the option but this not the fair price of the option. Cost of entering the market must be taken into account. Minimum entry fee for investors to get into market is given as,

$$B_1 = \inf\{B : V_1(B) \geq 0\}. \tag{4.8}$$

In order to get into market, writer is ready to pay B_1 which satisfies $B_1 \leq 0$ as $V_1(0) \geq 0$. Then, fair option price is

$$p_w = B_w - B_1. \tag{4.9}$$

At this price, writer is indifferent between accepting the option's obligations and sticking to the portfolio without the option. In conclusion, we have a definition for selecting one price in incomplete market framework.

Theorem 4.1.1 (**Replicating Portfolio**). *Assume that* $\mathbf{A}(B)$ *is a linear space. Option price is* $p_w = \hat{B}$ *if there exists a replicating portfolio* $\hat{\pi} \in \mathbf{A}(\hat{B})$ *such that* $(B^{\hat{\pi}}(T), y^{\hat{\pi}}(T)) = \mathbb{1}_{\{S(T)>E\}}(-E, 1)$.

Proof. By the linearity of $\mathbf{A}(B)$ it can be stated that $\pi = \tilde{\pi} + \hat{\pi}$ and $B = \tilde{B} + \hat{B}$. Here $\hat{\pi}$ shall be used to form replicating portfolio which must be equal to 4.4 at maturity. Therefore, the fair price for the option is initial endowment in the replicating portfolio. In other words, by using the strategy $\hat{\pi}$, investor would be able to create call option's payoff. In addition, $\tilde{\pi} = \pi - \hat{\pi}$ representing the rest of the holdings will be used to form an optimal portfolio consisting of stocks and bonds. As pointed above, option writer would charge minimum B_w to write and hedge the option. Based on the assumptions for V_j, we can now derive the amount needed for buying the option via linearity property as,

$$\begin{aligned}
0 &= V_w(B_w), \\
&= \sup_{\pi \in \mathbf{A}(B)} \mathbb{E}\Bigg[U\Bigg(B^{\pi}(T) + \mathbb{1}_{\{S(T)\leq E\}}\big(c(y^{\pi}(T), S(T)\big) \\
&\qquad\qquad + \mathbb{1}_{\{S(T)>E\}}\big((c(y^{\pi}(T) - 1, S(T)) + E\big)\Bigg)\Bigg] \\
&= \sup_{\tilde{\pi} \in \mathbf{A}(B_w - \hat{B})} \mathbb{E}\Bigg[U\Bigg(B^{\tilde{\pi}}(T) + B^{\hat{\pi}}(T) + \mathbb{1}_{\{S(T)\leq E\}}\big(c(y^{\tilde{\pi}}(T) + y^{\hat{\pi}}(T), S(T)\big) \\
&\qquad\qquad + \mathbb{1}_{\{S(T)>E\}}\big((c(y^{\tilde{\pi}}(T) + y^{\hat{\pi}}(T) - 1, S(T)) + E\big)\Bigg)\Bigg]
\end{aligned}$$

By the fact that $(B^{\hat{\pi}}(T), y^{\hat{\pi}}(T)) = \mathbb{1}_{\{S(T)>E\}}(-E, 1)$ then,

$$\begin{aligned}
&= \sup_{\tilde{\pi} \in \mathbf{A}(B_w - \hat{B})} \mathbb{E}\Bigg[U\Bigg(B^{\tilde{\pi}}(T) - \cancel{\mathbb{1}_{\{S(T)>E\}}(E)} + \mathbb{1}_{\{S(T)\leq E\}}\big(c(y^{\tilde{\pi}}(T) + \underbrace{y^{\hat{\pi}}(T)}_{0}, S(T)\big) \\
&\qquad\qquad + \mathbb{1}_{\{S(T)>E\}}\big((c(y^{\tilde{\pi}}(T), S(T)) + \cancel{\mathbb{1}_{\{S(T)>E\}}(E)}\big)\Bigg)\Bigg] \\
&= \sup_{\tilde{\pi} \in \mathbf{A}(\tilde{B})} \mathbb{E}\Bigg[U\Bigg(B^{\tilde{\pi}}(T) + c(y^{\tilde{\pi}}(T), S(T))\Bigg)\Bigg] \\
&= V_1(B_w - \hat{B})
\end{aligned} \tag{4.10}$$

It should be noted that call option is eliminated from the optimisation problem by using $\hat{\pi}$. Since B_1 is the minimum amount that makes $V_1(B) = 0$, we get $B_1 = B_w - \hat{B}$. So, the price is given as $\hat{B} = B_w - B_1 = p_w$. $\square$

It has been shown that option can be hedged if the replicating portfolio exists. Thus, we conclude that this model indeed reduces to Black-Scholes when transaction costs are not considered.

Corollary 4.1.1 (Hedging Strategy). *Investor who wishes to hedge the short position in the option contract shall use the strategy $\pi_h = \hat{\pi}$ with the initial endowment of $\hat{B}$. Therefore, two control problems presented below must be solved in order to find $\hat{B}$ and corresponding hedging strategy,*

$$V_w(B_w) = \sup_{\pi \in \mathbf{A}(B_w)} \mathbb{E}\Big[U\Big(B_w^{\pi}(T) + \mathbb{1}_{\{S(T)\leq E\}} c(y^{\pi}(T), S(T)) + \mathbb{1}_{\{S(T)>E\}} (c(y^{\pi}(T) - 1, S(T)) + E)\Big)\Big] \tag{4.11}$$

$$V_1(B_1) = \sup_{\tilde{\pi} \in \mathbf{A}(B_1)} \mathbb{E}\Big[U\big(B_1^{\tilde{\pi}}(T) + c(y^{\tilde{\pi}}(T), S(T))\big)\Big] \tag{4.12}$$

Therefore, the hedging strategy is $(\pi - \tilde{\pi})$ with an initial endowment of $B_w - B_1$.

4.2 Pricing with Transaction Costs

4.2.1 Optimal Control Problem

Indifference pricing in the case of transaction costs will be introduced. Authors of [14] have considered transaction costs that are proportional to the holdings transferred from stock to the cash account. Then, they have specified the conditions in which writer is indifferent between getting into market with or without the option. Main purpose is to derive a partial differential equation satisfied by the value functions that is defined in utility indifference fashion. Firstly, we consider that the investor has a portfolio consisting of a stock and a risk free asset. Dynamics of the stock price is given as,

$$dS(t) = \mu S(t)dt + \sigma S(t)d\mathbf{W}(t). \tag{4.13}$$

where μ, σ, r are all positive constant coefficients. Dynamics of the cash account paying interest at r is given as,

$$dB(t) = rB(t)dt - (1 + \Theta^b)y(t)S(t)dL(t) + (1 - \Theta^s)y(t)S(t)dM(t). \tag{4.14}$$

$\Theta^b, \Theta^s \in \mathbb{R}^+$ are transaction costs when buying and selling the stocks,respectively. So, transaction costs are reflected in the cash account whenever stocks are being traded. Here trading strategy corresponds to the cumulative number shares that are bought, denoted by $L(t)$ and sold, denoted by $M(t)$,

$$dy(t) = dL(t) - dM(t). \tag{4.15}$$

Cash value of the stocks under this market is given as,

$$c(y(t), S(t)) = (1 + \Theta^b)y(t)S(t), \text{ for } y(t) < 0, \tag{4.16}$$
$$c(y(t), S(t)) = (1 - \Theta^s)y(t)S(t), \text{ for } y(t) \geq 0. \tag{4.17}$$

As pointed out in the [12], the system of differential equations presented in 4.13, 4.14 and 4.15 has solution only in weak sense. Secondly, conditions for Utility Indifference Pricing will be constructed.

Definition 4.2.1 (Wealth Process). Wealth functions of the writer at maturity are given as,

$$W_1(T, B(T), y(T), S(T)) = B(T) + c(y(T), S(T)), \tag{4.18}$$
$$\begin{aligned} W_w(T, B(T), y(T), S(T)) = B(T) &+ \mathbb{1}_{\{S(T) \leq E\}}\big(c(y(T), S(T)\big) \\ &+ \mathbb{1}_{\{S(T) > E\}}\big((c(y(T) - 1, S(T)) + E\big). \end{aligned} \tag{4.19}$$

for the portfolios with option and without option, respectively.

Definition 4.2.2 (Admissible Strategies). Investor holdings $(B^\pi(t), y^\pi(t))$, solution to the system of equations 4.13—4.15, are influenced in response to control process $(L(t), M(t))$. Admissible strategy $(L(t), M(t)) \in \mathbf{A}(t, B, y, S)$ is any control policy that the triplet $(B(t), y(t), S(t))$ satisfies the following property for a constant $K \in \mathbb{R}$,

$$\begin{aligned} &(B^\pi(t), y^\pi(t), S(t)) \in \mathscr{S}_K, \text{ where } \mathscr{S}_K \text{ is defined as,} \\ &\mathscr{S}_K = \{(B, y, S) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+ : B + c(y, S) > -K\}. \end{aligned} \tag{4.20}$$

Finally, $L(0^-) = M(0^-) = 0$ is assumed. Set of admissible strategies shall be denoted by $\mathbf{A}$ in order to simplify the notations.

4.20 is similar to the solvency region from Davis and Norman [13]. However, shorting the stocks or borrowing from the bank is allowed up to K in this model. This is useful for two reasons. Firstly, this constraint refutes any policy that are surely not optimal. In other words, investor cannot run up infinitely huge debts since there is a finite credit limit. Secondly, this open set is later used to prove viscosity properties of the solution.

Definition 4.2.3 (Value Functions). Wealth functions are given in the form of 4.18 and 4.19. Therefore, for $j = 1, w$ and $(s, B, y, S) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+$, the value function is given as,

$$V_j(s, B, y, S) = \sup_{\pi \in \mathbf{A}} \mathbb{E}\Big[U\big(W_j(T, B^\pi(T), y^\pi(T), S(T)\big)\Big]. \tag{4.21}$$

Having defined the value functions for both portfolios, we are now interested in deriving Hamilton-Jacobi-Bellman equation for the optimal control problem that is just defined. It should be noted that same Hamiltonian is derived for both value functions. However, their terminal utilities are different.

Let control processes satisfy $L(t) = \int_0^t l(s)ds$ and $M(t) = \int_0^t m(s)ds$ where $l(s)$ and $m(s)$ are positive functions which are uniformly bounded by a fixed constant k. Hence, HJB equation for the value function is,

$$\sup_{0 \le l, m \le k} \left\{ \left(\frac{\partial V_j}{\partial y} (l - m) \right) + \left(\frac{\partial V_j}{\partial B} \big(rB - (1 + \Theta^b) Sl + (1 - \Theta^s) Sm \big) \right) + \frac{\partial V_j}{\partial s} + \frac{\partial V_j}{\partial S} \mu S + \frac{1}{2} \frac{\partial^2 V_j}{\partial S^2} \sigma^2 S^2 \right\} = 0. \tag{4.22}$$

We shall now arrange the terms of 4.22 in the following fashion,

$$\mathcal{B} V_j = \left(\frac{\partial V_j}{\partial y} - (1 + \Theta^b) S \frac{\partial V_j}{\partial B} \right). \tag{4.23}$$

$$\mathcal{S} V_j = \left(\frac{\partial V_j}{\partial y} - (1 - \Theta^s) S \frac{\partial V_j}{\partial B} \right). \tag{4.24}$$

$$\mathcal{NT} V_j = \frac{\partial V_j}{\partial s} + rB \frac{\partial V_j}{\partial B} + \frac{\partial V_j}{\partial s} + \frac{\partial V_j}{\partial S} \mu S + \frac{1}{2} \frac{\partial^2 V_j}{\partial S^2} \sigma^2 S^2. \tag{4.25}$$

Then, HJB becomes,

$$\sup_{0 \le l, m \le k} \left\{ \mathcal{B} V_j l - \mathcal{S} V_j m \right\} + \mathcal{NT} V_j = 0. \tag{4.26}$$

V_j is an increasing function of y and B as they provide additional wealth which increases the utility. That is, $\frac{\partial V_j}{\partial y}$ and $\frac{\partial V_j}{\partial B}$ must be positive. Therefore, optimality can be only achieved through the following ways,

$$l = \begin{cases} k, & \text{if } \frac{\partial V_j}{\partial y} \ge (1 + \Theta^b) S \frac{\partial V_j}{\partial B} \\ 0, & \text{if } \frac{\partial V_j}{\partial y} < (1 + \Theta^b) S \frac{\partial V_j}{\partial B} \end{cases} \tag{4.27}$$

$$m = \begin{cases} 0, & \text{if } \frac{\partial V_j}{\partial y} > (1 - \Theta^s) S \frac{\partial V_j}{\partial B} \\ k, & \text{if } \frac{\partial V_j}{\partial y} \le (1 - \Theta^s) S \frac{\partial V_j}{\partial B} \end{cases} \tag{4.28}$$

In fact, these expressions lead to three different scenarios of optimality,

1. $\mathcal{B} V_j \ge 0$ and $\mathcal{S} V_j > 0$, maximum is achieved when $l = k, m = 0$.

2. $\mathcal{B} V_j < 0$ and $\mathcal{S} V_j \le 0$ maximum is achieved when $l = 0, m = k$.

3. $\mathcal{B}V_j \leq 0$ and $\mathcal{S}V_j \geq 0$ maximum is achieved when $l = 0, m = 0$.

These optimality scenarios indicate that the control policy is singular because trading takes place at maximum possible rate or do not take place at all. Scenarios also suggest that state space is divided into three different regions which are Buy, Sell and No-Transaction regions. Buy and Sell Regions do not intersect since buying and selling at the same time is not optimal in this framework. Trading strategy is same with 4.2.2 when $k \to \infty$ and the state space is still divided into 3 regions. However, the transaction should occur in local time settings. Properties and boundaries of these regions shall be defined in following.

Definition 4.2.4 (**Buy Region**)**.** Buy Region and its boundary shall be denoted by $\mathfrak{B}$ and $\partial\mathfrak{B}$, respectively. Value function is constant along the trajectory of (s, B, y, S) as governed by the control process in buy region. That is,

$$V_j(s, B, y, S) = V_j(s, B - (1 + \Theta^b)S\Delta y_b, y + \Delta y_b, S) \tag{4.29}$$

Δy_b denotes the amount of shares bought to take the state to the boundary. If we divide 4.29 by Δy_b and let $\Delta y_b \to 0$, we get the condition for the boundary of Buy Region between No Transaction Region as,

$$\begin{aligned} 0 &= \lim_{\Delta y_b \to 0} \frac{V_j(s, B - (1 + \Theta^b)S\Delta y_b, y + \Delta y_b, S) - V_j(s, B, y, S)}{\Delta y_b} \\ &= \frac{\partial V_j}{\partial y} - \frac{\partial V_j}{\partial B}(1 + \Theta^b)S \\ &= \mathcal{B}V_j \end{aligned} \tag{4.30}$$

Definition 4.2.5 (**Sell Region**)**.** Sell Region and its boundary shall be denoted by $\mathfrak{S}$ and $\partial\mathfrak{S}$, respectively. Similar to Buy Region, the value function is also constant along the trajectory of (s, B, y, S) in Sell Region. That is,

$$V_j(s, B, y, S) = V_j(s, B + (1 - \Theta^s)S\Delta y_s, y - \Delta y_s, S) \tag{4.31}$$

Δy_s denotes the amount of shares sold to take the state to the boundary. If we divide 4.31 by Δy_s and let $\Delta y_s \to 0$, we get the condition for the boundary of Sell Region between No Transaction Region as,

$$\begin{aligned} 0 &= \lim_{\Delta y_s \to 0} \frac{V_j(s, B + (1 - \Theta^s)S\Delta y_s, y - \Delta y_s, S) - V_j(s, B, y, S)}{\Delta y_s} \\ 0 &= \frac{\partial V_j}{\partial y} - \frac{\partial V_j}{\partial B}(1 - \Theta^s)S \\ &= \mathcal{S}V_j \end{aligned} \tag{4.32}$$

Definition 4.2.6 (**No Transaction Region**)**.** No transaction region shall be denoted by $\mathfrak{N}$. Following inequalities hold for no transaction region,

$$\begin{aligned} V_j(s, B, y, S) &\geq V_j(s, B - (1 + \Theta^b)S\Delta y_b, y + \Delta y_b, S) \\ V_j(s, B, y, S) &\geq V_j(s, B + (1 - \Theta^s)S\Delta y_s, y - \Delta y_s, S) \end{aligned} \tag{4.33}$$

which implies,

$$\mathcal{B}V_j \leq 0 \ , \ \mathcal{S}V_j \geq 0 \tag{4.34}$$

By Dynamic Programming Principle V_j satisfies,

$$V_j(t, B, y, S) = \mathbb{E}^t\big[V(t+dt, B+dB(t), y, S+dS(t))\big] \tag{4.35}$$

Since we know that $dL(t) = dM(t) = 0$, the corresponding HJB equation is,

$$\mathcal{NT}V_j = 0. \tag{4.36}$$

Corollary 4.2.1 (Variational Inequality). *Equations derived in 4.30,4.32 and 4.36 suggests that for any $(s, B, y, S) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+$ the value function $V_j(s, B, y, S)$ satisfies the following variational inequality,*

$$\max\left\{\mathcal{B}V_j, -\mathcal{S}V_j, \mathcal{NT}V_j\right\} = 0 \tag{4.37}$$

It should be noted that if V_j is computed in No Transaction region, its value can be determined in $\mathfrak{B}$ and $\mathfrak{S}$ by using the boundaries given in 4.30 and 4.32. This is by the fact that V_j is continuous.

In conclusion, if $(s, B^*, y^*, S) \in \mathfrak{B}$, an instant transaction which moves the state to $\partial\mathfrak{B}$ occurs. Similar action takes place in Sell Regions also. If $(s, B^*, y^*, S) \in \mathfrak{N}$, it is drifted under the influence of stock process. Investors would like to stay in no transaction region as much as they can. Therefore, tradings after this point will happen in local time fashion.

4.2.2 Option Price Model

We shall give specifications to the problem and derive the indifference price. Exponential utility function shall be used to reduce the dimensionality of the problem. Let utility be defined as

$$U(x) = 1 - e^{-\gamma x}. \tag{4.38}$$

where γ is the coefficient of risk aversion $\gamma = -\dfrac{U''(x)}{U'(x)}$. Hence, risk aversion coefficient is independent of the wealth. This will allow us to reduce the dimensionality of the problem since weight in risky asset does not depend on the total wealth. Therefore, the term B shall be omitted from the problem by doing some manipulations explained later in this section. Given discount factor $\beta(T, t) = e^{-r(T-t)}$, the integral form of 4.14 is defined as,

$$B^\pi(T) = \frac{B}{\beta(T, s)} - \int_s^T \frac{(1+\Theta^b)S(t)}{\beta(T, t)} dL(t) + \int_s^T \frac{(1-\Theta^s)S(t)}{\beta(T, t)} dM(t) \tag{4.39}$$

After these settings, the value function becomes,

$$
\begin{aligned}
V_j(s,B,y,S) &= \sup_{\pi\in\mathbf{A}} \mathbb{E}\Big[1 - e^{-\gamma W_j(T,B^\pi(T),y^\pi(T),S(T))}\Big],\\
&= 1 - \inf_{\pi\in\mathbf{A}} \mathbb{E}\Big[e^{-\gamma W_j(T,B^\pi(T),y^\pi(T),S(T))}\Big],\\
&= 1 - \inf_{\pi\in\mathbf{A}} \mathbb{E}\Big[e^{\big(-\gamma B^\pi(T)\big)} e^{\big(-\gamma W_j(T,B^\pi(T),y^\pi(T),S(T)) + \gamma B^\pi(T)\big)}\Big],\\
&= 1 - \inf_{\pi\in\mathbf{A}} \mathbb{E}\Big[e^{\big(-\gamma B^\pi(T)\big)} e^{\big(-\gamma W_j(T,0,y^\pi(T),S(T))\big)}\Big]. \qquad (4.40)
\end{aligned}
$$

Here the following interchange will be used in order to ease the computation burden,

$$
\begin{aligned}
V(s,0,y,S) &= 1 - \inf_{\pi\in\mathbf{A}} \mathbb{E}\Big[e^{-\gamma\big(-\int_s^T \frac{(1+\Theta^b)S(t)}{\beta(T,t)}dL(t) + \int_s^T \frac{(1-\Theta^s)S(t)}{\beta(T,t)}dM(t)\big)}\\
&\qquad\qquad \times e^{-\gamma\big(W_j(T,0,y^\pi(T),S(T)\big)}\Big],\\
Q_j(s,y,S) &= 1 - V(s,0,y,S),\\
&= \inf_{\pi\in\mathbf{A}} \mathbb{E}\Big[e^{-\gamma\big(-\int_s^T \frac{(1+\Theta^b)S(t)}{\beta(T,t)}dL(t) + \int_s^T \frac{(1-\Theta^s)S(t)}{\beta(T,t)}dM(t)\big)}\\
&\qquad\qquad \times e^{-\gamma\big(W_j(T,0,y^\pi(T),S(T)\big)}\Big]. \qquad (4.41)
\end{aligned}
$$

where $Q_j : [0,T]\times\mathbb{R}\times\mathbb{R}^+ \to \mathbb{R}$. It should be noted that $Q_j(s,y,S)$ is a decreasing function of y and S. As a result, we can write the value function 4.40 as,

$$
\begin{aligned}
V_j(s,B,y,S) &= 1 - \inf_{\pi\in\mathbf{A}} \mathbb{E}\Big[e^{-\gamma\big(\frac{B}{\beta(T,s)} - \int_s^T \frac{(1+\Theta^b)S(t)}{\beta(T,t)}dL(t) + \int_s^T \frac{(1-\Theta^s)S(t)}{\beta(T,t)}dM(t)\big)}\\
&\qquad\qquad \times e^{-\gamma\big(W_j(T,0,y^\pi(T),S(T))\big)}\Big],\\
&= 1 - e^{\frac{-\gamma B}{\beta(T,s)}} Q_j(s,y,S). \qquad (4.42)
\end{aligned}
$$

Moreover, the variational inequality 4.37 for $Q_j(s,y,S)$ becomes,

$$
\begin{aligned}
\min\Big\{ &\frac{\partial Q_j}{\partial y} + \frac{\gamma(1+\Theta^b)S}{\beta(T,s)}Q_j, -\Big(\frac{\partial Q_j}{\partial y} + \frac{\gamma(1-\Theta^s)S}{\beta(T,s)}Q_j\Big),\\
&\frac{\partial Q_j}{\partial s} + \frac{\partial Q_j}{\partial S}\mu S + \frac{1}{2}\frac{\partial^2 Q_j}{\partial S^2}\sigma^2 S^2\Big\} = 0, \qquad (4.43)
\end{aligned}
$$

with the following boundary conditions,

$$
Q_1(T,y,S) = e^{-\gamma\big(W_1(T,0,y^\pi(T),S(T))\big)}. \qquad (4.44)
$$

$$
Q_w(T,y,S) = e^{-\gamma\big(W_w(T,0,y^\pi(T),S(T))\big)}. \qquad (4.45)
$$

Lastly, if $Q_j(s,y,S)$ is known in $\mathfrak{N}$, then it can be computed in $\mathfrak{B}$ or $\mathfrak{S}$ regions from the boundary conditions. We shall illustrate this point as following. Let $y_b = y + \Delta y_b$ and

$y_s = y - \Delta y_s$. Therefore, for all $y \leq y_b$

$$Q_j(s, y, S) = Q_j(s, y_b, S)e^{\left(- \frac{\gamma(1+\Theta^b)S}{\beta(T,s)}(y - y_b)\right)}. \tag{4.46}$$

Likewise, for all $y \geq y_s$,

$$Q_j(s, y, S) = Q_j(s, y_s, S)e^{\left(\frac{\gamma(1-\Theta^s)S}{\beta(T,s)}(y - y_s)\right)}. \tag{4.47}$$

Corollary 4.2.2 (Option Price with Transaction Costs). *Fair option price for the writer is given as,*

$$\begin{aligned} p_w(s, S) &= B_w - B_1, \\ &= \frac{\beta(T, s)}{\gamma} \ln\left(\frac{Q_w(s, 0, S)}{Q_1(s, 0, S)}\right). \end{aligned} \tag{4.48}$$

Proof. Following the argument in 4.7 and 4.8 ,the minimum fees for writer to play the market or accept the option are defined as

$$B_i = \inf\{B : V_i(s, B, y, S) \geq 0\} \ , \ i = 1, w. \tag{4.49}$$

Thus, we write the value function for fee required to get into market as,

$$\begin{aligned} V_1(s, B_1, y, S) &= 1 - e^{\frac{-\gamma B_1}{\beta(T,s)}} Q_1(s, y, S) \\ &= 0 \end{aligned} \tag{4.50}$$

Similarly, the value function for writing the option is,

$$\begin{aligned} V_w(s, B_w, y, S) &= 1 - e^{\frac{-\gamma B_w}{\beta(T,s)}} Q_w(s, y, S) \\ &= 0 \end{aligned} \tag{4.51}$$

In order to derive the fair option price, following arrangements have been made,

$$\begin{aligned} e^{\frac{-\gamma B_w}{\beta(T,s)}} Q_w(s, y, S) &= 1, \\ Q_w(s, y, S) &= e^{\frac{\gamma B_w}{\beta(T,s)}}, \\ B_w &= \ln(Q_w(s, y, S))\frac{\beta(T, s)}{\gamma}. \end{aligned} \tag{4.52}$$

Likewise, $B_1 = \ln(Q_1(s, y, S))\frac{\beta(T,s)}{\gamma}$. Thus, subtraction yields,

$$B_w - B_1 = \frac{\beta(T, s)}{\gamma}\big(\ln(Q_w(s, y, S)) - \ln(Q_1(s, y, S))\big). \tag{4.53}$$

Therefore, the option price is

$$p_w(s, S) = \frac{\beta(T, s)}{\gamma} \ln\left(\frac{Q_w(s, y, S)}{Q_1(s, y, S)}\right). \tag{4.54}$$

where y can take any value. □

4.2.3 Viscosity Property

Theorem 4.2.1 (Existence of the Viscosity Solution). *Let $\mathcal{O} \subset \mathbb{R}^3$ and x be the state vector of the variables t, B, y, S. Define $\hat{H}(x, \mathcal{W}, \mathcal{W}_x, \mathcal{W}_{xx})$ as a continous elliptic function such that*

$$\begin{aligned}\hat{H}(x, \mathcal{W}, \mathcal{W}_x, \mathcal{W}_{xx}) = \min\Bigg\{ & -\left(\frac{\partial \mathcal{W}}{\partial y} - (1+\Theta^b S\frac{\partial \mathcal{W}}{\partial B}\right), \frac{\partial \mathcal{W}}{\partial y} - (1-\Theta^s S\frac{\partial \mathcal{W}}{\partial B}, \\ & -\left(\frac{\partial \mathcal{W}}{\partial s} + rB\frac{\partial \mathcal{W}}{\partial B} + \frac{\partial \mathcal{W}}{\partial S}\mu S + \frac{1}{2}\frac{\partial^2 \mathcal{W}}{\partial S^2}\sigma^2 S^2\right)\Bigg\}, \\ = 0. & \end{aligned} \tag{4.55}$$

Particularly, $\mathcal{W} : [0, T] \times \mathcal{O} \to \mathbb{R}$ and $\phi \in \mathbf{C}^{1,2}([0, T] \times \mathcal{O})$ satisfy

$$\hat{H}(x_0, \mathcal{W}(x_0), \phi_x(x_0), \phi_{xx}(x_0)) \leq 0, \tag{4.56}$$

$$\hat{H}(x_0, \mathcal{W}(x_0), \phi_x(x_0), \phi_{xx}(x_0)) \geq 0. \tag{4.57}$$

whenever $\mathcal{W} - \phi$ attains its maximum and minimum values at $x_0 \in [0, T] \times \mathcal{O}$, respectively. Then, $\mathcal{W}$ is a viscosity solution of 4.55. Moreover, $V_1(s, B, y, S) = \mathcal{W}(s, B, y, S)$ and it is the viscosity solution on $[0, T] \times \mathscr{S}_K$.

Theorem 4.2.2 (Uniqueness of the Viscosity Solution). *Let $\mathscr{S}_K^0$ be the interior of the set 4.20 and the following inequalities hold,*

$$V^*(T, x) \leq V_*(T, x),\ \forall\, x \in \mathscr{S}_K^0, \tag{4.58}$$

$$V^*(t, B, y, 0) \leq V_*(t, B, y, 0)\ \forall\, (t, B, y, 0) \in [0, T] \times \mathscr{S}_K^0. \tag{4.59}$$

where V^* is an upper semicontinuous viscosity subsolution on $[0, T] \times \mathscr{S}_K^0$ and V_* is an lower semicontinuous viscosity supersolution of 4.55 on $[0, T] \times \mathscr{S}_K$. Therefore, the comparison theorem holds,

$$V^*(t, x) \leq V_*(t, x)\ \forall\, (t, x) \in [0, T] \times \mathscr{S}_K^0. \tag{4.60}$$

Although only the V_1 is specified, validity of the results are same for V_w as well. We refer to [14] for the proofs of these theorems.

4.3 Markov Chain Approximation

Markov Chain Approximation shall be utilised to solve the problem given in 4.43. Procedure follows the explicit approximation method given by [21] and the solution is produced by backward recursive algorithm on the value space. Table 4.1 includes discrete elements needed for the approximation.

Table 4.1: Discrete Scheme for the Model

Parameter Description	**Discrete Notation**	**Continuous Counterpart**
State Space	$\mathscr{S}_K^h := E_h \cap \mathscr{S}_K$	$\mathscr{S}_K$
Time	$t^h \in \mathbb{T} = \{0, \pm\Delta t, \pm 2\Delta t, ..., N\Delta t\}$	$s \in [0, T]$
Cash Account Process	$B^h(t^h + \Delta t) = B(t^h)e^{r\Delta t}$	$dB(t) = rB(t)dt$
Stock Price Process	$S^h(t^h + \Delta t) = S^h(t^h)e^{\mu\Delta t + \chi\sigma\sqrt{\Delta t}}$	$dS(t) = \mu S(t)dt + \sigma S(t)d\mathbf{W}(t)$
Value Function	$V_j^h(t^h, B^h, y^h, S^h)$	$V_j(s, B, y, S)$
Auxiliary Function	$Q_j^h(t^h, y^h, S^h)$	$Q_j(s, y, S)$

Time step is $\Delta t = \dfrac{T-s}{N}$ which is also equal to discretisation parameter h. Furthermore, χ is a random variable defined as,

$$\chi = \begin{cases} 1 & , \text{ with probability } \frac{1}{2} \\ -1 & , \text{ with probability } \frac{1}{2} \end{cases} \tag{4.61}$$

Discretised stock price process weakly converges to its continuous counterpart as argued by Cox & Rubinstein [10]. Therefore, locally consistent chain has been built. Dynamic Programming Principle in this scheme is,

$$V_j^h(t^h, B^h, y^h, S^h) = \mathbb{E}^{t^h}\left[V_j^h(t^h + \Delta t, B^h(t^h + \Delta t), y^h(t^h + \Delta t), S^h(t^h + \Delta t))\right] \tag{4.62}$$

This is replaced with the following as mandated by optimality scenarios defined in previous section,

$$\begin{aligned} V_j^h(t^h, B^h, y^h, S^h) = \max_{\Delta y}\Big\{ & \mathbb{E}^{t^h}\left[V_j^h(t^h + \Delta t, B^h - (1 + \Theta^b)S^h\Delta y, y^h + \Delta y, S^h(t^h + \Delta t))\right], \\ & \mathbb{E}^{t^h}\left[V_j^h(t^h + \Delta t, B^h + (1 - \Theta^s)S^h\Delta y, y^h - \Delta y, S^h(t^h + \Delta t))\right], \\ & \mathbb{E}^{t^h}\left[V_j^h(t^h + \Delta t, B^h(t^h + \Delta t), y^h, S^h(t^h + \Delta t))\right]\Big\} \end{aligned} \tag{4.63}$$

Hence, the discrete version of 4.37 becomes,

$$\begin{aligned} V_j^h(t^h, B^h, y^h, S^h) = \max\Big\{ & V_j^h(t^h, B^h - (1 + \Theta^b)S^h\Delta L^*, y^h + \Delta L^*, S^h), \\ & V_j^h(t^h, B^h + (1 - \Theta^s)S^h\Delta M^*, y^h - \Delta M^*, S^h), \\ & \mathbb{E}^{t^h}\left[V_j^h(t^h + \Delta t), B^h(t^h + \Delta t), y^h, S^h(t^h + \Delta t))\right]\Big\} \end{aligned} \tag{4.64}$$

where ΔL^* and ΔM^* optimal number of shares bought and sold, respectively. If the left hand side is carried to the right hand side, and the equation is divided by h, we would get the finite difference approximation of 4.37 for the each term. This is a direct application of the tools defined in Section 3.3.2. However, transition probabilities are tailored to the binomial settings.

Theorem 4.3.1 (Convergence to the Viscosity Solution). *Discretized scheme converges to the original value function as discretisation parameter gets smaller. That is,*

$$\lim_{h \to 0} V_j^h(t^h, B^h, y^h, S^h) = V_j(s, B, y, S), \tag{4.65}$$

locally uniformly.

We shall refer the reader to [2] and [21] for the proof. As from previous section, V_j^h can be written as below when utility is exponential,

$$V_j^h(t^h, B^h, y^h, S^h) = 1 - e^{-\gamma \frac{B^h}{\beta(N,t^h)}} Q_j^h(t^h, y^h, S^h). \tag{4.66}$$

Dynamic Programming Principle enables us to write

$$\begin{aligned} Q_j^h(t^h, y^h, S^h) = \min_{\Delta y} \Big\{ & \mathbb{E}^{t^h}\left[Q_j^h(t^h + \Delta t, y^h + \Delta y, S^h(t^h + \Delta t))\right] e^{\gamma \frac{(1+\Theta^b) S^h \Delta y}{\beta(N,t^h)}}, \\ & \mathbb{E}^{t^h}\left[Q_j^h(t^h + \Delta t, y^h - \Delta y, S^h(t^h + \Delta t))\right] e^{-\gamma \frac{(1-\Theta^s) S^h \Delta y}{\beta(N,t^h)}}, \\ & \mathbb{E}^{t^h}[Q_j^h(t^h + \Delta t, y^h, S^h(t^h + \Delta t))] \Big\} \end{aligned} \tag{4.67}$$

Therefore, the problem 4.43 characterized by $Q_j^h(t^h, y^h, S^h)$ in discrete setting becomes,

$$\begin{aligned} Q_j^h(t^h, y^h, S^h) = \min \Big\{ & Q_j^h(t^h, y^h + \Delta L^*, S^h) e^{\gamma \frac{(1+\Theta^b) S^h \Delta L^*}{\beta(N,t^h)}}, \\ & Q_j^h(t^h, y^h - \Delta M^*, S^h) e^{-\gamma \frac{(1-\Theta^s) S^h \Delta M^*}{\beta(N,t^h)}}, \\ & \mathbb{E}^{t^h}[Q_j^h(t^h + \Delta t, y^h, S^h(t^h + \Delta t))] \Big\} \end{aligned} \tag{4.68}$$

with the boundary conditions given below,

$$\begin{aligned} Q_1^h(T, y^h, S^h) &= e^{-\gamma W_1(T, 0, y^h, S^h)} \\ Q_w^h(T, y^h, S^h) &= e^{-\gamma W_w(T, 0, y^h, S^h)} \end{aligned} \tag{4.69}$$

Define $y_b^h = y^h + \Delta L^*$ and $y_s^h = y^h - \Delta M^*$. In the same way V_j is determined by boundary conditions in buy and sell regions , Q_j^h can be calculated in the following manner,

$$Q_j^h(t^h, y^h, S^h) = e^{\gamma \frac{(1+\Theta^b) S^h (y_b^h - y^h)}{\beta(N,t^h)}} Q_j^h(t^h, y_b^h, S^h), \ \forall\, y^h < y_b^h \tag{4.70}$$

$$Q_j^h(t^h, y^h, S^h) = e^{-\gamma \frac{(1-\Theta^s) S^h (y^h - y_s^h)}{\beta(N,t^h)}} Q_j^h(t^h, y_s^h, S^h), \ \forall\, y^h > y_s^h \tag{4.71}$$

Corollary 4.3.1 (Price Approximation). *Price given in 4.48 is turned its the discrete variant as,*

$$p_w^h(t^h, S^h) = \frac{\beta(N, t^h)}{\gamma} \ln\left(\frac{Q_w^h(t^h, 0, S^h)}{Q_1^h(t^h, 0, S^h)}\right) \tag{4.72}$$

Corollary 4.3.2 (Solution). *Solution algorithm is presented below.*

- ***Input***: $\Delta t, r, \theta^s, \theta^b, \gamma, T, E, S(0)$.
- *Set* $\Delta t = h$, $p^h(x,y) = \frac{1}{2}$.
- *Discretize State Space into* $\mathbb{TN} \times \mathbb{YN} \times \mathbb{SN}$ *where each set is defined as*
  - $\mathbb{TN} = \{0, \pm\Delta t, \pm 2\Delta t, \ldots, N\Delta t\}$, $N \in \mathbb{N}^+$.
  - $\mathbb{YN} = \{0, \pm\Delta y^h(t^h), \pm 2\Delta y^h(t^h), \ldots, \pm M_y \Delta y^h(t^h)\}$, $M_y \in \mathbb{N}^+$, $\Delta y^h(t^h) = y^h(t^h + \Delta t) - y^h(t^h)$.
  - $\mathbb{SN} = \{0, \Delta S^h(t^h), 2\Delta S^h(t^h), \ldots, M_S \Delta S^h(t^h)\}$, $M_S \in \mathbb{N}^+$, $\Delta S^h(t^h) = S^h(t^h + \Delta t) - S^h(t^h)$.
  - *Let* $0 \le n < N$, $-M_y \le m \le M_y$, $0 \le k \le M_S$ *where* $n, m, k \in \mathbb{N}$.
  - *Set* $Q_j(n,m,k) = Q_j^h(t^h, y^h, S^h)$ *and* $Q_j(N,m,k) = Q_j^h(T, y^h, S^h)$
- ***for n=N-1 to 0***
  1. $Q_j(n,m,k) = min\{\hat{Q}_j(n,m+1,k), \hat{Q}_j(n,m-1,k)\mathbb{E}[Q_j(n+1,m,k+1)]\}$.
  2. *Determine the transaction boundaries by using* $m-1, m, m+1$.
- ***end for***
- ***Output***: $Q_j^h(t^h, y^h, S^h)$ *for* $j = 1, w$. *Derive* $p_w^h(t^h, S^h)$ *as defined in 4.72.*

CHAPTER 5

NUMERICAL RESULTS

In this chapter we compute numerically the hedging strategy implied by the indifference utility approach expounded in the previous Chapter for a range of parameter values and evaluate its performance. All of the computation in this section is done in Octave/MATLAB. In the next section we overview and comment on the numerically computed option price and the implied hedging strategy. The sections following it study the performance of the hedging strategy.

In all of the numerical studies in this Chapter we take $r = 0$ (i.e., we are working with discounted prices) and the strike price $E = 1$. All times are in years.

5.1 Option Price and Hedging Strategy

Figure 5.1 shows the price of the call option as a function of the initial stock price computed using Markov chain approximation algorithm for the following parameter values:

$$\alpha = 5\%, \sigma = 20\%, T = 0.5, \gamma = 1, \Theta^b = \Theta^s = 0.1\%, S(0) = 1, \Delta t = 0.02. \tag{5.1}$$

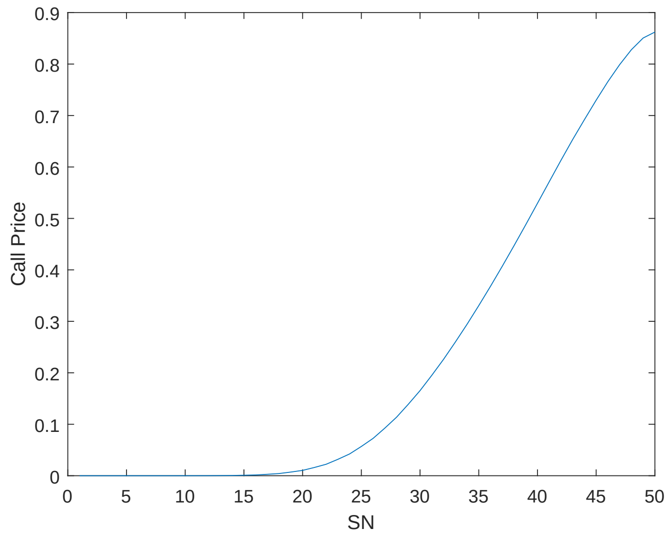

Figure 5.1: Call Prices versus Stock Prices.

The x-axis in this figure corresponds to the price grid points, 25^{th} point represents $S^h(0) = 1$, for which the option price equals

$$p_w(0, S(0)) = 0.0568. \tag{5.2}$$

Each increment on the x-axis corresponds to an increase of $e^{\sqrt{\Delta t}\sigma}$. We note that qualitatively the price graph looks similar to the European call option price under Black Scholes.

The hedging strategy is computed as the difference between the optimal controls corresponding to the value functions V_1 and V_w. As discussed in the introduction these optimal controls divide the state space of the price process into buy/no transaction/sell regions; we compute these regions for the approximating Markov chain using the iterative algorithm given in the previous Chapter. Several examples of these regions are shown in Figure 5.2, 5.5, 5.7 and 5.9. The area under blue line is buy region while the area above red line is sell region. No transaction region is the area between these lines. As suggested in [14] and recalled in the earlier chapter, the regions do not intersect. Sudden drop in y for $SN \geq 40$ is due to constraints on price dynamics at the boundary.

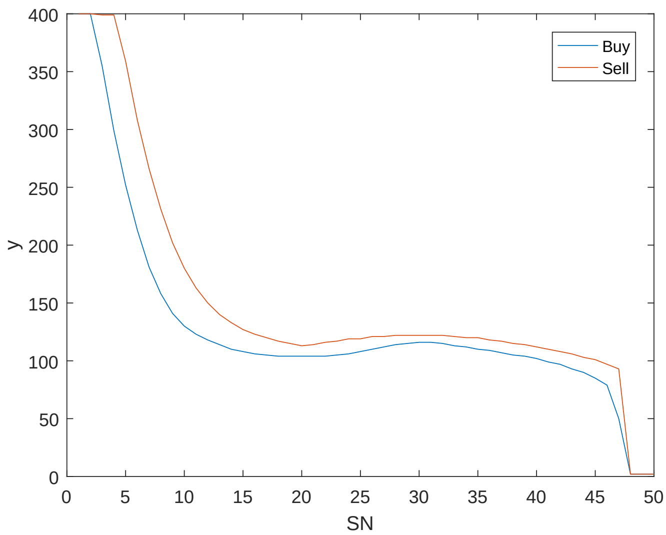

Figure 5.2: Regions.

The hedging strategy consists of the difference $\pi_w - \pi_1$ of the optimal policies for V_w and V_1. The difference between the buy/no transaction boundaries gives a sense of what the hedging strategy looks like, this is given in Figure 5.3. We note that this figure is similar to the Δ of a European call option under the Black Scholes framework except for the sudden drop for high underlying value, which is once again due to the constrained price dynamics imposed at very high prices to keep the numerical computations finite. The next section explains the performance measure used to test the performance of the hedging algorithm.

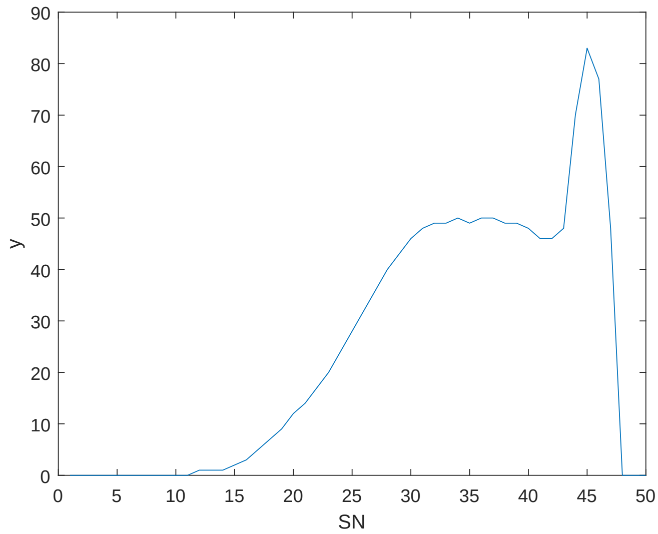

Figure 5.3: Hedging Strategy.

5.2 Conditional Mean of Hedging Error As a Function of Parameters

Let B^h and y^h denote the bond and stock positions given by the hedging algorithm. The hedging error at terminal time T is defined as follows:

$$H(T) = (c(y^h(T) - 1, S(T)) + E)\mathbb{1}_{\{S(T)>K\}} + c(y^h(T), S(T))\mathbb{1}_{\{S(T)\leq K\}} + p_w,$$

where we use $r = 0$. $H(T) = 0$ means a perfect hedge, $H(T) > 0$ means positive profit for the call writer and $H(T) < 0$ means that the writer will have to borrow $H(T)$ to complete the call transaction. $H(T)$ is a random variable whose distribution is not known- then a natural way to study is through simulation, and this is what we will do in this chapter. To simulate $H(T)$ we generate $K = 4000$ sample paths of the underlying security and apply the hedging algorithm to compute $H(T)$. Then, for example, $\mathbb{E}[H(T)]$ is approximated by

$$\frac{1}{K}\sum_{k=1}^{K} H_k(T),$$

where $H_k(T)$ is the hedging error computed for the k^{th} simulated price path. This algorithm applied to the parameter values listed in (5.1) gives $\mathbb{E}[H(T)] = 0.000613$, i.e., an average hedging error of almost 0 and 1.09% of the option price given in 5.2. Although this is a positive result for the option writer, the risky part of the hedging error for the writer is when it is negative, therefore it is important to measure how negative $H(T)$ can be. The same simulation results imply $P(H(T) < 0) = 45.51\%$, i.e., for the parameter values listed in (5.1) the hedging algorithm leads to a loss almost half the time. The expected loss given $H(T) < 0$ turns out to be

$$\mathbb{E}[H(T)|H(T) < 0] = -0.0218, \tag{5.3}$$

which is -38.43% of the option price, both of these values suggest that the hedging algorithm does have a significant risk of losing money. In light of these observations we have decided to measure the performance of the hedging algorithm by the conditional expected loss given in (5.3) as a percentage of the option price, i.e.,

$$\Xi(T) = \frac{\mathbb{E}[H(T)|H(T) < 0]}{p_w(0, S(0))}$$

which incorporates both the size of the loss and its probability of happening. The next sections study this performance measure as a function of the parameter values.

5.2.1 Transaction Costs

We shall now investigate the relation between hedging error and varying transaction costs. Buying and selling transaction costs are set to be equal. The rest of the parameter values are

$$\alpha = 5\%, \sigma = 20\%, r = 0\%, T = 0.5, \gamma = 1, S(0) = 1, E = 1, \Delta t = 0.02.$$

Having specified the input parameters, we exhibit the results of hedging error evaluation at Figure 5.4.

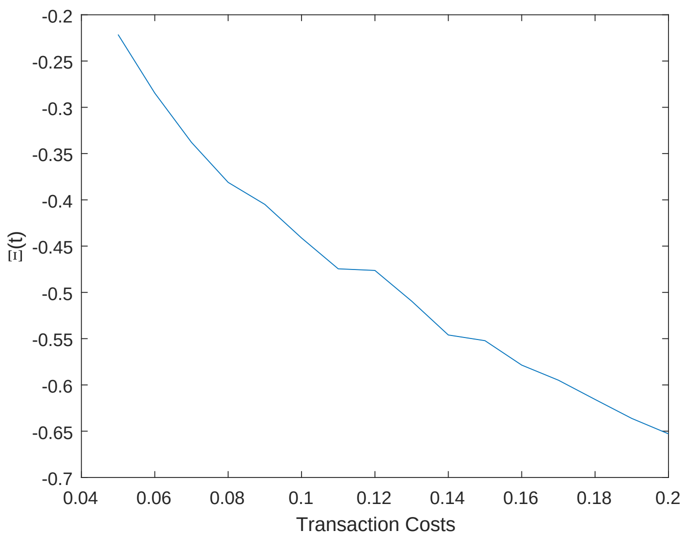

Figure 5.4: Hedging Error and Transaction Costs.

We observe that hedging error increases as transaction costs increase. It reaches up to 65% of the option price given for extremely high transaction fees. This is consistent with the fact that as transaction costs converge to 0 the model converges to the standard Black Scholes which has 0 hedging error. Another interpretation is as follows: increased transaction costs leaves lesser room to make transactions. Boundaries of Buy and Sell region moves away from each other as transaction costs increase. Therefore, No Transaction region expands. Ultimate optimal policy for investor is to make transactions as little as possible. This point is illustrated in the Figure 5.2.

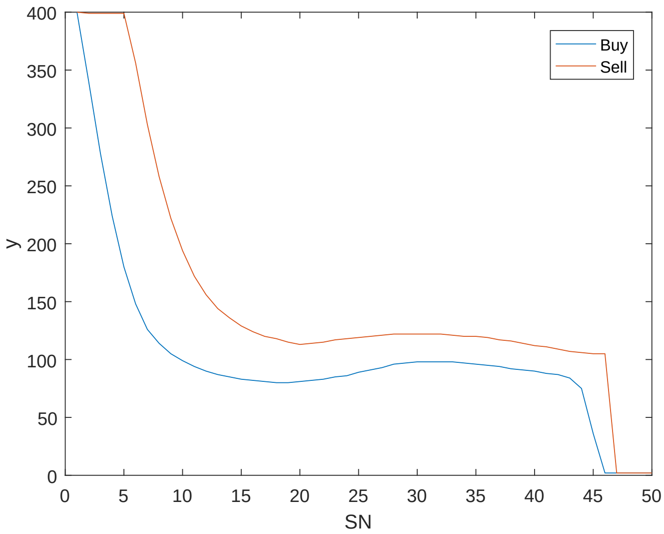

Figure 5.5: New Region for $\Theta^{b,s} = 0.005$.

Expansion of the No Transaction region can be seen when we compare Figure 5.2 to Figure 5.5. Although the premium charged for the option increases due to transaction costs, it is not enough to offset the losses caused by the lesser flexible trading strategy.

Table 5.1: Transaction Costs and Option Price

$\Theta^b = \Theta^s$	$\mathbf{p_w(0, S(0))}$
0.001	0.0568
0.01	0.0620
0.05	0.1042
0.1	0.1412
0.15	0.1769
0.2	0.2128

Table 5.1 shows the option price for different levels of transaction costs. As indicated above, the premium charged by the option writer increases with higher level of transaction costs.

5.2.2 Degree of Risk Aversion

Hedging Error against different levels of risk aversion, shown by γ, is tested with the parameters below,

$$\alpha = 5\%, \sigma = 20\%, r = 0\%, T = 0.5, \Theta^b = \Theta^s = 0.1\%, S(0) = 1, E = 1, \Delta t = 0.02.$$

The results are given in Figure 5.6. It is observed that hedging performance measure decreases as investors are more risk averse. This is explained by the fact that increase in risk aversion rate increases the premium charged by the option writer. Therefore, increase in option premium he receives is more likely to cover the loss from hedging errors. In addition, the distance between Buy and Sell regions is shorter if the investor is more risk averse. This enables the investor to trade more in the market as No Transaction region is shrunk.

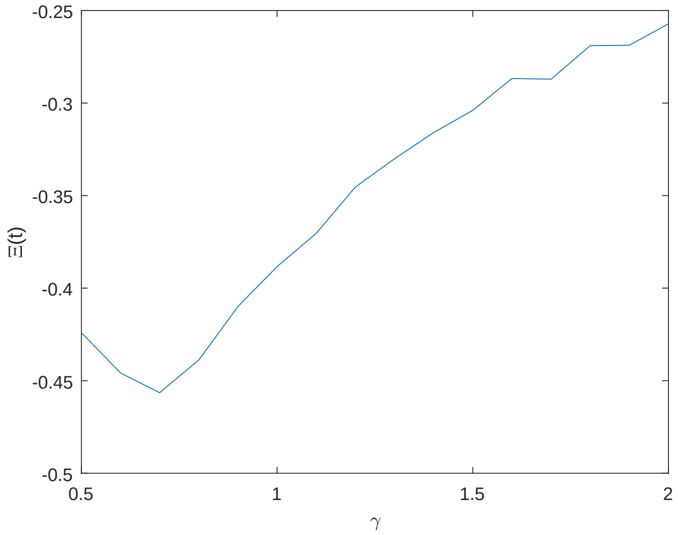

Figure 5.6: Hedging Error and Risk Aversion.

The effect of increase in risk aversion rate on the regions is provided in Figure 5.7. It suggests that No Transaction region becomes thinner for higher stock prices.

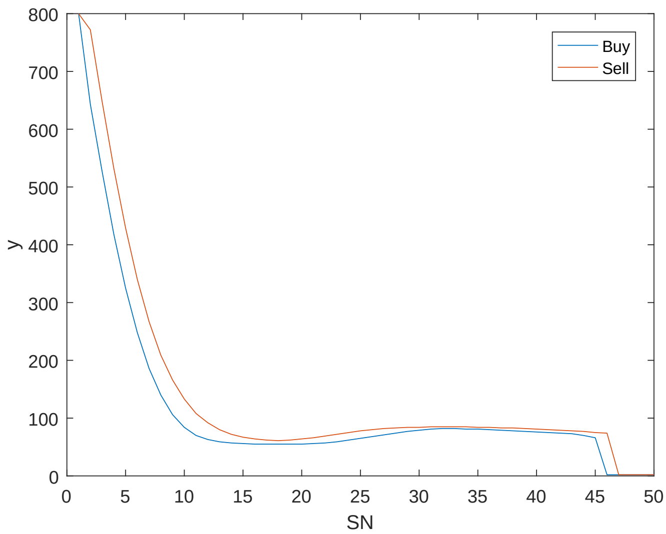

Figure 5.7: New Region for $\gamma = 2$.

The effect of changes in risk aversion rate on the option premium is presented on the Table 5.2. We see that option becomes more expensive as γ increases. However, it should be pointed out that those changes taking place in option prices are relatively small. Changes in transaction costs have stronger effect on the option price as indicated in Table 5.1.

Table 5.2: Risk Aversion and Option Price

γ	$\mathbf{p_w(0, S(0))}$
0.1	0.0564
0.5	0.0566
1	0.0568
1.5	0.0570
2	0.0571

5.2.3 Volatility

Relation between hedging error and volatility is investigated in this section. Measurement is made with the following designated values;

$$\alpha = 5\%, \gamma = 1, r = 0\%, T = 0.5, \Theta^b = \Theta^s = 0.1\%, S(0) = 1, E = 1, \Delta t = 0.02.$$

We provide the results in Figure 5.8. It is observed that hedging error is up to almost 65% of the option price for low volatility levels. Also, we see that hedging performance measure decreases as volatility increases. For a risk averse option writer, higher volatility causes an increase in the option price which he benefits from by receiving the premiums. Furthermore, he is less likely to incur losses since No Transaction region is contracted as indicated in Figure 5.9. This enables investor to make more adjustments in the portfolio.

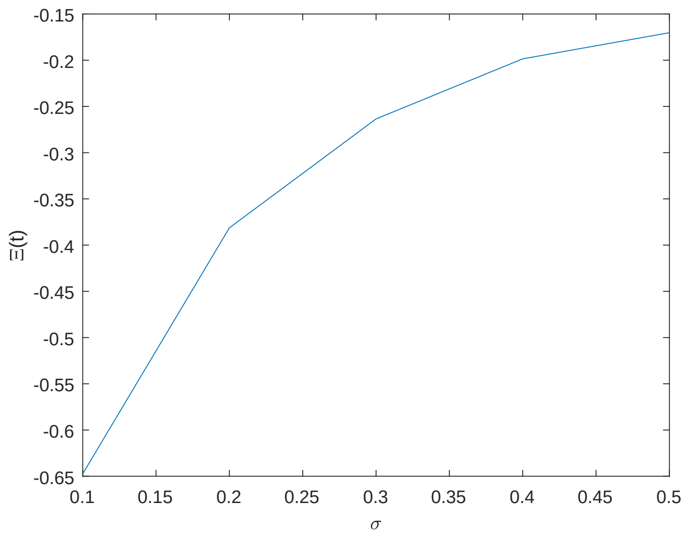

Figure 5.8: Hedging Error and Volatility.

A direct comparison of Figure 5.2 and 5.9 suggests that Buy and Sell regions move closer to each other as volatility increases. As pointed out before, this causes more transactions and higher option premiums for the writer.

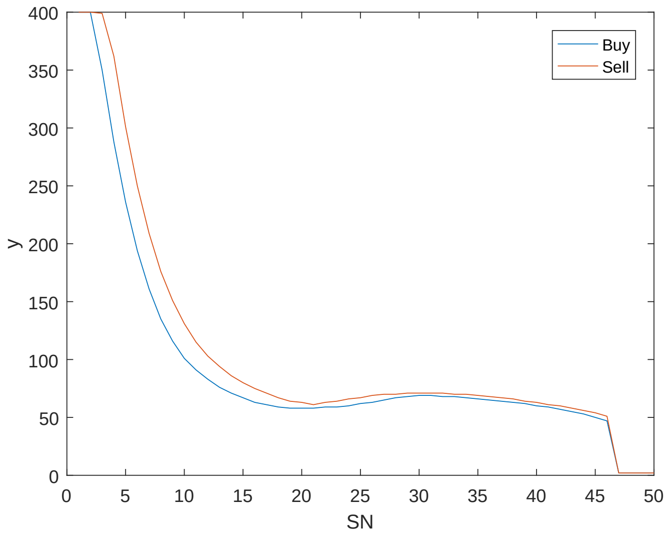

Figure 5.9: New Region for $\sigma = 0.5$.

Relation between volatility and the option prices is demonstrated at Table 5.3. Similar to the case when transaction costs are ignored, higher volatility rates cause option premiums to increase. It is observed that changes in volatility have a substantial effect on the option price. However, increase in volatility causes hedging performance measure to decrease unlike the increase in transaction costs.

Table 5.3: Volatility and Option Price

σ	$\mathbf{p_w(0, S(0))}$
0.1	0.0285
0.2	0.0568
0.3	0.0845
0.4	0.1118
0.5	0.1389

5.2.4 Moneyness of the Option

Hedging error is calculated for different initial stock prices. Computation is made with the following inputs,

$$\alpha = 5\%, \sigma = 20\%, r = 0\%, T = 0.5, \gamma = 1, E = 1, \Delta t = 0.02.$$

Results of the computation is displayed in Figure 5.10. We observe that hedging error is a decreasing function of stock prices. Hedging Error is large as 140% of the option price for out of the money options. Our interpretation is as follows: since out of the money options have 0 intrinsic value, their prices are lower than that of in the money options. Therefore, option writer would receive more premium which is a positive factor in decreasing the hedging error.

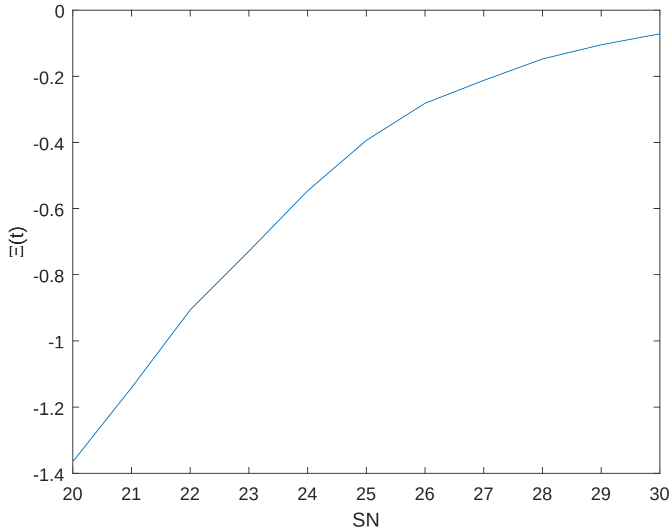

Figure 5.10: Hedging Error and Moneyness.

Option prices for different initial stock prices are computed and shown in Table 5.4. It is observed that initial stock prices have a significant effect on the option premium. Moreover, the amount charged for the option increases considerably as intrinsic value increases. We observe that this finding is consistent with the Black Scholes framework.

Table 5.4: Moneyness and Option Price

SN	$\mathbf{p_w(0, S(0))}$	**Moneyness**
20	0.0104	Out of the Money
21	0.0160	Out of the Money
22	0.0222	Out of the Money
23	0.0318	Out of the Money
24	0.0423	Out of the Money
25	0.0568	At the Money
26	0.0725	In the Money
27	0.0925	In the Money
28	0.1137	In the Money
29	0.1389	In the Money
30	0.1654	In the Money

Remark 5.2.1. *Numerical simulations show that $\Xi(T)$ is an increasing function of $\Theta^{b,s}$ and decreasing function of $\gamma, \sigma, S(0)$. Therefore, increasing the risk aversion parameter γ shall result in lower hedging error for any level of $\Theta^{b,s}$. This point is illustrated in Figure 5.11. When parameters are specified as below*

$$\alpha = 5\%, \sigma = 20\%, T = 0.5, S(0) = 1, \Delta t = 0.02,$$

$\Xi(T)$ is the blue line for $\gamma_1 = 1$ and is the red line for $\gamma_2 = 2$. The figure suggests that there is a significant decrease in hedging errors when γ is chosen appropriately.

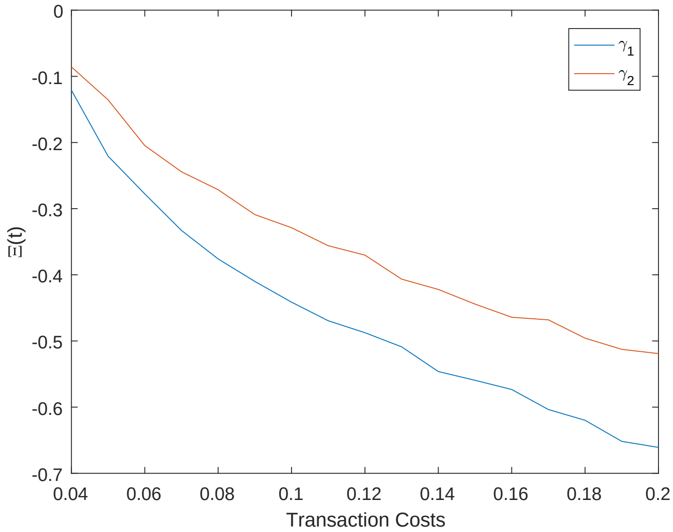

Figure 5.11: Hedging Error for Different Risk Aversion Rates and Transaction Costs. $(\gamma_1 = 1, \gamma_2 = 2)$

Same γ effect also is observed for σ as shown in Figure 5.12. For the following parameters,

$$\alpha = 5\%, T = 0.5, S(0) = 1, \Delta t = 0.02, \gamma_1 = 1, \gamma_2 = 2$$

we observe that increase in risk aversion rate causes hedging error to decrease for any level of volatility.

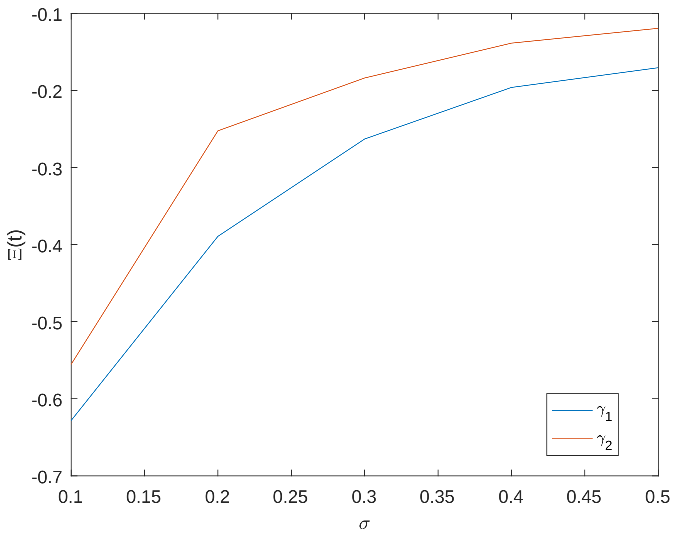

Figure 5.12: Hedging Error for Different Risk Aversion Rates and Volatility. $(\gamma_1 = 1, \gamma_2 = 2)$

CHAPTER 6

CONCLUSION

In this thesis, we evaluated the hedging performance of the utility based option pricing approach by measuring the expected conditional loss. As in [14], Markov Chain Approximation expounded in Kushner and Dupuis [21]) is adopted to compute the optimal control problem. We have preserved the exponential utility function used in Davis et al. [14] and carried out explicit approximations accordingly. Division of the state space into three regions is verified when numerical computation is implemented. The potential loss caused by hedging strategy is measured by expected loss given there is a loss. Our main finding is that, at least for a range parameter values (low volatility, high transaction costs and low risk aversion of the hedger) the hedging error can be significantly large as a percentage of the computed option price. A natural remedy for this is to choose the risk aversion parameter in a way that conditional expected loss is low; Figure 5.11 and 5.12 demonstrate this approach. On the positive side, the hedging algorithm, when the risk aversion parameter is chosen correctly, does provide a reasonable protection against loss arising from writing the call option.

As in [14] the present thesis works assumes that asset prices follow Black-Scholes dynamics and the computed hedging performance measures are made under this assumption. A well known fact is that actual asset price dynamics are not constant growth / volatility / interest rate. Since [14] the indifference utility framework has been generalized to a wider range of price dynamics (see our literature review in Chapter 2). A natural direction for future work is to extend our hedging study to these dynamics. In the present work the hedging performance is computed on simulated data; carrying out a similar analysis on actual asset price data is another direction for future research.

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