BACHIAN GRAVITY IN THREE DIMENSIONS

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ABSTRACT

BACHIAN GRAVITY IN THREE DIMENSIONS

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Modified theories in 3-dimensions such as the topologically massive gravity (TMG), new massive gravity (NMG) or Born-Infeld extension of NMG arise from the variations of diffeomorphism invariant actions; hence the resulting field equations are divergence free. Namely, the rank two tensor defining the field equations satisfy a Bianchi identity for all smooth metrics. However there are some recently constructed theories that do not identically satisfy Bianchi identities for all metrics, but only for the solutions of the theory. These are called on-shell consistent theories of which examples are the minimal massive gravity (MMG) and the exotic massive gravity (EMG). We work out the generic on-shell consistent model in 3-dimensions as a modified Einstein gravity theory which is based on the analog of the Bach tensor, hence we name it as the Bachian gravity. Conserved charges are found by using the linearization about maximally symmetric backgrounds for the Bañados-Teitelboim-Zanelli (BTZ)-black hole metric. It is complicated to solve the field equations of the gravity theory and hence very few solutions with only maximal symmetry are known. We use the projection formalism to obtain a reduction of the some relevant 2-tensors defining the field equations with the help of the Geroch's reduction method.

Keywords: 3-dimensional gravity, Bachian Gravity, Topologically Massive Gravity, New Massive Gravity, Exotic Massive Gravity,Symmetry reduction

ÖZ

ÜÇ BOYUTTA BACHIAN KÜTLE ÇEKİMİ

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Topolojik kütleli kütleçekim, Yeni kütleli kütleçekim veya Born-Infeld genişletilmiş Yeni kütleli kütleçekim gibi modifiye edilmiş 3 boyutlu teoriler difeomorfizmler altında değişmez kalan Etki'lerin varyasyonları sonucu elde edilirler ve ulaşılan alan denklemlerinin diverjansı sıfırdır. Yani bütün düzgün metrikler için, alan denklemlerini tanımlayan rank-2 tensörler Bianchi özdeşliğini sağlar. Bununla birlikte son zamanlarda geliştirilen bazı teoriler bütün metrikler için Bianchi özdeşliğini direkt sağlamak yerine, teorinin çözümü üzerinden sağlamaktadır. Bu tür teoriler "on-shell" tutarlı olarak adlandırılırlar. Örnek olarak Minimal kütleli kütleçekim ve Egzotik kütleli kütleçekim teorilerini verebiliriz. 3-boyutta Bach tensörünü göz önüne alarak ismini verdiğimiz Bachian kütleçekim teorisi, Einstein kütleçekim teorisinin bir modifikasyonu olarak 3-boyutlu kapsamlı on-shell tutarlı bir model olarak çözülmüştür. Korunumlu yükler, maksimum simetrik Bañados-Teitelboim-Zanelli BTZ kara deliği metriği etrafında linerizasyon kullanılarak bulunmuştur. Kütleçekim teorilerinin alan denklemlerini çözmenin zorluğu nedeniyle çok az sayıda ve sadece maksimum simetriye sahip çözümler bilinmektedir. Geroch indirgeme metodu yardımıyla alan denklemlerini tanımlayan bazı rank-2 tensörlerin indirgenmiş halleri projeksiyon for-

mulasyonu kullanılarak bulunmuştur.

Anahtar Kelimeler: 3-boyutlu Kütleçekim, Bachian Kütleçekim, Topolojik Kütleli Kütleçekim, Yeni Kütleli kütleçekim, Egzotik Kütleli Kütleçekim,Simetri indirgemesi

To my family and to people who unjustifiably prosecuted in Turkey

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LIST OF ABBREVIATIONS

ABBREVIATIONS	
GR	General Relativity
dS	de Sitter
AdS	Anti-de Sitter
TMG	Topologically Massive Gravity
NMG	New Massive Gravity
EMG	Exotic Massive Gravity

CHAPTER 1

INTRODUCTION: A BRIEF REVIEW OF GENERAL RELATIVITY

General relativity models gravity as a four dimensional manifold M [1] with certain desired properties that we shall discuss, whose metric g is determined from the Einstein's equations

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \frac{8\pi G_N}{c^4} T_{\mu\nu}, \tag{1.1}$$

where $R_{\mu\nu}$ is the Ricci tensor, R is the scalar curvature, Λ is the cosmological constant, $T_{\mu\nu}$ is the energy-momentum tensor and G_N is the Newton's constant while c is the speed of light. The left hand side is purely related to geometry, while the right hand side represents all possible matter distribution. The cosmological constant Λ was observed to be tiny but positive: in SI units $\Lambda = 10^{-52} m^{-2}$ hence it plays its major role in the global dynamics of the universe [1]. On the other hand, the numerical value of the "coupling constant" $\kappa := \frac{8\pi G_N}{c^4}$ is $\kappa = 2.1 \times 10^{-43} m/J$ which is again small but when it gets multiplied by possibly large $T_{\mu\nu}$ (as in the interior of a neutron star) whose unit is J/m^3 one gets a large effect. The fact that the left hand side of Eq.(1.1) satisfies the so called Bianchi identity $\nabla_\nu G^{\mu\nu} = 0$ for all smooth metrics, requires the so called the covariant conservation law $\nabla_\nu T^{\mu\nu} = 0$. The matter content of the Universe in small scales is very complicated, therefore there is no hope of solving Eq.(1.1) exactly. But for large scales the Universe is homogeneous and isotropic and the matter distribution can be modelled with some simple fluids.

The average observed density of the universe seems to be around $\rho \cong 9.9 \times 10^{-27} kg/m^3$ [2] which is related to T_{00} via $T_{00} = \rho c^2 \approx 10^{-9} J/m^3$. So when multiplied with the coupling constant κ one gets a value around the value of the cosmological constant

[1] In the next chapter we review the salient features of Riemannian geometry and relegate details of topology and manifolds to Appendix A and Appendix B.

Λ. These arguments suggest that at the large scales and/or outside matter source, the universe is an "Einstein space" satisfying

$$R_{\mu\nu} = \Lambda g_{\mu\nu}. \tag{1.2}$$

So a main part of understanding classical General Relativity is finding metrics that they obey Eq.(1.2); these are also called Einstein metrics. Unfortunately, even though there have been over 100 years of research on this equation, with many interesting solutions such as the rotating black holes or cosmological metrics, there is still no general procedure in finding exact solutions of these equations. For a compilation of exact solutions of General relativity, see the books [3, 4]. The reason is clear: this tensor equation is highly non-linear and without assuming symmetries one ends up with coupled non-linear partial differential equations which are in general too complicated to solve. So all the solutions known up to date have some symmetries. The fact that classical General relativity is so hard also makes the possible quantum version of the theory highly complicated. In fact, we have not yet been able to quantize General relativity: naive perturbation theory that works for other classical field theories such as electrodynamics fails as there appear new divergences at the one and two loop levels and beyond. These complications in the four dimensional General relativity led researchers to study gravity models in simpler settings of two and three dimensions. Research in lower dimensional gravity dates back to 1960s but received a renewed interest since 1980s. Some of the historical developments can be found in the book [5].

The subject of this thesis is the three dimensional gravity theories, not just Einstein theory, but its various generalizations that have received attention recently. The bulk of the thesis depends on our published work [6]. Motivation for studying three dimensional gravity theories will be explained in more detail in Chapter 3.

The lay-out of this thesis is as follows: In Chapter 2, some mathematical preliminaries that includes Riemannian geometry, hypersurfaces and Stokes' Theorem are briefly given. More technical details are explored in the Appendix starting from the notion of maps, topological spaces and manifolds. In Chapter 3, we discuss some known massive gravity theories such as the Topologically Massive Gravity (TMG), New Massive Gravity (NMG), Minimal Massive Gravity (MMG) in 3-dimensions. In

Chapter-4, we discuss the Exotic Massive Gravity (EMG) and Bachian Gravity as a general extension. This chapter is mostly based on our published work [6]. We also discuss for the developments in Exotic Massive Gravity in that chapter. In Chapter-5, we consider a 3-dimensional manifold with a time-like Killing vector field and a metric adapted to this Killing vector field; and study the decomposition of the relevant tensors such as the Ricci and Cotton tensors as well as the more complicated 2-tensors that appear in the field equations of other 3-dimensional gravity theories.

CHAPTER 2

MATHEMATICAL PRELIMINARIES

2.1 Basics of Riemannian Geometry

Following the discussion and notation of [7] let us consider a manifold M, if it is endowed with a metric g, which has the following properties at a point p on M:

i-) $g_p(U,V) = g_p(V,U)$, symmetric for all U, V in T_pM

ii-) $g_p(U,U) \geq 0$, positive definite, $g_p(U,U) = 0 \quad \Rightarrow \quad U = 0$,
then it is called a Riemann manifold and denoted as a pair (M, g). If (i) holds, but (ii) is related as

$$g_p(U,V) = 0, \quad \text{for any } U \in T_pM, \text{ then } V = 0, \tag{2.1}$$

M is called as a pseudo-Riemannian manifold.

The product between vectors and dual vectors is defined as a map in Appendix-B

$$\langle \quad , \quad \rangle : T_p^*M \times T_pM \to \mathbb{R}. \tag{2.2}$$

The metric g, which is a (0,2) tensor, gives an opportunity to construct an inner product of two vector fields. Let us say $U, V \in T_pM$, and the inner product is a map as

$$g_p : T_pM \otimes T_pM \to \mathbb{R} \tag{2.3}$$

The metric also forms an isomorphism between the tangent space and the dual tangent space, at each point on the manifold as

$$g_p(U, \quad) : T_pM \to \mathbb{R} \quad \Rightarrow \quad \text{a one-form } \omega_U := g_p(U, \quad) \in T_p^*M, \tag{2.4}$$

where the notation suggests that ω_U is the unique one-form obtained via the metric form the vector U.

The metric is symmetric 2-tensor, as such it can be diagonalized at each point p of M. The number of positive, negative or zero entries in this diagonal matrix is crucial. Let us note the possibilities:

i-) Riemann: all the eigenvalues are positive

ii-) Pseudo-Riemannian: One or more than one eigenvalues are negative

Note that if only one eigenvalue is negative, we also call it a Lorentzian metric. This is the relevant case for the space time of General Relativity and Minkowski spacetime. In the Lorentzian case, the vectors on the tangent space are divided into three classes as follows

i-) $g_p(U, U) > 0$, U is spacelike

ii-) $g_p(U, U) = 0$, U is lightlike(null)

iii-) $g_p(U, U) < 0$, U is timelike

On the manifold we need more structure than the metric to model gravity: next we discuss these structures.

2.1.1 Affine Connection

A derivative amounts to comparing a quantity at two distinct points. If that quantity is a scalar quantity, one does not need additional structure on the manifold to define the difference. But for any other quantities, such as vectors, tensors etc., the existing structure on the manifold is not sufficient and hence one must supply the manifold with some extra structure which will be broadly called the connection. In the differential structure of the manifold, a vector is defined on a tangent space at a specific point. To calculate a derivative of a vector field, it is transported between two points through a curve which is defined uniquely by some vector fields. Notice that this can be generalized to all type of tensor fields. A connection arises by the virtue of the tensor transportation.

Definition: Let $\chi(M)$ denote the space of vector fields on M. A map $\nabla : \chi(M) \times \chi(M) \to \chi(M)$, takes two vector fields and sends them to a one vector field, is called an affine connection with the following properties:

$$\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z, \tag{2.5}$$

$$\nabla_{(X+Y)}Z = \nabla_X Z + \nabla_Y Z, \tag{2.6}$$

$$\nabla_{(fX)}Y = f\nabla_X Y, \tag{2.7}$$

$$\nabla_X(fY) = X[f]Y + f\nabla_X Y, \tag{2.8}$$

where $X, Y, Z \in \chi(M)$ and $f \in C^\infty(M)$. First two properties demand the bilinearity, the third one is the so called tensorial property (namely the value of the X vector field at point p is relevant, but its extension around that point is not), the fourth property is the Leibniz condition with the definition $\nabla_X f = X[f]$. What is given here for vector fields is sufficient to generalize to form fields and tensor fields in general which we shall do below.

The connection coefficient is defined for a manifold with the coordinate system $x = \phi(p)$ and the coordinate basis $\{e_\mu\} = \{\partial/\partial x^\mu\}$ of the tangent space as

$$\nabla_{e_\nu} e_\mu := \nabla_\nu e_\mu =: e_\lambda \Gamma^\lambda_{\nu\mu}. \tag{2.9}$$

The action ∇ on a vector field W along the vector field V is

$$\begin{aligned}\nabla_V W &= V^\mu \nabla_{e_\mu}(W^\nu e_\nu) = V^\mu \left(e_\mu[W^\nu]\, e_\nu + W^\nu \nabla_{e_\mu} e_\nu\right) \\ &= V^\mu \left(\frac{\partial W^\lambda}{\partial x^\mu} + W^\nu \Gamma^\lambda_{\mu\nu}\right) e_\lambda = \left(V^\mu \nabla_\mu W^\lambda\right) e_\lambda.\end{aligned} \tag{2.10}$$

Here one can define;

$$\nabla_\mu W^\lambda \equiv \frac{\partial W^\lambda}{\partial x^\mu} + \Gamma^\lambda_{\mu\nu} W^\nu = \partial_\mu W^\lambda + \Gamma^\lambda_{\mu\nu} W^\nu. \tag{2.11}$$

The next topic, parallel transport plays an important role in the study of motion of point test particles on a manifold. We discuss it briefly.

2.1.2 Parallel Transport

A parallel transport of a vector is defined

$$\nabla_V X = 0 \tag{2.12}$$

where V is the tangent vector to a curve $c : (a, b) \to M$. Below the independent variable t parametrises the curve.

$$\begin{aligned}\nabla_V X &= V^\mu \nabla_{e_\mu} \left(X^\nu e_\nu\right) = V^\mu \left(e_\mu \left[X^v\right] e_\nu + X^v \nabla_{e_\mu} e_\nu\right) \\ &= V^\mu \left(\frac{\partial X^\lambda}{\partial x^\mu} + X^\nu \Gamma^\lambda_{\mu\nu}\right) e_\lambda\end{aligned} \tag{2.13}$$

$$\Rightarrow \quad \nabla_V X = \left(\frac{dx^\mu}{dt}\frac{\partial X^\lambda}{\partial x^\mu} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{dt} X^\nu\right) e_\lambda. \tag{2.14}$$

Here notice that $\frac{d}{dt}(X^\lambda) = \frac{dx^\mu}{dt}\frac{\partial}{\partial x^\mu}(X^\lambda) = \frac{dx^\mu}{dt}\frac{\partial X^\lambda}{\partial x^\mu}$ and so in the component form one has

$$\frac{\mathrm{d}X^\mu}{\mathrm{d}t} + \Gamma^\mu_{v\lambda}\frac{\mathrm{d}x^\nu(c(t))}{\mathrm{d}t} X^\lambda = 0. \tag{2.15}$$

There are certain curves that parallel transport their own tangent vectors; these are called geodesics and simply

$$\begin{aligned}0 = \nabla_V V &= V^\mu \nabla_\mu V^\lambda e_\lambda = V^\mu \left(\partial_\mu V^\lambda + \Gamma^\lambda_{\mu\nu} V^\nu\right) e_\lambda \\ &= \left(\frac{dx^\mu}{dt}\frac{\partial V^\lambda}{\partial x^\mu} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{dt} V^v\right) e_\lambda = \left(\frac{dV^\lambda}{dt} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\right) e_\lambda.\end{aligned} \tag{2.16}$$

Here realize that $\frac{dV^\lambda}{dt} = \frac{d}{dt}(V^\lambda) = \frac{d}{dt}(\frac{dx^\lambda}{dt}) = \frac{d^2x^\lambda}{dt^2}$. Now we arrive the geodesic equation;

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}t^2} + \Gamma^\mu_{\nu\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}t}\frac{\mathrm{d}x^\lambda}{\mathrm{d}t} = 0. \tag{2.17}$$

More properly this is an affinely parametrized geodesic equation, namely any other perturbation $t' = c_1 t + c_2$ with c_1 and c_2 keep the equation intact, but, for a non-affine parametrization of the form $t' = f(t)$ with some smooth function f, $\nabla_V = f$

2.1.3 The Covariant Derivative of Tensor Fields

Let us consider a pairing (or product) of a one-form $\omega \in \Omega^1(M)$ with a vector field $Y \in \chi(M)$,$\langle \omega, Y\rangle \in C^\infty(M)$. Using the affine connection of a function $\langle \omega, Y\rangle$ along the vector field $X \in \chi(M)$ will give us the covariant derivative of a one-form.

$$X[\langle \omega, Y\rangle] = \nabla_X[\langle \omega, Y\rangle] = \langle \nabla_X \omega, Y\rangle + \langle \omega, \nabla_X Y\rangle \tag{2.18}$$

$$\Rightarrow \quad X^\mu \partial_\mu(\omega_v Y^v) = (\nabla_X \omega)_v Y^v + \langle \omega, X^\mu \nabla_\mu Y^v\rangle \tag{2.19}$$

$$\Rightarrow \quad (\nabla_X \omega)_v Y^v = X^\mu \partial_\mu (\omega_v Y^v) - \omega_v X^\mu (\partial_\mu Y^v + \Gamma^\nu_{\mu\lambda} Y^\lambda)$$
$$= \left(X^\mu \partial_\mu \omega_\nu - X^\mu \omega_\lambda \Gamma^\lambda_{\mu\nu} \right) Y^v$$

$$(\nabla_X \omega)_v Y^v = X^\mu \partial_\mu \omega_\nu - X^\mu \omega_\lambda \Gamma^\lambda_{\mu\nu}. \tag{2.20}$$

For a basis vector $X = e_\mu$,

$$(\nabla_\mu \omega)_\nu = \partial_\mu \omega_\nu - \Gamma^\lambda_{\mu\nu} \omega_\lambda \tag{2.21}$$

and one can obtain connection coefficients by taking $\omega = dx^\nu$

$$\nabla_\mu \mathrm{d}x^\nu = -\Gamma^v_{\mu\lambda} \mathrm{d}x^\lambda. \tag{2.22}$$

The covariant derivative of a (p, q) tensor type is a straight forward generalization. As an example let us note the covariant derivative of a $(1, 1)$ tensor T:

$$\nabla_\nu T^\alpha_\beta = \partial_\nu T^\alpha_\beta + \Gamma^\alpha_{\nu\mu} T^\mu_\beta - \Gamma^\mu_{\nu\beta} T^\alpha_\mu. \tag{2.23}$$

It is important to analyze how connection coefficients transform under the coordinate changes. it does not transform as a tensor field of type $(1, 2)$. The connection coefficients $\Gamma^\lambda_{\mu\nu}$ apperantly look like a $(1, 2)$-tensor but this is not correct. At this stage they should be considered (without noting the symmetry issue in the lower indices) as n^3 functions for a manifold of dimension n. The fact that they are not components of a tensor is clear from the way they transform under coordinate transformations which we can work out as follows: Consider two overlapping charts. Let (U, ϕ) and (V, Ψ). $\{e_\mu\} = \{\partial/\partial x^\mu\}$ and $\{f_\alpha\} = \{\partial/\partial y^\alpha\}$ are the corresponding basis vectors adapted to the (U, ϕ) and (V, Ψ). Now, there are two connection coefficients such as

$$\nabla_{e_\nu} e_\mu = e_\lambda \Gamma^\lambda_{\nu\mu} \quad , \quad \nabla_{f_\alpha} f_\beta = \widetilde{\Gamma}^\gamma_{\alpha\beta} f_\gamma \tag{2.24}$$

and one can write $f_\alpha = \frac{\partial x^\mu}{\partial y^\alpha} e_\mu$.

$$\begin{aligned} \nabla_{f_\alpha} f_\beta = \nabla_{f_\alpha} \left(\frac{\partial x^\mu}{\partial y^\beta} e_\mu \right) &= \frac{\partial^2 x^\mu}{\partial y^\alpha \partial y^\beta} e_\mu + \frac{\partial x^\lambda}{\partial y^\alpha} \frac{\partial x^\mu}{\partial y^\beta} \nabla_{e_\lambda} e_\mu \\ &= \left(\frac{\partial^2 x^\nu}{\partial y^\alpha \partial y^\beta} + \frac{\partial x^\lambda}{\partial y^\alpha} \frac{\partial x^\mu}{\partial y^\beta} \Gamma^\nu_{\lambda\mu} \right) e_\nu \end{aligned} \tag{2.25}$$

$$\Rightarrow \quad \nabla_{f_\alpha} f_\beta = \widetilde{\Gamma}^\gamma_{\alpha\beta} f_\gamma = \widetilde{\Gamma}^\gamma_{\alpha\beta} \frac{\partial x^\nu}{\partial y^\alpha} e_\nu. \tag{2.26}$$

Now it is easy to realize the transformation of the connection coefficients by comparing two equations, Eq.(2.25) and Eq.(2.26)

$$\widetilde{\Gamma}^{\gamma}_{\alpha\beta} = \frac{\partial x^{\lambda}}{\partial y^{\alpha}}\frac{\partial x^{\mu}}{\partial y^{\beta}}\frac{\partial y^{\gamma}}{\partial x^{\nu}}\Gamma^{\nu}_{\lambda\mu} + \frac{\partial^2 x^{\nu}}{\partial y^{\alpha}\partial y^{\beta}}\frac{\partial y^{\gamma}}{\partial x^{\nu}}. \tag{2.27}$$

In GR, a specific connection, so called the metric compatible connection is used, next we discuss this.

2.1.4 The Metric Compatible Connection

In local coordinates, metric compatible connection is defined as $\nabla_{\lambda} g_{\mu\nu} = 0$ which yields explicitly

$$\partial_{\lambda} g_{\mu\nu} - \Gamma^{\kappa}_{\lambda\mu} g_{\kappa\nu} - \Gamma^{\kappa}_{\lambda\nu} g_{\kappa\mu} = 0. \tag{2.28}$$

In this aspect, the affine connection ∇ becomes a metric connection, in other words a metric compatible connection. In fact we can explicitly derive a formula for $\Gamma^{\lambda}_{\mu\nu}$ in terms of the metric with the following equations

$$\begin{aligned} \nabla_{\lambda} g_{\mu\nu} &= \partial_{\lambda} g_{\mu\nu} - \Gamma^{\kappa}_{\lambda\mu} g_{\kappa\nu} - \Gamma^{\kappa}_{\lambda\nu} g_{\kappa\mu} = 0, \\ \nabla_{\mu} g_{\nu\lambda} &= \partial_{\mu} g_{\nu\lambda} - \Gamma^{\kappa}_{\mu\nu} g_{\kappa\lambda} - \Gamma^{\kappa}_{\mu\lambda} g_{\kappa\nu} = 0, \\ \nabla_{\nu} g_{\lambda\mu} &= \partial_{\nu} g_{\lambda\mu} - \Gamma^{\kappa}_{\upsilon\lambda} g_{\kappa\mu} - \Gamma^{\kappa}_{\upsilon\mu} g_{\kappa\lambda} = 0. \end{aligned} \tag{2.29}$$

Subtracting the first equation from the other two equations, we arrive

$$\begin{aligned} \partial_{\mu} g_{\nu\lambda} + \partial_{\nu} g_{\mu\lambda} - \partial_{\lambda} g_{\nu\mu} &+ (\Gamma^{\kappa}_{\lambda\mu} - \Gamma^{\kappa}_{\mu\lambda}) g_{\kappa\nu} \\ &+ (\Gamma^{\kappa}_{\lambda\nu} - \Gamma^{\kappa}_{\nu\lambda}) g_{\kappa\mu} - (\Gamma^{\kappa}_{\mu\nu} - \Gamma^{\kappa}_{\nu\mu}) g_{\kappa\lambda} = 0 \end{aligned} \tag{2.30}$$

one defines the Torsion tensor as a $(1, 2)$ tensor as

$$T^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu}. \tag{2.31}$$

Note that even though the connection is not a tensor as we have shown, the difference of connections is a tensor since the non-homogeneous term in Eq.(2.27) drops out. After solving the above equation for $\Gamma^{\kappa}_{(\mu\nu)}$, one reaches at

$$\Gamma^{\kappa}_{(\mu\nu)} = \left\{ \begin{matrix} \kappa \\ \mu\nu \end{matrix} \right\} + \frac{1}{2}\left(T_{\nu}{}^{\kappa}{}_{\mu} + T_{\mu}{}^{\kappa}{}_{\nu}\right) \tag{2.32}$$

with $(\mu\nu)$ denoting symmetrization. Here we defined the connection coefficients known as the Christoffel symbols:

$$\left\{ \begin{matrix} \kappa \\ \mu\nu \end{matrix} \right\} = \frac{1}{2} g^{\kappa\lambda} \left(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu} \right). \tag{2.33}$$

As a result, making a simple manipulation on the Eq. (2.32), the connection coefficient Γ is found as

$$\Gamma^\kappa_{\mu\nu} = \Gamma^\kappa_{(\mu\nu)} + \Gamma^\kappa_{[\mu\nu]}, \quad \Gamma^\kappa_{[\mu\nu]} = \frac{1}{2} T^\kappa_{\mu\nu}$$

$$\Gamma^\kappa_{\mu\nu} = \left\{ \begin{matrix} \kappa \\ \mu\nu \end{matrix} \right\} + \frac{1}{2} \left(T_\nu{}^\kappa{}_\mu + T_\mu{}^\kappa{}_\nu + T^\kappa_{\mu\nu} \right). \tag{2.34}$$

The Contorsion tensor is defined as $K^\kappa_{\mu\nu} \equiv \frac{1}{2} \left(T_\nu{}^\kappa{}_\mu + T_\mu{}^\kappa{}_\nu + T^\kappa_{\mu\nu} \right)$. In GR, one also assumes that the Torsion is zero, therefore in what follows we do not use the symbol $\{{}^{\kappa}_{\mu\nu}\}$ but instead use $\Gamma^\kappa_{\mu\nu}$ to denote the metric compatible torsion-free connection which is called the "Levi-Civita connection". It is clear that the geometry simplified a lot with this choice of the connection.

2.1.5 Curvature and Torsion

Above, we gave the torsion tensor components in local coordinates. Here we define it more geometrically and also introduce the curvature tensor. A Torsion tensor is defined as a map

$$\begin{aligned} T : \chi(M) \otimes \chi(M) &\to \chi(M) \\ (X, Y) &\to T(X, Y) \end{aligned} \tag{2.35}$$

defined as

$$T(X, Y) \equiv \nabla_X Y - \nabla_Y X - [X, Y]. \tag{2.36}$$

The curvature tensor is defined as a map

$$R : \chi(M) \otimes \chi(M) \otimes \chi(M) \to \chi(M), \tag{2.37}$$

where

$$R(X, Y, Z) = R(X, Y)Z \equiv \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{2.38}$$

where $[X, Y]$ denotes the Lie bracket of two vector fields. The tensorial property of them can be shown rigourosly, which we do here.

$$R(fX, gY)(hZ) = \nabla_{fX}\nabla_{gY}(hZ) - \nabla_{gY}\nabla_{fX}(hZ) - \nabla_{[fX,gY]}(hZ), \tag{2.39}$$

with f, g smooth functions.

$$[fX, gY] = fX[g]Y - gY[f]X + fg[X, Y]. \tag{2.40}$$

$$\begin{aligned}
R(fX, gY)(hZ) &= f\nabla_X\{g\nabla_Y(hZ)\} - g\nabla_Y\{f\nabla_X(hZ)\} \\
&- fX[g]\nabla_Y(hZ) + gY[f]\nabla_X(hZ) - fg\nabla_{[X,Y]}(hZ) \\
&= fX[g]\nabla_Y(hZ) + fg\nabla_X\{Y[h]Z + h\nabla_Y Z\} \\
&- gY[f]\nabla_X(hZ) - gf\nabla_Y\{X[h]Z + h\nabla_X Z\} \\
&- fX[g]\nabla_Y(hZ) + gY[f]\nabla_X(hZ) - fg\nabla_{[X,Y]}(hZ) \\
&= fg\nabla_X\{Y[h]Z + h\nabla_Y Z\} - gf\nabla_Y\{X[h]Z + h\nabla_X Z\} \\
&- fg[X, Y][h]Z - fgh\nabla_{[X,Y]}(Z) \\
&= fgh\{\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z\} \\
&= fghR(X, Y)Z.
\end{aligned} \tag{2.41}$$

So this says that the curvature is tensorial. For the Torsion tensor, a similar calculation yields

$$\begin{aligned}
T(fX, gY) &= \nabla_{fX}(gY) - \nabla_{gY}(fX) - [fX, gY] \\
&= fg\{\nabla_X Y - \nabla_Y X - [X, Y]\} = fgT(X, Y).
\end{aligned} \tag{2.42}$$

Let us consider the coordinate basis $\{e_\mu\} = \{\frac{\partial}{\partial x^\mu}\}$ with its dual $\{dx^\mu\}$. For this basis $[e_\mu, e_\nu] = 0$. Now, it is easy to obtain component forms of the Torsion and Curvature tensors.

$$\begin{aligned}
T^\lambda_{\mu\nu} &= \left\langle \mathrm{d}x^\lambda, T\left(e_\mu, e_\nu\right)\right\rangle = \left\langle \mathrm{d}x^\lambda, \nabla_\mu e_\nu - \nabla_\nu e_\mu\right\rangle \\
&= \left\langle \mathrm{d}x^\lambda, \Gamma^\eta_{\mu\nu} e_\eta - \Gamma^\eta_{\nu\mu} e_\eta\right\rangle = \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}.
\end{aligned} \tag{2.43}$$

$$\begin{aligned}
R^\kappa{}_{\lambda\mu\nu} &= \left\langle \mathrm{d}x^\kappa, R\left(e_\mu, e_\nu\right) e_\lambda\right\rangle = \left\langle \mathrm{d}x^\kappa, \nabla_\mu\nabla_\nu e_\lambda - \nabla_\nu\nabla_\mu e_\lambda\right\rangle \\
&= \left\langle \mathrm{d}x^\kappa, \nabla_\mu\left(\Gamma^\eta_{\nu\lambda} e_\eta\right) - \nabla_\nu\left(\Gamma^\eta_{\mu\nu} e_\eta\right)\right\rangle \\
&= \left\langle \mathrm{d}x^\kappa, \left(\partial_\mu\Gamma^\eta_{v\lambda}\right) e_\eta + \Gamma^\eta_{v\lambda}\Gamma^\xi_{\mu\eta} - \left(\partial_\nu\Gamma^\eta_{\mu\lambda}\right) e_\eta - \Gamma^\eta_{\mu\lambda}\Gamma^\xi_{v\eta} e_\xi\right\rangle \\
&= \partial_\mu\Gamma^\kappa_{\nu\lambda} - \partial_\nu\Gamma^\kappa_{\mu\lambda} + \Gamma^\eta_{\nu\lambda}\Gamma^\kappa_{\mu\eta} - \Gamma^\eta_{\mu\lambda}\Gamma^\kappa_{\nu\eta}.
\end{aligned} \tag{2.44}$$

The torsion is a $(1, 2)$ and the curvature is a $(1, 3)$ tensor fields. For the metric compatible torsion-free connection $T = 0$ and the Curvature tensor is called the Riemann tensor.

2.1.6 The Ricci Tensor and the Scalar Curvature

The Ricci tensor is defined via the contraction of the Curvature as:

$$Ric(X, Y) \equiv \langle dx^\mu, R(e_\mu, Y)X\rangle \tag{2.45}$$

and in the component form, it reads

$$Ric_{\mu\nu} = Ric(e_\mu, e_\nu) = R^\lambda{}_{\mu\lambda\nu}. \tag{2.46}$$

Note that one writes $R_{\mu\nu} \equiv Ric_{\mu\nu}$. The scalar curvature $Scal$ is defined as

$$Scal \equiv g^{\mu\nu} Ric(e_\mu, e_\nu) = g^{\mu\nu} Ric_{\mu\nu} = g^{\mu\nu} R_{\mu\nu}, \tag{2.47}$$

and one introduces the notation $R = Scal$.

Bianchi Identities: Using the definition of the curvature tensor one can prove the following identities, whose proofs we omit here:

The first Bianchi identity;

$$R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0, \tag{2.48}$$

which in component form reads

$$R^\kappa{}_{\lambda\mu\nu} + R^\kappa{}_{\mu\nu\lambda} + R^\kappa{}_{\nu\lambda\mu} = 0. \tag{2.49}$$

The second Bianchi identity;

$$(\nabla_X R)(Y, Z)V + (\nabla_Z R)(X, Y)V + (\nabla_Y R)(Z, X)V = 0$$

$$\nabla_\kappa R^\xi{}_{\lambda\mu\nu} + \nabla_\mu R^\xi{}_{\lambda\nu\kappa} + \nabla_\nu R^\xi{}_{\lambda\kappa\mu} = 0. \tag{2.50}$$

Contracting, ξ and μ on the second Bianchi identity; one has

$$\nabla_\kappa R_{\lambda\nu} + \nabla_\mu R^\mu{}_{\lambda\nu\kappa} - \nabla_\nu R_{\lambda\kappa} = 0. \tag{2.51}$$

Another contraction yields

$$\nabla_\mu (R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R) = 0. \tag{2.52}$$

One defines the Einstein tensor as

$$\Rightarrow \quad G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R. \tag{2.53}$$

More geometrically, the covariant form of the Einstein tensor is a rank $(0, 2)$ tensor defined as

$$G = Ric - \frac{1}{2} Rg. \tag{2.54}$$

The fact that $\nabla_\mu G^{\mu\nu} = 0$ for any smooth metric is extremely crucial in General Relativity. In the exotic massive gravity theories discussed in this thesis, such an identity is not assumed for the $(0, 2)$ tensor defining the field equations. The consequences of this loss of "Bianchi identity" will be discussed.

2.2 Hypersurfaces

Hypersurfaces or codimension-one surfaces play an important role in General Relativity or other relativistic gravity theories. For example, to be able to count the dynamical degrees of freedom of the gravitational field, which we shall do in the next chapter, one has to choose a constant time slice and consider the induced metric and its derivative (momentum) on the surface. For this purpose, following Appendix-D of the book [8], we give here the details of the hypersurface geometry in generic n dimensions. A major part of the discussion is the projection of the tensors into the hypersurface and off the hypersurface. We carry out these and derive the Gauss's and Codazzi's equations. At the end of the chapter we discuss the Stoke's theorem that is used later in the conserved charge construction of exotic massive gravity.

For an n-dimensional manifold M, one can define an $n-1$-dimensional submanifold Σ and it is called a hypersurface. One way to define the hypersurface is to consider the level set of a single function f as

$$f(x) = c, \tag{2.55}$$

where c is a constant. The vector field ζ^μ;

$$\zeta^\mu = g^{\mu\nu} \nabla_\mu f \tag{2.56}$$

is orthogonal to all vectors that are defined on the surface Σ. There are three cases to specify this orthogonal vector;

i-)If ζ^{μ} is a timelike vector ($g(\zeta,\zeta) < 0, g_{\mu\nu}\zeta^{\mu}\zeta^{\nu} < 0$), then Σ is spacelike that means Σ "moves" along the time direction.[1]

ii-)If ζ^{μ} is a spacelike vector ($g(\zeta,\zeta) > 0, g_{\mu\nu}\zeta^{\mu}\zeta^{\nu} > 0$), then Σ is a timelike surface

iii-)If ζ^{μ} is a null vector ($g(\zeta,\zeta) = 0, g_{\mu\nu}\zeta^{\mu}\zeta^{\nu} = 0$), then Σ is a null surface.

For the timelike or spacelike cases, we can define

$$n^{\mu} := \pm \frac{\zeta^{\mu}}{\sqrt{|\zeta_{\mu}\zeta^{\mu}|}} \tag{2.57}$$

and it is called the normal vector with the magnitude

$$\begin{aligned} n^{\mu}n_{\mu} &= -1, \quad \text{for spacelike } \Sigma \\ n^{\mu}n_{\mu} &= +1, \quad \text{for timelike } \Sigma. \end{aligned}$$

The null case is subtle as the null vector is both orthogonal and parallel to the surface. Null hypersurfaces are relevant for the black hole event horizons, as we will not use them here in this thesis, we skip that part of the discussion which is done in detail [8].

2.2.1 Gaussian Normal Coordinates:

Let us assume that we choose a coordinate system on the hypersurface Σ, such that $y^i = \{y^1, \ldots, y^{n-1}\}$. For every point p on the Σ, one can define a geodesic whose tangent vector is n^{μ}(normal vector). The affine parameter of these geodesics is z, and it is unique after n^{μ} is normalized. There is always be a neighbourhood of a point p, let say q that is on the geodesic and not on the hypersurface Σ. One can reach that point by taking an affine parameter z as a coordinate component. As a result, the coordinate system that is valid on the manifold M at least locally, is defined as $\{z, y^1, \ldots, y^{n-1}\}$. This coordinate system is called the Gaussian normal coordinates and it may be not well defined globally on the manifold. $\{\partial_z, \partial_1, \ldots, \partial_{n-1}\}$ can be

[1] Note that we work with the mostly plus signature.

chosen as a basis vector fields and rename them as

$$\begin{aligned}(\partial_z)^\mu &= n^\mu,\\ (\partial_i)^\mu &= Y^\mu_{(i)}.\end{aligned} \tag{2.58}$$

We can now provide an explicit local form of the metric in Gaussian normal coordinates.

$$g(\partial_z, \partial_z) = g_{zz} = n_\mu n^\mu = \pm 1 = \sigma, \tag{2.59}$$

$$g(\partial_z, \partial_i) = g_{zi} = n_\mu Y^\mu_{(i)}. \tag{2.60}$$

To find $n_\mu Y^\mu_{(i)}$, let us take a directional covariant derivative along the z-coordinate;

$$\frac{D}{dz}(n_\mu Y^\mu_{(i)}) = n^\nu \nabla_\nu (n_\mu Y^\mu_{(i)}) = n^\nu \nabla_\nu n_\mu Y^\mu_{(i)} + n^\nu n_\mu \nabla_\nu Y^\mu_{(i)}. \tag{2.61}$$

From the geodesic equation, use $n^\nu \nabla_\nu n_\mu = 0$ to get

$$\frac{D}{dz}(n_\mu Y^\mu_{(i)}) = n^\nu n_\mu \nabla_\nu Y^\mu_{(i)},$$

$$\text{Note that,} \quad [n, Y_{(i)}]^\mu = 0 \quad \rightarrow \quad n^\nu \nabla_\nu Y^\mu_{(i)} = Y^\nu_{(i)} \nabla_\nu n^\mu$$

$$= n_\mu Y^\nu_{(i)} \nabla_\nu n^\mu = \frac{1}{2} Y^\nu_{(i)} \nabla_\nu (n_\mu n^\mu) = 0.$$

Therefore $n_\mu Y^\mu_{(i)} = 0$, since n_μ is orthogonal to all vectors on the Σ. Finally we have

$$g(\partial_i, \partial_j) = g_{ij} = \gamma_{ij} \tag{2.62}$$

and the line element reads

$$ds^2 = \sigma dz^2 + \gamma_{ij} dy^i dy^j, \tag{2.63}$$

clearly $\gamma_{ij} = \gamma_{ij}(z, y^i)$.

There is a natural map between the hypersurface and the manifold M that helps us to construct an induced metric and let us assume that Σ is defined by $z = z_*$ on M.

$$\begin{aligned}\phi &: \Sigma \rightarrow M\\ &: y^i \rightarrow x^\mu = (z_*, y^i).\end{aligned} \tag{2.64}$$

Now the pull-back of the metric g that is defined on M is simply

$$(\phi^* g)_{ij} = \gamma_{ij}. \tag{2.65}$$

It is natural to define a volume element on the hypersurface Σ by using the induced metric. A volume element of the whole manifold M is defined by the top form

$$\epsilon = \sqrt{|g|}dx^1 \wedge \cdots \wedge dx^n. \tag{2.66}$$

It is obvious that how to define a "volume element" of Σ whose metric is γ_{ij}.

$$\hat{\epsilon} = \sqrt{|\gamma|}dy^1 \wedge \cdots \wedge dy^{n-1}. \tag{2.67}$$

Here let us investigate the determinant of the metric g;

$$g_{\mu\nu} = \begin{pmatrix} \sigma & 0 \\ 0 & \gamma_{ij} \end{pmatrix} \Rightarrow \quad g = \sigma\gamma_{ij} = \pm\gamma_{ij}, \quad \sqrt{|g|} = \sqrt{|\gamma|}. \tag{2.68}$$

Therefore, we obtain the "volume element" of M as

$$\epsilon = \sqrt{|\gamma|}dz \wedge dy^1 \wedge \cdots \wedge dy^{n-1}. \tag{2.69}$$

The contraction of a volume element ϵ, with a normal vector n is;

$$\begin{aligned} \epsilon(n) &= \epsilon_{\mu_1\ldots\mu_n} dx^{\mu_1} \wedge \ldots dx^{\mu_n}(n^\lambda \partial_\lambda) \\ &= n^\lambda \epsilon_{\lambda\mu_2\ldots\mu_n} dx^{\mu_2} \wedge \ldots dx^{\mu_n}, \end{aligned} \tag{2.70}$$

where n^λ is the normal vector and with components , $n^\lambda = (1, 0, \ldots, 0)$.

$$\epsilon(n) = \epsilon_{\mu_2\ldots\mu_n} dx^{\mu_2} \wedge \ldots dx^{\mu_n} = \sqrt{|\gamma|}dy^1 \wedge \cdots \wedge dy^{n-1} = \hat{\epsilon}. \tag{2.71}$$

Then we can write the induced volume element in a component form,

$$\hat{\epsilon}_{\mu_1\cdots\mu_{n-1}} = n^\lambda \epsilon_{\lambda\mu_1\ldots\mu_{n-1}}. \tag{2.72}$$

2.2.2 Projection Tensor:

We have defined the hypersurface, next we would like to project tensors onto the surface and off the surface. The projection tensor is defined by

$$P_{\mu\nu} \equiv g_{\mu\nu} - \sigma n_\mu n_\nu, \tag{2.73}$$

where n^μ is the unit normal vector and $\sigma = n_\mu n^\mu$. For any vector $V \in T_pM$,

$$(P_{\mu\nu}V^\mu)n^\nu = \big[(g_{\mu\nu} - \sigma n_\mu n_\nu)V^\mu\big]n^\nu = 0, \tag{2.74}$$

which says that for any vector $V \in T_pM$, the projection operator projects a vector tangent to the surface. Another useful property of the projection tensor is acting as a metric for the vectors that are tangent to the surface, and by this reason projection tensor is also called the first fundamental form at the sufrace. For instance;

$$\begin{aligned} P_{\mu\nu}V^\mu W^\nu &= g_{\mu\nu}V^\mu W^\nu - \sigma n_\mu n_\nu V^\mu W^\nu \\ &= g_{\mu\nu}V^\mu W^\nu - \sigma(\underbrace{n_\mu V^\mu}_{=0})(\underbrace{n_\nu W^\nu}_{=0}) \\ &= g_{\mu\nu}V^\mu W^\nu. \end{aligned} \tag{2.75}$$

As expected, any power of a projection tensor is equal to itself since

$$P^\mu_\lambda P^\lambda_\nu = (\delta^\mu_\lambda - \sigma n^\mu n_\lambda)(\delta^\lambda_\nu - \sigma n^\lambda n_\nu) = \delta^\mu_\nu - \sigma n^\mu n_\nu = P^\mu_\nu. \tag{2.76}$$

The Riemann tensor measures the intrinsic curvature of a manifold. But for an embedded manifold in a higher dimensional space, the extrinsic curvature is relevant. The Lie derivative of the projection tensor along the normal vector will give us the extrinsic curvature of the hypersurface. Now let us define

$$K_{\mu\nu} \equiv \frac{1}{2}\mathcal{L}_n P_{\mu\nu}. \tag{2.77}$$

It is interesting to show that $K_{\mu\nu}$ is the twice projected Lie derivative of the metric with respect to the normal vector n. The Lie derivative of the metric tensor is $\mathcal{L}_n g_{\mu\nu} = \nabla_\mu n_\nu + \nabla_\nu n_\mu$.

$$\begin{aligned} P^\mu_\alpha P^\nu_\beta \mathcal{L}_n g_{\mu\nu} &= P^\mu_\alpha P^\nu_\beta(\nabla_\mu n_\nu + \nabla_\nu n_\mu) \\ &= (\delta^\mu_\alpha - \sigma n^\mu n_\alpha)(\delta^\nu_\beta - \sigma n^\nu n_\beta)(\nabla_\mu n_\nu + \nabla_\nu n_\mu) \\ &= \nabla_\alpha n_\beta + \nabla_\beta n_\alpha - \sigma n^\nu n_\beta \nabla_\alpha n_\nu - \underbrace{\sigma n^\nu n_\beta \nabla_\nu n_\alpha}_{-\sigma n_\beta a_\alpha} \\ &\quad - \underbrace{\sigma n^\mu n_\alpha \nabla_\mu n_\beta}_{-\sigma n_\alpha a_\beta} - \underbrace{\sigma n^\mu n_\alpha \nabla_\beta n_\mu}_{=0} \\ &\quad + \underbrace{n^\mu n^\nu n_\alpha n_\beta \nabla_\mu n_\nu}_{=0} + \underbrace{n^\mu n^\nu n_\alpha n_\beta \nabla_\nu n_\mu}_{=0}. \end{aligned} \tag{2.78}$$

$$\Rightarrow \quad \frac{1}{2}P^\mu_\alpha P^\nu_\beta \mathcal{L}_n g_{\mu\nu} = \nabla_{(\alpha}n_{\beta)} - \sigma n_{(\alpha}a_{\beta)} = \nabla_\alpha n_\beta - \sigma n_\alpha n_\beta. \tag{2.79}$$

Now let us show that $K_{\mu\nu} = \frac{1}{2}\mathcal{L}_n P_{\mu\nu} = \frac{1}{2}P_\mu^\alpha P_\nu^\beta \mathcal{L}_n g_{\alpha\beta}$.

$$\begin{aligned}\mathcal{L}_n P_{\mu\nu} &= n^\alpha \nabla_\alpha P_{\mu\nu} + P_{\alpha\nu}\nabla_\mu n^\alpha + P_{\mu\alpha}\nabla_\nu n^\alpha \\ &= -\sigma n^\alpha \nabla_\alpha (n_\mu n_\nu) + g_{\alpha\nu}\nabla_\mu n^\alpha + g_{\mu\alpha}\nabla_\nu n^\alpha \\ &\quad - \sigma n_\alpha n_\nu \nabla_\mu n^\alpha - \sigma n_\alpha n_\mu \nabla_\nu n^\alpha.\end{aligned} \tag{2.80}$$

Let us have a look at the following expression

$$n_\nu \nabla_\mu \underbrace{(n_\alpha n^\alpha)}_{=\sigma} = n_\nu n^\alpha \nabla_\mu n_\alpha + n_\nu n_\alpha \nabla_\mu n^\alpha = 2 n_\nu n^\alpha \nabla_\mu n_\alpha = 0,$$

$$\mathcal{L}_n P_{\mu\nu} = \nabla_\mu n_\nu + \nabla_\nu n_\mu - \sigma n_\mu a_\nu - \sigma n_\nu a_\mu,$$

$$\Rightarrow \quad \frac{1}{2}\mathcal{L}_n P_{\mu\nu} = \nabla_{(\mu} n_{\nu)} - \sigma n_{(\mu} a_{\nu)} = \nabla_\mu n_\nu - \sigma n_\mu n_\nu. \tag{2.81}$$

Therefore we obtain the desired result;

$$K_{\mu\nu} = \frac{1}{2}\mathcal{L}_n P_{\mu\nu} = \frac{1}{2}P_\mu^\alpha P_\nu^\beta \mathcal{L}_n g_{\alpha\beta} = P_\mu^\alpha P_\nu^\beta \nabla_\alpha n_\beta. \tag{2.82}$$

The extrinsic curvature $K_{\mu\nu}$ is a symmetric tensor. The contraction of the $K_{\mu\nu}$ with the normal vector n^μ is given by;

$$\begin{aligned}K_{\mu\nu} n^\mu &= (\nabla_\mu n_\nu + \nabla_\nu n_\mu - \sigma n_\mu a_\nu - \sigma n_\nu a_\mu) n^\mu \\ &= n^\mu \nabla_\mu n_\nu + \underbrace{n^\mu \nabla_\nu n_\mu}_{=0} - a_\nu - \sigma n_\nu a_\mu n^\mu \\ &= a_\nu - a_\nu - \underbrace{\sigma n_\nu n^\mu (n^\alpha \nabla_\alpha n_\mu)}_{=0} = 0.\end{aligned} \tag{2.83}$$

So it has no components orthogonal to the surface. Covariant derivative on the hypersurface Σ can be found by projecting the ordinary covariant derivative on the manifold. (Note: From now on we will use Latin indices instead of Greek indices due to the increasing number of indces.)

$$\hat{\nabla}_a V_b = P_a^c P_b^d \nabla_c V_d, \quad \hat{\nabla}_a T_{bc} = P_a^d P_b^e P_c^f \nabla_d T_{ed}, \quad \hat{\nabla}_a T^b_c = P_a^d P^b_e P_c^f \nabla_d T^e_f. \tag{2.84}$$

Now, using the definition of covariant derivative, let us find the Riemann tensor on the hypersurface.

$$[\hat{\nabla}_m, \hat{\nabla}_n] X_b = -\hat{R}^a{}_{bmn} X_a = \hat{\nabla}_m \hat{\nabla}_n X_b - \hat{\nabla}_n \hat{\nabla}_m X_b, \tag{2.85}$$

$$
\begin{aligned}
-\hat{R}^a{}_{bmn}X_a &= \hat{\nabla}_m\hat{\nabla}_n X_b - (m \leftrightarrow n) \\
&= P^t_m P^l_n P^e_b \nabla_t (P^r_l P^s_e \nabla_r X_s) - (m \leftrightarrow n) \\
&= P^t_m P^l_n P^e_b P^r_l P^s_e \nabla_t \nabla_r X_s + P^t_m P^l_n P^e_b \nabla_t (P^r_l P^s_e)(\nabla_r X_s) \\
&\quad - (m \leftrightarrow n) \\
&= P^t_m P^r_n P^s_b \nabla_t \nabla_r X_s + P^t_m P^l_n P^e_b (P^r_l \nabla_t P^s_e + P^s_e \nabla_t P^r_l)(\nabla_r X_s) \\
&\quad - (m \leftrightarrow n) \\
&= P^t_m P^r_n P^s_b \nabla_t \nabla_r X_s + P^t_m P^r_n P^e_b \nabla_t P^s_e (\nabla_r X_s) \\
&\quad + P^t_m P^l_n P^s_b \nabla_t P^r_l (\nabla_r X^s) - (m \leftrightarrow n) \\
&= P^t_m P^r_n P^s_b \nabla_t \nabla_r X_s + P^t_m P^r_n P^e_b \nabla_t (\delta^s_c - \sigma n_e n^s)(\nabla_r X_s) \\
&\quad + P^t_m P^l_n P^s_b \nabla_t (\delta^r_l - \sigma n_l n^r)(\nabla_r X_s) - (m \leftrightarrow n) \\
&= P^t_m P^r_n P^s_b \nabla_t \nabla_r X_s + P^t_m P^r_n P^e_b \nabla_t (-\sigma n_e n^s)(\nabla_r X_s) \\
&\quad + P^t_m P^l_n P^s_b \nabla_t (-\sigma n_l n^r)(\nabla_r X_s) - (m \leftrightarrow n) \\
&= P^t_m P^r_n P^s_b \nabla_t \nabla_r X_s - \underbrace{\sigma P^t_m P^r_n P^e_b n_e (\nabla_t n^s)(\nabla_r X_s)}_{P^e_b n_e = 0} \\
&\quad - \sigma P^t_m P^r_n P^e_b n^s (\nabla_t n_e)(\nabla_r X_s) - \underbrace{\sigma P^t_m P^l_n P^s_b n_l (\nabla_t n^r)(\nabla_r X_s)}_{P^l_n n_l = 0} \\
&\quad - \sigma P^t_m P^l_n P^s_b n^r (\nabla_t n_l)(\nabla_r X_s) - (m \leftrightarrow n) \\
&= P^t_m P^r_n P^s_b \nabla_t \nabla_r X_s - \sigma P^t_m P^r_n P^e_b n^s (\nabla_t n_e)(\nabla_r X_s) \\
&\quad - \sigma P^t_m P^l_n P^s_b n^r (\nabla_t n_l)(\nabla_r X_s).
\end{aligned}
\tag{2.86}
$$

Let us use,

$$
\Rightarrow \quad P^t_m P^e_b \nabla_t n_e = K_{mb}, \quad P^t_m P^l_n \nabla_t n_l = K_{mn}. \tag{2.87}
$$

$$
\begin{aligned}
-\hat{R}^a{}_{bmn}X_a &= P^t_m P^r_n P^s_b \nabla_t \nabla_r X_s - \sigma P^r_n n^s K_{mb} (\nabla_r X_s) \\
&\quad - \sigma P^s_b n^r K_{mn} (\nabla_r X_s) - (m \leftrightarrow n) \\
&= P^t_m P^r_n P^s_b \nabla_t \nabla_r X_s - \sigma P^r_n n^s K_{mb} (\nabla_r X_s) - (m \leftrightarrow n).
\end{aligned}
\tag{2.88}
$$

For any spatial tensor $n^s X_s = 0$;

$$
\begin{aligned}
\nabla_r(n_s X^s) = n_s \nabla_r X^s + X^s \nabla_r n_s = 0 \quad &\Rightarrow \quad n_s \nabla_r X^s = -X^s \nabla_r n_s \\
&= P^t_m P^r_n P^s_b \nabla_t \nabla_r X_s + \sigma P^r_n K_{mb} X^s \nabla_r n_s \\
&= P^t_m P^r_n P^s_b \nabla_t \nabla_r X_s + \sigma K_{mb} K_{ns} X^s \\
&= -P^t_m P^r_n P^s_b R_{astr} X^a + \sigma X^a K_{mb} K_{na} - \sigma X^a K_{nb} K_{ma}.
\end{aligned}
\tag{2.89}
$$

Let us say $X^a = P^a_b Y^b$;

$$
\begin{aligned}
-\hat{R}_{abmn} P^a_d Y^d = &- P^t_m P^r_n P^s_b R_{astr} P^a_d Y^d \\
&+ \sigma(K_{mb} K_{na} - K_{nb} K_{ma}) P^a_d Y^d,
\end{aligned}
$$

$$
-\hat{R}_{dbmn} Y^d = -P^t_m P^r_n P^s_b P^a_d R_{astr} Y^d + \sigma(K_{mb} K_{nd} - K_{nb} K_{md}) Y^d. \tag{2.90}
$$

Now we have reached the so called Gauss's equation

$$
\hat{R}_{dbmn} = P^t_m P^r_n P^s_b P^a_d R_{astr} + \sigma(K_{nb} K_{md} - K_{mb} K_{nd}),
$$

$$
\Rightarrow \quad \hat{R}_{abcd} = P^m_a P^n_b P^s_c P^t_d R_{mnst} + \sigma(K_{ac} K_{bd} - K_{ad} K_{bc}), \tag{2.91}
$$

$$
\hat{R}^a{}_{bcd} = P^a_m P^n_b P^s_c P^t_d R^m{}_{nst} + \sigma(K^a{}_c K_{bd} - K^a{}_d K_{bc}),
$$

$$
\hat{R}^a{}_{bcd} = \hat{R}_{bd} = P^a_m P^n_b P^s_a P^t_d R^m{}_{nst} + \sigma(K^a{}_a K_{bd} - K^a{}_d K_{ba}). \tag{2.92}
$$

Contracting yields the Ricci tensor on Σ

$$
\hat{R}_{bd} = P^s_m P^n_b P^t_d R^m{}_{nst} + \sigma(K K_{bd} - K^a{}_d K_{ba}), \tag{2.93}
$$

and one more contraction yields the scalar curvature.

$$
\hat{R} = P^{bd} \hat{R}_{bd} = P^{bd} P^s_m P^n_b P^t_d R^m{}_{nst} + \sigma(K P^{bd} K_{bd} - P^{bd} K^a{}_d K_{ba}), \tag{2.94}
$$

$$
\begin{aligned}
\hat{R} &= P^{bd} P^s_m P^n_b P^t_d R^m{}_{nst} + \sigma(K^2 - K^{ab}K_{ab}) \\
&= P^s_m P^{dn} P^t_d R^m{}_{nst} + \sigma(K^2 - K^{ab}K_{ab}) \\
&= P^s_m P^{nt} R^m{}_{nst} + \sigma(K^2 - K^{ab}K_{ab}) \\
&= (\delta^s_m - \sigma n^s n_m)(g^{nt} - \sigma n^n n^t) R^m{}_{nst} + \sigma(K^2 - K^{ab}K_{ab}) \\
&= (\delta^s_m g^{nt} - \sigma \delta^s_m n^n n^t - \sigma g^{nt} n^s n_m + n^s n_m n^n n^t) R^m{}_{nst} \\
&\quad + \sigma(K^2 - K^{ab}K_{ab}) \\
&= R - \sigma n^n n^t R_{nt} - \sigma n^s n_m R^m{}_s + \sigma(K^2 - K^{ab}K_{ab}) \\
&= R - \sigma 2 n^a n^b R_{ab} + \sigma(K^2 - K^{ab}K_{ab}).
\end{aligned} \tag{2.95}
$$

It is interesting to study the contraction of the Riemann tensor by a normal vector. After contraction is done, to reach the Codazzi's equation, projection of the contracted Riemann tensor will be achieved.

$$
R_{mnba} n^a = \nabla_m \nabla_n n_b - \nabla_n \nabla_m n_b, \tag{2.96}
$$

$$
\nabla_\mu n_\nu = K_{\mu\nu} + \sigma n_\mu a_\nu, \tag{2.97}
$$

$$
R_{mnba} n^a = \nabla_m (K_{nb} + \sigma n_n n_b) - (m \leftrightarrow n). \tag{2.98}
$$

$$
\begin{aligned}
P^m_c P^n_d P^b_e n^a R_{mnba} &= P^m_c P^n_d P^b_e \nabla_m K_{nb} + \sigma P^m_c P^n_d P^b_e \nabla_m (n_n n^s \nabla_s n_b) \\
&\quad - (c \leftrightarrow d) \\
&= P^m_c P^n_d P^b_e \nabla_m K_{nb} + \sigma P^m_c P^n_d P^b_e (\nabla_m (n_n n^s)(\nabla_s n_b)) \\
&\quad - (c \leftrightarrow d) \\
&= P^m_c P^n_d P^b_e \nabla_m K_{nb} + \sigma P^m_c P^n_d P^b_e n^s (\nabla_s n_b)(\nabla_m n_n) \\
&\quad - (c \leftrightarrow d) \\
&= \hat{\nabla}_c K_{de} + \sigma P^m_c P^n_d P^b_e (\nabla_s n_b)(\nabla_m n_n) - (c \leftrightarrow d) \\
&= \hat{\nabla}_c K_{de} + \sigma P^b_e (\nabla_s n_b) K_{cd} - (c \leftrightarrow d) \\
&= \hat{\nabla}_c K_{de} - \hat{\nabla}_d K_{ce}.
\end{aligned} \tag{2.99}
$$

$$
P^m_c P^n_d P^b_e n^a R_{mnba} = \hat{\nabla}_c K_{de} - \hat{\nabla}_d K_{ce}, \tag{2.100}
$$

$$(\delta_c^m - \sigma n^m n_c)(\delta_d^n - \sigma n^n n_d)(\delta_e^b - \sigma n^b n_e) n^a R_{mnba} = \hat{\nabla}_c K_{de} - \hat{\nabla}_d K_{ce}.$$

$$(\delta_c^m \delta_d^n - \sigma n^n n_d \delta_c^n - \sigma n^m n_c \delta_d^n + n^m n^n n_c n_d) \delta_e^b n^a R_{mnba} = \hat{\nabla}_c K_{de} - \hat{\nabla}_d K_{ce}$$

$$R_{cdea} n^a - \sigma n_d n^n n^a R_{cnea} - \sigma n_c n^m n^a R_{mdea} = \hat{\nabla}_c K_{de} - \hat{\nabla}_d K_{ce}. \tag{2.101}$$

Let us multiply above equation by g^{ce}, then

$$\Rightarrow \quad R_{ad} n^a - \sigma n_d n^n R_{na} n^a = \hat{\nabla}_c K_d{}^c - \hat{\nabla}_d K, \tag{2.102}$$

we get the Codazzi's equation

$$\Rightarrow \quad P_d^n R_{an} n^a = \hat{\nabla}_c K_d{}^c - \hat{\nabla}_d K. \tag{2.103}$$

2.3 Stokes' Theorem

Here we follow the discussion and notation of the related chapter of the book [7]. In a Euclidean space let us define the $r Adding higher order terms to the theory$-simplex $\bar{\sigma}_r = (p_0 p_1 \dots p_r)$ such as [2]

$$\begin{aligned} p_0 &= (0, 0, \dots, 0) \rightarrow \quad \text{the point at the origin} \\ p_1 &= (1, 0, \dots, 0) \\ &\dots \\ p_r &= (0, 0, \dots, 1) \end{aligned} \tag{2.104}$$

In a certain coordinate system $\bar{\sigma}_r$ is defined by

$$\overline{\sigma}_r = \left\{ \left(x^1, \dots, x^r \right) \in \mathbb{R}^r | x^\mu \geq 0, \sum_{\mu=1}^{r} x^\mu \leq 1 \right\}. \tag{2.105}$$

Now we can reach the volume-form ω on $\mathbb{R}^r$ as

$$\omega = a(x) dx^1 \wedge dx^2 \wedge \dots \wedge dx^r. \tag{2.106}$$

[2] Here we shall not go into the definitions of the relevant tools such as homology and homotopy theory used below: all these follow the book [7] very closerly.

and an integration of ω is defined

$$\int_{\overline{\sigma}_r} \omega \equiv \int_{\overline{\sigma}_r} a(x)\mathrm{d}x^1\mathrm{d}x^2 \dots \mathrm{d}x^r \tag{2.107}$$

We defined an r-simplex in $\mathbb{R}^r$ before. It is natural to construct a map $f : \sigma_r \to M$ (it is smooth and does not have to posses an inverse). Let us say $\{s_{r,i}\}$ is a set of r-simplex that is mapped from $\mathbb{R}^r$ to the manifold M. Using the set $\{s_{r,i}\}$, one can define an r-chain $c = \sum_i a_i s_{r,i}$, where $a_i \in \mathbb{R}$. Those r-chains form a group $C_r(M)$ on the manifold M. We have whole set-up onto the manifold M, and introducing r-chain, r-cycle and r-boundary is eligible on M. The boundary ∂s_r of an r-simplex s_r is an (r-1)-simplex with the help of the map $\partial : C_r(M) \to C_{r-1}(M)$. Let say c_r is an r-chain on M. For those c_r that satisfy $\partial c_r = 0$ are called an r-cycle and $c_r = \partial c_{r+1}$ are called an r-boundary.

Definition: $B_r(M)$ is a boundary group and its elements are r-chains that satisfies $c_r = \partial c_{r+1}$. $Z_r(M)$ is called cycle group that have elements such as r-chains without a boundary, $\partial c_r = 0$.
It is obvious that $B_r(M)$ is a subgroup of $Z_r(M)$, $B_r(M) \subset Z_r(M)$.

Definition: $H_r(M)$ is a homotopy group on M and defined by quotient space of $Z_r(M)$ by $B_r(M)$.

$$H_r(M) \equiv Z_r(M)/B_r(M). \tag{2.108}$$

We put all the tools on the table to define an integration of an r-form conclusively. Using the pull-back of an r-form ω with the help of the map $f : \sigma_r \to M$, ω is moved to the space $\mathbb{R}^r$ in which one can perform an integration as below.

$$\int_{s_r} d\omega = \int_{\bar{\sigma_r}} f^* \omega \tag{2.109}$$

Notice that the RHS of the equation is just an r-fold integration because it is performed in the space $\mathbb{R}^r$

Definition: Stokes' Theorem:

For an r-1-form $\omega \in \Omega^{r-1}(M)$, we have

$$\int_{s_r} d\omega = \int_{\bar{\sigma_r}} f^*(d\omega) = \int_{\bar{\sigma_r}} d(f^* \omega) \tag{2.110}$$

and using pull-back map for an r-1-form ω,

$$\int_{\partial s_r} \omega = \int_{\partial \bar{\sigma_r}} f^* \omega, \tag{2.111}$$

then we established the Stokes' theorem

$$\int_{s_r} d\omega = \int_{\partial s_r} \omega. \tag{2.112}$$

CHAPTER 3

THREE DIMENSIONAL GRAVITY THEORIES

Lower dimensional gravitational or non-gravitational theories have always been considered as both useful tools in understanding the four dimensional physics and actually effectively realizing them in the laboratory setting. In this context, especially lower dimensional quantum field theories have been widely studied; for example 2+1 dimensional Chern-Simons theories have applications in effectively 2-dimensional physical systems, such as the the quantum Hall effect. But as for gravity, the situation is more subtle for an unexpected reason: 2+1-dimensional General Relativity (GR) with or without a cosmological constant (Λ) has no local degrees of freedom, hence there are no gravitons, no gravitational waves and therefore 2+1-dimensional gravity, apparently cannot be a useful tool for the realistic gravity theory in four dimensions. To see that there is no nontrivial gravity in 2+1 dimensional GR, let us note the following: assuming the validity of Einstein's equations (say for $\Lambda = 0$), one has

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa T_{\mu\nu} \tag{3.1}$$

and in vacuum, $T_{\mu\nu} = 0$, $R_{\mu\nu} = 0$. In generic n-dimensions, algebraic decomposition of the Riemann tensor is given in terms of the Weyl tensor and the Ricci tensor and scalar as

$$\begin{aligned} R_{\lambda\mu\nu\kappa} =& C_{\lambda\mu\nu\kappa} + \frac{1}{(n-2)}(g_{\lambda\nu}R_{\mu\kappa} - g_{\lambda\kappa}R_{\mu\nu} - g_{\mu\nu}R_{\lambda\kappa} + g_{\mu\kappa}R_{\lambda\nu}) \\ &- \frac{R}{(n-1)(n-2)}\big(g_{\lambda\nu}g_{\mu\kappa} - g_{\lambda\kappa}g_{\mu\nu}\big). \end{aligned} \tag{3.2}$$

In vacuum for GR, $R_{\mu\nu} = 0$, $R = 0$, so $R_{\lambda\mu\nu\kappa} = C_{\lambda\mu\nu\kappa}$. But for $n = 2+1$, the Weyl tensor vanishes identically , hence the Riemann tensor vanishes, which means, outside the source the spacetime is flat. Since curvature encodes gravity, one has no local gravity in $n = 2+1$. For the $\Lambda \neq 0$ case, the above discussion can easily be extended to observe that the Riemann tensor is locally maximally symmetric

$$R_{\mu\alpha\nu\beta} = \Lambda \left(g_{\mu\nu} g_{\alpha\beta} - g_{\mu\beta} g_{\alpha\nu} \right). \tag{3.3}$$

For $\Lambda < 0$ the maximally symmetric spacetime is called the anti-de Sitter sapcetime, while for $\Lambda > 0$ it is the de Sitter spacetime. So we have just seen that in $2 + 1$ GR, outside the source, depending on Λ, the spacetime is either flat, de Sitter or anti-de Sitter. The source $T_{\mu\nu}$ cannot change this behaviour. In fact more concisely, in three dimensions the Riemann tensor and the Ricci (or the Einstein) tensor carry the same number of independent components, that is 6, and they are related as

$$R_{\mu\alpha\nu\beta} = \epsilon_{\mu\alpha\sigma} \epsilon_{\nu\beta\rho} G^{\sigma\rho}, \tag{3.4}$$

where $\epsilon_{\mu\alpha\sigma}$ is totally antisymmetric tensor. We prove this identity in the next chapter. It is clear from Eq. (3.4) that vanishing of $G^{\sigma\rho}$ yields flat spacetime.

Another way to see the local triviality of three dimensional Einstein's gravity is to directly count the number of degrees of freedom at a point in phase space following [5]. Consider the phase space of n dimensional GR, the relevant canonical variables are the spatial metric g_{ij} on a $n - 1$ dimensional hypersurface and the corresponding canonical momenta $\partial_t g_{ij}$ on the hypersurface. Being a symmetric tensor field, the spatial metric has $n(n-1)/2$ maximum possible independent components, the canonical momenta has the same number of components, adding to a maximum of $n(n - 1)$. But, there are n constraints coming from the Bianchi identity (1 Hamiltonian and $n - 1$ momentum constraints). In addition, using the coordinate transformations, we can gauge away n components of the metric hence the maximum possible number of degrees of freedom is $n(n-1) - 2n = n(n-3)$ which yields the expected 4 in four dimensions; that are 2 metric and 2 momenta degrees of freedom. But it yields 0 degree of freedom in 3 dimensions. This discussion is valid whether there is a cosmological constant or not. The number of degrees of freedom in 3-dimensions increase by adding higher order terms to the theory and the constraints become time-dependent. These facts eliminate the triviality of the 3-dimensional gravity.

All these show that no matter what a source has or does, it cannot change the gravitational field outside. Sometimes this triviality of $2 + 1$-dimensional gravity is summarized with the statement that $2 + 1$ GR has no non-trivial Newtonian limit. This is meant to say the following: in a well presented approximation scheme, say for $\Lambda = 0$ case, in generic $n > 2+1$ dimensions, one gets the Poisson equation for the Newtonian potential ϕ from GR as

$$\nabla^2 \phi = \kappa \rho, \tag{3.5}$$

but for $n = 2+1$, one does not obtain this equation, instead after gauge-fixing one gets $\phi = 0$ in sharp contrast to the 2 dimensional Newtonian gravity (i.e. the Poisson equation in 2-dimensions) which yields $\phi = \kappa \ln(\frac{r}{r_0})$ for a point source, which yields a $\frac{1}{r}$ force as expected.

One should keep in mind that this local triviality does not necessarily mean that the theory is globally trivial. In fact there have been highly influential works done on the global structure of gravity for point sources in $2+1$ dimensions in [9, 10]. These works initiated a closer scrutiny of the three dimensional gravity theories. Eventually, it turned out that when the cosmological constant Λ is negative, namely the spacetime is locally anti-de Sitter (AdS), the theory admits a black hole solution, so called the Bañados-Teitelboim-Zanelli (BTZ)-black hole [11]. This black hole was totally unexpected, because of the local simplicity discussion noted above, no one expected the existence of a black hole in this theory. In fact the BTZ black hole has almost all the properties of the four dimensional Kerr black hole: it has a mass m, angular momentum J, inner and outer event horizons and all the relevant thermodynamics associated with the horizon. There are of course several differences, one of which is that unlike the Kerr black hole of the four dimensions, the BTZ black hole does not have a curvature singularity; in addition it does not have a speed of light surface. The reason this black hole was found more than 75 years after GR was introduced is the following: this solution arises in a rather non-trivial way: the global AdS_3 is identified along some directions whose details are given in [11]. We should stress that the black hole solution does not exist for $\Lambda = 0$ or $\Lambda > 0$, namely the flat and the de Sitter cases. The fact that a negative cosmological constant is needed for the BTZ black hole to exist in Einstein's gravity was proven as a theorem in [12].

All the above discussion has been in the classical regime; in principle one uses lower dimensional theories as a tool to understand both the classical and quantum regimes of the realistic four dimensional theories. As for the quantum version of the three dimensional Einstein's gravity, there has been a lot of progress but the final answer has not been found. Earlier work [13] on the quantum version of the three dimensional GR showed that if the theory is formulated in terms of the dreiben and the spin connection [14], not the metric formulation, then up to a boundary term, the Einstein-Hilbert action with or without a cosmological constant can be recast as a non-Abelian Chern-Simons theory with a non-compact group. Once this is done, then since the latter theory is quantized, three dimensional gravity is also quantized. But there is a caveat, in the Chern-Simons formulation of the theory, one allows vanishing gauge fields, this in the gravity side corresponds to a vanishing dreiben which yields a degenerate metric which is not acceptable in gravity because one cannot consistently couple matter fields to a degenerate metric as the inverse does not exist. Therefore, currently the quantum version of the three dimensional Einstein's gravity has not been formulated.

This state of affairs and the fact that one would still like to define a gravity theory which is locally non-trivial, namely it has gravitons and gravitational waves, one resorts to other gravity theories that extends the three dimensional GR in one way or another. One of the first proposals of such a theory is the topologically massive gravity (TMG) [15], with the following action

$$I = \int d^3x \sqrt{-g} \left[\frac{1}{\kappa}(R - 2\Lambda) - \frac{1}{2\mu} \varepsilon^{\lambda\mu\nu} \Gamma^{\rho}{}_{\lambda\sigma} \left(\partial_\mu \Gamma^{\sigma}{}_{\rho\nu} + \frac{2}{3} \Gamma^{\sigma}{}_{\mu\beta} \Gamma^{\beta}{}_{\nu\rho} \right) \right], \tag{3.6}$$

μ is the so called topological mass; $\varepsilon_{012} = 1$ is a tensor density, and the Γ's are the Christoffel connections. A detailed canonical analysis of this theory, for $\Lambda = 0$ was done in [15] where it was shown that the theory describes a single massive spin-2 degree of freedom with mass $m = |\mu|$. The field equations coming from the variation of this theory in vacuum are

$$G_{\mu\nu} + \Lambda g_{\mu\nu} + \frac{1}{\mu} C_{\mu\nu} = 0, \tag{3.7}$$

where the "Cotton" tensor $C_{\mu\nu}$ is defined as

$$C_{\mu\nu} \equiv \epsilon_\mu{}^{\alpha\beta} \nabla_\alpha S_{\beta\nu}, \tag{3.8}$$

and $S_{\beta\nu} := R_{\beta\nu} - \frac{1}{4} g_{\beta\nu} R$ is the three dimensional Schouten tensor. It is important to note that $\epsilon^{\mu\alpha\beta}$ is a tensor not a tensor density. One can show that the Cotton tensor is symmetric, traceless and divergence-free: $C^{\mu\nu} = C^{\nu\mu}$, $g_{\mu\nu} C^{\mu\nu} = 0$ and $\nabla_\mu C^{\mu\nu} = 0$.

As these identities will be important later, let us show them. First let us prove that $C_{\mu\nu}$ is traceless;

$$g^{\mu\nu}C_{\mu\nu} = g^{\mu\nu}\epsilon_\mu{}^{\alpha\beta}\nabla_\alpha S_{\beta\nu} = \underbrace{\epsilon^{\nu\alpha\beta}}_{anti-sym}\nabla_\alpha \underbrace{S_{\beta\nu}}_{sym} = 0. \tag{3.9}$$

One can show that $C_{\mu\nu}$ is symmetric by multiplying it with the anti-symmetric tensor, $\epsilon^{\mu\nu\lambda}$;

$$\begin{aligned}\epsilon^{\mu\nu\lambda}C_{\mu\nu} &= \frac{1}{2}\epsilon^{\mu\nu\lambda}\big(C_{(\mu\nu)} + C_{[\mu\nu]}\big) = \underbrace{\epsilon^{\mu\nu\lambda}C_{(\mu\nu)}}_{=0} + \epsilon^{\mu\nu\lambda}C_{[\mu\nu]} \\ &= \frac{1}{2}\epsilon^{\mu\nu\lambda}(C_{\mu\nu} - C_{\nu\mu}) \\ &= \frac{1}{2}\big[\epsilon^{\mu\nu\lambda}\epsilon_\mu{}^{\alpha\beta}\nabla_\alpha S_{\beta\nu} - \epsilon^{\mu\nu\lambda}\epsilon_\nu{}^{\alpha\beta}\nabla_\alpha S_{\beta\mu}\big] \\ &= \frac{1}{2}\big(\nabla_\alpha S^{\alpha\lambda} - \nabla_\alpha S^{\alpha\lambda} - \nabla^\lambda S + \nabla^\lambda S\big) = 0.\end{aligned} \tag{3.10}$$

Finally let us prove that it is divergence-free

$$\nabla_\mu C^{\mu\nu} = \epsilon^{\mu\alpha\beta}\nabla_\mu\nabla_\alpha S_\beta{}^\nu. \tag{3.11}$$

The operator $\nabla_\mu\nabla_\alpha$ can be written as a sum of the symmetric and antisymmetric operators;

$$\nabla_\mu\nabla_\alpha := \frac{1}{2}(\nabla_\mu\nabla_\alpha + \nabla_\alpha\nabla_\mu) + \frac{1}{2}(\nabla_\mu\nabla_\alpha - \nabla_\alpha\nabla_\mu) \tag{3.12}$$

$$\begin{aligned}\nabla_\mu C^{\mu\nu} &= \frac{1}{2}\epsilon^{\mu\alpha\beta}(\nabla_\mu\nabla_\alpha - \nabla_\alpha\nabla_\mu)S_\beta{}^\nu = \frac{1}{2}\epsilon^{\mu\alpha\beta}[\nabla_\mu, \nabla_\alpha]S_\beta{}^\nu \\ &= \frac{1}{2}\epsilon^{\mu\alpha\beta}R_{\mu\alpha}{}^\nu{}_\lambda S_\beta{}^\lambda + \underbrace{\frac{1}{2}\epsilon^{\mu\alpha\beta}R_{\mu\alpha\beta}{}^\lambda S_\lambda{}^\nu}_{=0} = \frac{1}{2}\epsilon^{\mu\alpha\beta}R_{\mu\alpha}{}^\nu{}_\lambda S_\beta{}^\lambda.\end{aligned} \tag{3.13}$$

Next we use the algebraic decomposition of the Riemann tensor in 3-dimensions, which is

$$R_{\mu\alpha}{}^\nu{}_\lambda = \delta_\mu{}^\nu R_{\alpha\lambda} - g_{\mu\lambda}R_\alpha{}^\nu - \delta_\alpha{}^\nu R_{\mu\lambda} + g_{\alpha\lambda}R_\mu{}^\nu - \frac{R}{2}(\delta_\mu{}^\nu g_{\alpha\lambda} - g_{\mu\lambda}\delta_\alpha{}^\nu), \tag{3.14}$$

to get

$$\begin{aligned}\nabla_\mu C^{\mu\nu} &= \frac{1}{2}\Big[\epsilon^{\nu\alpha\beta}R_{\alpha\lambda}S_\beta{}^\lambda - \epsilon^{\mu\nu\beta}R_{\mu\lambda}S_\beta{}^\lambda\Big] = \epsilon^{\nu\alpha\beta}R_{\alpha\lambda}S_\beta{}^\lambda \\ &= \epsilon^{\nu\alpha\beta}R_{\alpha\lambda}(R_\beta{}^\lambda - \frac{1}{4}\delta_\beta{}^\lambda R) = \epsilon^{\nu\alpha\beta}R_{\alpha\lambda}R_\beta{}^\lambda = 0.\end{aligned} \tag{3.15}$$

We noted the degree of freedom structure of TMG for $\Lambda = 0$. The degree of freedom structure of TMG for $\Lambda \neq 0$ is quite non-trivial: for generic Λ, there is still a single massive degree of freedom with mass-square $m^2 = \mu^2 + \Lambda$ [16, 17], namely the cosmological constant and the mass parameter μ both appear in the graviton mass. On the other hand in AdS for $\Lambda \equiv -\frac{1}{l^2}$ where l is the radius of AdS_3, with $\mu l = 1$, the theory has no bulk degree of freedom. This theory is called Chiral Gravity [18, 19] to which we shall come back below. In addition to these interesting new possibility, it is easy to see that the TMG field equations (3.7) admit all solutions of Einstein's theory, including the BTZ black hole. This is because $G_{\mu\nu} = 0$, for any Einstein metric, $R_{\mu\nu} = 2\Lambda g_{\mu\nu}$. Of course in addition to these Einsteinian solutions, TMG admits a lot of solutions which are non-Einsteinian, namely $G_{\mu\nu} \neq 0$. Most of the solutions known to date have been compiled in [20].

The fact that TMG have both black holes and gravitons gave rise to an intensive research on this theory with refered to its classical and quantum nature. Classically it is a third order theory and much more complicated then the second order Einstein's gravity. In the quantum regime naive quantization techniques such as canonical quantizations or path-integral quantization are quite hopeless to carry out in this much more complicated theory. But after the discovery of anti-de Sitter/Conformal field theory (AdS/CFT) correspondence [21], another path to quantum gravity became possible. Briefly this path is as follows: define a gravity theory in an asymptotically AdS spacetime, the theory on the boundary will be conformal field theory. If this can be done consistently , then one interprets the boundary theory to be the quantum version of the bulk theory. Of course this requires both the bulk and boundary theories to be well-defined in the sense that they must be unitary without ghosts and tachyons. They must be casual. It is amusing to note that, much earlier than the work in AdS/CFT, Brown and Henneaux [22] found that asymptotic symmetries of AdS_3 is much larger then the bulk symmetries, in fact the corresponding boundary algebra is infinite dimensional and includes 2 copies of the Virasoro algebra which is suitable for a two dimensional conformal field theory. This Virasoro algebra has a central charge given as

$$c = \frac{3}{2}\frac{l}{G}. \tag{3.16}$$

It was shown [23] that in TMG, this boundary symmetry algebra still has two copies of Virasoro algebra with the different central charges given as

$$c_{R/L} = \frac{3l}{2G}\big(1 \pm \frac{1}{\mu l}\big). \tag{3.17}$$

Computation of the bulk and boundary excitations show that one has either the bulk or boundary excitations to be ghost-free for generic μl. This is dubbed as bulk-boundary unitarity clash which basically says that the theory cannot be quantized along the described lines above. Interestingly the chiral gravity limit is an exception if one takes $\mu l = 1$, then $c_L = 0$ and $c_R = \frac{3l}{G}$, one ends up with only the right moving sector on the boundary. This theory was conjectured to be valid both classically and quantum mechanically in [18, 19] . Of course to prove the conjecture one must find the corresponding boundary theory. Moreover, it was shown in [24] that exactly at the point $\mu l = 1$, the chiral point, there arises a classical solution with a negative total unbounded energy, the so called log-mode which ruins the stability of the vacuum. But it was shown in [20, 21] that the log mode found in [24] is an artifact of the linearization which was suggested in [19]. This issue is called linearization instability: linearized field equations of a non-linear theory such as TMG can have linear solutions which cannot be obtained from the linearization of any exact solution. This says that the perturbation theory in AdS_3 for chiral gravity must be done with care. So chiral gravity, as it now stands, is a possibly well-defined theory both classically and quantum mechanically.

There are various reasons to go beyond TMG: as we have seen TMG is a third derivative theory hence it is parity non-invariant which means it is highly different from the parity-invariant four dimensional General Realtivity. It has a single degree of freedom, one would like to have 2 degrees of freedom with ± 2 helicities as in four dimensions. Of course one would like to have massless graviton. But this is very difficult in 2+1-dimensions, hence in what follows, we shall discuss some well-known non trivial gravity theories in 2+1 dimensions, which are all massive gravity theories. The most obvious extension of Einstein's gravity is the quadratic theory with the action

$$I = \int d^3x \sqrt{-g} \left(\frac{1}{\kappa} \left(R - 2\Lambda_0 \right) + \alpha R^2 + \beta R_{\mu\nu} R^{\mu\nu} \right), \tag{3.18}$$

with α and β arbitrary dimensionful constants of this stage. Canonical analysis [25] shows that this theory has a massive spin-2 graviton and a massive spin-0 scalar graviton with masses

$$m_g^2 = -\frac{1}{\kappa\beta} - \frac{12\alpha\Lambda}{\beta} - 4\Lambda, m_s^2 = \frac{1}{\kappa(8\alpha + 3\beta)} - 4\Lambda \left(\frac{3\alpha + \beta}{8\alpha + 3\beta} \right), \tag{3.19}$$

where the effective cosmological constant Λ is given as

$$\Lambda = \frac{1}{4\kappa(3\alpha+\beta)}\left[1 \pm \sqrt{1-8\kappa\Lambda_0(3\alpha+\beta)}\right]. \tag{3.20}$$

Detailed analysis shows that both of these modes cannot be healty at the same parameter ranges.

The theory has no ghost or tachyon only either $\beta = 0$ or $8\alpha + 8\beta = 0$. For $\beta = 0$, spin-2 graviton is decoupled so we take $\beta \neq 0$. The choice is called the new massive gravity (NMG) [26, 27]. It is also valid for $\Lambda_0 = 0$.

NMG is the first known example of a parity invariant gravity theory with massive graviton without a ghost. There is also the Fierz-Pauli theory where one adds the term $\frac{m_g^2}{4}(h_{\mu\nu}h^{\mu\nu} - h^2)$ to the Einstein-Hilbert action but since $h_{\mu\nu} := g_{\mu\nu} - \bar{g}_{\mu\nu}$ is a "perturbation" defined with respect to the backround $\bar{g}_{\mu\nu}$, Fierz-Pauli theory is not diffeomorphism invariant. In fact a detailed study [28] shows that at the non-linear level, a ghost, so called Boulware-Deser ghost, arises in the Fierz-Pauli theory.

To possibly construct a dual conformal field theory to NMG, we must introduce a negative cosmological constant and re-study the bulk and boundary unitarity problem. The NMG action with a cosmogical constant with redefined parameters is

$$I = \frac{1}{\kappa^2}\int d^3x\sqrt{-g}\left[\sigma R - 2\lambda_0 m^2 + \frac{1}{m^2}\left(R_{\mu\nu}R^{\mu\nu} - \frac{3}{8}R^2\right)\right], \tag{3.21}$$

where we have taken the form given in [29]. The parameter $\sigma = \pm 1$ and λ_0 is the dimensionless bare cosmological parameter. Vacuum field equations show that there are two maximally symmetric vacua given by the effective cosmological constant,

$$\Lambda = \lambda m^2 = -2\left(\sigma \pm \sqrt{1+\lambda_0}\right) m^2, \tag{3.22}$$

hence one must have $\lambda_0 \geqslant -1$ for the existence of a maximally symmetric vacuum. The mass of the spin-2 graviton is obtained after a detailed analysis [25] to be

$$m_g^2 = (-\sigma + \frac{\lambda}{2})m^2. \tag{3.23}$$

The important issue is the following: even though the theory is unitary in the bulk, it has a negative central charge and cannot be unitary on the boundary. If one fixes the boundary theory, one loses unitarity in the bulk, hence this theory shares the same fate as the TMG theory for generic m. There has been extended works that try to improve NMG by adding judiciously chosen higher curvature terms. For example $O(R^3)$ and $O(R^4)$ terms were added [30] that are consistent with some requirements of ADS/CFT. But it turns out that these new additions do not remedy the problem. In fact in [31] a Born-Infeld type extension of NMG was given which has the added feature that it has a single maximally symmetric vacuum and a massive spin-2 excitation about it, but this theory also suffers from the bulk-boundary unitarity clash, as well as the infinite order theory introduced in [32]. A detailed study [29] showed that no theory that has the same particle content that is defined by an action based on the metric can be free of bulk and boundary unitarity clash. This interesting impasse led researcers to an interesting idea: instead of defining the theory based on the metric alone, can one construct on-shell consistent field equations? The idea is the following. Let us say that the vacuum field equations of the theory are defined as

$$\mathcal{E}^{\mu\nu} = 0, \tag{3.24}$$

but one does not have the Bianchi identity $\nabla_\mu \mathcal{E}^{\mu\nu} = 0$ for all smooth metrics, but only for those which solve the field equations (3.24). So one demands the weaker on-shell Bianchi-Identity

$$\nabla_\mu \mathcal{E}^{\mu\nu}\Big|_{\mathcal{E}^{\mu\nu}=0} = 0. \tag{3.25}$$

Such a requirement might seem to be too loose but that is not the case. In fact these so called "on shell consistent" theories are highly restricted. The first example of these theories is the Minimal Massive Gravity (MMG) [33] which was also consistently coupled to a matter source [34]. Source free field equations of MMG read

$$G_{\mu\nu} + \Lambda_0 g_{\mu\nu} + \frac{1}{\mu} C_{\mu\nu} + \frac{\gamma}{\mu^2} J_{\mu\nu} = 0, \tag{3.26}$$

which is a deformation of cosmological TMG with the $J_{\mu\nu}$ tensor given as

$$J^{\mu\nu} := \frac{1}{2} \epsilon^{\mu\rho\sigma} \epsilon^{\nu\alpha\beta} S_{\rho\alpha} S_{\sigma\beta}. \tag{3.27}$$

The divergence of the $J^{\mu\nu}$ tensor can be found to be

$$\nabla_\mu J^{\mu\nu} = \epsilon^{\nu\rho\sigma} S_\sigma{}^\tau C_{\rho\tau}, \tag{3.28}$$

which does not clearly vanish automatically, but MMG obeys the on-shell Bianchi identity. Let us show this

$$\nabla_\mu J^{\mu\nu} = \epsilon^{\nu\rho\sigma} S_\sigma{}^\tau C_{\rho\tau} = -\mu \epsilon^{\nu\rho\sigma} S_\sigma{}^\tau \left(G_{\rho\tau} + \Lambda_0 g_{\rho\tau} + \frac{\gamma}{\mu^2} J_{\rho\tau} \right), \tag{3.29}$$

each term vanish on its own due to symmetry.

With the addition of the J-tensor, the mass of the graviton is modified as [33, 35–37]

$$m_0^2 = \mu^2\left(1 + \frac{\gamma}{2\mu^2 l^2}\right)^2 + \frac{1}{l^2}, \quad \Lambda \equiv -\frac{1}{l^2} \tag{3.30}$$

but there is still a single massive graviton on the theory's parity-non invariant. Details of the MMG theory in various aspects have been worked out in [36, 37]. As for the bulk-boundary unitarity clash issue, the problem is subtle: If the theory is assumed to be defined by (3.26) and linearizations are carried out accordingly about the AdS vacuum, one finds that just like the TMG case, the theory can only be unitarity in the bulk and boundary at its chiral point [38]. But there is a second formulation of the theory in terms of the dreiben and auxilary fields which allows an action formulation. In this formulation bulk-boundary unitarity is achieved [39].

Deforming TMG with the J-tensor yields, as noted above, to a single massive degree of freedom with +2 or -2 helicity but not both. So one still would like to have the ± 2 helicities together. For that purpose MMG_2 was introduced in [35] which is defined by the following field equations

$$G_{\mu\nu} + \Lambda g_{\mu\nu} + \frac{1}{\mu} C_{\mu\nu} - \frac{1}{\mu^2} H_{\mu\nu} = 0, \tag{3.31}$$

where the $H_{\mu\nu}$ tensor is defined as the "curl" of the Cotton tensor via

$$H^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\alpha\beta}\nabla_\alpha C_\beta{}^\nu + \frac{1}{2}\epsilon^{\nu\alpha\beta}\nabla_\alpha C_\beta{}^\mu. \tag{3.32}$$

We will discuss this tensor (which we shall call the 3D Bach tensor due to its relevance to the higher dimensional Bach tensor) below, but let us note that the covariant divergence of this tensor reads

$$\nabla_\mu H^{\mu\nu} = \epsilon^{\nu\rho\sigma} S_\rho{}^\tau C_{\sigma\tau} = -\nabla_\mu J^{\mu\nu}, \tag{3.33}$$

so it does not vanish automatically, but clearly the sum of the $H^{\mu\nu}$ tensor and the $J^{\mu\nu}$ tensor is divergence free

$$\nabla_\mu (H^{\mu\nu} + J^{\mu\nu}) = 0. \tag{3.34}$$

Close scrutiny shows that separately $H^{\mu\nu}$ and $J^{\mu\nu}$ do not come from the variations of an action but their sum come from the variation of the purely quadratic part of NMG

$$\delta g \int d^3x \sqrt{-g}\left(R_{\mu\nu}R^{\mu\nu} - \frac{3}{8}R^2\right) \quad \Rightarrow \quad H^{\mu\nu} + J^{\mu\nu} = 0. \tag{3.35}$$

So deforming TMG with $J^{\mu\nu}$ yields MMG, while deforming TMG with $H^{\mu\nu}$ yields MMG_2. The computation of the masses and the central charges in this theory are done in [35]: the positive and negative helicity modes have different masses given as

$$m_i^2 = p_i^2 - \frac{1}{l^2} \quad i = 1, 2 \tag{3.36}$$

with

$$p^{1,2} = -\frac{m^2}{2\mu} \pm \sqrt{m^2 + \frac{m^4}{4\mu^2}}. \tag{3.37}$$

Note that, unlike MMG, MMG_2 has two massive helicities, but they have different masses. This is due to existence of the Cotton tensor which breaks parity due to its third derivative nature. To the best of our knowledge, four dimensional GR is parity invariant, but of course one can introduce tiny parity breaking terms that are not ruled out by experiments. One such attempt is the so called "Chern-Simons Modifications of GR" [40].

CHAPTER 4

GENERIC EXOTIC MASSIVE GRAVITY THEORIES

This chapter is mainly based on our work [6] and extends the details of the computations given there.

Three dimensional Einstein metrics are much simpler than the higher dimensional ones, as we discussed before; locally Einstein metrics are Riemann flat (or constant curvature) since in this dimension we have the following identity

$$R_{\mu\alpha\nu\beta} = \epsilon_{\mu\alpha\sigma}\epsilon_{\nu\beta\rho}G^{\sigma\rho}, \tag{4.1}$$

where $\epsilon_{\mu\alpha\sigma}$ is totally antisymmetric tensor and $G_{\rho\sigma}$ is the Einstein tensor $G_{\rho\sigma} = R_{\rho\sigma} - \frac{1}{2}g_{\rho\sigma}R$. One can show this as follows. For the signature $(-,+,+)$, the product of two totally anti-symmetric epsilon tensors can be written in terms of the products of the metric tensor as

$$\begin{aligned}\epsilon_{\mu\alpha\sigma}\epsilon_{\nu\beta\rho} =& - g_{\mu\nu}(g_{\alpha\beta}g_{\sigma\rho} - g_{\alpha\rho}g_{\sigma\beta}) + g_{\mu\beta}(g_{\alpha\nu}g_{\sigma\rho} - g_{\alpha\rho}g_{\sigma\nu}) \\ &- g_{\mu\rho}(g_{\alpha\nu}g_{\sigma\beta} - g_{\alpha\beta}g_{\sigma\nu}).\end{aligned} \tag{4.2}$$

Making use of this equation in Eq.(4.1), one arrives at

$$\begin{aligned}R_{\mu\alpha\nu\beta} = \epsilon_{\mu\alpha\sigma}\epsilon_{\nu\beta\rho}G^{\sigma\rho} =& g_{\mu\nu}R_{\alpha\beta} - g_{\mu\beta}R_{\alpha\nu} - g_{\alpha\nu}R_{\beta\mu} + g_{\alpha\beta}R_{\mu\nu} \\ &+ \frac{R}{2}(g_{\mu\beta}g_{\alpha\nu} - g_{\mu\nu}g_{\alpha\beta}),\end{aligned} \tag{4.3}$$

which is just the algebraic decomposition of the Riemann tensor in three dimensions (3.2). This basically says that in a vacuum, in Einstein's theory, $R_{\mu\nu} = 0 = R$, there is no gravity, and no gravitational wave or radiation. When a negative cosmological constant is introduced, local triviality is not lifted, but there is the all important Bañados-Teitelboim-Zanelli (BTZ) black hole [11] that can carry mass, spin and pretty much all the properties of its four-dimensional analog Kerr black hole, save the curvature singularity and the speed-of-light surface. So some of the Einstein metrics are highly nontrivial (when considered in $2+1$ GR) but one of course still needs local nontriviality, gravitation, gravitational waves *etc.* to be able to learn something from this lower-dimensional setting.

Fortunately, this can still be achieved with Einstein metrics but not as solutions to GR but as solutions to modified gravity theories, such as the topologically massive gravity (TMG) [15], new massive gravity (NMG) [26, 27] or Born-Infeld extension of NMG [31]. All these theories accommodate Einstein metrics and more general metrics that are not Einstein. But the good thing is that in these theories, perturbation about an Einstein metric can be interpreted as gravitons (usually massive) or gravitational waves. Hence these theories are much richer than Einstein's pure $2+1$ GR and simpler than the $3+1$ GR. The immediate aim is to be able to define and understand a version of quantum gravity in a $2+1$-dimensional setting. For this purpose, our current best hope is the AdS/CFT duality [21] which reduces the problem to a construction of a two-dimensional boundary conformal field theory for the AdS bulk of a given $3D$ theory.

4.1 3D Bach Tensor and On-shell Consistency

Let us go back to the discussion of Einstein metrics that was alluded to above: perhaps the next "nice" set of metrics are the ones conformally related to the Einstein metrics. Succinctly stated the problem is this: given a metric g (which is not necessarily Einstein) can one construct a metric, $\tilde{g} \equiv \Omega^2 g$, which is Einstein given that Ω is smooth and $\Omega > 0$? In $n-$dimensions, the generic necessary and sufficient conditions for such a metric $\tilde{g}$ to exist are too difficult to handle. But, in four dimensions the problem simplifies a little bit in the sense that the necessary condition is the vanishing of the so-called "Bach Tensor"

$$H_{\mu\nu} \equiv \big(\nabla^\alpha \nabla^\beta + \frac{1}{2} R^{\alpha\beta}\big) C_{\mu\alpha\nu\beta}, \tag{4.4}$$

where $C_{\mu\alpha\nu\beta}$ is the Weyl tensor. The Bach tensor is symmetric, traceless $H \equiv g^{\mu\nu} H_{\mu\nu} = 0$, divergence-free $\nabla^\mu H_{\mu\nu} = 0$ and conformally invariant (in four dimensions). Moreover, one can show that $H_{\mu\nu}$ comes from the variation of the action

$$S = \int d^4x\sqrt{-g}C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta}. \tag{4.5}$$

This so-called conformal gravity admits all the Einstein metrics as solutions, but there are non-Einstein solutions. Remarkably, with some simple (Neumann) boundary conditions, one can show that out of all the Bach flat manifolds, only Einstein manifolds can be selected [41] .

One can naturally wonder the simpler problem, that is, the problem of the conformal Einstein metrics in three-dimensions. As the Weyl tensor vanishes identically in three-dimensions, the naive dimensional continuation of the Bach tensor as defined by Eq.(4.4) to three dimensions does not yield any further information. But as was realized in [35, 42], using the 3-index Cotton tensor as a potential to the Weyl tensor yields a meaningful 3D Bach tensor. Recall that the n-dimensional Cotton tensor is[1]

$$C_{\alpha\mu\nu} = \nabla_\alpha R_{\mu\nu} - \nabla_\mu R_{\alpha\nu} - \frac{1}{2(n-1)}\big(g_{\mu\nu}\nabla_\alpha R - g_{\alpha\nu}\nabla_\mu R\big), \tag{4.6}$$

which is antisymmetric in the first two indices. This tensor is conformally invariant only in three dimensions. Using this, we define the analog of the n-dimensional Bach tensor as

$$H_{\mu\nu} \equiv \frac{1}{2}\nabla^\alpha C_{\alpha\mu\nu} + \frac{1}{2}R_{\alpha\beta}C_{\mu}{}^{\alpha}{}_{\nu}{}^{\beta}. \tag{4.7}$$

In particular, for $n = 3$, we can express the Cotton tensor in terms of the Cotton-York tensor ($C_{\mu\nu} \equiv \epsilon_\mu{}^{\sigma\rho}\nabla_\sigma S_{\rho\nu}$ with $S_{\mu\nu} = R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R$.) as

$$C^{\sigma\rho}{}_{\nu} = -\epsilon^{\sigma\rho\mu}C_{\mu\nu} \tag{4.8}$$

where

$$C_{\mu\nu} \equiv \frac{1}{2}\epsilon_\mu{}^{\alpha\beta}C_{\alpha\beta\nu}. \tag{4.9}$$

Therefore, the 3D Bach tensor can be defined as [2]

$$H_{\mu\nu} \equiv \frac{1}{2}\epsilon_\mu{}^{\alpha\beta}\nabla_\alpha C_{\beta\nu} + \frac{1}{2}\epsilon_\nu{}^{\alpha\beta}\nabla_\alpha C_{\beta\mu}. \tag{4.10}$$

The Cotton-York tensor plays the role of the Weyl tensor in 3D: namely it vanishes if and only if the metric is conformally flat. But an interesting situation arises in 3D: unlike the Weyl tensor (a four-index object) that does not come from the variation of an

[1] Note that one uses the same letter C for 3 different tensor, the rank-2 Cotton-York tensor, rank-3 Cotton tensor, and rank-4 Weyl tensor: the explicit indices remove any possible confusion.

[2] To conform with the original definition [35] where the tensor was denoted as $H_{\mu\nu}$, we drop an overall factor of 1/2.

action, the Cotton-York tensor does come from the variation of the topological Chern-Simons action and it behaves regularly: $\tilde{C}^{\mu\nu}(\tilde{g}) = \Omega^{-2}C^{\mu\nu}(g)$ under conformal transformations. This says that conformally flat metrics in 3D are conformally Einstein. So, the 3D Bach tensor vanishes for conformally Einstein metrics. It is possible that its vanishing can be a sufficient condition, which we do not know. What is interesting is that, even though $H_{\mu\nu}$ (4.10) is symmetric and traceless ($H \equiv g^{\mu\nu}H_{\mu\nu} = 0$), it is not divergence-free. In fact one has

$$\nabla_\mu H^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\alpha\beta}\nabla_\mu\nabla_\alpha C_\beta{}^\nu + \frac{1}{2}\epsilon^{\nu\alpha\beta}\nabla_\mu\nabla_\alpha C_\beta{}^\mu. \tag{4.11}$$

Since any tensor can be written in the form of the sum of symmetric and antisymmetric tensor, let us define two operators

$$S_{\mu\alpha} := \frac{1}{2}\big(\nabla_\mu\nabla_\alpha + \nabla_\alpha\nabla_\mu\big), \quad A_{\mu\alpha} := \frac{1}{2}\big(\nabla_\mu\nabla_\alpha - \nabla_\alpha\nabla_\mu\big), \tag{4.12}$$

with the following properties:

$$\begin{aligned} \nabla_\mu\nabla_\alpha = S_{\mu\alpha} + A_{\mu\alpha} \quad \rightarrow \quad \epsilon^{\mu\alpha\beta}\nabla_\mu\nabla_\alpha &= \underbrace{\epsilon^{\mu\alpha\beta}S_{\mu\alpha}}_{=0} + \epsilon^{\mu\alpha\beta}A_{\mu\alpha} \\ &= \frac{1}{2}\epsilon^{\mu\alpha\beta}\big(\nabla_\mu\nabla_\alpha - \nabla_\alpha\nabla_\mu\big). \end{aligned} \tag{4.13}$$

$$\Rightarrow \quad \frac{1}{2}\epsilon^{\mu\alpha\beta}\nabla_\mu\nabla_\alpha C_\beta{}^\nu = \frac{1}{4}\epsilon^{\mu\alpha\beta}\big(\nabla_\mu\nabla_\alpha - \nabla_\alpha\nabla_\mu\big)C_\beta{}^\nu \tag{4.14}$$

$$\begin{aligned} \nabla_\mu H^{\mu\nu} &= \frac{1}{4}\epsilon^{\mu\alpha\beta}\big(\nabla_\mu\nabla_\alpha - \nabla_\alpha\nabla_\mu\big)C_\beta{}^\nu + \frac{1}{2}\epsilon^{\nu\alpha\beta}\nabla_\mu\nabla_\alpha C_\beta{}^\mu \\ &= \frac{1}{4}\epsilon^{\mu\alpha\beta}\big[\nabla_\mu, \nabla_\alpha\big]C_\beta{}^\nu + \frac{1}{2}\epsilon^{\nu\alpha\beta}\nabla_\mu\nabla_\alpha C_\beta{}^\mu. \end{aligned} \tag{4.15}$$

Now since $\nabla_\mu C_\beta{}^\mu = 0$

$$\nabla_\mu\nabla_\alpha C_\beta{}^\mu = \big[\nabla_\mu, \nabla_\alpha\big]C_\beta{}^\mu + \underbrace{\nabla_\alpha\nabla_\mu C_\beta{}^\mu}_{=0} = \big[\nabla_\mu, \nabla_\alpha\big]C_\beta{}^\mu, \tag{4.16}$$

$$\Rightarrow \quad \nabla_\mu H^{\mu\nu} = \frac{1}{4}\underbrace{\epsilon^{\mu\alpha\beta}\big[\nabla_\mu, \nabla_\alpha\big]C_\beta{}^\nu}_{I} + \frac{1}{2}\underbrace{\epsilon^{\nu\alpha\beta}\big[\nabla_\mu, \nabla_\alpha\big]C_\beta{}^\mu}_{II}. \tag{4.17}$$

$$I : \epsilon^{\mu\alpha\beta}\big[\nabla_\mu, \nabla_\alpha\big]C_\beta{}^\nu = \epsilon^{\mu\alpha\beta}\big(R_{\mu\alpha}{}^\nu{}_\lambda C_\beta{}^\lambda + R_{\mu\alpha\beta}{}^\lambda C_\lambda{}^\nu\big), \epsilon^{\mu\alpha\beta}R_{\mu\alpha\beta}{}^\lambda = 0. \tag{4.18}$$

We use the 3-dimensional identity

$$R_{\lambda\mu\nu\kappa} = g_{\lambda\nu} R_{\mu\kappa} - g_{\lambda\kappa} R_{\mu\nu} - g_{\mu\nu} R_{\lambda\kappa} + g_{\mu\kappa} R_{\lambda\nu} - \frac{R}{2}\big(g_{\lambda\nu} g_{\mu\kappa} - g_{\lambda\kappa} g_{\mu\nu}\big), \quad (4.19)$$

to get

$$\Rightarrow \frac{1}{4}\epsilon^{\mu\alpha\beta}\big[\nabla_\mu, \nabla_\alpha\big] C_\beta{}^\nu = \frac{1}{2}\epsilon^{\nu\alpha\beta} R_{\alpha\lambda} C_\beta{}^\lambda. \quad (4.20)$$

The second term in (4.17) can be written as

$$\begin{aligned} II : \epsilon^{\nu\alpha\beta}\big[\nabla_\mu, \nabla_\alpha\big] C_\beta{}^\mu &= \epsilon^{\nu\alpha\beta}\big(R_{\mu\alpha}{}^\mu{}_\lambda C_\beta{}^\lambda + R_{\mu\alpha\beta}{}^\lambda C_\lambda{}^\mu\big) \\ &= \epsilon^{\nu\alpha\beta} R_{\alpha\lambda} C_\beta{}^\lambda + \epsilon^{\nu\alpha\beta} R_{\mu\alpha\beta}{}^\lambda C_\lambda{}^\mu, \end{aligned} \quad (4.21)$$

since

$$\epsilon^{\nu\alpha\beta} R_{\mu\alpha\beta}{}^\lambda C_\lambda{}^\mu = 0, \quad (4.22)$$

one has

$$\Rightarrow \quad \frac{1}{2}\epsilon^{\nu\alpha\beta}\big[\nabla_\mu, \nabla_\alpha\big] C_\beta{}^\mu = \frac{1}{2}\epsilon^{\nu\alpha\beta} R_{\alpha\lambda} C_\beta{}^\lambda. \quad (4.23)$$

Therefore, the divergence of the Bach tensor does not vanish:

$$\nabla_\mu H^{\mu\nu} = \epsilon^{\nu\alpha\beta} R_\alpha{}^\lambda C_{\beta\lambda} \neq 0, \quad (4.24)$$

but it vanishes for Einstein metrics and/or conformally flat or Einstein metrics. This also says that, the 3D Bach tensor cannot come from the variation of an action.

Bianchi identities are related to the diffeomorphism invariance of the action, therefore let us examine this issue. Consider an action under the infinitesimal coordinate transformation $x^{\alpha'} = x^\alpha - \xi^\alpha(x)$

$$S = \int d^n x \sqrt{-g(x)} \mathcal{L}(g(x), \partial g(x), \dots), \quad (4.25)$$

where g denotes the metric tensor field.

$$\begin{aligned} S' &= \int d^n x' \sqrt{-g'(x')} \mathcal{L}(g'(x'), \partial' g'(x'), \dots) \\ &= \int d^n x \sqrt{-g'(x)} \mathcal{L}(g'(x), \partial' g'(x), \dots), \end{aligned} \quad (4.26)$$

where in the second equality we relabelled x' to x. The variation of the action with respect to the metric yields

$$\delta S = S' - S = \int d^n x \frac{\delta}{\delta g_{\mu\nu}}(\sqrt{-g}\mathcal{L})\delta g^{\mu\nu} = \int d^n x \sqrt{-g}\Phi_{\mu\nu}\delta g^{\mu\nu} = 0. \quad (4.27)$$

Now consider the particular variation corresponding to the infinitesimal diffeomorphisms, then one has

$$\delta g^{\mu\nu} = \nabla^\mu \xi^\nu + \nabla^\nu \xi^\mu. \tag{4.28}$$

Plugging this expression in (4.27) one has

$$\begin{aligned} \delta S &= \int d^n x \sqrt{-g} \Phi_{\mu\nu} (\nabla^\mu \xi^\nu + \nabla^\nu \xi^\mu) \\ &= \int d^n x \sqrt{-g} 2 \Phi_{\mu\nu} \nabla^\mu \xi^\nu \\ &= \underbrace{2 \int d^n x \sqrt{-g} \nabla^\mu (\Phi_{\mu\nu} \xi^\nu)}_{\text{Boundary term=0}} - 2 \int d^n x \sqrt{-g} (\nabla^\mu \Phi_{\mu\nu}) \xi^\nu = 0. \end{aligned} \tag{4.29}$$

Note that we assumed variations that vanish rapidly, or so called compactly supported variations. This gives us a desired result that the variation of the diffeomorphism invariant action yields a divergence free field equations, $\nabla^\mu \Phi_{\mu\nu} = 0$, for all metrics. We have not used the field equations to derive this result.

4.2 Generalization of 3D Bach Tensor

Let us construct generalization of the 3D Bach tensor (4.10) to define on-shell conserved theories. Consider a 2-tensor $\mathcal{E}_{\mu\nu}$ that comes from the variation of an action such that it obeys the Bianchi identity $\nabla^\mu \mathcal{E}_{\mu\nu} = 0$; and assume that we have a symmetric 2-tensor $\Phi_{\mu\nu}$ that does not come from the variation of an action and hence does not obey the Bianchi identity $\nabla_\mu \Phi^{\mu\nu} \neq 0$. Now, consider the following potentially viable on-shell consistent equations

$$\mathcal{E}_{\mu\nu} + \frac{1}{\mu} \epsilon_\mu{}^{\alpha\beta} \nabla_\alpha \Phi_{\beta\nu} + \frac{k}{\mu^2} \epsilon_\mu{}^{\alpha\beta} \epsilon_\nu{}^{\sigma\rho} \Phi_{\alpha\sigma} \Phi_{\beta\rho} = 0, \tag{4.30}$$

with μ and k free parameters at this stage, but as we shall see, k will be fixed from consistency. One can certainly add more powers of $\Phi_{\mu\nu}$ but, as we shall comment later, these do not lead to consistent theories. Inspired by the construction of MMG, this form of the field equations was first introduced in [43], where the authors chose $\Phi_{\mu\nu} = C_{\mu\nu}$ to obtain Exotic Massive Gravity (EMG). The middle term is a generalization of the Bach tensor, while the last term is a generalization of the J tensor. The first and the third terms are symmetric under the interchange of indices μ and ν. The second one is symmetric only if

$$\nabla_\sigma \Phi = \nabla_\alpha \Phi_\sigma{}^\alpha, \tag{4.31}$$

which follows from

$$\begin{aligned}\epsilon^{\mu\nu\sigma}\epsilon_\mu{}^{\alpha\beta}\nabla_\alpha\Phi_{\beta\nu} &= -\big(g^{\nu\alpha}g^{\sigma\beta} - g^{\nu\beta}g^{\sigma\alpha}\big)\nabla_\alpha\Phi_{\beta\nu} \\ &= -\nabla^\nu\Phi_\nu{}^\sigma + \nabla^\sigma\phi = 0 \quad \rightarrow \quad \nabla^\sigma\phi = \nabla^\nu\Phi_\nu{}^\sigma\end{aligned} \tag{4.32}$$

where $\Phi \equiv g^{\mu\nu}\Phi_{\mu\nu}$. This is the first condition on the theory. Another condition comes from the vanishing of the covariant divergence of the field equations

$$\nabla_\nu\left(\mathcal{E}^{\mu\nu} + \frac{1}{\mu}\epsilon^{\mu\alpha\beta}\nabla_\alpha\Phi_\beta{}^\nu + \frac{k}{\mu^2}\epsilon^{\mu\alpha\beta}\epsilon^{\nu\sigma\rho}\Phi_{\alpha\sigma}\Phi_{\beta\rho}\right) = 0, \tag{4.33}$$

which explicitly reads

$$\Rightarrow \frac{1}{\mu}\underbrace{\epsilon^{\mu\alpha\beta}\nabla_\nu\nabla_\alpha\Phi_\beta{}^\nu}_{1} + \frac{k}{\mu^2}\epsilon^{\mu\alpha\beta}\epsilon^{\nu\sigma\rho}\big(\underbrace{\Phi_{\alpha\sigma}\nabla_\nu\Phi_{\beta\rho}}_{2} + \underbrace{\nabla_\nu\Phi_{\alpha\sigma}\Phi_{\beta\rho}}_{3}\big) = 0. \tag{4.34}$$

Using the identities;

$$\big[\nabla_\nu, \nabla_\alpha\big]\Phi_\beta{}^\nu = \nabla_\nu\nabla_\alpha\Phi_\beta{}^\nu - \nabla_\alpha\nabla_\nu\Phi_\beta{}^\nu \tag{4.35}$$

$$\big[\nabla_\nu, \nabla_\alpha\big]\Phi_\beta{}^\nu = R_{\nu\alpha\beta}{}^\lambda\Phi_\lambda{}^\nu + R_{\nu\alpha}{}^\nu{}_\lambda\Phi_\beta{}^\lambda, \tag{4.36}$$

let us calculate each term separately

$$\begin{aligned}1 \quad : \quad & \epsilon^{\mu\alpha\beta}\big[\nabla_\nu, \nabla_\alpha\big]\Phi_\beta{}^\nu + \epsilon^{\mu\alpha\beta}\nabla_\alpha\nabla_\nu\Phi_\beta{}^\nu \\ &= \epsilon^{\mu\alpha\beta}\big(R_{\nu\alpha\beta}{}^\lambda\Phi_\lambda{}^\nu + R_{\nu\alpha}{}^\nu{}_\lambda\Phi_\beta{}^\lambda\big) + \underbrace{\epsilon^{\mu\alpha\beta}\nabla_\alpha\nabla_\beta\Phi}_{=0} \\ &= \epsilon^{\mu\alpha\beta}R_{\nu\alpha\beta}{}^\lambda\Phi_\lambda{}^\nu + \epsilon^{\mu\alpha\beta}R_{\nu\alpha}{}^\nu{}_\lambda\Phi_\beta{}^\lambda \\ &= \epsilon^{\mu\alpha\beta}R_{\alpha\lambda}\Phi_\beta{}^\lambda.\end{aligned} \tag{4.37}$$

note that;

$$\begin{aligned}\epsilon^{\mu\alpha\beta}R_{\nu\alpha\beta}{}^\lambda\Phi_\lambda{}^\nu = \epsilon^{\mu\alpha\beta}\Big[& g_{\nu\beta}R_\alpha{}^\lambda - \delta_\nu{}^\lambda R_{\alpha\beta} + \delta_\alpha{}^\lambda R_{\nu\beta} - g_{\alpha\beta}R_\nu{}^\lambda \\ & - \frac{R}{2}\big(g_{\nu\beta}\delta_\alpha{}^\lambda + \delta_\nu{}^\lambda g_{\alpha\beta}\big)\Big]\Phi_\lambda{}^\nu \\ = \epsilon^{\mu\alpha\beta}\big(& R_{\alpha\nu}\Phi_\beta{}^\nu + R_{\beta\nu}\Phi_\alpha{}^\nu\big) = 0\end{aligned} \tag{4.38}$$

The second and the third term are identical,

$$\epsilon^{\mu\alpha\beta}\epsilon^{\nu\sigma\rho}\Phi_{\alpha\sigma}\nabla_\nu\Phi_{\beta\rho} = \epsilon^{\mu\alpha\beta}\epsilon^{\nu\sigma\rho}\nabla_\nu\Phi_{\alpha\sigma}\Phi_{\beta\rho}. \tag{4.39}$$

So we need to calculate;

$$\frac{2k}{\mu^2}\epsilon^{\mu\alpha\beta}\epsilon^{\nu\sigma\rho}\Phi_{\alpha\sigma}\nabla_\nu\Phi_{\beta\rho} = \frac{2k}{\mu^2}\epsilon^{\mu\alpha\beta}\Phi_{\alpha\sigma}\underbrace{\epsilon^{\nu\sigma\rho}\nabla_\nu\Phi_{\beta\rho}}_{-C_\beta{}^\sigma}. \tag{4.40}$$

To proceed further, we need to use the field equations. So here is what we get;

$$\begin{aligned}\nabla_\nu\left(\mathcal{E}^{\mu\nu} + \frac{1}{\mu}\epsilon^{\mu\alpha\beta}\nabla_\alpha\Phi_\beta{}^\nu + \frac{k}{\mu^2}\epsilon^{\mu\alpha\beta}\epsilon^{\nu\sigma\rho}\Phi_{\alpha\sigma}\Phi_{\beta\rho}\right)& \\ = \frac{1}{\mu}\epsilon^{\mu\alpha\beta}R_\alpha{}^\lambda\Phi_{\beta\lambda} - \frac{2k}{\mu^2}\epsilon^{\mu\alpha\beta}\Phi_{\alpha\sigma}C_\beta{}^\sigma& \\ = \frac{1}{\mu}\epsilon^{\mu\alpha\beta}\Phi_{\beta\lambda}\left[R_\alpha{}^\lambda + \frac{2k}{\mu}C_\alpha{}^\lambda\right] \neq 0.&\end{aligned} \tag{4.41}$$

$$\begin{aligned}\nabla_\nu\left(\mathcal{E}^{\mu\nu} + \frac{1}{\mu}\epsilon^{\mu\alpha\beta}\nabla_\alpha\Phi_{\beta\nu} + \frac{k}{\mu^2}\epsilon^{\mu\alpha\beta}\epsilon^{\nu\sigma\rho}\Phi_{\alpha\sigma}\Phi_{\beta\rho}\right)& \\ = \frac{1}{\mu}\epsilon^{\mu\alpha\beta}\Phi_{\beta\lambda}\left(R_\alpha{}^\lambda + \frac{2k}{\mu}\epsilon_\alpha{}^{\beta\gamma}\nabla_\beta\Phi_\gamma{}^\lambda\right).&\end{aligned} \tag{4.42}$$

Clearly this expression is not generically zero and the theory is generically inconsistent. But the explicit expression tells us that we must include Einstein's gravity in the $\mathcal{E}_{\mu\nu}$ tensor since in (4.42) the Ricci tensor appears. But the Ricci tensor is not divergence-free, so in $\mathcal{E}_{\mu\nu}$ we must have the Einstein tensor $G_{\mu\nu}$, in order to have any hope of constructing an on-shell consistent theory; hence, we choose

$$\mathcal{E}_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda_0 g_{\mu\nu}. \tag{4.43}$$

Therefore our theory reads

$$\mathcal{E}^{\mu\nu} + \frac{1}{\mu}C^{\mu\nu} + \frac{k}{\mu^2}L^{\mu\nu} = 0, \tag{4.44}$$

where $C^{\mu\nu}$ is the generalized Cotton tensor given as $C^{\mu\nu} = \epsilon^{\mu\alpha\beta}\nabla_\alpha\Phi_\beta{}^\nu$ and the $L-$ tensor is given $L^{\mu\nu} = \epsilon^{\mu\alpha\beta}\epsilon^{\nu\sigma\rho}\Phi_{\alpha\sigma}\Phi_{\beta\rho}$. From (4.30) we have

$$\Rightarrow \quad R_\alpha{}^\lambda = \frac{1}{2}\delta_\alpha{}^\lambda R - \Lambda\delta_\alpha{}^\lambda - \frac{1}{\mu}C_\alpha{}^\lambda - \frac{k}{\mu^2}L_\alpha{}^\lambda. \tag{4.45}$$

Inserting this result into the equation (4.41) yields

$$\begin{aligned}\nabla_\nu\left[\mathcal{E}^{\mu\nu}+\frac{1}{\mu}C^{\mu\nu}+\frac{k}{\mu^2}L^{\mu\nu}\right]\\ =\frac{1}{\mu}\epsilon^{\mu\alpha\beta}\Phi_{\beta\lambda}\left[\frac{1}{2}\delta_\alpha{}^\lambda R-\Lambda\delta_\alpha{}^\lambda-\frac{1}{\mu}C_\alpha{}^\lambda-\frac{k}{\mu^2}L_\alpha{}^\lambda+\frac{2k}{\mu}C_\alpha{}^\lambda\right].\end{aligned} \tag{4.46}$$

The first two terms vanish identically. The remaining terms are inhomogeneous in the powers μ, hence separate powers must vanish separately, which leads to $k=\frac{1}{2}$ for the Cotton generalized terms to vanish. Then one has

$$\nabla_\nu\left[\mathcal{E}^{\mu\nu}+\frac{1}{\mu}C^{\mu\nu}+\frac{k}{\mu^2}L^{\mu\nu}\right]=-\frac{k}{\mu^2}\epsilon^{\mu\alpha\beta}\Phi_{\beta\lambda}L_\alpha{}^\lambda. \tag{4.47}$$

We must check whether $\epsilon^{\mu\alpha\beta}\Phi_{\beta\lambda}L_\alpha{}^\lambda$ vanishes or not straightforward

$$\Rightarrow \quad \epsilon^{\mu\alpha\beta}\Phi_{\beta\lambda}L_\alpha{}^\lambda=\epsilon^{\mu\alpha\beta}\Phi_{\beta\lambda}\epsilon_\alpha{}^{\sigma\rho}\epsilon^{\lambda\kappa\gamma}\Phi_{\sigma\kappa}\Phi_{\rho\gamma}=0, \tag{4.48}$$

due to symmetry. Now we get the on-shell consistent theory (that is called a third way consistent 3D gravity) with the field equations explicitly given as

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R+\Lambda_0 g_{\mu\nu}+\frac{1}{\mu}\epsilon_\mu{}^{\alpha\beta}\nabla_\alpha\Phi_{\beta\nu}+\frac{1}{2\mu^2}\epsilon_\mu{}^{\alpha\beta}\epsilon_\nu{}^{\sigma\rho}\Phi_{\alpha\sigma}\Phi_{\beta\rho}=0, \tag{4.49}$$

with any $\Phi_{\mu\nu}=\Phi_{\nu\mu}$ satisfying $\nabla_\mu\Phi^\mu{}_\nu=\nabla_\nu\Phi$, is consistent. This is the main equation that we shall study in detail below.

The next obvious question is how to find a 2-tensor $\Phi_{\mu\nu}$ that satisfies the desired properties. This is also remarkably simple to answer: consider any diffeomorphism invariant action.

$$I=\int d^3x\sqrt{-g}\mathcal{L}, \tag{4.50}$$

vary it with respect to the metric to get a 2-tensor after dropping the boundary terms

$$\delta_g I=\int d^3x\Psi_{\mu\nu}\delta g^{\mu\nu}, \tag{4.51}$$

where we called this 2-tensor to be $\Psi_{\mu\nu}$. This is still not the $\Phi_{\mu\nu}$ that we are searching for even though, obviously it is a candidate if it is trace-free. But we can build $\Phi_{\mu\nu}$ from $\Psi_{\mu\nu}$ as follows. Let p be a constant, then

$$\Phi_{\mu\nu}\equiv\Psi_{\mu\nu}+pg_{\mu\nu}\Psi \quad , \quad \nabla_\mu\Psi^{\mu\nu}=0, \quad \Psi=g_{\mu\nu}\Psi^{\mu\nu}$$

$$g_{\mu\nu}\Phi^{\mu\nu} = \Phi = \Psi + 3p\Psi \quad \rightarrow \quad \Psi = \frac{1}{1+3p}\Phi \tag{4.52}$$

We already observe that $\nabla_\mu \Phi^{\mu\nu} = \nabla^\nu \Phi$, so

$$\nabla_\mu \Phi^{\mu\nu} = \nabla^\nu \Phi = p\nabla^\nu \Psi, \quad \rightarrow \quad \nabla^\nu \Phi = \frac{p}{1+3p}\nabla^\nu \Phi$$

$$\Rightarrow \quad \frac{p}{1+3p} = 1 \quad \rightarrow \quad p = -\frac{1}{2}. \tag{4.53}$$

and then one can choose [43]

$$\Phi_{\mu\nu} := \Psi_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\Psi \quad , \quad \Psi = g^{\mu\nu}\Psi_{\mu\nu}, \tag{4.54}$$

which satisfies the desired property $\nabla_\sigma \Phi = \nabla_\alpha \Phi_\sigma{}^\alpha$. Using the $\Psi_{\mu\nu}$ tensor, we can recast (4.49) as

$$\begin{aligned} E_{\mu\nu} :=& R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda_0 g_{\mu\nu} + \frac{1}{\mu}\epsilon_\mu{}^{\alpha\beta}\nabla_\alpha\big(\Psi_{\beta\nu} - \frac{1}{2} g_{\beta\nu}\Psi\big) \\ &+ \frac{1}{2\mu^2}\bigg(g_{\mu\nu}\big(\Psi^2_{\alpha\beta} - \frac{3}{4}\Psi^2\big) + \Psi_{\mu\nu}\Psi - 2\Psi_{\mu\alpha}\Psi_\nu{}^\alpha\bigg) = 0. \end{aligned} \tag{4.55}$$

So the summary of the above discussion is that we can deform Einstein's gravity with any covariantly conserved $\Psi_{\mu\nu}$ in such a way that we get a nontrivial on-shell-consistent theory. Let us stress that, even though $\nabla_\mu \Psi^{\mu\nu} = 0$, we only have $\nabla_\mu E^{\mu\nu} = 0$. As the "Bianchi identity" is not automatically satisfied, this theory does not have a variational formulation in terms of the metric only.

As we noted above, we could ask if further deformations of (4.49) or (4.55) which powers as $O(\Phi^3), O(\Phi^4)$ or more are possible. Namely, could they lead to on-shell consistent theories. In this most general formulation, we have not studied this problem but the answer seems to be this is unlikely because, in the simpler setting of MMG, with $\Phi_{\mu\nu} = S_{\mu\nu}$, it was shown in [42] that no further cubic or quadric or more deformation is possible. Moreover, it was also shown in that work that the second covariant divergence of the MMG field equations do not vanish automatically but vanish on-shell only. This is an other requirement for consistency.

The above discussion has been general, next we provide some examples of these theories by choosing the $\Psi_{\mu\nu}$ tensor from some well studied actions.

4.3 $\Psi_{\mu\nu}$ from Quadratic Gravity

On-shell consistency of the theory is certainly not a sufficient condition; keeping a possible quantum version of the theory in mind, one still needs to understand the spectrum (particle content) of the theory about its possible maximally symmetric vacua. In particular to emulate the four dimensional gravity, which is no spin-0 modes, we shall demand that in the spectrum of the theory, there are only massive spin-2 particles. To this end let us consider the generic quadratic action

$$I = \frac{1}{16\pi G}\int d^3x\sqrt{-g}\left(\sigma R + \alpha R^2 + \beta R_{\mu\nu}^2\right), \tag{4.56}$$

whose variation yields

$$\delta I = \frac{1}{16\pi G}\int d^3x\sqrt{-g}\,\Psi_{\mu\nu}\,\delta g^{\mu\nu}, \tag{4.57}$$

where [44]

$$\begin{aligned}\Psi_{\mu\nu} =& \sigma G_{\mu\nu} + \alpha\bigg(2RR_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R^2 + 2g_{\mu\nu}\Box R - 2\nabla_\mu\nabla_\nu R\bigg)\\ &+ \beta\bigg(\frac{3}{2}g_{\mu\nu}R_{\rho\sigma}R^{\rho\sigma} - 4R_\mu{}^\rho R_{\nu\rho} + \Box R_{\mu\nu}\\ &+ \frac{1}{2}g_{\mu\nu}\Box R - \nabla_\mu\nabla_\nu R + 3RR_{\mu\nu} - g_{\mu\nu}R^2\bigg).\end{aligned} \tag{4.58}$$

Since it is derived from the variation of an action, the tensor $\Psi_{\mu\nu}$ is symmetric, covariantly conserved, and therefore yields consistent field equations. We now consider the linearization around the AdS$_3$ spacetime as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \tag{4.59}$$

where the background AdS$_3$ metric satisfies

$$\begin{aligned}\bar{R}_{\mu\nu\rho\sigma} =& \Lambda\big(\bar{g}_{\mu\rho}\bar{g}_{\nu\sigma} - \bar{g}_{\mu\sigma}\bar{g}_{\nu\rho}\big), \quad \bar{R}_{\mu\nu} = 2\Lambda\bar{g}_{\mu\nu},\\ &\bar{R} = 6\Lambda, \quad \bar{G}_{\mu\nu} = -\Lambda\bar{g}_{\mu\nu},\end{aligned} \tag{4.60}$$

and the tensor $h_{\mu\nu}$ describes the perturbations around the AdS$_3$ background. The linearized versions of the Ricci tensor, Ricci scalar and the cosmological Einstein tensor are given, respectively, by [3]

$$R^L_{\mu\nu} = \bar{\nabla}^\rho \bar{\nabla}_{(\mu} h_{\nu)\rho} - \frac{1}{2}\bar{\Box} h_{\mu\nu} - \frac{1}{2}\bar{\nabla}_\mu \bar{\nabla}_\nu h,$$
$$R^L = -\bar{\Box} h + \bar{\nabla}^\rho \bar{\nabla}^\sigma h_{\rho\sigma} - 2\Lambda h, \tag{4.61}$$
$$\mathcal{G}_{\mu\nu} \equiv (G_{\mu\nu} + \Lambda g_{\mu\nu})^L = R^L_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu} R^L - 2\Lambda h_{\mu\nu}.$$

Under the linearization (4.59), the background value of the tensor $\Psi_{\mu\nu}$ is given by

$$\bar{\Psi}_{\mu\nu} = a\bar{g}_{\mu\nu}, \qquad a = -\Lambda\sigma + 2\Lambda^2\big(3\alpha + \beta\big), \tag{4.62}$$

and its linearization yields

$$\begin{aligned}\Psi^L_{\mu\nu} \;&=\; \bar{\sigma}\mathcal{G}_{\mu\nu} + \big(2\alpha + \beta\big)\Big(\bar{g}_{\mu\nu}\bar{\Box} - \bar{\nabla}_\mu \bar{\nabla}_\nu + 2\Lambda \bar{g}_{\mu\nu}\Big) R^L \\ &\quad + \beta\Big(\bar{\Box}\mathcal{G}_{\mu\nu} - \Lambda \bar{g}_{\mu\nu} R^L\Big) + a h_{\mu\nu},\end{aligned} \tag{4.63}$$

with

$$\bar{\sigma} = \sigma + 12\Lambda\alpha + 2\Lambda\beta. \tag{4.64}$$

The linearization of its trace $\Psi^L \equiv (g^{\mu\nu}\Psi_{\mu\nu})^L$, can be computed as

$$\Psi^L = \Big(4\alpha + \frac{3}{2}\beta\Big)\bar{\Box} R^L + \Big(-\frac{\sigma}{2} + 2\Lambda\big(3\alpha + \beta\big)\Big) R^L. \tag{4.65}$$

We have to constrain the parameters (σ, α, β) in such a way that there are only massive spin-2 exitations in the theory. We shall do that discussion below. But before that, let us note an important isuue: The quadratic theory we discussed here captures a lot of the physics of more general theories, the so called $f(Ricci)$ theories. In fact in [29]

$$\begin{aligned}I = \int d^3x\sqrt{-g}\Big[\tilde{\sigma}\left(R - 2\tilde{\lambda}_0\right) + \tilde{\alpha}R^2 + \tilde{\beta}R^2_{\mu\nu} \\ + a_1 R^\mu_\nu R^\nu_\rho R^\rho_\mu + a_2 R R^2_{\mu\nu} + a_3 R^3\Big],\end{aligned} \tag{4.66}$$

and the quadratic action

$$I = \int d^3x\sqrt{-g}\left[\sigma\left(R - 2\lambda_0\right) + \alpha R^2 + \beta R^2_{ab}\right], \tag{4.67}$$

[3] Derivations of some of these expressions are highly lengthy, hence we do not depict them here, but simply quot the final results [44]

have the same linearized equations if their parameters are related by the following equations [29]

$$\begin{aligned}
\sigma &= \tilde{\sigma} - 12\Lambda^2 \left(a_1 + 3a_2 + 9a_3\right), \\
\lambda_0 &= \frac{\tilde{\sigma}}{\sigma}\tilde{\lambda}_0 + \Lambda\left(1 - \frac{\tilde{\sigma}}{\sigma}\right), \\
\alpha &= \tilde{\alpha} + 2\Lambda\left(2a_2 + 9a_3\right), \\
\beta &= \tilde{\beta} + 6\Lambda\left(a_1 + a_2\right).
\end{aligned} \tag{4.68}$$

Introduction of a cosmological constant λ_0 is in the equivalent quadratic action (4.67), yields a term proportional to the metric tensor in $\Psi_{\mu\nu}$ (4.59), which as a result shifts the parameter Λ_0 in the field equations (4.55). The change in the parameter Λ_0 is not of much importance in our subsequent discussion.

4.4 Bachian Gravity

To remove the possible spin-0 modes, let us study the linearized equation. For this purpose, we consider the trace of the field equations

$$R - 6\Lambda_0 + \frac{1}{\mu^2}\left(\Phi^2 - \Phi_{\mu\nu}\Phi^{\mu\nu}\right) = 0, \tag{4.69}$$

which, in terms of the $\Psi_{\mu\nu}$ tensor,

$$\Phi_{\mu\nu} = \Psi_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\Psi \quad , \quad \Phi = -\frac{1}{2}\Psi$$

$$\Phi^2_{\mu\nu} = \Phi_{\mu\nu}\Phi^{\mu\nu} = \Psi^2_{\mu\nu} - \frac{1}{4}\Psi^2$$

can be recast as

$$R - 6\Lambda_0 + \frac{1}{\mu^2}\left(\frac{1}{2}\Psi^2 - \Psi^2_{\mu\nu}\right) = 0. \tag{4.70}$$

First of all let us find the first order linearization of the trace of the field equation; $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ and $g^{\mu\nu} = \bar{g}^{\mu\nu} - h^{\mu\nu}$, around the background metric $\bar{g}_{\mu\nu}$, and small perturbation $h_{\mu\nu}$:

$$\delta\Psi^2 = 2\Psi\delta\Psi \qquad \Rightarrow \qquad (\Psi^2)^L = 2\bar{\Psi}\Psi^L, \tag{4.71}$$

$$\delta\Psi^2_{\mu\nu} = \delta(\Psi_{\mu\nu}\Psi^{\mu\nu}) = \delta\Psi_{\mu\nu}\Psi^{\mu\nu} + \Psi_{\mu\nu}\delta\Psi^{\mu\nu}. \tag{4.72}$$

Here we need to calculate $\delta\Psi^{\mu\nu}$;

$$\delta\Psi^{\mu\nu} = \delta\big(g^{\mu\alpha}g^{\nu\beta}\Psi_{\alpha\beta}\big) = \delta g^{\mu\alpha}\Psi^{\nu}{}_{\alpha} + \delta g^{\nu\beta}\Psi^{\mu}{}_{\beta} + g^{\mu\alpha}g^{\nu\beta}\delta\Psi_{\alpha\beta},$$

$$\Rightarrow \qquad \delta\Psi^2_{\mu\nu} = 2\delta\Psi_{\mu\nu}\Psi^{\mu\nu} + 2\Psi_{\mu\nu}\Psi^{\nu}{}_{\alpha}\delta g^{\mu\alpha},$$

$$\big(\Psi^2_{\mu\nu}\big)^L = 2\bar{\Psi}^{\mu\nu}(\Psi_{\mu\nu})^L - 2\bar{\Psi}_{\mu\nu}\bar{\Psi}^{\nu}{}_{\alpha}h^{\mu\alpha}. \tag{4.73}$$

So the linearized trace equation becomes;

$$R^L + \frac{1}{\mu^2}\Bigg(\bar{\Psi}\Psi^L + 2\bar{\Psi}_{\mu\nu}\bar{\Psi}^{\mu}{}_{\alpha}h^{\nu\alpha} - 2\bar{\Psi}^{\mu\nu}\Psi^L_{\mu\nu}\Bigg) = 0, \tag{4.74}$$

where

$$\bar{\Psi}_{\mu\nu} = a\bar{g}_{\mu\nu} \quad \Rightarrow \quad \bar{\Psi} = 3a,$$

$$R^L + \frac{1}{\mu^2}\Bigg(3a\Psi^L + 2a^2h - 2a\bar{g}^{\mu\nu}\Psi^L_{\mu\nu}\Bigg) = 0. \tag{4.75}$$

Now, we are suppose to find the terms, Ψ^L and $\bar{g}^{\mu\nu}\Psi^L_{\mu\nu}$ which can be done as follows

$$\delta\Psi = \delta\big(g^{\mu\nu}\Psi_{\mu\nu}\big) = \delta g^{\mu\nu}\Psi_{\mu\nu} + g^{\mu\nu}\delta\Psi_{\mu\nu},$$

$$\Psi^L = -h^{\mu\nu}\bar{\Psi}_{\mu\nu} + \bar{g}^{\mu\nu}\Psi^L_{\mu\nu},$$
$$\bar{\Psi}_{\mu\nu} = a\bar{g}_{\mu\nu} \Rightarrow -h^{\mu\nu}\bar{\Psi}_{\mu\nu} = -h^{\mu\nu}a\bar{g}_{\mu\nu} = -ah$$

so one has

$$\Psi^L = -ah + \bar{g}^{\mu\nu}\Psi^L_{\mu\nu} \qquad \Rightarrow \quad \bar{g}^{\mu\nu}\Psi^L_{\mu\nu} = \Psi^L + ah$$

yielding

$$\Rightarrow \quad R^L + \frac{1}{\mu^2}\Bigg(3a\Psi^L + 2a^2h - 2a\big(\Psi^L + ah\big)\Bigg) = R^L + \frac{a}{\mu^2}\Psi^L = 0. \tag{4.76}$$

Linearization of the $\Psi^L_{\mu\nu}$ is given by;

$$\begin{aligned}\Psi^L_{\mu\nu} &= \bar{\sigma}\mathcal{G}^L_{\mu\nu} + \big(2\alpha+\beta\big)\Bigg(\bar{g}_{\mu\nu}\bar{\Box} - \bar{\nabla}_\mu\bar{\nabla}_\nu + 2\Lambda\bar{g}_{\mu\nu}\Bigg)R^L \\ &\quad +\beta\Bigg(\bar{\Box}\mathcal{G}^L_{\mu\nu} - \Lambda\bar{g}_{\mu\nu}R^L\Bigg) + h_{\mu\nu}\Bigg(-\Lambda\sigma + 2\Lambda^2\big(3\alpha+\beta\big)\Bigg),\end{aligned}$$

with

$$\bar{\sigma} = \sigma + 12\Lambda\alpha + 2\lambda\beta \quad , \quad a = -\Lambda\sigma + 2\Lambda^2\big(3\alpha + \beta\big)$$

$$\begin{aligned} \bar{g}^{\mu\nu}\Psi^L_{\mu\nu} &= \bar{\sigma}\bar{g}^{\mu\nu}\mathcal{G}^L_{\mu\nu} + \big(2\alpha + \beta\big)\big(2\bar{\Box} + 6\Lambda\big)R^L \\ &\quad +\beta\Bigg(\bar{\Box}(\bar{g}^{\mu\nu}\mathcal{G}^L_{\mu\nu}) - 3\Lambda R^L\Bigg) + ah. \end{aligned}$$

Recalling

$$\mathcal{G}^L_{\mu\nu} = R^L_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R^L - 2\Lambda h_{\mu\nu},$$

$$\delta R = \delta\big(g^{\mu\nu}R_{\mu\nu}\big) = \delta g_{\mu\nu}R_{\mu\nu} + g_{\mu\nu}\delta R_{\mu\nu}$$

$$R^L = -h^{\mu\nu}\bar{R}_{\mu\nu} + \bar{g}^{\mu\nu}R^L_{\mu\nu} = -2\Lambda h + \bar{g}^{\mu\nu}R^L_{\mu\nu} \to \bar{g}^{\mu\nu}R^L_{\mu\nu} = R^L + 2\Lambda h \tag{4.77}$$

$$\bar{g}^{\mu\nu}\mathcal{G}^L_{\mu\nu} = \bar{g}^{\mu\nu}R^L_{\mu\nu} - \frac{3}{2}R^L - 2\Lambda h = R^L + 2\Lambda h - \frac{3}{2}R^L - 2\Lambda h = -\frac{1}{2}R^L$$

$$\begin{aligned} \bar{g}^{\mu\nu}\Psi^L_{\mu\nu} &= \bar{\sigma}\bar{g}^{\mu\nu}\mathcal{G}^L_{\mu\nu} + \big(2\alpha + \beta\big)\big(2\bar{\Box} + 6\Lambda\big)R^L \\ &\quad + \beta\Bigg(\bar{\Box}(\bar{g}^{\mu\nu}\mathcal{G}^L_{\mu\nu}) - 3\Lambda R^L\Bigg) + ah \\ &= -\frac{\bar{\sigma}}{2}R^L + \big(2\alpha + \beta\big)\big(2\bar{\Box} + 6\Lambda\big)R^L \\ &\quad + \beta\big(-\frac{1}{2}\bar{\Box}R^L - 3\Lambda R^L\big) + ah \\ &= -\frac{1}{2}\big(\sigma + 12\Lambda\alpha + 2\Lambda\beta\big)R^L + \big(2\alpha + \beta\big)\big(2\bar{\Box} + 6\Lambda\big)R^L \\ &\qquad + \beta\big(-\frac{1}{2}\bar{\Box}R^L - 3\Lambda R^L\big) + ah. \end{aligned} \tag{4.78}$$

Finally we have

$$\begin{aligned} \Psi^L &= -ah + \bar{g}^{\mu\nu}\Psi^L_{\mu\nu} \\ &= -\frac{1}{2}\big(\sigma + 12\Lambda\alpha + 2\Lambda\beta\big)R^L + \big(2\alpha + \beta\big)\big(2\bar{\Box} + 6\Lambda\big)R^L \\ &\quad + \beta\big(-\frac{1}{2}\bar{\Box}R^L - 3\Lambda R^L\big) \\ &= \Bigg(4\alpha + \frac{3}{2}\beta\Bigg)\bar{\Box}R^L + \Bigg(-\frac{\sigma}{2} + 2\Lambda\big(3\alpha + \beta\big)\Bigg)R^L, \end{aligned} \tag{4.79}$$

and the linearized trace equation reads

$$R^L + \frac{a}{\mu^2}\Bigg[\Bigg(4\alpha + \frac{3}{2}\beta\Bigg)\bar{\Box}R^L + \Bigg(-\frac{\sigma}{2} + 2\Lambda\big(3\alpha + \beta\big)\Bigg)R^L\Bigg] = 0. \tag{4.80}$$

Recalling that (4.61) $R^L \sim \bar{\Box}h$, this equation is already a fourth order wave equation of the form with the operator

$$(c_1\bar{\Box}^2 + c_2\bar{\Box} + c_3)h = 0. \tag{4.81}$$

Of course in this form it will have spin-0 modes that we would like to avoid unless we set $c_1 = 0$ and $c_2 = 0$. Setting the coefficient of the $\bar{\Box}R^L$ term to zero, yields two possibilities:

$$4\alpha + \frac{3}{2}\beta = 0, \quad \text{or} \quad a = \Lambda\big(-\sigma + 6\Lambda\alpha + 2\Lambda\beta\big) = 0. \tag{4.82}$$

In both cases, we have $R^L = 0$, and as a result we can choose the compatible transverse-traceless (TT) gauge ($\bar{\nabla}^\mu h_{\mu\nu} = 0 = h$).

Having studied the linearization of the trace equation and the constraints coming from the absence of the scalar mode, we can now linearize the full field equations (4.49) to find the particle content of the theory and their masses. The background value the tensor $\Phi_{\mu\nu}$ is given as

$$\bar{\Phi}_{\mu\nu} = -\frac{a}{2}\bar{g}_{\mu\nu}, \tag{4.83}$$

and its linearization yields

$$\Phi^L_{\mu\nu} = \Psi^L_{\mu\nu} - \frac{1}{2}h_{\mu\nu}\bar{\Psi} - \frac{1}{2}\bar{g}_{\mu\nu}\Psi^L. \tag{4.84}$$

The vacuum equation determining the effective cosmological constants is

$$\Lambda_0 - \Lambda - \frac{a^2}{4\mu^2} = 0, \tag{4.85}$$

where, of course, a is given in (4.62). The linearization of the field equations can be obtained as follows

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda_0 g_{\mu\nu} + \frac{1}{\mu}\epsilon_\mu{}^{\alpha\beta}\nabla_\alpha\Phi_{\beta\nu} + \frac{1}{2\mu^2}\epsilon_\mu{}^{\alpha\beta}\epsilon_\nu{}^{\rho\sigma}\Phi_{\alpha\rho}\Phi_{\beta\sigma} = 0 \tag{4.86}$$

with any $\nabla_\mu\Phi_\nu{}^\mu = \nabla_\nu\Phi$ and $\Phi_{\mu\nu} = \Phi_{\nu\mu}$.

$$\Phi_{\mu\nu} = \Psi_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\Psi \quad , \quad \Psi = g^{\mu\nu}\Psi_{\mu\nu} \tag{4.87}$$

Now, study the linearization of each term in the field eqautions:

$$\begin{aligned}
\epsilon_\mu{}^{\alpha\beta}\epsilon_\nu{}^{\rho\sigma}\Phi_{\alpha\rho}\Phi_{\beta\sigma} \quad = \quad & \big(-g_{\mu\nu}g^{\alpha\rho}g^{\beta\sigma} + g_{\mu\nu}g^{\alpha\sigma}g^{\beta\rho}\big)\Phi_{\alpha\rho}\Phi_{\beta\sigma} \\
& + \big(-g_\mu{}^\rho g^{\alpha\sigma}g_\nu{}^\beta + g_\mu{}^\rho g^{\beta\sigma}g_\nu{}^\alpha\big)\Phi_{\alpha\rho}\Phi_{\beta\sigma} \\
& + \big(-g_\nu{}^\alpha g^{\beta\rho}g_\mu{}^\sigma + g_\nu{}^\beta g^{\alpha\rho}g_\mu{}^\sigma\big)\Phi_{\alpha\rho}\Phi_{\beta\sigma}
\end{aligned}$$

$$\Rightarrow \quad \epsilon_\mu{}^{\alpha\beta}\epsilon_\nu{}^{\rho\sigma}\Phi_{\alpha\rho}\Phi_{\beta\sigma} = -g_{\mu\nu}\big(\Phi^2 - \Phi^2_{\alpha\beta}\big) + 2\Phi_{\mu\nu}\Phi - 2\Phi_{\mu\alpha}\Phi_\nu{}^\alpha.$$

$$g^{\mu\nu}\epsilon_\mu{}^{\alpha\beta}\epsilon_\nu{}^{\rho\sigma}\Phi_{\alpha\rho}\Phi_{\beta\sigma} = \epsilon_\mu{}^{\alpha\beta}\epsilon^{\mu\rho\sigma}\Phi_{\alpha\rho}\Phi_{\beta\sigma}$$

$$\Rightarrow \quad \epsilon_\mu{}^{\alpha\beta}\epsilon^{\mu\rho\sigma}\Phi_{\alpha\rho}\Phi_{\beta\sigma} = \Phi^2_{\alpha\beta} - \Phi^2. \tag{4.88}$$

We list the linearization of the relevant tensors here:

$$\delta\big(\Phi_{\alpha\beta}\Phi^{\alpha\beta}\big) = \delta\big(g^{\mu\alpha}g^{\nu\beta}\Phi_{\alpha\beta}\Phi_{\mu\nu}\big) = 2\Phi^{\alpha\beta}\delta\Phi_{\alpha\beta} + 2\delta g^{\mu\alpha}\Phi_{\alpha\beta}\Phi_\mu{}^\beta$$

$$\delta\big(\Phi_\mu{}^\rho\Phi_{\nu\rho}\big) = \delta\big(g^{\alpha\rho}\Phi_{\mu\alpha}\Phi_{\nu\rho}\big) = \delta g^{\alpha\rho}\Phi_{\mu\alpha}\Phi_{\nu\rho} + \Phi_\nu{}^\alpha\delta\Phi_{\mu\alpha} + \Phi_\mu{}^\rho\delta\Phi_{\nu\rho}$$

$$\begin{aligned}\delta\big(\epsilon_\mu{}^{\alpha\beta}\epsilon_\nu{}^{\rho\sigma}\Phi_{\alpha\rho}\Phi_{\beta\sigma}\big) = &-\big(\Phi^2 - \Phi^2_{\alpha\beta}\big)\delta g_{\mu\nu} - g_{\mu\nu}\Big(2\Phi\delta\Phi - \delta\big(\Phi_{\alpha\beta}\Phi^{\alpha\beta}\big)\Big)\\ &+ 2\Phi\delta\Phi_{\mu\nu} + 2\Phi_{\mu\nu}\delta\Phi - 2\delta\big(\Phi_\mu{}^\alpha\Phi_{\nu\alpha}\big).\end{aligned}$$

$$\begin{aligned}\big(\epsilon_\mu{}^{\alpha\beta}\epsilon_\nu{}^{\rho\sigma}\Phi_{\alpha\rho}\Phi_{\beta\sigma}\big)^L = &-\big(\Phi^2 - \Phi^2_{\alpha\beta}\big)h_{\mu\nu}\\ &- \bar{g}_{\mu\nu}\big(2\bar{\Phi}\Phi^L - 2\bar{\Phi}^{\alpha\beta}\Phi^L_{\alpha\beta} + 2\bar{\Phi}_{\alpha\beta}\bar{\Phi}_\mu{}^\beta h^{\mu\alpha}\big)\\ &+ 2\bar{\Phi}\Phi^L_{\mu\nu} + 2\bar{\Phi}_{\mu\nu}\Phi^L + 2h^{\alpha\rho}\bar{\Phi}_{\mu\alpha}\bar{\Phi}_{\nu\rho}\\ &- 2\bar{\Phi}_\nu{}^\alpha\Phi^L_{\mu\alpha} - 2\bar{\Phi}_\mu{}^\rho\Phi^L_{\nu\rho}.\end{aligned} \tag{4.89}$$

We now plug the background tensors given

$$\rightarrow \quad \bar{\Phi}_{\mu\nu} = c\bar{g}_{\mu\nu} \quad , \quad \bar{\Phi} = 3c \quad , \quad \bar{\Phi}_{\mu\nu}\bar{\Phi}^{\mu\nu} = 3c^2, \tag{4.90}$$

to arrive at

$$\begin{aligned}\big(\epsilon_\mu{}^{\alpha\beta}\epsilon_\nu{}^{\rho\sigma}\Phi_{\alpha\rho}\Phi_{\beta\sigma}\big)^L =& 2c\Phi^L_{\mu\nu} - 4c\bar{g}_{\mu\nu}\Phi^L - 4c^2h_{\mu\nu}\\ &- 2c^2\bar{g}_{\mu\nu}h + 2c\bar{g}_{\mu\nu}\bar{g}^{\alpha\beta}\Phi^L_{\alpha\beta}\end{aligned}$$

$$\delta\Phi = \delta(g^{\alpha\beta}\Phi_{\alpha\beta}) = \delta g^{\alpha\beta}\Phi_{\alpha\beta} + g^{\alpha\beta}\delta\Phi_{\alpha\beta}$$

$$\begin{aligned}\Phi^L = -h^{\alpha\beta}\bar{\Phi}_{\alpha\beta} + \bar{g}^{\alpha\beta}\Phi^L_{\alpha\beta}, \quad \Phi^L = -ch + \bar{g}^{\alpha\beta}\Phi^L_{\alpha\beta},\\ \bar{g}^{\alpha\beta}\Phi^L_{\alpha\beta} = \Phi^L + ch.\end{aligned} \tag{4.91}$$

Now using $\bar{g}^{\alpha\beta}\Phi^L_{\alpha\beta} = \Phi^L + ch$, $\big(\epsilon_\mu{}^{\alpha\beta}\epsilon_\nu{}^{\rho\sigma}\Phi_{\alpha\rho}\Phi_{\beta\sigma}\big)^L$ the quadratic terms becomes

$$\big(\epsilon_\mu{}^{\alpha\beta}\epsilon_\nu{}^{\rho\sigma}\Phi_{\alpha\rho}\Phi_{\beta\sigma}\big)^L = 2c\Phi^L_{\mu\nu} - 2c\bar{g}_{\mu\nu}\Phi^L - 4c^2 h_{\mu\nu}. \tag{4.92}$$

We can move on the linearization of the generalized Cotton term. Linearization of the epsilon tensor reads

$$\delta\epsilon^{\mu\nu\rho} = \delta\big(\frac{1}{\sqrt{-g}}\varepsilon^{\mu\nu\rho}\big) = -\frac{1}{2}\epsilon^{\mu\nu\rho}g^{\alpha\beta}\delta g_{\alpha\beta}, \tag{4.93}$$

hence

$$(\epsilon^{\mu\nu\rho})^L = -\frac{1}{2}\bar{\epsilon}^{\mu\nu\rho}h. \tag{4.94}$$

$$\big(\epsilon_\mu{}^{\alpha\beta}\nabla_\alpha\Phi_{\beta\nu}\big)^L = \big(g_{\mu\lambda}\epsilon^{\lambda\alpha\beta}\nabla_\alpha\Phi_{\beta\nu}\big)^L,$$

$$\begin{aligned}\big(g_{\mu\lambda}\epsilon^{\lambda\alpha\beta}\nabla_\alpha\Phi_{\beta\nu}\big)^L =& h_{\mu\lambda}\bar{\epsilon}^{\lambda\alpha\beta}\bar{\nabla}_\alpha\bar{\Phi}_{\beta\nu} - \frac{1}{2}\bar{g}_{\mu\lambda}\bar{\epsilon}^{\lambda\alpha\beta}h\bar{\nabla}_\alpha\bar{\Phi}_{\beta\nu}\\ &+ \bar{\epsilon}_\mu{}^{\alpha\beta}(\bar{\nabla}_\alpha\bar{\Phi}_{\beta\nu})^L,\end{aligned}$$

$$\begin{aligned}\big(\epsilon_\mu{}^{\alpha\beta}\nabla_\alpha\Phi_{\beta\nu}\big)^L =& h_{\mu\lambda}\bar{\epsilon}^{\lambda\alpha\beta}\bar{\nabla}_\alpha\bar{\Phi}_{\beta\nu} - \frac{1}{2}\bar{g}_{\mu\lambda}\bar{\epsilon}^{\lambda\alpha\beta}h\bar{\nabla}_\alpha\bar{\Phi}_{\beta\nu}\\ &+ \bar{\epsilon}_\mu{}^{\alpha\beta}\big[\bar{\nabla}_\alpha\Phi^L_{\beta\nu} - (\Gamma^\rho_{\alpha\beta})^L\bar{\Phi}_{\rho\nu} - (\Gamma^\rho_{\alpha\nu})^L\bar{\Phi}_{\beta\rho}\big]\end{aligned} \tag{4.95}$$

$\bar{\nabla}_\alpha\bar{\Phi}_{\beta\nu} = 0$, since $\bar{\Phi}_{\beta\nu} = c\bar{g}_{\beta\nu}$

$$\big(\epsilon_\mu{}^{\alpha\beta}\nabla_\alpha\Phi_{\beta\nu}\big)^L = \bar{\epsilon}_\mu{}^{\alpha\beta}\bar{\nabla}_\alpha\Phi^L_{\beta\nu} - \bar{\epsilon}_\mu{}^{\alpha\beta}(\Gamma^\lambda_{\alpha\nu})^L\bar{\Phi}_{\beta\lambda}. \tag{4.96}$$

We can easily write everything in terms of the $\Psi-$ tensor. $\Phi_{\mu\nu} = \Psi_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\Psi$, $\Phi = -\frac{1}{2}\Psi$, $\bar{\Phi}_{\mu\nu} = c\bar{g}_{\mu\nu}$, $\bar{\Phi} = 3c$ and $\bar{\Psi} = -6c$.

$$\Phi^L_{\mu\nu} = \Psi^L_{\mu\nu} - \frac{1}{2}h_{\mu\nu}\bar{\Psi} - \frac{1}{2}\bar{g}_{\mu\nu}\Psi^L, \tag{4.97}$$

$$\begin{aligned}\big(\epsilon_\mu{}^{\alpha\beta}\epsilon_\nu{}^{\rho\sigma}\Phi_{\alpha\rho}\Phi_{\beta\sigma}\big)^L &= 2c\big(\Psi^L_{\mu\nu} - \frac{1}{2}h_{\mu\nu}\bar{\Psi} - \frac{1}{2}\bar{g}_{\mu\nu}\Psi^L\big)\\ &\quad - 2c\bar{g}_{\mu\nu}\big(-\frac{1}{2}\Psi^L\big) - 4c^2h_{\mu\nu}\\ &= 2c\big(\Psi^L_{\mu\nu} + ch_{\mu\nu}\big).\end{aligned} \tag{4.98}$$

We can now collect all the pieces together

$$\begin{aligned}\mathcal{G}^L_{\mu\nu} + (\Lambda_0 - \Lambda)h_{\mu\nu} + \frac{1}{\mu}\bar{\epsilon}_\mu{}^{\alpha\beta}\bar{\nabla}_\alpha\Phi^L_{\beta\nu} - \frac{1}{\mu}\bar{\epsilon}_\mu{}^{\alpha\beta}(\Gamma^\lambda_{\alpha\nu})^L\bar{\Phi}_{\beta\lambda}&\\ + \frac{c}{\mu^2}\big(\Psi^L_{\mu\nu} + ch_{\mu\nu}\big) = 0,&\end{aligned} \tag{4.99}$$

where

$$\begin{aligned}\bar{\epsilon}_\mu{}^{\alpha\beta}(\Gamma^\lambda_{\alpha\nu})^L\bar{\Phi}_{\beta\lambda} &= \bar{\epsilon}_\mu{}^{\alpha\beta}(\Gamma^\lambda_{\alpha\nu})^L c\bar{g}_{\beta\lambda} = c\bar{\epsilon}_\mu{}^\alpha{}_\beta(\Gamma^\beta_{\alpha\nu})^L \\ &= c\bar{\epsilon}_\mu{}^\alpha{}_\beta\frac{1}{2}\bar{g}^{\beta\lambda}\big(\bar{\nabla}_\alpha h_{\lambda\nu}+\bar{\nabla}_\nu h_{\alpha\lambda}-\bar{\nabla}h_{\alpha\nu}\big) \\ &= \frac{c}{2}\bar{\epsilon}_\mu{}^{\alpha\lambda}\big(\bar{\nabla}_\alpha h_{\lambda\nu}-\bar{\nabla}h_{\alpha\nu}\big) = c\bar{\epsilon}_\mu{}^{\alpha\lambda}\bar{\nabla}_\alpha h_{\lambda\nu}, \qquad (4.100)\end{aligned}$$

and

$$\bar{\nabla}_\alpha\Phi^L_{\beta\nu} = \bar{\nabla}_\alpha\big(\Psi^L_{\beta\nu}+3ch_{\beta\nu}-\frac{1}{2}\bar{g}_{\beta\nu}\Psi^L\big). \qquad (4.101)$$

So finally we arrive at the linearized equation.

$$\begin{aligned}\mathcal{G}_{\mu\nu} + \left(\Lambda_0-\Lambda+\frac{a^2}{4\mu^2}\right)h_{\mu\nu} - \frac{a}{2\mu^2}\Psi^L_{\mu\nu} \\ +\frac{1}{\mu}\bar{\epsilon}_{(\mu}{}^{\alpha\beta}\bar{\nabla}_{|\alpha}\Psi^L_{\beta|\nu)} - \frac{a}{\mu}\bar{\epsilon}_{(\mu}{}^{\alpha\beta}\bar{\nabla}_{|\alpha}h_{\beta|\nu)} = 0.\end{aligned} \qquad (4.102)$$

Here round brackets denote symmetrization with a factor of $\frac{1}{2}$ and the vertical lines exclude the indices inside. As it stands, this equation is highly cumbersome and so one cannot see the excitations that it describes. We need further simplifications. We have seen that the field equations are compatible with the transverse-traceless (TT) guage, that is $\bar{\nabla}_\mu h^{\mu\nu}=0, h=0$. So we choose this gauge which reduces $\Psi^L_{\mu\nu}$ to

$$\Psi^L_{\mu\nu} = \bar{\sigma}\mathcal{G}_{\mu\nu}+\beta\left(\bar{\Box}\mathcal{G}_{\mu\nu}-\Lambda\bar{g}_{\mu\nu}R^L\right)+ah_{\mu\nu}, \qquad (4.103)$$

plugging this to (4.102) one options a fifth-order equation in $h_{\mu\nu}$:

$$\big(1-\frac{\bar{\sigma}a}{2\mu^2}\big)\mathcal{G}_{\mu\nu}+\frac{\bar{\sigma}}{\mu}\bar{\epsilon}_\mu{}^{\alpha\beta}\bar{\nabla}_\alpha\mathcal{G}_{\beta\nu}-\frac{\beta a}{2\mu^2}\bar{\Box}\mathcal{G}_{\mu\nu}+\frac{\beta}{\mu}\bar{\epsilon}_\mu{}^{\alpha\beta}\bar{\nabla}_\alpha\bar{\Box}\mathcal{G}_{\beta\nu}=0, \qquad (4.104)$$

which is a linear equation of coupled relativistic (massive) fields which we need to decouple. In order to identify the spin-2 modes, we introduce the mutually commuting operators that was introduced in [18]

$$\begin{aligned}\big(\mathcal{D}^{L/R}\big)_\mu{}^\nu &:= \delta_\mu{}^\nu \pm \ell\bar{\epsilon}_\mu{}^{\alpha\nu}\bar{\nabla}_\alpha, \\ \big(\mathcal{D}^{p_i}\big)_\mu{}^\nu &:= \delta_\mu{}^\nu+\frac{1}{p_i}\bar{\epsilon}_\mu{}^{\alpha\nu}\bar{\nabla}_\alpha, \qquad i=1,2,3,\end{aligned} \qquad (4.105)$$

where p_i are to be determined below. In the TT gauge, we have $\bar{\nabla}^\rho\bar{\nabla}_\mu h_{\rho\nu} = -\frac{3}{\ell^2}h_{\mu\nu}$ and the linearized cosmological Einstein tensor can be written as

$$\mathcal{G}_{\mu\nu} = -\frac{1}{2}\big(\bar{\Box}+\frac{2}{\ell^2}\big)h_{\mu\nu} = \frac{1}{2\ell^2}\big(\mathcal{D}^L\mathcal{D}^R h\big)_{\mu\nu}. \qquad (4.106)$$

For the remaining three operators, one can show the following identity

$$\begin{aligned}\left(\mathcal{D}^{p_1}\mathcal{D}^{p_2}\mathcal{D}^{p_3}h\right)_{\mu\nu} =& h_{\mu\nu} + \left(\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}\right)\bar{\epsilon}_\mu{}^{\alpha\beta}\bar{\nabla}_\alpha h_{\beta\nu} \\ &+ \frac{1}{p_1p_2p_3}\bar{\epsilon}_\mu{}^{\alpha\beta}\bar{\nabla}_\alpha\left(\bar{\Box}+\frac{3}{\ell^2}\right)h_{\beta\nu} \\ &+ \left(\frac{1}{p_1p_2}+\frac{1}{p_1p_3}+\frac{1}{p_2p_3}\right)(\bar{\Box}+\frac{3}{\ell^2})h_{\mu\nu}.\end{aligned} \tag{4.107}$$

Since all the operators mutually commute, it is now easy to apply all of them to $h_{\mu\nu}$, which yields

$$\begin{aligned}\frac{1}{2\ell^2}\left(\mathcal{D}^L\mathcal{D}^R\mathcal{D}^{p_1}\mathcal{D}^{p_2}\mathcal{D}^{p_3}h\right)_{\mu\nu} =& \mathcal{G}_{\mu\nu} + \left(\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}\right)\bar{\epsilon}_\mu{}^{\alpha\beta}\bar{\nabla}_\alpha \mathcal{G}_{\beta\nu} \\ &+ \frac{1}{p_1p_2p_3}\bar{\epsilon}_\mu{}^{\alpha\beta}\bar{\nabla}_\alpha\left(\bar{\Box}+\frac{3}{\ell^2}\right)\mathcal{G}_{\beta\nu} \\ &+ \left(\frac{1}{p_1p_2}+\frac{1}{p_1p_3}+\frac{1}{p_2p_3}\right)(\bar{\Box}+\frac{3}{\ell^2})\mathcal{G}_{\mu\nu}.\end{aligned} \tag{4.108}$$

By inspection, one can see that the linearized equations (4.104) can be written in this form if the parameters (p_1, p_2, p_3) are chosen such that

$$\begin{aligned}p_1+p_2+p_3 &= -\frac{a}{2\mu}, \\ p_1p_2+p_1p_3+p_2p_3 &= \frac{\bar{\sigma}}{\beta}-\frac{3}{\ell^2}, \\ p_1p_2p_3 &= \frac{2\mu^2-\bar{\sigma}a}{2\beta\mu}+\frac{3a}{2\mu\ell^2}.\end{aligned} \tag{4.109}$$

For generic values of the parameters, there is one set of real roots for (p_1, p_2, p_3). One can solve these equations but the explicit solution is not particularly useful as the expressions are lengthy. Since the operators defined in (4.105) commute[4], the most general solution for the equation (4.108) can be written as a sum of all solutions

$$h_{\mu\nu} = h^L_{\mu\nu} + h^R_{\mu\nu} + h^{m_1}_{\mu\nu} + h^{m_2}_{\mu\nu} + h^{m_3}_{\mu\nu}, \tag{4.110}$$

where

$$(\mathcal{D}^L h^L)_{\mu\nu} = 0, \quad (\mathcal{D}^R h^R)_{\mu\nu} = 0, \quad (\mathcal{D}^{p_i} h^{m_i})_{\mu\nu} = 0, \quad i = 1,2,3. \tag{4.111}$$

Since $\left(\mathcal{D}^L\mathcal{D}^R h\right)_{\mu\nu} = 0$ implies $\mathcal{G}_{\mu\nu} = 0$, $h^L_{\mu\nu}$ and $h^R_{\mu\nu}$ are the two massless excitations in the theory. But these are the modes that already exist in Einstein's theory, so they

[4] Note that when two linear operators $\mathcal{D}_1$ and $\mathcal{D}_2$ degenerate, namely, $\mathcal{D}_1\mathcal{D}_2\phi = 0 \rightarrow \mathcal{D}_1^2\phi = 0$, the most general solution can be obtained as $\mathcal{D}_1\phi_1 = 0$,$\mathcal{D}_1\phi_2 = \phi_1$, as $\phi = \phi_1 + \phi_2$.

are pure gauge modes in the bulk. With the help of the following equation

$$\left(\mathcal{D}^{-p}\mathcal{D}^{p}h\right)_{\mu\nu} = -\frac{1}{p^2}\left(\bar{\Box} + \frac{3}{\ell^2} - p^2\right) h_{\mu\nu}, \tag{4.112}$$

it is easy to see that the remaining solutions describe massive excitations with the masses

$$m_i^2 = p_i^2 - \frac{1}{\ell^2}. \tag{4.113}$$

Since we have a real set of solutions for (p_1, p_2, p_3), the Breitenlohner-Freedman bound $m_i^2 \geq -\frac{1}{\ell^2}$ [45] is automatically satisfied and we have three nontachyonic massive excitations.

The good news it that we have eliminated the massive spin-0 mode and retain only the massive spin-2 modes. But this does not yet say that the remaining modes are valid from the point of quantum theory. In fact it turns out that one cannot avoid ghosts in this construction.

4.5 Conserved Charges

Since we have the linearized equations at our disposal and since the theory admits the BTZ blackhole metric (the massive, rotating solution which is an Einstein space), we can calculate the conserved charges of this metric (or any other blackhole solution) in the Bachian theory. For this purpose, one still needs to develop a lot of machinery which in itself would constitute another topic which is beyond the scope of this thesis. So here we basically review briefly the Killing charge construction of Abbott-Deser [46] as generated to the general gravity theories by Deser-Tekin [44, 47]. For more details on the conserved charges, see the recent review of the Abbott-Deser-Tekin construction [48]

Having identified the spin-2 modes in the theory, we now compute the energy and the angular momentum of the BTZ black hole by using the Abbott-Deser-Tekin technique [46, 47]. For a spacetime metric $g_{\mu\nu}$ having asymptotically the same Killing symmetries as the background space, one can define ”conserved charges” from the matter coupled linearized field equations symbolically read

$$\mathcal{O}(\overline{g})_{\mu\nu\alpha\beta}h^{\alpha\beta} = \kappa\, T_{\mu\nu}. \tag{4.114}$$

For each background Killing vector $\bar{\xi}_\mu$, satisfying $\bar{\nabla}_{(\mu}\xi_{\nu)} = 0$, a conserved current can be formed as

$$\sqrt{-\bar{g}}\overline{\nabla}_\mu\left(\bar{\xi}_\nu T^{\mu\nu}\right) = \partial_\mu\left(\sqrt{-\bar{g}}\bar{\xi}_\nu T^{\mu\nu}\right) = 0. \tag{4.115}$$

By applying Stokes' theorem, one obtains an expression for the conserved global charges

$$Q(\bar{\xi}) = \int_{\mathcal{M}} d^{n-1}x\sqrt{-\bar{g}}\bar{\xi}_\nu T^{0\nu} = \int_\Sigma d\Sigma_i \mathcal{F}^{0i}, \tag{4.116}$$

where $\mathcal{M}$ is the $(n-1)$-dimensional spatial manifold, Σ is its boundary and the antisymmetric tensor $\mathcal{F}^{\mu\nu}$ satisfies $T^{\mu\nu}\xi_\nu = \overline{\nabla}_\nu\mathcal{F}^{\mu\nu}$. Charge expressions for the $\mathcal{G}$, $\epsilon\nabla\mathcal{G}$ and $\Box\mathcal{G}$ terms in the linearized field equations (4.104) were obtained in [46], [47] and [49] respectively. For the $\epsilon\nabla\Box\mathcal{G}$ term, one can make use of the equation

$$\begin{aligned} 2\bar{\xi}^\nu\bar{\epsilon}_\mu{}^{\alpha\beta}\bar{\nabla}_\alpha\bar{\Box}\mathcal{G}_{\beta\nu} =& \overline{\nabla}_\alpha\left\{\bar{\epsilon}^{\mu\alpha\beta}\overline{\Box}\mathcal{G}_{\nu\beta}\bar{\xi}^\nu + \bar{\epsilon}^{\nu\alpha}_{\beta}\overline{\Box}\mathcal{G}^{\mu\beta}\bar{\xi}_\nu + \bar{\epsilon}^{\mu\nu\beta}\overline{\Box}\mathcal{G}^\alpha_\beta\bar{\xi}_\nu\right\} \\ &+ X_\beta\overline{\Box}\mathcal{G}^{\mu\beta}, \end{aligned} \tag{4.117}$$

and the final result can be written as

$$Q(\bar{\xi}) = \frac{1}{2\pi G_3}\oint_{\partial\Sigma}\sqrt{-\bar{g}}\,dl_i\,q^{0i}(\bar{\xi}), \tag{4.118}$$

where

$$\begin{aligned} q^{0i}(\bar{\xi}) =& \left(1 - \frac{\bar{\sigma}a}{2\mu^2}\right)q^{0i}_{(1)}(\bar{\xi}) + \frac{\bar{\sigma}}{2\mu}\left[q^{0i}_{(1)}(\overline{X}) + q^{0i}_{(2)}(\bar{\xi})\right] \\ &- \frac{\beta a}{2\mu^2}q^{0i}_{(3)}(\bar{\xi}) + \frac{\beta}{2\mu}\left[q^{0i}_{(3)}(\overline{X}) + q^{0i}_{(4)}(\bar{\xi})\right]. \\ q^{0i}_{(1)}(\bar{\xi}) =& \bar{\xi}_\nu\overline{\nabla}^0 h^{i\nu} - \bar{\xi}_\nu\overline{\nabla}^i h^{0\nu} + \bar{\xi}^0\overline{\nabla}^i h - \bar{\xi}^i\overline{\nabla}^0 h \\ &+ h^{0\nu}\overline{\nabla}^i\bar{\xi}_\nu - h^{i\nu}\overline{\nabla}^0\bar{\xi}_\nu + \bar{\xi}^i\overline{\nabla}_\nu h^{0\nu} - \bar{\xi}^0\overline{\nabla}_\nu h^{i\nu} + h\overline{\nabla}^0\bar{\xi}^i, \\ q^{0i}_{(2)}(\bar{\xi}) =& \bar{\epsilon}^{0i\beta}\mathcal{G}_{\nu\beta}\bar{\xi}^\nu + \bar{\epsilon}^{\nu i\beta}\mathcal{G}^0_\beta\bar{\xi}_\nu + \bar{\epsilon}^{0\nu\beta}\mathcal{G}^i_\beta\bar{\xi}_\nu, \\ q^{0i}_{(3)}(\bar{\xi}) =& \bar{\xi}_\nu\overline{\nabla}^i\mathcal{G}^{0\nu} - \bar{\xi}_\nu\overline{\nabla}^0\mathcal{G}^{i\nu} - \mathcal{G}^{0\nu}\overline{\nabla}^i\bar{\xi}_\nu + \mathcal{G}^{i\nu}\overline{\nabla}^0\bar{\xi}_\nu, \\ q^{0i}_{(4)}(\bar{\xi}) =& \bar{\epsilon}^{0i\beta}\overline{\Box}\mathcal{G}_{\nu\beta}\bar{\xi}^\nu + \bar{\epsilon}^{\nu i\beta}\overline{\Box}\mathcal{G}^0_\beta\bar{\xi}_\nu + \bar{\epsilon}^{0\nu\beta}\overline{\Box}\mathcal{G}^i_\beta\bar{\xi}_\nu, \end{aligned} \tag{4.119}$$

and $\bar{X}^\beta = \epsilon^{\alpha\nu\beta}\overline{\nabla}_\alpha\bar{\xi}_\nu$ is also a background Killing vector.

Let us now apply the above construction to find the charges of the rotating BTZ black hole in this theory. BTZ is locally AdS_3 and hence it is a solution of the theory once the cosmological constant is adjusted. In the usual (t, r, ϕ) coordinates, the metric reads

$$ds^2 = \left(mG_3 + \Lambda r^2\right)dt^2 - jdtd\phi + r^2 d\phi^2 + \frac{dr^2}{-mG_3 - \Lambda r^2 + \frac{j^2}{4r^2}}, \tag{4.120}$$

where the background metric is found by setting $m = 0$ and $j = 0$ as

$$ds^2 = \Lambda r^2 dt^2 + r^2 d\phi^2 - \frac{dr^2}{\Lambda r^2}. \tag{4.121}$$

In the asymptotic region, the linearized cosmological Einstein tensor vanishes $\mathcal{G}_{\mu\nu} = 0$ and only $q^{\mu i}_{(1)}$ terms in (4.119) contribute. Killing vectors $\bar{\xi}^\mu = -\left(\frac{\partial}{\partial t}\right)^\mu$ and $\bar{\xi}^\mu = \left(\frac{\partial}{\partial \phi}\right)^\mu$ yield the energy and the angular momentum, respectively, as

$$E = \frac{1}{G_3}\left[\left(1 - \frac{\bar{\sigma} a}{2\mu^2}\right) m + \frac{j\Lambda\bar{\sigma}}{\mu}\right], \quad J = \frac{1}{G_3}\left[\left(1 - \frac{\bar{\sigma} a}{2\mu^2}\right) j - \frac{m\bar{\sigma}}{\mu}\right] \tag{4.122}$$

4.6 Further Developments in Exotic Massive Gravity

Here we briefly summarize some of the further developments in exotic massive gravity. One particularly interesting issue is the matter coupling of the theory. As we do not possess a Bianchi identity matter coupling is highly non-trivial. But how consistent coupling can be done was worked out [43]. The resulting equations are

$$G_{\mu\nu} + \frac{1}{\mu} C_{\mu\nu} - \frac{1}{m^2} H_{\mu\nu} + \frac{1}{m^4} L_{\mu\nu} = \Theta_{\mu\nu}(T) \tag{4.123}$$

where $\Theta_{\mu\nu}(T)$ is a complicated effective "energy-momentum" tensor which is on-shell consistent and defined by

$$\begin{aligned}\Theta_{\mu\nu}(T) =& \frac{\lambda}{\mu}\hat{T}_{\mu\nu} - \frac{\lambda}{m^2}\epsilon_\mu{}^{\rho\sigma}\nabla_\rho \hat{T}_{\nu\sigma} + \frac{2\lambda}{m^4}\epsilon_\mu{}^{\rho\sigma}\epsilon_\nu{}^{\lambda\tau} C_{\rho\lambda}\hat{T}_{\sigma\tau} \\ &- \frac{\lambda^2}{m^4}\epsilon_\mu{}^{\rho\sigma}\epsilon_\nu{}^{\lambda\tau}\hat{T}_{\rho\lambda}\hat{T}_{\sigma\tau}\end{aligned} \tag{4.124}$$

where $\hat{T}_{\mu\nu} = T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T$ and $\nabla_\mu T^{\mu\nu} = 0$. Here λ is a coupling constant which appears linearly and quadratically on the right-hand side. The Cotton, Schouten, H and L-tensors are defined

$$C_{\mu\nu} \equiv \epsilon_\mu{}^{\rho\sigma}\nabla_\rho S_{\sigma\nu} \quad , H_{\mu\nu} \equiv \epsilon_\mu{}^{\rho\sigma}\nabla_\rho C_{\sigma\nu} \quad , L_{\mu\nu} \equiv \frac{1}{2}\epsilon_\mu{}^{\rho\sigma}\epsilon_\nu{}^{\lambda\tau} C_{\rho\lambda} C_{\sigma\tau} \tag{4.125}$$

and $S_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R$.

Recently [50] it was shown that even though the theory has a massive spin-2 ghost, it is a causal theory. The causality proof was given using the Shapiro time delay ideas in the presence of a shockwave geometry which is a solution to the theory.

In [51, 52], non-trivial solutions of the exotic massive gravity have been found. In [53], a unitary extension of the exotic massive gravity as a bigeometry has been constructed.

CHAPTER 5

SYMMETRY REDUCTION VIA THE GEROCH METHOD

5.1 Reduction of the Various tensors under a Killing Symmetry

The field equations of the gravity theories discussed in the previous chapter are highly complicated; and hence, very few solutions, usually with maximal symmetry, are known. The existence of a symmetry can be coordinate invariantly defined by the existence of Killing vector fields. Let us assume that there is a single Killing vector field in the spacetime manifold (M, g), then we can ask if such a Killing vector field can be used to define a hypersurface Σ (of codimension-one). In general, without for the restriction such an invariantly defined Σ does not exist as a submanifold [54]. Naively this is because of the following: say G be the 1-dimensional group generated by the assumed Killing vector field, then the invariantly defined space is the quotient space M/G (see Appendix B for details) which in general is not a manifold, let alone a codimension-one hypersurface. The proper conditions are summarized as a theorem in [55] which we write here for completeness, but without proof which is beyond the scope of the thesis. The theorem is called the quotient manifold theorem:

Theorem: Suppose G is a Lie group acting smoothly, freely and properly on the smooth manifold M. Then the orbit space M/G is a topological manifold if dimension equal to dim(M) − dim(G) and has unique smooth structure with the property that the quotient map $\pi : M \to M/G$ is a smooth submersion.

Geroch [56, 57] developped this so called projection formalism and invariant quantities needed to describe the geometry of M/G. In [54] details of this construction is given. The invariant quantities are the twist and the norm of the Killing vector field as well as the metric on M/G if it is a manifold.

In this chapter, following [58], we make the first attempt of reducing the relevant tensors, Ricci, Cotton, $J_{\mu\nu}$, $H_{\mu\nu}$ under the assumption of a time-like Killing vector field. The ensuring discussion is still complicated and gave rise to new solutions in the core of the topologically massive gravity(TMG) [58]. We will start in this general setting and later restrict our ansatz to the case of vanishing twist which is to be defined below.

5.1.1 The Stationary Metric

Assuming the existence of a time-like Killing vector field K, we can choose coordinates adapted to this Killing vector field such that local coordinates are x^0, x^1, x^2 and the Killing vector field reads

$$K \equiv \frac{\partial}{\partial x^0}. \tag{5.1}$$

Then the metric components $g_{\mu\nu} = g_{\mu\nu}(x^1, x^2)$ do not depend on x^0 and the line element reads

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = g_{00}(dx^0)^2 + 2g_{0i}dx^0dx^i + g_{ij}dx^idx^j. \tag{5.2}$$

Defining

$$\omega^2 = g_{00} \quad , \quad A_i = \frac{g_{0i}}{g_{00}} \quad , \quad h_{ij} = g_{ij} - \frac{g_{0i}g_{0j}}{g_{00}}, \tag{5.3}$$

we can recast the line element in the compact form

$$\begin{aligned} ds^2 &= \omega^2\left[(dx^0)^2 + 2\frac{g_{0i}}{\omega^2}dx^0dx^i + \left(\frac{g_{0i}}{\omega^2}dx^i\right)^2\right] + g_{ij}dx^idx^j - \frac{g_{0i}g_{0j}}{\omega^4}dx^idx^j \\ &= \omega^2\left[dx^0 + \frac{g_{0i}}{\omega^2}\right]^2 + \left(g_{ij} - \frac{g_{0i}g_{0j}}{\omega^4}\right)dx^idx^j \\ &= \omega^2\left(dx^0 + A_idx^i\right)^2 + h_{ij}dx^idx^j. \end{aligned} \tag{5.4}$$

This will be the form that we shall employ in this chapter. Let us stress that ω^2, A_i, h_{ij} depend on x^1 and x^2 but not on x^0. These are 6 functions that represent the symmetric $g_{\mu\nu}$ with 6 independent entries.

As matrices, we have the metric and its inverse:

$$g_{\mu\nu} = \begin{bmatrix} \omega^2 & \omega^2 A_j \\ \omega^2 A_i & h_{ij} + \omega^2 A_iA_j \end{bmatrix}, \; g^{\mu\nu} = \begin{bmatrix} \frac{(1+\omega^2A^2)}{\omega^2} & -A^k \\ -A^j & h^{jk} \end{bmatrix}, \tag{5.5}$$

with $\det(g_{\mu\nu}) = \omega^2 \det(h_{ij})$.

Note that h_{ij} is the metric of the spatial part and as such it is used for raising and lowering of the spatial indices such as

$$A^i = h^{ji} A_j, \quad h_{ik} h^{kj} = \delta_i^j. \tag{5.6}$$

5.1.2 Coordinate or Gauge Transformation

We have chosen the coordinates (x^0, x^i) but close inspection shows that the following "new coordinates" leave the w^2 and h_{ij} intact while change only the A_i field. This fact can be used to simplify the equations, hence we work this out. Let $(\tilde{x}^0, \tilde{x}^i)$ be the new coordinates, then we define the coordinate transformation as:

$$\tilde{x}^0 = x^0 + T(x^i) \tag{5.7}$$

$$\tilde{x}^i = x^i. \tag{5.8}$$

It is straightforward to see that any covariant field equation is invariant (since it is of the form $\mathcal{E}^{\mu\nu} = 0$) under these transformations. Notice that the metric also stays stationary as shown below;

$$g_{\mu\nu} = \tilde{g}_{\alpha\beta} \frac{\partial \tilde{x}^\alpha}{\partial x^\mu} \frac{\partial \tilde{x}^\beta}{\partial x^\nu},$$

$$g_{00} = \tilde{g}_{00} \;\rightarrow\; \tilde{\omega} = \omega,$$

$$\begin{aligned} g_{0i} =& \tilde{g}_{00} \frac{\partial T(x^i)}{\partial x^i} + \tilde{g}_{0j} \delta_i^j \\ & \tilde{A}_i = A_i - \partial_i T(x^i) \end{aligned}$$

$$g_{ij} = \tilde{g}_{\alpha\beta} \frac{\partial \tilde{x}^\alpha}{\partial x^i} \frac{\partial \tilde{x}^\beta}{\partial x^j} \;\rightarrow\; h_{ij} = \tilde{h}_{ij}. \tag{5.9}$$

It can be easily seen that w^2, h_{ij} and $F_{ij} = \partial_i A_j - \partial_j A_i \equiv \epsilon_{ij} \Omega$ remain invariant. Note that the last equation defines the scalar function Ω, and ϵ_{ij} is a spatial tensor (not density), defined in terms of the ε_{ij} symbol as

$$\epsilon_{ij} = \sqrt{h} \varepsilon_{ij} \;,\; \epsilon^{ij} = \frac{1}{\sqrt{h}} \varepsilon^{ij}. \tag{5.10}$$

5.1.3 Scalar Twist

Following the analogous expression in four (or higher) dimensions [59], we can define the scalar twist of the Killing vector field as

$$\rho \equiv \frac{\epsilon_{\mu\nu\lambda} K^\mu \nabla^\mu K^\lambda}{K_\sigma K^\sigma}. \tag{5.11}$$

Since $K^\mu = (1,0,0)$, we have $K_\sigma K^\sigma = g_{00} = w^2$ and

$$\begin{aligned} \rho &= \frac{\epsilon_{0ij} K^0 \nabla^i K^j}{\omega^2} \\ &= \frac{\epsilon_{0ij} g^{i\mu} \nabla_\mu K^j}{\omega^2} = \frac{\epsilon_{0ij} g^{i\mu} \Gamma^j_{\mu 0}}{\omega^2} \\ &= \frac{\epsilon_{0ij}}{\omega^2} (g^{i0} \Gamma^j_{00} + g^{ik} \Gamma^j_{k0}). \end{aligned} \tag{5.12}$$

To simplify the expression we need to work out the Christoffel symbols which can be computed. We list the non-vanishing components here

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\lambda} \big(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu} \big)$$

$$\Gamma^0_{00} = \frac{1}{2} A^i \partial_i \omega^2 \quad , \quad \Gamma^i_{00} = -\frac{1}{2} \partial^i \omega^2, \tag{5.13}$$

$$\Gamma^0_{i0} = \frac{1}{2} \big[A_i A^j \partial_j \omega^2 + \omega^2 A^j F_{ji} + \frac{1}{\omega^2} \partial_i \omega^2 \big], \tag{5.14}$$

$$\Gamma^i_{j0} = \frac{1}{2} \big[\omega^2 F_j{}^i - A_j \partial^i \omega^2 \big], \tag{5.15}$$

$$\begin{aligned} \Gamma^0_{ij} =& \frac{1}{2} \big(\hat{\nabla}_i A_j + \hat{\nabla}_j A_i \big) + \frac{1}{2\omega^2} \big(A_j \partial_i \omega^2 + A_i \partial_j \omega^2 \big) \\ &+ \frac{1}{2} A^k \big(A_j A_i \partial_k \omega^2 - \omega^2 A_i F_{jk} - \omega^2 A_j F_{ik} \big), \end{aligned} \tag{5.16}$$

$$\Gamma^i_{jk} = \hat{\Gamma}^i_{jk} + \frac{1}{2} \big(\omega^2 A_k F_j{}^i + \omega^2 A_j F_k{}^i - A_k A_j \partial^i \omega^2 \big), \tag{5.17}$$

$$\Rightarrow \quad \Gamma^k_{ij} = \hat{\Gamma}^k_{ij} + \Theta^k_{ij}, \quad \Theta^k_{ij} = \frac{1}{2} \big(\omega^2 A_k F_j{}^i + \omega^2 A_j F_k{}^i - A_k A_j \partial^i \omega^2 \big). \tag{5.18}$$

Using the relevant connections in (5.12) and the fact that $\epsilon_{0ij} = \omega \varepsilon_{ij}$, we get

$$\rho = \frac{1}{2} \omega \varepsilon_{ij} F^{ij} \tag{5.19}$$

and since $\Omega = \frac{1}{2}\varepsilon_{ij}F^{ij}$, the scalar twist is related to the function Ω as

$$\rho = \omega\Omega. \tag{5.20}$$

Also, let us note that

$$F_{ij}F^{ij} = 2\Omega^2 = 2\frac{\rho^2}{\omega^2}, \tag{5.21}$$

and

$$F^{ik}F^{j}{}_{k} = \frac{h^{ij}}{\omega^2}\rho^2. \tag{5.22}$$

It is clear from the above construction that, just like ω, the twist ρ is gauge invariant under the transformations (5.7),(5.8).

Before calculating the Ricci tensor ($R_{\mu\nu}$), the Cotton tensor ($C_{\mu\nu}$), the $J_{\mu\nu}$ and $H_{\mu\nu}$ tensors, we search for the components of these tensors if they can be written in terms of invariant quantities ρ and ω. Under the change of coordinates, a rank (0,2) tensor transform as

$$\tilde{B}_{\mu\nu} = \frac{\partial x^\alpha}{\partial\tilde{x}^\mu}\frac{\partial x^\beta}{\partial\tilde{x}^\nu}B_{\alpha\beta}, \tag{5.23}$$

and for particular case of (5.7),(5.8), one has

$$\tilde{B}_{00} = \frac{\partial x^\alpha}{\partial\tilde{x}^0}\frac{\partial x^\beta}{\partial\tilde{x}^0}B_{\alpha\beta} = B_{00}, \tag{5.24}$$

$$\tilde{B}_{i0} = \frac{\partial x^\alpha}{\partial\tilde{x}^i}\frac{\partial x^\beta}{\partial\tilde{x}^0}B_{\alpha\beta} = B_{i0} - \frac{\partial T}{\partial\tilde{x}^i}B_{00}, \tag{5.25}$$

$$\tilde{B}_{ij} = \frac{\partial x^\alpha}{\partial\tilde{x}^i}\frac{\partial x^\beta}{\partial\tilde{x}^j}B_{\alpha\beta} = B_{ij} - \frac{\partial T}{\partial\tilde{x}^j}B_{i0} + \frac{\partial T}{\partial\tilde{x}^i}\frac{\partial T}{\partial\tilde{x}^j}B_{00}, \tag{5.26}$$

$$\tilde{B}^i_0 = \frac{\partial\tilde{x}^i}{\partial x^\alpha}\frac{\partial x^\beta}{\partial\tilde{x}^0}B^\alpha_\beta = B^i_0, \tag{5.27}$$

and finally

$$\tilde{B}^{ij} = \frac{\partial\tilde{x}^i}{\partial x^\alpha}\frac{\partial\tilde{x}^j}{\partial x^\beta}B^{\alpha\beta} = B^{ij}. \tag{5.28}$$

So the B_{00}, $B^i{}_0$, B^{ij} components are gauge invariant. This means any gauge can be used to compute them. Also, no gauge-noninvariant object, such as A_i should appear them.

5.1.4 Ricci Tensor and The Scalar Curvature

As we have seen above, R_{00}, R_0^j and R^{ij} are the components of the Ricci tensor that can be written in terms of the invariant quantities ω and ρ. Let us calculate these explicitly.

The Riemann tensor is

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\sigma\nu} - \partial_\nu \Gamma^{\rho}_{\sigma\mu} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\sigma\nu} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\sigma\mu}. \tag{5.29}$$

So one has

$$\begin{aligned} R_{00} &= R^{\rho}{}_{0\rho 0} = R^{i}{}_{0i0} \\ &= \partial_i \Gamma^{i}_{00} + \Gamma^{i}_{i0}\Gamma^{0}_{00} + \Gamma^{i}_{ij}\Gamma^{j}_{00} - \Gamma^{i}_{00}\Gamma^{0}_{i0} - \Gamma^{i}_{j0}\Gamma^{j}_{i0}, \end{aligned} \tag{5.30}$$

which yields

$$R_{00} = -w\hat{\Box}\omega + \frac{1}{4}\omega^4 F_{ij}F^{ij} = -\omega\hat{\Box}\omega + \frac{1}{2}\omega^2\rho^2, \tag{5.31}$$

where $\hat{\Box} := h^{ij}\hat{\nabla}_i\hat{\nabla}_j$ and $\hat{\nabla}_i$ is the covariant derivative compatible with the spatial metric h_{ij}.

Next we have

$$\begin{aligned} R_{i0} &= R^{\lambda}{}_{i\lambda 0} = R^{j}{}_{ij0}. \\ &= \partial_j \Gamma^{j}_{i0} + \Gamma^{j}_{j0}\Gamma^{0}_{i0} + \Gamma^{j}_{jk}\Gamma^{k}_{i0} - \Gamma^{j}_{00}\Gamma^{0}_{ij} - \Gamma^{j}_{0k}\Gamma^{k}_{ij}, \end{aligned} \tag{5.32}$$

which reads

$$\begin{aligned} R_{i0} &= -\frac{1}{2}A_i\hat{\Box}\omega^2 + \frac{3}{4}F_{ij}\hat{\nabla}^j\omega^2 + \frac{1}{2}\omega^2\hat{\nabla}^j F_{ij} \\ &\quad + \frac{1}{4}\omega^{-2}\hat{\nabla}_j\omega^2\hat{\nabla}^j\omega^2 + \frac{1}{4}\omega^4 A_i F_{jk}F^{jk} \\ &= -\omega A_i\hat{\Box}\omega + \frac{3}{4}F_{ij}\hat{\nabla}^j\omega^2 + \frac{1}{2}\omega^2\hat{\nabla}^j F_{ij} + \frac{1}{4}\omega^4 A_i F_{jk}F^{jk} \\ &= -\omega A_i\hat{\Box}\omega + \frac{1}{2}A_i\omega^2\rho^2 + \frac{\epsilon_{ik}}{2\omega}\partial^k(\omega^2\rho), \end{aligned} \tag{5.33}$$

as expected gauge invariant parts (A_i) appear for this component. We need to raise the spatial index to get

$$R^{j}{}_{0} = g^{j\alpha}R_{\alpha 0} = g^{j0}R_{00} + g^{ji}R_{i0}, \tag{5.34}$$

which reads

$$R^j_0 = g^{j\mu} R_{\mu 0} = \frac{3}{2} w \hat{\nabla}_k \omega F^{jk} + \frac{1}{2}\omega^2 \hat{\nabla}_k F^{jk} = \frac{\epsilon^{jk}}{2\omega}\partial_k(\omega^2\rho), \tag{5.35}$$

which is gauge invariant and compared to (5.33) the latter expression is simpler. Next we have

$$R_{ij} = R^\lambda{}_{i\lambda j} = R^k{}_{ikj} + R^0{}_{i0j}, \tag{5.36}$$

and we need to compute the following components

$$R^0{}_{i0j} = -\partial_j \Gamma^0_{i0} + \Gamma^0_{00}\Gamma^0_{ij} + \Gamma^0_{0k}\Gamma^k_{ij} - \Gamma^0_{j0}\Gamma^0_{i0} - \Gamma^0_{jk}\Gamma^k_{i0}, \tag{5.37}$$

$$R^k{}_{ikj} = \partial_k \Gamma^k_{ij} - \partial_j \Gamma^k_{ik} + \Gamma^k_{k0}\Gamma^0_{ij} + \Gamma^k_{kl}\Gamma^l_{ij} - \Gamma^k_{j0}\Gamma^0_{ik} - \Gamma^k_{jl}\Gamma^l_{ik}, \tag{5.38}$$

After some long calculations, we arrive at

$$\begin{aligned} R_{ij} &= \hat{R}_{ij} - \omega A_i A_j \hat{\Box}\omega - \frac{1}{\omega}\hat{\nabla}_j \hat{\nabla}_i \omega + \frac{3}{4} A_i F_{jk} \hat{\nabla}^k \omega^2 \\ &\quad + \frac{3}{4} A_j F_{ik} \hat{\nabla}^k \omega^2 + \frac{1}{2}\omega^2 A_j \hat{\nabla}_k F_i{}^k + \frac{1}{2}\omega^2 A_i \hat{\nabla}_k F_j{}^k \\ &\quad - \frac{1}{2}\omega^2 F_i{}^k F_{jk} + \frac{1}{4}\omega^4 A_i A_j F_{kl} F^{kl} \\ &= \frac{1}{2}\hat{R} h_{ij} - \omega A_i A_j \hat{\Box}\omega - \frac{1}{\omega}\hat{\nabla}_j \hat{\nabla}_i \omega + A_i \frac{\epsilon_{jk}}{2\omega}\partial^k(\omega^2\rho) \\ &\quad + A_j \frac{\epsilon_{ik}}{2\omega}\partial^k(\omega^2\rho) - \frac{1}{2} h_{ij}\rho^2 + \frac{1}{2} A_i A_j \omega^2 \rho^2, \end{aligned} \tag{5.39}$$

where again h_{ij} refers to the 2 dimensional spatial metric. Raising the two indices up, we get

$$\begin{aligned} R^{ij} = g^{i\mu} g^{j\nu} R_{\mu\nu} &= \hat{R}^{ij} - \frac{1}{\omega}\hat{\nabla}^i \hat{\nabla}^j \omega - \frac{1}{2}\omega^2 F^{ik} F^j{}_k \\ &= \frac{1}{2}\hat{R} h^{ij} - \frac{1}{\omega}\hat{\nabla}^i \hat{\nabla}^j \omega - \frac{1}{2} h^{ij}\rho^2, \end{aligned} \tag{5.40}$$

which is a gauge invariant expression. Finally for the scalar curvature, we have

$$R = g^{\mu\nu} R_{\mu\nu} = g^{0\nu} R_{0\nu} + g^{i\nu} R_{i\nu} = g^{00} R_{00} + g^{0j} R_{0j} + g^{i0} R_{i0} + g^{ij} R_{ij}, \tag{5.41}$$

which reduces to

$$R = \hat{R} - \frac{2}{\omega}\hat{\Box}\omega - \frac{1}{2}\rho^2. \tag{5.42}$$

which is clearly gauge invariant. The computations so far can be used to study solutions with a Killing vector in 3D Einstein's gravity, but we shall not do that, instead we shall study the reduction of the other rank-2 tensors.

5.1.5 Reductions of the Cotton Tensor $C_{\mu\nu}$, and the $J_{\mu\nu}$ and $H_{\mu\nu}$ tensors

Next we carry out the similar reduction to these tensors. Recall that the Cotton tensor is defined as[1]

$$C^{\mu\nu} = \epsilon^{\mu\alpha\beta}\nabla_\alpha S_\beta{}^\nu \quad \text{with} \quad S_{\mu\nu} = R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R. \tag{5.43}$$

Some components of the Schouten tensor that we shall use are

$$S_{00} = -\frac{1}{2}\omega\hat{\Box}\omega + \frac{5}{8}\omega^2\rho^2 - \frac{1}{4}\omega^2\hat{R}, \tag{5.44}$$

$$S_{i0} = -\frac{1}{2}A_i\omega\hat{\Box}\omega + \frac{5}{8}A_i\omega^2\rho^2 - \frac{1}{4}\omega^2 A_i\hat{R} + \frac{1}{2\omega}\epsilon_{il}\partial^l(\omega^2\rho), \tag{5.45}$$

and

$$\begin{aligned} S_{ij} =& \frac{1}{4}\hat{R}h_{ij} - \frac{1}{2}A_iA_j\omega\hat{\Box}\omega - \frac{1}{\omega}\hat{\nabla}_j\hat{\nabla}_i\omega - \frac{3}{8}\rho^2 h_{ij} \\ &+ \frac{5}{8}A_iA_j\omega^2\rho^2 + \frac{1}{2}h_{ij}\frac{1}{\omega}\hat{\Box}\omega - \frac{1}{4}\omega^2A_iA_j\hat{R} \\ &+ \frac{1}{2\omega}A_i\epsilon_{jk}\partial^k\left(\omega^2\rho\right) + \frac{1}{2\omega}A_j\epsilon_{ik}\partial^k\left(\omega^2\rho\right). \end{aligned} \tag{5.46}$$

we have

$$\begin{aligned} C^0{}_0 &= \epsilon^{0\alpha\beta}\nabla_\alpha S_{\beta 0} = \epsilon^{0i\beta}\nabla_i S_{\beta 0} = \epsilon^{0ij}\nabla_i S_{j0} = \frac{\epsilon^{ij}}{\omega}\nabla_i S_{j0} \\ &= \frac{\epsilon^{ij}}{\omega}\left[\partial_i S_{j0} - \Gamma^0_{ij}S_{00} - \Gamma^k_{ij}S_{k0} - \Gamma^0_{i0}S_{j0} - \Gamma^k_{i0}S_{jk}\right], \end{aligned}$$

$$\Rightarrow \quad \Gamma^k_{ij} = \hat{\Gamma}^k_{ij} + \Theta^k_{ij}, \quad \hat{\nabla}_i S_{j0} = \partial_i S_{j0} - \hat{\Gamma}^k_{ij}S_{k0}. \tag{5.47}$$

Symmetric terms in the parenthesis vanish due to the antisymmetric tensor ϵ^{ij}. Then $C^0{}_0$ becomes

$$C^0{}_0 = \frac{\epsilon^{ij}}{\omega}\left[\hat{\nabla}_i S_{j0} - \Gamma^0_{i0}S_{j0} - \Gamma^k_{i0}S_{jk}\right], \tag{5.48}$$

$$C^0{}_0 = \frac{\rho}{\omega^2}\hat{\nabla}_i\omega\hat{\nabla}^i\omega - \frac{1}{\omega}\hat{\nabla}_i\rho\hat{\nabla}^i\omega - \frac{3\rho}{2\omega}\hat{\Box}\omega - \frac{1}{2}\hat{\Box}\rho - \frac{1}{2}\rho\hat{R} + \rho^3. \tag{5.49}$$

$$\begin{aligned} C_{k0} &= g_{k\alpha}C^\alpha{}_0 = g_{k0}C^0{}_0 + g_{ki}C^i{}_0 = \omega^2A_kC^0{}_0 + h_{ki}C^i{}_0 + \omega^2A_kA_iC^i{}_0 \\ &= \omega^2A_kC^0{}_0 + h_{ki}C^i{}_0. \end{aligned} \tag{5.50}$$

[1] Reduction of the Ricci tensor and the Cotton tensor was worked out in [58].

The explicit computation of all the components of the Cotton tensor is extremely tedious. But one can use the previously observed fact that in the gauge invariant components $C_{00}, C^i{}_0, C^{ij}$, no A_i term should appear. For the sake of keeping the expressions simple, we shall drop the terms that have the products of A_i's such as A_iA_j. This is because, later on we shall compute $H_{\mu\nu}$ which involve the derivative of $C_{\mu\nu}$: if there is one A_i in the expression, its derivative yields an F_{ij}, but terms like A_iA_j will disappear at the end from the gauge-invariant expressions. Whenever we drop terms involving A_i, we shall add $\mathcal{O}(A_i)$ meaning that the expression is simplified by dropping these terms.

$$\begin{aligned} C_{k0} =& A_k\Big(- \omega\hat{\nabla}_l\rho\hat{\nabla}^l\omega + \rho\hat{\nabla}_l\omega\hat{\nabla}^l\omega - \frac{3}{2}\omega\rho\hat{\Box}\omega \\ & - \frac{1}{2}\omega^2\hat{\Box}\rho - \frac{1}{2}\omega^2\rho\hat{R} + \omega^2\rho^3\Big) \\ & + \frac{\epsilon^{ij}}{2\omega}h_{ik}\hat{\nabla}_j\left(\frac{3}{2}\omega^2\rho^2 + \hat{\nabla}_l\omega\hat{\nabla}^l\omega - \frac{1}{2}\omega^2\hat{R} - \omega\hat{\Box}\omega\right). \end{aligned} \tag{5.51}$$

$$\begin{aligned} C^0{}_k &= \epsilon^{0\alpha\beta}\nabla_\alpha S_{\beta k} = \epsilon^{0ij}\nabla_i S_{jk} \\ &= \frac{\epsilon^{ij}}{\omega}\nabla_i S_{jk} = \frac{\epsilon^{ij}}{\omega}\left[\hat{\nabla}_i S_{jk} - \Gamma^0_{ik}S_{j0} - \Theta^l_{ik}S_{jl}\right] \end{aligned}$$

$$\begin{aligned} C^0{}_k =& \frac{\epsilon^{ij}}{\omega}\Big[h_{jk}\hat{\nabla}_i\big(\frac{1}{4}\hat{R} - \frac{3}{8}\rho^2 + \frac{1}{2\omega}\hat{\Box}\omega\big) + \frac{1}{\omega^2}\hat{\nabla}_i\omega\hat{\nabla}_k\hat{\nabla}_j\omega \\ & - \frac{1}{\omega}\hat{\nabla}_i\hat{\nabla}_k\hat{\nabla}_j\omega - \frac{1}{\omega}\hat{\nabla}_k\hat{\nabla}_i\hat{\nabla}_j\omega\Big] \\ & + \frac{3\rho}{4\omega^3}\epsilon_{kl}\partial^l(\omega^2\rho). \end{aligned} \tag{5.52}$$

$$\begin{aligned} C^i{}_j &= \epsilon^{i\alpha\beta}\nabla_\alpha S_{\beta j} = \frac{\epsilon^{ik}}{\omega}\left[\nabla_k S_{0j} - \nabla_0 S_{kj}\right] \\ &= \frac{\epsilon^{ik}}{\omega}\left[\hat{\nabla}_k S_{0j} - \Gamma^0_{kj}S_{00} - \Theta^l_{kj}S_{0l} + \Gamma^0_{0j}S_{k0} + \Gamma^l_{0j}S_{kl}\right] \end{aligned}$$

$$\begin{aligned} C^i{}_j =& - \frac{1}{\omega}\hat{\nabla}_j\omega\hat{\nabla}^i\rho - \frac{1}{\omega}\hat{\nabla}_j\rho\hat{\nabla}^i\omega - \frac{\rho}{\omega^2}\hat{\nabla}_j\omega\hat{\nabla}^i\omega \\ & - \frac{1}{2}\hat{\nabla}_j\hat{\nabla}^i\rho - \frac{\rho}{\omega}\hat{\nabla}_j\hat{\nabla}^i\omega + \frac{\rho}{2\omega}\hat{\nabla}^i\hat{\nabla}_j\omega \\ & + \delta^i_j\big(- \frac{1}{2}\rho^3 + \frac{1}{4}\rho\hat{R} + \frac{3}{2\omega}\hat{\nabla}_k\omega\hat{\nabla}^k\rho + \frac{\rho}{\omega}\hat{\Box}\omega + \frac{1}{2}\hat{\Box}\rho\big). \end{aligned} \tag{5.53}$$

$$C_{ij} = g_{i\mu}C^\mu{}_j = g_{i0}C^0{}_j + g_{ik}C^k{}_j = h_{ik}C^k{}_j, \tag{5.54}$$

$$
\begin{aligned}
C_{ij} =& h_{ij}\bigg(-\frac{1}{2}\rho^3 + \frac{1}{4}\rho\hat{R} + \frac{3}{2\omega}\hat{\nabla}_k\omega\hat{\nabla}^k\rho + \frac{1}{2}\hat{\Box}\rho + \frac{\rho}{\omega}\hat{\Box}\omega\bigg) \\
& - \frac{1}{\omega}\big(\hat{\nabla}_i\rho\hat{\nabla}_j\omega + \hat{\nabla}_i\omega\hat{\nabla}_j\rho\big) - \frac{\rho}{\omega^2}\hat{\nabla}_i\omega\hat{\nabla}_j\omega \\
& - \frac{1}{2}\hat{\nabla}_i\hat{\nabla}_j\rho - \frac{\rho}{2\omega}\hat{\nabla}_i\hat{\nabla}_j\omega.
\end{aligned} \tag{5.55}
$$

$$
C_{00} = \omega^2\rho^3 + \rho\hat{\nabla}_k\omega\hat{\nabla}^k\omega - \omega\hat{\nabla}_k\omega\hat{\nabla}^k\rho - \frac{3}{2}\omega\rho\hat{\Box}\omega - \frac{\omega^2}{2}\big(\hat{\Box}\rho + \hat{R}\rho\big). \tag{5.56}
$$

$$
\begin{aligned}
C^i{}_0 &= \epsilon^{i\alpha\beta}\nabla_\alpha S_{\beta 0} = \epsilon^{i0j}\nabla_0 S_{j0} + \epsilon^{ij0}\nabla_j S_{00} \\
&= -\frac{1}{\sqrt{g}}\varepsilon^{i0j}\nabla_0 S_{j0} + \frac{1}{\sqrt{g}}\varepsilon^{ij0}\nabla_j S_{00} \\
&= -\frac{1}{\omega\sqrt{h}}\varepsilon^{ij}\nabla_0 S_{j0} + \frac{1}{\omega\sqrt{h}}\varepsilon^{ij}\nabla_j S_{00} \\
&= \frac{1}{\omega}\epsilon^{ij}\big(\nabla_j S_{00} - \nabla_0 S_{j0}\big)
\end{aligned}
$$

$$
C^i_0 = \frac{\epsilon^{ik}}{2\omega}\hat{\nabla}_k\Big[\frac{3}{2}\omega^2\rho^2 + \hat{\nabla}_l\omega\hat{\nabla}^l\omega - \frac{1}{2}\omega^2\hat{R} - \omega\hat{\Box}\omega\Big]. \tag{5.57}
$$

$$
\begin{aligned}
C^{ij} =& h^{ij}\Big[\frac{3}{2\omega}\hat{\nabla}_k\omega\hat{\nabla}^k\rho + \frac{\rho}{\omega}\hat{\Box}\omega + \frac{1}{2}\hat{\Box}\rho - \frac{1}{2}\rho^3 + \frac{1}{4}\rho\hat{R}\Big] - \frac{1}{2}\hat{\nabla}^i\hat{\nabla}^j\rho \\
& - \frac{\rho}{2\omega}\hat{\nabla}^i\hat{\nabla}^j\omega - \frac{\rho}{\omega^2}\hat{\nabla}^i\omega\hat{\nabla}^j\omega - \frac{1}{\omega}\big(\hat{\nabla}^i\omega\hat{\nabla}^j\rho + \hat{\nabla}^j\omega\hat{\nabla}^i\rho\big).
\end{aligned} \tag{5.58}
$$

Now we get all the components of the Cotton tensor that have only invariant quantities.

Without going into details, let us give the results of the invariant components of the J-tensor, defined as

$$
J^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\rho\sigma}\epsilon^{\nu\tau\alpha}S_{\rho\tau}S_{\sigma\alpha} \tag{5.59}
$$

$$
\begin{aligned}
J_{00} =& \omega\hat{\Box}\omega\bigg(\frac{1}{4}\hat{R} - \frac{3}{8}\rho^2 + \frac{1}{2\omega}\hat{\Box}\omega\bigg) - \omega^2\bigg(\frac{1}{4}\hat{R} - \frac{3}{8}\rho^2 + \frac{1}{2\omega}\hat{\Box}\omega\bigg)^2 \\
& - \frac{1}{2}\big(\hat{\Box}\omega\big)^2 + \frac{1}{2}\hat{\nabla}^k\hat{\nabla}_l\omega\hat{\nabla}^l\hat{\nabla}_k\omega,
\end{aligned} \tag{5.60}
$$

$$
J^i_0 = \frac{\epsilon^{ij}}{2\omega}\bigg[\bigg(\frac{1}{4}\hat{R} - \frac{3}{8}\rho^2 + \frac{1}{2\omega}\hat{\Box}\omega\bigg)\hat{\nabla}_j(\omega^2\rho) - \frac{1}{\omega}\hat{\nabla}^k(\omega^2\rho)\hat{\nabla}_k\hat{\nabla}_j\omega\bigg], \tag{5.61}
$$

$$
\begin{aligned}
J^{ij} =& \frac{1}{\omega^2}\Bigg[h^{ij}\bigg(-\frac{1}{4}(\hat{\Box}\omega)^2+\frac{1}{8}\omega\rho^2\hat{\Box}\omega-\frac{1}{4}\omega^2\rho^2\hat{R}^2+\frac{15}{64}\omega^2\rho^4+\frac{1}{16}\omega^2\hat{R}^2\bigg)\\
&+\bigg(\frac{1}{2}\hat{\Box}\omega-\frac{5}{8}\omega\rho^2+\frac{1}{4}\omega\hat{R}\bigg)\hat{\nabla}^i\hat{\nabla}^j\omega+\frac{1}{4}\omega^2\hat{\nabla}^i\rho\hat{\nabla}^j\rho+\rho^2\hat{\nabla}^i\omega\hat{\nabla}^j\omega\\
&+\frac{\omega\rho}{2}\big(\hat{\nabla}^i\rho\hat{\nabla}^j\omega+\hat{\nabla}^i\omega\hat{\nabla}^j\rho\big)\Bigg].
\end{aligned}
\tag{5.62}
$$

Now we can move on with the H-tensor:

$$
H^{\mu\nu}=\epsilon^{\mu\alpha\beta}\nabla_\alpha C_\beta{}^\nu,
\tag{5.63}
$$

and give the final results:

$$
\begin{aligned}
H^{ij} =& h^{ij}\frac{1}{\omega}\hat{\nabla}^k\Bigg[\frac{1}{2\omega}\hat{\nabla}_k\bigg(\frac{3}{2}\omega^2\rho^2-\frac{1}{2}\omega^2\hat{R}-\omega\hat{\Box}\omega+\hat{\nabla}_l\omega\hat{\nabla}^l\omega\bigg)\Bigg]\\
&-\frac{1}{\omega}\hat{\nabla}^j\Bigg[\frac{1}{2\omega}\hat{\nabla}^i\bigg(\frac{3}{2}\omega^2\rho^2-\frac{1}{2}\omega^2\hat{R}-\omega\hat{\Box}\omega+\hat{\nabla}_l\omega\hat{\nabla}^l\omega\bigg)\Bigg]\\
&+\frac{1}{\omega}\hat{\nabla}^j\omega\hat{\nabla}^i\bigg(\frac{1}{4}\hat{R}-\frac{3}{8}\rho^2+\frac{1}{2\omega}\hat{\Box}\omega\bigg)\\
&+\frac{1}{\omega^3}\bigg(\hat{\nabla}^j\omega\hat{\nabla}^i\omega\hat{\Box}\omega-\hat{\nabla}^j\omega\hat{\nabla}_k\omega\hat{\nabla}^k\hat{\nabla}^i\omega\bigg)\\
&-\frac{1}{\omega^2}\bigg(\hat{\nabla}^j\omega\hat{\nabla}^i\hat{\Box}\omega+\hat{\nabla}^j\omega\hat{\nabla}^k\hat{\nabla}^i\hat{\nabla}_k\omega+2\hat{\nabla}^j\omega\hat{\Box}\hat{\nabla}^i\omega\bigg)\\
&-\frac{\rho}{4\omega}\hat{\nabla}^j\omega\hat{\nabla}^i\rho+\frac{\rho}{2\omega}\hat{\nabla}^j\rho\hat{\nabla}^i\omega-\frac{\rho^2}{\omega^2}\hat{\nabla}^j\omega\hat{\nabla}^i\omega\\
&+\frac{\rho}{4}\hat{\nabla}^j\hat{\nabla}^i\rho+\frac{\rho^2}{4\omega}\hat{\nabla}^j\hat{\nabla}^i\omega\\
&+h^{ij}\bigg(-\frac{\rho^2}{\omega^2}\hat{\nabla}_k\omega\hat{\nabla}^k\omega-\frac{5\rho}{4\omega}\hat{\nabla}_k\omega\hat{\nabla}^k\rho+\frac{1}{8}\rho^2\hat{R}-\frac{3}{4}\rho^4-\frac{1}{4}\rho\hat{\Box}\rho\bigg),
\end{aligned}
\tag{5.64}
$$

$$
\begin{aligned}
H^i{}_0 =& \epsilon^{ij}\Bigg[\frac{9}{4}\rho^3\hat{\nabla}_j\omega+\frac{3\rho}{\omega}\hat{\nabla}_j\hat{\nabla}_k\omega\hat{\nabla}^k\omega-\frac{1}{2}\hat{\nabla}_k\omega\hat{\nabla}_j\hat{\nabla}^k\rho-\frac{5\rho}{4\omega}\hat{\nabla}_j\omega\hat{\Box}\omega\\
&-\rho\hat{R}\hat{\nabla}_j\omega+\frac{15}{4}\omega\rho^2\hat{\nabla}_j\rho-\frac{5}{8}\omega\rho\hat{\nabla}_j\hat{R}-\frac{7}{4}\rho\hat{\nabla}_j\hat{\Box}\omega+\frac{2}{\omega}\hat{\nabla}_j\rho\hat{\nabla}_k\omega\hat{\nabla}^k\omega\\
&-\hat{\nabla}_j\omega\hat{\Box}\rho-\frac{1}{2\omega}\hat{\nabla}_j\omega\hat{\nabla}_k\omega\hat{\nabla}^k\rho+\frac{\rho}{\omega^2}\hat{\nabla}_j\omega\hat{\nabla}_k\omega\hat{\nabla}^k\omega-\hat{\nabla}_j\hat{\nabla}_k\omega\hat{\nabla}^k\rho\\
&-\frac{3}{2}\hat{\nabla}_j\rho\hat{\Box}\omega-\frac{1}{2}\omega\hat{\nabla}_j\hat{\Box}\rho-\frac{1}{2}\omega\hat{R}\hat{\nabla}_j\rho\Bigg],
\end{aligned}
\tag{5.65}
$$

$$
\begin{aligned}
H_{00} =& \omega^2\hat{\nabla}^k\Bigg[-\frac{1}{2\omega^2}\hat{\nabla}_k\bigg(\frac{3}{2}\omega^2\rho^2+\hat{\nabla}_l\omega\hat{\nabla}^l\omega-\frac{1}{2}\omega^2\hat{R}-\omega\hat{\Box}\omega\bigg)\Bigg]\\
&+\frac{5}{4}\rho^2\hat{\nabla}_k\omega\hat{\nabla}^k\omega-\frac{3}{2}\omega\rho\hat{\nabla}_k\omega\hat{\nabla}^k\rho-\frac{9}{4}\omega\rho^2\hat{\Box}\omega\\
&-\frac{3}{4}\omega^2\rho\hat{\Box}\rho-\frac{3}{4}\omega^2\rho^2\hat{R}+\frac{3}{2}\omega^2\rho^4.
\end{aligned}
\tag{5.66}
$$

CHAPTER 6

CONCLUSIONS

At large scales or outside a matter source, all the solutions of Einstein equation are Einstein spaces with the metric $R_{\mu\nu} = \Lambda g_{\mu\nu}$. Einstein equation is non-linear and without symmetries it is a set of coupled non-linear partial differential equations that are too complicated to obtain exact solutions. Besides arriving an exact solution, quantum version of the theory has not been formulated. Up to now we have not able to have the quantize General Relativity in four dimensions. Lower dimensional gravity theories have become a hope to the researchers for finding a quantum version of a gravity theory. In three dimensions, General Relativity does not have local degrees of freedom. Without cosmological constant the spacetime is flat: the Riemann tensor vanishes. For the $\Lambda \neq 0$ case, locally the solution is either de Sitter (dS) with $\Lambda > 0$, or anti-de Sitter $\Lambda < 0$. AdS case can be non-trivial globally, admits the first known solution called BTZ-black hole. There are several extensions of Einstein theory with non-trivial local dynamics in 3D: there are TMG, NMG and Born-Infeld extension of NMG. To reach the quantum gravity, these theories are studied using anti-de Sitter/Conformal field theory (AdS/CFT) correspondence. Unfortunately these theories suffer from the bulk-boundary unitarity clash which should not exist for a viable theory. In search for other theories, on shell consistent theories were proposed such as the MMG and the EMG that are only divergence free under the condition

$$\nabla_\mu \mathcal{E}^{\mu\nu}\Big|_{\mathcal{E}^{\mu\nu}=0} = 0. \tag{6.1}$$

We studied the generic on-shell consistent Exotic massive gravity theory which we called the Bachian gravity. We define the 3D Bach tensor as

$$H_{\mu\nu} \equiv \frac{1}{2}\epsilon_\mu{}^{\alpha\beta}\nabla_\alpha C_{\beta\nu} + \frac{1}{2}\epsilon_\nu{}^{\alpha\beta}\nabla_\alpha C_{\beta\mu}, \tag{6.2}$$

that does not come from the variation of an action.

On shell consistent theory is constructed with a 2-tensor $\mathcal{E}^{\mu\nu}$ that comes from the variation of an action, generalization of the Bach and J-tensors

$$\mathcal{E}_{\mu\nu} + \frac{1}{\mu}\epsilon_\mu{}^{\alpha\beta}\nabla_\alpha\Phi_{\beta\nu} + \frac{k}{\mu^2}\epsilon_\mu{}^{\alpha\beta}\epsilon_\nu{}^{\sigma\rho}\Phi_{\alpha\sigma}\Phi_{\beta\rho} = 0, \tag{6.3}$$

where $\Phi_{\mu\nu}$ is a symmetric 2-tensor and does not come from any action, $\nabla_\mu\Phi^{\mu\nu} \neq 0$. With the Choosing

$$\mathcal{E}_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda_0 g_{\mu\nu}, \tag{6.4}$$

we obtain the field equations given as

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda_0 g_{\mu\nu} + \frac{1}{\mu}\epsilon_\mu{}^{\alpha\beta}\nabla_\alpha\Phi_{\beta\nu} + \frac{1}{2\mu^2}\epsilon_\mu{}^{\alpha\beta}\epsilon_\nu{}^{\sigma\rho}\Phi_{\alpha\sigma}\Phi_{\beta\rho} = 0. \tag{6.5}$$

After identification of the $\Phi_{\mu\nu} := \Psi_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\Psi$, with covariantly conserved $\Psi_{\mu\nu}$, we finally get the latest form of the field equations

$$\begin{aligned} E_{\mu\nu} :=& R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda_0 g_{\mu\nu} + \frac{1}{\mu}\epsilon_\mu{}^{\alpha\beta}\nabla_\alpha\big(\Psi_{\beta\nu} - \frac{1}{2}g_{\beta\nu}\Psi\big) \\ &+ \frac{1}{2\mu^2}\bigg(g_{\mu\nu}\big(\Psi^2_{\alpha\beta} - \frac{3}{4}\Psi^2\big) + \Psi_{\mu\nu}\Psi - 2\Psi_{\mu\alpha}\Psi_\nu{}^\alpha\bigg) = 0. \end{aligned} \tag{6.6}$$

The spectrum of the theory was investigated with the help of the linearization about maximally symmetric vacua and Killing charge construction is presented.

As mentioned earlier, obtaining a solution of the gravity theories is hard to find. A symmetry is defined by the existence of a Killing vector field which can be employed to reach a solution. Projection formalism on the spacetime manifold (M, g) with a single Killing vector field is introduced as an option to solve this compelling problem. Projection formalism needs invariant quantities such as the twist and the norm of the Killing vector field. We work out the reduction of the tensors, Ricci, Cotton, $J_{\mu\nu}$ and $H_{\mu\nu}$.

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APPENDIX A

MAPS AND TOPOLOGICAL SPACES

[1]

A.1 Maps

A map can be defined as a connection between two sets. It is a sort of a rule of assigning one or some elements of one set to the other. Let X and Y be sets, we may write

$$f : X \to Y. \tag{A.1}$$

The set X becomes the domain of the map and Y is the range of the map. The domain and the range could be the whole set or the subsets of the corresponding sets. One can define the image of the map as the Y itself or a subset of Y. Let $y \in Y$ and $f(X) = \{y = f(x) | y \in Y \quad \text{and} \quad x \in X\}$. The inverse image of the map is defined as, $f^{-1}(Y) = \{x = f^{-1}(y) | x \in X \quad \text{and} \quad y \in Y\}$.

$$f^{-1} : Y \to X. \tag{A.2}$$

[1] Both Appendix A and Appendix B follow the book [7] very closely and we do not claim any originality. The material in these Appendicies were directly or indirectly used in the understanding of the spacetime as a manifold and all the structures that come with it.

A.1.1 Properties of the Maps

i-) **Injective map:** A map is called injective if an element in the domain set is assigned only one element in the range set by the rule of the map, $f(x) \neq f(x')$ for $x \neq x'$
ii-) **Surjective map:** A map is called surjective if every element in the range set has corresponding element in the domain set, $imf(X) = Y$
iii-) **Bijective map:** If the map is injective and surjective then it is called a bijective map.
iv-) **Inverse map:** Let f be a bijective map between two sets X and Y, $f : X \to Y$. Due to bijectivity one can define an inverse map $f^{-1} : Y \to X$ that is also bijective.
v-) **Composite map:** Let f and g be two maps which are defined as $f : X \to Y$ and $g : Y \to Z$. The composite map is constructed such as : $g \circ f : X \to Z$.

An algebraic structure may be formed on the sets. This structure could be addition or product. Structure preserving map $f : X \to Y$, is called a *homomorphism*. For instance, $f(a + b) = f(a) + f(b)$ or $f(ab) = f(a)f(b)$. In addition to that, if homomorphic map is also bijective, the map becomes an *isomorphism* and the two sets are *isomorphic* to each other, and one writes $X \cong Y$.

A.1.2 Equivalence Class

Equivalence classes are basically mutually disjoint subsets of the set X and the notation is $[s]$.

Definition: $\sim$ is an equivalance relation if it satisfies the following conditions:
i-) Reflexivity: $s \sim s$
ii-) Symmetry: If $s \sim t$, then $t \sim s$
iii-) Transitivity: If $s \sim t$ and $t \sim q \Rightarrow s \sim q$

Definition: Let $\sim$ be an equivalence relation on the set X. One can define a subset $[s]$ such as

$$[s] := \{t \in X | s \sim t\}, \tag{A.3}$$

and it is called an equivalence class of s. Since equivalence classes are supposed to be a mutually disjoint subsets, the intersection of two different classes are either empty set, $[s] \cap [t] = \emptyset$.

Definition: Consider a set X. A power set of X, $\mathcal{P}(X)$, is a set that contains all the subsets of X.

Definition: Consider a set X and its equivalence classes $[s]$. These are subsets of X and also elements of the power set of X, $[s] \in \mathcal{P}(x)$ by definition. The quotient space is a power set $\mathcal{P}(x)$ which is defined by

$$X_{/\sim} := \{[s] \in \mathcal{P}(x) | s \in X\}. \tag{A.4}$$

Remark: Any element s in $[s]$ is called the *representative* of a class $[s]$ and the union of representatives is isomorphic to the main set, $R \cong X_{/\sim}$.

A.2 Topological Spaces

Definition: Let X be a set. A topology on X is a power set $T \subseteq \mathcal{P}(X)$ that satisfies the following requirements:

i-) $\emptyset \in T$ and $X \in T$.

ii-) Intersection of any two subsets of T must be the element of T, e.g., $U, V \subseteq T$ and $U \cap V \in T$.

iii-) Union of finite number of subsets again must be element of T, e.g., $C \subseteq T \Rightarrow \bigcup C \in T$.

Then the pair (X, T) is called a *topological space*.

Informally speaking, a topology can be constructed on almost all sets. Because it is one of the weakest structure that is defined on a set. For instance, every set accept empty set ($\emptyset$) and the set itself as a subset. The topology is called *chaotic* topology if it has elements the empty set and the set itself, $T = \{\emptyset, X\}$. If the topology contains all the subsets of a set on which topology is constructed, then this is called the *discrete* topology.

Another useful topology type can be seen like this: Consider a real line as a set, $\mathbb{R}$. The standard topology consist of the all open intervals and their unions. It can be extended to any dimension, d.

$$\mathbb{R} := \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{d-times}. \tag{A.5}$$

However we need one more definition to construct standard topology.

Definition: The *open ball* of radius r around the point x,

$$B_r(x) := \left\{ y \in \mathbb{R}^d | \sqrt{\sum_{i=1}^{d} (y_i - x_i)^2} < r \right\}, \tag{A.6}$$

where $r \in \mathbb{R}^+ := \{r \in \mathbb{R} | r > 0\}$, and $x_i, y_i \in \mathbb{R}$. Then the *standard topology* on $\mathbb{R}$ is defined by;

$$U \in T_{std} :\Leftrightarrow \forall p \in U : \exists r \in \mathbb{R}^+ : B_r(p) \subseteq U. \tag{A.7}$$

Definition: Consider a map $f : X \to Y$ and topologies on the sets , (X, T_X) and (Y, T_Y). The map f is continuous if it satisfies the following;

$$\forall S \in T_Y, \quad preim_f(S) \in T_X, \tag{A.8}$$

where $preim_f(S) := \{x \in X : f(x) \in S\} \subseteq X$. Here note that $preim_f(S) \subseteq X$ and $im_f(U) \subseteq Y$ are open sets.

Definition: Consider an open or closed subset, N, of the topological space (X, T). N may contain at least one or more open sets U_i. Then N is a *neighbourhood* of a point $x \in U_i$. If N is an open subset, then it is called an *open neighbourhood* of $x \in U_i$.

Definition: Let x and x' are arbitrary two points in the topological space (X, T) with two neighbourhoods U_x and $U_{x'}$. For the case $U_x \cap U_{x'} = \emptyset$, (X, T) is called a *Hausdorff* space.

Definition: Let (X, T) be a toplogical space. A subset U of (X, T) is called *closed* if its complement in X is an open set.

A.2.1 Compactness and Paracompactness

Let (X, T) be a topological space. One can consider some subsets of X whose union is X such as $\bigcup U_i = X$. Then it is said to be a *cover* of X. For the subsets U_i are open, they become an *open cover* of a topological space. Now every open cover U_i can have finite subsets, V_j, that is also a cover. A family of V_j is called a *subcover* of X.

Definition: A topological space (X, T) is called *compact* if every open cover has a finite subcover.

Definition: A topological space (X, T) is called *paracompact* if every open cover has an open refinement.

Here a refinement R of an open cover is defined as a subcover of an open cover, e.g.,

$$\forall P \in R : \quad \exists V \in U_i : P \subseteq V. \tag{A.9}$$

A.2.2 Connectedness and Path-connectedness

Definition: A topological space (X, T) is *connected* unless there exist two subsets such as $X = X_1 \cup X_2$ and $X_1 \cap X_2 = \emptyset$.

Definition: Consider a pair of points $x_1, x_2 \in X$ and a continuous curve $\gamma : [0, 1] \to X$ such that $\gamma(0) = x_1$ and $\gamma(1) = x_2$. If this construction is applied to the every pair of points in the topological space, then it is called a *path-connected*.

A.2.3 Homeomorphism

For some topological spaces, continuous deformation is possible. Informally speaking, two spaces are equivalent if one can deform one space to another.

Definition: Let $f : X_1 \to X_2$ be a map and it is continuous and has an inverse which is also continuous. Then the map f is called a *homeomorphism* and two topological spaces X_1 and X_2 are *homeomorphic* to each other. In other words they are topologically equivalent.

The equivalance between spaces allows us to put all topologically equivalant spaces into one equivalence classes. A Topological invariant means that there is some quantity which is conserved under homoemorphisms. This quantity could be an algebraic structure, compactness, connectedness etc. Hence homeomorphism is a structure preserving map between topological spaces.

APPENDIX B

MANIFOLDS AND TENSOR FIELDS

B.1 Manifolds:

An m-dimensional manifold is a topological space that is locally homeomorphic to $\mathbb{R}^m$. Local homeomorphism allows us to introduce coordinate systems on some regions of the manifold. The important point is that a manifold globally does not have to be homeomorphic to $\mathbb{R}^m$. Hence one has to put different coordinate systems for the different regions of the manifold.

Definition: Let M be an m-dimensional topological manifold. $U_i \in M$ are open sets that satisfy;

i-) $\bigcup U_i = M$

ii-) $U_i \cap U_j \neq \emptyset$

One can define a homeomorphism ϕ_i between $U_i \in M$ and $U_i' \in \mathbb{R}^m$

$$\phi_i : U_i \rightarrow U_i' \in \mathbb{R}^m. \tag{B.1}$$

Then the pair (U_i, ϕ_i) is called a *chart* and the whole family of charts $\{(U_i, \phi_i)\}$ is called an *atlas*. It is legitimate to construct a map between two charts.

$$\Psi_{ij} : U_i' \rightarrow U_j' \quad , \quad U_i', U_j' \in \mathbb{R}^m. \tag{B.2}$$

Every constructed chart can be seen as a coordinate system. Therefore the map Ψ_{ij} becomes a coordinate transformation. The number of differentiability of Ψ_{ij} states the differentiability of the manifold. For instance, if Ψ_{ij} is infinitely differentiable then we have infinitely differentiable manifold, denoted as C^{∞}. The explicit form of the map Ψ_{ij} can be written by

$$\Psi_{ij} = \phi_j \circ \phi_i^{-1}. \tag{B.3}$$

We can cover the manifold using different family of charts that form another atlas. Let us assume A_1 consist of the open set family $\{(U_i, \phi_i)\}$ and A_2 of the $\{(V_i, \psi_i)\}$. If the map between these two atlases is differentiable then two atlases are said to be compatible.

Let M be a topological manifold and the family of open sets $\{U_i\}$ cover the manifold M. It is known that each U_i is homeomorphic to an open set of $\mathbb{R}^m$. Now suppose that each U_i is homoemorphic to an open set other than $\mathbb{R}^m$, say that is $\mathcal{H}^m \equiv \{(x^1, \ldots, x^m) \in \mathbb{R}^m | x^m \geq 0\}$. A manifold that is covered by such kind of open sets is said to be manifold with a boundary. The boundary of M is the set of points which are mapped to points with $x^m = 0$, and denoted by ∂M. The dimension of the boundary is one dimension less than the manifold, $dim(\partial M) = m - 1$. Let us say that there are two charts constructed by the maps

$$\phi_i : U_i \to \mathcal{H}^m \quad \text{and} \quad \phi_j : U_j \to \mathcal{H}^m. \tag{B.4}$$

One can define a map between two charts such as

$$\Psi_{ij} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j). \tag{B.5}$$

The smoothness of the map Ψ_{ij} is guaranteed when Ψ_{ij} is C^{∞} in an open set $\mathbb{R}^m$.

A product manifold can be defined after building up the individual manifolds. Let M and N be manifolds with $dim(M) = m$ and $dim(N) = n$. M and N have their own atlases such as $\{(U_i, \phi_i)\}$ and $\{(V_j, \psi_j)\}$. Now, constructed product manifold is $(m + n)$-dimensional and its atlas is $\{(U_i, \phi_i), (V_j, \psi_j)\}$. Any point on the product manifold is mapped by the homeomorphism $(\phi_i(p), \psi_j(q)) \in \mathbb{R}^{m+n}$, where $p \in M$ and $q \in N$. Notice that U_i and V_j are the open sets on the corresponding manifold, $U_i \in M$ and $V_j \in N$.

The torus T^2 is a well known example of the product manifold. It can be constructed by two S^1 manifolds:

$$T^2 = S^1 \times S^1. \tag{B.6}$$

It can be generalized to construct a n-Torus that is n-dimensional product manifold.

$$T^n = \underbrace{S^1 \times S^1 \times \cdots \times S^1}_{n}. \tag{B.7}$$

Definition: Let M and N are manifolds with charts (U, ϕ) and (V, ψ) with

$$\phi : U \to \mathbb{R}^m \quad , \quad \psi : V \to \mathbb{R}^n. \tag{B.8}$$

Let us define a map f such as

$$f : M \to N, \tag{B.9}$$

with $p \in U$ and $f(p) \in N$. We know that $\phi(p) \in \mathbb{R}^m$ and $\psi(f(p)) \in \mathbb{R}^n$. If the map

$$\psi \circ f \circ \phi^{-1} : \mathbb{R}^m \to \mathbb{R}^n \tag{B.10}$$

is C^∞-differentiable then f is said to be differentiable at point $p \in M$.

What we have done is simple: First we put a coordinate system on the manifolds M and N by defining the homeomorphism $\phi : U \to \mathbb{R}^m$ and $\psi : V \to \mathbb{R}^n$ where U and V are the open sets, $U \subset M$ and $V \subset N$. Then we see that the map f is differential if $\psi \circ f \circ \phi^{-1} : \mathbb{R}^m \to \mathbb{R}^n$ is C^∞-differentiable. Once we have a differentiable map between two manifolds, it enables us to construct a calculus on manifolds.

Definition: Let $f : M \to N$ be a homeomorphism. If the map $\psi \circ f \circ \phi^{-1} : \mathbb{R}^m \to \mathbb{R}^n$ is invertible and both $\psi \circ f \circ \phi^{-1}$ and $\phi \circ f^{-1} \circ \psi^{-1}$ are C^∞, f is called a diffeomorphism and the manifolds M and N are diffeomorphic to each other. In other words diffeomorphism f is a differential structure preserving map.

We have said that, when topological spaces are homeomorphic to each other, it is possible to deform one space to another continuously. Now we can insert this concept to the diffeomorphic manifolds as diffeomorphism enables to deform one differentiable manifold to another smoothly. It easy to conclude that diffeomorphic spaces are the same manifold.

Active and passive transformations have much of use in physics. Let us see these transformations in the framework of the manifolds. We say that a manifold can have more than one differentiable structure and it is characterized by the differential map between manifolds, say f. Now let say we have a set of diffeomorphisms between the same manifold, $f : M \to M$. Using chart (U, ϕ) with the coordinate map $\phi : U \to \mathbb{R}^m$, a point $p \in U$ can be moved to $\mathbb{R}^m$ to define a set of coordinates $\phi(p) \in \mathbb{R}^m$. Picking up one of the diffeomorphisms, f , from many, one can create an another coordinate values for a point $f(p) \in U$ by $\phi\big(f(p)\big) \in \mathbb{R}^m$. Now we have two coordinate values and a tool to transform one to another. The transformation can be formed by the map

$$\phi \circ f \circ \phi^{-1} : \mathbb{R}^m \to \mathbb{R}^m. \tag{B.11}$$

This transformation corresponds to active transformation in physics such as rotating a vector without changing the coordinate axis. Passive transformation is constructed using two different charts (U, ϕ) and (V, ψ) that are overlapping, $U \cap V \neq \emptyset$. Say we have a point $p \in U \cap V$. Now there are two coordinate maps ϕ and ψ and coordinate systems, $\phi(p) \in \mathbb{R}^m$ and $\psi(p) \in \mathbb{R}^m$. Again we use the diffeomorphism f to construct a coordinate transformation,

$$\psi \circ f \circ \phi^{-1} : \mathbb{R}^m \to \mathbb{R}^m. \tag{B.12}$$

This transformation is called a passive transformation. For instance, keeping the vector fixed and rotate the coordinate axis. As a result there is a set of diffeomorphism $f : M \to M$ and this set forms a group of transformation.

B.1.1 Curves and Functions:

Consider an open interval in $\mathbb{R}^1$ such as (a, b). A map $c : (a, b) \to M$ defines an open curve in an m-dimensional manifold, M. A closed curve can be seen by using a map

$c : S^1 \to M$. Now we have a curve on the manifold and we construct a coordinate representation of the curve with the chart (U, ϕ) on the manifold and the map

$$\phi \circ c : \mathbb{R} \to \mathbb{R}^m. \tag{B.13}$$

A function on the manifold is a smooth map

$$f : M \to \mathbb{R}. \tag{B.14}$$

In order to get a coordinate representation of the function, we again use a chart (U, ϕ) on the manifold and it is defined as

$$f \circ \phi^{-1} : \mathbb{R}^m \to \mathbb{R}. \tag{B.15}$$

B.1.2 Vectors:

We can use a curve and a function on the manifold to give a rigorous definiton of the vector. What we are supposed to do is the following,

1. Define a curve on the manifold using an open interval (a, b) that contains zero, $a < 0 < b$

$$c : (a, b) \to M. \tag{B.16}$$

2. Define a function $f : M \to \mathbb{R}$

3. and a chart on the manifold (U, ϕ)

$$\phi \circ c : (a, b) \to \mathbb{R}^m, \tag{B.17}$$

$$f \circ \phi^{-1} : \mathbb{R}^m \to \mathbb{R}. \tag{B.18}$$

Here remember that the coordinate representation of the points on the curve

$$x^\mu\big(c(t)\big) \equiv \phi \circ c : (a,b) \to \mathbb{R}^m, \tag{B.19}$$

and a function along the curve c

$$f\big(c(t)\big) \equiv f \circ \phi^{-1} : \mathbb{R}^m \to \mathbb{R}, \tag{B.20}$$

where $t \in (a,b)$.

Definition: A directional derivative of a function along the curve is

$$\frac{df\big(c(t)\big)}{dt} = \frac{\partial f}{\partial x^\mu}\frac{dx^\mu\big(c(t)\big)}{dt}\Bigg|_{t=0} \tag{B.21}$$

and a vector or tangent vector is defined as

$$\frac{d}{dt} = X^\mu \frac{\partial}{\partial x^\mu}, \tag{B.22}$$

where $X^\mu = \frac{dx^\mu}{dt}$, is a component of a vector and $e_\mu = \frac{\partial}{\partial x^\mu}$ is a basis.

Definition: Tangent space is a vector space that consists of all the tangent vectors at a point $p \in M$ and it is denoted by T_pM. Dimension of the tangent space is equal to the dimension of the manifold, $dim(T_pM) = dim(M)$.

B.1.3 One-forms:

One-forms are the elements of the cotangent space which is the dual space of the tangent space denoted by T_p^*M. In general the one-form is a linear function of a vector.

$$\omega : T_pM \to \mathbb{R} \tag{B.23}$$

$\omega(V) \in \mathbb{R}$ where $\omega \in T_p^*M$ and $V \in T_pM$.

We have seen that $\frac{d}{dt} = V^{\mu}\partial_{\mu}$ is a vector. Now one-form can be written as

$$df = \frac{\partial f}{\partial x^{\mu}} dx^{\mu} \quad , \quad \frac{\partial f}{\partial x^{\mu}} = f_{,\mu} : \text{component of a one-form}, \quad dx^{\mu} : \text{basis one-form} \tag{B.24}$$

From the map $\omega : T_pM \rightarrow \mathbb{R}$, it is easy to see that one-form takes a vector from tangent space as variable and gives a real value as a result.

$$df\big(\frac{d}{dt}\big) = \frac{\partial f}{\partial x^{\nu}} dx^{\nu}\big(X^{\mu}\frac{\partial}{\partial x^{\mu}}\big) = X^{\mu}\frac{\partial f}{\partial x^{\nu}} \underbrace{dx^{\nu}\big(\frac{\partial}{\partial x^{\mu}}\big)}_{\delta^{\nu}_{\mu}} = X^{\mu}\frac{\partial f}{\partial x^{\mu}} \in \mathbb{R}. \tag{B.25}$$

Here, we have used dx^{μ} as a dual basis, $dx^{\nu}\big(\frac{\partial}{\partial x^{\mu}}\big) = \delta^{\nu}_{\mu}$.

Let us take a general one-form ω and a vector V to define an inner product. Inner product is a bilinear map such as

$$\big(\quad , \quad \big) : T^{*}_{p}M \times T_pM \rightarrow \mathbb{R}, \tag{B.26}$$

$$\omega = \omega_{\mu} dx^{\mu} \quad , \quad V = V^{\nu}\frac{\partial}{\partial x^{\nu}}, \tag{B.27}$$

$$\big(\omega, V\big) = \big(\omega_{\mu} dx^{\mu}, V^{\nu}\frac{\partial}{\partial x^{\nu}}\big) = \omega_{\mu}V^{\nu}\big(dx^{\mu}, \frac{\partial}{\partial x^{\nu}}\big) = \omega_{\mu}V^{\nu}\delta^{\nu}_{\mu} = \omega_{\mu}V^{\mu}. \tag{B.28}$$

As a note, basis vectors $e_{\mu} = \frac{\partial}{\partial x^{\mu}}$ and basis one-forms $e^{\mu} = dx^{\mu}$ are called the coordinate basis. One can define non-coordinate basis $\{e_i\}$ and $\{e^i\}$ which are not related to the coordinates

B.1.4 Tensors:

A tensor is a multilinear map of vectors and one-forms. A (q, r) type tensor T is defined at a point $p \in M$ as

$$T : \underbrace{T_p^*M \otimes T_p^*M \otimes \cdots \otimes T_p^*M}_{q} \underbrace{T_pM \otimes T_pM \otimes \cdots \otimes T_pM}_{r} \to \mathbb{R}. \tag{B.29}$$

In this notation a vector is a $(1, 0)$ type tensor, a one-form is a $(0, 1)$ type tensor and a scalar is a $(0, 0)$ type tensor.

Let us suppose that T is a (q, r) type tensor.

$$T = T^{\mu_1 \ldots \mu_q}{}_{\nu_1 \ldots \nu_r} \frac{\partial}{\partial x^{\mu_1}} \cdots \frac{\partial}{\partial x^{\mu_q}} dx^{\nu_1} \ldots dx^{\nu_r}. \tag{B.30}$$

$$T\big(\underbrace{\omega, \alpha, \ldots}_{q}; \underbrace{V, W, \ldots}_{r}\big) = T^{\mu_1 \ldots \mu_q}{}_{\nu_1 \ldots \nu_r} \omega_{\mu_1} \alpha_{\mu_2} \ldots \beta_{\mu_q} V^{\nu_1} W^{\nu_2} \ldots X^{\nu_r}, \tag{B.31}$$

where $\omega, \alpha, \ldots, \beta \in T_p^*M$ and $V, W, \ldots, X \in T_pM$.

B.1.5 Tensor Field:

A vector is a linear map from cotangent space to the $\mathbb{R}$-space. If this map is defined at every point of the manifold then we have a map over the whole manifold and it is called a vector field. Similarly, one-form is a linear map from the tangent space to the $\mathbb{R}$-space. One-form field is a defined map for every point of the manifold. In general, for every point of the manifold defined multilinear maps from tangent and cotangent space to $\mathbb{R}$-space form a tensor field.

B.1.6 Push-forward and Pull-back:

Let M and N be manifolds and the map f is defined as

$$f : M \to N. \tag{B.32}$$

The corresponding charts of the manifolds are $(U, \phi) \in M$ and $(V, \psi) \in N$. Now define a function $g : N \to \mathbb{R}$. We know how a vector acts on a function, e.g.;

$$Xg = \frac{dg}{dt} = \frac{dx^\mu}{dt}\frac{\partial g}{\partial x^\mu} \quad \in \mathbb{R}. \tag{B.33}$$

In this set up, one can define a differential map f_*;

$$f_* : T_pM \to T_{f(p)}N, \tag{B.34}$$

where T_pM and $T_{f(p)}N$ are the corresponding tangent spaces of M and N. Let us move on with the vector $V \in T_pM$ that acts on a function $\big(g \circ f\big)(x)$ to give a real number $V\big[g \circ f\big] \in \mathbb{R}$.

Definition: Push-forward of a vector $V \in T_pM$ is defined as the following

$$\big(f_*V\big)[g] \equiv V\big[g \circ f\big], \tag{B.35}$$

where $\big(f_*V\big)$ is a vector that is defined at the point $f(p) \in N$ and acts on a function $g : N \to \mathbb{R}$. This action can be seen more clearly by using the charts of the manifolds, $(U, \phi) \in M$ and $(V, \psi) \in N$.

$$\big(f_*V\big)[g] = \big(f_*V\big)\big[g \circ \psi^{-1}(y)\big],$$

$$V\big[g \circ f\big] = V\big[g \circ f \circ \phi^{-1}(x)\big], \tag{B.36}$$

$$\Rightarrow \quad \big(f_*V\big)\big[g \circ \psi^{-1}(y)\big] \equiv V\big[g \circ f \circ \phi^{-1}(x)\big]. \tag{B.37}$$

Informally speaking, push-forward of a vector works like this: Instead of acting a vector to a function that is defined in the manifold that contains a corresponding vector, move a vector to another manifold with the help of a differential map between those manifolds and act a vector to a function that is defined on the manifold.

Definition: Pull-back of a one-form is defined with the differential map f^* as

$$f^* : T^*_{f(p)}N \to T^*_pM, \tag{B.38}$$

$$\big(f^*\omega\big)(V) = \omega\big(f_*V\big), \tag{B.39}$$

where $\omega \in T^*_{f(p)}N$ and $\big(f_*V\big)$ is a push-forward vector on the tangent space of the N at a point $f(p)$.

B.1.7 Lie Derivatives:

The picture so far is like this: There is a parametrized curve on the manifold, $x(t)$ and the tangent vector correponds to this curve, $X = X^\mu \frac{\partial}{\partial x^\mu}$. From this point it is easy to obtain a tensor field as an assigned tensor on every point of the manifold. Here is the thing, we can get an integral curve going backward from that path.

Let suppose that $x(t)$ is a parametrized curve and the coordinate representation of it is $x^\mu(t)$. That basically means using chart (U, ϕ), μth component of $\phi\big(x(t)\big)$ is $x^\mu(t)$. Consider a vector $X = X^\mu \frac{\partial}{\partial x^\mu}$ with the component

$$X^\mu\big(x(t)\big) = \frac{dx^\mu}{dt}, \tag{B.40}$$

and it is a system of ordinary differential equations (ODEs). The solution of these ODEs with the initial conditions $x_0^\mu = x^\mu(0)$, which are the coordinates of a curve at $t = 0$, provides us the desired integral curve of the vector field X. The existence and the uniqueness of the ODEs ensure that there is a single curve for a specific point on the manifold. From the uniqueness of the ODEs we are able to write a curve more rigorously such as

$$\sigma : \mathbb{R} \times M \to M \tag{B.41}$$

and σ is called the flow generated by the vector field X as a solution of the ODE:

$$\frac{d}{dt}\sigma^\mu(t, x) = X^\mu\big(\sigma(t, x)\big). \tag{B.42}$$

Let assume that there are two vector fields and two curves which are generated by these vector fields.

$$\frac{d\sigma^\mu(s,x)}{ds} = X^\mu\big(\sigma(s,x)\big), \tag{B.43}$$

$$\frac{d\tau^\mu(t,x)}{dt} = Y^\mu\big(\tau(t,x)\big). \tag{B.44}$$

To get a geometrical approach of the Lie derivative of the vector fields, we can carry out the following steps:

1. Take a tangent vector of the curve $\tau(t,x)$ at an intersecting point x of two curves $\tau(t,x)$ and $\sigma(s,x)$

$$Y\Big|_x \in T_x M. \tag{B.45}$$

2. Go through along curve $\sigma(s,x)$ at a point $\sigma_\epsilon(x)$ and take a tangent vector,

$$Y\Big|_{\sigma(x)} \in T_{\sigma(x)} M, \tag{B.46}$$

by using the map

$$\sigma_\epsilon : T_x M \to T_{\sigma(x)} M. \tag{B.47}$$

The map σ_ϵ can be seen as simple coordinate transformation, e.g.,

$$x'^\mu = x^\mu + \epsilon X^\mu(x). \tag{B.48}$$

Now we have a different tangent vector than $Y\big|_x$ because $Y\big|_{\sigma(x)}$ is a tangent vector at the point $\sigma_\epsilon(x)$. We need to compare two vectors by definition of the derivative. Therefore, $Y\big|_{\sigma(x)}$ is supposed to move the point x. It can be done by the map:

$$\sigma_{-\epsilon} : T_{\sigma_\epsilon(x)} M \to T_x M, \tag{B.49}$$

and the resulting vector, $(\sigma_{-\epsilon})Y\big|_{\sigma(x)}$ is a vector at the point x.

The Lie derivative of a vector field Y along the curve σ that is generated by a vector field X is defined by

$$\mathcal{L}_X Y = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \big[\big(\sigma_{-\epsilon}\big) Y \big|_{\sigma(x)} - Y\big|_x \big]. \tag{B.50}$$

Coordinate-induced view of the Lie derivative of a vector field Y along X is the following: Consider a chart (U, ϕ) on the manifold M and two vector fields, $X = X^\mu \frac{\partial}{\partial x^\mu}$ and $Y = Y^\mu \frac{\partial}{\partial x^\mu}$.

$$\sigma_\epsilon(x) = x^\mu + \epsilon X^\mu(x),$$

$$Y\big|_{\sigma(x)} = Y^\mu\big(x^\nu + \epsilon X^\nu(x)\big) \frac{\partial}{\partial x^\mu}\Big|_{x+\epsilon X}, \tag{B.51}$$

Make a Taylor series expansion,

$$Y_{\sigma(x)} \cong \big[Y^\mu(x) + \epsilon X^\mu(x) \partial_\nu Y^\mu(x)\big] \frac{\partial}{\partial x^\mu}\Big|_{x+\epsilon X}. \tag{B.52}$$

$\big(\sigma_{-\epsilon}\big) Y\big|_{\sigma(x)}$ is nothing but just a coordinate transformation such as;

$$\frac{\partial}{\partial x^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial}{\partial x'^\nu},$$

$$x^\nu = x'^\nu - \epsilon X^\nu(x). \tag{B.53}$$

$$\begin{aligned}
\big(\sigma_{-\epsilon}\big) Y\big|_{\sigma(x)} &= \big[Y^\mu(x) + \epsilon X^\lambda(x) \partial_\lambda Y^\mu(x)\big] \frac{\partial}{\partial x^\mu} \big(x^\nu - \epsilon X^\nu(x)\big) \frac{\partial}{\partial x^\nu}\Big|_x \\
&= \big[Y^\mu(x) + \epsilon X^\lambda(x) \partial_\lambda Y^\mu(x)\big] \big[\delta^\nu_\mu - \epsilon \partial_\mu X^\nu(x)\ \big] \frac{\partial}{\partial x^\nu}\Big|_x \\
&= Y^\mu(x) \frac{\partial}{\partial x^\mu}\Big|_x + \epsilon \big[X^\mu(x) \partial_\mu Y^\nu(x) - Y^\mu(x) \partial_\mu X^\nu(x)\big] \frac{\partial}{\partial x^\nu}\Big|_x \\
&+ \mathcal{O}(\epsilon^2).
\end{aligned} \tag{B.54}$$

From there it follows that

$$
\begin{aligned}
\mathcal{L}_X Y = &\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left\{Y^\mu(x)\frac{\partial}{\partial x^\mu}\Big|_x + \epsilon\big[X^\mu(x)\partial_\mu Y^\nu(x) - Y^\mu(x)\partial_\mu X^\nu(x)\big]\frac{\partial}{\partial x^\nu}\Big|_x\right.\\
&\left. - Y^\mu(x)\frac{\partial}{\partial x^\nu}\Big|_x\right\}\\
= &\big[X^\mu(x)\partial_\mu Y^\nu(x) - Y^\mu(x)\partial_\mu X^\nu(x)\big]\frac{\partial}{\partial x^\nu}\Big|_x.
\end{aligned}
\tag{B.55}
$$

Hence;

$$
\mathcal{L}_X Y = \big(X^\mu\partial_\mu Y^\nu - Y^\mu\partial_\mu X^\nu\big)\frac{\partial}{\partial x^\nu}. \tag{B.56}
$$

Notice that $\mathcal{L}_X Y$ is itself a vector field with the component

$$
\big(\mathcal{L}_X Y\big)^\nu = X^\mu\partial_\mu Y^\nu - Y^\mu\partial_\mu X^\nu. \tag{B.57}
$$

One can easily show that

$$
\mathcal{L}_X Y = \big[X, Y\big] = XY - YX, \tag{B.58}
$$

where $\big[X, Y\big]$ is called a Lie Bracket.

Let us focus now the geometrical meaning of the Lie bracket. There are two flows $\sigma(s,x)$ and $\tau(t,x)$ that are generated by vector fields X and Y respectively. First move along the curve σ by the amount of ϵ and then move along τ by δ.

$$
\begin{aligned}
\tau^\mu\big(\delta, \sigma(\epsilon, x)\big) &= \tau^\mu\big(\delta, x^\nu + \epsilon X^\nu(x)\big)\\
&= x^\mu + \epsilon X^\mu(x) + \delta Y^\mu\big(x^\nu + \epsilon X^\nu(x)\big)\\
&= x^\mu + \epsilon X^\mu(x) + \delta Y^\mu(x) + \epsilon\delta X^\nu(x)\partial_\nu Y^\mu(x).
\end{aligned}
\tag{B.59}
$$

Now, let us move in the reverse order, first move along a curve τ by the amount of δ and then move along σ by ϵ,

$$
\begin{aligned}
\sigma^\mu\big(\epsilon, \tau(\delta, x)\big) &= \sigma^\mu\big(\epsilon, x^\nu + \delta Y^\nu(x)\big)\\
&= x^\mu + \delta Y^\mu(x) + \epsilon X^\mu\big(x^\nu + \delta Y^\nu(x)\big)\\
&= x^\mu + \delta Y^\mu(x) + \epsilon X^\mu(x) + \epsilon\delta Y^\nu(x)\partial_\nu X^\mu(x).
\end{aligned}
\tag{B.60}
$$

It is easy to see that two motions do not have to end up at the same point. To see this, look for the difference of two paths:

$$\begin{aligned}\tau^{\mu}\big(\delta,\sigma(\epsilon,x)\big)-\sigma^{\mu}\big(\epsilon,\tau(\delta,x)\big)&=\epsilon\delta\big(X^{\nu}\partial_{\nu}Y^{\mu}-Y^{\nu}\partial_{\nu}X^{\mu}\big)\\&=\epsilon\delta\big[X,Y\big]^{\mu}.\end{aligned}\tag{B.61}$$

Reaching the same point requires that the Lie bracket of two vector fields should be zero.

$$\mathcal{L}_X Y=\big[X,Y\big]=0,$$

$$\Rightarrow\quad \sigma\big(s,\tau(t,x)\big)=\tau\big(t,\sigma(s,x)\big).\tag{B.62}$$

The Lie derivative of a one-form ω along a vector field X can be calculated by a similar manner.

$$\mathcal{L}_X\omega\equiv\lim_{\epsilon\to 0}\frac{1}{\epsilon}\bigg[\sigma_\epsilon\omega\Big|_{\sigma_\epsilon(x)}-\omega\Big|_x\bigg],\tag{B.63}$$

$$\sigma_\epsilon\omega\Big|_{\sigma_\epsilon(x)}=\omega_\mu(x)dx^\mu+\epsilon\big[X^\nu(x)\partial_\nu\omega_\mu(x)+\partial_\mu X^\nu\omega_\nu(x)\big]dx^\mu,$$

$$\omega\Big|_x=\omega_\mu(x)dx^\mu,$$

$$\Rightarrow\quad \mathcal{L}_X\omega=\big(X^\nu\partial_\nu\omega_\mu+\omega_\nu\partial_\mu X^\nu\big)dx^\mu.\tag{B.64}$$

The Lie derivative of a function f along a vector field X is

$$\begin{aligned}\mathcal{L}_X f&=\lim_{\epsilon\to 0}\frac{1}{\epsilon}\big[f\big(\sigma_\epsilon(x)\big)-f(x)\big]\\&=\lim_{\epsilon\to 0}\frac{1}{\epsilon}\big[f\big(x^\mu+\epsilon X^\mu(x)\big)-f(x^\mu)\big]\\&=X^\mu(x)\frac{\partial f}{\partial x^\mu}=Xf.\end{aligned}\tag{B.65}$$

The Lie derivative of any rank tensor can be found by using one of the properties of a Lie derivative.

$$\mathcal{L}_X\big(T_1\otimes T_2\big)=\big(\mathcal{L}_X T_1\big)\otimes T_2+T_1\otimes\big(\mathcal{L}_X T_2\big),\tag{B.66}$$

where T_1 and T_2 are arbitrary tensor fields of any rank. As an example, consider a $\binom{1}{1}$-tensor field $T=T_\mu{}^\nu dx^\mu\otimes e_\nu$

$$\mathcal{L}_X T=X\big(T_\mu{}^\nu\big)dx^\mu\otimes e_\nu+T_\mu{}^\nu\big(\mathcal{L}_X dx^\mu\big)\otimes e_\nu+T_\mu{}^\nu dx^\mu\otimes\big(\mathcal{L}_X e_\nu\big).\tag{B.67}$$

B.1.8 Differential Forms:

Definition: Differential form is a totally anti-symmetric $\binom{0}{r}$-rank tensor and called also an r-form.

Definition: Wedge product $\wedge$ of r one-forms defined by

$$dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_r} = \sum_{p \in perm} sgn(p) dx^{\mu_{p(1)}} \wedge dx^{\mu_{p(2)}} \wedge \cdots \wedge dx^{\mu_{p(r)}}, \quad \text{(B.68)}$$

where it constitutes a totally anti-symmetric tensor product.

Some examples:

i-) $dx^{\mu} \wedge dx^{\nu} = dx^{\mu} \otimes dx^{\nu} - dx^{\nu} \otimes dx^{\mu}$,

ii-)

$$\begin{aligned} dx^{\mu} \wedge dx^{\nu} \wedge dx^{\sigma} &= dx^{\mu} \otimes dx^{\nu} \otimes dx^{\sigma} + dx^{\nu} \otimes dx^{\sigma} \otimes dx^{\mu} \\ &+ dx^{\sigma} \otimes dx^{\mu} \otimes dx^{\nu} - dx^{\mu} \otimes dx^{\sigma} \otimes dx^{\nu} \\ &- dx^{\sigma} \otimes dx^{\nu} \otimes dx^{\mu} - dx^{\nu} \otimes dx^{\mu} \otimes dx^{\sigma}. \end{aligned} \quad \text{(B.69)}$$

Let us denote $\Omega_p^r(M)$ is the vector space of r-forms, then any r-form $\omega \in \Omega_p^r(M)$ is

$$\omega = \frac{1}{r!} \omega_{\mu_1 \mu_2 \dots \mu_r} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_r}, \quad \text{(B.70)}$$

where $\omega_{\mu_1 \mu_2 \dots \mu_r}$ are totally anti-symmetric tensor components.

The dimension of the vector space of r-forms is

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}, \quad \text{(B.71)}$$

where n is the dimension of the manifold. From this it is easy to check that the dimensions are equal for the spaces $\Omega_p^r(M)$ and $\Omega_p^{n-r}(M)$, $dim\big(\Omega_p^r(M)\big) = dim\big(\Omega_p^{n-r}(M)\big)$.

$$\binom{n}{r} = \binom{n}{n-r} = \frac{n!}{r!(n-r)!}. \quad \text{(B.72)}$$

Definition: The exterior product of a q-form and an r-form is a map such as

$$\wedge : \Omega_p^q(M) \times \Omega_p^r(M) \to \Omega_p^{q+r}(M). \quad \text{(B.73)}$$

Let $\omega \in \Omega_p^q(M)$ and $\xi \in \Omega_p^r(M)$ and the exterior product of these forms $(\omega \wedge \xi \in \Omega_p^{q+r}(M))$. The action of the $(q+r)$-form $(\omega \wedge \xi)$ on $(q+r)$-vectors is defined by

$$(\omega \wedge \xi)(V_1, \ldots, V_{q+r}) = \frac{1}{q!r!} \sum_{p \in perm} sign(p) \omega(V_{p(1)}, \ldots V_{p(q)}) \times \xi(V_{p(q+1)}, \ldots, V_{p(q+r)}). \tag{B.74}$$

Using exterior product, an algebra is defined

$$\Omega_p^*(M) \equiv \Omega_p^0(M) \oplus \Omega_p^1(M) \oplus \cdots \oplus \Omega_p^m(M), \tag{B.75}$$

where $\Omega_p^*(M)$ is the vector space of all differential forms at p. For comparing an r-form field, one may assign an r-form smoothly at each point on a manifold and it is denoted as $\Omega^r(M)$.

Definition: The exterior derivative, d, on an r-form is defined by

$$d\omega = \frac{1}{r!} \big(\frac{\partial}{\partial x^\nu} \omega_{\mu_1 \mu_2 \ldots \mu_r}\big) dx^\nu \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r},$$

where

$$\omega = \frac{1}{r!} \omega_{\mu_1 \mu_2 \ldots \mu_r} dx^\nu \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r}. \tag{B.76}$$

Now, straightforward calculation of the exterior derivative of an r-form $\omega \in \Omega^r(M)$ leads to,

$$d\omega(X_1, \ldots, X_{r+1}) = \sum_{i=1}^{r} (-1)^{i+1} X_i \omega(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_{r+1}) + \sum_{i<j} (-1)^{i+j} \omega\big([X_i, X_j], X_1, .., X_{i-1}, X_{i+1}, \ldots , X_{j-1}, X_{j+1}, .., X_{r+1}\big). \tag{B.77}$$

Another important property of the exterior derivative is

$$d^2 = 0. \tag{B.78}$$

To prove this result, let us take r-form $\omega \in \Omega^r(M)$;

$$\omega = \frac{1}{r!} \omega_{\mu_1 \mu_2 \ldots \mu_r} dx^\nu \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r},$$

$$d\omega = \frac{1}{r!}\frac{\partial \omega_{\mu_1 \ldots \mu_r}}{\partial x^\nu} dx^\nu \wedge dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_r},$$

$$d^2\omega = \frac{1}{r!}\frac{\partial^2 \omega_{\mu_1 \ldots \mu_r}}{\partial x^\lambda \partial x^\nu} dx^\lambda \wedge dx^\nu \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r}. \quad \text{(B.79)}$$

Here we notice that $\frac{\partial^2}{\partial x^\lambda \partial x^\nu}$ is symmetric while $dx^\lambda \wedge dx^\nu$ is anti-symmetric and multiplication of symmetric and anti-symmetric objects gives zero.

Definition: Let ω be an r-form, $\omega \in \Omega^r(M)$. ω is called a closed r-form if $d\omega = 0$. In addition to this, it is an exact r-form if there exist $\psi \in \Omega^{r-1}(M)$ such that $\omega = d\psi$.

Beside the exterior product, there is also the interior product of forms.

Definition: The Interior product is a map,

$$i_x : \Omega^r(M) \to \Omega^{r-1}(M), \quad \text{(B.80)}$$

and it is defined by an r-form $\omega \in \Omega^r(M)$

$$i_x\omega(X_1, \ldots, X_{r-1}) \equiv \omega(X, X_1, \ldots, X_{r-1}), \quad \text{(B.81)}$$

where $X \in \mathcal{X}(M)$.

B.1.9 Integration of Differential Forms:

In an m-dimensional connected manifold M, let U_i and U_j be charts such that $U_i \cap U_j \neq$. For the common point of the charts, $p \in U_i \cap U_j$, the tangent space T_pM can be spanned by two different basis, $\{e_\mu\} = \{\frac{\partial}{\partial x^\mu}\}$ and $\{\tilde{e_\alpha}\} = \left(\frac{\partial}{\partial y^\alpha}\right)$. The transformation between the basis is defined

$$\tilde{e_\alpha} = \frac{\partial x^\mu}{\partial y^\alpha} e_\mu, \quad \text{(B.82)}$$

and $J = \det\left(\frac{\partial x^\mu}{\partial y^\alpha}\right)$.

Definition: If $J > 0$ for any overlapping charts U_i and U_j, then the manifold M is called orientable. Otherwise, $J < 0$, it is called a non-orientable.

The important point is that only for an orientable manifold an integration of a differential form is constructed. There exist non-vanishing m-form on an m-dimensional orientable manifold and it is called a volume element. One can say that two volume-forms are equivalent if there is a positive function $h \in \mathcal{F}(M)$ such that $\omega = h\omega'$. The manifold may have right-handed orientation and left handed orientation which depends on the definition of a function h. If h is negative-definite then the manifold is left handed and for h is positive-definite the manifold is right-handed.

Integration of Forms:

A manifold is a paracompact topological space from definition. Paracompactness provides that any open covering $\{U_i\}$ has a finite subcover. Using paracompactness of the manifold, once the integral of a function over U_i is defined, the integral can be calculated over the whole manifold. This is possible by using the property which is called partition of unity.

Definition: The family of differentiable functions $\epsilon_i(p)$ is called a partition function of unity if it satisfies the following conditions:

i-) $0 \leqslant \epsilon_i(p) \leqslant 1$

ii-) $\epsilon_i(p) = 0$ if $p \notin U_i$

iii-) $\epsilon_1(p) + \epsilon_2(p) + \cdots = 1$ for any point $p \in M$.

The conditions enable us to write

$$f(p) = \sum_i f(p)\epsilon_i(p) = \sum_i f_i(p). \tag{B.83}$$

From the paracompactness of the manifold the summation has a finite number of elements. Now consider a function $f : M \to \mathbb{R}$ and a volume element ω. We define the integration as

$$\int_{U_i} f\omega \equiv \int_{\phi(U_i)} f\big(\phi_i^{-1}(x)\big)h\big(\phi_i^{-1}(x)\big)dx^1dx^2\dots dx^m, \tag{B.84}$$

where $\phi : M \to \mathbb{R}^m$ is a chart map corresponding to the open set U_i. Using the partition of unity the integral becomes

$$\int_M f\omega \equiv \sum_i \int_{U_i} f_i\omega. \tag{B.85}$$

Notice that the result of the integral is independent of the chosen atlas and the coordinates.

B.1.10 Lie Groups and Lie Algebras:

Definition: A Lie Group G is a differentiable manifold with the differential group operations;

$$\begin{aligned} \bullet \quad : \quad & G \times G \to G \\ & (g_1, g_2) \mapsto g_1 \cdot g_2. \end{aligned} \tag{B.86}$$

$$\begin{aligned} inv \quad : \quad & G \to G \\ & g \mapsto g^{-1}. \end{aligned} \tag{B.87}$$

G has the all group operations such as, unit element e, $(e \cdot g = g)$, every element has an inverse, associativity and closure.

Some of the familiar Lie groups are general Lie groups $GL(n, \mathbb{R})$ or $GL(n, \mathbb{C})$ and their subgroups. The elements of these groups are matrices and the operations are matrix multipilication and matrix inverse.

Let us introduce some interesting subgroups of $GL(n, \mathbb{R})$

-Orthogonal Group: $O(n) = \{M \in GL(n, \mathbb{R}) | MM^T = M^T M = \mathcal{I}_n\}$.

-Special Linear group: $SL(n, \mathbb{R}) = \{M \in GL(n, \mathbb{R}) | \text{det} M = 1\}$.

-Special Orthogonal Group: $SO(n) = O(n) \cap SL(n, \mathbb{R})$.

-Lorentz Group: $O(1, 3) = \{M \in GL(4, \mathbb{R}) | M\eta M^T = \eta\}$

where η is the Minkowski metric, $\eta = \text{diag}(-1, 1, 1, 1)$.

Theorem: Every closed subgroup H of a Lie group G is a Lie subgroup.

Definition: The coset space G/H is a manifold that is not necessarily a Lie group and it is constructed with an equivalance class $g \sim g'$ and a Lie subgroup H of G

such that

$$g' = gh, \tag{B.88}$$

with a set

$$[g] = \{gh|h \in H\}. \tag{B.89}$$

If $ghg^{-1} \in H$, that means H is a normal subgroup of G, then the coset G/H is a Lie group. In other words the elements of the coset space G/H are the equivalance classes, $[g]$. Hence notice that the group structure is preserved. That is to say, group operations, multiplication and inverse element are maintained.

$$[g][g'] = [gg'], \tag{B.90}$$

$$[g]^{-1} = [g^{-1}], \tag{B.91}$$

where $[g], [g'] \in G/H$.

Definition: The map $L_a : G \to G$ is called the left translation and defined by

$$L_a = ag, \tag{B.92}$$

where G is a Lie group and $a, g \in G$. L_a is a diffeomorphism and one can define an induced map

$$L_{a*} : T_gG \to T_{ag}G. \tag{B.93}$$

Definition: The left-invariant vector field in a given Lie group G is defined

$$L_{a*}X\Big|_g = X\Big|_{ag}. \tag{B.94}$$

To understand the definition better let us introduce coordinate representation of the vector field as,

$$X\Big|_g = X^\mu(g)\frac{\partial}{\partial x^\mu(g)} \tag{B.95}$$

is a vector field at the point g and

$$L_{a*}X\Big|_g = X^\mu(g)\frac{\partial x^\nu(ag)}{\partial x^\mu(g)}\frac{\partial}{\partial x^\nu}\Big|_{ag} = X^\nu(ag)\frac{\partial}{\partial x^\nu}\Big|_{ag} \tag{B.96}$$

is a vector field at the point that is left-translated.

Consider a unique left-invariant vector field X_V and

$$X_V\Big|_g = L_{g*}V \tag{B.97}$$

where $V \in T_eG$, $g \in G$. It is straightforward to show that

$$X_V\Big|_{ag} = L_{ag*}V = \big(L_aL_g\big)_*V = L_{a*}L_{g*}V = L_{a*}X_V\Big|_g \tag{B.98}$$

by using the property of the induced map such as $\big(g \circ f\big)_* = g_* \circ f_*$

Here there is a left-invariant vector field X_V that is constructed by the vector $V \in T_eG$. Hence, it is possible to define a unique vector $V = X\Big|_e \in T_eG$ by using a left-invariant vector field X. Using the uniqueness above, we may say that there is a set of left-invariant vector fields on G, denoted by $\mathfrak{g}$. One can easily see that the map $T_eG \to \mathfrak{g}$ is an isomorphism and we conclude that the set of left-invariant vector fields isomorphic to T_eG

$$\mathfrak{g} \cong T_eG \tag{B.99}$$

and also $\mathfrak{g}$ is a vector space with the same dimension of G, $dim\mathfrak{g} = dimG$.

Consider two vector fields $X, Y \in \mathfrak{g}$. If we apply L_{a*} to the Lie bracket of two vector field,

$$L_{a*}\big[X,Y\big]\Big|_g = \Big[L_{a*}X\Big|_g, L_{a*}Y\Big|_g\Big] = \big[X,Y\big]\Big|_{ag} \tag{B.100}$$

is also an element of $\mathfrak{g}$, $\big[X,Y\big]\Big|_{ag} \in \mathfrak{g}$.

In the vector $X_V\Big|_g = gV$ is

$$L_{g*}V = gV$$

and

$$\big[X_V, Y_W\big]\Big|_g = L_{g*}\big[V,W\big] = g\big[V,W\big]. \tag{B.101}$$

Definition: Lie algebra of a Lie group G

$$\big[\quad , \quad\big] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \tag{B.102}$$

for the left-invariant vector fields $\mathfrak{g}$.

B.1.11 The one parameter subgroup:

Let us consider a curve on the manifold G.

$$\phi : \mathbb{R} \to G. \tag{B.103}$$

We can define a vector field X through the curve ϕ.

$$\frac{d\phi^\mu(t)}{dt} = X^\mu\big(\phi(t)\big). \tag{B.104}$$

Let us check that a vector field X is left-invariant. It is obvious that a vector field $\frac{d}{dt}$ is left-invariant vector field on $\mathbb{R}$,

$$(L_t)_* \frac{d}{dt}\Big|_0 = \frac{d}{dt}\Big|_t. \tag{B.105}$$

Using a diffeomorphism ϕ, one can define a push-forward map:

$$\phi_* : T_t\mathbb{R} \to T_{\phi(t)}G. \tag{B.106}$$

The pushed-forward vector fields are

$$\phi_* \frac{d}{dt}\Big|_0 = \frac{d\phi^\mu(t)}{dt}\Big|_0 \frac{\partial}{\partial g^\mu}\Big|_e = X\Big|_e$$

$$\phi_* \frac{d}{dt}\Big|_t = \frac{d\phi^\mu(t)}{dt}\Big|_t \frac{\partial}{\partial g^\mu}\Big|_g = X\Big|_g. \tag{B.107}$$

Using equation-(126) and the commutativity of the map $\phi L_t = L_g \phi$, equation-(129) becomes

$$\big(\phi L_t\big)_* \frac{d}{dt}\Big|_0 = \phi_* L_{t*} \frac{d}{dt}\Big|_0 = X\Big|_g$$

and

$$\phi_* L_{t*} \frac{d}{dt}\Big|_0 = L_{g*}\phi_* \frac{d}{dt}\Big|_0 = L_{g*}X\Big|_e = X\Big|_g. \tag{B.108}$$

As a result a vector field X through a curve ϕ is a left-invariant vector field, $X \in \mathfrak{g}$.

Definition: A curve $\phi : \mathbb{R} \to G$ is a one-parameter subgroup of G if it satisfies the following conditions:

i-) $\phi(t)\phi(s) = \phi(t, s)$,

ii-) $\phi(0) = e$,

iii-) $\phi^{-1}(t) = \phi(-t)$.

This one-parameter subgroup may be Abelian even if G is non-Abelian.

$$\phi(t)\phi(s) = \phi(t+s) = \phi(s+t) = \phi(s)\phi(t). \tag{B.109}$$

We can conclude that any left-invariant vector field $X \in \mathfrak{g}$ defines a one-parameter subgroup of G. To see this let us define the exponential map:

$$\exp : T_eG \to G, \tag{B.110}$$

$$\Rightarrow \quad \exp V \equiv \phi_V(1), \tag{B.111}$$

where $V \in T_eG$ and G is a Lie group and ϕ_V is a one-parameter subgroup of G that corresponds to the left-invariant vector field $X_V\Big|_g = L_{g*}V$. By using the exponential map a one-parameter subgroup $\phi_V(t)$, $t \in \mathbb{R}$, is constructed by $X_V\Big|_g = L_{g*}V$ as

$$\exp(tV) = \phi_V(t) = \mathcal{I}_n + tV + \frac{t^2}{2!}V^2 + \cdots + \frac{t^n}{n!}V^n + \ldots \tag{B.112}$$

where $V \in T_eG$.

B.1.12 Frames and Structure Equation:

Consider a Lie group G as an n-dimensional manifold. A basis at each point of the manifold can be constructed as

$$X_\mu\Big|_g = L_{g*}V_\mu. \tag{B.113}$$

Here $\{V_\mu\}$ is the basis at a point e, $V_\mu \in T_eG$ and $X_\mu\Big|_e = V_\mu$. $\{X_\mu\}$ are the n linearly independent left-invariant vector fields and it is defined at each point on the manifold G. Hence $\{X_\mu\}$ is called the frame of basis for G. We know that the Lie bracket of two basis vectors are again a left-invariant vector field $\big[X_\mu, X_\nu\big]\Big|_g \in \mathfrak{g}$. One can write

$$\big[X_\mu, X_\nu\big] = C_{\mu\nu}{}^\lambda X_\lambda, \tag{B.114}$$

where $C_{\mu\nu}{}^\lambda$ are the structure constants of the Lie group and using $C_{\mu\nu}{}^\lambda$, a Lie group is constructed.

After basis vector fields are determined, the dual basis of that can be introduced , $\{\theta_\mu\}$, $\langle \theta^\mu, X_\nu \rangle = \delta^\mu_\nu$. The basis left-invariant one forms satisfy the Maurer-Cartan's structure equation,

$$d\theta^\mu = -\frac{1}{2} C_{\nu\lambda}{}^\mu \theta^\nu \wedge \theta^\lambda. \tag{B.115}$$

Definition: A Lie-algebra-valued one-form is defined

$$\theta : T_g G \to T_e G, \tag{B.116}$$

by

$$\theta : X \mapsto \left(L_{g^{-1}}\right)_* X = \left(L_g\right)_*^{-1} X, \quad X \in T_g G, \tag{B.117}$$

where θ is called a canonical one-form or Maurer-Cartan form on G.

Theorem: Let $\{V_\mu\}$ is the basis of $T_e G$ and $\{\theta^\mu\}$ is the basis one-form of $T_e^* G$. The canonical one-form θ can be written

$$\theta = V_\mu \otimes \theta^\mu \tag{B.118}$$

and θ satisfies

$$d\theta + \frac{1}{2}\left[\theta \wedge \theta\right] = 0, \tag{B.119}$$

where

$$d\theta \equiv V_\mu \otimes d\theta^\mu, \tag{B.120}$$

and

$$\left[\theta \wedge \theta\right] \equiv \left[V_\mu, V_\nu\right] \otimes \theta^\mu \wedge \theta^\nu. \tag{B.121}$$

Proof: $Y = Y^\mu X_\mu \in T_g G$ and $\{X_\mu\}$ is the set of frames at the point $g \in G$, $X_\mu\Big|_g = L_{g*} V_\mu$

$$\theta(Y) = Y^\mu \theta(X_\mu) = Y^\mu \left(L_{g*}\right)^{-1}\left[L_{g*} V_\mu\right] = Y^\mu V_\mu. \tag{B.122}$$

Let us compare this result with the theorem $\theta = V_\mu \otimes \theta^\mu$

$$\left(V_\mu \otimes \theta^\mu\right)(Y) = \left(V_\mu \otimes \theta^\mu\right)\left(Y^\nu X_\nu\right) = Y^\nu V_\mu \theta^\mu(X_\nu) = Y^\nu V_\mu \delta^\mu_\nu = Y^\mu V_\mu$$

$$d\theta + \frac{1}{2}\left[\theta \wedge \theta\right] = 0.$$

Using the Maurer-Cartan's structure equation

$$d\theta = V_\mu \otimes d\theta^\mu = -\frac{1}{2} V_\mu \otimes C_{\nu\lambda}{}^\mu \theta^\nu \wedge \theta^\lambda,$$

$$\big[\theta \wedge \theta\big] \equiv \big[V_\mu, V_\nu\big] \otimes \theta^\mu \wedge \theta^\nu = C_{\mu\nu}{}^\lambda V_\lambda \otimes \theta^\mu \wedge \theta^\nu,$$

$$\Rightarrow d\theta + \frac{1}{2}\big[\theta \wedge \theta\big] = -\frac{1}{2} V_\mu \otimes C_{\nu\lambda}{}^\mu \theta^\nu \wedge \theta^\lambda + \frac{1}{2} C_{\nu\lambda}{}^\mu V_\mu \otimes \theta^\nu \wedge \theta^\lambda = 0. \quad \text{(B.123)}$$

B.1.13 The action of Lie groups on manifolds:

Let us assume that G is a Lie group and M is a manifold.

Definition:A differential map $\sigma : G \times M \to M$ is called an action of G on M if it satisfies the following conditions

i-) $\sigma(e,p) = p$, $\forall p \in M$,

ii-) $\sigma\big(g_1, \sigma(g_2,p)\big) = \sigma(g_1 g_2, p)$ Here notice that $\sigma(g_2,p) \in M$ is a point on the manifold.

Notation: $\sigma(g,p) = gp$ and $\sigma\big(g_1, \sigma(g_2,p)\big) = g_1(g_2 p) = (g_1 g_2)p$.

The action is called transitive if it satisfies

$$\sigma(g, p_1) = p_2, \quad \text{(B.124)}$$

where $p_1, p_2 \in M$ and $g \in G$. If the only element of G is the identity element, e, that satisfies the following

$$\sigma(g,p) = p, \quad \text{(B.125)}$$

then the action is called free.

Finally, if the trivial action on M is done by only the identity element

$$\sigma(g,p) = p \quad \forall p \in M \quad \Rightarrow g = e, \quad \text{(B.126)}$$

then the action is called an effective action.

B.1.14 Orbits and Isotropy groups:

Consider an action of the Lie group G on a manifold M.

$$\sigma : G \times M \to M. \tag{B.127}$$

Take an element of M, let say $p \in M$, the orbit of p is defined as a subset of M.

$$G_p = \{\sigma(g, p) | g \in G \quad \text{and} \quad p \in M\}. \tag{B.128}$$

Here notice that if the action is transitive then the orbit of $p \in M$ becomes M itself.

One of the special subgroups of G is important. If there is a point on M, $p \in M$, and the action leaves the point invariant such as

$$\sigma(g, p) = p, \tag{B.129}$$

then the isotropy group of $p \in M$ is defined by

$$H(p) = \{g \in G | \sigma(g, p) = p\}, \tag{B.130}$$

where $H(p)$ is also called the stabilizer or little group of p. A familiar example of the isotropy group is $SO(2)$ for the manifold $M = \mathbb{R}^3$ and $G = SO(3)$. For the point $p = (0, 0, 1) \in \mathbb{R}^3$ the isotropy group is the set of notations about the z-axis.

Theorem: Let M be a manifold and G is a Lie group that acts on M. The isotropy group $H(p)$ for any $p \in M$ is a Lie subgroup.

One of the important result of the isotropy group can be seen by contracting a coset space with a Lie group G and a subgroup $H(p)$. Now consider a Lie group G acting on a manifold M and an isotropy group $H(p)$. If the coset space $G/H(p)$ with the dimension, $dimG/H = dimG - dimH$, has certain requirements (e.g., $G/H(p)$ compact), $G/H(p)$ is homeomorphic to M.

B.1.15 Induced Vector Fields:

Consider a Lie group G and a manifold M. G is a manifold itself. Taking an element from the manifold G, e.g., $V \in T_eG$ and generated left-invariant vector field X_V by

V, one can induce a vector field in M with the help of action of G on M. The flow in M can be generated by the action

$$\sigma(t,x) = \exp(tV)x \quad , \quad x \in M, \tag{B.131}$$

and it is obvious that σ is a one-parameter group of transformations.

Definition: The induced vector field is defined

$$V^{\#}\Big|_x = \frac{d}{dt}\exp(tV)x\Big|_{t=0}, \tag{B.132}$$

by the map $\# : T_eG \to \mathcal{X}(M)$.

B.1.16 The adjoint representation:

Definition: Let G be a Lie group. The adjoint representation of G is defined, $a \in G$

$$ad_a : g \mapsto aga^{-1} \tag{B.133}$$

by the homomorphism $ad_a : G \to G$.

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PUBLICATION

- *"Bachian Gravity in Three Dimensions"*. Phys. Rev. D **98**, 104021 (2018). Gökhan Alkaç, Mustafa Tek and Bayram Tekin.

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