AN AFFINE TERM STRUCTURE MODEL FOR TURKISH INTEREST RATE SWAP MARKET: DO SWAPS SPAN VOLATILITY RISK?

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF APPLIED MATHEMATICS
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

ALİ GENCAY ÖZBEKLER

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF MASTER OF SCIENCE
IN
FINANCIAL MATHEMATICS

AUGUST 2019

Approval of the thesis:

AN AFFINE TERM STRUCTURE MODEL FOR TURKISH INTEREST RATE SWAP MARKET: DO SWAPS SPAN VOLATILITY RISK?

submitted by **ALİ GENCAY ÖZBEKLER** in partial fulfillment of the requirements for the degree of **Master of Science in Financial Mathematics Department, Middle East Technical University** by,

Prof. Dr. Ömür Uğur
Director, Graduate School of **Applied Mathematics** __________

Prof. Dr. Ayşe Sevtap Kestel
Head of Department, **Financial Mathematics** __________

Prof. Dr. Ömür Uğur
Supervisor, **Scientific Computing, METU** __________

Dr. İrem Talaslı
Co-supervisor, **Central Bank of the Republic of Turkey** __________

Examining Committee Members:

Assoc. Prof. Dr. Ceylan Talu Yozgatlıgil
Department of Statistics, METU __________

Prof. Dr. Ömür Uğur
Scientific Computing, METU __________

Assoc. Prof. Dr. İbrahim Ünalmış
Department of Business Administration, TED University __________

Date: __________

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Last Name: ALİ GENCAY ÖZBEKLER

Signature :

ABSTRACT

AN AFFINE TERM STRUCTURE MODEL FOR TURKISH INTEREST RATE SWAP MARKET: DO SWAPS SPAN VOLATILITY RISK?

Özbekler, Ali Gencay
M.S., Department of Financial Mathematics
Supervisor : Prof. Dr. Ömür Uğur
Co-Supervisor : Dr. İrem Talaslı

August 2019, 49 pages

We follow novel procedure of [4] to assess presence of unspanned stochastic volatility (USV) phenomenon in the Turkish lira interest rate swap (IRS) market. The estimations reveal that IRS yield curve dynamics fail to span volatility in IRS market and thus volatility risk cannot be hedged using only IRS instruments. The major implication of USV is then used to investigate the systemic volatility in domestic markets. In this scope, we employ USV condition as a specification for affine term structure (AFTS) models. Comparing AFTS models with stochastic and constant volatility, we find that three-factor constant volatility model provides more robust estimation results in terms of both volatility and yield fitting.

Keywords: Affine Term Structure Models, Term Premia, Stochastic Volatility, Systemic Volatility, Spanning Hypothesis

ÖZ

TÜRKİYE FAİZ SWAPI PİYASALARI İÇİN AFFİNE VADE PRİMİ MODELİ:
FAİZ TAKASI VOLATİLİTE RİSKİNİ HEDGE ETMEK AMAÇLI
KULLANILABİLİR Mİ?

Özbekler, Ali Gencay
Yüksek Lisans, Finansal Matematik Bölümü
Tez Yöneticisi : Prof. Dr. Ömür Uğur
Ortak Tez Yöneticisi : Dr. İrem Talaslı

Ağustos 2019, 49 sayfa

Bu çalışmada, [4] çalışmasında önerilen yöntem kullanılarak Türk lirası faiz swapı (IRS) piyasası için kapsanmamış rassal volatilite (USV) olgusu değerlendirilmiştir. Tahminler, IRS verim eğrisi dinamiklerinin IRS piyasasındaki oynaklığı kapsamadığı ve bu nedenle oynaklık riskinin yalnızca IRS enstrümanları kullanılarak "hedge" edilemeyeceğini ortaya koymaktadır. Ayrıca, USV sonuçları yurtiçi piyasalarda sistemik riskin varlığına ilişkin çıkarım yapabilmemize imkan sağlamaktadır. Bu kapsamda, USV koşulları afin vade yapısı (AFTS) modelleri açısından gösterge olarak kullanılmaktadır. Sonuçlar, rassal ve sabit volatilite AFTS modelleri karşılaştırıldığında üç-faktörlü sabit volatilite modelinin hem oynaklık hem de getiri tahmini açısından daha güçlü sonuçlar verdiğine işaret etmektedir.

Anahtar Kelimeler: Afin Vade Yapısı Modeli, Vade Primi, Rassal Oynaklık, Sistemik Oynaklık, Kapsama Hipotezi

ACKNOWLEDGMENTS

I would like to express my very great appreciation to my thesis supervisors Prof. Dr. Ömür Uğur and Dr. İrem Talaslı for their patient guidance, enthusiastic encouragement and valuable advices during the development and preparation of this thesis. Their willingness to give the time and to share their experiences has brightened my path.

I especially thank Ceylan Yozgatlıgil and İbrahim Ünalmış for being the examining committee members and providing their insightful comments and suggestions.

I would like to thank the Central Bank of the Republic of Turkey for allowing me the time for my MSc studies.

I am grateful to my family for their endless support for me to be who I am. They have never stopped believing in me. I wish to thank my wife Merve for her endless love and support. Everything is easier when she is with me.

TABLE OF CONTENTS

LIST OF TABLES

TABLES

LIST OF FIGURES

FIGURES

LIST OF ABBREVIATIONS

AFTS	Affine Term Structure
GLS	Generalized Least Squares
IRS	Interest Rate Swap
LIBOR	London Interbank Borrowing Rate
ODE	Ordinary Differential Equation
OLS	Ordinary Least Squares
PC	Principal Component
PCA	Principal Component Analysis
QV	Quadratic Variation
SDE	Stochastic Differential Equation
SDF	Stochastic Discount Factor
TRY	Turkish lira
USV	Unspanned Stochastic Volatility
VAR	Vector Auto-Regressive

CHAPTER 1

INTRODUCTION

The term structure of interest rates, which could be explained as the relationship between yields and maturities, draw great attention in economics and finance. Through studying the source of and the determinants of the term structure, economists try to provide a complete overview of interest rates. As a starting point, there has been strand literature accumulated in the construction of theoretical framework on term structure modeling. Since the 1930s, models with simple equilibrium structure to stochastic processes developed in defining the relationship between yields and their term to maturities.

One of the pioneering works that builds the theoretical framework of term structure modeling was proposed by Irving Fisher, in the scope of well-known expectations theory [28]. The expectations hypothesis points out that long term interest rates coincide with the expected short term rates through the investment horizon. Since expectations play a major role in the consistent term structure modeling of interest rates, even though expected short term rates deviate from its general trend, they tend to revert to their mean. The expectation hypothesis states that longer maturity rates are simply an average of expected future rates of the following subsequent periods. Thus, the hypothesis makes no distinction between different maturity assets in terms of their generated returns, and therefore no additional compensations, and default-free bonds are perfect substitutes on all maturities.

Another theory for the term structure of interest rates centers around the notion of rational expectations hypothesis ([49]) and states that market prices are efficient in incorporating economic agents' expectations concerning all available information

([47], [45], [56]). The foundation of rational expectations paves the way for employing martingale theory in the context of term structure modeling by [54] and usage of stochastic processes in the evaluation of spot rate modeling (in [22]) that became one of the main pillars of modern term structure models. In this framework, studies such as [56] and [31] apply rational expectations theory to the term structure of interest rates and find that moving averages representation of interest rates is fundamental and provides some evidence on interest rates to be linearly weighted combinations of different maturity rates. On the contrary, the expectations hypothesis fails to find empirical support in terms of constant risk and term premium and no uncertainty assumptions. Therefore, a term premium could appear, when there is a deviation from the hypothesis. Some preliminary studies such as [48] emphasize the importance of risk premiums by finding reasonable stable premiums on the medium-term interest rates with different maturities using the notion of Hicksian risk aversion. This notion is a well-known contribution of Hicks in [34], which is amended in a simple model in hypothesizing that the long-term rate tends to exceed the value implied by the average of expected future rates by a premium. Thus, differences in expected and actual returns could be interpreted as rewards for risks. Although, the premise models such as the liquidity preference hypothesis of John Hicks in [34] explains expected returns always increase with maturity and therefore the risk premium is hypothesized in an increasing pattern with maturity, [26] finds that the ordering of expected returns across maturities changes through time and this implies changes in the ordering of risks. This behavior of expected returns is inconsistent with previous simple term structure models. Moreover, the findings of [26] support that the expected term premium is time-varying and its sign is subject to the changes of the respect to economic cycles: positive in boom whereas negative in recession periods.

Actually, the term premium literature is also benefited from the liquidity preference hypothesis of Hicks (in [34]) and the market segmentation hypothesis of Culbertson (in [20]). The liquidity preference hypothesis asserts that premium would appear because future rates are not known with certainty and thus the actual yield of long-term securities is uncertain and the holders of securities would require compensation for bearing the uncertainty risk given the risk aversion. On the other hand, the market segmentation hypothesis contends the Hicksian theory by stating that the shorter and longer maturity markets are essentially independent, which implies that the market for bonds of different maturities is isolated from each other, so that the explanations using expectations are theoretically unsatisfactory and lack of empirical relevance. Although, both the liquidity preference hypothesis and the market segmentation hypothesis are empirically rejected by successive studies (see [45]) and thus [26] attempts to validate or to deny those hypotheses led the accumulation of strand literature in term structure models.

The term structure of interest rates is important for economic agents in representing market pricing of risk and general expectations on the economy regarding different time horizons. The interest rates in an economy are anchored by central banks or monetary authorities using short rates. Thus, the longer rates are out of the direct control horizon of monetary policy through the transmission mechanism. Since uncertainty regarding economic conditions usually increases with maturity, longer-term interest rates are subject to risks stem from this uncertainty. Therefore, it is important to assess the dynamics for term structure, roughly a compensation for risk, for all economic agents in terms of whether policymaking or portfolio management purposes.

The policymakers use term premium models as an input for their decision-making process in terms of market perception of risk and economic prospects. From the perspective of central banking, the term premium carries expectations on future short term rates and thus implies the expected monetary policy stance. Also, the term premium is essential for treasury financing operations while determining the financing strategy and the balance between short and long term issuances. Besides, evaluating the term structure is important for investors and portfolio managers in making investment decisions. Investors generally use interest rate contingent assets, such as derivatives, in hedging their portfolio risks. Thus, it is important to illuminate the future path of short term rates for hedging strategies.

The term structure of interest rates could be obtained by comparing realized long term rates and its corresponding expected forward short term rates. Even though, there are a plethora attempts in decomposing yield curves' factors including term premium, the analysis of term premium is not straightforward since term premium is not directly observable nor is it measurable and it requires some assumptions to be extracted from the market rates (in [40]).

In the literature, the term structure of interest rates is exploited to evaluate yield curve movements in an arbitrage-free framework. No arbitrage property in finance necessitates the prices of securities to be equal when their level of risks are the same. Thus, the pricing procedure of fixed income securities could be simplified with the assumption of the no-arbitrage condition in a predetermined volatility and correlation constraints across the cross-section of yields. The recent arbitrage-free term premium models could be classified under two major categories: the affine terms structure models and the macro-finance models. In the affine approach, the yield curve is assumed to be affected by a set of unobservable factors and the framework satisfies no-arbitrage conditions. Whereas, macro-finance models emerge from macroeconomics, which takes into account the effect of economic conditions in factor modeling. Although both models require yields to be affine functions of sets of risk factors which are assumed to have Gaussian distributions, so-called affine Gaussian term structure models, some of the studies perceive risk factors as latent variables ([24], [21]) and others use macroeconomic variables in addition to unobservable risk factors ([7], [1])).

In this context, according to their risk factors' specifications, term structure models can be classified into two groups. Firstly, affine term structure models imply that bond yields are a linear function of a set of risk factors, and it is accepted models that rely exclusively on yields as input data in the scope of this paper. Although affine models are computationally costly, they have become popular because of their analytical tractability. Affine models are developed as an eclectic model in the finance literature. After the first use of martingale property ([54]) and stochastic processes ([22]) in the context of interest rate models, following the development of similar argument for option pricing by the seminal paper of Black and Scholes ([11]), Vasicek in [57] characterizes the term structure of interest rates in an efficient market framework. [57] models the term structure as an affine function of spot short rate factor, a single factor, in a Markov setting while conforming the dominating theories of the expectation hypothesis, the liquidity preference hypothesis and the market segmentation hypothesis at that time. Also, [57] defines the difference between the forward rates and expected spot rates as the liquidity premium, while this is referred to as the term premium by [50]. The single factor model implies that longer-maturity yields are determined by the average of short term yields and the risk premium, which hinges on the variance of instantaneous rate and thus the market price of risk ([51]). The market price of risk is the marginal compensation for bearing an additional unit of risk. In single-factor models, bond prices depend only on a single short rate and as a consequence the price changes are perfectly correlated across the maturities, which led a short term rate to completely span whole term structure and bond prices are related to the path of interest rates ([57], [17]). Cox et al. (1985) in [17] overcome those issues by extending the affine structure to a two-factor squared Gaussian model framework while asserting that the mean and variance is proportional to the state vector for a given stochastic process. Due to the increased popularity of interest rate contingent securities, several specifications are proposed for modeling term structure. Heath, et al. in [32] exhibit a multifactor model, which provides consistent structure with the observed term structure of interest rates and any specified volatility pattern.

Also, [35] presents a term structure model with a time-varying drift parameter by extending fundamental one-factor models. Most of the contemporary affine term structure models benefit from the complete characterization of [24]. In [24], it is suggested that models with affine bond yields by presenting a consistent and arbitrage-free multifactor model of the term structure of interest rates in which factors, both the factors extracted from market yields and unobservable latent factors, follow stochastic Markov process with stochastic volatility and jump-diffusion state processes. The direction, which is paved by [24], is further extended by [21]. The structural differences and relative goodness-of-fits of affine term structure models are explored in their seminal study. [21] posits that there is a tradeoff between flexibility in modeling the conditional correlations and volatilities of the risk factor within the family of affine models. They also characterize the affine specifications by diversifying multifactor affine models with augmented stochastic volatility parameters. Although one of the most striking features of affine term structure models is their analytical tractability, it comes with a cost of strict linear restrictions. Recent advances help to ease some of the restrictive assumptions, even if there is plenty of space open to further exploration in the term premium literature (see [38], [58] and [12]).

The second major group of term structure models is arranged under the macro-finance group, in which macroeconomic variables and financial market indicators are used in addition to the latent risk factors. Since [26] argued that the expected term premium is time-varying and could be related to the economic cycles, there has been strand literature accumulated to expose the determinants of term premium and yield curve structure. In their seminal paper, [7] employs a special case of [24] in a vector autoregressive (VAR) model. They incorporate macroeconomic variables as risk factors such that the pricing kernel is represented as a factor in the model and find that those macro variables can explain 85% of the total variation in yields. Some increasingly popular models attempt to make a connection between the joint evaluation of yields including macroeconomic and financial variables in affine modeling framework (see [8], [58], [30] and [19]). In macro-finance models, some papers suggest that expectations play a major role in term premium models and therefore market players' consensus on several macroeconomic indicators may be related to term premium developments. While the results are more than sufficient to support the hypothesis, they also encourage further development in the literature of the term structure of interest rates ([37], [38] and [53]).

In addition to term structure models, various studies attempt to identify the main factors that impact interest rates. For instance, to explain the variation observed in returns Litterman and Scheinkman (1991) in [42] employ an alternative approach to determine common factors and claim that most of this variation is attributable to yield curve factors, namely level, steepness, and curvature. Since the term structure models are accepted to incorporate sufficient information to complete fixed income markets, this invokes the questioning of the spanning hypothesis which states that the yield curve contains all of the necessary information for forecasting or estimating yields. The notion of spanning hypothesis is also exploited to assess whether volatility spans by the yield curve factors. The relationship between yield variation and yield curve factors nest important implications. If common factors of yields can explain a significant part of the yield volatility, one can hedge its portfolio against volatility using solely those securities. Thus, it is crucial to be investigated not only for portfolio management but also for market monitoring purposes.

In this thesis, we firstly address the presence of volatility spanning conditions in Turkish lira interest rate swap (IRS) returns following the methodology of [4]. They propose a procedure, where quadratic variation could be approximated by realized volatility in a relatively higher frequency such as intraday and daily frequency. Thus, the volatility spanning hypothesis could be tested using the factors and volatility by employing realized volatility data. The most striking result of volatility spanning hypothesis is that since fixed income markets are assumed to be complete due to incorporating all the available information, if variation in the swap rates are explained by yield curve factors, meaning that the IRS market is complete, and the volatility can be hedged using only the IRS instruments. Our empirical results show that volatility does not span by the IRS market and the market is incomplete. This phenomenon is called as unspanned stochastic volatility (USV) by [15]. The results are consistent with the existing literature ([15] and [4]) in addressing USV in fixed income markets. Hence, we propose a new indicator for systemic volatility for the Turkish lira IRS market using the residuals from the spanning hypothesis. The proposed systemic volatility indicator could be used to assess the vulnerability of the market to possible sudden shocks.

We also introduce two different affine term structure models: one with a stochastic volatility ([18]) and one is with constant volatility ([59]) component. Then, we use the USV phenomenon as a specification test for model to suggest and find out that the affine term structure model without stochastic volatility provides a more robust and reliable results in representing realized rate dynamics.

The plan of our paper is structured as follows. In Chapter 2, we introduce spanning hypothesis approachs and primarily focus on unspanned stochastic volatility studies. In Chapter 3, we provide methodological information regarding affine term structure models with and without stochastic volatility and USV testing procedure.Also, information regarding data and the method that we use in estimating volatilities are explained, in Chapter 3. Then, we give empirical results on realized data and affine term structure fitted data on USV procedure in Chapter 4. In Chapter 5, we give a brief conclusion.

CHAPTER 2

SPANNING HYPOTHESIS

Financial market participants attempt to identify the main factors that impact interest rates. In their seminal paper of [42], Litterman and Scheinkman (1991) apply an alternative approach in determining the common factors, which explain the variation in US Treasury bond returns and find out that most of the variation could be attributable to yield curve factors, namely level, steepness, and curvature. Since the term structure models are accepted as incorporating sufficient information to complete fixed income markets, this raises the spanning hypothesis: the yield curve contains all the necessary information for forecasting or estimating yields. Following this assumption, there appears the need to extract information from the cross-section of returns. In this framework, the principal component analysis (PCA) is used to find out the common factors of the yield curve. PCA is a dimension reduction technique, which is employed to analyze and explore large data sets easier and reduced dimensions are representing most of the information while all components are orthogonal, linearly independent, to each other. Although the principal components are extracted in a mechanical way, the first three principal components of the cross-section of yields correspond to the level, steepness, and curvature of the yield curve in order. Thus, principal components are reliable indicators in representing information regarding the yield curve dynamics.

2.1 Unspanned Macroeconomic Variables

The term structure models are accepted as incorporating sufficient information to complete fixed income markets. Then, affine models using principal components as yield curve factors are contributed the popularity of term structure models due to their convenience in practice (see [24], [25], [21] and [52]). Those advances in the yield curve and term premium literature, help the rise of spanning hypothesis; that is, yield curve contains all the information for forecasting or estimating yields. This notion evokes the spanning hypothesis to become a benchmark model for testing the relevance of not only yield curve dynamics but also macroeconomic factors in extracting contemporaneous relationships and prediction of fixed income yields. This implication led to a pile of macro-finance affine term structure models ([7], [8], [6], [55] and [10]) in which macroeconomic variable factors are spanned by the yield curve.

On the contrary, this macro-spanning structure posits that strong and counterfactual restrictions on the joint distribution of bond yields and the macroeconomic indicators and the relationship between macro-variables and term premiums ([36]). Thus, there is a growing consensus on questioning spanning hypothesis; whether those macro-finance models are successful improving output in addition to yield curve dynamics (see [44], [29], [36], [14] and [23]) states that the cross-section of yields fail to explain almost half of the variation in the bond risk premia and only a limited part of unexplained variation could be detected by macroeconomic factors. Similarly, [36] investigated the impact of unspanned macro risks on bond risk premiums and find out that unspanned real economic activity and inflation have a detrimental effects on term premiums but not on the bond pricing. Thus, they develop a canonical affine model such that macro factors are not perfectly spanned by yield curve dynamics.

2.2 Unspanned Stochastic Volatility

The spanning hypothesis is also used for assessing the completeness of a fixed income market in terms of volatility forecasting and volatility hedging purposes. Under complete market conditions of the term structure models, fixed income securities are assumed to be redundant ([46]) and they could also be used to hedge volatility risk. As it is emphasized by [4], the major property of affine term structure models is that quadratic variation of bond yields of each maturity are linear combinations of the term structure of yields. Thus, the volatility of the yields could be hedged solely through bonds.

First of all, [43] examined the level of interest rate volatility and the shape of the yield curve. They discuss a model in which the shape of the yield curve is measured by yield spreads of butterflies, which is the curvature factor, and they exhibit a strong link with volatility. Similarly, [27] finds out that swaption and swap straddle positions could be hedged solely by LIBOR bonds, therefore this concludes completeness in the bond market. On the contrary, [5] posits that volatility risk cannot be hedged via trading bonds only. Also, Collin-Dufresne and Goldstein (2002) in [15] find that using swap rates, caps and floors, there is a weak linkage between changes in swap rates and at-the-money straddles. Their results suggest that there is at least one state variable which affects the volatility but not rate innovations. Thus, portfolios are exposed to volatility risk and swap markets are found to be incomplete ([15] and [39]). Then, they adapt equity-derivative literature, where volatility risk cannot be hedged solely by underlying stock ([33]), to fixed income derivatives and permit fixed-income derivatives to be non-redundant securities by specifying joint dynamics of rates and state variables that drive volatilities. This restriction is called unspanned stochastic volatility (USV). They identify a class of affine models that can exhibit USV. However, even if the affine framework could provide closed-form solutions for bonds, bond yields do not relate to volatility state. As a consequence, bond prices are not sufficient to hedge volatility risk nor do they identify affine class models with USV restrictions In this setting both bonds and derivatives are required for hedging against the volatility.

The novel procedure in [15] reveals that only a limited portion of straddle returns could be explained by the changes in the term structure of swap rates and most of this unexplained portion of straddle returns could be represented by a single factor. Thus, the risk factor, which is a deterministic volatility indicator, does not relate to interest rate changes. This outcome could be generalized to that fixed income markets are incomplete. Although in most of the affine term structure models ([24]) spot rate is a function of unobserved variables and implies that bonds are sufficient to complete fixed income markets, [15] proposes a restriction that yields do not depend on volatility variables. In this case, such securities are not redundant and cannot be used to hedge volatility. Similarly, [41] finds out that USV is valid for the cap market in a quadratic term structure model. [4] highlights the spanning conditions for conditional variances and propose a framework for instantaneous volatility and quadratic variation. Their derivations have implications in enabling discrete-time data in testing the contemporaneous affine yield variation spanning condition at a reasonably high frequency (daily or intraday). Then, the semi-martingale property of bond prices under physical probability measure implies that changes in yields over relatively short periods could be remissible in their contributions to instantaneous yield variation and thus quadratic variation. Thus, we can use realized volatility as a consistent approximate of quadratic variation. In this framework, [4] concludes that the US Treasury bond market curve fails to span realized volatility and the bond market is incomplete in hedging volatility risk.

Despite the fact that USV models are resolving the unfitting in the broad class of affine models, there is a tradeoff between the accuracy of the affine model and its representation power of volatility. [18] finds out that spanned models capture the cross-section of yields well but not the volatility while unspanned models fit volatility at the expense of a poorer fitting to the cross-section of yields. Thus, the practitioner should make a decision between the accuracy of the yield curve and volatility modeling purposes.

CHAPTER 3

METHODOLOGY

3.1 Testing Volatility Spanning Condition

In this thesis, the volatility span hypothesis of fixed income securities is tested using realized volatility of yields following [4].

The fundamental structure of affine term structure models states that short term rates are an affine function of latent and/or observed factors ([24]). Let $y_0(t)$ be the short term rate and $X(t)$ be the state variable. Then,

$$y_0(t) = \delta_0 + \sum_{i=1}^{N} \delta_i x_i(t) = \delta_0 + \delta_X^{'} X(t). \tag{3.1}$$

Following [17], it is assumed that risk-neutral probability measure Q, the state vector X is governed by the following stochastic differential equation (SDE):

$$dX(t) = \kappa(\Theta - X(t)) + \Sigma\sqrt{S(t)}dW^Q(t), \tag{3.2}$$

where $S(t)$ is an affine function of state variables, $S(t) = \alpha_i + \beta_i^{'} X(t)$, where α_i is a scalar and β_i is an $N \times 1$ vector; $S(t)$ indicates that SDE has not only stochastic drift but also stochastic diffusion parameter Σ. In this setting zero-coupon bond prices at time t with maturity τ could be obtained using (3.3):

$$P(t, \tau) = \exp(A(\tau) - B(\tau)^{'} X(t)), \tag{3.3}$$

where $A(\tau)$ is a scalar function and $B(\tau)$ is $N \times 1$ vector of functions solving the ordinary differential equations (ODE).

The relationship between bond prices and yield to maturity could be expressed as $P(t, \tau) = \exp(-\tau y_\tau(t))$, which in turn could be used to make connection with (3.3) that

$$y_\tau(t) = -\frac{A(\tau)}{\tau} + \frac{B(\tau)^{'}}{\tau}X(t). \tag{3.4}$$

For $J \times 1$ vector of zero coupon yields, we assume that the number of state variables, N, are less than the observed bond yields, $J \geq N$. Simply, (3.4) can be transformed as a system of equations:

$$Y(t) = -A + B^{'}X(t), \tag{3.5}$$

where $N \times 1$ vectors $Y(t) = \left\{y_{\tau_j}(t), j = 1, ..., J\right\}$, $A = \left\{\frac{A(\tau_j)}{\tau_j}, j = 1, ...J\right\}$ and $N \times J$ matrix $B = \left\{\frac{B(\tau_j)}{\tau_j}, j = 1, ...J\right\}$. In the popular affine model of [24], B is taken as full-ranked, and thus all factors are assumed to have contemporaneous relationship with the bond prices. We can transform (3.5) to

$$X(t) = (BB^{'})^{-1}B(Y(t) + A) = C + (BB^{'})^{-1}BY(t), \tag{3.6}$$

where we define $C = (BB^{'})A$.

Also, using Ito's Lemma and (3.4), the SDE of the yield process y_τ follows a process with drift and diffusion. Since our aim is to focus on volatility, drift parameter is assumed to have the form $\mu_{y_\tau} = \mu_{y_\tau}\big(X(t), t\big)$ and hence,

$$dy_\tau(t) = \mu_{y_\tau}(X(t), t)dt + \frac{B(\tau)^{'}}{\tau}\Sigma\sqrt{S(t)}dW^Q(t). \tag{3.7}$$

By the diffusion parameter, it is easy to obtain instantaneous variation of yield, namely instantaneous volatility:

$$V_{y_\tau}(t) = \frac{B(\tau)'}{\tau}\Sigma S(t)\Sigma'\frac{B(\tau)}{\tau}. \tag{3.8}$$

[4] posits that since the affine model follows a stochastic volatility pattern, $S(t)$ matrix is also an affine function of the state vector $X(t)$. In a similar fashion, (3.6) implies that $X(t)$ is also an affine function of $Y(t)$. Thus, it is convenient to assert that instantaneous volatility is an affine function of the yields in the following form,

$$V_{y_\tau}(t) = a_{\tau,0} + \sum_{j=1}^{J} a_{\tau,j}y_{\tau_j}(t). \tag{3.9}$$

Instantaneous variation of yields could be mapped to contemporaneous cross section of yields in an affine setting as it is represented in (3.9). This relationship is viable across the maturity spectrum and is called as the volatility spanning condition. [4] asserts that this volatility spanning condition is valid for both affine and quadratic term structure models. Since, quadratic models and affine models are isomorphic to each other in terms of generating volatility and in case of extended state vectors quadratic term structure models could be incorporated by affine models ([2]). Thus, allowing sufficient number of state vectors in turn led [4] to cover quadratic models.

This instantaneous volatility framework enables us to move from discrete frequency basis to test volatility spanning condition in high frequency setting. The quadratic variation (QV) process of observed yields can be defined as:

$$QV_{y_\tau}(t) = \int_0^t V_{y_\tau}(s)ds. \tag{3.10}$$

Then, the model can be reduced to allow testing daily or intraday time increments. We will let $h > 0$ be an intraday time increment in the period of $[t+h, t]$. The quadratic variation between the time period could be approximated as $QV_{y_\tau}(t+h, h) = QV_{y_\tau}(t+h) - QV_{y_\tau}(t)$ and $\bar{y}_{\tau_j}(t) = \frac{1}{h}\int_y^{t+h} Y_{\tau_j}(s)ds$. Therefore, (3.9) becomes:

$$QV_{y_\tau}(t+h, h) = a_{\tau,0} + \sum_{j=1}^{J} a_{\tau,j}\bar{y}_{\tau_j}(t+h, t). \tag{3.11}$$

Here (3.11) defines the contemporaneous affine yield variation spanning condition. The right side of (3.11) is said to be approximated by intraday or at least frequency closing yields in daily basis. Since, the quadratic variation cannot be directly observed in bond prices, most of the previous studies ([15], [39] and [41]) tried to expose this relationship using volatility embedded derivative prices. However, [4] proposes a proxy for quadratic variation that could be obtained from observed prices and yields.

[4] highlights the spanning condition for conditional variances and propose a framework for instantaneous volatility and quadratic variation. Their derivations have implications in enabling discrete time data in testing the relation contemporaneous affine yield variation spanning condition at a reasonable high frequency (daily or intraday). Then, the semi-martingale property of bond prices under physical probability measure implies that the changes in yields over relatively short periods, as dt^2 tends

to zero, could be remissible in their contributions to instantaneous yield variation and thus quadratic variation. Therefore, we can approximate the realized volatility with quadratic variation, using bond prices' martingale approximation property under physical probability. This relation enables us to test spanning condition using realized volatility.

The approximate martingale relationship of bond prices under P-measure is

$$\mathbb{E}_t^P\left[y_\tau(t+\frac{ih}{n})\right] = y_\tau(t), \quad i = 1, ...n \tag{3.12}$$

So that the conditional variance could be given by

$$Var_t^P\big[y_\tau(t+h)\big] = \mathbb{E}_t^P\Bigg[\sum_{i=1,\ldots,n}\bigg(y_\tau\big(t+\frac{ih}{n}\big) - y_\tau\big(t+\frac{(i-1)h}{n}\big)\bigg)^2\Bigg]. \tag{3.13}$$

Note that (3.13) is assumed to hold for any n, thus as n goes to infinity variance equation approximate to the quadratic variation. This relationship can be interpreted as

$$Var_t^P\big[y_\tau(t+h)\big] = \mathbb{E}_t^P\big[QV_{y_\tau}(t+h,h)\big]. \tag{3.14}$$

Hence, by combining equation (3.11) and (3.14), we obtain

$$Var_t^P\big[y_\tau(t+h)\big] = a_{\tau,0} + \sum_{j=1}^{J} a_{\tau,j}\mathbb{E}_t^P\big[\bar{y}_{\tau_j}(t+h,t)\big]. \tag{3.15}$$

This approximation is assumed to be valid only under P-measure. It should be highlighted that the moments of the estimation under P-measure and Q-measure would be subject to some disparity over discrete steps that are defined as intraday or daily time intervals. This discrepancy is the major result of differences in the diffusion setting of two measures.

By the assumption of affine representation under P-measure, we can present predictive yield variation spanning condition. Since future expected yields can be interpreted as a linear function of the current level of yields, we get

$$Var_t^P\big[y_\tau(t+h)\big] = b_{\tau,0} + \sum_{j=1}^{J} b_{\tau,j}\bar{y}_{\tau_j}(t). \tag{3.16}$$

Although conceptually (3.16) seems to be a robust predictor of yield variation, it should be noted that the test result will not be as powerful as (3.11). Since expectations on future volatility is a conditional and an ex-ante concept, as being forward looking in its nature, the left hand side of (3.16) is an ex-post measure, namely realized, and subject to a larger innovation component.

3.1.1 Measuring Realized Volatility

The contemporaneous affine yield variation spanning condition requires us to use intraday or daily realized volatility in the testing procedure. We follow a model-free approach in estimating the realized volatilities in the scope of volatility span hypothesis. The variation is computed basically by taking sum of squared changes in predetermined discrete periods. For illustrative purposes, we annualize those daily realized volatility by taking 252-working day calendar year and obtain

$$v^2_{y_\tau}(t+h,h) = \frac{1}{h}\sum_{i=1}^{n}\left(y_\tau\left(t+\frac{ih}{n}\right) - y_\tau\left(t+\frac{(i-1)h}{n}\right)\right)^2. \tag{3.17}$$

Here, we take sampling frequency as $\frac{h}{n}$ for realized volatility y_τ in the period $[t, t+h]$. As emphasized before, realized volatility converges to quadratic variation as discrete time intervals become smaller. Hence, using realized intraday volatility measure we construct (3.11) to test volatility spanning conditions.

3.1.2 Testing Contemporaneous Volatility Spanning Condition

The volatility spanning condition constructed in (3.11) implies that using a linear regression the condition could be tested by using

$$QV_{y_\tau}(t+h,h) = a_{\tau,0} + \sum_{j=1}^{J} a_{\tau,j}\bar{y}_{\tau_j}(t+h,t) + \epsilon(t+h,h), \tag{3.18}$$

where $\epsilon(t+h,h)$ denotes the error term. In a hypothetical case, where there exist no measurement errors the residual term tends to be zero, but in practice measurement of yields, the transformation of yields to zero-coupon returns and the calculation of instantaneous volatility are subject to error. This is why there appears the residual term $\epsilon(t+h,h)$ in the equation (3.18).

As shown in the previous sections, we could use realized volatility in the approximation of quadratic variation of yields. Also, yield levels across different maturities tend to have quite high correlation; thus, analyzing yields contemporaneous impact on volatility indicator is embedded to violate fundamental assumptions of linear regressions. To alleviate this problem, we apply one of the most common dimension reduction techniques, the principal component analysis. The principal components are mechanically linearly independent from each other and as a result of this property, we can have robust estimations. Although [4] employed the first six principal components in analyzing US treasury bonds' volatility spanning condition, we prefer to use only the three principal components; those are accepted as bearing level, steepness and curvature factors of the yield curve.

$$v_{y_\tau}^2(t,h) = \beta_0 + \sum_{k=1}^{3} \beta_k PC_k(t,h) + \epsilon(t,h), \tag{3.19}$$

where, $\epsilon(t,h)$ denotes the error term.

Least squares model is given by (3.19) corresponds to the volatility spanning regression. The linear combination of yields is assumed to cover most of the information that is incorporated by the cross section of yields. While, (3.19) uses yields in volatility spanning hypothesis, we prefer (3.19) in our estimation due to reduced dimensionality of cross sections and also, some of the statistical problems, such as multicollinearity, is resolved by using mechanically orthogonal principal components. Thus, we employ (3.19) in testing the contemporaneous volatility spanning condition.

3.2 Affine Term Structure Models

In this thesis, we follow a new estimation procedure proposed by Creal and Wu (2015) in [18] for affine term structure models. Their method is flexible to use for both spanned and unspanned stochastic models. The USV models impose the restriction that yields do not depend on volatility factors ([15]). Even though spanned models solely require an optimization of the parameter space, estimation of unspanned models are more demanding due to unknown closed form likelihood functions. This problem of USV models is resolved by [3]. In their seminal paper, a new method is developed for estimating closed form maximum likelihood estimators. Also, [18] proposes the expected maximization algorithm should be used in affine term structure estimation procedure.

First of all, [18] depicts an affine term structure model with stochastic volatility similar to the studies such as [24] and [21], which is also sufficient to cover unspanned models. The model is described as covering $X^1 \times 1$ vector of state variables x^1 that have Gaussian distribution and its diffusion parameter consists of $X^2 \times 1$ vector of state variables x^2. In this model, state variables follow vector autoregressive (VAR) model with conditional heteroscedasticity.

$$\begin{aligned} x^1_{t+1} = \mu^Q_{x^1} + \Phi^Q_{x^1} x^1_t + \Phi^Q_{x^1 x^2} x^2_t + \sum_{x^1 x^2} \xi^Q_{x^2,t+1} + \xi^Q_{x^1,t+1}, \\ \xi^Q_{x^1,t+1} \sim N\big(0, \Sigma_{x^1} \Sigma^{'}_{x^1}\big), \\ \Sigma_{x^1} \Sigma^{'}_{x^1} = \Sigma_{0,x^1} \Sigma^{'}_{0,x^1} + \sum_{i=1}^{X^2} \Sigma_{i,x^1} \Sigma^{'}_{i,x^1} x^2_{i,t}, \\ \xi^Q_{x^2,t+1} = x^2_{t+1} - \mathbb{E}^Q(x^2_{t+1} | \mathcal{L}_t), \end{aligned} \tag{3.20}$$

where $\mathcal{L}_t$ captures all available information at time t. Following [17] the state factor variables for diffusion are affine transformation of the multivariate process, which can be stated as

$$x^2_{t+1} = \mu_{x^2} + \Sigma_{x^2} \omega_{t+1}, \tag{3.21}$$

$$\omega_{i,t+1} \sim Gamma\big(v^Q_{x^2,i} + z^Q_{i,t+1}, 1\big), \quad i = 1, ..., X^2, \tag{3.22}$$

$$z^Q_{i,t+1} \sim Poisson\big(e^{'}_i \Sigma^{-1}_{x^2} \Phi^Q_{x^2} \Sigma_{x^2} \omega_t\big), \quad i = 1, ..., X^2. \tag{3.23}$$

In the distribution of $z^Q_{i,t+1}$ in (3.23), $e^{'}_i$ denotes transpose of the ith column of the $X^2 \times X^2$ identity matrix I_{X^2}.

Since we handle the model using zero coupon rates, we can obtain the bond price by discounting the short rate as follows:

$$P_t^n = \mathbb{E}^Q\big[exp(-r_t)P_{t+1}^{n-1}\big].$$

By the affinity assumption, the short term interest rate is the linear function of factors:

$$r_t = \delta_0 + \delta_{1,x^2}^{'} x_t^2 + \delta_{2,x^2}^{'} x_t^1.$$

Hence, bond prices are exponentially affine functions of state variable under Q-measure are determined by

$$P_t^n = exp\big(\bar{a}_n + \bar{b}_{n,x^2}^{'} x_t^2 + \bar{b}_{n,x^1}^{'} x_t^1\big).$$

The bond loadings following [18] are then given by:

$$\begin{aligned}\bar{a}_n = -\delta_0 + \bar{a}_{n-1} + \mu_{x^1}^{Q'}\bar{b}_{n-1,x^1} + \big[\mu_{x^2} + \Phi_{x^1}^{Q}\mu_{x^2} + \textstyle\sum_{x^2} v_{x^2}^{Q}\big]'\bar{b}_{n-1,x^2}^{'} + \\ \frac{1}{2}\bar{b}_{n-1,x^1}^{'}\textstyle\sum_{0,x^1}\sum_{0,x^1}^{'}\bar{b}_{n-1,x^1} + \\ \mu_{x^2}^{'}\Phi_{x^2}^{Q'}\textstyle\sum_{x^2}^{-1'}\left(I_{X^2} - \big[\mathrm{diag}(\iota_{x^2} - \sum_{x^2}^{'}\bar{b}_{n-1,x^1x^2})\big]^{-1}\right)\sum_{x^2}^{'}\bar{b}_{n-1,x^1x^2} - \\ v_{x^2}^{Q'}\big[\log(\iota_{x^2} - \textstyle\sum_{x^2}^{'}\bar{b}_{n-1,x^1x^2}) + \iota_{x^2} - \sum_{x^2}^{'}\bar{b}_{n-1,x^1x^2}\big],\end{aligned} \tag{3.24}$$

and

$$\begin{aligned}\bar{b}_{n,x^2} = -\delta_{1,x^2} + \Phi_{x^1x^2}^{Q'}\bar{b}_{n-1,x^1} + \Phi_{x^2}^{Q'}\bar{b}_{n-1,x^2} + \\ \frac{1}{2}\big(I_{X^2}\bigotimes \bar{b}_{n-1,x^1}^{'}\big)\textstyle\sum_{x^1}\sum_{x^1}^{'}\big(\iota_{X^2}\bigotimes \bar{b}_{n-1,x^1}^{'}\big) - \\ \Phi_{x^2}^{Q'}\textstyle\sum_{x^2}^{-1'}\left(I_{X^2} - \big[\mathrm{diag}(\iota_{X^2} - \sum_{x^2}^{'}\bar{b}_{n-1,x^1x^2})\big]^{-1}\right)\sum_{x^2}^{'}\bar{b}_{n-1,x^1x^2},\end{aligned} \tag{3.25}$$

where

$$\bar{b}_{n,x^1} = -\delta_{1,x^1} + \Phi_{x^1}^{Q'}\bar{b}_{n-1,x^1}. \tag{3.26}$$

Those equations, (3.24) to (3.26), have initial values: $\bar{a}_1 = -\delta_0$, $\bar{b}_{1,x^1} = -\delta_{1,x^1}$ and $\bar{b}_{1,x^2} = -\delta_{1,x^2}$. The matrix $\sum_{x^1}\sum_{x^1}^{'}$ is a block diagonal matrix with elements $\sum_{i,x^1}\sum_{i,x^1}^{'}$, where $i = 1, ..., X^2$ denotes the number of volatility factors. Also, $\bar{b}_{n-1,x^1x^2} = \sum_{x^1,x^2}^{'}\bar{b}_{n-1,x^1} + \bar{b}_{n-1,x^2}$. This is also stated by [18] that the condition must be satisfied for factor loadings: $\sum_{x^2}^{'}\bar{b}_{n-1,x^1,x^2} < 1$ for each volatility factor.

Then, bond yields can be demonstrated using the bond loadings with the help of the relationship between bond prices and yields $y_t^n = -\dfrac{1}{n}\log(P_t^n)$.

$$y_t^n = a_n + b_{n,x^2}^{'} x_t^2 + b_{n,x^1}^{'} x_t^1, \tag{3.27}$$

where $a_n = \frac{1}{n}\bar{a}_n$, $b_{n,x^2} = \frac{1}{n}\bar{b}_{n,x^2}$ and $b_{n,x^1} = \frac{1}{n}\bar{b}_{n,x^1}$.

3.2.1 Unspanned Stochastic Volatility Models

Using the unspanned stochastic volatility model restrictions that are proposed by [15], we can simplify the bond loadings in (3.27) by setting volatility state vector as zero, $\bar{b}_{n,x^2} = 0$, across cross section of maturities. Thus, we get

$$\bar{a}_n = -\delta_0 + \bar{a}_{n-1} + \mu_{x^1}^{Q^{'}} \bar{b}_{n-1,x^1} + \frac{1}{2}\bar{b}^{'}_{n-1,x^1} \Sigma_{0,x^1} \Sigma^{'}_{0,x^1} \bar{b}_{n-1,x^1}, \tag{3.28}$$

and

$$\bar{b}_{n,x^1} = -\delta_{1,x^1} + \Phi_{x^1}^{Q^{'}} \bar{b}_{n-1,x^1}. \tag{3.29}$$

3.2.2 Physical Dynamics

[18] constructs the state variable dynamics that have the same structure under both P-measure and Q-measure. The state variables again follow a VAR model with conditional heteroscedasticity similar to that of under Q-measure:

$$\begin{aligned} x_{t+1}^1 = \mu_{x^1} + \Phi_{x^1} x_t^1 + \Phi_{x^1 x^2} x_t^2 + \textstyle\sum_{x^1 x^2} \xi_{x^2,t+1} + \xi_{x^1,t+1}, \\ \xi_{x^1,t+1} \sim N\big(0, \textstyle\sum_{x^1,t} \textstyle\sum^{'}_{x^1,t}\big), \\ \textstyle\sum_{x^1,t} \textstyle\sum^{'}_{x^1,t} = \textstyle\sum_{0,x^1} \textstyle\sum^{'}_{0,x^1} + \displaystyle\sum_{i=1}^{X^2} \textstyle\sum_{i,x^1} \textstyle\sum^{'}_{i,x^1} x_{i,t}^2, \\ \xi_{x^2,t+1} = x_{t+1}^2 - \mathbb{E}\big(x_{t+1}^2 | \mathcal{L}_t\big). \end{aligned} \tag{3.30}$$

In this structure Gaussian state variables are also dependent on non-Gaussian state variables in VAR equation through $\Phi_{x^1 x^2}$ and $\sum_{x^1 x^2}$. Also, the conditional mean and variance are different under P-measure and Q-measure become

$$x_{t+1}^2 = \mu_{x^2} + \textstyle\sum_{x^2} \omega_{t+1}, \tag{3.31}$$

where

$$\omega_{i,t+1} \sim Gamma\big(v_{x^2,i} + z_{i,t+1}, 1\big), \quad i = 1, ..., X^2, \tag{3.32}$$

and

$$z_{i,t+1} \sim Poisson\big(e_i^{'} \textstyle\sum_{x^2}^{-1} \Phi_{x^2} \textstyle\sum_{x^2} \omega_t\big), \quad i = 1, ..., X^2, \tag{3.33}$$

with $v_{x^2} = (v_{x^2,1}, ..., v_{x^2,X^2})$ as shape parameters. The autocorrelation of x_{t+1}^2 is controlled by Φ_{x^1} and also $\sum_{x^2}$ is a scale matrix and its lower bound is constrained by μ_{x^2}.

The sufficiency conditions for $x_t^2 \geqslant 0$ are $\mu_{x^2} \geqslant 0$; $\sum_{x^2} \geqslant 0$; $\sum_{x^2}^{-1}\Phi_{x^2}\sum_{x^2} \geqslant 0$. The Feller condition in discrete time, which is $v_{x^2,i} > 0$, guarantees the process of volatility state variablesin order to avoid their lower bound.

The formulation

$$\mathbb{E}\left(x_{t+1}^2|\mathcal{L}_t\right) = (I_{X^2} - \Phi_{x^2})\mu_{x^2} + \textstyle\sum_{x^2} v_{x^2} + \Phi_{x^2} x_t^2$$

gives us the conditional mean of state variables in diffusion, which is also affine. In addition, we use

$$V\left(x_{t+1}^2|\mathcal{L}_t\right) = \textstyle\sum_{x^2} \operatorname{diag}(v_{x^2} - 2\sum_{x^2}^{-1}\Phi_{x^2}\mu_{x^2})\sum_{x^2}^{'} + \sum_{x^2} \operatorname{diag}(2\sum_{x^2}^{-1}\Phi_{x^2}\mu_{x^2})\sum_{x^2}^{'}$$

as the conditional variance of volatility factors.

3.2.3 Stochastic Discount Factor

Spanned Models

The agents are compensated for holding fixed income securities. The compensation for risk defines the relationship between P-measure and Q-measure via stochastic discount factor (SDF). In the following equation $\mathbb{E}\left[P_{t+1}^n e^{-r_t}\right] = \mathbb{E}\left[P_{t+1}^n M_{t+1}\right]$, where M_{t+1} denotes SDF, the no-arbitrage assumptions requires $M_{t+1} > 0$. For intuition, we focus on the compensation for the risk, the log of SDE, $m_{t+1} = \log(M_{t+1})$ and decompose the compensation parameters as

$$m_{t+1} = -r_t + \frac{1}{2}\lambda_{x^1t}^{'}\lambda_{x^1t} - \lambda_{x^1t}^{'}\eta_{x^1,t+1} - \lambda_{\omega t}^{'}\eta_{\omega,t+1} - \lambda_{zt}^{'}\eta_{z,t+1}. \tag{3.34}$$

The compensation for risk consists of three parts, namely Gaussian risk, gamma risk and non-Gaussian risk. In (3.34), $\eta_{i,t+1}$ corresponds to standardized shocks with zero mean and unit variance; λ_{it} denotes the price of risk for risk components. The notion of compensation for risk covers Gaussian shocks in $\eta_{x^1,t+1}$, gamma shocks in $\eta_{\omega,t+1}$ and non-Gaussian shocks in $\eta_{z,t+1}$ parameters.

The market price of risk actually corresponds to the compensation per unit of risk bearing. [18] gives the prices of risks as the following:

$$\lambda_{x^1t} = V\left(x_{t+1}^1|\mathcal{I}_t, x_{t+1}^2, z_{t+1}\right)^{-1/2}\left[\mathbb{E}\left(x_{t+1}^1|\mathcal{I}_t, x_{t+1}^2, z_{t+1}\right) - \mathbb{E}^Q\left(x_{t+1}^1|\mathcal{I}_t, x_{t+1}^2, z_{t+1}\right)\right],$$

$$\lambda_{\omega t} = V\big(\omega_{t+1}|\mathcal{I}_t, z_{t+1}\big)^{-1/2}\Big[\mathbb{E}\big(\omega_{t+1}|\mathcal{I}_t, z_{t+1}\big) - \mathbb{E}^Q\big(\omega_{t+1}|\mathcal{I}_t, z_{t+1}\big)\Big],$$

$$\lambda_{zt} = V\big(z_{t+1}|\mathcal{I}_t\big)^{-1/2}\Big[\mathbb{E}\big(z_{t+1}|\mathcal{I}_t\big) - \mathbb{E}^Q\big(z_{t+1}|\mathcal{I}_t\big)\Big].$$

They are defined as the standardized deviations between mean levels under P-measure and Q-measure.

Unspanned Models

The implemented restriction in the unspanned models helps us to simplify the risk compensation. Since USV models put some restrictions on the volatility state vector and on the stochastic diffusion parameter, (3.34) can be reduced by taking the gamma shock, $\eta_{\omega,t+1}$, and non-Gaussian shock, $\eta_{z,t+1}$, as zero. The volatility parameters are then written as unused while estimating likelihood functions. The compensation is

$$m_{t+1} = -r_t + \frac{1}{2}\lambda'_{x^1t}\lambda_{x^1t} - \lambda'_{x^1t}\eta_{x^1,t+1}.$$

The compensation parameter m_{t+1} in the USV models is then depended on price for risk of drift factors and Gaussian shocks.

3.2.4 State Space Representation

[18] define x_t as $N_1 \times 1$ vector of state factors, which cover both spanned $x_t = (x_t^{1'}, x_t^{2'})'$ and unspanned $x_t = (x_t^1)$ stochastic volatility models within affine framework. If the number of observed yields are higher than number of factors, all yields could not be priced without error. Thus, the yields, those observed with error or not, are separated.

Assume that there are N different maturities for yields y_t^n, where $n = (n_1, \ldots, n_N)$ and $Y_t = A + Bx_t$ for A and B are N dimensional vectors. Due to the fact that $N > N_1$, only $Y_t^{(1)} = S_{Y_1}Y_1$ could be priced without error while the other yields, for N_2 maturities, where $N_2 = N - N_1$, $Y_t^{(2)} = S_{Y_2}Y_2$ are exposed to Gaussian measurement errors. In this framework, the equations are given as

$$Y_t^{(1)} = A_1 + B_1 x_t, \tag{3.35}$$

and

$$Y_t^{(2)} = A_2 + B_2 x_t + \psi_t, \quad \psi_t \sim N(0, \Omega), \tag{3.36}$$

where $A_i = S_{Y_i} A$ and $B_i = S_{Y_i} B$ for $i = 1, 2$. Then, the representation for state space is completed using (3.30) – (3.33).

3.2.5 Estimation Procedure

[18] also proposes an estimation methodology using the concept of least squares. In the scope of estimation procedure, firstly the parameter set θ is separated according to whether those used in the bond loading calculations or not. Then bond loadings and spanned factors are solved as $x_t = B_1^{-1}(Y_t^{(1)} - A_1)$ given the parameters. In this procedure, the spanned factors do not represented by principal components and the yields $Y^{(1)}$ are not subject to measurement error, which let us extract factors, directly. Those properties of and the parameters $(\mu_{x^1}, \Phi_{x^1}, \Phi_{x^1 x^2})$ under P-measure are solved in a Gaussian VAR framework using least squares regressions.

The likelihood function for spanned models given θ parameters is stated as:

$$\begin{aligned} p(Y_{1:T}; \theta) = p\big(Y_{1:T}^{(2)} | Y_{1:T}^{(1)}; \theta\big) p\big(Y_{1:T}^{(1)}; \theta\big) = \prod_{t=1}^{T} p\big(Y_{1:T}^{(2)} | Y_{1:T}^{(1)}; \theta\big) |J(\theta)|^{-T} \\ \times \prod_{t=1}^{T} p\big(x_t^1 | x_t^2, \mathcal{L}_{t-1}; \theta\big) \prod_{t=1}^{T} \prod_{i=1}^{X^2} p\big(x_{it}^2 | \mathcal{L}_{t-1}; \theta\big), \end{aligned} \tag{3.37}$$

where $J(\theta)$ is defined as the Jacobian transformation from state vectors $x_t = (x_t^{1'}, x_t^{2'})'$, to $Y_t^{(1)}$. In this framework, the Jacobian, $J(\theta)$, is from x_t^1 and x_t^2 to $Y_t^{(1)}$. In this section, we refer conditional log likelihood function as $\mathcal{L}(\theta)$, where $\mathcal{L}(\theta) = \log p(Y_{1:T}; \theta)$.

[18] argues that if the model is defined as given in (3.30) - (3.33), (3.35) and (3.36) concentrated likelihood, $\max_{\theta_m} \mathcal{L}\big(\hat{\theta}_m(\theta_m), \theta_m\big)$, where $\theta_c = \big(\mu_{x^1}, \Phi_{x^1}, \Phi_{x^1 x^2}, \Omega\big)$ and the remaining parameters are grouped in θ_m, which solves the maximum likelihood function $\hat{\theta} = \operatorname{argmax}_\theta \mathcal{L}(\theta)$. Under P-measure, we solve for (3.30) via $\max_{\theta_c} \mathcal{L}\big(\theta_m, \theta_c\big)$ using generalized least squares (GLS) and (3.36) using OLS to obtain covariance matrix Ω. Then, they propose a novel estimation procedure using concentrated likelihood.

The procedure is constructed by maximization of concentrated likelihood function. For given parameter set θ_m the likelihood is formed as follows:

1. Bond loadings, A and B, and spanned factors are solved: $x_t = B_1^{-1}(Y_t^{(1)} - A_1)$;
2. Given state vectors x_t^1 and x_t^2, calculate $\sum_{x^1x^2}$ and $\xi_{x^1,t+1}$ in (3.30). Then modify (3.30) to get

$$x_{t+1}^1 - \sum_{x^1x^2}\xi_{x^2,t+1} = \mu_{x^1} + \Phi_{x^1}x_t^1 + \Phi_{x^1x^2}x_t^2 + \xi_{x^1,t+1};$$

3. The variance-covariance matrix is also computed in concentrated form:

$$\hat{\Omega}(\theta_m) = \frac{1}{T-1}\sum_{t=2}^{T}\Big(Y_t^{(2)} - A_2 - B_2x_t\Big) \times \Big(Y_t^{(2)} - A_2 - B_2x_t\Big)';$$

   and using GLS estimation obtain $\hat{\mu}_{x^1}(\theta_m)$, $\hat{\Phi}_{x^1}(\theta_m)$, and $\hat{\Phi}_{x^1,x^2}(\theta_m)$.

4. Finally, replace the estimated parameters into log-likelihood function;

$$\hat{\theta}_c(\theta_m) = \Big(\hat{\mu}_{x^1}(\theta_m), \hat{\Phi}_{x^1}(\theta_m), \hat{\Phi}_{x^1x^2}(\theta_m), \hat{\Omega}(\theta_m)\Big).$$

3.3 Affine Term Structure Models with Constant Volatility

In the previous section, we depict the affine model structure with volatility state variable, this model permits both spanned and unspanned stochastic volatility. In this section, we restrict the affine model not to allow stochastic volatility and thus there is no volatility state variable. Following [59] and [9] , we type a homoscedastic affine term structure model in discrete time. Those models let macroeconomic variables to span bond prices but, since we focus on AFTS models, our model only includes latent factors.

The dynamics for Gausian state variables are given in a VAR form:

$$x_{t+1}^1 = \mu_{x^1}^Q + \Phi_{x^1}^Q x_t^1 + \sum_{x^1}\xi_{x^1,t+1}^Q, \quad \xi_{x^1,t+1} \sim N\big(0, \sum_{x^1,t}\sum_{x^1,t}'\big),$$
$$\sum_{x^1,t}\sum_{x^1,t}' = \sum_{0,x^1}\sum_{0,x^1}' + \sum_{i=1}^{X^2}\sum_{i,x^1}\sum_{i,x^1}' x_{i,t}^2, \quad \xi_{x^2,t+1} = x_{t+1}^2 - \mathbb{E}\big(x_{t+1}^2|\mathcal{L}_t\big). \tag{3.38}$$

The short term interest rate is a linear function of x_t^1 is

$$r_t = \delta_0 + \delta_{1,x^1}^{'} x_t^1. \tag{3.39}$$

Then, bond prices are exponentially affine functions of the state variable under Q-measure becomes

$$P_t^n = exp\big(\bar{a}_n + \bar{b}_{n,x^1}^{'} x_t^1\big). \tag{3.40}$$

The connection between P-measure and Q-measure is sustained by SDF following the procedure that simplifying the bond loading equations, (3.24) to (3.26), using the identity that $\lambda_t = \Sigma_{0,x^1}^{-1}\big(\lambda_0 + \lambda_1 x_t^1\big)$. Thus we get $\mu_{x^1}^Q = \mu_{x^1}^P - \Sigma_{0,x^1}\lambda_0$ and $\Phi_{x^1}^Q = \Phi_{x^1}^P - \Sigma_{0,x^1}\lambda_1$ under Q-measure. Then, bond loadings are transformed to

$$\bar{a}_n = -\delta_0 + \bar{a}_{n-1} + \left(\mu_{x^1}^P - \Sigma_{0,x^1}\lambda_0\right)^{'} \bar{b}_{n-1,x^1} + \frac{1}{2}\bar{b}_{n-1,x^1}^{'}\Sigma_{0,x^1}\Sigma_{0,x^1}^{'}\bar{b}_{n-1,x^1}, \tag{3.41}$$

and

$$\bar{b}_{n,x^1} = -\delta_{1,x^1} + \left(\Phi_{x^1}^P - \Sigma_{0,x^1}\lambda_1\right)^{'} \bar{b}_{n-1,x^1}. \tag{3.42}$$

The initial values are again $\bar{a}_1 = -\delta_0$ and $\bar{b}_{1,x^1} = -\delta_{1,x^1}$.

As mentioned before, the affine term structure models price exactly the number of assets equal to number of latent variables. When we take number of yield maturities as N and number of latent factor variables asN_1, when $N > N_1$, only N_1 of N securities could be priced with no error while the other yield vectors $N_2 = N - N_1$, are exposed to Gaussian measurement error. Thus, similar to [13], we assume that those maturities in N_2 are observed with measurement error. Then, the likelihood function, given by [7] is simplified by our restriction that there is no observed macroeconomic variable in state variable dynamics. The likelihood function becomes

$$\mathcal{L}(\theta) = \prod_{t=2}^{T} p\left(Y_t^{(1)} | Y_{t-1}^{(1)}; \theta\right); \tag{3.43}$$

and therefore,

$$\begin{aligned} \log\big(\mathcal{L}(\theta)\big) = -(T-1)\log|J(\theta)| - (T-1)\frac{1}{2}log|\Sigma_{0,x^1}\Sigma_{0,x^1}^{'}| \\ -\frac{1}{2}\sum_{t=2}^{T}\left(x_{t+1}^1 - \mu_{x^1}^Q - \Phi_{x^1}^Q x_t^1\right)^{'}\left(\Sigma_{0,x^1}\Sigma_{0,x^1}^{'}\right)^{-1}\left(x_{t+1}^1 - \mu_{x^1}^Q - \Phi_{x^1}^Q x_t^1\right) \\ -\frac{T-1}{2}\log\sum_{i=1}^{N_2=N-N_1}\sigma_i^2 - \frac{1}{2}\sum_{t=2}^{T}\sum_{i=1}^{N_2=N-N_1}\frac{\psi_{i,t}^2}{\sigma_i^2} \end{aligned} \tag{3.44}$$

Here, this gives us the log-likelihood function of affine term structure models without volatility state variable.

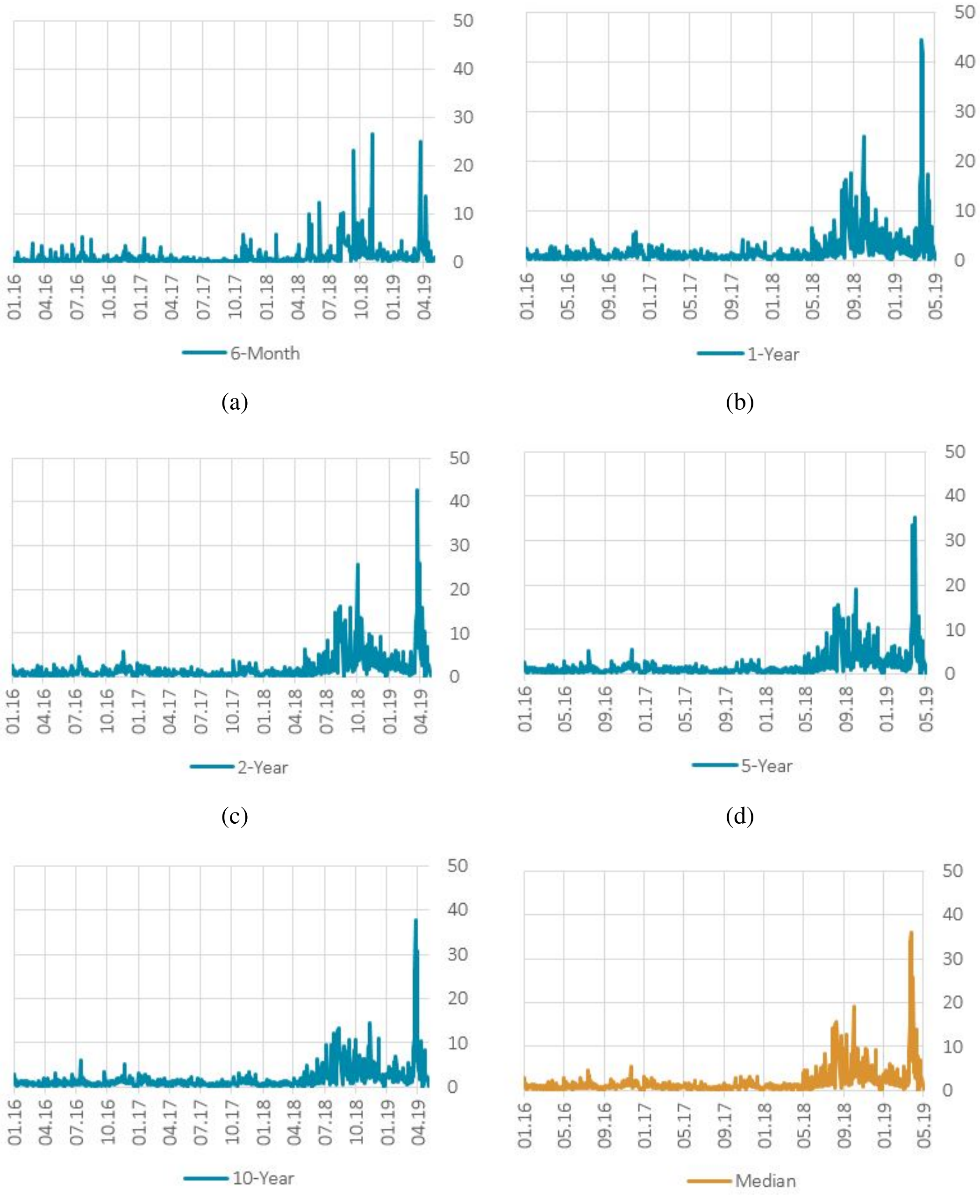
3.4 Data

In this thesis, we use intraday Turkish lira interest rate swap (IRS) data with various maturities during January 2016 and June 2019 period. Due to liquid interest rate derivatives market, IRS instruments has gained great importance for investors. Thus, the estimation results regarding TRY IRS market are representative in terms of not only swap market but also general wellbeing of the domestic financial markets. Basically, IRS is a tool that enables two parties to exchange their fixed interest rate and floating interest rate cash flows. Although reasons of engaging IRS transactions are out of the scope of this thesis, in short, investors prefer to engage IRS transactions for hedging and speculative reasons. For instance, in case of a deposit bank that has liabilities, which bear relatively floating interest rate, such as short term deposits, and assets with fixed interest rate, e.g. long term loans. In that case, financial institution involves in an IRS transaction by paying fixed rate and receiving floating rate in order to hedge itself against interest rate mismatch. Also, for example, an investor that has an expectation of decrease in future interest rates, then he/she could speculate through engaging IRS by paying floating rate and receiving fixed rate transaction.

Firstly, we obtain IRS market data for Turkey via Thompson Reuters (Refinitiv) Datascope. According to the quality of the data we prefer to use 6-month, 1-year, 2-year, 5-year and 10-year IRS maturities in our analysis. We aggregate high frequency data into 1-hour periods, where we use closing levels across maturities for each hour between 10:00 and 17:00 in 24-hour basis.

In (3.17), we depict how to construct realized volatility using swap rates. Intraday volatility of swap rates are given in Figure 3.1. The volatility pattern reveals the increasing tension in the domestic financial markets since the 1st quarter of 2018. Actually, the distress in Turkish markets are gradually elevated after a credit rating agency, Moody's, downgraded Turkey's sovereign rating by justifying erosion in the institutional strength and becoming more exposed to external risks by stressing wide current account deficits, large external debt and high political risks in March 2018. Since then uncertainty in domestic markets has resulted in volatility to rise at unprecedented levels in following periods, especially in 2nd and 3rd quarters of 2018. The major part of the market tension in 2018 disappeared after Central Bank of Turkey‘s (CBRT) accumulated interest rate hike in September 2018. The intraday market volatility converged to its long term average shortly after CBRT's policy reaction. The highest level of intraday volatility is observed on March 2019 for most of the maturity dimensions. This is due to Turkish lira liquidity squeeze in offshore markets.

Moreover, we compare some important events in the estimation period in Figure 3.2. The first important date is the first working day after the coup attempt in 2016. The second date corresponds to sanction decision of the US against Turkey and the last date is due to the squeeze in offshore Turkish lira liquidity. The short term volatility, 6-month, seems to have less affected by market tension while volatility levels of 1-year and longer maturities reflected a higher level of sensitivity. Although, all major events unearth uncertainty in terms of interest rates, a significant shock in liquidity conditions found to be more effective in volatility.



(a) (b) (c) (d) (e) (f)

Figure 3.1: Intraday Swap Rate Volatility

Figure 3.2: Intraday Swap Rate Volatility Path

CHAPTER 4

RESULTS

In this thesis, we explore whether volatility spans in the TRY IRS market. Thus, we choose to employ the estimation procedure that is proposed by [4]. In the methodology section, we give the details regarding how an affine model structure could be transformed to the test volatility span using linear regressions. Prior studies (such as [15] and [41]) are heavily depended on volatility embedded derivatives such as straddles or interest rate caps to analyze spanning condition of yields. Following the robust methodology of [4], we could analyze the volatility spanning condition of the IRS market in Turkey using the actual data. In their methodology, they proposed to take time steps dt as small as possible, thus when computing the quadratic variation, variance dt^2 of the yield becomes omissible and realized volatility could be used to assess the spanning condition.

To test the contemporaneous volatility spanning condition, we estimate using (3.19). In (3.19), the dependent variable is the intraday realized volatility of IRS returns. We give the results of intraday realized volatility estimation in the data section for 6-month, 1-year, 2-year, 5-year, and 10-year maturities. After obtaining realized volatilities in daily basis, we aggregate intraday yields as closing levels at 16:00 - 17:00 period that is assumed to correspond daily yield for our analysis. As emphasized before the standard version of spanning regression, (3.18), contains a contemporary level of yields and thus it is subjected to violation of basic assumptions, multicollinearity, of ordinary least squares estimation. Hence, to resolve the multicollinearity issue and to reduce the dimensionality of the cross-section of yields, we firstly extract the principal components (PCs) of the yields. The extracted components are mechanically

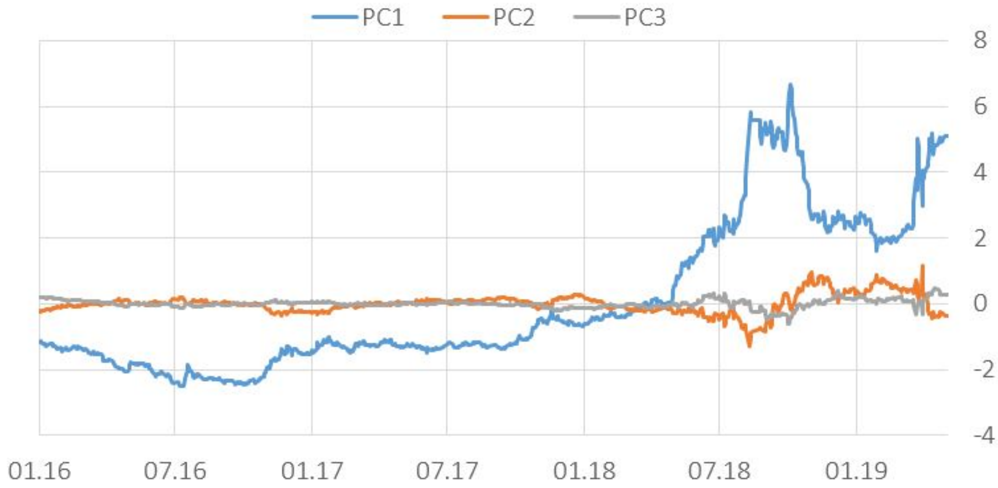


Figure 4.1: Principal Components

orthogonal to each other and thus they are not subject to multicollinearity problem. The first three PCs of yields are associated with, respectively, slope, steepness, and curvature factors of the yield curve and accepted as reflecting the major part of the total variation ([42]). The results of the principal component analysis are given in Table 4.1.

Table 4.1: Principal Component Analysis for Realized Data

	Marginal Contribution	Variance Coverage Ratio
PC1	97.92%	97.92%
PC2	1.68%	99.60%
PC3	0.33%	99.93%
PC4	0.06%	99.98%
PC5	0.03%	100.00%

Results suggest that the slope, steepness and curvature factors of the yield curve are sufficient to explain almost all of the variation in IRS yields. Thus, although [4] prefers to use a larger number of factors in volatility span testing procedure of US Treasury bond yields, we propose the regression, namely (3.19), to contain only the first three principal components as independent variables.

Prior to contemporaneous volatility spanning procedure, we firstly explore the direct relationship between intraday yield volatility in the IRS markets and the principal components of the cross section of yields. According to Figure 4.2 - Figure 4.4,

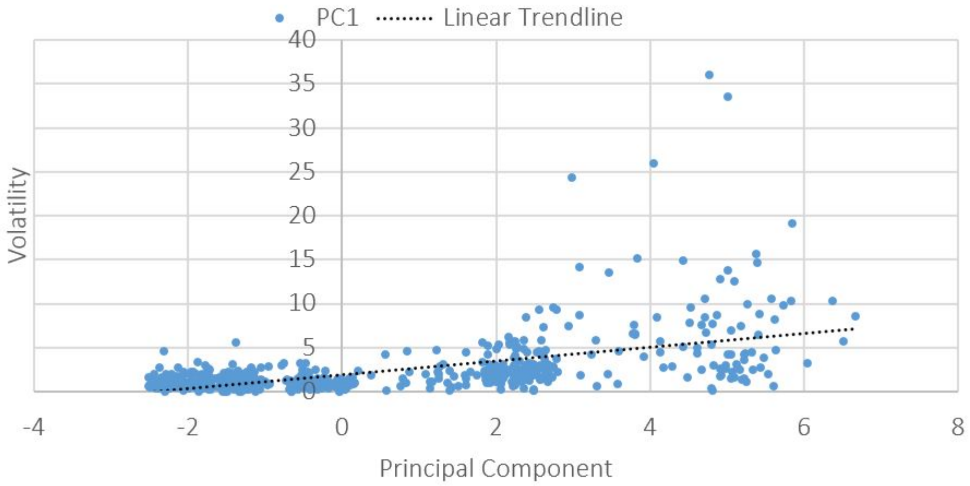


Figure 4.2: Principal Components and Volatility

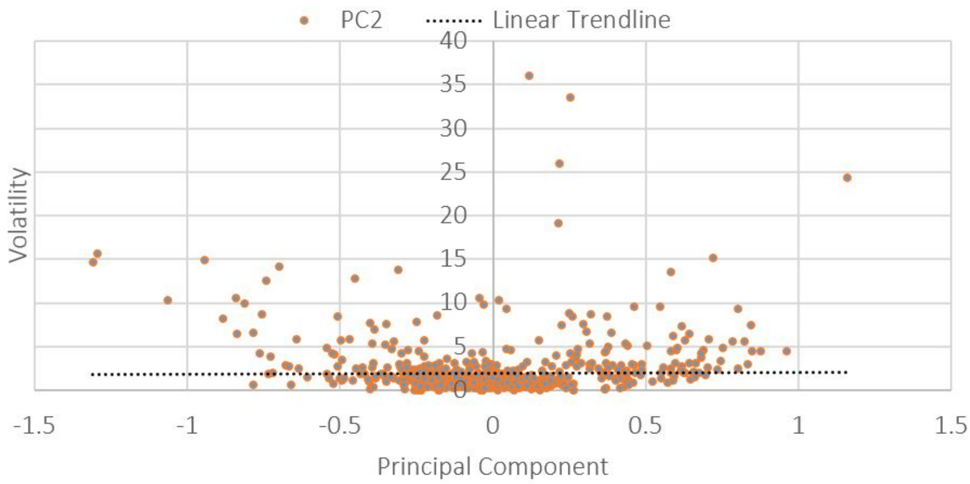


Figure 4.3: Principal Components and Volatility

only PC1, namely the level factor, demonstrates a slight positive relationship between volatility, while other factors seem to be unrelated to the volatility. Although, steepness, and curvature factors are visually not related to the volatility, those factors' importance could arise in the regression framework.

Moreover, another pillar of linear regression points to the fact that there must be no unit root in the volatility or latent factor series. In the data section, it is obvious that realized volatility series are stationary by their mean reverting nature and we need to check the stationarity of yield curve factors. However, by both graphical inspection Figure 4.1 and Dickey-Fuller unit root test results indicate that the first principal component, namely level factor has unit root and has a first difference stationary process, where Augmented Dickey-Fuller test statistic results show that PC1 has failed

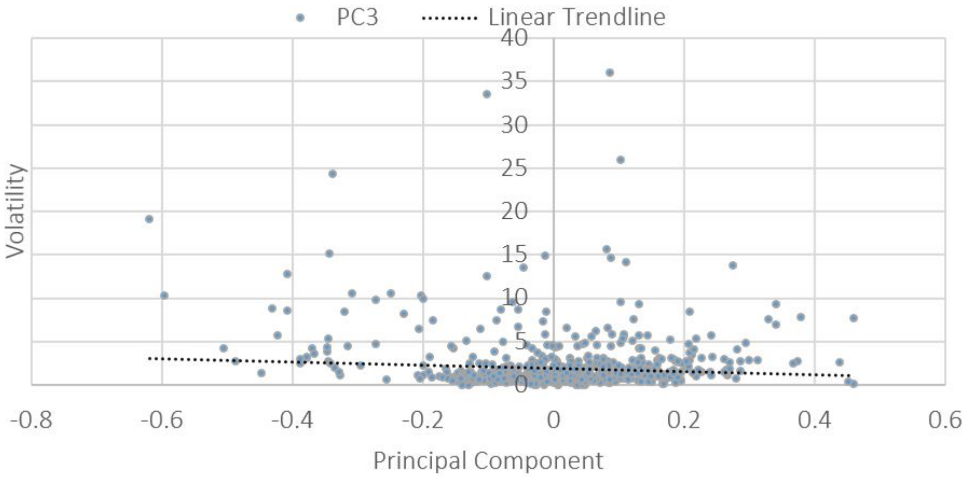


Figure 4.4: Principal Components and Volatility

to reject the null hypothesis indicating there is a unit root with 94% probability, while the results for first difference reject the unit root hypothesis with 100% probability. Thus, in the scope of this thesis, we transform (3.19) using the first differences of yield curve factors as explanatory variables.

The linear regression is estimated using ordinary least squares as follows:

$$v^2_{y_\tau}(t,h) = \beta_0 + \sum_{k=1}^{3} \beta_k \triangle PC_k(t,h) + \epsilon(t,h) \tag{4.1}$$

The results are given in Table 4.2.

One of the most striking results of linear volatility spanning regression is that yield curve factors can barely explain 15% of the quadratic variation of the yields. The outcome supports the existing literature that yields do not span volatility in fixed income markets and thus there is unspanned stochastic volatility in the Turkish lira IRS market returns. This could be interpreted as interest rate swaps have a very limited ability in hedging against the volatility that is embedded in the market. In addition to that the proportion of volatility, which is explained by yield curve factors decreases as maturity increases. Thus, it would be convenient to relate shocks in the yield curve factors only with the shorter-term volatility indicators. We find that the coefficient of level factor β_1 is statistically significant for all maturities, while the slope coefficient β_2 is significant except for 6-month maturity. This is an intriguing outcome since the steepness of the yield curve is generally associated with the duration, the indicator for

Table 4.2: Regression Results of (4.1) on Realized Data

	β_0	β_1	β_2	β_3	R^2
$v^2_{y_\tau}, \tau = 6M$	1.066278 (0.123)*	9.579756 (4.668)**	17.68531 (14.596)	11.91107 (14.498)	14.96%
$v^2_{y_\tau}, \tau = 1Y$	1.952741 (0.214)*	11.53067 (3.412)*	18.95969 (6.023)*	15.92402 (11.197)	9.34%
$v^2_{y_\tau}, \tau = 2Y$	1.975724 (0.224)*	11.29174 (3.174)*	17.10105 (6.538)*	16.25567 (10.258)	9.53%
$v^2_{y_\tau}, \tau = 5Y$	1.912554 (0.215)*	9.450793 (2.551)*	14.99637 (6.011)**	15.09404 (7.616)**	7.42%
$v^2_{y_\tau}, \tau = 10Y$	1.816984 (0.193)*	7.616979 (2.105)*	12.42617 (5.233)**	14.43748 (6.955)**	6.32%

*(1) The results in the parenthesis indicates Newey-West heteroscedasticity and autocorrelation robust standard errors.(2) *, **, *** show 1%, 5% and 10% statistically significant coefficients, respectively.*

measuring the sensitivity of bond prices to the changes of the interest rate. Duration sensitivity therefore increases with the maturity and its effect on shorter-term bonds are considered to be minor. As a result of this relationship between duration and maturity, it is logical to have an insignificant β_2 coefficient at the short term maturity.

In addition, [43] argues that interest rate volatility is associated with the curvature factor. Thus, prior to obtaining estimation results, our expectation that the curvature factor is the most significant and potent indicator in terms of its effect on realized volatility. Conversely, to the argument of [43], our findings indicate that the coefficient of curvature factor β_3 is statistically insignificant for most of the maturity dimensions and only significant for longer-term maturities, 5-year, and 10-year. This finding can also be related to the characteristics of the yield curve. Since the curvature characteristics of the yield curve are mainly associated with medium to longer-term maturities, our findings are consistent with the widely observed dynamics. To sum up, yield curve dynamics appear to be not directly effective on yield volatility.

In addition to that the yield curve does not span volatility in the Turkish lira IRS market, we seek for any general pattern in the residuals of the regression results presented in Table 4.2. This attempt is in line with [15] and [4], where they both find out that yield variation shows a strong interrelation across different maturities that are not

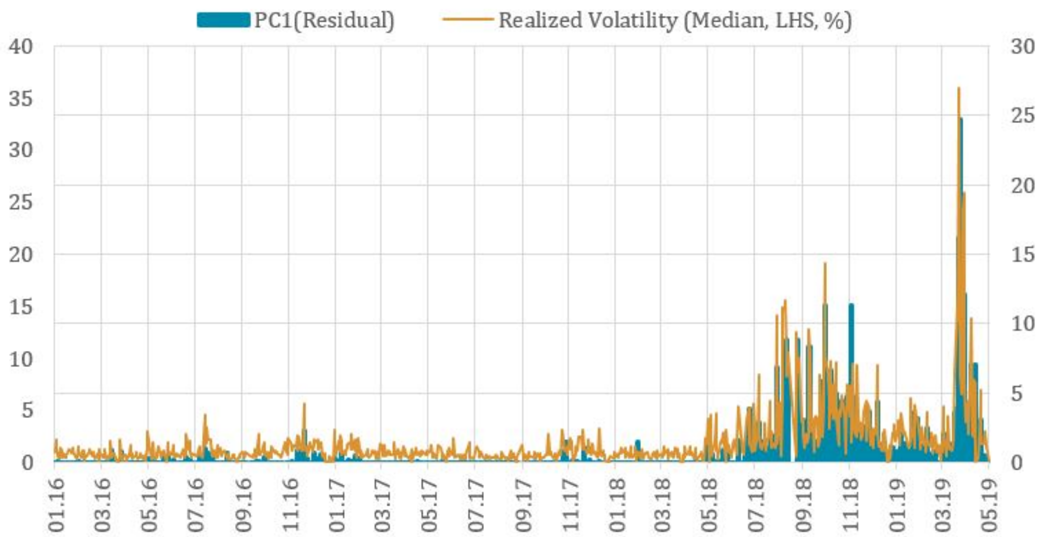


Figure 4.5: PC1 of Residual and Median Volatility Across Maturities

related to yield levels. To explore this relationship, we use principal component analysis on error terms across the maturity spectrum and interpret the variance coverage ratios of PCs. The results of the principal component analysis are given in Table 4.3.

Table 4.3: Principal Component Analysis for Error Terms in Volatility Equation

	Marginal Contribution	Variance Coverage Ratio
PC1	82.20%	82.20%
PC2	12.24%	94.44%
PC3	4.74%	99.18%
PC4	0.70%	99.88%
PC5	0.12%	100.00%

Our findings are consistent with the prior studies that roughly 80% of the variation in the model error terms, (4.1), could be explained by only the first principal component. Thus, there is a presence of a common factor, which is not linearly related to the yield curve dynamics and affecting yield variation across the maturity spectrum. Also, [4] points out that although the first principal component explains most of the variation, volatility innovations do not show perfect correlation, and thus there should be multiple stochastic volatility factors required to contain its characteristics.

The first principal component of residuals from yield variation and median volatility level follow similar trends across the estimation period (Figure 4.5). Then, we could interpret the 1^{st} principal component as systemic volatility using the definition of spanning hypothesis. As mentioned before spanning hypothesis assumes that the term structure models are accepted as incorporating sufficient information to complete fixed income markets. Therefore, yield curve dynamics represent all the information for replicating assets. Since all the unsystematic shocks are reflected in the yield curve components and thus a linear combination of those factors could be associated with the unsystematic part of the yield variation. Then, due to the fact that volatility is divided into two parts unsystematic and systemic, the residuals from (4.1) could be good indicators for systemic volatility. As the principal component analysis of residuals and high correlation with median volatility shows that we could treat the major principal component of residuals as a reliable indicator for systemic risk, or volatility.

In the previous chapter, we introduce two different affine term structure models, one with stochastic volatility component and one with constant volatility components. One of the main goals of this thesis is to assess which affine model should be preferred if we use unspanned stochastic volatility phenomenon as a specification test for term structure models.

4.1 Affine Term structure Model with Stochastic Volatility

[18] proposes an affine term structure model with stochastic volatility which is similar to widely accepted studies of [24] and [21]. Then, in the scope of this thesis, we apply an affine model with three latent factors that has one stochastic volatility factor using Turkish lira interest rate swap returns. After, affine model estimation, we repeat the testing procedure of the volatility spanning hypothesis that is explained in the previous section. We first obtain the latent factors from fitted data and seek for the same relationship with intraday volatility. The volatility testing results for affine model with stochastic volatility are depicted in Table 4.4.

According to the results given in Table 4.4, latent factors of the yield curve are barely have any significant effect on the yield variation. The variation in 1-year maturity

Table 4.4: Regression Results of Affine Model with Stochastic Volatility (4.1)

	β_0	β_1	β_2	β_3	R^2
$v^2_{y_\tau}, \tau = 6M$	1.138101	-0.078876	3.04	2.82	0.21%
	(0.129)*	(0.235)	(3.72)	(2.37)	
$v^2_{y_\tau}, \tau = 1Y$	2.039425	-0.047258	-2.1	-5.8	0.15%
	(0.237)*	(0.256)	(3.27)*	(3.58)	
$v^2_{y_\tau}, \tau = 2Y$	2.060109	0.084087	-3.97	-7.52	0.29%
	(0.243)*	(0.399)	(3.48)	(4.2)***	
$v^2_{y_\tau}, \tau = 5Y$	1.982666	0.086627	-3.92	-7.94	0.35%
	(0.228)*	(0.48)	(3.4)	(4.39)***	
$v^2_{y_\tau}, \tau = 10Y$	1.872577	0.166833	-3.21	-6.09	0.4%
	(0.202)*	(0.411)	(2.95)	(3.92)	

*(1) The results in the parenthesis indicates Newey-West heteroscedasticity and autocorrelation robust standard errors.(2) *, **, *** show 1%, 5% and 10% statistically significant coefficients, respectively.*

is only affected by the steepness factor, while the curvature factor is found to be effective in 2-year and 5-year levels. Even if some yield curve components could be linearly dependent to yield volatility, the signs of their coefficients contradict with the OLS results using realized data (see Table 4.2). The results of realized data show that all yield curve factors are positively related to the volatility, contrary to the affine model with stochastic volatility results that indicate a negative relationship between steepness, curvature factors of yield curve dynamics and variation in the yields.

Also, a low level of R^2 in OLS results shows that in the affine model less than 1% of the total variation could be explained using the cross-section of yields. Even though, the level of explained proportion is limited compared to the original regression results in Table 4.2, the presence of USV is consistent with the regression results using realized data.

We then seek the presence of systemic factors in the residuals of spanning regression. We again employ principal component analysis in an exploration of systemic factors. The PCA results are presented in Table 4.5.

As shown in Table 4.5, only one component in the error term could explain the rest of the variation from the spanning regression. Although the existing literature emphasizes the fact that there is at least one dominant factor regardless of the maturity

Table 4.5: Principal Component Analysis for Error Terms in Volatility Equation

	Marginal Contribution	Variance Coverage Ratio
PC1	100.00%	100.00%
PC2	0.00%	100.00%
PC3	0.00%	100.00%
PC4	0.00%	100.00%
PC5	0.00%	100.00%

spectrum, there is only one single hidden factor found to be effective in volatility. This is the main result of restricting stochastic volatility on a single factor in the affine framework and therefore volatility innovations show perfect correlation.

The outcome could be misleading since the prior studies find that there should be multiple stochastic volatility factors required to contain its characteristics. Thus, while the regression results ensure that unspanned stochastic volatility condition still holds, the specification check of affine term structure model with one stochastic volatility factor fails to represent some of the major characteristics in representing P-measure dynamics. This could be concluded as affine model with three latent variables and stochastic volatility for Turkish lira IRS markets has weak results in terms of its robustness with realized volatility dynamics, therefore it would be misleading to depend on this model especially in volatility modeling of the swap rates.

4.2 Affine Term structure Model with Constant Volatility

In the previous section, we have shown the volatility spanning test results of the affine model structure with the volatility state variable. In this section, we restrict the affine model not to allow stochastic volatility and thus there is no volatility state variable. Following [59] and [9], we apply a homoscedastic affine term structure model in discrete time which assumes the model to be only affected by the latent variables in the trend parameter.

Then we construct the testing procedure following [4] as mentioned before. In testing whether volatility spans the Turkish lira IRS market we use the fitted yields from an affine model without any stochastic volatility factor.

As shown in the Table 4.6, at most 16% of the total yield variation is explained through the spanning hypothesis, which indicates there is unspanned stochastic volatility in the IRS market using affine model outcome. Similar to the testing results in the realized yield data, the explained proportion of yield variation decreases as maturity increases. Also, the coefficient of the level factor is significant across all of the maturity spectra while the steepness factor is found to be ineffective for short term, 6-month, maturity. This finding is in line with the results of realized data shown in Table 4.2 and we could relate this outcome with the sensitivity of prices, and thus yields, to the duration, which is expected to be ignorable in short term securities. It is also important to mention that the curvature factor is effective on longer-term maturities, 5-year, and 10-year. The results of testing volatility spanning conditions in affine term structure models with constant volatility are found to be analogous with the results on realized data in the general.

Table 4.6: Regression Results of Affine Model with Constant Volatility (4.1)

	β_0	β_1	β_2	β_3	R^2
$v^2_{y_\tau}, \tau = 6M$	1.05943	10.66061	18.49447	11.04967	15.88%
	(0.13)*	(6.181)***	(16.122)	(14.489)	
$v^2_{y_\tau}, \tau = 1Y$	1.949208	12.20996	17.9046	11.63761	9.17%
	(0.214)*	(3.676)*	(5.735)*	(8.044)	
$v^2_{y_\tau}, \tau = 2Y$	1.97325	11.83911	15.53958	11.18911	9.17%
	(0.224)*	(3.357)*	(5.907)*	(7.312)	
$v^2_{y_\tau}, \tau = 5Y$	1.909593	9.968756	13.58779	10.46978	7.08%
	(0.215)*	(2.723)*	(5.275)**	(5.721)***	
$v^2_{y_\tau}, \tau = 10Y$	1.812908	8.205961	11.28976	10.35079	6.03%
	(0.192)*	(2.318)*	(4.727)**	(5.399)**	

*(1) The results in the parenthesis indicates Newey-West heteroscedasticity and autocorrelation robust standard errors.(2) *, **, *** show 1%, 5% and 10% statistically significant coefficients, respectively.*

Moreover, the structure of residual parameters is investigated to determine the main factor in the unexplained proportion of the volatility spanning condition. The principal component analysis results of residuals are shown in Table 4.7.

The findings, given in Table 4.7, are consistent with the prior studies and P-measure dynamics in that roughly 80% of the variation in the model error terms could be

Table 4.7: Principal Component Analysis for Error Terms in Volatility Equation

	Marginal Contribution	Variance Coverage Ratio
PC1	82.22%	82.22%
PC2	12.24%	94.46%
PC3	4.72%	99.18%
PC4	0.70%	99.88%
PC5	0.12%	100.00%

explained by only the first principal component. The yield variation is not related to the maturity in its innovations, and therefore, there exists a common factor, which is interpreted as systemic volatility in the scope of this paper.

As a result of the testing procedure, an affine model with constant volatiliy provides similar findings with the P-measure dynamics and we can conclude that to model the Turkish lira IRS market, this restricted version of the affine model is more preferable comparing to the model with volatility factor.

CHAPTER 5

CONCLUSION

Since the seminal paper of [42], a pile of literature employed spanning hypothesis on not only yield curve factors but also macro-finance indicators and volatility of underlying asset returns. The notion of spanning hypothesis asserts that yield curve factors incorporate all the available information in the market and thus it is expected, by those factors, to explain the variation in yields.

In the scope of this thesis, we follow the novel testing procedure of volatility spanning by [4]. The findings using intraday volatility indicators show that only a limited proportion of return variation could be explained by yield curve dynamics. Our results support the existing literature that the Turkish lira interest rate swap markets do not span volatility and thus incomplete. The incompleteness of IRS markets implies that volatility in the swap returns cannot be hedged by only using IRS instruments, while the residuals of the unexplained proportion of yield variation provide us evidence and a measure on the presence of systemic volatility in IRS market.

The systemic volatility is important for both policymakers and investors. From the policymakers' perspective, the presence of systemic volatility constitutes an externality for the transmission mechanism of policy decisions regarding financial markets. Besides, investors that are interested in fixed income markets are subject to systemic risk and they need to adjust their positions in terms of uncertainty that is arising from systemic volatility. Thus, the systemic volatility indicator that is generated from the spanning hypothesis is a complementary tool for market monitoring purposes.

In the scope of this thesis, we compare affine term structure models with diffusion factor, [18], and with a constant volatility component, [59] in the Turkish lira IRS market and treat the presence and structure of USV as a specification test. According to the spanning test results, although both the model with stochastic diffusion parameter and the model with constant volatility validate the presence of USV condition in the IRS market, fitted rates, the model without a stochastic diffusion process provides more analogous dynamics with the observed data in terms of systemic component of the yield variations. Thus, it is advised to use constant volatility AFTS models in the Turkish lira IRS market. This result is consistent with the literature that standard affine models with a volatility state variable are found to be unrelated to the quadratic variation ([16]). Thus, it is proposed to use an unspanned volatility factor in the scope of affine model that factor affecting volatility and unraleted to bond prices.

The volatility spanning hypothesis in the affine framework is, indeed, able to yield much more than what is presented here. Investigation of the sources and directionality of systemic volatility are important but they are beyond the scope of this thesis. Also, although the presence of USV makes it harder to obtain closed-form bond price projections, it provides the opportunity for suggesting new restrictions on state variable dynamics and those could be used in volatility modeling.

REFERENCES

[1] M. Abrahams, T. Adrian, R. K. Crump, E. Moench, and R. Yu, Decomposing real and nominal yield curves, Journal of Monetary Economics, 84(C), pp. 182–200, 2016.

[2] D. H. Ahn and R. F. Dittmar, Quadratic Term Structure Models: Theory and Evidence, Review of Financial Studies, 15(1), pp. 243–288, 2002.

[3] Y. Ait-Sahalia and R. L. Kimmel, Estimating affine multifactor term structure models using closed-form likelihood expansions, Journal of Financial Economics, 98(1), pp. 113–144, 2010.

[4] T. G. Andersen and L. Benzoni, Do bonds span volatility risk in the u.s. treasury market? a specification test for affine term structure models, The Journal of Finance, 65(2), pp. 603–653, 2010.

[5] J. Andreasen, P. Collin-Dufresne, and W. Shi, An arbitrage model of the term structure of interest rates with stochastic volatility, Gsia working papers, Carnegie Mellon University, Tepper School of Business, 1997.

[6] A. Ang, S. Dong, and M. Piazzesi, No-arbitrage taylor rules, NBER Working Papers 13448, National Bureau of Economic Research, Inc, 2007.

[7] A. Ang and M. Piazzesi, A no-arbitrage vector autoregression of term structure dynamics with macroeconomic and latent variables, Journal of Monetary Economics, 50(4), pp. 745–787, 2003.

[8] A. Ang, M. Piazzesi, and M. Wei, What does the yield curve tell us about gdp growth?, Journal of Econometrics, 131(1), pp. 359 – 403, 2006.

[9] M. D. Bauer, G. D. Rudebusch, and J. C. Wu, Term premia and inflation uncertainty: Empirical evidence from an international panel dataset: Comment, American Economic Review, 104(1), pp. 323–37, 2014.

[10] R. Bikbov and M. Chernov, No-arbitrage macroeconomic determinants of the yield curve, Journal of Econometrics, 159(1), pp. 166–182, 2010.

[11] F. Black and M. Scholes, The pricing of options and corporate liabilities, Journal of Political Economy, 81(3), pp. 637–54, 1973.

[12] J. Y. Campbell, A. Sunderam, and L. M. Viceira, Inflation bets or deflation hedges? the changing risks of nominal bonds, Critical Finance Review, 6, pp. 263–301, 2017.

[13] R.-R. Chen and L. Scott, Maximum likelihood estimation for a multifactor equilibrium model of the term structure of interest rates, The Journal of Fixed Income, 3(3), pp. 14–31, 1993.

[14] A. Cieslak and P. Povala, Expected returns in treasury bonds, Review of Financial Studies, 28(10), pp. 2859–2901, 2015.

[15] P. Collin-Dufresne and R. S. Goldstein, Do bonds span the fixed income markets? theory and evidence for unspanned stochastic volatility, Journal of Finance, 57(4), pp. 1685–1730, 2002.

[16] P. Collin-Dufresne, R. S. Goldstein, and C. S. Jones, Can interest rate volatility be extracted from the cross section of bond yields?, Journal of Financial Economics, 94(1), pp. 47 – 66, 2009.

[17] J. Cox, J. Ingersoll, and S. Ross, A theory of the term structure of interest rates, Econometrica, 53(2), pp. 385–407, 1985.

[18] D. Creal and J. C. Wu, Estimation of affine term structure models with spanned or unspanned stochastic volatility, Journal of Econometrics, 185(1), pp. 60–81, 2015.

[19] R. K. Crump, S. Eusepi, and E. Moench, The term structure of expectations and bond yields, Staff Reports 775, Federal Reserve Bank of New York, 2016.

[20] J. M. Culbertson, The term structure of interest rates, The Quarterly Journal of Economics, 71(4), pp. 485–517, 1957.

[21] Q. Dai and K. J. Singleton, Specification analysis of affine term structure models, The Journal of Finance, 55(5), pp. 1943–1978, 2000.

[22] S. Diller, Expectations in the Term Structure of Interest Rates, in *Economic Forecasts and Expectations: Analysis of Forecasting Behavior and Performance*, NBER Chapters, pp. 112–166, National Bureau of Economic Research, Inc, 1969.

[23] G. R. Duffee, Information in (and not in) the Term Structure, Review of Financial Studies, 24(9), pp. 2895–2934, 2011.

[24] D. Duffie and R. Kan, A yield-factor model of interest rates, Mathematical Finance, 6(4), pp. 379–406, 1996.

[25] D. Duffie, J. Pan, and K. Singleton, Transform analysis and asset pricing for affine jump-diffusions, Econometrica, 68(6), pp. 1343–1376, 2000.

[26] E. Fama and R. R. Bliss, The information in long-maturity forward rates, American Economic Review, 77(4), pp. 680–92, 1987.

[27] R. Fan, A. Gupta, and P. Ritchken, Hedging in the possible presence of unspanned stochastic volatility: Evidence from swaption markets, The Journal of Finance, 58(5), pp. 2219–2248, 2003.

[28] I. Fisher, *The Theory of Interest*, New York: The Macmillan Co., 1930.

[29] R. Greenwood and D. Vayanos, Bond Supply and Excess Bond Returns, Review of Financial Studies, 27(3), pp. 663–713, 2014.

[30] R. Gürkaynak and J. H. Wright, Macroeconomics and the term structure, Journal of Economic Literature, 50(2), pp. 331–67, 2012.

[31] L. P. Hansen and T. J. Sargent, Exact linear rational expectations models: specification and estimation, Technical report, 1981.

[32] D. Heath, R. Jarrow, and A. Morton, Bond pricing and the term structure of interest rates: A new methodology for contingent claims valuation, Econometrica, 60(1), pp. 77–105, 1992.

[33] S. L. Heston, A closed-form solution for options with stochastic volatility with applications to bond and currency options, The Review of Financial Studies, 6(2), pp. 327–343, 1993.

[34] J. R. Hicks, *Value and Capital*, Oxford University Press, 1939.

[35] J. Hull and A. White, Pricing interest-rate-derivative securities, Review of Financial Studies, 3(4), pp. 573–92, 1990.

[36] S. Joslin, M. Priebsch, and K. Singleton, Risk premiums in dynamic term structure models with unspanned macro risks, Journal of Finance, 69(3), pp. 1197–1233, 2014.

[37] D. H. Kim and A. Orphanides, Term structure estimation with survey data on interest rate forecasts, Journal of Financial and Quantitative Analysis, 47(01), pp. 241–272, 2012.

[38] D. H. Kim and J. H. Wright, An arbitrage-free three-factor term structure model and the recent behavior of long-term yields and distant-horizon forward rates, Technical report, 2005.

[39] R. L. Kimmel, Modeling the term structure of interest rates: A new approach, Journal of Financial Economics, 72(1), pp. 143 – 183, 2004.

[40] E. Kopp and P. D. Williams, A Macroeconomic Approach to the Term Premium, IMF Working Papers 18/140, International Monetary Fund, 2018.

[41] H. Li and F. Zhao, Unspanned stochastic volatility: Evidence from hedging interest rate derivatives, The Journal of Finance, 61(1), pp. 341–378, 2006.

[42] R. B. Litterman and J. Scheinkman, Common factors affecting bond returns, The Journal of Fixed Income, 1(1), pp. 54–61, 1991.

[43] R. B. Litterman, J. Scheinkman, and L. Weiss, Volatility and the yield curve, The Journal of Fixed Income, 1(1), pp. 49–53, 1991.

[44] S. Ludvigson and S. Ng, Macro factors in bond risk premia, Review of Financial Studies, 22(12), pp. 5027–5067, 2009.

[45] B. G. Malkiel, The term structure of interest rates, The American Economic Review, 54(3), pp. 532–543, 1964.

[46] L. Martellini, P. Priaulet, and S. Priaulet, *Fixed-Income Securities: Valuation, Risk Management and Portfolio Strategies*, Wiley, 2003.

[47] D. J. Meiselman, *The Term Structure of Interest Rates*, Englewood Cliffs, N.J.:Prentice Hall, 1962.

[48] F. Modigliani and R. J. Shiller, Inflation, Rational Expectations and the Term Structure of Interest Rates, Economica, 40(157), pp. 12–43, 1973.

[49] J. F. Muth, Rational expectations and the theory of price movements, Econometrica, 29(3), pp. 315–335, 1961.

[50] C. R. Nelson, Estimation of term premiums from average yield differentials in the term structure of interest rates, Econometrica, 40(2), pp. 277–287, 1972.

[51] Norges Bank, On fixed income investments, Staff report, Norges Bank Investment Management, 2011.

[52] M. Piazzesi, Chapter 12 - affine term structure models, in Y. Ait-Sahalia and L. P. Hansen, editors, *Handbook of Financial Econometrics: Tools and Techniques*, volume 1 of *Handbooks in Finance*, pp. 691 – 766, North-Holland, San Diego, 2010.

[53] M. Piazzesi and M. Schneider, Trend and cycle in bond premia, Technical report, 2009.

[54] R. Roll, Interest-rate risk and the term structure of interest rates: Comment, Journal of Political Economy, 74(6), pp. 629–631, 1966.

[55] G. D. Rudebusch and T. Wu, A macro-finance model of the term structure, monetary policy and the economy*, The Economic Journal, 118(530), pp. 906–926, 2008.

[56] T. Sargent, Rational expectations and the term structure of interest rates, Journal of Money, Credit and Banking, 4(1), pp. 74–97, 1972.

[57] O. Vasicek, An equilibrium characterization of the term structure, Journal of Financial Economics, 5(2), pp. 177 – 188, 1977.

[58] J. Wachter, A consumption-based model of the term structure of interest rates, Journal of Financial Economics, 79(2), pp. 365–399, 2006.

[59] J. H. Wright, Term premia and inflation uncertainty: Empirical evidence from an international panel dataset, The American Economic Review, 101(4), pp. 1514–1534, 2011.