THE COMPRESSIBLE EULER SYSTEM AND ITS NUMERICAL ANALYSIS

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ABSTRACT

THE COMPRESSIBLE EULER SYSTEM AND ITS NUMERICAL ANALYSIS

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In this thesis we analyze the compressible Euler equations in one and two dimensions. For this purpose, we firstly consider a particular form of this system, namely the inviscid Burgers equation, which can be derived by imposing vanishing pressure to the Euler system. The inviscid Burgers equation leads us to understand the idea behind discontinuous solutions such as shock and rarefaction waves. A brief analysis of smooth and weak solutions with necessary conditions for choosing physically meaningful solutions among the others, entropy and Rankine-Hugonoit conditions are studied in the first part of this work.

In the second part, the derivation of the compressible Euler equations is demonstrated in one dimension where the thermodynamic aspects are given to understand the nature of the Euler system. Furthermore, in order to illustrate the model numerically, the stability analysis of three different methods, namely Lax Friedrich, two step Lax Wendroff, and two step MacCormack methods, are examined in one dimensional case. We use Sod shock tube problem to test numerical methods since analytic solution of this problem exists. We finalize this work by a particular illustration of the

Euler model in two dimensional case by applying the Lax Friedrich's method with a short concluding remark.

Keywords: Compressible Euler equations, Finite difference method, Shock tube

ÖZ

SIKIŞTIRILABİLİR EULER SİSTEMİ VE SAYISAL ANALİZİ

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Bu tezde, sıkıştırılabilir Euler denklemlerini bir ve iki boyutta analiz ettik. Bu amaç doğrultusunda, öncelikle bu sistemin özel bir formu olan, Euler sisteminde basınç kaldırılarak elde edilebilen vizkositesiz Burgers denklemini inceledik. Vizkozitesiz Burger denklemi şok ve seyrelti dalgaları gibi süreksiz çözümlerin arkasındaki ana fikri anlamamıza yol açar. Bu çalışmanın ilk kısmında, düzgün ve zayıf çözümler ve fiziksel olarak anlamlı çözümleri seçmek için gerekli olan entropi ve Rankine-Hugonoit koşullarının kısa analizi çalışıldı.

İkinci kısımda, Euler sisteminin yapısını anlamak için termodinamik kabullerin verildiği, bir boyutta sıkıştırabilir Euler denklemlerinin çıkarılışı yapıldı. Ayrıca, sayısal olarak açıklamak için, bir boyutta üç farklı sayısal yöntemin kararlılığı çalışıldı. Sayısal yöntemleri denemek için analitik çözümü olan Sod şok tüp problemini kullandık. Bu çalışmayı, Lax Friedrich's yöntemini iki boyutta Euler modeline uygulayarak noktaladık.

Anahtar Kelimeler: Sıkıştırılabilir Euler denklemleri, Sonlu fark metodu, Şok tüp

To my parents, Sevim and Fahrettin,
my sister, Özge

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FVM	Finite Volume Method
FDM	Finite Difference Method
1D	one-dimensional
2D	two-dimensional
$\vec{F}$	flux function
$\vec{U}$	conserved quantity
$\vec{E}$	energy
u	velocity
ρ	density
p	pressure
s	characteristic speed
h	enthalpy
R	gas constant
T	temperature
δ	adiabatic constant
c_v, c_p	specific heats
k	Boltzmann's constant
α	degrees of freedom
x, y	Cartesian coordinates

CHAPTER 1

INTRODUCTION

Hyperbolic systems of conservation laws are non-linear systems of partial differential equations. These systems naturally arise in science and engineering problems that contain compressible flows and conserved quantities. The compressible Euler equations are a particular example of time-dependent non-linear hyperbolic conservation laws. The Euler equations identify laws of compressible, inviscid, perfect gases in stationary coordinates. In 1757, the form of compressible Euler equations consisted of momentum and continuity equations are published by Euler in [21]. Pierre-Simon Laplace added an adiabatic condition in a letter in 1816.[6] Finally, the necessity of conservation of energy was introduced in the second half of the 19th century. The general form of the Euler equations can be written in the form

$$\partial_t U + \partial_x (F(U)) = 0, \tag{11}$$

where U and $F(U)$ are the vectors of conserved quantities and fluxes with

$$U = \begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ (E + p)u \end{bmatrix}, \tag{12}$$

where ρ, p, u, and E are density, pressure, velocity, and total energy, respectively. The Euler equations have become widely used as a test problem in the interest of testing accuracy of numerical schemes while deriving the new numerical schemes. The numerical solutions have been studied in various papers of Yee [25], Roe [20], and Jiang and Shu [15].

The structure of the thesis is as follows:

In Chapter 2, we start by introducing hyperbolic systems and conservation laws. Since inviscid Burgers equation is a particular example of the Euler equations, we attempt to construct solution of it. Then, we introduce the concept of strong solutions, weak solutions with necessary conditions entropy, and Rankine-Hugonoit jump relations to obtain physically meaningful solutions.

In Chapter 3, we give the derivation of the compressible Euler equations. Then, the thermodynamics considerations, entropy and Rangine-Hugoniot conditions are introduced. Conservative and non-conservative formulations are studied for one dimensional Euler equations. In addition to discontinuous solution such as shock and rarefaction waves, in compressible Euler equations there is one more discontinuity which is called contact discontinuity is studied. The Riemann problem and shock tube problem is defined for the Euler equations.

In Chapter 4, analysis of three numerical schemes that we apply to approximate the solution of Sod shock tube problem in one dimension are performed. In two dimensional case the Riemann problem is solved by first order Lax Friedrich's method.

CHAPTER 2

HYPERBOLIC CONSERVATION LAWS

2.1 Hyperbolic Systems

Many of partial differential equations arise from science and engineering problems. These problems contain conserved quantities such as mass, momentum, energy. As a result of the nature of these problems we have nonlinear or linear systems of partial differential equation. To obtain these systems we use physical laws like conservation laws and thermodynamics laws. In this study, we are interested in the structure of solutions of hyperbolic conservation laws with initial data. The following formula is a general form of a systems of conservation law;

$$\partial_t u + \sum_{j=1}^{n} \partial_{x_j} f_j(u) = 0, \quad t > 0, \tag{21}$$

where $x = (x_1, ..., x_n) \in \mathbb{R}^n$ and $u = (u_1, ..., u_p)^T$ is a vector valued function, and $f_j = f(f_{1j}, ..., f_{nj})^T$ are called flux functions of system. When we say u is conserved that means it should be a constant with respect to time.

We define hyperbolic systems of conservation law as follows. Let

$$A_j(u) = \left(\frac{\partial f_{ij}}{\partial u_k}(u) \right)_{1 \leqslant i,k \leqslant n} \tag{22}$$

be the Jacobian matrix of $f_j(u)$. The system which is given by equation (21) is called hyperbolic if matrix

$$A(u, a) = \sum_{j=1}^{n} a_j A_j(u) \tag{23}$$

diagonalizable that is there is a complete set of n linearly independent corresponding eigenvectors for any u and any $a = (a_1, ..., a_n) \in \mathbb{R}^n$ and has n real eigenvalues. In

addition if all eigenvalues are distinct, system (21) is said to be strictly hyperbolic.

2.1.1 Conservation Laws

2.1.1.1 Integral and Differential Forms

Derivation of Euler equations by conservation laws is the issue of the next chapter. Here we only introduce the basic idea behind how conservation laws arise in gas dynamics. We start with the equation of conservation of mass in one dimensional case.

Denoting the gas density by $\rho(t,x)$, we assume that the density and the velocity are constant in the tube. The cross sectional area in one dimensional case is of the form $[x_1, x_2]$ and the density is per unit mass in per unit volume. It follows that, we integrate density in any section, the total mass in this section is

$$\int_{x_1}^{x_2} \rho(t,x)dx. \tag{24}$$

Denoting the velocity by $u(t,x)$, the mass flux at time t and point x is

$$\rho(t,x)u(t,x) \tag{25}$$

cause fluid must has the velocity to flow. The rate of change of the mass in $[x_1, x_2]$ is given by

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho(t,x)dx = \rho(t,x_1)u(t,x_1) - \rho(t,x_2)u(t,x_2) \tag{26}$$

which we call integral form of the conservation law. The only thing we need for obtaining differential form is to integrate equation (26) in time from t_1 to t_2, that is,

$$\int_{x_1}^{x_2} \rho(t_2,x)dx - \int_{x_1}^{x_2} \rho(t_1,x)dx = \int_{t_1}^{t_2} \rho(t,x_1)u(t,x_1) - \int_{t_1}^{t_2} \rho(t,x_2)u(t,x_2). \tag{27}$$

Assume that $\rho(t,x)$ and $u(t,x)$ are differentiable. By the fundamental theorem of calculus

$$\rho(t_2,x) - \rho(t_1,x) = \int_{t_1}^{t_2} \partial_t \rho(t,x)dx \tag{28}$$

and

$$\rho(t,x_2)u(t,x_2) - \rho(t,x_1)u(t,x_1) = \int_{x_1}^{x_2} \partial_x(\rho(t,x)u(t,x))dx. \tag{29}$$

After substitution (28) and (29) into (27) it follows that

$$\int_{t_1}^{t_2} \int_{x_1}^{x_2} \{\partial_t \rho(t,x) + \partial_x \rho(t,x) u(t,x)\} \, dx \, dt = 0. \tag{210}$$

Since limits of the integrations are arbitrary, the inequality has to be satisfied for any $[x_1, x_2]$ and $[t_1, t_2]$, thus integrand must be zero. As a result conservation of mass yields

$$\partial_t \rho + \partial_x (u\rho) = 0, \tag{211}$$

which is differential form of the conservation law.

2.1.2 Scalar Conservation Laws

We study preliminaries of Euler equations in this subsection. First we study the scalar case, that is $n = 1$. The most common notation for the scalar conservation law is

$$\partial_t u + \partial_x (f(u)) = 0, \tag{212}$$

where u represents the conserved quantity and f is the flux function of fluid flow.

2.1.2.1 Strong Solutions

We take into account the following initial value problem,

$$\begin{aligned} &\partial_t u + \partial_x (f(u)) = 0, \quad t > 0, \quad x \in \mathbb{R} \\ &u(0,x) = u_0(x), \quad x \in \mathbb{R} \end{aligned} \tag{213}$$

and let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. It follows that $\partial_x (f(u)) = \partial_u f(u) \partial_x u$ by the chain rule. Now equation (213) becomes

$$\begin{aligned} &\partial_t u + \partial_u f(u) \partial_x u = 0, \quad t > 0, \quad x \in \mathbb{R} \\ &u(0,x) = u_0(x), \quad x \in \mathbb{R}. \end{aligned} \tag{214}$$

The solution (214) is called a strong solution if it is differentiable and it satisfies initial value problem given in (213). We can write equation (214) in the quasilinear form

$$\partial_t u + b(u) \partial_x u = 0, \tag{215}$$

where $b(u) = \partial_u f(u)$. Let u be a strong solution of (214) and initial data u_0 is a differentiable. We use the method of characteristics for reducing partial differential equation to a systems of ordinary differential equations. A solution of system,

$$\begin{aligned} \partial_t x &= b(u(t, x(t))), \\ x(0) &= x_0, \end{aligned} \tag{216}$$

gives us equation of the characteristic curve through the point $(0, x_0)$. Along this curve it can be easily shown that u is constant because

$$\begin{aligned} \partial_t u(t, x(t)) &= \partial_t u(t, x(t)) + \partial_t x \partial_x u(t, x(t)), \\ &= \partial_t u + b(u)\partial_x u = 0. \end{aligned} \tag{217}$$

As a result we can say that the characteristic curves are straight lines defined by equation

$$x = x_0 + b(u_0(x_0))t = 0, \tag{218}$$

and this provides us to define smooth solutions by initial data.

Now we examine two different cases that depend on $b(u_0(x))$. Let us define the form of slope of the characteristics by

$$m_i = \frac{1}{b(u_0(x_i))}. \tag{219}$$

Case 1: $x \to b(u_0(x))$ is increasing

In this case slopes of the characteristics are decreasing. Furthermore, the characteristics cannot intersect. The solution can be defined for all t which is greater than zero.

Case 2: $x \to b(u_0(x))$ is decreasing

In this case slopes of characteristics are increasing which means characteristics intersect at some point. However at the intersection point characteristics solution cannot take both values $u_0(x_1)$ and $u_0(x_2)$. Furthermore we cannot define the strong solution for all the time $t > 0$. That is the reason we need to define weak solutions to extend the concept of solution.

2.1.2.2 The Linear Advection Equation

The fundamental example of scalar conservation law is the advection equation. If $f(u) = bu$ where b is a constant then we call the following equation

$$\partial_t u + b\partial_x u = 0,$$

as a linear advection equation. It is a quasilinear partial differential equation. Linear advection equation with initial data having proper domain has a simple solution. We consider initial value problem or Cauchy problem of linear advection equation

$$\begin{aligned} &\partial_t u + \partial_x f(u) = 0, \quad -\infty < x < \infty, \quad t \geqslant 0, \\ &u(0,x) = u_0(x) \quad -\infty < x < \infty. \end{aligned} \tag{220}$$

The solution by characteristics take the form

$$u(t,x) = u_0(x - bt), t \geqslant 0.$$

This solution is the simplest example of a wave solution. Here b is called wave speed, $u(t,x)$ is called wave and characteristics lines $x - bt =$ constant are called wavefronts.

2.1.2.3 Burgers Equation

Scalar equation

$$\partial_t u + \partial_x(f(u)) = 0,$$

is nonlinear whenever $f(u)$ is a non-linear function of u. Burgers equation is the simplest nonlinear equation and the most common problem used for numerical implementations in scalar conservation laws. Original equation studied by Burgers is

$$\partial_t u + u\partial_x u = \epsilon \partial_{xx} u$$

where the right hand side of the equation is a viscous term and $\epsilon > 0$ is the constant of viscosity. Inviscid Burgers equation is

$$\partial_t u + u\partial_x u = 0,$$

where $f(u) = \frac{1}{2}u^2$, and the viscous term is zero.

2.1.2.4 Shock Information

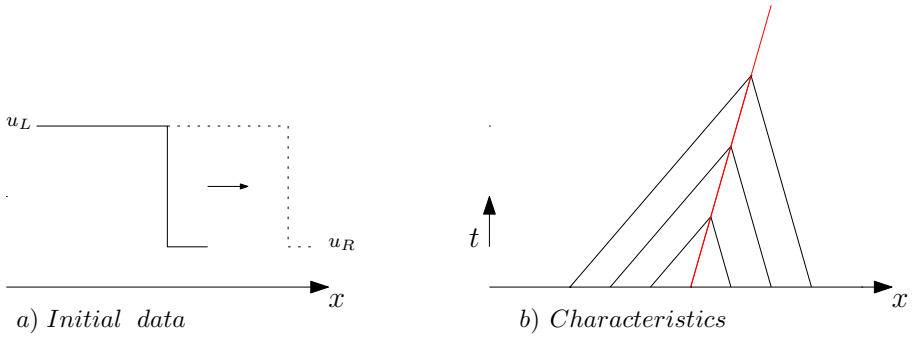
For a simple explanation of shock waves, we consider characteristics of Burgers equations defined by

$$\frac{dx}{dt} = u(t,x).$$

The solution is constant along characteristics for Burgers equation. As a result we can write characteristic

$$x(t) = u_0(x_0)t + x_0$$

where $u_0(x) = u(0, x)$ and $x_0 = x(0)$. It means that characteristics are straight lines and they can intersect or may not cover the entire (t, x)space. When characteristics intersect our solution has an infinite slope, waves break and we get a shock form. In Riemann problem, if the relation between particle speeds is $u_L > u_R$ we end up with shock waves and it's characteristics illustrated in Figure 2.1(b).



a) Initial data *b) Characteristics*

Figure 2.1: Characteristics for the initial data $u_L > u_R$

2.1.3 Rarefaction Waves

In Riemann problem, other possibility for the relation between particle speeds is $u_L < u_R$. In this case, characteristics may not intersect; however, they may not cover the whole space. Furthermore, characteristics separate regions of uniform flow which is illustrated in Figure 2.2(a). Rarefaction fan ensures that separated region can be filled and it is in Figure 2.2(b).

Definition 1 (Rarefaction fan)**.** *A rarefaction wave in which all characteristics arise from a single point in the xt plane is called rarefaction fan.*

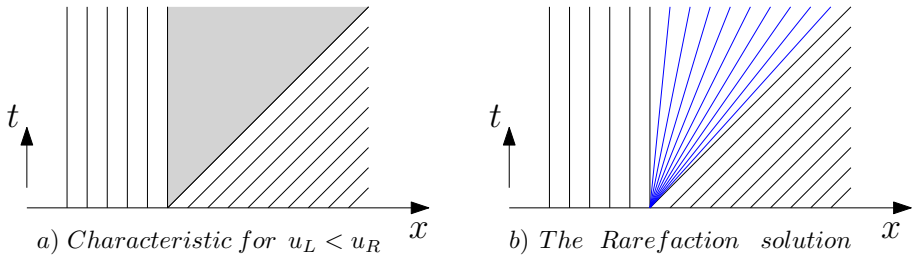
Definition 2. *Solution of the initial value problem for convex flux with initial conditions*

$$u_0(x) = u_L \quad \textit{for} \quad x < 0, \quad u_0(x) = u_R \quad \textit{for} \quad x > 0, \tag{221}$$

will be in the following form:

If $u_L < u_R$, then solution is a rarefaction wave and has the form

- *If* $x/t \leqslant f'(u_L)$, *then* $u(t,x) = u_L$.
- *If* $x/t \geqslant f'(u_R)$, *then* $u(t,x) = u_R$.
- *If* $f'(u_L) \leqslant x/t \leqslant f'(u_R)$, *then* $f'(u(t,x)) = x/t$.



a) Characteristic for $u_L < u_R$ *b) The Rarefaction solution*

Figure 2.2: The rarefaction waves

2.1.3.1 Weak Solutions

We are looking for a way to give a meaning to discontinuous solution of following initial value problem

$$\begin{aligned} &\partial_t u + \partial_x(f(u)) = 0, \quad t > 0, \quad x \in \mathbb{R}, \\ &u(0,x) = u_0(x), \quad x \in \mathbb{R}. \end{aligned} \tag{222}$$

Idea behind the weak solution is to arrange partial differential equation by using smooth test function. Sobolov spaces necessary to introduce weak formulation of differential equations.[7] We set this smooth function as $\phi(t,x)$, which is differentiable and has a compact support. Having a compact support means $\phi(t,x)$ is zero for some certain space variables such that $|x| > H$. We start to rearrange equation by multiplying partial differential equation by test function $\phi(t,x)$, take integration as needed and use integration by parts to get

$$\int\limits_{-H<x<H} \int_0^\infty u\partial_t\phi + f(u)\partial_x\phi dt\, dx + \int\limits_{-H<x<H} u(0,x)\phi(x)dx = 0.$$

As a result we do not have derivatives of u and f anymore which yields less smoothness. This way we reduce the smoothness requirement to find a solution.

Definition 3. *The function $u(t,x)$ is called a weak solution of conservation law if the following*

$$\int_0^\infty \int_{-\infty}^\infty u\partial_t\phi + f(u)\partial_x\phi dx\, dt + \int_{-\infty}^\infty u(0,x)\phi(x)dx = 0,$$

holds for all smooth functions ϕ with compact support.

Proposition 1. *Strong solutions are also weak solutions. A weak solution which is continuous and piecewise differentiable is also a strong solution.*

The advantage of the weak solution is that it contains discontinuities. However the weak solutions may not unique which can be considered as disadvantage of them. Since we deal with physical problems we want our solution to be physically meaningful. In the following, we provide some conditions to obtain more accurate solution.

2.1.3.2 Riemann Problem

An initial value problem that is composed of a conservation law together with a piecewise constant data having a single discontinuity is called Riemann problem. The following initial value problem

$$\begin{aligned} &\partial_t u + u\partial_x u = 0, \quad x \in \mathbb{R}, \qquad\qquad t \in \mathbb{R}_+, \\ &u(0,x) = \begin{cases} u_L & \text{if} \qquad x < 0, \\ u_R & \text{if} \qquad x > 0, \end{cases} \end{aligned} \tag{223}$$

is a Riemann problem for classical (inviscid) Burgers equation. The relation between u_L and u_R identifies form of solution.

Case 1: $(u_L > u_R)$
In this case solution is given by

$$u(t,x) = \begin{cases} u_L & \text{if} \qquad x - st < 0, \\ u_R & \text{if} \qquad x - st > 0, \end{cases}$$

where

$$s = (u_L + u_R)/2$$

represents characteristic (shock) speed. The left hand side waves move faster than the right hand sides and this is the reason that we end up with a shock. To verify that taller waves move faster than shorter waves, we consider general scalar conservation law

$$\partial_t u(x,t) + \partial_u f(u(x,t)) = \partial_t u(x,t) + \partial_x f \partial_x u = 0.$$

From characteristic equations the relation between speed of the solution and flux function

$$\frac{dx}{dt} = f'(u)$$

can be obtained. If we apply for Burgers equation, it follows that

$$\frac{dx}{dt} = u,$$

which illustrates that taller waves move faster.

Case 2: $(u_L < u_R)$

The solution given in the first case is also a solution for this case but we get an area without characteristic; see Figure 2.2(a). This is physically not possible. Our aim is to obtain physically meaningful solution. Because of that we need rarefaction waves. Defining solution by the following form

$$u(t,x) = \begin{cases} u_L & \text{if} \quad x < u_L t, \\ x/t & \text{if} \quad u_L t < x < u_R t, \\ u_R & \text{if} \quad u_R t < x, \end{cases} \tag{224}$$

we cover the area with characteristics and complete solution form.

2.1.3.3 Rankine-Hugoniot Jump Condition

Rankine Hugoniot jump condition provides us to control the discontinuity along characteristics and work at any jump since at a different shock speed we cannot obtain the weak solution.

Definition 4. *Rankine-Hugoniot jump condition is defined by*

$$s = \frac{f(u_L) - f(u_R)}{u_L - u_R} = \frac{[f]}{[u]},$$

where s is the shock speed (the speed at discontinuity travel); u_L, u_R are initial values given in problem, and $f(u)$ is the arbitrary flux. The notation $[.]$ represents jump across the discontinuity.

2.1.3.4 Entropy Conditions

Entropy is a measure of disorder i.e, how much energy is not available to do work. That unavailable energy is of interest in thermodynamics. We examined that the weak solution may not need to be unique. Besides Rankine-Hugoniot condition an additional condition is required.

Definition 5. *A discontinuity propagating with the speed s given by Rankine-Hugoniot jump condition satisfies the entropy condition if*

$$f'(u_L) > s > f'(u_R) \tag{225}$$

holds.

Entropy condition (225) reduces to the form $u_L > u_R$ for the convex functions.

Definition 6. *The function f is convex in the domain I if and only if*

$$\frac{f(u) - f(a)}{u - a} \leqslant \frac{f(b) - f(a)}{b - a} \leqslant \frac{f(b) - f(u)}{b - u}$$

for all $a < u < b \in I$.

In limit sense, if we set $u \to a$ in the first inequality and $u \to b$ on the second inequality then we obtain

$$f'(a) \leqslant \frac{f(b) - f(a)}{b - a} \leqslant f'(b)$$

for all $a < b$. Basically, we observe that convexity implies that f' is a strictly increasing function. That means solution u satisfies entropy condition $f'(u_L) > s > f'(u_R)$ if and only if $u_L > u_R$ for the convex functions. The geometric explanation of this condition is illustrated in Figure 2.3.

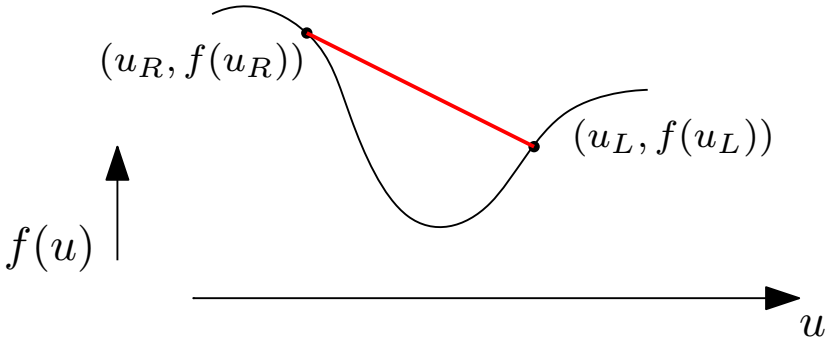


Figure 2.3: Physically acceptable shock under entropy condition

CHAPTER 3

COMPRESSIBLE EULER EQUATIONS

3.1 Compressibility

The compressibility of a fluid is the measure of the change in density that is produced in fluid by a specified change in pressure. In general, gases are highly compressible while most liquids have a very low compressibility. Although there is no incompressible fluids in real life, we use this term with such flows that the change in density with pressure is so small as to be negligible. This case usually happens with liquids. There are some factors which make change in pressure like changes in velocity in the flow. Changes in pressure affect the fluid's density, thus influencing fluid compressibility. Applications of compressible fluid theory are used in the design problems of high speed aerodynamics, rocket and missile propulsion, steam and gas turbines.

3.1.1 Derivation of Conservation Laws for the Euler Equations

Basic equations of fluid dynamics that govern fluid motion are derived from conservation of mass, conservation of momentum, and conservation of energy. In the following section, we make derivation these conservation laws.

3.1.1.1 Conservation of Mass

Basic idea of conservation of mass can be formulated by the following relation :

Rate of change of mass of fluid in control volume with respect to time :
Rate mass flow enters control volume - Rate mass flow leaves control volume

In the following, we use divergence theorem to investigate integral approach of mass conservation. Suppose $u(t,x)$ be the velocity of fluid particles passing through the point p in the control volume V. We denote fluid density and mass of fluid by $\rho(t,x)$ and $m(t)$, respectively. To express whole mass we take volume integral of density, that is per unit mass per unit volume, i.e,

$$m(t) = \iiint \rho(t,x)dV. \tag{31}$$

The rate of change of mass in control volume with respect to time is given by

$$\frac{\partial m}{\partial t} = \iiint \frac{\partial \rho}{\partial t}dV. \tag{32}$$

The flow fluid must have a velocity to be able to flow. Since we consider velocity component passing through the surface, we need to take projection of velocity into the normal of surface. Let S represent surface area. Mass flow per unit time is denoted by $\rho(t,x)\ \vec{u}.\vec{n}$. To express the whole mass flow we need to take surface integral. The rate of mass flow is of the form

$$\iint \rho\ \vec{u}.\vec{n}\ dS. \tag{33}$$

Next, we apply divergence theorem to the surface integral (33), to get

$$-\iiint \nabla(\rho\ \vec{u})dV = \iiint \frac{\partial \rho}{\partial t}dV \tag{34}$$

As a result of equation (34), if we assume that ρ and $\vec{u}$ are continuous, we can write the following equation

$$\nabla(\rho\ \vec{u}) + \frac{\partial \rho}{\partial t} = 0 \tag{35}$$

which is called the continuity equation.

3.1.1.2 Conservation of Momentum

When we compress the gas by applying a force in the region where the density is higher than neighbourhood, then it spreads out and pushes the gas particular to neighbourhood. As a result we lower the density in this region and raise the density in neighbourhood. This variation in density causes changes in the velocity.

From Newton's law of motion we know that momentum is a conserved quantity while the velocity is not. Factors that affect momentum are pressure and momentum flux.

Depending on problem, in addition to these terms, we may have gravitational force, too. Like we did in conservation of mass, we suppose that $u(t,x)$ and $\rho(t,x)$ represent velocity and density, respectively. Moreover V denotes control volume and S denotes surface area. Density of momentum is product of the density and the velocity. To find total momentum in control volume, we take volume integral of density of momentum, that is $\rho(t,x)u(t,x)$ and momentum flux is $(\rho(t,x)u(t,x))u(t,x)$. Since pressure also affects momentum, total momentum in control volume is

$$\iint_S (\rho u^2 + p)dS. \tag{36}$$

Moreover, the rate of change of momentum with respect to time is

$$\iiint \frac{\partial(\rho u)}{\partial t} dV. \tag{37}$$

The rate of change of momentum with respect to time in control volume is equal to the total force applying on the surface area. We apply divergence theorem and we substitute what we find in (37), that is,

$$\iiint \left[\quad \frac{\partial(\rho(t,x)u(t,x))}{\partial t} + \nabla(\rho u^2 + p) \quad \right] dV = 0. \tag{38}$$

Following (38), if we suppose $\rho, u, and \quad p$ are all smooth then we can express

$$\frac{\partial(\rho(t,x)u(t,x))}{\partial t} + \nabla(\rho u^2 + p) = 0, \tag{39}$$

which is differential form of momentum equation.

3.1.1.3 Conservation of Energy

To complete derivation of Euler equations, we consider finally the conserved quantity, energy. The total energy of the fluid is the sum of internal energy and kinetic energy. For total energy we use the notation E. Kinetic energy is associated with the motion of the fluid particles with the velocity $u(t,x)$, and mass of the fluid. The internal energy arises from translational, rotational, and vibrational energy of motion of the individual molecules. Indeed we have

$$E = \frac{1}{2}\rho u^2 + \rho e. \tag{310}$$

The first term in (310) is the kinetic energy, while ρe is the internal energy. We call e as *specific internal energy* that is internal energy per unit mass and it is a function of density and pressure. As done for the laws of conservation we show that, the time rate of change of energy in control volume is equal to sum of the flux of energy that acrosses surface and the net rate of work that is done by the pressure. The law of conservation of energy is

$$\frac{\partial}{\partial t}\iiint\left(\frac{1}{2}\rho u^2+\rho e\right)dV=-\iint EundS-\iint pundS. \tag{311}$$

We apply the Green's theorem and the expression (311) becomes

$$\frac{\partial}{\partial t}\iiint\left(\frac{1}{2}\rho u^2+\rho e\right)dV+\iiint\nabla((E+p)u)dV=0. \tag{312}$$

If we suppose $\rho, u,$ and p are all smooth, then we can express the equation (312) by

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2+\rho e\right)+\nabla((E+p)u)=0, \tag{313}$$

which is the last equation of the Euler system.

3.1.1.4 The One Dimensional Compressible Euler Equations

In the previous part, we obtained three equations. Putting them together gives the system of the Euler equations

$$\begin{bmatrix}\rho\\ \rho u\\ E\end{bmatrix}_t+\begin{bmatrix}\rho u\\ \rho u^2+p\\ (E+p)u\end{bmatrix}_x=0. \tag{314}$$

Or equivalently,

$$U_t+F(U)_x=0, \tag{315}$$

where

$$U=\begin{bmatrix}\rho\\ \rho u\\ E\end{bmatrix}\quad\text{and}\quad F=\begin{bmatrix}\rho u\\ \rho u^2+p\\ (E+p)u\end{bmatrix}. \tag{316}$$

We use second notation when we specify conservative formulation of the Euler equations.

3.1.2 Thermodynamic Aspects

3.1.2.1 Equation of State

We derived three equations for four unknowns in the previous section. To obtain a unique solution of this system of equations, we need additional information. These equations of states contain two more equations with one more unknown. The equations of state determines type of fluid like distinguishing air from engine oil.

3.1.2.2 Perfect Gas

For compressible flows there is an interaction between microscopic and macroscopic levels. Velocity and kinetic energy are macroscopic properties of a system, while internal energy is average microscopic property of the system. Mechanical properties describe macroscopic properties and thermodynamic properties describe microscopic properties. Density and pressure can be considered either mechanical or thermodynamic. The fluid that satisfies thermal equation of state (ideal gas law) and caloric equation of state is called a perfect gas. The ideal gas law is given by

$$p = \rho R T, \tag{317}$$

where R is a gas constant which differs for different gases and T is temperature. Moreover, the caloric equation of state is

$$e = c_v T, \tag{318}$$

where c_v is specific heat at a constant volume. We define the specific heat at a constant pressure as

$$c_p = \frac{h}{T}. \tag{319}$$

Now we introduce the state variable enthalpy which is defined by

$$h = e + \frac{p}{\rho} \tag{320}$$

and for a perfect gas we have

$$h = e + RT. \tag{321}$$

If we substitute e and h in (321), then we obtain;

$$R = c_p - cv. \tag{322}$$

The equation of the state of perfect gas ends up to depend on the ratio of specific heats and it is called adiabatic exponent, that is,

$$\delta = \frac{c_p}{c_v}. \tag{323}$$

We stated that the internal energy arises from rotational, translational and vibrational energy of motion of the molecules.These terminologies are called as degrees of freedom. Generally α states degrees of freedom. It follows that, the internal energy is given by

$$e = \frac{\alpha}{2} nkT, \tag{324}$$

where n is the number of molecules per unit mass, k is the Boltzmann's constant. Actually the product $nk = R$ is the gas constant. If we compare the equations (318) and (319) with the equation (324), it follows that

$$c_v = \frac{\alpha}{2} R \tag{325}$$

and we can rewrite c_p as

$$c_p = (1 + \frac{\alpha}{2})R. \tag{326}$$

As a result, by (325) and (326) we get

$$\delta = \frac{c_p}{c_v} = \frac{\alpha + 2}{\alpha}. \tag{327}$$

From the ideal gas law, we know that $T = \dfrac{p}{R\rho}$. Finally using this term in (318) we can state the internal energy by

$$e = c_v T = \frac{c_v}{R} \frac{p}{\rho} = \frac{p}{(\delta - 1)\rho}. \tag{328}$$

Concluding, the equation of state for perfect gas commonly uses the following form;

$$E = \frac{p}{\delta - 1} + \frac{1}{2}\rho u^2. \tag{329}$$

3.1.2.3 Entropy

The first law of thermodynamics states that the total energy in the isolated system remains constant. This is the idea we have used to obtain the equation of conservation of energy. Any physical system is obliged to satisfy the second law of thermodynamics. It can be expressed as follows: The total entropy of system never decreases. Indeed, entropy is the measure of disorder or loss information and randomness in the system. More specifically, zero entropy states excellent knowledge of the microscopic states particles of the gas particles. The more entropy increases, the more uncertainty about the microscopic states of the gas particles increases. Entropy per unit mass for perfect gas is given by

$$s = c_v \log(p/\rho^\delta) + \text{constant}. \tag{330}$$

Solving equality (330) for pressure gives

$$p = ke^{s/c_v}\rho^\delta, \tag{331}$$

where *k* is constant. Entropy maintains constant on each particle path for smooth flow. However, entropy may jump to a higher value if a particle goes through or crosses a shock. This situation is important because if we try to obtain conservation law for entropy, then entropy will not be conserved through shocks.

3.1.2.4 Rankine-Hugoniot Conditions

The following equations are known as Rankine-Hugoniot conditions for the Euler system:

$$\begin{aligned} s(\rho_1 - \rho_2) &= m_2 - m_1, \\ s(m_2 - m_1) &= \frac{m_2^2}{\rho_2} + p_2 - \frac{m_1^2}{\rho_1} - p_1, \\ s(\rho_2 E_2 - \rho_1 E_1) &= m_2 H_2 - H_1 m_1. \end{aligned} \tag{332}$$

Here by sub-indexes we denote left and right of shock. We give some of useful relations that can be derived from the relations in (332). We consider stationary shock that is $s = 0$. We have $m_1 = m_2$ from first equation of (332) and $H_1 = H_2$ from third equation of (332). Using definition $\rho = mu$ and simplify second equation of (332)

by $m_1 = m_2$ we obtain

$$u_1 - u_2 = \frac{c_2^2}{\delta u_2} - \frac{c_1^2}{\delta u_1}.$$

By the definition of enthalpy $h = \dfrac{c^2}{(\delta \breve{} 1)}$ we can write $H_1 = H_2$ as

$$\frac{1}{2}u_1^2 + \frac{c_1^2}{(\delta - 1)} = \frac{1}{2}u_2^2 + \frac{c_2^2}{(\delta - 1)}.$$

The condition $u = c$ is called the sonic condition and it is denoted by $u = c_*$. Now we can rewrite last equation as

$$\frac{1}{2}u_1^2 + \frac{c_1^2}{(\delta - 1)} = \frac{1}{2}u_2^2 + \frac{c_2^2}{(\delta - 1)} = \frac{\delta + 1}{(\delta - 1)}c_*^2.$$

3.1.3 Conservative Formulation

The formulation that we have obtained from the conservation laws is called conservative formulation. For one dimensional Euler equations differential form is of the form

$$U_t + F(U)_x = 0, \tag{333}$$

where U and $F(U)$ are given by

$$U = \begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ (E + p)u \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}.$$

The unknowns ρ, p, u, and E are density, pressure, velocity, and total energy, respectively, where

$$E = \frac{1}{2}\rho u^2 + \rho e.$$

Here e is the specific internal energy. For perfect gases we showed that

$$e = \frac{p}{(\delta - 1)\rho},$$

where $\delta = c_p/c_v$ which is the ratio of specific heats. The conservative formulation (333) can be rewritten in quasilinear form

$$U_t + A(U)U_x = 0, \tag{334}$$

where the coefficient matrix

$$A(U) = \frac{\partial F}{\partial U} = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \frac{\partial f_1}{\partial u_3} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \frac{\partial f_2}{\partial u_3} \\ \frac{\partial f_3}{\partial u_1} & \frac{\partial f_3}{\partial u_2} & \frac{\partial f_3}{\partial u_3} \end{bmatrix},$$

is the Jacobian matrix. We need to express all components of F which is flux vector in terms of the components of U. First we can easily notice that $f_1 = u_2 = \rho u$. Because both f_2 and f_3 contain pressure we need to express p in terms of components of the vector U. Using the total energy and the internal energy formulations for ideal gas, it follows that

$$p = e(\delta - 1)\rho \tag{335}$$

and

$$e = \frac{1}{\rho}\left(E - \frac{1}{2}\frac{u^2}{\rho}\right).$$

Substituting e into first equation (335), we obtain pressure in the following form

$$p = (\delta - 1)\left(u_3 - \frac{1}{2}\frac{u_2^2}{u_1}\right).$$

Now we can write flux vector

$$F = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ (E+p)u \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} u_2 \\ \frac{u_2^2}{u_1} + (\delta - 1)\left(u_3 - \frac{1}{2}\frac{u_2^2}{u_1}\right) \\ \frac{u_2}{u_1}\left[u_3 + (\delta - 1)\left(u_3 - \frac{1}{2}\frac{u_2^2}{u_1}\right)\right] \end{bmatrix}. \tag{336}$$

It follows that

$$A(U) = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{2}(\delta - 3)(\frac{u_2}{u_1})^2 & (3-\delta)(\frac{u_2}{u_1}) & (\delta - 1) \\ -\frac{\delta u_2 u_3}{u_1^2} + (\delta - 1)(\frac{u_2}{u_1})^3 & \frac{\delta u_3}{u_1} - \frac{3}{2}(\delta - 1)(\frac{u_2}{u_1})^2 & \delta(\frac{u_2}{u_1}) \end{bmatrix}.$$

In order to find eigenvalues of $A(U)$, we substitute the sound speed a given by

$$a^2\rho = \delta p$$

into matrix $A(U)$. Thus the Jacobian matrix $A(U)$ in terms of the sound speed a and the velocity u reads as

$$A(U) = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2}(\delta - 3)u^2 & (3-\delta)u & (\delta - 1) \\ \frac{1}{2}(\delta - 2)u^3 - \frac{a^2 u}{\delta - 1} & \frac{3-2\delta}{2}u^2 + \frac{a^2}{\delta - 1} & \delta u \end{bmatrix}.$$

Now we consider eigenvalues of the Jacobian matrix $A(U)$. Characteristic polynomial of A satisfies

$$|A - \lambda I| = 0,$$

that is,

$$(\lambda - u)[(\delta u - \lambda)(2u - \delta u - \lambda) + (-a^2 - (\delta - 1)u^2 + (\delta - 1)\delta u^2) + (\delta u^2(\delta - 1))] = 0.$$

Since $(\lambda - u)$ is a common factor, we have $\lambda_1 = u$ as an eigenvalue of A. To obtain other eigenvalues we simplify, characteristic polynomial by cancelling the term $(\lambda - u)$. It follows that

$$\lambda^2 - 2u\lambda + u^2 - a^2 = 0.$$

The real roots

$$\lambda_2 = u - a, \quad \lambda_3 = u + a,$$

are remaining eigenvalues. Thus corresponding eigenvectors are

$$E^{(1)} = \begin{bmatrix} 1 \\ u \\ \frac{1}{2}u^2 \end{bmatrix}, \quad E^{(2)} = \begin{bmatrix} 1 \\ u - a \\ H - ua \end{bmatrix}, \quad E^{(3)} = \begin{bmatrix} 1 \\ u + a \\ H + ua \end{bmatrix}.$$

We can show that all eigenvalues are real and eigenvectors linearly independent. In other words, one dimensional Euler equations for perfect gases are hyperbolic. In addition if the sound speed remains positive equations will be strictly hyperbolic.

3.1.4 Non-Conservative Formulation

Instead of using conserved variables, we can obtain the Euler equations in terms of non-conservative forms. For smooth solutions, both formulations are the same. However for solutions that come across with shocks, non-conservative formulation gives incorrect solutions. Main advantage of non-conservative formulation is that its easier to work and analyze equations.

3.1.4.1 The Euler Equations in Primitive Variables

Instead of conserved variable, we try to obtain the system containing variables ρ, u, and p. For obtaining the system we expand derivatives in equations that we obtain

from conservation laws, it follows that

$$\rho_t + u\rho_x + \rho u_x = 0,$$

from conservation of mass. Similarly we expand derivatives in momentum equation and use equation

$$u_t + uu_x + \frac{1}{\rho}p_x = 0.$$

From energy equation

$$p_t + \rho a^2 u_x + up_x = 0.$$

By these three equations we can write the system in quasi-linear form

$$W_t + A(W)W_x = 0,$$

where

$$W = \begin{bmatrix} \rho \\ u \\ p \end{bmatrix} \quad , \quad A(W) = \begin{bmatrix} u & \rho & 0 \\ 0 & u & \frac{1}{\rho} \\ 0 & \rho a^2 & u \end{bmatrix}.$$

Eigenvalues of this system can be computed easily which are

$$\lambda^{(1)} = u - a, \quad \lambda^{(2)} = u, \quad \lambda^{(3)} = u + a,$$

and corresponding eigenvalues

$$E^{(1)} = \begin{bmatrix} -\frac{\rho}{a} \\ 1 \\ -\rho a \end{bmatrix}, \quad E^{(2)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad E^{(3)} = \begin{bmatrix} \frac{\rho}{a} \\ 1 \\ \rho a, \end{bmatrix}$$

where a is the speed sound of perfect gas we have defined earlier. Now we consider characteristics fields that are obtained by eigenvalues whether linearly degenerate or genuinely nonlinear.

Genuinely Nonlinear and Linearly Degenerate Fields

Consider hyperbolic system

$$U_t + WU_x = 0.$$

The k-th characteristic field of this system is defined by the corresponding $\lambda^{(k)}$-th eigenvalue or $E^{(k)}$-th eigenvector. A k-th characteristic field is said to be linearly degenerate if

$$\frac{\partial \lambda^{(k)}(U)}{\partial U} E^{(k)}(U) = 0,$$

for all U. This is important because discontinuity in linearly degenerate fields is called contact discontinuity. A k-th characteristic field is said to be genuinely nonlinear if

$$\frac{\partial \lambda^{(k)}(U)}{\partial U} E^{(k)}(U) \neq 0,$$

for all U. For one dimensional Euler equation in primitive variables we obtain eigenvalues and corresponding eigenvectors. Moreover we can find characteristic fields of those three different eigenvalues. By the following calculations;

$$\begin{aligned}
\tfrac{\partial \lambda^{(1)}}{\partial U} E^{(1)} &= \begin{bmatrix} \frac{\partial (u-a)}{\partial \rho} \\ \frac{\partial (u-a)}{\partial u} \\ \frac{\partial (u-a)}{\partial p} \end{bmatrix} \begin{bmatrix} \frac{-\rho}{a} \\ 1 \\ -\rho a \end{bmatrix} = \begin{bmatrix} \frac{a}{2\rho} \\ 1 \\ -\frac{a}{2p} \end{bmatrix} \begin{bmatrix} \frac{-\rho}{a} \\ 1 \\ -\rho a \end{bmatrix} = \tfrac{\delta+1}{2} \neq 0, \\
\tfrac{\partial \lambda^{(2)}}{\partial U} E^{(2)} &= \begin{bmatrix} \frac{\partial u}{\partial \rho} \\ \frac{\partial u}{\partial u} \\ \frac{\partial u}{\partial p} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 0, \qquad (337) \\
\tfrac{\partial \lambda^{(3)}}{\partial U} E^{(3)} &= \begin{bmatrix} \frac{\partial (u+a)}{\partial \rho} \\ \frac{\partial (u+a)}{\partial u} \\ \frac{\partial (u+a)}{\partial p} \end{bmatrix} \begin{bmatrix} \frac{\rho}{a} \\ 1 \\ \rho a \end{bmatrix} = \begin{bmatrix} \frac{-a}{2\rho} \\ 1 \\ \frac{a}{2p} \end{bmatrix} \begin{bmatrix} \frac{\rho}{a} \\ 1 \\ \rho a \end{bmatrix} = \tfrac{\delta+1}{2} \neq 0,
\end{aligned}$$

it follows that $\lambda^{(1)}$ and $\lambda^{(3)}$ characteristic fields are genuinely nonlinear while $\lambda^{(2)}$ is linearly degenerate.

Contact Discontinuity

In gas dynamics we have three different kind of nonlinear waves; shocks, rarefaction, and contact discontinues. In the first section we examined shocks and rarefaction waves. Contact discontinuity occurs when characteristic field is linearly degenerate like in the Euler equations. In this case, fluid has the same characteristic speed and

pressure on both sides of contact discontinuities while density varies. Therefore contact discontinuities move with fluid while shock passes the discontinuities. Variation in the density can happen at two different temperatures.

Riemann problem for the one dimensional compressible Euler equations

In the first Chapter, we introduce the form of Riemann problem which is an initial value problem with equation obtained from conservation laws. Riemann problem for one dimensional Euler equations is

$$U_t + F(U)_x = 0, \quad x \in \mathbb{R}, \quad t > 0,$$

$$U = \begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ (E+p)u \end{bmatrix}, \tag{338}$$

with initial conditions

$$U(0,x) = U_0(x) = \begin{cases} U_L & \text{if} \quad x < 0, \\ U_R & \text{if} \quad x > 0. \end{cases} \tag{339}$$

In this notation the vector U contains conserved variables; however, while solving Riemann problem, mostly we use the vector $W^T = (\rho, u, p)^T$ of primitive variables. Therefore when we use primitive variables form, initial data consists of ${W_L}^T = (\rho_L, u_L, p_L)^T$ and ${W_R}^T = (\rho_R, u_R, p_R)^T$.

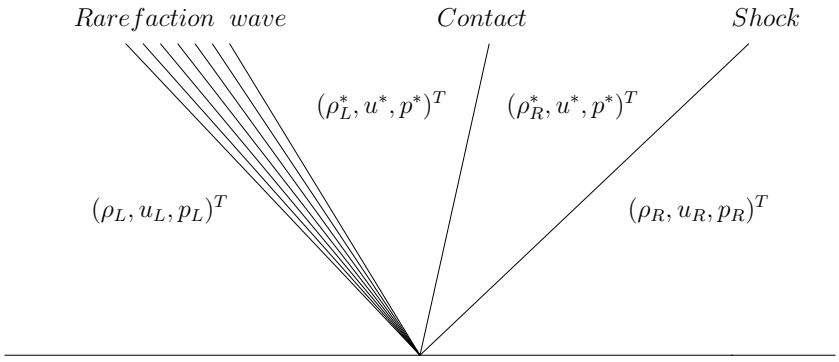


Figure 3.1: General form of solution to the Riemann problem

Riemann problem for two dimensional compressible Euler equations

Riemann problem for two dimensional Euler equations is of the form

$$U_t + F(U)_x + G(U)_y = 0, \quad x \in \mathbb{R}, \quad y \in \mathbb{R}, \quad t > 0,$$

$$U = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ E \end{bmatrix}, \; F = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ (E+p)u \end{bmatrix}, \text{ and } G = \begin{bmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ (E+p)v \end{bmatrix} \tag{340}$$

with initial conditions

$$U(0,x) = U_0(x) = \begin{cases} U_L & \text{if} \quad x < 0, \\ U_R & \text{if} \quad x > 0. \end{cases} \tag{341}$$

In two dimension pressure is defined by $p = (\gamma - 1)\left(E - \frac{1}{2}\rho(u^2 + v^2)\right)$.

Shock Tube Problem

Shock tube problem is a special case of Riemann problem since initial velocity is taken as zero everywhere. Consider a one dimensional long tube that is divided into two regions by a diagram. Tube can be filled with the same gas, but at different pressure or with two different gases. Along tube we are ignoring viscous and diffusive effects. In Figure 3.2, an illustration of shock tube is given.

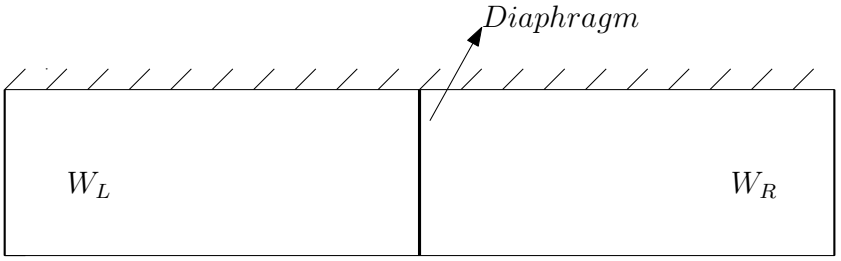


Figure 3.2: Shock tube at the initial statement

CHAPTER 4

FINITE DIFFERENCE AND FINITE VOLUME FOR CONSERVATION LAWS

In this chapter we deal with numerical methods and appropriate numerical schemes for conservation law of the form

$$\partial_t U + \partial_x(F(U)) = 0, \tag{41}$$

where U is the conserved quantity, F the conserved flux function, and x a vector of spatial coordinates. Governing equations of hyperbolic systems are expressing of conservation laws. In physically relevant problems like Euler equations, the flux function F does not depend only x and t, it depends the conserved quantity U that is $F = F(U(x,t))$.

In non-linear hyperbolic conservation laws, we deal with discontinuous solutions such as shock or rarefaction waves. At that point, conservation form of the equations are extremely important. Lax Wendroff Theorem states that, conservative numerical method converges to the weak solution of the problem, if method is convergent.[15]On the contrary, a non-conservative method does not converge to correct solution and fails at shock waves.

4.0.1 Finite Difference Methods

When finite difference methods are used to solve (41), it leads to new problems. A finite difference procedure contains replacing derivatives with finite differences by using Taylor expansion. Because of this, solution is excepted to be smooth, however, in scalar conservation laws we use weak solutions which are not smooth. Near discontinuities, standard finite difference methods are expected to give poor results. In

addition, since the weak solutions are not unique, method might converge to wrong solution. By wrong solution we mean the weak solution which does not satisfy entropy conditions. Difficulties that we have mentioned above can be solved by using conservative numerical methods.

Definition 7. *A conservative scheme for the scalar conservation law* (41) *is a numerical method of the form*

$$U_i^{n+1} = U_i^n + \tfrac{\Delta t}{\Delta x}(F_{i-1/2} - F_{i+1/2}),$$

where

$$F_{i+1/2} = F_{i+1/2}(U_{i-l_L}^n, ..., U_{i+l_R}^n),$$

with l_L, l_R are two non-negative integers, $F_{i+1/2}$ is the numerical flux which is an approximation to the physical flux $F(U)$ in the equation (41)*.*

General Formulation

First we define a mesh in the (x, t) plane with mesh width Δx and Δt which is the step length in time variable. The points (x_i, t_n) are defined by the following

$$\begin{aligned} x_i &= a + i\Delta x, \quad i = 0, ..., N, \\ t_n &= n\Delta t, \quad n = 0, 1, ..., \\ x_{i+1/2} &= (i + 1/2)\Delta x, \end{aligned}$$

where $x_{i+1/2}$ represent intermediate points. In the first Chapter, we introduced the integral form of conservation law that describes the rate of change in time of integrals. Here we have

$$\bar{U}_i^n \approx \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} U(t_n, x)dx = \frac{1}{\Delta x}\int_{C_i} U(t_n, x)dx,$$

which defines approximation of the average of $U(t_n,)$ on $[x_{i-1/2}, x_{i+1/2})$. Generally in the finite difference method we focus on producing approximations to the true solution $U(t_n, x_i)$; however, when we deal with conservation laws we consider producing approximations to the average values.

4.0.2 Finite Volume Methods

Different from the finite difference method, in finite volume method we divide the geometric domain into finite volumes or cells. Finite volume method is used to solve

computational fluid dynamics. And one of the most important properties of the finite volume method is to derive the boundary conditions. The unknown variables are evaluated at the cells not at the boundary faces.

General Formulation

First we define a grid in the (x, t) plane with discrete points (x_i, t_n) defined by the following

$$\begin{aligned} &x_i = x_L + (i + 1/2)\Delta x, \quad i = 0, ..., N, \quad \text{where} \quad \Delta x = \tfrac{x_R - x_L}{N+1}, \\ &t_n = n\Delta t, \quad n = 0, 1, ..., \\ &x_{i-1/2} = x_L + i\Delta x, \quad i = 0, ..., N + 1 \end{aligned}$$

in the domain $[x_R, x_L]$. We denote the grid cells or control volumes by

$$C_i = [x_{i-1/2}, x_{i+1/2}),$$

which is also illustrated in the following Figure 4.1.

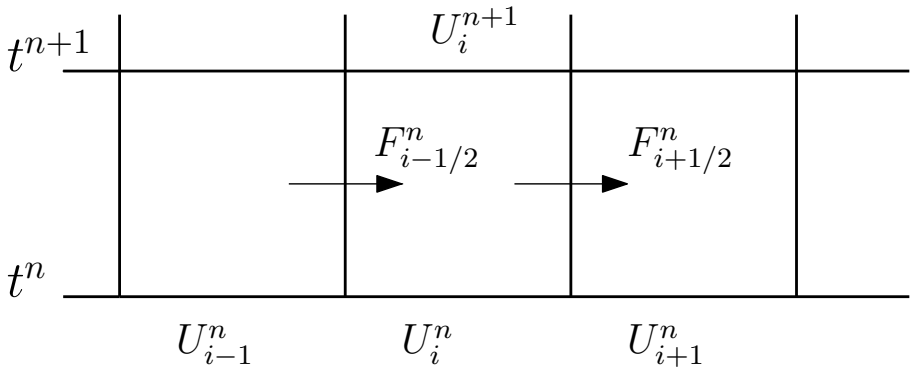


Figure 4.1: Illustration of finite volume methods

Idea Behind the Conservative Methods

The finite volume method is suitable for different types of conservation laws; elliptic, parabolic, hyperbolic. Like we have emphasized in the finite difference method, working with cell averages while driving numerical methods provides us that the numerical method is conservative. The solution that we have obtained from numerical solution behaves like true solution. In the following, we briefly examine numerical methods of the conservative forms. We consider the integral form of the conservation

law

$$\partial_t \int_{C_i} U(t,x)dx = F(U(t,x_{i-1/2})) - F(U(t,x_{i+1/2})). \tag{42}$$

The approximation of the cell averages is defined by $\bar{U}_i^n$ over the i^{th} interval and at time level t_n. Setting $\bar{U}_i^{n+1}$ as the next time level, in order to approximate it, we integrate the expression (42) from t_n to t_{n+1}. It follows that

$$\int_{C_i} U(t_{n+1},x)dx - \int_{C_i} U(t_n,x)dx = \int_{t_n}^{t_{n+1}} F(U(t,x_{i-1/2}))dt - \int_{t_n}^{t_{n+1}} F(U(t,x_{i+1/2}))dt.$$

Dividing both sides with Δx and rearranging the terms, we get

$$\begin{aligned} \frac{1}{\Delta x}\int_{C_i} U(t_{n+1},x)dx &= \frac{1}{\Delta x}\int_{C_i} U(t_n,x)dx \\ &\quad - \frac{1}{\Delta x}\left[\int_{t_n}^{t_{n+1}} F(U(t,x_{i-1/2}))dt + \int_{t_n}^{t_{n+1}} F(U(t,x_{i+1/2}))dt\right]. \end{aligned} \tag{43}$$

At this stage, we have a problem since $F(t,x_{i\pm1/2})$ varies with time along C_i and we do not have exact solution to calculate the right hand side time integrals. But we can define approximation to the average flux which is denoted by $\bar{F}_{i+1/2}^n$, that is,

$$\bar{F}_{i+1/2}^n \approx \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} F(U(t,x_{i+1/2}))dt.$$

The only thing that we need to do here is to approximate the above flux based on the values $\bar{U}_i^n$. Since we work with hyperbolic problems, we have finite speed. Thus, we set $\bar{F}_{i+1/2}^n$ which can be obtained by

$$\bar{F}_{i+1/2}^n = \mathscr{F}(\bar{U}_i^n, \bar{U}_{i+1}^n), \tag{44}$$

where $\mathscr{F}$ is some numerical flux function. Using the equation (43) with the form of $\bar{F}_{i+1/2}^n$ given in (44), it follows that

$$\begin{aligned} &\frac{1}{\Delta x}\int_{C_i} U(t_{n+1},x)dx \\ &\quad = \frac{1}{\Delta x}\left[\int_{C_i} U(t_n,x)dx - \int_{t_n}^{t_{n+1}} F(U(t,x_{i-1/2}))dt + \int_{t_n}^{t_{n+1}} F(U(t,x_{i+1/2}))dt\right]. \end{aligned} \tag{45}$$

Therefore, as a result of (45), we can write our numerical method of the form

$$\bar{U}_i^{n+1} = \bar{U}_i^n - \frac{\Delta x}{\Delta t}\left[\mathscr{F}(\bar{U}_i^n, \bar{U}_{i+1}^n) - \mathscr{F}(\bar{U}_{i-1}^n, \bar{U}_i^n)\right], \tag{46}$$

which is an explicit method. This is the numerical form that we have already derived by conservative method. The method given by (46) can be written as

$$\frac{\bar{U}_i^{n+1} - \bar{U}_i^n}{\Delta x} + \frac{F_{i+1/2}^n - F_{i-1/2}^n}{\Delta t} = 0, \tag{47}$$

which is a finite difference approximation to the conservation law

$$\partial_t U + \partial_x (F(u)) = 0.$$

Similarly many numerical methods can be equivalently written as finite difference approximations to this equation or as finite volume methods.

The CFL Condition

When we solve a problem by using the numerical methods, we have to provide numerical approximations to be convergent and stable. In 1928, Richard Courant, Kurt Friedrichs, and Hans Lewy described the condition in their paper.[21] After then, the condition is called the CFL condition, which is given in the following definition.

Definition 8 ([13])**.** *A numerical method can be convergent only if its numerical domain of dependence contains the true domain of the partial differential equation, at least in the limit as Δt and Δx go to zero.*

A simple example is given for physical interpretation of the CFL condition.

Example

Consider the linear advection equation that we have already examined in the first chapter. The exact solution of the advection is $u(t, x) = u_0(x - at)$. That means characteristics travel along the solution. Characteristics are straight lines which are parallel to each other. Suppose that we use the explicit method which is given by the equation (46) to the advection equation with positive wave speed $b > 0$. In the explicit method we express approximation value $\bar{U}_i^{n+1}$ by the values of $\bar{U}_{i-1}^n$, $\bar{U}_i^n$, and $\bar{U}_{i+1}^n$.

In Figure 4.2, we have two different cases; one of them satisfies CFL condition and the other one does not. In Figure 4.2 (a), $a\Delta t < \Delta x$. Defining the flux at $x_{i-1/2}$ in terms of the values $\bar{U}_{i-1}^n$, $\bar{U}_i^n$ makes sense. On the other hand, in Figure $4.2(b)$ we have larger time step such that $\Delta x < a\Delta t$. The flux at $x_{i-1/2}$ depends on the value

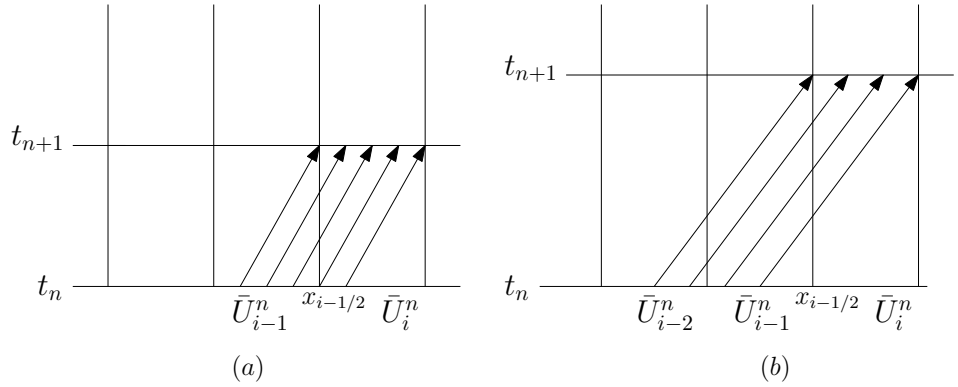


Figure 4.2: Characteristics for the advection equation

$\bar{U}^n_{i-2}$; however, in explicit method this term does not exist. That means we need a new well average for the approximation value $\bar{U}^{n+1}_i$. If we used the explicit method that given by (46) for large time, then our method would be unstable.

4.0.3 First Order Numerical Methods for One Dimension

4.0.3.1 The Lax Friedrich's Method

In order to analyze the Euler system, we firstly examine the Lax Friedrich's method. The Lax Friedrich's scheme is an explicit scheme obtained using forward difference in time and central difference in space. The scheme is amplified by taking average of u^n_j over the neighborhood cells

$$\frac{U^{n+1}_j - \frac{1}{2}(U^n_{j-1} + U^n_{j+1})}{\Delta t} = -\frac{F(U^n_{j+1}) - F(U^n_{j-1})}{2\Delta x}.$$

After leaving the U^{n+1}_j alone at the left hand side, we have the following scheme

$$U^{n+1}_j = \frac{1}{2}(U^n_{j-1} + U^n_{j+1}) - \frac{\Delta t}{2\Delta x}(F(U^n_{j+1}) - F(U^n_{j-1})), \tag{48}$$

which is called the Lax Friedrich's scheme. The illustration of the first order scheme in time and central in space given in Figure 4.3.

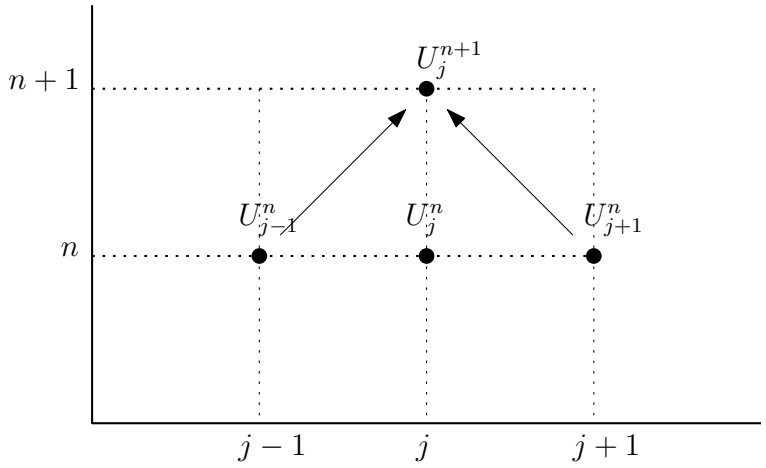


Figure 4.3: Lax Friedrich's Diagram

Von Nuemann Stability Analysis of the Lax Friedrich's Method

In order to solve any problem numerically, we must investigate the stability of the schemes for the solution of the problem. Instead of investigating stability, we consider the following linearized form

$$\partial_t U + A\partial_x(U) = 0, \tag{49}$$

where A is constant. If the stability of the scheme applied to (49) is provided with Von Neumann method then the scheme is called linearly Von Neumann stable. We analyze the linear Von Neumann stability of the Lax Friedrich's scheme. Lax Friedrich's scheme for the linear equation (49) becomes

$$U_j^{n+1} = \frac{1}{2}(U_{j-1}^n + U_{j+1}^n) - A\left(\frac{\Delta t}{2\Delta x}(U_{j+1}^n - U_{j-1}^n)\right). \tag{410}$$

The Von Neumann stability is based on the Fourier series while defining approximate solution. We suppose that our solution is of the form

$$U_j^n = \xi^n e^{i(j\Delta x)}. \tag{411}$$

Substituting (411) into (410), we obtain

$$\begin{aligned} \xi^{n+1} =& \xi^n\Big[\frac{e^{inh} + e^{inh}}{2} - \frac{A\mu(e^{inh} - e^{inh})}{2}\Big] \\ =& \xi^n[\cosh(inh) - A\mu\sinh(inh)] \end{aligned} \tag{412}$$

where $\mu = \Delta t/\Delta x = \eta/h$ and $i^2 = -1$. Recall that

$$\cosh(ix) = \cos(x), \quad \text{and} \quad \sinh(ix) = i\sin(x) \tag{413}$$

Thus the equation given by the (412) is equal to

$$\xi^{n+1} = \xi^n[\cos(nh) - iA\mu\sin(nh)]. \tag{414}$$

The amplification factor $G = \xi^{n+1}/\xi^n$ must satisfy the stability condition which is $|G| \leqslant 1$. This condition ensures error remaining bounded, that means our scheme is stable. Returning to our amplification factor $G = \cos(nh) - iA\mu\sin(nh)$, we will check whether $|G| \leqslant 1$ or not. It follows that

$$\begin{aligned} G &= [\cos(nh) - iA\mu\sin(nh)], \\ |G|^2 &= \cos^2 nh + A^2\mu^2\sin^2 nh, \\ |G|^2 &= 1 - \sin^2 nh(1 - A^2\mu^2) \leqslant 1, \end{aligned} \tag{415}$$

for $h \geqslant 0$ if and only if $|A|\mu \leqslant 1$, that is $|A|\Delta t \leqslant \Delta x$. This inequality is the CFL condition that we have mentioned in section $4.0.2$. As a result, the Lax-Friedrich's scheme is conditionally stable.

4.0.4 Second Order Numerical Methods for One Dimension

4.0.4.1 Two Step Lax-Wendroff Method

Two step Lax-Wendroff method is the second method that will be investigated. In this method, firstly half steps are calculated for time steps and space variables. The approximate value at next time step is defined by using the values at previous and half-steps. The method has second order accuracy in time and space. The two steps of the Lax-Wendroff scheme is introduced by the following equations:

First step:

$$\begin{aligned} U_{i+1/2}^{n+1/2} &= \frac{1}{2}\left(U_i^n + U_{i+1}^n\right) - \frac{\Delta t}{2\Delta x}\left[F(U_{i+1}^n) - F(U_i^n)\right] \\ U_{i-1/2}^{n+1/2} &= \frac{1}{2}\left(U_i^n + U_{i-1}^n\right) - \frac{\Delta t}{2\Delta x}\left[F(U_i^n) - F(U_{i-1}^n)\right] \end{aligned} \tag{416}$$

Second step :

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}\left[F(U_{i+1/2}^{n+1/2}) - F(U_{i-1/2}^{n+1/2})\right]. \tag{417}$$

Von Neumann Stability Analysis of the Lax Wendroff Method

Stability of the scheme that applied to solution of the linear form will be studied by Von Neumann analysis. Therefore defining two steps for linear flux provides us to reduce the number of the step size. The only thing we need to do is substituting the equations (416) into (417). We obtain

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{2\Delta x}\left[U_{i+1}^n - U_{i-1}^n\right] + \frac{(\Delta t)^2}{2(\Delta x)^2}\left[U_{i-1}^n - 2U_i^n + U_{i+1}^n\right], \tag{418}$$

which is the one step Lax Wendroff method for linear flux. Suppose solution of the linear form is $U_j^n = \xi^n e^{injh}$, after substituting solution into scheme (418) and dividing both sides with e^{injh}, it follows that

$$\xi^{n+1} = \xi^n\left[1 - iA\frac{\Delta t}{\Delta x}\sin(nh) + \frac{(\Delta t)^2}{2(\Delta x)^2}A^2(\cos(nh) - 1)\right]. \tag{419}$$

The amplification factor for the Lax-Wendroff becomes

$$\begin{aligned} G &= \left[1 - iA\mu\sin(nh) + \mu^2A^2(\cos(nh) - 1)\right] \\ |G|^2 &= \left[1 + \mu^2A^2(\cos(nh) - 1)\right]^2 + A^2\mu^2\sin^2(nh) \\ &= 1 - A^2\mu^2(1 - A^2\mu^2)(1 - \cos(nh))^2, \end{aligned} \tag{420}$$

where $\mu = \frac{\Delta t}{\Delta x}$. The norm of the amplification factor is less than one if and only if $1 - \mu^2A^2$ is greater or equal than zero. Furthermore the Lax-Wendroff method is conditionally $|A|\mu \leqslant 1$ stable.

4.0.4.2 The MacCormack's Method

Another method that we will analyze is the MacCormack's method. It is a two step (predictor-corrector type) method.

Predictor step :

$$U_i^* = U_i^n - \frac{\Delta t}{\Delta x}\left[F(U_{i+1}^n) - F(U_i^n)\right],$$

Corrector step :

$$U_i^{n+1} = \frac{1}{2}\left(U_i^n + U_i^*\right) - \frac{\Delta t}{2\Delta x}\left[F(U_i^*) - F(U_{i-1}^*)\right].$$

Von Neumann Stability Analysis of the MacCormack's Method

We apply linear Von Neumann stability analysis for all the schemes that we used in this work. We write our scheme for a linear advection flux. The MacCormack scheme for linear flux is identical to the Lax Wendroff scheme so that the stability of these two schemes has the same feature. Since we studied the Von Neumann stability of the Lax-Wendroff scheme and showed that it conditionally stable, so does MacCormack scheme.

4.0.5 First Order Numerical Methods for Two Dimension

4.0.5.1 The Lax Friedrich's Method

In the previous chapter, two dimensional form of the Euler's equation is given by

$$U_t + F(U)_x + G(U)_y = 0. \tag{421}$$

The Lax Friedrich's scheme for two dimensional Euler's equation is also first order scheme like one dimensional case. The scheme is given by

$$\begin{aligned} U_{i,j}^{n+1} =& \frac{1}{4}\left(U_{i-1,j}^n + U_{i+1,j}^n + U_{i,j-1}^n + U_{i,j+1}^n\right) \\ & - \frac{\Delta t}{2\Delta x}\left(F_{i+1,j} - F_{i-1,j}\right) - \frac{\Delta t}{2\Delta y}\left(G_{i,j+1} - G_{i,j-1}\right) \\ & i = 1, 2, \ldots, N_x - 1, \quad j = 1, 2, \ldots, N_y - 1, \end{aligned} \tag{422}$$

where N_x and N_y are the number of grid cells in x and y directions, respectively.

Von Neumann Stability Analysis of the Lax Friedrich's Method

The stability analysis for two dimension case has the same process with one dimension case. First, we linearize the system given in (421) then investigate stability of the numerical scheme for linear system. The linearized system is

$$U_t + A_1 U_x + A_2 U_y = 0, \tag{423}$$

where A_1 and A_2 are constants. Secondly, we assume the solution can be written as a Fourier series in two space variables. In x direction we choose i as a discretization index. However, here we use k instead of i to prevent possible confusion. It follows that

$$U_{k,j}^n = \xi^n e^{i(\mu_1 k\Delta x + \mu_2 j\Delta y)}, \tag{424}$$

where $i = \sqrt{-1}$, μ_1 and μ_2 are wave numbers in x and y directions. The Lax Friedrich's scheme becomes

$$\begin{aligned} U_{k,j}^{n+1} =& \frac{1}{4}\left(U_{k-1,j}^n + U_{k+1,j}^n + U_{k,j-1}^n + U_{k,j+1}^n\right) \\ & - \frac{\Delta t}{2\Delta x} A_1 \left(U_{k+1,j} - U_{k-1,j}\right) - \frac{\Delta t}{2\Delta y} A_2 \left(U_{k,j+1} - U_{k,j-1}\right) \end{aligned} \tag{425}$$

for the linearized system that is given by (423). Inserting the solution (424) into (423) and dividing both sides by $e^{i(\mu_1 k\Delta x + \mu_2 j\Delta y)}$, we obtain

$$\begin{aligned} \xi^{n+1} =& \xi^n \left[\frac{1}{4}\left(e^{-i\mu_1\Delta x} + e^{i\mu_1\Delta x} + e^{-i\mu_2\Delta y} + e^{i\mu_2\Delta y}\right)\right] \\ & - \xi^n \left[\frac{\Delta t}{2\Delta x}\left(e^{i\mu_1\Delta x} - e^{-i\mu_1\Delta x}\right) + \frac{\Delta t}{2\Delta y}\left(e^{i\mu_2\Delta y} - e^{-i\mu_2\Delta y}\right)\right]. \end{aligned} \tag{426}$$

Using the hyperbolic functions properties given in (49), equality (426) becomes

$$\begin{aligned} \xi^{n+1} =& \xi^n \left[\frac{1}{2}\left(\cos(\mu_1\Delta x) + \cos(\mu_2\Delta x)\right)\right] \\ & - \xi^n i \left[\frac{\Delta t}{\Delta x}\sin(\mu_1\Delta x) + \frac{\Delta t}{\Delta y}\sin(\mu_2\Delta y\right]. \end{aligned} \tag{427}$$

Thus the amplification factor is

$$G = \frac{1}{2}\left(\cos(\mu_1\Delta x) + \cos(\mu_2\Delta x) - i\left[\frac{\Delta t}{\Delta x}\sin(\mu_1\Delta x) + \frac{\Delta t}{\Delta y}\sin(\mu_2\Delta y)\right]\right. . \tag{428}$$

The norm of the amplification factor is

$$
\begin{aligned}
|G|^2 &= \left[\frac{1}{2}\left(\cos(\mu_1\Delta x)+\cos(\mu_2\Delta x)\right)\right]^2+\left[\frac{\Delta t}{\Delta x}\sin(\mu_1\Delta x)+\frac{\Delta t}{\Delta y}\sin(\mu_2\Delta y)\right]^2\\
&= \left[\frac{1}{4}\left(\cos^2(\mu_1\Delta x)+2\cos(\mu_1\Delta x)\cos(\mu_2\Delta x)+\cos^2(\mu_2\Delta x)\right)\right]\\
&\quad+\left[(\Delta t)^2\left(\frac{1}{(\Delta x)^2}\sin^2(\mu_1\Delta x)+\frac{2}{\Delta x\Delta y}\sin(\mu_1\Delta x)\sin(\mu_2\Delta y)+\frac{1}{\Delta y}\sin^2(\mu_2\Delta y)\right)\right]
\end{aligned}
$$

Using the inequality $2xy \leqslant x^2+y^2$, we obtain that

$$
|G|^2 \leqslant \frac{1}{2}\left[2-\sin^2(\mu_1\Delta x)\left(1-\frac{2(\Delta t)^2}{(\Delta x)^2}\right)-\sin^2(\mu_2\Delta y)\left(1-\frac{2(\Delta t)^2}{(\Delta y)^2}\right)\right]
$$

is less than one if and only if $1-\mu^2A^2$ is greater or equal than zero. Furthermore the Lax Friedrich's method is conditionally $|A|\mu \leqslant 1$ stable.

4.1 Numerical Solutions of the Compressible Euler Equations

In this section we study shock tube problem in one dimension and two dimension, separately, by using finite difference methods. We compare numerical solutions for the Sod shock tube problem with the analytic solution obtained by Sod [21]. Lax-Friedrich's, Lax-Wendroff, and MacCormack methods with the initial values which are defined by Sod are studied.

4.1.0.1 One Dimensional Case

In one dimensional case we discuss the solution of the Sod shock tube problem by using finite difference methods. The Sod shock tube problem is a Riemann problem and used as a test problem for computational fluid. The initial values taken as defined by Sod in [21]. We take the initial data as

$$
(\rho,u,p)_{t=0} = \begin{cases} (1.0,0.0,1.0) \quad \text{if} \qquad 0 \leqslant x \leqslant 0.5, \\ (0.125,0.0,0.1) \quad \text{if} \qquad 0.5 < x \leqslant 1. \end{cases} \tag{429}
$$

The Dirichlet type boundary conditions are taken for the conserved quantities. It is specified by the initial values

$$(\rho, u, p) = \begin{cases} (1.0, 0.0, 1.0) & \text{at} \quad x = 0, \\ (0.125, 0.0, 0.1) & \text{at} \quad x = 1, \end{cases} \tag{430}$$

with the ratio of the specific heat was $\gamma = 1.4$. In all of the calculations boundary is discretized by 755 grid points for the one dimensional case. The CFL condition is taken 0.5 for Lax-Friedrich's and two step Lax-Wendroff methods; however, for the MacCormack's method CFL is taken 0.66 because it breaks down for the CFL values less than 0.652.

The results of the numerical tests are illustrated in Figure (4.4), Figure (4.5), and Figure (4.6) for density, velocity, pressure, and energy for one dimensional Sod shock tube problem. It can be observed by figures that Lax-Friedrich's and two step Lax-Wendroff methods behave in the same manner; besides Lax-Friedrich's method shock wave, contact discontinuity and rarefaction wave rarely exist.

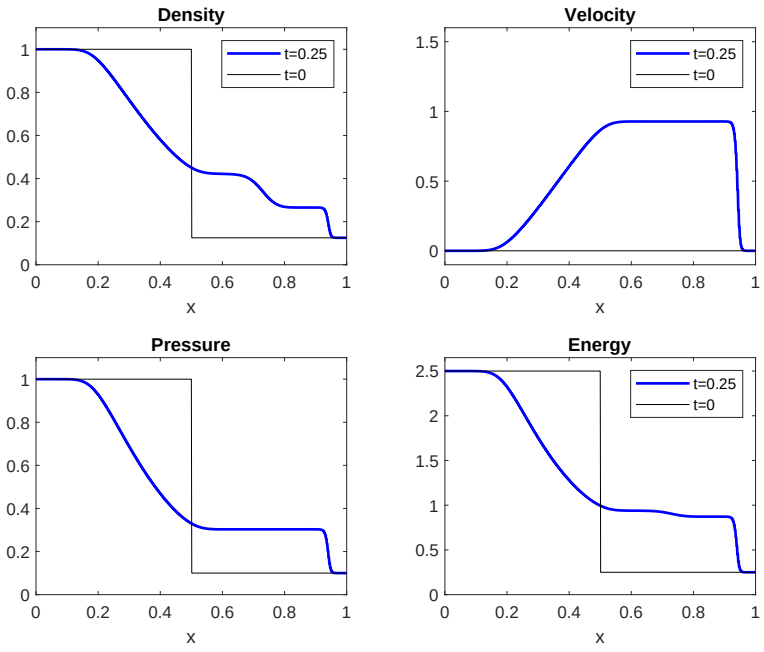


Figure 4.4: Lax-Friedrich's method solutions for Sod shock tube

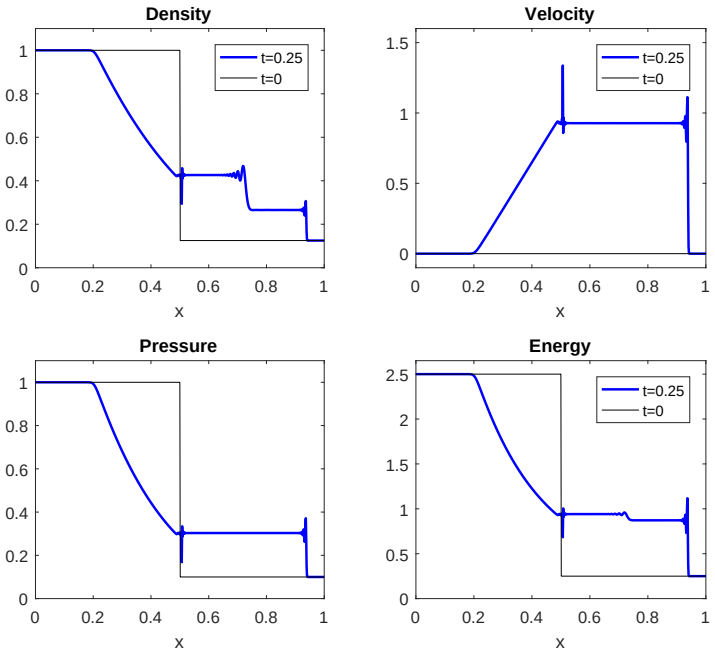


Figure 4.5: Two step Lax-Wendroff method solutions for Sod shock tube

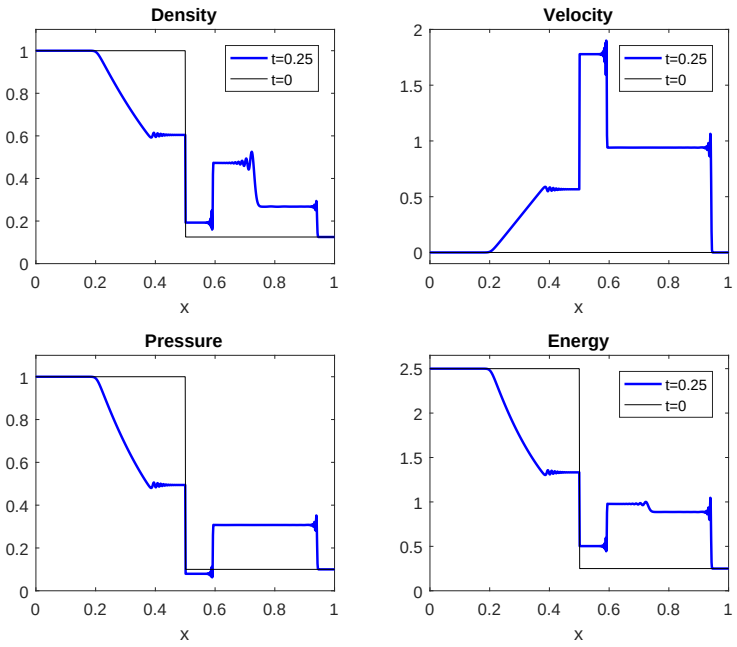


Figure 4.6: MacCormack's method solutions for Sod shock tube

4.1.0.2 Two Dimensional Case

In two dimensional case we discuss the solution of Riemann problem by using first order finite difference method. We solve two dimensional Riemann problem over the computational domain. The following initial values

$$(\rho,u,v,p)_{t=0}=\begin{cases}(0.138,1.206,1.206,0.029) & \text{if} \quad x\leqslant 0.5 \quad \text{and} \quad y\leqslant 0.5,\\ (0.5323,0.0,1.206,0.3) & \text{if} \quad x\geqslant 0.5 \quad \text{and} \quad y\leqslant 0.5,\\ (0.5323,1.206,0.0,0.3) & \text{if} \quad x\leqslant 0.5 \quad \text{and} \quad y\geqslant 0.5,\\ (1.5,0.0,0.0,1.5) & \text{if} \quad x\geqslant 0.5 \quad \text{and} \quad y\geqslant 0.5,\end{cases} \tag{431}$$

were studied. Dirichlet type boundary conditions are taken for the conserved quantities. It is specified by the initial values

$$(\rho,u,p)=\begin{cases}(0.138,1.206,1.206,0.029) & \text{at} \quad x=0,\\ (0.5323,0.0,1.206,0.3) & \text{at} \quad x=1,\\ (0.5323,1.206,0.0,0.3) & \text{at} \quad y=0,\\ (1.5,0.0,0.0,1.5) & \text{at} \quad y=1,\end{cases} \tag{432}$$

with the ratio of the specific heat is $\gamma=1.4$. The boundary is discretized by $200x200$ grid points for two dimensional case. The CFL condition is taken 0.75 for Lax-Friedrich's method. Numerical results are given for the time 0.3.

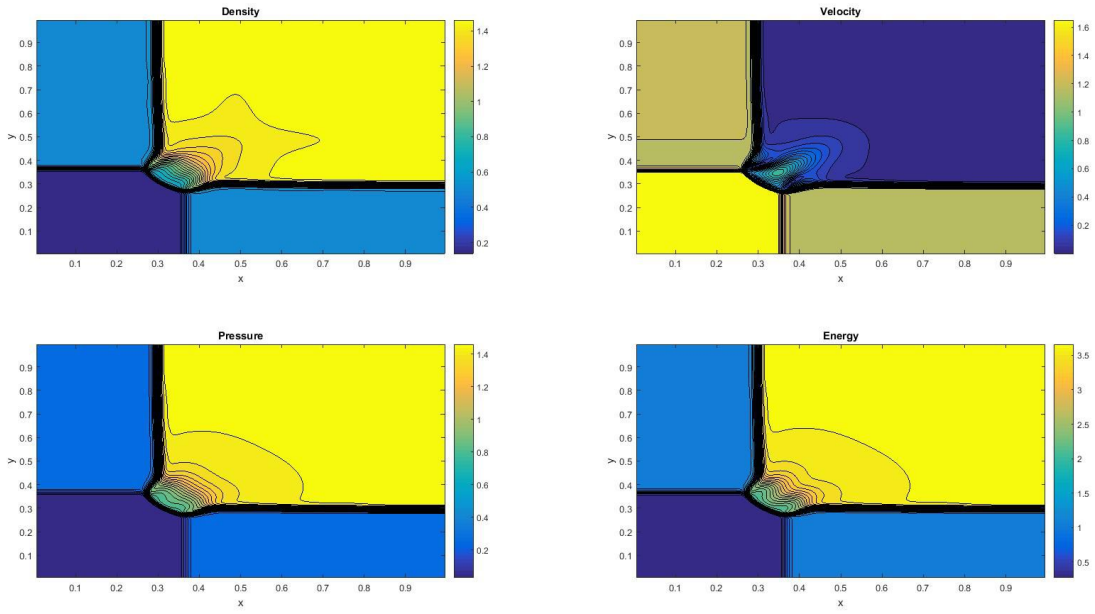


Figure 4.7: Lax Friedrich's solution for two dimensional Riemann problem

CHAPTER 5

CONCLUSION

This thesis is mainly focused on the derivation of conservative equations of the compressible Euler system and provides a numerical study by means of finite difference and finite volume approximations in one and two dimensional cases. The compressible Euler system is a system of non-linear hyperbolic conservation laws consisting of second law of thermodynamics. In gas dynamics, the effect of body forces and viscous stress can be neglected; that is, dropping these terms from the Navier-Stokes equations gives a hyperbolic system of conservation laws. Analogously, by imposing zero pressure to the compressible Euler equations, a particular form of this system so called the inviscid Burgers equation, known as one of the simplest non-linear partial differential equation, can be derived. In the first part, we examine the inviscid Burgers equation to understand the idea behind discontinuous solutions such as shock and rarefaction waves. Beside, we analyze smooth and weak solutions with necessary conditions for choosing physically meaningful solutions among the others, entropy conditions and Rankine-Hugonoit jump relation in the first part of this work.

In the second part, we extended our analysis in the first part to the compressible Euler system. Mainly, the derivation of the compressible Euler equations is firstly presented in one dimensional form where the thermodynamic aspects are given to understand the nature of the Euler system. In addition to shock and rarefaction waves, while dealing with the Euler system, there exists one more important phenomena which is called contact discontinuity. We provide the Riemann problem and consider a well-known test problem used for compressible fluid flows, so called Sod shock tube problem in one dimensional case. Moreover, for numerical part, we start by analyzing the stability of the Lax Friedrich, two step Lax Wendroff and two step MacCormack

methods in one dimensional form. Finally, we illustrate some numerical results of the model in two dimensional case by applying the Lax Friedrich's method.

- The Lax Friedrich's scheme is the most diffusive method as expected.
- The solution obtained by two step Lax-Wendroff and MacCormack scheme oscillate at the contact discontinuity.
- The rarefaction results of two step Lax-Wendroff and MacCormack scheme are similar and more accurate than Lax-Friedrich's scheme.
- Two steps MacCormack scheme breaks down for CFL$< 0.6518.$
- As we excepted, the solutions of second order methods more accurate than Lax Friedrich's method.
- In two dimension, although the accuracy of Lax-Friedrich's scheme is first order we obtain good results.

The three schemes for the Euler equations were compared for Sod shock tube problem. From this study, the following conclusions may be drawn:
Lax-Friedrich's and two steps Lax-Wendroff methods behave in the same manner; besides Lax-Friedrich's method shock wave, contact discontinuity and rarefaction wave rarely exists.

REFERENCES

[1] C. O. Bennett and J. E. Myers. *Momentum, heat, and mass transfer*. McGraw-Hill, 1962.

[2] A. J. Chorin, J. E. Marsden, and J. E. Marsden. *A mathematical introduction to fluid mechanics*, volume 3. Springer, 1990.

[3] R. Courant and K. O. Friedrichs. *Supersonic flow and shock waves*, volume 21. Springer Science & Business Media, 1999.

[4] B. Després. *Numerical Methods for Eulerian and Lagrangian Conservation Laws*. Birkhäuser, 2017.

[5] M. Feistauer, J. Felcman, J. Felcman, and I. Straškraba. *Mathematical and computational methods for compressible flow*. Oxford University Press on Demand, 2003.

[6] B. S. Finn. Laplace and the speed of sound. *Isis*, 55(1):7–19, 1964.

[7] C. Grossmann, H.-G. Roos, and M. Stynes. *Numerical treatment of partial differential equations*, volume 154. Springer, 2007.

[8] J. S. Hesthaven. *Numerical methods for conservation laws: From analysis to algorithms*, volume 18. SIAM, 2018.

[9] D. Kröner. *Numerical schemes for conservation laws*. Wiley, 1997.

[10] C. B. Laney. *Computational gasdynamics*. Cambridge University Press, 1998.

[11] P. Lax and B. Wendroff. Systems of conservation laws. *Communications on Pure and Applied mathematics*, 13(2):217–237, 1960.

[12] P. D. Lax. *Hyperbolic systems of conservation laws and the mathematical theory of shock waves*, volume 11. SIAM, 1973.

[13] R. J. LeVeque. *Finite volume methods for hyperbolic problems*, volume 31. Cambridge University Press, 2002.

[14] H. W. Liepmann and A. Roshko. *Elements of gasdynamics*. Courier Corporation, 2001.

[15] M.-S. Liou. A sequel to ausm: Ausm+. *Journal of Computational Physics*, 129(2):364–382, 1996.

[16] K. Masatsuka. *I do Like CFD, vol. 1*, volume 1. Lulu. com, 2013.

[17] R. E. Meyer. *Introduction to mathematical fluid dynamics*, volume 24. Courier Corporation, 1982.

[18] P. H. Oosthuizen and W. E. Carscallen. *Compressible fluid flow*, volume 179.

[19] P. H. Oosthuizen and W. E. Carscallen. *Introduction to compressible fluid flow*. CRC press, 2013.

[20] P. L. Roe. Approximate riemann solvers, parameter vectors, and difference schemes. *Journal of Computational Physics*, 43(2):357–372, 1981.

[21] G. A. Sod. A survey of several finite difference methods for systems of nonlinear hyperbolic conservation laws. *Journal of Computational Physics*, 27(1):1–31, 1978.

[22] E. F. Toro. *Riemann solvers and numerical methods for fluid dynamics: a practical introduction*. Springer Science & Business Media, 2013.

[23] W. G. Vincenti and C. H. Kruger. *Introduction to physical gas dynamics*, volume 246. Wiley New York, 1965.

[24] P. Wesseling. *Principles of computational fluid dynamics*, volume 29. Springer Science & Business Media, 2009.

[25] H. C. Yee. Upwind and symmetric shock-capturing schemes. *NASA Technical Memorandum*, 1987.