CALCULATIONS OF THE ROOTS OF CLASSICAL ORTHOGONAL POLYNOMIALS: AN APPLICATION TO GAUSSIAN QUADRATURE

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ABSTRACT

CALCULATIONS OF THE ROOTS OF CLASSICAL ORTHOGONAL POLYNOMIALS: AN APPLICATION TO GAUSSIAN QUADRATURE

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This thesis focuses on classical orthogonal polynomials namely Jacobi, Laguerre and Hermite polynomials and a method to calculate the roots of these polynomials is constructed. The roots are expressed as the eigenvalues of a tridiagonal matrix whose coefficients depend on the recurrence formula for the classical orthogonal polynomials. These approximations of roots are used as method of computation of Gaussian quadratures. Then the discussion of the numerical results are then introduced to deduce the efficiency of the method.

Keywords: Gaussian Quadrature, Orthogonal functions, Jacobi, Hermite, Laguerre and Legendre polynomials , classical orthogonal polynomials,equation of hypergeometric type

ÖZ

KLASİK ORTHOGONAL POLİNOMLARIN KÖKLERİNİN HESAPLANMASI: GAUSS KARELEME YÖNTEMİYLE SAYISAL İNTEGRASYON İÇİN BİR UYGULAMA

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Bu tez, Jacobi, Laguerre ve Hermite polinomları gibi klasik ortogonal polinomlara ağırlık vermekte ve bu polinomların köklerinin hesaplanması için bir yöntem oluşturulmaktadır. Kökler klasik ortogonal polinomlar için rekürrens formülüne bağlı olan bir tridiagonal matrisin özdeğerleridir. Köklerin bu yaklaşımları, Gauss karelemeyi hesaplama yöntemi olarak kullanılmıştır. Daha sonra yöntemin verimliliğinin çıkarılması için sayısal sonuçlar tanıtılmıştır.

Anahtar Kelimeler: Gauss kareleme, hipergeometrik tipte denklem, klasik dik polinomlar, Jacobi,Laguerre ve Hermite polinomları

To my family

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EHT	Equation of Hypergeometric type

CHAPTER 1

INTRODUCTION

Numerical integration, also called quadrature, is the approximate computation of an integral of a given function using numerical techniques. There are several reasons for carrying out numerical integration. One of the reasons is computer applications and embedded systems need numerical integration to compute the data obtained by sampling when the integration of the function is known only at certain points. Another case when the numerical integration is preferred is when it is difficult or impossible to integrate the function. The most famous of numerical integration are Newton-Cotes and Gaussian quadrature [1] .
The idea of Gaussian quadrature is to give ourselves the freedom to choose not only the weighting coefficients, but also the location of the abscissas at which the function is to be evaluated: They will no longer be equally spaced. Thus, we will have twice the number of degrees of freedom at our disposal; it will turn out that we can achieve Gaussian quadrature formulas whose order is, essentially, twice that of the Newton-Cotes formula with the same number of function evaluations [2].
In practice, it is common to prefer lower order methods over higher order and Newton-Cotes over Gaussian. However, Sermutlu compared Gaussian and Newton-Cotes methods with each other at given orders and compared higher order and lower order methods within each category and concluded that Gaussian method at the highest order gives more accurate results than all other methods. In other words, Gaussian is superior to Newton–Cotes and higher orders are superior to lower orders, as the number of points increase [3].

A.H.Stroud and Don Secrest said that advantages of Gaussian quadrature formula are that they converge under conditions which are most always realizable in practice and the order of precision with reference to the degree of the polynomial for which a specific formula is exact is much greater than that for the corresponding Newton-Cotes formula [4].

The special functions is a huge area of mathematics which includes trigonometric, exponential and logarithmic functions and also it extended to include beta, gamma functions and classes of classical orthogonal polynomials[5]. These special functions have many and varied applications in pure mathematics and numerous applied sciences such as astronomy, heat conduction, electric circuits, quantum mechanics, electrostatic interpretation and mathematical statistics [5, 6, 7, 8, 9, 10] .
The special functions satisfy the second order differential equation of the form

$$\sigma(z)y'' + \tau(z)y' + \lambda y = 0, z \in \mathbb{C} \tag{1.1}$$

where λ is constant, $\sigma(z)$ is at most a quadratic polynomial and $\tau(z)$ is a linear polynomial. The roots of special functions are related to resonances in mechanical and electromagnetic systems, arise in quantum mechanical calculations, and are the nodes of Gaussian quadrature formulas[10]. Their role as the nodes of Gaussian quadrature formulas motivates much of the interest in the numerical computation of the roots of special functions defined by equations of the form (1.1). Moreover, the zeros of the Jacobi polynomials play an interesting role in a problem of Stieltjes concerning electrostatic equilibrium.
Each family of classical orthogonal polynomials — Legendre polynomials, Hermite polynomials, Laguerre polynomials — is associated with a nonnegative weight function w and a collection of Gaussian quadrature rules, one for each positive integer n, of the form

$$\int_a^b \varphi(t)w(t)dt = \sum_{j=1}^{n} \varphi(t_j)w(j) \tag{1.2}$$

The Gaussian quadrature rule (1.2) is exact when φ is a polynomial of degree less than $2n$. The nodes $t_1, \ldots, t_n$ are the roots of a polynomial p of degree n that satisfies an equation of the form (1.1), and, at least in the case of the classical orthogonal polynomials, the weights $w_1, \ldots, w_n$ can be calculated from the values of the derivatives of p at the nodes $t_1, \ldots, t_n$ [10]. In 1984, Gauss constructed Gauss-Legendre quadrature rule using hypergeometric functions and continued fractions [11], and in

1826, Jacobi noted the quadrature nodes where precisely the roots of the Legendre polynomial of degree n [12].

Moreover, Andreas Glaser and Vladimir Rokhlin showed that the Gauss quadrature nodes are the simple roots of an associated orthogonal polynomials [13]. Thus, we could say zeros of classical special functions play an important role in the mathematical sciences, being related to Gaussian quadratures.

In last decades, many methods are proposed to calculate the n-point Gaussian quadrature and in [14], Newton's method is combined with a scheme, based on asymptotic formulas, for evaluating Legendre polynomials of arbitrary orders and arguments. A similar approach is taken in [15], in which asymptotic formulas for Jacobi polynomials are used to evaluate Newton iterates which converge to the nodes of Gauss-Jacobi quadrature rules. The most widely known method for computing Gauss quadrature nodes and weights is the Golub-Welsh algorithm [16], which exploits the three term recurrence relations satisfied by all real orthogonal polynomials and the eigenvalues of symmetric tridiagonal matrices in order to calculate the nodes of an n-point Gaussian quadrature formula . The scheme of [17] uses a rapidly convergent fixed point method to calculate the roots of solutions of quite general second order differential equations of the form (1.1). Unlike Newton's method, the fixed point scheme of [10, 17], is guaranteed to converge. This approach has the disadvantage, though, that the regions of monotonicity of the coefficient q must be explicitly known. The schemes of [14, 15, 17, 18, 19] all have the property that any particular quadrature node and its corresponding weight can be calculated independently of the others, making them suitable for parallelization.

Moreover, in 2007, Glasier-Liu-Rokhlin [13] developed an algorithm which computes all the nodes and weights of the n-point quadrature rule in very effective way that can compute one million quadrature nodes in a handful of seconds. The Glaser-Liu-Rokhlin method [13] combines the Prufer transform with the classical Taylor series method for the solution of ordinary differential equations. It computes n roots in $O(n)$ operations and is more general than the schemes [14, 15, 18] in that it applies to special functions defined by second order differential equations of the form

$$r_0(t)y''(t) + p_0(t)y'(t) + q_0(t)y(t) = 0 \tag{1.3}$$

where p_0, q_0 and r_0 are polynomials of degree less than or equal to two. This class includes the classical orthogonal polynomials, Bessel functions, prolate spheroidal wave functions, etc. The Glaser-Liu-Rokhlin algorithm does, however, suffer from several disadvantages. It is typically slower than the methods discussed above, and is unsuitable for parallel implementation since the roots must be computed in sequential order[10].

Gaussian quadrature rule is used in many areas of mathematics and some of them are mentioned below. Jayan Sarada and K.V.Nagaraja presented gaussian quadrature method for numerical integration over triangular, parallelogram and quadrilateral elements with linear sides [20].

Takashi Amisaki presented a numerical integration method for estimating the area under the curve (AUC) over the infinite time interval. This method is based on the Gauss–Laguerre quadrature and produces AUC estimates over the infinite time interval without extrapolation in a usual sense [22].
Ping Yang, Xue-Wen Feng, Wen-Jun Liang and Kai-SuWu used Gaussian-Laguerre quadrature formula which is less nodes and higher precision numerical integral formula to approximate integral term of IBBRP (The inverse black body radiation problem) [23].

The thesis is organized as follows. In Chapter 2, we give the necessary knowledge about special functions and introduce Jacobi, Laguerre and Hermite polynomials with hypergeometric type and recurrence relations of classical orthogonal polynomials. In chapter 3, we give the method to find the roots of classical orthogonal polynomials and some roots and weights are tabulated. Numerical applications of the Gaussian quadrature and their results are given in Chapter 4.

CHAPTER 2

CLASSICAL ORTHOGONAL POLYNOMIALS

In this chapter, we review some basic and remarkable properties of the classical orthogonal polynomials which are very well known in literature. Here, we present the results without their proofs and refer the reader to Nikiforov and Uvarov [28] for more about the special functions.

2.1 Equation of Hypergeometric type

Special functions including the classical orthogonal polynomials namely Hermite, Laguerre, Legendre, Chebyshev and Jacobi and hypergeometric functions are all generated by differential equation of the form

$$\sigma(x)y'' + \tau(x)y' + \lambda y = 0 \tag{2.1}$$

where λ is constant, $\sigma(x)$ is at most a quadratic and $\tau(x)$ is a linear polynomial. Here, x denotes a complex variable, in general. (2.1) is called as an equation of hypergeometric type (EHT). First, we consider more general equation of type

$$u'' + \frac{\tilde{\tau}(x)}{\sigma(x)}u' + \frac{\tilde{\sigma}(x)}{\sigma^2(x)}u = 0 \tag{2.2}$$

where $\sigma(x)$ and $\tilde{\sigma}(x)$ is at most quadratic and $\tilde{\tau}(x)$ is a linear polynomial. Now, introducing the transformation

$$u = \varphi(x)y \tag{2.3}$$

and substituting into (2.2), we get

$$y'' + (2\frac{\varphi'}{\varphi} + \frac{\tilde{\tau}}{\sigma})y' + (\frac{\tilde{\sigma}}{\sigma^2} + \frac{\tilde{\tau}}{\sigma}\frac{\varphi'}{\varphi} + \frac{\varphi''}{\varphi})y = 0 \tag{2.4}$$

in which the coefficient of y' can be made a rational function $\frac{\tau(x)}{\sigma(x)}$, where $\tau(x)$ is a linear polynomial. Then, we have

$$2\frac{\varphi'(x)}{\varphi(x)} = \frac{\tau(x) - \tilde{\tau}(x)}{\sigma(x)}$$

and

$$\frac{\varphi'}{\varphi} = \frac{p(x)}{\sigma(x)}, \qquad p(x) := \frac{1}{2}[\tau(x) - \tilde{\tau}(x)]. \tag{2.5}$$

Here, $p(x)$ is a polynomial of degree at most 1. Since

$$\frac{\varphi''}{\varphi} = \left(\frac{\varphi'}{\varphi}\right)' + \left(\frac{\varphi'}{\varphi}\right)^2 = \left(\frac{p}{\sigma}\right)' + \left(\frac{p}{\sigma}\right)^2$$

equation (2.2) now takes the form

$$y'' + \frac{\tau(x)}{\sigma(x)}y' + \frac{\tilde{\tilde{\sigma}}(x)}{\sigma^2(x)}y(x) = 0. \tag{2.6}$$

Here, $\tilde{\tilde{\sigma}}(x)$ is a quadratic polynomial,

$$\tau(x) = 2p(x) + \tilde{\tau}(x) \tag{2.7}$$

and

$$\tilde{\tilde{\sigma}}(x) = p^2(x) + [\tilde{\tau}(x) - \sigma'(x)]p(x) + \sigma(x)p'(x) + \tilde{\sigma}(x). \tag{2.8}$$

Equation (2.6) is the same type of equation (2.2) so that type of the generalized EHT in (2.2) remains invariant under the substitution $u = \varphi(x)y$, where the transformation doesn't modify the type of the given differential equation. The coefficients $\tau(x)$ and $\tilde{\tilde{\sigma}}(x)$ in the transformed equation (2.6) depend on the selection of $p(x)$. Therefore, we may choose $p(x)$ such that $\tilde{\tilde{\sigma}}(x)$ is a multiple of σ , i.e

$$\tilde{\tilde{\sigma}}(x) = \lambda\sigma(x), \qquad \lambda \in \mathbb{C}. \tag{2.9}$$

Then, equation (2.6) is

$$y'' + \frac{\tau(x)}{\sigma(x)}y' + \frac{\lambda}{\sigma(x)}y = 0$$

which is equal to EHT in (2.1).

2.2 Polynomial solution of EHT

The EHT has solutions such that $y(x) = y_n(x)$, where $y_n(x)$ is a polynomial of degree n and depends on the variable x for particular values of λ given by the formula:

$$\lambda = \lambda_n = -n\tau' - \frac{1}{2}n(n-1)\sigma''. \tag{2.10}$$

2.2.1 Rodrigues Theorem

Theorem 1. *The polynomial solutions of the EHT can be constructed by a formula*

$$y_n(x) = \frac{B_n}{\rho(x)} \frac{d^n}{dx^n}[\sigma^n(x)\rho(x)] \tag{2.11}$$

known as the Rodrigues formula. B_n *is called a normalization constant.*

We can write the EHT in a self adjoint form

$$\frac{d}{dx}[\sigma(x)\rho(x)\frac{dy}{dx}] + \lambda\sigma(x)y = 0$$

where $\rho(x)$ satisfies the separable first order ODE

$$[\sigma(x)\rho(x)]' = \tau(x)\rho(x). \tag{2.12}$$

Solving equation (2.12), we obtain possible forms of $\rho(x)$ as

$$\begin{array}{lll} (b-x)^\alpha(x+a)^\beta & \text{for} & \sigma(x) = (b-x)(x-a) \\ (x-a)^\alpha e^{\beta x} & \text{for} & \sigma(x) = x-a \\ e^{\alpha x^2+\beta x} & \text{for} & \sigma(x) = 1 \end{array}$$

corresponding to possible degrees of $\sigma(x)$ where a, b, α, β are constants.
Now, we can reduce $\sigma(x)$ and $\rho(x)$ to the canonical forms:

$$\begin{array}{lll} (1-x)^\alpha(x+1)^\beta & \text{for} & \sigma(x) = 1-x^2 \\ x^\alpha e^{-x} & \text{for} & \sigma(x) = x \\ e^{-x^2} & \text{for} & \sigma(x) = 1 \end{array} \tag{2.13}$$

2.3 Orthogonality of Polynomial Solutions of EHT

Theorem 2. *Suppose the coefficients in EHT are such that*

$$\sigma(x)\rho(x)x^j|_{x=a,b} = 0, \quad j = 0, 1, ... \tag{2.14}$$

where a *and* b *are the end points of a finite or infinite interval. Then, the polynomial solutions* $y = p_n(x)$ *constitute an orthogonal sequence of real functions of the real argument x*

$$p_0(x), p_1(x), p_2(x), \ldots, p_m(x), \ldots, p_n(x), \ldots.$$

In other words,

$$\int_a^b \rho(x)p_n(x)p_m(x)dx = \begin{cases} 0 & \text{if} \quad m \neq n \\ \mathcal{N}_n^2 & \text{if} \quad m = n \end{cases} \tag{2.15}$$

where $\mathcal{N}_n$ is called a normalization constant and $\rho(x)$ is called a weighting function.

2.4 General Properties of Orthogonal Polynomials

The classical orthogonal polynomials namely Jacobi, Legendre, Laguerre and Hermite which are polynomial solutions of hypergeometric differential equations in (2.1) are obtained by choosing the values for $\sigma(z)$, $\tau(z)$ and λ in (2.1).

2.4.1 Jacobi polynomials

Jacobi Polynomials are denoted by ${P_n}^{(\alpha,\beta)}(x)$. Suppose $\sigma(x) = (1-x^2)$ and $\rho(x) = (1-x)^\alpha(1+x)^\beta$.
Then (2.12) reads as

$$[(1-x^2)(1-x)^\alpha(1+x)^\beta]' = \tau(x)(1-x)^\alpha(1+x)^\beta$$

and we obtain

$$\tau(x) = \beta - \alpha - (\alpha+\beta+2)x.$$

Also, (2.10) becomes $\lambda_n = n(n+\alpha+\beta+1)$.

The EHT

$$(1-x^2)y'' + [\beta - \alpha - (\alpha+\beta+2)x]y' + n(n+\alpha+\beta+1)y = 0 \tag{2.16}$$

is known as Jacobi differential equation, and their polynomial solutions are Jacobi polynomials. Jacobi polynomial is expressed by the Rodrigues formula

${P_n}^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}[(1-x)^{n+\alpha}(1+x)^{n+\beta}]$

$= a_n x^n + b_n x^{n-1} + \ldots$

where the normalization constant B_n is taken as $B_n = \frac{(-1)^n}{2^n n!}$ for historical reasons. a_n is the leading and b_n is the sub-leading coefficient of the Jacobi Polynomial $P_n^{(\alpha,\beta)}$. The coefficients are expressed by the below formula:

$$a_n = \frac{(\alpha+\beta+1+n)_n}{2^n n!} \tag{2.17}$$

and

$$\frac{b_n}{a_n} = \frac{n(\alpha-\beta)}{\alpha+\beta+2n} \tag{2.18}$$

where the symbol $(t)_n = t(t+1)\cdots(t+n-1)$ is called Pochammer's symbol for any t. Thus here

$$(\alpha+\beta+1+n)_n = (\alpha+\beta+1+n)(\alpha+\beta+2+n)\cdots(\alpha+\beta+2n).$$

Classical Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ are orthogonal. From the hypothesis of the Theorem 2,

$$\sigma(x)\rho(x)x^j|_{x=a,b} = (1-x)^{\alpha+1}(x+1)^{\beta+1}x^j|_{x=a,b} = 0, \quad j = 0,1,2,\ldots$$

is satisfied if $(a,b) = (-1,1)$ provided that $\alpha > -1$ and $\beta > -1$.

The normalization factor of Jacobi polynomial is $\mathcal{N}_n$ where

$$\mathcal{N}_n^2 = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1))}{n!(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)} \tag{2.19}$$

and the Gamma function is defined by

$$\Gamma = \int_0^\infty t^{x-1}e^t dt.$$

satisfies the recurrence relation

$$\Gamma(x+1) = x\Gamma(x).$$

2.4.1.1 Legendre polynomials

The subclass of Jacobi polynomials when $\alpha = \beta = 0$ is called Legendre polynomial. Legendre polynomials are given

$$P_n{}^{(0,0)} = P_n = \frac{(-1)^n}{2^n n!}\frac{d^n}{dx^n}[(1-x^2)^n] \tag{2.20}$$

and satisfy the differential equation of the form

$$(1-x^2)P_n'' - 2xP_n' + n(n+1)P_n = 0. \tag{2.21}$$

Legendre polynomials are orthogonal and the normalization constant of Legendre polynomial is $\mathcal{N}_n$ where

$$\mathcal{N}_n^2 = \frac{2}{2n+1}. \tag{2.22}$$

2.4.1.2 Chebyshev polynomials

The subclass the Jacobi polynomials with $\alpha = \beta = -\frac{1}{2}$ is called Chebyshev polynomials. Chebyshev polynomials are

$$T_n := \delta_n P_n^{(-\frac{1}{2},-\frac{1}{2})}, \quad n \in \mathbb{N} \tag{2.23}$$

where

$$\delta_n = \frac{n!\sqrt{\pi}}{\Gamma(n+\frac{1}{2})}$$

Consequently, they are solutions of differential equation

$$(1-x^2)T_n'' - xT_n' + n^2T_n(x) = 0. \tag{2.24}$$

By taking $\alpha = \beta = -\frac{1}{2}$, we get the orthogonality for Chebyshev polynomials as

$$\int_{-1}^{1} T_n(x)T_m(x)\sqrt{\frac{1}{1-x^2}}dx = \mathcal{N}_n^2\delta_{mn}$$

where

$$\mathcal{N}_n^2 = \begin{cases} \pi & , n = 0 \\ \frac{\pi}{2} & , n > 0 \end{cases} \tag{2.25}$$

2.4.2 Laguerre polynomials

Laguerre polynomials are denoted by $L_n^{\alpha}(x)$.
Take $\rho(x) = x^{\alpha}e^{-x}$ and $\sigma(x) = x$. Then, $\tau(x) = \alpha + 1 - x$, $\lambda_n = n$ and the corresponding polynomials satisfy

$$xy'' + (\alpha + 1 - x)y' + ny = 0. \tag{2.26}$$

The Rodrigues formula for the Laguerre polynomial is

$$L_n^{\alpha}(x) = \frac{1}{n!} e^x x^{-\alpha} \frac{d^n}{dx^n}[x^{n+\alpha} e^{-x}] = a_n x^n + b_n x^{n-1} + \dots, \quad n = 0, 1, 2, \dots$$

where

$$B_n = \frac{1}{n!},$$

$$a_n = \frac{(-1)^n}{n!} \tag{2.27}$$

and

$$\frac{b_n}{a_n} = (-n)(n + \alpha). \tag{2.28}$$

The Laguerre polynomials are orthogonal on the interval $(a, b) = (0, \infty)$ since

$$\sigma(x)\rho(x)x^j|_{x=a,b} = x^{\alpha+1+j} e^{-x}|_{x=a,b} = 0, \quad j = 0, 1, 2, \cdots$$

provided that $\alpha > -1$

The normalization factor of Laguerre polynomial is $\mathcal{N}_n$ where

$$\mathcal{N}_n^2 = \frac{1}{n!} \Gamma(\alpha + 1 + n). \tag{2.29}$$

2.4.3 Hermite Polynomials

Hermite polynomials are denoted by $H_n(x)$.
Let $\rho(x) = e^{-x^2}$ and $\sigma(x) = 1$. Now $\lambda_n = 2n$, $\tau(x) = -2x$ and the corresponding polynomials satisfy the equation

$$y'' - 2xy' + 2ny = 0. \tag{2.30}$$

The Rodriques formula for Hermite polynomial is

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}[e^{-x^2}] = a_n x^n + b_n x^{n-1} + \cdots + constant, \quad n = 0, 1, 2, \dots \tag{2.31}$$

where

$$B_n = (-1)^n, a_n = 2^n, \frac{b_n}{a_n} = 0.$$

$H_n(x)$ are orthogonal over the whole real line $(a, b) = (-\infty, \infty)$ because

$$\sigma(x)\rho(x)x^j|_{x=a,b} = x^j e^{-x^2}|_{x=a,b} = 0, \quad j = 0, 1, 2, \ldots .$$

The normalization constant of Hermite polynomial is $\mathcal{N}_n$ where

$$\mathcal{N}_n^2 = 2^n n! \sqrt{\pi}. \tag{2.32}$$

Theorem 3. ***Darboux- Cristoffel Formula:***
For each classical orthogonal polynomial $p_n(x)$, there exists a finite sum formula of the form;

$$\sum_{i=0}^{n} \frac{1}{\mathcal{N}_i^2} p_i(x)p_i(y) = \frac{\mathcal{A}_n(x,y)}{y-x} \tag{2.33}$$

where

$$\mathcal{A}_i(x,y) = \frac{\alpha_i}{\mathcal{N}_i^2} \begin{vmatrix} p_i(x) & p_i(y) \\ p_{i+1}(x) & p_{i+1}(y) \end{vmatrix} = \frac{-1}{\mathcal{N}_i^2} \frac{a_i}{a_{i+1}} [p_{i+1}(x)p_i(y) - p_i(x)p_{i+1}(y)],$$

a_i is the leading coefficient of p_i and $\alpha_i = \frac{a_i}{a_{i+1}}$.

Theorem 4. ***The roots of classical orthogonal polynomials:***
The polynomials $p_n(x)$ has exactly n real and distinct roots $x_1 < x_2 < x_3 < \cdots < x_n$ lying on the interval (a, b).

Proof. Let x_j be a zero of $p_n(x)$, in other words, $p_n(x_j) = 0$. Suppose $p_n(x)$ has t changes of sign on (a, b). Clearly, $0 \leq t \leq n$. Let us define the polynomials

$$q_t(x) = \begin{cases} \pm 1 & \text{for} \quad t = 0 \\ \prod_{j=1}^{k}(x - x_j) & \text{for} \quad 0 < t \leq n \end{cases}$$

in which $x_j \in (a, b)$ are the points where $p_n(x)$ changes sign.

$$q_t(x) = (x - x_1)(x - x_2) \cdots (x - x_t)$$

Now observe that $q_t(x)p_n(x)$ does not change sign for $x \in (a, b)$ and is always positive. Then, we have

$$\int_a^b \rho(x)q_t(x)p_n(x)dx \neq 0$$

since the integrand is positive. Therefore, we see that $t = n$ because $p_n(x)$ is orthogonal to all polynomials degree less than n, making the integral zero. □

Theorem 5. *The roots of classical orthogonal polynomials are interlaced. The polynomial $p_n(x)$ has exactly one root between any two roots x_j and x_{j+1} of $p_{n+1}(x)$.*

Proof. Let us consider the limiting case of Darboux-Cristoffel Formula as $y \to x$. Now, if we apply L'Hospital rule, we see that

$$\lim_{y\to x} \frac{\mathcal{A}_n(x,y)}{y-x} = \lim_{y\to x} \frac{\frac{\partial(\mathcal{A}_n)}{\partial y}}{1} = \frac{-1}{\mathcal{N}_n^2} \frac{a_n}{a_{n+1}} \lim_{y\to x} [p_{n+1}(x)p_n'(y) - p_n(x)p_{n+1}'(y)] =$$
$$\frac{-1}{\mathcal{N}_n^2} \frac{a_n}{a_{n+1}} [p_{n+1}(x)p_n'(x) - p_n(x)p_{n+1}'(x)] = \sum_{k=0}^{n} \frac{1}{\mathcal{N}_k^2} [p_k(x)]^2.$$

Let $x = x_j$, where x_j is a root of $p_{n+1}(x)$, i.e $p_{n+1}(x_j) = 0$. Then

$$p_n(x_j)p_{n+1}'(x_j) = \mathcal{N}_n^2 \frac{a_{n+1}}{a_n} \sum_{k=0}^{n} \frac{1}{\mathcal{N}_k^2} [p_k(x_j)]^2.$$

Obviously, the sign of right hand side is independent of x_j and depends only on the sign of the ratio $\frac{a_{n+1}}{a_n}$. However, signs of $p_{n+1}'(x)$ changes between x_j and x_{j+1} since p_{n+1} has a local extreme value there. That is, $p_n(x)$ should also change signs accordingly, which means it has at least one zero between x_j and x_{j+1}. However, since there are n subintervals (x_j, x_{j+1}) on (a, b) and there exists at least one root of $p_n(x)$ in each subinterval, we conclude that, there is exactly one zero between any two successive roots of $p_{n+1}(x)$. □

2.5 Recurrence equations for Classical Orthogonal Polynomials $p_n(x)$

Theorem 6. *The following relation holds for Classical Orthogonal Polynomials:*

$$\alpha_n p_{n+1}(x) = (x - \beta_n)p_n(x) - \gamma_n p_{n-1}(x) \tag{2.34}$$

for $n = 1, 2, ...$ $\quad p_0(x) = 1$ *where the coefficients*

$$\alpha_n = \frac{a_n}{a_{n+1}}, \qquad \beta_n = \frac{b_n}{a_n} - \frac{b_{n+1}}{a_{n+1}}, \qquad \gamma_n = \frac{\mathcal{N}_n^2}{\mathcal{N}_{n-1}^2} \frac{a_{n-1}}{a_n} \tag{2.35}$$

are expressed by normalization constant, the leading and sub-leading coefficients.

Proof. By the expansion formula,

$$xp_n(x) = \sum_{k=0}^{n+1} c_{kn} p_k(x) = c_{0n}p_0(x) + c_{1n}p_1(x) + ... + c_{n+1,n}p_{n+1}(x) \tag{2.36}$$

we get,

$$\int_a^b xp_n(x)p_m(x)\rho(x)dx = \sum_{k=0}^{n+1} c_{kn} \int_a^b p_k(x)p_m(x)\rho(x)dx = c_{mn}\mathcal{N}_m^2, \qquad (2.37)$$

$$m = 0, 1, \ldots, n+1$$

Thus, we get

$$c_{mn} = \frac{1}{\mathcal{N}_m^2} \int_a^b xp_n(x)p_m(x)\sigma(x)dx,$$

so the first linear combination constants are given by the formula

$$c_{kn} = \frac{1}{\mathcal{N}_k^2} \int_a^b \rho(x)p_n(x)xp_k(x)dx, \quad k = 0, 1, \ldots, n+1 \qquad (2.38)$$

However, we know that $xp_k(x)$ is polynomial of degree $k+1$. Thus, $c_{kn} = 0$ when $k+1 < n$ and $k < n-1$, Hence, from (2.39), we write

$$xp_n(x) = c_{n-1,n}p_{n-1}(x) + c_{n,n}p_n(x) + c_{n+1,n}p_{n+1}(x) \qquad (2.39)$$

which is nothing but the relation is

$$\alpha_n = c_{n+1,n}, \quad \beta_n = c_{n,n}, \quad \gamma_n = c_{n-1,n}.$$

The integral in (2.40) is unchanged when the indices k and n are interchanged. It follows then that,

$$\mathcal{N}_k^2 c_{kn} = \mathcal{N}_n^2 c_{nk} \qquad (2.40)$$

Putting $k = n-1$, we obtain

$$\mathcal{N}_{n-1}^2 c_{n-1,n} = \mathcal{N}_n^2 c_{n,n-1}$$

which implies that

$$\mathcal{N}_{n-1}^2 \gamma_n = \mathcal{N}_n^2 \alpha_{n-1}$$

and, hence

$$\gamma_n = \frac{\mathcal{N}_n^2}{\mathcal{N}_{n-1}^2}\alpha_{n-1}. \qquad (2.41)$$

On the other hand, since

$$p_n(x) = a_n x^n + b_n x^{n-1} \ldots, \quad a_n \neq 0$$

the relation in (2.42) leads to

$$a_n x^{n+1} + b_n x^n + ... = \alpha_n a_{n+1} x^{n+1} + (\alpha_n b_{n+1} + \beta_n a_n) x^n + ...$$

from which we get,

$$\alpha_n = \frac{a_n}{a_{n+1}} \tag{2.42}$$

and

$$b_n = \frac{a_n}{a_{n+1}} b_{n+1} + \beta_n a_n$$

which implies that

$$\beta_n = \frac{b_n}{a_n} - \frac{b_{n+1}}{a_{n+1}} \tag{2.43}$$

Now, gives that

$$\gamma_n = \frac{\mathcal{N}_n^2}{\mathcal{N}_{n-1}^2} \frac{a_{n-1}}{a_n} \tag{2.44}$$

□

As shown, if we know the coefficients a_n and b_n of first two leading order terms of COP $p_n(x)$, and the squared norm $\mathcal{N}_n^2$, then the classical orthogonal polynomials can be determined recursively, Now, we evaluate α_n, β_n and γ_n for Hermite, Laguerre and Jacobi polynomials separately and then we get the recurrence relation of them.

2.5.1 Hermite

We know the values of leading coefficients and normalization factor, which are

$$a_n = 2^n, \quad \frac{b_n}{a_n} = 0, \quad \mathcal{N}_n^2 = 2^n n! \sqrt{\pi}.$$

Then,

$$\alpha_n = \frac{2^n}{2^{n+1}} = \frac{1}{2}, \tag{2.45}$$

$$\beta_n = 0, \tag{2.46}$$

$$\gamma_n = \frac{2^n n! \sqrt{\pi}}{2^{n-1}(n-1)! \sqrt{\pi}} = 2n\frac{1}{2} = n. \tag{2.47}$$

Thus, the recursion formula for Hermite polynomial:

$$xH_n(x) = \frac{1}{2}H_{n+1}(x) + nH_{n-1}(x)$$

or

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x) \tag{2.48}$$

with $H_0(x) = 1$ and $H_1(x) = 2x$.

2.5.2 Laguerre

We know the values of leading coefficients and normalization factor, which are

$$a_n = \frac{-(1)^n}{n!}, \quad \frac{b_n}{a_n} = -n(n+\alpha), \quad \mathcal{N}_n^2 = \frac{1}{n!}\Gamma(\alpha+1+n).$$

Then,

$$\alpha_n = n+1, \tag{2.49}$$

$$\beta_n = 2n+1+\alpha \tag{2.50}$$

$$\gamma_n = \frac{\frac{1}{n!}\Gamma(\alpha+n+1)}{\frac{1}{(n-1)!}\Gamma(\alpha+n)}n = \frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+n)} = \alpha+n. \tag{2.51}$$

Thus, the recursion formula of Laguerre polynomials is

$$(n+1)L_{n+1}^{\alpha}(x) = (\alpha+2n+x-1)L_n^{\alpha}(x) - (\alpha+n)L_{n-1}^{\alpha}(x) \tag{2.52}$$

where

$$L_0^{\alpha}(x) = 1 \quad and \quad L_1^{\alpha}(x) = \alpha+1-x.$$

2.5.3 Jacobi

For Jacobi Polynomial since the calculations are too long we give the recursion formula directly.

$$\begin{aligned} 2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)P_{n+1}^{(\alpha,\beta)}(x) = \\ (2n+\alpha+\beta+1)[(2n+\alpha+\beta+2)(2n+\alpha+\beta)x+\alpha^2-\beta^2]P_n^{(\alpha,\beta)} \\ -2(n+\alpha)(n+\beta)(2n+\alpha+\beta+2)P_{n-1}^{(\alpha,\beta)} \end{aligned} \tag{2.53}$$

where

$$P_0^{(\alpha,\beta)} = 1 \qquad and \quad P_1^{(\alpha,\beta)} = \frac{1}{2}(\alpha+\beta+2)x + \frac{1}{2}(\alpha-\beta)$$

2.5.3.1 Legendre

By putting $\alpha = \beta = 0$ in (2.56), we get

$$2(n+1)(n+1)2nP_{n+1}(x) = (2n+1)(2n+2)(2n)xP_n(x) - 2n^2(2n+2)P_{n-1}(x)$$

Now, if we eliminate $4n(n+1)$ from both sides

$$(n+1)P_{n+1}(x) = (2n+1)xP_n - nP_{n-1}(x)$$

is the recursion formula for Legendre polynomial.

2.5.3.2 Chebyshev

The Chebyshev polynomials of the first kind are related to the Jacobi polynomials with $\alpha = \beta = -\frac{1}{2}$. Then, the recursion formula is

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x).$$

CHAPTER 3

ROOTS OF CLASSICAL ORTHOGONAL POLYNOMIALS

Recurrence relation for the Classical Orthogonal Polynomials are given by

$$\alpha_n p_{n+1}(x) = (x - \beta_n) p_n(x) - \gamma_n p_{n-1}(x) \quad for \quad n = 1, 2, \ldots, \quad p_0(x) = 1$$

We want to find the roots of classical orthogonal polynomial $p_{n+1}(x)$. Here, we run the recursion for Jacobi, Laguerre and Hermite polynomials for $n = 0, 1, 2, \ldots, N$. Then we get a linear system of the form $(T - xI) = b$. In [26], it is shown that T is symmetric if the polynomials are orthonormal. If T is not symmetric, then we may perform a diagonal similarity transformation which will yield a symmetric tridiagonal matrix R. However, here we prefer to use orthonormal polynomials. So, we transform classical orthogonal polynomials to orthonormal polynomial by the equation $\psi_n(x) = \frac{1}{\mathcal{N}_n^2} p_n(x)$.

Theorem 7. *[27] Recurrence relation formula given by*

$$\alpha_k p_{k+1}(x) + (\beta_k - x) p_k(x) + \alpha_{k-1} p_{k-1}(x) = 0, \quad k = 1, 2, \ldots \tag{3.1}$$

is true for three consecutive orthonormal polynomials with $p_{-1}(x) = 0$ *,* $p_0(x) = \frac{1}{\mathcal{N}_0}$ *where the coefficients*

$$\alpha_k = \frac{a_k}{a_{k-1}} \frac{\mathcal{N}_{k+1}}{\mathcal{N}_k} \quad and \quad \beta_k = \frac{b_{k-1}}{a_{k-1}} - \frac{b_k}{a_k}$$

are expressed by leading and subleading terms of orthonormal polynomial and by the normalization constant.

Recursion (3.1) will be used to determine the zeros of $p_k(x)$ and by running the recursion from $k = 0, 1, \ldots, n$, we get [24]

$$\alpha_0 p_1(x) + (\beta_0 - x) p_0(x) = 0$$

$$\alpha_1 p_2(x) + (\beta_1 - x)p_1(x) + \alpha_0 p_0(x) = 0$$

$$\alpha_2 p_3(x) + (\beta_2 - x)p_2(x) + \alpha_1 p_1(x) = 0$$

$$\vdots$$

$$\alpha_{n-1} p_n(x) + (\beta_{n-1} - x)p_{n-1}(x) + \alpha_{n-2} p_{n-2}(x) = 0$$

$$\alpha_n p_{n+1}(x) + (\beta_n - x)p_n(x) + \alpha_{n-1} p_{n-1}(x) = 0$$

or in the matrix form,

$$\begin{bmatrix} \beta_0 - x & \alpha_0 & 0 & \ldots & & 0 \\ \alpha_0 & \beta_1 - x & \alpha_1 & 0 & \ldots & \\ 0 & \alpha_1 & \beta_2 - x & \alpha_2 & & \ldots \\ & & & & & \\ & & & & & \alpha_{n-1} \\ 0 & & & \ldots & \alpha_{n-1} & \beta_n - x \end{bmatrix} \begin{bmatrix} p_0(x) \\ p_1(x) \\ p_2(x) \\ \vdots \\ \\ p_n(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \\ \\ r(x) \end{bmatrix}. \quad (3.2)$$

where $r(x) = -\alpha_n p_{n+1}(x)$ which implies a homogeneous one when $p_{n+1}(x) = 0$
Then we have, $[\mathbf{R} - x\mathbf{I}]\mathbf{P} = 0$ where

$$R = \begin{bmatrix} \beta_0 & \alpha_0 & 0 & \ldots & & 0 \\ \alpha_0 & \beta_1 & \alpha_1 & \ldots & & \\ 0 & \alpha_1 & \beta_2 & \ldots & & 0 \\ \vdots & & \ldots & & & \\ & & \ldots & & & \alpha_{n-1} \\ 0 & \ldots & & 0 & \alpha_{n-1} & \beta_n \end{bmatrix}, P = \begin{bmatrix} p_0(x) \\ p_1(x) \\ \vdots \\ p_{n-1}(x) \\ p_n(x) \end{bmatrix} \text{ and}$$

I is the identity matrix.
Since the zeros of $p_n(x)$ and $p_{n+1}(x)$ are interlaced **P** can not be a zero vector when $p_{n+1}(x) = 0$. Thus, $[\mathbf{R} - x\mathbf{I}] = 0$. But such x values are the eigenvalues of matrix **R** and also roots of the p_{n+1}. Thus,we see that the nodes of the Gaussian quadrature

rule are formed by the eigenvalues of the three-diagonal Jacobi matrix **R**.

Now, we need to find the weights. Since,

$$(p_m, p_k) = \int_a^b p_m(x)p_k(x)w(x)dx = \delta_{mk}, \quad 0 \le m, k \le n-1$$

and Gaussian quadrature is exact for $p_0, p_1, \ldots, p_N$ we have

$$\delta_{mk} = (p_m, p_k) = \sum_{j=1}^{n} p_m(x)p_k(x)w_j.$$

In the matrix form this expression is

$$PWP^T = I, \tag{3.3}$$

where W =diag$(w_1, w_2, \ldots, w_n)$ and $P = [p(x_0), p(x_1), p(x_2), \ldots, p(x_n)]$. In other words, the columns of the matrix P are the scaled eigenvectors of the matrix R. From equation (3.3), it follows that P is invertible, hence

$$W = P^{-1}(P^T)^{-1} = (P^T P)^{-1}$$

Thus,

$$W^{-1} = P^T P$$

which implies that

$$\frac{1}{w_j} = \sum_{k=0}^{n-1}(p_k(x_j))^2 = ||p(x_j)||^2 \tag{3.4}$$

where $||\cdot||$ is the Euclidean norm. By the way, the eigenvector $v^j = [v_1^j, v_2^j, \ldots, v_n^j]$ of the matrix **R** is given and there exits a constant A such that

$$v_j = AP(x_j) = A[p_0(x_j), p_1(x_j), \ldots, p_n(x_j)].$$

It follows that

$$p_0 = \frac{1}{\mathcal{N}_0}.$$

Hence

$$v_1^j = Ap_0 = \frac{A}{\mathcal{N}_0},$$

and

$$A = v_1^j \mathcal{N}_0.$$

Therefore,

$$P(x_j) = \frac{1}{A} v^j = \frac{1}{v_1^j \mathcal{N}_0} v^j.$$

Now, we get the weights w_j associated with the nodes x_j as

$$w_j = \frac{1}{||P(x_j)||^2} = \mathcal{N}_0^2 \frac{(v_1^j)^2}{||v^j||^2} \tag{3.5}$$

3.1 Hermite Polynomial

The normalization factor of Hermite polynomial is;

$$\mathcal{N}_n^2 = 2^n n! \sqrt{\pi}$$

The recurrence relation is of the form;

$$H_{n+1}(x) = -2nH_{n-1}(x) + 2xH_n(x)$$

Then, we have to normalize Hermite Polynomial and get the recurrence relation for the normalized Hermite polynomial. That is;

$$\Psi_n(x) = \frac{1}{\sqrt{\mathcal{N}_n^2}} H_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} H_n(x), \qquad n = 0, 1, ...$$

Then, if we substitute $\Psi_n(x)$ into the three term recursion, it takes the form

$$H_{n+1}(x) = \sqrt{2^{n+1}(n+1)!\sqrt{\pi}}\Psi_{n+1}(x) = 2x\sqrt{2^n n!\sqrt{\pi}}\Psi_n(x) - \\ -2n\sqrt{2^{n-1}(n-1)!\sqrt{\pi}}\Psi_{n-1}(x)$$

Now if we eliminate the same terms from the given equation above, we have

$$\sqrt{4n(n+1)\Psi_{n+1}(x)} = 2x\sqrt{2n}\Psi_n(x) - 2n\Psi_{n-1}(x)$$

and then by dividing both sides of the equation by 2 and $\sqrt{n}$ get the recurrence relation for orthonormal Hermite;

$$\sqrt{n}\Psi_{n-1}(x) - \sqrt{2}x\Psi_n(x) + \sqrt{n+1}\Psi_{n+1}(x) = 0, \quad n = 0, 1, 2, \ldots, N$$

Then, by (3.1) and (3.2) and we wrote these equations in matrix form

$$\begin{bmatrix} -\sqrt{2}x & \sqrt{1} & 0 & ... & 0 \\ \sqrt{1} & -\sqrt{2}x & \sqrt{2} & ... & \\ 0 & \sqrt{2} & ... & & \\ ... & & ... & -\sqrt{2}x & \sqrt{N} \\ 0 & ... & 0 & \sqrt{N} & -\sqrt{2}x \end{bmatrix} \begin{bmatrix} \Psi_0(x) \\ \Psi_1(x) \\ \vdots \\ \Psi_{N-1}(x) \\ \Psi_N(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ r(x) \end{bmatrix} \tag{3.6}$$

Here we put the value $-\sqrt{N+1}\Psi_{N+1}(x)$ on the other side of the equation and express it by $r(x)$. Now, the system is homogeneous as $\Psi_{N+1}(x) = 0$. The matrix R is given below.

$$R = \begin{bmatrix} 0 & \sqrt{1} & 0 & & ... & & 0 \\ \sqrt{1} & 0 & \sqrt{2} & & ... & & \\ 0 & \sqrt{2} & & & ... & & 0 \\ & & ... & & & & \\ & & ... & & & 0 & \sqrt{N} \\ 0 & ... & & 0 & & \sqrt{N} & 0 \end{bmatrix} \tag{3.7}$$

R is tridiagonal symmetric as seen above. Then we face with the standard eigenvalue problem $Rx = \lambda x$. Now the eigenvalues λ_m of matrix R are connected to the zeros of Hermite polynomial by given formula below;

$$\lambda_m = \sqrt{2}x_m \qquad for \quad m = 0, 1, \ldots, N \tag{3.8}$$

To make sure that the eigenvalues of matrix R is related to the roots of Hermite polynomial with the formula (3.3);

$$H_2(x) = 4x^2 - 2 = 0 \quad implies \quad that \quad 4x^2 - 2 = 0 \Rightarrow x^2 = \frac{1}{2} \Rightarrow x_{1,2} = \pm\frac{1}{\sqrt{2}}$$

Now, eigenvalue of R^{2x2} is

$$R = \begin{bmatrix} 0 & \sqrt{1} \\ \sqrt{1} & 0 \end{bmatrix} \Rightarrow |R - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} \Rightarrow \lambda^2 - 1 = 0 \Rightarrow \lambda_{1,2} = \pm 1$$

Then by the formula (3.3), we get

$$x_{1,2} = \frac{\lambda_{1,2}}{\sqrt{2}} \Rightarrow x_{1,2} = \frac{\pm 1}{\sqrt{2}}$$

as desired. MATLAB trideigs function is applied to calculate the eigenvalues and eigenvectors of the symmetric tridiagonal matrix R. Weights are related to the eigenvectors of a tridiagonal matrix and calculated by the given formula in (3.5). Thus,the roots and weights of Hermite polynomials are tabulated for 2,4,8,16 and 20th degrees of Hermite.

Table 3.1: Roots and weights for quadratic Hermite

i	x_i	w_i
1	$-0.707\,106\,781\,186\,54$	$0.886\,226\,925\,452\,76$
2	$0.707\,106\,781\,186\,54$	$0.886\,226\,925\,452\,76$

Table 3.2: Roots and weights for fourth degree Hermite

i	x_i	w_i
1	$-1.650\,680\,123\,885\,8$	$0.081\,312\,835\,447\,245$
2	$-0.524\,647\,623\,275\,29$	$0.804\,914\,090\,005\,51$
3	$0.524\,647\,623\,275\,29$	$0.804\,914\,090\,005\,51$
4	$1.650\,680\,123\,885\,8$	$0.081\,312\,835\,447\,245$

Table 3.3: Roots and weights for 8th degree Hermite

i	x_i	w_i
1	$-2.930\,637\,420\,257\,2$	$0.001\,996\,040\,722\,11$
2	$-1.981\,656\,756\,695\,8$	$0.017\,077\,983\,007\,41$
3	$-1.157\,193\,712\,446\,8$	$0.207\,802\,325\,814\,89$
4	$-0.381\,186\,990\,207\,32$	$0.661\,147\,012\,558\,24$
5	$0.381\,186\,990\,207\,32$	$0.661\,147\,012\,558\,24$
6	$1.157\,193\,712\,446\,8$	$0.207\,802\,325\,814\,89$
7	$1.981\,656\,756\,695\,8$	$0.017\,077\,983\,007\,41$
8	$2.930\,637\,420\,257\,2$	$0.001\,996\,040\,722\,11$

Table 3.4: Roots and weights for 16th degree Hermite

i	x_i	w_i
1	$-4.688\,738\,939\,305\,8$	$0.000\,000\,232\,098\,084$
2	$-3.869\,447\,904\,860\,1$	$0.000\,027\,118\,600\,925$
3	$-3.176\,999\,161\,98$	$0.000\,932\,284\,008\,624$
4	$-2.546\,202\,157\,847\,5$	$0.661\,147\,012\,558\,24$
5	$-1.951\,787\,990\,916\,3$	$0.012\,880\,311\,535\,51$
6	$-1.380\,258\,539\,198\,9$	$0.083\,810\,041\,398\,986$
7	$-0.822\,951\,449\,144\,66$	$0.280\,647\,458\,528\,53$
8	$-0.273\,481\,046\,138\,2$	$0.507\,929\,479\,016\,61$
9	$0.273\,481\,046\,138\,2$	$0.507\,929\,479\,016\,61$
10	$0.822\,951\,449\,144\,66$	$0.280\,647\,458\,528\,53$
11	$1.380\,258\,539\,198\,9$	$0.083\,810\,041\,398\,986$
12	$1.951\,787\,990\,916\,3$	$0.012\,880\,311\,535\,51$
13	$2.546\,202\,157\,847\,5$	$0.661\,147\,012\,558\,24$
14	$3.176\,999\,161\,98$	$0.000\,932\,284\,008\,624$
15	$3.869\,447\,904\,860\,1$	$0.000\,027\,118\,600\,925$
16	$4.688\,738\,939\,305\,8$	$0.000\,000\,232\,098\,084$

Table 3.5: Roots and weights for 20th degree Hermite

i	x_i	w_i
1	−5.387 480 890 011 2	0.000 000 000 002 229 39
2	−4.603 682 449 550 7	0.000 000 004 399 340 992
3	−3.944 764 040 115 6	0.000 001 086 069 370
4	−3.347 854 567 383 25	0.000 078 025 564 785
5	−2.788 806 058 428 1	0.000 228 338 636 016
6	−2.254 974 002 089 3	0.003 243 773 342 238
7	−1.738 537 712 116 6	0.024 810 520 887 46
8	−1.234 076 215 395 3	0.109 017 206 020 02
9	−0.737 473 728 545 39	0.286 675 505 362 83
10	−0.245 340 708 300 9	0.462 243 669 600 61
11	0.245 340 708 300 9	0.462 243 669 600 61
12	0.737 473 728 545 39	0.286 675 505 362 83
13	1.234 076 215 395 3	0.109 017 206 020 02
14	1.738 537 712 116 6	0.024 810 520 887 46
15	2.254 974 002 089 3	0.003 243 773 342 238
16	2.788 806 058 428 1	0.000 228 338 636 016
17	3.347 854 567 383 25	0.000 078 025 564 785
18	3.944 764 040 115 6	0.000 001 086 069 370
19	4.603 682 449 550 7	0.000 000 004 399 340 992
20	5.387 480 890 011 2	0.000 000 000 002 229 39

3.2 Laguerre Polynomial

The normalization factor of Laguerre polynomial is;

$$\mathcal{N}_n^2 = \frac{1}{n!}\Gamma(\alpha + 1 + n)$$

The recurrence relation is of the form;

$$(n+1)L_{n+1}^{\alpha}(x) = (\alpha + 2n + x - 1)L_n^{\alpha}(x) - (\alpha + n)L_{n-1}^{\alpha}(x)$$

Then,we have to normalize Laguerre Polynomial and get the recurrence relation for the normalized Laguerre polynomial. That is;

$$\Psi_n^\alpha(x) = \frac{1}{\sqrt{\mathcal{N}_n^2}} L_n^\alpha(x) = \frac{1}{\sqrt{\frac{\Gamma(\alpha+1+n)}{n!}}} L_n^\alpha(x)$$

Then,

$$L_n^\alpha(x) = \Psi_n(x)^\alpha \sqrt{\frac{\Gamma(\alpha+n+1)}{n!}}$$

Then, if we substitute ${}_n^\alpha(x)$ in the three term recursion, it becomes

$$(n+1)\Psi_{n+1}^\alpha(x)\sqrt{\frac{\Gamma(\alpha+n+2)}{(n+1)!}} = (\alpha+2n+x-1)\Psi_n^\alpha(x)\sqrt{\frac{\Gamma(\alpha+n+1)}{n!}}$$
$$-(\alpha+n)\Psi_{n-1}^\alpha(x)\sqrt{\frac{\Gamma(\alpha+n)}{(n-1)!}}$$

Next, we continue to eliminate the same terms from the sides of the equation.

$$\sqrt{\frac{(n+1)(\alpha+n)(\alpha+n+1)}{n}}\Psi_{n+1}^\alpha(x) = (\alpha+2n+x-1)\sqrt{\frac{(\alpha+n)}{n}}\Psi_n^\alpha(x) -$$
$$-(\alpha+n)\Psi_{n-1}^\alpha(x)$$

Finally, the orthonormal Laguerre recurrence relation takes the form

$$\sqrt{n(n+\alpha)}\Psi_{n-1}^\alpha(x) + (\alpha+2n+x-1)\Psi_n^\alpha(x) + \\ + \sqrt{(\alpha+n+1)(n+1)}\Psi_{n+1}^\alpha(x) = 0 \tag{3.9}$$

Then, by (3.1 and (3.2) we get the matrix form, but before it let us define the elements of matrix by $r_{i,j}$ where $i, j = 0, 1, 2, ..., N$ such that

$$r_{i,j} = -(2i+1+\alpha-x) \qquad if \qquad i = j$$

$$r_{i,j} = \sqrt{i}\sqrt{i+\alpha} \qquad if \qquad i = j+1$$

$$r_{i,j} = \sqrt{j}\sqrt{j+\alpha} \qquad if \qquad i = j-1$$

,

$$otherwise, \qquad r_{i,j} = 0$$

From the above notation, we see that

$$r_{0,0} = -(1 + \alpha - x)$$

$$r_{0,1} = \sqrt{1}\sqrt{1+\alpha}$$

$$r_{1,0} = \sqrt{1}\sqrt{1+\alpha}$$

$$\dots$$

$$r_{(N-1),N} = r_{N,(N-1)} = \sqrt{N}\sqrt{N+\alpha}$$

$$r_{N,N} = -(2N + 1 + \alpha - x)$$

Now, by writing the elements of matrix, we get

$$\begin{bmatrix} r_{0,0} & r_{0,1} & 0 & & \dots & 0 \\ r_{1,0} & r_{1,1} & r_{1,2} & & \dots & \\ 0 & r_{2,1} & r_{2,2} & & \dots & \dots \\ & & \dots & & & \\ & & & & \dots & \\ & & & & \dots & r_{N-1,N} \\ 0 & & \dots & 0 & r_{N,N-1} & r_{N,N} \end{bmatrix} \begin{bmatrix} \Psi_0^\alpha(x) \\ \Psi_1^\alpha(x) \\ \\ \vdots \\ \\ \Psi_{N-1}^\alpha(x) \\ \Psi_N^\alpha(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \\ \vdots \\ \\ 0 \\ r(x) \end{bmatrix} \qquad (3.10)$$

where $r(x) = -\sqrt{N+1}\Psi_{N+1}(x)$ Now, the system is homogeneous as $\Psi_{N+1}(x) = 0$. Then the matrix R is

$$R = \begin{bmatrix} -(1+\alpha) & \sqrt{1}\sqrt{1+\alpha} & 0 & ... & & 0 \\ \sqrt{1}\sqrt{1+\alpha} & -(3+\alpha) & \sqrt{2}\sqrt{2+\alpha} & & ... & \\ 0 & \sqrt{2}\sqrt{2+\alpha} & -(5+\alpha) & & ... & \\ & & ... & & & \\ & & & ... & & \\ & & & & ... & \sqrt{N}\sqrt{N+\alpha} \\ 0 & ... & & & \sqrt{N}\sqrt{N+\alpha} & -(2N+1+\alpha) \end{bmatrix} \tag{3.11}$$

R is tridiagonal symmetric as $r_{i,j} = r_{j,i}$ for $i, j = 0, 1, \ldots, N$. Then we face with the standard eigenvalue problem $Rx = \lambda x$. Now the eigenvalues λ_m of matrix R which are associated to the zeros of Laguerre polynomial by given formula below;

$$\lambda_m = -x_m \qquad for \quad m = 0, 1, 2, \ldots, N \tag{3.12}$$

To find the roots as a real number, we made the calculations for $\alpha = 0$. Then, when $\alpha = 0$, we get the matrix of the form;

$$R = \begin{bmatrix} -1 & 1 & 0 & .... & & & 0 \\ 1 & -3 & 2 & 0 & .... & & \\ 0 & 2 & -5 & 3 & & .... & \\ & & & & & & \\ & & & & & & \\ & & & & & -(2N-1) & N \\ 0 & & .... & & & N & -(2N+1) \end{bmatrix}$$

To make sure that the eigenvalues of matrix R is related to the roots of Laguerre polynomial with the formula (3.7) for $\alpha = 0$;

$$L_2(x) = \frac{1}{2}(x^2 - 4x + 2) = 0$$

implies that $x_{1,2} = \frac{4 \pm \sqrt{16-8}}{2} = 2 \pm \sqrt{2}$ Now, the eigenvalues of $\mathbb{R}^{2x2}$ is

$$R = \begin{bmatrix} -1 & 1 \\ 1 & 3 \end{bmatrix} \Rightarrow |R - \lambda I| = \begin{vmatrix} -\lambda - 1 & 1 \\ 1 & -\lambda - 3 \end{vmatrix} \Rightarrow \lambda^2 + 4\lambda + 2 = 0$$

$$\Rightarrow \lambda_{1,2} = -2 \pm \sqrt{2}$$

Then by the formula (3.7), we get

$$x_{1,2} = -\lambda_{1,2} \Rightarrow x_{1,2} = 2 \pm \sqrt{2}$$

as desired.

MATLAB trideigs function is applied to compute the eigenvalues and the eigenvectors of the symmetric tridiagonal matrix R.Weights are calculated by the given formula in (3.5).

The roots and weights of Laguerre polynomials are tabulated for the degrees $n = 2$, $n = 4$, $n = 8$ and $n = 20$ for $\alpha = 0$.

Table 3.6: Roots and weights of Laguerre Polynomial for $n = 2$ and $\alpha = 0$

i	x_i	w_i
1	0.585 786 437 626 91	0.853 553 390 593 27
2	3.414 213 562 373 1	0.146 446 609 406 73

Table 3.7: Roots and weights of Laguerre Polynomial for $n = 4$ and $\alpha = 0$

i	x_i	w_i
1	0.322 547 689 619 39	0.603 154 104 341 63
2	1.745 761 101 158 4	0.357 418 692 437 8
3	4.536 620 296 921 1	0.038 887 908 515 005
4	9.395 070 912 301 1	0.005 392 947 055 613

Table 3.8: Roots and weights of Laguerre Polynomial for $n = 8$ and $\alpha = 0$

i	x_i	w_i
1	$0.170\,279\,632\,305$	$0.369\,188\,589\,341\,64$
2	$0.903\,701\,776\,799\,4$	$0.418\,786\,780\,814\,34$
3	$2.251\,086\,629\,866\,1$	$0.175\,794\,986\,637\,17$
4	$4.266\,700\,170\,287\,7$	$0.033\,343\,492\,261\,216$
5	$7.045\,905\,402\,393\,5$	$0.002\,794\,536\,235\,22$
6	$10.758\,516\,010\,181$	$0.000\,907\,650\,877\,335$
7	$15.740\,678\,641\,28$	$0.000\,008\,485\,746\,716$
8	$22.863\,131\,736\,889$	$0.000\,000\,104\,800\,117$

Table 3.9: Roots and weights of Laguerre Polynomial for $n = 20$ and $\alpha = 0$

i	x_i	w_i
1	$0.070\,539\,889\,692$	$0.168\,746\,801\,851\,1$
2	$0.372\,126\,818\,001\,6$	$0.291\,254\,362\,006\,1$
3	$0.916\,582\,102\,483\,3$	$0.266\,686\,102\,867$
4	$1.707\,306\,531\,028\,3$	$0.166\,002\,453\,269\,5$
5	$2.749\,199\,255\,309\,4$	$0.074\,826\,064\,668\,792$
6	$4.048\,925\,313\,851$	$0.024\,964\,417\,309\,28$
7	$5.615\,174\,970\,861\,6$	$0.006\,202\,550\,844\,572$
8	$7.459\,017\,453\,671$	$0.001\,144\,962\,386\,476\,9$
9	$9.594\,392\,869\,581$	$0.001\,557\,417\,730\,278\,1$
10	$12.038\,802\,546\,964$	$0.000\,154\,014\,408\,652$
11	$14.814\,293\,442\,631$	$0.000\,010\,864\,863\,665$
12	$17.948\,895\,520\,519$	$0.000\,005\,330\,120\,909\,5$
13	$21.478\,788\,240\,29$	$0.000\,000\,175\,798\,179\,051$
14	$25.451\,702\,793\,187$	$0.000\,000\,037\,255\,024\,02$
15	$29.932\,554\,631\,701$	$0.000\,000\,047\,675\,292\,51$
16	$35.013\,434\,240\,48$	$0.000\,000\,003\,372\,844\,243$
17	$40.833\,057\,056\,729$	$0.000\,000\,000\,001\,155\,014$
18	$47.619\,994\,047\,347$	$0.000\,000\,000\,000\,001\,539$
19	$55.810\,795\,750\,064$	$0.000\,000\,000\,000\,005\,286$
20	$66.524\,416\,525\,616$	$0.000\,000\,000\,000\,000\,165$

Now, we evaluated the roots of Laguerre polynomial for any chosen α and we chose the value $\alpha = 1$. Then for the given matrix R in (3.6) we put $\alpha = 1$ and get the new symmetric tridiagonal matrix which is given below.

$$R = \begin{bmatrix} -2 & 1 & 0 & \dots & & & 0 \\ 1 & -4 & 2 & 0 & \dots & & \\ 0 & 2 & -6 & 3 & & \dots & \\ . & & & & & & \\ . & & & & & & \\ . & & & & & & \\ & & & & & -2N & \sqrt{N(N+1)} \\ 0 & & \dots & & & \sqrt{N(N+1)} & -(2N+2) \end{bmatrix}$$

Then by running the MATLAB program for $n = 2$, $n = 4$, $n = 16$ for $\alpha = 1$ we get the eigenvalues and weights related to the eigenvectors by formula (3.5) are tabulated below.

Table 3.10: Roots and weights of Laguerre Polynomial for $n = 2$ and $\alpha = 1$

i	x_i	w_i
1	1.267 949 192 431 1	0.788 675 134 594 81
2	4.732 050 807 568 9	0.211 324 865 405 19

Table 3.11: Roots and weights of Laguerre Polynomial for $n = 4$ and $\alpha = 1$

i	x_i	w_i
1	0.743 291 927 981 43	0.446 870 593 218 78
2	2.571 635 007 646 3	0.477 635 772 363 87
3	5.731 178 751 689 1	0.074 177 784 731 05
4	10.953 894 312 683	0.001 315 849 686 303

Table 3.12: Roots and weights of Laguerre Polynomial for $n = 16$ and $\alpha = 1$

i	x_i	w_i
1	$0.216\,140\,305\,239\,5$	$0.063\,277\,332\,879\,539$
2	$0.726\,388\,243\,251\,8$	$0.231\,090\,461\,520\,72$
3	$1.533\,593\,160\,373\,5$	$0.316\,933\,542\,164$
4	$2.644\,970\,998\,611\,9$	$0.237\,894\,217\,875\,2$
5	$4.070\,978\,160\,880\,2$	$0.110\,272\,743\,359\,25$
6	$5.825\,855\,515\,106$	$0.033\,089\,588\,356\,4$
7	$7.928\,504\,185\,306$	$0.006\,525\,716\,407\,792$
8	$10.403\,808\,289\,95$	$0.084\,271\,143\,679\,2$
9	$13.284\,661\,070\,707$	$0.006\,996\,190\,955\,56$
10	$16.615\,173\,216\,867$	$0.000\,361\,208\,604\,213$
11	$20.456\,006\,020\,027$	$0.000\,011\,015\,412\,514$
12	$24.893\,847\,025\,352$	$0.000\,001\,835\,682\,512$
13	$30.059\,862\,920\,203$	$0.000\,000\,148\,148\,886$
14	$36.170\,694\,543\,68$	$0.000\,000\,047\,348\,453\,5$
15	$43.640\,365\,184\,177$	$0.000\,000\,004\,084\,146$
16	$53.529\,151\,160\,268$	$0.000\,000\,000\,036\,239$

3.3 Jacobi Polynomial

The normalization factor of Jacobi polynomial $\mathcal{N}_n$ where

$$\mathcal{N}_n^2 = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1))}{n!(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)}$$

The recurrence relation is;

$$2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)P_{n+1}^{(\alpha,\beta)}(x) =$$

$$(2n+\alpha+\beta+1)[(2n+\alpha+\beta+2)(2n+\alpha+\beta)x+\alpha^2-\beta^2]P_n^{(\alpha,\beta)}$$

$$-2(n+\alpha)(n+\beta)(2n+\alpha+\beta+2)P_{n-1}^{(\alpha,\beta)}$$

Then,

$$\Psi_n^{(\alpha,\beta})(x) = \frac{1}{\sqrt{\mathcal{N}_n^2}}P_n^{(\alpha,\beta)}(x)$$

that is,

$$P_n^{(\alpha,\beta)}(x) = \Psi_n^{(\alpha,\beta)}(x)\sqrt{\frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1))}{n!(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)}}. \tag{3.13}$$

Now if we replace (3.1) into the recurrence relation of Jacobi, we get

$$2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)\times$$
$$\sqrt{\frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+2)\Gamma(n+\beta+2))}{(n+1)!(2n+\alpha+\beta+3)\Gamma(n+\alpha+\beta+2)}}.\Psi_{n+1}^{(\alpha,\beta)}(x) =$$
$$(2n+\alpha+\beta+1)[(2n+\alpha+\beta+2)(2n+\alpha+\beta)x+\alpha^2-\beta^2]\times$$
$$\sqrt{\frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1))}{n!(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)}}\Psi_n^{(\alpha,\beta)}(x)-$$
$$-2(n+\alpha)(n+\beta)(2n+\alpha+\beta+2)\times$$
$$\sqrt{\frac{2^{\alpha+\beta+1}\Gamma(n+\alpha)\Gamma(n+\beta))}{(n-1)!(2n+\alpha+\beta-1)\Gamma(n+\alpha+\beta)}}\Psi_{n-1}^{(\alpha,\beta)}(x).$$

Then, since $\Gamma(x+1) = x\Gamma(x)$ if we eliminate same terms from both side, we get

$$2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)\times$$

$$\sqrt{\frac{(n+\alpha)(n+1+\alpha)(n+\beta)(n+1+\beta)}{n(n+1)(2n+\alpha+\beta+3)(n+\alpha+\beta)(n+\alpha+\beta+1)}}\Psi_{n+1}^{(\alpha,\beta)}(x) =$$

$$(2n+\alpha+\beta+1)[(2n+\alpha+\beta+2)(2n+\alpha+\beta)x+\alpha^2-\beta^2]\times$$

$$\sqrt{\frac{(n+\alpha)(n+\beta)}{n(2n+\alpha+\beta+1)(n+\alpha+\beta)}}\Psi_n^{(\alpha,\beta)}(x)-$$

$$-2(n+\alpha)(n+\beta)(2n+\alpha+\beta+2)\sqrt{\frac{1}{(2n+\alpha+\beta-1)}}\Psi_{n-1}^{(\alpha,\beta)}(x)$$

After that,

$$2(2n+\alpha+\beta)\sqrt{\frac{(n+\alpha+1)(n+\beta+1)(n+1)(n+\alpha+\beta+1)}{n(2n+\alpha+\beta+3)(n+\alpha+\beta)}}\Psi_{n+1}^{(\alpha,\beta)}(x) =$$

$$((\alpha+\beta+2n)(\alpha+\beta+2n+2)x+\alpha^2-\beta^2)\sqrt{\frac{(2n+\alpha+\beta+1)}{n(n+\alpha+\beta)}}\Psi_n^{(\alpha,\beta)}(x) -$$

$$-2(2n+\alpha+\beta+2)\sqrt{\frac{(n+\alpha)(n+\beta)}{2n+\alpha+\beta-1}}\Psi_{n-1}^{(\alpha,\beta)}(x)$$

Then,

$$(2n+\alpha+\beta)\sqrt{\frac{(n+\alpha+1)(n+\beta+1)(n+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+3)(2n+\alpha+\beta+1)}}\Psi_{n+1}^{(\alpha,\beta)}(x) -$$

$$-\frac{1}{2}((\alpha+\beta+2n)(\alpha+\beta+2n+2)x+\alpha^2-\beta^2)\Psi_n^{(\alpha,\beta)}(x) +$$

$$+(2n+\alpha+\beta+2)\sqrt{\frac{(n+\alpha)(n+\beta)n(n+\alpha+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta+1)}}\Psi_{n-1}^{(\alpha,\beta)}(x) = 0$$

Finally, we get the recurrence relation for orthonormal Jacobi polynomial;

$$\frac{1}{2n+\alpha+\beta}\sqrt{\frac{n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta-1)}}\Psi_{n-1}^{(\alpha,\beta)}(x)- \tag{3.14}$$

$$-\frac{1}{2}[x+\frac{\alpha^2-\beta^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}]\Psi_{n}^{(\alpha,\beta)}(x)$$

$$+\frac{1}{(2n+\alpha+\beta+2)}\sqrt{\frac{(n+\alpha+1)(n+\beta+1)(n+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+3)}}\Psi_{n+1}^{(\alpha,\beta)}(x)=0$$

Then,by (3.1) and (3.2) we can write this N+1 equations as an inhomogeneous linear system. Let us define the elements of matrix by $r_{i,j}$ where $i,j=0,1,2,\ldots,N$ such that

$$r_{i,j}=-\frac{1}{2}[x+\frac{\alpha^2-\beta^2}{(2i+\alpha+\beta)(2i+\alpha+\beta+2)}] \qquad if \qquad i=j$$

$$r_{i,j}=\frac{1}{2j+\alpha+\beta}\sqrt{\frac{j(j+\alpha)(j+\beta)(j+\alpha+\beta)}{(2j+\alpha+\beta+1)(2j+\alpha+\beta-1)}} \qquad if \qquad i=j-1$$

$$r_{i,j}=\frac{1}{2i+\alpha+\beta}\sqrt{\frac{i(i+\alpha)(i+\beta)(i+\alpha+\beta)}{(2i+\alpha+\beta+1)(2i+\alpha+\beta-1)}} \qquad if \qquad i=j+1,$$

$$otherwise \qquad r_{i,j}=0$$

Now, by running the above notation we get the elements of matrix ,

$$r_{0,0}=-\frac{1}{2}[x+\frac{\alpha^2-\beta^2}{(\alpha+\beta+2)(\alpha+\beta)}]$$

$$r_{0,1} = \frac{1}{\alpha+\beta+2}\sqrt{\frac{(\alpha+1)(\beta+1)}{\alpha+\beta+3}} = r_{1,0}$$

$$r_{1,1} = -\frac{1}{2}[x + \frac{\alpha^2-\beta^2}{(\alpha+\beta+2)(\alpha+\beta+4)}]$$

$$\vdots$$

$$r_{N-1,N} = r_{N,N-1} = \frac{1}{2N+\alpha+\beta}\sqrt{\frac{N(N+\alpha)(N+\beta)(N+\alpha+\beta)}{(2N+\alpha+\beta+1)(2N+\alpha+\beta-1)}}$$

$$r_{N,N} = -\frac{1}{2}[x + \frac{\alpha^2-\beta^2}{(2N+\alpha+\beta)(2N+\alpha+\beta+2)}]$$

$$r(x) = \frac{1}{(2N+\alpha+\beta+2)}\sqrt{\frac{(N+\alpha+1)(N+\beta+1)(N+1)(N+\alpha+\beta+1)}{(2N+\alpha+\beta+1)(2N+\alpha+\beta+3)}}\cdot \\ \cdot\Psi_{N+1}^{\alpha,\beta}(x)$$

Now, by writing the elements of matrix, we get

$$\begin{bmatrix} r_{0,0} & r_{0,1} & 0 & \dots & 0 \\ r_{1,0} & r_{1,1} & r_{1,2} & \dots & \\ 0 & r_{2,1} & r_{2,2} & \dots & \dots \\ & & \dots & & \\ & & & \dots & \\ & & \dots\ \ 0 & r_{N,N-1} & r_{N,N} \end{bmatrix} \begin{bmatrix} \Psi_0^{\alpha,\beta}(x) \\ \Psi_1^{\alpha,\beta}(x) \\ \vdots \\ \Psi_{N-1}^{\alpha,\beta}(x) \\ \Psi_N^{\alpha,\beta}(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ r(x) \end{bmatrix} \tag{3.15}$$

It implies homogeneous equation since $\Psi_{N+1}(x) = 0$. Therefore, we see a standard matrix eigenvalue problem for symmetric tridiagonal matrix R and $Rx = \lambda x$
Let us define the elements $m_{i,j}$ for $i, j = 0, 1, 2, \dots, N$ of matrix R by the formulas

$$m_{i,j} = \frac{\alpha^2-\beta^2}{(2i+\alpha+\beta)(2i+\alpha+\beta+2)} \qquad if \qquad i = j$$

$$m_{i,j} = \frac{2}{\alpha+\beta+2j}\sqrt{\frac{j(\alpha+j)(\beta+j)(\alpha+\beta+j)}{(\alpha+\beta+3+2j+1)(\alpha+\beta+2j-1)}} \qquad if \qquad i=j-1$$

$$m_{i,j} = \frac{2}{\alpha+\beta+2i}\sqrt{\frac{i(\alpha+i)(\beta+i)(\alpha+\beta+i)}{(\alpha+\beta+2i+1)(\alpha+\beta+2i-1)}} \qquad if \qquad i=j+1$$

$$otherwise \qquad m_{i,j}=0$$

From the above formula, we could derive the elements of matrix R:

$$m_{0,0} = \frac{\alpha^2-\beta^2}{(\alpha+\beta+2)(\alpha+\beta)}$$

$$m_{0,1} = m_{1,0} = \frac{2}{\alpha+\beta+2}\sqrt{\frac{(\alpha+1)(\beta+1)}{\alpha+\beta+3}}$$

$$m_{1,1} = \frac{\alpha^2-\beta^2}{(\alpha+\beta+2)(\alpha+\beta+4)}$$

$$m_{1,2} = m_{2,1} = \frac{2}{\alpha+\beta+4}\sqrt{\frac{2(\alpha+2)(\beta+2)(\alpha+\beta+2)}{(\alpha+\beta+3)(\alpha+\beta+5)}}$$

$$\vdots$$

$$m_{N,N-1} = m_{N-1,N} = \frac{2}{\alpha+\beta+2N}\sqrt{\frac{N(\alpha+N)(\beta+N)(\alpha+\beta+N)}{(\alpha+\beta+2N+1)(\alpha+\beta+2N-1)}}$$

$$m_{N,N} = \frac{\alpha^2-\beta^2}{(2N+\alpha+\beta)(2N+\alpha+\beta+2)}$$

$$R = \begin{bmatrix} m_{0,0} & m_{0,1} & 0... & & 0 \\ m_{1,0} & m_{1,1} & m_{1,2} & ... & \\ 0 & m_{2,1} & & ... & \\ & & & & \\ & & ... & & \\ & & & & m_{N-1,N} \\ 0 & ... & & m_{N,N-1} & m_{N,N} \end{bmatrix} \tag{3.16}$$

where the eigenvalues λ_m of matrix R are related to the zeros of Jacobi polynomial by the formula

$$\lambda_m = \frac{1}{2} x_m \tag{3.17}$$

for $m = 0, 1, \ldots, N$. For Jacobi polynomials , we calculated the roots of the case when $\alpha = \beta = 0$, that is Legendre Polynomial. The recurrence relation for orthonormal Legendre Polynomial is;

$$\frac{n}{\sqrt{(2n-1)(2n+1)}} \Psi_{n-1}(x) - x\Psi_n(x) + \frac{n+1}{\sqrt{(2n+1)(2n+3)}} \Psi_{n+1}(x) = 0$$

From Jacobi matrix R,when we make $\alpha = \beta = 0$, we get the matrix

$$R = \begin{bmatrix} 0 & \frac{1}{\sqrt{3}} & 0 & \ldots & & 0 \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{15}} & & \ldots & \\ & & \ldots & & & \\ & & & \ldots & & \\ & & & & \ldots & \frac{N}{\sqrt{(2N-1)2N+1)}} \\ & & & \ldots & \frac{N}{\sqrt{(2N-1)(2N+1)}} & 0 \end{bmatrix}$$

Then, by MATLAB, we compute the eigenvalues and eigenvectors of symmetric tridiagonal matrix. Therefore, the roots and weights of Jacobi polynomial when $\alpha = \beta = 0$ is tabulated for $n = 2$, $n = 4$, $n = 8$, $n = 16$ and $n = 20$.

Table 3.13: Roots and weights for quadratic Legendre

i	x_i	w_i
1	$-0.577\,350\,269\,189\,63$	1
2	$0.577\,350\,269\,189\,63$	1

Table 3.14: Roots and weights of Legendre for $n = 4$

i	x_i	w_i
1	$-0.861\,136\,311\,594\,05$	$0.347\,854\,845\,137\,45$
2	$-0.339\,981\,043\,584\,86$	$0.652\,145\,154\,862\,55$
3	$0.339\,981\,043\,584\,86$	$0.652\,145\,154\,862\,55$
4	$0.861\,136\,311\,594\,05$	$0.347\,854\,845\,137\,45$

Table 3.15: Roots and weights of Legendre for $n = 8$

i	x_i	w_i
1	$-0.960\,289\,856\,497\,54$	$0.101\,228\,536\,290\,38$
2	$-0.796\,666\,477\,413\,63$	$0.222\,381\,034\,453\,4$
3	$-0.525\,532\,409\,916\,33$	$0.313\,706\,645\,877\,89$
4	$-0.183\,434\,642\,495\,65$	$0.362\,683\,783\,378\,36$
5	$0.183\,434\,642\,495\,65$	$0.362\,683\,783\,378\,36$
6	$0.525\,532\,409\,916\,33$	$0.313\,706\,645\,877\,89$
7	$0.796\,666\,477\,413\,63$	$0.222\,381\,034\,453\,4$
8	$0.960\,289\,856\,497\,54$	$0.101\,228\,536\,290\,38$

Table 3.16: Roots and weights of Legendre for $n = 16$

i	x_i	w_i
1	$-0.989\,400\,934\,991\,65$	$0.027\,152\,459\,411\,75$
2	$-0.944\,575\,023\,073\,23$	$0.062\,253\,523\,938\,65$
3	$-0.865\,631\,202\,387\,83$	$0.095\,158\,511\,682\,5$
4	$-0.755\,404\,408\,355$	$0.124\,628\,971\,255\,5$
5	$-0.617\,876\,244\,402\,64$	$0.149\,595\,988\,816\,58$
6	$-0.458\,016\,777\,657\,23$	$0.169\,156\,519\,395$
7	$-0.281\,603\,550\,779\,26$	$0.182\,603\,415\,044$
8	$-0.095\,012\,509\,837\,64$	$0.189\,450\,610\,455\,1$
9	$0.095\,012\,509\,837\,64$	$0.189\,450\,610\,455\,1$
10	$0.281\,603\,550\,779\,26$	$0.182\,603\,415\,044\,9$
11	$0.458\,016\,777\,657\,23$	$0.169\,156\,519\,395$
12	$0.617\,876\,244\,402\,64$	$0.149\,595\,988\,816\,58$
13	$0.755\,404\,408\,355$	$0.124\,628\,971\,255\,5$
14	$0.865\,631\,202\,387\,83$	$0.095\,158\,511\,682\,5$
15	$0.944\,575\,023\,073\,23$	$0.062\,253\,523\,938\,65$
16	$0.989\,400\,934\,991\,65$	$0.027\,152\,459\,411\,75$

Table 3.17: Roots and weights of Legendre for $n = 20$

i	x_i	w_i
1	−0.993 128 599 185 1	0.017 614 007 139 15
2	−0.963 971 927 277 91	0.040 601 429 800 39
3	−0.912 234 428 251 33	0.062 672 048 334 11
4	−0.839 116 971 822 22	0.083 276 741 576 7
5	−0.746 331 906 460 15	0.101 930 119 817 24
6	−0.636 053 680 726 52	0.118 194 531 961 5
7	−0.510 867 001 950 83	0.131 688 638 449 18
8	−0.373 706 088 715 42	0.142 096 109 318 4
9	−0.227 785 851 141 65	0.149 172 986 472 6
10	−0.076 526 521 133 5	0.152 753 387 130 73
11	0.076 526 521 133 5	0.152 753 387 130 73
12	0.227 785 851 141 65	0.149 172 986 472 6
13	0.373 706 088 715 42	0.142 096 109 318 4
14	0.510 867 001 950 83	0.131 688 638 449 18
15	0.636 053 680 726 52	0.118 194 531 961 5
16	0.746 331 906 460 15	0.101 930 119 817 24
17	0.839 116 971 822 22	0.083 276 741 576 7
18	0.912 234 428 251 33	0.062 672 048 334 11
19	0.963 971 927 277 91	0.040 014 298 003 9
20	0.993 128 599 185 1	0.017 614 007 139 15

The special case of Jacobi polynomial is Chebyshev, that is when $\alpha = -\frac{1}{2}$ and $\beta = -\frac{1}{2}$. Chebyshev recurrence relation is given by;

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

By putting the values $\alpha = -\frac{1}{2}$ and $\beta = -\frac{1}{2}$, we get the matrix

$$R = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & 0 & \dots & & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{2} & & \dots & \\ & & \dots & & & \\ & & & \dots & & \\ & & & & \dots & \frac{1}{2} \\ & & & \dots & \frac{1}{2} & 0 \end{bmatrix}$$

Then by MATLAB, we find the eigenvalue and eigenvector of symmetric matrix R. Thus, the roots and weights of Chebyshev polynomial for $n = 2$, $n = 4$, $n = 10$ and $n = 20$ is given;

Table 3.18: Roots and weights of Chebyshev for $n = 2$

i	x_i	w_i
1	$-0.707\,106\,781\,186\,55$	$1.570\,796\,326\,794\,9$
2	$0.707\,106\,781\,186\,55$	$1.570\,796\,326\,794\,9$

Table 3.19: Roots and weights of Chebyshev for $n = 4$

i	x_i	w_i
1	$-0.923\,879\,532\,511\,29$	$0.785\,398\,163\,397\,45$
2	$-0.382\,683\,432\,365\,09$	$0.785\,398\,163\,397\,45$
3	$0.382\,683\,432\,365\,09$	$0.785\,398\,163\,397\,4$
4	$0.923\,879\,532\,511\,29$	$0.785\,398\,163\,397\,4$

Table 3.20: Roots and weights of Chevyshev for $n = 10$

i	x_i	w_i
1	$-0.987\,688\,340\,595\,14$	$0.314\,159\,265\,359$
2	$-0.891\,006\,524\,188\,37$	$0.314\,159\,265\,359$
3	$-0.707\,106\,781\,186\,55$	$0.314\,159\,265\,359$
4	$-0.453\,990\,499\,739\,55$	$0.314\,159\,265\,359$
5	$-0.156\,434\,465\,040\,23$	$0.314\,159\,265\,359$
6	$0.156\,434\,465\,040\,23$	$0.314\,159\,265\,359$
7	$0.453\,990\,499\,739\,55$	$0.314\,159\,265\,359$
8	$0.707\,106\,781\,186\,55$	$0.314\,159\,265\,359$
9	$0.891\,006\,524\,188\,37$	$0.314\,159\,265\,359$
10	$0.987\,688\,340\,951\,4$	$0.314\,159\,265\,359$

Table 3.21: Roots and weights of Chevyshev for $n = 20$

i	x_i	w_i
1	$-0.996\,917\,333\,733\,1$	$0.157\,079\,632\,679\,5$
2	$-0.972\,369\,920\,397\,68$	$0.157\,079\,632\,679\,5$
3	$-0.923\,879\,532\,511\,3$	$0.157\,079\,632\,679\,5$
4	$-0.852\,640\,643\,540\,9$	$0.157\,079\,632\,679\,5$
5	$-0.760\,405\,965\,600\,03$	$0.157\,079\,632\,679\,5$
6	$-0.649\,480\,483\,301\,8$	$0.157\,079\,632\,679\,5$
7	$-0.522\,498\,564\,715\,95$	$0.157\,079\,632\,679\,5$
8	$-0.382\,683\,432\,365\,1$	$0.157\,079\,632\,679\,5$
9	$-0.233\,445\,363\,855\,91$	$0.157\,079\,632\,679\,5$
10	$-0.078\,459\,095\,727\,845$	$0.157\,079\,632\,679\,5$
11	$0.078\,459\,095\,727\,845$	$0.157\,079\,632\,679\,5$
12	$0.233\,445\,363\,855\,91$	$0.157\,079\,632\,679\,5$
13	$0.382\,683\,432\,365\,1$	$0.157\,079\,632\,679\,5$
14	$0.522\,498\,564\,715\,95$	$0.157\,079\,632\,679\,5$
15	$0.649\,448\,048\,330\,18$	$0.157\,079\,632\,679\,5$
16	$0.760\,405\,965\,600\,03$	$0.157\,079\,632\,679\,5$
17	$0.852\,640\,164\,354\,09$	$0.157\,079\,632\,679\,5$
18	$0.923\,879\,532\,511\,3$	$0.157\,079\,632\,679\,5$
19	$0.972\,369\,920\,397\,68$	$0.157\,079\,632\,679\,5$
20	$0.996\,917\,333\,733\,1$	$0.157\,079\,632\,679\,5$

Then,the roots and weights of the Jacobi polynomial for any value of α and β is evaluated and tabulated. We choose $\alpha = 5$, $\beta = 5$ and $n = 2$, $n = 4$, $n = 15$.

Table 3.22: Roots and weights of Jacobi Polynomial for $\alpha = 5$, $\beta = 5$ and $n = 2$

i	x_i	w_i
1	$-0.277\,350\,098\,112\,62$	$0.369\,408\,369\,408\,37$
2	$0.277\,350\,098\,112\,62$	$0.369\,408\,369\,408\,37$

Table 3.23: Roots and weights of Jacobi Polynomial for $\alpha = 5$, $\beta = 5$ and $n = 4$

i	x_i	w_i
1	$-0.561\,847\,368\,494\,95$	$0.052\,616\,584\,453\,57$
2	$-0.193\,051\,057\,976\,62$	$0.316\,791\,784\,954\,8$
3	$0.193\,051\,057\,976\,62$	$0.316\,791\,784\,954\,8$
4	$0.561\,847\,368\,494\,95$	$0.052\,616\,584\,453\,57$

Table 3.24: Roots and weights of Jacobi Polynomial for $\alpha = 5$, $\beta = 5$ and $n = 15$

i	x_i	w_i
1	$-0.908\,034\,257\,028\,03$	$0.000\,130\,019\,381\,05$
2	$-0.820\,744\,339\,238\,36$	$0.000\,357\,712\,207\,39$
3	$-0.715\,202\,146\,167\,3$	$0.003\,175\,440\,500\,013$
4	$-0.593\,208\,000\,788\,75$	$0.014\,792\,883\,231\,19$
5	$-0.457\,444\,147\,406\,5$	$0.043\,788\,675\,144\,85$
6	$-0.311\,024\,826\,301\,76$	$0.090\,571\,365\,472\,4$
7	$-0.157\,349\,220\,922\,56$	$0.137\,727\,032\,762\,21$
8	$0.000\,000\,000\,000$	$0.157\,964\,516\,304\,4$
9	$0.157\,922\,092\,256$	$0.137\,727\,032\,762\,21$
10	$0.311\,024\,826\,301\,76$	$0.090\,571\,365\,472\,4$
11	$0.457\,444\,147\,406\,5$	$0.043\,788\,675\,144\,85$
12	$0.593\,208\,000\,788\,75$	$0.014\,792\,883\,231\,19$
13	$0.715\,202\,146\,167\,3$	$0.003\,175\,440\,500\,013$
14	$0.820\,744\,339\,238\,36$	$0.000\,357\,712\,207\,39$
15	$0.908\,034\,257\,028\,03$	$0.000\,130\,019\,381\,05$

CHAPTER 4

APPLICATION TO GAUSSIAN QUADRATURE

Gaussian quadrature is applied to integrate the integrals which are impossible or difficult to compute and the main purpose of it is to maximizing the degree of exactness and so that it chooses the interpolation points cautiously. Its main aim is to compute the integrals given below;

$$I(f) = \int_a^b f(x)w(x)dx \tag{4.1}$$

with the weight function $w(x)$ on the interval $[a, b]$.

4.1 Definition of Gaussian Quadrature

Let $x_1 < x_2 < x_3 < \ldots < x_n$, $x_j \in [a, b]$, $j = 1, 2, 3, \ldots, n$ be the interpolating nodes. Then by the rule

$$I(f) = \int_a^b f(x)w(x)dx = \int_a^b w(x) \sum_{j=1}^{n} f(x_j)L_j(x) = \sum_{j=1}^{n} w_j f(x_j) \tag{4.2}$$

where

$$w_j = \int_a^b L_j(x)w(x)dx \qquad and \qquad L_j(x) = \prod_{k=1,k\neq j}^{n} \frac{(x - x_k)}{(x_j - x_k)} \tag{4.3}$$

is called Gaussian quadrature.

Theorem 8. *[24] Suppose $\{p_k(x)\}_{k=0}^{n}$ consists of orthogonal polynomials which satisfy the inner product*

$$(f, g) = \int_a^b f(x)g(x)w(x)dx$$

Suppose $p_n(x_j) = 0$ for the value x_j where $j = 1, 2, \ldots, N$. Then we say that the quadrature rule given in (4.2) and (4.3) is exact for polynomials of degree equal to $2n - 1$.

Proof. Let $f(x)$ be a polynomial of degree less than or equal to $2n - 1$.
Denote $f(x) = p_n(x)t(x) + r(x)$, where $t(x)$ and $r(x)$ has a degree less than or equal to $n - 1$. Now, since we know that any polynomial is orthogonal to every polynomial of lower degree, $p_n(x)$ is orthogonal to every polynomial of degree less than or equal to $n - 1$.
Thus,

$$I(f) = I(tp_n + r) = \int_a^b t(x)p_n(x)w(x)dx + \int_a^b r(x)w(x)dx = \int_a^b r(x)w(x)dx.$$

Now, $f(x_j) = p_n(x_j)t_(x_j) + r(x_j) = r(x_j)$ since x_j 's are the zeros of $p_n(x)$. We obtain,

$$I(f) = Q(r) = \sum_{j=1}^{n} r(x_j)w_j = \sum_{j=1}^{n} w_j[p_n(x_j)t(x_j) + r(x_j)] = \sum_{j=1}^{n} w_j f(x_j) = Q(f).$$

Here, we see that the Gaussian Quadrature is exact for all polynomials with degree less than or equal to $2n - 1$. □

4.2 Error in Gaussian Quadrature

Theorem 9. *[24] Let $p(x_j) = 0$ for $j = 1, 2, \ldots, n$. Suppose $f \in \mathbb{C}^{2n}[a, b]$. Then the error $E(f)$ for Gaussian quadrature in (4.2) and (4.3) is*

$$E(f) = \frac{f^{(2n)}(\zeta)}{(2n)!} \int_a^b p_n^2(x)w(x)dx \tag{4.4}$$

for $\zeta \in (a, b)$, where $w(x)$ is a weight function and $p_n(x)$ is a monic polynomial of degree n.

Proof. To prove the above theorem we make use of Hermite Interpolation rule. Then, let $h_{2n-1}(x)$ of degree $2n - 1$ be a unique polynomial with

$$h_{2n-1}(x_j) = f(x_j) \qquad , \qquad h'_{2n-1}(x_j) = f'(x_j)$$

Moreover, let $\theta \in (a, b)$ with

$$f(x) = h_{2n-1}(x) + \frac{f^{(2n)}(\theta)}{(2n)!}(x - x_1)^2 \cdots (x - x_n)^2.$$

Note that $(x - x_1) \cdots (x - x_n) = p_n(x)$ as p_n is monic and $p_n(x_j) = 0$. Thus,

$$\int_a^b f(x)w(x)dx = \int_a^b h_{2n-1}(x)w(x)dx + \int_a^b \frac{f^{(2n)}(\theta)}{(2n)!} p_n^2(x)w(x)dx$$

The quadrature rule is exact for $h_{2n-1}(x)$.
Thus,

$$\int_a^b h_{2n-1}(x)w(x)dx = Q(h_{2n-1}) = \sum_{j=1}^{n} h_{2n-1}(x_j)w_j = \sum_{j=1}^{n} f(x_j)w_j = Q(f)$$

Also, we know that $p_n^2(x)w(x) \geq 0$ on $[a, b]$. Then by applying the MVT, we obtain

$$\int_a^b \frac{f^{(2n)}(\theta)}{(2n)!} p_n^2(x)dx = \frac{f^{(2n)}(\zeta)}{(2n)!} \int_a^b p_n^2(x)w(x)dx$$

for some $\zeta \in (a, b)$. □

4.3 Fundamental theorem of Gaussian

It implies that the nodes of the Gaussian quadrature in the interval (a,b) and along with the weight function $w(x)$ are exactly the roots of the orthogonal polynomials $p_n(x)$ for the same weight function and interval.[25] Thus,we can conclude that the roots of the classical orthogonal polynomials that we get in the previous chapter are the nodes of Gaussian quadrature rule.

4.4 Example

4.4.1 Hermite Polynomial

Gaussian Quadrature by using the roots x_i and weights w_i of Hermite Polynomial is given by

$$\int_{-\infty}^{\infty} e^{-x^2} f(x)dx = \sum_{j=1}^{n} w_i f(x_i).$$

For example

$$I = \int_{-\infty}^{\infty} e^{-x^2} \cos(x) dx. \tag{4.5}$$

The exact value of the integral is $I = \frac{\sqrt{\pi}}{e^{-\frac{1}{4}}} = 1.3803884470432.....$. Then by using the formula in (4.5) to evaluate I approximately, and taking n

Table 4.1: Example by using the roots and weights of Hermite

n	I_n
4	1.380 329 757 161 3
8	1.380 388 447 031 7
9	1.380 388 447 043 3
10	1.380 388 447 043 2
15	1.380 388 447 043 2

Here, we see that the result of the integral has 14 digit accuracy.

4.4.2 Laguerre Polynomial

Gaussian Quadrature by using the roots x_i and weights w_i of Laguerre Polynomial is given by

$$\int_{0}^{\infty} x^{\alpha} e^{-x} f(x) dx = \sum_{j=1}^{n} w_i f(x_i). \tag{4.6}$$

For the case $\alpha = 0$, consider the example

$$I = \int_{0}^{\infty} e^{-x} e^{(sin(\frac{x}{10})-cos(\frac{x}{10}))} dx = \sum_{j=1}^{n} w_i f(x_i)$$

The exact value of the integral is 0.41330875870848.

Then by using the roots and weight of Laguerre when $\alpha = 0$ to evaluate I approximately, and taking n

Table 4.2: Example by using the roots and weights of Laguerre for $\alpha = 0$

n	I_n
4	0.413 308 737 784 09
8	0.413 308 758 710 23
10	0.413 308 758 708 52
11	0.413 308 758 708 48

The integral has 14 digit accuracy. Here, also we see that the Gaussian quadrature converges to the exact value of the integral.
Now for the case $\alpha = -\frac{1}{2}$, consider the example

$$I = \int_0^\infty x^{-\frac{1}{2}}(x^{10} + 5x^4 + 20)e^{-x}dx. \tag{4.7}$$

The exact value of the integral is not known so here we show that Gaussian quadrature is also used for the integrals which are impossible to be solved. Then by using the formula in (4.6) to evaluate I approximately, and taking n

Table 4.3: Example by using the roots and weights of Laguerre for $\alpha = -\frac{1}{2}$

n	I_n
4	901 737.757 378 55
8	1 127 090.863 333 7
10	1 133 371.996 667 8
11	1 133 371.996 667 8
20	1 133 371.996 667 8

The integral has 14 digit accuracy and it converges to the value 1133371.9966678

4.4.3 Jacobi Polynomial

Gaussian Quadrature by using the roots x_i and weights w_i of Jacobi Polynomial is given by

$$\int_{-1}^{1}(1-x)^{\alpha}(x-1)^{\beta}f(x)dx = \sum_{j=1}^{n} w_i f(x_i) \tag{4.8}$$

Consider first Legendre Polynomial,i.e., put $\alpha = \beta = 0$ in Jacobi polynomial. Then, we have as an example

$$I = \int_{-1}^{1} \frac{1}{x^2+1}dx \tag{4.9}$$

The exact value of the integral is $I = \frac{\pi}{2} = 1.57079632679....$
Then by using the formula in (4.9) to evaluate I approximately, and taking

Table 4.4: Example by using the roots and weights of Legendre

n	I_n
4	1.568 627 450 98
8	1.570 794 412 54
9	1.570 796 655 94
10	1.570 796 270 22
12	1.570 796 325 12
13	1.570 796 327 08
14	1.570 796 326 74
16	1.570 796 326 79
17	1.570 796 326 79
100	1.570 796 326 79

Here, we see that the values are all has 11 digit accuracy after n=16 partition.
Now, consider Chebyshev Polynomial where $\alpha = \beta = -\frac{1}{2}$. Then, we have as an example

$$I = \int_{-1}^{1} \frac{e^x}{\sqrt{1-x^2}}dx \tag{4.10}$$

The exact value of the integral is not known, so we are going to evaluate the integral by Gaussian quadrature using the nodes and weights we calculated before. Then by using the formula in (4.9) to evaluate I approximately, and taking

Table 4.5: Example by using the roots and weights of Chebyshev

n	I_n
2	$3.960\,266\,053$
3	$3.977\,321\,961$
4	$3.977\,462\,635$
6	$3.977\,463\,261$
7	$3.977\,463\,261$
8	$3.977\,463\,261$
50	$3.977\,463\,261$

Here, we see that the values are all has 10 digit accuracy after n=6 so that we can conclude the Gaussian quadratures converges to the exact value of the integral.Thus, for the integrals which are really difficult to evaluate we can use Gaussian quadrature method.

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