CALCULATIONS OF THE ROOTS OF CLASSICAL ORTHOGONAL
POLYNOMIALS: AN APPLICATION TO GAUSSIAN QUADRATURE

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ABSTRACT

CALCULATIONS OF THE ROOTS OF CLASSICAL ORTHOGONAL POLYNOMIALS: AN APPLICATION TO GAUSSIAN QUADRATURE

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This thesis focuses on classical orthogonal polynomials namely Jacobi, Laguerre and Hermite polynomials and a method to calculate the roots of these polynomials is constructed. The roots are expressed as the eigenvalues of a tridiagonal matrix whose coefficients depend on the recurrence formula for the classical orthogonal polynomials. These approximations of roots are used as method of computation of Gaussian quadratures. Then the discussion of the numerical results are then introduced to deduce the efficiency of the method.

Keywords: Gaussian Quadrature, Orthogonal functions, Jacobi, Hermite, Laguerre and Legendre polynomials, classical orthogonal polynomials, equation of hypergeometric type
ÖZ

KLASİK ORTHOGONAL POLİNOMLARIN KÖKLERİNİN HESAPLANMASI: GAUSS KARELEME YÖNTEMIYLE SAYİSAL İNTEGRASYON İÇİN BİR UYGULAMA

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Anahtar Kelimeler: Gauss kareleme, hipergeometrik tıpte denklem, klasik dik polinomlar, Jacobi,Laguerre ve Hermite polinomları
To my family
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Numerical integration, also called quadrature, is the approximate computation of an integral of a given function using numerical techniques. There are several reasons for carrying out numerical integration. One of the reasons is computer applications and embedded systems need numerical integration to compute the data obtained by sampling when the integration of the function is known only at certain points. Another case when the numerical integration is preferred is when it is difficult or impossible to integrate the function. The most famous of numerical integration are Newton-Cotes and Gaussian quadrature.

The idea of Gaussian quadrature is to give ourselves the freedom to choose not only the weighting coefficients, but also the location of the abscissas at which the function is to be evaluated: They will no longer be equally spaced. Thus, we will have twice the number of degrees of freedom at our disposal; it will turn out that we can achieve Gaussian quadrature formulas whose order is, essentially, twice that of the Newton-Cotes formula with the same number of function evaluations.

In practice, it is common to prefer lower order methods over higher order and Newton-Cotes over Gaussian. However, Sermutlu compared Gaussian and Newton-Cotes methods with each other at given orders and compared higher order and lower order methods within each category and concluded that Gaussian method at the highest order gives more accurate results than all other methods. In other words, Gaussian is superior to Newton–Cotes and higher orders are superior to lower orders, as the number of points increase.

A.H. Stroud and Don Secrest said that advantages of Gaussian quadrature formula are that they converge under conditions which are most always realizable in practice.
and the order of precision with reference to the degree of the polynomial for which a specific formula is exact is much greater than that for the corresponding Newton-Cotes formula [4].

The special functions is a huge area of mathematics which includes trigonometric, exponential and logarithmic functions and also it extended to include beta, gamma functions and classes of classical orthogonal polynomials[5]. These special functions have many and varied applications in pure mathematics and numerous applied sciences such as astronomy, heat conduction, electric circuits, quantum mechanics, electrostatic interpretation and mathematical statistics [5, 6, 7, 8, 9, 10].

The special functions satisfy the second order differential equation of the form

$$\sigma(z)y'' + \tau(z)y' + \lambda y = 0, \ z \in \mathbb{C} \quad (1.1)$$

where $\lambda$ is constant, $\sigma(z)$ is at most a quadratic polynomial and $\tau(z)$ is a linear polynomial. The roots of special functions are related to resonances in mechanical and electromagnetic systems, arise in quantum mechanical calculations, and are the nodes of Gaussian quadrature formulas[10]. Their role as the nodes of Gaussian quadrature formulas motivates much of the interest in the numerical computation of the roots of special functions defined by equations of the form (1.1). Moreover, the zeros of the Jacobi polynomials play an interesting role in a problem of Stieltjes concerning electrostatic equilibrium.

Each family of classical orthogonal polynomials — Legendre polynomials, Hermite polynomials, Laguerre polynomials — is associated with a nonnegative weight function $w$ and a collection of Gaussian quadrature rules, one for each positive integer $n$, of the form

$$\int_a^b \varphi(t)w(t)dt = \sum_{j=1}^{n} \varphi(t_j)w(j) \quad (1.2)$$

The Gaussian quadrature rule (1.2) is exact when $\varphi$ is a polynomial of degree less than $2n$. The nodes $t_1, \ldots, t_n$ are the roots of a polynomial $p$ of degree $n$ that satisfies an equation of the form (1.1), and, at least in the case of the classical orthogonal polynomials, the weights $w_1, \ldots, w_n$ can be calculated from the values of the derivatives of $p$ at the nodes $t_1, \ldots, t_n$ [10]. In 1984, Gauss constructed Gauss-Legendre quadrature rule using hypergeometric functions and continued fractions [11], and in
1826, Jacobi noted the quadrature nodes where precisely the roots of the Legendre polynomial of degree $n$ \[12\].

Moreover, Andreas Glaser and Vladimir Rokhlin showed that the Gauss quadrature nodes are the simple roots of an associated orthogonal polynomials \[13\]. Thus, we could say zeros of classical special functions play an important role in the mathematical sciences, being related to Gaussian quadratures.

In last decades, many methods are proposed to calculate the $n$-point Gaussian quadrature and in \[14\], Newton’s method is combined with a scheme, based on asymptotic formulas, for evaluating Legendre polynomials of arbitrary orders and arguments. A similar approach is taken in \[15\], in which asymptotic formulas for Jacobi polynomials are used to evaluate Newton iterates which converge to the nodes of Gauss-Jacobi quadrature rules. The most widely known method for computing Gauss quadrature nodes and weights is the Golub-Welsh algorithm \[16\], which exploits the three term recurrence relations satisfied by all real orthogonal polynomials and the eigenvalues of symmetric tridiagonal matrices in order to calculate the nodes of an $n$-point Gaussian quadrature formula. The scheme of \[17\] uses a rapidly convergent fixed point method to calculate the roots of solutions of quite general second order differential equations of the form (1.1). Unlike Newton’s method, the fixed point scheme of \[10, 17\], is guaranteed to converge. This approach has the disadvantage, though, that the regions of monotonicity of the coefficient $q$ must be explicitly known. The schemes of \[14, 15, 17, 18, 19\] all have the property that any particular quadrature node and its corresponding weight can be calculated independently of the others, making them suitable for parallelization.

Moreover, in 2007, Glasier-Liu-Rokhlin \[13\] developed an algorithm which computes all the nodes and weights of the $n$-point quadrature rule in very effective way that can compute one million quadrature nodes in a handful of seconds. The Glaser-Liu-Rokhlin method \[13\] combines the Prufer transform with the classical Taylor series method for the solution of ordinary differential equations. It computes $n$ roots in $O(n)$ operations and is more general than the schemes \[14, 15, 18\] in that it applies to special functions defined by second order differential equations of the form

$$r_0(t)y''(t) + p_0(t)y'(t) + q_0(t)y(t) = 0 \quad (1.3)$$
where $p_0, q_0$ and $r_0$ are polynomials of degree less than or equal to two. This class includes the classical orthogonal polynomials, Bessel functions, prolate spheroidal wave functions, etc. The Glaser-Liu-Rokhlin algorithm does, however, suffer from several disadvantages. It is typically slower than the methods discussed above, and is unsuitable for parallel implementation since the roots must be computed in sequential order[10].

Gaussian quadrature rule is used in many areas of mathematics and some of them are mentioned below. Jayan Sarada and K.V.Nagaraja presented gaussian quadrature method for numerical integration over triangular, parallelogram and quadrilateral elements with linear sides [20].

Takashi Amisaki presented a numerical integration method for estimating the area under the curve (AUC) over the infinite time interval. This method is based on the Gauss–Laguerre quadrature and produces AUC estimates over the infinite time interval without extrapolation in a usual sense [22].

Ping Yang, Xue-Wen Feng, Wen-Jun Liang and Kai-SuWu used Gaussian-Laguerre quadrature formula which is less nodes and higher precision numerical integral formula to approximate integral term of IBBRP (The inverse black body radiation problem) [23].

The thesis is organized as follows. In Chapter 2, we give the necessary knowledge about special functions and introduce Jacobi, Laguerre and Hermite polynomials with hypergeometric type and recurrence relations of classical orthogonal polynomials. In chapter 3, we give the method to find the roots of classical orthogonal polynomials and some roots and weights are tabulated. Numerical applications of the Gaussian quadrature and their results are given in Chapter 4.
CHAPTER 2

CLASSICAL ORTHOGONAL POLYNOMIALS

In this chapter, we review some basic and remarkable properties of the classical orthogonal polynomials which are very well known in literature. Here, we present the results without their proofs and refer the reader to Nikiforov and Uvarov [28] for more about the special functions.

2.1 Equation of Hypergeometric type

Special functions including the classical orthogonal polynomials namely Hermite, Laguerre, Legendre, Chebyshev and Jacobi and hypergeometric functions are all generated by differential equation of the form

$$\sigma(x)y'' + \tau(x)y' + \lambda y = 0$$  \hspace{1cm} (2.1)

where $\lambda$ is constant, $\sigma(x)$ is at most a quadratic and $\tau(x)$ is a linear polynomial. Here, $x$ denotes a complex variable, in general. (2.1) is called as an equation of hypergeometric type (EHT). First, we consider more general equation of type

$$u'' + \frac{\tilde{\tau}(x)}{\sigma(x)}u' + \frac{\tilde{\sigma}(x)}{\sigma^2(x)}u = 0$$ \hspace{1cm} (2.2)

where $\sigma(x)$ and $\tilde{\sigma}(x)$ is at most quadratic and $\tilde{\tau}(x)$ is a linear polynomial. Now, introducing the transformation

$$u = \varphi(x)y$$ \hspace{1cm} (2.3)

and substituting into (2.2), we get

$$y'' + \left(2\frac{\varphi'}{\varphi} + \frac{\tilde{\tau}}{\sigma}\right)y' + \left(\frac{\tilde{\sigma}}{\sigma^2} + \frac{\tilde{\tau}}{\sigma} \frac{\varphi'}{\varphi} + \frac{\varphi''}{\varphi}\right)y = 0$$ \hspace{1cm} (2.4)
in which the coefficient of \( y' \) can be made a rational function \( \frac{\tau(x)}{\sigma(x)} \), where \( \tau(x) \) is a linear polynomial. Then, we have

\[
2 \frac{\varphi'(x)}{\varphi(x)} = \frac{\tau(x) - \tilde{\tau}(x)}{\sigma(x)}
\]

and

\[
\frac{\varphi'}{\varphi} = \frac{p(x)}{\sigma(x)}, \quad p(x) := \frac{1}{2} [\tau(x) - \tilde{\tau}(x)]. \tag{2.5}
\]

Here, \( p(x) \) is a polynomial of degree at most 1. Since

\[
\frac{\varphi''}{\varphi} = \left(\frac{\varphi'}{\varphi}\right)' + \left(\frac{\varphi'}{\varphi}\right)^2 = \left(\frac{p}{\sigma}\right)' + \left(\frac{p}{\sigma}\right)^2
\]

equation (2.2) now takes the form

\[
y'' + \frac{\tau(x)}{\sigma(x)} y' + \frac{\tilde{\sigma}(x)}{\sigma^2(x)} y(x) = 0. \tag{2.6}
\]

Here, \( \tilde{\sigma}(x) \) is a quadratic polynomial,

\[
\tau(x) = 2p(x) + \tilde{\tau}(x) \tag{2.7}
\]

and

\[
\tilde{\sigma}(x) = p^2(x) + [\tilde{\tau}(x) - \sigma'(x)]p(x) + \sigma(x)p'(x) + \tilde{\sigma}(x). \tag{2.8}
\]

Equation (2.6) is the same type of equation (2.2) so that type of the generalized EHT in (2.2) remains invariant under the substitution \( u = \varphi(x) y \), where the transformation doesn’t modify the type of the given differential equation. The coefficients \( \tau(x) \) and \( \tilde{\sigma}(x) \) in the transformed equation (2.6) depend on the selection of \( p(x) \). Therefore, we may choose \( p(x) \) such that \( \tilde{\sigma}(x) \) is a multiple of \( \sigma \), i.e

\[
\tilde{\sigma}(x) = \lambda \sigma(x), \quad \lambda \in \mathbb{C}. \tag{2.9}
\]

Then, equation (2.6) is

\[
y'' + \frac{\tau(x)}{\sigma(x)} y' + \frac{\lambda}{\sigma(x)} y = 0
\]

which is equal to EHT in (2.1).

### 2.2 Polynomial solution of EHT

The EHT has solutions such that \( y(x) = y_n(x) \), where \( y_n(x) \) is a polynomial of degree \( n \) and depends on the variable \( x \) for particular values of \( \lambda \) given by the formula:

\[
\lambda = \lambda_n = -n \tau' - \frac{1}{2} n(n - 1) \sigma''. \tag{2.10}
\]
2.2.1 Rodrigues Theorem

**Theorem 1.** The polynomial solutions of the EHT can be constructed by a formula

\[ y_n(x) = \frac{B_n}{\rho(x)} \frac{d^n}{dx^n} [\sigma^n(x) \rho(x)] \]  

(2.11)

known as the Rodrigues formula. \( B_n \) is called a normalization constant.

We can write the EHT in a self adjoint form

\[ \frac{d}{dx} [\sigma(x) \rho(x) \frac{dy}{dx}] + \lambda \sigma(x) y = 0 \]

where \( \rho(x) \) satisfies the separable first order ODE

\[ [\sigma(x) \rho(x)]' = \tau(x) \rho(x). \]  

(2.12)

Solving equation (2.12), we obtain possible forms of \( \rho(x) \) as

\[
\begin{align*}
(b - x)^\alpha (x + a)^\beta & \quad \text{for } \sigma(x) = (b - x)(x - a) \\
(x - a)^\alpha e^{\beta x} & \quad \text{for } \sigma(x) = x - a \\
e^{\alpha x^2 + \beta x} & \quad \text{for } \sigma(x) = 1
\end{align*}
\]

(corresponding to possible degrees of \( \sigma(x) \) where \( a, b, \alpha, \beta \) are constants.

Now, we can reduce \( \sigma(x) \) and \( \rho(x) \) to the canonical forms:

\[
\begin{align*}
(1 - x)^\alpha (x + 1)^\beta & \quad \text{for } \sigma(x) = 1 - x^2 \\
x^\alpha e^{-x} & \quad \text{for } \sigma(x) = x \\
e^{-x^2} & \quad \text{for } \sigma(x) = 1
\end{align*}
\]  

(2.13)

2.3 Orthogonality of Polynomial Solutions of EHT

**Theorem 2.** Suppose the coefficients in EHT are such that

\[ \sigma(x) \rho(x) x^j \bigg|_{x=a,b} = 0, \quad j = 0, 1, ... \]  

(2.14)

where \( a \) and \( b \) are the end points of a finite or infinite interval. Then, the polynomial solutions \( y = p_n(x) \) constitute an orthogonal sequence of real functions of the real argument \( x \)

\[ p_0(x), p_1(x), p_2(x), \ldots, p_m(x), \ldots, p_n(x), \ldots \]
In other words,

\[ \int_a^b \rho(x)p_n(x)p_m(x)dx = \begin{cases} 
0 & \text{if } m \neq n \\
N_n^2 & \text{if } m = n
\end{cases} \tag{2.15} \]

where \( N_n \) is called a normalization constant and \( \rho(x) \) is called a weighting function.

### 2.4 General Properties of Orthogonal Polynomials

The classical orthogonal polynomials namely Jacobi, Legendre, Laguerre and Hermite which are polynomial solutions of hypergeometric differential equations in (2.1) are obtained by choosing the values for \( \sigma(z) \), \( \tau(z) \) and \( \lambda \) in (2.1).

#### 2.4.1 Jacobi Polynomials

Jacobi Polynomials are denoted by \( P_n^{(\alpha,\beta)}(x) \). Suppose \( \sigma(x) = (1 - x^2) \) and \( \rho(x) = (1 - x)^\alpha(1 + x)^\beta \).

Then (2.12) reads as

\[ \left[(1 - x^2)(1 - x)^\alpha(1 + x)^\beta\right]' = \tau(x)(1 - x)^\alpha(1 + x)^\beta \]

and we obtain

\[ \tau(x) = \beta - \alpha - (\alpha + \beta + 2)x. \]

Also, (2.10) becomes \( \lambda_n = n(n + \alpha + \beta + 1) \).

The EHT

\[ (1 - x^2)y'' + [\beta - \alpha - (\alpha + \beta + 2)x]y' + n(n + \alpha + \beta + 1)y = 0 \tag{2.16} \]

is known as Jacobi differential equation, and their polynomial solutions are Jacobi polynomials. Jacobi polynomial is expressed by the Rodrigues formula

\[ P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!} (1 - x)^{-\alpha}(1 + x)^{-\beta} \frac{d^n}{dx^n} [(1 - x)^{n+\alpha}(1 + x)^{n+\beta}] \]

\[ = a_n x^n + b_n x^{n-1} + \ldots \]
where the normalization constant $B_n$ is taken as $B_n = \frac{(-1)^n}{2^n n!}$ for historical reasons. $a_n$ is the leading and $b_n$ is the sub-leading coefficient of the Jacobi Polynomial $P^{(\alpha,\beta)}_n$. The coefficients are expressed by the below formula:

$$a_n = \frac{(\alpha + \beta + 1 + n)_n}{2^n n!}$$

and

$$b_n = \frac{n(\alpha - \beta)}{\alpha + \beta + 2n}$$

where the symbol $(t)_n = t(t+1)\cdots(t+n-1)$ is called Pochammer’s symbol for any $t$. Thus here

$$(\alpha + \beta + 1 + n)_n = (\alpha + \beta + 1 + n)(\alpha + \beta + 2 + n)\cdots(\alpha + \beta + 2n).$$

Classical Jacobi polynomials $P^{(\alpha,\beta)}_n(x)$ are orthogonal. From the hypothesis of the

Theorem 2,

$$\sigma(x)\rho(x)x^j|_{x=a,b} = (1-x)^{\alpha+1}(x+1)^{\beta+1}x^j|_{x=a,b} = 0, \quad j = 0, 1, 2, \ldots$$

is satisfied if $(a,b) = (-1,1)$ provided that $\alpha > -1$ and $\beta > -1$.

The normalization factor of Jacobi polynomial is $\mathcal{N}_n$ where

$$\mathcal{N}_n^2 = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)}$$

and the Gamma function is defined by

$$\Gamma = \int_0^\infty t^{x-1} e^t dt.$$ 

satisfies the recurrence relation

$$\Gamma(x+1) = x\Gamma(x).$$

2.4.1.1 Legendre polynomials

The subclass of Jacobi polynomials when $\alpha = \beta = 0$ is called Legendre polynomial. Legendre polynomials are given

$$P_n^{(0,0)} = P_n = \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n}[(1-x^2)^n]$$

(2.20)
and satisfy the differential equation of the form

\[(1 - x^2)P_n'' - 2xP_n' + n(n + 1)P_n = 0.\]  \hspace{1cm} (2.21)

Legendre polynomials are orthogonal and the normalization constant of Legendre polynomial is \(N_n\) where

\[N_n^2 = \frac{2}{2n + 1}.\]  \hspace{1cm} (2.22)

### 2.4.1.2 Chebyshev polynomials

The subclass the Jacobi polynomials with \(\alpha = \beta = -\frac{1}{2}\) is called Chebyshev polynomials. Chebyshev polynomials are

\[T_n := \delta_n P_n^{(-\frac{1}{2},-\frac{1}{2})}, \quad n \in \mathbb{N}\]  \hspace{1cm} (2.23)

where

\[\delta_n = \frac{n!\sqrt{\pi}}{\Gamma(n + \frac{1}{2})}\]

Consequently, they are solutions of differential equation

\[(1 - x^2)T''_n - xT'_n + n^2T_n(x) = 0.\]  \hspace{1cm} (2.24)

By taking \(\alpha = \beta = -\frac{1}{2}\), we get the orthogonality for Chebyshev polynomials as

\[\int_{-1}^{1} T_n(x)T_m(x)\sqrt{\frac{1}{1 - x^2}}dx = N_n^2\delta_{mn}\]

where

\[N_n^2 = \begin{cases} 
\pi, & n = 0 \\
\frac{\pi}{2}, & n > 0
\end{cases}\]  \hspace{1cm} (2.25)

### 2.4.2 Laguerre polynomials

Laguerre polynomials are denoted by \(L_n^\alpha(x)\).

Take \(\rho(x) = x^\alpha e^{-x}\) and \(\sigma(x) = x\). Then, \(\tau(x) = \alpha + 1 - x\), \(\lambda_n = n\) and the corresponding polynomials satisfy

\[xy'' + (\alpha + 1 - x)y' + ny = 0.\]  \hspace{1cm} (2.26)
The Rodrigues formula for the Laguerre polynomial is
\[ L_\alpha^n(x) = \frac{1}{n!} e^x x^{-\alpha} \frac{d^n}{dx^n}[x^{n+\alpha} e^{-x}] = a_n x^n + b_n x^{n-1} + \ldots, \quad n = 0, 1, 2, \ldots \]
where
\[ B_n = \frac{1}{n!}, \]
\[ a_n = \frac{(-1)^n}{n!} \]  \hspace{1cm} (2.27)
and
\[ \frac{b_n}{a_n} = (-n)(n + \alpha). \]  \hspace{1cm} (2.28)

The Laguerre polynomials are orthogonal on the interval \((a, b) = (0, \infty)\) since
\[ \sigma(x)\rho(x)x^j |_{x=a,b} = x^\alpha + 1 + j e^{-x} |_{x=a,b} = 0, \quad j = 0, 1, 2, \ldots \]
provided that \(\alpha > -1\)

The normalization factor of Laguerre polynomial is \(N_n\) where
\[ N_n^2 = \frac{1}{n!} \Gamma(\alpha + 1 + n). \]  \hspace{1cm} (2.29)

### 2.4.3 Hermite Polynomials

Hermite polynomials are denoted by \(H_n(x)\).

Let \(\rho(x) = e^{-x^2}\) and \(\sigma(x) = 1\). Now \(\lambda_n = 2n\), \(\tau(x) = -2x\) and the corresponding polynomials satisfy the equation
\[ y'' - 2xy' + 2ny = 0. \]  \hspace{1cm} (2.30)

The Rodrigues formula for Hermite polynomial is
\[ H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}[e^{-x^2}] = a_n x^n + b_n x^{n-1} + \cdots + \text{constant}, \quad n = 0, 1, 2, \ldots \]  \hspace{1cm} (2.31)
where
\[ B_n = (-1)^n, a_n = 2^n, \frac{b_n}{a_n} = 0. \]
$H_n(x)$ are orthogonal over the whole real line $(a, b) = (-\infty, \infty)$ because

$$\sigma(x)\rho(x)x^j|_{x=a,b} = x^je^{-x^2}|_{x=a,b} = 0, \quad j = 0, 1, 2, \ldots.$$ 

The normalization constant of Hermite polynomial is $N_n$ where

$$N_n^2 = 2^n n! \sqrt{\pi}. \quad (2.32)$$

**Theorem 3. Darboux-Cristoffel Formula:**

For each classical orthogonal polynomial $p_n(x)$, there exists a finite sum formula of the form;

$$\sum_{i=0}^{n} \frac{1}{N_i^2} p_i(x)p_i(y) = \frac{A_n(x,y)}{y-x} \quad (2.33)$$

where

$$A_i(x,y) = \frac{\alpha_i}{N_i^2} \begin{vmatrix} p_i(x) & p_i(y) \\ p_{i+1}(x) & p_{i+1}(y) \end{vmatrix} = -\frac{1}{N_i^2} \frac{\alpha_i}{a_{i+1}} [p_{i+1}(x)p_i(y) - p_i(x)p_{i+1}(y)],$$

$\alpha_i$ is the leading coefficient of $p_i$ and $\alpha_i = \frac{a_i}{a_{i+1}}$.

**Theorem 4. The roots of classical orthogonal polynomials:**

The polynomials $p_n(x)$ has exactly $n$ real and distinct roots $x_1 < x_2 < x_3 < \cdots < x_n$ lying on the interval $(a, b)$.

**Proof.** Let $x_j$ be a zero of $p_n(x)$, in other words, $p_n(x_j) = 0$. Suppose $p_n(x)$ has $t$ changes of sign on $(a, b)$. Clearly, $0 \leq t \leq n$. Let us define the polynomials

$$q_t(x) = \begin{cases} \pm 1 & \text{for } t = 0 \\ \prod_{j=1}^{k} (x - x_j) & \text{for } 0 < t \leq n \end{cases}$$

in which $x_j \in (a, b)$ are the points where $p_n(x)$ changes sign.

$$q_t(x) = (x - x_1)(x - x_2) \cdots (x - x_t)$$

Now observe that $q_t(x)p_n(x)$ does not change sign for $x \in (a, b)$ and is always positive. Then, we have

$$\int_a^b \rho(x)q_t(x)p_n(x)dx \neq 0$$

since the integrand is positive. Therefore, we see that $t = n$ because $p_n(x)$ is orthogonal to all polynomials degree less than $n$, making the integral zero.  \hfill \Box
**Theorem 5.** The roots of classical orthogonal polynomials are interlaced. The polynomial \( p_n(x) \) has exactly one root between any two roots \( x_j \) and \( x_{j+1} \) of \( p_{n+1}(x) \).

**Proof.** Let us consider the limiting case of Darboux-Cristoffel Formula as \( y \to x \). Now, if we apply L'Hospital rule, we see that

\[
\lim_{y \to x} \frac{\mathcal{A}_n(x,y)}{y-x} = \lim_{y \to x} \frac{\partial (\mathcal{A}_n)}{\partial y} \frac{a_n}{1} \lim_{y \to x} \left[ p_{n+1}(x)p_n'(y) - p_n(x)p_{n+1}'(y) \right] =
\]

\[
\frac{-1}{N_n^2} a_n \frac{1}{a_{n+1}} \left[ p_{n+1}(x)p_n'(x) - p_n(x)p_{n+1}'(x) \right] = \sum_{k=0}^{n} \frac{1}{N_k^2} [p_k(x)]^2.
\]

Let \( x = x_j \), where \( x_j \) is a root of \( p_{n+1}(x) \), i.e \( p_{n+1}(x_j) = 0 \). Then

\[
p_n(x_j)p_{n+1}'(x_j) = N_n^2 \frac{a_{n+1}}{a_n} \sum_{k=0}^{n} \frac{1}{N_k^2} [p_k(x_j)]^2.
\]

Obviously, the sign of right hand side is independent of \( x_j \) and depends only on the sign of the ratio \( \frac{a_{n+1}}{a_n} \). However, signs of \( p_{n+1}'(x) \) changes between \( x_j \) and \( x_{j+1} \) since \( p_{n+1} \) has a local extreme value there. That is, \( p_n(x) \) should also change signs accordingly, which means it has at least one zero between \( x_j \) and \( x_{j+1} \). However, since there are \( n \) subintervals \( (x_j, x_{j+1}) \) on \( (a, b) \) and there exists at least one root of \( p_n(x) \) in each subinterval, we conclude that, there is exactly one zero between any two successive roots of \( p_{n+1}(x) \). \( \square \)

### 2.5 Recurrence equations for Classical Orthogonal Polynomials \( p_n(x) \)

**Theorem 6.** The following relation holds for Classical Orthogonal Polynomials:

\[
\alpha_n p_{n+1}(x) = (x - \beta_n) p_n(x) - \gamma_n p_{n-1}(x) \tag{2.34}
\]

for \( n = 1, 2, \ldots \) \( p_0(x) = 1 \) where the coefficients

\[
\alpha_n = \frac{a_n}{a_{n+1}}, \quad \beta_n = \frac{b_n}{a_n} - \frac{b_{n+1}}{a_{n+1}}, \quad \gamma_n = \frac{N_n^2}{N_{n-1}^2} \frac{a_{n-1}}{a_n} \tag{2.35}
\]

are expressed by normalization constant, the leading and sub-leading coefficients.

**Proof.** By the expansion formula,

\[
x p_n(x) = \sum_{k=0}^{n+1} c_k n p_k(x) = c_0 n p_0(x) + c_1 n p_1(x) + \ldots + c_{n+1} n p_{n+1}(x) \tag{2.36}
\]
we get,
\[
\int_{a}^{b} x^p n(x)p_m(x)\rho(x)dx = \sum_{k=0}^{n+1} c_{kn} \int_{a}^{b} p_k(x)p_m(x)\rho(x)dx = c_{mn} N_m^2, \quad m = 0, 1, \ldots, n + 1
\]

Thus, we get
\[
c_{mn} = \frac{1}{N_m^2} \int_{a}^{b} x^p n(x)p_m(x)\sigma(x)dx,
\]
so the first linear combination constants are given by the formula
\[
c_{kn} = \frac{1}{N_k^2} \int_{a}^{b} \rho(x) p_n(x)x^p k(x)dx, \quad k = 0, 1, \ldots, n + 1
\]

However, we know that \(x^p k(x)\) is polynomial of degree \(k + 1\). Thus, \(c_{kn} = 0\) when \(k + 1 < n\) and \(k < n - 1\). Hence, from (2.39), we write
\[
x^p n(x) = c_{n-1,n} p_{n-1}(x) + c_{n,n} p_n(x) + c_{n+1,n} p_{n+1}(x)
\]
which is nothing but the relation is
\[
\alpha_n = c_{n+1,n}, \quad \beta_n = c_{n,n}, \quad \gamma_n = c_{n-1,n}.
\]
The integral in (2.40) is unchanged when the indices \(k\) and \(n\) are interchanged. It follows then that,
\[
N_k^2 c_{kn} = N_n^2 c_{nk}
\]
Putting \(k = n - 1\), we obtain
\[
N_{n-1}^2 c_{n-1,n} = N_n^2 c_{n,n-1}
\]
which implies that
\[
N_{n-1}^2 \gamma_n = N_n^2 \alpha_{n-1}
\]
and, hence
\[
\gamma_n = \frac{N_n^2}{N_{n-1}^2} \alpha_{n-1}. \quad (2.41)
\]
On the other hand, since
\[
p_n(x) = a_n x^n + b_n x^{n-1} \ldots, \quad a_n \neq 0
\]
the relation in (2.42) leads to
\[ a_n x^{n+1} + b_n x^n + \ldots = \alpha_n a_{n+1} x^{n+1} + (\alpha_n b_n + \beta_n a_n) x^n + \ldots \]
from which we get,
\[ \alpha_n = \frac{a_n}{a_{n+1}} \tag{2.42} \]
and
\[ b_n = \frac{a_n}{a_{n+1}} b_{n+1} + \beta_n a_n \]
which implies that
\[ \beta_n = \frac{b_n}{a_n} - \frac{b_{n+1}}{a_{n+1}} \tag{2.43} \]
Now, gives that
\[ \gamma_n = \frac{N_n^2}{N_{n-1}^2} \frac{a_{n-1}}{a_n} \tag{2.44} \]
\[ \square \]
As shown, if we know the coefficients \( a_n \) and \( b_n \) of first two leading order terms of COP \( p_n(x) \), and the squared norm \( N_n^2 \), then the classical orthogonal polynomials can be determined recursively. Now, we evaluate \( \alpha_n \), \( \beta_n \) and \( \gamma_n \) for Hermite, Laguerre and Jacobi polynomials separately and then we get the recurrence relation of them.

### 2.5.1 Hermite

We know the values of leading coefficients and normalization factor, which are
\[ a_n = 2^n, \quad b_n = 0, \quad N_n^2 = 2^n n! \sqrt{\pi}. \]
Then,
\[ \alpha_n = \frac{2^n}{2^{n+1}} = \frac{1}{2}, \tag{2.45} \]
\[ \beta_n = 0, \tag{2.46} \]
\[ \gamma_n = \frac{2^n n! \sqrt{\pi}}{2^{n-1} (n-1)! \sqrt{\pi}} = 2^n \frac{1}{2} = n. \tag{2.47} \]
Thus, the recursion formula for Hermite polynomial:

\[ xH_n(x) = \frac{1}{2}H_{n+1}(x) + nH_{n-1}(x) \]

or

\[ H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x) \]  

(2.48)

with \( H_0(x) = 1 \) and \( H_1(x) = 2x \).

### 2.5.2 Laguerre

We know the values of leading coefficients and normalization factor, which are

\[ a_n = \frac{(-1)^n}{n!}, \quad b_n = -n(n + \alpha), \quad \mathcal{N}_n^2 = \frac{1}{n!} \Gamma(\alpha + 1 + n). \]

Then,

\[ \alpha_n = n + 1, \]  

(2.49)

\[ \beta_n = 2n + 1 + \alpha \]  

(2.50)

\[ \gamma_n = \frac{1}{n!} \Gamma(\alpha + n + 1) \]  

(2.51)

Thus, the recursion formula of Laguerre polynomials is

\[ (n + 1)L_{n+1}^\alpha(x) = (\alpha + 2n + x - 1)L_n^\alpha(x) - (\alpha + n)L_{n-1}^\alpha(x) \]  

(2.52)

where

\[ L_0^\alpha(x) = 1 \quad \text{and} \quad L_1^\alpha(x) = \alpha + 1 - x. \]

### 2.5.3 Jacobi

For Jacobi Polynomial since the calculations are too long we give the recursion formula directly.

\[
2(n + 1)(n + \alpha + \beta + 1)(2n + \alpha + \beta)P_{n+1}^{(\alpha, \beta)}(x) = \\
(2n + \alpha + \beta + 1)\left( (2n + \alpha + \beta + 2)(2n + \alpha + \beta)x + \alpha^2 - \beta^2 \right)P_n^{(\alpha, \beta)} \\
-2(n + \alpha)(n + \beta)(2n + \alpha + \beta + 2)P_{n-1}^{(\alpha, \beta)} \]  

(2.53)
where
\[ P_0^{(\alpha,\beta)} = 1 \quad \text{and} \quad P_1^{(\alpha,\beta)} = \frac{1}{2}(\alpha + \beta + 2)x + \frac{1}{2}(\alpha - \beta) \]

### 2.5.3.1 Legendre

By putting \( \alpha = \beta = 0 \) in (2.56), we get
\[ 2(n+1)(n+1)2nP_{n+1}(x) = (2n+1)(2n+2)(2n)xP_n(x) - 2n^2(2n+2)P_{n-1}(x) \]

Now, if we eliminate \( 4n(n+1) \) from both sides
\[ (n+1)P_{n+1}(x) = (2n+1)xP_n - nP_{n-1}(x) \]

is the recursion formula for Legendre polynomial.

### 2.5.3.2 Chebyshev

The Chebyshev polynomials of the first kind are related to the Jacobi polynomials with \( \alpha = \beta = -\frac{1}{2} \). Then, the recursion formula is
\[ T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x). \]
CHAPTER 3

ROOTS OF CLASSICAL ORTHOGONAL POLYNOMIALS

Recurrence relation for the Classical Orthogonal Polynomials are given by

$$\alpha_np_{n+1}(x) = (x - \beta_n)p_n(x) - \gamma_np_{n-1}(x) \quad for \quad n = 1, 2, \ldots, \quad p_0(x) = 1$$

We want to find the roots of classical orthogonal polynomial $p_{n+1}(x)$. Here, we run
the recursion for Jacobi, Laguerre and Hermite polynomials for $n = 0, 1, 2, \ldots, N$. Then we get a linear system of the form $(T - xI) = b$. In [26], it is shown that $T$ is
symmetric if the polynomials are orthonormal. If $T$ is not symmetric, then we may
perform a diagonal similarity transformation which will yield a symmetric tridiagonal
matrix $R$. However, here we prefer to use orthonormal polynomials. So, we transform
classical orthogonal polynomials to orthonormal polynomial by the equation $\psi_n(x) = \frac{1}{\sqrt{\pi}}p_n(x)$.

**Theorem 7.** [27] Recurrence relation formula given by

$$\alpha_kp_{k+1}(x) + (\beta_k - x)p_k(x) + \alpha_{k-1}p_{k-1}(x) = 0, \quad k = 1, 2, \ldots \quad (3.1)$$

is true for three consecutive orthonormal polynomials with $p_{-1}(x) = 0$, $p_0(x) = \frac{1}{\sqrt{\pi}}$
where the coefficients

$$\alpha_k = \frac{a_k}{a_{k-1}} \frac{N_{k+1}}{N_k} \quad \text{and} \quad \beta_k = \frac{b_{k-1}}{a_{k-1}} - \frac{b_k}{a_k}$$

are expressed by leading and subleading terms of orthonormal polynomial and by the
normalization constant.

Recursion (3.1) will be used to determine the zeros of $p_k(x)$ and by running the re-
cursion from $k = 0, 1, \ldots, n$, we get [24]

$$\alpha_0p_1(x) + (\beta_0 - x)p_0(x) = 0$$
\[
\begin{align*}
\alpha_1 p_2(x) + (\beta_1 - x)p_1(x) + \alpha_0 p_0(x) &= 0 \\
\alpha_2 p_3(x) + (\beta_2 - x)p_2(x) + \alpha_1 p_1(x) &= 0 \\
\vdots
\end{align*}
\]

\[
\begin{align*}
\alpha_{n-1} p_n(x) + (\beta_{n-1} - x)p_{n-1}(x) + \alpha_{n-2} p_{n-2}(x) &= 0 \\
\alpha_n p_{n+1}(x) + (\beta_n - x)p_n(x) + \alpha_{n-1} p_{n-1}(x) &= 0
\end{align*}
\]

or in the matrix form,

\[
\begin{bmatrix}
\beta_0 - x & \alpha_0 & 0 & \ldots & 0 \\
\alpha_0 & \beta_1 - x & \alpha_1 & 0 & \ldots \\
0 & \alpha_1 & \beta_2 - x & \alpha_2 & \ldots \\
0 & \ldots & \ldots & \ldots & \ldots \\
0 & \ldots & \alpha_{n-1} & \beta_{n-1} - x & \alpha_{n-1}
\end{bmatrix}
\begin{bmatrix}
p_0(x) \\
p_1(x) \\
p_2(x) \\
\vdots \\
p_n(x)
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0 \\
\vdots \\
r(x)
\end{bmatrix}.
\] (3.2)

where \(r(x) = -\alpha_n p_{n+1}(x)\) which implies a homogeneous one when \(p_{n+1}(x) = 0\).

Then we have, \([R - xI]P = 0\) where

\[
R = 
\begin{bmatrix}
\beta_0 & \alpha_0 & 0 & \ldots & 0 \\
\alpha_0 & \beta_1 & \alpha_1 & \ldots & \\
0 & \alpha_1 & \beta_2 & \ldots & 0 \\
\vdots & \ldots & \ldots & \ldots & \vdots \\
0 & \ldots & \ldots & \alpha_{n-1} & \beta_{n-1} \\
0 & \ldots & 0 & \alpha_{n-1} & \beta_{n}
\end{bmatrix},
\quad P = 
\begin{bmatrix}
p_0(x) \\
p_1(x) \\
\vdots \\
p_{n-1}(x) \\
p_n(x)
\end{bmatrix}
\]

\(I\) is the identity matrix.

Since the zeros of \(p_n(x)\) and \(p_{n+1}(x)\) are interlaced \(P\) can not be a zero vector when \(p_{n+1}(x) = 0\). Thus, \([R - xI] = 0\). But such \(x\) values are the eigenvalues of matrix \(R\) and also roots of the \(p_{n+1}\). Thus, we see that the nodes of the Gaussian quadrature...
rule are formed by the eigenvalues of the three-diagonal Jacobi matrix $\mathbf{R}$. Now, we need to find the weights. Since,

$$ (p_m, p_k) = \int_a^b p_m(x)p_k(x)w(x)dx = \delta_{mk}, \quad 0 \leq m, k \leq n - 1 $$

and Gaussian quadrature is exact for $p_0, p_1, \ldots, p_N$ we have

$$ \delta_{mk} = (p_m, p_k) = \sum_{j=1}^n p_m(x)p_k(x)w_j. $$

In the matrix form this expression is

$$ \mathbf{PWP}^T = \mathbf{I}, \quad (3.3) $$

where $\mathbf{W} = \text{diag}(w_1, w_2, \ldots, w_n)$ and $\mathbf{P} = [p(x_0), p(x_1), p(x_2), \ldots, p(x_n)]$. In other words, the columns of the matrix $\mathbf{P}$ are the scaled eigenvectors of the matrix $\mathbf{R}$. From equation (3.3), it follows that $\mathbf{P}$ is invertible, hence

$$ \mathbf{W} = \mathbf{P}^{-1}(\mathbf{P}^T)^{-1} = (\mathbf{P}^T \mathbf{P})^{-1} $$

Thus,

$$ (\mathbf{W}^{-1} = \mathbf{P}^T \mathbf{P} $$

which implies that

$$ \frac{1}{w_j} = \sum_{k=0}^{n-1}(p_k(x_j))^2 = ||p(x_j)||^2 \quad (3.4) $$

where $|| \cdot ||$ is the Euclidean norm. By the way, the eigenvector $v_j = [v^j_1, v^j_2, \ldots, v^j_n]$ of the matrix $\mathbf{R}$ is given and there exits a constant $A$ such that

$$ v_j = A\mathbf{P}(x_j) = A[p_0(x_j), p_1(x_j), \ldots, p_n(x_j)]. $$

It follows that

$$ p_0 = \frac{1}{N_0}. $$

Hence

$$ v^j_1 = A p_0 = A, $$

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and

\[ A = v_i^j N_0. \]

Therefore,

\[ P(x_j) = \frac{1}{A} v^j = \frac{1}{v_i^j N_0} v^j. \]

Now, we get the weights \( w_j \) associated with the nodes \( x_j \) as

\[ w_j = \frac{1}{||P(x_j)||^2} = \frac{N_0^2 (v_i^j)^2}{||v_j||^2}, \tag{3.5} \]

### 3.1 Hermite Polynomial

The normalization factor of Hermite polynomial is;

\[ N_n^2 = 2^n n! \sqrt{\pi} \]

The recurrence relation is of the form;

\[ H_{n+1}(x) = -2n H_{n-1}(x) + 2x H_n(x) \]

Then, we have to normalize Hermite Polynomial and get the recurrence relation for the normalized Hermite polynomial. That is;

\[ \Psi_n(x) = \frac{1}{\sqrt{N_n^2}} H_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} H_n(x), \quad n = 0, 1, \ldots \]

Then, if we substitute \( \Psi_n(x) \) into the three term recursion, it takes the form

\[ H_{n+1}(x) = \sqrt{2^{n+1}(n+1)! \sqrt{\pi}} \Psi_{n+1}(x) - 2x \sqrt{2^n n! \sqrt{\pi}} \Psi_n(x) - 2n \sqrt{2^{n-1}(n-1)! \sqrt{\pi}} \Psi_{n-1}(x) \]

Now if we eliminate the same terms from the given equation above, we have

\[ \sqrt{4n(n+1)} \Psi_{n+1}(x) = 2x \sqrt{2n} \Psi_n(x) - 2n \Psi_{n-1}(x) \]

and then by dividing both sides of the equation by 2 and \( \sqrt{n} \) get the recurrence relation for orthonormal Hermite;

\[ \sqrt{n} \Psi_{n-1}(x) - \sqrt{2} x \Psi_n(x) + \sqrt{n+1} \Psi_{n+1}(x) = 0, \quad n = 0, 1, 2, \ldots, N \]
Then, by (3.1) and (3.2) and we wrote these equations in matrix form

\[
\begin{bmatrix}
-\sqrt{2}x & \sqrt{1} & 0 & \ldots & 0 \\
\sqrt{1} & -\sqrt{2}x & \sqrt{2} & \ldots & \\
0 & \sqrt{2} & \ldots & \\
\ldots & \ldots & -\sqrt{2}x & \sqrt{N} & \\
0 & \ldots & 0 & \sqrt{N} & -\sqrt{2}x
\end{bmatrix}
\begin{bmatrix}
\Psi_0(x) \\
\Psi_1(x) \\
\vdots \\
\Psi_{N-1}(x) \\
\Psi_N(x)
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
r(x)
\end{bmatrix}
\] (3.6)

Here we put the value \(-\sqrt{N+1}\Psi_{N+1}(x)\) on the other side of the equation and express it by \(r(x)\). Now, the system is homogeneous as \(\Psi_{N+1}(x) = 0\). The matrix \(R\) is given below.

\[
R = \begin{bmatrix}
0 & \sqrt{1} & 0 & \ldots & 0 \\
\sqrt{1} & 0 & \sqrt{2} & \ldots & \\
0 & \sqrt{2} & \ldots & 0 & \\
\ldots & \ldots & 0 & \sqrt{N} & \\
0 & \ldots & 0 & \sqrt{N} & 0
\end{bmatrix}
\] (3.7)

\(R\) is tridiagonal symmetric as seen above. Then we face with the standard eigenvalue problem \(Rx = \lambda x\). Now the eigenvalues \(\lambda_m\) of matrix \(R\) are connected to the zeros of Hermite polynomial by given formula below:

\[
\lambda_m = \sqrt{2}x_m \quad for \quad m = 0, 1, \ldots, N
\] (3.8)
To make sure that the eigenvalues of matrix $R$ is related to the roots of Hermite polynomial with the formula (3.3):

$$H_2(x) = 4x^2 - 2 = 0 \quad \text{implies that} \quad 4x^2 - 2 = 0 \Rightarrow x^2 = \frac{1}{2} \Rightarrow x_{1,2} = \pm \frac{1}{\sqrt{2}}$$

Now, eigenvalue of $R^{2^2}$ is

$$R = \begin{bmatrix} 0 & \sqrt{1} \\ \sqrt{1} & 0 \end{bmatrix} \Rightarrow |R - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} \Rightarrow \lambda^2 - 1 = 0 \Rightarrow \lambda_{1,2} = \pm 1$$

Then by the formula (3.3), we get

$$x_{1,2} = \frac{\lambda_{1,2}}{\sqrt{2}} \Rightarrow x_{1,2} = \pm \frac{1}{\sqrt{2}}$$

as desired. MATLAB trideigs function is applied to calculate the eigenvalues and eigenvectors of the symmetric tridiagonal matrix $R$. Weights are related to the eigenvectors of a tridiagonal matrix and calculated by the given formula in (3.5). Thus, the roots and weights of Hermite polynomials are tabulated for 2, 4, 8, 16, and 20th degrees of Hermite.

**Table 3.1: Roots and weights for quadratic Hermite**

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<thead>
<tr>
<th>$i$</th>
<th>$x_i$</th>
<th>$w_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.707 106 781 186 54</td>
<td>0.886 226 925 452 76</td>
</tr>
<tr>
<td>2</td>
<td>0.707 106 781 186 54</td>
<td>0.886 226 925 452 76</td>
</tr>
</tbody>
</table>
Table 3.2: Roots and weights for fourth degree Hermite

<table>
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<th>$x_i$</th>
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</tr>
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<tbody>
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<td>1</td>
<td>-1.650 680 123 885 8</td>
<td>0.081 312 835 447 245</td>
</tr>
<tr>
<td>2</td>
<td>-0.524 647 623 275 29</td>
<td>0.804 914 090 005 51</td>
</tr>
<tr>
<td>3</td>
<td>0.524 647 623 275 29</td>
<td>0.804 914 090 005 51</td>
</tr>
<tr>
<td>4</td>
<td>1.650 680 123 885 8</td>
<td>0.081 312 835 447 245</td>
</tr>
</tbody>
</table>

Table 3.3: Roots and weights for 8th degree Hermite

<table>
<thead>
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<th>$i$</th>
<th>$x_i$</th>
<th>$w_i$</th>
</tr>
</thead>
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<td>1</td>
<td>-2.930 637 420 257 2</td>
<td>0.001 996 040 722 11</td>
</tr>
<tr>
<td>2</td>
<td>-1.981 656 756 695 8</td>
<td>0.017 077 983 007 41</td>
</tr>
<tr>
<td>3</td>
<td>-1.157 193 712 446 8</td>
<td>0.207 802 325 814 89</td>
</tr>
<tr>
<td>4</td>
<td>-0.381 186 990 207 32</td>
<td>0.661 147 012 558 24</td>
</tr>
<tr>
<td>5</td>
<td>0.381 186 990 207 32</td>
<td>0.661 147 012 558 24</td>
</tr>
<tr>
<td>6</td>
<td>1.157 193 712 446 8</td>
<td>0.207 802 325 814 89</td>
</tr>
<tr>
<td>7</td>
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<td>0.017 077 983 007 41</td>
</tr>
<tr>
<td>8</td>
<td>2.930 637 420 257 2</td>
<td>0.001 996 040 722 11</td>
</tr>
</tbody>
</table>
Table 3.4: Roots and weights for 16th degree Hermite

<table>
<thead>
<tr>
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<th>$w_i$</th>
</tr>
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<tbody>
<tr>
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<td>−4.688 738 939 305 8</td>
<td>0.000 000 232 098 084</td>
</tr>
<tr>
<td>2</td>
<td>−3.869 447 904 860 1</td>
<td>0.000 027 118 600 925</td>
</tr>
<tr>
<td>3</td>
<td>−3.176 999 161 98</td>
<td>0.000 932 284 008 624</td>
</tr>
<tr>
<td>4</td>
<td>−2.546 202 157 847 5</td>
<td>0.661 147 012 558 24</td>
</tr>
<tr>
<td>5</td>
<td>−1.951 787 990 916 3</td>
<td>0.012 880 311 535 51</td>
</tr>
<tr>
<td>6</td>
<td>−1.380 258 539 198 9</td>
<td>0.083 810 041 398 986</td>
</tr>
<tr>
<td>7</td>
<td>−0.822 951 449 144 66</td>
<td>0.280 647 458 528 53</td>
</tr>
<tr>
<td>8</td>
<td>−0.273 481 046 138 2</td>
<td>0.507 929 479 016 61</td>
</tr>
<tr>
<td>9</td>
<td>0.273 481 046 138 2</td>
<td>0.507 929 479 016 61</td>
</tr>
<tr>
<td>10</td>
<td>0.822 951 449 144 66</td>
<td>0.280 647 458 528 53</td>
</tr>
<tr>
<td>11</td>
<td>1.380 258 539 198 9</td>
<td>0.083 810 041 398 986</td>
</tr>
<tr>
<td>12</td>
<td>1.951 787 990 916 3</td>
<td>0.012 880 311 535 51</td>
</tr>
<tr>
<td>13</td>
<td>2.546 202 157 847 5</td>
<td>0.661 147 012 558 24</td>
</tr>
<tr>
<td>14</td>
<td>3.176 999 161 98</td>
<td>0.000 932 284 008 624</td>
</tr>
<tr>
<td>15</td>
<td>3.869 447 904 860 1</td>
<td>0.000 027 118 600 925</td>
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<td>0.000 000 232 098 084</td>
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### Table 3.5: Roots and weights for 20th degree Hermite

<table>
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<td>(0.00000000022939)</td>
</tr>
<tr>
<td>2</td>
<td>(-4.6036824495507)</td>
<td>(0.0000004399340992)</td>
</tr>
<tr>
<td>3</td>
<td>(-3.9447640401156)</td>
<td>(0.000001086069370)</td>
</tr>
<tr>
<td>4</td>
<td>(-3.34785456738325)</td>
<td>(0.000078025564785)</td>
</tr>
<tr>
<td>5</td>
<td>(-2.7888060584281)</td>
<td>(0.000228338636016)</td>
</tr>
<tr>
<td>6</td>
<td>(-2.2549740020893)</td>
<td>(0.003243773342238)</td>
</tr>
<tr>
<td>7</td>
<td>(-1.7385377121166)</td>
<td>(0.02481052088746)</td>
</tr>
<tr>
<td>8</td>
<td>(-1.2340762153953)</td>
<td>(0.10901720602002)</td>
</tr>
<tr>
<td>9</td>
<td>(-0.73747372854539)</td>
<td>(0.28667550536283)</td>
</tr>
<tr>
<td>10</td>
<td>(-0.2453407083009)</td>
<td>(0.46224366960061)</td>
</tr>
<tr>
<td>11</td>
<td>(0.2453407083009)</td>
<td>(0.46224366960061)</td>
</tr>
<tr>
<td>12</td>
<td>(0.73747372854539)</td>
<td>(0.28667550536283)</td>
</tr>
<tr>
<td>13</td>
<td>(1.2340762153953)</td>
<td>(0.10901720602002)</td>
</tr>
<tr>
<td>14</td>
<td>(1.7385377121166)</td>
<td>(0.02481052088746)</td>
</tr>
<tr>
<td>15</td>
<td>(2.2549740020893)</td>
<td>(0.003243773342238)</td>
</tr>
<tr>
<td>16</td>
<td>(2.7888060584281)</td>
<td>(0.000228338636016)</td>
</tr>
<tr>
<td>17</td>
<td>(3.34785456738325)</td>
<td>(0.000078025564785)</td>
</tr>
<tr>
<td>18</td>
<td>(3.9447640401156)</td>
<td>(0.000001086069370)</td>
</tr>
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<td>19</td>
<td>(4.6036824495507)</td>
<td>(0.00000004399340992)</td>
</tr>
<tr>
<td>20</td>
<td>(5.3874808900112)</td>
<td>(0.00000000022939)</td>
</tr>
</tbody>
</table>

### 3.2 Laguerre Polynomial

The normalization factor of Laguerre polynomial is:

\[
N_n^2 = \frac{1}{n!} \Gamma(\alpha + 1 + n)
\]

The recurrence relation is of the form:

\[
(n + 1)L_{n+1}^\alpha(x) = (\alpha + 2n + x - 1)L_n^\alpha(x) - (\alpha + n)L_{n-1}^\alpha(x)
\]
Then, we have to normalize Laguerre Polynomial and get the recurrence relation for the normalized Laguerre polynomial. That is;

\[ \Psi_n^\alpha(x) = \frac{1}{\sqrt{N_n^2}} L_n^\alpha(x) = \frac{1}{\sqrt{\Gamma(\alpha+1+n)/n!}} L_n^\alpha(x) \]

Then,

\[ L_n^\alpha(x) = \Psi_n(x)^\alpha \sqrt{\Gamma(\alpha+n+1)/n!} \]

Then, if we substitute \( \alpha_n(x) \) in the three term recursion, it becomes

\[ (n + 1)\Psi_{n+1}^\alpha(x) \sqrt{\frac{\Gamma(\alpha + n + 2)}{(n+1)!}} = (\alpha + 2n + x - 1)\Psi_n^\alpha(x) \sqrt{\frac{\Gamma(\alpha + n + 1)}{n!}} - (\alpha + n)\Psi_{n-1}^\alpha(x) \sqrt{\frac{\Gamma(\alpha + n)}{(n-1)!}} \]

Next, we continue to eliminate the same terms from the sides of the equation.

\[ \sqrt{\frac{(n+1)(\alpha+n)(\alpha+n+1)}{n}} \Psi_{n+1}^\alpha(x) = (\alpha + 2n + x - 1)\Psi_n^\alpha(x) \sqrt{\frac{\Gamma(\alpha + n)}{n}} \Psi_{n-1}^\alpha(x) - (\alpha + n)\Psi_{n-1}^\alpha(x) \]

Finally, the orthonormal Laguerre recurrence relation takes the form

\[ \sqrt{n(n+\alpha)}\Psi_{n-1}^\alpha(x) + (\alpha + 2n + x - 1)\Psi_n^\alpha(x) + \sqrt{(\alpha + n + 1)(n+1)}\Psi_{n+1}^\alpha(x) = 0 \] (3.9)

Then, by (3.1 and 3.2) we get the matrix form, but before it let us define the elements of matrix by \( r_{i,j} \) where \( i, j = 0, 1, 2, ..., N \) such that

\[ r_{i,j} = -(2i + 1 + \alpha - x) \quad if \quad i = j \]

\[ r_{i,j} = \sqrt{i}\sqrt{i+\alpha} \quad if \quad i = j + 1 \]

\[ r_{i,j} = \sqrt{j}\sqrt{j+\alpha} \quad if \quad i = j - 1 \]
otherwise, \( r_{i,j} = 0 \)

From the above notation, we see that

\[
\begin{align*}
  r_{0,0} &= -(1 + \alpha - x) \\
  r_{0,1} &= \sqrt{1} \sqrt{1 + \alpha} \\
  r_{1,0} &= \sqrt{1} \sqrt{1 + \alpha} \\
  \vdots \\
  r_{(N-1),N} &= r_{N,(N-1)} = \sqrt{N} \sqrt{N + \alpha} \\
  r_{N,N} &= -(2N + 1 + \alpha - x)
\end{align*}
\]

Now, by writing the elements of matrix, we get

\[
\begin{bmatrix}
  r_{0,0} & r_{0,1} & 0 & \ldots & 0 \\
  r_{1,0} & r_{1,1} & r_{1,2} & \ldots & \vdots \\
  0 & r_{2,1} & r_{2,2} & \ldots & \vdots \\
  \vdots & \ddots & \ddots & \ddots & \vdots \\
  0 & \ldots & 0 & r_{N-1,N} & r_{N,N}
\end{bmatrix}
\begin{bmatrix}
  \Psi_0^\alpha(x) \\
  \Psi_1^\alpha(x) \\
  \vdots \\
  \Psi_{N-1}^\alpha(x) \\
  \Psi_N^\alpha(x)
\end{bmatrix}
= 
\begin{bmatrix}
  0 \\
  0 \\
  \vdots \\
  0 \\
  r(x)
\end{bmatrix}
\]

(3.10)

where \( r(x) = -\sqrt{N + 1} \Psi_{N+1}(x) \) Now, the system is homogeneous as \( \Psi_{N+1}(x) = 0 \).

Then the matrix R is
\[
R = \begin{bmatrix}
-(1 + \alpha) & \sqrt{1 + \alpha} & 0 & \ldots & 0 \\
\sqrt{1 + \alpha} & -(3 + \alpha) & \sqrt{2 + \alpha} & \ldots \\
0 & \sqrt{2 + \alpha} & -(5 + \alpha) & \ldots \\
& & \ddots & \ddots & \ddots \\
0 & \ldots & \sqrt{N \sqrt{N + \alpha}} & -(2N + 1 + \alpha)
\end{bmatrix}
\]
(3.11)

\text{R is tridiagonal symmetric as } r_{i,j} = r_{j,i} \text{ for } i, j = 0, 1, \ldots, N. \text{ Then we face with the standard eigenvalue problem } Rx = \lambda x. \text{ Now the eigenvalues } \lambda_m \text{ of matrix } R \text{ which are associated to the zeros of Laguerre polynomial by given formula below;}

\[
\lambda_m = -x_m \quad \text{for } m = 0, 1, 2, \ldots, N
\]
(3.12)

To find the roots as a real number, we made the calculations for } \alpha = 0. \text{ Then, when } \alpha = 0, \text{ we get the matrix of the form;}

\[
R = \begin{bmatrix}
-1 & 1 & 0 & \ldots & 0 \\
1 & -3 & 2 & 0 & \ldots \\
0 & 2 & -5 & 3 & \ldots \\
& & \ddots & \ddots & \ddots \\
0 & \ldots & -(2N - 1) & N & -(2N + 1)
\end{bmatrix}
\]

To make sure that the eigenvalues of matrix } R \text{ is related to the roots of Laguerre polynomial with the formula (3.7) for } \alpha = 0;

\[
L_2(x) = \frac{1}{2}(x^2 - 4x + 2) = 0
\]

30
implies that \( x_{1,2} = \frac{4 \pm \sqrt{16 - 8}}{2} = 2 \pm \sqrt{2} \)

Now, the eigenvalues of \( R^{2 \times 2} \) is

\[
R = \begin{pmatrix}
-1 & 1 \\
1 & 3
\end{pmatrix} \Rightarrow |R - \lambda I| = \begin{vmatrix}
-\lambda - 1 & 1 \\
1 & -\lambda - 3
\end{vmatrix} \Rightarrow \lambda^2 + 4\lambda + 2 = 0
\]
\[
\Rightarrow \lambda_{1,2} = -2 \pm \sqrt{2}
\]

Then by the formula (3.7), we get

\[
x_{1,2} = -\lambda_{1,2} \Rightarrow x_{1,2} = 2 \pm \sqrt{2}
\]
as desired.

MATLAB trideigs function is applied to compute the eigenvalues and the eigenvectors of the symmetric tridiagonal matrix \( R \). Weights are calculated by the given formula in (3.5).

The roots and weights of Laguerre polynomials are tabulated for the degrees \( n = 2 \), \( n = 4 \), \( n = 8 \) and \( n = 20 \) for \( \alpha = 0 \).

Table 3.6: Roots and weights of Laguerre Polynomial for \( n = 2 \) and \( \alpha = 0 \)

<table>
<thead>
<tr>
<th>( i )</th>
<th>( x_i )</th>
<th>( w_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.58578643762691</td>
<td>0.85355339059327</td>
</tr>
<tr>
<td>2</td>
<td>3.4142135623731</td>
<td>0.14644660940673</td>
</tr>
</tbody>
</table>

Table 3.7: Roots and weights of Laguerre Polynomial for \( n = 4 \) and \( \alpha = 0 \)

<table>
<thead>
<tr>
<th>( i )</th>
<th>( x_i )</th>
<th>( w_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.32254768961939</td>
<td>0.60315410434163</td>
</tr>
<tr>
<td>2</td>
<td>1.7457611011584</td>
<td>0.3574186924378</td>
</tr>
<tr>
<td>3</td>
<td>4.5366202969211</td>
<td>0.038887908515005</td>
</tr>
<tr>
<td>4</td>
<td>9.3950709123011</td>
<td>0.005392947055613</td>
</tr>
</tbody>
</table>
Table 3.8: Roots and weights of Laguerre Polynomial for $n = 8$ and $\alpha = 0$

<table>
<thead>
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</tr>
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<tbody>
<tr>
<td>1</td>
<td>0.170 279 632 305</td>
<td>0.369 188 589 341 64</td>
</tr>
<tr>
<td>2</td>
<td>0.903 701 776 799 4</td>
<td>0.418 786 780 814 34</td>
</tr>
<tr>
<td>3</td>
<td>2.251 086 629 866 1</td>
<td>0.175 794 986 637 17</td>
</tr>
<tr>
<td>4</td>
<td>4.266 700 170 287 7</td>
<td>0.033 343 492 261 216</td>
</tr>
<tr>
<td>5</td>
<td>7.045 905 402 393 5</td>
<td>0.002 794 536 235 22</td>
</tr>
<tr>
<td>6</td>
<td>10.758 516 010 181</td>
<td>0.000 907 650 877 335</td>
</tr>
<tr>
<td>7</td>
<td>15.740 678 641 28</td>
<td>0.000 008 485 746 716</td>
</tr>
<tr>
<td>8</td>
<td>22.863 131 736 889</td>
<td>0.000 000 104 800 117</td>
</tr>
</tbody>
</table>
Table 3.9: Roots and weights of Laguerre Polynomial for $n = 20$ and $\alpha = 0$

<table>
<thead>
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<td>1</td>
<td>0.070 539 889 692</td>
<td>0.168 746 801 851 1</td>
</tr>
<tr>
<td>2</td>
<td>0.372 126 818 001 6</td>
<td>0.291 254 362 006 1</td>
</tr>
<tr>
<td>3</td>
<td>0.916 582 102 483 3</td>
<td>0.266 686 102 867</td>
</tr>
<tr>
<td>4</td>
<td>1.707 306 531 028 3</td>
<td>0.166 002 453 269 5</td>
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<tr>
<td>5</td>
<td>2.749 199 255 309 4</td>
<td>0.074 826 064 668 792</td>
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<tr>
<td>6</td>
<td>4.048 925 313 851</td>
<td>0.024 964 417 309 28</td>
</tr>
<tr>
<td>7</td>
<td>5.615 174 970 861 6</td>
<td>0.006 202 550 844 572</td>
</tr>
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<td>7.459 017 453 671</td>
<td>0.001 144 962 386 476 9</td>
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<td>9.594 392 869 581</td>
<td>0.001 557 417 730 278 1</td>
</tr>
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<td>12.038 802 546 964</td>
<td>0.000 154 014 408 652</td>
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<td>0.000 010 864 863 665</td>
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<td>17.948 895 520 519</td>
<td>0.000 005 330 129 909 5</td>
</tr>
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<td>21.478 788 240 29</td>
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</tr>
<tr>
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<td>25.451 702 793 187</td>
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<td>29.932 554 631 701</td>
<td>0.000 000 047 675 292 51</td>
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<td>16</td>
<td>35.013 434 240 48</td>
<td>0.000 000 003 372 844 243</td>
</tr>
<tr>
<td>17</td>
<td>40.833 057 056 729</td>
<td>0.000 000 000 001 155 014</td>
</tr>
<tr>
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<td>47.619 994 047 347</td>
<td>0.000 000 000 000 153 939</td>
</tr>
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<td>19</td>
<td>55.810 795 750 064</td>
<td>0.000 000 000 000 005 286</td>
</tr>
<tr>
<td>20</td>
<td>66.524 416 525 616</td>
<td>0.000 000 000 000 000 165</td>
</tr>
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</table>
Now, we evaluated the roots of Laguerre polynomial for any chosen $\alpha$ and we chose the value $\alpha = 1$. Then for the given matrix $R$ in (3.6) we put $\alpha = 1$ and get the new symmetric tridiagonal matrix which is given below.

$$
R = \begin{pmatrix}
-2 & 1 & 0 & \ldots & 0 \\
1 & -4 & 2 & 0 & \ldots \\
0 & 2 & -6 & 3 & \ldots \\
& & & & \\
& & & & \\
0 & \ldots & \sqrt{N(N+1)} & -2N & \sqrt{N(N+1)} & -(2N + 2)
\end{pmatrix}
$$

Then by running the MATLAB program for $n = 2$, $n = 4$, $n = 16$ for $\alpha = 1$ we get the eigenvalues and weights related to the eigenvectors by formula (3.5) are tabulated below.

**Table 3.10: Roots and weights of Laguerre Polynomial for $n = 2$ and $\alpha = 1$**

<table>
<thead>
<tr>
<th>$i$</th>
<th>$x_i$</th>
<th>$w_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.267 949 192 431 1</td>
<td>0.788 675 134 594 81</td>
</tr>
<tr>
<td>2</td>
<td>4.732 050 807 568 9</td>
<td>0.211 324 865 405 19</td>
</tr>
</tbody>
</table>

**Table 3.11: Roots and weights of Laguerre Polynomial for $n = 4$ and $\alpha = 1$**

<table>
<thead>
<tr>
<th>$i$</th>
<th>$x_i$</th>
<th>$w_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.743 291 927 981 43</td>
<td>0.446 870 593 218 78</td>
</tr>
<tr>
<td>2</td>
<td>2.571 635 007 646 3</td>
<td>0.477 635 772 363 87</td>
</tr>
<tr>
<td>3</td>
<td>5.731 178 751 689 1</td>
<td>0.074 177 784 731 05</td>
</tr>
<tr>
<td>4</td>
<td>10.953 894 312 683</td>
<td>0.001 315 849 686 303</td>
</tr>
</tbody>
</table>
Table 3.12: Roots and weights of Laguerre Polynomial for \( n = 16 \) and \( \alpha = 1 \)

<table>
<thead>
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<th>( x_i )</th>
<th>( w_i )</th>
</tr>
</thead>
<tbody>
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<td>0.2161403052395</td>
<td>0.063277332879539</td>
</tr>
<tr>
<td>2</td>
<td>0.7263882432518</td>
<td>0.23109046152072</td>
</tr>
<tr>
<td>3</td>
<td>1.5335931603735</td>
<td>0.316933542164</td>
</tr>
<tr>
<td>4</td>
<td>2.6449709986119</td>
<td>0.2378942178752</td>
</tr>
<tr>
<td>5</td>
<td>4.0709781608802</td>
<td>0.11027274335925</td>
</tr>
<tr>
<td>6</td>
<td>5.825855515106</td>
<td>0.0330895883564</td>
</tr>
<tr>
<td>7</td>
<td>7.928504185306</td>
<td>0.006525716407792</td>
</tr>
<tr>
<td>8</td>
<td>10.40380828995</td>
<td>0.0842711436792</td>
</tr>
<tr>
<td>9</td>
<td>13.284661070707</td>
<td>0.00699619095556</td>
</tr>
<tr>
<td>10</td>
<td>16.615173216867</td>
<td>0.000361208604213</td>
</tr>
<tr>
<td>11</td>
<td>20.456006020027</td>
<td>0.000011015412514</td>
</tr>
<tr>
<td>12</td>
<td>24.893847025352</td>
<td>0.000001835682512</td>
</tr>
<tr>
<td>13</td>
<td>30.059862920203</td>
<td>0.000000148148886</td>
</tr>
<tr>
<td>14</td>
<td>36.17069454368</td>
<td>0.0000000473484535</td>
</tr>
<tr>
<td>15</td>
<td>43.640365184177</td>
<td>0.000000004084146</td>
</tr>
<tr>
<td>16</td>
<td>53.529151160268</td>
<td>0.000000000036239</td>
</tr>
</tbody>
</table>
3.3 Jacobi Polynomial

The normalization factor of Jacobi polynomial $N_n$ where

$$N_n^2 = \frac{2^{\alpha + \beta + 1} \Gamma(n + \alpha + 1) \Gamma(n + \beta + 1)}{n!(2n + \alpha + \beta + 1) \Gamma(n + \alpha + \beta + 1)}$$

The recurrence relation is:

$$2(n+1)(n+\alpha + \beta + 1)(2n + \alpha + \beta)P_{n+1}^{(\alpha,\beta)}(x) =$$

$$(2n + \alpha + \beta + 1)(2n + \alpha + \beta + 2)(2n + \alpha + \beta)x + \alpha^2 - \beta^2]P_n^{(\alpha,\beta)} - 2(n + \alpha)(n + \beta)(2n + \alpha + \beta + 2)P_{n-1}^{(\alpha,\beta)}$$

Then,

$$\Psi_n^{(\alpha,\beta)}(x) = \frac{1}{\sqrt{N_n^2}} P_n^{(\alpha,\beta)}(x)$$

that is,

$$P_n^{(\alpha,\beta)}(x) = \Psi_n^{(\alpha,\beta)}(x) \sqrt{\frac{2^{\alpha + \beta + 1} \Gamma(n + \alpha + 1) \Gamma(n + \beta + 1)}{n!(2n + \alpha + \beta + 1) \Gamma(n + \alpha + \beta + 1)}}. \quad (3.13)$$

Now if we replace (3.1) into the recurrence relation of Jacobi, we get

$$2(n+1)(n+\alpha + \beta + 1)(2n + \alpha + \beta) \times$$

$$\sqrt{\frac{2^{\alpha + \beta + 1} \Gamma(n + \alpha + 2) \Gamma(n + \beta + 2)}{(n + 1)!(2n + \alpha + \beta + 3) \Gamma(n + \alpha + \beta + 2)}} \Psi_{n+1}^{(\alpha,\beta)}(x) =$$

$$(2n + \alpha + \beta + 1)(2n + \alpha + \beta + 2)(2n + \alpha + \beta)x + \alpha^2 - \beta^2] \times$$

$$\sqrt{\frac{2^{\alpha + \beta + 1} \Gamma(n + \alpha + 1) \Gamma(n + \beta + 1)}{n!(2n + \alpha + \beta + 1) \Gamma(n + \alpha + \beta + 1)}} \Psi_n^{(\alpha,\beta)}(x) -$$

$$-2(n + \alpha)(n + \beta)(2n + \alpha + \beta + 2) \times$$

$$\sqrt{\frac{2^{\alpha + \beta + 1} \Gamma(n + \alpha) \Gamma(n + \beta)}{(n - 1)!(2n + \alpha + \beta - 1) \Gamma(n + \alpha + \beta)}} \Psi_{n-1}^{(\alpha,\beta)}(x).$$
Then, since $\Gamma(x + 1) = x\Gamma(x)$ if we eliminate same terms from both side, we get

$$2(n + 1)(n + \alpha + \beta + 1)(2n + \alpha + \beta) \times$$

$$\sqrt{\frac{(n + \alpha)(n + 1 + \alpha)(n + \beta)(n + 1 + \beta)}{n(n + 1)(2n + \alpha + \beta + 3)(n + \alpha + \beta)(n + \alpha + \beta + 1)}} \Psi^{(\alpha,\beta)}_{n+1}(x) =$$

$$(2n + \alpha + \beta + 1)[(2n + \alpha + \beta + 2)(2n + \alpha + \beta)x + \alpha^2 - \beta^2] \times$$

$$\sqrt{\frac{(n + \alpha)(n + \beta)}{n(2n + \alpha + \beta + 1)(n + \alpha + \beta)}} \Psi^{(\alpha,\beta)}_n(x) -$$

$$-2(n + \alpha)(n + \beta)(2n + \alpha + \beta + 2) \sqrt{\frac{1}{(2n + \alpha + \beta - 1)}} \Psi^{(\alpha,\beta)}_{n-1}(x)$$

After that,

$$2(2n + \alpha + \beta) \sqrt{\frac{(n + \alpha + 1)(n + \beta + 1)(n + 1)(n + \alpha + \beta + 1)}{n(2n + \alpha + \beta + 3)(n + \alpha + \beta)}} \Psi^{(\alpha,\beta)}_{n+1}(x) =$$

$$((\alpha + \beta + 2n)(\alpha + \beta + 2n + 2)x + \alpha^2 - \beta^2) \sqrt{\frac{(2n + \alpha + \beta + 1)}{n(n + \alpha + \beta)}} \Psi^{(\alpha,\beta)}_n(x) -$$

$$-2(2n + \alpha + \beta + 2) \sqrt{\frac{(n + \alpha)(n + \beta)}{2n + \alpha + \beta - 1}} \Psi^{(\alpha,\beta)}_{n-1}(x)$$

Then,

$$(2n + \alpha + \beta) \sqrt{\frac{(n + \alpha + 1)(n + \beta + 1)(n + 1)(n + \alpha + \beta + 1)}{(2n + \alpha + \beta + 3)(2n + \alpha + \beta + 1)}} \Psi^{(\alpha,\beta)}_{n+1}(x) -$$

$$-\frac{1}{2}((\alpha + \beta + 2n)(\alpha + \beta + 2n + 2)x + \alpha^2 - \beta^2)\Psi^{(\alpha,\beta)}_n(x) +$$

$$+(2n + \alpha + \beta + 2) \sqrt{\frac{(n + \alpha)(n + \beta)n(n + \alpha + \beta)}{(2n + \alpha + \beta - 1)(2n + \alpha + \beta + 1)}} \Psi^{(\alpha,\beta)}_{n-1}(x) = 0$$
Finally, we get the recurrence relation for orthonormal Jacobi polynomial:

\[
\frac{1}{2n + \alpha + \beta} \sqrt{\frac{n(n + \alpha)(n + \beta)(n + \alpha + \beta)}{(2n + \alpha + \beta + 1)(2n + \alpha + \beta - 1)}} \Psi_{n-1}^{(\alpha,\beta)}(x) - \\
\frac{1}{2} [x + \frac{\alpha^2 - \beta^2}{(2n + \alpha + \beta)(2n + \alpha + \beta + 2)}] \Psi_{n}^{(\alpha,\beta)}(x) \\
+ \frac{1}{(2n + \alpha + \beta + 2)} \sqrt{\frac{(n + \alpha + 1)(n + \beta + 1)(n + 1)(n + \alpha + \beta + 1)}{(2n + \alpha + \beta + 1)(2n + \alpha + \beta + 3)}} \Psi_{n+1}^{(\alpha,\beta)}(x) = 0
\]  

(3.14)

Then, by (3.1) and (3.2) we can write this N+1 equations as an inhomogeneous linear system. Let us define the elements of matrix by \( r_{i,j} \) where \( i, j = 0, 1, 2, \ldots, N \) such that

\[
r_{i,j} = -\frac{1}{2} [x + \frac{\alpha^2 - \beta^2}{(2i + \alpha + \beta)(2i + \alpha + \beta + 2)}] \quad if \quad i = j
\]

\[
r_{i,j} = \frac{1}{2j + \alpha + \beta} \sqrt{\frac{j(j + \alpha)(j + \beta)(j + \alpha + \beta)}{(2j + \alpha + \beta + 1)(2j + \alpha + \beta - 1)}} \quad if \quad i = j - 1
\]

\[
r_{i,j} = \frac{1}{2i + \alpha + \beta} \sqrt{\frac{i(i + \alpha)(i + \beta)(i + \alpha + \beta)}{(2i + \alpha + \beta + 1)(2i + \alpha + \beta - 1)}} \quad if \quad i = j + 1,
\]

otherwise \( r_{i,j} = 0 \)

Now, by running the above notation we get the elements of matrix ,

\[
r_{0,0} = -\frac{1}{2} [x + \frac{\alpha^2 - \beta^2}{(\alpha + \beta + 2)(\alpha + \beta)}]
\]
\[
\begin{align*}
    r_{0,1} &= \frac{1}{\alpha + \beta + 2} \sqrt{\frac{(\alpha + 1)(\beta + 1)}{\alpha + \beta + 3}} = r_{1,0} \\
    r_{1,1} &= -\frac{1}{2} [x + \frac{\alpha^2 - \beta^2}{(\alpha + \beta + 2)(\alpha + \beta + 4)}] \\
    &\vdots
\end{align*}
\]

\[
\begin{align*}
    r_{N-1,N} = r_{N,N-1} &= \frac{1}{2N + \alpha + \beta} \sqrt{\frac{N(N + \alpha)(N + \beta)(N + \alpha + \beta)}{(2N + \alpha + \beta + 1)(2N + \alpha + \beta - 1)}} \\
    r_{N,N} &= -\frac{1}{2} [x + \frac{\alpha^2 - \beta^2}{(2N + \alpha + \beta)(2N + \alpha + \beta + 2)}] \\
    r(x) &= \frac{1}{(2N + \alpha + \beta + 2)} \sqrt{\frac{(N + \alpha + 1)(N + \beta + 1)(N + 1)(N + \alpha + \beta + 1)}{(2N + \alpha + \beta + 1)(2N + \alpha + \beta + 3)}} \\
    \cdot \Psi_{N+1}^{\alpha,\beta}(x)
\end{align*}
\]

Now, by writing the elements of matrix, we get

\[
\begin{bmatrix}
    r_{0,0} & r_{0,1} & 0 & \ldots & 0 \\
    r_{1,0} & r_{1,1} & r_{1,2} & \ldots & \vdots \\
    0 & r_{2,1} & r_{2,2} & \ldots & \vdots \\
    \vdots & \ddots & \ddots & \ddots & \vdots \\
    \ldots & \ldots & 0 & r_{N,N-1} & r_{N,N}
\end{bmatrix}
\begin{bmatrix}
    \Psi_{0}^{\alpha,\beta}(x) \\
    \Psi_{1}^{\alpha,\beta}(x) \\
    \vdots \\
    \Psi_{N-1}^{\alpha,\beta}(x) \\
    \Psi_{N}^{\alpha,\beta}(x)
\end{bmatrix} =
\begin{bmatrix}
    0 \\
    0 \\
    \vdots \\
    0 \\
    r(x)
\end{bmatrix}
\]

(3.15)

It implies homogeneous equation since \( \Psi_{N+1}(x) = 0 \). Therefore, we see a standard matrix eigenvalue problem for symmetric tridiagonal matrix \( R \) and \( Rx = \lambda x \)

Let us define the elements \( m_{i,j} \) for \( i, j = 0, 1, 2, \ldots, N \) of matrix \( R \) by the formulas

\[
m_{i,j} = \frac{\alpha^2 - \beta^2}{(2i + \alpha + \beta)(2i + \alpha + \beta + 2)} \quad \text{if} \quad i = j
\]
\[ m_{ij} = \frac{2}{\alpha + \beta + 2j} \sqrt{\frac{j(\alpha + j)(\beta + j)(\alpha + \beta + j)}{(\alpha + \beta + 3 + 2j + 1)(\alpha + \beta + 2j - 1)}} \quad \text{if} \quad i = j - 1 \]

\[ m_{ij} = \frac{2}{\alpha + \beta + 2i} \sqrt{\frac{i(\alpha + i)(\beta + i)(\alpha + \beta + i)}{(\alpha + \beta + 2i + 1)(\alpha + \beta + 2i - 1)}} \quad \text{if} \quad i = j + 1 \]

\[ \text{otherwise} \quad m_{ij} = 0 \]

From the above formula, we could derive the elements of matrix \( R \):

\[ m_{0,0} = \frac{\alpha^2 - \beta^2}{(\alpha + \beta + 2)(\alpha + \beta)} \]

\[ m_{0,1} = m_{1,0} = \frac{2}{\alpha + \beta + 2} \sqrt{\frac{(\alpha + 1)(\beta + 1)}{\alpha + \beta + 3}} \]

\[ m_{1,1} = \frac{\alpha^2 - \beta^2}{(\alpha + \beta + 2)(\alpha + \beta + 4)} \]

\[ m_{1,2} = m_{2,1} = \frac{2}{\alpha + \beta + 4} \sqrt{\frac{2(\alpha + 2)(\beta + 2)(\alpha + \beta + 2)}{(\alpha + \beta + 3)(\alpha + \beta + 5)}} \]

\[ \vdots \]

\[ m_{N,N-1} = m_{N-1,N} = \frac{2}{\alpha + \beta + 2N} \sqrt{\frac{N(\alpha + N)(\beta + N)(\alpha + \beta + N)}{(\alpha + \beta + 2N + 1)(\alpha + \beta + 2N - 1)}} \]

\[ m_{N,N} = \frac{\alpha^2 - \beta^2}{(2N + \alpha + \beta)(2N + \alpha + \beta + 2)} \]

\[
R = \begin{bmatrix}
  m_{0,0} & m_{0,1} & 0 & \ldots & 0 \\
  m_{1,0} & m_{1,1} & m_{1,2} & \ldots \\
  0 & m_{2,1} & \ldots \\
  \vdots & \vdots & \ddots & \ddots \\
  0 & \ldots & \ldots & m_{N-1,N} & m_{N,N}
\end{bmatrix}
\] (3.16)
where the eigenvalues $\lambda_m$ of matrix $R$ are related to the zeros of Jacobi polynomial by the formula

$$\lambda_m = \frac{1}{2} x_m$$

(3.17)

for $m = 0, 1, \ldots, N$. For Jacobi polynomials, we calculated the roots of the case when $\alpha = \beta = 0$, that is Legendre Polynomial. The recurrence relation for orthonormal Legendre Polynomial is;

$$\frac{n}{\sqrt{(2n-1)(2n+1)}} \Psi_{n-1}(x) - x \Psi_n(x) + \frac{n + 1}{\sqrt{(2n+1)(2n+3)}} \Psi_{n+1}(x) = 0$$

From Jacobi matrix $R$, when we make $\alpha = \beta = 0$, we get the matrix

$$R = \begin{bmatrix}
0 & \frac{1}{\sqrt{3}} & 0 & \cdots & 0 & \cdots \\
\frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{15}} & \cdots & \cdots & \cdots \\
0 & \frac{2}{\sqrt{15}} & \frac{2}{\sqrt{35}} & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \cdots \\
0 & \cdots & \cdots & \frac{N}{\sqrt{(2N-1)(2N+1)}} & 0 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \frac{N}{\sqrt{(2N-1)(2N+1)}}
\end{bmatrix}$$

Then, by MATLAB, we compute the eigenvalues and eigenvectors of symmetric tridiagonal matrix. Therefore, the roots and weights of Jacobi polynomial when $\alpha = \beta = 0$ is tabulated for $n = 2$, $n = 4$, $n = 8$, $n = 16$ and $n = 20$.

Table 3.13: Roots and weights for quadratic Legendre

<table>
<thead>
<tr>
<th>$i$</th>
<th>$x_i$</th>
<th>$w_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.577 350 269 189 63</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>0.577 350 269 189 63</td>
<td>1</td>
</tr>
</tbody>
</table>
### Table 3.14: Roots and weights of Legendre for $n = 4$

<table>
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<tr>
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</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.861 136 311 594 05</td>
<td>0.347 854 845 137 45</td>
</tr>
<tr>
<td>2</td>
<td>-0.339 981 043 584 86</td>
<td>0.652 145 154 862 55</td>
</tr>
<tr>
<td>3</td>
<td>0.339 981 043 584 86</td>
<td>0.652 145 154 862 55</td>
</tr>
<tr>
<td>4</td>
<td>0.861 136 311 594 05</td>
<td>0.347 854 845 137 45</td>
</tr>
</tbody>
</table>

### Table 3.15: Roots and weights of Legendre for $n = 8$

<table>
<thead>
<tr>
<th>$i$</th>
<th>$x_i$</th>
<th>$w_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.960 289 856 497 54</td>
<td>0.101 228 536 290 38</td>
</tr>
<tr>
<td>2</td>
<td>-0.796 666 477 413 63</td>
<td>0.222 381 034 453 4</td>
</tr>
<tr>
<td>3</td>
<td>-0.525 532 409 916 33</td>
<td>0.313 706 645 877 89</td>
</tr>
<tr>
<td>4</td>
<td>-0.183 434 642 495 65</td>
<td>0.362 683 783 378 36</td>
</tr>
<tr>
<td>5</td>
<td>0.183 434 642 495 65</td>
<td>0.362 683 783 378 36</td>
</tr>
<tr>
<td>6</td>
<td>0.525 532 409 916 33</td>
<td>0.313 706 645 877 89</td>
</tr>
<tr>
<td>7</td>
<td>0.796 666 477 413 63</td>
<td>0.222 381 034 453 4</td>
</tr>
<tr>
<td>8</td>
<td>0.960 289 856 497 54</td>
<td>0.101 228 536 290 38</td>
</tr>
</tbody>
</table>
Table 3.16: Roots and weights of Legendre for \( n = 16 \)

<table>
<thead>
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<th>( i )</th>
<th>( x_i )</th>
<th>( w_i )</th>
</tr>
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<tbody>
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<td>1</td>
<td>(-0.989) 000 934 991 65</td>
<td>(0.02715245941175)</td>
</tr>
<tr>
<td>2</td>
<td>(-0.944) 575 023 073 23</td>
<td>(0.06225352393865)</td>
</tr>
<tr>
<td>3</td>
<td>(-0.865) 631 202 387 83</td>
<td>(0.0951585116825)</td>
</tr>
<tr>
<td>4</td>
<td>(-0.755) 404 408 355</td>
<td>(0.1246289712555)</td>
</tr>
<tr>
<td>5</td>
<td>(-0.617) 876 244 402 64</td>
<td>(0.14959598881658)</td>
</tr>
<tr>
<td>6</td>
<td>(-0.458) 016 777 657 23</td>
<td>(0.169156519395)</td>
</tr>
<tr>
<td>7</td>
<td>(-0.281) 603 550 779 26</td>
<td>(0.182603415044)</td>
</tr>
<tr>
<td>8</td>
<td>(-0.095) 012 509 837 64</td>
<td>(0.1894506104551)</td>
</tr>
<tr>
<td>9</td>
<td>(0.095) 012 509 837 64</td>
<td>(0.1894506104551)</td>
</tr>
<tr>
<td>10</td>
<td>(0.281) 603 550 779 26</td>
<td>(0.1826034150449)</td>
</tr>
<tr>
<td>11</td>
<td>(0.458) 016 777 657 23</td>
<td>(0.169156519395)</td>
</tr>
<tr>
<td>12</td>
<td>(0.617) 876 244 402 64</td>
<td>(0.14959598881658)</td>
</tr>
<tr>
<td>13</td>
<td>(0.755) 404 408 355</td>
<td>(0.1246289712555)</td>
</tr>
<tr>
<td>14</td>
<td>(0.865) 631 202 387 83</td>
<td>(0.0951585116825)</td>
</tr>
<tr>
<td>15</td>
<td>(0.944) 575 023 073 23</td>
<td>(0.06225352393865)</td>
</tr>
<tr>
<td>16</td>
<td>(0.989) 400 934 991 65</td>
<td>(0.02715245941175)</td>
</tr>
</tbody>
</table>
Table 3.17: Roots and weights of Legendre for $n = 20$

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<td>-0.9931285991851</td>
<td>0.01761400713915</td>
</tr>
<tr>
<td>2</td>
<td>-0.96397192727791</td>
<td>0.04060142980039</td>
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<tr>
<td>3</td>
<td>-0.91223442825133</td>
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<td>0.14209610931844</td>
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<td>-0.22778585114165</td>
<td>0.1491729864726</td>
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<td>10</td>
<td>-0.0765265211335</td>
<td>0.15275338713073</td>
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<tr>
<td>11</td>
<td>0.0765265211335</td>
<td>0.15275338713073</td>
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<td>0.1491729864726</td>
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<td>19</td>
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</tr>
<tr>
<td>20</td>
<td>0.9931285991851</td>
<td>0.01761400713915</td>
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</table>
The special case of Jacobi polynomial is Chebyshev, that is when $\alpha = -\frac{1}{2}$ and $\beta = -\frac{1}{2}$. Chebyshev recurrence relation is given by:

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

By putting the values $\alpha = -\frac{1}{2}$ and $\beta = -\frac{1}{2}$, we get the matrix

$$R = \begin{bmatrix}
0 & \frac{1}{\sqrt{2}} & 0 & \ldots & 0 \\
\frac{1}{\sqrt{2}} & 0 & \frac{1}{2} & \ldots \\
 & \ldots & \ddots & \ddots \\
 & \ldots & \ddots & \frac{1}{2} \\
 & \ldots & \ldots & 0 & 1/2
\end{bmatrix}$$

Then by MATLAB, we find the eigenvalue and eigenvector of symmetric matrix $R$. Thus, the roots and weights of Chebyshev polynomial for $n = 2$, $n = 4$, $n = 10$ and $n = 20$ is given;

<table>
<thead>
<tr>
<th>Table 3.18: Roots and weights of Chebyshev for $n = 2$</th>
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</thead>
<tbody>
<tr>
<td>$i$</td>
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<tr>
<td>-----</td>
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<tr>
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<table>
<thead>
<tr>
<th>Table 3.19: Roots and weights of Chebyshev for $n = 4$</th>
</tr>
</thead>
<tbody>
<tr>
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</tr>
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<tr>
<td>3</td>
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<td>4</td>
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</tbody>
</table>
Table 3.20: Roots and weights of Chevyshev for $n = 10$

<table>
<thead>
<tr>
<th>$i$</th>
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</thead>
<tbody>
<tr>
<td>1</td>
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</tr>
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<td>2</td>
<td>$-0.891,006,524,188,37$</td>
<td>0.314,159,265,359</td>
</tr>
<tr>
<td>3</td>
<td>$-0.707,106,781,186,55$</td>
<td>0.314,159,265,359</td>
</tr>
<tr>
<td>4</td>
<td>$-0.453,990,499,739,55$</td>
<td>0.314,159,265,359</td>
</tr>
<tr>
<td>5</td>
<td>$-0.156,434,465,040,23$</td>
<td>0.314,159,265,359</td>
</tr>
<tr>
<td>6</td>
<td>0.156,434,465,040,23</td>
<td>0.314,159,265,359</td>
</tr>
<tr>
<td>7</td>
<td>0.453,990,499,739,55</td>
<td>0.314,159,265,359</td>
</tr>
<tr>
<td>8</td>
<td>0.707,106,781,186,55</td>
<td>0.314,159,265,359</td>
</tr>
<tr>
<td>9</td>
<td>0.891,006,524,188,37</td>
<td>0.314,159,265,359</td>
</tr>
<tr>
<td>10</td>
<td>0.987,688,340,951,4</td>
<td>0.314,159,265,359</td>
</tr>
</tbody>
</table>
Table 3.21: Roots and weights of Chebyshev for $n = 20$

<table>
<thead>
<tr>
<th>$i$</th>
<th>$x_i$</th>
<th>$w_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.996 917 333 733 1</td>
<td>0.157 079 632 679 5</td>
</tr>
<tr>
<td>2</td>
<td>-0.972 369 920 397 68</td>
<td>0.157 079 632 679 5</td>
</tr>
<tr>
<td>3</td>
<td>-0.923 879 532 511 3</td>
<td>0.157 079 632 679 5</td>
</tr>
<tr>
<td>4</td>
<td>-0.852 640 643 540 9</td>
<td>0.157 079 632 679 5</td>
</tr>
<tr>
<td>5</td>
<td>-0.760 405 965 600 03</td>
<td>0.157 079 632 679 5</td>
</tr>
<tr>
<td>6</td>
<td>-0.649 480 483 301 8</td>
<td>0.157 079 632 679 5</td>
</tr>
<tr>
<td>7</td>
<td>-0.522 498 564 715 95</td>
<td>0.157 079 632 679 5</td>
</tr>
<tr>
<td>8</td>
<td>-0.382 683 432 365 1</td>
<td>0.157 079 632 679 5</td>
</tr>
<tr>
<td>9</td>
<td>-0.233 445 363 855 91</td>
<td>0.157 079 632 679 5</td>
</tr>
<tr>
<td>10</td>
<td>-0.078 459 095 727 845</td>
<td>0.157 079 632 679 5</td>
</tr>
<tr>
<td>11</td>
<td>0.078 459 095 727 845</td>
<td>0.157 079 632 679 5</td>
</tr>
<tr>
<td>12</td>
<td>0.233 445 363 855 91</td>
<td>0.157 079 632 679 5</td>
</tr>
<tr>
<td>13</td>
<td>0.382 683 432 365 1</td>
<td>0.157 079 632 679 5</td>
</tr>
<tr>
<td>14</td>
<td>0.522 498 564 715 95</td>
<td>0.157 079 632 679 5</td>
</tr>
<tr>
<td>15</td>
<td>0.649 448 048 330 18</td>
<td>0.157 079 632 679 5</td>
</tr>
<tr>
<td>16</td>
<td>0.760 405 965 600 03</td>
<td>0.157 079 632 679 5</td>
</tr>
<tr>
<td>17</td>
<td>0.852 640 164 354 09</td>
<td>0.157 079 632 679 5</td>
</tr>
<tr>
<td>18</td>
<td>0.923 879 532 511 3</td>
<td>0.157 079 632 679 5</td>
</tr>
<tr>
<td>19</td>
<td>0.972 369 920 397 68</td>
<td>0.157 079 632 679 5</td>
</tr>
<tr>
<td>20</td>
<td>0.996 917 333 733 1</td>
<td>0.157 079 632 679 5</td>
</tr>
</tbody>
</table>

Then, the roots and weights of the Jacobi polynomial for any value of $\alpha$ and $\beta$ is evaluated and tabulated. We choose $\alpha = 5$, $\beta = 5$ and $n = 2, n = 4, n = 15$.  

47
Table 3.22: Roots and weights of Jacobi Polynomial for $\alpha = 5$, $\beta = 5$ and $n = 2$

<table>
<thead>
<tr>
<th>$i$</th>
<th>$x_i$</th>
<th>$w_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$-0.277,350,098,112,62$</td>
<td>$0.369,408,369,408,37$</td>
</tr>
<tr>
<td>2</td>
<td>$0.277,350,098,112,62$</td>
<td>$0.369,408,369,408,37$</td>
</tr>
</tbody>
</table>

Table 3.23: Roots and weights of Jacobi Polynomial for $\alpha = 5$, $\beta = 5$ and $n = 4$

<table>
<thead>
<tr>
<th>$i$</th>
<th>$x_i$</th>
<th>$w_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$-0.561,847,368,494,95$</td>
<td>$0.052,616,584,453,57$</td>
</tr>
<tr>
<td>2</td>
<td>$-0.193,051,057,976,62$</td>
<td>$0.316,791,784,954,8$</td>
</tr>
<tr>
<td>3</td>
<td>$0.193,051,057,976,62$</td>
<td>$0.316,791,784,954,8$</td>
</tr>
<tr>
<td>4</td>
<td>$0.561,847,368,494,95$</td>
<td>$0.052,616,584,453,57$</td>
</tr>
</tbody>
</table>
Table 3.24: Roots and weights of Jacobi Polynomial for $\alpha = 5$, $\beta = 5$ and $n = 15$

<table>
<thead>
<tr>
<th>$i$</th>
<th>$x_i$</th>
<th>$w_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.908 034 257 028 03</td>
<td>0.000 130 019 381 05</td>
</tr>
<tr>
<td>2</td>
<td>-0.820 744 339 238 36</td>
<td>0.000 357 712 207 39</td>
</tr>
<tr>
<td>3</td>
<td>-0.715 202 146 167 3</td>
<td>0.003 175 440 500 013</td>
</tr>
<tr>
<td>4</td>
<td>-0.593 208 000 788 75</td>
<td>0.014 792 883 231 19</td>
</tr>
<tr>
<td>5</td>
<td>-0.457 444 147 406 5</td>
<td>0.043 788 675 144 85</td>
</tr>
<tr>
<td>6</td>
<td>-0.311 024 826 301 76</td>
<td>0.090 571 365 472 4</td>
</tr>
<tr>
<td>7</td>
<td>-0.157 349 220 922 56</td>
<td>0.137 727 032 762 21</td>
</tr>
<tr>
<td>8</td>
<td>0.000 000 000 000</td>
<td>0.157 964 516 304 4</td>
</tr>
<tr>
<td>9</td>
<td>0.157 922 092 256</td>
<td>0.137 727 032 762 21</td>
</tr>
<tr>
<td>10</td>
<td>0.311 024 826 301 76</td>
<td>0.090 571 365 472 4</td>
</tr>
<tr>
<td>11</td>
<td>0.457 444 147 406 5</td>
<td>0.043 788 675 144 85</td>
</tr>
<tr>
<td>12</td>
<td>0.593 208 000 788 75</td>
<td>0.014 792 883 231 19</td>
</tr>
<tr>
<td>13</td>
<td>0.715 202 146 167 3</td>
<td>0.003 175 440 500 013</td>
</tr>
<tr>
<td>14</td>
<td>0.820 744 339 238 36</td>
<td>0.000 357 712 207 39</td>
</tr>
<tr>
<td>15</td>
<td>0.908 034 257 028 03</td>
<td>0.000 130 019 381 05</td>
</tr>
</tbody>
</table>
Gaussian quadrature is applied to integrate the integrals which are impossible or difficult to compute and the main purpose of it is to maximizing the degree of exactness and so that it chooses the interpolation points cautiously. Its main aim is to compute the integrals given below;

\[
I(f) = \int_a^b f(x)w(x)dx \tag{4.1}
\]

with the weight function \(w(x)\) on the interval \([a,b]\).

### 4.1 Definition of Gaussian Quadrature

Let \(x_1 < x_2 < x_3 < \ldots < x_n\), \(x_j \in [a,b], j = 1, 2, 3, \ldots, n\) be the interpolating nodes. Then by the rule

\[
I(f) = \int_a^b f(x)w(x)dx = \int_a^b w(x) \sum_{j=1}^{n} f(x_j)L_j(x) = \sum_{j=1}^{n} w_j f(x_j) \tag{4.2}
\]

where

\[
w_j = \int_a^b L_j(x)w(x)dx \quad \text{and} \quad L_j(x) = \prod_{k=1, k \neq j}^{n} \frac{x - x_k}{x_j - x_k} \tag{4.3}
\]

is called Gaussian quadrature.

**Theorem 8.** \cite{24} Suppose \(\{p_k(x)\}_{k=0}^{n}\) consists of orthogonal polynomials which satisfy the inner product

\[
(f, g) = \int_a^b f(x)g(x)w(x)dx
\]
Suppose \( p_n(x_j) = 0 \) for the value \( x_j \) where \( j = 1, 2, \ldots, N \). Then we say that the quadrature rule given in (4.2) and (4.3) is exact for polynomials of degree equal to \( 2n - 1 \).

**Proof.** Let \( f(x) \) be a polynomial of degree less than or equal to \( 2n - 1 \).

Denote \( f(x) = p_n(x)t(x) + r(x) \), where \( t(x) \) and \( r(x) \) has a degree less than or equal to \( n - 1 \). Now, since we know that any polynomial is orthogonal to every polynomial of lower degree, \( p_n(x) \) is orthogonal to every polynomial of degree less than or equal to \( n - 1 \).

Thus,

\[
I(f) = I(tp_n + r) = \int_a^b t(x)p_n(x)w(x)dx + \int_a^b r(x)w(x)dx = \int_a^b r(x)w(x)dx.
\]

Now, \( f(x_j) = p_n(x_j)t(x_j) + r(x_j) = r(x_j) \) since \( x_j \)’s are the zeros of \( p_n(x) \). We obtain,

\[
I(f) = Q(r) = \sum_{j=1}^{n} r(x_j)w_j = \sum_{j=1}^{n} w_j[p_n(x_j)t(x_j) + r(x_j)] = \sum_{j=1}^{n} w_j f(x_j) = Q(f).
\]

Here, we see that the Gaussian Quadrature is exact for all polynomials with degree less than or equal to \( 2n - 1 \). \( \square \)

### 4.2 Error in Gaussian Quadrature

**Theorem 9.** [24] Let \( p(x_j) = 0 \) for \( j = 1, 2, \ldots, n \). Suppose \( f \in \mathbb{C}^{2n}[a, b] \). Then the error \( E(f) \) for Gaussian quadrature in (4.2) and (4.3) is

\[
E(f) = \frac{f^{(2n)}(\zeta)}{(2n)!} \int_a^b p_n^2(x)w(x)dx \tag{4.4}
\]

for \( \zeta \in (a, b) \), where \( w(x) \) is a weight function and \( p_n(x) \) is a monic polynomial of degree \( n \).

**Proof.** To prove the above theorem we make use of Hermite Interpolation rule. Then, let \( h_{2n-1}(x) \) of degree \( 2n - 1 \) be a unique polynomial with

\[
h_{2n-1}(x_j) = f(x_j) \quad , \quad h'_{2n-1}(x_j) = f'(x_j)
\]
Moreover, let $\theta \in (a, b)$ with

$$f(x) = h_{2n-1}(x) + \frac{f^{(2n)}(\theta)}{(2n)!}(x - x_1)^2 \cdots (x - x_n)^2.$$ 

Note that $(x - x_1) \cdots (x - x_n) = p_n(x)$ as $p_n$ is monic and $p_n(x_j) = 0$. Thus,

$$\int_a^b f(x)w(x)dx = \int_a^b h_{2n-1}(x)w(x)dx + \int_a^b \frac{f^{(2n)}(\theta)}{(2n)!}p_n^2(x)w(x)dx$$

The quadrature rule is exact for $h_{2n-1}(x)$.

Thus,

$$\int_a^b h_{2n-1}(x)w(x)dx = Q(h_{2n-1}) = \sum_{j=1}^n h_{2n-1}(x_j)w_j = \sum_{j=1}^n f(x_j)w_j = Q(f)$$

Also, we know that $p_n^2(x)w(x) \geq 0$ on $[a, b]$. Then by applying the MVT, we obtain

$$\int_a^b f^{(2n)}(\theta) \frac{(2n)!}{(2n)!} p_n^2(x)dx = \frac{f^{(2n)}(\zeta)}{(2n)!} \int_a^b p_n^2(x)w(x)dx$$

for some $\zeta \in (a, b)$. 

4.3 Fundamental theorem of Gaussian

It implies that the nodes of the Gaussian quadrature in the interval $(a, b)$ and along with the weight function $w(x)$ are exactly the roots of the orthogonal polynomials $p_n(x)$ for the same weight function and interval.\textsuperscript{[25]} Thus, we can conclude that the roots of the classical orthogonal polynomials that we get in the previous chapter are the nodes of Gaussian quadrature rule.

4.4 Example

4.4.1 Hermite Polynomial

Gaussian Quadrature by using the roots $x_i$ and weights $w_i$ of Hermite Polynomial is given by

$$\int_{-\infty}^{\infty} e^{-x^2} f(x)dx = \sum_{j=1}^n w_if(x_i).$$
For example

\[ I = \int_{-\infty}^{\infty} e^{-x^2} \cos(x) dx. \quad (4.5) \]

The exact value of the integral is \( I = \frac{\sqrt{\pi}}{e^{\frac{1}{4}}} = 1.3803884470432 \ldots \). Then by using the formula in (4.5) to evaluate \( I \) approximately, and taking \( n \)

<table>
<thead>
<tr>
<th>( n )</th>
<th>( I_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>1.380 329 757 161 3</td>
</tr>
<tr>
<td>8</td>
<td>1.380 388 447 031 7</td>
</tr>
<tr>
<td>9</td>
<td>1.380 388 447 043 3</td>
</tr>
<tr>
<td>10</td>
<td>1.380 388 447 043 2</td>
</tr>
<tr>
<td>15</td>
<td>1.380 388 447 043 2</td>
</tr>
</tbody>
</table>

Here, we see that the result of the integral has 14 digit accuracy.

### 4.4.2 Laguerre Polynomial

Gaussian Quadrature by using the roots \( x_i \) and weights \( w_i \) of Laguerre Polynomial is given by

\[ \int_{0}^{\infty} x^\alpha e^{-x} f(x) dx = \sum_{j=1}^{n} w_i f(x_i). \quad (4.6) \]

For the case \( \alpha = 0 \), consider the example

\[ I = \int_{0}^{\infty} e^{-x} e^{(\sin(\frac{x}{\alpha}) - \cos(\frac{x}{\alpha}))} dx = \sum_{j=1}^{n} w_i f(x_i) \]

The exact value of the integral is 0.41330875870848.

Then by using the roots and weight of Laguerre when \( \alpha = 0 \) to evaluate \( I \) approximately, and taking \( n \)
Table 4.2: Example by using the roots and weights of Laguerre for $\alpha = 0$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$I_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0.413 308 737 784 09</td>
</tr>
<tr>
<td>8</td>
<td>0.413 308 758 710 23</td>
</tr>
<tr>
<td>10</td>
<td>0.413 308 758 708 52</td>
</tr>
<tr>
<td>11</td>
<td>0.413 308 758 708 48</td>
</tr>
</tbody>
</table>

The integral has 14 digit accuracy. Here, also we see that the Gaussian quadrature converges to the exact value of the integral.

Now for the case $\alpha = -\frac{1}{2}$, consider the example

$$I = \int_0^{\infty} x^{-\frac{1}{2}}(x^{10} + 5x^4 + 20)e^{-x}dx.$$ \hspace{1cm} (4.7)

The exact value of the integral is not known so here we show that Gaussian quadrature is also used for the integrals which are impossible to be solved. Then by using the formula in (4.6) to evaluate $I$ approximately, and taking $n$

Table 4.3: Example by using the roots and weights of Laguerre for $\alpha = -\frac{1}{2}$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$I_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>901 737.757 378 55</td>
</tr>
<tr>
<td>8</td>
<td>1 127 090.863 333 7</td>
</tr>
<tr>
<td>10</td>
<td>1 133 371.996 667 8</td>
</tr>
<tr>
<td>11</td>
<td>1 133 371.996 667 8</td>
</tr>
<tr>
<td>20</td>
<td>1 133 371.996 667 8</td>
</tr>
</tbody>
</table>

The integral has 14 digit accuracy and it converges to the value 1133371.9966678
4.4.3 Jacobi Polynomial

Gaussian Quadrature by using the roots $x_i$ and weights $w_i$ of Jacobi Polynomial is given by

$$\int_{-1}^{1} (1-x)^\alpha (x-1)^\beta f(x)dx = \sum_{j=1}^{n} w_i f(x_i) \quad (4.8)$$

Consider first Legendre Polynomial, i.e., put $\alpha = \beta = 0$ in Jacobi polynomial. Then, we have as an example

$$I = \int_{-1}^{1} \frac{1}{x^2 + 1} dx \quad (4.9)$$

The exact value of the integral is $I = \frac{\pi}{2} = 1.57079632679\ldots$.

Then by using the formula in (4.9) to evaluate $I$ approximately, and taking

<table>
<thead>
<tr>
<th>$n$</th>
<th>$I_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>1.568 627 450 98</td>
</tr>
<tr>
<td>8</td>
<td>1.570 794 412 54</td>
</tr>
<tr>
<td>9</td>
<td>1.570 796 655 94</td>
</tr>
<tr>
<td>10</td>
<td>1.570 796 270 22</td>
</tr>
<tr>
<td>12</td>
<td>1.570 796 325 12</td>
</tr>
<tr>
<td>13</td>
<td>1.570 796 327 08</td>
</tr>
<tr>
<td>14</td>
<td>1.570 796 326 74</td>
</tr>
<tr>
<td>16</td>
<td>1.570 796 326 79</td>
</tr>
<tr>
<td>17</td>
<td>1.570 796 326 79</td>
</tr>
<tr>
<td>100</td>
<td>1.570 796 326 79</td>
</tr>
</tbody>
</table>

Here, we see that the values are all with 11 digit accuracy after $n=16$ partition.

Now, consider Chebyshev Polynomial where $\alpha = \beta = -\frac{1}{2}$. Then, we have as an example

$$I = \int_{-1}^{1} \frac{e^x}{\sqrt{1-x^2}} dx \quad (4.10)$$
The exact value of the integral is not known, so we are going to evaluate the integral by Gaussian quadrature using the nodes and weights we calculated before. Then by using the formula in (4.9) to evaluate $I$ approximately, and taking

Table 4.5: Example by using the roots and weights of Chebyshev

<table>
<thead>
<tr>
<th>$n$</th>
<th>$I_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3.960 266 053</td>
</tr>
<tr>
<td>3</td>
<td>3.977 321 961</td>
</tr>
<tr>
<td>4</td>
<td>3.977 462 635</td>
</tr>
<tr>
<td>6</td>
<td>3.977 463 261</td>
</tr>
<tr>
<td>7</td>
<td>3.977 463 261</td>
</tr>
<tr>
<td>8</td>
<td>3.977 463 261</td>
</tr>
<tr>
<td>50</td>
<td>3.977 463 261</td>
</tr>
</tbody>
</table>

Here, we see that the values are all has 10 digit accuracy after $n=6$ so that we can conclude the Gaussian quadratures converges to the exact value of the integral. Thus, for the integrals which are really difficult to evaluate we can use Gaussian quadrature method.
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