TIGHT CONTACT STRUCTURES ON HYPERBOLIC THREE-MANIFOLDS

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ABSTRACT

TIGHT CONTACT STRUCTURES ON HYPERBOLIC THREE-MANIFOLDS

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In this dissertation, we study tight contact structures on hyperbolic 3-manifolds and homology spheres. We build a family of infinitely many hyperbolic 3-manifolds admitting tight contact structures. To put it more explicitly, we consider a certain infinite family of surface bundles over the circle whose monodromies are taken from some collection of pseudo-Anosov diffeomorphisms. We show the existence of tight contact structure on every closed 3-manifold obtained via rational r-surgery along a section of any member of the family except one r. Consequently, we obtain infinitely many hyperbolic closed 3-manifolds admitting tight contact structures.

Moreover, we construct infinitely many contractible 4-manifolds bounded by a homology sphere as generalized Mazur type manifolds built by Akbulut and Kirby. Specifically, the construction is formed by a 4-dimensional 2-handlebody where infinitely many of them have hyperbolic Stein fillable boundaries.

Keywords: Tight contact structure, Open Book, Hyperbolic, Homology Sphere

ÖZ

HİPERBOLİK ÜÇ-MANİFOLDLAR ÜZERİNDEKİ SIKI KONTAKT YAPILAR

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Bu tezde, hiperbolik 3-manifoldlar ve homoloji küreleri üzerindeki sıkı kontakt yapıları çalıştık. Üzerinde sıkı kontakt yapılar olan sonsuz sayıda hiperbolik 3-manifold ailesi inşa ettik. Daha açık ifade etmek gerekirse, monodromisi bazı pseudo-Anosov topluluğundan alınan, sonsuz sayıda, belirli bir çember üzerindeki yüzey demeti ailesini dikkate aldık. Bir r haricinde, ailenin herhangi bir öğesinin bir kesitine rasyonel r-ameliyat yapılarak elde edilen her 3-manifold üzerinde sıkı kontakt yapının varlığını gösterdik. Sonuç olarak, üzerinde sıkı kontakt yapılar olan sonsuz tane hiperbolik kapalı 3-manifold elde ettik.

Dahası, Akbulut ve Kirby tarafından oluşturulan genelleştirilmiş Mazur tip manifoldlarına benzer şekilde, bir homoloji küresi tarafından sınırlanan sonsuz tane büzülebilir 4-manifold inşa ettik. Yapı, özellikle, sonsuz tanesi hiperbolik Stein doldurulabilir sınıra sahip olan 4-boyutlu 2-kulplu cisimden meydana gelmektedir.

Anahtar Kelimeler: Sıkı Kontakt Yapı, Açık Kitap, Hiperbolik, Homoloji Küresi

To my family

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CHAPTER 1

INTRODUCTION

Although contact geometry has been working areas of physicists since the 19th century, it has been studied as a major field in mathematics after Lutz and Martinet [39] showed the existence of contact structures in all 3-manifolds. Besides having many applications in physics, such as geometrical optics, classical mechanics etc., it has also played a key role in mathematics, especially in low dimensional topology, for example, to prove the property P conjecture by Kronheimer and Mrowka [32]. Together with the classical result of Eliashberg [12] that guarantees the existence of overtwisted contact structures and the examples of non-existence of tight contact structures [21, 36] in some closed 3-manifolds, it was affirmed that tight contact structures have more leverage to understanding the topology of the underlying manifold.

One of the most useful topological aspects to study 3-manifolds is open book decomposition, which is relevant to contact structures. Giroux [27] completed the result of Thurston and Winkelnkemper [52] saying that every open book admits a contact structure, by a pioneering correspondence of contact structures on a 3-manifold up to isotopy and open book decompositions up to positive stabilization.

Many other techniques have been developed to investigate tight contact structures. As an illustration, one can give some special complex compact submanifolds of complex spaces, so-called Stein manifolds named by Karl Stein, which imply tightness on the boundaries. Besides that, Ozsváth and Szabó have brought a different approach to contact and symplectic topology with many more applications. They established powerful algebraic tools; one of them is called contact invariant defined in [44], which assures that the contact structure on a 3-manifold is tight if the corresponding contact invariant does not vanish.

Moreover, one can restrict attention to geometric aspects of 3-manifolds to study contact structures on them. Since tight contact structures on the connected sum of two 3-manifolds decompose uniquely as tight contact structures on each component, one can focus on contact structures on irreducible 3-manifolds. As a result of Thurston's groundbreaking conjecture which is called geometrization conjecture, irreducible 3-manifolds are characterized as Seifert fibered or toroidal or hyperbolic manifolds. In two independent work [6, 30], Colin and Honda-Kazez-Matić showed the existence of tight contact structures on toroidal 3-manifolds which means that it contains an incompressible torus. Besides, the existence problem of tight contact structures on Seifert fibered manifolds is completed by Lisca and Stipsicz. They in [37], proved that a closed oriented Seifert fibered 3-manifold admits a tight contact structure if and only if it is not obtained via $(2q-1)-$surgery along the $(2, 2q+1)$ torus knot in the 3-sphere S^3 for $q \geq 1$. However, whether every hyperbolic 3-manifold admits a tight contact structure or not is still an open problem. Many mathematicians have investigated tightness in the hyperbolic world which is the generic case in dimension three (see [5, 31, 49], etc.). Etgü in [18] also explored infinitely many hyperbolic 3-manifolds that carry tight contact structures. His construction uses Dehn surgeries along sections of hyperbolic torus bundles over the circle S^1. The first part of this thesis generalizes these ideas for surface bundles over S^1 with fiber genus at least two. The thesis is organized as follows:

In Chapter 2, we give background about contact 3-manifolds, mapping class groups, open book decompositions and Giroux correspondence, Lefschetz fibrations and Stein manifolds, Heegaard Floer homology and finally hyperbolic 3-manifolds.

In Chapter 3 , we consider closed connected oriented surface Σ_g with genus g at least two and Σ_g-bundles over the circle M_ϕ with monodromy $\phi = t_{a_1}^m t_{a_2} \cdots t_{a_{2g}} t_{a_{2g+1}}^n$ where a_i's are simple closed curves as in Figure 3.1 and t_{a_i} indicates the Dehn twist along the curve a_i. Denote by $M_\phi(r)$ the resulting manifold of rational r-surgery along a section in M_ϕ. This construction gives a family of infinitely many hyperbolic 3-manifolds admitting tight contact structures as a corollary of the following theorems:

Theorem 1.0.1. *$M_\phi(r)$ is hyperbolic 3-manifold for all but finitely many $m \in \mathbb{Z}$ and $r \in \mathbb{R}$.*

Theorem 1.0.2. *$M_\phi(r)$ admits a tight contact structure for any positive integers m, n and for any rational number $r \neq 2g - 1$.*

On the other hand, Gabai [25] discovered that if the first Betti number b_1 of a 3-manifold is positive , it admits taut foliations which lead to construct infinitely many tight contact structure by Eliashberg and Thurston [17]. Hence, it is purposeful to investigate the tightness of contact structures on homology spheres. In this manner, in Chapter 4, we built infinitely many contractible 4-dimensional handlebodies which have homology spheres as boundary similar to generalized Mazur manifolds introduced by Akbulut and Kirby [2, 40]. Specifically, let $W^{\pm}(l, k, n)$ be the 4-dimensional manifold formed by one 0-handle, one 1-handle and one 2-handle attached along the curve γ in Figure 4.2 with framing k. Then, we show that $W^{\pm}(l, k, n)$ is contractible with homology spheres as boundary for all $k, l \in \mathbb{Z}$ and $n \in \mathbb{N}$. In addition, the boundary of the minus version is hyperbolic and Stein fillable.

CHAPTER 2

BACKGROUND

2.1 Contact 3-Manifolds

In this section, we will recall some background from contact geometry and topology. The reader can see [26] for more details. A *hyperplane distribution* ξ in a $(2n+1)$-dimensional manifold M is a subbundle of the tangent bundle TM such that at every point p of M the intersection of tangent space T_pM with ξ is a $2n$-dimensional subspace of T_pM. In dimension 3, sometimes a hyperplane distribution is called a *plane field*. Given a hyperplane distribution ξ in M and a point $p \in M$, there exist a neighborhood of p, say U, and a 1-form α defined on U such that $\ker \alpha = \xi|_U$.

Definition 2.1.1. *A contact structure ξ on a $(2n+1)$-dimensional manifold M is defined as a hyperplane distribution which is maximally nonintegrable; i.e., $\alpha \wedge (d\alpha)^n \neq 0$ for all 1-form α with $\ker \alpha = \xi$ locally or globally. Here, α is called a contact form and the pair (M, ξ) a contact manifold.*

If one can find a 1-form α defined on all M having the property $\ker \alpha = \xi$, then ξ will be called *coorientable*. Given a coorientable contact structure $\ker \alpha = \xi$, α is called a *contact form*. A coorientable contact structure $\xi = \ker \alpha$ is called *positive* in M if $\alpha \wedge (d\alpha)^n > 0$ with respect to the given orientation on M. In what follows, any contact structure will be positive and coorientable and for the rest of the dissertation, (M, ξ) will denote a contact 3-manifold unless otherwise stated. Now, we will focus on contact 3-manifolds and give some examples to be more illustrative.

Example 2.1.2. *Consider the standard Cartesian coordinates (x, y, z) in $\mathbb{R}^3$ and the 1-form $\alpha = dz + xdy$. It can be shown that $\alpha \wedge d\alpha = dx \wedge dy \wedge dz \neq 0$. $\xi_{std} = \ker \alpha$ is*

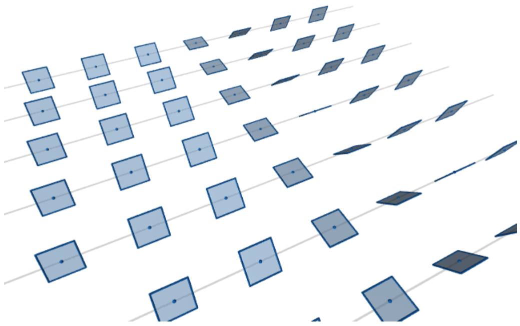


Figure 2.1: Standard contact structure in $\mathbb{R}^3$. Pictured by Patrick Massot.

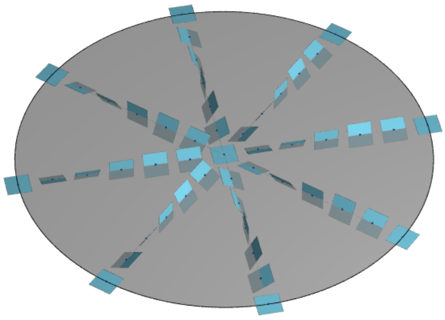


Figure 2.2: The contact structure ξ_{ot} in $\mathbb{R}^3$. Pictured by Patrick Massot.

named the standard contact structure on $\mathbb{R}^3$*. One can see the picture of* ξ_{std} *in Figure 2.1.*

Example 2.1.3. *Consider the coordinates* $(x_1, y_1, x_2, y_2) \in S^3 \subset \mathbb{R}^3$ *and the* 1*-form* $\alpha = x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2$ *as restricted to* S^3*. Then it can be checked that* α *is a contact form on* S^3*. The contact form* $\xi_{std} = \ker \alpha$ *is called the standard contact structure on* S^3*.*

Example 2.1.4. *Another example of contact structure on* $\mathbb{R}^3$ *can be given by* $\xi_{ot} = \ker(cos(r)dz + rsin(r)d\theta)$ *where* (r, θ, z) *is the cylindrical coordinate in* $\mathbb{R}^3$ *(see Figure 2.2).*

Example 2.1.5. $S^1 \times S^2$ *admits a contact structure given by the kernel of the* 1*-form* $\alpha = zd\theta + xdy - ydx$ *where* $\theta \in S^1$ *and* $(x, y, z) \in S^2 \subset \mathbb{R}^3$*.* $\xi_{std} = \ker \alpha$ *is also called the standard contact structure on* $S^1 \times S^2$*.*

There is a dichotomy among contact structures: *tight* vs. *overtwisted*. If one can find an embedded disk D in a contact 3-manifold (M, ξ) such that $T_p(\partial D) \subset \xi_p$ at every point of $p \in \partial D$ and the framings of ∂D coming from ξ and D coincide then ξ is called an overtwisted contact structure. Here, D is called an *overtwisted disk*. Otherwise ξ will be called a tight contact structure. The notion of "framing" will be addressed again in Subsection 2.1.1.

Definition 2.1.6. *Two contact* 3*-manifolds* (M_1, ξ_1) *and* (M_2, ξ_2) *are contactomorphic if there is a diffeomorphism* ϕ *between* M_1 *and* M_2 *such that* $\phi_*(\xi_1) = \xi_2$*. If there is a contactomorphism between two contact manifolds* (M, ξ_1) *and* (M, ξ_2) *which is isotopic to the identity, we say that* ξ_1 *and* ξ_2 *are isotopic.*

Remark 2.1.7. *While Examples 2.1.2, 2.1.3 and 2.1.5 are tight contact structures, the Example 2.1.4 is an example of an overtwisted contact structure and the the disk given in Figure 2.2 is an overtwisted disk in* (M, ξ_{ot})*. In fact, Examples 2.1.2, 2.1.3 and 2.1.5 are the unique tight contact structures up to isotopy, on* $\mathbb{R}^3$, S^3 *and* $S^1 \times S^2$*, respectively (see [26], Theorem 4.10.1(a)).*

The restriction of ξ_{std} in Example 2.1.3 onto $S^3 \setminus \{*\}$ is contactomorphic to the standard contact structure on $\mathbb{R}^3$. But the contact structures in Example 2.1.2 and Example 2.1.4 are not contactomorphic.

Theorem 2.1.8. *[12] Every closed compact* 3*-manifold admits a contact structure. More precisely, any plane field is homotopic to an overtwisted contact structure.*

We don't have the similar existence result of tight contact structures on closed compact 3-manifolds.

Theorem 2.1.9. *[21] There exist a closed compact* 3*-manifold which does not admit any tight contact structure.*

All contact structures locally look the same. We restrict the following theorem to dimension 3.

Theorem 2.1.10. *(Darboux) Given a pair* (M, ξ) *and a point* $p \in M$*, there exist a neighborhood* U *of* p *and a contactomorphism* $\phi : (U, \xi|_U) \to (\mathbb{R}^3, \xi_{std})$ *such that* $\phi(p) = (0, 0, 0)$.

We state some definitions and theorems to be prepared for Subsection 2.1.2.

Definition 2.1.11. *Given an embedded surface* Σ *in* (M, ξ)*, the characteristic foliation induced on* Σ*, denoted as* Σ_ξ*, is the singular* 1*-dimensional foliation, defined as* $(\Sigma_\xi)_x = \xi_x \cap T_x\Sigma$ *at each point* x *of* Σ*.*

It is easy to see that there are finitely many points x at which $(\Sigma_\xi)_x = T_x\Sigma$ otherwise $(\Sigma_\xi)_x$ is a line in $T_x\Sigma$. Characteristic foliation on a surface holds information about a neighborhood of the surface.

Theorem 2.1.12. *[19] Let* Σ *and* Σ' *be embedded surfaces in* (M, ξ) *and* (M', ξ')*, respectively. If* ϕ *is a diffeomorphism between* Σ *and* Σ' *preserving characteristic foliations on them, then* ϕ *can be extended to a contactomorphism between some neighborhoods of surfaces.*

Definition 2.1.13. *A vector field in* (M, ξ) *is called contact if the flow of the vector field preserves* ξ*; i.e.,*$(\phi_t)_*(\xi) = \xi$ *where* $(\phi_t)_*$ *is the induced map of time t flow* ϕ_t *on tangent bundles.*

Definition 2.1.14. *Let* Σ *be a surface in* (M, ξ)*. If there is a contact vector field transverse to* ξ *on* Σ*, the surface is called a convex surface.*

Definition 2.1.15. *Given a convex surface* Σ *in* (M, ξ)*, the set*

$$\{w \in \Sigma : X(w) \in \xi_w\}$$

is called dividing set of Σ *where* X *is a contact vector field transverse to* Σ*.*

Note that a dividing set of a surface Σ, generally denoted by Γ_Σ, forms a union of smooth 1-submanifolds; i.e, curves. These curves are called *dividing curves*. It's a fact that different contact vector fields transverse to a surface give rise to the same isotopy type of corresponding dividing sets.

To determine tightness of a neighborhood of a convex surface, it's enough to interpret the dividing set.

Theorem 2.1.16. Giroux Criterion *[29]*
Let Σ *be an embedded surface in* (M, ξ)*. If* $\Sigma \neq S^2$*, then* Σ *has a tight neighborhood if and only if all dividing curves are homotopically nontrivial. In addition,* S^2 *has a tight neighborhood if and only if* Γ_{S^2} *is connected.*

2.1.1 Knots in Contact 3-Manifolds

As knots play an important role in low dimensional topology, they carry lots of information in contact topology too. There are two different kind of knots in contact topology. In this subsection, we will give some definitions, properties and invariants of these knots. We refer the reader to [20] for more details.

Definition 2.1.17. *A knot L in a contact 3-manifold (M, ξ) is Legendrian if $TL \subset \xi$. An oriented knot T in (M, ξ) is called transverse if ξ is positively transverse to the tangent line of T at every point of T.*

Let's first study Legendrian knots in $(\mathbb{R}^3, \xi_{std})$. We will focus on transverse knots later in Subsection 3.2.1. Given a Legendrian knot L, fix a parametrization of it:

$$\phi : S^1 \to \mathbb{R}^3, \phi(t) = (x(t), y(t), z(t)).$$

Since L is tangent to ξ_{std}, $\phi'(t) \in \xi_{std} = \ker(dz + xdy)$ which means that

$$z'(t) + x(t)y'(t) = 0. \tag{2.1}$$

There are two projections of Legendrian knots: front projection and Lagrangian projection. We focus on the front projection of L which is $\Pi(L)$ where $\Pi : \mathbb{R}^3 \to \mathbb{R}^2, \Pi(x, y, z) = (y, z)$. Combining with the parametrization ϕ, we have $\phi_\Pi : S^1 \to \mathbb{R}^2, \phi_\Pi(t) = (y(t), z(t))$. If $y'(t)$ vanishes, so does $z'(t)$ by the Equation 2.1. Assuming that ϕ is an immersion (in fact embedding) implies that $y'(t)$ can not be zero since ϕ_Π must be also an immersion. Hence, there is no vertical tangencies in a front projection. On the other hand, front projection gives all information about the Legendrian knot by $-x(t) = \frac{z'(t)}{y'(t)}$ as long as $y'(t) \neq 0$. One can easily check that $y'(t)$ is zero only at isolated points, so called cusps. We choose that positive x-axis points into the page by convention. Another fact about front projection is that at each crossing the part having more positive slope under-crosses the other part. One can see cusps and the rule of crossing in Figure 2.3.

A *framing* of a knot (smooth) K is a trivialization of the normal bundle νK of K. We define *Seifert framing* as the trivialization of νK given by push-off of K along a Seifert surface of K. A Legendrian knot L in (M, ξ) has a canonical framing which is called *contact framing*; i.e., the trivialization defined by a normal vector field to ξ.

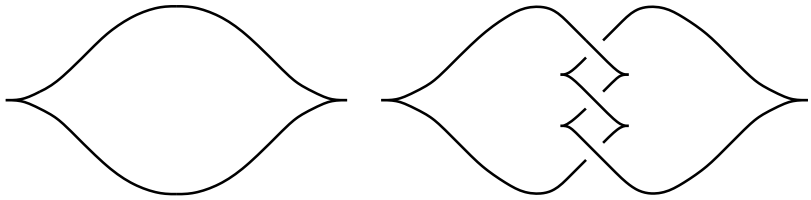


Figure 2.3: Legendrian representations of unknot (on the left) and left-handed trefoil (on the right).

If L is *null-homologous*; i.e., if there is a surface Σ bounded by L, $\xi|_\Sigma$ is the trivial bundle since every (oriented) plane bundle on a surface with boundary is trivial. This trivialization yields a trivialization of L, say $\nu(L)$. Now we define some classical invariants of Legendrian knots.

Definition 2.1.18. *Thurston-Bennequin number of a Legendrian null-homologous knot L, denoted by $tb(L)$, is the linking number $lk(L', L'')$ where L' and L'' are push-offs of L with respect to Seifert framing and contact framing, respectively. Let L also be oriented and let v be a nonzero vector field in the trivialization $\nu(L)$ defined above. The rotation number of L is defined as the winding number of the vector field v, denoted by $rot(L)$.*

Given an (oriented) Legendrian knot L in $(\mathbb{R}^3, \xi_{std})$, the Thurston-Bennequin number can be computed as $tb(L) = wr(L) - \frac{1}{2}c(L)$ where $wr(L)$ is the writhe of L; i.e., the sum of the signs of the crossings, and $c(L)$ is the number of cusps in the front projection of L. We can also interpret the rotation number from the front projection as $rot(L) = \frac{1}{2}(d - u)$ where d (respectively u) is the number of down (respectively up) cusps.

Definition 2.1.19. *Let L be an oriented Legendrian knot in (M, ξ). A positive (resp. negative) stabilization of L is obtained by adding one down cusp (resp. up cusp) in the front projection as in Figure 2.4.*

A *Legendrian band* is a band in the front projection such that one boundary component is Legendrian and the other one is a push-off of it in the z-direction. Given two oriented Legendrian knot L_1 and L_2, the *Legendrian connected sum* of them is the connected sum of the knots by a Legendrian band as in Figure 2.5. If the orientation matches, then the Legendrian sum is called a *Legendrian handle addition*, otherwise

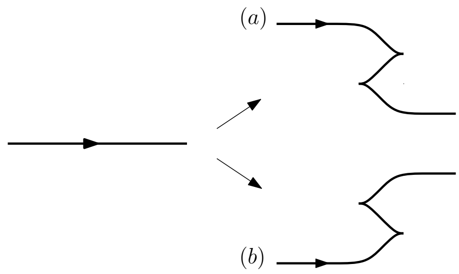


Figure 2.4: (a) Adding a down cusp to a Legendrian arc. (b) Adding an up cusp to a Legendrian arc.

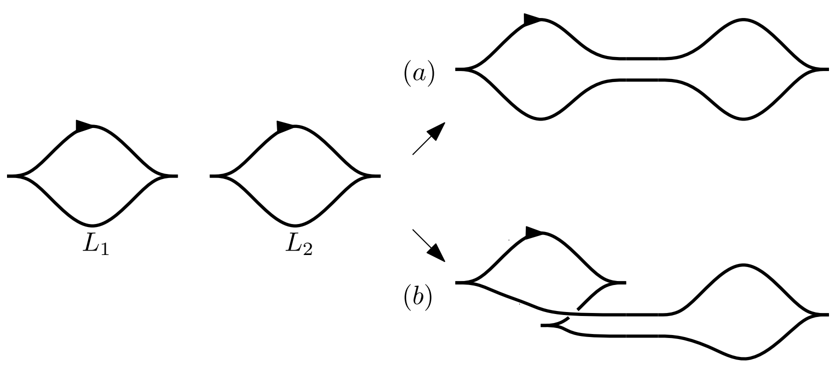


Figure 2.5: (a) Legendrian handle addition of L_1 over L_2. (b) Legendrian handle substraction of L_1 over L_2.

it is called a *Legendrian handle substraction*. One can find the details in [10].

Two knots K_1 and K_2 are *isotopic* means that there is a smooth map $i : S^1 \times I \to M$ such that $i(S^1 \times \{0\}) = K_1$, $i(S^1 \times \{1\}) = K_2$ and $i(S^1 \times \{t\})$ is a knot in M for all t.

Theorem 2.1.20. *Every knot K in a contact manifold (M, ξ) can be C^0-approximated to an isotopic Legendrian knot and an isotopic transverse knot positively (or negatively) with respect to ξ (ξ is cooriented.).*

2.1.2 Contact Surgery

Let K be a knot in a 3-manifold M and let $\nu(K)$ be the normal bundle of K. A *Dehn surgery* along K in M is basically the operation of removing $\nu(K)$ and gluing a solid torus by identifying the boundary of the solid torus with the boundary of the complement $M \setminus \nu(K)$ via an orientation reversing diffeomorphism. More precisely, let $f : \partial(S^1 \times \mathbb{D}^2) \to \partial(M \setminus \nu(K))$ and let μ be the meridian (i.e., the curve bounding a disk) of $\partial(M \setminus \nu(K))$. Clearly μ is a generator of the first homology group $H_1(\partial(M \setminus \nu(K))) \simeq \mathbb{Z} \oplus \mathbb{Z}$. However, the other generator, so-called *longitude*, may not be canonical. Let λ be the preferred longitude and let f be defined by $f(m) = p\mu + q\lambda$ where m is the meridian of $\partial(S^1 \times \mathbb{D}^2)$. Note that it is enough to be given the coprime pair (p, q) to determine f and hence the surgery, when the longitude is fixed. The surgery is called $(\frac{p}{q})$*-surgery along the knot* K *in* M.

Given a null-homologous knot K, there is a canonical framing called *the Seifert framing* of K that is the push-off of K along a Seifert surface for K. There is also a canonical framing for a given Legendrian knot L in a contact 3-manifold, *the contact framing* of L, which is defined by the push-off of the knot along a vector field transverse to the contact planes. Note that these framings do not depend on the chosen Seifert surface and the transverse vector field.

Definition 2.1.21. *If* K *is a null-homologous knot in a* 3*-manifold* M *and* λ *is chosen with respect to the Seifert framing of* K *then a Dehn surgery which is defined as above is called a topological surgery. If* K *is a Legendrian knot and* λ *is a push-off of* K *with respect to the contact framing, Dehn surgery is called contact surgery.*

Contact (-1)-surgery is sometimes called *Legendrian surgery*. It is easy to check that contact (r)-surgery along a Legendrian knot L is topologically equivalent to topological $(r + tb(L))$-surgery along L. The next theorem illustrates the importance of surgery theory.

Theorem 2.1.22. *[33, 54] Every closed connected oriented* 3*-manifold can be obtained by a Dehn surgery along a link in* S^3.

The previous theorem has a contact version too.

Theorem 2.1.23. *[9] Every closed connected oriented contact* 3*-manifold* (M, ξ) *can be obtained by a sequence of contact* (± 1)*-surgeries along a Legendrian link in* (S^3, ξ_{std}).

In [8], Ding and Geiges introduced contact (r)-surgery along a Legendrian knot explicitly. We will give the details about this definition.

Theorem 2.1.24. ***Contact Neighborhood Theorem***
If (L_1, M_1) and (L_2, M_2) are diffeomorphic to each other where L_i is a Legendrian submanifold of the contact manifold M_i for each $i = 1, 2$, then they have contactomorphic neighborhoods.

Let $L \subset (M, \xi)$ be a Legendrian knot. Consider the manifold M' obtained via contact (r)-surgery along L where $r = \frac{p}{q} \in \mathbb{Q}$. We will define a contact structure ξ' on M' as follows:

Let $\nu(L)$ be a tubular neighborhood of L. According to the Contact Neighborhood Theorem, $(\nu(L), \xi|_{\nu(L)})$ is contactomorphic to $(S^1 \times \mathbb{D}^2, \zeta = \ker(cos\theta dx - sin\theta dy))$ where $\theta \in S^1$, $(x, y) \in \mathbb{R}^2$ and L is identified with $S^1 \times \{(0, 0)\}$. Let N_δ be the solid torus $S^1 \times \{(x, y) : x^2 + y^2 \leq \delta^2\}$ in $S^1 \times \mathbb{D}^2$. It can be checked that the vector field $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ is a contact vector field and is transverse to $\xi|_{\partial N_\delta}$ and hence ∂N_δ is a convex surface. Then one can talk about the dividing set on it which can be easily computed as

$$\{(\theta, \pm\delta sin\theta, \pm\delta cos\theta) : \theta \in S^1\}.$$

It is easy to observe that the canonical longitudes on ∂N_δ determined by the contact framing are the dividing curves. Let us fix the positive one as λ. According to Giroux's criterion (see Theorem 2.1.16), the boundary torus of N_δ has a tight neighborhood since it has no homotopically trivial dividing curves.

Let $f : (N_b, \zeta) \to (M, \xi)$ be a contact embedding identifying the core of N_b with L. Let $\phi : N \to N$ be an orientation preserving diffeomorphism fixing boundary components such that $\phi(\mu) = p\mu + q\lambda$ where $N = N_b \setminus int(N_a)$ with $a < \delta < b$. One can construct M' as $(M \setminus f(N_a)) \cup N_b$ by identifying boundaries with $x \in \partial N_b \sim f(\phi(x)) \in \partial(M \setminus f(N_a))$.

If $p \neq 0$, then the contact structure $\phi^{-1}(\zeta)$ can be extended to a contact structure ξ' on M' since ∂N_δ has nonzero slope and by Honda (see Theorem 2.3, [29]) it guarantees a tight contact structure on N_b.

If $p = 0$, the contact structure ξ' can be obtained by using Lutz twist. We refer the reader to [9] for details.

Theorem 2.1.25. *[9] Every contact (r)-surgery can be converted into a sequence of* *contact* (± 1)*-surgeries.*

A *contact surgery diagram* is a Legendrian link diagram accompanied by a rational number for each link component which shows the contact surgery along the link in (S^3, ξ_{std}) with given rational number coefficients. Since $(S^3 \setminus \{*\}, \xi_{std})$ is contactomorphic to the standard contact structure on $\mathbb{R}^3$, the surgery can be thought in $(\mathbb{R}^3, \xi)$. In [11], it is given an explicit algorithm of converting a contact surgery diagram into a diagram that consists of only (± 1)'s as contact surgery coefficient. The details of this algorithm is given below where we again exclude the case of contact (0)-surgery.

Consider contact (r)-surgery along a Legendrian knot L in (S^3, ξ_{std}) with $r < 0$. Let

$$r = r_1 + 1 - \cfrac{1}{r_2 - \cfrac{1}{\dots - \frac{1}{r_n}}}$$

where $r_i \leq -2$ for all $i \in \{1, 2, \dots, n\}$. We take a front projection of L, say L_1, drawn with extra $|r_1 + 2|$ zigzags as indicated in Figure 2.6. Similarly, let L_i be the Legendrian copy of L_{i-1} with additional $|r_i + 2|$ zigzags for $i \in \{2, \dots, n\}$. Then the sequence of contact (-1)-surgeries along the Legendrian knots $L_1, L_2, \dots, L_n$ coincides with the contact (r)-surgery along L.

Let $r = \frac{p}{q} > 0$ and let $n \in \mathbb{N}$ such that $q - np < 0$. Then contact $(+1)$-surgeries along Legendrian push-offs of L, say $L_1, L_2, \dots, L_n$, together with contact $(\frac{p}{q-np})$-surgery along L correspond to contact (r)-surgery. Since $\frac{p}{q-np}$ is negative, this surgery coefficient can be converted into -1's as above.

Note that contact $(\frac{1}{k})$-surgery along a Legendrian knot for some integer k results in a unique contact structure; however, except these surgery coefficients, the choice of left or right zigzags may give rise to non-contactomorphic contact structures.

Example 2.1.26. *Contact* $(+1)$*-surgery along the Legendrian unknot on the left in Figure 2.6 , gives* $S^1 \times S^2$ *with the standard contact structure.*

2.2 Mapping Class Groups

Consider a compact oriented connected surface Σ_g^n of genus g with n boundary components. Let $Homeo^+(\Sigma_g^n, \partial\Sigma_g^n)$ denote the group of orientation-preserving home-

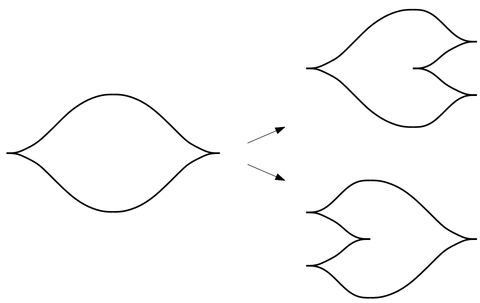


Figure 2.6: A Legendrian unknot and its copies with an additional zigzag.

omorphisms of Σ_g^n which are the identity on $\partial\Sigma_g^n$. The set of isotopy classes of $Homeo^+(\Sigma_g^n, \partial\Sigma_g^n)$ is called the *mapping class group* of Σ_g^n, denoted by $MCG(\Sigma_g^n)$.

Definition 2.2.1. *Given a simple closed curve c on a compact oriented surface Σ, a tubular neighborhood of c, say N, in Σ is homeomorphic to an annulus $S^1 \times [0,1]$. Fix a homeomorphism ϕ from N to an annulus. Let $f : S^1 \times [0,1] \to S^1 \times [0,1]$ be a map defined by $f(e^{i\theta}, t) = (e^{i(\theta+2\pi)}, t)$ where $e^{i\theta} \in S^1 \subset \mathbb{C}$. A positive Dehn twist about the curve c is a homeomorphism $t_c : \Sigma \to \Sigma$ defined as follows:*

$$t_c(x) = \begin{cases} \phi^{-1} \circ f \circ \phi(x) & x \in N \\ x & x \in \Sigma \setminus N \end{cases}$$

By a negative Dehn twist, we mean -1 power of a positive Dehn twist.

Theorem 2.2.2. *[34] Every element of $MCG(\Sigma)$ can be represented by a product of Dehn twists.*

2.3 Open Book Decomposition and Giroux Correspondence

Throughout this section, M will be a closed oriented 3-manifold.

Definition 2.3.1. *A pair (Σ, ϕ) is called an abstract open book decomposition of M if Σ is a compact oriented surface with boundary and ϕ is a self-diffeomorphism of Σ which is the identity on a neighborhood of the boundary. ϕ is named as the monodromy of the open book decomposition.*

Definition 2.3.2. *We say a pair (B, π) is an open book decomposition of M if B is an oriented link in M and complement of B in M is a fibration π over S^1 such that* *every fiber is a Seifert surface for B. Here, B is called the binding and the Seifert surface is called the page of the open book decomposition.*

Denote by $M_{(\Sigma,\phi)}$ the *surface bundle over circle* (or *mapping torus*) with a fiber Σ and the monodromy $\phi : \Sigma \to \Sigma$; i.e., the quotient space $\Sigma \times [0,1]/\sim$ where $\sim$ is the relation defined by $(x, 0) \sim (\phi(x), 1)$ for all $x \in \Sigma$. Given an abstract open book decomposition (Σ, ϕ) of M, consider the corresponding mapping torus $M_{(\Sigma,\phi)}$. It is obvious that $\partial M_{(\Sigma,\phi)} = \coprod_{|\partial\Sigma|} S^1 \times S^1$ where the first S^1 indicates a boundary component of Σ and the second one is the base of $M_{(\Sigma,\phi)}$. Then fill each boundary component of $M_{(\Sigma,\phi)}$ with a solid torus $S^1 \times \mathbb{D}^2$ via a diffeomorphism $\partial(S^1 \times \mathbb{D}^2) \to S^1 \times S^1$ sending the longitude $S^1 \times \{*\}$ to the corresponding boundary component of Σ and the meridian to the base S^1. Note that the resulted closed 3-manifold is diffeomorphic to M and the cores of the glued solid tori in M form the binding of an open book decomposition. Also observe that the complement of the cores in M is diffeomorphic to $M_{(\Sigma,\phi)}$.

Definition 2.3.3. *A positive stabilization of an abstract open book decomposition (Σ, ϕ) is defined as the new open book decomposition $(\Sigma', \phi \circ t_a)$ where Σ' is obtained by attaching a 1-handle to Σ, and a is a simple closed curve on Σ' intersecting the cocore of the 1-handle exactly once. Similarly, $(\Sigma', \phi \circ t_a^{-1})$ is called a negative stabilization of (Σ, ϕ).*

Theorem 2.3.4. *[4] Any closed oriented 3-manifold admits an open book decomposition.*

Definition 2.3.5. *Let ξ be a contact structure on M. We say that ξ is supported by an open book decomposition (B, π) of M if there is a contact 1-form α for ξ such that $d\alpha$ is a positive area form on each page of (B, π) and α is positive on the binding B with respect to the orientation on it givn by the pages.*

Let L be a Legendrian knot in (M, ξ). One can show that there is an open book decomposition supporting ξ such that L sits on a page of the open book and the framing given by the page and by ξ agree.

Theorem 2.3.6. *[52] Any open book decomposition of M supports a contact structure.*

In the construction of the previous theorem, Thurston and Winkelnkemper show that the binding of an open book supporting a contact structure ξ, is transverse to ξ. In the following theorem, Giroux completes the missing direction of the theorem of Thurston and Winkelnkemper.

Theorem 2.3.7. ***Giroux Correspondence*** *[27]*
Given a 3-manifold M, there is a one-to-one correspondence between open book decompositions on M up to positive stabilizations and contact structures on M up to isotopies.

2.4 Lefschetz Fibrations and Stein Manifolds

In this section, otherwise stated, X will be a compact oriented 4-manifold. X is a complex manifold if there is an atlas of charts (U_i, ϕ_i) where U_i is homeomorphic to $\mathbb{C}^2$ and ϕ_i is a holomorphic map; i.e., complex differentiable map at every point of U_i.

Definition 2.4.1. *A map $f : X \to C$ where C is a compact oriented surface is called a Lefschetz fibration if*

- *there are finitely many critical values $c_1, c_2, \ldots c_n$ in the interior of C, having unique critical point p_i for each i and*
- *there exist charts around p_i and c_i for each i matching with the orientations of X and C where, in the restriction of f on the charts, we have $f(z_1, z_2) = z_1 z_2$.*

Remark 2.4.2. *A Lefschetz fibration is sometimes called positive Lefschetz fibration because of the condition that charts must agree with the orientations of total and base spaces. If the condition is taken out then $f : X \to C$ in the previous definition is called achiral Lefschetz fibration.*

Remark 2.4.3. *By Sard's theorem, $f^{-1}(c) = \Sigma_g$ is a compact oriented surface (possibly with boundary) for a fixed g for all non-critical values $c \in C$. Any such surface is called a regular fiber.*

Let $f : X \to \mathbb{D}^2$ be a Lefschetz fibration over a 2-disk and let $\{d_1, d_2, \ldots, d_n\}$ be the set of critical values in $int(\mathbb{D}^2)$ as indicated in Figure 2.7 and p_i be the critical point

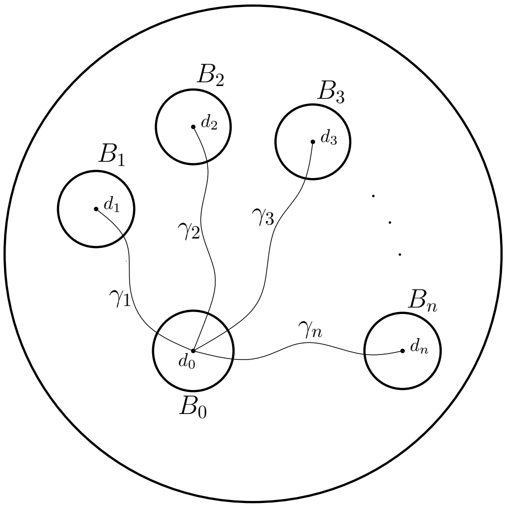


Figure 2.7: The base of a Lefschetz fibration from X onto $\mathbb{D}^2$ with critical values $d_1, d_2, \dots, d_n$ surrounded by disks $B_1, B_2, \dots, B_n$, respectively, and a regular value d_0 circled by a disk B_0.

in X corresponding to d_i for all i. Fix a regular value d_0 and consider arcs γ_i from d_0 to d_i and pairwise disjoint small disks B_i with center d_i for all $i \in \{0, 1, \dots, n\}$ (see Figure 2.7). Then $f^{-1}(B_0)$ is diffeomorphic to the $\Sigma \times \mathbb{D}^2$ where Σ is a regular fiber and $f^{-1}(B_0 \cup \nu(\gamma_1) \cup B_1)$ is diffeomorphic to $\Sigma \times \mathbb{D}^2 \cup H_1^2$ where H_1^2 is a 4-dimensional 2-handle attached to $\Sigma \times \mathbb{D}^2$ along an embedded simple closed curve α_1 in a fiber with framing -1 relative to the framing coming from $\Sigma \times \mathbb{D}^2$. Here, α_1 is called a *vanishing cycle*. Additionally, one can show that $f^{-1}(\partial(B_0 \cup \nu(\gamma_1) \cup B_1))$ is diffemorphic to Σ-bundle over S^1 with monodromy t_{α_1}. More generally, $f^{-1}(B_0 \cup \bigcup_{i=1}^{n} (\nu(\gamma_i) \cup B_i))$ is diffeomorphic to $\Sigma \times \mathbb{D}^2 \cup \bigcup_{i=1}^{n} H_i^2$ where each H_i^2 is attached to $\Sigma \times \mathbb{D}^2$ along α_i with the counterclockwise order appearing in Figure 2.7 (so with the order from n to 1). Similarly, $f^{-1}(\partial(B_0 \cup \bigcup_{i=1}^{n} (\nu(\gamma_i) \cup B_i)))$ is the Σ-bundle over S^1 with monodromy $t_{\alpha_1} t_{\alpha_2} \dots t_{\alpha_n}$. One can also show that ∂X admits the open book decomposition $(\Sigma, t_{\alpha_1} t_{\alpha_2} \dots t_{\alpha_n})$.

Definition 2.4.4. *A Lefschetz fibration $f : X \to C$ is called allowable if all vanishing cycles are homologically non-trivial. A positive allowable Lefschetz fibration over $\mathbb{D}^2$ having fibers with boundary is called a PALF.*

Definition 2.4.5. *A complex manifold X of any dimension is holomorphically convex* *if given a compact subset $K \subset X$, the holomorphically convex hull of K; i.e.,*

$$\bar{K} = \{z \in X : |f(z)| \leq \sup_{w \in X} f(w), for\ all\ holomorphic\ function\ of\ X\},$$

is again compact. X is called holomorphically separable if for any pair of points x, y of X, we can find a holomorphic function f such that $f(x) \neq f(y)$.

Definition 2.4.6. *A complex manifold X of complex dimension n is called a Stein manifold if it is holomorphically convex and holomorphically separable.*

Remark 2.4.7. *By Eliashberg and Gromov [16], every Stein manifold can be properly biholomorhically embedded in $\mathbb{C}^N$, for some positive integer N. Given a complex manifold X of complex dimension 2, if there is a proper biholomorphic embedding $X \hookrightarrow \mathbb{C}^N$, then X is called a Stein surface.*

The next theorem points out the the relation between Lefschetz fibrations and Stein surfaces.

Theorem 2.4.8. *[3, 38, 1] Every compact Stein surface with boundary admits infinitely many PALF.*

Conversely, any PALF is also a Stein manifold.

Definition 2.4.9. *A contact 3-manifold (M, ξ) is Stein fillable if there is a Stein manifold (X, J) where J is the complex structure such that $\partial X = M$ and the induced contact structure from J; i.e., $J(T(\partial X)) \cap T(\partial X)$, is contactomorphic to ξ.*

Theorem 2.4.10. *[27] A contact 3-manifold (M, ξ) is Stein fillable if and only if it has a compatible open book decomposition (Σ, ϕ) such that ϕ can be written as a product of positive Dehn twists.*

Example 2.4.11. *(A, id) is a compatible open book of $(S^1 \times S^2, \xi_{std})$ in Example 2.1.5 where A is the annulus and id denotes the identity map. Thus, $(S^1 \times S^2, \xi_{std})$ is Stein fillable. Also, (S^3, ξ_{std}) in Example 2.1.3, admits a unique Stein filling which is diffeomorphic to $\mathbb{D}^4$ equipped with the standard complex structure (see [13, 15]).*

Theorem 2.4.12. *[14, 55] Legendrian surgery preserves Stein fillability. More precisely, given a Stein manifold X, the cobordism obtained by attaching a 2-handle along a Legendrian knot in the contact boundary with framing -1 relative to the contact framing extends the Stein structure.*

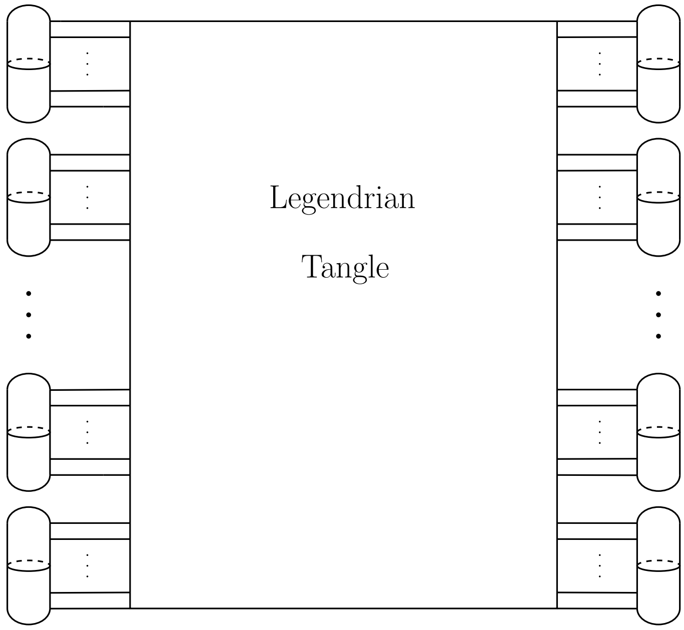


Figure 2.8: A Legendrian link diagram in standard form.

Eliashberg gave a topological handlebody description of 4-dimensional Stein manifolds.

Theorem 2.4.13. *[14] An oriented open 4-manifold X is a Stein manifold if and only if it is the interior of a 4-manifold obtained by one 0-handle, 1-handles and 2-handles such that each 2-handle is attached to a Legendrian knot with framing -1 relative to the canonical framing of the knot.*

Gompf [28] determined the picture of the handlebody diagram of a Stein manifold. A *Legendrian link diagram* in standard form, as indicated in Figure 2.8, consists of n 1-handles (represented by horizontal ball pairs), n pairs of horizontal distinguished segments corresponding to each ball pair and a *Legendrian tangle*; i.e., union of Legendrian arcs and knots, with endpoints touching the segments. As in Legendrian knots, the Thurston Bennequin number of a link component L of a Legendrian tangle is given by the formula $tb(L) = wr(L) - \frac{1}{2}c(L)$.

Proposition 2.4.14. *[28] An oriented compact connected 4-manifold X is a Stein surface if and only if it admits a handlebody diagram formed by a Legendrian link diagram in standard form equipped with 2-handles attached to link components L_i's with framing $tb(L_i) - 1$.*

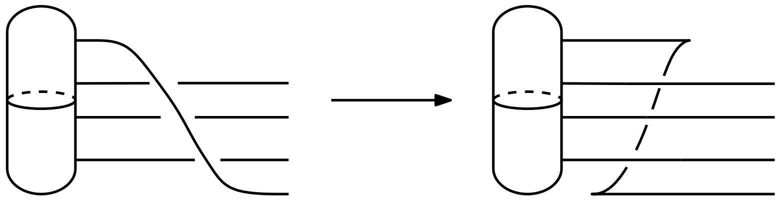


Figure 2.9: A move in a Legendrian link diagram of a Stein manifold.

Gompf [28] also defined some moves for the handlebody diagram of a Stein manifold. In particular, as in Figure 2.9, one can swing a strand of a Legendrian tangle in a Legendrian link diagram in standard form. Note that this move increases the Thurston-Bennequin number of the link component that owns the strands by one, but it does not change the Stein structure of the diagram.

2.5 Heegaard Floer Homology

In the early 2000s, Ozsváth and Szábó defined powerful algebraic tools of low dimensional objects (see [43, 42, 44, 45]). In this section, we will review an invariant of a closed 3-manifold called Heegaard Floer homology. We give some fundamental definitions and theorems of Heegaard Floer homology.

Given a closed oriented connected 3-manifold M, it is assigned four variants of Heegaard Floer homology groups, $\widehat{HF}$, HF^+, HF^- and HF^∞. We will focus on $\widehat{HF}(M)$, a finitely generated Abelian group associated to M.

A *genus g-handlebody* is a 3-dimensional handlebody consisting of one 0-handle and g 1-handles. A decomposition $M = H_0 \underset{\Sigma}{\cup} H_1$ where each H_i is a genus g-handlebody glued to each other by a diffeomorphism of $\partial H_0 = \partial H_1 = \Sigma$, is called a *Heegaard decomposition of genus* g of M.

Theorem 2.5.1. *[48] Every closed connected orientable* 3*-manifold admits a Heegaard decomposition.*

One can find the proof of the theorem in [47]. Now suppose $M = H_0 \underset{\Sigma}{\cup} H_1$ is a Heegaard decomposition of genus g. Then the triple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is called a *Heegaard diagram* if $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are two systems of disjoint simple closed curves $\alpha_1, \alpha_2, \ldots, \alpha_g$ and $\beta_1, \beta_2, \ldots, \beta_g$ respectively and the curves α_i's (resp. β_j's) form homologically independent sets in ∂H_0 (resp. ∂H_1). Curves satisfying these conditions are called *attaching circles*. Let $Sym^g(\Sigma)$ be the unordered g-tuples consisting of points in Σ and let $\mathbb{T}_{\boldsymbol{\alpha}} = \alpha_1 \times \cdots \times \alpha_g$, $\mathbb{T}_{\boldsymbol{\beta}} = \beta_1 \times \cdots \times \beta_g$ in $Sym^g(\Sigma)$.

Fix a Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ of M and a point z in Σ away from α_i's and β_j's. Consider $\widehat{CF}(M)$ as the free Abelian group generated by the intersection points in $\mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ with $\mathbb{Z}/2\mathbb{Z}$ coefficients. Given $x, y \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$, *a holomorphic disk* connecting x and y is a holomorphic map $\Phi : \mathbb{D}^2 \to Sym^g(\Sigma)$ where $\Phi(-i) = x$, $\Phi(i) = y$, $\Phi(\{Re(z) \geq 0\}) \subset \mathbb{T}_{\boldsymbol{\alpha}}$ and $\Phi(\{Re(z) \leq 0\}) \subset \mathbb{T}_{\boldsymbol{\beta}}$. Here, we regard the complex structure on $Sym^g(\Sigma)$ inherited from Σ. Finally, we roughly define *differential map* $\partial : \widehat{CF}(M) \to \widehat{CF}(M)$ as $\partial(x) = \sum_{y_i \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}} n_i y_i$ where n_i is the number of holomorphic disks connecting x and y that don't involve z. Note that $\widehat{CF}(M)$ forms a chain complex equipped with ∂.

Definition 2.5.2. *The homology group of the chain complex $\widehat{CF}(M)$ is called the (hatted) Heegaard Floer homology of M, denoted by $\widehat{HF}(M)$.*

There is a notion called $spin^c$ structure on the tangent bundle of a manifold that can be illustrated as a generalization of orientation. In [53], Turaev showed that there is a one-to-one correspondence between $spin^c$ structures on a closed connected oriented 3-manifold M and equivalence classes of non-vanishing vector fields on M.

Remark 2.5.3. *Every intersection point $x \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ corresponds to a $spin^c$ structure t_x. If there is a holomorphic disk between two intersection points, they have the same $spin^c$ structures. This yields a chain complex $\widehat{CF}(M, t)$ having the homology group $\widehat{HF}(M, t)$. One can see that $\widehat{HF}(M)$ splits as $\bigoplus_{t \in Spin^c(M)} \widehat{HF}(M, t)$ where $Spin^c(M)$ denotes the set of $spin^c$ structures on M. We refer to the lecture notes [46] for details.*

$\widehat{HF}(M)$ determines an invariant for contact structures on M. Given a contact structure ξ, this gives a $spin^c$ structure t_ξ defined as the equivalence class of a nonzero vector field transverse to ξ everywhere. In [44], Ozsváth and Szábó describe an invariant called *contact invariant* or *contact class* $c(M, \xi)$ as an element of $\widehat{HF}(-M, t_\xi)$, where $-M$ denotes the reversed oriented M, in the following manner:

Ozsváth and Szábó also introduced *knot Floer homology* which is an extension of Heegaard Floer homology. This time, fix two points w and z disjoint from a Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ of M. This yields a knot K in M which is obtained by union of push-offs of two arcs connecting w and z in Σ such that one lies in a tubular neighborhood of α_i's and so does the other in a tubular neighborhood of β_j's. Define the chain complex $\widehat{CFK}(M, K)$ similar to $\widehat{CF}(M)$. Now, differential counts the holomorphic disks disjoint from w and z. $\widehat{CFK}(M, K)$ has a $\mathbb{Z}$-filtration. Let $\widehat{CFK}(M, K, n)$ be the subcomplex with filtration level less than or equal to n. Then it corresponds to the homology group $\widehat{HFK}(M, K, n)$ so-called knot Floer homology of (M, K) with filtration $\leq n$.

Theorem 2.5.4. *[44] If K is a fibered knot of genus g in M, then*

$$\widehat{HFK}(-M, K, -g) \simeq \mathbb{Z}.$$

The reader can find the proof and details about knot Floer homology in [44].

Definition 2.5.5. *Let K be a fibered knot of genus g and let ξ be the contact structure supported by the open book of K. Then the homology class of the image of the generator in $\widehat{HFK}(-M, K, -g)$ under the inclusion*

$$\iota : \widehat{CFK}(-M, K, -g) \to \widehat{CF}(-M)$$

is called the contact class $c(M, \xi)$. Note that it can be also seen in $\widehat{HF}(-M, t_\xi)$.

The contact class holds much information about the contact structure.

Theorem 2.5.6. *[44] If ξ is an overtwisted contact structure in M, then $c(M, \xi)$ vanishes.*

Theorem 2.5.7. *[44] If (M, ξ) is a Stein fillable contact manifold then $c(M, \xi)$ does not vanish, hence ξ is tight. In particular $\widehat{HF}(S^3)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and the contact class on it is nonzero.*

Lemma 2.5.8. *[35] Let ξ be the contact structure on $S^1 \times S^2$ obtained via contact $(+1)$-surgery along the Legendrian unknot with Thurston bennequin number -1 in (S^3, ξ_{std}). Then $c(S^1 \times S^2, \xi)$ is nonzero.*

Let M_1 be a 3-manifold obtained via n-surgery along a knot K in M_0 where $n \in \mathbb{Z}$. It is easy to see that there is a cobordism W from M_0 to M_1 which is constructed

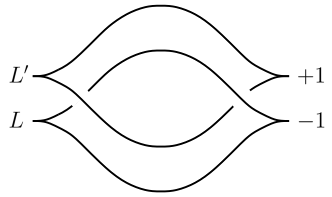


Figure 2.10: A cancelling pair. A Legendrian unknot and its Legendrian push-off.

by attaching a 2-handle along K with framing n to $M_0 \times I$. Suppose that s is a $spin^c$ structure on W such that $s|_{M_0} = t_0$ and $s|_{M_1} = t_1$. Then (W, s) associates a homomorphism

$$F_{W,s} : \widehat{HF}(M_0, t_0) \to \widehat{HF}(M_1, t_1).$$

Let L be a Legendrian knot in (M, ξ) and let $(M_L^{\pm}, \xi_L^{\pm})$ be the result of contact (± 1)-surgery along L.

Theorem 2.5.9. *[35, 44] $F_{-W}(c(M, \xi)) = c(M_L^+, \xi_L^+)$ where $-W$ be the reversed oriented cobordism induced by the surgery and*

$$F_{-W} = \sum_{s \in Spin^c(-W)} F_{-W,s} : \widehat{HF}(-M, s|_{-M}) \to \widehat{HF}(-M_L^+, s|_{-M_L^+}).$$

So we have the following immediate corollary:

Corollary 2.5.10. *If $c(M, \xi)$ is nonzero, so is $c(M_L^-, \xi_L^-)$.*

Proof. Consider a Legendrian push-off of L, say L', in (M_L^-, ξ_L^-) and apply contact $(+1)$-surgery along L' (see Figure 2.10). By Proposition 8 of [8], the contact surgeries cancel each other which means that we get (M, ξ). Hence we have a homomorphism $F_{-W} : \widehat{HF}(-M_L^-) \to \widehat{HF}(-M)$ sending $c(M_L^-, \xi_L^-)$ to $c(M, \xi)$ by the previous theorem. This finishes the proof. □

2.6 Hyperbolic 3-Manifolds

In this section, we review some basics of hyperbolic 3-manifolds and some groundbreaking theorems.

A *hyperbolic* 3-*manifold* M is a Riemannian 3-manifold with negative constant sectional curvature. Unless otherwise stated, a hyperbolic manifold M will be assumed to have a complete metric giving a finite volume.

2.6.1 Nielsen-Thurston Classification

An isotopy class of a compact orientable surface self-homeomorphism was characterized by Nielsen-Thurston. We focus on the classification of homeomorphisms on a closed surface. We give definitions of the classes and some backgrounds before stating the classification.

A *measured foliation* on a closed surface Σ is a singular foliation $\mathcal{F}$ with a transverse measure (see [24] and [50] for details).

Definition 2.6.1. *Let f be an element of $MCG(\Sigma)$ where Σ is a closed orientable surface. Then f is called;*

- *periodic if f^k is the identity of $MCG(\Sigma)$ for some k.*
- *reducible if there exists a finite set of disjoint homotopically nontrivial simple closed curves on Σ fixed by f.*
- *pseudo-Anosov if there are a transverse pair of measured foliations on Σ, say $\mathcal{F}_u$ and $\mathcal{F}_s$, and a real number $\lambda > 1$ such that*

$$f(\mathcal{F}_u) = \lambda \mathcal{F}_u, f(\mathcal{F}_s) = \frac{1}{\lambda}\mathcal{F}_s.$$

Theorem 2.6.2. *[41, 50] A self-homeomorphism f of a closed surface Σ is isotopic to either periodic or reducible or pseudo-Anosov homeomorphism. Additionally, a pseudo-Anosov homeomorphism is neither periodic nor reducible.*

When Thurston classified diffeomorphism of surfaces, he proved the following theorem:

Theorem 2.6.3. *[50] Let M_f be a surface bundle over the circle with monodromy f. Then,*

$$f \text{ is pseudo-Anosov} \Leftrightarrow M_f \text{ is a hyperbolic 3-manifold.}$$

Now, we state a fundamental theorem of Thurston which is called the Hyperbolic Dehn Surgery Theorem. A complete hyperbolic 3-manifold with finite volume is called *cusped* if it has no boundary. A cusped hyperbolic 3-manifold has n-cusps if it is the interior of a compact manifold with boundary a disjoint union of n-tori.

Theorem 2.6.4. *[51] Let M be a cusped hyperbolic 3-manifold with n-cusps. Let T_i's be the tori that form the boundary of a compact manifold whose interior is M where $i \in \{1, \ldots, n\}$. Denote by $M(r_1, r_2, \ldots, r_n)$ the manifold which is obtained from M by filling T_i with slopes $r_i = \frac{p_i}{q_i}$. Then $M(r_1, r_2, \ldots, r_n)$ is also a hyperbolic 3-manifold except finite values for r_i.*

CHAPTER 3

TIGHT CONTACT STRUCTURES ON SOME HYPERBOLIC 3-MANIFOLDS

3.1 Family of Infinitely Many Hyperbolic 3-Manifolds

In this section, we will build some hyperbolic 3-manifolds based on the work of Thurston [50, 51] and Fathi [23].

Consider the closed oriented surface Σ_g with $g \geq 2$ and the simple closed curves $a_1, a_2, \cdots, a_{2g+1}$ on Σ_g as shown in Figure 3.1.

Let $\phi \in MCG(\Sigma_g)$ be the isotopy class of the homeomorphism

$$t_{a_1}^m t_{a_2} \cdots t_{a_{2g}} t_{a_{2g+1}}^n. \tag{3.1}$$

Let M_ϕ stand for the Σ_g-bundle with monodromy ϕ. Since ϕ has a fixed point for each $m, n \in \mathbb{Z}$, we can ensure that M_ϕ has a section. Denote by $M_\phi(r)$ the surgered manifold obtained via rational r-surgery along a section of M_ϕ. Now we can state the main theorem of this section.

Theorem 3.1.1. *$M_\phi(r)$ is a hyperbolic 3-manifold for all $m \in \mathbb{Z}$ and $r \in \mathbb{R}$ but finitely many of m's and r's.*

In 1987, Fathi [23] gave a way to construct pseudo-Anosov homeomorphisms as products of Dehn twists.

Let α and β be two simple closed curves in a compact oriented surface Σ. We will denote $\iota(\alpha, \beta)$ the geometric intersection number of the representatives of α and β in minimal positions.

Fact 3.1.2. *For any two simple closed curves α and β, we have*

$$\iota(\alpha, \beta) = 0 \iff t_\alpha(\beta) = \beta$$

Proposition 3.1.3. *[22] For any two simple closed curves α and β with $\iota(\alpha, \beta) = 1$, we have $t_\alpha t_\beta(\alpha) = \beta$.*

Definition 3.1.4. *We shall say that a set of simple closed curves $\alpha_1, \alpha_2, \cdots, \alpha_k$ fills a compact oriented surface Σ if $\Sigma \setminus \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ is the disjoint union of some topological disks and annuli.*

Theorem 3.1.5. *[23] Let $f \in MCG(\Sigma)$ and let α be a simple closed curve in Σ. If the orbit set of α under f; i.e., $\{f^i(\alpha) \mid i \in \{0, 1, \ldots\}\}$, fills Σ, then $t_\alpha^m f$ is a pseudo-Anosov diffeomorphism except for at most 7 consecutive values of m.*

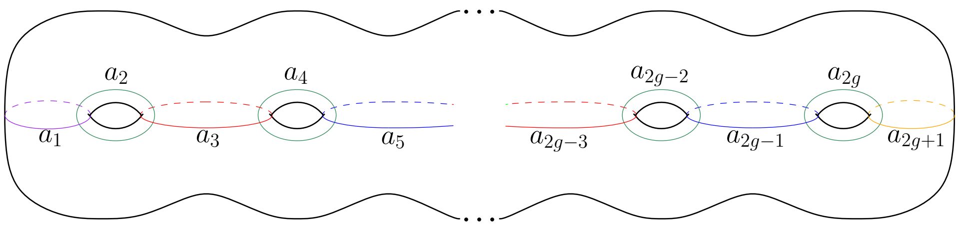


Figure 3.1: Simple closed curves on the surface Σ_g.

Lemma 3.1.6. *Let ϕ be the class in $MCG(\Sigma_g)$ as described in (3.1) above. Then ϕ is pseudo-Anosov for any integer n and for all but at most 7 consecutive values of m.*

Proof. Let α represent the curve a_1 and let f be the product of Dehn twists

$$t_{a_1} t_{a_2} \cdots t_{a_{2g}} t^n_{a_{2g+1}}.$$

Then, regarding Fact 3.1.2 and Proposition 3.1.3, one can conclude that

$f(\alpha) = t_{a_1} t_{a_2} \cdots t_{a_{2g}} t^n_{a_{2g+1}}(a_1) = t_{a_1} t_{a_2}(a_1) = a_2$,
$f^2(\alpha) = t_{a_1} t_{a_2} \cdots t_{a_{2g}} t^n_{a_{2g+1}}(a_2) = t_{a_1} t_{a_2} t_{a_3}(a_2) = t_{a_1}(a_3) = a_3$, and inductively,

$f^i(\alpha) = t_{a_1} t_{a_2} \cdots t_{a_{2g}} t^n_{a_{2g+1}}(a_i) = t_{a_1} t_{a_2} \cdots t_{a_i} t_{a_{i+1}}(a_i) = t_{a_1} t_{a_2} \cdots t_{a_{i-1}}(a_{i+1}) = a_{i+1}$ for all $i \in 1, 2, \ldots, 2g-1$.

It is clear that $\Sigma_g \setminus \{a_1, a_2, \ldots a_{2g}\} = \Sigma_g \setminus \{\alpha, f(\alpha), \ldots, f^{2g-1}(\alpha)\}$ is a topological disk. Hence the orbit set of α under f fills Σ_g. By Theorem 3.1.5, we know that $t_\alpha^m f$ is pseudo-Anosov homeomorphism for all integers n and for all but at most 7 consequtive values of m.

□

Proof of Theorem 3.1.1 . As stated in Theorem 2.6.3, all M_ϕ's are hyperbolic 3-manifolds since ϕ is a pseudo-Anosov homeomorphism when $m, n \in \mathbb{Z}$ except at most 7 consequtive values of m. According to Hyperbolic Dehn Surgery Theorem 2.6.4, we can say all $M_\phi(r)$'s are hyperbolic 3-manifolds by avoiding exceptional slopes r and 7 successive integers m.

□

3.2 A Family of Infinitely Many Tight Contact Manifolds

3.2.1 Transverse Surgery and Its Relation with Contact Surgery

In this subsection, we shall give definitions of admissible and inadmissible transverse surgery and their relations with contact surgery due to [5] and [7].

Given a transverse knot T in a contact 3-manifold (M, ξ), there is a neighborhood N of T such that N is contactomorphic to $S^1 \times \{r \leq a\}$ in

$$(S^1 \times \mathbb{R}^2, \xi_0 = \ker(cos f(r)dz + f(r) sin f(r) d\theta))$$

for some a where $z \in S^1$, $(r, \theta) \in \mathbb{R}^2$, $f : [0, \infty) \to [0, \pi)$ is an increasing onto map, and the contactomorphism sends T onto $\{r = 0\}$. So the characteristic foliation on a torus $\{r = r_0\}$ with respect to ξ_0 is the union of leaves $f(r_0)\mu - cot(f(r_0))\lambda$ where μ is the meridian and λ is the longitude given by $\{\theta = \theta_0\}$ for some θ_0. Hence the slope of the leaves of the characteristic foliation on $\{r = r_0\}$ is $\frac{-cot(f(r_0))}{f(r_0)}$ which means linear.

Definition 3.2.1. ***Contact cut***

Let M be a 3-manifold with torus boundary Σ. Consider a smooth free S^1-action *on Σ and the quotient of M, say M', under the action. Then the orbits of the action are embedded curves on Σ with some slope r. Notice that M' can be topologically obtained via r-Dehn filling along Σ. Now, let ξ be a contact structure on M such that the slope of the characteristic foliation Σ_ξ is r. Since the slope is linear, one can find a diffeomorphism from Σ to a torus in $(S^1 \times \mathbb{R}^2, \xi_0)$ preserving the characteristic foliations. By Theorem 2.1.12, there is a contactomorphism between a neighborhood of Σ, say N, in (M, ξ) and a neighborhood of the torus in $(S^1 \times \mathbb{R}^2, \xi_0)$, say $N_{s,r} = \overline{N_s - N_r}$ where s is is the slope of the boundary of the solid torus N_s and ∂N_r is contactomorphic to Σ.*

Now, extend the S^1-action on N such that each level torus in N is invariant under the action. Suppose that the contact form α in N describing ξ is invariant under the action. Hence, the induced contact form α' under the action defines the contact structure ξ' on M'. Here, (M', ξ') is called the contact cut of (M, ξ) along Σ.

Definition 3.2.2. ***Admissible transverse surgery***
Let T be a transverse knot in (M, ξ). Consider the solid torus N_s in $(S^1 \times \mathbb{R}^2, \xi_0)$ contactomorphic to a neighborhood N of T where N_s has the boundary torus T_s with slope s. Let $M_T(r)$ be the manifold obtained via topological r-surgery along T where $r \in (-\infty, s)$. The contact structure $\xi_T(r)$ on $M_T(r)$ is defined as follows:

Since the slope of the characteristic foliation is an increasing function, one can think that the solid torus N_r is included in N_s where r is the slope of the leaves of the characteristic foliation in ∂N_r. It is easy to define a free S^1-action on ∂N_r such that the orbits are the leaves of the characteristic foliation. After contact cut in $(\overline{M - \partial N_r}, \xi|_{\overline{M - \partial N_r}})$ along ∂N_r, the resulting contact manifold $(M_T(r), \xi_T(r))$ is called the admissible transverse r-surgery along T in (M, ξ).

Theorem 3.2.3. *[5] Given a Legendrian knot $L \subset (M, \xi)$, the obtained contact structure after Legendrian surgery along L can be obtained as an admissible transverse surgery along a transversal push-off of L. On the contrary, if T is a transverse knot in (M, ξ), the admissible transverse r-surgery along T with $r < s$ where s is the slope of the boundary of the neighborhood of T as mentioned above, can be obtained via Legendrian surgery along some Legendrian link in the neighborhood.*

In [7], Conway found an equivalence between inadmissible transverse surgery, defined below, and positive contact surgery.

Definition 3.2.4. ***Indmissible transverse surgery***
Let T be a transverse knot in (M, ξ) and let N be a neighborhood of T which is contactomorphic to a solid torus in $(S^1 \times \mathbb{R}^2, \xi_0 = \ker(cosrdz + rsinrd\theta))$ as in Definition 3.2.2, in this case, the slope of curves in the characteristic foliation is $-\infty$; i.e., meridional and it is taken r instead of $f(r)$ for simplicity. Fix a framing λ of T whose image under the contactomorphism is $S^1 \times \{(0,0)\}$. Then, glue a thickened torus, with the contact structure ξ_0 on it, to $\overline{M - \partial N}$ such that contact planes twist out at some slope b. Finally, performing contact cut on the boundary gives a new contact manifold which is called an inadmissible transverse surgery along T.

Theorem 3.2.5. *[7] Inadmissible transverse r-surgery along a transverse knot T corresponds to the contact (r)-surgery on a Legendrian push-off of T where all stabilizations are chosen to be negative as shown in the section.*

Recall that if a knot is the binding (resp. a binding component) of an open book decomposition, then it is called *fibered* (resp. *integrally fibered*) knot. *The genus* of a null-homologous knot K in a 3-manifold is the minimum number of genus of any Seifert surface for K.

Theorem 3.2.6. *[7] Given an integrally fibered transverse knot T where the contact structure ξ supported by T is tight (resp. has nonzero contact invariant), the inadmissible transverse r-surgery along T for $r > 2g-1$, where g is the genus of T, also admits a tight contact structure (resp. nonzero contact invariant).*

3.2.2 Main Theorem

Theorem 3.2.7. *Assume that $M_\phi(r)$ is a 3-manifold as constructed in the Section 3.1. Then $M_\phi(r)$ admits tight contact structure for any positive integers m, n and for any rational number $r \neq 2g - 1$.*

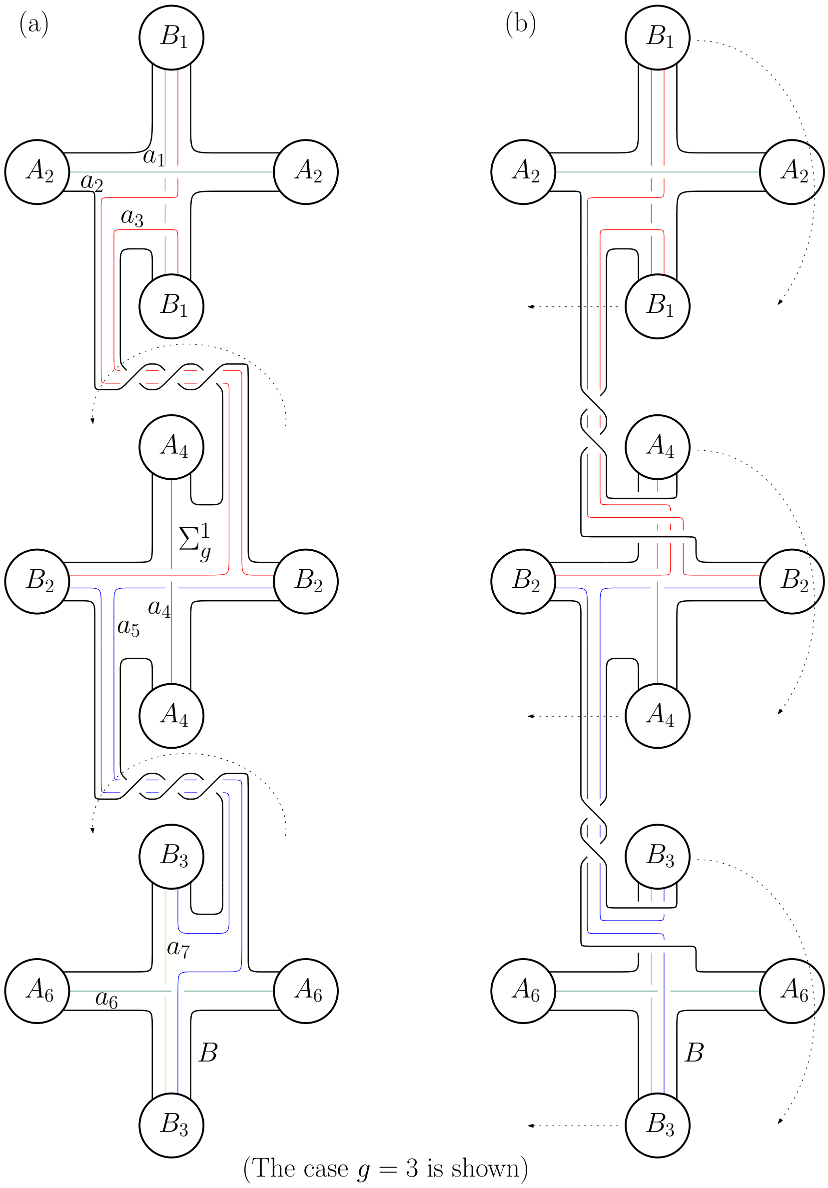


Figure 3.2: (a) A Lefschetz fibration structure on X_ϕ. (b) Another handle description of X_ϕ obtained by flipping the connecting bands in (a).

Proof. We will analyze the proof with respect to the genus g of the fiber Σ_g. First assume $g \geq 3$ odd. Note that conjugation of the monodromy by any class of $MCG(\Sigma_g)$

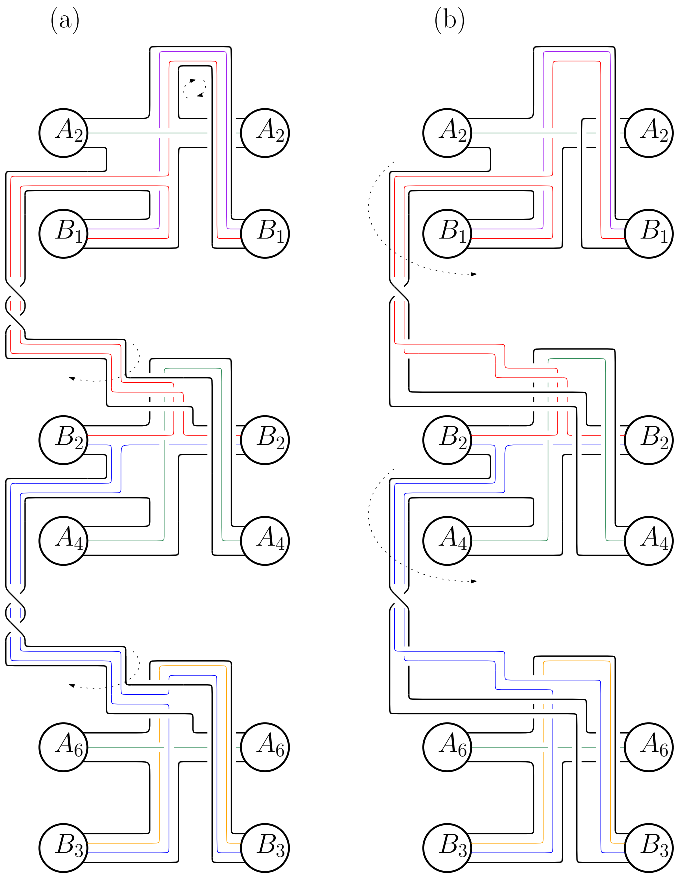


Figure 3.3: Other handle diagrams for X_ϕ.

does not change the mapping torus up to diffeomorphism. Since

$$t_{a_2}\cdots t_{a_{2g}} t^n_{a_{2g+1}} t^m_{a_1} = t^{-m}_{a_1}\phi\, t^m_{a_1}$$

we may replace ϕ in (3.1) with the mapping class $t_{a_2}\cdots t_{a_{2g}} t^n_{a_{2g+1}} t^m_{a_1}$. Also observe that $M_\phi(r)$ can be also obtained from a Dehn surgery on the binding of an open book decomposition whose page is Σ^1_g (punctured Σ_g) and monodromy can be still assumed to be $\phi \in MCG(\Sigma^1_g)$. We will construct the required contact structure ξ on $M_\phi(r)$ via Dehn surgery on the open book decomposition (Σ^1_g, ϕ) along its binding.

By Theorem 2.4.10, the contact structure, say ξ_0, (before the surgery along the binding) supported by (Σ^1_g, ϕ) is Stein fillable. More precisely, consider the handlebody diagram of the smooth 4-manifold X_ϕ given in Figure 3.2-(a) (in the case of genus 3) with "$2g$" 1-handles and "$m + n + 2g - 1$" 2-handles. Note that Figure 3.2-(a) describes a Lefschetz fibration structure on X_ϕ for $g = 3$ with a regular fiber Σ^1_g and the vanishing cycles $a_1, a_2, ..., a_{2g+1}$. There are n copies for a_{2g+1} and m copies for a_1

(not drawn for simplicity). All coefficients are -1 with respect to the framing given by the page Σ_g^1. We remark that no handle is attached along the binding of the induced open book (Σ_g^1, ϕ) on the boundary ∂X_ϕ which is realized as B in Figure 3.2-(a).

Next starting from the topological description in Figure 3.2-(a) of X_ϕ, we'll get a diagram describing a Stein structure on X_ϕ inducing ξ_0 as follows: First we flip the twisted bands over the 1-handles as pointed out in Figure 3.2-(a) and get Figure 3.2-(b). Figure 3.3-(a) gives another handle description of X_ϕ obtained by moving the feet of 1-handles as indicated by the dotted arrows in Figure 3.2-(b). Then flip the bands as shown in Figure 3.3-(a) to get rid of one more left half twist for each band (see Figure 3.3-(b)), and obtain Figure 3.4-(a) by flipping the connecting bands over the feet of 1-handles suggested by the dotted arrows in Figure 3.3-(b). Figure 3.4-(b) defines a Stein structure on X_ϕ obtained by putting the attaching circles in part (a) into Legendrian positions, where a Legendrian realization L_0 of B in the tight contact boundary ∂X_ϕ is also provided. All coefficients (except on L_0) are -1 with respect to Thurston-Bennequin (contact) framing in ∂X_ϕ and no handle is attached along L_0. Note that $tb(L_0) = 2$ (the case $g = 3$ is shown). In the general case, $tb(L_0) = g - 1$. Finally, we use the trick in Figure 2.9 to obtain a Legendrian representation L of B with $tb(L) = 2g - 1$ (see Figure 3.5). Note that Figure 3.5 describes the same Stein structure on X_ϕ as in Figure 3.4-(b).

Now if $g \geq 2$ is even, we replace the monodromy ϕ with $t^n_{a_{2g+1}} t_{a_2} \cdots t_{a_{2g}} t^m_{a_1}$ since

$$t^n_{a_{2g+1}} t^{-m}_{a_1} \phi \, t^{-n}_{a_{2g+1}} t^m_{a_1} = t^n_{a_{2g+1}} t_{a_2} \cdots t_{a_{2g}} t^m_{a_1}.$$

Then starting from the handlebody diagram given in Figure 3.6-(a) (where the case $g = 4$ is shown) and following the moves as in the case of odd genus, one can get Figure 3.6-(b) describing a Stein structure realizing a Legendrian representation L with $tb(L) = 2g - 1$ as in Figure 3.5. One should note that we need to consider different monodromies (but still giving the same mapping torus) depending on the parity of g to make the contact and the page framing on any attaching circle coincide.

Now (in any case of g) we first (Legendrian) slide (Stein) 2-handle corresponding a_3 over the ones represented by the curves $a_1, a_5, a_7, ..., a_{2g+1}$, and then cancel the 2-handles represented by $a_5, a_7, ..., a_{2g-1}$ with the corresponding 1-handles. Second, we

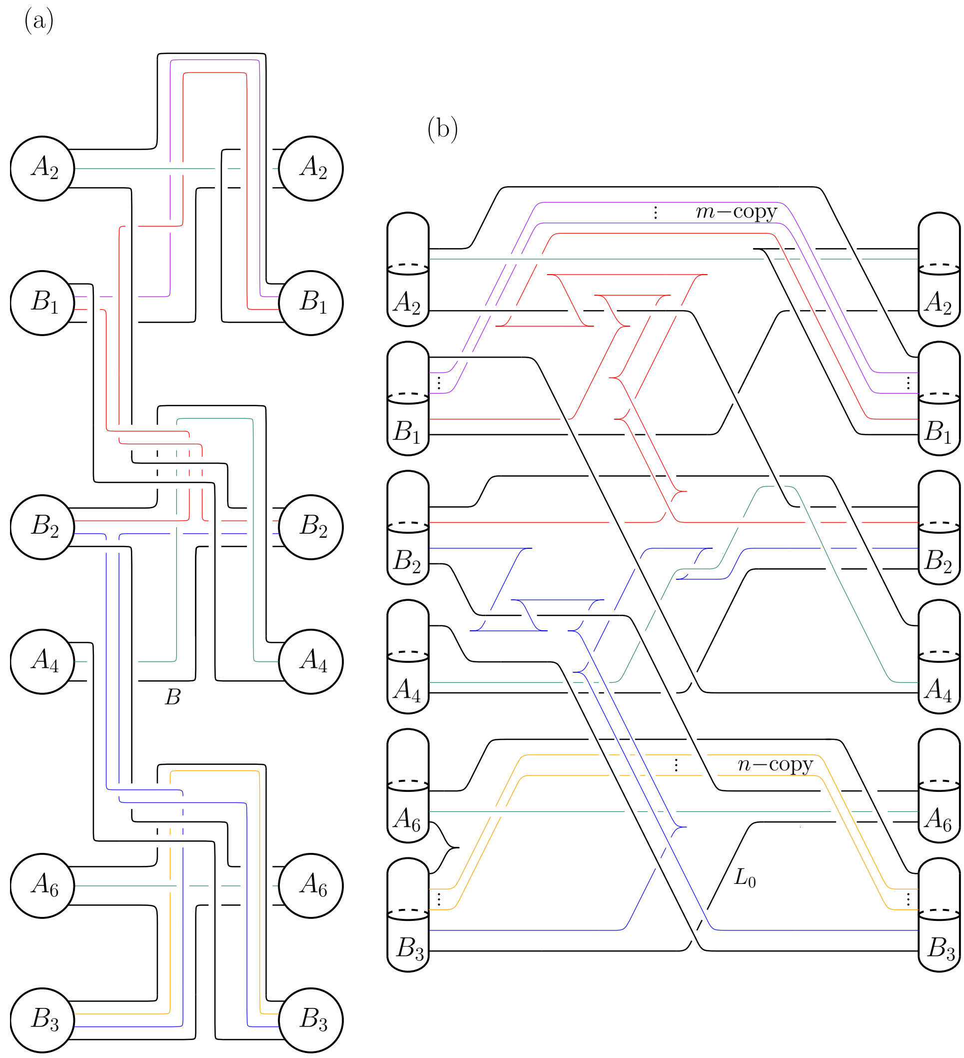


Figure 3.4: (a) The last handle diagram for X_ϕ. (b) The Stein structure on X_ϕ obtained from the Lefschetz fibration structure in (a).

(Legendrian) slide 2-handles represented by the curves a_1 and a_{2g+1} over a fixed one (chosen from each family in Figure 3.5 / Figure 3.6-(b)), and then cancel 1-handles B_1 and B_g with the chosen 2-handles corresponding a_1 and a_{2g+1} respectively. Also we cancel each 1-handle A_i with the 2-handle corresponding the curve a_i for each i even. As a result, we obtain another (but equivalent) Stein structure on X_ϕ which can be also considered as the contact surgery diagram for ξ_0 on ∂X_ϕ. Finally, we set $r' = r - 2g + 1$ and perform contact (r')-surgery along $L \subset (\partial X_\phi, \xi_0)$ to get a contact structure ξ on $M_\phi(r)$ whose diagram is given in Figure 3.7 (where we use continued fractions).

First suppose $r' = r - 2g + 1 < 0$. Any contact surgery with negative contact

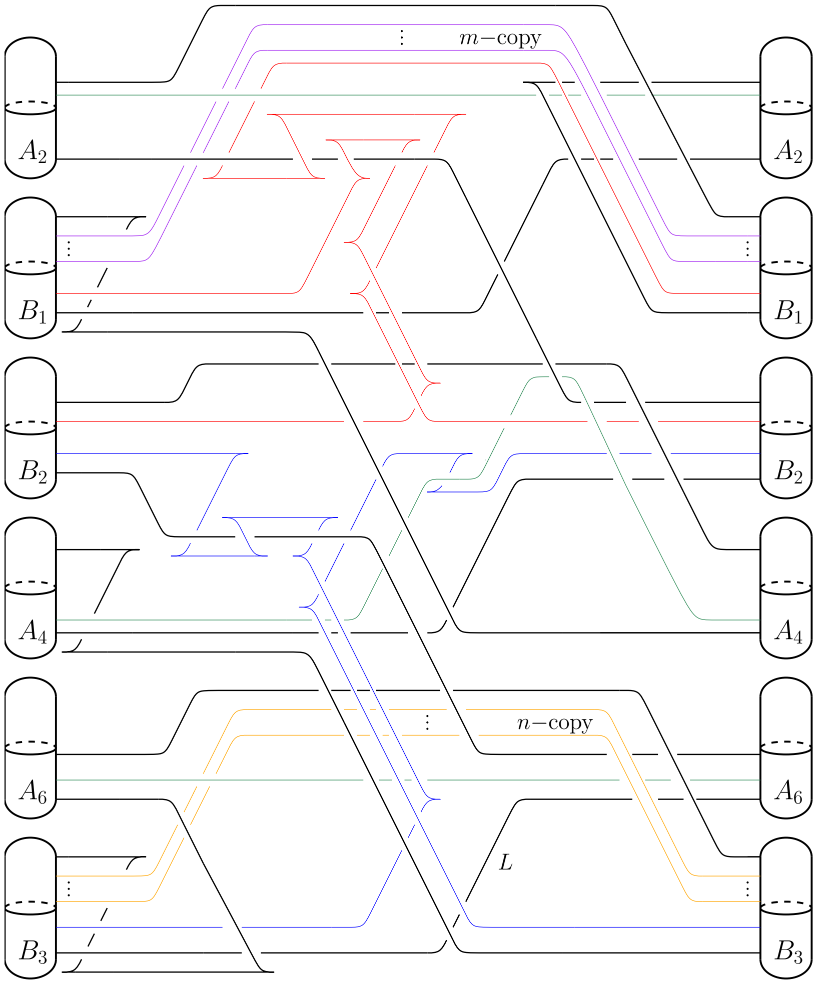


Figure 3.5: The same Stein structure on X_ϕ as in Figure 3.4-(b).

framing can be converted to a sequence of contact (-1)-surgeries and (-1)-surgeries preserve Stein fillability by Theorem 2.4.12. Thus $(M_\phi(r), \xi)$ is Stein fillable hence tight according to Theorem 2.5.7.

Now let $r' = r - 2g + 1 > 0$. By Thurston-Winkelnkemper construction in Theorem 2.3.6, they also showed that the binding B is transverse to the contact structure supported by the open book decomposition, i.e it is an integrally transverse fibered knot. Also since ∂X_ϕ is Stein fillable, ξ_0 has nonzero contact invariant according to Ozsváth and Szabó, see Theorem 2.5.7. By Theorem 3.2.5, we know that contact (r')-surgery corresponds to inadmissible transverse r-surgery along B. As a result of Conway's work (see Theorem 3.2.6), it is concluded that $(M_\phi(r), \xi)$ has nonzero contact invariant (hence tight). This finishes the proof of Theorem 3.2.7.

□

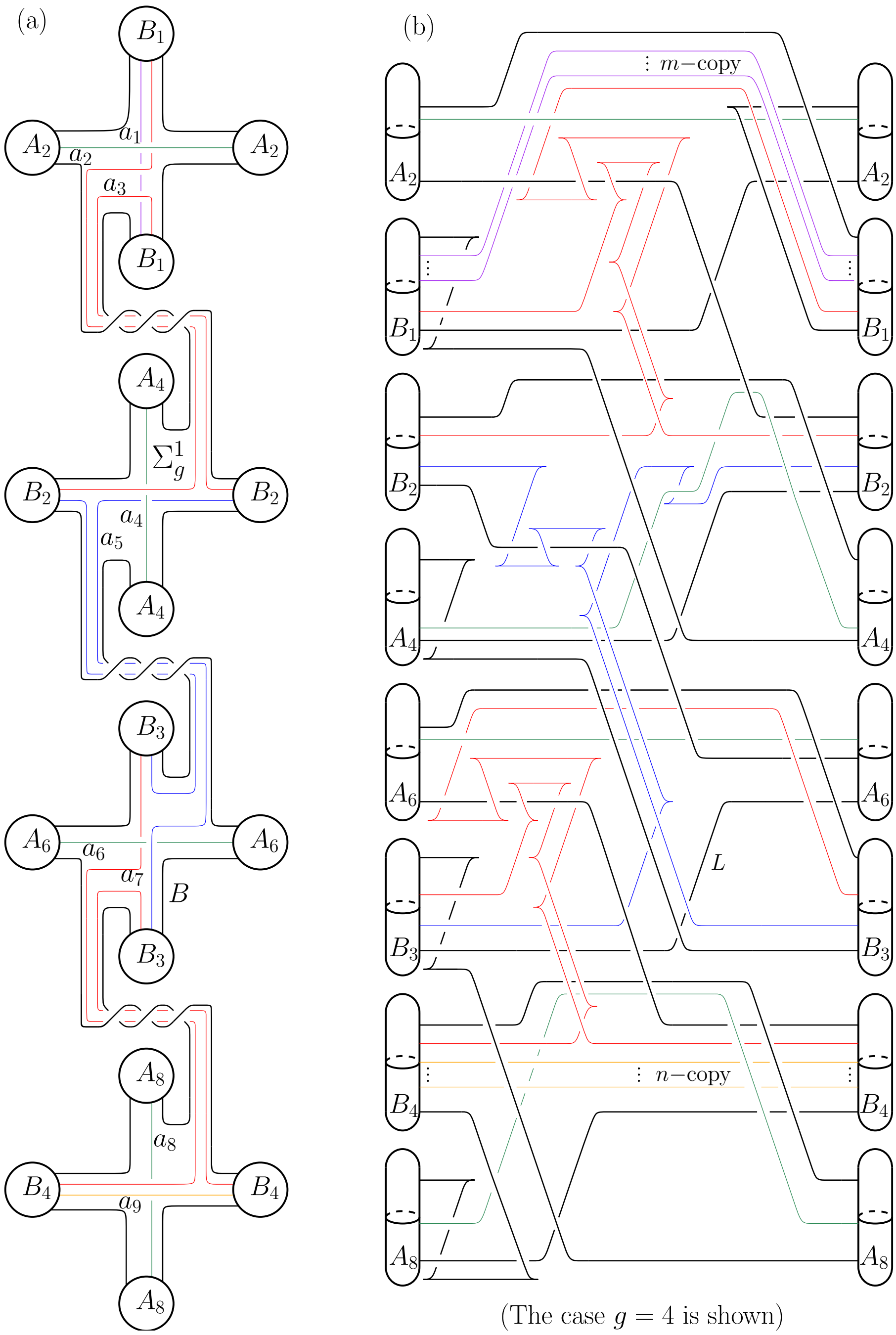


Figure 3.6: (a) A Lefschetz fibration structure on X_ϕ when $g = 4$. (b) The Stein structure on X_ϕ obtained from the Lefschetz fibration structure in (a).

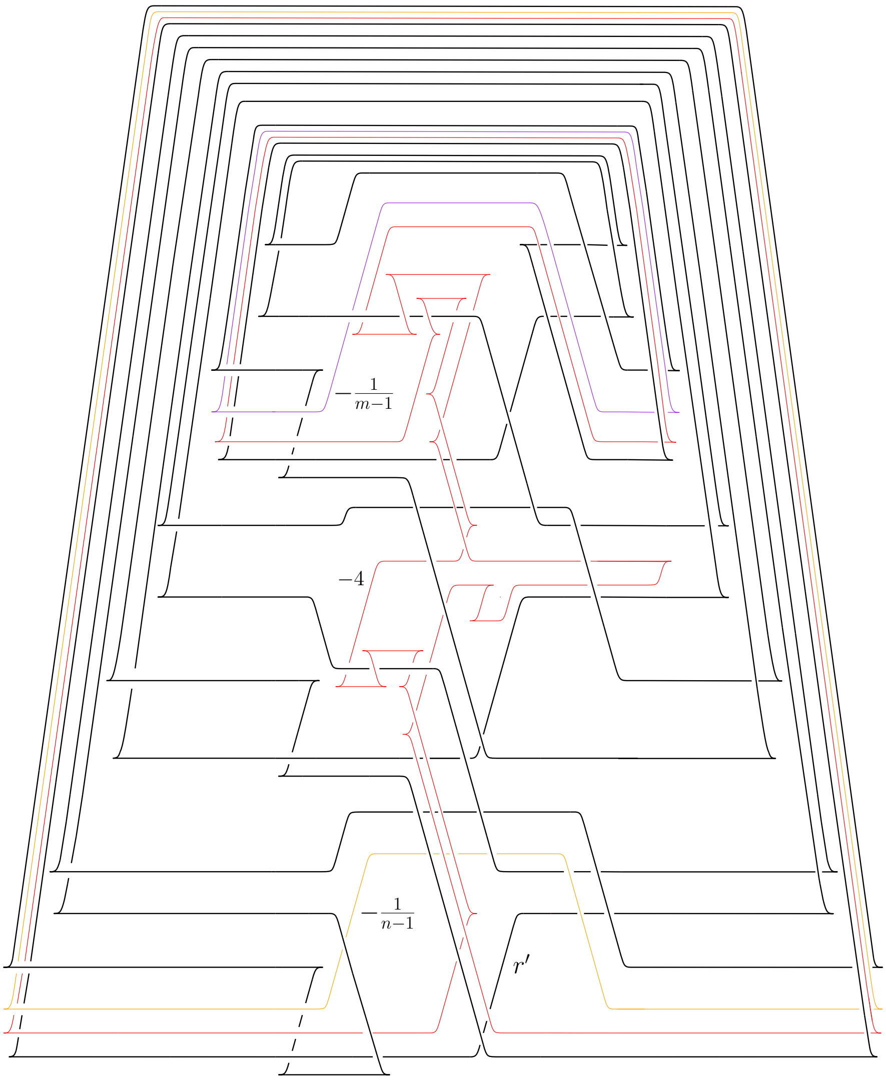


Figure 3.7: The contact 3-manifold $(M_\phi(r), \xi)$. (The case $g = 3$ is shown.)

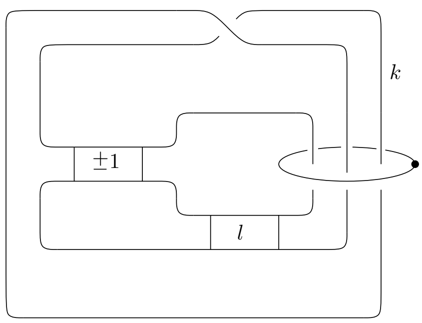
CHAPTER 4

TIGHT CONTACT STRUCTURES ON SOME HOMOLOGY SPHERES

In this chapter, some homology spheres admitting tight contact structures are exhibited and we will mention about its importance. There is a close relation between foliation theory and contact geometry.

A *taut foliation* is a foliation such that there exists a circle intersecting with every fiber transversely. Gabai in [25] showed that if the first Betti number b_1 of an irreducible 3-manifold is positive, then the manifold admits a taut foliation. Eliashberg and Thurston in [17], gave a way to construct fillable (hence tight) contact structures on a manifold admitting taut foliation. This leaves us the question whether every homology sphere admits a tight contact structure or not.

In [2], Akbulut and Kirby generalized Mazur manifolds (see [40]) which are contractible 4-manifolds bounded by homology spheres different from 3-sphere. Their construction includes one 0-handle, one 1-handle and one 2-handle which is shown in Figure 4.1. Here, $k \in \mathbb{Z}$ and $l \in \mathbb{Z}$ indicates l full twist (left or right depending on the sign of l).



$W^{\pm}(l, k)$

Figure 4.1: Generalized Mazur Manifolds constructed by Akbulut and Kirby.

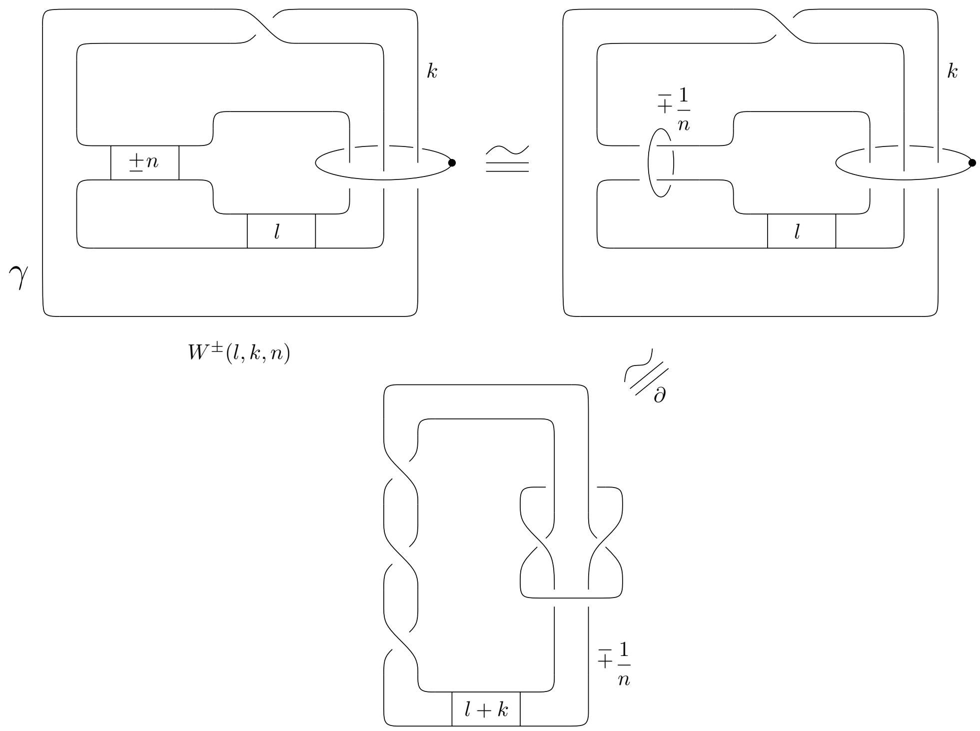


Figure 4.2: Equivalent Kirby diagrams of $W^{\pm}(l,k,n)$ and a Kirby diagram of $\partial(W^{\pm}(l,k,n))$.

We will construct a generalization of these type of manifolds. Let $W^{\pm}(l,k,n)$ be a 4-manifold obtained by one 0-handle, one 1-handle and one 2-handle as indicated in Figure 4.2 where $n \in \mathbb{N}$ and $k, l \in \mathbb{Z}$.

Proposition 4.0.8. *$W^{\pm}(l,k,n)$ is diffeomorphic to $W^{\pm}(l+1,k-1,n)$ for all l,k,n.*

Proof. First we add one 1-handle and one 2-handle to the diagram of $W^{\pm}(l,k,n)$ which is diffeomorphic to $W^{\pm}(l,k,n)$. Then we slide the 2-handle, the 1-handle and the 2-handle attached afterwards respectively as shown in Figure 4.3. Finally we get the the diagram of $W^{\pm}(l+1,k-1,n)$. □

Proposition 4.0.9. *$W^{\pm}(l,k,n)$ is a contractible 4-manifold for all l,k,n.*

Proof. The proof is based on the proof of Proposition 1 in [40]. Let W_1 be the handlebody consisting one 0-handle and one 1-handle. Then it is diffeomorphic to $S^1 \times \mathbb{D}^3$. Consider the 4-dimensional 2-handle $H_2 = \mathbb{D}^2 \times \mathbb{D}^2$ and the attaching circle γ in $\partial(S^1 \times \mathbb{D}^3) = S^1 \times S^2$ which is the image of the circle $\gamma' = \partial\mathbb{D}^2 \times \{*\}$ by an em-

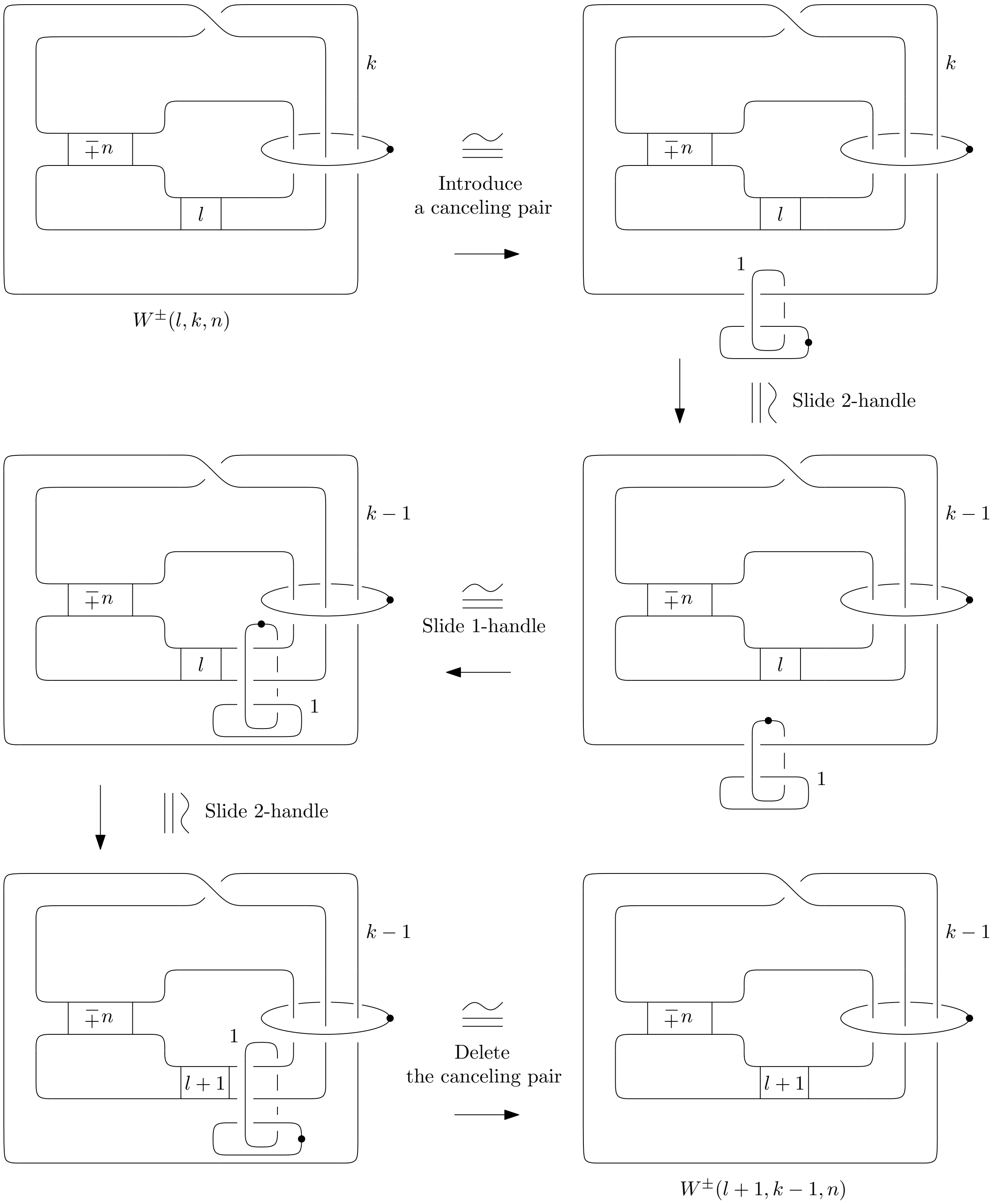


Figure 4.3: A sequence of handle slides of 1- and 2-handles.

bedding $\phi : \partial\mathbb{D}^2 \times \mathbb{D}^2 \hookrightarrow \partial W_1$ with framing k as in Figure 4.2. One can see that γ is a generator of $H_1(S^1 \times S^2)$. Now consider

$\gamma \times I \subset \partial W_1 \times I \subset \partial(W_1 \times I) = S^1 \times S^3 \subset W_1 \times I = S^1 \times \mathbb{D}^4$,
$\gamma' \times I \subset \partial H_2 \times I \subset \partial(H_2 \times I) = S^4 \subset H_2 \times I = \mathbb{D}^5$ and extension of ϕ to $\phi' : (\partial\mathbb{D}^2 \times \mathbb{D}^2) \times I \hookrightarrow \partial W_1 \times I$ as identity of the I component. So the thickened $W^{\pm}(l,k,n)$ can be obtained by

$$W^{\pm}(l,k,n) \times I = W_1 \times I \underset{\phi'}{\cup} H_2 \times I = S^1 \times \mathbb{D}^4 \underset{\phi'}{\cup} \mathbb{D}^5.$$

Since the embedding of γ' in S^4 and as a generator, the embedding of γ in $S^1 \times S^3$ are unique, it can be taken the unknot, the generator of $H_1(S^1 \times S^2)$, as the embedding in $S^1 \times S^3$ instead of γ. Thus, we get $W^{\pm}(l,k,n) \times I \cong \mathbb{D}^5$ which finishes the proof. □

Recall that M is an *integral homology sphere* if $H_i(M,\mathbb{Z}) \simeq H_i(S^3,\mathbb{Z})$ for all i.

Proposition 4.0.10. *$\partial(W^{\pm}(l,k,n))$ is an integral homology sphere for all l,k,n.*

Proof. Fix l,k,n and let $W^{\pm}$ denote $W^{\pm}(l,k,n)$ and $M^{\pm}$ denote $\partial(W^{\pm}(l,k,n))$ for simplicity. First we observe that:

$\partial(W^{\pm} \times I) \cong \partial\mathbb{D}^5 = S^4$ by the proof of the previous proposition and

$\partial(W^{\pm} \times I) = (W^{\pm} \times \{0\}) \cup (M^{\pm} \times I) \cup (W^{\pm} \times \{1\})$ which is diffeomorphic to $W^{\pm} \underset{M^{\pm}}{\cup} W^{\pm}$ obtained by identifying the boundaries with identity.

Set $X^{\pm} := W^{\pm} \underset{M^{\pm}}{\cup} W^{\pm} \cong S^4$ and let $X^{\pm} = A \cup B$ where A (resp. B) is the union of the first (resp. second) $W^{\pm}$ component in $X^{\pm}$ with the collar neighborhood of $M^{\pm}$ in the other $W^{\pm}$. Thus A and B are both diffeomorphic to $W^{\pm}$. Applying Mayer-Vietoris sequence to $X^{\pm} = A \cup B$, we get:

$0 \to H_4(X^{\pm}) \to H_3(M^{\pm}) \to H_3(W^{\pm}) \oplus H_3(W^{\pm}) \to H_3(X^{\pm}) \to H_2(M^{\pm}) \to H_2(W^{\pm}) \oplus H_2(W^{\pm}) \to H_2(X^{\pm}) \to H_1(M^{\pm}) \to H_1(W^{\pm}) \oplus H_1(W^{\pm}) \to H_1(X^{\pm}) \to H_0(M^{\pm}) \to H_0(W^{\pm}) \oplus H_0(W^{\pm}) \to H_0(X^{\pm}) \to 0.$

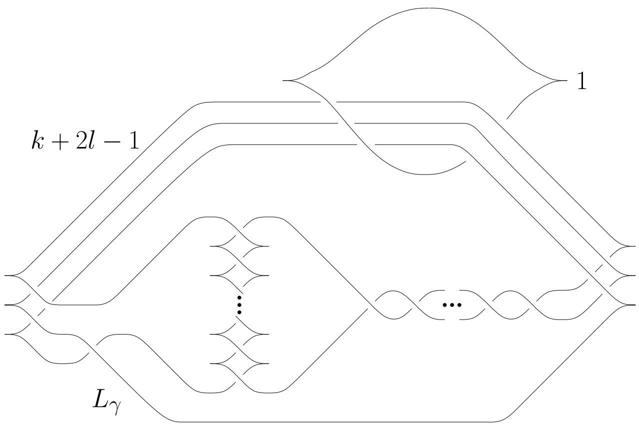


Figure 4.4: Stein fillable contact structure of $\partial(W^-(l,k,n))$ for $l \geq 0$ and $k \leq 0$.

Hence,

$$H_i(M^{\pm}) = \begin{cases} \mathbb{Z} & i = 0,3 \\ 0 & otherwise \end{cases}$$

which implies that $M^{\pm}$ is an integral homology sphere for all l, k, n.

□

Theorem 4.0.11. *$\partial(W^-(l,k,n))$ admits a Stein fillable contact structure for all l, k, n.*

Proof. First assume that $l \geq 0$ and $k \leq 0$. Turning the topological diagram of $\partial(W^-(l,k,n))$ into a contact surgery diagram, we get a Legendrian realization L_γ of γ with $tb(L_\gamma) = 1 - 2l$ as in Figure 4.4. So the contact framing of L_γ must be $k + 2l - 1$ which is negative. Then the contact surgery on L_γ can be converted into a sequence of (-1)-surgeries. Since $(+1)$-surgery along the unknot gives the unique Stein fillable contact structure in $S^1 \times S^2$ and Legendrian surgery preserves fillability, the contact structure given in Figure 4.4 is Stein fillable.

Secondly, let $l < 0$ or $k > 0$. Through Proposition 4.0.8, we can find a diffeomorphic copy $\partial(W^-(l', k', n))$ of $\partial(W^-(l,k,n))$ such that $l' \geq 0$ and $k' \leq 0$. This leads to interpret a Stein fillable contact structure on $\partial(W^-(l,k,n))$, by following the first step of the proof.

□

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