NEGATIVE BENDING OF STRAIN HARDENING CURVED BEAM IN THE ELASTOPLASTIC STATES OF STRESS

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ABSTRACT

NEGATIVE BENDING OF STRAIN HARDENING CURVED BEAM IN THE ELASTOPLASTIC STATES OF STRESS

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A narrow rectangular cross-section curved beam subjected to negative bending is considered. Plane stress and plane strain analytical solutions are derived for partially plastic deformation of the beam under pure bending. Linearly hardening material behavior is assumed. Elastic and two stages of elastic-plastic deformations are studied using Tresca's yield criteria and its associated flow rule. Elastic, partially-plastic and residual stresses are calculated. While the results of the studies in which elastoplastic bending of curved beam under positive bending has been published in the literature, the residual stresses have not been estimated and reported. This study is carried out considering the case of negative bending and provides the missing information in the literature. In addition, the results for positive bending given in the literature are regenerated by the use of a different nonlinear system solution technique and the corresponding residual stresses are evaluated.

Beam deforms elastically as long as the load falls below the elastic limit. If the elastic limit is exceeded, plastic deformation commences at the inner surface ($r = a$). Further increase in the applied load leads to formation of an other plastic region at

outer surface ($r = b$). As the bending moment is further increased, the two plastic regions spread into the elastic region. It is observed that the stresses and deformation in cases of positive and negative bending exhibit similar behavior. It is also observed that the deformation in the cross-section of the beam is negligibly small.

Keywords: Curved beam, pure bending, elastic-plastic deformations, Tresca's yield criteria, residual stresses.

ÖZ

DOĞRUSAL SERTLEŞEN MALZEME DAVRANIŞI GÖSTEREN EĞRİ KİRİŞİN ELASTOPLASTİK GERİLME ALTINDA NEGATİF YÖNDE EĞİLMESİ

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Dar dikdörtgen kesitli bir eğri kiriş göz önünde bulundurulmuştur. Kirişin basit eğilme altında kısmen plastik deformasyonu için düzlemsel gerilme ve düzlemsel gerinim analitik çözümleri yapılmıştır. Lineer sertleşen malzeme davranışı varsayılmıştır. Elastik ve elastik-plastik deformasyonların iki aşaması, Tresca akma kriteri ve ilgili akma kuralı kullanılarak incelenmiştir. Elastik, kısmen plastik ve artık gerilmeler hesaplanmıştır. Pozitif eğilme altındaki eğri kirişin elastoplastik eğilmesini içeren çalışmaların sonuçları literatürde yayınlanırken, artık gerilmeler hesaplanmamış ve yayınlanmamıştır. Bu çalışma, negatif eğilme durumu göz önüne alınarak yapılmış ve literatürde konu ile ilgili eksikler tamamlanmıştır. Ek olarak, literatürde verilen pozitif eğilme sonuçları, farklı lineer olmayan sistem çözüm teknikleri kullanılarak yeniden üretilmiş ve ilgili artık gerilmeler hesaplanmıştır.

Elastik limit aşılmadığı sürece, kiriş elastik olarak deforme olur. Elastik limit aşıldığında, plastik deformasyon iç yüzeyde başlar ($r = a$). Uygulanan yükün daha fazla artırılması, dış yüzeyde başka bir plastik bölgenin oluşmasına yol açar ($r = b$). Eğilme

momenti daha fazla arttırılırsa, iki plastik bölge aradaki elastik bölgeye doğru ilerler. Pozitif ve negatif eğilme durumlarındaki gerilmelerin ve deformasyonun benzer davranış sergilediği gözlemlenmiştir. Ayrıca, kiriş kesitindeki deformasyonun ihmal edilebilir derecede küçük olduğu görülmüştür.

Anahtar Kelimeler: Eğri kiriş, basit eğilme, elastik-plastik deformasyonlar, Tresca akma kriteri, artık gerilmeler.

To My Family

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CHAPTER 1

INTRODUCTION

1.1 General Aspects

The research on the prediction of elastic, partially plastic, thermal and residual stresses in basic mechanical structures such as prismatic bars, thin plates, shafts, disks, tubes, pressure vessels, spheres, and cylinders under different loading conditions has been the subject of a large number of investigations due to their significance in various branches of engineering and in daily life. Plane stress and plane strain solutions of the problems corresponding to these structures are available in the literature. The mechanical response under certain loading is related to boundary condition, loading pattern and initial geometry of the structure. Researchers have also spent prevalent effort for the analysis of basic structures made of advanced materials such as functionally graded and composite. Solutions involving elastic stress state, elastic-plastic stress states, fully plastic stress state, thickness variability and material non-homogeneity appear in recent research articles. Tresca's and Von Mises' yield criteria are generally used to treat plastic part of elastoplastic deformation. Analytical solutions for relatively simple geometries are achieved by using Tresca's yield conditions which provide a nonlinear problem consists of a combination of linear differential equations. The criteria of yielding is mentioned in Chapter 2. Furthermore, the material behavior can be presumed as perfectly plastic, linear hardening or nonlinear hardening beyond elastic limit. Perfectly plastic and linear hardening models can be treated analytically by Tresca's yield criteria whereas nonlinear hardening material behavior must be solved numerically. Nevertheless, actual behavior of a structure can roughly be described by linear models. This is because almost all structures behave in some nonlinear aspect prior to reaching their ultimate limit. Nonlinear response of such structures are captured for better system characterization in modern developments of computational mechanics.

Analytical solutions of most of the basic structures in different states of stress are available in advanced textbooks. Comprehensive treatments of a large number of basic structures in the elastic state of stress by Timoshenko [1], Timoshenko and Goodier [2], Rees [3], Boresi et al. [4], Ugural and Fenster [5], Boley and Weiner [6], and Timoshenko and Gere [7], and in the plastic state of stress by Johnson and Mellor [8], Mendelson [9], Hill [10], and Nadai [11] are available in literature.

The analyses in elastic-plastic stress states are as important as those in the elastic states, because an elastoplastic study helps to calculate residual stresses and to assess the advantages of leaving residual stresses on the system. A structure is subjected to an internal pressure so that it becomes partially plastic, then the pressure is released and the residual stresses occur to increase the pressure capacity. This process is called as autofrettage which has been widely used for several years in various industries for automotive, aerospace and defence systems and in construction of fuel injection systems, pressure vessels, tank gun barrels, etc. to increase operating pressures. For a pressure vessel subjected to immense pressure, the largest tensile stress exists at the internal parts. To withstand this tensile stress, compressive residual stresses are produced at the inside of the vessel by autofrettage. In this way, autofrettage improves strength of the pressure vessel. Moreover, it delays the formation of crack and extends the lifetime of the material. It is used for design of pressure vessels to resist high internal pressures. The residual stress analysis is carried out conveniently by using Tresca' yield criterion for several types of structures and loading conditions.

1.2 Literature Review

Investigations in related literature are classified into four categories in this thesis according to the type of structures. The first category includes the studies concerned with analysis of the elastic and elastic-plastic behavior of curved beams subjected to mechanical and thermal loading. The researches regarding behavior of curved panel structures under thermal loading form the second one. Then, the works that represent the solution for rotating disks and shafts generate the third category and the studies related to cylindrical elements, tubes and vessels under thermal load effects comprise the fourth category.

1.2.1 Curved Beams

Deformation behavior of curved bars and methods to increase the material limits under varying loads have received attention by researchers. Elastic and partially plastic solutions are achieved under states of both plain stress and plane strain for different material behavior such as ideally plastic, linear strain hardening, non-work hardening and linear kinematic hardening. The models used in the articles that are mentioned in this thesis are generally based on the Tresca's yield criterion. Elastic analysis of a curved bar under pure bending conditions was treated by Timoshenko and Goodier [2]. In this study, a state of plane strain was assumed. The articles authored by Shaffer and House Jr. [12], [13], and [14] are the pioneering ones in defining, formulating and deriving a partially plastic solution to the curved beam problem. Shaffer and House Jr. presumed a narrow rectangular cross-section curved beam subjected to couples at its end sections. They assumed a state of plane strain and considered ideally plastic material behavior. The beam deforms and as a result stresses develop in the beam. As the elastic limit is exceeded, plastic deformation begins at the inner surface. While this plastic region spreads into the beam with increasing bending moment, another plastic region appear at the outer surface. Thereafter, the beam is composed of an inner plastic, an elastic and an outer plastic region.

In a later work, Dadras [15] obtained an analytical solution of elastoplastic pure bending of a linear strain hardening curved beam under plane strain supposition. Dadras considered a narrow rectangular cross-section curved beam subjected to positive couples at its end sections. Because the nonlinear equations for a general hardening case could not be solved analytically, only a linear hardening case was analyzed. As in the result of analysis for ideally plastic curved beam under plane strain assumption by Shaffer and House Jr. [12], the linear strain hardening solution assuming a state of plane strain by Dadras [15] showed that the plastic deformation commences at the inner surface and spreads into the beam with increasing moment. At a critical load, another plastic region begins to form at the outer surface. Thereafter, the slender beam becomes composed of an inner plastic region, an elastic region, and an outer plastic region. Then, the two plastic regions expand over the elastic zone as the couple moment increases further. Eraslan and Arslan [17] studied an analytical treatment of elastic-plastic bending of a curved beam. Plane stress and plane strain analytical solutions for curved beam subjected to positive couples at its end sections were presented and linear strain hardening material behavior was presumed. It was demonstrated that plane stress and plane strain solutions coincide with the elastic and in the elastic-plastic deformation stages. It was also observed that the changes in the dimensions of the beam as it deforms are negligibly small. Eraslan and Arslan [51] also solved the non-linear strain hardening curved beam problem numerically. A computational model was developed for solution of a slender curved beam in purely elastic and partially plastic stress states. The model was based on the von Mises' yield criterion and a state of plane stress was presumed. A second order nonlinear differential equation was obtained to describe the deformation behavior of the beam. A shooting technique using Newton iterations were used for the numerical integration of the governing equation. Recently, the authors have derived the closed form solution to the bending of a non-linearly hardening curved bar [53]. An analytical solution to the partially plastic deformation was derived and a state of plane strain was assumed. The results were verified in comparison to the linear hardening solution available in the literature [17] and the plane strain linearly hardening solutions were extended to nonlinear hardening. Dryden [16] also derived a solution by letting the modulus of elasticity vary in the radial direction. Plane stress analytical solution was derived.

Beyond solutions for a homogeneous curved bar under pure bending, the literature also comprises researches regarding solutions of composite and non-homogeneous beams. As an example for solution of composite bars, yielding of two-layer curved bars under pure bending was investigated by Arslan and Sülü [22]. It was assumed that both layers were in a state of plane stress. Analytical expressions were derived for the bending moments concerning to the elastic limit leading to plastic flow. The results showed that plastic deformation depends on material properties and dimensions of the layers and yielding may occur at the inner or outer surface of the bar or at the interface between two layers. Moreover, yielding may initiate at the outer and interface surfaces simultaneously. It was known that yielding always emerges at the inner surface of a homogeneous bar but this study demonstrated that yielding may emerge at different locations in a composite bar. Composition of a functionally graded material (FGM) is also non-homogeneous. Properties of these types of materials like modulus of elasticity, Poisson's ratio, and thermal expansion coefficient may vary throughout the material. This non-homogeneity in the material enables lower stresses and as a result higher strength of the structure. Elastic and elastoplastic behavior of a functionally graded curved bar subjected to pure bending have been studied by several researchers. Wang and Liu [20] extended Dryden's [16] study by considering an elastic curved bar with functionally graded layers. Bending of graded curved bars at elastic limits was also studied by Arslan and Eraslan [21]. Analytical and computational models were developed to predict the stress response under pure bending in elastic and partially plastic stress states. The modulus of elasticity in the analytical model and both the modulus of elasticity and the hardening parameter in the computational model of the bar material were pressumed to vary in the radial direction. The analytical model was based on Tresca's yield criterion, its associated flow rule and ideal plastic material behavior, and the computational one was based on von Mises' yield criterion, total deformation theory and a Swift type nonlinear hardening law. The results indicated that the variation of material properties affects the deformation behavior. Whereas yielding begins at the inner surface in homogeneous bars, yielding may commence at the inner, at the outer or at both surfaces in graded ones. This work also extended Dryden's [16] to provide analytical and numerical analyses of partially plastic stress states under pure bending.

The analysis of deformation behavior of curved beams have also appeared in very recent research articles. A new computational approach was presented to determine the elastic field of deep curved beams with mixed boundary conditions by Ahmed and Ghosh [23]. The study was based on a displacement-function equilibrium method and the displacement function was expressed in terms of the radial and circumferential displacement components. These components governed by a single partial differential equation of equilibrium. It was shown that this method is effective to predict the elastic field of structural members of curved geometries with mixed boundary conditions accurately. An analytical study on the elastic-plastic pure bending of a linear kinematic hardening curved beam was performed by Fazlali, Arghavani and Eskandari [25]. Although exact plane plasticity solution requires solution of a system of equations, this study was based on the hyperbolic strain distribution on the cross section which derives a simple approximate solution. Hull, Perez and Cox [26] studied an analytical model of a curved beam with a T shaped cross section. They derived an analytical dynamic model of a closed circular beam. The new model includes in-plane and out-of-plane vibrations derived by continuous media expressions. Sarria, Gimena, Gonzaga, Goñi and Gimena [27] examined formulation and solution of curved beams with elastic supports. The authors presented the general system of differential equations which can be solved by either numerical or analytical methods, and a new numerical process called the Finite Transfer Method in order to solve a linear system of ordinary differential equations. It was shown that the numerical solution agrees in the exact analytical solution. Dehrouyeh studied [24] on the thermally induced non-linear response of functionally graded beams with rectangular cross section. The response of shear-deformable slightly curved beams in monotonic loading and unloading were examined analytically.

1.2.2 Curved Panels

Researchers also investigated deformation behavior of curved panels under various loads. Elastic and elastoplastic states of a radially heated thick-walled cylindrically curved panel were studied by Arslan, Mack and Gamer [28] and Arslan and Mack [29], respectively. In these studies, the basic equations of a curved panel with homogeneous thickness were derived in a plane strain state and under pure bending conditions. The stresses occurring for a heated inner or outer surface were analyzed in detail, and the yield criteria of Von Mises' and Tresca for the elastic and elastic-plastic states were used. Furthermore, the residual stresses after cooling were obtained [29]. The same problem was discussed in a state of generalized plane strain by Haskul, Arslan and Mack [30]. The stresses and elastic limits of a cylindrical curved panel subjected to a radial temperature gradient was discussed by analytical means. The stresses occurring for both a positive and a negative temperature gradient were analyzed, and it was found that the yield limit according to Von Mises' may be reached either at the inner surface, the outer surface or at both surfaces simultaneously with increasing temperature gradient. Moreover, it was shown that the loads at the supports and the weight of panel are decreased using a graded panel as compared to a homogeneous panel.

1.2.3 Rotating Disks and Shafts

Solid disks are another engineering structures providing high transferable loads. These structures were studied analytically in elastic, elastoplastic and fully plastic states by researchers. The first meaningful solution for solid disk problem was derived by Gamer. Gamer [31], [33], [34] demonstrated that the stress distribution in a rotating disk given in several articles and textbooks on plasticity was not accurate because the corresponding deformation was not consistent with the continuity requirements at the elastoplastic interface. Gamer [32] [33], [34] produced a compatible solution for a strain hardening material considering the fact that the plastic core of the disk includes two parts with different yield conditions. The rotating solid disk in the fully plastic and elastic-plastic states were studied by Gamer [33] and [32], respectively. Linear strain-hardening material was assumed in the fully plastic state. The plastic zone was consists of two parts and it was shown that the boundary between these regions approaches a limit for unbounded growth of the angular velocity. Residual stress after the standstill was also discussed. The usual statically determinate stress distribution was recovered by specializing the results to perfectly plastic material [32]. Elastic unloading of a disk after plastic deformation by a circular heat source was studied by Gamer and Mack [35]. It comprised that the source during heating is removed, a plastic region spreads around the source absorbed by an unloaded region. Residual stresses and secondary plastic flow limits in nonlinearly strain hardening rotating shafts was studied by Eraslan and Mack [36]. A computational procedure to estimate the residual stress distributions and the limit angular speeds to avoid secondary plastic deformation were given based on von Mises' yield condition. Newton iterations were used to solve the boundary value problem for the governing nonlinear differential equation.

There also appear solutions of non-homogeneous solid shaft and disk structures in the literature in addition to homogeneous ones. Elastic and elastoplastic deformations of functionally graded structures have been studied by several researchers. Akis and Eraslan [37] obtained exact solution of rotating FGM shaft problem in the elastoplastic state of stress. Plane strain analytical solutions to estimate purely elastic, partially plastic and fully plastic deformation behavior were studied. The modulus of elasticity was assumed to vary non-linearly in the radial direction. By considering different material compositions, it was demonstrated that both elastic and elastoplastic responses are affected significantly by the material non-homogeneity. Furthermore, the graded yield limit changes the stress and deformation expressions in the plastic regions. Elastic–plastic stresses of rotating functionally graded discs were analyzed by Çallıoğlu, Sayer and Demir [38]. The discs were assumed to be subjected to constant angular velocity and non-work hardening material yielding behavior was considered. Elasticity modulus, density and yield strength of the disc were considered to vary radially. Different angular velocities were considered to get the distribution of the plastic region. It was seen that the analytical and numerical results are in good agreement. It was also seen that yielding region expands throughout the outer surface with increasing angular velocity, and the tangential and radial stresses increase as the centrifugal force increases.

1.2.4 Cylindrical Elements, Tubes, and Vessels

Analytical solutions to elastic, elastic-plastic, thermal and residual stresses for deformation behavior of tubes are also available in literature. A thick-walled tube under pressure was performed in purely elastic stress state by Timoshenko [1], Timoshenko and Goodier [2], Ugural and Fenster [5], and Boresi et al. [4], in the fully plastic stress state by Boresi et al. [4], Mendelson [9], and Nadai [11], and in the elastic-plastic stress state by Parker [40] and Perry and Aboudi [41]. Analytical solution to thermal loading and unloading of a cylinder subjected to periodic surface heating was studied by Eraslan and Apatay [44]. Transient temperature distribution in the cylinder was obtained by Duhamel's theorem. The generalized plane strain in the axial direction and linearly hardening material behavior were presupposed. An elastic and two plastic regions were expressed for the cylinder with free ends. The model was compared to a purely numerical solution and equilibrium, interface, and boundary conditions were satisfied in every stage of deformation. Elastic-plastic deformation of a centrally heated cylinder was examined by Gamer and Orcan [39]. A perfectly plastic cylinder with uniform temperature was considered. The core temperature was increased up to the value that the elastic region vanished and it was seen that plastic collapse does not occur for these temperatures. Analyses of composite tubes are included in articles. For instance, stress distributions in energy generating two-layer tubes subjected to free and radially constrained boundary conditions were considered by Eraslan, Sener and Argeso [42]. Linearly hardening material behavior was assumed in the analysis. Elastic solutions of different materials showed that eight different plastic regions appear during elastoplastic deformation stages. The stress response for perfectly plastic material are deduced from the linearly hardening material expressions. The literature also includes studies for solutions of non-homogeneous pressure vessels. Analytical solutions to elastic functionally graded cylindrical and spherical pressure vessels were examined by Eraslan and Akis [43]. In this study, analytical solutions to earlier models [45] regarding elastic analysis of graded cylindrical and spherical pressure vessels and an annular disk were obtained.

CHAPTER 2

THEORY

2.1 Stress-Strain Relation

Tensile strength is the ability to support axial load without rupture and it is determined by the tensile test which is the most important experiment to determine stress-strain relation of a material [10]. An increasing tensile load P is applied to the material specimen. Mainly there are two types of stress-strain curves which are engineering and true stress strain. The engineering stress-strain curve is generally used in design application if there is no expectation of deformation due to strain and hence original cross section and original length are used. Then, the nominal stress $\sigma_n = P/A_0$, where A_0 is the original cross sectional area, and the engineering strain $\epsilon = (L - L_0)/L_0$, where L_0 the original length, are evaluated to draw the stress-strain diagram. In the computation of true stress the actual area is used. As the length increases, the cross sectional area decreases. The true stress is $\sigma_t = P/A_i$, where A_i is the actual area of the cross-section.

Engineering stress-strain diagram of low-carbon steel which is a ductile material is shown in Fig. (2.1). The strain at first increase proportionally to the stress and the specimen returns to its original length on removal of the stress. Then, stress is less than or equal to yield stress, i.e., $\sigma \leq \sigma_y$ and the slope of the curve is equal to modulus of elasticity, E and Hooke's law, $\sigma = E\epsilon$ is valid here. Upon loading beyond the yield limit, the material begins to yield, and the resultant strain increases more quickly than the corresponding stress in yielding region. The elongation after yielding is started is 200 times as large as deformation before yield. Stress remains constant over a large range of strain after onset of yielding for structural steel (Fig. (2.1)). Stress continues to increase with strain until the ultimate strength, σ_u is reached. The increase in stress is due to material strain hardening, hence this region is called strain hardening. Beyond ultimate limit, necking region takes place and the stress decreases until the specimen ruptures with the stress σ_b. If load is removed, elastic strain ϵ^e is recovered and permanent plastic strain, ϵ^p remains. Therefore, the total strain is considered as total of elastic and plastic strain components, i.e, $\epsilon = \epsilon^e + \epsilon^p$ [5].

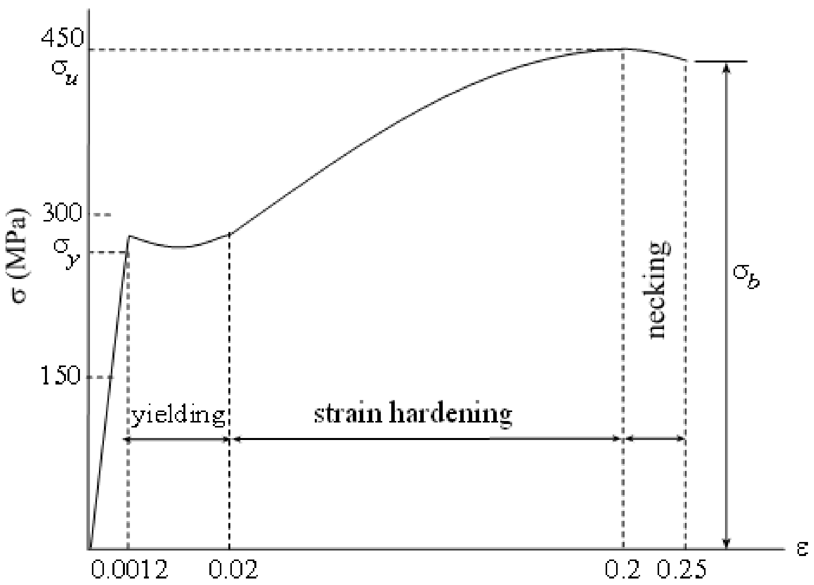


Figure 2.1: Engineering stress-strain diagram for low-carbon steel

A typical engineering stress-strain curve for brittle materials is shown in Fig. (2.2). Brittle materials have very low percentage of elongation and break suddenly under stress at a point just beyond the elastic limit. Therefore, the ultimate strength and breaking strength are the same.

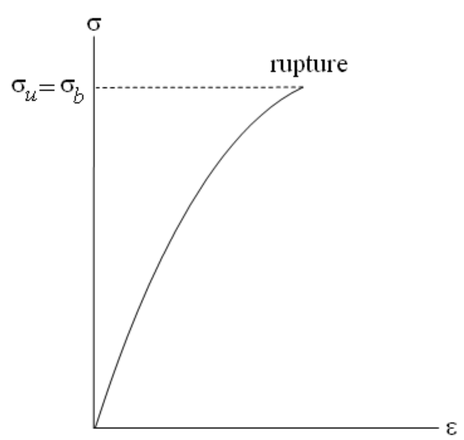


Figure 2.2: Engineering stress-strain diagram for brittle materials

There are two types of assumptions to perform analytical solutions based on Tresca's yield criterion. These are idealized models of material behavior which are elastic and perfectly plastic, and elastic and strain hardening. As it can be seen in Fig. (2.3) (a) for elastic and perfectly plastic materials, when the yield point is reached, plastic deformation begins. Elastic and strain hardening materials obey Hooke's law in the elastic region and begins to flow at the yield point. This material behavior is shown in Fig. (2.3) (b). Most of the ductile material behave in this manner. The analytical solutions may not be performed in the state of linear hardening material behavior based on Tresca's yield criterion. In such situations, perfectly plastic behavior is considered.

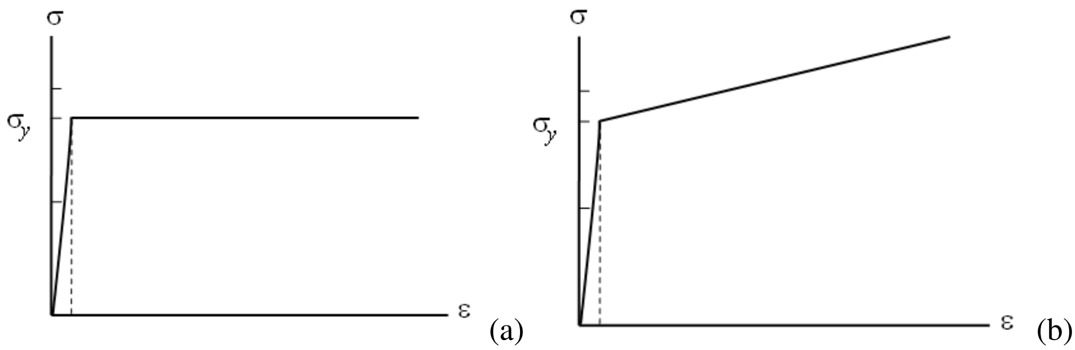


Figure 2.3: Idealized material behavior for states of (a) elastic and perfectly plastic, (b) elastic and strain hardening.

2.2 General Criteria of Yielding

Criteria of yielding describe the limit of elasticity under any stress combination [10]. The material behavior is defined by using any yielding criterion for the stresses larger than yield stress. Plastic yielding is based on the magnitude of the three principal applied stresses. It does not depend on the directions of these principal stresses as the isotropic material is considered. There are several criteria to define the yielding of solids, i.e. Tresca's criterion, von Mises' criterion, maximum stress theory, maximum strain theory, etc. Tresca's and von Mises' yield criteria are useful for the initial yield of isotropic materials.

2.2.1 Tresca's Yield Criterion

This criterion was firstly proposed by Tresca [46]. It is also called as maximum shear theory. Tresca obtained that if the maximum shear stress reaches the maximum shear stress at yielding, yielding will take place in a simple tension test. For the principal stresses, $\sigma_1 > \sigma_2 > \sigma_3$, Tresca's yield condition is

$$\pm\frac{1}{2}(\sigma_1 - \sigma_3) = \tau_{\max}. \tag{2.1}$$

The maximum shear stress at yielding point is $\tau_{\max} = \sigma_y/2$ in the simple tension. Hence, yielding begins as any of the below conditions is satisfied according to Tresca's criterion[9]:

$$\sigma_1 - \sigma_3 = \pm\sigma_y, \tag{2.2}$$

$$\sigma_1 - \sigma_2 = \pm\sigma_y, \tag{2.3}$$

$$\sigma_2 - \sigma_3 = \pm\sigma_y. \tag{2.4}$$

This yield criterion is generally used for ductile materials [3]. Then, it provides a closed form solution for the theoretical treatment of the yielding problem.

2.2.2 Von Mises' Yield Criterion

This criterion is attributed to Von Mises [50]. It is also called as distortion energy theory. In this theory, yielding will occur at any coordinate of the material when the distortional strain energy per unit volume in a state of combined stress becomes equal to that associated with yielding in a simple tension test [5]. For the principal stress, $\sigma_1 > \sigma_2 > \sigma_3$, the Von Mises' yield criterion is

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2 = 2\sigma_y^2. \tag{2.5}$$

Yielding condition is not affected by any additional amount to stresses as only the differences of the principal stresses are concerned in Eq. (2.5) [5]. This yield criterion is commonly used for ductile materials [5].

2.2.3 Maximum Normal Stress Theory

This theory is proposed by Rankiene [51]. Failure takes place when the major principal stress reaches that which caused fracture in a simple tension test.

2.2.4 Maximum Normal Strain Theory

It is suggested by Saint-venant. It assumes that failure occurs when the maximum strain reaches to the strain at yield point in the tensile test [10].

2.2.5 Tsai-Hill Theory

Hill [10] extended the yield criterion of von-Mises [50] for isotropic materials to anisotropic materials and Tsai [48], [49] extended it for anisotropic materials to a unidirectional lamina. Total strain energy in a body is composed of two parts as distortion energy and dilation energy.

CHAPTER 3

STATEMENT OF THE PROBLEM

Beam is a basic mechanical structure which resists load, transverse to the longitudinal axis, primarily by providing resistance against bending. Analytical solution of a beam provides determination of stresses and displacements developed in the beam under the specified condition of loading. In this study, a narrow rectangular cross-section curved beam is considered to be subjected to couples at its end sections. Plane stress and plane strain analytical solutions are derived for partially plastic deformation of the beam under pure bending in both negative and positive bending cases. The geometry and the loading for negative bending case is shown at the considered coordinate system in Fig. (3.1).

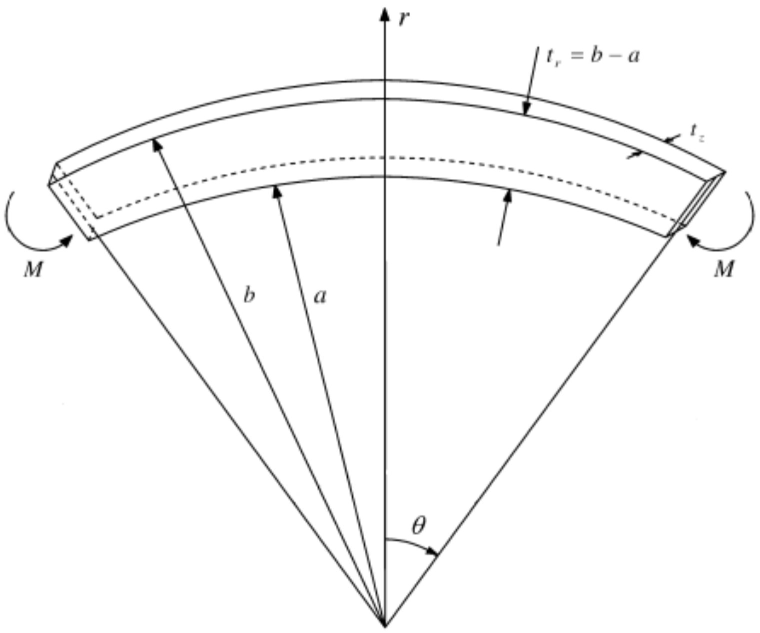


Figure 3.1: The geometry and the coordinate system of the curved beam under negative bending

Moreover, linearly hardening material behavior is assumed. Elastic and two stages of elastic-plastic deformations are studied using Tresca's yield criteria and its associated flow rule.

Beam deforms elastically below the elastic limit load. When the elastic limit is exceeded, plastic deformation commences at the inner surface ($r = a$). In the first stage of elastic-plastic deformation, the beam consists of an inner plastic region and an outer elastic region. Further increase in the applied load leads to formation of an other plastic region at outer surface ($r = b$). Then, the beam consists of an inner plastic region, an elastic region and an outer plastic region in the second stage. As the bending moment is further increased, the two plastic regions spread into the elastic region. A single differential equation that governs the elastoplastic behavior of the bar has been derived by using the equations of the generalized Hooke's law, the equation of equilibrium and the compatibility relation.The radial stresses, circumferential stresses, axial stresses and the radial displacement are calculated analytically. The calculations are performed for the plastic range because the results allow to investigate the level of the residual stresses which is the difference between the stress distribution for elastoplastic and elastic behavior. For this purpose, the residual stresses remaining in the beam after removing the load at plastic limit for stages I and II are also evaluated. The numerical solutions of the elastic and plastic deformations are obtained by solving the nonlinear systems of equations associated with the boundary and interface conditions by using Newton's Method with two versions for each stages. In the first version, the evaluations are performed by solving all of the boundary and interface conditions simultaneously, and in the second version, by reducing the number of equations from the linear systems of equations and solving the remaining nonlinear system. The convergence analysis comparing Versions I and II is performed for the state of positive bending.

CHAPTER 4

ANALYTICAL SOLUTIONS

4.1 Basic Equations

The total strain equations with the usage of the generalized Hooke's law of elastic strain are

$$\epsilon_r = \epsilon_r^p + \frac{1}{E}\left[\sigma_r - \nu\left(\sigma_\theta + \sigma_z\right)\right], \tag{4.1}$$

$$\epsilon_\theta = \epsilon_\theta^p + \frac{1}{E}\left[\sigma_\theta - \nu\left(\sigma_r + \sigma_z\right)\right], \tag{4.2}$$

$$\epsilon_z = \epsilon_z^p + \frac{1}{E}\left[\sigma_z - \nu\left(\sigma_r + \sigma_\theta\right)\right]. \tag{4.3}$$

where E is the modulus of elasticity, σ_i the normal stress component, ν the Poisson's ratio, ϵ_i the normal strain component, and ϵ_i^p the normal plastic strain component. The stresses and strains are functions of the radial direction, r and $\tau_{r\theta} = 0$, then the strain - displacement relations read

$$\epsilon_r = \frac{du}{dr}, \tag{4.4}$$

$$\epsilon_\theta = \frac{u}{r} + \frac{1}{r}\frac{dv}{d\theta}. \tag{4.5}$$

$$\gamma_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r} + \frac{\partial v}{\partial r} = 0. \tag{4.6}$$

in which u and v are the radial and tangential components of the displacement vector, $\tau_{r\theta}$ the shearing stress and $\gamma_{r\theta}$ the shearing strain The equilibrium equation is

$$\sigma_\theta = \frac{d}{dr}(r\sigma_r). \tag{4.7}$$

Compatibility relation is

$$\frac{1}{r}\frac{d}{dr}\left(r^2\frac{d\epsilon_\theta}{dr}\right)-\frac{d\epsilon_r}{dr}=0. \tag{4.8}$$

Integration of the compatibility relation can be found in Appendix (A). It is:

$$\frac{d}{dr}(r\epsilon_\theta)-\epsilon_r=\phi. \tag{4.9}$$

The derivation of radial and tangential components of the displacement are shown in Appendix. (B). These components are as follows:

$$u(r,\theta)=r\epsilon_\theta-A_1r-A_2\cos\theta, \tag{4.10}$$

$$v=A_1r\theta+A_2\sin\theta, \tag{4.11}$$

where A_1 and A_2 are arbitrary constants. Rewriting Eq. (4.10),

$$r\epsilon_\theta=u+A_1r+A_2\cos\theta. \tag{4.12}$$

Substituting Eqs. (4.4) and (4.12) in Eq. (4.9), $A_1=\phi$ is determined, hence the displacement equations turn into

$$u(r,\theta)=r\epsilon_\theta-\phi r-A_2\cos\theta, \tag{4.13}$$

$$v=\phi r\theta+A_2\sin\theta. \tag{4.14}$$

The equations for the strain and displacement are valid for both elastic and plastic regions and the plastic strain vanishes in the elastic region.

4.2 Governing Differential Equation

Combining the strain equations, Eqs. (4.1) and (4.2), and Eq. (4.7), Eq. (4.9), the differential equation is obtained as

$$r^2\frac{d^2\sigma_r}{dr^2}+3r\frac{d\sigma_r}{dr}-\nu r\frac{d\sigma_z}{dr}-E\left(\epsilon_r^p-\epsilon_\theta^p-r\frac{d\epsilon_\theta^p}{dr}\right)=E\phi. \tag{4.15}$$

This equation governs the partially plastic response of the curved beam under pure bending and valid for both plane stress and plane strain states.

4.3 Plane Stress

4.3.1 Elastic

The plastic strains vanish in purely elastic deformations, i.e., $\epsilon_i^p = 0$ and in the plane stress solution $\sigma_z = 0$. Writing differential equation, Eq. (4.15) with $\epsilon_r^p = 0$, $\epsilon_\theta^p = 0$ and $\sigma_z = 0$;

$$r^2 \frac{d^2\sigma_r}{dr^2} + 3r\frac{d\sigma_r}{dr} = E\phi. \tag{4.16}$$

Solution of the differential equation, Eq. (4.16), gives the radial stress component, σ_r. It is:

$$\sigma_r = \frac{C_1}{r^2} + C_2 + \frac{1}{2}E\phi \ln r, \tag{4.17}$$

where C_1 and C_2 are integration constants. From the equilibrium equation, Eq. (4.7) the circumferential stress component, σ_θ is determined as;

$$\sigma_\theta = -\frac{C_1}{r^2} + C_2 + \frac{1}{2}E\phi(1 + \ln r). \tag{4.18}$$

Finally, by using the equation of strain, Eq. (4.2) and Eq. (4.13), the radial displacement becomes

$$u = \frac{1}{E}\left[-\frac{1}{r}(1+\nu)C_1 + (1-\nu)C_2 r\right] - \left[1 - (1-\nu)\ln r\right]\frac{\phi r}{2} - A_2\cos\theta. \tag{4.19}$$

4.3.2 Plastic Region I

The stress state in this region is

$$\sigma_z(=0) > \sigma_r > \sigma_\theta, \tag{4.20}$$

and Tresca's yield criterion reads

$$\sigma_y = \sigma_z - \sigma_\theta = -\sigma_\theta. \tag{4.21}$$

Because there is no predeformation, the associated flow rule gives

$$\epsilon_z^p = -\epsilon_\theta^p \text{ and } \epsilon_r^p = 0. \tag{4.22}$$

From the equivalence of plastic work increment,

$$\epsilon_z^p = \epsilon_{EQ}, \tag{4.23}$$

where ϵ_{EQ} is the equivalent plastic strain. The yield stress for a linearly hardening material is

$$\sigma_y = \sigma_0 \left(1 + \eta\epsilon_{EQ}\right), \tag{4.24}$$

in which σ_0 is the uniaxial yield limit and η the hardening parameter. Then combining Eqs. (4.21) and (4.24), ϵ_{EQ} becomes

$$\epsilon_{EQ} = \left(\frac{\sigma_y}{\sigma_0} - 1\right)\frac{1}{\eta} = \left(-\frac{\sigma_\theta}{\sigma_0} - 1\right)\frac{1}{\eta}. \tag{4.25}$$

By using Eq. (4.7) with Eqs. (4.22) and (4.23), ϵ_θ^p and ϵ_z^p become

$$\epsilon_z^p = -\epsilon_\theta^p = \left[-\frac{1}{\sigma_0}\frac{d}{dr}(r\sigma_r) - 1\right]\frac{1}{\eta}. \tag{4.26}$$

Substitution of the plastic strains, Eq. (4.26), $\epsilon_r^p = 0$, and $\sigma_z = 0$ in the governing equation, Eq. (4.15) leads to

$$r^2\frac{d^2\sigma_r}{dr^2} + 3r\frac{d\sigma_r}{dr} + \frac{\sigma_r}{1+H} = \frac{EH\phi}{1+H} + \frac{\sigma_0}{1+H}, \tag{4.27}$$

where the normalized hardening parameter $H = \eta\sigma_0/E$. Solving the differential equation, Eq. (4.27)

$$\sigma_r = C_3 r^{-1-W} + C_4 r^{-1+W} + EH\phi - \sigma_0, \tag{4.28}$$

in which $W = \sqrt{H/(1+H)}$. From the equilibrium equation, Eq. (4.7) the tangential stress is obtained as;

$$\sigma_\theta = -WC_3 r^{-1-W} + WC_4 r^{-1+W} + EH\phi - \sigma_0. \tag{4.29}$$

The plastic strain components are achieved by Eqs. (4.26) and (4.28) as

$$\epsilon_z^p = -\epsilon_\theta^p = \frac{W}{EH}\left[\frac{C_3}{r^{1+W}} - \frac{C_4}{r^{1-W}}\right] - \phi. \tag{4.30}$$

Combining Eqs. (4.2) and (4.30) and substituting in Eq. (4.13), the radial displacement becomes

$$\begin{aligned} u \;=\; & \frac{1}{EH}\left\{-\left[W + H(W+\nu)\right]C_3 r^{-W} + \left[W + H(W-\nu)\right]C_4 r^{W}\right. \\ & \left.-H(1-\nu)\sigma_0 r\right\} + H\phi(1-\nu)r - A_2\cos\theta. \end{aligned} \tag{4.31}$$

4.3.3 Plastic Region II

The stress state in this region is

$$\sigma_\theta > \sigma_z (= 0) > \sigma_r, \tag{4.32}$$

and Tresca's yield criterion reads

$$\sigma_y = \sigma_\theta - \sigma_r. \tag{4.33}$$

The plastic strains and the equivalent plastic strain with the help of the associated flow rule:

$$\epsilon_\theta^p = -\epsilon_r^p = \epsilon_{EQ} \text{ and } \epsilon_z^p = 0. \tag{4.34}$$

The yield stress is

$$\sigma_y = \sigma_0 \left(1 + \eta \epsilon_{EQ}\right), \tag{4.35}$$

then

$$\epsilon_{EQ} = \left(\frac{\sigma_\theta - \sigma_r}{\sigma_0} - 1\right)\frac{1}{\eta}. \tag{4.36}$$

By using Eq. (4.7) with Eqs. (4.34) and (4.36), ϵ_r^p and ϵ_θ^p become

$$\epsilon_\theta^p = -\epsilon_r^p = \left\{\frac{1}{\sigma_0}\left[\frac{d}{dr}(r\sigma_r) - \sigma_r\right] - 1\right\}\frac{1}{\eta}. \tag{4.37}$$

Substituting Eq. (4.37) and $\sigma_z = 0$ in the differential equation, Eq. (4.15)

$$r^2\frac{d^2\sigma_r}{dr^2} + 3r\frac{d\sigma_r}{dr} = \frac{EH\phi}{1+H} + \frac{2\sigma_0}{1+H}, \tag{4.38}$$

and solving the differential equation, Eq. (4.38);

$$\sigma_r = \frac{C_5}{r^2} + C_6 + \frac{EH\phi \ln r}{2\left(1+H\right)} + \frac{\sigma_0 \ln r}{1+H}. \tag{4.39}$$

From the equilibrium equation, Eq. (4.7), the tangential stress becomes

$$\sigma_\theta = -\frac{C_5}{r^2} + C_6 + \frac{EH\phi(1+\ln r)}{2\left(1+H\right)} + \frac{\sigma_0(1+\ln r)}{1+H}. \tag{4.40}$$

The plastic strain components are obtained by substituting Eq. (4.39) in Eq. (4.37) as

$$\epsilon_\theta^p = -\epsilon_r^p = -\frac{1}{EH}\left[\frac{2C_5}{r^2} + \frac{\sigma_0 H}{1+H}\right] + \frac{\phi}{2\left(1+H\right)}. \tag{4.41}$$

Combining Eqs. (4.2) and (4.41) and substituting in Eq. (4.13), the radial displacement becomes

$$\begin{aligned} u &= \frac{1}{E}\left\{-\frac{[2+H(1+\nu)]\,C_5}{Hr}+(1-\nu)\,C_6 r+\frac{\sigma_0(1-\nu)\,r\ln r}{1+H}\right\} \\ &\quad -\frac{\phi\,[1+H-H(1-\nu)\ln r]\,r}{2(1+H)}-A_2\cos\theta. \end{aligned} \tag{4.42}$$

Analytical solutions for positive bending are also given in Appendix (C).

4.3.4 Summary of the equations of plane stress analytical solution

4.3.4.1 Elastic Region

Governing differential equation:

$$r^2\frac{d^2\sigma_r}{dr^2}+3r\frac{d\sigma_r}{dr}=E\phi, \tag{4.43}$$

solution of Eq. (4.43):

$$\sigma_r=\frac{C_1}{r^2}+C_2+\frac{1}{2}E\phi\ln r, \tag{4.44}$$

solution for the circumferential stress component:

$$\sigma_\theta=-\frac{C_1}{r^2}+C_2+\frac{1}{2}E\phi(1+\ln r), \tag{4.45}$$

and solution for the radial displacement:

$$u=\frac{1}{E}\left[-\frac{1}{r}(1+\nu)C_1+(1-\nu)C_2 r\right]-[1-(1-\nu)\ln r]\,\frac{\phi r}{2}-A_2\cos\theta. \tag{4.46}$$

4.3.4.2 Plastic Region I

Governing differential equation:

$$r^2\frac{d^2\sigma_r}{dr^2}+3r\frac{d\sigma_r}{dr}+\frac{\sigma_r}{1+H}=\frac{EH\phi}{1+H}-\frac{\sigma_0}{1+H}, \tag{4.47}$$

solution of Eq. (4.47):

$$\sigma_r=C_3 r^{-1-W}+C_4 r^{-1+W}+EH\phi-\sigma_0, \tag{4.48}$$

solution for the circumferential stress component:

$$\sigma_\theta=-WC_3 r^{-1-W}+WC_4 r^{-1+W}+EH\phi-\sigma_0, \tag{4.49}$$

solution for the radial displacement:

$$
\begin{aligned}
u \;=\; & \frac{1}{EH}\left\{-\left[W+H(W+\nu)\right]C_3 r^{-W}+\left[W+H(W-\nu)\right]C_4 r^{W}\right. \\
& \left.-H(1-\nu)\sigma_0 r\right\}+H\phi(1-\nu)r-A_2\cos\theta,
\end{aligned} \tag{4.50}
$$

then, solutions for the plastic strains in axial and circumferential direction:

$$
\epsilon_z^p=-\epsilon_\theta^p=\frac{W}{EH}\left[\frac{C_3}{r^{1+W}}-\frac{C_4}{r^{1-W}}\right]-\phi. \tag{4.51}
$$

4.3.4.3 Plastic Region II

Governing differential equation:

$$
r^2\frac{d^2\sigma_r}{dr^2}+3r\frac{d\sigma_r}{dr}=\frac{EH\phi}{1+H}+\frac{2\sigma_0}{1+H}, \tag{4.52}
$$

solution of Eq. (4.52):

$$
\sigma_r=\frac{C_5}{r^2}+C_6+\frac{EH\phi\ln r}{2\,(1+H)}+\frac{\sigma_0\ln r}{1+H}, \tag{4.53}
$$

solution for the circumferential stress component:

$$
\sigma_\theta=-\frac{C_5}{r^2}+C_6+\frac{EH\phi(1+\ln r)}{2\,(1+H)}+\frac{\sigma_0(1+\ln r)}{1+H}, \tag{4.54}
$$

solution for the radial displacement:

$$
\begin{aligned}
u \;=\; & \frac{1}{E}\left\{-\frac{\left[2+H\,(1+\nu)\right]C_5}{Hr}+(1-\nu)\,C_6 r+\frac{\sigma_0\,(1-\nu)\,r\ln r}{1+H}\right\} \\
& -\frac{\phi\left[1+H-H\,(1-\nu)\ln r\right]r}{2\,(1+H)}-A_2\cos\theta,
\end{aligned} \tag{4.55}
$$

and finally, solutions for the plastic strains in circumferential and radial direction:

$$
\epsilon_\theta^p=-\epsilon_r^p=-\frac{1}{EH}\left[\frac{2C_5}{r^2}+\frac{\sigma_0 H}{1+H}\right]+\frac{\phi}{2\,(1+H)}. \tag{4.56}
$$

4.4 Plane Strain

4.4.1 Elastic

The plastic strains vanish in the elastic region, i.e., $\epsilon_i^p=0$. In the plane strain solution $\epsilon_z=0$. Solving Eq. (4.3) for σ_z

$$
\sigma_z=\nu\left(\sigma_r+\sigma_\theta\right), \tag{4.57}
$$

and with the equilibrium equation, Eq. (4.7), the axial stress component, σ_z becomes

$$\sigma_z = \nu \left[\sigma_r + \frac{d}{dr}(r\sigma_r) \right]. \tag{4.58}$$

Substituting $\epsilon_r^p = 0$, $\epsilon_\theta^p = 0$ and Eq. (4.58) in the differential equation, Eq. (4.15);

$$r^2 \frac{d^2\sigma_r}{dr^2} + 3r\frac{d\sigma_r}{dr} = \frac{E\phi}{1-\nu^2}. \tag{4.59}$$

Solving the differential equation, Eq. (4.59) for σ_r :

$$\sigma_r = \frac{A}{r^2} + \left[B + \frac{E\phi(\ln 2 + \ln(1-\nu^2))}{1-\nu^2} \right] + \frac{E\phi \ln r}{2(1-\nu^2)}, \tag{4.60}$$

or simply

$$\sigma_r = \frac{C_1}{r^2} + C_2 + \frac{E\phi \ln r}{2(1-\nu^2)}. \tag{4.61}$$

From the equilibrium equation, Eq. (4.7) the tangential stress is obtained as;

$$\sigma_\theta = -\frac{C_1}{r^2} + C_2 + \frac{E\phi(1+\ln r)}{2(1-\nu^2)}. \tag{4.62}$$

The axial stress component is determined by substituting Eqs. (4.61) in Eq. (4.57) as

$$\sigma_z = 2\nu C_2 + \frac{E\nu\phi}{2(1-\nu^2)} + \frac{E\nu\phi \ln r}{1-\nu^2}. \tag{4.63}$$

By substituting the strains, Eq. (4.2) and Eq. (4.58) in Eq. (4.13), the radial displacement becomes

$$\begin{aligned} u \;=\; & \frac{1}{E}\left[-\frac{1}{r}(1+\nu)C_1 + (1+\nu)(1-2\nu)C_2 r \right] \\ & -\frac{[1-\nu-(1-2\nu)\ln r]\,\phi r}{2(1-\nu)} - A_2\cos\theta. \end{aligned} \tag{4.64}$$

4.4.2 Plastic Region I

The stress state in this region is

$$\sigma_r > \sigma_z > \sigma_\theta, \tag{4.65}$$

and Tresca's yield criterion reads

$$\sigma_y = \sigma_r - \sigma_\theta, \tag{4.66}$$

From the equivalence of plastic work increment,

$$\epsilon_{EQ} = \epsilon_r^p = -\epsilon_\theta^p \text{ and } \epsilon_z^p = 0, \tag{4.67}$$

The yield stress is

$$\sigma_y = \sigma_0 \left(1 + \eta \epsilon_{EQ}\right), \tag{4.68}$$

then combining Eqs. (4.66) and (4.68)

$$\epsilon_{EQ} = \left(\frac{\sigma_r - \sigma_\theta}{\sigma_0} - 1 \right) \frac{1}{\eta}. \tag{4.69}$$

Solving Eq. (4.3) for σ_z by using Eq. (4.7)

$$\sigma_z = \nu \left[\sigma_r + \frac{d}{dr}(r\sigma_r) \right]. \tag{4.70}$$

By using Eq. (4.7) with Eq. (4.67), ϵ_θ^p and ϵ_r^p become

$$\epsilon_r^p = -\epsilon_\theta^p = \left\{ \frac{1}{\sigma_0} \left[\sigma_r - \frac{d}{dr}(r\sigma_r) \right] - 1 \right\} \frac{1}{\eta}. \tag{4.71}$$

Substituting Eqs. (4.71) and (4.58) in the governing equation, Eq. (4.15)

$$r^2 \frac{d^2\sigma_r}{dr^2} + 3r \frac{d\sigma_r}{dr} = \frac{EH\phi}{1 + H(1-\nu^2)} - \frac{2\sigma_0}{1 + H(1-\nu^2)}. \tag{4.72}$$

Solving the differential equation, Eq. (4.72);

$$\sigma_r = \frac{C_3}{r^2} + C_4 + \frac{EH\phi \ln r}{2\left[1 + H(1-\nu^2)\right]} - \frac{\sigma_0 \ln r}{1 + H(1-\nu^2)}. \tag{4.73}$$

With the equilibrium equation, Eq. (4.7), the tangential stress becomes;

$$\sigma_\theta = -\frac{C_3}{r^2} + C_4 + \frac{EH\phi(1 + \ln r)}{2\left[1 + H(1-\nu^2)\right]} - \frac{\sigma_0(1 + \ln r)}{1 + H(1-\nu^2)}. \tag{4.74}$$

The axial stress component is determined by substituting Eqs. (4.73) in Eq. (4.70) as

$$\sigma_z = 2\nu C_4 - \frac{\nu\sigma_0(1 + 2\ln r)}{1 + H(1-\nu^2)} + \frac{EH\nu\phi(1 + 2\ln r)}{2\left[1 + H(1-\nu^2)\right]}. \tag{4.75}$$

The plastic strain components are determined by substituting Eq. (4.73) in Eq. (4.71) as

$$\epsilon_r^p = -\epsilon_\theta^p = \frac{1}{E} \left[\frac{2C_3}{Hr^2} - \frac{(1-\nu^2)\sigma_0}{1 + H(1-\nu^2)} \right] - \frac{\phi}{2\left[1 + H(1-\nu^2)\right]}. \tag{4.76}$$

Combining Eqs. (4.2) and (4.76) and substituting in Eq. (4.13), the displacement becomes

$$\begin{aligned} u \;=\; & \frac{1}{E} \left\{ -\frac{\left[2 + H(1+\nu)\right] C_3}{Hr} + (1+\nu)(1-2\nu)C_4 r \right. \\ & \left. - \frac{\sigma_0(1+\nu)(1-2\nu) r \ln r}{1 + H(1-\nu^2)} \right\} \\ & - \left[\frac{1 + H(1-\nu^2) - H(1+\nu)(1-2\nu)\ln r}{1 + H(1-\nu^2)} \right] \frac{\phi r}{2} \\ & - A_2 \cos\theta. \end{aligned} \tag{4.77}$$

4.4.2.1 Plastic Region II

The stress state in this region is

$$\sigma_\theta > \sigma_z > \sigma_r. \tag{4.78}$$

and Tresca's yield criterion reads

$$\sigma_y = \sigma_\theta - \sigma_r, \tag{4.79}$$

The plastic strains and the equivalent plastic strain are

$$\epsilon_\theta^p = -\epsilon_r^p = \epsilon_{EQ} \text{ and } \epsilon_z^p = 0. \tag{4.80}$$

Solving Eq. (4.3) for σ_z by using Eq. (4.7)

$$\sigma_z = \nu \left[\sigma_r + \frac{d}{dr}(r\sigma_r)\right]. \tag{4.81}$$

The yield stress is

$$\sigma_y = \sigma_0 \left(1 + \eta\epsilon_{EQ}\right), \tag{4.82}$$

then

$$\epsilon_{EQ} = \left(\frac{\sigma_\theta - \sigma_r}{\sigma_0} - 1\right)\frac{1}{\eta}. \tag{4.83}$$

By using Eqs. (4.7) and (4.83) ϵ_r^p and ϵ_θ^p become

$$\epsilon_\theta^p = -\epsilon_r^p = \left\{\frac{1}{\sigma_0}\left[\frac{d}{dr}(r\sigma_r) - \sigma_r\right] - 1\right\}\frac{1}{\eta}. \tag{4.84}$$

Substituting Eqs. (4.84) and (4.58) in the differential equation, Eq. (4.15)

$$r^2\frac{d^2\sigma_r}{dr^2} + 3r\frac{d\sigma_r}{dr} = \frac{EH\phi}{1 + H(1-\nu^2)} + \frac{2\sigma_0}{1 + H(1-\nu^2)}. \tag{4.85}$$

Solving the differential equation, Eq. (4.85);

$$\sigma_r = \frac{C_5}{r^2} + C_6 + \frac{EH\phi\ln r}{2\left[1 + H(1-\nu^2)\right]} + \frac{\sigma_0 \ln r}{1 + H(1-\nu^2)}. \tag{4.86}$$

From the equilibrium equation, Eq. (4.7) we get σ_θ as;

$$\sigma_\theta = -\frac{C_5}{r^2} + C_6 + \frac{EH\phi(1 + \ln r)}{2\left[1 + H(1-\nu^2)\right]} + \frac{\sigma_0(1 + \ln r)}{1 + H(1-\nu^2)}. \tag{4.87}$$

The axial stress component is determined by substituting Eqs. (4.86) in Eq. (4.70) as

$$\sigma_z = 2\nu C_6 + \frac{\nu\sigma_0(1 + 2\ln r)}{1 + H(1-\nu^2)} + \frac{EH\nu\phi(1 + 2\ln r)}{2\left[1 + H(1-\nu^2)\right]}. \tag{4.88}$$

With Eq. (4.84), the plastic strain components become

$$\epsilon_r^p = -\epsilon_\theta^p = \frac{1}{E}\left[\frac{2C_5}{Hr^2} + \frac{(1-\nu^2)\sigma_0}{1+H(1-\nu^2)}\right] - \frac{\phi}{2\left[1+H(1-\nu^2)\right]}. \tag{4.89}$$

From the Eq. (4.13) the displacement takes the form of

$$\begin{aligned} u \;=\; & \frac{1}{E}\left\{-\frac{\left[2+H(1+\nu)\right]C_5}{Hr} + (1+\nu)(1-2\nu)C_6 r\right. \\ & \left. + \frac{\sigma_0(1+\nu)(1-2\nu)r\ln r}{1+H(1-\nu^2)}\right\} \\ & - \left[\frac{1+H(1-\nu^2) - H(1+\nu)(1-2\nu)\ln r}{1+H(1-\nu^2)}\right]\frac{\phi r}{2} \\ & - A_2\cos\theta. \end{aligned} \tag{4.90}$$

Analytical solutions for positive bending are also given in Appendix (C).

4.4.3 Summary of the equations of plane strain analytical solution

4.4.3.1 Elastic Region

Governing differential equation:

$$r^2\frac{d^2\sigma_r}{dr^2} + 3r\frac{d\sigma_r}{dr} = \frac{E\phi}{1-\nu^2}, \tag{4.91}$$

solution of Eq. (4.91):

$$\sigma_r = \frac{C_1}{r^2} + C_2 + \frac{E\phi\ln r}{2(1-\nu^2)}, \tag{4.92}$$

solution for the circumferential stress component:

$$\sigma_\theta = -\frac{C_1}{r^2} + C_2 + \frac{E\phi(1+\ln r)}{2(1-\nu^2)}, \tag{4.93}$$

solution for the axial stress component:

$$\sigma_z = 2\nu C_2 + \frac{E\nu\phi}{2(1-\nu^2)} + \frac{E\nu\phi\ln r}{1-\nu^2}, \tag{4.94}$$

and solution for the radial displacement:

$$\begin{aligned} u \;=\; & \frac{1}{E}\left[-\frac{1}{r}(1+\nu)C_1 + (1+\nu)(1-2\nu)C_2 r\right] \\ & - \frac{\left[1-\nu-(1-2\nu)\ln r\right]\phi r}{2(1-\nu)} - A_2\cos\theta. \end{aligned} \tag{4.95}$$

4.4.3.2 Plastic Region I

Governing differential equation:

$$r^2\frac{d^2\sigma_r}{dr^2}+3r\frac{d\sigma_r}{dr}=\frac{EH\phi}{1+H(1-\nu^2)}-\frac{2\sigma_0}{1+H(1-\nu^2)}, \tag{4.96}$$

solution of Eq. (4.96):

$$\sigma_r=\frac{C_3}{r^2}+C_4+\frac{EH\phi\ln r}{2\left[1+H(1-\nu^2)\right]}-\frac{\sigma_0\ln r}{1+H(1-\nu^2)}, \tag{4.97}$$

solution for the circumferential stress component:

$$\sigma_\theta=-\frac{C_3}{r^2}+C_4+\frac{EH\phi(1+\ln r)}{2\left[1+H(1-\nu^2)\right]}-\frac{\sigma_0(1+\ln r)}{1+H(1-\nu^2)}, \tag{4.98}$$

solution for the axial stress component:

$$\sigma_z=2\nu C_4-\frac{\nu\sigma_0(1+2\ln r)}{1+H(1-\nu^2)}+\frac{EH\nu\phi(1+2\ln r)}{2\left[1+H(1-\nu^2)\right]}, \tag{4.99}$$

solution for the radial displacement:

$$\begin{aligned} u \;=\; & \frac{1}{E}\left\{-\frac{\left[2+H(1+\nu)\right]C_3}{Hr}+(1+\nu)(1-2\nu)C_4 r\right. \\ & \left.-\frac{\sigma_0(1+\nu)(1-2\nu)r\ln r}{1+H(1-\nu^2)}\right\} \\ & -\left[\frac{1+H(1-\nu^2)-H(1+\nu)(1-2\nu)\ln r}{1+H(1-\nu^2)}\right]\frac{\phi r}{2} \\ & -A_2\cos\theta, \end{aligned} \tag{4.100}$$

then, solutions for the plastic strains in circumferential and radial direction:

$$-\epsilon_\theta^p=\epsilon_r^p=\frac{1}{E}\left[\frac{2C_3}{Hr^2}-\frac{(1-\nu^2)\sigma_0}{1+H(1-\nu^2)}\right]-\frac{\phi}{2\left[1+H(1-\nu^2)\right]}. \tag{4.101}$$

4.4.3.3 Plastic Region II

Governing differential equation:

$$r^2\frac{d^2\sigma_r}{dr^2}+3r\frac{d\sigma_r}{dr}=\frac{EH\phi}{1+H(1-\nu^2)}+\frac{2\sigma_0}{1+H(1-\nu^2)}, \tag{4.102}$$

solution of Eq. (4.102):

$$\sigma_r=\frac{C_5}{r^2}+C_6+\frac{EH\phi\ln r}{2\left[1+H(1-\nu^2)\right]}+\frac{\sigma_0\ln r}{1+H(1-\nu^2)}, \tag{4.103}$$

solution for the circumferential stress component:

$$\sigma_\theta = -\frac{C_5}{r^2} + C_6 + \frac{EH\phi(1+\ln r)}{2\left[1+H(1-\nu^2)\right]} + \frac{\sigma_0(1+\ln r)}{1+H(1-\nu^2)}, \tag{4.104}$$

solution for the axial stress component:

$$\sigma_z = 2\nu C_6 + \frac{\nu\sigma_0(1+2\ln r)}{1+H(1-\nu^2)} + \frac{EH\nu\phi(1+2\ln r)}{2\left[1+H(1-\nu^2)\right]}, \tag{4.105}$$

solution for the radial displacement:

$$\begin{aligned} u \;=\; & \frac{1}{E}\left\{-\frac{\left[2+H(1+\nu)\right]C_5}{Hr} + (1+\nu)(1-2\nu)C_6 r \right. \\ & \left. + \frac{\sigma_0(1+\nu)(1-2\nu)r\ln r}{1+H(1-\nu^2)}\right\} \\ & - \left[\frac{1+H(1-\nu^2)-H(1+\nu)(1-2\nu)\ln r}{1+H(1-\nu^2)}\right]\frac{\phi r}{2} \\ & - A_2\cos\theta, \end{aligned} \tag{4.106}$$

and finally, solutions for the plastic strains in radial and circumferential direction:

$$\epsilon_r^p = -\epsilon_\theta^p = \frac{1}{E}\left[\frac{2C_5}{Hr^2} + \frac{(1-\nu^2)\sigma_0}{1+H(1-\nu^2)}\right] - \frac{\phi}{2\left[1+H(1-\nu^2)\right]}. \tag{4.107}$$

CHAPTER 5

RESULTS AND DISCUSSION

The following data are used in the numerical solutions: $E = 200$ GPa, $\sigma_0 = 250$ MPa, $\nu = 0.3$ and the dimensionless variables for radial coordinate $\overline{r} = r/a$, bending moment $\overline{M} = M/(\sigma_0 \times b^2 \times t)$, normal stress $\overline{\sigma}_j = \sigma_j/\sigma_0$, normal strain $\overline{\epsilon}_j = \epsilon_j E/\sigma_0$, and displacement $\overline{u} = uE/b\sigma_0$ where a and b are inner and outer diameters, t is the thickness of the beam. In the calculations, $b = 1.0$ and $t = 1.0$.

5.1 Plane Stress

5.1.1 Elastic

Initially, the entire beam is stress free. The beam deforms elastically under bending moment $M \leq M_E$ and becomes partially plastic when $M \geqslant M_E$. Therefore, M_E is called the elastic limit load.

The distribution of the stresses and displacement are described by Eqs. (4.17)-(4.19). In the stress equations, there are three unknowns which are C_1, C_2 and ϕ. For evaluation of these unknowns, three linearly independent conditions are available. These are:

$$\int_a^b \sigma_\theta r dr = M, \tag{5.1}$$

$$\sigma_r(a) = \sigma_r(b) = 0, \tag{5.2}$$

where M is the couple moment per unit width.

The solution of the integral, Eq. (5.1):

$$C_1 \ln(b/a) + \frac{1}{2}C_2(b^2 - a^2) + \frac{1}{8}E\phi\left[-a^2(1 + 2\ln a) + b^2(1 + 2\ln b)\right] = M. \tag{5.3}$$

With the help of the above equations, it is possible to get the analytical expressions for the unknowns. Combining Eqs. (5.2) and (5.3), C_1, C_2 and ϕ are obtained as

$$C_1 = \frac{4M}{N}a^2b^2 \ln(b/a), \tag{5.4}$$

$$C_2 = -\frac{4M}{N}(b^2 \ln b - a^2 \ln a), \tag{5.5}$$

$$\phi = -\frac{8M}{EN}(a^2 - b^2), \tag{5.6}$$

in which

$$N = (a^2 - b^2)^2 - 4a^2b^2[\ln(b/a)]^2. \tag{5.7}$$

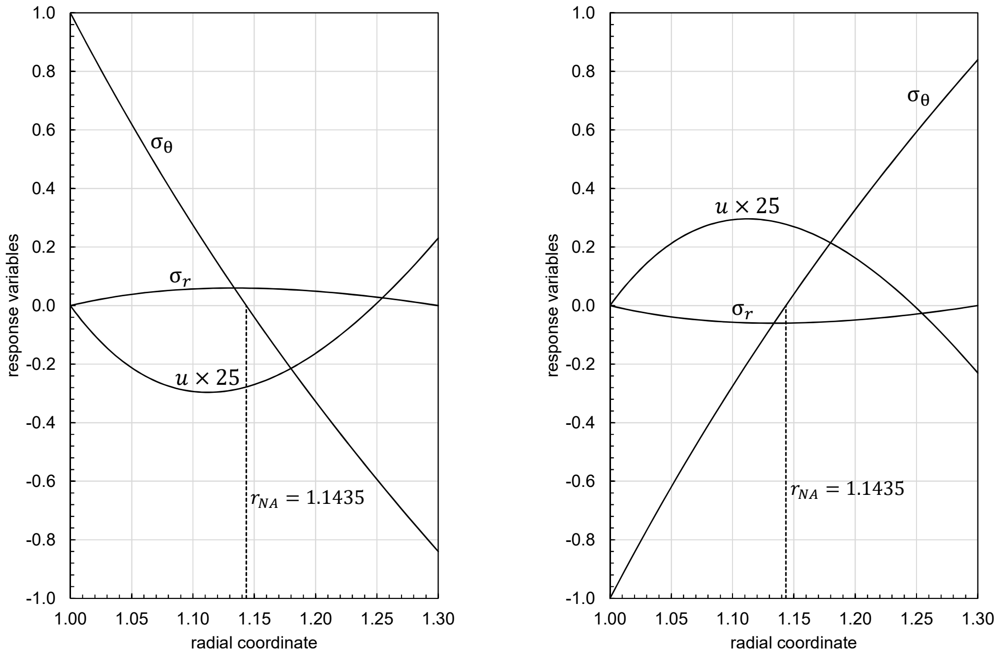
The displacement equation, Eq. (4.19), contains the fourth unknown A_2 that has to be determined. The displacement will be equal to zero assuming that the beam is rigidly fixed at $r = a$, $\theta = 0$, i.e.

$$u(a, 0) = \frac{1}{E}\left[-\frac{1}{a}(1 + \nu)C_1 + (1 - \nu)C_2a\right] - [1 - (1 - \nu)\ln a]\frac{\phi a}{2} - A_2 = 0. \tag{5.8}$$

Solving Eq. (5.8) for A_2

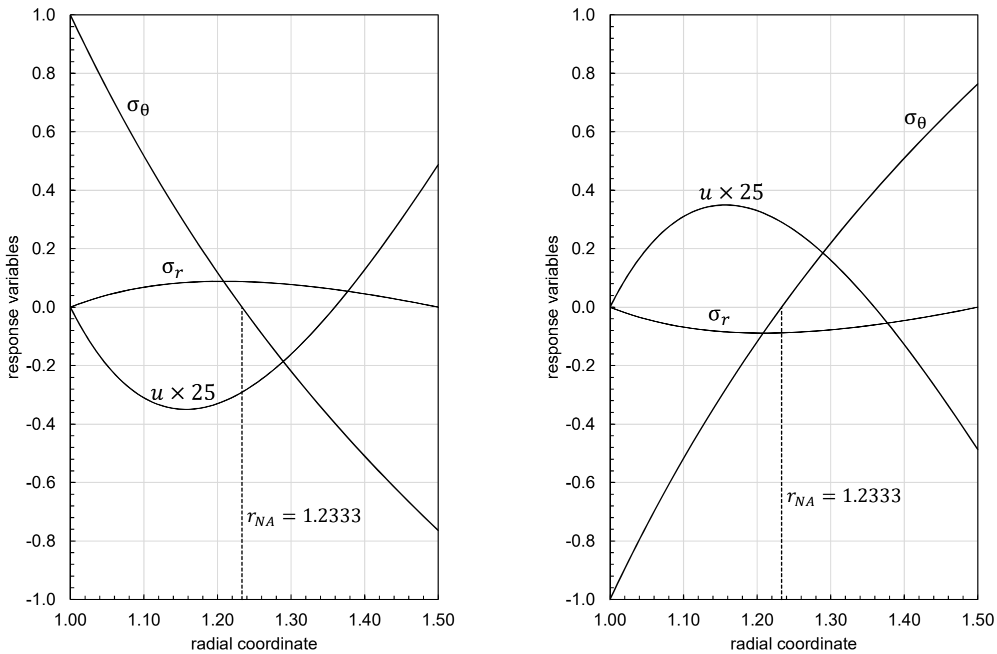
$$A_2 = \frac{1}{E}\left[-\frac{1}{a}(1 + \nu)C_1 + (1 - \nu)C_2a\right] - [1 - (1 - \nu)\ln a]\frac{\phi a}{2}. \tag{5.9}$$

The elastic limit load for a beam of $b/a = 1.3$ is calculated as $\overline{M}_E = 8.0999 \times 10^{-3}$. The corresponding integration constants C_1, C_2, ϕ, and A_2 are calculated as $\overline{C}_1 = 1.3332$, $\overline{C}_2 = -1.3332$, $\phi = 8.7656 \times 10^{-3}$, and $A_2 = -7.7043 \times 10^{-3}$ with the dimensionless form of $\overline{C}_i = C_i/\sigma_0$. The elastic limit load for a beam of $b/a = 1.5$ is $\overline{M}_E = 1.6027 \times 10^{-2}$. Then, the unknowns $\overline{C}_1, \overline{C}_2, \phi$, and A_2 take the numerical values $\overline{C}_1 = 7.0566 \times 10^{-1}$, $\overline{C}_2 = -7.0566 \times 10^{-1}$, $\phi = 5.4386 \times 10^{-3}$, $A_2 = -4.4590 \times 10^{-3}$.



(a) (b)

Figure 5.1: The stress response under the elastic limit load of a beam of b/a=1.3 for (a) positive bending case, (b) negative bending case.



(a) (b)

Figure 5.2: The stress response under the elastic limit load of a beam of b/a=1.5 for (a) positive bending case, (b) negative bending case.

The stress response of the beam at the plane of symmetry, i.e. $\theta = 0$ under elastic limit load $\overline{M} = \overline{M}_E$ is plotted for $b/a = 1.3$ in Fig. (5.1) (a) and (b) for positive and negative bending cases, respectively, and for $b/a = 1.5$ in Fig. (5.2) (a) and (b). The solution of $\sigma_\theta(r_{NA}) = 0$ is achieved by Newton-Raphson method and the neutral axis are calculated for the beam of $b/a = 1.3$ and $b/a = 1.5$ as $\overline{r}_{NA} = 1.1435$ and 1.2333, respectively.

5.1.2 First Stage of Elastic-Plastic Solution

The curved beam consists of a plastic region in

$$a \leq r \leq r_1,$$

and an elastic region in

$$r_1 \leq r \leq b,$$

where r_1 is the plastic-elastic border radius.

The distribution of the stresses, displacement and plastic strains are described by Eqs. (4.28)-(4.31). In the first stage of elastic-plastic solution, it is required to solve elastic region and plastic region I together. $C_1, C_2, C_3, C_4, A_2, \phi$ and r_1 are the unknowns that should be evaluated to solve this problem. Corresponding seven boundary and interface conditions are available to determine these unknowns. These are:

$$\begin{aligned}
u^p(a,0) &= 0, \\
\sigma_r^p(a) &= 0, \\
\sigma_0 &= -\sigma_\theta^e(r_1), \\
\sigma_r^e(b) &= 0, \\
\sigma_r^p(r_1) &= \sigma_r^e(r_1), \\
u^p(r_1,\theta) &= u^e(r_1,\theta), \\
\int_a^b \sigma_\theta r dr &= \int_a^{r_1} \sigma_\theta^p r dr + \int_{r_1}^b \sigma_\theta^e r dr = M.
\end{aligned}$$

In explicit forms:

$$u^p(a,0) = 0 : -A_2 - \frac{[W + H(W+\nu)]\, C_3}{EHa^W} + \frac{[W + H(W-\nu)]\, a^W C_4}{EH} \tag{5.10}$$
$$+H(1-\nu)a\phi = \frac{\sigma_0 a(1-\nu)}{E},$$

$$\sigma_r^p(a) = 0 : \frac{C_3}{a^{1+W}} + \frac{C_4}{a^{1-W}} + EH\phi = \sigma_0, \tag{5.11}$$

$$\sigma_0 = -\sigma_\theta^e(r_1) : -\frac{C_1}{r_1^2} + C_2 + \frac{E(1+\ln r_1)}{2}\phi = -\sigma_0, \tag{5.12}$$

$$\sigma_r^e(b) = 0 : \frac{C_1}{b^2} + C_2 + \frac{E\ln b}{2}\phi = 0, \tag{5.13}$$

$$\sigma_r^p(r_1) = \sigma_r^e(r_1) : -\frac{C_1}{r_1^2} - C_2 + \frac{C_3}{r_1^{1+W}} + \frac{C_4}{r_1^{1-W}} + E\left[H - \frac{1}{2}\ln r_1\right]\phi = \sigma_0, \tag{5.14}$$

$$\begin{aligned} u^p(r_1,\theta) &= u^e(r_1,\theta) : \frac{H(1+\nu)C_1}{r_1} - H(1-\nu)r_1C_2 \\ &\quad -\frac{[W+H(W+\nu)]\,C_3}{r_1^W} + [W+H(W-\nu)]\,r_1^W C_4 \\ &\quad +[1+2H(1-\nu)-(1-\nu)\ln r_1]\,\frac{EHr_1\phi}{2} \\ &= Hr_1(1-\nu)\sigma_0, \end{aligned} \tag{5.15}$$

$$\begin{aligned} \int_a^b \sigma_\theta r dr = M : \ln(r_1/b)C_1 + \frac{1}{2}(b^2-r_1^2)C_2 + \frac{W(a^{1-W}-r_1^{1-W})C_3}{1-W} \\ +\frac{W(r_1^{1+W}-a^{1+W})C_4}{1+W} + \big[-4a^2H + b^2(1+2\ln b) \\ -r_1^2(1-4H+2\ln r_1)\big]\,\frac{E\phi}{8} = M + (r_1^2-a^2)\frac{\sigma_0}{2}. \end{aligned} \tag{5.16}$$

The beam becomes partially plastic under the load $M \geq M_E$. The first stage of elasto-plastic deformation is in the range $M_E \leq M \leq M_I$ with $\overline{M}_I = 9.5488 \times 10^{-3}$ for a beam of $b/a = 1.3$. The curved beam consists of a plastic region in $a \leq r \leq r_1$ and an elastic region in $r_1 \leq r \leq b$, the plastic region is governed by Tresca's yield criterion $\sigma_y = -\sigma_\theta$ in negative bending case. The hardening parameter, H is taken as 0.2 for $b/a = 1.3$ and $b/a = 1.5$ in the plane stress calculations. The integration constants are calculated for $\overline{M} = \overline{M}_I$. As seen in Fig. (5.3) (b), the plastic deformation begins at $r/a = 1$ and $r/a = 1.3$ is also critical location as $\overline{\sigma}_\theta - \overline{\sigma}_r = 1$. In the comprehensive study by Eraslan and Arslan [17], the optimization method was used to solve the systems of nonlinear equations with MINPACK library [52]-[54]. In this thesis, all of the above equations obtained for the boundary and interface conditions are solved simultaneously by using subroutıne NLNF which solves non-linear system of equations using Newton method included in NONLIN library. Subroutines are supplied for the nonlinear equations to estimate the unknowns $C_1, C_2, C_3, C_4, A_2, \phi$ and r_1. The definition of NONLIN solver and the subroutine NLNF can be found in Appendix (D) and the subroutines used in calculations are available in Appendix (E). The solution using all of the nonlinear equations will be called Version I in this thesis. Thereafter, the unknowns are found as $\overline{C}_1 = 1.5986$, $\overline{C}_2 = -1.5986$, $\overline{C}_3 = 1.2153$, $\overline{C}_4 = -2.0868$, $A_2 = -9.2294 \times 10^{-3}$, $\phi = 1.0493 \times 10^{-2}$, and $\overline{r}_1 = 1.0218$. The unknowns are calculated for a beam of $b/a = 1.5$ under the load $\overline{M}_I = 2.0561 \times 10^{-2}$ as $\overline{C}_1 = 9.4093 \times 10^{-1}$, $\overline{C}_2 = -9.4093 \times 10^{-1}$, $\overline{C}_3 = 7.9433 \times 10^{-1}$, $\overline{C}_4 = -1.2266$, $A_2 = -5.9250 \times 10^{-3}$, $\phi = 7.2076 \times 10^{-3}$, and $\overline{r}_1 = 1.0510$. However, by giving a careful consideration for these seven equations, it can be seen that the constants $C_1, C_2, C_3,$ and C_4 can be solved by a linear system of the Eqs.(5.10), (5.11), (5.12), and (5.13). After solving the four linear equations and obtaining the definitions for $C_1, C_2, C_3,$ and C_4, the number of equations is reduced from seven to three. The remaining three constants A_2, ϕ and r_1 are calculated by solving the three nonlinear equations which are Eqs.(5.14), (5.15), and (5.16). Then, the results for the unknowns A_2, ϕ and r_1 are substituted in the equations for C_1, C_2, C_3, C_4 and hence all of the unknowns are obtained. The solution achieved in this way will be called Version II in this thesis. Solving the linear equations, Eqs.(5.10), (5.11), (5.12), and (5.13), $C_1, C_2, C_3,$ and C_4 become

$$C_1 = \frac{b^2 r_1^2 \sigma_0}{b^2 + r_1^2} + \frac{E\phi b^2 r_1^2 \left[1 + \ln(r_1/b)\right]}{2\left(b^2 + r_1^2\right)}, \tag{5.17}$$

$$C_2 = -\frac{r_1^2 \sigma_0}{b^2 + r_1^2} - \frac{E\phi(r_1^2 + b^2 \ln b + r_1^2 \ln r_1)}{2\left(b^2 + r_1^2\right)}, \tag{5.18}$$

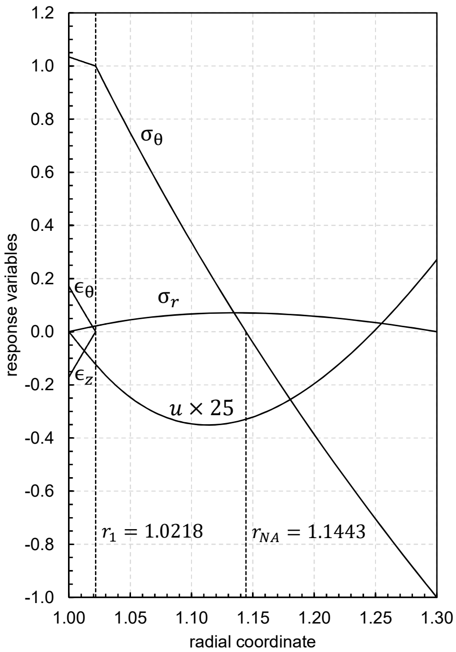
$$\begin{aligned} C_3 \;=\; & -\frac{a^W A_2 E H}{2(1+H)W} + \frac{a^{1+W}\left[W - H(1-W)\right]\sigma_0}{2(1+H)W} \\ & -\frac{a^{1+W} E H \phi \left[W - H(1-W)\right]}{2(1+H)W}, \end{aligned} \tag{5.19}$$

$$\begin{aligned} C_4 \;=\; & \frac{a^{-W} A_2 E H}{2(1+H)W} + \frac{a^{1-W}\left[W + H(1+W)\right]\sigma_0}{2(1+H)W} \\ & -\frac{a^{1-W} E H \phi \left[W + H(1+W)\right]}{2(1+H)W}. \end{aligned} \tag{5.20}$$

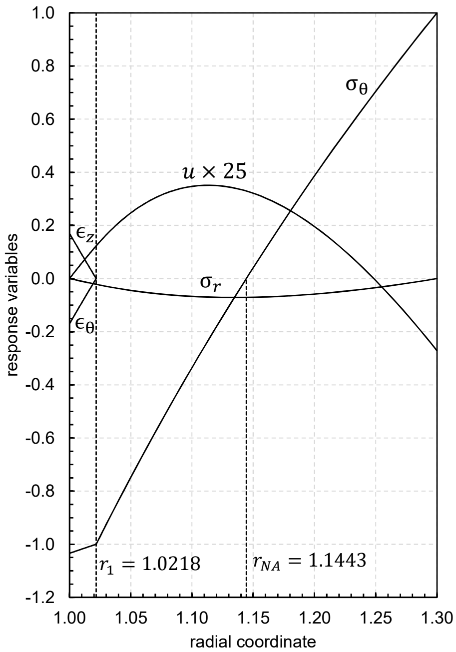
The remaining constants A_2, ϕ and r_1 are calculated solving the nonlinear systems of equations, Eqs.(5.14), (5.15), and (5.16). The results for the beam of $b/a = 1.3$ are $A_2 = -9.2294 \times 10^{-3}$, $\phi = 1.0493 \times 10^{-2}$, and $\overline{r}_1 = 1.0218$, and for the beam of $b/a = 1.5$ $A_2 = -5.9250 \times 10^{-3}$, $\phi = 7.2076 \times 10^{-3}$, and $\overline{r}_1 = 1.0510$. Substituting the solution of the unknowns A_2, ϕ and r_1 in Eqs.(5.17), (5.18), (5.19) and (5.20) $\overline{C}_1, \overline{C}_2, \overline{C}_3,$ and $\overline{C}_4$ for $b/a = 1.3$ become $\overline{C}_1 = 1.5986$, $\overline{C}_2 = -1.5986$, $\overline{C}_3 = 1.2153$, $\overline{C}_4 = -2.0868$,and for $b/a = 1.5$ $\overline{C}_1 = 9.4093 \times 10^{-1}$, $\overline{C}_2 = -9.4093 \times 10^{-1}$, $\overline{C}_3 = 7.9433 \times 10^{-1}, \overline{C}_4 = -1.2266$.

The stress response of the beam at the plane of symmetry, i.e. $\theta = 0$ under $\overline{M} = \overline{M}_I$ is plotted for a beam of $b/a = 1.3$ in Fig. (5.3) (a) and (b) and for $b/a = 1.5$ in Fig. (5.5) (a) and (b). The solution of $\sigma_\theta(r_{NA}) = 0$ is achieved by Newton-Raphson method and the neutral axis are calculated for the beam of $b/a = 1.3$ and $b/a = 1.5$ as $\overline{r}_{NA} = 1.1443$ and 1.2364, respectively.

The residual stress remaining in the structure after the unloading is simply obtained by the difference of the stress distribution for elastic-plastic behavior and the one for elastic behavior both evaluated at the load under consideration. By applying this evaluation on Stage I, i.e. taking difference of the stresses for elastic-plastic and elastic behavior under $\overline{M}_I = 9.5488 \times 10^{-3}$ for a beam of $b/a = 1.3$ and $\overline{M}_I = 2.0561 \times 10^{-2}$ for a beam of $b/a = 1.5$, the unknowns for the residual stresses are obtained for the beam $b/a = 1.3$ as $\overline{C}_1 = 1.5717$, $\overline{C}_2 = -1.5717$, $\phi = 1.0334 \times 10^{-2}$, $A_2 = -9.0825 \times 10^{-3}$, and for the beam $b/a = 1.5$ as $\overline{C}_1 = 9.0527 \times 10^{-1}$, $\overline{C}_2 = -9.0527 \times 10^{-1}$, $\phi = 6.9771 \times 10^{-3}$, $A_2 = -5.7205 \times 10^{-3}$ and the corresponding graphs can be seen in Fig. (5.4) (a) and (b), and Fig. (5.6) (a) and (b), respectively.

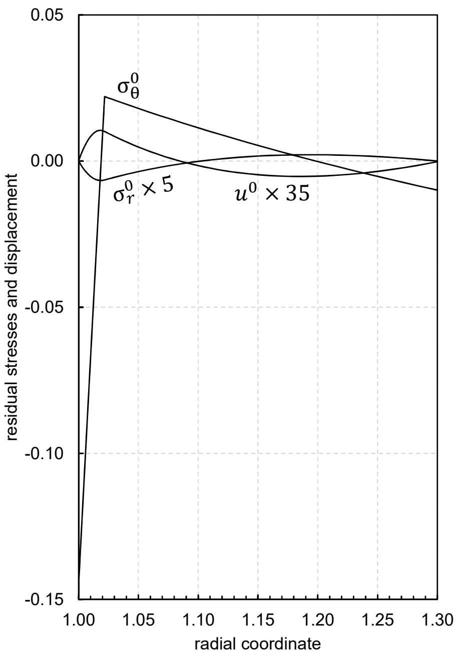


(a)

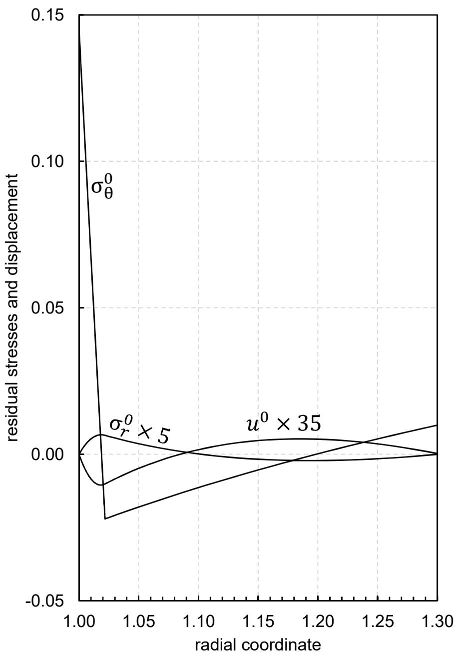


(b)

Figure 5.3: The stress response under $\overline{M} = \overline{M}_I$ of a beam of b/a=1.3 for (a) positive bending case, (b) negative bending case.

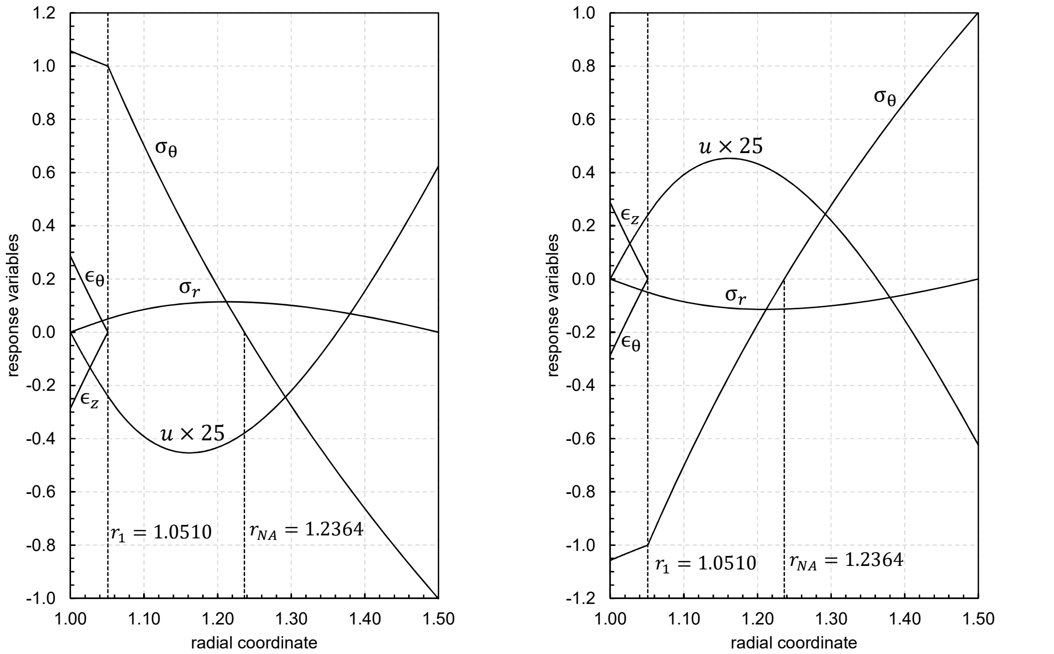


(a)



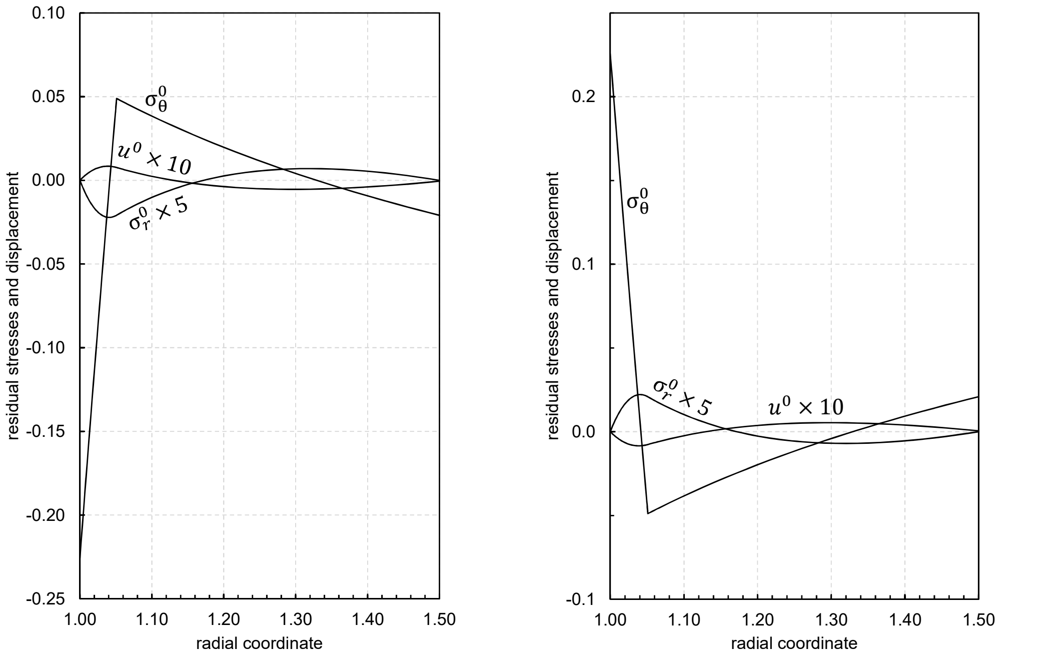
(b)

Figure 5.4: The residual stresses of a beam of b/a=1.3 in stage I for (a) positive bending case, (b) negative bending case.



(a) (b)

Figure 5.5: The stress response under $\overline{M} = \overline{M}_I$ of a beam of b/a=1.5 for (a) positive bending case, (b) negative bending case.



(a) (b)

Figure 5.6: The residual stresses of a beam of b/a=1.5 in stage I for (a) positive bending case, (b) negative bending case.

5.1.3 Second Stage of Elastic-Plastic Solution

The curved beam consists of an inner plastic region (region I) in

$$a \leq r \leq r_1,$$

an elastic region in

$$r_1 \leq r \leq r_2,$$

and an outer plastic region (region II) in

$$r_2 \leq r \leq b,$$

where r_1 and r_2 are the elastic-plastic border radii. The distribution of the stresses, displacement and plastic strains are described by Eqs. (4.39)-(4.43). In the second stage of elastic-plastic solution, it is required to solve elastic region, plastic region I and plastic region II together. $C_1, C_2, C_3, C_4, C_5, C_6, A_2, \phi$, r_1, and r_2 are the unknowns that should be evaluated to solve this problem. Ten boundary and interface conditions are available to determine these unknowns. These are:

$$\begin{aligned}
u^p(a,0) &= 0, \\
\sigma_r^p(a) &= 0, \\
\sigma_r^p(r_1) &= \sigma_r^e(r_1), \\
u^p(r_1,\theta) &= u^e(r_1,\theta), \\
\sigma_0 &= -\sigma_\theta^e(r_1), \\
\sigma_\theta^e(r_2) - \sigma_r^e(r_2) &= \sigma_0, \\
\sigma_r^e(r_2) &= \sigma_r^p(r_2), \\
u^e(r_2,\theta) &= u^p(r_2,\theta), \\
\sigma_r^p(b) &= 0, \\
\int_a^b \sigma_\theta r dr &= \int_a^{r_1} \sigma_\theta^p r dr + \int_{r_1}^{r_2} \sigma_\theta^e r dr + \int_{r_2}^{b} \sigma_\theta^p r dr = M.
\end{aligned}$$

In explicit forms:

$$\begin{aligned}
u^p(a,0) = 0 : -A_2 - \frac{[W + H(W+\nu)]\, C_3}{EHa^W} \\
+\frac{[W + H(W-\nu)]\, a^W C_4}{EH} + H(1-\nu)a\phi = \frac{\sigma_0 a(1-\nu)}{E},
\end{aligned} \tag{5.21}$$

$$\sigma_r^p(a) = 0 : \frac{C_3}{a^{1+W}} + \frac{C_4}{a^{1-W}} + EH\phi = \sigma_0, \tag{5.22}$$

$$\sigma_r^p(r_1) = \sigma_r^e(r_1) : -\frac{C_1}{r_1^2} - C_2 + \frac{C_3}{r_1^{1+W}} + \frac{C_4}{r_1^{1-W}} + E\left[H - \frac{1}{2}\ln r_1\right]\phi = \sigma_0, \tag{5.23}$$

$$\begin{aligned} u^p(r_1,\theta) = u^e(r_1,\theta) : \frac{H(1+\nu)C_1}{r_1} - H(1-\nu)r_1C_2 \\ -\frac{[W + H(W+\nu)]\,C_3}{r_1^W} + [W + H(W-\nu)]\,r_1^W C_4 \\ + [1 + 2H(1-\nu) - (1-\nu)\ln r_1]\,\frac{EHr_1\phi}{2} \\ = Hr_1(1-\nu)\sigma_0, \end{aligned} \tag{5.24}$$

$$\sigma_0 = -\sigma_\theta^e(r_1) : -\frac{C_1}{r_1^2} + C_2 + \frac{E(1+\ln r_1)}{2}\phi = -\sigma_0, \tag{5.25}$$

$$\sigma_r^e(r_2) - \sigma_\theta^e(r_2) = \sigma_0 : -\frac{2C_1}{r_2^2} + \frac{E\phi}{2} = \sigma_0, \tag{5.26}$$

$$\sigma_r^e(r_2) = \sigma_r^p(r_2) : \frac{C_1}{r_2^2} + C_2 - \frac{C_5}{r_2^2} - C_6 + \frac{E\ln r_2}{2\,(1+H)}\phi = \frac{\ln r_2}{1+H}\sigma_0, \tag{5.27}$$

$$\begin{aligned} u^e(r_2,\theta) = u^p(r_2,\theta) : -\frac{(1+\nu)\,C_1}{Er_2} + \frac{(1-\nu)\,r_2C_2}{E} \\ +\frac{[2 + H\,(1+\nu)]\,C_5}{EHr_2} - \frac{(1-\nu)\,r_2C_6}{E} \\ +\frac{(1-\nu)\,r_2\ln r_2}{2\,(1+H)}\phi = \frac{(1-\nu)\,r_2\ln r_2}{E\,(1+H)}\sigma_0, \end{aligned} \tag{5.28}$$

$$\sigma_r^p(b) = 0 : 2\,(1+H)\,C_5 + 2b^2\,(1+H)\,C_6 + b^2EH\phi\ln b = -2b^2\sigma_0\ln b, \tag{5.29}$$

$$\begin{aligned} \int_a^b \sigma_\theta r dr = M : \ln(r_1/r_2)C_1 + \frac{1}{2}(r_2^2 - r_1^2)C_2 + \frac{W(a^{1-W} - r_1^{1-W})C_3}{1-W} \\ +\frac{W(r_1^{1+W} - a^{1+W})C_4}{1+W} + \ln(r_2/b)C_5 + \frac{1}{2}(b^2 - r_2^2)C_6 \\ + \left[-4a^2H(1+H) + b^2H(1+2\ln b) - r_1^2(1+H)(1-4H+2\ln r_1)\right. \\ \left. + r_2^2\,(1+2\ln r_2)\right]\frac{E\phi}{8\,(1+H)} \\ = M - \left[2a^2\,(1+H) + b^2(1+2\ln b) - 2r_1^2\,(1+H) - r_2^2(1+2\ln r_2)\right]\frac{\sigma_0}{4\,(1+H)}. \end{aligned} \tag{5.30}$$

The second stage of elastoplastic deformation begins when $M \geq M_I$ with $\overline{M} = 1.4 \times 10^{-2}$ for a beam of $b/a = 1.3$. The curved beam consists of an inner plastic region (region I) in $a \leq r \leq r_1$, an elastic region in $r_1 \leq r \leq r_2$, and an outer plastic region (region II) in $r_2 \leq r \leq b$. The outer plastic region is governed by Tresca's yield criterion $\sigma_y = \sigma_\theta - \sigma_r$. Fig. (5.7) (b) shows the consequent distributions at the second stage under the load $M \geq M_I$. By solving above ten equations for boundary and interface conditions simultaneously using Version I solution method, the unknowns are calculated as $\overline{C}_1 = 3.6684$, $\overline{C}_2 = -3.6282$, $\overline{C}_3 = 2.5110$, $\overline{C}_4 = -5.5283$, $\overline{C}_5 = 6.1140 \times 10^{-1}$, $\overline{C}_6 = -6.1140 \times 10^{-1}$, $A_2 = -2.1050 \times 10^{-2}$, $\phi = 2.3896 \times 10^{-2}$, $\overline{r}_1 = 1.0885$, and $\overline{r}_2 = 1.2036$. The unknowns are calculated for a beam of $b/a = 1.5$ under the load $\overline{M} = 3.6 \times 10^{-2}$ as $\overline{C}_1 = 3.8477$, $\overline{C}_2 = -3.6475$, $\overline{C}_3 = 2.6858$, $\overline{C}_4 = -6.5939$, $\overline{C}_5 = 6.4128 \times 10^{-1}$, $\overline{C}_6 = -6.4128 \times 10^{-1}$, $A_2 = -2.3790 \times 10^{-2}$, $\phi = 2.8925 \times 10^{-2}$, $\overline{r}_1 = 1.1835$, and $\overline{r}_2 = 1.2799$. By applying Version II solution for stage II, as the constants C_1, C_2, C_3, and C_4 can be solved by a linear system of the Eqs. (5.21), (5.22), (5.23), and (5.24), C_1, C_2, C_3, and C_4 become

$$\begin{aligned} C_1 = & -\frac{E\phi r_1^2}{4} - \frac{a^W r_1^{1-W}(H+W+HW)(A_2EH + aH\sigma_0 - aW\sigma_0)}{4\,(1+H)\,HW} \\ & -\frac{a^{W+1} r_1^{1-W}(H+W+HW)(-W\sigma_0 - EH\phi + EW\phi + EHW\phi)}{4\,(1+H)\,W} \\ & +\frac{a^{-W} r_1^{1+W}(-H+W+HW)(-A_2EH - aH\sigma_0 - aW\sigma_0)}{4\,(1+H)\,HW} \\ & +\frac{a^{1-W} r_1^{1+W}(-H+W+HW)(-W\sigma_0 + EH\phi + EW\phi + EHW\phi)}{4\,(1+H)\,W}, \end{aligned} \tag{5.31}$$

$$\begin{aligned} C_2 = & -\sigma_0 + \frac{E\phi}{4}(1 + 4H - 2\ln r_1) \\ & +\frac{a^{-W} r_1^{-1-W} A_2 E\left[a^{2W}(-H+W+HW) + r_1^{2W}(H+W+HW)\right]}{4\,(1+H)\,W} \\ & +\frac{a^{1-W} r_1^{-1-W}\sigma_0\left[-a^{2W}(-H+W+HW)^2 + r_1^{2W}(H+W+HW)^2\right]}{4\,(1+H)\,HW} \\ & +\frac{a^{1-W} r_1^{-1-W}E\phi\left[a^{2W}(-H+W+HW)^2 - r_1^{2W}(H+W+HW)^2\right]}{4\,(1+H)\,W}, \end{aligned} \tag{5.32}$$

$$\begin{aligned} C_3 = & -\frac{a^W A_2 EH}{2\,(1+H)\,W} + \frac{a^{1+W}\sigma_0(-H+W+HW)}{2\,(1+H)\,W} \\ & -\frac{a^{1+W}EH\phi(-H+W+HW)}{2\,(1+H)\,W}, \end{aligned} \tag{5.33}$$

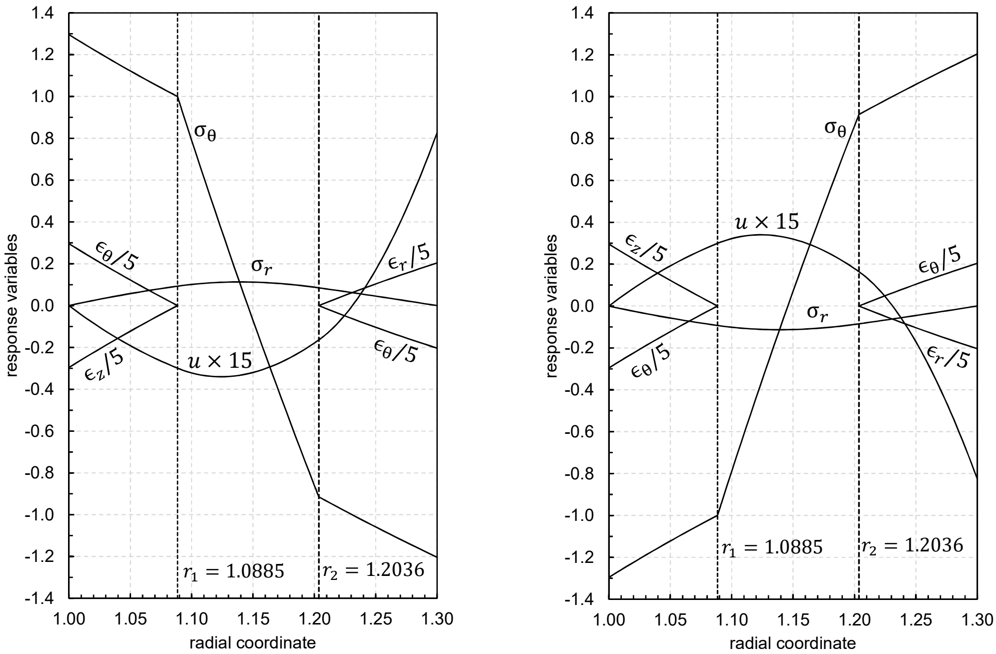
$$
\begin{aligned}
C_4 \quad = \quad & \frac{a^{-W} A_2 E H}{2(1+H) W} + \frac{a^{1-W} \sigma_0 (H + W + HW)}{2(1+H) W} \\
& - \frac{a^{1-W} E H \phi (H + W + HW)}{2(1+H) W}.
\end{aligned} \tag{5.34}
$$

Thus the number of equations is reduced from ten to six and the unknowns C_5, C_6, A_2, ϕ, r_1, and r_2 are obtained from Eqs. (5.25), (5.26), (5.27), (5.28), (5.29) and (5.30) by using Newton's Method. The results of the unknowns for the beam of $b/a = 1.3$ are $\overline{C}_5 = 6.1140 \times 10^{-1}$, $\overline{C}_6 = -6.1140 \times 10^{-1}$, $A_2 = -2.1050 \times 10^{-2}$, $\phi = 2.3896 \times 10^{-2}$, $\overline{r}_1 = 1.0885$, and $\overline{r}_2 = 1.2036$.and for the beam of $b/a = 1.5$ $\overline{C}_5 = 6.4128 \times 10^{-1}$, $\overline{C}_6 = -6.4128 \times 10^{-1}$, $A_2 = -2.3790 \times 10^{-2}$, $\phi = 2.8925 \times 10^{-2}$, $\overline{r}_1 = 1.1835$, and $\overline{r}_2 = 1.2799$. Substituting the solution of the unknowns C_5, C_6, A_2, ϕ, r_1, and r_2 in Eqs.(5.31), (5.32), (5.33) and (5.34) $\overline{C}_1, \overline{C}_2, \overline{C}_3$, and $\overline{C}_4$ for $b/a = 1.3$ become $\overline{C}_1 = 3.6684$, $\overline{C}_2 = -3.6282$, $\overline{C}_3 = 2.5110$, $\overline{C}_4 = -5.5283$, and for $b/a = 1.5$ $\overline{C}_1 = 3.8477$, $\overline{C}_2 = -3.6475$, $\overline{C}_3 = 2.6858$, $\overline{C}_4 = -6.5939$.

The constants C_5 and C_6 can also be solved by a linear system of Eqs. (5.27) and (5.29). However, the solutions for these two unknowns are so complex and difficult to solve, and in order not to cause an error in calculations and in order to take accurate results, these equations have been included in the system.

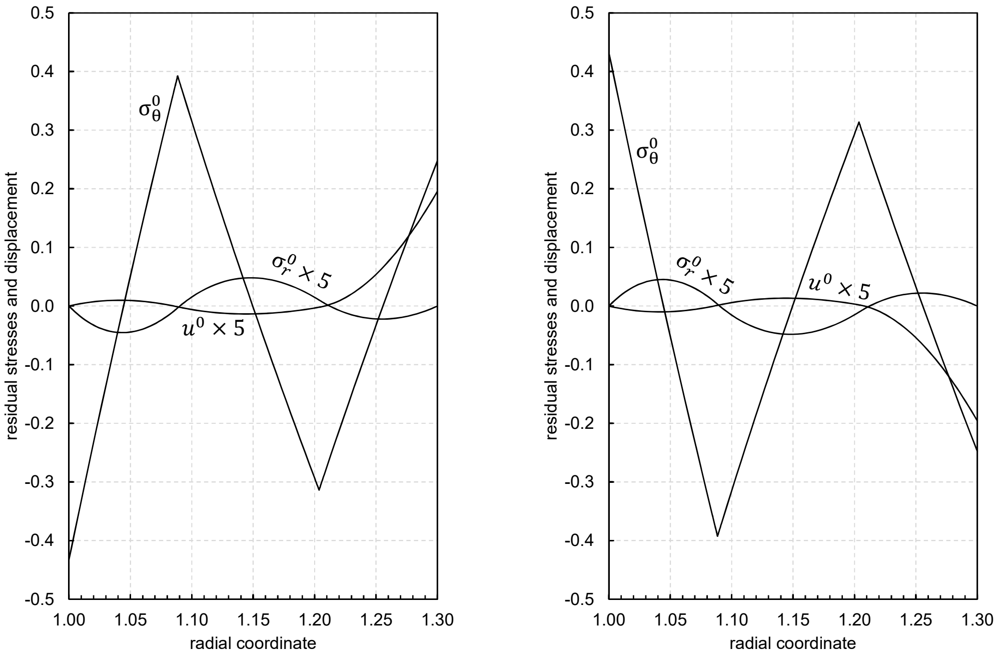
The stress response of the beam at the plane of symmetry, i.e. $\theta = 0$ under $\overline{M}$ is plotted for $b/a = 1.3$ in Fig. (5.7) (a) and (b) and for $b/a = 1.5$ in Fig. (5.9) (a) and (b). The neutral axis in the elastic region are calculated with $\sigma_\theta(r_{NA}) = 0$ for the beam of $b/a = 1.3$ and $b/a = 1.5$ as $\overline{r}_{NA} = 1.1457$ and 1.2351, respectively.

To obtain the residual stresses, taking difference of the stresses for elastic-plastic and elastic behavior under $\overline{M} = 1.4 \times 10^{-2}$ for a beam of $b/a = 1.3$ and $\overline{M} = 3.6 \times 10^{-2}$ for a beam of $b/a = 1.5$, the unknowns for the residual stresses are obtained for the beam $b/a = 1.3$ as $\overline{C}_1 = 2.3044$, $\overline{C}_2 = -2.3044$, $\phi = 1.5151 \times 10^{-2}$, $A_2 = -1.3316 \times 10^{-2}$, and for the beam $b/a = 1.5$ as $\overline{C}_1 = 1.5850$, $\overline{C}_2 = -1.5850$, $\phi = 1.2216 \times 10^{-2}$, $A_2 = -1.0016 \times 10^{-2}$, and the corresponding graphs can be seen in Fig. (5.8) (a) and (b), and Fig. (5.10) (a) and (b), respectively.



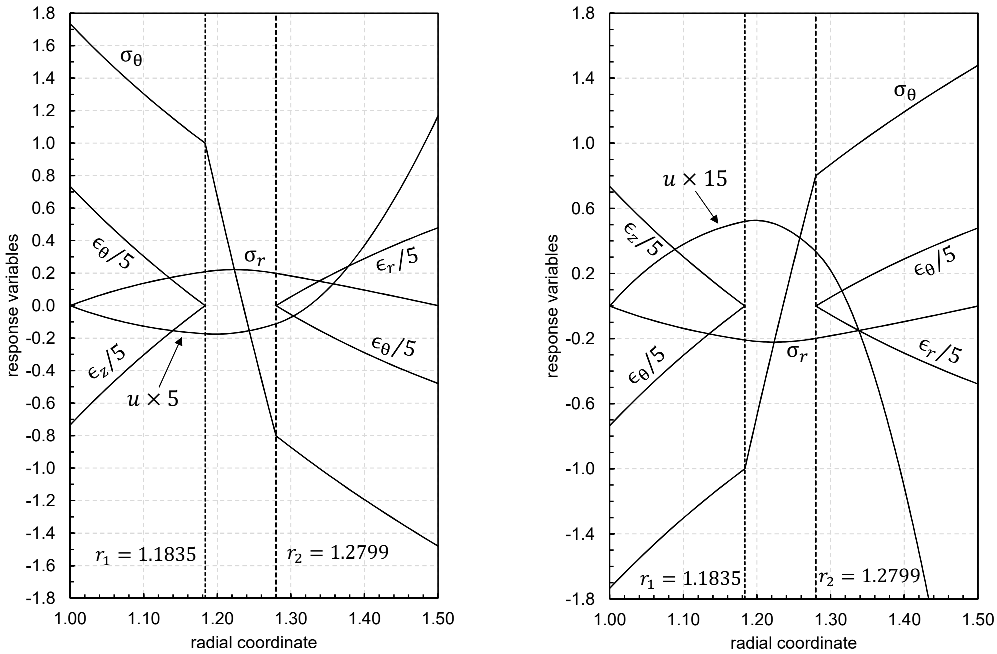
(a) (b)

Figure 5.7: The stress response under $\overline{M}$ of a beam of b/a=1.3 for (a) positive bending case, (b) negative bending case.



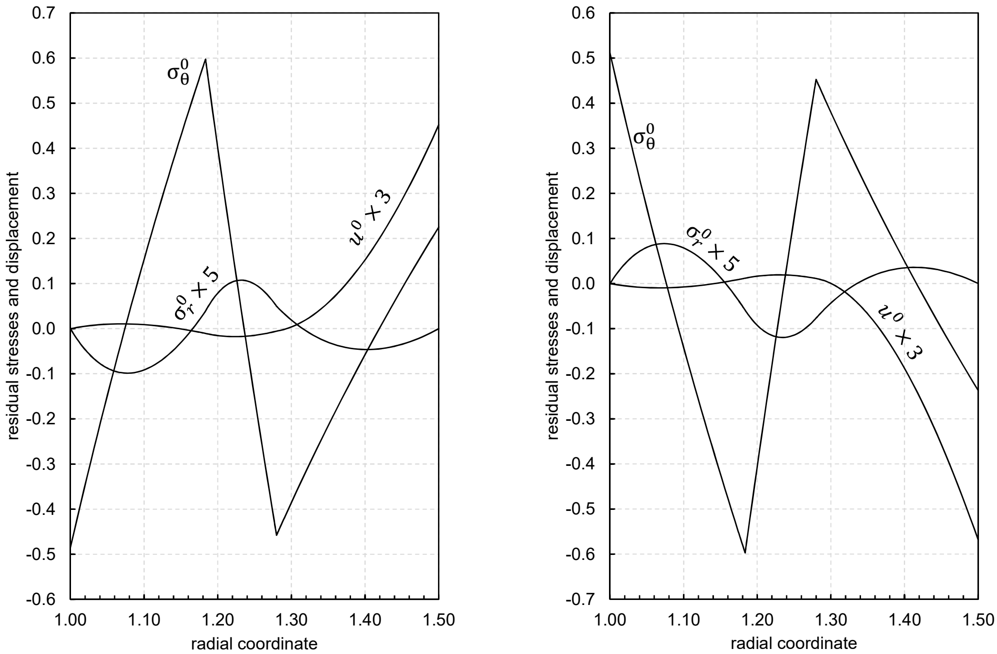
(a) (b)

Figure 5.8: The residual stresses of a beam of b/a=1.3 in stage II for (a) positive bending case, (b) negative bending case.



(a) (b)

Figure 5.9: The stress response under $\overline{M}$ of a beam of b/a=1.5 for (a) positive bending case, (b) negative bending case.



(a) (b)

Figure 5.10: The residual stresses of a beam of b/a=1.5 in stage II for (a) positive bending case, (b) negative bending case.

5.2 Plane Strain

5.2.1 Elastic

The distribution of the stresses and displacement are described by Eqs. (4.61)-(4.64). In the stress equations, there are three unknowns which are C_1, C_2 and ϕ. For evaluation of these unknowns three linearly independent conditions are available. These are:

$$\int_a^b \sigma_\theta r dr = M, \tag{5.35}$$

$$\sigma_r(a) = \sigma_r(b) = 0. \tag{5.36}$$

The solution of the integral in Eq. (5.35) becomes

$$C_1 \ln(a/b) + \frac{1}{2}C_2(b^2 - a^2) - \frac{E\phi[a^2(1 + 2\ln a) - b^2(1 + 2\ln b)]}{8(1 - \nu^2)} = M. \tag{5.37}$$

Combining Eqs. (5.36) and (5.37), the analytical expressions for the unknowns C_1, C_2 and ϕ are obtained as

$$C_1 = \frac{4M}{N} a^2 b^2 \ln(b/a), \tag{5.38}$$

$$C_2 = -\frac{4M}{N}(b^2 \ln b - a^2 \ln a), \tag{5.39}$$

$$\phi = -\frac{8M}{EN}(a^2 - b^2)(1 - \nu^2), \tag{5.40}$$

in which

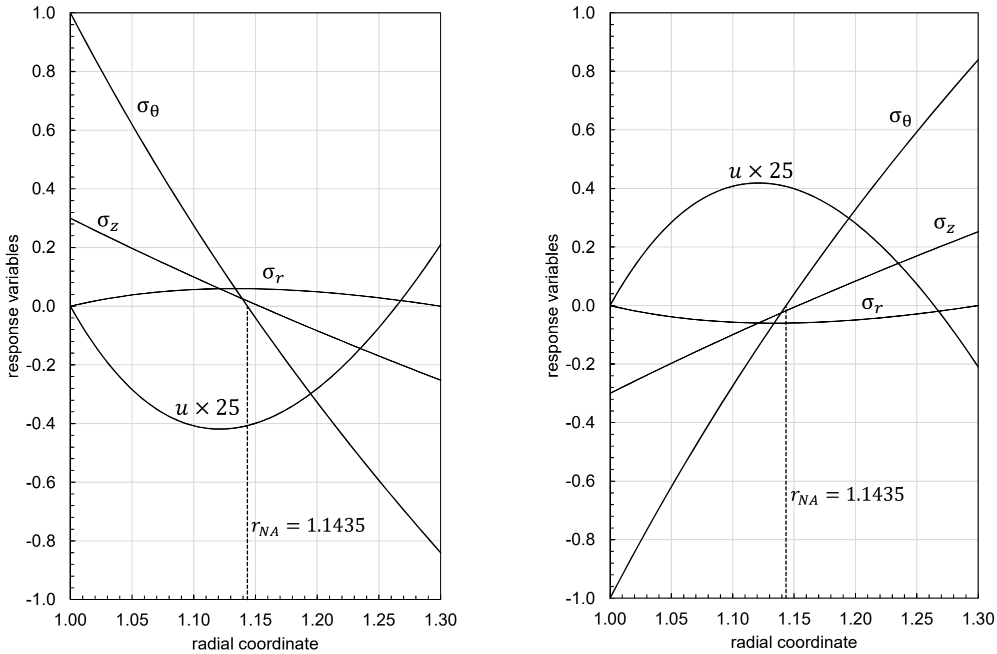
$$N = (a^2 - b^2)^2 - 4a^2b^2[\ln(b/a)]^2. \tag{5.41}$$

The displacement, Eq. (4.64), will be equal to zero assuming that the beam is rigidly fixed at $r = a$, $\theta = 0$, i.e.

$$\begin{aligned} u(a, 0) = \frac{1}{E}\left[-\frac{1}{a}(1 + \nu)C_1 + (1 + \nu)(1 - 2\nu)aC_2\right] \\ -\frac{[1 - \nu - (1 - 2\nu)\ln a]\,\phi a}{2(1 - \nu)} - A_2 = 0. \end{aligned} \tag{5.42}$$

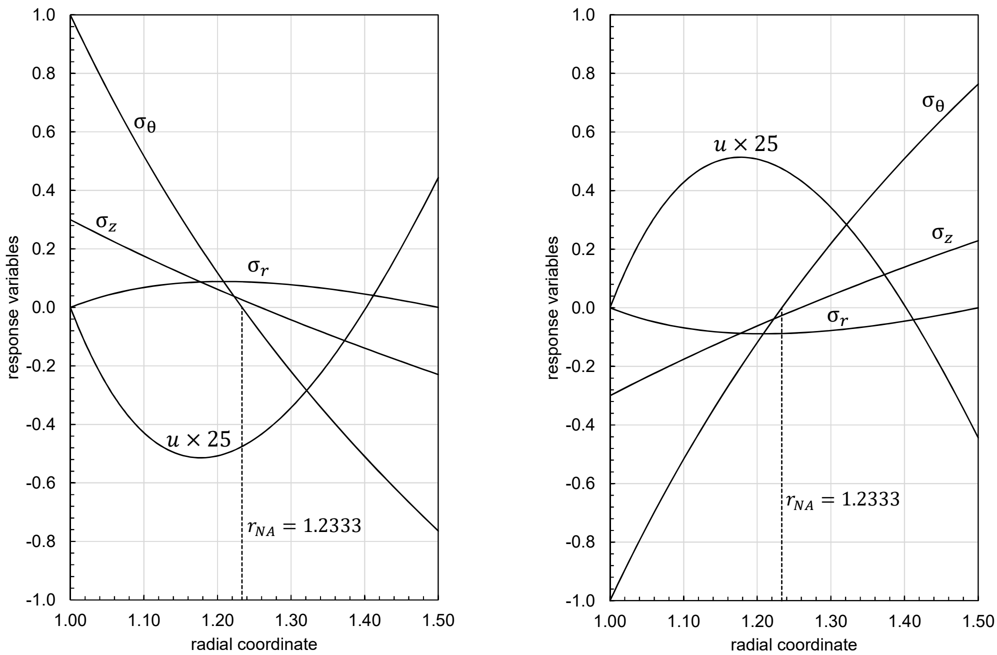
Solving Eq. (5.42) for A_2

$$\begin{aligned} A_2 &= \frac{1}{E}\left[-\frac{1}{a}(1 + \nu)C_1 + (1 + \nu)(1 - 2\nu)aC_2\right] \\ &\quad -\frac{[1 - \nu - (1 - 2\nu)\ln a]\,\phi a}{2(1 - \nu)}. \end{aligned} \tag{5.43}$$



(a) (b)

Figure 5.11: The stress response under the elastic limit load of a beam of b/a=1.3 for (a) positive bending case, (b) negative bending case.



(a) (b)

Figure 5.12: The stress response under the elastic limit load of a beam of b/a=1.5 for (a) positive bending case, (b) negative bending case.

The beam deforms elastically under $M \leq M_E$. The elastic limit load for a beam of $b/a = 1.3$ is $\overline{M}_E = 8.0999 \times 10^{-3}$. The corresponding integration constants C_1, C_2, ϕ, and A_2 are calculated as $\overline{C}_1 = 1.3332$, $\overline{C}_2 = -1.3332$, $\phi = 7.9767 \times 10^{-3}$, and $A_2 = -7.0110 \times 10^{-3}$. The elastic limit load for a beam of $b/a = 1.5$ is $\overline{M}_E = 1.6028 \times 10^{-2}$. The unknowns C_1, C_2, ϕ, and A_2 take the numerical values $\overline{C}_1 = 7.0566 \times 10^{-1}$, $\overline{C}_2 = -7.0566 \times 10^{-1}$, $\phi = 4.9491 \times 10^{-3}$, and $A_2 = -4.0577 \times 10^{-3}$. The stress response of the beam at the plane of symmetry, i.e. $\theta = 0$ under elastic limit load $\overline{M} = \overline{M}_E$ is plotted for $b/a = 1.3$ in Fig. (5.11) (a) and (b), and for $b/a = 1.5$ in Fig. (5.12) (a) and (b). The solution of $\sigma_\theta(r_{NA}) = 0$ is achieved by Newton-Raphson method and the neutral axis are calculated for the beam of $b/a = 1.3$ and $b/a = 1.5$ as $\overline{r}_{NA} = 1.1435$ and 1.2333, respectively.

5.2.2 First Stage of Elastic-Plastic Solution

The curved beam consists of a plastic region in

$$a \leq r \leq r_1,$$

and an elastic region in

$$r_1 \leq r \leq b,$$

where r_1 is the plastic-elastic border radius.The distribution of the stresses, displacement and plastic strains are described by Eqs. (4.73)-(4.78). In the first stage of elastic-plastic solution, it is required to solve elastic region and plastic region I together. $C_1, C_2, C_3, C_4, A_2, \phi$ and r_1 are the unknowns that should be evaluated to solve this problem. Seven boundary and interface conditions are available to determine these unknowns. These are:

$$\begin{aligned} u^p(a, 0) &= 0, \\ \sigma_r^p(a) &= 0, \end{aligned}$$

$$
\begin{aligned}
\sigma_r^p(r_1) &= \sigma_r^e(r_1), \\
u^p(r_1,\theta) &= u^e(r_1,\theta), \\
\sigma_r^e(r_1) - \sigma_\theta^e(r_1) &= \sigma_0, \\
\sigma_r^e(b) &= 0, \\
\int_a^b \sigma_\theta r dr &= \int_a^{r_1} \sigma_\theta^p r dr + \int_{r_1}^b \sigma_\theta^e r dr = M.
\end{aligned}
$$

In explicit forms:

$$
\begin{aligned}
u^p(a,0) &= 0 : -A_2 - \frac{[2+H(1+\nu)]\,C_3}{EHa} + \frac{1}{E}(1+\nu)(1-2\nu)aC_4 \\
&\quad - \left[\frac{1+H(1-\nu^2) - H(1+\nu)(1-2\nu)\ln a}{1+H(1-\nu^2)}\right]\frac{a\phi}{2} \\
&= \left[\frac{(1+\nu)(1-2\nu)a\ln a}{E\,[1+H(1-\nu^2)]}\right]\sigma_0,
\end{aligned}
\tag{5.44}
$$

$$
\begin{aligned}
\sigma_r^p(a) &= 0 : \frac{C_3}{a^2} + C_4 + \left[\frac{EH\ln a}{2\,[1+H(1-\nu^2)]}\right]\phi \\
&= \left[\frac{\ln a}{1+H(1-\nu^2)}\right]\sigma_0,
\end{aligned}
\tag{5.45}
$$

$$
\begin{aligned}
\sigma_r^p(r_1) &= \sigma_r^e(r_1) : \frac{C_1}{r_1^2} + C_2 - \frac{C_3}{r_1^2} - C_4 \\
&\quad + \left[\frac{E\ln r_1}{2(1-\nu^2)\,[1+H(1-\nu^2)]}\right]\phi \\
&= -\left[\frac{\ln r_1}{1+H(1-\nu^2)}\right]\sigma_0,
\end{aligned}
\tag{5.46}
$$

$$
\begin{aligned}
u^p(r_1,\theta) &= u^e(r_1,\theta) : \frac{-(1+\nu)C_1}{Er_1} + \frac{1}{E}(1+\nu)(1-2\nu)r_1C_2 \\
&\quad + \frac{[2+H(1+\nu)]\,C_3}{EHr_1} - \frac{1}{E}(1+\nu)(1-2\nu)r_1C_4 \\
&\quad + \left\{\frac{(1-2\nu)r_1\ln r_1}{2(1-\nu)\,[1+H(1-\nu^2)]}\right\}\phi \\
&= -\left\{\frac{(1+\nu)(1-2\nu)r_1\ln r_1}{E\,[1+H(1-\nu^2)]}\right\}\sigma_0,
\end{aligned}
\tag{5.47}
$$

$$
\sigma_\theta^e(r_1) - \sigma_r^e(r_1) = -\sigma_0 : -\frac{2C_1}{r_1^2} + \frac{E\phi}{2(1-\nu^2)} = -\sigma_0, \tag{5.48}
$$

$$
\sigma_r^e(b) = 0 : \frac{C_1}{b^2} + C_2 + \frac{E\ln b\phi}{2(1-\nu^2)} = 0, \tag{5.49}
$$

$$\int_a^b \sigma_\theta r dr = M : \ln(r_1/b)C_1 + \frac{1}{2}(b^2 - r_1^2)C_2 + \ln(a/r_1)C_3 \tag{5.50}$$

$$+\frac{1}{2}(r_1^2 - a^2)C_4 + \frac{E}{8(1-\nu^2)\left[1+H(1-\nu^2)\right]}\left\{-a^2 H(1-\nu^2)(1+2\ln a)\right.$$

$$\left. +b^2\left[1+H(1-\nu^2)\right](1+2\ln b) - r_1^2(1+2\ln r_1)\right\}\phi$$

$$= M - \left\{\frac{a^2(1+2\ln a) - r_1^2(1+2\ln r_1)}{4\left[1+H(1-\nu^2)\right]}\right\}\sigma_0$$

The beam becomes partially plastic under the load $M \geq M_E$. The first stage of elastoplastic deformation is in the range $M_E \leq M \leq M_I$ with $\overline{M}_I = 9.5574 \times 10^{-3}$ for a beam of $b/a = 1.3$. The curved beam consists of a plastic region in $a \leq r \leq r_1$ and an elastic region in $r_1 \leq r \leq b$,the plastic region is governed by Tresca's yield criterion $\sigma_y = -\sigma_\theta$. The hardening parameter H is taken as 0.2 for $b/a = 1.3$ and $b/a = 1.5$. By solving above seven equations simultaneously by using Version I solution, the unknowns are obtained as $\overline{C}_1 = 1.5975$, $\overline{C}_2 = -1.5975$, $\overline{C}_3 = 2.4598 \times 10^{-1}$, $\overline{C}_4 = -4.6821 \times 10^{-1}$, $A_2 = -8.3954 \times 10^{-3}$, $\phi = 9.5438 \times 10^{-3}$, and $\overline{r}_1 = 1.0195$.

The unknowns are calculated for a beam of $b/a = 1.5$ under $\overline{M}_I = 2.0561 \times 10^{-2}$ as $\overline{C}_1 = 9.3527 \times 10^{-1}$, $\overline{C}_2 = -9.3527 \times 10^{-1}$, $\overline{C}_3 = 1.4401 \times 10^{-1}$, $\overline{C}_4 = -4.8796 \times 10^{-1}$, $A_2 = -5.3669 \times 10^{-3}$, $\phi = 6.5260 \times 10^{-3}$, and $\overline{r}_1 = 1.0430$.

As the constants C_1, C_2, C_3, and C_4 can be solved by a linear system of the Eqs. (5.44), (5.45), (5.48), and (5.49), Version II solution is applied to determine the equations for C_1, C_2, C_3, and C_4. These are:

$$C_1 = \frac{r_1^2\sigma_0}{2} + \frac{E\phi r_1^2}{4(1-\nu^2)}, \tag{5.51}$$

$$C_2 = -\frac{r_1^2\left[2(1-\nu^2)\sigma_0 + E\phi\right]}{4b^2(1-\nu^2)} - \frac{E\phi\ln b}{2(1-\nu^2)}, \tag{5.52}$$

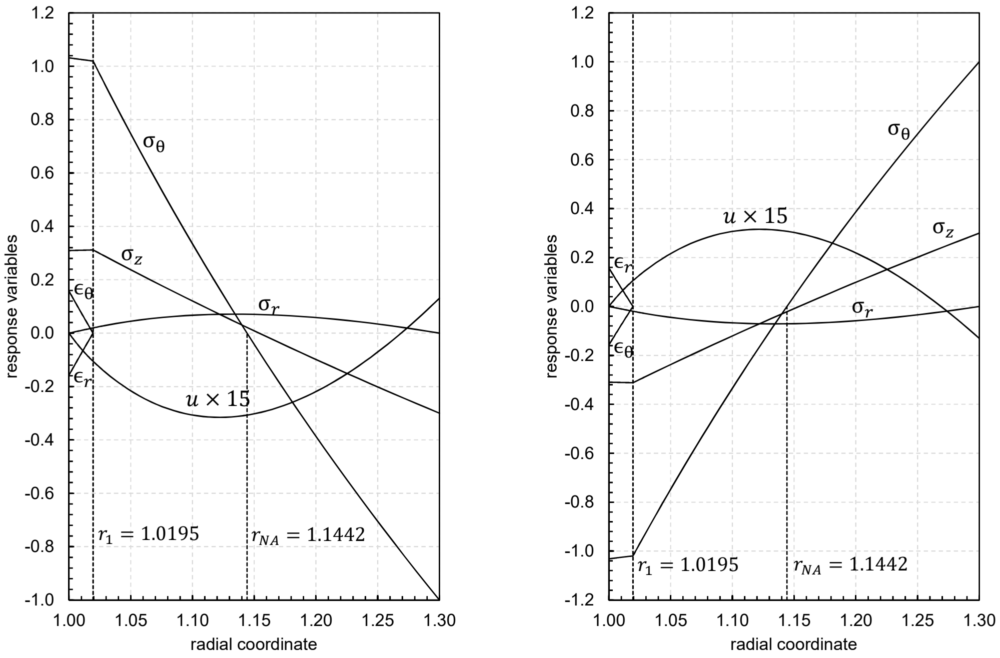
$$C_3 = -\frac{A_2 E H a}{2\left[1+H(1-\nu^2)\right]} - \frac{E H\phi a^2}{4\left[1+H(1-\nu^2)\right]}, \tag{5.53}$$

$$C_4 = \frac{EH(2A_2 + \phi a)}{4a\left[1+H(1-\nu^2)\right]} + \frac{(2\sigma_0 - EH\phi)\ln a}{2\left[1+H(1-\nu^2)\right]}. \tag{5.54}$$

Thus the number of equations is reduced from seven to three and the remaining unknowns A_2, ϕ and r_1 can be calculated from Eqs. (5.46), (5.47), and (5.50). The results of the unknowns for the beam of $b/a = 1.3$ are $A_2 = -8.3954 \times 10^{-3}$, $\phi = 9.5438 \times 10^{-3}$, and $\overline{r}_1 = 1.0195$, and for the beam of $b/a = 1.5$ $A_2 = -5.3669 \times 10^{-3}$, $\phi = 6.5260 \times 10^{-3}$, and $\overline{r}_1 = 1.0430$. Substituting the solution of the unknowns A_2, ϕ and r_1 in Eqs. (5.51), (5.52), (5.53), and (5.54); $\overline{C}_1, \overline{C}_2, \overline{C}_3$, and $\overline{C}_4$ for $b/a = 1.3$ become $\overline{C}_1 = 1.5975$, $\overline{C}_2 = -1.5975$, $\overline{C}_3 = 2.4598 \times 10^{-1}$, $\overline{C}_4 = -4.6821 \times 10^{-1}$and for $b/a = 1.5$ $\overline{C}_1 = 9.3527 \times 10^{-1}$, $\overline{C}_2 = -9.3527 \times 10^{-1}$, $\overline{C}_3 = 1.4401 \times 10^{-1}$, $\overline{C}_4 = -4.8796 \times 10^{-1}$.

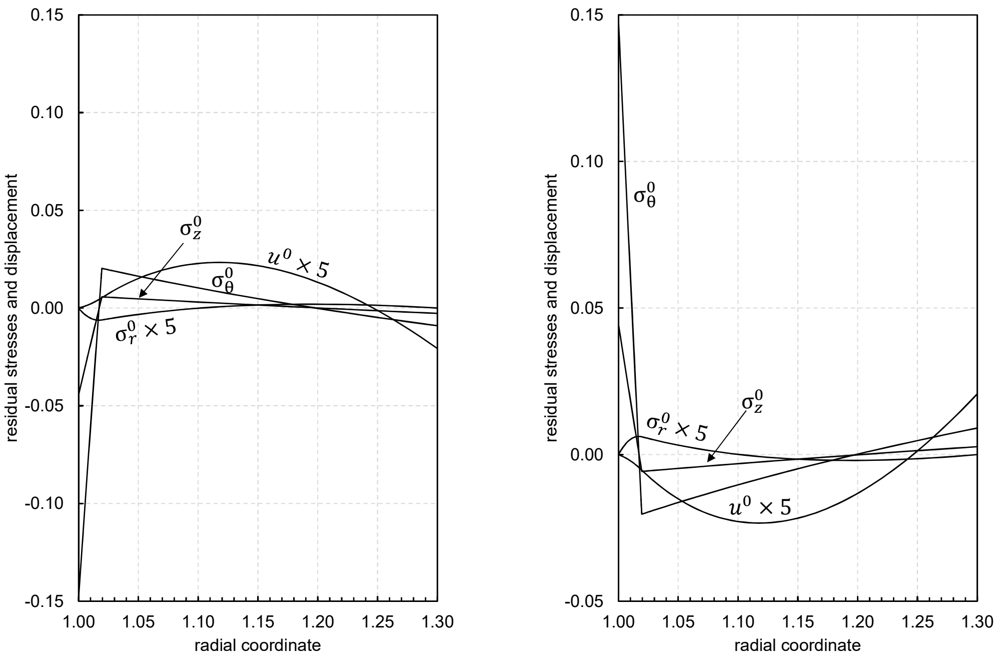
The stress response of the beam at the plane of symmetry, i.e. $\theta = 0$ under $\overline{M} = \overline{M}_I$ is plotted for $b/a = 1.3$ in Fig. (5.13) (a) and (b), and for $b/a = 1.5$ in Fig. (5.15) (a) and (b). The neutral axis are calculated with $\sigma_\theta(r_{NA}) = 0$ for the beam of $b/a = 1.3$ and $b/a = 1.5$ as $\overline{r}_{NA} = 1.1442$ and 1.2359, respectively.

To obtain the residual stresses, taking difference of the stresses for elastic-plastic and elastic behavior under $\overline{M}_I = 9.5574 \times 10^{-3}$ for a beam of $b/a = 1.3$ and $\overline{M}_I = 2.0561 \times 10^{-2}$ for a beam of $b/a = 1.5$, the unknowns for the residual stresses are obtained for the beam $b/a = 1.3$ as $\overline{C}_1 = 1.5731$, $\overline{C}_2 = -1.5731$, $\phi = 9.4121 \times 10^{-3}$, $A_2 = -8.2725 \times 10^{-3}$, and for the beam $b/a = 1.5$ as $\overline{C}_1 = 9.0527 \times 10^{-1}$, $\overline{C}_2 = -9.0527 \times 10^{-1}$, $\phi = 6.3492 \times 10^{-3}$, $A_2 = -5.2056 \times 10^{-3}$, and the corresponding graphs can be seen in Fig. (5.14) (a) and (b), and Fig. (5.16) (a) and (b), respectively.



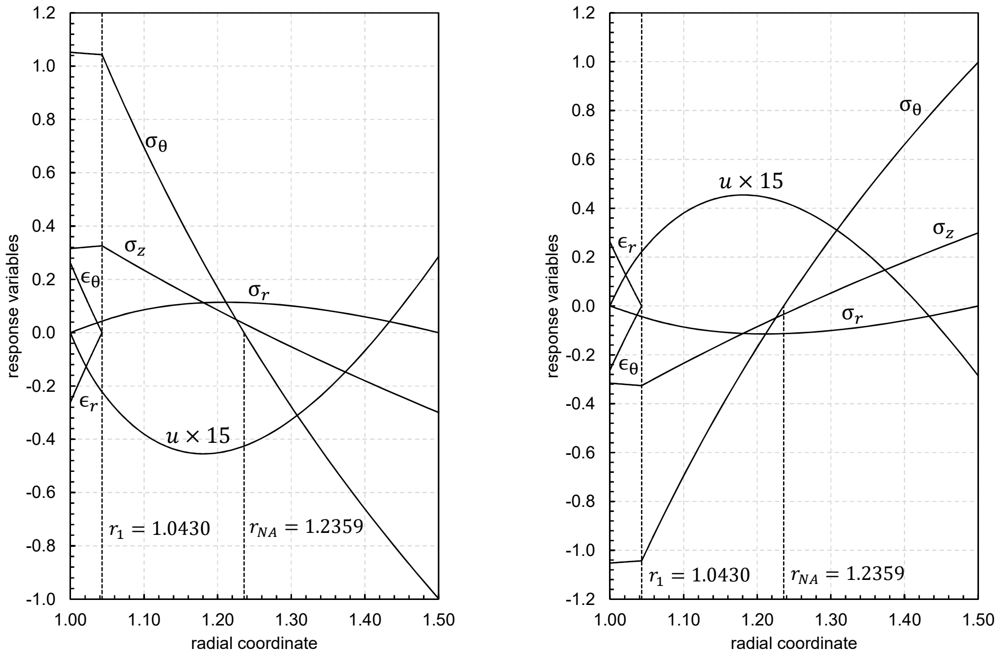
(a) (b)

Figure 5.13: The stress response under $\overline{M} = \overline{M}_I$ of a beam of b/a=1.3 for (a) positive bending case, (b) negative bending case.



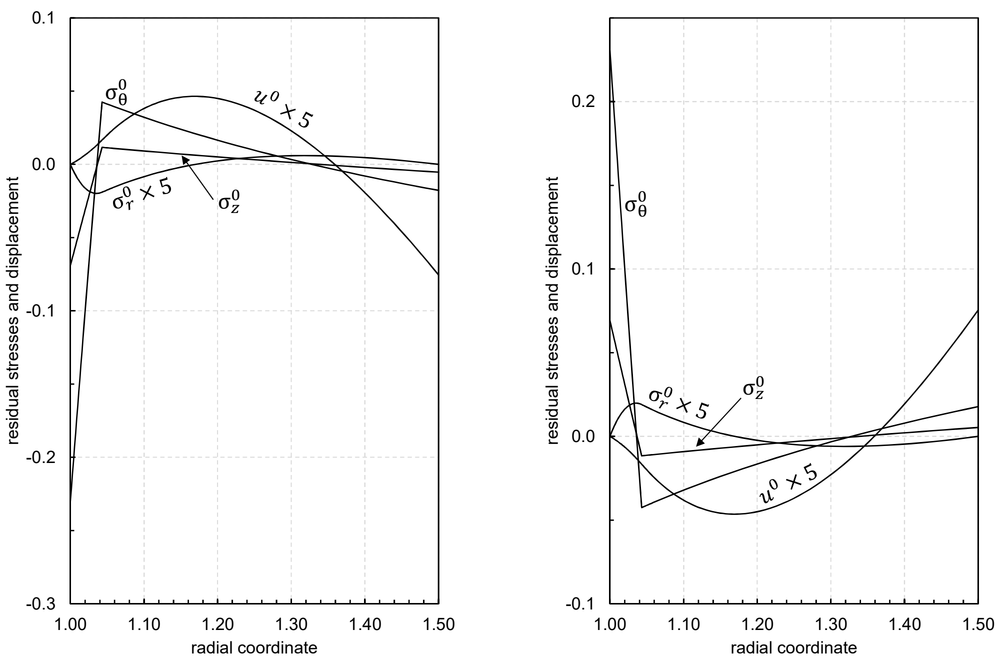
(a) (b)

Figure 5.14: The residual stresses of a beam of b/a=1.3 in stage I for (a) positive bending case, (b) negative bending case.



(a) (b)

Figure 5.15: The stress response under $\overline{M} = \overline{M}_I$ of a beam of b/a=1.5 for (a) positive bending case, (b) negative bending case.



(a) (b)

Figure 5.16: The residual stresses of a beam of b/a=1.5 in stage I for (a) positive bending case, (b) negative bending case.

5.2.3 Second Stage of Elastic-Plastic Solution

The curved beam consists of an inner plastic region (region I) in

$$a \leq r \leq r_1,$$

an elastic region in

$$r_1 \leq r \leq r_2,$$

and an outer plastic region (region II) in

$$r_2 \leq r \leq b.$$

where r_1 and r_2 are the elastic-plastic border radii.The distribution of the stresses, displacement and plastic strains are described by Eqs. (4.86)-(4.91). In the second stage of elastic-plastic solution, it is required to solve elastic region, plastic region I and plastic region II together. $C_1, C_2, C_3, C_4, C_5, C_6, A_2, \phi$, r_1, and r_2 are the unknowns that should be evaluated to solve this problem. Ten boundary and interface conditions are available to determine these unknowns. These are:

$$\begin{aligned}
u^p(a,0) &= 0, \\
\sigma_r^p(a) &= 0, \\
\sigma_r^p(r_1) &= \sigma_r^e(r_1), \\
u^p(r_1,\theta) &= u^e(r_1,\theta), \\
\sigma_r^e(r_1) - \sigma_\theta^e(r_1) &= -\sigma_0, \\
\sigma_\theta^e(r_2) - \sigma_r^e(r_2) &= \sigma_0, \\
\sigma_r^e(r_2) &= \sigma_r^p(r_2), \\
u^e(r_2,\theta) &= u^p(r_2,\theta), \\
\sigma_r^p(b) &= 0, \\
\int_a^b \sigma_\theta r dr &= \int_a^{r_1} \sigma_\theta^p r dr + \int_{r_1}^{r_2} \sigma_\theta^e r dr + \int_{r_2}^{b} \sigma_\theta^p r dr = M.
\end{aligned}$$

In explicit forms:

$$\begin{aligned}
u^p(a,0) &= 0 : -A_2 - \frac{[2+H(1+\nu)]\, C_3}{EHa} + \frac{1}{E}(1+\nu)(1-2\nu)aC_4 \\
&\quad - \left[\frac{1+H(1-\nu^2) - H(1+\nu)(1-2\nu)\ln a}{1+H(1-\nu^2)}\right]\frac{a\phi}{2} \\
&= \left[\frac{(1+\nu)(1-2\nu)a\ln a}{E\left[1+H(1-\nu^2)\right]}\right]\sigma_0,
\end{aligned} \tag{5.55}$$

$$\begin{aligned}\sigma_r^p(a) &= 0 : \frac{C_3}{a^2} + C_4 + \left[\frac{EH\ln a}{2\left[1+H(1-\nu^2)\right]}\right]\phi \\ &= \left[\frac{\ln a}{1+H(1-\nu^2)}\right]\sigma_0,\end{aligned} \tag{5.56}$$

$$\begin{aligned}\sigma_r^p(r_1) &= \sigma_r^e(r_1) : \frac{C_1}{r_1^2} + C_2 - \frac{C_3}{r_1^2} - C_4 \\ &\quad + \left[\frac{E\ln r_1}{2(1-\nu^2)\left[1+H(1-\nu^2)\right]}\right]\phi \\ &= -\left[\frac{\ln r_1}{1+H(1-\nu^2)}\right]\sigma_0,\end{aligned} \tag{5.57}$$

$$\begin{aligned}u^p(r_1,\theta) &= u^e(r_1,\theta) : \frac{-(1+\nu)C_1}{Er_1} + \frac{1}{E}(1+\nu)(1-2\nu)r_1C_2 \\ &\quad + \frac{\left[2+H(1+\nu)\right]C_3}{EHr_1} - \frac{1}{E}(1+\nu)(1-2\nu)r_1C_4 \\ &\quad + \left\{\frac{(1-2\nu)r_1\ln r_1}{2(1-\nu)\left[1+H(1-\nu^2)\right]}\right\}\phi \\ &= -\left\{\frac{(1+\nu)(1-2\nu)r_1\ln r_1}{E\left[1+H(1-\nu^2)\right]}\right\}\sigma_0,\end{aligned} \tag{5.58}$$

$$\sigma_\theta^e(r_1) - \sigma_r^e(r_1) = -\sigma_0 : -\frac{2C_1}{r_1^2} + \frac{E\phi}{2(1-\nu^2)} = -\sigma_0, \tag{5.59}$$

$$\sigma_r^e(r_2) - \sigma_\theta^e(r_2) = -\sigma_0 : \frac{2C_1}{r_2^2} - \frac{E\phi}{2(1-\nu^2)} = -\sigma_0, \tag{5.60}$$

$$\begin{aligned}\sigma_r^e(r_2) &= \sigma_r^p(r_2) : \frac{C_1}{r_2^2} + C_2 - \frac{C_5}{r_2^2} - C_6 \\ &\quad + \left\{\frac{E\ln r_2}{2(1-\nu^2)\left[1+H(1-\nu^2)\right]}\right\}\phi \\ &= \left[\frac{\ln r_2}{1+H(1-\nu^2)}\right]\sigma_0\end{aligned} \tag{5.61}$$

$$\begin{aligned}u^e(r_2,\theta) &= u^p(r_2,\theta) : -\frac{(1+\nu)C_1}{Er_2} + \frac{1}{E}(1+\nu)(1-2\nu)r_2C_2 \\ &\quad + \frac{\left[1+H(1+\nu)\right]C_5}{EHr_2} - \frac{1}{E}(1+\nu)(1-2\nu)r_2C_6 \\ &\quad + \left[\frac{(1-2\nu)r_2\ln r_2}{2(1-\nu)\left[1+H(1-\nu^2)\right]}\right]\phi \\ &= \left[\frac{(1+\nu)(1-2\nu)r_2\ln r_2}{1+H(1-\nu^2)}\right]\frac{\sigma_0}{E},\end{aligned} \tag{5.62}$$

$$\begin{aligned}\sigma_r^p(b) &= 0 : \frac{C_5}{b^2} + C_6 + \left\{ \frac{EH \ln b}{2\left[1 + H(1-\nu^2)\right]} \right\} \phi \\ &= -\left[\frac{\ln b}{1 + H(1-\nu^2)} \right] \sigma_0,\end{aligned} \tag{5.63}$$

$$\begin{aligned}&\int_a^b \sigma_\theta r dr = M : \ln(r_1/r_2) C_1 + \frac{1}{2}(r_2^2 - r_1^2) C_2 + \ln(a/r_1) C_3 \\ &+\frac{1}{2}(r_1^2 - a^2) C_4 + \ln(r_2/b) C_5 + \frac{1}{2}(b^2 - r_2^2) C_6 + \frac{E}{8(1-\nu^2)\left[1 + H(1-\nu^2)\right]} \\ &\left\{ -a^2 H(1-\nu^2)(1 + 2\ln a) + b^2 H(1-\nu^2)(1 + 2\ln b) - r_1^2(1 + 2\ln r_1) \right. \\ &\left. + r_2^2(1 + 2\ln r_2) \right\} \phi \\ = \; & M - \left\{ \frac{a^2(1 + 2\ln a) + b^2(1 + 2\ln b) - r_1^2(1 + 2\ln r_1) - r_2^2(1 + 2\ln r_2)}{4\left[1 + H(1-\nu^2)\right]} \right\} \sigma_0.\end{aligned} \tag{5.64}$$

The second stage of elastoplastic deformation begins when $M \geq M_I$ with $\overline{M} = 1.4387 \times 10^{-2}$ for a beam of $b/a = 1.3$. The curved beam consists of an inner plastic region (region I) in $a \leq r \leq r_1$, an elastic region in $r_1 \leq r \leq r_2$, and an outer plastic region (region II) in $r_2 \leq r \leq b$. The outer plastic region is governed by Tresca's yield criterion $\sigma_y = \sigma_\theta - \sigma_r$. Fig. (5.17) (b) shows the consequent distributions at the second stage under the load $M \geq M_I$. By solving above ten equations for boundary and interface conditions simultaneously using Version I solution method, the unknowns are calculated as $\overline{C}_1 = 3.9106$, $\overline{C}_2 = -3.8608$, $\overline{C}_3 = 6.0214 \times 10^{-1}$, $\overline{C}_4 = -8.2685 \times 10^{-1}$, $\overline{C}_5 = 6.0214 \times 10^{-1}$, $\overline{C}_6 = -6.0214 \times 10^{-1}$, $A_2 = -2.0506 \times 10^{-2}$, $\phi = 2.3244 \times 10^{-2}$, $\overline{r}_1 = 1.0855$, and $\overline{r}_2 = 1.1975$.

The unknowns are calculated for a beam of $b/a = 1.5$ under the load $\overline{M} = 3.4 \times 10^{-2}$ as $\overline{C}_1 = 3.1745$, $\overline{C}_2 = -3.0172$, $\overline{C}_3 = 4.8879 \times 10^{-1}$, $\overline{C}_4 = -8.4082 \times 10^{-1}$, $\overline{C}_5 = 4.8879 \times 10^{-1}$, $\overline{C}_6 = -4.8879 \times 10^{-1}$, $A_2 = -1.8145 \times 10^{-2}$, $\phi = 2.1936 \times 10^{-2}$, $\overline{r}_1 = 1.1586$, and $\overline{r}_2 = 1.2857$.

By applying Version II solution for stage II, as the constants C_1, C_2, C_3, and C_4 can be solved by a linear system of the Eqs. (5.55), 5.56), (5.57), and (5.58), C_1, C_2, C_3, and C_4 become

$$C_1 = -\frac{A_2 E a}{2(1-\nu^2)} - \frac{E \phi a^2}{4(1-\nu^2)}, \tag{5.65}$$

$$C_2 = \frac{aE(2A_2+\phi a)}{4r_1^2(1-\nu^2)\left[1+H(1-\nu^2)\right]} + \frac{EH(2A_2+\phi a)}{4a\left[1+H(1-\nu^2)\right]}, \tag{5.66}$$
$$+\frac{(2\sigma_0 - EH\phi)\ln a}{2\left[1+H(1-\nu^2)\right]} - \frac{\left[2(1-\nu^2)\sigma_0 + E\phi\right]\ln r_1}{2(1-\nu^2)\left[1+H(1-\nu^2)\right]}$$

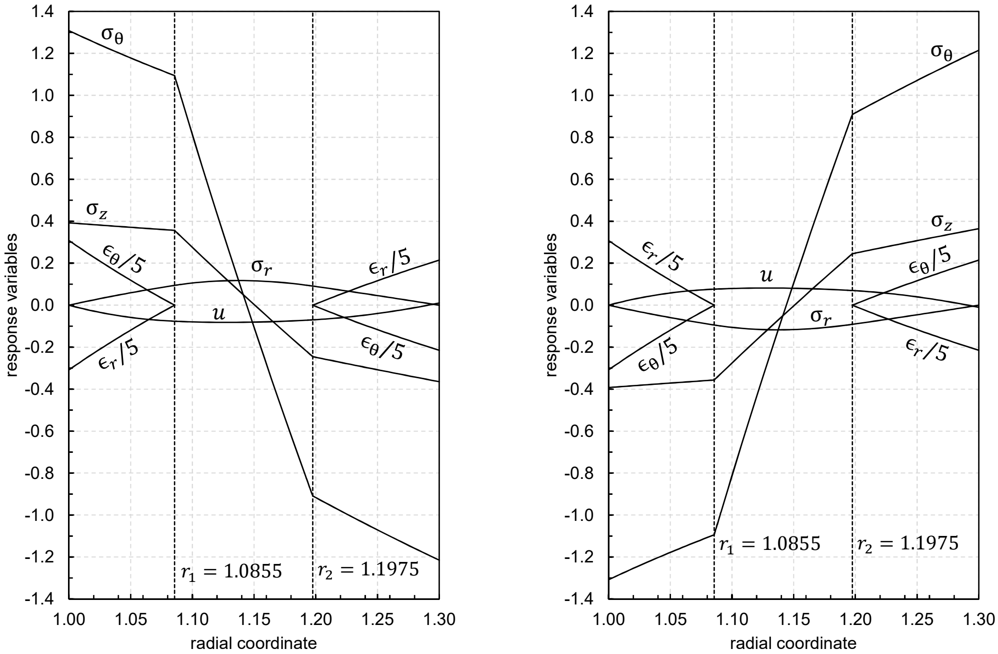
$$C_3 = -\frac{A_2 EHa}{2\left[1+H(1-\nu^2)\right]} - \frac{a^2 EH\phi}{4\left[1+H(1-\nu^2)\right]}, \tag{5.67}$$

$$C_4 = \frac{A_2 EH}{2a\left[1+H(1-\nu^2)\right]} + \frac{EH\phi}{4\left[1+H(1-\nu^2)\right]} \tag{5.68}$$
$$+\frac{\sigma_0 \ln a}{1+H(1-\nu^2)} - \frac{EH\phi \ln a}{2\left[1+H(1-\nu^2)\right]}.$$

Thus the number of equations is reduced from ten to six and the unknowns C_5, C_6, A_2, ϕ, r_1, and r_2 are obtained from Eqs. (5.59), (5.60), (5.61), (5.62), (5.63) and (5.64). The results of the unknowns for the beam of $b/a = 1.3$ are $\overline{C}_5 = 6.0214 \times 10^{-1}$, $\overline{C}_6 = -6.0214 \times 10^{-1}$, $A_2 = -2.0506 \times 10^{-2}$, $\phi = 2.3244 \times 10^{-2}$, $\overline{r}_1 = 1.0855$, and $\overline{r}_2 = 1.1975$.and for the beam of $b/a = 1.5$ $\overline{C}_5 = 4.8879 \times 10^{-1}$, $\overline{C}_6 = -4.8879 \times 10^{-1}$, $A_2 = -1.8145 \times 10^{-2}$, $\phi = 2.1936 \times 10^{-2}$, $\overline{r}_1 = 1.1586$, and $\overline{r}_2 = 1.2857$. Substituting the solution of the unknowns C_5, C_6, A_2, ϕ, r_1, and r_2 in Eqs. (5.65), (5.66), (5.67) and (5.68); $\overline{C}_1, \overline{C}_2, \overline{C}_3$, and $\overline{C}_4$ for $b/a = 1.3$ become $\overline{C}_1 = 3.9106$, $\overline{C}_2 = -3.8608$, $\overline{C}_3 = 6.0214 \times 10^{-1}$, $\overline{C}_4 = -8.2685 \times 10^{-1}$,and for $b/a = 1.5$ $\overline{C}_1 = 3.1745$, $\overline{C}_2 = -3.0172$, $\overline{C}_3 = 4.8879 \times 10^{-1}$, $\overline{C}_4 = -8.4082 \times 10^{-1}$.

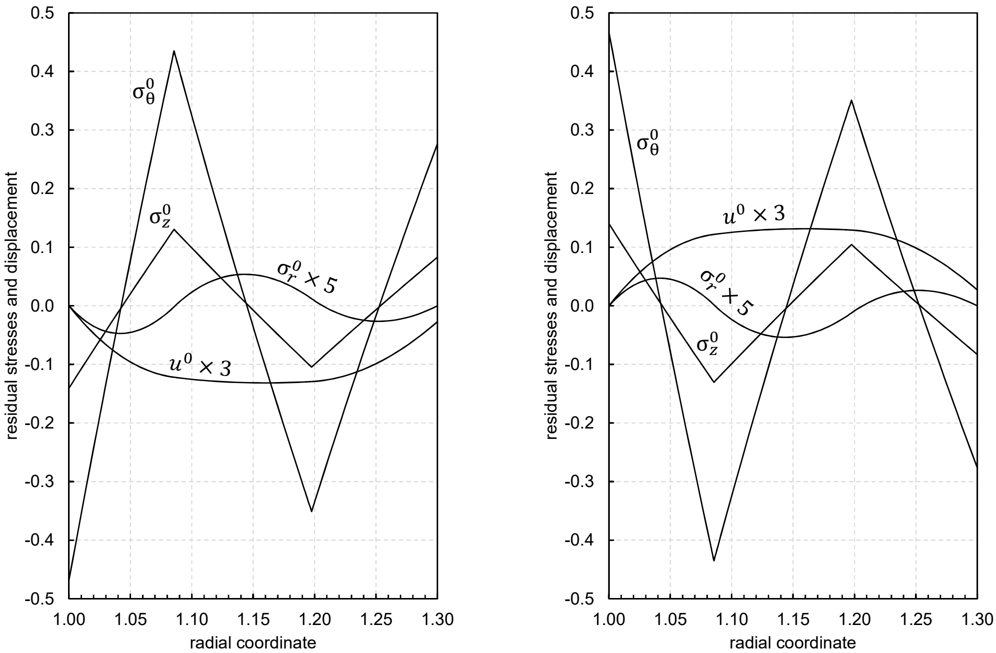
The stress response of the beam at the plane of symmetry, i.e. $\theta = 0$ under the load $\overline{M}$ is plotted for $b/a = 1.3$ in Fig. (5.17) (a) and (b) and Fig. (5.19) (a) and (b) for $b/a = 1.5$ in Fig. 4(c). The neutral axis are calculated with the solution of $\sigma_\theta(r_{NA}) = 0$ by Newton-Raphson method for the beam of $b/a = 1.3$ and $b/a = 1.5$ as $\overline{r}_{NA} = 1.1440$ and 1.2306, respectively.

To obtain the residual stresses, taking difference of the stresses for elastic-plastic and elastic behavior under $\overline{M} = 1.4387 \times 10^{-2}$ for a beam of $b/a = 1.3$ and $\overline{M} = 3.4 \times 10^{-2}$ for a beam of $b/a = 1.5$, the unknowns are calculated for the beam $b/a = 1.3$ as $\overline{C}_1 = 2.3681$, $\overline{C}_2 = -2.3681$, $\phi = 1.4168 \times 10^{-2}$, $A_2 = -1.2453 \times 10^{-2}$, and for the beam $b/a = 1.5$ as $\overline{C}_1 = 1.4970$, $\overline{C}_2 = -1.4970$, $\phi = 1.0499 \times 10^{-2}$, $A_2 = -8.6081 \times 10^{-3}$, and the corresponding graphs can be seen in Fig. (5.18) (a) and (b), and Fig. (5.20) (a) and (b), respectively.



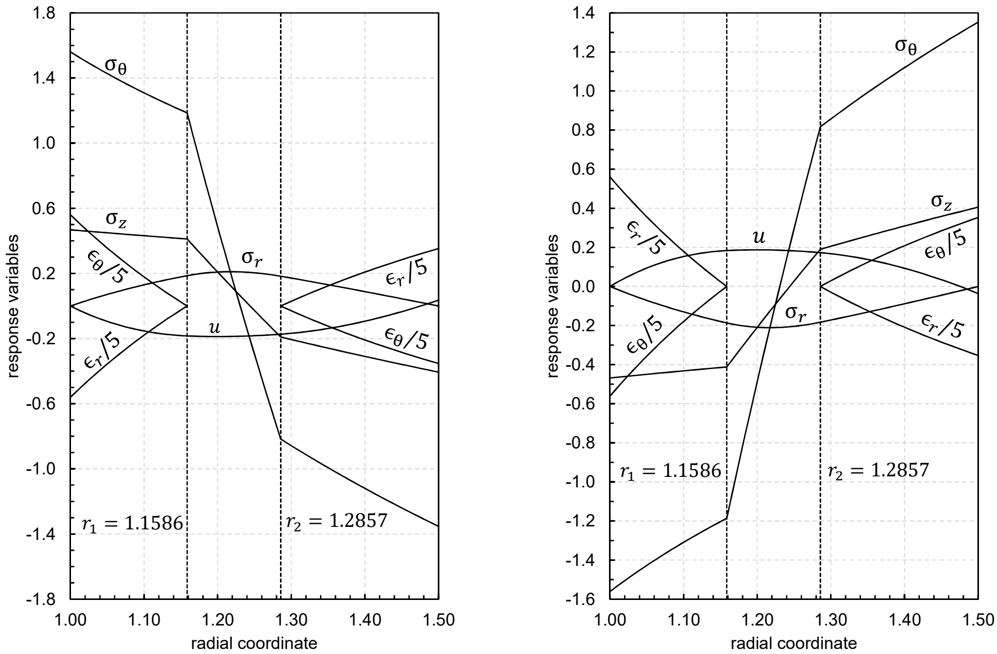
(a) (b)

Figure 5.17: The stress response under $\overline{M}$ of a beam of b/a=1.3 for (a) positive bending case, (b) negative bending case.



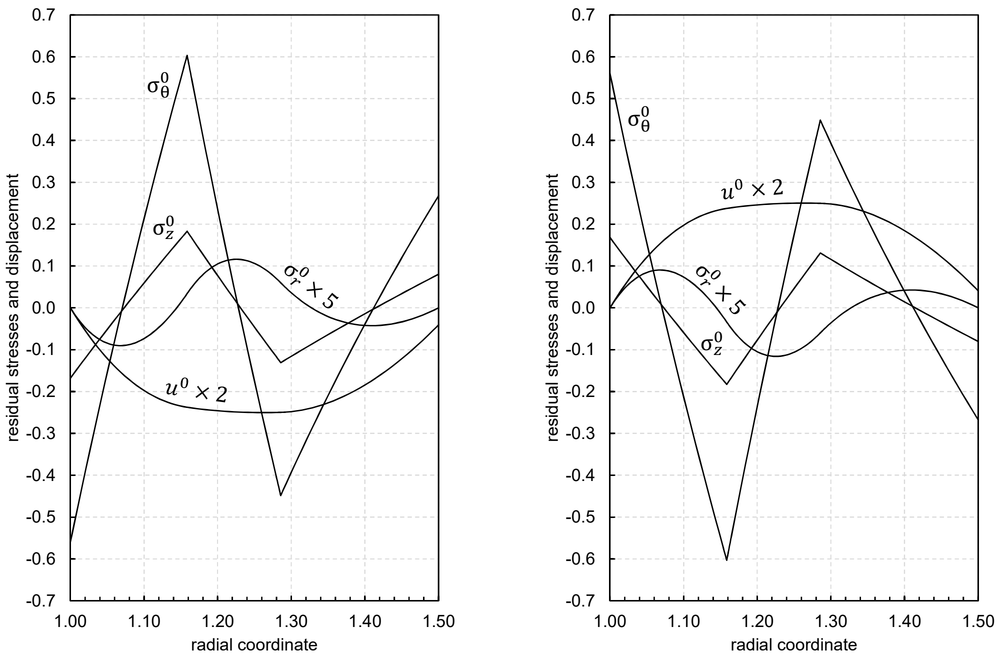
(a) (b)

Figure 5.18: The residual stresses of a beam of b/a=1.3 in stage II for (a) positive bending case, (b) negative bending case.



(a) (b)

Figure 5.19: The stress response under $\overline{M}$ of a beam of b/a=1.5 for (a) positive bending case, (b) negative bending case.



(a) (b)

Figure 5.20: The residual stresses of a beam of b/a=1.5 in stage II for (a) positive bending case, (b) negative bending case.

5.3 Comparisons

The results of the analytical solutions to elastoplastic deformation of a strain hardening curved beam based on Tresca's yield criterion in the states of plane stress and plane strain and in the cases of negative and positive bending are performed and the calculations and the corresponding graphs show that the solutions of negative bending case are similar to those of positive bending case under the same values of bending moment for different values of b/a ratio and in the plane stress and plane strain states. The results show similarity excepted that the stresses and displacement have opposite signs. The residual stresses in negative and positive bending cases are also similar. In plane stress solution, the first stage ends under $\overline{M}_I = 9.5488 \times 10^{-3}$ while the same limit is $\overline{M}_I = 9.5574 \times 10^{-3}$ in plane strain solution for a beam of $b/a = 1.3$. The elastic-plastic border radii, as shown in Figs. (5.3) and (5.13), are $\overline{r}_1 = 1.0218$ and 1.0195 for plane stress and plane strain solutions, respectively and neutral axis, r_{NA} are almost the same. This means that the plastic core by plane stress solution moves further at the end of the first stage. Moreover, σ_r, σ_θ and residual stress values are also similar in both solutions whereas smaller displacement values are obtained by plane stress solution. In the second stage of elastoplastic deformation, two of the solutions result in similar distributions again and larger displacement values are obtained by plane strain solution. Plane stress solution offers $\epsilon_r^p = 0$ because of the flow rule in the inner plastic region, and the plastic strain levels are compatible in both plastic regions. These results are also valid for the solutions of positive bending case, and the negative and positive bending solutions give the same absolute values of distributions for stresses and displacements in both states of plane stress and plane strain. In summary, plane stress solution is confirmed with plane strain solution in the elastic and elastic-plastic deformation stages in negative and positive bending conditions.

5.4 Convergence Analysis

As stated in previous sections of this chapter, the integration constants in the solutions of the plane stress and plane strain states are calculated by Version I solution which solves the nonlinear systems of all the equations using Newton's Method and by Version II solution in which the number of equations is reduced by solving some of the linear equations simultaneously and which solves the remaining nonlinear system. The convergence analysis is achieved for the positive bending case by comparing Version I and Version II solutions for a beam of $b/a = 1.3$.

By setting initial guesses for the unknowns closer to the desired root, and setting the number of the maximum iteration as 100, in the solutions of plastic region I for plane stress and plane strain states, the iteration for Version I converges until the convergence tolerance of 1.00 with the number of correct digits 9 while Version II converges with the tolerance 1.0×10^{-7} and 6 correct digits as seen in Table (5.1). However, the iteration in Version II converges quicker than Version I for tolerance 1.0×10^{-7}. In the solutions of plastic region II, the iterations for both versions converge until convergence tolerance takes the value 1.00. In Version I and II, iterations converge with the correct digits 10 and 8, respectively. Until tolerance becomes 1.0×10^{-5}, Version II converges much quicker but after exceeding this value, Version I converges quicker.

To sum up, convergence analysis indicates the reduction of the number of equations in solving nonlinear systems is unpractical because the iterations require small tolerances to converge with the required digits, but it reduces the number of iterations and makes the iteration converge quickly within definite tolerances.

NUMBER OF MAXIMUM ITERATION

		TOLERANCE	1.00E-10	1.00E-09	1.00E-08	1.00E-07	1.00E-06	1.00E-05	1.00E-04
STATE OF PLANE STRESS	REGION I	VERSION I	100	100	100	100	31	19	12
STATE OF PLANE STRESS	REGION I	VERSION II	40	37	35	32	-	-	-
STATE OF PLANE STRESS	REGION II	VERSION I	100	100	100	100	100	98	19
STATE OF PLANE STRESS	REGION II	VERSION II	100	100	100	100	92	27	20
STATE OF PLANE STRAIN	REGION I	VERSION I	100	100	100	100	23	12	12
STATE OF PLANE STRAIN	REGION I	VERSION II	32	30	28	26	-	-	-
STATE OF PLANE STRAIN	REGION II	VERSION I	100	100	100	100	100	45	18
STATE OF PLANE STRAIN	REGION II	VERSION II	100	100	100	100	100	17	17

Table 5.1: Number of Maximum Iterations for Convergence Analysis

CHAPTER 6

CONCLUSION

A narrow rectangular cross-section curved beam subjected to negative couples at its end sections is considered and analytical solutions for partially plastic deformation of the beam under pure bending are performed for plane stress and plane strain assumptions. Elastic and two stages of elastic-plastic deformations are studied for linearly hardening material behavior using Tresca's yield criterion and its associated flow rule which permit the analytical treatment and result in linear differential equations. Tresca's flow rule also provides conservative results in the elastoplastic solutions. Thereafter, a single differential equation, Eq. (4.15) governing the elastoplastic behavior of a curved beam under pure bending has been obtained using the generalized Hooke's law, the equation of equilibrium and the compatibility relation. Elastic, partially-plastic and residual stresses are calculated for all stages of deformations. The negative bending case is studied and the results for positive bending available in the literature [17] are regenerated by the use of different nonlinear system solution technique. Beam deforms elastically below the elastic limit load. If the load exceeds the elastic limit, plastic deformation commences at the inner surface. As the bending moment is further increased, this plastic region spreads into the beam in the radial direction. When it reaches at a critical load, second plastic region occurs at the outer surface. The two plastic regions expand over the elastic region with the increasing load. In the first stage of elastic-plastic deformation, the beam consists of an inner plastic region and an outer elastic region, and it consists of an inner plastic region, an elastic region and an outer plastic region in the second stage. The radial stresses, circumferential stresses, axial stresses, radial displacement, and the corresponding strains are calculated analytically. The calculations are performed for the plastic range to investigate the residual stresses. For this purpose, the residual stresses remaining in the beam after removing the load at plastic limit for stages I and II are also evaluated. The residual stresses are evaluated for both positive and negative bending cases. The numerical solutions of the elastic and plastic deformations are obtained by solving the nonlinear systems of equations associated with the boundary and interface conditions by using Newton's Method with two versions for each stages. In the first version, the evaluations are performed by solving all of the boundary and interface conditions simultaneously, and in the second version, by reducing the number of equations solving the linear systems of equations and solving the remaining nonlinear system. The convergence analysis comparing versions I and II is performed for the state of positive bending. The results for the case of negative bending moment are compared with the ones for the case of positive bending. It is observed that the stresses and deformation in cases of positive and negative bending exhibit similar behavior. It is also observed that the deformation in the cross-section is negligibly small. Then, it is determined that the plane stress and plane strain solutions give identical stress and displacement distributions. The results also indicate that the tangential stress components for elastic and elastoplastic solutions give higher absolute magnitudes than the radial and axial stress components. The radial stress components are compressive, and highest at the center, and the tangential stress components are compressive at inner radius and tensile at the outer radius, and highest where the plastic deformation begins. Then, the magnitude of the tangential residual stresses is higher than that of the radial and axial residual stresses.

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APPENDIX A

INTEGRATION OF EQ. (4.9)

Compatibility relation is

$$\frac{1}{r}\frac{d}{dr}\left(r^2\frac{d\epsilon_\theta}{dr}\right) - \frac{d\epsilon_r}{dr} = 0, \tag{A.1}$$

multiplying by r

$$\frac{d}{dr}\left(r^2\frac{d\epsilon_\theta}{dr}\right) - r\frac{d\epsilon_r}{dr} = 0. \tag{A.2}$$

Note that

$$\frac{d}{dr}\left(r\epsilon_\theta\right) = r\frac{d\epsilon_\theta}{dr} + \epsilon_\theta. \tag{A.3}$$

Multiplying Eq. (A.3) by r

$$r\frac{d}{dr}\left(r\epsilon_\theta\right) = r^2\frac{d\epsilon_\theta}{dr} + r\epsilon_\theta. \tag{A.4}$$

or

$$r^2\frac{d\epsilon_\theta}{dr} = r\frac{d}{dr}\left(r\epsilon_\theta\right) - r\epsilon_\theta. \tag{A.5}$$

Substituting Eq. (A.5) in Eq. (A.1)

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\left(r\epsilon_\theta\right) - r\epsilon_\theta\right) - \frac{d\epsilon_r}{dr} = 0, \tag{A.6}$$

$$\frac{1}{r}\left[\frac{d}{dr}\left(r\epsilon_\theta\right) + r\frac{d^2}{dr^2}(r\epsilon_\theta) - \frac{d}{dr}\left(r\epsilon_\theta\right)\right] - \frac{d\epsilon_r}{dr} = 0, \tag{A.7}$$

$$\frac{d^2}{dr^2}\left(r\epsilon_\theta\right) - \frac{d\epsilon_r}{dr} = 0, \tag{A.8}$$

Integrating Eq. (A.8)

$$\frac{d}{dr}\left(r\epsilon_\theta\right) - \epsilon_r = \phi. \tag{A.9}$$

APPENDIX B

DERIVATION OF EQS. (4.10) AND (4.11)

The strain-displacement relations:

$$\epsilon_r = \frac{\partial u}{\partial r}, \tag{B.1}$$

$$\epsilon_\theta = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta}, \tag{B.2}$$

$$\gamma_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r} + \frac{\partial v}{\partial r} = 0. \tag{B.3}$$

Eq. (B.2) can be written multiplying by r as

$$\epsilon_\theta r - u = \frac{\partial v}{\partial \theta}. \tag{B.4}$$

Differentiating Eq. (B.4) w.r.t. θ

$$-\frac{\partial u}{\partial \theta} = \frac{\partial^2 u}{\partial \theta^2} = F_1(\theta). \tag{B.5}$$

Substituting Eq. (B.5) in Eq. (B.3)

$$r\frac{\partial v}{\partial r} - v = F_1(\theta), \tag{B.6}$$

then

$$v = rF_2(\theta) - F_1(\theta). \tag{B.7}$$

Taking derivatives of both sides w.r.t. θ twice

$$\frac{\partial^2 v}{\partial \theta^2} = r\frac{\partial^2 F_2(\theta)}{\partial \theta^2} - \frac{\partial^2 F_1(\theta)}{\partial \theta^2}. \tag{B.8}$$

Substituting Eq. (B.5) in Eq. (B.8)

$$\frac{\partial^2 F_2(\theta)}{\partial \theta^2} = 0, \tag{B.9}$$

$$\frac{\partial^2 F_1(\theta)}{\partial \theta^2} = -F_1(\theta). \tag{B.10}$$

Integrating the Eqs. (B.9) and (B.10)

$$F_2(\theta) = A_1\theta, \tag{B.11}$$

$$F_1(\theta) = -A_2 \sin\theta - A_3 \cos\theta. \tag{B.12}$$

Substituting Eqs. (B.11) and (B.12) in Eq. (B.7)

$$v = A_1 r\theta + A_2 \sin\theta + A_3 \cos\theta. \tag{B.13}$$

Because of symmetry conditions, i.e. $v_{+\theta} = -v_{-\theta}$, we get $A_3 = 0$. The tangential displacement component becomes

$$v = A_1 r\theta + A_2 \sin\theta. \tag{B.14}$$

Substituting Eq. (B.14) in Eq. (B.2), the radial displacement component gets

$$u = r\epsilon_\theta - A_1 r - A_2 \cos\theta. \tag{B.15}$$

APPENDIX C

SOLUTION FOR POSITIVE BENDING

C.1 Plane stress

C.1.1 Elastic Region

The normal plastic strains vanish in the elastic region, i.e., $\epsilon_i^p = 0$ and in the plane stress solution $\sigma_z = 0$. Writing differential equation in Eq. (4.15) with $\epsilon_r^p = 0$, $\epsilon_\theta^p = 0$ and $\sigma_z = 0$;

$$r^2 \frac{d^2\sigma_r}{dr^2} + 3r\frac{d\sigma_r}{dr} = E\phi. \tag{C.1}$$

Solving the differential equation, Eq. (C.1);

$$\sigma_r = \frac{C_1}{r^2} + C_2 + \frac{1}{2}E\phi \ln r. \tag{C.2}$$

With the help of the equilibrium equation, Eq. (4.7), σ_θ becomes;

$$\sigma_\theta = -\frac{C_1}{r^2} + C_2 + \frac{1}{2}E\phi(1 + \ln r). \tag{C.3}$$

By using Hooke's law in Eq. (4.2) with Eq. (4.13), the displacement becomes

$$u = \frac{1}{E}\left[-\frac{1}{r}(1+\nu)C_1 + (1-\nu)C_2 r\right] - \left[1 - (1-\nu)\ln r\right]\frac{\phi r}{2} - A_2 \cos\theta. \tag{C.4}$$

In the stress equations, there are three unknowns which are C_1, C_2 and ϕ. For evaluation of these unknowns, three linearly independent conditions are available. These are:

$$\int_a^b \sigma_\theta r dr = -M, \tag{C.5}$$

$$\sigma_r(a) = \sigma_r(b) = 0. \tag{C.6}$$

Eq. (C.5) becomes

$$C_1 \ln(b/a) + \frac{1}{2}C_2(b^2 - a^2) + \frac{1}{8}E\phi \left[-a^2(1 + 2\ln a) + b^2(1 + 2\ln b)\right] = -M. \tag{C.7}$$

Combining Eqs. (C.6) and (C.7), the unknown integration constants C_1, C_2 and ϕ are obtained as

$$C_1 = -\frac{4M}{N}a^2b^2 \ln(b/a), \tag{C.8}$$

$$C_2 = \frac{4M}{N}(b^2 \ln b - a^2 \ln a), \tag{C.9}$$

$$\phi = \frac{8M}{EN}(a^2 - b^2), \tag{C.10}$$

in which

$$N = (a^2 - b^2)^2 - 4a^2b^2[\ln(b/a)]^2. \tag{C.11}$$

The displacement will be equal to zero assuming that the beam is rigidly fixed at $r = a$, $\theta = 0$, i.e.

$$u(a, 0) = \frac{1}{E}\left[-\frac{1}{a}(1 + \nu)C_1 + (1 - \nu)C_2a\right] - [1 - (1 - \nu)\ln a]\frac{\phi a}{2} - A_2 = 0. \tag{C.12}$$

Solving Eq. (C.12) for A_2

$$A_2 = \frac{1}{E}\left[-\frac{1}{a}(1 + \nu)C_1 + (1 - \nu)C_2a\right] - [1 - (1 - \nu)\ln a]\frac{\phi a}{2}. \tag{C.13}$$

C.1.2 Plastic Region I

The stress state in this region is

$$\sigma_\theta > \sigma_r > \sigma_z (= 0), \tag{C.14}$$

and Tresca's yield criterion reads

$$\sigma_y = \sigma_\theta - \sigma_z = \sigma_\theta. \tag{C.15}$$

The associated flow rule gives

$$\epsilon_\theta^p = -\epsilon_z^p \text{ and } \epsilon_r^p = 0. \tag{C.16}$$

From the equivalence of plastic work increment, the plastic tangential normal strain becomes

$$\epsilon_\theta^p = \epsilon_{EQ}, \tag{C.17}$$

where ϵ_{EQ} is the equivalent plastic strain. The yield stress for a linearly hardening material

$$\sigma_y = \sigma_0 \left(1 + \eta \epsilon_{EQ}\right). \tag{C.18}$$

Then combining Eqs. (C.15) and (C.18)

$$\epsilon_{EQ} = \left(\frac{\sigma_\theta}{\sigma_0} - 1\right)\frac{1}{\eta}. \tag{C.19}$$

By using Eq. (4.7) with Eqs. (C.16) and (C.17), ϵ_θ^p and ϵ_z^p become

$$\epsilon_\theta^p = -\epsilon_z^p = \left[\frac{1}{\sigma_0}\frac{d}{dr}(r\sigma_r) - 1\right]\frac{1}{\eta}. \tag{C.20}$$

Substituting Eq. (C.20), $\sigma_z = 0$ and $\epsilon_r^p = 0$ in the differential equation, Eq. (4.15)

$$r^2\frac{d^2\sigma_r}{dr^2} + 3r\frac{d\sigma_r}{dr} + \frac{\sigma_r}{1+H} = \frac{EH\phi}{1+H} + \frac{\sigma_0}{1+H}, \tag{C.21}$$

where the hardening parameter $H = \eta\sigma_0/E$. Solving the differential equation, Eq. (C.21) for σ_r

$$\sigma_r = C_3 r^{-1-W} + C_4 r^{-1+W} + EH\phi + \sigma_0, \tag{C.22}$$

in which $W = \sqrt{H/(1+H)}$. From the equilibrium equation, Eq. (4.7), σ_θ becomes;

$$\sigma_\theta = -WC_3 r^{-1-W} + WC_4 r^{-1+W} + EH\phi + \sigma_0. \tag{C.23}$$

The plastic strain components are obtained by Eqs. (C.20) and (C.22) as

$$\epsilon_\theta^p = -\epsilon_z^p = \frac{W}{EH}\left[-\frac{C_3}{r^{1+W}} + \frac{C_4}{r^{1-W}}\right] + \phi. \tag{C.24}$$

Combining Eqs. (4.2) and (C.24) and substituting in Eq. (4.13), the displacement becomes

$$\begin{aligned} u \;=\; & \frac{1}{EH}\left\{-\left[W + H(W+\nu)\right]C_3 r^{-W} + \left[W + H(W-\nu)\right]C_4 r^{W}\right. \qquad \text{(C.25)} \\ & \left. + H(1-\nu)\sigma_0 r\right\} + H\phi(1-\nu)r - A_2\cos\theta. \end{aligned}$$

The curved beam consists of a plastic region in

$$a \le r \le r_1,$$

and an elastic region in

$$r_1 \le r \le b,$$

where r_1 is the plastic-elastic border radius. The similar boundary and interface conditions to the negative bending solution are used to determine the unknowns in the stage I calculations.

C.1.3 Plastic Region II

The stress state in this region:

$$\sigma_r > \sigma_z (= 0) > \sigma_\theta, \tag{C.26}$$

and Tresca's yield criterion reads

$$\sigma_y = \sigma_r - \sigma_\theta. \tag{C.27}$$

The plastic normal strains and the equivalent plastic strain are

$$\epsilon_r^p = -\epsilon_\theta^p = \epsilon_{EQ} \text{ and } \epsilon_z^p = 0. \tag{C.28}$$

The yield stress is

$$\sigma_y = \sigma_0 \left(1 + \eta \epsilon_{EQ}\right), \tag{C.29}$$

then

$$\epsilon_{EQ} = \left(\frac{\sigma_r - \sigma_\theta}{\sigma_0} - 1\right) \frac{1}{\eta}. \tag{C.30}$$

By using Eq. (4.7) with Eqs. (C.28) and (C.30), ϵ_r^p and ϵ_θ^p become

$$\epsilon_r^p = -\epsilon_\theta^p = \left\{ \frac{1}{\sigma_0} \left[\sigma_r - \frac{d}{dr}(r\sigma_r) \right] - 1 \right\} \frac{1}{\eta}. \tag{C.31}$$

Substituting Eq. (C.31) and $\sigma_z = 0$ in the differential equation, Eq. (4.15)

$$r^2 \frac{d^2\sigma_r}{dr^2} + 3r\frac{d\sigma_r}{dr} = \frac{EH\phi}{1+H} - \frac{2\sigma_0}{1+H}. \tag{C.32}$$

Solving the differential equation, Eq. (C.32);

$$\sigma_r = \frac{C_5}{r^2} + C_6 + \frac{EH\phi \ln r}{2\left(1+H\right)} - \frac{\sigma_0 \ln r}{1+H}. \tag{C.33}$$

From the equilibrium equation, Eq. (4.7) σ_θ becomes

$$\sigma_\theta = -\frac{C_5}{r^2} + C_6 + \frac{EH\phi(1+\ln r)}{2(1+H)} - \frac{\sigma_0(1+\ln r)}{1+H}. \tag{C.34}$$

The plastic strain components are obtained by substituting Eq. (C.33) in Eq. (C.31) as

$$\epsilon_r^p = -\epsilon_\theta^p = \frac{1}{EH}\left[\frac{2C_5}{r^2} - \frac{H\sigma_0}{1+H}\right] - \frac{\phi}{2(1+H)}. \tag{C.35}$$

Combining Eqs. (4.2) and (C.35) and substituting in Eq. (4.13), the displacement becomes

$$\begin{aligned} u \;=\; & \frac{1}{E}\left\{-\frac{[2+H(1+\nu)]\,C_5}{Hr} + (1-\nu)\,C_6 r - \frac{\sigma_0(1-\nu)\,r\ln r}{1+H}\right\} \\ & -\frac{\phi\,[1+H-H(1-\nu)\ln r]\,r}{2(1+H)} - A_2\cos\theta. \end{aligned} \tag{C.36}$$

The curved beam consists of an inner plastic region (region I) in

$$a \le r \le r_1,$$

an elastic region in

$$r_1 \le r \le r_2,$$

and an outer plastic region (region II) in

$$r_2 \le r \le b,$$

where r_1 and r_2 are the elastic-plastic border radius. The similar boundary and interface conditions to the negative bending solution are used to determine the unknowns in the stage II calculations.

C.2 Plane strain

C.2.1 Elastic Region

The normal plastic strains vanish in the elastic region, i.e., $\epsilon_i^p = 0$. In the plane strain solution $\epsilon_z = 0$. Solving Eq. (4.3) for σ_z

$$\sigma_z = \nu\,(\sigma_r + \sigma_\theta)\,, \tag{C.37}$$

and with the equilibrium equation, Eq. (4.7)

$$\sigma_z = \nu \left[\sigma_r + \frac{d}{dr}(r\sigma_r) \right]. \tag{C.38}$$

Solving the differential equation, Eq. (4.15) with $\epsilon_r^p = 0$, $\epsilon_\theta^p = 0$ and Eq. (C.38);

$$r^2 \frac{d^2\sigma_r}{dr^2} + 3r\frac{d\sigma_r}{dr} = \frac{E\phi}{1-\nu^2}. \tag{C.39}$$

Solving differential equation, Eq. (C.39) for σ_r :

$$\sigma_r = \frac{A}{r^2} + \left[B + \frac{E\phi(\ln 2 + \ln(1-\nu^2))}{1-\nu^2} \right] + \frac{E\phi \ln r}{2(1-\nu^2)}, \tag{C.40}$$

or simply

$$\sigma_r = \frac{C_1}{r^2} + C_2 + \frac{E\phi \ln r}{2(1-\nu^2)}. \tag{C.41}$$

From the equilibrium equation, Eq. (4.7), σ_θ becomes;

$$\sigma_\theta = -\frac{C_1}{r^2} + C_2 + \frac{E\phi(1+\ln r)}{2(1-\nu^2)}. \tag{C.42}$$

By substituting the equations of Hooke's law, Eq. (4.2) and Eq. (C.38), in Eq. (4.13), the displacement becomes

$$\begin{aligned} u \;=\; & \frac{1}{E}\left[-\frac{1}{r}(1+\nu)C_1 + (1+\nu)(1-2\nu)C_2 r \right] \\ & -\frac{[1-\nu-(1-2\nu)\ln r]\,\phi r}{2(1-\nu)} - A_2 \cos\theta. \end{aligned} \tag{C.43}$$

In the stress equations, there are three unknowns which are C_1, C_2 and ϕ. For evaluation of these unknowns, three linearly independent conditions are available. These are:

$$\int_a^b \sigma_\theta r dr = -M, \tag{C.44}$$

$$\sigma_r(a) = \sigma_r(b) = 0. \tag{C.45}$$

Eq. (C.44) becomes

$$C_1 \ln(a/b) + \frac{1}{2}C_2(b^2 - a^2) - \frac{E\phi[a^2(1+2\ln a) - b^2(1+2\ln b)]}{8(1-\nu^2)} = -M. \tag{C.46}$$

Combining Eqs. (C.45) and (C.46), the unknown integration constants C_1, C_2 and ϕ are obtained as

$$C_1 = -\frac{4M}{N} a^2 b^2 \ln(b/a), \tag{C.47}$$

$$C_2 = \frac{4M}{N}(b^2 \ln b - a^2 \ln a), \tag{C.48}$$

$$\phi = \frac{8M}{EN}(a^2 - b^2)(1 - \nu^2), \tag{C.49}$$

in which

$$N = (a^2 - b^2)^2 - 4a^2b^2[\ln(b/a)]^2. \tag{C.50}$$

The displacement will be equal to zero assuming that the beam is rigidly fixed at $r = a$ and $\theta = 0$, i.e.

$$\begin{aligned} u(a,0) = \frac{1}{E}\left[-\frac{1}{a}(1+\nu)C_1 + (1+\nu)(1-2\nu)aC_2\right] \\ -\frac{[1-\nu-(1-2\nu)\ln a]\,\phi a}{2(1-\nu)} - A_2 = 0. \end{aligned} \tag{C.51}$$

Solving Eq. (C.51) for A_2

$$\begin{aligned} A_2 &= \frac{1}{E}\left[-\frac{1}{a}(1+\nu)C_1 + (1+\nu)(1-2\nu)aC_2\right] \\ &\quad -\frac{[1-\nu-(1-2\nu)\ln a]\,\phi a}{2(1-\nu)}. \end{aligned} \tag{C.52}$$

C.2.2 Plastic Region I

The stress state in this region is

$$\sigma_\theta > \sigma_z > \sigma_r, \tag{C.53}$$

and Tresca's yield criterion reads

$$\sigma_y = \sigma_\theta - \sigma_r, \tag{C.54}$$

From the equivalence of plastic work increment,

$$\epsilon_{EQ} = \epsilon_\theta^p = -\epsilon_r^p \text{ and } \epsilon_z^p = 0, \tag{C.55}$$

where ϵ_{EQ} is the equivalent plastic strain. The yield stress is

$$\sigma_y = \sigma_0\left(1 + \eta\epsilon_{EQ}\right), \tag{C.56}$$

then, combining Eqs. (C.54) and (C.56)

$$\epsilon_{EQ} = \left(\frac{\sigma_\theta - \sigma_r}{\sigma_0} - 1\right)\frac{1}{\eta}. \tag{C.57}$$

By using Eq. (4.7) with Eq. (C.55), ϵ_θ^p and ϵ_r^p become

$$\epsilon_\theta^p = -\epsilon_r^p = \left\{ \frac{1}{\sigma_0} \left[\frac{d}{dr}(r\sigma_r) - \sigma_r \right] - 1 \right\} \frac{1}{\eta}. \tag{C.58}$$

Substituting Eqs. (C.58) and (C.38) in differential equation, Eq. (4.15)

$$r^2 \frac{d^2\sigma_r}{dr^2} + 3r\frac{d\sigma_r}{dr} = \frac{EH\phi}{1 + H(1-\nu^2)} + \frac{2\sigma_0}{1 + H(1-\nu^2)}. \tag{C.59}$$

Solving the differential equation, Eq. (C.59) for σ_r;

$$\sigma_r = \frac{C_3}{r^2} + C_4 + \frac{EH\phi \ln r}{2\left[1 + H(1-\nu^2)\right]} + \frac{\sigma_0 \ln r}{1 + H(1-\nu^2)}. \tag{C.60}$$

From the equilibrium equation, Eq. (4.7), σ_θ becomes;

$$\sigma_\theta = -\frac{C_3}{r^2} + C_4 + \frac{EH\phi(1 + \ln r)}{2\left[1 + H(1-\nu^2)\right]} + \frac{\sigma_0(1 + \ln r)}{1 + H(1-\nu^2)}. \tag{C.61}$$

The plastic strain components are obtained by substituting Eq. (C.60) in Eq. (C.58)

$$\epsilon_\theta^p = -\epsilon_r^p = -\frac{1}{E}\left[\frac{2C_3}{Hr^2} + \frac{(1-\nu^2)\sigma_0}{1 + H(1-\nu^2)}\right] + \frac{\phi}{2\left[1 + H(1-\nu^2)\right]}. \tag{C.62}$$

Combining Eqs. (4.2) and (C.62) and substituting in Eq. (4.13), the displacement becomes

$$\begin{aligned} u \;=\; & \frac{1}{E}\left\{-\frac{\left[2 + H(1+\nu)\right]C_3}{Hr} + (1+\nu)(1-2\nu)C_4 r \right. \\ & \left. + \frac{\sigma_0(1+\nu)(1-2\nu) r \ln r}{1 + H(1-\nu^2)}\right\} \\ & - \left[\frac{1 + H(1-\nu^2) - H(1+\nu)(1-2\nu)\ln r}{1 + H(1-\nu^2)}\right]\frac{\phi r}{2} \\ & - A_2 \cos\theta. \end{aligned} \tag{C.63}$$

The curved beam consists of a plastic region in

$$a \leq r \leq r_1,$$

and an elastic region in

$$r_1 \leq r \leq b,$$

where r_1 is the plastic-elastic border radius. The similar boundary and interface conditions to the negative bending solution are used to determine the unknowns in the stage I calculations.

C.2.3 Plastic Region II

The stress state in this region:

$$\sigma_r > \sigma_z > \sigma_\theta. \tag{C.64}$$

and Tresca's yield criterion reads

$$\sigma_y = \sigma_r - \sigma_\theta, \tag{C.65}$$

The plastic normal strains and the equivalent plastic strain are

$$\epsilon_r^p = -\epsilon_\theta^p = \epsilon_{EQ} \text{ and } \epsilon_z^p = 0. \tag{C.66}$$

The yield stress is

$$\sigma_y = \sigma_0 \left(1 + \eta\epsilon_{EQ}\right), \tag{C.67}$$

then

$$\epsilon_{EQ} = \left(\frac{\sigma_r - \sigma_\theta}{\sigma_0} - 1\right)\frac{1}{\eta}. \tag{C.68}$$

By using Eqs. (4.7) and (C.68) ϵ_r^p and ϵ_θ^p become

$$\epsilon_r^p = -\epsilon_\theta^p = \left\{\frac{1}{\sigma_0}\left[\sigma_r - \frac{d}{dr}(r\sigma_r)\right] - 1\right\}\frac{1}{\eta}. \tag{C.69}$$

Substituting Eqs. (C.69) and (C.38) in differential equation, Eq. (4.15)

$$r^2\frac{d^2\sigma_r}{dr^2} + 3r\frac{d\sigma_r}{dr} = \frac{EH\phi}{1 + H(1-\nu^2)} - \frac{2\sigma_0}{1 + H(1-\nu^2)}. \tag{C.70}$$

Solving the differential equation, Eq. (C.70) for σ_r;

$$\sigma_r = \frac{C_5}{r^2} + C_6 + \frac{EH\phi\ln r}{2\left[1 + H(1-\nu^2)\right]} - \frac{\sigma_0 \ln r}{1 + H(1-\nu^2)}. \tag{C.71}$$

From the equilibrium equation, Eq. (4.7) we get σ_θ as;

$$\sigma_\theta = -\frac{C_5}{r^2} + C_6 + \frac{EH\phi(1 + \ln r)}{2\left[1 + H(1-\nu^2)\right]} - \frac{\sigma_0(1 + \ln r)}{1 + H(1-\nu^2)}. \tag{C.72}$$

The plastic strain components are obtained by Eq. (C.69) as

$$\epsilon_r^p = -\epsilon_\theta^p = \frac{1}{E}\left[\frac{2C_5}{Hr^2} + \frac{(1-\nu^2)\sigma_0}{1 + H(1-\nu^2)}\right] - \frac{\phi}{2\left[1 + H(1-\nu^2)\right]}. \tag{C.73}$$

From the Eq. (4.13) the displacement becomes

$$
\begin{aligned}
u \;=\; & \frac{1}{E}\left\{-\frac{\left[2+H(1+\nu)\right]C_5}{Hr}+(1+\nu)(1-2\nu)C_6 r\right. \\
& \left.-\frac{\sigma_0(1+\nu)(1-2\nu)r\ln r}{1+H(1-\nu^2)}\right\} \\
& -\left[\frac{1+H(1-\nu^2)-H(1+\nu)(1-2\nu)\ln r}{1+H(1-\nu^2)}\right]\frac{\phi r}{2} \\
& -A_2\cos\theta.
\end{aligned}
\tag{C.74}
$$

The curved beam consists of an inner plastic region (region I) in

$$a \leq r \leq r_1,$$

an elastic region in

$$r_1 \leq r \leq r_2,$$

and an outer plastic region (region II) in

$$r_2 \leq r \leq b.$$

where r_1 and r_2 are the elastic-plastic border radius. The similar boundary and interface conditions to the negative bending solution are used to determine the unknowns in the stage II calculations.

The analytical solutions and results of elastic-plastic bending of the curved beam under positive bending can be found in the comprehensive research by Eraslan and Aslan [17].

APPENDIX D

NONLIN PACKAGE

NONLIN is a FORTRAN Subroutine library to solve nonlinear systems of algebraic equations by Newton's Method. The program was coded by Ahmet N. Eraslan in 1993. The entire library consists of 8 FORTAN Subroutines. These are NLNFJ, NLNF, NLNBJ, NLNB, NLNTJ, NLNT, JAC and SOLVE. NONLIN is linked to LINPACK to solve linear systems of algebraic equations having full or banded Jacobian matrices. Linear systems having Tri-Diagonal Jacobian matrices are solved by Subroutine SOLVE contained in the package. Brief explanations for the Subroutines in the package are as the following.

NLNFJ: This subroutine solves n-nonlinear simultaneous equations (having a full Jacobian matrix) using Newton's method. The analytical Jacobian matrix should be provided thru Subroutine FI by the user.

NLNF: This Subroutine solves n-nonlinear simultaneous equations (having a full Jacobian matrix) using Newton's method. The Jacobian matrix is generated internally by making n-extra calls to the user supplied routine FI.

NLNBJ: This Subroutine solves n-nonlinear simultaneous equations (having a banded Jacobian matrix) using Newton's method. The analytical Jacobian matrix should be provided thru subroutine FI by the user.

NLNB: This Subroutine solves n-nonlinear simultaneous equations (having a banded Jacobian matrix) using Newton's method. The Jacobian matrix is generated internally by making n-extra calls to the user supplied routine FI.

NLNTJ: This Subroutine solves n-nonlinear simultaneous equations (having a tri-diagonal Jacobian matrix) using Newton's method. The analytical Jacobian matrix should be provided thru subroutine FI by the user.

NLNT: This Subroutine solves n-nonlinear simultaneous equations (having a tri-diagonal Jacobian matrix) using Newton's method. The Jacobian matrix is generated internally by making n-extra calls to the user supplied routine FI.

JAC: This Subroutine approximates the Jacobian matrix using forward differences.

SOLVE: This subroutine solves a system of n linear simultaneous equations having a Tri-Diagonal coefficient matrix.

In this study Subroutine NLNF is used throughout. The parameters and their meanings are given below.

```
C     ========================================================
      SUBROUTINE NLNF (FI, X, N, ITMAX, TOL, IPVT, W, LW)
C     ========================================================
      IMPLICIT DOUBLE PRECISION (A-H , O-Z)
      DIMENSION X(N), IPVT(N), W(LW)
      EXTERNAL FI
C-----------------------------------------------------------------------
C  SUBROUTINE NLNF SOLVES N-NONLINEAR SIMULTANEOUS EQUATIONS
C  (HAVING A FULL JACOBIAN MATRIX) USING NEWTON METHOD.
C  THE JACOBIAN MATRIX IS GENERATED INTERNALLY BY MAKING N-EXTRA
C  CALLS TO THE USER SUPPLIED ROUTINE FI.
C
C  PARAMETER LIST:
C  ---------------
C  FI            : THE NAME OF THE USER SUPPLIED SUBROUTINE WHICH
C                  PROVIDES THE FUNCTION VALUES (F(I), I=1,N).
C                  THIS SUBROUTINE SHOULD HAVE THE FORM:
C
C                  SUBROUTINE FI(F, X, N)
C                  DOUBLE PRECISION F, X
C                  INTEGER N
C                  DIMENSION F(N), X(N)
C                  F(1) = .....
C                  F(2) = .....
C                  ....
```

```
C                  F(N) = .....
C                  RETURN
C                  END
C
C  X              : SOLUTION VECTOR OF LENGTH N. WHEN NLNF IS CALLED
C                   X SHOULD CARRY THE INITIAL ESTIMATES.
C  N              : NUMBER OF SIMULTANEOUS EQUATIONS.
C  ITMAX          : MAXIMUM NUMBER OF ITERATIONS ALLOWED. ON RETURN ITMAX
C                   CARRIES THE NUMBER OF ITERATIONS PERFORMED TO HIT
C                   THE GIVEN ERROR BOUND.
C  TOL            : ERROR TOLERANCE FOR TERMINATING NEWTON ITERATIONS.
C                   MAX (X(I,[K]) - X(I,[K-1])) < TOL IS IMPOSED. WHERE K
C                   IS THE ITERATION COUNTER.
C  IPVT           : AN INTEGER WORK ARRAY OF LENGTH N.
C  W              : A DOUBLE PRECISION WORK ARRAY OF LENGTH LW. ON RETURN
C                   THE FIRST N LOCATIONS OF W CONTAIN MOST RECENT F
C                   VALUES.
C  LW             : DIMENSION OF W EQUAL TO 1+N*(N + 3).
C
C    AHMET N. ERASLAN
C    30 - 6 - 1993
C-----------------------------------------------------------------------
```

APPENDIX E

SUBROUTINES USED IN THIS THESIS

```
C     ----------------------------------------------------
      SUBROUTINE ELAST (R, C1, C2, PHI, A2, SIGR, SIGT, U)
C     ----------------------------------------------------
C     SUBROUTINE ELAST CALCULATES DIMENSIONLESS STRESS COMPONENTS SIGR,
C     SIGT, AND DIMENSIONLESS RADIAL DISPLACEMENT U IN THE ELASTIC
C     REGION OF THE PLANE STRESS SOLUTION.
C
C     ARGUMENT LIST:
C     --------------
C     R          : RADIAL COORDINATE,
C     C1, C2     : INTEGRATION CONSTANTS,
C     PHI, A2    : ADDITIONAL CONSTANTS IN THE STRESS EXPRESSIONS,
C     SIGR       : RADIAL STRESS COMPONENT,
C     SIGT       : CIRCUMFERENTIAL STRESS COMPONENT,
C     U          : RADIAL DISPLACEMENT.
C
C     COMMON BLOCK PROP:
C     ------------------
C     E          : MODULUS OF ELASTICITY,
C     NU         : POISSON'S RATIO,
C     SIG0       : YIELD STRENGTH,
C     THETA      : CIRCUMFERENTIAL ANGLE,
C     A          : INNER RADIUS OF THE ARC,
C     B          : OUTER RADIUS OF THE ARC,
C     H          : HARDENING PARAMETER,
C     WH         : FUNCTION OF HARDENING PARAMETER,
C     M          : BENDING MOMENT.
C
C     GAMZE CAKIR AND AHMET N. ERASLAN, MAY 2018,
C     DEPARTMENT OF ENGINEERING SCIENCES,
```

```
C     MIDDLE EAST TECHNICAL UNIVERSITY, ANKARA.
C
      IMPLICIT NONE
      DOUBLE PRECISION SIGR, SIGT, U, C1, C2, R, SIG0, E, PHI,
     1                 NU, A2, THETA, A, B, R1, M, H, WH
      COMMON /PROP/ E, NU, SIG0, THETA, A, B, H, WH, M
      SIGR = C1/R**2 + C2 + E*PHI*DLOG(R)/2.0
      SIGT = -C1/R**2 + C2 + E*PHI*(1.0D0 + DLOG(R))/2.0
      U    = 1.0/E*(-1.0/R*(1.0D0 + NU)*C1 + (1.0D0 - NU)*C2*R)
     1     - (1.0D0 - (1.0D0 - NU)*DLOG(R))*PHI*R/2.0 - A2*DCOS(THETA)
      SIGR = SIGR/SIG0
      SIGT = SIGT/SIG0
      U    = U*E/(SIG0*B)
      RETURN
      END
C

C     ------------------------------------------------------------
      SUBROUTINE PLAST (R, C3, C4, PHI, A2, SIGR, SIGT, U, ET, EZ)
C     ------------------------------------------------------------
C     SUBROUTINE PLAST CALCULATES DIMENSIONLESS STRESS COMPONENTS SIGR,
C     SIGT, DIMENSIONLESS RADIAL DISPLACEMENT U AND NORMALIZED PLASTIC
C     STRAINS ET AND EZ IN THE PLASTIC REGION I OF THE PLANE STRESS SOLUTION.
C
C     ARGUMENT LIST:
C     --------------
C     R          : RADIAL COORDINATE,
C     C3, C4     : INTEGRATION CONSTANTS,
C     PHI, A2    : ADDITIONAL CONSTANTS IN THE STRESS EXPRESSIONS,
C     SIGR       : RADIAL STRESS COMPONENT,
C     SIGT       : CIRCUMFERENTIAL STRESS COMPONENT,
C     U          : RADIAL DISPLACEMENT,
C     ET         : PLASTIC STRAIN IN CIRCUMFERENTIAL DIRECTION,
C     EZ         : PLASTIC STRAIN IN AXIAL DIRECTION.
C
C     GAMZE CAKIR AND AHMET N. ERASLAN, MAY 2018,
C     DEPARTMENT OF ENGINEERING SCIENCES,
C     MIDDLE EAST TECHNICAL UNIVERSITY, ANKARA.
C
      IMPLICIT NONE
      DOUBLE PRECISION SIGR, SIGT, U, ET, EZ, C3, C4, R, SIG0, E, PHI,
```

```
     1                 NU, A2, THETA, A, B, R1, H, WH, M, T1, T2
      COMMON /PROP/ E, NU, SIG0, THETA, A, B, H, WH, M
      T1   = -1.0D0 - WH
      T2   = -1.0D0 + WH
      SIGR = C3*R**T1 + C4*R**T2 - SIG0 + E*H*PHI
      SIGT = - C3*WH*R**T1 + C4*WH*R**T2 - SIG0 + E*H*PHI
      U    = 1.0/(E*H)*(-(WH + H*(WH + NU))*C3*R**(-WH) + (WH + H*(WH
     1        - NU))*C4*R**WH - H*(1.0D0 - NU)*SIG0*R) + H*PHI*(1.0D0
     2        - NU)*R - A2*DCOS(THETA)
      EZ   = WH/(E*H)*(C3*R**T1 - C4*R**T2) - PHI
      ET   = - EZ
      SIGR = SIGR/SIG0
      SIGT = SIGT/SIG0
      U    = U*E/(SIG0*B)
      EZ   = EZ*E/SIG0
      ET   = ET*E/SIG0
C
      RETURN
      END
C

C     --------------------------------------------------------------
      SUBROUTINE PLAST2 (R, C5, C6, PHI, A2, SIGR, SIGT, U, ET, ER)
C     --------------------------------------------------------------
C     SUBROUTINE PLAST2 CALCULATES DIMENSIONLESS STRESS COMPONENTS SIGR,
C     SIGT, DIMENSIONLESS RADIAL DISPLACEMENT U AND NORMALIZED PLASTIC
C     STRAINS ET AND ER IN THE PLASTIC REGION II OF THE PLANE STRESS SOLUTION.
C
C     ARGUMENT LIST:
C     --------------
C     R          : RADIAL COORDINATE,
C     C5, C6     : INTEGRATION CONSTANTS,
C     PHI, A2    : ADDITIONAL CONSTANTS IN THE STRESS EXPRESSIONS,
C     SIGR       : RADIAL STRESS COMPONENT,
C     SIGT       : CIRCUMFERENTIAL STRESS COMPONENT,
C     U          : RADIAL DISPLACEMENT,
C     ET         : PLASTIC STRAIN IN CIRCUMFERENTIAL DIRECTION,
C     ER         : PLASTIC STRAIN IN RADIAL DIRECTION.
C
C     GAMZE CAKIR AND AHMET N. ERASLAN, MAY 2018,
C     DEPARTMENT OF ENGINEERING SCIENCES,
```

```
C     MIDDLE EAST TECHNICAL UNIVERSITY, ANKARA.
C
      IMPLICIT NONE
      DOUBLE PRECISION SIGR, SIGT, U, ET, ER, C5, C6, R, SIG0, E, PHI,
     1                  NU, A2, THETA, A, B, R1, H, WH, M, T1, T2
       COMMON /PROP/ E, NU, SIG0, THETA, A, B, H, WH, M
       T1   = 1.0D0 - NU
       T2   = 1.0D0 + H
       SIGR = C6 + C5/R**2 + SIG0*DLOG(R)/T2 + E*H*PHI*DLOG(R)/(2.0*T2)
       SIGT = C6 - C5/R**2 + SIG0*(1.0D0 + DLOG(R))/T2 + E*H*PHI*(1.0D0
     1       + DLOG(R))/(2.0*T2)
       U    = 1.0/E*(-(2.0D0 + H*(1.0D0 + NU))*C5/(H*R) + T1*C6*R
     1       + R*SIG0*T1*DLOG(R)/T2) - PHI*R*(1.0D0 + H - H*T1*DLOG(R))
     2       /(2.0*T2) - A2*DCOS(THETA)
      ET   = - 1.0/(E*H)*(2.0*C5/R**2 + H*SIG0/T2) + PHI/(2.0*T2)
      ER   = - ET
      SIGR = SIGR/SIG0
      SIGT = SIGT/SIG0
      U    = U*E/(SIG0*B)
      ET   = ET*E/SIG0
      ER   = ER*E/SIG0
      RETURN
      END
C

C     ----------------------
      SUBROUTINE FI(F, X, N)
C     ----------------------
C     STAGE 1
C     VERSION 1
C     SUBROUTINE FI SUPPLIES THE NONLINEAR EQUATIONS F(1) = 0, F(2) =0,...,
C     F(N) = 0 TO THE SOLVER 'NLNF' IN ORDER TO ESTIMATE THE UNKNOWNS C1,
C     C2, C3, C4, A2, PHI, AND R1. ROUTINE NLNF USES INTERNALLY GENERATED
C     JACOBIAN MATRIX. SUBROUTINE FI IS USED IN THE FIRST STAGE ELASTIC-
C     PLASTIC DEFORMATION OF THE BAR. FI MUST BE DECLARED EXTERNAL IN THE
C     CALLING PROGRAM.
C
C     ARGUMENT LIST:
C     --------------
C     F          : A VECTOR OF LENGTH N DEFINING THE NONLINEAR EQUATIONS
C                  F(1) = 0, F(2) =0,...,F(N) = 0 TO THE SOLVER 'NLNF',
```

```
C     X          : A VECTOR OF LENGTH N WHICH CARRIES THE CURRENT VALUES
C                  OF THE UNKNOWNS C1, C2, C3, C4, A2, PHI, AND R1,
C     N          : NUMBER OF NONLINEAR SIMULTANEOUS EQUATIONS.
C
C     GAMZE CAKIR AND AHMET N. ERASLAN, MAY 2018,
C     DEPARTMENT OF ENGINEERING SCIENCES,
C     MIDDLE EAST TECHNICAL UNIVERSITY, ANKARA.
C
      IMPLICIT NONE
      INTEGER N
      DOUBLE PRECISION F, X, B, A, NU, E, SIG0, M, H, WH, ET, EZ,
     1                  THETA, T1, T2, SIGR1, SIGR2, SIGT1, SIGT2, U1,
     2                  U2, SIGR, SIGT, U, C1, C2, C3, C4, A2, PHI, R1
      DIMENSION F(N), X(N)
      COMMON /PROP/ E, NU, SIG0, THETA, A, B, H, WH, M
C
      C1   = X(1)
      C2   = X(2)
      C3   = X(3)
      C4   = X(4)
      A2   = X(5)
      PHI  = X(6)
      R1   = X(7)
C
      T1   = (1.0D0 - WH)
      T2   = (1.0D0 + WH)
C
      CALL PLAST(A, C3, C4, PHI, A2, SIGR, SIGT, U, ET, EZ)
      F(1) = U
      F(2) = SIGR
      CALL PLAST(R1, C3, C4, PHI, A2, SIGR1, SIGT1, U1, ET, EZ)
      CALL ELAST(R1, C1, C2, PHI, A2, SIGR2, SIGT2, U2)
      F(3) = SIGR1 - SIGR2
      F(4) = U1 - U2
      F(5) = SIGT2 + 1.0D0
      CALL ELAST(B, C1, C2, PHI, A2, SIGR, SIGT, U)
      F(6) = SIGR
      F(7) = C1*DLOG(R1/B) + 0.5*C2*(B**2 - R1**2) + C3*(A**T1 - R1**T1)
     1       *WH/T1 + C4*(R1**T2 - A**T2)*WH/T2 + 0.125*E*PHI*(-4.0*A**2
     2       *H + B**2*(1.0D0 + 2.0*DLOG(B)) - R1**2*(1.0D0 - 4.0*H
```

```
     3       + 2.0*DLOG(R1))) - M - 0.5*(R1**2 - A**2)*SIG0
      F(7) = F(7)/(B**2*SIG0)
C
      RETURN
      END
C

C     ----------------------
      SUBROUTINE FI(F, X, N)
C     ----------------------
C     STAGE 1
C     VERSION 2
C     ALTERNATIVE VERSION OF ROUTINE FI ABOVE. FOUR OF THE UNNKOWNS: C1,
C     C2, C3 AND C4 AMONG THE SEVEN NONLINEAR EQUATIONS ARE ELIMINATED
C     AND THE REMAINING THREE: A2, PHI, R1 ARE SOLVED BY NLNF. WHEN
C     CONVERGENCE IS OBTAINED C1, C2, C3 AND C4 ARE CALCULATED IN TERMS
C     OF A2, PHI, R1. FI MUST BE DECLARED EXTERNAL IN THE CALLING
C     PROGRAM.
C
C     ARGUMENT LIST:
C     --------------
C     F          : A VECTOR OF LENGTH N DEFINING THE NONLINEAR EQUATIONS
C                  F(1) = 0, F(2) =0, F(3) = 0 TO THE SOLVER 'NLNF',
C     X          : A VECTOR OF LENGTH N WHICH CARRIES THE CURRENT VALUES
C                  OF THE UNKNOWNS A2, PHI, AND R1,
C     N          : NUMBER OF NONLINEAR SIMULTANEOUS EQUATIONS.
C
C     GAMZE CAKIR AND AHMET N. ERASLAN, MAY 2018,
C     DEPARTMENT OF ENGINEERING SCIENCES,
C     MIDDLE EAST TECHNICAL UNIVERSITY, ANKARA.
C
      IMPLICIT NONE
      INTEGER N
      DOUBLE PRECISION F, X, B, A, NU, E, SIG0, M, H, WH, ET, EZ,
     1                 THETA, T1, T2, SIGR1, SIGR2, SIGT1, SIGT2, U1,
     2                 U2, SIGR, SIGT, U, C1, C2, C3, C4, A2, PHI, R1
      DIMENSION F(N), X(N)
      COMMON /PROP/ E, NU, SIG0, THETA, A, B, H, WH, M
C
      A2   = X(1)
      PHI  = X(2)
```

```
      R1   = X(3)
      C1   = B**2.0*R1**2.0*SIG0/(B**2.0 + R1**2.0) + B**2.0*E
     1        *R1**2.0*PHI*(1.0D0 + DLOG(R1/B))/(2.0*B**2.0 + 2.0
     2        *R1**2.0)
      C2   = -R1**2.0*SIG0/(B**2.0 + R1**2.0) - E*PHI*(R1**2.0
     1        + B**2.0*DLOG(B) + R1**2.0*DLOG(R1))/(2.0*B**2.0 + 2.0
     2        *R1**2.0)
      C3   = - A**WH*A2*E*H/(2.0*WH + 2.0*H*WH) + A**(1.0D0 + WH)*SIG0
     1        *(WH - H*(1.0D0 - WH))/(2.0*WH + 2.0*H*WH) - A**(1.0D0
     2        + WH)*E*H*PHI*(WH - H*(1.0D0 - WH))/(2.0*WH + 2.0
     3        *H*WH)
      C4   = A**(-WH)*A2*E*H/(2.0*WH + 2.0*H*WH) + A**(1.0D0 - WH)
     1        *SIG0*(WH + H*(1.0D0 + WH))/(2.0*WH + 2.0*H*WH)
     2        - A**(1.0D0 - WH)*E*H*PHI*(WH + H*(1.0D0 + WH))/(2.0
     3        * WH + 2.0*H*WH)
C
      T1   = (1.0D0 - WH)
      T2   = (1.0D0 + WH)
C
      CALL PLAST(R1, C3, C4, PHI, A2, SIGR1, SIGT1, U1, ET, EZ)
      CALL ELAST(R1, C1, C2, PHI, A2, SIGR2, SIGT2, U2)
      F(1) = SIGR1 - SIGR2
      F(2) = U1 - U2
      F(3) = C1*DLOG(R1/B) + 0.5*C2*(B**2 - R1**2) + C3*(A**T1 - R1**T1)
     1       *WH/T1 + C4*(R1**T2 - A**T2)*WH/T2 + 0.125*E*PHI*(- 4.0*A**2
     2       *H + B**2*(1.0D0 + 2.0*DLOG(B)) - R1**2*(1.0D0 - 4.0*H
     3       + 2.0*DLOG(R1))) - M - 0.5*(R1**2 - A**2)*SIG0
      F(3) = F(3)/(B**2*SIG0)
C
      RETURN
      END
C

C     ----------------------
      SUBROUTINE FI(F, X, N)
C     ----------------------
C     STAGE 2
C     VERSION 1
C     SUBROUTINE FI SUPPLIES THE NONLINEAR EQUATIONS F(1) = 0, F(2) =0,...,
C     F(N) = 0 TO THE SOLVER 'NLNF' IN ORDER TO ESTIMATE THE UNKNOWNS C1,
C     C2, C3, C4, C5, C6, A2, PHI, R1, AND R2. ROUTINE NLNF USES INTERNALLY
```

```
C     GENERATED JACOBIAN MATRIX. SUBROUTINE FI IS USED IN THE SECOND STAGE
C     ELASTIC-PLASTIC DEFORMATION OF THE BAR. FI MUST BE DECLARED EXTERNAL
C     IN THE CALLING PROGRAM.
C
C     ARGUMENT LIST:
C     --------------
C     F          : A VECTOR OF LENGTH N DEFINING THE NONLINEAR EQUATIONS
C                  F(1) = 0, F(2) = 0,...,F(N) = 0 TO THE SOLVER 'NLNF',
C     X          : A VECTOR OF LENGTH N WHICH CARRIES THE CURRENT VALUES
C                  OF THE UNKNOWNS C1, C2, C3, C4, C5, C6, A2, PHI, R1, AND
C                  R2,
C     N          : NUMBER OF NONLINEAR SIMULTANEOUS EQUATIONS.
C
C     GAMZE CAKIR AND AHMET N. ERASLAN, MAY 2018,
C     DEPARTMENT OF ENGINEERING SCIENCES,
C     MIDDLE EAST TECHNICAL UNIVERSITY, ANKARA.
C
      IMPLICIT NONE
      INTEGER N
      DOUBLE PRECISION F, X, B, A, NU, E, SIG0, M, H, WH, ET, EZ, ER,
     1                 THETA, T1, T2, SIGR1, SIGR2, SIGT1, SIGT2, U1,
     2                 U2, SIGR, SIGT, U, C1, C2, C3, C4, C5, C6, A2,
     3                 PHI, R1, R2
      DIMENSION F(N), X(N)
      COMMON /PROP/ E, NU, SIG0, THETA, A, B, H, WH, M
C
      C1   = X(1)
      C2   = X(2)
      C3   = X(3)
      C4   = X(4)
      C5   = X(5)
      C6   = X(6)
      A2   = X(7)
      PHI  = X(8)
      R1   = X(9)
      R2   = X(10)
C
      T1   = (1.0D0 - WH)
      T2   = (1.0D0 + WH)
C
```

```
      CALL PLAST(A, C3, C4, PHI, A2, SIGR1, SIGT1, U1, ET, EZ)
      F(1) = U1
      F(2) = SIGR1
      CALL PLAST(R1, C3, C4, PHI, A2, SIGR1, SIGT1, U1, ET, EZ)
      CALL ELAST(R1, C1, C2, PHI, A2, SIGR2, SIGT2, U2)
      F(3) = SIGR1 - SIGR2
      F(4) = U1 - U2
      F(5) = SIGT2 + 1.0D0
      CALL ELAST(R2, C1, C2, PHI, A2, SIGR, SIGT, U)
      F(6) = SIGT - SIGR - 1.0D0
      CALL PLAST2(R2, C5, C6, PHI, A2, SIGR1, SIGT1, U1, ET, ER)
      CALL ELAST(R2, C1, C2, PHI, A2, SIGR2, SIGT2, U2)
      F(7) = SIGR1 - SIGR2
      F(8) = U1 - U2
      CALL PLAST2(B, C5, C6, PHI, A2, SIGR, SIGT, U, ET, ER)
      F(9) = SIGR
      F(10)= DLOG(R1/R2)*C1 + C2*(R2**2 - R1**2)/2.0
     1       + C3*WH*(A**T1 - R1**T1)/T1 + C4*WH*(R1**T2 - A**T2)/T2
     2       + C5*DLOG(R2/B) + C6*(B**2 - R2**2)/2.0 + (-4.0*A**2*H
     3       *(1.0D0 + H) + B**2*H*(1.0D0 + 2*DLOG(B)) - R1**2*(1.0D0
     4       + H)*(1.0D0 - 4.0*H + 2.0*DLOG(R1)) + R2**2*(1.0D0 + 2.0
     5       *DLOG(R2)))*E*PHI/(8.0*(1.0D0 + H)) - M + SIG0*(2.0*A**2
     6       *(1.0D0 + H) + B**2*(1.0D0 + 2*DLOG(B)) - 2.0*R1**2*(1.0D0
     7       + H) - R2**2*(1.0D0 + 2*DLOG(R2)))/(4.0*(1.0D0 + H))
      F(10)= F(10)/(B**2*SIG0)
C
      RETURN
      END
C

C     ----------------------
      SUBROUTINE FI(F, X, N)
C     ----------------------
C     STAGE 2
C     VERSION 2
C     ALTERNATIVE VERSION OF ROUTINE FI ABOVE. FOUR OF THE UNKNOWNS: C1,
C     C2, C3 AND C4 AMONG THE TEN NONLINEAR EQUATIONS ARE ELIMINATED
C     AND THE REMAINING SIX: C5, C6, A2, PHI, R1, R2 ARE SOLVED BY NLNF.
C     WHEN CONVERGENCE IS OBTAINED C1, C2, C3 AND C4 ARE CALCULATED IN TERMS
C     OF C5, C6, A2, PHI, R1, R2. FI MUST BE DECLARED EXTERNAL IN THE CALLING
C     PROGRAM.
```

```
C
C     ARGUMENT LIST:
C     --------------
C     F          : A VECTOR OF LENGTH N DEFINING THE NONLINEAR EQUATIONS
C                  F(1) = 0, F(2) = 0,...,F(N) = 0 TO THE SOLVER 'NLNF',
C     X          : A VECTOR OF LENGTH N WHICH CARRIES THE CURRENT VALUES
C                  OF THE UNKNOWNS C5, C6, A2, PHI, R1, AND R2,
C     N          : NUMBER OF NONLINEAR SIMULTANEOUS EQUATIONS.
C
C     GAMZE CAKIR AND AHMET N. ERASLAN, MAY 2018,
C     DEPARTMENT OF ENGINEERING SCIENCES,
C     MIDDLE EAST TECHNICAL UNIVERSITY, ANKARA.
C
      IMPLICIT NONE
      INTEGER N
      DOUBLE PRECISION F, X, B, A, NU, E, SIG0, M, H, WH, ET, EZ, ER,
     1                 THETA, T1, T2, SIGR1, SIGR2, SIGT1, SIGT2, U1,
     2                 U2, SIGR, SIGT, U, C1, C2, C3, C4, C5, C6, A2,
     3                 PHI, R1, R2
      DIMENSION F(N), X(N)
      COMMON /PROP/ E, NU, SIG0, THETA, A, B, H, WH, M
C
      C5   = X(1)
      C6   = X(2)
      A2   = X(3)
      PHI  = X(4)
      R1   = X(5)
      R2   = X(6)
      C1   = -(E*PHI*R1**2)/4.0 - (A**(1.0D0 + WH)*R1**(1.0D0 - WH)*(H + WH
     1       + H*WH)*(-(E*H*PHI) + E*PHI*WH+E*H*PHI*WH - SIG0*WH))/(4.0
     2       *(1.0D0 + H)*WH) + (A**(1.0D0 -WH)*R1**(1.0D0 +WH)*(-H + WH
     3       + H*WH)*(E*H*PHI + E*PHI*WH + E*H*PHI*WH - SIG0*WH))/(4.0
     4       *(1.0D0 + H)*WH) + (R1**(1.0D0 + WH)*(- H + WH + H*WH)*(-(A2*E
     5       *H) - A*H*SIG0 - A*SIG0*WH))/(4.0*A**WH*H*(1.0D0 + H)*WH) -
     6       (A**WH*R1**(1.0D0 - WH)*(H + WH + H*WH)*(A2*E*H + A*H*SIG0
     7       - A*SIG0*WH))/(4.0*H*(1.0D0 + H)*WH)
      C2   = -SIG0 + (A2*E*R1**(-1.0D0 - WH)*(A**(2.0*WH)*(- H + WH + H*WH)
     1       + R1**(2.0*WH)*(H + WH + H*WH)))/(4.0*A**WH*(1.0D0 + H)*WH)
     2       + (A**(1.0D0 - WH)*R1**(-1.0D0 - WH)*SIG0*(-(A**(2*WH)*(- H
     3       + WH + H*WH)**2) + R1**(2.0*WH)*(H + WH + H*WH)**2))/(4.0*H
```

```
     4        *(1.0D0 + H)*WH) + (A**(1.0D0 - WH)*E*PHI*R1**(-1.0D0 - WH)
     5        *(A**(2.0*WH)*(-H + WH + H*WH)**2 - R1**(2.0*WH)*(H + WH
     6        + H*WH)**2))/(4.0*(1.0D0 + H)*WH) + (E*PHI*(1.0D0 + 4.0*H
     7        - 2.0*DLOG(R1)))/4.0
      C3   = -A**WH *A2*E*H/(2.0*(1.0D0 + H)*WH) + A**(1.0D0 + WH) *SIG0*(-H
     1        + WH + WH*H)/(2.0*(1.0D0 + H)*WH) - A**(1.0D0 + WH) *E*H
     2        *PHI*(-H + WH + WH*H)/(2.0*(1.0D0 + H)*WH)
      C4   = A**(-WH) *A2*E*H/(2.0*(1.0D0 + H)*WH) + A**(1.0D0 - WH) *SIG0
     1        *(H + WH + WH*H)/(2.0*(1.0D0 + H)*WH) - A**(1.0D0 - WH) *E*H
     2        *PHI*(H + WH + WH*H)/(2.0*(1.0D0 + H)*WH)
C
      T1   = (1.0D0 - WH)
      T2   = (1.0D0 + WH)
C
      CALL ELAST(R1, C1, C2, PHI, A2, SIGR, SIGT, U)
      F(1) = SIGT + 1.0D0
      CALL ELAST(R2, C1, C2, PHI, A2, SIGR, SIGT, U)
      F(2) = SIGT - SIGR - 1.0D0
      CALL PLAST2(R2, C5, C6, PHI, A2, SIGR1, SIGT1, U1, ET, ER)
      CALL ELAST(R2, C1, C2, PHI, A2, SIGR2, SIGT2, U2)
      F(3) = SIGR1 - SIGR2
      F(4) = U1 - U2
      CALL PLAST2(B, C5, C6, PHI, A2, SIGR, SIGT, U, ET, ER)
      F(5) = SIGR
      F(6) = DLOG(R1/R2)*C1 + C2*(R2**2 - R1**2)/2.0
     1       + C3*WH*(A**T1 - R1**T1)/T1 + C4*WH*(R1**T2 - A**T2)/T2
     2       + C5*DLOG(R2/B) + C6*(B**2 - R2**2)/2.0 + (-4.0*A**2*H
     3       *(1.0D0 + H) + B**2*H*(1.0D0 + 2*DLOG(B)) - R1**2*(1.0D0
     4       + H)*(1.0D0 - 4.0*H + 2.0*DLOG(R1)) + R2**2*(1.0D0 + 2.0
     5       *DLOG(R2)))*E*PHI/(8.0*(1.0D0 + H)) - M + SIG0*(2.0*A**2
     6       *(1.0D0 + H) + B**2*(1.0D0 + 2*DLOG(B)) - 2.0*R1**2*(1.0D0
     7       + H) - R2**2*(1.0D0 + 2*DLOG(R2)))/(4.0*(1.0D0 + H))
      F(6) = F(6)/(B**2*SIG0)
C
      RETURN
      END
C

C     ---------------------------------------------------------
      SUBROUTINE ELAST (R, C1, C2, PHI, A2, SIGR, SIGT, SIGZ, U)
C     ---------------------------------------------------------
```

```
C     SUBROUTINE ELAST CALCULATES DIMENSIONLESS STRESS COMPONENTS SIGR,
C     SIGT, SIGZ AND DIMENSIONLESS RADIAL DISPLACEMENT U IN THE ELASTIC
C     REGION OF THE PLANE STRAIN SOLUTION.
C
C     ARGUMENT LIST:
C     --------------
C     R          : RADIAL COORDINATE,
C     C1, C2     : INTEGRATION CONSTANTS,
C     PHI, A2    : ADDITIONAL CONSTANTS IN THE STRESS EXPRESSIONS,
C     SIGR       : RADIAL STRESS COMPONENT,
C     SIGT       : CIRCUMFERENTIAL STRESS COMPONENT,
C     SIGZ       : AXIAL STRESS COMPONENT,
C     U          : RADIAL DISPLACEMENT.
C
C     COMMON BLOCK PROP:
C     ------------------
C     E          : MODULUS OF ELASTICITY,
C     NU         : POISSON'S RATIO,
C     SIG0       : YIELD STRENGTH,
C     THETA      : CIRCUMFERENTIAL ANGLE,
C     A          : INNER RADIUS OF THE ARC,
C     B          : OUTER RADIUS OF THE ARC,
C     H          : HARDENING PARAMETER,
C     M          : BENDING MOMENT.
C
C     GAMZE CAKIR AND AHMET N. ERASLAN, MAY 2018,
C     DEPARTMENT OF ENGINEERING SCIENCES,
C     MIDDLE EAST TECHNICAL UNIVERSITY, ANKARA.
C
      IMPLICIT NONE
      DOUBLE PRECISION SIGR, SIGT, SIGZ, U, C1, C2, R, SIG0, E, PHI,
     1             \qquad  NU, A2, THETA, A, B, R1, M, H, T1, T2, T3
      COMMON /PROP/ E, NU, SIG0, THETA, A, B, H, M
      SIGR = C1/R**2 + C2 + E*PHI*DLOG(R)/(2.0*(1.0D0-NU**2))
      SIGT = -C1/R**2 + C2 + E*PHI*(1.0D0 + DLOG(R))/(2.0*(1.0D0-NU**2))
      SIGZ = 2.0*NU*C2 + E*NU*PHI/(2.0*(1.0D0-NU**2)) + E*NU*PHI*DLOG(R)
     1       /(1.0D0-NU**2)
      U    = 1.0/E*(-(1.0D0 + NU)*C1/R + (1.0D0 + NU)*(1.0 - 2.0*NU)
     1       *C2*R) - (1.0D0 - NU - (1.0D0 - 2.0*NU)*DLOG(R))*PHI*R
     2       /(2.0*(1.0D0 - NU)) - A2*DCOS(THETA)
```

```
C
      SIGR = SIGR/SIG0
      SIGT = SIGT/SIG0
      SIGZ = SIGZ/SIG0
      U    = U*E/(SIG0*B)
C
      RETURN
      END
C

C     -----------------------------------------------------------------
      SUBROUTINE PLAST (R, C3, C4, PHI, A2, SIGR, SIGT, SIGZ, U, ET, ER)
C     -----------------------------------------------------------------
C     SUBROUTINE PLAST CALCULATES DIMENSIONLESS STRESS COMPONENTS SIGR,
C     SIGT, SIGZ, DIMENSIONLESS RADIAL DISPLACEMENT U AND NORMALIZED PLASTIC
C     STRAINS ET AND ER IN THE PLASTIC REGION I OF THE PLANE STRAIN SOLUTION.
C
C     ARGUMENT LIST:
C     --------------
C     R          : RADIAL COORDINATE,
C     C3, C4     : INTEGRATION CONSTANTS,
C     PHI, A2    : ADDITIONAL CONSTANTS IN THE STRESS EXPRESSIONS,
C     SIGR       : RADIAL STRESS COMPONENT,
C     SIGT       : CIRCUMFERENTIAL STRESS COMPONENT,
C     SIGZ       : AXIAL STRESS COMPONENT,
C     U          : RADIAL DISPLACEMENT,
C     ET         : PLASTIC STRAIN IN CIRCUMFERENTIAL DIRECTION,
C     ER         : PLASTIC STRAIN IN RADIAL DIRECTION.
C
C     GAMZE CAKIR AND AHMET N. ERASLAN, MAY 2018,
C     DEPARTMENT OF ENGINEERING SCIENCES,
C     MIDDLE EAST TECHNICAL UNIVERSITY, ANKARA.
C
      IMPLICIT NONE
      DOUBLE PRECISION SIGR, SIGT, SIGZ, U, ET, ER, C3, C4, R, SIG0, E,
     1                 PHI, NU, A2, THETA, A, B, R1, H, M, T1, T2, T3
      COMMON /PROP/ E, NU, SIG0, THETA, A, B, H, M
      T1   = 1.0D0 - NU**2
      T2   = 1.0D0 - 2*NU
      T3   = 1.0D0 + NU
      SIGR = C4 + C3/R**2 - SIG0*DLOG(R)/(1.0D0 + H*T1)
```

```
     1       + E*H*PHI*DLOG(R)/(2.0*(1.0D0 + H*T1))
      SIGT = C4 - C3/R**2 - SIG0*(1.0D0 + DLOG(R))/(1.0D0 + H*T1)
     1       + E*H*PHI*(1.0D0 + DLOG(R))/(2.0*(1.0D0 + H*T1))
      SIGZ = 2.0*C4*NU - NU*SIG0*(1.0D0 + 2.0*DLOG(R))/(1.0D0 + H*T1)
     1       + E*H*PHI*NU*(1.0D0 + 2.0*DLOG(R))/(2.0*(1.0D0 + H*T1))
      U    = 1.0/E*(C4*R*T2*T3 - C3*(2.0D0 + H*T3)/(H*R)
     1       - R*T2*T3*SIG0*DLOG(R)/(1.0D0 + H*T1)) - A2*DCOS(THETA)
     2       - R*PHI/2.0*(1.0D0 + H*T1 - H*T2*T3*DLOG(R))/(1.0D0 + H*T1)
      ER   = (2.0*C3/(H*R**2) - T1*SIG0/(1.0D0 + H*T1))/E
     1       - PHI/(2.0*(1.0D0 + H*T1))
      ET   = -ER
C
      SIGR = SIGR/SIG0
      SIGT = SIGT/SIG0
      SIGZ = SIGZ/SIG0
      U    = U*E/(SIG0*B)
      ET   = ET*E/SIG0
      ER   = ER*E/SIG0
C
      RETURN
      END
C

C     ------------------------------------------------------------------
      SUBROUTINE PLAST2(R, C5, C6, PHI, A2, SIGR, SIGT, SIGZ, U, ET, ER)
C     ------------------------------------------------------------------
C     SUBROUTINE PLAST2 CALCULATES DIMENSIONLESS STRESS COMPONENTS SIGR,
C     SIGT, SIGZ, DIMENSIONLESS RADIAL DISPLACEMENT U AND NORMALIZED
C     PLASTIC STRAINS ET AND ER IN THE PLASTIC REGION II OF THE PLANE
C     STRAIN SOLUTION.
C
C     ARGUMENT LIST:
C     --------------
C     R          : RADIAL COORDINATE,
C     C5, C6     : INTEGRATION CONSTANTS,
C     PHI, A2    : ADDITIONAL CONSTANTS IN THE STRESS EXPRESSIONS,
C     SIGR       : RADIAL STRESS COMPONENT,
C     SIGT       : CIRCUMFERENTIAL STRESS COMPONENT,
C     SIGZ       : AXIAL STRESS COMPONENT,
C     U          : RADIAL DISPLACEMENT,
C     ET         : PLASTIC STRAIN IN CIRCUMFERENTIAL DIRECTION,
```

```
C     ER        : PLASTIC STRAIN IN RADIAL DIRECTION.
C
C     GAMZE CAKIR AND AHMET N. ERASLAN, MAY 2018,
C     DEPARTMENT OF ENGINEERING SCIENCES,
C     MIDDLE EAST TECHNICAL UNIVERSITY, ANKARA.
C
      IMPLICIT NONE
      DOUBLE PRECISION SIGR, SIGT, SIGZ, U, ET, ER, C5, C6, R, SIG0, E,
     1                 PHI, NU, A2, THETA, A, B, R1, H, M, T1, T2, T3
      COMMON /PROP/ E, NU, SIG0, THETA, A, B, H, M
      T1   = 1.0D0 - NU**2
      T2   = 1.0D0 - 2.0*NU
      T3   = 1.0D0 + NU
      SIGR = C6 + C5/R**2 + SIG0*DLOG(R)/(1.0D0 + H*T1)
     1       + E*H*PHI*DLOG(R)/(2.0*(1.0D0 + H*T1))
      SIGT = C6 - C5/R**2 + SIG0*(1.0D0 + DLOG(R))/(1.0D0 + H*T1)
     1       + E*H*PHI*(1.0D0 + DLOG(R))/(2.0*(1.0D0 + H*T1))
      SIGZ = 2.0*C6*NU + NU*(SIG0/(1.0D0 + H*T1) + (E*H*PHI)
     1       /(2.0*(1.0D0 + H*T1)) + (2.0*SIG0*DLOG(R))/(1.0D0 + H*T1)
     2       + (E*H*PHI*DLOG(R))/(1.0D0 + H*T1))
      U    = C6*R*T2*T3/E - C5*(2.0D0 + H*T3)/(E*H*R) - A2*DCOS(THETA)
     1       + R*T2*T3*SIG0*DLOG(R)/(E*(1.0D0 + H*T1))
     2       - 0.5*R*PHI*(1.0D0 + H*T1 - H*T2*T3*DLOG(R))/(1.0D0 + H*T1)
      ER   = 2.0*C5/(H*R**2)/E + T1*SIG0/(1.0D0 + H*T1)/E
     1       - PHI/(2.0*(1.0D0 + H*T1))
      ET   = -ER
C
      SIGR = SIGR/SIG0
      SIGT = SIGT/SIG0
      SIGZ = SIGZ/SIG0
      U    = U*E/(SIG0*B)
      ET   = ET*E/SIG0
      ER   = ER*E/SIG0
C
      RETURN
      END

C     ----------------------
      SUBROUTINE FI(F, X, N)
C     ----------------------
C     STAGE 1
```

```
C     VERSION 1
C     SUBROUTINE FI SUPPLIES THE NONLINEAR EQUATIONS F(1) = 0, F(2) = 0,...,
C     F(N) = 0 TO THE SOLVER 'NLNF' IN ORDER TO ESTIMATE THE UNKNOWNS C1,
C     C2, C3, C4, A2, PHI, AND R1. ROUTINE NLNF USES INTERNALLY GENERATED
C     JACOBIAN MATRIX. SUBROUTINE FI IS USED IN THE FIRST STAGE ELASTIC-
C     PLASTIC DEFORMATION OF THE BAR. FI MUST BE DECLARED EXTERNAL IN THE
C     CALLING PROGRAM.
C
C     ARGUMENT LIST:
C     --------------
C     F          : A VECTOR OF LENGTH N DEFINING THE NONLINEAR EQUATIONS
C                  F(1) = 0, F(2) =0,...,F(N) = 0 TO THE SOLVER 'NLNF',
C     X          : A VECTOR OF LENGTH N WHICH CARRIES THE CURRENT VALUES
C                  OF THE UNKNOWNS C1, C2, C3, C4, A2, PHI, AND R1,
C     N          : NUMBER OF NONLINEAR SIMULTANEOUS EQUATIONS.
C
C     GAMZE CAKIR AND AHMET N. ERASLAN, MAY 2018,
C     DEPARTMENT OF ENGINEERING SCIENCES,
C     MIDDLE EAST TECHNICAL UNIVERSITY, ANKARA.
C
      IMPLICIT NONE
      INTEGER N
      DOUBLE PRECISION F, X, B, A, NU, E, SIG0, M, H, ET, ER, THETA, T1,
     1                 T2, SIGR1, SIGR2, SIGT1, SIGT2, SIGZ, U1, U2,
     2                 SIGR, SIGT, U, C1, C2, C3, C4, A2, PHI, R1
      DIMENSION F(N), X(N)
      COMMON /PROP/ E, NU, SIG0, THETA, A, B, H, M
C
      C1   = X(1)
      C2   = X(2)
      C3   = X(3)
      C4   = X(4)
      A2   = X(5)
      PHI  = X(6)
      R1   = X(7)
C
      T1   = 1.0D0 - NU**2
      T2   = 1.0D0 + H*T1
C
      CALL PLAST(A, C3, C4, PHI, A2, SIGR1, SIGT1, SIGZ, U1, ET, ER)
```

```
      F(1) = U1
      F(2) = SIGR1
      CALL PLAST(R1, C3, C4, PHI, A2, SIGR1, SIGT1, SIGZ, U1, ET, ER)
      CALL ELAST(R1, C1, C2, PHI, A2, SIGR2, SIGT2, SIGZ, U2)
      F(3) = SIGR1 - SIGR2
      F(4) = U1 - U2
      F(5) = - SIGT2 + SIGR2 - 1.0D0
      CALL ELAST(B, C1, C2, PHI, A2, SIGR, SIGT, SIGZ, U)
      F(6) = SIGR
      F(7) = C1*DLOG(R1/B) + C2*(B**2 - R1**2)/2.0 + C3*DLOG(A/R1)
             + C4*(-A**2 + R1**2)/2.0D0 + E*PHI/(8.0*T1*T2)*(-A**2.0*H*T1
     2       *(1.0D0 + 2.0*DLOG(A)) + B**2*T2*(1.0D0 + 2.0*DLOG(B))
     3       - R1**2.0*(1.0D0 + 2.0*DLOG(R1))) - M + SIG0*A**2*(1.0D0
     4       + 2*DLOG(A))/(4.0*T2)- SIG0*R1**2*(1.0D0 + 2*DLOG(R1))
     5       /(4.0*T2)
      F(7) = F(7)/(B**2*SIG0)
C
      RETURN
      END
C

C     ----------------------
      SUBROUTINE FI(F, X, N)
C     ----------------------
C     STAGE 1
C     VERSION 2
C     ALTERNATIVE VERSION OF ROUTINE FI ABOVE. FOUR OF THE UNNKOWNS: C1,
C     C2, C3 AND C4 AMONG THE SEVEN NONLINEAR EQUATIONS ARE ELIMINATED
C     AND THE REMAINING THREE: A2, PHI, R1 ARE SOLVED BY NLNF. WHEN
C     CONVERGENCE IS OBTAINED C1, C2, C3 AND C4 ARE CALCULATED IN TERMS
C     OF A2, PHI, R1. FI MUST BE DECLARED EXTERNAL IN THE CALLING PROGRAM.
C
C     ARGUMENT LIST:
C     --------------
C     F          : A VECTOR OF LENGTH N (=3) DEFINING THE NONLINEAR
C                  EQUATIONS F(1) = 0, F(2) =0, F(3) = 0 TO THE SOLVER
C                 'NLNF',
C     X          : A VECTOR OF LENGTH N WHICH CARRIES THE CURRENT
C                  VALUES OF THE UNKNOWNS A2, PHI, AND R1,
C     N          : NUMBER OF NONLINEAR SIMULTANEOUS EQUATIONS.
C
```

```
C     GAMZE CAKIR AND AHMET N. ERASLAN, MAY 2018,
C     DEPARTMENT OF ENGINEERING SCIENCES,
C     MIDDLE EAST TECHNICAL UNIVERSITY, ANKARA.
C
      IMPLICIT NONE
      INTEGER N
      DOUBLE PRECISION F, X, B, A, NU, E, SIG0, M, H, ET, ER, THETA, T1,
     1                 T2, SIGR1, SIGR2, SIGT1, SIGT2, SIGZ, U1, U2,
     2                 SIGR, SIGT, U, C1, C2, C3, C4, A2, PHI, R1
      DIMENSION F(N), X(N)
      COMMON /PROP/ E, NU, SIG0, THETA, A, B, H, M
C
      A2  = X(1)
      PHI = X(2)
      R1  = X(3)
      C1  = R1**2*SIG0/2.0 + E*PHI*R1**2/(4.0*(1.0D0 - NU**2))
      C2  = -R1**2*(2.0*(1.0D0 - NU**2)*SIG0 + E*PHI)
     1      /(4.0*B**2*(1.0D0 - NU**2))
     2      - E*PHI*DLOG(B)/(2.0*(1.0D0 - NU**2))
      C3  = - A2*E*H*A/(2.0*(1.0D0 + H*(1.0D0 - NU**2)))
     1      - E*H*PHI*A**2.0/(4.0*(1.0D0 + H*(1.0D0 - NU**2)))
      C4  = E*H*(2.0*A2 + A*PHI)/(4.0*A*(1.0D0 + H*(1.0D0 - NU**2)))
     1      + (2.0*SIG0 - E*H*PHI)*DLOG(A)/(2.0*(1.0D0 + H*(1.0D0
     2      - NU**2)))
C
      T1  = 1.0D0 - NU**2
      T2  = 1.0D0 + H*T1
C
      CALL PLAST(R1, C3, C4, PHI, A2, SIGR1, SIGT1, SIGZ, U1, ET, ER)
      CALL ELAST(R1, C1, C2, PHI, A2, SIGR2, SIGT2, SIGZ, U2)
      F(1) = SIGR1 - SIGR2
      F(2) = U1 - U2
      F(3) = C1*DLOG(R1/B) + C2*(B**2 - R1**2)/2.0 + C3*DLOG(A/R1)
            + C4*(-A**2 + R1**2)/2.0D0 + E*PHI/(8.0*T1*T2)*(-A**2.0*H*T1
     2      *(1.0D0 + 2.0*DLOG(A)) + B**2*T2*(1.0D0 + 2.0*DLOG(B))
     3      - R1**2.0*(1.0D0 + 2.0*DLOG(R1))) - M + SIG0*A**2*(1.0D0
     4      + 2*DLOG(A))/(4.0*T2)- SIG0*R1**2*(1.0D0 + 2*DLOG(R1))
     5      /(4.0*T2)
      F(3) = F(3)/(B**2*SIG0)
C
```

```
      RETURN
      END
C

C     ----------------------
      SUBROUTINE FI(F, X, N)
C     ----------------------
C     STAGE 2
C     VERSION 1
C     SUBROUTINE FI SUPPLIES THE NONLINEAR EQUATIONS F(1) = 0, F(2) =0,...
C     F(N) = 0 TO THE SOLVER 'NLNF' IN ORDER TO ESTIMATE THE UNKNOWNS C1,
C     C2, C3, C4, C5, C6, A2, PHI, R1, AND R2. ROUTINE NLNF USES INTERNALLY
C     GENERATED JACOBIAN MATRIX. SUBROUTINE FI IS USED IN THE SECOND STAGE
C     ELASTIC-PLASTIC DEFORMATION OF THE BAR. FI MUST BE DECLARED
C     EXTERNAL IN THE CALLING PROGRAM.
C     ARGUMENT LIST:
C     --------------
C     F          : A VECTOR OF LENGTH N DEFINING THE NONLINEAR EQUATIONS
C                  F(1) = 0, F(2) = 0,...,F(N) = 0 TO THE SOLVER 'NLNF',
C     X          : A VECTOR OF LENGTH N WHICH CARRIES THE CURRENT VALUES
C                  OF THE UNKNOWNS C1, C2, C3, C4, C5, C6, A2, PHI, R1, AND
C                  R2,
C     N          : NUMBER OF NONLINEAR SIMULTANEOUS EQUATIONS.
C
C     GAMZE CAKIR AND AHMET N. ERASLAN, MAY 2018,
C     DEPARTMENT OF ENGINEERING SCIENCES,
C     MIDDLE EAST TECHNICAL UNIVERSITY, ANKARA.
C
      IMPLICIT NONE
      INTEGER N
      DOUBLE PRECISION F, X, B, A, NU, E, SIG0, M, H, ET, ER, THETA, T1,
     1                 T2, SIGR1, SIGR2, SIGT1, SIGT2, SIGZ, U1, U2,
     2                 SIGR, SIGT, U, C1, C2, C3, C4, C5, C6, A2, PHI,
     3                 R1, R2
      DIMENSION F(N), X(N)
      COMMON /PROP/ E, NU, SIG0, THETA, A, B, H, M
C
      C1   = X(1)
      C2   = X(2)
      C3   = X(3)
      C4   = X(4)
```

```
      C5   = X(5)
      C6   = X(6)
      A2   = X(7)
      PHI  = X(8)
      R1   = X(9)
      R2   = X(10)
C
      T1   = 1.0D0 - NU**2
      T2   = 1.0D0 + H*T1
C
      CALL PLAST(A, C3, C4, PHI, A2, SIGR, SIGT, SIGZ, U, ET, ER)
      F(1) = U
      F(2) = SIGR
      CALL PLAST(R1, C3, C4, PHI, A2, SIGR1, SIGT1, SIGZ, U1, ET, ER)
      CALL ELAST(R1, C1, C2, PHI, A2, SIGR2, SIGT2, SIGZ, U2)
      F(3) = SIGR1 - SIGR2
      F(4) = U1 - U2
      F(5) = SIGR2 - SIGT2 - 1.0D0
      CALL ELAST(R2, C1, C2, PHI, A2, SIGR, SIGT, SIGZ, U)
      F(6) = SIGT - SIGR - 1.0D0
      CALL PLAST2(R2, C5, C6, PHI, A2, SIGR1, SIGT1, SIGZ, U1, ET, ER)
      CALL ELAST(R2, C1, C2, PHI, A2, SIGR2, SIGT2, SIGZ, U2)
      F(7) = SIGR1 - SIGR2
      F(8) = U1 - U2
      CALL PLAST2(B, C5, C6, PHI, A2, SIGR, SIGT, SIGZ, U, ET, ER)
      F(9) = SIGR
      F(10)= C1*DLOG(R1/R2) + C2*(-R1**2 + R2**2)/2.0 + C3*DLOG(A/R1)
            + C4*(-A**2 + R1**2)/2.0 + C5*DLOG(R2/B) + C6*(B**2 - R2**2)
     2      /2.0 - M + SIG0*A**2*(1.0D0 + 2*DLOG(A))/(4.0*T2)
     3      + SIG0*B**2*(1.0D0 + 2*DLOG(B))/(4.0*T2) - SIG0*R1**2*(1.0D0
     4      + 2*DLOG(R1))/(4.0*T2) - SIG0*R2**2*(1.0D0 + 2*DLOG(R2))
     5      /(4.0*T2) - E*PHI*A**2*H*T1 *(1.0D0 + 2*DLOG(A))
     6      /(8.0*T1*T2) + E*PHI*B**2*H*T1*(1.0D0 + 2*DLOG(B))
     7      /(8.0*T1*T2) - E*PHI*R1**2*(1.0D0 + 2*DLOG(R1))/(8.0*T1*T2)
     8      + E*PHI*R2**2*(1.0D0 + 2*DLOG(R2))/(8.0*T1*T2)
      F(10)= F(10)/(B**2*SIG0)
C
      RETURN
      END
C
```

```
C     ----------------------
      SUBROUTINE FI(F, X, N)
C     ----------------------
C     STAGE 2
C     VERSION 2
C     ALTERNATIVE VERSION OF ROUTINE FI ABOVE. FOUR OF THE UNKNOWNS: C1,
C     C2, C3 AND C4 AMONG THE TEN NONLINEAR EQUATIONS ARE ELIMINATED
C     AND THE REMAINING SIX: C5, C6, A2, PHI, R1, R2 ARE SOLVED BY NLNF.
C     WHEN CONVERGENCE IS OBTAINED C1, C2, C3 AND C4 ARE CALCULATED IN TERMS
C     OF C5, C6, A2, PHI, R1, R2. FI MUST BE DECLARED EXTERNAL IN THE
C     CALLING PROGRAM.
C     ARGUMENT LIST:
C     --------------
C     F          : A VECTOR OF LENGTH N DEFINING THE NONLINEAR EQUATIONS
C                  F(1) = 0, F(2) = 0,...,F(N) = 0 TO THE SOLVER 'NLNF',
C     X          : A VECTOR OF LENGTH N WHICH CARRIES THE CURRENT VALUES
C                  OF THE UNKNOWNS C5, C6, A2, PHI, R1, AND R2,
C     N          : NUMBER OF NONLINEAR SIMULTANEOUS EQUATIONS.
C
C     GAMZE CAKIR AND AHMET N. ERASLAN, MAY 2018,
C     DEPARTMENT OF ENGINEERING SCIENCES,
C     MIDDLE EAST TECHNICAL UNIVERSITY, ANKARA.
C
      IMPLICIT NONE
      INTEGER N
      DOUBLE PRECISION F, X, B, A, NU, E, SIG0, M, H, ET, ER, THETA, T1,
     1                 T2, SIGR1, SIGR2, SIGT1, SIGT2, SIGZ, U1, U2,
     2                 SIGR, SIGT, U, C1, C2, C3, C4, C5, C6, A2, PHI,
     3                 R1, R2
      DIMENSION F(N), X(N)
      COMMON /PROP/ E, NU, SIG0, THETA, A, B, H, M
C
      C5   = X(1)
      C6   = X(2)
      A2   = X(3)
      PHI  = X(4)
      R1   = X(5)
      R2   = X(6)
      C1   = - A2*A*E/(2.0*(1.0D0 - NU**2)) - E*A**2.0*PHI/(4.0*(1.0D0
     1       - NU**2))
```

```
      C2   = E*H*(2.0*A2 + A*PHI)/(4.0*A*(1.0D0 + H*(1.0D0 - NU**2)))
     1       + A*E*(2.0*A2 + A*PHI)/(4.0*R1**2*(1.0D0 - NU**2)*(1.0D0
     2       + H*(1.0D0 - NU**2))) + (2.0*SIG0 - E*H*PHI)*DLOG(A)
     3       /(2.0*(1.0D0 + H*(1.0D0 - NU**2))) - (2.0*(1.0D0 - NU**2)
     4       *SIG0 + E*PHI)*DLOG(R1)/(2.0*(1.0D0 - NU**2)*(1.0D0
     5       + H*(1.0D0 - NU**2)))
      C3   = -(A*A2*E*H/(2.0D0 + 2.0*H*(1.0D0 - NU**2)))
     1       - A**2*E*H*PHI/(4.0D0 + 4.0*H*(1.0D0 - NU**2))
      C4   = A2*E*H/(2.0*A*(1.0D0 + H*(1.0D0 - NU**2)))
     1       + E*H*PHI/(4.0*(1.0D0 + H*(1.0D0 - NU**2)))
     2       + SIG0*DLOG(A)/(1.0D0 + H*(1.0D0 - NU**2))
     3       - E*H*PHI*DLOG(A)/(2.0*(1.0D0 + H*(1.0D0 - NU**2)))
C
      T1   = 1.0D0 - NU**2
      T2   = 1.0D0 + H*T1
C
      CALL ELAST(R1, C1, C2, PHI, A2, SIGR, SIGT, SIGZ, U)
      F(1) = SIGR - SIGT - 1.0D0
      CALL ELAST(R2, C1, C2, PHI, A2, SIGR, SIGT, SIGZ, U)
      F(2) = SIGT - SIGR - 1.0D0
      CALL PLAST2(R2, C5, C6, PHI, A2, SIGR1, SIGT1, SIGZ, U1, ET, ER)
      CALL ELAST(R2, C1, C2, PHI, A2, SIGR2, SIGT2, SIGZ, U2)
      F(3) = SIGR1 - SIGR2
      F(4) = U1 - U2
      CALL PLAST2(B, C5, C6, PHI, A2, SIGR, SIGT, SIGZ, U, ET, ER)
      F(5) = SIGR
      F(6) = C1*DLOG(R1/R2) + C2*(-R1**2 + R2**2)/2.0 + C3*DLOG(A/R1)
             + C4*(-A**2 + R1**2)/2.0 + C5*DLOG(R2/B) + C6*(B**2 - R2**2)
     2       /2.0 - M + SIG0*A**2*(1.0D0 + 2*DLOG(A))/(4.0*T2) + SIG0
     3       *B**2*(1.0D0 + 2*DLOG(B))/(4.0*T2) - SIG0*R1**2*(1.0D0
     4       + 2.0*DLOG(R1))/(4.0*T2) - SIG0*R2**2*(1.0D0 + 2.0*DLOG(R2)
     5       )/(4.0*T2) - E*PHI*A**2*H*T1 *(1.0D0 + 2*DLOG(A))/(8.0*T1
     6       *T2) + E*PHI*B**2*H*T1*(1.0D0 + 2*DLOG(B))/(8.0*T1*T2) - E
     7       *PHI*R1**2*(1.0D0 + 2.0*DLOG(R1))/(8.0*T1*T2) + E*PHI*R2**2
     8       *(1.0D0 + 2.0*DLOG(R2))/(8.0*T1*T2)
      F(6) = F(6)/(B**2*SIG0)
C
      RETURN
      END
C
```