MODELING INTEREST RATES MOVING IN A BAND

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ABSTRACT

MODELING INTEREST RATES MOVING IN A BAND

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It is not uncommon to observe interest rates or currencies to move in a band or being subject to an upper and/or lower bound set by national central banks. The Turkish Central Bank is using the interest rate corridor system actively in tandem with the liquidity policy to fine–tune the short rate in the TRY money market. Bond pricing models relying on a single factor use the short rate as the sole determinant of the entire yield curve. It would be a big mistake to ignore the fact that the short rate in Turkey is moving in a corridor, while pricing bonds using the short rate as the single factor. In this work, we try to establish a one factor yield curve model, where the interest rate is modeled as Vasicek process. The closed-form bond price is the main contribution of the novel approach devised in the thesis. Furthermore, mean reversion and normality tests of the time series justifies the usage of Vasicek process as the underlying interest rate model.

Keywords: Interest rate model, Vasicek model, Monte Carlo simulation, interest rate corridor

ÖZ

BİR BANT İÇERİSİNDE HAREKET EDEN FAİZLERİN MODELLENMESİ

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Merkez bankaları tarafından kontrol edilen faiz oranlarının veya para birimlerinin bir bant içinde hareket etmesi söz konusu olabilmektedir. Türkiye Cumhuriyet Merkez Bankası, TL para piyasasındaki kısa vadeli faizlerde ince ayarların yapılabilmesi için likidite politikasıyla birlikte aktif olarak faiz koridoru sistemini uygulamaktadır. Tek faktöre dayanan tahvil fiyatlama modelleri, kısa vadeli getirileri tüm verim eğrisinin tek belirleyicisi olarak kullanmaktadır. Türkiye'deki kısa vadeli faizin koridorda hareket ettiğini göz ardı etmek bono fiyatlarının hatalı olmasına yol açmaktadır. Bu çalışmada faiz oranının Vasicek süreci olarak modellendiği tek faktörlü bir verim eğrisi modeli oluşturmaya çalışıyoruz. Tahvil fiyatının analitik olarak hesaplanması, tezin literatüre en önemli katkısıdır. Zaman serilerinin ortalamaya geri dönüşü ve normalliği test edilerek faizlerin Vasicek süreci ile modellenebileceği gösterilmiştir.

Anahtar Kelimeler: Faiz modeli, Vasicek modeli, Monte Carlo simulasyonu, faiz koridoru

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TABLE OF CONTENTS

APPENDICES

LIST OF TABLES

TABLES

LIST OF FIGURES

FIGURES

CHAPTER 1

INTRODUCTION

The overnight interest rates are set by the Central Bank of Turkey (CBRT) in the inflation targeting framework. The overnight interest rates are indeed the main policy tool in controlling inflation; hence it is called “the policy rate”. At the beginning, the CBRT used to set the overnight interest rate equal to the policy rate, but as the volatility of capital inflows have increased, the CBRT started the overnight interest rate to fluctuate within the so–called “interest rate corridor”. This can be achieved through open market operations (OMO). In this setting, in a one–factor model approach, the short rate is confined within a band, and this fact needs to be considered while pricing bonds, or other interest rate derivatives.

The main output of the thesis is a closed form solution of a bond price for the Vasicek interest rate model, a version of the Ornstein–Uhlenbeck process [37], as the benchmark short–term interest rate in the money market. This benchmark interest rate model is the “one-factor” in the jargon of the affine term-structure models (see for instance [10, 11, 12, 17, 22, 25, 46]. A more comprehensive result would be produced, in case one had used a “multi-factor” interest rate model (see for instance [33]). Yet, in that case, we doubt that the complexity of the model would allow the researcher to come up with a closed form solution. Another alternative could be using a general HJM framework to model the short–rate, which again is not a very promising approach to find a closed form solution.
The Central Bank is employing the inflation targeting (IT) framework as the monetary policy strategy. At the initial stages of the IT, the CBRT has set the policy interest rates in its monthly meetings. However, the policy rate evolved into average funding

rate in the latest stages of the IT framework, as discussed in detail in Chapter 3. At this stage the average funding rate, the new benchmark rate for the monetary policy is calculated as the weighted average of the policy rate, which is the weekly repo rate and the upper band of the interest rate corridor, which was the overnight lending rate and lately it became the Late Liquidity Window (LLW) lending rate. The respective weights are actually the amount of liquidity given to the financial system from each instrument. Thanks to the corridor and the accompanying liquidity management operations, the CBRT is keeping the money market interest rates, mostly proxied by the overnight rate in Borsa Istanbul (BIST) Interbank Repo and Reverse Repo Market, within the corridor.

When it comes to modeling the spikes, it is a simple extension of our model. First of all the spikes could be detected as discussed in [19]. At the next step, the parameters of the remaining continuous model can be estimated through a simple AR(1) model and the parameters of the Vasicek model may be retrieved as discussed in detail in Chapter 3. Then, the distribution of the spikes can be fitted and the impact of average size of the spikes on the bond prices may be added exogenously. Modeling the jumps in the short–rate is extremely challenging, for that one has to model the discrete movements in the bands and the policy rate. In other words, the short–rate is not independent of the corridor parameters. Indeed, an upward pressure in the market interest rate implies an upward pressure for the bands and the policy rate, as well. This means that one has to model the jumps in the short-term interest rate and the interest rates under the control of the CBRT together. Another complication is that our approach to find the closed form solution does not reconcile with jumps in the interest rate, because then we would lose the Markov property of the sets we have defined. The details of this are discussed in Chapter 2, where we derive our closed form solution formula. Clearly there are important applications of jump-diffusion models on pricing bonds. However, what we like to stress is that, the bands would be creating a complication for producing a closed form solution in a jump–diffusion model.

As discussed in Chapter 3, we are justifying the usage of the Vasicek model, by the means of the mean reversion test. The relevant regression to apply our test shows that we are rejecting the H_0 of no mean–reversion at 90% level of significance for the whole sample, because the relevant test statistics is 6.3155 and exceeds the critical value of 6.2103. The case of normality for the overnight interest rate cannot be established for the whole sample due to the spikes throughout the whole sample. However, in subsamples without spikes we are able to establish normality based on Jarque–Bera test.

Previous papers about Turkish interest rates focus on yield curve estimation, determinants of interest rates and applications of interest rate models. However, to the best of our knowledge, ours is the only work specifically focusing on to the interest rates controlled by the CBRT.

The literature has very few papers addressing interest rate movements in a corridor. [5] aim at retrieving the term structure based on the short-rate governed by a jump–diffusion model controlled by Bank of England. They have managed to write a partial differential difference equation (PDDE) for the discount bonds. However, they do not solve the PDDE, nor do they provide any estimation result. The authors only present an illustrative example based on simulation techniques. [6] focuses on term structure modeling under alternative official regimes. They again derive a PDDE, where the short rate is subject to an upper and lower band and following a discussion of the monetary policy frameworks of the UK, US, Germany and France for controlling the short term interest rates, the authors apply the model to countries' frameworks. The application is done by solving the PDDE for specific cases numerically. [21] a develops a simulated maximum likelihood (SML) estimator in order to estimate a jump–diffusion model of short rates moving in a corridor. The method is a variant of Monte–Carlo simulation techniques and the author employs a finite difference scheme to fit the term structure. The results are applied on interest rate data from German money market. All in all these three works employ either numerical solutions for specific cases, or simulation techniques to price bonds based on a short-rate governed by a jump diffusion process and moving in a corridor. [14] derive an approximate bond price formula by a weighted series of Jacobi polynomials.

Another example of an asset moving within bands due to the control of central banks is the exchange rate. Specifically, Hungarian Forint against Euro moved within a corridor for some time. As seen in the following graph, the upper and lower bands have displayed one large and one small jump throughout the data period, and Forint has always been close to the lower band, i.e. was relatively strong against Euro thanks to the interventions of the Hungarian Central Bank as seen in Figure 1.1.

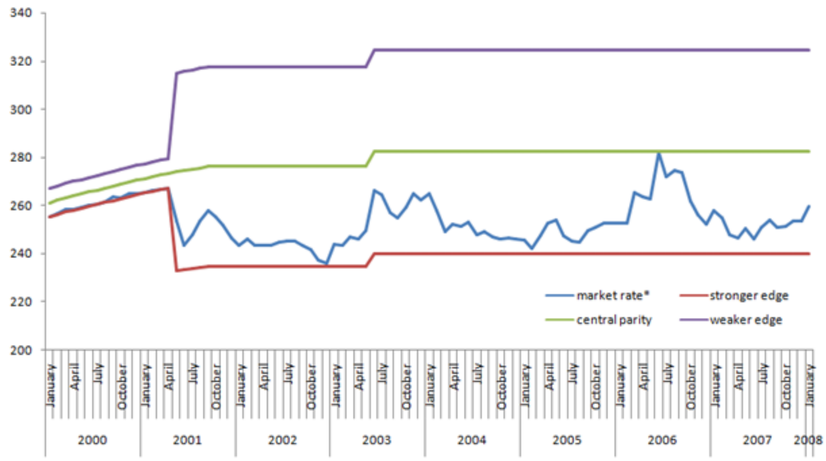


Figure 1.1: Euro/Forint Corridor

The Swiss National Bank had also been active in the FX market for some time in order to prevent appreciation of Swiss Franc against Euro by introducing a lower band. Swiss National Bank managed to preserve the lower band for more than three years at the same level before the policy was abandoned at the end of 2014 (see Figure 1.2)

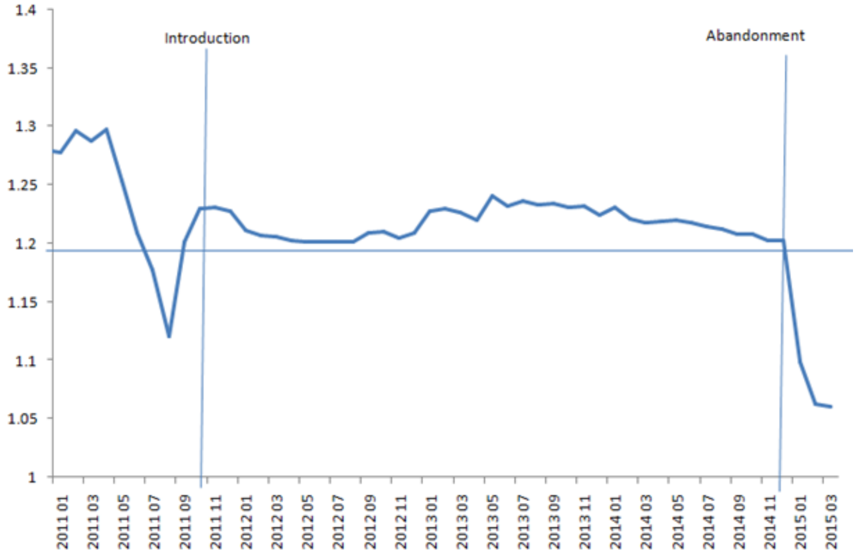


Figure 1.2: Euro/Swiss Franc Lower Band

We see that there are examples of active interest rate or exchange rate corridors under the control of central banks. Other examples of quantities moving within an upper and lower band might be inventories or temperature in an insurance contract. For the former, a firm may stop buying when the level of inventory reaches the level U and starts buying when it reaches a level $L < U$. Regarding the latter, we may want to price an insurance contract when the temperature remains outside a pre–specified band for some time.

All in all, this thesis is based on the corridor system accompanied by active liquidity management, which is implemented by the CBRT. The applications can be generalized to other examples. In our case, we first justify the usage of the Vasicek model by showing mean-reversion and normality. In Chapter 2, we derive the formula of the closed form solution a bond price where the underlying interest rate model is Vasicek, but it is moving below an active upper band. In Chapter 3, we introduce the data and summarize the monetary policy framework of the CBRT. Furthermore, we present the estimation results and the Monte Carlo simulation results for the case of both the upper and lower band are active. We conclude in Chapter 4.

CHAPTER 2

CLOSED FORM BOND PRICE SOLUTION

Some definitions and theorems in this chapter are retrieved from [2, 7, 9, 13, 15, 24, 26, 27, 28, 29, 30, 31, 32, 39, 41, 42, 45]

2.1 MOTIVATION

In general a bond price is described as:

$$B(t,T) = \mathbb{E}_x\left(e^{-\int_t^T R(s)ds}\right),$$

where, t is the current time and T is the maturity. $R(s)$ is the instantaneous interest rate, called the short rate, as well. The functional form of $R(s)$ may vary. For our data set we model $R(s)$ to be a functional form of $r(s)$, another short–rate process. Incorporating the constants L and U we can write $R(s)$ as follows:

$$R(s) = \begin{cases} L, & \text{if} \quad r(s) \leq L, \\ r(s), & \text{if} \quad L < r(s) < U, \\ U, & \text{if} \quad r(s) \geq U. \end{cases}$$

The rationale behind this model is that the central bank does not interfere with the short–rate market as long as $L < r(s) < U$, but resorts to its operational tools available whenever the short–rate market dynamics imply $r(s) \geq U$, and the intervention enables the central bank to dictate $r(s) = U = R(s)$ (similar for $r(s) \leq L$).

Our model does not include jumps in the data generating process of the O/N interest

rates.
We can create the entire term structure at any time t by using bond prices. Therefore, having a closed form bond price formula is equivalent to construct the whole term structure.
In general, affine term structure models have closed form solution. Vasicek, Dothan, CIR are examples of affine term structure models with explicit bond pricing formulas. However, being a special case of the OU process, Vasicek model resulting in Gaussian and Markovian short–rate process makes the model easily tractable.
Although normality implies that the short–rate can become negative with positive probability, the Turkish O/N interest rate market implies highly persistent parameter values, so that this drawback is negligible and can be opt for the tractability of the model. Furthermore, since the general level of interest rates is much higher than zero, the probability of negative interest rates becomes further negligible. Yet, in the aftermath of the great recession, we have witnessed nominal interest rates, or even central bank policy rates to be negative from time to time.
After this motivation, we can write the Vasicek model of $r(s)$ as follows:

$$dr_t = \alpha(\xi - r_t)dt + \sigma dW_t. \tag{2.1}$$

We assume this SDE to be valid in the real world so that we can convert it to the risk neutral world by observing that $\widetilde{W}_t = W_t + \lambda t$ by the Girsanov theorem, where λ is the parameter of the market price of risk (see [40]).

$$\begin{aligned} dr_t &= \alpha(\xi - r_t)dt + \sigma(d\widetilde{W}_t - \lambda dt), \\ &= (\alpha\xi - \alpha r_t - \lambda\sigma)dt + \sigma d\widetilde{W}_t, \\ &= \alpha\left(\xi - \tfrac{\lambda\sigma}{\alpha} - r_t\right)dt + \sigma d\widetilde{W}_t, \\ &= dr_t = \alpha(\beta - r_t)dt + \sigma d\widetilde{W}_t, \qquad (2.2) \\ &\text{by letting} \quad \xi - \tfrac{\lambda\sigma}{\alpha} = \beta. \end{aligned}$$

In this equation; β is the asymptotic, or the long–run mean and α is the “mean reversion rate” i.e., the rate determining how fast $r(s)$ reverts to its long–term mean after a shock or perturbation hits the system.
When it comes to the solution of the SDE that governs $r(s)$, let $y_t = e^{\alpha t}r_t$ and apply

Ito's Lemma to this function to get:

$$\begin{aligned}
dy_t &= \alpha e^{\alpha t} r_t dt + e^{\alpha t} dr_t, \\
&= \alpha e^{\alpha t} r_t dt + e^{\alpha t}\left[\alpha(\beta - r_t)dt + \sigma d\widetilde{W}_t\right], \\
&= \alpha e^{\alpha t} r_t dt + e^{\alpha t}\alpha\beta dt - e^{\alpha t}\alpha r_t dt + e^{\alpha t}\sigma d\widetilde{W}_t.
\end{aligned}$$

Thus

$$dy_t = e^{\alpha t}(\alpha\beta dt + \sigma d\widetilde{W}_t), \tag{2.3}$$

and in integral form:

$$y_t = y_0 + \alpha\beta \int_0^t e^{\alpha s} ds + \sigma \int_0^t e^{\alpha s} d\widetilde{W}_s. \tag{2.4}$$

observe that $y_0 = r_0$ and $r_t = e^{-\alpha t} y_t$, so by multiplying both sides of (2.4) by $e^{-\alpha t}$ we get (under the risk–neutral measure):

$$\begin{aligned}
e^{-\alpha t} y_t &= e^{-\alpha t} r_0 + \alpha\beta e^{-\alpha t} \textstyle\int_0^t e^{\alpha s} ds + \sigma e^{-\alpha t} \int_0^t e^{\alpha s} d\widetilde{W}_s, \\
r_t &= e^{-\alpha t} r_0 + \alpha\beta e^{-\alpha t}\left[\tfrac{1}{\alpha} e^{\alpha s}\big|_0^t\right] + \sigma \textstyle\int_0^t e^{-\alpha(t-s)} d\widetilde{W}_s, \\
r_t &= r_0 e^{-\alpha t} + \beta e^{-\alpha t}\left(e^{\alpha t} - 1\right) + \sigma \textstyle\int_0^t e^{-\alpha(t-s)} d\widetilde{W}_s, \\
r_t &= \beta + (r_0 - \beta) e^{-\alpha t} + \sigma \textstyle\int_0^t e^{-\alpha(t-s)} d\widetilde{W}_s. \qquad (2.5)
\end{aligned}$$

and (2.5) is recognized as the solution of (2.1).
In general, if r is deterministic, or non–random, $\int_0^t r(u)dW(u)$ is Gaussian by the features of Ito integrals. Normality of the Vasicek model adds a great deal of tractability to the model.
When we look closer to the solution we see that:

$$\mathbb{E}(r_t|r_0) = \beta + (r_0 - \beta)e^{-\alpha t} = r_0 e^{-\alpha t} + \beta\left(1 - e^{-\alpha t}\right), \tag{2.6}$$

since $\mathbb{E}\left[\sigma \int_0^t e^{\alpha(t-s)} d\widetilde{W}_s\right] = 0$. Now

$$\text{Var}(r_t|r_0) = \mathbb{E}\left[\left(\sigma \int_0^t e^{-\alpha(t-s)} d\widetilde{W}_s\right)^2\right], \tag{2.7}$$

using Ito isometry to obtain

$$\text{Var}(r_t|r_0) = \sigma^2 \mathbb{E}\left[\int_0^t e^{-2\alpha(t-s)} ds\right] = \frac{\sigma^2}{2\alpha}\left(1 - e^{-2\alpha t}\right). \tag{2.8}$$

Observe that for a positive variance we must have

$$1 - e^{-2\alpha t} > 0 \Rightarrow e^{-2\alpha t} < 1 \Rightarrow -2\alpha t < 0 \Rightarrow 2\alpha t > 0.$$

Since $2t$ is always positive we must have $\alpha > 0$ for a correctly calibrated model.

From the above analysis,

$$(r_t|r_0) \sim N\left[\beta + (r_0 - \beta)e^{-\alpha t}, \frac{\sigma^2}{2\alpha}\left(1 - e^{-2\alpha t}\right)\right]. \tag{2.9}$$

Now, in order to derive the bond price formula, we solve the SDE (2.2). Using the substitution $\widetilde{r}_t = r_t - \beta$ we obtain:

$$d\widetilde{r}_t = -\alpha \widetilde{r}_t dt + \sigma d\widetilde{W}_t. \tag{2.10}$$

Then the solution of the OU process (2.10) is:

$$\widetilde{r}_t = \widetilde{r}_0 e^{-\alpha t} + \sigma e^{-\alpha t}\int_0^t e^{\alpha s} d\widetilde{W}_s. \tag{2.11}$$

Moreover, observe that

$$\text{Var}\left(\widetilde{r}_t\right) = \sigma^2 e^{-2\alpha t}\int_0^t e^{2\alpha s} ds,$$

so that

$$\widetilde{r}_t \sim N\left[\widetilde{r}_0 e^{-\alpha t}, \frac{\sigma^2}{2\alpha}\left(1 - e^{-2\alpha t}\right)\right]. \tag{2.12}$$

Since $\widetilde{r}_t$ is Markovian, the bond price $B(t,T)$ can be expressed as an expectation, conditional on the current value

$$B(t,T) = \mathbb{E}^Q\left(e^{-\int_t^T \widetilde{r}_s ds}.1|F_t\right), = \mathbb{E}^Q\left(e^{-\int_t^T \widetilde{r}_s ds}|\widetilde{r}_t = x\right). \tag{2.13}$$

Stationarity of the stochastic process defined above enables us to set $\theta = T - t$ and we get $\mathbb{E}_x\left(e^{-\int_0^\theta \widetilde{r}_s ds}\right)$.

By noting that $\widetilde{r}_s$ is normally distributed, we have that $\int_0^\theta \widetilde{r}_s ds$ is normally distributed as well. In this case, $e^{-\int_0^\theta \widetilde{r}_s ds}$ is log–normally distributed and has a well known link with the underlying normally distributed variable:

$$\mathbb{E}_x\left(e^{-\int_0^\theta \widetilde{r}_s ds}\right) = e^{-\mathbb{E}_x\left(e^{\int_0^\theta \widetilde{r}_s ds}\right) + \frac{1}{2}Var_x\left(\int_0^\theta \widetilde{r}_s ds\right)}. \tag{2.14}$$

In other words, we can rewrite the bond price in the form of an exponential function.

Now, from (2.11),

$$\mathbb{E}_x\left(\int_0^\theta \widetilde{r}_s ds\right) = \int_0^\theta \mathbb{E}_x(\widetilde{r}_s) ds = \int_0^\theta x e^{-\alpha s} ds = \frac{x}{\alpha}\left(1 - e^{-\alpha\theta}\right), \tag{2.15}$$

where the first equality follows from the linearity of the expectation operator.

Next we calculate the conditional variance:

$$\text{Var}_x\left(\int_0^\theta \widetilde{r}_s ds\right) = \text{Cov}_x\left(\int_0^\theta \widetilde{r}_s ds, \int_0^\theta \widetilde{r}_s ds\right). \tag{2.16}$$

Before this, we calculate a version of the above expression as follows using (2.12):

$$\begin{aligned}\text{Cov}(\widetilde{r}_t, \widetilde{r}_u) &= \mathbb{E}\left(\sigma^2 e^{-\alpha(u+t)} \int_0^t e^{\alpha s} d\widetilde{W}_s \int_0^u e^{\alpha s} d\widetilde{W}_s\right), \\ &= \sigma^2 e^{-\alpha(u+t)} \int_0^{u\wedge v} e^{2\alpha s}. \end{aligned} \tag{2.17}$$

Due to the definition of covariance of Ito integral where $u \wedge v$ denotes the minimum of u and t,

$$\text{Cov}(\widetilde{r}_t, \widetilde{r}_u) = \frac{\sigma^2}{2\alpha} e^{-\alpha(u+t)} \left[e^{2\alpha(t\wedge u)} - 1\right]. \tag{2.18}$$

Now again

$$\begin{aligned}\text{Var}_x\left(\int_0^\theta \widetilde{r}_s ds\right) &= \text{Cov}_x\left(\int_0^\theta \widetilde{r}_s ds, \int_0^\theta \widetilde{r}_s ds\right), \\ &= \mathbb{E}\left(\int_0^\theta \widetilde{r}_u du - \mathbb{E}\left[\int_0^\theta \widetilde{r}_u du\right]\right)\left(\int_0^\theta \widetilde{r}_t dt - \mathbb{E}\left[\int_0^\theta \widetilde{r}_t dt\right]\right), \end{aligned} \tag{2.19}$$

which is doable because the variable s is a dummy variable and can be replaced by u and t as above.

$$\begin{aligned}\text{Var}_t &= \int_0^t \int_0^t \mathbb{E}\left[(\widetilde{r}_u - \mathbb{E}[\widetilde{r}_u])(\widetilde{r}_t - \mathbb{E}[\widetilde{r}_t])\right] dudt, \\ &= \int_0^\theta \int_0^\theta \text{Cov}(\widetilde{r}_u, \widetilde{r}_t) dudt.\end{aligned}$$

For $\quad 0 < u < t$, we have,

$$\begin{aligned}\text{Var}_t &= 2\int_0^\theta \left[\int_0^t \sigma^2 e^{-\alpha(t+u)} \tfrac{e^{2\alpha u}-1}{2\alpha} du\right] dt, \\ &= 2\int_0^\theta \int_0^t \left[\tfrac{\sigma^2 e^{-\alpha t} e^{\alpha u}}{2\alpha} - \tfrac{\sigma^2 e^{-\alpha t} e^{-\alpha u}}{2\alpha}\right] dudt, \\ &= 2\int_0^\theta \left[\tfrac{\sigma^2 e^{-\alpha t} e^{\alpha u}}{2\alpha^2} + \tfrac{\sigma^2 e^{-\alpha t} e^{-\alpha u}}{2\alpha^2}\right]\Big|_0^t dt, \\ &= 2\int_0^\theta \left[\tfrac{\sigma^2}{2\alpha^2} + \tfrac{\sigma^2}{2\alpha^2} e^{-2\alpha t} - 2\tfrac{\sigma^2}{2\alpha^2} e^{-\alpha t}\right] dt, \\ &= \tfrac{\sigma^2}{\alpha^2} \int_0^\theta \left[1 + e^{-2\alpha t} - 4e^{-\alpha t}\right] dt, \\ &= \tfrac{\sigma^2}{\alpha^2}\left[t - \tfrac{e^{-2\alpha t}}{2\alpha} + 4\tfrac{e^{-\alpha t}}{\alpha}\right]\Big|_0^\theta, \\ &= \tfrac{\sigma^2}{\alpha^2}\left[\theta - \tfrac{e^{-2\alpha\theta}}{2\alpha} + 4\tfrac{e^{-\alpha\theta}}{2\alpha} - \tfrac{3}{2\alpha}\right], \\ &= \tfrac{\sigma^2}{2\alpha^3}\left[2\alpha\theta - e^{-2\alpha\theta} + 4e^{-\alpha\theta} - 3\right]. \end{aligned} \tag{2.20}$$

Previously, we have set $\widetilde{r}_t = r_t - \beta$, so that

$$\mathbb{E}\left[-\int_t^T r_u du | r_t\right] = \mathbb{E}\left[-\int_t^T [\widetilde{r}_u + \beta] du\right],$$

and since

$$\mathbb{E}\left[\int_t^T \widetilde{r}_u du\right] = \frac{\widetilde{r}_t}{\alpha}\left(1 - e^{-\alpha(T-t)}\right),$$

we get

$$\mathbb{E}\left[-\int_t^T r_u du | r_t\right] = -\frac{r_t - \beta}{\alpha}\left(1 - e^{-\alpha(T-t)}\right) - \beta(T-t). \tag{2.21}$$

Furthermore,

$$\begin{aligned}\text{Var}\left[-\textstyle\int_t^T r_u du | r_t\right] &= \text{Var}\left[\textstyle\int_t^T r_u du | r_t\right] = \text{Var}\left[\textstyle\int_t^T \widetilde{r}_u du\right],\\ &= \tfrac{\sigma^2}{2\alpha^3}\left[2\alpha(T-t) - 3 + 4e^{-\alpha(T-t)} - e^{-2\alpha(T-t)}\right].\end{aligned} \tag{2.22}$$

From the Ito integral representation of r_t, we realize that the defining process for the short–rate is also Markovian. Thus,

$$B(t,T) = \mathbb{E}\left[-\int_t^T r_u du | F_t\right] = \mathbb{E}\left[-\int_t^T r_u du | r_t\right],$$

we can write

$$B(t,T,r_t) = \mathbb{E}\left[-\int_t^T r_u du | r_t\right] = \mathbb{E}\left[-\int_t^T r_u(r_t) du\right],$$

so that r_u is a function of r_t.

If we combine the previous steps, we arrive at the closed form bond price as follows:

$$\begin{aligned}B(t,T,r_t) &= \exp\left(\mathbb{E}\left[-\textstyle\int_t^T r_u(r_t)du\right] + \tfrac{1}{2}\text{Var}\left[-\textstyle\int_t^T r_u(r_t)du\right]\right)\\ &= \exp\big[-\tfrac{r_t-\beta}{\alpha}\left(1 - e^{-\alpha(T-t)}\right) - \beta(T-t)\\ &\quad + \tfrac{\sigma^2}{2\alpha^3}\left[2\alpha(T-t) - 3 + 4e^{-\alpha(T-t)} - e^{-2\alpha(T-t)}\right]\big]\\ &= \exp\big[-\tfrac{1-e^{-\alpha(T-t)}}{\alpha} + \beta\tfrac{1-e^{-\alpha(T-t)}}{\alpha} - (T-t)\\ &\quad - \tfrac{\sigma^2}{2\alpha^2}\tfrac{1-e^{-\alpha(T-t)}}{\alpha} + \tfrac{\sigma^2}{2\alpha^2}(T-t) - \tfrac{\sigma^2}{4\alpha}\tfrac{\left(1-2e^{-\alpha(T-t)}+e^{-2\alpha(T-t)}\right)}{\alpha^2}\big]\\ &= \exp\big[-A(t,T)r_t + \beta\left(A(t,T) - (T-t)\right)\\ &\quad - \tfrac{\sigma^2}{2\alpha^2}A(t,T) + \tfrac{\sigma^2}{2\alpha^2}(T-t) - \tfrac{\sigma^2}{4\alpha}[A(t,T)]^2\big]\\ &= \exp\left(-A(t,T)r_t + D(t,T)\right),\end{aligned} \tag{2.23}$$

where

$$A(t,T) = \frac{1 - e^{-\alpha(T-t)}}{\alpha},$$

and

$$D(t,T) = \left[\beta - \frac{\sigma^2}{2\alpha^2}\right]\left(A(t,T) - (T-t)\right) - \frac{\sigma^2[A(t,T)]^2}{4\alpha},$$

for detailed alternative derivations of the Vasicek bond prices see [35].
We observe that for all t, the yield defined as $-\ln\left(\frac{B(t,T,r_t)}{T-t}\right)$ obtained from the bond price formula is affine in r_t. That is why Vasicek model is a member of the set of affine term structure models, or it has an exponential affine bond price.
We have established that in case the upper bound U is never binding throughout the life of the bond, we have a plain Vasicek model at hand, and as we have established above, there is a closed form solution for the bond price formula in this special case. However, this is not the case in the Turkish O/N money market, which is the proxy for the short–rate money market in the model. Assuming that $r_0 < U$ and $r(s) \geq L$ until maturity, in our case, the bond price in our problem is:

$$\mathbb{E}\left(e^{-\int_0^T R(s)ds}|r_0\right),$$

and can be written as

$$\mathbb{E}\big(e^{-\int_0^{\tau_1} R(s)ds+\int_{\tau_1}^T R(s)ds\mathbb{I}_{\{r_s\geq U\}}+\int_{\tau_1}^T R(s)ds\mathbb{I}_{\{r_s<U\}}}|r_0\big),$$

where τ_1 is the first hitting time of r_s to U.
If we condition the second and third terms in the above expression on τ_1 we get:

$$\mathbb{E}\left[e^{-\int_0^{\tau_1} R(s)ds}|r_0,\tau_1\right]\mathbb{E}\left[e^{-\int_{\tau_1}^T U\mathbb{I}_{\{r_s\geq U\}}ds}|r_0,\tau_1\right]\mathbb{E}\left[e^{-\int_{\tau_1}^T r_s\mathbb{I}_{\{r_s<U\}}ds}|r_0,\tau_1\right]. \quad (2.24)$$

In other words, the bond price can be decomposed into three conditionally independent expressions. The first expression can further be conditioned on τ_1 and $R_s = r_s$. Notice that within this set $r_0 < U$ [1], so that

$$\mathbb{E}\left(e^{-\int_0^{\tau_1} r(s)ds}|\tau_1,r_0\right),$$

by the tower property of conditional expectation.
This expression is nothing but the closed form bond pricing formula derived previously for the Vasicek model, conditional on τ_1 and can be rewritten as:

$$\int_0^T \mathbb{E}\left(e^{-\int_0^{\tau_1} r(s)ds}|r_0\right)dF(\tau_1).$$

Apparently, we need the probability density function of τ_1, however, for a general process there is no closed form solution for the pdf of the first hitting time, but we are

[1] Note that this is a harmless assumption because if $r_0 \geq U$ then the first two conditional expectations can be combined. So, by letting $r_0 < U$ we address a more general case.

only able to derive its characteristic exponent which can be inverted by expanding into series using numerical methods. Luckily, in our case we can derive the first hitting time analytically. The derivation of the characteristic exponent is given in Section 2.2:

2.2 DERIVATION OF PDF OF THE HITTING TIME DISTRIBUTION OF OU PROCESS

The OU Process SDE is

$$dX_t = -\lambda X_t dt + \sigma dB_t, \quad X_0 = x_0.$$

Letting $Y_t = e^{\lambda t} X_t, Y_0 = x_0$ and applying Ito formula yields

$$\begin{gathered}
dY_t = \lambda e^{\lambda t} X_t dt + e^{\lambda t}[-\lambda X_t dt + \sigma dB_t] = \sigma e^{\lambda t} dB_t, \\
Y_t = Y_0 + \int_0^t \sigma e^{\lambda s} dB_s, \\
X_t = x_0 e^{-\lambda t} + \int_0^t \sigma e^{-\lambda(t-s)} dB_s, \\
\mathbb{E}(X_t) = x_0 e^{-\lambda t}, \\
Var(X_t) = \int_0^t \sigma^2 e^{-2\lambda(t-s)} ds = \sigma^2 e^{-2\lambda t} \int_0^t e^{2\lambda s} ds \\
= \sigma^2 e^{-2\lambda t} \left(\tfrac{1}{2\lambda} e^{2\lambda s} \Big|_0^t \right) = \sigma^2 \left(\tfrac{1-e^{-2\lambda t}}{2\lambda} \right), \\
X_t \sim N\left(x_0 e^{-\lambda t}, \sigma^2 \left(\tfrac{1-e^{-2\lambda t}}{2\lambda} \right) \right). \qquad (2.25)
\end{gathered}$$

Let $m = x_0 e^{-\lambda t}$ and $\Sigma^2 = \sigma^2 \left(\frac{1-e^{-2\lambda t}}{2\lambda} \right)$. After these we want to make a digression on the derivation of the acteristic exponent of Brownian motion with drift. The relevant SDE is:

$$\begin{gathered}
dX_t = \mu dt + \sigma dW_t, \\
X_t = x_0 + \mu t + \sigma W_t.
\end{gathered}$$

Let $x_0 = 0$ and define $\tau_s = inf[t \geq 0 : \Sigma W_t + mt \geq a]$ be the first passage time of X_t.

First see that $M_t = e^{\sigma W_t - \frac{1}{2}\sigma^2 t}$ is a Martingale (see [32]). Now check

$$M_t = e^{\sigma W_t - \frac{1}{2}\sigma^2 t \pm \sigma m t} = e^{\sigma(W_t + mt)} e^{\left[-\frac{\sigma^2}{2} - m\sigma\right] t},$$

stopping the sequence at a stopping time, by Doob's optimal sampling theorem:

$$\mathbb{E}(M_t) = 1 = \mathbb{E}\left(e^{\sigma(w_{t\wedge\tau_a})+m(t\wedge\tau_a)}e^{-\left[\frac{1}{2}\sigma^2+m\sigma\right](t\wedge\tau_a)}\right),$$

as $\lim_{t\to\infty}$ we have

$$1 = \mathbb{E}\left(e^{\sigma(w_{\tau_a})+m\tau_a}e^{-\left[\frac{1}{2}\sigma^2+m\sigma\right]\tau_a}\right),$$

but $w_{\tau_a} + \mu\tau_a = a$

$$e^{-\sigma a} = \mathbb{E}\left(e^{-\left[\frac{1}{2}\sigma^2+m\sigma\right]\tau_a}\right). \tag{2.26}$$

At this point, we use a trick to find the characteristic exponent of τ_a:
Let

$$\xi = \frac{1}{2}\sigma^2 + m\sigma,$$

which implies

$$\begin{gathered}
\sigma^2 + m\sigma - 2\xi = 0, \\
\sigma = -m + \sqrt{m^2 + 2\xi}, \\
(\sigma + m)^2 - m^2 - 2\xi = 0 \quad , \\
(\sigma + m)^2 = m^2 + 2\xi.
\end{gathered}$$

So by 2.26:

$$\mathbb{E}\left[e^{-\xi\tau_a}\right] = e^{-a\left[\sqrt{m^2+2\xi}-m\right]}.$$

$\xi = -\frac{m^2}{2}$ becomes branch point of the complex valued function $h(\xi) = e^{-a\left[\sqrt{m^2+2\xi}-m\right]}$.
Using analytic extension we can replace ξ by $-i\theta$ and we obtain:

$$h(-i\theta) = \varphi_{\tau_a} = \mathbb{E}e^{i\theta\tau_a} = e^{-a\left[\sqrt{m^2-2i\theta}-m\right]},$$

which is the characteristic function with characteristic exponent

$$\psi(\theta) = \sqrt{m^2 - 2i\theta} - m.$$

Then using the inversion formula we can compute the pdf of τ_s as:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\varphi(t)dt.$$

Note that $\mathbb{E}e^{i\theta\tau_a}$ is the characteristic function of τ_a, by definition. In 2.26, on the right hand side we have an expectation and on the left hand side there is no expectation. We employ the trick of setting $\xi = \frac{1}{2}\sigma^2 + m\sigma$ and solve σ as a function of ξ so that we can replace σ on the left hand side with a function of ξ and m where m is known and is the parameter of the characteristic function. So the left hand side has no expectation and it is a function of ξ, the parameter of the characteristic function of τ_a. Further note that we replace ξ with θ as the parameter and we get:

$$\varphi_{\tau_a}(\theta) = e^{-a\left[\sqrt{m^2-2i\theta}-m\right]},$$

and

$$\psi(\theta) = \sqrt{m^2 - 2i\theta} - m.$$

Another caveat is that one needs to be careful with the calibration, so that U, m, σ, a are different quantities. Now, if we come back to the inversion, first note that we can do the analytical extension to where $\{x \in \mathbb{C} : Re(x) > \frac{-m^2}{2}\}$ (see Figure 2.1).

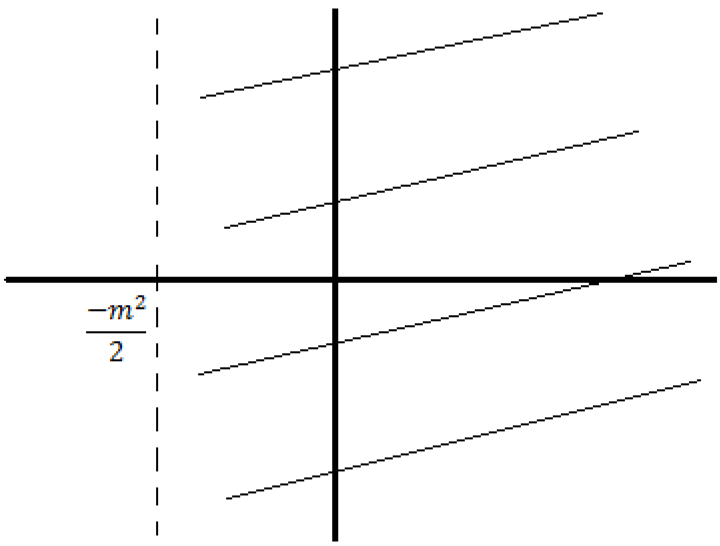


Figure 2.1: Analytical extension

In this case, there are no singularities and the characteristic function in the inversion formula multiplied by $e^{-i\theta x}$ converges in all the domain it is defined above as a Taylor series:

$$
\begin{aligned}
f_{\tau_a}(x) &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\theta x} e^{-a\left[\sqrt{m^2-2i\theta}\right]-m} d\theta, \\
f_{\tau_a}(x) &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\left(i\theta x + a\left[\sqrt{m^2-2i\theta}\right]-m\right)} d\theta, \\
f_{\tau_a}(x) &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} 1 - \left(i\theta x + a\left[\sqrt{m^2-2i\theta}\right] - m\right) \\
&\quad + \tfrac{\left(i\theta x + a\left[\sqrt{m^2-2i\theta}\right]-m\right)^2}{2!} + \ldots d\theta.
\end{aligned}
$$

So one can achieve Taylor expansion and integrate term by term. Here we skip the remaining terms after some finite number according to the determination of the upper bound for the error. If the series is alternating, the upper bound is the first term which is not used in the summation. However, for non–alternating series finding an upper bound may not be easy. Although for the general case, inversion of the hitting time distribution of any process is the only possibility for the recovery of the pdf, for the OU process, we have a closed form solution of the hitting time distribution. For this define:

$$
\mu_a(dx) = \frac{a}{\sqrt{2\pi x^3}} e^{ma} e^{-\frac{1}{2}\left(s^2 x^{-1} + m^2 x\right)} dx,
$$

on $x > 0$ to show that:

$$
\int_0^{\infty} e^{-\lambda x} \mu_a(dx) = e^{-a\left(\sqrt{m^2+2\lambda}-m\right)},
$$

and for $\lambda = -i\theta$

$$
e^{-a\left(\sqrt{m^2+2\lambda}-m\right)} = e^{-a\Phi(\theta)},
$$

i.e. the characteristic function of the desired hitting rime distribution so that by the definition of characteristic function, we may conclude that $\mu_a(dx)$ is the pdf of the hitting time distribution. Now we have:

$$
\int_0^{\infty} e^{-\lambda x} \mu_a(dx),
$$

after multiplying $\mu_a(dx)$ with $e^{\pm a\left(\sqrt{m^2+2\lambda}\right)}$ we have

$$
\begin{aligned}
& e^{ma} e^{-a\left(\sqrt{m^2+2\lambda}\right)} \int_0^{\infty} \tfrac{a}{\sqrt{2\pi x^3}} e^{-\frac{1}{2}\left(\frac{a^2}{x} + m^2 x - 2a\sqrt{m^2+2\lambda} + 2\lambda x\right)} dx, \\
&= e^{ma - a\sqrt{m^2+2\lambda}} \int_0^{\infty} \tfrac{a}{\sqrt{2\pi x^3}} e^{-\frac{1}{2}\left[\frac{a}{\sqrt{x}} - \sqrt{(m^2+2\lambda)x}\right]^2} dx. \qquad (2.27)
\end{aligned}
$$

Now apply the following change of variables:

$$
\begin{aligned}
\tfrac{a}{\sqrt{x}} &= \sqrt{(2\lambda + m^2)u}, \\
x &= \tfrac{a^2}{(2\lambda+m^2)u}, dx = \tfrac{-a^2}{(2\lambda+m^2)u^2} du,
\end{aligned}
$$

to get

$$e^{ma-a\sqrt{m^2+2\lambda}}\int_0^\infty\left\{\frac{1}{\sqrt{2\pi}}\frac{a}{\left[\left[\frac{a^2}{(2\lambda+m^2)u}\right]^3\right]^{\frac{1}{2}}}\right.$$

$$\left.e^{-\frac{1}{2}\left[\sqrt{(2\lambda+m^2)u}-\frac{a}{\sqrt{u}}\right]^2}\frac{-a^2}{u^2(2\lambda+m^2)}\right\}du,$$

$$e^{ma-a\sqrt{m^2+2\lambda}}\int_0^\infty\left\{\frac{1}{\sqrt{2\pi}}\frac{2\lambda+m^2)u^{\frac{3}{2}}}{\sqrt{(2\lambda+m^2)u^2}}e^{-\frac{1}{2}\left[\sqrt{(2\lambda+m^2)u}-\frac{a}{\sqrt{u}}\right]^2}\right\}du,$$

$$e^{ma-a\sqrt{m^2+2\lambda}}\int_0^\infty\left\{\frac{\sqrt{(2\lambda+m^2)}}{\sqrt{2\pi u}}e^{-\frac{1}{2}\left[\frac{a}{\sqrt{u}}-\sqrt{(2\lambda+m^2)u}\right]^2}\right\}. \tag{2.28}$$

Now observe that taking the arithmetic averages of the equations (2.27) and (2.28)

$$\int_0^\infty e^{-\lambda x}\mu_a(dx)=\frac{1}{2}e^{am-a\sqrt{m^2+2\lambda}},$$

$$=\int_0^\infty\left[\frac{a}{\sqrt{2\pi x^3}}+\frac{\sqrt{(2\lambda+m^2)}}{\sqrt{2\pi x}}\right]e^{-\frac{1}{2}\left[\frac{a}{\sqrt{x}}-\sqrt{(m^2+2\lambda)x}\right]^2}dx. \tag{2.29}$$

Let us change the variable of integration as follows:

$$K=\frac{a}{\sqrt{x}}-\sqrt{(m^2+2\lambda)}x,$$

so that

$$dK=-\left[\frac{a}{2\sqrt{x^3}}+\frac{\sqrt{(2\lambda+m^2)}}{2\sqrt{x}}\right]dx.$$

Furthermore the new limits of integration are: $K\to\infty$ as $x\to 0$ and $K\to-\infty$ as $x\to\infty$ because if we rewrite K as:

$$\frac{a}{\sqrt{x}}-\sqrt{A}\sqrt{x},$$

where $A=\sqrt{(2\lambda+m^2)}$ as $x\to 0$, $\frac{a}{\sqrt{x}}\to 0$ and $\sqrt{x}\to 0$ so that $K\to\infty$ and $\frac{a}{\sqrt{x}}\to 0$ as $x\to\infty$ but now $-\sqrt{A}\sqrt{x}\to-\infty$. Therefore, with the new limits and the integrator:

$$\int_0^\infty e^{-\lambda x}\mu_a(dx)=e^{ma-a\sqrt{m^2+2\lambda}}\int_{-\infty}^\infty-\frac{1}{2\sqrt{\pi}}e^{-\frac{1}{2}K^2}dK,$$

$$\int_0^\infty e^{-\lambda x}\mu_a(dx)=e^{ma-a\sqrt{m^2+2\lambda}},$$

and for $\lambda=-i\theta$

$$\int_0^\infty e^{-\lambda x}\mu_a(dx)=e^{ma-a\sqrt{m^2-2i\theta}}=e^{a\Phi(\theta)},$$

where $\Phi(\theta)$ is the characteristic exponent of the OU first hitting time as shown before, so that $\mu_a(dx)$ is the corresponding pdf. This establishes the closed form solution for the first component in (2.24)

2.3 DERIVING THE REMAINING

First we define three intervals that will be useful in bond pricing:
$I_1 = [0, \tau_1)$, $I_2 = [\tau_1, \tau_2)$ and $I_3 = [\tau_2, T]$ where τ_1 is the first hitting time discussed above, τ_2 is the total time spent by r above U and T is the maturity. In I_1, the variable s keeps record of the real time whereas we introduce a time change in I_2 and I_3 such that s does not represent the real time but corresponds to the time spent by r only in the respective set. Observe that

$$|I_1|+|I_2|+|I_3| = T,$$

Next, we can move to the second expression:

$$\mathbb{E}\left[e^{-\int_{\tau_1}^{T} U\mathbb{I}_{\{r_s \geq U\}}ds}|\tau_1\right] = \mathbb{E}\left[e^{-U\int_{\tau_1}^{T} \mathbb{I}_{\{r_s \geq U\}}ds}|\tau_1\right]. \tag{2.30}$$

given τ_1 it is clear that $r_{\tau_1} = U$ by the definition of hitting time. In this vein, given τ_1 define:

$$\tau_2 = \int_{\tau_1}^{T} \mathbb{I}_{\{r_s \geq U\}}ds,$$

is the time spent by the underlying Vasicek model above the level U, where the interest rate will be forced to U by the central bank monetary policy tools. In the literature this is classified as a Sojourn time problem ([43, 44]). Again, in case we condition on the time spent above U, i.e.τ_2, we can rewrite the expectation in (2.30) as follows (already conditioned on τ_1):

$$\mathbb{E}\left(e^{-U\tau_2}|\tau_1\right) = \int_{\tau_1}^{T} e^{-U\tau_2}g(\tau_2)d\tau_2, \tag{2.31}$$

where $g(\tau_2)$ is the pdf of the random variable τ_2. Observe that in this set we fix the value of the interest rate to U. The random variable τ_2 has a generalized arcsin distribution. We will prove this in the sequel, after giving some definitions and a lengthy digression.

First consider a Lévy Process on the time interval $[0, t]$ and introduce the time it spends on the upper half interval $[0, \infty)$ as:

$$A_t = \int_{0}^{t} \mathbb{I}_{\{X_s \geq 0\}}ds. \tag{2.32}$$

We will do the proof for the case $U = 0$, but it can be generalized for all values of U in the support of the process at hand. Next, we introduce the supremum process as:

$$S_t = \sup\{0 \vee X_s : 0 \leq s \leq t\},$$

for a general Lévy process X_s.

This definition, we will further use for the introduction of the instant of the last supremum for X_s:

$$G_t = \text{sup}\{s < t : X_s = S_s\},$$

The following theorem links A_t and G_t which could be found in [8].

Theorem 2.1. *Assume that* $\mathbb{P}(X_t \geq 0) = p \in (0,1)$ *for all* $t > 0$*. Then for every* $t > 0$ *the random variables* $t^{-1}A_t$ *and* $t^{-1}G_t$ *are both distributed according to the generalized arcsin law with parameter* p*.*

Proof. Recall that for $p \in (0,1)$, the generalized arcsin law with parameter p is the probability distribution on $[0,1]$ given by

$$\frac{s^{p-1}(1-s)^{-p}}{\Gamma(p)\Gamma(1-p)}ds \qquad (0 < s < 1). \tag{2.33}$$

This is a special case for the $beta(\alpha, \beta)$ distribution specifically it is equal to $beta(p, 1-p)$, where $B(\alpha, \beta)$ is the beta distribution:

$$\frac{\Gamma(\alpha, \beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1} \qquad 0 \leq x \leq 1. \tag{2.34}$$

Both distributions have $[0,1]$ as their support. If we make the following change of variables

$$x = \frac{y-a}{b-a},$$

then

$$\begin{gathered} Y \sim \tfrac{\Gamma(\alpha,\beta)}{\Gamma(\alpha)\Gamma(\beta)} \left(\tfrac{y-a}{b-a}\right)^{\alpha-1} \left(1 - \tfrac{y-a}{b-a}\right)^{\beta-1} \left|\tfrac{dx}{dy}\right|, \\ Y \sim B(\alpha,\beta) \tfrac{(y-a)^{\alpha-1}}{(b-a)^{\alpha-1}} \tfrac{(b-y)^{\beta-1}}{(b-a)^{\beta-1}} \tfrac{1}{b-a}, \\ Y \sim B(\alpha,\beta) \tfrac{(y-a)^{\alpha-1}(b-y)^{\beta-1}}{(b-a)^{\alpha+\beta-1}} \qquad \text{for} \quad a \leq y \leq b. \end{gathered} \tag{2.35}$$

This is the general form fof the beta function. So, we have shown that, as a special case of the beta distribution, the arcsin distribution can be extended to any interval other than $[0,1]$, especially to $[\tau_1, \tau_2]$, as described above. $\square$

Now that we have established the usefulness of the generalized arcsin distribution in our case, we would like to give the proof for the distribution of A_t being the generalized arcsin distribution. For this, first we present the following Lemma:

Lemma 2.2 (Spare Anderson's Identity)**.** *For every $t > 0$, A_t and G_t have the same law.*

Proof. The first part of the proof is achieved for the random walk using induction. We want to show that for $r \in \mathbb{Z}$, $0 \leq r \leq n$, the number of A_r permutations with exactly r non–negative partial sums in $S_1, S_2, \ldots, S_n$ (excluding $S_0 = 0$), is the same as the number B_r of permutations in which the last maximum among these partial sums occurs at the place r.

Here $S_0 = 0$ and $S_R = x_1 + \cdots + x_R$ are the partial sums for the random variables $x_i, i = 1, \ldots, n$ that comprise a random walk, i.e.taking the value 1 or -1 with some given probability. For the proof by induction, first let $n = 1$:

$x_1 > 0$ implies $A_1 = B_1 = 1$ and $A_0 = B_0 = 0$, while

$x_1 \leq 0$ implies $A_1 = B_1 = 0$ and $A_0 = B_0 = 1$.

Assume that the assertion holds for $n - 1 \geq 1$. Denote by $A_r^{(k)}$ and $B_r^{(k)}$ the numbers corresponding to A_r and B_r when the n–tuple $(x_1, \ldots, x_n)$ is replaced by the $n-1$–tuple obtained by omitting x_k .The induction hypothesis asserts that:

$$A_r^{(k)} = B_r^{(k)},$$

for $1 \leq k \leq n$ and $r = 0, \ldots, n-1$

(a) Suppose

$$S_R = x_1 + \cdots + x_n \leq 0,$$

In this case the $n!$ permutations of $(x_1, \ldots, x_n)$ are obtained by choosing the element x_k at the last place and permitting the remaining $(n-1)$ elements, then clearly the number of non–negative partial sums and the index of the last mathematical term depends only on the first $n-1$ elements. Thus:

$$A_r = \sum_{k=1}^{n} A_r^{(k)}, \qquad B_r = \sum_{k=1}^{n} B_r^{(k)},$$

so $A_r = B_r$ by the induction hypothesis.

(b) In case

$$S_R = x_1 + \cdots + x_n > 0,$$

Now, since the n^{th} partial sum is positive, then

$$A_r = \sum_{k=1}^{n} A_{r-1}^{(k)},$$

An analogous recursion formula for B_r is achieved by considering the permutations $(x_k, x_j, \ldots, x_{j_{n-1}})$ starting with x_k. Clearly, the last maximum occurs at place $r, (1 \leq r \leq n)$ if and only if the last maximum of the partial sums for $(x_{j+1}, \ldots, x_{j_{n-1}})$ occurs at the place $r-1$. Thus

$$B_r = \sum_{k=1}^{n} B_{r-1}^{(k)}.$$

So again $A_r = B_r$.

□

For a detailed explanation see [16].

After establishing the equality for the discrete case, we need to extend the result to the continuous Lévy processes. Assume that x is not a compound Poisson process, so $\mathbb{P}(X_t = 0) = 0$ for almost every t. Using the right continuity of the path and the theorem of dominated convergence, we can show that $n^{-1}B_n$ converges to A_1 almost surely. On the other hand, one can check from the fact of the paths being right continuous with left limits that

$$\limsup n^{-1} A_n \leq G_1,$$

almost surely and similarly since the local suprema of X are distinct, we have

$$\liminf n^{-1} A_n \geq G_1,$$

almost surely. So A_1 and G_1 have the same distribution as claimed by (2.2). Finally, the compound Poisson case follows by approximations, using the process $(x_t + \epsilon_t, t \in [0,1])$ and letting ϵ tend to 0^+.

Now we can pursue with the proof of (2.1). By (2.2) it is enough to check that $t^{-1}G_t$ has the generalized arcsin distribution. For this, we will exploit a fluctuation identity, where

$$G_\tau = G_{\tau_\xi} = \sup\{t < \tau(\xi) : X_t = S_t\},$$

with $\tau = \tau_\xi$ an exponential random time with parameter $\xi > 0$, which is independent of the Lévy process. The identity reads:

$$\begin{aligned}\mathbb{E}\left(\exp\{-\lambda G_{\tau_\xi}\}\right) &= \int_0^\infty \xi e^{-\xi t}\mathbb{E}\left(\exp\{-\lambda G_t\}\right) dt,\\ &= \exp\left(\int_0^\infty \left(e^{-\lambda t}-1\right) t^{-1} e^{-\xi t}\mathbb{P}(X_t \geq U) dt\right). \qquad (2.36)\end{aligned}$$

Remember, for the process at hand i.e.,OU, the interest rates have the mean for $r_0 = U$:

$$\beta + (U-\beta)e^{-\alpha t},$$

and the variance

$$\frac{\alpha^2}{2\alpha}\left(1-e^{2-\alpha t}\right).$$

This means that $\mathbb{P}(X_t \geq U)$ in 2.36 is not constant due to the $(U-\beta)e^{-\alpha t}$ term. Although this term vanishes asymptotically, for a bond having a finite maturity, this term stays. On the other hand, its effect decreases as U is close to β; t and α are both increasing.

In Turkey, the historical difference between U and β are in general small, although there are periods, where the upper bound is far from the long term mean β. Furthermore, Turkish interest rates exhibit fast mean reversion. Considering these, we have decided to model $\mathbb{P}(X_t \geq U) = p$ where p is a constant. Accordingly the formula is:

$$\exp\left(p\int_0^\infty \left(e^{-\lambda t}-1\right) t^{-1} e^{-\xi t} dt\right). \qquad (2.37)$$

We can rewrite (2.37) in a simpler form by using the identity known as the Frullani integral:

Lemma 2.3 (Frullani integral). *$\forall \alpha, \beta > 0$ and $z \in \mathbb{C}$ such that $Re(z) \leq 0$ we have*

$$\frac{1}{\left(1-\frac{z}{\alpha}\right)^\beta} = \exp\left(-\int_0^\infty \left(1-e^{zx}\right)\beta x^{-1} e^{-\alpha x} dx\right).$$

Proof. (i) For any function f such that f' exist and continuous, $f(0), f(\infty)$ are finite for $\infty > b > a > 0$ we have

$$\int_0^\infty \frac{f(ax)-f(bx)}{x} dx = \int_0^\infty \int_a^b f'(yx) dy dx,$$

Notice that

$$\int_a^b f'(yx) dy dx = f(bx) - f(ax),$$

and since x is a constant we have

$$\int_a^b f'(yx)dy = \frac{f(bx)-f(ax)}{x},$$

As the boundaries are measurable using Fubini's theorem yields

$$\int_a^b \int_0^\infty f'(yx)dydx,$$

$u = yx$, $du = ydx$

$$\begin{aligned} &= -\textstyle\int_a^b dy \int_0^\infty f'(u)\frac{du}{y}, \\ &= -\textstyle\int_a^b \frac{1}{y}dy[f(\infty)-f(0)], \\ &[f(0)-f(\infty)]\ln\left(\tfrac{b}{a}\right). \end{aligned} \tag{2.38}$$

(ii) Choose $f(x) = e^{-x}$, $\quad a = \alpha > 0$, $\quad b = \alpha - z$, $\quad z < 0$,

$$\begin{aligned} &\textstyle\int_0^\infty \frac{f(ax)-f(bx)}{x}dx, \\ &= \textstyle\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}dx, \\ &= \textstyle\int_0^\infty \frac{e^{-ax}-e^{(-\alpha+z)x}}{x}dx, \\ &= \textstyle\int_0^\infty \frac{e^{-ax}}{x}\left(1-e^{zx}\right)dx, \end{aligned} \tag{2.39}$$

From (2.38)

$$\begin{aligned} &\textstyle\int_0^\infty \frac{f(ax)-f(bx)}{x}dx, \\ &= \ln\left(\tfrac{b}{a}\right)[f(0)-f(\infty)], \\ &= \ln\left(\tfrac{\alpha-z}{\alpha}\right)[1-0], \\ &= \ln\left(1-\tfrac{z}{\alpha}\right). \end{aligned} \tag{2.40}$$

Since (2.39) and (2.40) are equal we have

$$\int_0^\infty \frac{e^{-ax}}{x}\left(1-e^{zx}\right)dx = \ln\left(1-\frac{z}{\alpha}\right). \tag{2.41}$$

For $\beta > 0$ multiply each sides of 2.41 by $-\beta$, we obtain

$$-\beta\int_0^\infty \frac{e^{-ax}}{x}\left(1-e^{zx}\right)dx, = -\beta\ln\left(1-\frac{z}{\alpha}\right),$$

and taking exponential yields the desired result:

$$\exp\left(-\beta\int_0^\infty \frac{e^{-ax}}{x}\left(1-e^{zx}\right)dx\right) = \frac{1}{\left(1-\frac{z}{\alpha}\right)^\beta}.$$

The result is proven for $z \in \mathbb{R}$,$(z < 0)$. It also holds for $z \in \mathbb{C}$.

The formula is:

$$\exp\left(-\int_0^\infty (1 - e^{zx})\,\beta x^{-1} e^{-\alpha x} dx\right) = \frac{1}{\left(1 - \frac{z}{\alpha}\right)^{\beta}}.$$

the left hand side can be rewritten as

$$\exp\left[\beta\left(-\int_0^\infty (1 - e^{zx})\,x^{-1} e^{-\alpha x} dx\right)\right] = \frac{1}{\left(1 - \frac{z}{\alpha}\right)^{\beta}},$$

In our problem we have:

$$\exp\left[p\left(-\int_0^\infty \left(e^{-\lambda t} - 1\right) t^{-1} e^{-\xi t} dt\right)\right],$$

so that for $\beta = p, z = -\lambda, x = t$ and $\alpha = \xi$ we have $p > 0, \xi > 0$ and since $\lambda > 0$, $-\lambda$ is of the form $Re(z) < 0$ so that (2.3) applies:

$$\exp\left[p\left(-\int_0^\infty \left(e^{-\lambda t} - 1\right) t^{-1} e^{-\xi t} dt\right)\right] = \frac{1}{\left(1 + \frac{\lambda}{\xi}\right)^{p}} = \left(\frac{\xi}{\xi + \lambda}\right)^{p}. \tag{2.42}$$

(2.42) can be inverted using the double Laplace transform for almost every $t > 0$ as follows:

$$\begin{aligned}
\mathcal{L}_\lambda^{-1}\mathcal{L}_\xi^{-1}\{\xi^p(\xi+\lambda)^{-p}\} &= \textstyle\int_0^\infty \int_0^\infty \xi^p(\xi+\lambda)^{-p} e^{\xi t + \lambda s} d\lambda ds,\\
&= \textstyle\int_0^\infty \xi^p e^{\xi t} \int_0^\infty \left((\xi+\lambda)^{-p} e^{\lambda s} d\lambda\right) d\xi,\\
&= \textstyle\int_0^\infty \xi^p e^{\xi(t-s)} \frac{s^{p-1}}{\Gamma(1-p)},\\
&= \textstyle\frac{s^{p-1}}{\Gamma(1-p)} \int_0^\infty \xi^p e^{\xi(t-s)} d\xi,\\
&= \textstyle\frac{s^{p-1}}{\Gamma(p)} \frac{(t-s)^{-p}}{\Gamma(1-p)}.
\end{aligned} \tag{2.43}$$

Since the process $t \to G_t$ is right continuous, the presented result holds $\forall t > 0$ □

Next, we proceed to the last set of the interest rate movements where r_s is the collection of all periods after the first hitting time where the interest rate is below U. Conditional on τ_1 and τ_2, the time spent by the process is $T - \tau_1 - \tau_2$, i.e. a conditional constant. In other words, there is no need to derive the distribution of the time spent in this set given T, τ_1 and τ_2.

In this set, we model the interest rate as a shifted folded normal distribution

$$R(s) = U - |U - r(s)| = \begin{cases} r(s), & \text{if} \quad r(s) \leq U \\ 2U - r(s), & \text{if} \quad r(s) > U, \end{cases}$$

where $r(s)$ is the underlying Vasicek process. Clearly with $r(s) > 2U, R(s)$ in the last set can have negative values. Yet, for $\beta < U$ and for a large value of α, i.e. a fast mean reverting series renders $r(s) > 2U$ very unlikely. Even if there is a slight probability of interest rates turning negative, following Great Recession, this is not an unlikely phenomenon.

This functional form of $R(s)$ guarantees that interest rates are always at, or below the level U, as required in this set.

Remember that we have already accounted the periods of $r(s)$ exceeding U in the previous set. Hence the model to keep the interest rates below U. So, the pricing of bond in this set reads:

$$\mathbb{E}\left[\exp\left(-\int_{T-\tau_1-\tau_2}^{T}[U-|U-r(s)|]ds\big|\tau_1,\tau_2\right)\right]. \tag{2.44}$$

This formula can be simplified as

$$\mathbb{E}\left[\exp\left(-\int_{T-\tau_1-\tau_2}^{T}Uds\big|\tau_1,\tau_2\right)\right]\mathbb{E}\left[\exp\left(\int_{T-\tau_1-\tau_2}^{T}|U-r_s|ds\big|\tau_1,\tau_2\right)\right]. \tag{2.45}$$

For the second conditional expectation, we see that the expression $|U - r_s|$ has a folded normal, or sometimes called as half–normal distribution. Its pdf is:

$$\frac{1}{\sigma\sqrt{2\pi}}\left[e^{\frac{-(x-\mu)^2}{2\sigma^2}}+e^{\frac{-(x+\mu)^2}{2\sigma^2}}\right], \tag{2.46}$$

where μ and σ are the mean and variance of the underlying normal distribution, respectively.

As we have already derived the distribution of $r(s)$ as:

$$(r_s|r_0) \sim N\left(\beta+(r_0-\beta)e^{-\alpha t}, \frac{\sigma^2}{2\alpha}\left(1-e^{-2\alpha t}\right)\right),$$

Mean of the shifted distribution is:

$$\begin{aligned}
\mu &= U-\beta-(U-\beta)e^{-\alpha t},\\
\mu &= U-\beta-Ue^{-\alpha t}+\beta e^{-\alpha t},\\
\mu &= U\left(1-e^{-\alpha t}\right)-\beta\left(1-e^{-\alpha t}\right),\\
\mu &= (U-\beta)\left(1-e^{-\alpha t}\right),
\end{aligned}$$

in (2.46).

As of now, we have completely established our approach to divide the interest rate process into different subsets and calculating bond prices theoretically in a closed form way. On the other hand, we need to stitch together everything we have done so far and analyze the tractability of the solution.

First of all, we would like to sort out one complication that we have not accounted for previously. The maturity of the bond price is T, a finite time whereas, the first hitting time of $r(s)$ has a support of $[0, \infty)$. In other words, it might well be that the interest rate stays below U from the outset until T, so that the bond price becomes simply one described by the Vasicek model. This eventually must be added to the bond price as an additional term. All in all, the bond price formula yields:

$$\begin{aligned}
B(0,T) = \mathbb{E}\left[\mathbb{E}\left[\exp\left(-\textstyle\int_0^{\tau_1} r(s)ds \middle| \tau_1 \leq T\right)\right]\mathbb{P}(\tau_1 \leq T)\right] \times \\
\mathbb{E}\left[\mathbb{E}\left[\exp\left(-\textstyle\int_0^{\tau_2} U ds \middle| \tau_2\right)\mathbb{P}(\tau_2) \middle| \tau_1 \leq T\right)\right]\mathbb{P}(\tau_1 \leq T)\right] \times \\
\mathbb{E}\left[\mathbb{E}\left[\mathbb{E}\left[\exp\left(-\textstyle\int_{T-\tau_1-\tau_2}^{T}[U - |U - r_s|]ds \middle| \tau_1 \leq T, \tau_2\right)\mathbb{P}(\tau_2)\right]\right]\right]\mathbb{P}(\tau_1 \leq T)\right] \\
+\mathbb{E}\left[\exp\left(-\textstyle\int_0^{T} r(s)ds \middle| \tau_1 > T\right)\right]\mathbb{P}(\tau_1 > T). \qquad (2.47)
\end{aligned}$$

Again the last term arises since there is a positive probability that the bond expires before the interest rate hits the upper bound U.

When we look at the first line, it is a conditional expectation on τ_1. Given τ_1, the inner most expectation is the closed form solution of a Vasicek bond price of which the formula presented previously.

For $t = 0$ and $T = \tau_1$

$$\mathbb{E}\left[\exp\left(-\int_0^{\tau_1} r(s)ds \middle| \tau_1 \leq T\right)\right] = e^{-A(0,\tau_1)r_0 + D(0,\tau_1)}, \qquad (2.48)$$

where

$$A(0,\tau_1) = \frac{1 - \exp(-\alpha\tau_1)}{\alpha},$$

and

$$D(0,\tau_1) = \left[\beta - \frac{\sigma^2}{2\alpha^2}\right](A(0,\tau_1) - \tau_1) - \frac{\sigma^2 (A(0,\tau_1))^2}{4\alpha}.$$

Regarding the outer expectation remember that for $f(\tau_1)$, i.e. pdf of the first hitting distribution, we have

$$f_{\tau_1}(x) = \frac{a}{\sqrt{2\pi x^3}} e^{ma} e^{-\frac{1}{2}\left(s^2 x^{-1} + m^2 x\right)} dx, \qquad (2.49)$$

for $x > 0$.

So, the first line becomes

$$\int_0^T e^{-A(0,\tau_1)r_0 + D(0,\tau_1)} f_{\tau_1} d\tau_1,$$

where A, D and f_{τ_1} are as stated above. Furthermore, now $m = \frac{\beta}{\sigma}$ and $a = U$.
Evidently this is a very challenging exercise, because the integral is extremely difficult to evaluate, however, Matlab's "integral" function can be used. For a numerically safer way, we may rewrite f_{τ_1} in a slightly different form by letting $s = \sqrt{x}$ and $\frac{s}{m} = \nu$.
This transformation leads to the following:

$$f_{\tau_1}(x) = \sqrt{\frac{\lambda}{2\pi x^3}} \exp\left[-\lambda \frac{(x-\nu)^2}{2\nu^2 x}\right],$$

setting $s = \sqrt{x} \Rightarrow s^2 = \lambda$ and $s > 0$

$$\begin{aligned} f_{\tau_1}(x) &= \sqrt{\tfrac{\lambda}{2\pi x^3}} \exp\left[-\tfrac{1}{2}\left(\tfrac{s^2(x^2-2\nu x+\nu^2)}{\nu^2 x}\right)\right], \\ &= \sqrt{\tfrac{\lambda}{2\pi x^3}} \exp\left[-\tfrac{1}{2}\left(\tfrac{x}{\nu^2} - \tfrac{2}{\nu} + \tfrac{1}{x}\right)\right], \end{aligned}$$

letting $\nu = \frac{x}{m}$

$$\begin{aligned} &= \sqrt{\tfrac{\lambda}{2\pi x^3}} \exp\left[-\tfrac{1}{2}\left(m^2 x - 2ms + \tfrac{s^2}{x}\right)\right], \\ f_{\tau_1}(x) &= \tfrac{a}{\sqrt{2\pi x^3}} e^{ma} e^{-\frac{1}{2}\left(s^2 x^{-1} + m^2 x\right)} dx, \end{aligned}$$

as given in (2.49).
Now we can proceed to the second line in the closed-form solution:

$$\mathbb{E}\left[\mathbb{E}\left[\exp\left(-\int_{\tau_1}^{\tau_2} U ds \middle| \tau_2\right) \mathbb{P}(\tau_2) \middle| \tau_1 \leq T\right)\right] \mathbb{P}(\tau_1 \leq T),$$

or equally

$$\int_0^T \left[\int_{\tau_1}^T e^{-U(\tau_2-\tau_1)} g(\tau_2) d\tau_2\right] f(\tau_1) d\tau_1, \tag{2.50}$$

where

$$g(\tau_2) = \tau_2^{p-1}(T-\tau_2)^{-p}[\Gamma(p)\Gamma(1-p)]^{-1},$$

as mentioned previously. It can be rewritten as:

$$\frac{\sin(p\pi)}{\pi} s^{p-1}(1-s)^{-p}, \tag{2.51}$$

If we examine 2.50 carefully, the inner integral can be rewritten as:

$$\int_{\tau_1}^T e^{U\tau_1 - U\tau_2} g(\tau_2) d\tau_2 = e^{U\tau_1} \int_{\tau_1}^T e^{-U\tau_2} g(\tau_2) d\tau_2, \tag{2.52}$$

so we have

$$\int_{\tau_1}^T e^{-U\tau_2}, g(\tau_2) d\tau_2,$$

as the moment generating function of the arcsin distribution.

We have previously mentioned that the arcsin distribution is a version of the beta distribution; specifically $beta(p, 1-p)$. It is well–known that the beta distribution has the moment generating function called "confluent hypergeometric function of the first kind":

$$1+\sum_{k=1}^{\infty}\left(\prod_{r=0}^{k-1}\frac{\alpha+r}{\alpha+\beta+r}\right)\frac{t^k}{k!}, \tag{2.53}$$

for $beta(\alpha, \beta)$ and the mgf of the arcsin distribution is:

$$\sum_{n=0}^{\infty}\left(\prod_{j=0}^{n-1}\frac{2j+1}{2j+2}\right)\frac{t^n}{n!}. \tag{2.54}$$

Obviously, neither (2.53) nor (2.54) can be computed by hand. For (2.53) Matlab's "hypergeom" function can be used, and there is no built–in function for (2.54). Instead, we can directly compute (2.50) by Matlab's "integral 2" function which is a more direct approach to the problem at hand.

Next we move on to the third expectation which is

$$\mathbb{E}\left[\mathbb{E}\left[\mathbb{E}\left[\exp\left(-\int_{T-\tau_1-\tau_2}^{T}[U-|U-r_s|]ds\middle|\tau_1\leq T,\tau_2\right)\mathbb{P}(\tau_2)\right]\right]\mathbb{P}(\tau_1\leq T)\right],$$

concentrating on the inner–most expectation:

$$\mathbb{E}\left[\exp\left(-\int_{T-\tau_1-\tau_2}^{T}[U-|U-r_s|]ds\middle|\tau_1\leq T,\tau_2\right)\mathbb{P}(\tau_2)\right],$$

can be written as:

$$\mathbb{E}\left[\exp(-U)\left[\exp\left(-\int_{T-\tau_1-\tau_2}^{T\tau_2}[-|U-r_s|]ds\middle|\tau_1\leq T,\tau_2\right)\mathbb{P}(\tau_2)\right]\right],$$

and letting $|U-r_s| = y$ and observing that $y \sim FN$, which is the folded normal distribution we can write the expectation as an integral:

$$\int_0^\infty \exp(-U)\exp\left(\int_{T-\tau_1-\tau_2}^{T} yds\right)f_y dy,$$
$$\exp(-U)\int_0^\infty \exp\left(\int_{T-\tau_1-\tau_2}^{T} yds\right)f_y dy,$$

where

$$f_y(y)=\frac{1}{\sigma\sqrt{2\pi}}\left[e^{\frac{-(x-\mu)^2}{2\sigma^2}}+e^{\frac{-(x+\mu)^2}{2\sigma^2}}\right],$$

and μ, σ are the parameters of the underlying normal distribution which in our case is

$$(r_s|r_0)\sim N\left(\beta+(r_0-\beta)e^{-\alpha t},\frac{\sigma^2}{2\alpha}\left(1-e^{-2\alpha t}\right)\right),$$

and the mean must be shifted by U as by the functional form of $|U - r_s|$ (see [7, 18]). The remaining outer integrals are more standard to evaluate numerically as in the previous steps since we already know $g(\tau_2)$ and $f(\tau_1)$.

All in all, Matlab's "integral 3" function is the appropriate way to compute the triple integral, which is extremely difficult to solve by hand.

The last integral in the closed form solution enters the formula additively. It represents the case, where the underlying Vasicek short rate $r(s)$ never hits the upper boundary until the end of T. In this case we have $\tau_1 \in (T, \infty)$ and the bond price corresponds to the closed form solution of the Vasicek bond prices. This last expectation can be written as the following integral:

$$\int_T^\infty \mathbb{E}\left[\exp\left(-\int_0^T r(s)\right)\right] f_{\tau_1} d\tau_1, \tag{2.55}$$

and the expectation inside the integral is simply $B(0,T)$ as given below:

$$B(0,T) = e^{-A(0,T)r_0 + D(0,T)}, \tag{2.56}$$

where

$$A(0,T) = \frac{1 - \exp(-\alpha T)}{\alpha},$$

and

$$D(0,T) = \left[\beta - \frac{\sigma^2}{2\alpha^2}\right](A(0,T) - T) - \frac{\sigma^2 (A(0,T))^2}{4\alpha},$$

so the integral becomes

$$\int_T^\infty B(0,T) f_{\tau_1} d\tau_1 = B(0,T)\int_T^\infty f_{\tau_1} d\tau_1, \tag{2.57}$$

and this is simply

$$B(0,T)\mathbb{P}(\tau_1 > T). \tag{2.58}$$

Both components can be computed numerically without much burden.

Now, as a simple extension, the version of the closed form solution ((2.47)) where only the lower bound is active will be presented by assuming that $r(s) \leq U$ until the maturity of the bond.

$$\begin{aligned} B(0,T) = {} & \mathbb{E}\left[\mathbb{E}\left[\exp\left(-\textstyle\int_0^{\tau_1} r(s)ds \middle| \tau_1 \leq T\right)\right]\mathbb{P}(\tau_1 \leq T)\right] \times \\ & \mathbb{E}\left[\mathbb{E}\left[\exp\left(-\textstyle\int_0^{\tau_2} L ds \middle| \tau_2\right)\mathbb{P}(\tau_2) \middle| \tau_1 \leq T\right)\right]\mathbb{P}(\tau_1 \leq T)\right] \times \\ & \mathbb{E}\left[\mathbb{E}\left[\mathbb{E}\left[\exp\left(-\textstyle\int_{T-\tau_1-\tau_2}^{T} [L + |L - r_s|] ds \middle| \tau_1 \leq T, \tau_2\right)(1 - \mathbb{P}(\tau_2))\right]\right]\right]\mathbb{P}(\tau_1 \leq T)\right] \\ & + \mathbb{E}\left[\exp\left(-\textstyle\int_0^T r(s)ds \middle| \tau_1 > T\right)\right]\mathbb{P}(\tau_1 > T), \end{aligned} \tag{2.59}$$

where now τ_1 is the first hitting time to the lower L and τ_2 represents the time the process would spend below L in the absence of central bank intervention.
In this case, the density of τ_1 remains the same, whereas one needs to use $[1 - \mathbb{P}(\tau_2)]$ because τ_2 represents the time the process spends above L.
It is crucial to conduct robustness tests in order to verify the validity of the analytic formula. There may be various alternatives to check robustness. One may start with the fit to real data.

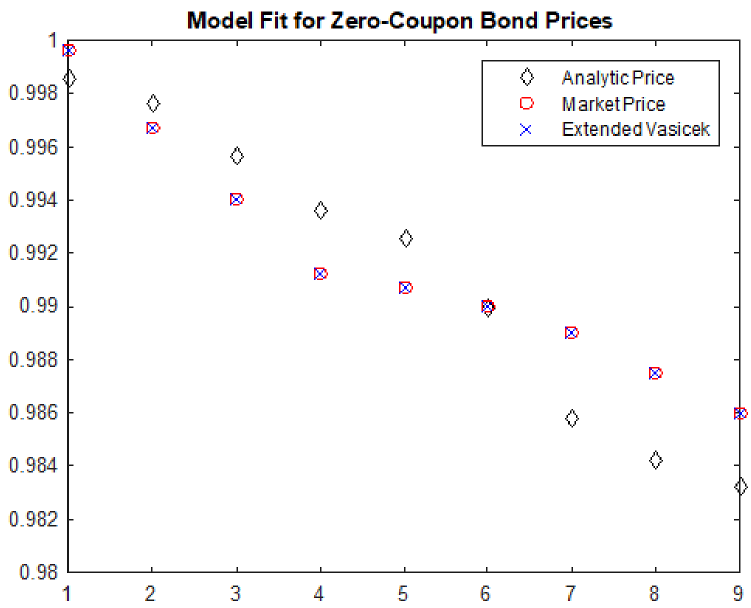


Figure 2.2: An Example of the Fit Performance

Figure 2.2 shows how good is the fit to real zero–coupon bond prices. The circles represent real values, whereas diamonds stand for the theoretical prices derived from the closed form solution proposed in (2.47). The crosses within the circles correspond to the extended Vasicek model, also called, Ho-Lee model ([20]). The crosses exactly match the real bond prices, in that the idea of [20] is to fit the initial term structure exactly by introducing a time–varying θ parameter. However, as time passes the fit to the new term structure generally fails.
This shows that it may be useful to estimate the time varying parameter θ implied by the Ho–Lee's approach. However, one still needs to compute the volatility parameter using the Kalman Filter since in the Ho–Lee model, volatility is not assumed to be time varying.

Another interesting robustness exercise is to check whether the formula converges to the closed from solution of the Vasicek bond price as U is increased. Figure 2.3 confirms this point, as U becomes larger and larger, the bond prices converge to the expected level and stay there.

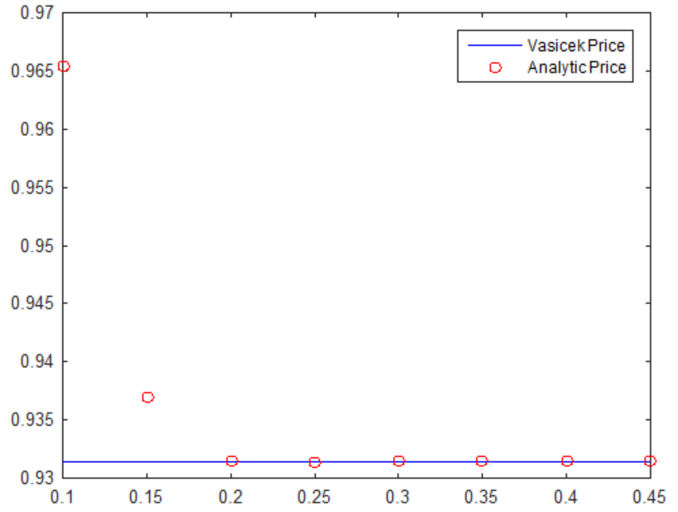


Figure 2.3: Convergence Results

The next analysis checks whether the theoretical prices remain within the confidence interval of a Monte Carlo exercise where the number of simulations increases gradually (Figure 2.4).

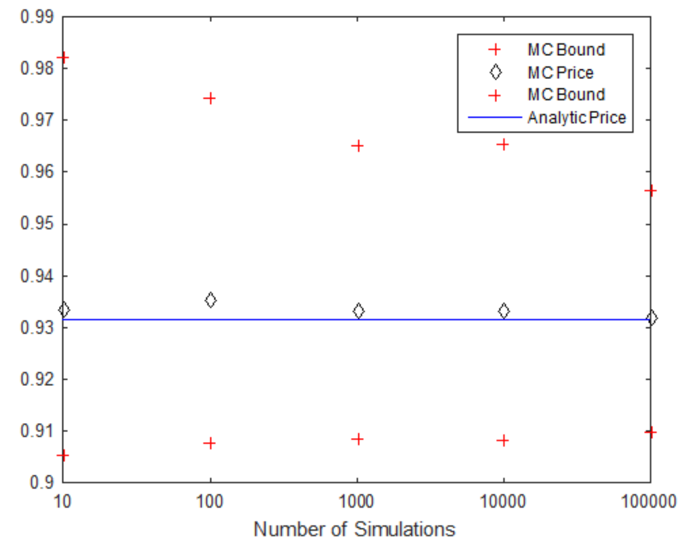


Figure 2.4: Monte Carlo Simulation

Evidently, the analytic prices fall within the confidence interval. All in all, the analytical price passes the robustness analysis hence reliability is demonstrated.

CHAPTER 3

DATA AND ESTIMATION RESULTS

We have specified 04 April 2011–31 August 2017 (1615 observations) as the data. In this period one–week repo rate has been specified as the policy rate. This whole period is characterized by an enlarging liquidity deficit around a negative linear trend. Indeed, the era of "liquidity surplus" ceased in the middle of 2008 (Figure 3.1).

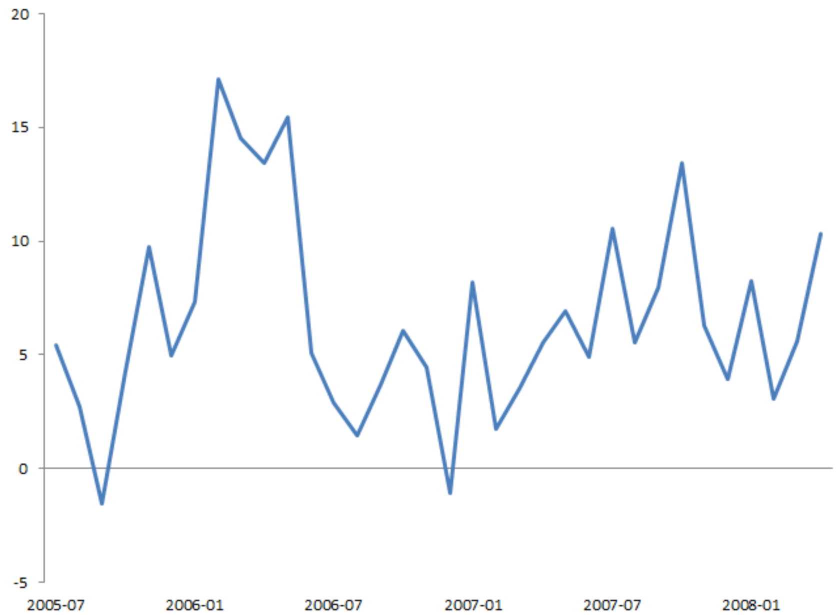


Figure 3.1: Banking Liquidity Deficit

The reason for the liquidity surplus was the 2001 banking and currency crisis, where state bank duty losses were assumed by the Under secretariat of Treasury. The Treasury injected capital to state banks, but in reality it funded this injection with a loan from the Central Bank of Turkey. However, this loan was provided in lieu of Treasury bonds put into the Open Market Operations (OMO) portfolio of the Central Bank. All of those bonds have matured by now.

Another source for excess liquidity was the direct intervention of the Central Bank in foreign exchange markets. The Turkish Central Bank's initial reaction to capital inflows in the aftermath of the 2001 crisis was purchasing foreign currency (FX) in order to stabilize Turkish lira (TL) and boost its reserves. There are a lot of pull and push factors for fast capital inflows. The most important push factor is the liquidity glut in the global economy. With retrospect, this is continuing until now. Over the sample period, but particularly following the 2008-2009 global financial crisis, all the emerging markets have benefited from the abundance of global liquidity and dovish monetary policy. Based on the Emerging Portfolio Fund Research (EPFR) data, foreign portfolio flows to emerging markets have increased exponentially (Figure 3.2).As a result of global liquidity, emerging bond markets also displayed a similar.

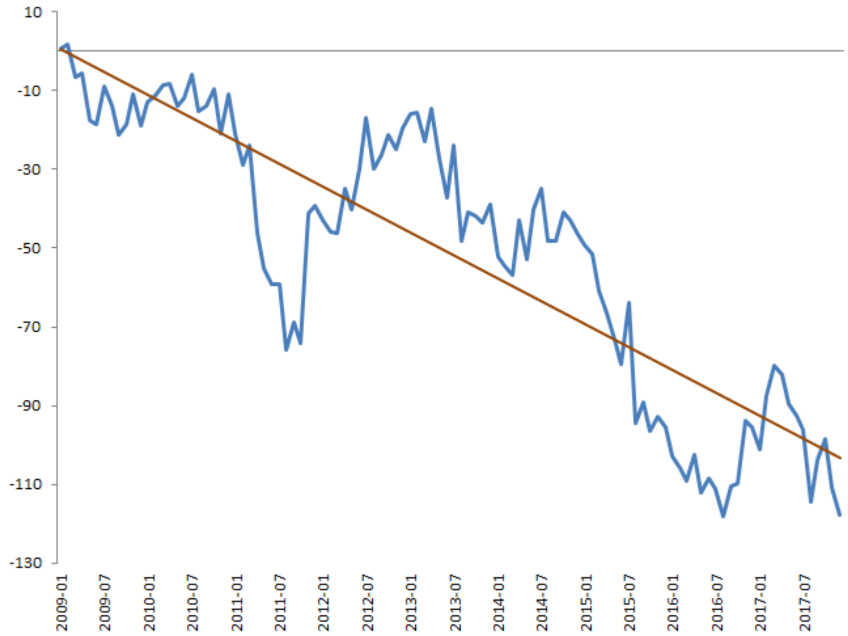
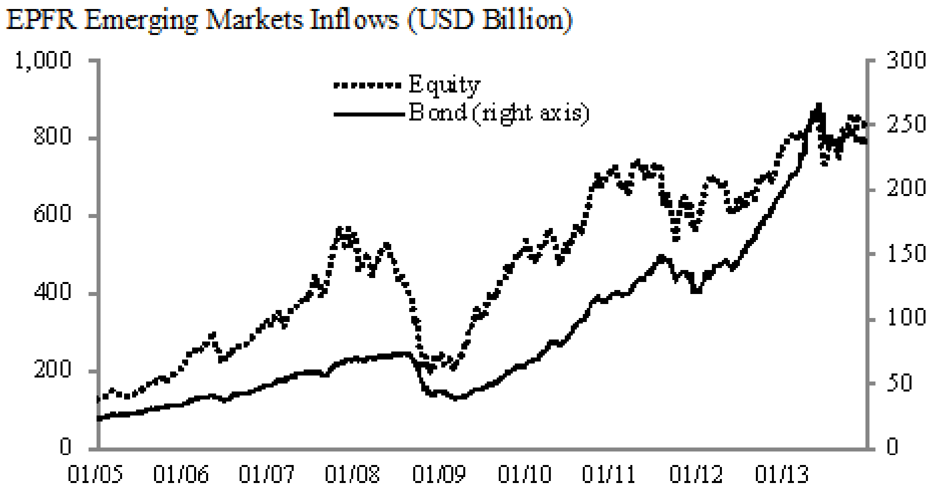


Figure 3.2: EPFR Data

Regarding the pull factors, over the sample period, Turkey has been granted investment grade by Fitch and Moody's making the country eligible for investment by large funds. The economy is growing fast and experienced disinflation especially during the 2003–2008 period, where capital inflows were especially strong. The following figure reveals the large liquidity surplus in the financial sector during July 2005–April

2008 arising as a result of the pull and push factors just mentioned (Figure 3.3) .



Source: Emerging Portfolio Fund Research (EPFR). The figure shows the cumulative fund flows into equity and bond markets in emerging market economies.

Figure 3.3: Banking Liquidity Surplus

On the other hand, the side effect of FX purchases was creating excess liquidity in the TL market. The disadvantage of this development was a less effective policy interest rate, which is the main policy tool for the Central Bank. Furthermore, strong portfolio inflows were associated with a fast credit growth, which was raising financial stability concerns and fueling an increasing current account deficit. In order to counteract this cycle, the Central Bank resorted to required reserve ratio (RRR) (see [4]). Central banks use reserve requirements to reach financial stability goal in the following ways, as [36] noted: They can increase the reserve requirements to curb credit growth in the boom - part of the business cycle, in the event of an economic downturn reduce reserve requirements in order to leverage reserve buffers accumulated during the boom, and encourage the banking sector to increase lending to non-financial corporations. Therefore, reserve requirements can be used as a counter-cyclical policy instrument to mitigate credit fluctuations in the financial sector and thus stabilize the real economy. The reserve requirements serve as a macroprudential tool since the end of 2010. In particular, the Central Bank has gradually increased the weighted average of RRR from 5% to 13% between October 2010 and April 2011 in such a way to encourage banks to extend the maturity of their liabilities.

In terms of FX interventions, the Central Bank used to make outright purchases until 15.02.2006. The total purchases from 12.05.2003 to that time have reached a total of almost 25 billion USD dollars (Table 3.1).

Table 3.1: FX interventions

Date	Intervention (Purchase)	Intervention (Sale)
12.05.2003	62	
21.05.2003	517	
09.06.2003	566	
18.07.2003	938	
10.09.2003	704	
25.09.2003	1442	
16.02.2004	1283	
27.01.2005	1347	
09.03.2005	2361	
03.06.2005	2056	
22.07.2005	2366	
04.10.2005	3271	
18.11.2005	3164	
15.02.2006	5441	
13.06.2006		494
23.06.2006		763
26.06.2006		848
18.10.2011		525
30.12.2011		1865
02.01.2012		525
03.01.2012		326
04.01.2012		155
23.01.2014		3151

Briefly, the Central Bank ceased purchasing FX and started to use RRR as a new tool. Yet, Turkey still faced strong capital inflows, which created appreciation bias for TL and jeopardizing financial stability. As a remedy, the Central Bank slashed the policy rate, which was then the overnight borrowing rate, to very low levels. The decisions of the CBRT about conduct of monetary policy have proven to be efficient within time. This is extremely important because wrong monetary policy decisions can be detrimental for the economy.

As discussed in [23] in general, it is assumed that the Bank of Japan (BoJ) made several mistakes that may have added and extended the negative effects of stock and real estate bubble bursts. Since the monetary policy was related to inflation and asset prices, BoJ began braking the money supply in the late 1980s, which may have contributed to the bursting of equity and real estate bubbles. Later when equity values fell BoJ continued to increase interest rates because real estate values were still rising. Higher interest rates helped to the fall of land prices, but at the same time helped the economy enter a downward trend. In 1991, when stock prices and land prices fell, the BoJ largely reversed the course and began to lower interest rates. But it was too late, a liquidity trap had already been identified, and a credit crunch followed.

During ensuing periods, capital inflows slowed down and the Central Bank now stood at the selling side of FX market, as evident from the table. The Bank sold circa 8 billion US dollars between 13.06.2006 and 23.01.2014. In the meantime, the liquidity surplus turned into a deficit. At those times, the policy rate first switched to overnight lending rate and then to weekly repo rate (Figure 3.4).

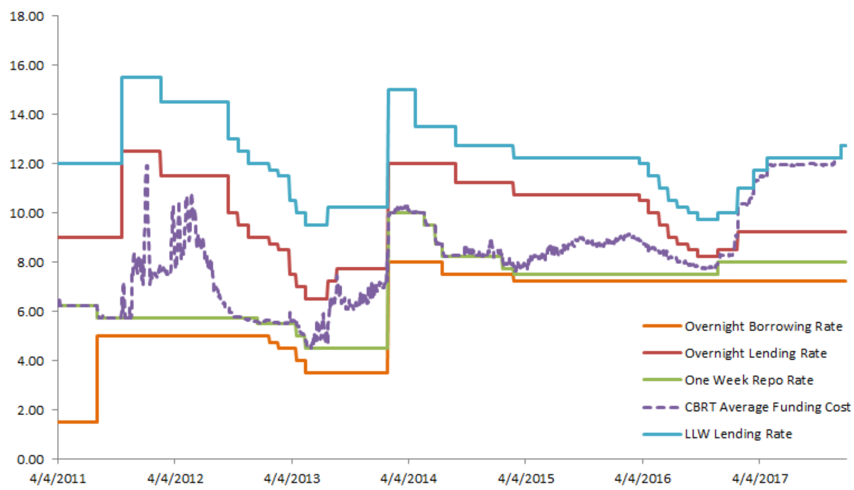


Figure 3.4: Corridor Data

3.4 displays the movement of the CBRT average funding cost from 4–4–2011 to 29–12–2017 on a daily basis. The CBRT average funding cost is a weighted average of interest rates at which the eligible parties borrow TL from the CBRT at different maturities and markets. From the beginning of the data period to mid-December 2016, the CBRT average funding cost was computed using the one week repo rate and overnight lending rate weighted by the outstanding amount of each maturity. From the end of 2016 to today (February 2018), the CBRT average funding cost is being computed using the one week repo rate and late liquidity window lending rate. However, while the computation is changing, the CBRT average funding cost always stays within some bands, i.e. in the interest rate corridor.

The active usage of the interest rate corridor is introduced subsequently. Before that the CBRT was applying conventional inflation targeting by controlling a single policy interest rate. The overnight borrowing rate was the policy rate at the times with liquidity surplus. When on the other hand, the overnight liquidity turned into negative, the overnight lending rate became the new policy rate until 2011.This era was marked by passive liquidity management.

Since the end of 2010, the CBRT has implemented a monetary policy strategy in which it is using more than one interest rate in a wide corridor, and specified the one-week repo rate as the new policy rate.

In this strategy, the funding composition (liquidity policy) becomes almost as important as the CBRT interest rates in terms of the stance of the monetary policy. In order to be able to correctly interpret the changes in the CBRT policy, it is important to first understand the practical functioning of the monetary policy. Thus, it will be possible to interpret things in a healthy way, such as how short-term interest rates are determined, the meaning of changes in fund composition and which interest rate is more important in terms of monetary transmission. The reason for this policy change was the huge global liquidity that forced many emerging market economies changing monetary policy frameworks for financial stability difficulties arising from volatile capital flows. In this context, CBRT is using unconventional policy instruments to complement financial stability to target price stability. Liquidity management is actively used with a wide interest rate corridor to correct extreme volatility in short-term credits due to large capital inflows. As a result, CBRT's average funding rate became more volatile.

The broad interest rate corridor is a tool developed by the CBRT to mitigate financial stability and price stability in the period of global volatility. After the global crisis, developing countries' markets have become extremely sensitive to global monetary policies. Sudden changes in capital flows and risk appetite can quickly affect domestic financial conditions, threatening macroeconomic stability by reducing predictability in the economy. This situation necessitated a tool that would enable timely response to sudden changes in global risk appetite and liquidity conditions. In this respect, the CBRT has designed and implemented a monetary policy strategy in which the broad interest rate corridor and active liquidity policy that have been used together since the end of 2010.

In a conventional corridor policy, while all short–term interest rates move in tandem, in the new policy implemented by the CBRT, different short-term interest rates within the corridor can move into different directions at differing magnitudes. Therefore, it may be confusing to define the stance of the monetary policy. The convention in this framework is to define the tightness of the monetary policy in terms of the average funding cost and/or the overnight interest rate coming about in the Borsa Istanbul (BIST) money market. The latter is the average interest rate at which banks are willing to lend each other for a maturity of up to seven days.

In countries under inflation targeting regime, short term interest rate is the main means of monetary policy. Although there are technically different applications between countries, central banks determine the short-term interest rate mainly by changing their balance sheet sizes. This can be achieved through the most direct way of buying and selling bonds, as well as by providing short-term debt (liquidity) to financial institutions in the way the CBRT does. At present, the CBRT does this through weekly and overnight repo transactions.

The net liquidity position of the market is important for the operational framework. Turkey faced liquidity surplus after the 2001 crisis during banks' balance sheet restructuring process, the resulting quantitative easing provided banking system a liquidity surplus. Since 2010, the central bank's funding rates have played a more important role in the monetary policy stance, since the CBT has been in a net lending position to the market, so that short term liquidity constraints become binding for the financial system, increasing the effectiveness of CBRT's policy rate. Hence, the operational framework described here is designed to reflect the situation.

As described before, average funding rate represents the weighted average interest rate of the short-term liquidity that the central bank gives to the market from various channels. During times when all the liquidity demand of the banks is provided through one week repo rate, BIST money market rates converge to the CBRT's policy rate, whereas, tighter liquidity policy, i.e. partial funding through overnight lending rate, leads the money market rates to be close to the overnight lending rate, which was the upper band of the liquidity corridor and this became late liquidity window lending rate after the end of 2016. It should be clear that the interest rates moving discretely on the above graph may be changed on the meetings of the Monetary Policy Committee and declared to the public after the decision. The decision is a result of an interaction of many macroeconomic and financial variables followed by the CBRT in the pursuit to fulfill its targets. On the other hand, the CBRT can also change the funding composition between MPC meetings, i.e., the share of the weekly repo in the total funding. In the traditional corridor application, once the policy rate is declared, central bank open market operations are set so that all short-term market interest rates are close to this level, so the funding composition assumes a passive role in terms of monetary policy. However, the funding composition of the CBRT since 2011 has a significant impact on the monetary policy stance. Short-term market interest rates can reach marginal funding rate under tight liquidity policy, while weekly repo interest rate is achieved when full funding is provided with weekly repo. Therefore, the tightness of the monetary policy can be changed flexibly without changing the announced policy rates. In sum, the interest rates determined by the CBRT on a monthly basis constitute the basic parameters of the monetary policy, while the funding composition and liquidity policy are important in terms of short-term actual policy stance. When the policy rates and the funding composition are taken as a whole, the average funding cost and the market interest rates may differ, so that one needs to be careful which one to follow in order to judge the monetary policy stance. The former is completely controlled by the CBRT, whereas the latter is affected my market players' actions, as well. However, CBRT has a strong influence on money market rates due to no-arbitrage constraints. As a result, all the basic short-term interest rates that may be important for monetary transmission can be represented by these two interest rates. Therefore, it can be said that the stance of the monetary policy will remain the same unless these two rates, which summarize the corridor and liquidity policy of the CBRT, remain unchanged. In sum, it is possible to say that if the policy stance of the CBRT is to be expressed with a single interest rate, this is somewhere between the BIST money market rate and the average funding rate of the CBRT. Both rates directly affect the funding costs of banks; they play an important role in the pricing of credit and deposit rates and in the monetary transmission mechanism.

For the purposes of the thesis, we are using the BIST money market rate as the underlying short interest rate process. Since our aim is to find a closed-form bond price solution in the one-factor Vasicek model, BIST money market rate is the factor designating the whole term structure of the interest rates. On the other hand, we would like to justify that the data displays mean-reversion as in the Vasicek model. For this purpose we employ the test proposed in [38]. The relevant regression result shows that we are rejecting the H_0 at 90% level of significance, because the relevant test statistics is 6.3155 and exceeds the critical value of 6.2103. Details are given in Table 3.2.

Table 3.2: Test Results

Variable	**Coefficient**	**Std. Error**	**t-Statistics**	**Prob.**
C	-0.046101	0.020246	-2.277009	0.0229
OVERNIGHT	0.006205	0.002469	2.513074	0.0121
R-Squared	0.003903	Mean dependent var		0.003569
Adjusted R-Squared	0.003285	S.D. dependent var		0.176668
S.E. of regression	0.176377	Akaike info criterion		-0.631145
Sum squared resid	50.14757	Schwarz criterion		-0.624471
Log likelihood	511.3343	Hannan-Quinn criter.		-0.628668
F-statistic	6.315540	Durbin-Watson stat.		1.703928
Prob (F-statistic)	0.012065			

The case of normality for the overnight interest rate cannot be established for the whole sample due to the spikes and jumps throughout the whole sample. For this we need to have a sample without jumps and spikes. The jumps are a result of the changes in at least one of the bands, whereas spikes are a short term and mean reverting deviation in interest rates in order to reduce the excess volatility in domestic currency. As an example, 2 January 2015-24 February 2015 is a tranquil period without spikes and the bands are constant. The following result shows that we fail to reject normality in this sample based on the Jarque-Bera test (Table 3.3).

Table 3.3: Normality Test Results

	OVERNIGHT
Mean	8.112368
Median	7.975000
Maximum	8.530000
Minimum	7.770000
Std. Dev.	0.254956
Skewness	0.339753
Kurtosis	1.567458
Jarque-Bera	3.980348
Probability	0.136672

We have done the same analysis for some other subsamples with the explained features and normality is the typical outcome of the test. So, normality and mean-reversion justify our approach to use the Vasicek approach as the short rate in our one factor model. Now, I would like to derive the link between the Vasicek interest rate model and $AR(1)$ process. This is necessary because having the data, we can simply estimate an $AR(1)$ model and retrieve the parameters of the Vasicek model:

$$dS_t = \theta(\mu - S_t)dt + \sigma dB_t,$$

and discretization yields

$$\begin{aligned}
S_t - S_{t-1} &= \theta\mu\Delta t + \theta S_t\Delta t + \sigma(B_t - B_{t-1}), \\
S_t(1-\theta(\Delta t)) &= \theta\mu\Delta t + S_{t-1} + \sigma(B_t - B_{t-1}), \\
S_t(1-\theta(\Delta t)) &= \theta\mu\Delta t + S_{t-1} + \sigma\sqrt{t}\epsilon_t, \quad \epsilon_t \sim N(0,1), \\
S_t &= \tfrac{\theta\mu\Delta t}{1+\theta\Delta t} + \tfrac{1}{1+\theta\Delta t} + u_t, \quad u_t \sim N(0,1), \\
S_t &= \alpha + \beta S_{t-1} + u_t. \qquad (3.1)
\end{aligned}$$

For stability $|\beta| < 1$ hence

$$-1 < \frac{1}{1+\theta\Delta t} < 1,$$

implies that $\sigma\Delta t < 1$ so that $\theta < 0$ must hold since Δt is always positive.

For daily data we can take $\Delta t = 1$ so that

$$S_t = \frac{\theta\mu}{1+\theta} + \frac{1}{1+\theta} + u_t.$$

Now, we can revert to the original parameters as follows:

$$\beta = \frac{1}{1+\theta} = \frac{1}{\beta} = 1 + \theta = \theta = \frac{1}{\beta} - 1 = \theta = \frac{1-\beta}{\beta}, \tag{3.2}$$

and

$$\alpha = \tfrac{\theta\mu}{1+\theta} = \theta\mu\beta = \tfrac{1-\beta}{\beta}\mu\beta,$$
$$\mu = \tfrac{\alpha}{1-\beta}. \tag{3.3}$$

So upon estimating α and β we can revert back to the original variables in the mean od the SDE. As for the variance, the standard error of the regression divided by $\sqrt{t}$ will provide the instantaneous σ which is assumed to be constant.

We have the following estimations for the intercept (C) and the slope $(AOFF(-1))$. Results are portrayed in Table 3.4.

Table 3.4: Parameter Estimation

Variable	**Coefficient**	**Std. Error**	**t-Statistics**	**Prob.**
C	0.025356	0.018139	1.397819	0.1624
AOFF(-1)	0.997376	0.002151	463.7013	0.0000
R-Squared	0.992174	Mean dependent var		8.208881
Adjusted R-Squared	0.992169	S.D. dependent var		1.952209
S.E. of regression	0.172752	Akaike info criterion		-0.672743
Sum squared resid	50.61411	Schwarz criterion		-0.666339
Log likelihood	573.1589	Hannan-Quinn criter.		-0.670372
F-statistic	215018.9	Durbin-Watson stat.		1.706747
Prob (F-statistic)	0.000000			

Following this derivation, we restore the model parameters for the whole sample:

$$\alpha = 0.002631, \beta = 9.66311, \sigma = 0.22533.$$

Alternatively we have run a Kalman filter code (see appendix A) to compute time varying parameters. As could be followed from appendix C4 although the parameters are varying, they are stable around the OLS estimates presented above.

The parameters reveal that the policy rate converges to the long-term mean of 9.66, by adjusting a one unit of deviation from that mean at a rate of 0.0026 percentage points on a daily basis.

Figure 3.5 pertains to the Monte Carlo simulation of the general case. In this approach both the upper and the lower band are active. In other words, the Central Bank does not allow for the money market interest rates to exceed the upper band and fall below the lower band using liquidity operations. First of all, we would like to present the movement of the interest rates when U and L are constant throughout the maturity of the bond (month $= 10$ in this case). Maturity in terms of days (the x-axis values) are produces assuming 22 working days in a month. The parameters used are:

$$U = 0.1375; \text{upper band}$$
$$L = 0.0575; \text{lower band}$$
$$r_0 = 0.08; \text{initial level of the interest rate}$$
$$\beta = 0.1; \text{long–term interest rate}$$
$$\alpha = 0.2; \text{mean–reversion parameter}$$
$$\sigma = 0.1; \text{volatility of the interest rates}$$
$$\text{month} = 10; \text{maturity of the bond in terms of months}$$

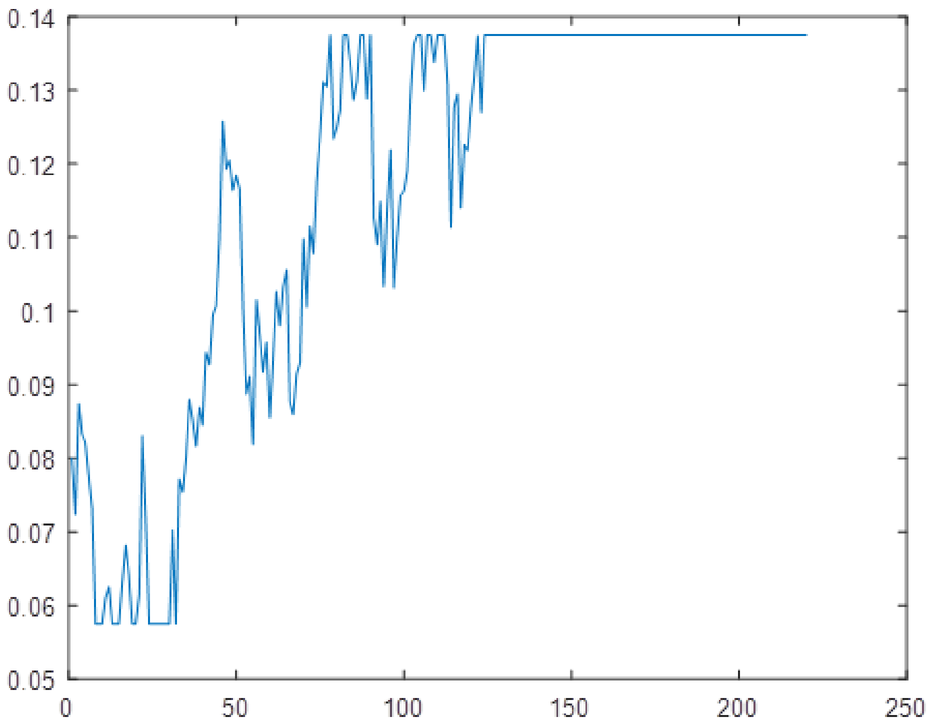


Figure 3.5: Sample Path (Fix)

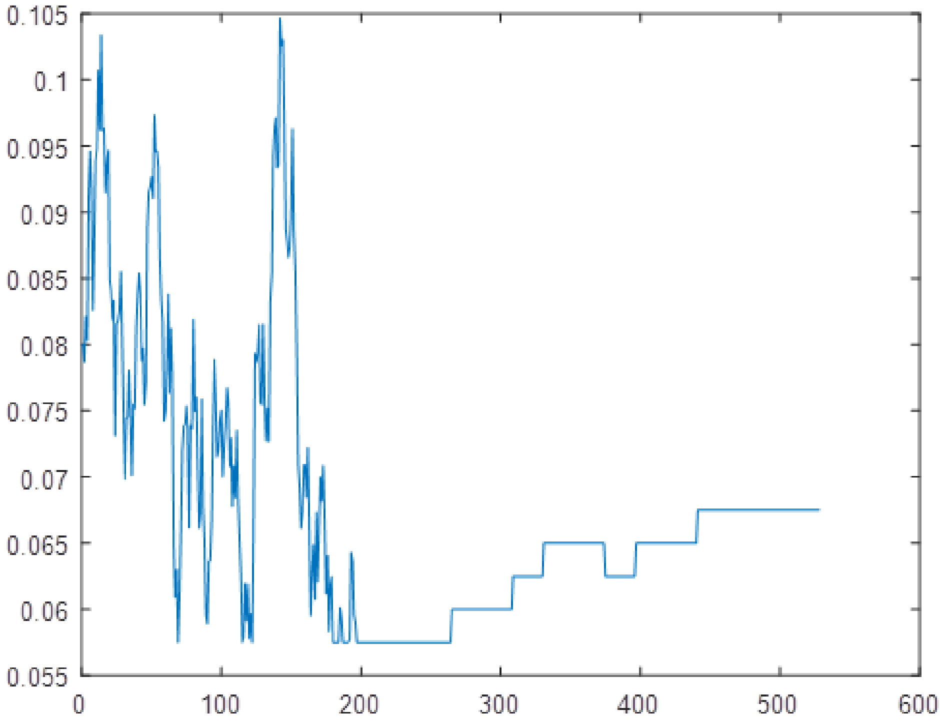


Figure 3.6: Sample Path (Flexible)

Figure 3.6, on the other hand, displays a sample path when both the upper and lower bands are flexible. In the following figure we have extended the maturity to 24 months. The figure is quite interesting, where for almost first 10 months, the short-rate never hits a barrier, but then it moves in a discrete manner equal to the lower band. This means that, without Central Bank intervention, the actual money market rate would be lower than the lower bands that have been updated six times until maturity. In other words, in this sample path the MPC so to say has adjusted the interest rates six times in the last 14 months. In this simulation all the remaining parameters except the maturity are the same. However, now U and L are only the initial values for the upper and lower bands, respectively. With some positive probability, both the upper and lower band can increase and decrease by 25 basis points simultaneously at the end of the month, i.e. at the assumed MPC days. In this example we have fixed the changes in the interest rates as 25 basis points and we are adjusting both bands simultaneously by the same amount. Furthermore, the current changes in interest rates are independent of the previous rate hike or cuts. These assumptions can be enriched easily, however for the purposes of bond pricing we doubt that a more complex movement in the policy rate would have a material impact on the price of the bond produced by the Monte Carlo simulation.

CHAPTER 4

CONCLUSION

Application of a unit root test to overnight interest rate in the Turkish money market verifies mean–reversion of the Turkish short rates. Mean reversion, conjoined with normality justifies our approach to model the underlying short-rate process as Vasicek.

After putting this, we derive the closed form solution for the case of the underlying Vasicek model restricted below the upper bound U. Specifically, we exploit the Markov property of the continuous Vasicek model. For bond pricing, the key observation is that the level of the interest rate at a specific time is not important. Instead, we divide the domain of the interest rate into three mutually exclusive subsets. In the first set, the interest rate starts at a level below U and ends up at U after some random time. This random time, i.e. the first hitting time, may be larger than the maturity T of the bond. In this eventuality, the model turns into the plain Vasicek model. However, for the case of the first hitting time being below T, we can divide the remaining time until T into two further subsets.

The first of such subset entails the total time the process would spend above U, in the absence of central bank policy. The distribution of this time is known and the interest rate in this subset simply equals U. The last subset accounts for the remaining time. In this subset the interest rate is always below U, and there we model the short rate appropriately.

On top of these, we make bond pricing for the case of an active upper and lower band and changing levels of the bands using simulation techniques. This way, we are able to price bonds of longer maturity and under a more realistic alternative of non-constant bands.

For further studies alternative short–rate models, possibly with non–normal distributions can be used. For instance, researchers may apply the CIR model in case the short rate can be fitted as a chi-squared distribution. Our approach extends to this case, as well. However, there is no closed form solution for the first hitting time and it can be approximated as a series.

We should not forget that ours is a one factor model and increasing the number of factors would probably improve the fit of the model. However, increasing the number of factors comes with the cost of losing the analytical solution.

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APPENDIX A

MATLAB CODES

```
%-------------------------------------------------------
general_case.m

%This code gives the bond price
for the general case,
where both the uppper
and lower bands exist,
unlike the analytical solution
 where only one band was active.
Clearly, r_0 and beta must
lie within the bands,
in order the model to make sense.
Furthermore, alpha>0 must hold
for mean-reversion to take place.
The outputs are: the bond
price for the case
with two bands; the bond price
for the case
with no bands,
i.e., the Vasicek bond price and the
percentage difference between
both prices.
%------------------------------------------------------
```

```
clearvars; clc


%Set the parameters of the model
U_init=0.1375;
L_init=0.0575;
r_0=0.08;
beta=0.1;
alpha=1;
sigma=0.05;
month=24;
MPC=month-1;

xx=rand(MPC,1);
%The number of paths I have set as
%roughly the number of trading days
%whatever the value T is.
path=22*month;
kk=1:22:path+1;
kk(end)=path;
kk=kk(2:end);

U(1:22)=U_init;
L(1:22)=L_init;

for hh=1:MPC

if xx(hh)>0.90
U(kk(hh):kk(hh+1))=U(kk(hh)-1)+...
0.0025
L(kk(hh):kk(hh+1))=L(kk(hh)-1)+...
0.0025
elseif  xx(hh)<0.10
```

```
U(kk(hh):kk(hh+1))=U(kk(hh)-1)-...
0.0025
L(kk(hh):kk(hh+1))=L(kk(hh)-1)-...
0.0025
else
U(kk(hh):kk(hh+1))=U(kk(hh)-1)
L(kk(hh):kk(hh+1))=L(kk(hh)-1)
end
end

%Number of simulations may vary
num_simul=1000;

T=month/12;
dt=T/path;

r(1)=r_0;
R(1)=r(1);


for i=2:path

r(i)=r(i-1)-alpha*(r(i-1)-beta)*...
dt+sigma*randn/10;
R(i)=min(max(L(i),r(i)),U(i));

end

discount_R(j)=exp(-T*sum(R)/path);
discount_r(j)=exp(-T*sum(r)/path);

end
```

```
B_T_R=sum(discount_R)/num_simul
B_T_r=sum(discount_r)/num_simul
difference= 100*[(B_T_R/B_T_r)-1]

%------------------------------------------------------
%------------------------------------------------------
oujd_simulation.m

%This code gives the trajectories
 of a Vasicek process
with and/or without jumps.
Although the model in the
thesis does not incorporate jumps,
  simulations
can be enriched by adding a jump
diffusion case,
in case of an extension.
A closed form solution with jumps
is extremely
challenging and simulation might be
the only option
to price a bond, where the interest
rates moving in
a band exhibit jumps.

%-----------------------------------------------------
function X =
oujd_simulation(ntraj, T, x0, P)
%OUJD_SIMULATION Simulate
trajectories of a OUJD model.
%   X = OUJD_SIMULATION (NTRAJ,T,X0,P)
 returns a vector
of NTRAJ realizations of the
```

```
Ornstein Uhlenbeck
Jump Diffusion (OUJD) process:
% dX = (alpha - beta*X)*dt +
 sigma*dB
+ N(mu,gamma)*dN(lambda)
% over a time period 0,1,...,T
and an initial value X0.
%   The timestep dt is set to 1.
%   The Euler scheme is used.
%   P = [ALPHA,BETA,SIGMA,MU,GAMMA,LAMBDA]
is the parameter vector.
%
%   Sample use:
% r = oujd_simulation
(10,1000,5,[.5,.1,.2,4,1,.01]);
% plot(r')

% Initialize output matrix
X = zeros(ntraj,T+1);
X(:,1) = repmat(x0,ntraj,1);

% Diffusion (normal) noise
r = randn(ntraj,T);

% Jump occurences
rjump = rand(ntraj,T);

alpha = P(1); beta = P(2);
sigma = P(3);
mu = P(4); gamma = P(5);
lambda = P(6);

for j=1:ntraj
```

```
for i=1:T
if rjump(j,i)>lambda
%No jump
X(j,i+1) = X(j,i) + alpha -...
beta*X(j,i)...
+ sigma*r(j,i);
else
% Jump
X(j,i+1) = X(j,i) + alpha -...
 beta*X(j,i) + mu...
+ sqrt(gamma^2+sigma^2)*r(j,i);
end
end
end

%---------------------------------------------
%---------------------------------------------
DF_Stats.m

%This code computes the asymptotic critical values
for the famous Dickey-Fuller test. The test checks
whether a time-series has a unit-root. It has three
versions: no mean and no trend; with mean and
no trend and lastly with mean and trend.
The code computes each version. Users can choose
desired critical levels.

%-----------------------------------------------
% Dickey Fuller Asymptotic Critical
values simulation
clear all
close all
clc
```

```
T=1;
N=1000;
N2=10000;
dt=T/N;
num=zeros(1,N2);
den=zeros(1,N2);
df_1=zeros(1,N2);
df_2=zeros(1,N2);
df_3=zeros(1,N2);
for k=1:N2
dW = zeros(1,N);
W = zeros(1,N);
s=zeros(1,N);
W_t = zeros(1,N);
% Z=zeros(1,N);
% Y= zeros(1,N);
%W(1) = dW(1);
s(1)=dt;

for j = 2:N
%dW(j) = sqrt(dt)*randn;
W(j) = W(j-1)+dW(j);
s(j) = j*dt;
end
W_m=W-mean(W);

for j = 1:N
%dW(j) = sqrt(dt)*randn;
W_t(j) = W(j)-(4-6*s(j))*mean(W)-(12*s(j)-6)
*mean(s.*W);
End

%W_t=W-(4-6*mean(s))*mean(W)-(12*mean(s)-6)
```

```
*mean(s.*W);

df_1(k)=(W(end)^2-W(1)^2-1)/(2*sqrt(mean(W.^2)));
df_2(k)=(W_m(end)^2-W_m(1)^2-1)/...
(2*sqrt(mean(W_m.^2)));
df_3(k)=(W_t(end)^2-W_t(1)^2-1)/...
(2*sqrt(mean(W_t.^2)));
end
% for i=1:T/N;
% W_m2=W-(4-6*i)*mean(W)-(12*i-6)*mean(W)*dt
% end
% for j=2:N
% Z(j) = (W_m(j-1));
% Y(j) = (W_m(j-1)) * dW(j);
% end
%
% Z = cumsum(Z);
% Y = cumsum(Y);
% den(k)=sqrt(dt*Z(N));
% num(k)= Y(N);
% df(k)=num(k)/den(k);
% end
%plot(W)
dfcritical_1=sort(df_1);
dfcritical_2=sort(df_2);
dfcritical_3=sort(df_3);
%plot(df)
%plot(dfcritical);
nbins=100;

hist(dfcritical_1,nbins)
figure
hist(dfcritical_2,nbins)
```

```
figure
hist(dfcritical_3,nbins)

% Critical values
alpha_1=dfcritical_1(0.1*N2)
alpha_2=dfcritical_2(0.1*N2)
alpha_3=dfcritical_3(0.1*N2)

%alpha2=dfcritical(0.025*N2);
%alpha3=dfcritical(0.05*N2)
%alpha=[alpha1,alpha2,alpha3]
%---------------------------------------------
%---------------------------------------------
```

OU_Stats.m

```
%This code computes the asymptotic
critical values for the OU mean
reversion test statistics. The test
checks whether a time-series is mean
reverting. It has one version, where the
series has no mean and no trend.
Users can choose desired critical levels.
%--------------------------------------------
% OU Asymptotic Critical values simulation

clear all
close all
clc
T=1;
N=100;
N2=1000;
dt=T/N;
```

```
df_1=zeros(1,N2);
for jj=1:100
for k=1:N2
dW = zeros(1,N);
W = zeros(1,N);

for j = 2:N
W(j) = W(j-1)+dW(j);
end

num=(0.5*(W(end)^2-W(1)^2-1)-...
mean(W)*W(end))^2;
den=(mean(W.^2))-(mean(W))^2;

df_1(k)=W(end)^2+num/den;
%df_1(k)=num/den

end

dfcritical_1=sort(df_1);
%randn(N,1)
% dW;
nbins=100;
%  for k=1:nbins
%   plot(W)
%   hold on
%  end

%  hist(dfcritical_1,nbins)
%  set(gca,'YTickLabel',{})
% % Critical values
alpha_1(jj)=dfcritical_1(0.9*N2);
alpha_2(jj)=dfcritical_1(0.95*N2);
```

```
alpha_3(jj)=dfcritical_1(0.99*N2) ;

end
%alpha2=dfcritical(0.025*N2);
%alpha3=dfcritical(0.05*N2)
%alpha=[alpha1,alpha2,alpha3]
median(alpha_1)
median(alpha_2)
median(alpha_3)

%------------------------------------------
error_type1_ard.m

%This code computes the Type-1 errors committed
by the OU mean reversion test.
As a yardstick, it compares
results with those of the Dickey-Fuller test statistics.
%---------------------------------------------
clear all; clc

sigma=2:0.1:4;

for jj=1:length(sigma)
npaths=100;

zz=randn(npaths,101);
r = 10+cumsum(zz,2);
t=1:size(r,2)-1;
c=ones(length(t),1);

x=zeros(size(r,1),1);
beta=zeros(size(r,1),1);
z=zeros(size(r,1),1);
```

```
%null hypothesis of a unit root
x(i)= adftest(r(i,:),'model','ARD',
'alpha',0.01);
%x(i)= adftest(r(i,:),'model','TS');
%the H_0 is that the series has unit root
%If a series is not
%growing, 'AR' and 'ARD'
models provide reasonable stationary
%alternatives to a unit-root process
 without drift.
The 'ARD' alternative has mean c/(1-a);
the 'AR' alternative has mean 0.
%'TS' means (trend stationary.
yy=(r(i,2:end)-r(i,1:end-1))';
xx=r(i,1:end-1)';

b=regress(yy,[c xx]);
theta(i)=-b(2);

XXX=fitlm(xx,yy);

std_err=XXX.CoefficientCovariance ;
s_e(i)=sqrt(std_err(2,2));
stat(i)=(theta(i)/s_e(i))^2;

%the critical values of asymptotic
 distribution of the null hypothesis
  of no mean reversion are:
6.2103, 8.1634 and 11.1673 at the
10%, 5% and 1% level of
%significance
```

```
if stat(i)>11.1673
z(i)=1;
else
z(i)=0;
end
DF_type1(jj)=sum(x,1);
OU_type1(jj)=sum(z,1);
end

end

DF_result=100*sum(DF_type1,2)/...
(npaths*length(DF_type1))
OU_result=100*sum(OU_type1,2)/...
(npaths*length(OU_type1))

%---------------------------------------------
error_type2_ard.m

%This code computes the Type-2 errors committed
by the OU mean reversion test . As a yardstick
, it compares the results with those of the Dickey-Fuller
test statistics.
%-----------------------------------------------

clear all; clc

sigma=2:0.1:4;
x_0=8.0:0.25:12;
npaths=10;

for jj=1:length(sigma)
for kk=1:length(x_0)
```

```
r = mrjd_sim2(npaths,100,x_0(kk),
[1,0.1,sigma(jj),0,0,0]);
t=1:size(r,2)-1;
c=ones(length(t),1);

x=zeros(size(r,1),1);
beta=zeros(size(r,1),1);
z=zeros(size(r,1),1);

%null hypothesis of a unit root
x(i)= adftest(r(i,:),'model','ARD',
'alpha',0.01);
%x(i)= adftest(r(i,:),'model','TS');
%the H_0 is that the series
has unit root.
%If a series is not
%growing, 'AR' and 'ARD' models
provide reasonable stationary
%alternatives to a unit-root process
without drift.
The 'ARD' alternative has mean c/(1-a);
the 'AR' alternative has mean 0.
% 'TS' means (trend stationary.
yy=(r(i,2:end)-r(i,1:end-1))';
xx=r(i,1:end-1)';

b=regress(yy,[c xx]);
theta(i)=-b(2);

XXX=fitlm(xx,yy);

std_err=XXX.CoefficientCovariance ;
```

```
s_e(i)=sqrt(std_err(2,2));
stat(i)=(theta(i)/s_e(i))^2;

%the critical values of
asymptotic distribution
of the null hypothesis of
%no mean reversion are:
6.2103, 8.1634 and 11.1673 at the
10%, 5% and 1% level of
%significance

if stat(i)>12.8488
z(i)=1;
else
z(i)=0;
end

DF_type2(jj,kk)=sum(x,1);
OU_type2(jj,kk)=sum(z,1);

end


end
end

DF_result=100*sum(sum(DF_type2,1),2)/
(size(DF_type2,1)*...
size(DF_type2,2)*npaths)
OU_result=100*...
sum(sum(OU_type2,1),2)/
(size(OU_type2,1)*size(OU_type2,2)*npaths)
%-------------------------------------------
%--------------------------------------------
```

integrals.m

```
%This code actually computes the
bond price for differing maturities
for the Vasicek model.
It must be carefully calibrated
due to numerical issues.
The calibration in this very code
comes from the
estimation of the model parameters,
and an initial
state of r_0, which is below beta
(and necessarily U).

%---------------------------------------------
%Part 1
%---------------------------------------------
clear all;
close all;
clc

alpha=0.263;
beta=0.0966311;
sigma=0.0419;
U=0.1275;
%set r_0 below beta (and U)
r_0=0.09;
T=10;

% m=beta/sigma;
% lambda=U^2;
% nu=sqrt(lambda)/m;
```

```
a=U;
c=alpha+(beta-r_0)/sigma;
%c=exp(-alpha*T)*beta/sigma

%A_s=sym(A);
D= @(x)(beta-sigma^2/(2*alpha^2))*...
((1-exp(-alpha*x))/alpha-x)
-(sigma^2*((1-exp(-alpha*x))/alpha).^2)/...
(4*alpha);
%D_s=sym(D);

%f = @(x) sqrt(lambda./(2*pi*x.^3)).*...
exp(-lambda*(x-nu).^2.
/(2*nu^2.*x));
%f=  @(x) (U./sqrt(2*pi.*(x.^3)))*...
exp(U*beta).*
exp(-0.5*((U./sqrt(x)).^2)
+x.*(beta)^2);

f = @(x) sqrt(a^2./(2*sigma^2*pi*x.^3))

H= @(x) exp((beta-sigma^2/(2*alpha^2))*...
((1-exp(-alpha*x))/alpha-x)-...
(sigma^2*((1-exp(-alpha*x))/alpha).^2)/...
(4*alpha)-(1-exp(-alpha*x))/alpha*r_0)...
.*sqrt(a^2./(2*sigma^2*pi*x.^3)).*...
exp(-((a-c*x).^2./(2*sigma^2*x)));

I = integral(H,0,T);

%---------------------------------------
% %Part 2
%---------------------------------------
```

```
p=(1-normcdf(U,beta,
sqrt(sigma^2/(2*alpha))));
%(1/U^2)*[exp(U*T)-1]-exp(-U*T)/
(2*U^2)*[exp(2*U*T)-1]
% zzz= @(x,y) exp(-U*(y-x));

% ymax = @(x) T-x;
% fun = @(x,y) [exp(-U*(y-x))].*...
[(betapdf(y/T,1-p,p))/T]...
%.*sqrt(a^2./(2*sigma^2*pi*x.^3)).*
exp(-((a-c*x).^2./(2*sigma^2*x)));
% yyy = @(y) (betapdf(y/T,1-p,p));
% II = integral2(fun,0,T,0,ymax,
'RelTol',1e-3);
% deneme=integral(yyy,0,T);
II=exp(-U*p*T);
%-----------------------------------------
% %Part 3
%-----------------------------------------
% ymin = @(x) x;
% fun = @(x,y,z) exp(-U-abs(U-z)).*
normpdf(beta,sigma^2/(2*alpha))....
%     *sqrt(a^2./(2*sigma^2*pi*x.^3)).*
exp(-((a-c*x).^2./(2*sigma^2*x)))...
% .*(x+(T-x).*betapdf(y,p,1-p));
% III=integral3(fun,0,T,ymin,T,-inf,inf);
%G= @(x) exp((T*(1-p))*
(beta-sigma^2/(2*alpha^2))
% (sigma^2*((1-exp(-alpha*x))/alpha).^2)/
(4*alpha)-(1-exp(-alpha*x))/alpha*r_0)
%III=integral(G,0,(1-p)*T)
%here, we can approximate the
above integral as follows:
```

```
III=exp(-beta*(1-p)*T);
%-------------------------------------------
% %Part 4
%-------------------------------------------
B= exp((beta-sigma^2/(2*alpha^2))*...
((1-exp(-alpha*T))/alpha-T)-...
(sigma^2*((1-exp(-alpha*T))/alpha).^2)/
(4*alpha)-(1-exp(-alpha*T))/alpha*r_0);
M = integral(f,T,inf);

IV=B*M;
%---------------------------------------
% %All Together
%---------------------------------------

Price=I*II*III+IV

%----------------------------------------
Vasicek_Bond.m

%This function computes the bond pric
e for differing maturities for the
Vasicek model. It first finds the
bond prices for the closed form
solution; then for the SDE and finally
for the simulation. At the end
a graph is produced showing that
all the solutions coincide.
%----------------------------------------

function Vasicek_Bond(alpha,beta,
sigma,r_0)
```

```
if nargin<4
r_0=.07;
alpha=.3;
beta=.08;
sigma=.01;
end

step=1; % Yearly frequency
MM=[0:step:10]'; % Maturity matrix

%% Analytical solution
B=(1-exp(-alpha*MM))/alpha;
A=(beta-sigma^2/(2*alpha^2))*...
(B-MM)-(sigma^2*B.^2)/(4*alpha);
P=exp(A-B*r_0);

%% ODE solution
function dy=vasicek_ode(t,y)
dy(1,1)=(1/2)*sigma^2*y(2)^2...
-beta*alpha*y(2);
end

[~,y]=ode45(@vasicek_ode,
MM,[0 0]);
A_ode=y(:,1); B_ode=y(:,2);
P_ode=exp(A_ode-B_ode*r_0);

%% Simulation solution
delta_t=step/50;
no_sim=1000; no_per=max(MM)/delta_t;
rsim=zeros(no_sim,no_per);
drsim=zeros(no_sim,no_per);
rsim(:,1)=r;
```

```
for j=2:no_per
dW=randn(no_sim,1);
drsim(:,j)=alpha*(beta-rsim(:,j-1))*...
delta_t+sigma*sqrt(delta_t)*dW;
rsim(:,j)=rsim(:,j-1)+drsim(:,j);
end
P_sim=ones(length(MM),1);
for i=2:length(MM)
P_sim(i)=mean(exp(-delta_t*...
sum(rsim(:,1:MM(i)/delta_t),2)));
end

%% Plots
figure
plot(MM,P,'b',MM,P_ode,'r o',MM,
P_sim,'g *')
legend('Analytical Price',
'ODE Price','Simulation Price')
xlabel('Maturity')
ylabel('Price')
end
%---------------------------------------

%---------------------------------------
OU_AR.m

%This code estimates an AR(1)
 regression, and calculates
 the model parameters via
the transformation presented
 in the thesis. This way,
 we can extract the mean-reversion
rate, the long-run mean and
```

```
 the volatility of the
interest rate model, that will
 be further used
in calculating the closed-form solution.
%----------------------------------------

cdata=xlsread('forecast_data.xlsx',
'Sheet1','B3:C3615');

%the SDE is of the form: dS=
 theta(mu-S)*dt+sigma*dW
y=cdata(2:end,2);
y_L=cdata(1:end-1,2);
c=ones(size(y,1),1);
x=[c y_L];

[B,a1,a2,a3,stats] = regress(z,x);

%here we revert back to
the original variables
theta= (1-B(2))/B(2);
sigma=stats(4)/sqrt(size(y,1));

%-------------------------------------------
fit_graph.m
%This code calculates the
fitted bond prices
and plots them together
with the bond prices
realized in the market,
with a maturity close to the
%--------------------------------------------
```

```
cdata=xlsread('forecast_data.xlsx',
'Sheet1','B3:C3615');

%the SDE is of the form:
dS= theta(mu-S)*dt+sigma*dW
y=cdata(2:end,2);
y_L=cdata(1:end-1,2);
c=ones(size(y,1),1);
x=[c y_L];

[B,a1,a2,a3,stats] = regress(y,x);

B=[B(1)/100,B(2)/100,
stats(4)/10000,105/252];

alpha= (1-B(2))/B(2);
sigma=B(3)/sqrt(B(4));
r_0=0.08;
U=0.1275;

market=[0.992,0.984,0.989,
0.9826,0.981];
T=[28/365,29/365,27/365,
31/365,33/365];

for i=1:length(market)
price(i)= int1.m;
end

plot(price,'x')
hold on
plot(market,'o')
```

```
axis tight

set(gca,'xtick',1:1:length(market))

%--------------------------------------------
%--------------------------------------------
TVP_Koridor.m: This code is
used to calculate the intercept
and the slope of the underlying
Vasicek model written in discrete
time. The parameters found here
can then be used to compute the
original parameters in the
continuous case.
The code provides confidence intervals,
as well.
%--------------------------------------------
clear all;close all;clc
% a(t)=T*a(t-1)+c(t)+R*w(t)
% y(t)=Z(t)a(t)+e(t)
% E[w(t)w(t)']=Q
% E[e(t)e(t)']=H
% E[w(t)e(t)']=0
% Notation of Harvey (1990)

echo on

echo off

[ndata,list,alldata] =
xlsread('data.xlsx');
date=ndata(:,1);
mhith=ndata(:,2);
```

```
a=ndata(:,3);
b=ndata(:,4);

% sample period of April 2011
to August 2011

k=2; % number of parameters to
be estimated (no constant case)
y=mhith;
x=[a b];
n=size(y,1);
T=eye(k); % Parameters follow
a random walk process

% T=[0.9 0;
% 0 0.9];

R=eye(k);

Q=[0.056 0;
0 0.012];

H=25;

% initials of state (xcorr0) and
 its var-cov (Pcorr0)

xcorr0=[2.7 0.5]';
% full sample OLS coefficients

% Pcorr0=[0.075589742 0;
%         0 0.00377070];
```

```
Pcorr0=[0.07 0;
0 0.004];

% Forward pass: filter
for i=1:n;

Z=[ a(i) b(i)];

if i==1
% time update ("predict") for i=1
xpred(:,i)=T*xcorr0;
Ppred{i}=T*Pcorr0*T'+R*Q*R';

% measurement update ("correct")
for i=1

F=Z*Ppred{i}*Z'+H;
K=Ppred{i}*Z'*inv(F);
xcorr(:,i)=xpred(:,i)+K*...
(y(i)-Z*xpred(:,i));
Pcorr{i}=Ppred{i}-K*Z*Ppred{i};
Pcorr_trace(i)=trace(Pcorr{i});

p_a(1)=xcorr(1,i);
p_b(1)=xcorr(2,i);
mhith_est(1)=a(i)*p_a(i)'+...
b(i)*p_b(i)';
else

% time update ("predict")
xpred(:,i)=T*xcorr(:,i-1);
Ppred{i}=T*Pcorr{i-1}*T'+R*Q*R';
```

```
% measurement update ("correct")

F=Z*Ppred{i}*Z'+H;
K=Ppred{i}*Z'*inv(F);
xcorr(:,i)=xpred(:,i)+K*...
(y(i)-Z*xpred(:,i));
Pcorr{i}=Ppred{i}-K*Z*Ppred{i};
Pcorr_trace(i)=trace(Pcorr{i});

p_a(i)=xcorr(1,i);
p_b(i)=xcorr(2,i);

mhith_est(i)=a(i)*...
b(i)'+b(i)*p_b(i)';

end
end

mhith_est_kontrol=
diag(xcorr'*x');
kontrol=
sum(mhith_est'-mhith_est_kontrol)

% Backward pass: smooth
% Rauch-Tung-Striebel
Two-Pass Smoother
(Fixed-Interval Smoother)

xsmooth = xcorr;
Psmooth = Pcorr;

for j=n-1:-1:1
```

```
A{j} = Pcorr{j}*T'*inv(Ppred{j+1});
xsmooth(:,j) =...
xsmooth(:,j) + A{j}*(xsmooth(:,j+1)...
- xpred(:,j+1));
Psmooth{j} = Psmooth{j}+A{j}*...
(Psmooth{j+1}- Ppred{j+1})*A{j}';

end;

FilteredParameters=xcorr';
SmoothedParameters=xsmooth'

figure(1)
subplot(2,1,1)
plot(Pcorr_trace,'r')
title('Trace of Pcorr_trace')
subplot(2,1,2)
plot(Psmooth_trace,'r')
title('Trace of Psmooth_trace')

% figure(3)
% plot(xcorr(1,:),'k')
% hold on
% plot(xsmooth(1,:),'r')
% title('Time-varying intercept')
% legend('filtered','smoothed')
%
%
% figure(4)
% plot(xcorr(2,:),'k')
% hold on
% plot(xsmooth(2,:),'r')
% title('Time-varying slope')
```

```
% legend('filtered','smoothed')

figure(2)
subplot(2,1,1)
plot(xcorr(1,:),'k')
hold on
plot(xsmooth(1,:),'r')
title('Time-varying intercept')
legend('filtered','smoothed')
subplot(2,1,2)
plot(xcorr(2,:),'k')
hold on
plot(xsmooth(2,:),'r')
title('Time-varying slope')
legend('filtered','smoothed')

figure(3)
plot(y,'k')
hold on
plot(mhith_est,'r')
xlabel('time')


resid=y'-mhith_est;

figure(4)
plot(resid,'k')
title('Hata terimleri')

rmse=sqrt(mean(resid.^2))

hort=mean(resid);
hstd=std(resid);
```

```
hskew=skewness(resid);
hkurt=kurtosis(resid);

figure(5)
hist(resid)
title('resid terimleri(Kalman)')
text(-7,4.0,['Ortalama= ',
num2str(hort)])
text(-7,3.75,['St. sapma= ',
num2str(hstd)])
text(-7,3.5,['Skewness= ',
num2str(hskew)])
text(-7,3.25,['Kurtosis= ',
num2str(hkurt)])

for i=1:n;
Psmooth_a(i)=Psmooth{i}(1,1);
Psmooth_b(i)=Psmooth{i}(2,2);
end

lower_a=xsmooth(1,:)-2*...
sqrt(Psmooth_a);
upper_a=xsmooth(1,:)+2*...
sqrt(Psmooth_a);

lower_b=xsmooth(2,:)-2*...
sqrt(Psmooth_b);
upper_b=xsmooth(2,:)+2*...
sqrt(Psmooth_b);

figure(6)
plot(xsmooth(1,:),'k')
hold on
```

```
plot(lower_a,'r')
hold on
plot(upper_a,'r')
title('CI for the intercept')


figure(7)
plot(xsmooth(2,:),'k')
hold on
plot(lower_b,'r')
hold on
plot(upper_b,'r')
title('CI for the intercept')
```

APPENDIX B

MATLAB OUTPUTS

```
Closed form bond price

alpha=0.263;
beta=0.0966311;
sigma=0.0419;
U=0.1275;
%set r_0 below beta (and U)
r_0=0.09;
T=10;

I = 0.9731
II = 0.6852
III = 0.5067
IV = 1.2197e-210
Price = 0.3378
```

```
Parameter estimations
```

```
B =
0.0584
-0.0180

a1 =

-0.0040    0.1208
-0.0802    0.0442

a2 =

0.4626    0.5605    0.9480   -0.5762   -1.1654    0.1951
2.2656   -0.4843    0.2305    0.6712    0.0814    1.3077
-0.5042    0.2377    0.0825   -0.6864    0.0457   -0.9722
0.1311    0.5378    1.1398   -0.7839   -1.3578   -1.6591
2.6315    1.1954    0.6412   -0.5997   -0.4048    0.5162
-1.0929   -2.5824    0.7578   -0.4534    0.2844    0.0732
0.3119   -0.7470    1.4387    0.5244    1.3053    0.0924
-0.2484    1.4397    1.2669    0.8713   -0.3653   -0.9016
1.3521    0.6345    0.2254   -2.0425    0.9592    0.7632
0.0965    0.2235   -1.3307   -0.4129   -1.6216    0.5027
1.2941    0.5419    1.4389   -1.4440   -0.3292   -1.3118
0.3980    1.1930   -1.0184    1.2368    0.2110    0.0236
0.7712   -0.9563   -2.4356    1.5563   -2.3976    1.2364
-1.6700    0.4669    0.4848   -0.7123    0.3268   -0.1044
-0.6547   -1.8677    0.1632    0.8792   -0.6045    2.0548
-1.2740    0.2174   -0.5669    0.5173   -1.0659    0.0344
-1.4438    0.1182    0.1939    0.3647    1.2364   -0.2225
0.9851    1.2832    1.4068    0.9041    0.2836   -1.2477
1.0414   -0.7481    0.3275    0.7185   -0.7363    0.0872
-0.1437    0.5947   -0.2722    0.2096   -0.3184    0.6229
-0.9599    0.9649    0.1399   -0.7404   -1.4036    1.8943
1.1017   -0.7106   -1.6361    1.5385   -0.9815   -0.8435
```

1.8180	0.3600	-1.4106	-1.0457	1.4249	-0.1586
0.6723	0.0652	-0.9962	-0.0478	1.9891	0.1103
-0.8181	-0.1492	-1.0845	-0.3338	0.7222	0.3591
-0.3973	-0.4666	2.2696	1.0277	-0.7347	0.4458
0.7435	-0.6205	0.5277	1.1732	0.3928	1.4926
0.0663	-1.1210	0.9965	1.2768	-0.5272	0.6906
1.0507	-0.4004	-0.2942	1.3883	-0.6616	-0.5560
-0.9880	-0.8893	-0.6966	-1.4407	1.0367	2.4470
1.5932	0.0093	0.8463	0.4314	1.9886	-0.6000
-0.3100	1.5927	0.4106	0.0237	0.2391	0.4681
-0.6761	-0.3193	-0.4108	-0.6054	1.0003	1.5051
-1.1264	1.0071	-1.7532	0.2280	-0.7009	1.5393
0.9189	0.4836	2.1710	-0.2952	-0.2199	-1.0317
-0.3369	1.3614	1.9809	-1.2037	-0.3042	-0.1139
-0.2374	-0.0227	-1.1077	-0.7413	1.3632	-0.7424
-0.3425	-0.0730	1.3077	-0.9150	-0.4631	-0.0889
-1.0017	-0.1220	-0.3078	-0.8440	0.0637	-0.1244
-0.6495	-1.1865	-0.5839	-0.0798	0.6299	-1.2417
-0.0707	2.0970	-0.9927	0.2273	-0.4228	0.4452
2.1994	-1.4647	-0.3986	-0.9642	-0.8884	0.4345
-0.4186	-0.0529	0.2069	-1.4968	-0.1653	-0.1816
0.8367	-0.3364	-1.4139	-0.9818	-0.6843	-0.4136
-0.9409	0.1114	0.1860	0.6647	-0.4949	-0.2980
0.9289	-1.4073	-1.5138	0.8687	0.9176	-2.1797
0.8947	0.8497	-0.0971	-1.4264	-0.1128	-1.0777
0.7136	-0.0415	-0.9256	0.4189	0.8509	-0.8477
0.3753	-0.2014	1.3114	-0.4409	-1.3618	0.2064
-1.8107	1.4551	-0.7158	-0.8954	-0.0284	0.2445
-0.0580	0.6562	-0.1770	0.3352	-1.2139	-0.4004
0.6320	1.7238	0.4608	-1.3766	0.0520	0.6234
-1.8020	1.2866	-0.1949	-1.3118	-1.1475	0.2602
-0.3359	-0.5623	-0.5446	-0.3173	0.3848	0.8719
1.2369	-0.2125	-1.7216	-2.2866	0.8463	-0.0402

0.0223	1.2814	-0.3124	-0.7177	-0.4271	0.1219
1.3814	-0.4301	1.1796	-1.2951	0.2045	0.6442
-0.3548	-0.0359	0.4092	-1.7214	0.8785	0.0373
1.2755	-0.4204	-1.1464	-0.8107	0.6543	1.0951
0.1471	0.4576	1.0183	0.0880	0.0521	-0.6294
0.6994	1.0883	-0.1421	-0.2975	-1.6310	0.5050
-1.4369	-0.6165	0.7703	2.2268	-0.3705	-0.1383
-1.4080	-0.4565	0.1547	-0.2224	-1.9508	0.4655
0.5610	0.9957	-2.2985	-0.8652	-0.2863	-0.8234
-0.3479	1.7064	-0.9673	0.7548	0.9197	0.8507
-1.7877	0.9249	1.6566	0.6150	0.4722	-2.4061
-0.3695	-1.3766	-0.2141	-0.9452	2.4379	-0.5906
-0.3151	0.0009	-0.5796	-0.0695	-0.2430	2.2122
-0.8174	0.3429	0.3179	-1.1537	0.7397	1.6136
-1.1938	0.0891	0.5920	-0.3175	0.1127	0.0431
0.2871	-0.5050	-0.7188	-0.2920	-0.9147	-0.9850
0.3549	-1.7130	0.3550	2.1326	0.0869	-0.5252
-0.7868	1.2883	-1.2509	0.6308	-0.5449	-0.6393
1.3752	1.0498	-0.6173	-1.1849	-0.8780	0.6086
-0.5028	0.7161	-0.5779	-1.3898	-0.5477	1.6537
-1.9392	-0.3743	0.2344	-0.6510	-0.1554	-0.1610
0.1113	-1.2764	1.9369	0.3850	2.6761	-0.0225
0.8391	0.6510	0.5624	1.8814	-0.5858	-0.1312
-1.8969	-0.8534	-0.9039	-0.6545	-0.6489	1.2804
1.9633	-0.1865	-1.6237	-0.1346	-0.6758	-0.2927
-0.3235	1.2834	-0.7048	-1.5629	0.0476	0.3505
-0.7876	0.9485	1.5448	1.2773	1.1739	-0.9869
0.3358	-0.3331	0.5008	0.5739	-1.1708	0.7938
-0.5222	0.4028	-1.0776	0.8828	-2.3920	-1.2416
0.9929	-0.6481	1.3931	0.7276	0.0177	0.5224
1.3242	-0.7372	-0.8614	-2.0878	1.0253	-0.9254
0.4218	-0.1473	0.0162	1.2104	-2.4196	3.1773
0.0069	-1.2545	-0.6744	-0.0787	0.8728	0.1777

-1.6770	1.9550	0.0353	-0.5905	1.2526	-0.7935
-0.6841	0.4707	-0.6589	-0.4182	0.1611	-0.1488
0.0720	-0.7520	-0.6946	1.1256	-1.9389	1.3080
-1.9727	0.0432	-0.6930	1.8926	0.1078	0.0018
-0.9649	0.9869	2.0839	0.9800	0.5899	-1.5363
0.2088	-0.1778	-2.0436	1.8034	0.8404	0.8804
-1.2104	-0.1982	1.1123	-0.5764	0.7090	-0.1977
-0.7063	0.8112	-0.4692	-1.4860	0.7372	-0.9090
0.3517	1.4783	0.5919	-0.3207	1.1286	1.4590
0.2062	0.8485	1.0433	-1.0450	-1.2076	0.1789
0.1388	0.1347	-1.4731	-1.0516	1.0983	-1.0462
2.4634	0.5525	0.8129	-0.3188	-1.7634	1.7126
0.6201	0.0327	-0.2518	-1.4644	0.1348	1.0848
-0.6744	-0.4228	-0.9964	-0.8358	0.3514	-1.3233
0.0961	-0.6182	0.0267	-0.0591	1.1116	0.2956
1.1852	-0.7383	-0.0649	0.1388	-0.9866	0.8001
1.7984	-0.7962	0.3371	-0.7724	0.2388	-0.7578
0.6540	0.0637	0.4202	-0.2177	1.3557	0.4562
-0.7734	-1.5626	-0.2919	0.0018	0.3779	-0.1075
0.7362	-0.0519	0.7842	-0.3345	0.7694	-0.1833
0.5730	-1.4830	0.3105	0.7367	-0.0293	-1.0041
0.5336	-0.1946	1.2906	-0.9051	-0.5682	1.4047
-0.0440	-0.8856	1.4100	0.5077	1.0520	0.4648
-0.1900	0.2009	-0.5091	1.5133	0.6002	-0.0294
0.1300	0.9948	-0.2350	1.6481	1.0956	0.2278
-0.5757	-1.6844	1.1050	-0.5512	-0.4618	1.4403
0.6394	-0.6697	2.0198	0.5429	0.0952	-0.2480
-0.0432	-0.6993	1.2599	-0.9695	1.0703	-0.6569
1.3594	-0.4110	-0.0155	0.5254	-1.0716	0.4504
0.7300	-0.1532	0.2379	-1.6606	0.6257	0.9192
0.2291	-0.8967	1.6290	-1.6465	0.4821	-0.3864
1.3506	-0.2009	-1.1470	-0.8403	1.5639	-1.2453
0.8527	1.0275	-0.1845	-1.7799	1.9456	-1.6474

0.3389	0.1390	-0.7579	0.5841	1.2736	-1.0521
-0.8951	-0.4774	1.4481	0.0811	-1.1404	1.5969
1.3874	0.3473	-1.2681	1.0121	-1.5749	0.9677
-0.1420	0.5605	-0.4557	0.3023	-1.2641	-1.1754
-0.3214	-1.7675	0.1030	0.6668	0.7930	-0.1254
0.0674	-0.0928	0.3545	-0.9812	-0.5263	0.3428
1.0936	2.0184	-0.3812	0.9246	-1.0291	-0.6450
0.1112	0.0211	-0.9776	-0.0287	-0.1283	0.3774
-1.1141	0.1992	-0.2731	-0.6309	-0.0722	-0.7011
-0.2189	1.5970	-0.2892	1.1851	-1.3322	-1.0635
-0.7712	-1.2283	-1.1487	0.7362	0.0939	-0.8989
-0.2203	0.9171	-1.2748	-0.1848	-1.0645	0.6685
-1.2166	-0.3079	0.2722	0.3087	-0.7750	-0.4250
0.4213	0.2839	0.7144	-0.3586	0.9553	-0.3441
-0.3298	0.6211	0.5381	-0.8089	-0.5374	0.0011
-0.8468	-0.2005	0.0093	0.2622	-0.6555	-0.9484
-0.5273	-0.6033	-0.6003	-0.2518	1.5481	-0.4821
1.3149	0.6882	-1.6129	-0.3552	1.0067	0.6990
-1.2088	-0.1170	-0.9309	0.4006	0.4288	0.9992
0.5498	0.1388	-0.5633	-0.4622	0.3244	-0.0353
0.1799	0.6192	-0.2518	-0.2858	-0.8912	0.2569
-1.1727	0.9087	0.2787	1.7032	-0.6266	-0.7556
0.1038	-0.2504	0.2393	-2.2981	0.8329	-0.4306
-0.9004	-0.9393	0.8377	-0.6684	0.0617	-0.3651
0.8912	0.4829	0.3409	-1.2197	-0.5168	0.5627
0.5512	1.3234	-1.3856	2.1030	2.1139	-0.2380
-0.1082	-0.6350	0.8470	-0.2944	-0.6285	1.1796
-0.9593	0.5451	-1.0822	1.6719	-0.6234	-1.0314
-0.4626	1.4248	-0.4225	0.1532	-0.4475	0.3606
-0.4556	-0.0911	-0.7079	0.0823	-2.3443	-0.6294
0.3872	0.3590	0.6678	-1.1532	0.6244	0.0729
1.2616	-0.0945	-1.7721	-1.0101	1.0671	0.1463
0.7457	-0.8951	0.6573	-0.0109	0.5299	-2.1235

```
-0.4428   -0.7456   -0.7566   -0.5948   -1.1924   -0.1143
-1.8428   -1.1794   -0.8392    2.1133   -1.1802   -0.6603
-0.6505   -0.1032   -1.3134   -1.1814    0.9373    2.0158
0.4020    1.8343   -0.8004    1.4534   -0.6398    1.0107
-1.0460    0.2032   -1.2187    0.4073    1.2005   -0.1720
0.0544    1.6469   -0.8826   -0.3279   -1.1442   -0.5484
1.4990   -0.9230    1.1877   -0.9975    0.5985   -0.1728
1.6054   -0.4451   -0.2449   -0.6892   -0.9542    0.2731
0.1509    1.7762   -1.0738   -0.6287   -0.4780    1.0399
1.7905   -1.3873    1.1964   -1.3026   -0.6346   -0.0962
0.5404   -1.2349    0.5513    1.5933    1.2998    0.8683
0.7172    2.0638    0.3439   -0.8362    0.7875   -0.2975

stats = 0.0003    0.3236    0.5696    1.0065
mu = 0.0574
theta = -56.4484
sigma = 0.0318

---------------------------------------------------

Difference between bond prices
with and w/out band

B_T_R = 0.8306
B_T_r = 0.8337
difference = -0.3716
```

APPENDIX C

MATLAB FIGURES

C.1 VASICEK BOND PRICE

$r_0 = .07; \alpha = .3; \beta = .08; \sigma = .01;$

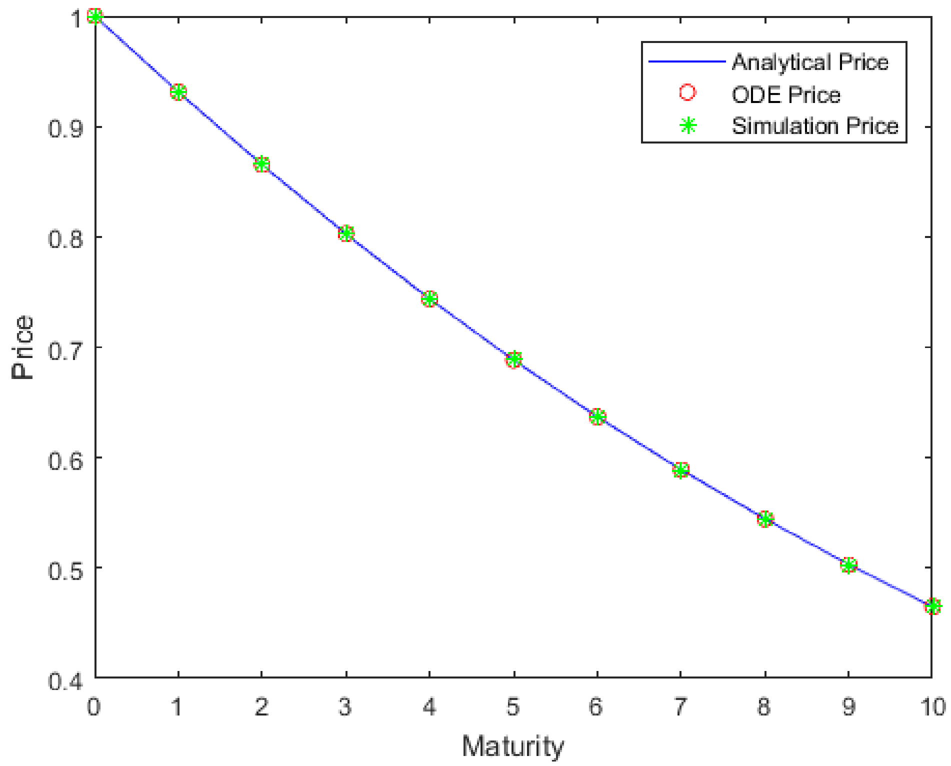

Figure C.1: Vasicek bond price

C.2 ESTIMATES OF TIME VARYING PARAMETERS

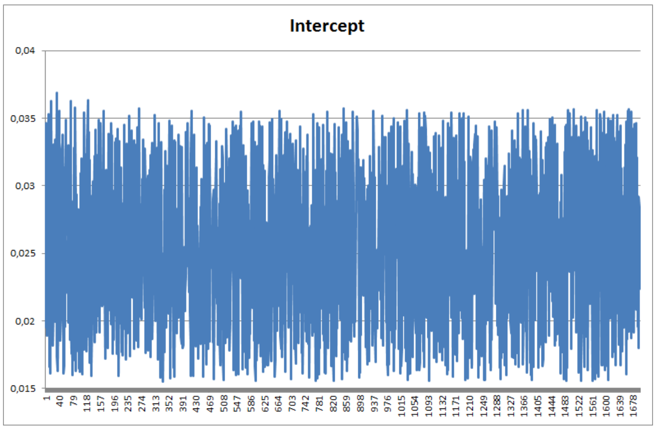

Figure C.2: Estimates of Time Varying Parameters (Intercept)

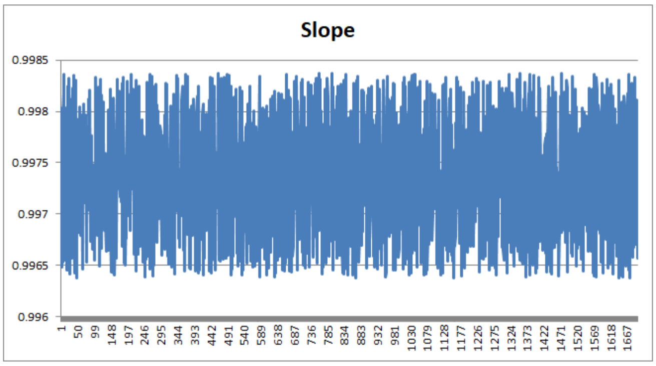

Figure C.3: Estimates of Time Varying Parameters (Slope)

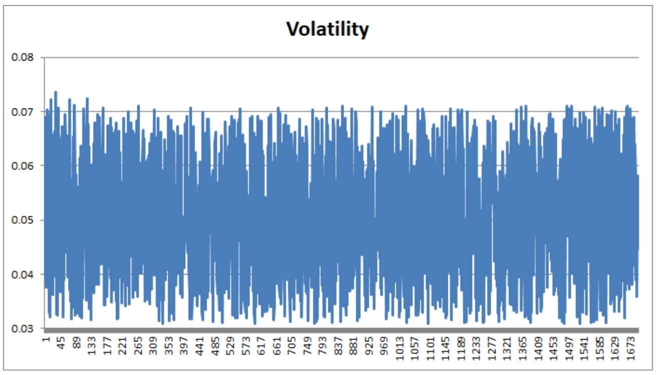

Figure C.4: Estimates of Time Varying Parameters (Volatility)

CURRICULUM VITAE

PERSONAL INFORMATION

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EDUCATION

Degree	Institution	Year of Graduation
M.S.	Financial Mathematics, University of Chicago	2009
M.A.	Economics, Sabancı University	2005
B.S.	Business Administration, Koç University	2002

PROFESSIONAL EXPERIENCE

Year	Place	Enrollment
2017–Present	Central Bank of the Republic of Turkey	Economist
2016–2017	European Central Bank	Economist
2014–2016	Central Bank of the Republic of Turkey	Economist
2014	European Bank for Reconstruction and Development	Analyst
2005–2014	Central Bank of the Republic of Turkey	Economist
2002–2003	Global Securities	Analyst

PUBLICATIONS

Journal Publications

- Ilalan, Deniz, and Özgür Özel. Non-linear unit root testing with arctangent trend: Simulation and applications in finance. Cogent Mathematics and Statistics 5.1 (2018), 1458555.

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