PRICING SPREAD AND BASKET OPTIONS UNDER MARKOV-MODULATED MODELS

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF APPLIED MATHEMATICS
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

SİNEM KOZPINAR

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY
IN
FINANCIAL MATHEMATICS

SEPTEMBER 2018

Approval of the thesis:

PRICING SPREAD AND BASKET OPTIONS UNDER MARKOV-MODULATED MODELS

submitted by **SİNEM KOZPINAR** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy in Financial Mathematics Department, Middle East Technical University** by,

Prof. Dr. Ömür Uğur
Director, Graduate School of **Applied Mathematics** ______________

Prof. Dr. A. Sevtap Selçuk-Kestel
Head of Department, **Financial Mathematics** ______________

Prof. Dr. Ömür Uğur
Supervisor, **Scientific Computing, METU** ______________

Assoc. Prof. Dr. Zehra Eksi-Altay
Co-supervisor, **Institute for Statistics and Mathematics, WU** ______________

Examining Committee Members:

Prof. Dr. A. Sevtap Selçuk-Kestel
Institute of Applied Mathematics, METU ______________

Prof. Dr. Ömür Uğur
Institute of Applied Mathematics, METU ______________

Prof. Dr. Tolga Omay
Department of Economics, Atılım University ______________

Assoc. Prof. Dr. Ceylan Talu Yozgatlıgil
Department of Statistics, METU ______________

Assoc. Prof. Dr. Kasırga Yıldırak
Department of Actuarial Science, Hacettepe University ______________

Date: ______________

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Last Name: SİNEM KOZPINAR

Signature :

ABSTRACT

PRICING SPREAD AND BASKET OPTIONS UNDER MARKOV-MODULATED MODELS

Kozpınar, Sinem
Ph.D., Department of Financial Mathematics
Supervisor : Prof. Dr. Ömür Uğur
Co-Supervisor : Assoc. Prof. Dr. Zehra Eksi-Altay

September 2018, 108 pages

This thesis first aims to study the evaluation of spread and basket options under the classical Markov-modulated framework, for which a transition in the Markov process leads to a switch in the model parameters. In this regard, we provide approximations to the exact option prices based on ideas from the literature without regime switching.

We start with pricing spread options when risky assets follow Markov-modulated geometric Brownian motions (MMGBMs). In this context, we focus on the regime-switching version of Kirk's formula. For that reason, a change of numeraire technique is introduced which allows to associate the spread option price with the value of a European call option. Since the underlying asset of this European call follows a MMGBM for relatively small strikes, we evaluate the spread option by using Markov-modulated Black-Scholes formula. Then, we discuss the valuation of spread options when the underlying asset prices are driven by Markov-modulated Lévy processes (MMLPs). Under this modeling set-up, we approximate the spread option price by means of an accurate lower bound, which is obtained via a univariate Fourier inversion. For this method, we only require the joint characteristic function; and therefore, our approximation becomes valid for many regime-switching models.

Afterwards, we concentrate on the valuation of basket options for which we provide lower and upper bounds considering the MMLP framework. We first obtain an accurate lower bound by using a univariate Fourier inversion combined with an optimization procedure. However, this optimization procedure increases the computational cost. Therefore, we then derive faster analogous bounds by using the arithmetic-geometric mean inequality and univariate Fourier inversion without an optimization. As in the case of spread options, the approaches we followed for basket options are applicable to several MMLPs under which the joint characteristic functions of the underlying assets are known analytically.

Furthermore in this thesis we aim to price spread and basket options under a more generalized framework, in which a transition in the Markov process may induce a switch in the parameters as well as synchronous jumps in the asset prices. For this purpose, we extend the results obtained under the classical MMLP framework, which does not take the synchronous jumps into account, to this generalized framework.

Finally, in order to verify the accuracy of proposed approximations presented in this thesis, we include several numerical experiments.

Keywords: Regime-switching, Spread options, Kirk's formula, Basket options, Fourier inversion, Synchronous jumps.

ÖZ

MARKOV KİPLEMELİ MODELLER ALTINDA SPREAD VE BASKET OPSİYONLARININ FİYATLANDIRILMASI

Kozpınar, Sinem
Doktora, Finansal Matematik Bölümü
Tez Yöneticisi : Prof. Dr. Ömür Uğur
Ortak Tez Yöneticisi : Doç. Dr. Zehra Eksi-Altay

September 2018, 108 sayfa

Bu tez, öncelikle, Markov sürecindeki bir geçişin sadece model parametrelerinde değişmeye sebep olduğu klasik Markov kiplemeli modeller çerçevesinde spread ve basket opsiyonlarının fiyatlandırılmasını incelemeyi amaçlamıştır. Bu bağlamda, rejim değişimi olmayan modeller için literatürde geliştirilen fikirler baz alınarak yaklaşımlar önerilmiştir.

İlk olarak, riskli varlıkların Markov kiplemeli geometrik Brownian hareketini (MKG BH) takip ettiği varsayılarak, spread opsiyonlarının fiyatlanması incelenmiştir. Bu çerçevede, Kirk's formülünün rejim değişimli versiyonuna odaklanılmış; dolayısıyla, spread opsiyonunun bir Avrupa tipi alım opsiyonu ile fiyatlanmasına olanak tanıyan numeraire değişim tekniği önerilmiştir. Oldukça küçük kullanım fiyatları için, söz konusu alım opsiyonunun dayanak varlık fiyatı MKGBH'e göre modellendiğinden dolayı, spread opsiyonu Markov kiplemeli Black-Scholes formülü kullanarak fiyatlandırılmıştır. Ardından, spread opsiyonlarının değerlemesi, dayanak varlık fiyatlarını Markov kiplemeli Lévy süreçleri (MKLS) ile modelleyerek gerçekleştirilmiştir. Bu modelleme baz alınarak, tek değişkenli bir Fourier inversiyonu kullanılmış, böylece spread opsiyonunun gerçek değerine oldukça yakın olan bir alt sınır elde edilmiştir. Sadece birleşik karakteristik fonksiyonlara ihtiyaç duyulduğundan, yaklaşımımız rejim değişimli birçok modele uygulanabilmektedir.

Daha sonra, MKLS'i göz önünde bulundurularak basket opsiyonlarının değerlemesine odaklanılmış ve opsiyonun gerçek değerine oldukça yakın olan alt ve üst sınırlar belirlenmiştir. İlk olarak, tek değişkenli Fourier inversiyonu ve bir optimizasyon prosedürü kullanarak opsiyonun gerçek değeri için bir alt sınır elde edilmiştir. Bu optimizasyon prosedürü hesaplama maliyetini arttırdığından dolayı, aritmetik-geometrik ortalama eşitsizliği ve optimizasyonsuz tek değişkenli Fourier inversiyonu kullanarak, daha hızlı bir şekilde elde edilen sınır fiyatları belirlenmiştir. Spread opsiyonununda olduğu gibi, basket opsiyonun fiyatlaması için izlediğimiz bütün yaklaşımlar dayanak varlıklara ait birleşik karakteristik fonksiyonunun bilinmesini gerektirmekte, bu ise bahsi geçen yöntemlerin birçok MKLS'ne uygulanmasına imkan vermektedir.

Ayrıca, bu tezde spread ve basket opsiyonlarının Markov sürecindeki bir geçişin sadece parametrelerde bir değişmeye değil, aynı zamanda varlık fiyatlarında senkronize sıçramalara da sebep olduğu daha genel bir çerçeveye göre fiyatlandırılması da amaçlanmıştır. Bu amaç doğrultusunda, senkronize sıçramaları hesaba katmayan klasik MKLS çerçevesi altında elde edilen tüm sonuçlar, bu genel çerçeveye göre de ele alınmıştır.

Son olarak, bu tez kapsamında elde edilen tüm yaklaşımların doğruluğunu kontrol etmek için birçok nümerik örnek sunulmuştur.

Anahtar Kelimeler: Rejim-değişimi, Spread opsiyonları, Kirk's formülü, Basket opsiyonları, Fourier inversiyonu, Senkronize sıçramalar.

To My Family

ACKNOWLEDGMENTS

I would like to express my great appreciation to my thesis supervisor Prof. Dr. Ömür Uğur for his guidance and valuable advices during the development and preparation of this thesis.

I would also like to thank to my co-supervisor Assoc. Prof. Dr. Zehra Eksi-Altay for her valuable supports and guidance throughout my work.

I owe my sincere appreciation to Assoc. Prof. Dr. Yeliz Yolcu-Okur because of her great enthusiastic encouragement and advices. She has been very supportive ever since the days I began master program in Financial Mathematics at Institute of Applied Mathematics (IAM).

I would also like to express my deepest gratitude to Prof. Dr. Griselda Deelstra. Her willingness to support and her invaluable guidance has brightened my path during the development of my thesis. It has been a privilege and a great experience to work with her. I am also grateful for the financial support she provided during my research visits at Université Libre de Bruxelles (ULB). I wish to extend my warmest thanks to Dr. Matthieu Simon who has shared his knowledge generously with me. I feel myself very lucky and delighted to interact with Prof. Dr. Griselda Deelstra and Dr. Matthieu Simon.

Special thanks to my committee members, Prof. Dr. A. Sevtap Selçuk-Kestel, Prof. Dr. Tolga Omay, Assoc. Prof. Dr. Ceylan Talu Yozgatlıgil and Assoc. Prof. Dr. Kasırga Yıldırak for their insightful comments and suggestions.

I am really thankful to Assoc. Prof. Dr. Azize Hayfavi and Prof. Dr. Bülent Karasözen for their friendly supports and encouragements.

I am grateful to my lovely friend Hacer Öz Bakan for her unfailing support, patience, motivations and invaluable friendship. I owe her a lot since she always supported me in my hardest times and did not let me give up.

Furthermore, my profound thanks to my roommates Ahmet Sınak, Meral Şimşek, Özge Tekin, Cansu Evcin and Hasan Bartu Yünüak for their friendship and cheers all the time. Without their encouragement it would be harder to finish my thesis. Also, Ahmet deserves my special thanks for providing a unique enjoyable working environment during my IAM life.

I would like to express my appreciations to the members of the Institute of Applied

Mathematics for their kindness.

Finally, I eternally grateful for my mother Esma, my father Mustafa and my sister Dilara for their unconditional love throughout my life. Without their support, it would have been impossible for me to finish my thesis. All my accomplishments are thanks to their love and I dedicate this dissertation to them.

TABLE OF CONTENTS

LIST OF TABLES

TABLES

LIST OF FIGURES

FIGURES

LIST OF ABBREVIATIONS

MMLP	Markov-modulated Lévy process
GBM	Geometric Brownian motion
MMGBM	Markov-modulated geometric Brownian motion
MC	Monte Carlo
CTMP	Continuous-time Markov process
FFT	Fast Fourier transform
i.i.d.	Independent and identically distributed
$\mathbb{R}^N$	N- dimensional Euclidean space
$\mathbb{C}^N$	N- dimensional complex vector space
CI	Confidence interval
$o(.)$	Little-o notation
$\mathbf{1}$	Column vector of ones
$\mathbf{0}$	Column vector of zeros
$\mathbb{1}_A$	Indicator function of A

CHAPTER 1

INTRODUCTION

1.1 Motivation and Literature Review

Many attempts have been made to discuss regime-switching models for the purpose of, e.g., identifying the impact of business cycles in the market. The model parameters are allowed to switch at certain times by means of a Markov process whose states represent the different regimes of the economy. To be more precise, whenever the state of the underlying Markov process (i.e. the regime in the market) changes, the model parameters are modulated according to the new state. In this thesis, we will first focus upon the pricing of spread and basket options under this classical Markov-modulated framework. Considering the fact that these options do not generally have closed-form prices under the dynamics with regime-switching, we provide approximations to the exact option prices based on ideas from the literature without regime-switching. One of the main contributions of this thesis lies in the derivation of these pricing formulas. Based on our knowledge the valuation of spread and basket options under this Markov-modulated framework has not been studied so far.

Regime-switching models have been extensively used in the literature, and even when concentrating only on option pricing, it is impossible to give an exhaustive overview. Among these papers dedicated to option pricing in the context of Markov-modulated Brownian motions (MMBMs), we refer to Buffington and Elliot [7] who price European vanilla and American options by using Black-Scholes arguments, Boyle and Draviam [6] who evaluate Asian and lookback options by partial differential equations and Zhu et al. [42] who deal with pricing European vanilla options via a Fourier transform method. Furthermore, Elliott et al. [19] propose the regime-switching version of Esscher transform with the aim of completing the market and then examine the valuation of European vanilla options considering the ideas of Buffington and Elliot [7].

This paper, additionally, provides a detailed overview of the literature on regime-switching models and their applications in finance. There exists also a wide range of studies that consider Markov-modulated Lévy processes (MMLP) for the dynamics of the risky assets. For example, Konikov and Madan [25] assume a two-state Markov-modulated Variance Gamma process and evaluate European vanilla options by computing the characteristic function of the log-returns of the underlying. Elliott and Osakwe [18] extend the study of Konikov and Madan to arbitrary number of states. Under a regime-switching version of Merton jump-diffusion model, Ramponi [32] investigates the valuation of forward starting options via Fourier transform method. Tour et al. [35] concentrate upon the valuation of contracts such as Bermudan, American and barrier options in a regime-switching model of time-changed Lévy processes; and for this purpose, they use the Fourier cosine expansion (COS) method.

All these papers given above regard option pricing problems with a single underlying. As far as we know, there are not so many papers focusing upon multivariate option pricing in a regime-switching framework, especially not in a MMLP setting. In the MMBM setting, Yoon et al. [40] introduce analytical pricing formulas using the occupation times for the valuation of quanto and exchange options. Chen et al. [13] price European-type quanto options by assuming that forward interest rates are driven by a Markov-modulated HJM model, the foreign stock prices follow the regime-switching version of a jump-diffusion and the spot FX rate is modeled by a geometric Brownian motion (GBM). More recently, Deelstra and Simon [16] study the pricing of exchange and quanto options in a MMLP framework whereas Fan and Wang [20] focus on valuing correlation options under a regime-switching stochastic interest rate model. In these last two papers, the valuation of the options are carried out with a fast Fourier transform (FFT) method.

The framework of regime-switching models, where transitions to another phase can happen with only a change in the model parameters, can be generalized if we allow the asset prices to jump synchronously in the case of a regime change. That is, a transition in the Markov process does not only yield a switch in the parameters, but can also yield a jump in the asset prices. All these approaches given above are carried out by assuming that a transition in the Markov process induces the modulation of parameters, and none of them considers the synchronous jumps in the asset prices. To our knowledge, Chourdakis [14] was the first to investigate Markov-modulated Lévy processes with synchronous jumps with the goal of option pricing and to show by numerical examples that this regime-switching model can be successful in capturing asymmetric volatility skews. Hainaut and Colwell [22] consider a regime-switching version of Merton's structural model to evaluate the default risk, assuming that the asset dynamics jump synchronously whenever a transition occurs in the underlying Markov process. As argued in this paper, synchronous jumps can model the events that lead to an immediate change in the price dynamics, such as economic downturn, terrorist attacks or natural catastrophes. Hainaut and Colwell [22] also provide econometric evidence that Lévy-based regime-switching models with synchronous jumps lead a good fit to the historical time series.

In a regime-switching framework with synchronous jumps, Chourdakis [14] concentrates upon pricing vanilla options and exotic contingent claims like barrier, Bermudan and American options. Since no research results are yet available for the valuation of spread and basket options even in a Markov-modulated framework without synchronous jumps, the incorporation of these jumps make another significant contributions to the literature on evaluation of such options. There is, however, a vast literature when the underlying assets of the options are modeled without regime shifts. Note that there is generally no closed-form pricing formula for these options, even not in a Gaussian setting.

Spread options, whose payoff is based on the difference of two asset's prices, are very popular among practitioners due to its great variety of applications in different types of markets, such as energy, commodity and equity markets. As regards to its evaluation under the GBM setting, Margrabe [29] obtains a Black-Scholes-like pricing formula in the case of zero strikes (exchange options). In this noteworthy paper, the value of the first asset in terms of the second asset is treated as a new underlying, which can still be identified by a GBM. Therefore, the pricing problem of exchange option is reduced to the evaluation of a European option. This study is one of the exceptions yielding a closed-form solution for spread options.

Extending the GBM case to the non-zero strike prices, in which case we do not have any closed-form solution, leads us to the approximation methods such as in Kirk [24] and Venkatramanan and Alexander [36]. Being frequently applied by practitioners due to its tractability, the former study follows the approach of Margrabe [29]: when the sum of the second asset and strike price is defined as the numeraire process, the value of the first asset in terms of this numeraire becomes approximately log-normal for small strikes. Therefore, the evaluation of spread options are carried out by using Black-Scholes formula, which points out the tractability of Kirk's approximation. However, up to our knowledge, a detailed derivation of this pricing formula is not provided by Kirk. The latter study approximates the value of the spread option by summing two compound options and also by using Kirk's formula, whose detailed derivation is also provided by means of stochastic differential equations.

In this thesis, we favor Venkatramanan and Alexander [36] for their approach introduced for the derivation of Kirk's formula. Amongst others, we also refer to Carmona and Durrleman [10] who give lower and upper bounds by using some trigonometric functions, Deelstra et al. [15] who propose to use moment matching techniques, and Bjerksund and Stensland [5] who derive a lower bound relating a power function of the second asset with the first asset.

When the underlying prices evolve according to jump-diffusion processes, for instance, Cheang and Chiarella [12] and Benth et al. [4] focus on the valuation of exchange options via a Margrabe-type formula. Differently, Dempster and Hong [17] and Hurd and Zhou [23] concentrate upon the numerical valuation of spread options via a two-dimensional FFT. Recently, Caldana and Fusai [8] propose a very accurate lower bound extending the study of [5] to non-Gaussian models. This approach can be applied whenever the joint characteristic function of the underlying log-returns is explicitly known; furthermore, it uses only a univariate Fourier inversion regarding the exponential form of asset prices.

Basket options, written on the linear combination of two or more assets, are also widely used in many financial markets and, as a consequence, there exist many papers studying the evaluation of basket options by adapting different approaches. Regarding a GBM setting, see e.g. Vorst [37] for the use of arithmetic-geometric mean inequality, Deelstra et al. [15] for the application of moment matching techniques, Zhou and Wang [41] for approximating the basket distribution with a log-extended-skew-Normal distribution, Alexander and Venkatramanan [1] for an analytical approximation based on the sum of several compound-exchange options.

In the line of a non-Gaussian setting, Linders and Stassen [26] consider a multivariate Variance Gamma model for the evaluation of basket options, in which the dependence between different assets are implied by a common time change. By conditioning on this common time change, they introduce some lower and upper bounds to the true option price. Under a local volatility jump-diffusion model, Xu and Zheng [38] provide an approximation for basket options by conditioning on a normal and Poisson variable.

All methods mentioned above are model-dependent, which limits their applicability. For the line of model-free approximations, see, for instance, Caldana et al. [9]. This interesting paper provides very useful lower and upper bounds for the prices of basket options in a wide range of models for which the joint characteristic function of the log-returns is known explicitly. These results can be obtained for baskets with no constraints on the signs of the underlying assets and by using only a univariate Fourier inversion based on the exponential form of asset prices. In particular, Caldana et al. [9] first obtain a lower bound through a set based on the logarithm of the geometric average of the weighted assets. Then, they propose new lower and upper bounds and an approximate pricing formula for the fair price of the basket option, which follow from the arithmetic-geometric mean inequality proposed by Vorst [37]. This paper [9] also provides a good background on the valuation of basket options when the underlying assets are modeled without regime-switching. For a detailed overview of the literature, we refer the interested readers to this paper.

1.2 Aim of the Thesis

This thesis first aims to study the evaluation of spread and basket options under the regime-switching models without synchronous jumps. In this context, we start with pricing spread options in a Markov-modulated geometric Brownian motion (MM GBM) setting without synchronous jumps. For this purpose we provide a Markov-modulated Kirk's formula, by generalizing the approach of Alexander and Venkatramanan [36] to a framework with regime shifts. Since their original work requires the use of Black-Scholes formula, we also regard the Markov-modulated Black-Scholes arguments given in Buffington and Elliott [7] and Elliott et al. [19].

Then, we price spread and basket options when risky asset prices are driven by the exponential of Lévy-based regime-switching models without sychronous jumps. We therefore will extend the approaches of Caldana and Fusai [8] and Caldana et al. [9] to the framework in which the underlying assets are assumed to follow a MMLP framework without synchronous jumps. More precisely, we provide a lower bound for the price of spread options by adopting the ideas of [8]. We further address the pricing problem of basket options in this MMLP framework. Inspired by [9], we first provide a lower bound by defining a set based on the geometric average of the weighted assets, which can be obtained via a univariate Fourier inversion and an optimization procedure. Afterwards, we study the analogous bounds and approximate price estimate implied by the arithmetic-geometric mean inequality. Finally, we discuss pricing of these options by generalizing the results obtained in the classical MMLP framework for which synchronous jumps do not take place. Namely, we derive pricing formulas inspired by the studies of [8] and [9], but now considering a MMLP framework with synchronous jumps motivated by Chourdakis [14] and Hainaut and Colwell [22].

In order to verify the accuracy of all approximations presented in this thesis, we include several numerical experiments. The accuracy of the Markov-modulated Kirk's formula has been verified by comparing with the Monte Carlo simulations. In this context, we also report the lower bound price in order to examine its performance. Furthermore, we test the tightness of spread and basket bounds under several MMLPs both without and with synchronous jumps. In particular, we study regime-switching versions of models introduced in Ballotta and Bonfiglioli [3] in which a common systematic component is used to construct dependency between the different assets, and in Mai et. al. [28], in which a multivariate Kou model with dependence between the jump components of the different assets is constructed by means of a stochastic time change.

Most of the contributions of this thesis is due to the derivation of Kirk's formula in a MMGBM setting without synchronous jumps and to the generalisation of the results of [8] and [9] to MMLP frameworks without and with synchronous jumps. Furthermore, we want to mention that our MMLP model with synchronous jumps generalizes the ones in [14] and [22] in various aspects. These papers only model the regime shifts with synchronous jumps whereas we also take into account the possibility of occuring no synchronous jumps after a transition. The unobservable driving forces create an interesting correlation between the asset price processes. Moreover, since we are interested in multivariate option pricing like spread and basket options, we present a multidimensional framework for modeling different asset prices simultaneously which allows also the correlation among the asset prices that is not due to the underlying Markov chain.

1.3 Plan of the Thesis

The paper is organized as follows. Chapter 2 is devoted to the preliminaries including fundamental definitions and results used in the valuation of spread and basket options. In Chapter 3, we derive Markov-modulated Kirk's formula by means of stochastic differential equations, formulated under a MMGBM setting without synchronus jumps. Chapter 4 first starts with the model set-up for MMLPs without synchronus jumps and then introduces a lower bound to the spread option prices, by extending the approach of [8] to these Lévy-based framework. Chapter 5 extends the modeling framework given in Chapter 4 for the basket options and provide the lower and upper bounds generalizing the ideas of [9] to the present setting. Following two chapters we examine the valuation of spread and basket options under a MMLP framework with synchronous jumps. More precisely, Chapter 6 generalizes the spread bounds obtained in Chapter 4 to the regime-switching framework with synchronous jumps; and, Chapter 7 introduces the basket bounds analogous to those proposed in Chapter 5. In

Chapter 8, several numerical examples are discussed in order to show the accuracy of the proposed approximations. The last chapter concludes the thesis.

CHAPTER 2

PRELIMINARIES

In this chapter, we first give a basic terminology in connection with the continuous-time Markov process (CTMP), based on Norris [31, Chapter 2], Grimmett and Stirzaker [21, Section 6.9] and Ross [33, Chapter 5]. These will provide a better understanding about the mathematical framework developed for pricing spread and basket options. After getting a general idea about the Markov processes, we investigate the approach given in Buffington and Elliott [7] for a Markov-modulated Black-Scholes formula, Carr and Madan [11] for option pricing with fast Fourier transform (FFT) method and Deelstra and Simon [16] for a useful result that is very crucial in the development of Chapters 4 and 5.

2.1 A Brief Introduction to Continuous-Time Markov Processes

This section is devoted to the fundamental definitions and results related to the CTMPs. Mainly, we will focus on the properties of transition probability matrices $P(t)$ and generators Q.

To start with, we give the following notions.

Definition 2.1. A process $\{M(t)\}_{t\geq 0}$ which takes its values in a countable state space $\mathcal{S}$ is said to be a CTMP if the following condition holds:

$$\begin{aligned}\mathbb{P}\left(M(t_k) = j \mid M(t_1) = l_1, M(t_2) = l_2, \ldots, M(t_{k-1}) = l_{k-1}\right)& \\ = \mathbb{P}\left(M(t_k) = j \mid M(t_{k-1}) = l_{k-1}\right)&\end{aligned}$$

for all $j, l_1, \ldots, l_{k-1} \in \mathcal{S}$ and $t_1 < t_2 < \cdots < t_k$.

Definition 2.2. Assume that $M(t)$ is a CTMP with the state space $\mathcal{S}$. Then, the transition probability $p_{lj}(s,t)$ is defined as

$$p_{lj}(s,t) = \mathbb{P}\left(M(t) = j \mid M(s) = l\right), \quad (j, l \in \mathcal{S} \text{ and } s \leq t).$$

If $p_{lj}(s,t) = p_{lj}(0, t-s)$ for all $j, l \in \mathcal{S}$ and $s \leq t$, the Markov process $M(t)$ is said to be *homogeneous* and $p_{lj}(s,t)$ is therefore denoted by $p_{lj}(t-s)$. For the sake of completeness, p_l denotes the initial probability $\mathbb{P}\left(M(0) = l\right)$ for all $l \in \mathcal{S}$.

All CTMPs considered hereafter are assumed to be homogeneous and have a finite state space $\mathcal{S}$ with cardinality $|\mathcal{S}| = N$.

Consider a $N \times N$ matrix $P(t)$ whose (l, j)'th entry is given by the transition probability $p_{lj}(t)$. The following theorem gives some properties of this matrix $P(t)$:

Theorem 2.1. *Let $M(t)$ be a CTMP with the countable state space $\mathcal{S}$ and consider the matrix $P(t) = (p_{lj}(t))_{N \times N}$. Then, the following three conditions hold:*

- *$P(0) = I$, where I is the $N \times N$ identity matrix.*
- *$P(t)$ is a stochastic matrix, namely $p_{lj}(t) \geq 0$ for all l, j, and $\sum_{j \in \mathcal{S}} p_{lj}(t) = 1$ for all l.*
- *$P(t+s) = P(t)P(s)$ for all $s, t \leq 0$, which is also known as Chapman-Kolmogorov equation.*

Next, following the arguments of Ross [33], an alternative characterization of a CTMP as well as the construction of its generator (also known as the rate matrix or Q-matrix) are given.

Theorem 2.2. *A CTMP defined on the state space $\mathcal{S}$ is a stochastic process, in which the following assertions hold whenever entering a state l:*

(i) The holding time in the current state l, namely the time spent in this state before transition to another state, is exponentially distributed with some parameter a_l. Furthermore, this random variable is independent of the next transition.

(ii) *If the process moves from state* l*, it goes to a different state* j *with some probability* $\tilde{p}_{lj}$*, formulated as*

$$\tilde{p}_{ll} = 0, \qquad \sum_{l \neq j} \tilde{p}_{lj} = 1.$$

Noting that a_l is also called *the rate of moving from state* l, the following theorem states the generator of a CTMP.

Theorem 2.3. *If* $M(t)$ *is a CTMP with the countable state space* $\mathcal{S}$*, the leaving rates* a_l *and probabilites* $\tilde{p}_{lj}$ *(*$l, j \in \mathcal{S}$*), the generator matrix* $Q := (q_{lj})_{N \times N}$ *is defined as follows:*

(i) $q_{lj} = a_l \tilde{p}_{lj}, \quad l \neq j,$

(ii) $q_{ll} = -a_l.$

With this construction, q_{lj} *turns to be the rate of transition from state* l *to state* j*. Additionally, the diagonals of the generator matrix* Q *describe the holding times: Since* $q_{ll} = -a_l$*, the holding time in state* l *is now exponentially distributed with* $-q_{ll}$*.*

In the following, the transition rates are related with the transition probabilites by invoking a crucial assumption: for small $y > 0$, the number of transitions in the time interval $(t, t + y)$ is more than one with a probability of $o(y)$.

Theorem 2.4. *Suppose that* $M(t)$ *is a CTMP with the transition probability matrix* $P(t)$ *and generator* Q*. Then, as* $y \downarrow 0$ *and for all* $l \neq j$*:*

(i) $p_{lj}(y) = q_{lj} y + o(y).$

(ii) $p_{ll}(y) = 1 + q_{ll} y + o(y).$

In other words, for small values of y *the transition probabilities turn out be approximately linear in* y*.*

By using the above result and stochasticity of the transition probability matrix $P(t)$, the properties of the generator Q are summarized in the following lemma:

Lemma 2.1. *A square matrix $Q = (q_{lj})_{N\times N}$ is called a Q-matrix or the generator of a CTMP if and only if the following properties hold for all $l, j = 1, \ldots, N$:*

(i) $0 \le -q_{ll} < \infty$,

(ii) $q_{lj} \ge 0$ *for* $l \ne j$,

(iii) $\sum_{j=1}^{N} q_{lj} = 0$.

Lastly, Kolmogorov's backward and forward equations are introduced, which are one of most noteworthy results in the context of CTMPs. In particular, these results indicate how the transition probability matrix $P(t)$ can be obtained from the generator matrix Q.

Theorem 2.5. *Let the process $M(t)$ be a CTMP with the finite state space $\mathcal{S}$, transition probability matrix $P(t)$ and generator Q. Also, consider that e^A denotes the matrix exponential of a finite-dimensional square matrix A.*

Then, having the unique solution $P(t) = e^{Qt}$, the followings hold:

(i) (Kolmogorov's Backward Equation)

$$\frac{d}{dt}P(t) = QP(t), \quad P(0) = I.$$

(ii) (Kolmogorov's Forward Equation)

$$\frac{d}{dt}P(t) = P(t)Q, \quad P(0) = I.$$

2.2 Markov-Modulated Black-Scholes Formula

In this section, we focus on the approach of Buffington and Elliott [7], developed for the valuation of vanilla options under a MMGBM setting without synchronous jumps. In particular, they present the regime-switching version of Black-Scholes formula by conditioning on the whole trajectory of Markov process. Noting that their pricing methodology will be very crucial for our Markov-modulated Kirk's formula, we briefly review their arguments in the sequel.

Consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ on a finite time horizon $[0, T]$, where $\mathbb{Q}$ is the risk-neutral probability measure, and suppose a homogeneous continuous-time Markov process $\{M(t)\}_{t\in[0,T]}$ with a finite-state space $\mathcal{S} = \{1, 2, \ldots, N\}$. In this context, the authors assume that the risk-neutral dynamics of the risky asset $\{S(t)\}_{t\in[0,T]}$ is modelled by

$$\frac{dS(t)}{S(t)} = r(M(t))dt + \sigma(M(t))dW(t), \quad S(0) = s,$$

or in exponential form,

$$S(t) = se^{\int_0^t \left(r(M(u)) - \frac{1}{2}[\sigma(M(u))]^2\right)du + \int_0^t \sigma(M(u))dW(u)},$$

where $W(t)$ is a Brownian motion independent of $M(t)$, $r(M(t))$ is the risk-free interest rate and $\sigma(M(t))$ is the volatility of the asset, both of which are characterized by the underlying Markov process $M(t)$:

$$r(M(t)) = \sum_{j=0}^{N} r_j \mathbb{1}_{\{M(t)=j\}}, \quad \sigma(M(t)) = \sum_{j=0}^{N} \sigma_j \mathbb{1}_{\{M(t)=j\}},$$

with r_j and σ_j are positive constants for each $j = 1, 2, \ldots, N$.

Regarding the above dynamics, they consider the pricing problem

$$V(0) = \mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}\left(S(T) - K\right)^+\right],$$

where T is the maturity, K is the strike and $U(T)$ is the integrated process defined by $U(T) := \int_0^T r(M(s))ds$. In order to price, they favor the law of total expectation by conditioning on the whole trajectory of $M(t)$:

$$V(0) = \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}\left(S(T) - K\right)^+ \mid \mathbb{F}^M(T)\right]\right],$$

where $\mathbb{F}^M(T)$ represents the σ-algebra generated by the Markov process $M(t),\ 0 \leq t \leq T$. As pointed out in [7], with conditioning on $\mathbb{F}^M(T)$ the values of integrated processes $U(T) = \int_0^T r(M(s))ds$ and $\int_0^T [\sigma(M(s))]^2\, ds$ are known in advance, and hence the inner conditional expectation $\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}\left(S(T) - K\right)^+ \mid \mathbb{F}^M(T)\right]$ is computed by using the Black-Scholes formula

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}\left(S(T) - K\right)^+ \mid \mathbb{F}^M(T)\right] = sN(d_1) - Ke^{-U(T)}N(d_2),$$

where

$$d_1 = \left(\int_0^T [\sigma(M(u))]^2\, du\right)^{-1/2} \left[\log\left(\frac{s}{K}\right) + U(T) + \frac{1}{2}\int_0^T [\sigma(M(u))]^2\, du\right],$$

$$d_2 = d_1 - \left(\int_0^T [\sigma(M(u))]^2 du\right)^{1/2}.$$

Note that these arguments will be very useful in Chapter 3.

2.3 A Useful Result from Deelstra and Simon

In this section, we visit a noteworthy result from Deelstra and Simon [16], proposed to obtain the joint characteristic function of Markov-modulated Lévy processes (MMLPs) without synchronous jumps. This result will be frequently used for developing the theoretical framework in Chapter 4 and Chapter 5. Also, it is worth mentioning that an analogous result will be derived for the MMPLs with synchronous jumps (see Appendix A).

Lemma 2.2. *Suppose that $M(t)$ is a continuous-time Markov process defined by a finite state space $\mathcal{S} = \{1, 2, \ldots, N\}$, generator Q and initial probability vector $\mathbf{p} = [p_1, p_2, \ldots, p_N]$ with $p_j = \mathbb{P}(M(0) = j)$. Assume that $\mathbf{X}(t) = (X_1(t) \ldots, X_n(t))$ is an n-dimensional MMLP without synchronous jumps, whose dynamics in state k is identified by the n-dimensional Lévy process $\mathbf{Y}_k(t) = (Y_{1k}(t), \ldots, Y_{nk}(t))$ that has the characteristic exponent $\Phi_k(\mathbf{u})$:*

$$\mathbb{E}^{\mathbb{Q}}\left[e^{i\langle \mathbf{u}, \mathbf{Y}_k(t)\rangle}\right] = e^{-\Phi_k(\mathbf{u})t},$$

with $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{C}^N$.

Under the assumption that Φ_j is known analytically and $C(t) = \int_0^t c(M(s))\, ds$ is a Markov-modulated drift process with

$$c(M(t)) = \sum_{j=0}^{N} c_j \mathbb{1}_{M(t)=j},$$

($c_j,\ j = 1, \ldots, N$, being some constants), then for all $\boldsymbol{a} \in \mathbb{C}^n$ and $t \geq 0$:

$$\mathbb{E}\left[e^{C(t)+\langle \boldsymbol{a}, \boldsymbol{X}(t)\rangle}\right] = \boldsymbol{p} e^{(-A+Q)t}\mathbf{1},$$

where $\mathbf{1}$ is the column vector of ones and A is the diagonal matrix with $A_{jj} = \Phi_j(-i\boldsymbol{a}) - c_j$.

2.4 Option Pricing with FFT

In this section, we review Carr-Madan approach [11] developed for the evaluation of European call options under a framework without regime-switching. In this interesting paper, the authors propose to use the FFT under the models for which the characteristic function of log-returns is known analytically. Although the main concern in this approach is the valuation of vanilla options, their idea will motivate us to obtain pricing formulas for spread and basket options under the Markov-modulated models. Therefore, in the following we introduce their pricing methodology.

Consider a European call option with underlying asset S, maturity T and strike K. Denoting $X(T) = \log(S(T))$ and $k = \log(K)$, the fair price of the option at time $t = 0$ is given as

$$\begin{aligned} V(0) &= \mathbb{E}^{\mathbb{Q}}[e^{-rT}(e^{X(T)} - e^k)^+] \\ &= e^{-rT}\int_k^{\infty}(e^x - e^k)f(x)dx, \end{aligned}$$

where $\mathbb{Q}$ is the risk-neutral probability measure, r is the risk-free interest rate and $f(x)$ is the density function of $X(T)$. Carr and Madan point out that since $\lim_{k\to-\infty} V(0) = S(0)$, $V(0)$ is not square-integrable, and therefore Fourier transform method cannot be directly applied. For that reason, they introduce a modified price $\tilde{V}(0) = e^{\delta k}V(0)$ which turns out to be square integrable with a suitable choice of damping factor $\delta > 0$. Then, by considering the Fourier transform of $\tilde{V}(0)$,

$$\Psi_T(\gamma; K) = \int_{-\infty}^{\infty} e^{i\gamma k}\tilde{V}(0)dk = \int_{-\infty}^{\infty} e^{i\gamma k}\int_k^{\infty} e^{-rT}(e^x - e^k)f(x)dxdk,$$

the option price is formulated as

$$V(0) = \frac{e^{-\delta k}}{2\pi}\int_{-\infty}^{\infty} e^{-i\gamma k}\Psi_T(\gamma; K)d\gamma = \frac{e^{-\delta k}}{\pi}\int_0^{\infty} e^{-i\gamma k}\Psi_T(\gamma; K)d\gamma,$$

as a result of inverse transform. As they mentioned, the last equality is based on the fact that since the option price $V(0)$ is real, the imaginary part of $\Psi_T(\gamma; K)$ is an odd function and its real part is even. Lastly, by using Fubini arguments, the function $\Psi_T(\gamma; K)$ is written in terms of the characteristic function of $X(T)$; and hence, the option price is obtained. For more discussion about the derivation of the pricing formula and the choice of the damping factor, we refer to Carr and Madan [11]. It is important to mention that the authors also present an efficient FFT algorithm for computing the corresponding option price numerically.

CHAPTER 3

PRICING SPREAD OPTIONS UNDER A MARKOV-MODULATED GEOMETRIC BROWNIAN MOTION WITHOUT SYNCHRONOUS JUMPS

This chapter is devoted to the valuation of spread options when the evolution of underlying assets is identified within a Markov-modulated GBM (MMGBM) without synchronous jumps. In this context, whenever the state of the Markov process change, only a switch in the model parameters will occur according to the visited state. It is worth mentioning that when risky assets follow a framework without regime-switching, closed-form pricing formulas are generally not available for spread options, even in a GBM setting.

Therefore, one can discuss the pricing problem without regime-switching in the line of analytical approximations or numerical methods. This study concentrates on the derivation of an analytical approximation, namely the Markov-modulated Kirk's formula, whose version without regime-switching draws many attention from practitioners.

3.1 The Market Model

This section is devoted to the market dynamics under a regime-switching framework without synchronous jumps. We suppose a complete probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ on a finite time horizon $[0, T]$, where $\mathbb{Q}$ is defined as the risk-neutral probability measure. We also consider a homogeneous continuous-time Markov process $\{M(t)\}_{t\in[0,T]}$ with a finite-state space $\mathcal{S} = \{1, 2, \dots, N\}$, generator $Q = (q_{ij})_{N\times N}$ and initial probability vector $\mathbf{p} = [p_1\, p_2\, \dots p_N] \in \mathbb{R}^N$. Remember from Section 2.1 that under a framework without synchronous jumps, q_{lj} denotes the constant transition rate from state l to j and $p_l = \mathbb{P}\left(M(0) = l\right)$. Unless otherwise stated, all dynamics in this chapter will be given under the risk-neutral probability measure $\mathbb{Q}$.

When considering the continuous-time financial market, we define a money market account $\{S_0(t)\}_{t\in[0,T]}$ and two risky assets $\{S_1(t)\}_{t\in[0,T]}$, $\{S_2(t)\}_{t\in[0,T]}$ whose dynamics are given in the following: let the interest rate process $r(M(t))$ be Markov-modulated, i.e.,

$$r(M(t)) = \sum_{j=1}^{N} r_j \mathbb{1}_{\{M(t)=j\}},$$

with $r_j > 0$ being constant for each $j = 1, 2, \dots, N$. For such a specification, the dynamics of the money market account $S_0(t)$ is described as

$$dS_0(t) = r(M(t))S_0(t)dt, \quad S_0(0) = 1,$$

or equivalently,

$$S_0(t) = e^{U(t)} \quad \text{with} \quad U(t) = \int_0^t r(M(s))ds.$$

Moreover, define two volatility processes $\sigma_i(M(t))$, $i = 1, 2$, characterized by the underlying Markov process $M(t)$ as follows:

$$\sigma_i(M(t)) = \sum_{j=1}^{N} \sigma_{ij} \mathbb{1}_{\{M(t)=j\}},$$

where $\sigma_{ij} > 0$ is constant for each $j = 1, 2, \dots, N$ and $i = 1, 2$. We also define $\sigma_i^{\max} = \max\limits_{1\le j\le N} \sigma_{ij} < \infty$ for each $i = 1, 2$.

Therefore, price processes $S_1(t)$ and $S_2(t)$ are identified by the following MMGBMs:

$$\frac{dS_1(t)}{S_1(t)} = r(M(t))dt + \sigma_1(M(t))dW_1(t), \quad S_1(0) = s_1,$$
$$\frac{dS_2(t)}{S_2(t)} = r(M(t))dt + \sigma_2(M(t))dW_2(t), \quad S_2(0) = s_2,$$

where $\{W_1(t)\}_{t\in[0,T]}$ and $\{W_2(t)\}_{t\in[0,T]}$ are two standard Brownian motions with $d\langle W_1, W_2\rangle(t) = \rho(M(t))dt$. Here, $\rho(M(t))$ represents the correlation between the two Brownian motions $W_1(t)$ and $W_2(t)$ so that

$$\rho(M(t)) = \sum_{j=1}^{N} \rho_j \mathbb{1}_{\{M(t)=j\}},$$

with $\rho_j \in (-1, 1)$ for $j = 1, 2, \ldots, N$. Importantly, the Markov process $M(t)$ is assumed to be independent from the two Brownian motions $W_1(t)$ and $W_2(t)$. We also want to remark that $dW_2(t)$ can be rewritten as

$$\rho(M(t))dW_1(t) + \sqrt{1 - [\rho(M(t))]^2}dB(t),$$

where $\{B(t)\}_{t\in[0,T]}$ is a Brownian motion independent of $W_1(t)$. More explicitly, the dynamics of $S_1(t)$ and $S_2(t)$ are driven by the stochastic differential equations

$$\frac{dS_1(t)}{S_1(t)} = r(M(t))dt + \sigma_1(M(t))dW_1(t), \tag{3.1}$$

$$\frac{dS_2(t)}{S_2(t)} = r(M(t))dt + \sigma_2(M(t))\Big(\rho(M(t))dW_1(t) + \sqrt{1 - [\rho(M(t))]^2}dB(t)\Big). \tag{3.2}$$

If we regard the processes $Z_1(t) = \log(S_1(t)/S_1(0))$ and $Z_2(t) = \log(S_2(t)/S_2(0))$, the celebrated Itô formula leads to

$$Z_1(t) = \Lambda_1(t) + X_1(t) \quad \text{and} \quad Z_2(t) = \Lambda_2(t) + X_2(t),$$

where

$$\Lambda_1(t) = \int_0^t \left(r(M(u)) - \frac{1}{2}[\sigma_1(M(u))]^2\right) du,$$

$$\Lambda_2(t) = \int_0^t \left(r(M(u)) - \frac{1}{2}[\sigma_2(M(u))]^2\right) du,$$

$$X_1(t) = \int_0^t \sigma_1(M(u))dW_1(u),$$

$$X_2(t) = \int_0^t \sigma_2(M(u))\Big(\rho(M(u))dW_1(u) + \sqrt{1 - [\rho(M(u))]^2}dB(u)\Big).$$

Hence, the dynamics of the risky assets $S_i(t)$, $i = 1, 2$, can also be expressed in the exponential form:

$$S_i(t) = S_i(0)e^{Z_i(t)} = s_i e^{\Lambda_i(t)+X_i(t)}.$$

Lastly, we identify the information carried out by Brownian motions $W_1(t)$, $B(t)$ and the Markov process $M(t)$, which will be required when pricing spread options. Let σ-algebras $\{\mathcal{F}^{W_1}(t)\}_{t\in[0,T]}$, $\{\mathcal{F}^{B}(t)\}_{t\in[0,T]}$, $\{\mathcal{F}^{M}(t)\}_{t\in[0,T]}$ define the filtrations generated by $W_1(t)$, $B(t)$ and $M(t)$, and let $\mathcal{G}(t) := \mathcal{F}^{W_1}(t) \vee \mathcal{F}^{B}(t) \vee \mathcal{F}^{M}(t)$ denote the enlarged filtration for all $t \in [0, T]$.

Giving these set-up, we now introduce the Markov-modulated Kirk's approximation technique.

3.2 Pricing Spread Options via Kirk's Approximation Technique

In this section, we establish the regime-switching valuation of spread options via Kirk's approximation technique. But before giving the perspective for the corresponding regime-switching framework, we find beneficial to briefly discuss the classical Kirk's formula with a non-Markovian set-up.

3.2.1 An Overwiew under GBMs without Regime Switching

Consider a spread option whose payoff is given by $h(S_1(T), S_2(T)) = (S_1(T) - S_2(T) - K)^+$, where S_1 and S_2 are the underlying asset prices governed by GBMs, T is the maturity and K is the strike price of the option. Assume also that regime-switching is not allowed in the market. Then, the fair price of the option under the martingale measure $\mathbb{Q}$ is defined as

$$V(0) = e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[(S_1(T) - S_2(T) - K)^+\right], \tag{3.3}$$

with r being the constant risk-free interest rate. Equivalently,

$$V(0) = e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[(S_2(T) + K)\left(\frac{S_1(T)}{S_2(T) + K} - 1\right)^+\right].$$

But as commonly mentioned in the literature, there is (in general) no closed-form solution for this pricing problem (3.3), even when the underlying assets are governed by GBMs. For the exception that yields a closed-form solution in the GBM setting, one should regard the celebrated work of Margrabe [29]. Since the Kirk's approximation technique is inspired by [29], we will first recall Margrabe's approach for the sake of completeness. To briefly state, the author shows that the pricing problem under a measure $\hat{\mathbb{Q}}$, which is defined as $\frac{d\hat{\mathbb{Q}}}{d\mathbb{Q}} = e^{-\frac{1}{2}\sigma_2^2 T + \sigma_2 W_2(T)}$, is reduced to

$$V(0) = s_2\mathbb{E}^{\hat{\mathbb{Q}}}\left[\left(\frac{S_1(T)}{S_2(T)} - 1\right)^+\right].$$

In other words, spread options with zero strike (exchange option) can be priced by means of a European call whose underlying is $S_1(t)/S_2(t)$ and strike is 1. Since the process $S_1(t)/S_2(t)$ is still log-normal, one can then easily evaluate the corresponding European call option by the well known Black-Scholes formula:

$$V(0) = s_1 N(d_1) - s_2 N(d_2),$$

where

$$d_1 = \left(\frac{\log(s_1/s_2) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \right), \; d_2 = d_1 - \sigma\sqrt{T},$$
$$\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}.$$

On the other hand, when $K > 0$, we regard the process $S_1(t)/(S_2(t) + Ke^{-r(T-t)})$ as the underlying, which is not driven by a GBM anymore. For this case, Kirk [24] introduces a pricing formula for spread options, relying on Margrabe's approach when $K \ll S_2(t)$. His approximation is based on the fact that for considerably small strikes $K \ll S_2(t)$, the process $S_1(t)/(S_2(t) + Ke^{-r(T-t)})$ can be considered as log-normal. Therefore, the spread option price can be given by a Black-Scholes type formula, in which $S_1(t)/(S_2(t) + Ke^{-r(T-t)})$ is the underlying asset and strike is 1.

But as far as we know, the derivation of the formula is not carried out explicitly by Kirk. For this reason, Venkatramanan and Alexander [36] provide a derivation of Kirk's pricing formula. In particular, they examine the dynamics of the price process $S_1(t)/(S_2(t) + Ke^{-r(T-t)})$ and confirm that it is approximately log-normal when $K \ll S_2(t)$. By defining a new equivalent probability measure $\hat{\mathbb{Q}}$ with $\frac{d\hat{\mathbb{Q}}}{d\mathbb{Q}} = e^{-\frac{1}{2}\tilde{\sigma}_2^2 T + \tilde{\sigma}_2 W_2(T)}$, spread options are then valued in terms of the expectation

$$V(0) \approx (s_2 + Ke^{-r(T-t)})\mathbb{E}^{\hat{\mathbb{Q}}}\left[\left(\frac{S_1(T)}{S_2(T) + K} - 1 \right)^+ \right],$$

implying the following Kirk's formula:

$$V(0) \approx s_1 N(d_1) - (s_2 + Ke^{-rT})e^{(\tilde{r}-r)T} N(d_2),$$

where

$$d_1 = \left[\frac{\log(s_1/(s_2 + Ke^{-rT})) + \left(r - \tilde{r} + \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}} \right], \; d_2 = d_1 - \sigma\sqrt{T},$$
$$\sigma = \sqrt{(\sigma_1)^2 + (\tilde{\sigma}_2)^2 - 2\rho\sigma_1\tilde{\sigma}_2}$$
$$\tilde{r} = \frac{s_2}{(s_2 + Ke^{-rT})} r$$
$$\tilde{\sigma}_2 = \frac{s_2}{(s_2 + Ke^{-rT})} \sigma_2.$$

For more on Kirk's pricing formula, we also refer to studies [5, 10] and references therein.

3.2.2 Extension to GBMs with Regime Switching

When we allow the regime shifts in the market, to the best our knowledge there is no Markov-modulated Kirk's formula in the existing literature. This section is devoted to adapt the abovementioned arguments for MMGBMs, motivated by Venkatramanan and Alexander [36]. Briefly, we apply a change of measure under the fact that the model dynamics of $S_1(t)/(S_2(t)+Ke^{-(U(T)-U(t))})$ is approximated by a MMGBM for strikes $K \ll S_2(t)$. Hence, we derive a Black-Scholes type formula for the valuation of spread options, inspired by the study of Buffington and Elliott [7] and Elliott et al. [19].

Let $h(S_1(T), S_2(T)) = (S_1(T) - S_2(T) - K)^+$ be the payoff of a spread call written on S_1 and S_2 with maturity T and strike K. Then, pricing problem is reduced to the computation of the expectation

$$\begin{aligned} V(0) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}\left(S_1(T) - S_2(T) - K\right)^+\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}(S_2(T)+K)\left(\frac{S_1(T)}{S_2(T)+K} - 1\right)^+\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}(S_2(T)+K)\left(Y(T)-1\right)^+\right], \end{aligned}$$

where $Y(t) = S_1(t)/(S_2(t)+Ke^{-(U(T)-U(t))})$ for $t \in [0,T]$. From this representation, it is clear that if one can find a measure that eliminates the process $(S_2(T)+K)$ from the above expectation, the pricing problem is reduced to the valuation of a European option.

3.2.2.1 A Change of Measure

Therefore, we specify a new probability measure $\hat{\mathbb{Q}} \sim \mathbb{Q}$ which will enable us to express the value of a spread call in terms of a European option price.

In the following proposition, we introduce the Markov-modulated dynamics of the process $S_2(t) + Ke^{-(U(T)-U(t))}$ for $t \in [0,T]$.

Proposition 3.1. *The process* $S_2(t) + Ke^{-(U(T)-U(t))}$, $t \in [0,T]$, *evolves according* *to the following stochastic differential equation with regime-switching:*

$$\frac{d(S_2(t) + Ke^{-(U(T)-U(t))})}{S_2(t) + Ke^{-(U(T)-U(t))}} \approx \\ \tilde{r}(M(t))dt + \tilde{\sigma}_2(M(t))\Big(\rho(M(t))dW_1(t) + \sqrt{1-[\rho(M(t))]^2}dB(t)\Big), \quad (3.4)$$

where

$$\tilde{r}(M(t)) := \frac{S_2(t)}{S_2(t) + Ke^{(U(T)-U(t))}} r(M(t)),$$
$$\tilde{\sigma}_2(M(t)) := \frac{S_2(t)}{S_2(t) + Ke^{-(U(T)-U(t))}} \sigma_2(M(t)).$$

Proof. Notice that $S_2(t) + Ke^{-(U(T)-U(t))}$ has the following dynamics:

$$d(S_2(t) + Ke^{-(U(T)-U(t))}) = dS_2(t) + r(M(t))Ke^{-(U(T)-U(t))}dt \\ = S_2(t)\left[r(M(t))dt + \sigma_2(M(t))\Big(\rho(M(t))dW_1(t) + \sqrt{1-[\rho(M(t))]^2}dB(t)\Big)\right] \\ + r(M(t))Ke^{-(U(T)-U(t))}dt.$$

Since $Ke^{-(U(T)-U(t))}/(S_2(t)+Ke^{-(U(T)-U(t))}) \approx 0$ within each regime, we then have

$$\frac{d(S_2(t) + Ke^{-(U(T)-U(t))})}{S_2(t) + Ke^{-(U(T)-U(t))}} \approx \frac{S_2(t)}{S_2(t) + Ke^{-(U(T)-U(t))}}\Bigg[r(M(t))dt \\ + \sigma_2(M(t))\Big(\rho(M(t))dW_1(t) + \sqrt{1-[\rho(M(t))]^2}dB(t)\Big)\Bigg].$$

Finally if we define two processes $\tilde{r}(M(t))$ and $\tilde{\sigma}_2(M(t))$ as follows

$$\tilde{r}(M(t)) := \frac{S_2(t)}{S_2(t) + Ke^{(U(T)-U(t))}} r(M(t)),$$
$$\tilde{\sigma}_2(M(t)) := \frac{S_2(t)}{S_2(t) + Ke^{-(U(T)-U(t))}} \sigma_2(M(t)),$$

we confirm

$$\frac{d(S_2(t) + Ke^{-(U(T)-U(t))})}{S_2(t) + Ke^{-(U(T)-U(t))}} \approx \\ \tilde{r}(M(t))dt + \tilde{\sigma}_2(M(t))\Big(\rho(M(t))dW_1(t) + \sqrt{1-[\rho(M(t))]^2}dB(t)\Big).$$

□

Remark 3.1. Since we assume that $K \ll S_2(t)$, $S_2(t)/\left(S_2(t)+Ke^{(U(T)-U(t))}\right) \approx 1$, and hence the processes $\tilde{r}(M(t))$ and $\tilde{\sigma}_2(M(t))$ are approximately constant within each regime.

Remark 3.2. In the light of Proposition 3.1,

$$
\begin{aligned}
d&\left(e^{-\int_0^t \tilde{r}(M(u))du}\left(S_2(t)+Ke^{-(U(T)-U(t))}\right)\right)\\
&\approx -\tilde{r}(M(t))e^{-\int_0^t \tilde{r}(M(u))du}\left(S_2(t)+Ke^{-(U(T)-U(t))}\right)dt\\
&\quad + e^{-\int_0^t \tilde{r}(M(u))du}\left(S_2(t)+Ke^{-(U(T)-U(t))}\right)\\
&\quad \times\left(\tilde{r}(M(t))dt+\tilde{\sigma}_2(M(t))\left(\rho(M(t))dW_1(t)+\sqrt{1-[\rho(M(t))]^2}dB(t)\right)\right)\\
&= e^{-\int_0^t \tilde{r}(M(u))du}\left(S_2(t)+Ke^{-(U(T)-U(t))}\right)\tilde{\sigma}_2(M(t))\\
&\quad \times\left(\rho(M(t)dW_1(t)+\sqrt{1-[\rho(M(t))]^2}dB(t)\right)\\
&= e^{-\int_0^t \tilde{r}(M(u))du}\left(S_2(t)+Ke^{-(U(T)-U(t))}\right)\tilde{\sigma}_2(M(t))dW_2(t),
\end{aligned}
$$

where

$$dW_2(t)=\rho(M(t))dW_1(t)+\sqrt{1-[\rho(M(t))]^2}dB(t).$$

Thus, the process $e^{-\int_0^t \tilde{r}(M(u))du}\left(S_2(t)+Ke^{(U(T)-U(t))}\right)$ is approximately a local martingale. Note that since the process $[\tilde{\sigma}_2(M(t))]^2$ is bounded due to $\sigma_i^{\max} < \infty$ and $S_2(t)/\left(S_2(t)+Ke^{(U(T)-U(t))}\right) \approx 1$, the Novikov condition holds; namely,

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-\frac{1}{2}\int_0^T[\tilde{\sigma}_2(M(t))]^2dt}\right] < \infty.$$

Then, $e^{-\int_0^t \tilde{r}(M(u))du}\left(S_2(t)+Ke^{(U(T)-U(t))}\right)$ appears to be a martingale under the assumption $K \ll S_2(t)$. This result will be used to determine the Radon-Nikodym derivative in the sequel.

By means of Proposition 3.1, we will identify the dynamics of the process $Y(t) = S_1(t)/(S_2(t)+Ke^{-(U(T)-U(t))})$ for strikes $K \ll S_2(t)$.

Proposition 3.2. *The price process $Y(t) = S_1(t)/(S_2(t)+Ke^{-(U(T)-U(t))})$ for $t \in [0,T]$ is expressed by the following Markov-modulated dynamics:*

$$
\begin{aligned}
\frac{dY(t)}{Y(t)} &\approx \left(r(M(t))-\tilde{r}(M(t))+\tilde{\sigma}_2^2(M(t))-\sigma_1(M(t))\tilde{\sigma}_2(M(t))\rho(M(t))\right)dt\\
&+\left(\sigma_1(M(t))-\tilde{\sigma}_2(M(t))\rho(M(t))\right)dW_1(t)-\tilde{\sigma}_2(M(t))\sqrt{1-[\rho(M(t))]^2}dB(t).
\end{aligned}
$$

Proof. Define $C(t) := S_2(t) + Ke^{-(U(T)-U(t))}$ for simplicity. Then, by a direct application of two-dimensional Itô formula, we have

$$dY(t) = \frac{\partial Y}{\partial C}dC(t) + \frac{\partial Y}{\partial S_1}dS_1(t) + \frac{1}{2}\frac{\partial^2 Y}{\partial C^2}d[C,C](t) + \frac{1}{2}\frac{\partial^2 Y}{\partial S_1^2}d[S_1,S_1](t) + \frac{\partial^2 Y}{\partial C S_1}d[C,S_1](t).$$

Considering (3.1) and (3.4),

$$dY(t) \approx \frac{S_1(t)}{C(t)}\Big(r(M(t))dt + \sigma_1(M(t))dW_1(t)\Big) - \frac{S_1(t)}{C^2(t)}C(t)\bigg(\tilde{r}(M(t))dt + \tilde{\sigma}_2(M(t))\Big(\rho(M(t))dW_1(t) + \sqrt{1-[\rho(M(t))]^2}dB(t)\Big)\bigg) + S_1(t)\frac{C^2(t)}{C^3(t)}[\tilde{\sigma}_2(M(t))]^2dt - S_1(t)\frac{C(t)}{C^2(t)}\sigma_1(M(t))\tilde{\sigma}_2(M(t))\rho(M(t))dt.$$

Since $Y(t) = S_1(t)/(S_2(t) + Ke^{-(U(T)-U(t))})$, we conclude that

$$\frac{dY(t)}{Y(t)} \approx \Big(r(M(t)) - \tilde{r}(M(t)) + [\tilde{\sigma}_2(M(t))]^2 - \sigma_1(M(t))\tilde{\sigma}_2(M(t))\rho(M(t))\Big)dt + \Big(\sigma_1(M(t)) - \tilde{\sigma}_2(M(t))\rho(M(t))\Big)dW_1(t) - \tilde{\sigma}_2(M(t))\sqrt{1-[\rho(M(t))]^2}dB(t).$$

□

Hence, Proposition 3.2 points out that the price process $Y(t)$ given $M(T)$ is approximately log-normal, and driven by two independent Brownian motions $W_1(t)$ and $B(t)$. More precisely, $\log(Y(t)/Y(0))$ given $M(T)$ is a normal random variable with mean

$$\int_0^t \Big(r(M(t)) - \tilde{r}(M(t)) - \frac{1}{2}\left([\sigma_1(M(t))]^2 - [\tilde{\sigma}_2(M(t))]^2\right)\Big)dt$$

and variance

$$\int_0^t \Big([\sigma_1(M(t))]^2 + [\tilde{\sigma}_2(M(t))]^2 - 2\sigma_1(M(t))\tilde{\sigma}_2(M(t))\rho(M(t))\Big)dt.$$

Hereafter we introduce the following change of measure, which will be convenient in the sequel. Based on Remark 3.2, define

$$\Gamma(t) := e^{-\int_0^t \tilde{r}(M(u))du}\frac{S_2(t) + Ke^{-(U(T)-U(t))}}{s_2 + Ke^{-U(T)}}$$

for $t \in [0,T]$. Then, we consider the following Radon-Nikodym derivative:

$$\frac{d\hat{\mathbb{Q}}}{d\mathbb{Q}} = \Gamma(T) = e^{-\int_0^T \tilde{r}(M(u))du} \frac{S_2(T)+K}{s_2 + Ke^{-U(T)}}, \tag{3.5}$$

where $\hat{\mathbb{Q}}$ is the new probability measure equivalent to $\mathbb{Q}$. By applying Girsanov theorem for Markov-modulated processes,

$$\hat{W}_1(t) = W_1(t) - \int_0^t \tilde{\sigma}_2(M(u))\rho(M(u))du,$$
$$\hat{B}(t) = B(t) - \int_0^t \tilde{\sigma}_2(M(u))\sqrt{1-[\rho(M(u))]^2}du$$

are two standard Brownian motions under $\hat{\mathbb{Q}}$ [39].

In the following proposition, we give $\hat{\mathbb{Q}}$-dynamics of the price process $Y(t)$.

Proposition 3.3. *Under the probability measure* $\hat{\mathbb{Q}}$,

$$\frac{dY(t)}{Y(t)} \approx \Big(r(M(t)) - \tilde{r}(M(t))\Big)dt + \sigma(M(t))dW(t),$$

where $W(t)$ *is a* $\hat{\mathbb{Q}}$*-Brownian motion and*

$$\sigma(M(t)) = \sqrt{[\sigma_1(M(t))]^2 + [\tilde{\sigma}_2(M(t))]^2 - 2\rho(M(t))\sigma_1(M(t))\tilde{\sigma}_2(M(t))}.$$

Proof. By straightforward calculations, we have

$$\frac{dY(t)}{Y(t)} \approx \big(r(M(t)) - \tilde{r}(M(t))\big)dt + \big(\sigma_1(M(t)) - \tilde{\sigma}_2(M(t))\rho(M(t))\big)d\hat{W}_1(t) - \tilde{\sigma}_2(M(t))\sqrt{1-[\rho(M(t))]^2}d\hat{B}(t),$$

where

$$\hat{W}_1(t) = W_1(t) - \int_0^t \tilde{\sigma}_2(M(u))\rho(M(u))du,$$
$$\hat{B}(t) = B(t) - \int_0^t \tilde{\sigma}_2(M(u))\sqrt{1-[\rho(M(u))]^2}du.$$

Let

$$\sigma(M(t))dW(t) = \Big(\sigma_1(M(t)) - \tilde{\sigma}_2(M(t))\rho(M(t))\Big)d\hat{W}_1(t) - \tilde{\sigma}_2(M(t))\sqrt{1-[\rho(M(t))]^2}d\hat{B}(t).$$

As a result of Lévy characterization theorem, $W(t)$ is a $\hat{\mathbb{Q}}$-Brownian motion if

$$\left[\frac{\sigma_1(M(t)) - \tilde{\sigma}_2(M(t))\rho(M(t))}{\sigma(M(t))}\right]^2 dt + \left[\frac{\tilde{\sigma}_2(M(t))\sqrt{1-[\rho(M(t))]^2}}{\sigma(M(t))}\right]^2 dt = dt.$$

Hence,

$$\sigma(M(t)) = \sqrt{[\sigma_1(M(t))]^2 + [\tilde{\sigma}_2(M(t))]^2 - 2\rho(M(t))\sigma_1(M(t))\tilde{\sigma}_2(M(t))}.$$

□

Notice that under new measure $\hat{\mathbb{Q}}$, process $Y(t)$ given $M(T)$ is still approximately log-normal. But now it is a MMGBM with drift $r(M(t)) - \tilde{r}(M(t))$ and volatility $\sigma(M(t))$.

Below, we will examine the pricing problem under the new probability measure $\hat{\mathbb{Q}}$, inspired by the study of Buffington and Elliott [7] and Elliott et al. [19]. The results we obtain may be considered as the main contributions of this chapter.

3.2.2.2 Pricing under New Measure $\hat{\mathbb{Q}}$

As our aim is to price the spread options under the new measure $\hat{\mathbb{Q}} \sim \mathbb{Q}$, given in (3.5), we note that by applying Bayes formula

$$\begin{aligned} V(0) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}(S_2(T)+K)\,(Y(T)-1)^+\right] \\ &= \mathbb{E}^{\hat{\mathbb{Q}}}\left[\frac{d\mathbb{Q}}{d\hat{\mathbb{Q}}}e^{-U(T)}(S_2(T)+K)\,(Y(T)-1)^+\right] \\ &= \mathbb{E}^{\hat{\mathbb{Q}}}\left[(s_2+Ke^{-U(T)})e^{\int_0^T(\tilde{r}(M(u))-r(M(u))du}\,(Y(T)-1)^+\right]. \end{aligned}$$

This pricing problem adresses to the fact that when we know the whole trajectory of Markov process $M(t)$, we know the values of $s_2 + Ke^{-U(T)}$, $\int_0^T(\tilde{r}(M(u)) - r(M(u)))du$ and $\int_0^T \sigma(M(u))^2 du$ in advance, and thus the value of spread option given $M(T)$ is expressed with the price of a European call whose underlying is $Y(t)$ and strike is 1. The expected value of this conditional price then gives the required spread price.

To be more precise, law of total expectation implies that $V(0)$ can be rewritten as

$$V(0) = \mathbb{E}^{\hat{\mathbb{Q}}}\left[\mathbb{E}^{\hat{\mathbb{Q}}}\left[(s_2+Ke^{-U(T)})e^{\int_0^T(\tilde{r}(M(u))-r(M(u))du}\,(Y(T)-1)^+ \mid \mathcal{F}^M(T)\right]\right].$$

Here, the conditional expectation with respect to $\mathcal{F}^M(T)$ represents the price of the corresponding European call option. Since the process $Y(t)$ follows a MMGBM under the probability measure $\hat{\mathbb{Q}}$, we can evaluate the European option by the well-known Black-Scholes formula.

Therefore, as suggested in Buffington and Elliott [7] and Elliott et al. [19], we should take the second expectation over $\int_0^T (\tilde{r}(M(u)) - r(M(u)))du$ and $\int_0^T \sigma(M(u))^2 du$. Theorem 3.1 infers the corresponding approximate price for spread options, which may be considered as the main result of this chapter.

Theorem 3.1. *Considering the dynamics given in Proposition 3.3, the price of the spread option given $\mathcal{F}^M(T)$,*

$$V(0, U(T)) := \mathbb{E}^{\hat{\mathbb{Q}}} \left[(s_2 + Ke^{-U(T)}) e^{\int_0^T (\tilde{r}(M(u)) - r(M(u))du} (Y(T) - 1)^+ \mid \mathcal{F}^M(T) \right],$$

is calculated as

$$V(0, U(T)) = s_1 N(d_1) - (s_2 + Ke^{-U(T)}) e^{\int_0^T (\tilde{r}(M(u)) - r(M(u))du} N(d_2), \tag{3.6}$$

where N denotes the standard Normal cumulative distribution function, and

$$\begin{aligned} d_1 = \left(\int_0^T \sigma^2(M(u))du \right)^{-1/2} \Bigg[\log \left(\frac{s_1}{s_2 + Ke^{-(U(T))}} \right) - \int_0^T (\tilde{r}(M(u)) - r(M(u)))du \\ + \frac{1}{2} \int_0^T \sigma^2(M(u))du \Bigg], \\ d_2 = d_1 - \left(\int_0^T \sigma^2(M(u))du \right)^{1/2}, \\ \sigma(M(t)) = \sqrt{[\sigma_1(M(t))]^2 + [\tilde{\sigma}_2(M(t))]^2 - 2\rho(M(t))\sigma_1(M(t))\tilde{\sigma}_2(M(t))}. \end{aligned}$$

Furthermore,

$$V(0) = \mathbb{E}^{\hat{\mathbb{Q}}} \left[s_1 N(d_1) - (s_2 + Ke^{-U(T)}) e^{\int_0^T (\tilde{r}(M(u)) - r(M(u))du} N(d_2) \right]. \tag{3.7}$$

CHAPTER 4

PRICING SPREAD OPTIONS UNDER A MARKOV-MODULATED LÉVY MODEL WITHOUT SYNCHRONOUS JUMPS

In this chapter, we study the valuation of spread options when the price dynamics of risky assets are ruled by Markov-modulated Lévy processes (MMLPs) without synchronous jumps. For this purpose, we favor the study of Caldana and Fusai [8], which concentrates on only the models without regime switching. Under our modeling set-up, we propose a lower bound to the fair price of the option, by simply replacing the true exercise region with a set very close to it. This set basically relates a power function of the second asset with the first asset; its closeness to the exercise region will be ensured by the high precision of the lower bound under different regime-switching models (see Chapter 8). Importantly, this corresponding lower bound is obtained by using only a one-dimensional fast Fourier transform (FFT), which can be applied whenever the joint characteristic function of log returns are known analytically. Therefore, our methodology is very flexible in the sense that it can be employed to several regime-switching models. Note that in the GBM setting this pricing methodology can be considered a good alternative to the Kirk's approximation technique, since the joint characteristic function of the two-dimensional GBM is explicitly known.

4.1 The Market Model

This section is devoted to the model dynamics under a regime-switching framework without synchronous jumps, and we will closely follow the modeling framework of

Deelstra and Simon [16]. Recall that with a framework without sychronous jumps, a state change in the Markov process will lead to only a switch in the model parameters.

As in Chapter 3, we consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ on a finite time horizon $[0, T]$ with $\mathbb{Q}$ being the risk-neutral probability measure. We also suppose a homogeneous continuous-time Markov process $\{M(t)\}_{t\in[0,T]}$ with a finite-state space $\mathcal{S} = \{1, 2, \ldots, N\}$, generator $Q = (q_{ij})_{N\times N}$ and initial probability vector $\mathbf{p} = [p_1\, p_2\, \ldots p_N] \in \mathbb{R}^N$.

The money market account $\{S_0(t)\}_{t\in[0,T]}$ has the same dynamics as given in Chapter 3, i.e.,

$$dS_0(t) = r(M(t))S_0(t)dt, \quad S_0(0) = 1,$$

where $r(M(t))$ is the Markov-modulated interest rate process defined by

$$r(M(t)) = \sum_{k=1}^{N} r_k \mathbb{1}_{\{M(t)=k\}},$$

with $r_l > 0$ being constant within each regime l. For a vectoral notation, denote $\mathbf{r} = (r_1, r_2, \ldots, r_N)^\top$, where $\boldsymbol{u}^\top$ denotes the transpose of a vector $\boldsymbol{u}$. Equivalently,

$$S_0(t) = e^{U(t)} \quad \text{with} \quad U(t) = \int_0^t r(M(s))ds.$$

Furthermore, we assume that the 2-dimensional process $\mathbf{S}(t) = (S_1(t), S_2(t))$ of asset prices is governed by the exponential of a Markov-modulated drift $\mathbf{\Lambda}(t)$ and a Lévy process $\mathbf{X}(t)$ (with zero drift). More precisely, $\mathbf{\Lambda}(t) = (\Lambda_1(t), \Lambda_2(t))$ is a 2-dimensional drift process given by

$$\Lambda_j(t) = \int_0^t \mu_j(M(s))ds, \quad j = 1, 2,$$

where

$$\mu_j(M(t)) = \sum_{k=1}^{N} \mu_{jk} \mathbb{1}_{\{M(t)=k\}},$$

with $\boldsymbol{\mu_j} = (\mu_{j1}, \mu_{j2}, \ldots, \mu_{jN})^\top$. Additionally, $\mathbf{X}(t) = (X_1(t), X_2(t))$ is a 2-dimensional MMLP whose behaviour within k'th regime is characterized by the 2-dimensional Lévy process $\mathbf{Y}_k = (Y_{1k}, Y_{2k})$, which has the following characteristic exponent $\Phi_k(\mathbf{u})$:

$$\mathbb{E}^{\mathbb{Q}}[e^{i\langle \mathbf{u}, \mathbf{Y}_k(t)\rangle}] = e^{-\Phi_k(\mathbf{u})t} \tag{4.1}$$

for any $\mathbf{u} = (u_1, u_2) \in \mathbb{C}^2$. Here, $\mathbf{Y}_1(t), \mathbf{Y}_2(t), \ldots, \mathbf{Y}_N(t)$ and the Markov process $M(t)$ are all mutually independent.

Since in the sequel, we also need the vector of characteristic exponents, we introduce the notation

$$\mathbf{\Phi}(\mathbf{u}) = (\Phi_1(\mathbf{u}), \Phi_2(\mathbf{u}), \ldots, \Phi_N(\mathbf{u}))^\top,$$

where $\mathbf{u} = (u_1, u_2) \in \mathbb{C}^2$. Note that when $M = k$, we consider the characteristic exponent $\Phi_k(u)$.

Under the dynamics given above, the asset price process $\mathbf{S}(t) = (S_1(t), S_2(t))$ evolves according to

$$S_j(t) = S_j(0)e^{\Lambda_j(t)+X_j(t)}, \quad S_j(0) = s_j,$$

where $j = 1, 2$.

We begin with presenting a very useful result from Deelstra and Simon [16], which shows the drift condition required for the martingality of discounted asset prices. The proof of this lemma is based on another result from Deelstra and Simon [16], proven for the exact valuation of exchange and quanto options in a MMLP framework without synchronous jumps (see Lemma 2.2 in Preliminaries). In this chapter, we will use these two results very often so as to obtain a lower bound to the exact price of spread options.

Lemma 4.1 (Deelstra and Simon [16])**.** *The discounted asset price process $\tilde{S}_j(t) = e^{-U(t)}S_j(t)$ becomes a $\mathbb{Q}$-martingale if the following condition holds:*

$$\boldsymbol{\mu_j} = \mathbf{r} + \mathbf{\Phi}(-ie_j), \quad j = 1, 2, \tag{4.2}$$

where e_j is the j'th element of the canonical basis of $\mathbb{R}^2$.

4.2 Pricing via Lower Bound

In this section, our main objective is to approximate the spread option price by means of a lower bound, as in Caldana and Fusai [8], but generalizing to MMLPs without synchronous jumps.

Recall that the fair price of a two-asset spread option $V(0)$ at time $t = 0$ is given as

$$V(0) = \mathbb{E}^{\mathbb{Q}}[e^{-U(T)}(S_1(T) - S_2(T) - K)^+],$$

where K is the strike price of the option and $U(t) = \int_0^t r(M(s))ds$.

For the derivation of a lower bound under a Markov-modulated framework, we extend the idea given in Caldana and Fusai [8] to the regime-switching models. Therefore, we define a set H inspired from Bjerksund and Stensland [5]:

$$H = \left\{ \omega \in \Omega : \frac{S_1(T)}{S_2^{\alpha}(T)} > \frac{e^k}{\mathbb{E}^{\mathbb{Q}}[S_2^{\alpha}(T)]} \right\},$$

where the free parameters α and k are chosen to be

$$\alpha = \frac{F_2(0,T)}{F_2(0,T) + K}, \tag{4.3}$$

$$k = \log\left(F_2(0,T) + K\right), \tag{4.4}$$

with $F_2(0,T) = \mathbb{E}^{\mathbb{Q}}\left[S_2(T)\right]$ being the forward price of the second asset.

We consider that on the event H,

$$(S_1(T) - S_2(T) - K)\mathbb{1}_{(H)} \leq (S_1(T) - S_2(T) - K)^+$$

implying that

$$V_K^{k,\alpha}(0) := \mathbb{E}^{\mathbb{Q}}[e^{-U(T)}(S_1(T) - S_2(T) - K)\mathbb{1}_{(H)}] \leq \mathbb{E}^{\mathbb{Q}}[e^{-U(T)}(S_1(T) - S_2(T) - K)^+]. \tag{4.5}$$

As a result, $V_K^{k,\alpha}(0)$ can be viewed as a lower bound to the exact option price defined on the set H.

Before providing an explicit expression to the lower bound, we first comment on the set H. In fact, one can optimize the spread price $V_K^{k,\alpha}(0)$ with respect to the parameters α and k by leaving them unspecified. However, Caldana and Fusai [8] remark that their lower bound for which α and k are chosen as in (4.3)-(4.4) and only the models without regime-switching are considered, turns out to be very tight. Therefore, they point out that the maximization procedure with respect to α and k is not needed in practice. They also confirm their arguments by examining how close the shape of the corresponding set is to the true exercise region. Regarding our generalization to the regime-switching models, we will see in Chapter 8 that also our lower bound $V_K^{k,\alpha}(0)$ with this suitable choice of α and k addresses very tight approximations to the true opton price.

Since the lower bound given in (4.5) should be calculated through the set H, we rewrite H more explicitly by considering the following remarks:

Remark 4.1. (i) By using Lemma 2.2 with $\mathbf{a} = (0, 1)$ and $C(t) = \Lambda_2(t)$, we verify that

$$F_2(0,T) = \mathbb{E}^{\mathbb{Q}}[S_2(T)] = s_2\mathbb{E}^{\mathbb{Q}}[e^{\Lambda_2(T)+X_2(T)}] = s_2\mathbf{p}e^{(Q-A)T}\mathbf{1}, \tag{4.6}$$

where $\mathbf{1}$ is the column vector of ones and A is the diagonal matrix with $A_{jj} = \Phi_j(0, -i) - \mu_{2j}$. Using (4.6), the chosen parameters α and k can therefore be rewritten as:

$$\alpha = \frac{s_2\mathbf{p}e^{(Q-A)T}\mathbf{1}}{S_2(0)\mathbf{p}e^{(Q-A)T}\mathbf{1} + K}, \tag{4.7}$$

$$k = \log\left(s_2\mathbf{p}e^{(Q-A)T}\mathbf{1} + K\right). \tag{4.8}$$

(ii) Similarly, the expectation $\mathbb{E}^{\mathbb{Q}}[S_2^\alpha(T)]$ can be calculated by Lemma 2.2 with $\mathbf{a} = (0, \alpha)$ and $C(t) = \alpha\Lambda_2(t)$:

$$\mathbb{E}^{\mathbb{Q}}[S_2^\alpha(T)] = s_2^\alpha\mathbb{E}^{\mathbb{Q}}[e^{\alpha\Lambda_2(T)+\alpha X_2(T)}] = s_2^\alpha\mathbf{p}e^{(Q-B)T}\mathbf{1},$$

where B is the diagonal matrix with $B_{jj} = \Phi_j(0, -i\alpha) - \alpha\mu_{2j}$.

In the light of these remark, the set H can be rewritten as:

$$\begin{aligned} H &= \left\{\omega : \ln\left(S_1(T)\right) - \alpha\ln\left(S_2(T)\right) > k - \ln\left(s_2^\alpha\mathbf{p}e^{(Q-D)T}\mathbf{1}\right)\right\} \\ &= \left\{\omega : \tilde{X}_1(T) - \alpha\tilde{X}_2(T) > k - \ln\left(s_2^\alpha\mathbf{p}e^{(Q-D)T}\mathbf{1}\right)\right\}, \end{aligned} \tag{4.9}$$

where $\tilde{X}_1(T) = \ln(s_1) + \Lambda_1(T) + X_1(T)$ and $\tilde{X}_2(T) = \ln(s_2) + \Lambda_2(T) + X_2(T)$.

We now present the explicit computation of the lower bound by the following theorem.

Theorem 4.1. *The approximate price of the spread option $V_K^{k,\alpha}(0)$ is given as:*

$$V_K^{k,\alpha}(0) = \left(\frac{e^{-\delta k}}{\pi}\int_0^\infty e^{-i\gamma k}\Psi_T(\gamma;\delta,\alpha,K)d\gamma\right)^+,$$

where δ is the damping factor, α and k are given by (4.7) and (4.8), respectively, and

$$\Psi_T(\gamma;\delta,\alpha,K) = \frac{\exp\left\{(\delta+i\gamma)\ln\left(s_2^{\alpha}\mathbf{p}e^{(Q-B)T}\mathbf{1}\right)\right\}}{\delta+i\gamma}\times$$
$$\Bigg(\exp\Big\{(1+\delta+i\gamma)\ln(s_1)-\alpha(\delta+i\gamma)\ln(s_2)\Big\}\mathbf{p}e^{(Q-G_1)T}\mathbf{1}$$
$$-\exp\Big\{(\delta+i\gamma)\ln(s_1)+(1-\alpha(\delta+i\gamma))\ln(s_2)\Big\}\mathbf{p}e^{(Q-G_2)T}\mathbf{1}$$
$$-K\exp\Big\{(\delta+i\gamma)\ln(s_1)-\alpha(\delta+i\gamma)\ln(s_2)\Big\}\mathbf{p}e^{(Q-G_3)T}\mathbf{1}\Bigg),$$

with $\boldsymbol{a}_1 = (1+\delta+i\gamma, -\alpha(\delta+i\gamma))$, $\boldsymbol{a}_2 = (\delta+i\gamma, 1-\alpha(\delta+i\gamma))$, $\boldsymbol{a}_3 = (\delta+i\gamma, -\alpha(\delta+i\gamma))$ and

$$\begin{aligned}
B &= diag\left(\mathbf{\Phi}(0,-i\alpha)-\alpha\boldsymbol{\mu}_2\right),\\
G_1 &= diag\Big(\mathbf{\Phi}(-i\boldsymbol{a}_1)+\boldsymbol{r}-(1+\delta+i\gamma)\boldsymbol{\mu}_1+\alpha(\delta+i\gamma)\boldsymbol{\mu}_2\Big),\\
G_2 &= diag\Big(\mathbf{\Phi}(-i\boldsymbol{a}_2)+\boldsymbol{r}-(\delta+i\gamma)\boldsymbol{\mu}_1-(1-\alpha(\delta+i\gamma))\boldsymbol{\mu}_2\Big),\\
G_3 &= diag\Big(\mathbf{\Phi}(-i\boldsymbol{a}_3)+\boldsymbol{r}-(\delta+i\gamma)\boldsymbol{\mu}_1+\alpha(\delta+i\gamma)\boldsymbol{\mu}_2\Big).
\end{aligned}$$

Proof. Our aim is to compute the following integral:

$$\begin{aligned}
V_K^{k,\alpha}(0) &= \mathbb{E}^{\mathbb{Q}}[e^{-U(T)}(S_1(T)-S_2(T)-K)\mathbb{1}_{(H)}]\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-u}(e^{\tilde{x}_1}-e^{\tilde{x}_2}-K)\mathbb{1}_{(H)}f(\tilde{x}_1,\tilde{x}_2,u)d\tilde{x}_1d\tilde{x}_2du
\end{aligned}$$

where $f(\tilde{x}_1,\tilde{x}_2,u)$ is the density function of $(\ln(s_1)+\Lambda_1(T)+X_1(T), \ln(s_2)+\Lambda_2(T)+X_2(T), U(T))$.

Following Carr and Madan [11],

$$V_K^{k,\alpha}(0) = \frac{e^{-\delta k}}{\pi}\int_0^{\infty} e^{-i\gamma k}\Psi_T(\gamma;\delta,\alpha,K)d\gamma$$

where δ is the damping factor, the parameters α and k are defined in (4.7) and (4.8), respectively, and

$$\begin{aligned}
\Psi_T(\gamma;\delta,\alpha,K) &= \int_{-\infty}^{\infty} e^{(\delta+i\gamma)k}\mathbb{E}^{\mathbb{Q}}[e^{-U(T)}(S_1(T)-S_2(T)-K)\mathbb{1}_{(H)}]dk\\
&= \int_{-\infty}^{\infty} e^{(\delta+i\gamma)k}\int_{\mathbb{R}^3} e^{-u}(e^{\tilde{x}_1}-e^{\tilde{x}_2}-K)\mathbb{1}_{(H)}f(\tilde{x}_1,\tilde{x}_2,u)d\tilde{x}_1d\tilde{x}_2dudk,
\end{aligned}$$

being the Fourier transform of $V_K^{k,\alpha}(0)$.

In order to determine $\Psi_T(\gamma;\delta,\alpha,K)$, we first take the set H into account for the integration bounds. Next, we apply Fubini theorem and compute the integral with respect to k. These arguments yield to

$$\begin{aligned}
&\Psi_T(\gamma;\delta,\alpha,K)\\
&=\int_{\mathbb{R}^3}\int_{\alpha\tilde{x}_2+k-\ln(s_2^\alpha \mathbf{p}e^{(Q-B)T}\mathbf{1})}^{\infty} e^{-u+(\delta+i\gamma)k}(e^{\tilde{x}_1}-e^{\tilde{x}_2}-K)f(\tilde{x}_1,\tilde{x}_2,u)d\tilde{x}_1 d\tilde{x}_2 du dk\\
&=\int_{\mathbb{R}^3}\int_{-\infty}^{\tilde{x}_1-\alpha\tilde{x}_2+\ln(s_2^\alpha \mathbf{p}e^{(Q-B)T}\mathbf{1})} e^{-u+(\delta+i\gamma)k}(e^{\tilde{x}_1}-e^{\tilde{x}_2}-K)f(\tilde{x}_1,\tilde{x}_2,u)dk d\tilde{x}_1 d\tilde{x}_2 du
\end{aligned}$$

By elaborating the product, we have

$$\begin{aligned}
&\Psi_T(\gamma;\delta,\alpha,K)\\
&=\int_{\mathbb{R}^3}\frac{e^{-u+(\delta+i\gamma)(\tilde{x}_1-\alpha\tilde{x}_2+\ln(s_2^\alpha \mathbf{p}e^{(Q-B)T}\mathbf{1}))}}{\delta+i\gamma}(e^{\tilde{x}_1}-e^{\tilde{x}_2}-K)f(\tilde{x}_1,\tilde{x}_2,u)d\tilde{x}_1 d\tilde{x}_2 du\\
&=\frac{e^{(\delta+i\gamma)\ln(s_2^\alpha \mathbf{p}e^{(Q-B)T}\mathbf{1})}}{\delta+i\gamma}\Bigg(\int_{\mathbb{R}^3} e^{-u+\tilde{x}_1(1+\delta+i\gamma)-\alpha(\delta+i\gamma)\tilde{x}_2}f(\tilde{x}_1,\tilde{x}_2,u)d\tilde{x}_1 d\tilde{x}_2 du\\
&\qquad -\int_{\mathbb{R}^3} e^{-u+\tilde{x}_1(\delta+i\gamma)+(1-\alpha(\delta+i\gamma))\tilde{x}_2}f(\tilde{x}_1,\tilde{x}_2,u)d\tilde{x}_1 d\tilde{x}_2 du\\
&\qquad -K\int_{\mathbb{R}^3} e^{-u+\tilde{x}_1(\delta+i\gamma)-\alpha(\delta+i\gamma)\tilde{x}_2}f(\tilde{x}_1,\tilde{x}_2,u)d\tilde{x}_1 d\tilde{x}_2 du\Bigg).
\end{aligned}$$

Notice that since the resulting triple integrals are indeed expectations, $\Psi_T(\gamma;\delta,\alpha,K)$ can be expressed as:

$$\begin{aligned}
\Psi_T(\gamma;\delta,\alpha,K) &= \frac{e^{(\delta+i\gamma)\ln(s_2^\alpha \mathbf{p}e^{(Q-B)T}\mathbf{1})}}{\delta+i\gamma}\Bigg(\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)+\tilde{X}_1(T)(1+\delta+i\gamma)-\alpha(\delta+i\gamma)\tilde{X}_2(T)}\right]\\
&-\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)+\tilde{X}_1(T)(\delta+i\gamma)+(1-\alpha(\delta+i\gamma))\tilde{X}_2(T)}\right]-K\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)+\tilde{X}_1(T)(\delta+i\gamma)-\alpha(\delta+i\gamma)\tilde{X}_2(T)}\right]\Bigg).
\end{aligned}$$

Here, the first expectation given above is evaluated as follows:

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)+\tilde{X}_1(T)(1+\delta+i\gamma)-\alpha(\delta+i\gamma)\tilde{X}_2(T)}\right]\\
&=\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)+(1+\delta+i\gamma)(\ln(s_1)+\Lambda_1(T)+X_1(T))-\alpha(\delta+i\gamma)(\ln(s_2)+\Lambda_2(T)+X_2(T))}\right]\\
&=e^{(1+\delta+i\gamma)\ln(s_1)-\alpha(\delta+i\gamma)\ln(s_2)}\mathbf{p}e^{(Q-G_1)T}\mathbf{1},
\end{aligned}$$

where

$$G_1=\operatorname{diag}\Big(\boldsymbol{\Phi}(-i\boldsymbol{a}_1)+\boldsymbol{r}-(1+\delta+i\gamma)\boldsymbol{\mu}_1+\alpha(\delta+i\gamma)\boldsymbol{\mu}_2\Big),$$

which is easily calculated by Lemma 2.2 with

$$C(t) = (1 + \delta + i\gamma)\Lambda_1(t) - \alpha(\delta + i\gamma)\Lambda_2(t) - U(t)$$

and $\mathbf{a} = (1 + \delta + i\gamma, -\alpha(\delta + i\gamma))$.

Similarly,

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)+(\delta+i\gamma)\tilde{X}_1(T)+(1-\alpha(\delta+i\gamma))\tilde{X}_2(T)}\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)+(\delta+i\gamma)(\ln(s_1)+\Lambda_1(T)+X_1(T))+(1-\alpha(\delta+i\gamma))(\ln(s_2)+\Lambda_2(T)+X_2(T))}\right] \\
&= e^{(\delta+i\gamma)\ln(s_1)+(1-\alpha(\delta+i\gamma))\ln(s_2)}\mathbf{p}e^{(Q-G_2)T}\mathbf{1},
\end{aligned}$$

where

$$G_2 = \text{diag}\Big(\mathbf{\Phi}(-i\boldsymbol{a}_2) + \boldsymbol{r} - (\delta + i\gamma)\boldsymbol{\mu}_1 - (1 - \alpha(\delta + i\gamma))\boldsymbol{\mu}_2\Big),$$

as a result of Lemma 2.2 with

$$C(t) = (\delta + i\gamma)\Lambda_1(t) + (1 - \alpha(\delta + i\gamma))\Lambda_2(t) - U(t)$$

and $\mathbf{a} = (\delta + i\gamma, (1 - \alpha(\delta + i\gamma)))$.

Finally, by the application of Lemma 2.2 with

$$C(t) = (\delta + i\gamma)\Lambda_1(t) - \alpha(\delta + i\gamma)\Lambda_2(t) - U(t)$$

and $\mathbf{a} = (\delta + i\gamma, -\alpha(\delta + i\gamma))$, the last expectation turns out to be equal to

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)+(\delta+i\gamma)\tilde{X}_1(T)-\alpha(\delta+i\gamma)\tilde{X}_2(T)}\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)+(\delta+i\gamma)(\ln(s_1)+\Lambda_1(T)+X_1(T))-\alpha(\delta+i\gamma)(\ln(s_2)+\Lambda_2(T)+X_2(T))}\right] \\
&= e^{(\delta+i\gamma)\ln(s_1)-\alpha(\delta+i\gamma)\ln(s_2)}\mathbf{p}e^{(Q-G_3)T}\mathbf{1},
\end{aligned}$$

where

$$G_3 = \text{diag}\Big(\mathbf{\Phi}(-i\boldsymbol{a}_3) + \boldsymbol{r} - (\delta + i\gamma)\boldsymbol{\mu}_1 + \alpha(\delta + i\gamma)\boldsymbol{\mu}_2\Big).$$

Using the results given above, we obtain the desired lower bound $V_K^{k,\alpha}(0)$.

It is worth mentioning that by means of the positive part in the formula we guarantee non-negative prices, especially for out-of-the-money options. $\square$

CHAPTER 5

PRICING BASKET OPTIONS UNDER A MARKOV-MODULATED LÉVY MODEL WITHOUT SYNCHRONOUS JUMPS

In this chapter, we introduce an approximation technique for the valuation of basket options assuming a multivariate MMLP framework without synchronous jumps. This pricing methodology generalizes the study of Caldana et al. [9] which only considers the continuous-time models without regime-switching.

More precisely, we first aim to derive a lower bound on a set based on the geometric average of underlyings. As in Chapter 4, this lower bound is expressed via a univariate Fourier inversion under the assumption that the joint characteristic function of the multivariate log-returns is known explicitly within each state. Since we only require the joint characteristic function to be available, our approximation is very managable in the sense that it becomes valid for many different types of financial models with regime-switching. In spite of this advantage, the calculation of this lower bound adresses an optimization procedure which can increase the computational cost. Therefore, we propose the analogous bounds and approximate price estimate implied by the arithmetic-geometric mean inequality.

5.1 The Market Model

In this chapter, we adapt the modeling framework given in Chapter 4 for the case of n underlying assets in the sequel.

Consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ endowed with the risk-neutral probability measure $\mathbb{Q}$, a homogeneous continuous-time Markov process $\{M(t)\}_{t\in[0,T]}$ and a financial market comprising of a money market account$\{S_0(t)\}_{t\in[0,T]}$ and n underlying assets $\{S_j(t)\}_{t\in[0,T]}$, $j = 1, 2, \dots, n$. Assuming that the Markov process $M(t)$ and the money market account $S_0(t)$ have the same dynamics as given in Chapter 4, we specify the modeling framework of risky assets.

Let $\mathbf{S}(t) = (S_1(t), \dots, S_n(t))$ be the n-dimensional price vector where each component $S_j(t)$ is described by the exponential of a one-dimensional drift $\Lambda_j(t)$ and Lévy process $X_j(t)$. That is,

$$S_j(t) = S_j(0)e^{\Lambda_j(t)+X_j(t)}, \quad S_j(0) = s_j,$$

where $\mathbf{X}(t) = (X_1(t) \dots, X_n(t))$ is a n-dimensional MMLP such that when $M = k$, $\mathbf{X}$ is identified by the n-dimensional Lévy process $\mathbf{Y}_k = (Y_{1k}, \dots, Y_{nk})$ that has the characteristic exponent $\Phi_k(\mathbf{u})$:

$$\mathbb{E}^{\mathbb{Q}}\left[e^{i\langle \mathbf{u}, \mathbf{Y}_k(t)\rangle}\right] = e^{-\Phi_k(\mathbf{u})t}, \tag{5.1}$$

with $\mathbf{u} = (u_1, u_2, \dots, u_n) \in \mathbb{C}^n$. Herewith, the drift process is given as

$$\Lambda_j(t) = \int_0^t \mu_j(M(s))ds, \quad j = 1, 2, \dots, n,$$

where

$$\mu_j(M(t)) = \sum_{k=1}^{N} \mu_{jk}\mathbb{1}_{\{M(t)=k\}},$$

with $\boldsymbol{\mu_j} = (\mu_{j1}, \mu_{j2}, \dots, \mu_{jN})^\top$ for each j. Importantly, we assume that all source of randomness, namely, $\mathbf{Y}_1(t), \mathbf{Y}_2(t), \dots, \mathbf{Y}_N(t)$ and $M(t)$, are mutually independent.

Considering the vector of characteristic exponents

$$\boldsymbol{\Phi}(\mathbf{u}) = (\Phi_1(\mathbf{u}), \Phi_2(\mathbf{u}), \dots, \Phi_N(\mathbf{u}))^\top,$$

we now establish the martingale condition:

$$\boldsymbol{\mu_j} = \mathbf{r} + \boldsymbol{\Phi}(-ie_j), \quad j = 1, 2, \dots, n, \tag{5.2}$$

where e_j is the jth element of the canonical basis of $\mathbb{R}^n$. Notice that this condition is a straightforward generalization of the one given in Lemma 4.1, if we use Lemma 2.2 with $C(t) = \Lambda_j(t) - U(t)$ and $\mathbf{a} = e_j$.

In the following sections, we introduce some lower and upper bounds to the fair price of basket options, which can be applied to many different types of financial models with regime-switching.

5.2 Basket Option Pricing by the Use of a Lower Bound

In this section, we derive a lower bound to the fair price of basket options by using the geometric average of the underlyings, where we generalize the study of Caldana et al. [9] to the proposed MMLP framework.

Consider the vector of weights $\mathbf{w} = (w_1, w_2, \ldots, w_n) \in \mathbb{R}^n$ and a value process $A_n(T)$ given by

$$A_n(T) = \sum_{j=1}^{n} w_j S_j(T).$$

The fair price (at time $t = 0$) of the basket option $V^{\text{Basket}}(0)$ with this underlying basket is well-known to be equal to

$$V^{\text{Basket}}(0) = \mathbb{E}^{\mathbb{Q}} \left[e^{-U(T)} (A_n(T) - K)^+ \right],$$

where K is the strike price and T is the maturity.

Following the idea of Caldana et al. [9], we define the set $\mathcal{G}(x)$ with $x \in \mathbb{R}$

$$\mathcal{G}(x) := \{\omega \in \Omega : H_n(T) > x\},$$

where $H_n(T)$ refers to the logarithm of the geometric average $G_n(T)$:

$$H_n(T) = \ln G_n(T) \quad \text{with} \quad G_n(T) = \prod_{j=1}^{n} S_j(T)^{w_j}. \tag{5.3}$$

Similarly as in Caldana et al. [9], we consider that the following inequality holds true for all $x \in \mathbb{R}$

$$V^{\text{Basket}}(0) \geq \mathbb{E}^{\mathbb{Q}} \left[e^{-U(T)} (A_n(T) - K) \mathbb{1}_{\mathcal{G}(x)} \right]^+ =: V_K^{\mathcal{G}}(0, x), \tag{5.4}$$

and therefore

$$V^{\text{Basket}}(0) \geq \max_{x \in \mathbb{R}} V_K^{\mathcal{G}}(0, x) =: V_K^{\mathcal{G}}(0, x^*) =: V_K^{\mathcal{G}}(0), \tag{5.5}$$

where $V_K^{\mathcal{G}}(0)$ is considered as a lower bound to the exact option price and the maximum is attained at x^*.

In the following theorem, we present the explicit computation of $V_K^{\mathcal{G}}(0)$ in the settings of this chapter. We first note that if we define $R_j(T)$, for $j = 1, 2, \ldots, n$, as the log-return over $[0, T]$, namely,

$$R_j(T) = \ln\left(\frac{S_j(T)}{s_j}\right) = \Lambda_j(T) + X_j(T), \tag{5.6}$$

then the process $H_n(T)$ can be rewritten as:

$$H_n(T) = \sum_{j=1}^{n} w_j R_j(T) + H_n(0), \tag{5.7}$$

where $H_n(0) = \sum_{j=1}^{n} w_j \ln s_j$ with $s_j = S_j(0)$.

Theorem 5.1. *The lower bound price $V_K^{\mathcal{G}}(0)$ is obtained by*

$$V_K^{\mathcal{G}}(0) = \max_{x \in \mathbb{R}} V_K^{\mathcal{G}}(0, x),$$

with

$$V_K^{\mathcal{G}}(0, x) = \left(\frac{e^{-\delta x}}{\pi} \int_0^{\infty} e^{-i\gamma x} \Psi_T(\gamma; \delta, K) d\gamma\right)^+,$$

where δ is the damping factor, $\boldsymbol{a}_j = (\delta + i\gamma)\boldsymbol{w} + \boldsymbol{e}_j$ for $j = 1, 2, \ldots, n$, $\boldsymbol{b} = (\delta + i\gamma)\boldsymbol{w}$ and $\Psi_T(\gamma; \delta, K) = \Psi_T^1(\gamma; \delta, K) - \Psi_T^2(\gamma; \delta, K)$ with

$$\begin{aligned}
\Psi_T^1(\gamma; \delta, K) &= \frac{e^{(\delta+i\gamma)H_n(0)}}{\delta + i\gamma} \sum_{j=1}^{n} w_j s_j \boldsymbol{p} e^{(Q - D_j)T} \mathbf{1}, \\
\Psi_T^2(\gamma; \delta, K) &= K \frac{e^{(\delta+i\gamma)H_n(0)}}{\delta + i\gamma} \boldsymbol{p} e^{(Q-L)T} \mathbf{1}, \\
H_n(0) &= \sum_{j=1}^{n} w_j \ln s_j, \\
D_j &= \mathit{diag}\left(\Phi(-i\boldsymbol{a}_j) + \boldsymbol{r} - \boldsymbol{\mu}_j - (\delta + i\gamma)\sum_{l=1}^{n} w_l \boldsymbol{\mu}_l\right), \quad j = 1, \ldots, n, \\
L &= \mathit{diag}\left(\Phi(-i\boldsymbol{b}) + \boldsymbol{r} - (\delta + i\gamma)\sum_{l=1}^{n} w_l \boldsymbol{\mu}_l\right).
\end{aligned}$$

Proof. As in Carr and Madan [11], we easily get the expression

$$V_K^{\mathcal{G}}(0, x) = \left(\frac{e^{-\delta x}}{\pi} \int_0^{\infty} e^{-i\gamma x} \Psi_T(\gamma; \delta, K) d\gamma\right)^+,$$

where $\Psi_T(\gamma;\delta,K)$ is the Fourier transform of $V_K^{\mathcal{G}}(0,x)$ defined as

$$\begin{aligned}
\Psi_T(\gamma;\delta,K) &= \int_{-\infty}^{\infty} e^{i\gamma x+\delta x}\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}(A_n(T)-K)\mathbb{1}_{\mathcal{G}(x)}\right]dx \\
&= \int_{-\infty}^{\infty} e^{(\delta+i\gamma)x}\sum_{j=1}^{n} w_j\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}S_j(T)\mathbb{1}_{\mathcal{G}(x)}\right]dx \\
&\quad - \int_{-\infty}^{\infty} e^{(\delta+i\gamma)x}K\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}\mathbb{1}_{\mathcal{G}(x)}\right]dx \\
&:= \Psi_T^1(\gamma;\delta,K) - \Psi_T^2(\gamma;\delta,K).
\end{aligned}$$

Denoting the density function of $(R_j(T), H_n(T), U(T))$ by $f(r_j,h_n,u)$, both integrals can be easily calculated. Indeed, let us first concentrate upon the first integral $\Psi_T^1(\gamma;\delta,K)$ considering equations (5.6), (5.7) and use the following equalities:

$$\begin{aligned}
\Psi_T^1(\gamma;\delta,K) &= \int_{-\infty}^{\infty} e^{(\delta+i\gamma)x}\sum_{j=1}^{n} w_j\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)+\ln s_j+\Lambda_j(T)+X_j(T)}\mathbb{1}_{\mathcal{G}(x)}\right]dx \\
&= \int_{-\infty}^{\infty} e^{(\delta+i\gamma)x}\left(\sum_{j=1}^{n} w_j s_j\int_{-\infty}^{\infty}\int_{x}^{\infty}\int_{-\infty}^{\infty} e^{-u+r_j}f(r_j,h_n,u)dr_j dh_n du\right)dx \\
&= \sum_{j=1}^{n} w_j s_j\int_{\mathbb{R}^3}\int_{-\infty}^{h_n} e^{(\delta+i\gamma)x}e^{-u+r_j}f(r_j,h_n,u)dx dr_j dh_n du \\
&= \frac{1}{\delta+i\gamma}\sum_{j=1}^{n} w_j s_j\int_{\mathbb{R}^3} e^{(\delta+i\gamma)h_n-u+r_j}f(r_j,h_n,u)dr_j dh_n du \\
&= \frac{1}{\delta+i\gamma}\sum_{j=1}^{n} w_j s_j\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)+R_j(T)+(\delta+i\gamma)H_n(T)}\right].
\end{aligned}$$

The use of Lemma 2.2 with $C(t) = -U(t) + \Lambda_j(t) + (\delta+i\gamma)\sum_{l=1}^{n} w_l\Lambda_l(t)$ and $\mathbf{a}_j = (\delta+i\gamma)\mathbf{w} + \boldsymbol{e}_j$ leads then to an explicit expression of the expectation in the last equality:

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)+R_j(T)+(\delta+i\gamma)H_n(T)}\right] = e^{(\delta+i\gamma)H_n(0)}\boldsymbol{p}e^{(Q-D_j)T}\mathbf{1}.$$

In the same way, $\Psi_T^2(\gamma;\delta,K)$ can be reformulated as

$$\begin{aligned}
\Psi_T^2(\gamma;\delta,K) &= \frac{K}{\delta+i\gamma}\int_{\mathbb{R}^3} e^{(\delta+i\gamma)h_n-u}f(r_j,h_n,u)dr_j dh_n du \\
&= \frac{K}{\delta+i\gamma}\mathbb{E}^{\mathbb{Q}}\left[e^{(\delta+i\gamma)H_n(T)-U(T)}\right].
\end{aligned}$$

Using Lemma 2.2 with $C(t) = -U(t) + (\delta+i\gamma)\sum_{l=1}^{n} w_l\Lambda_l(t)$ and $\boldsymbol{b} = (\delta+i\gamma)\mathbf{w}$, we find

$$\mathbb{E}^{\mathbb{Q}}\left[e^{(\delta+i\gamma)H_n(T)-U(T)}\right] = e^{(\delta+i\gamma)H_n(0)}\boldsymbol{p}e^{(Q-L)T}\mathbf{1}.$$

This yields the announced expression for $V_K^{\mathcal{G}}(0,x)$. Maximizing $V_K^{\mathcal{G}}(0,x)$ over the values of x, we get the desired result. $\square$

5.3 Basket Option Pricing by the Arithmetic-Geometric Mean Inequality

When we follow the optimization procedure given in the previous section, evaluation of basket options with several underlyings can be rather time demanding. Therefore, Caldana et al. [9] also proposed fast approximations considering the arithmetic-geometric mean inequality. Therefore, we aim to extend these approximations to the basket option price as regards to the proposed Lévy-based regime-switching framework.

Consider that J^{pos} and J^{neg} are the sets of indices corresponding to positive and negative weights, respectively. Then,

$$A_n(T) = \sum_{k \in J^{\text{pos}}} w_k S_k(T) - \sum_{k \in J^{\text{neg}}} |w_k| S_k(T) = b^{\text{pos}} A_n^{\text{pos}}(T) - b^{\text{neg}} A_n^{\text{neg}}(T),$$

where

$$A_n^{\text{pos}}(T) = \frac{\sum_{k \in J^{\text{pos}}} w_k S_k(T)}{\sum_{k \in J^{\text{pos}}} w_k}, \quad b^{pos} = \sum_{k \in J^{\text{pos}}} w_k,$$

and

$$A_n^{\text{neg}}(T) = \frac{\sum_{k \in J^{\text{neg}}} |w_k| S_k(T)}{\sum_{k \in J^{\text{neg}}} |w_k|}, \quad b^{\text{neg}} = \sum_{k \in J^{\text{neg}}} |w_k|.$$

Moreover, let $\boldsymbol{w}^{\text{pos}}$ be the vector whose kth component w_k^{pos} is:

$$w_k^{\text{pos}} = \begin{cases} w_k / \sum_{k \in J^{\text{pos}}} w_k & \text{if} \quad k \in J^{\text{pos}}, \\ 0 & \text{if} \quad k \in J^{\text{neg}}. \end{cases}$$

Similarly, consider the vector $\boldsymbol{w}^{\text{neg}}$ to be the vector whose k'th component w_k^{neg} is given by:

$$w_k^{\text{neg}} = \begin{cases} |w_k| / \sum_{k \in J^{\text{neg}}} |w_k| & \text{if} \quad k \in J^{\text{neg}}, \\ 0 & \text{if} \quad k \in J^{\text{pos}}. \end{cases}$$

Additionally, assume that

$$G_n^{\text{pos}}(T) = \Pi_{k \in J^{\text{pos}}} S_k(T)^{w_k^{\text{pos}}}, \quad G_n^{\text{neg}}(T) = \Pi_{k \in J^{\text{neg}}} S_k(T)^{w_k^{\text{neg}}},$$

and define

$$H_n^{\text{pos}}(T) = \ln G_n^{\text{pos}}(T), \quad H_n^{\text{neg}}(T) = \ln G_n^{\text{neg}}(T).$$

In the present framework, we now focus on a lower bound $L_K^{AG}(0)$, an upper bound $U_K^{AG}(0)$ and an approximation $C_K^{AG}(0)$, all inspired by the study of Caldana et al. [9]. Indeed, if we consider the arithmetic-geometric mean inequality $G_n(T) \leq A_n(T)$, the bounds $L_K^{AG}(0)$, $U_K^{AG}(0)$ and the approximation $C_K^{AG}(0)$ can be defined by the following formulas, which are straightforward generalizations of the results given in Caldana et al. [9]:

$$\begin{aligned} L_K^{AG}(0) =& \mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}(b^{\text{pos}}G_n^{\text{pos}}(T) - b^{\text{neg}}G_n^{\text{neg}}(T) - K)^+\right] \\ &+ b^{\text{neg}}\left(\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}G_n^{\text{neg}}(T)\right] - \mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}A_n^{\text{neg}}(T)\right]\right), \\ U_K^{AG}(0) =& \mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}(b^{\text{pos}}G_n^{\text{pos}}(T) - b^{\text{neg}}G_n^{\text{neg}}(T) - K)^+\right] \\ &+ b^{\text{pos}}\left(\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}A_n^{\text{pos}}(T)\right] - \mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}G_n^{\text{pos}}(T)\right]\right), \\ C_K^{AG}(0) =& \mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}(b^{\text{pos}}G_n^{\text{pos}}(T) - b^{\text{neg}}G_n^{\text{neg}}(T) - K^*)^+\right], \end{aligned}$$

where $L_K^{AG}(0) \leq C_K^{AG}(0) \leq U_K^{AG}(0)$ and the strike price K^* is defined by

$$\begin{aligned} K^* = K - \mathbb{E}^{\mathbb{Q}}\left[b^{\text{pos}}A_n^{\text{pos}}(T)\right] + \mathbb{E}^{\mathbb{Q}}\left[b^{\text{pos}}G_n^{\text{pos}}(T)\right] \\ + \mathbb{E}^{\mathbb{Q}}\left[b^{\text{neg}}A_n^{\text{neg}}(T)\right] - \mathbb{E}^{\mathbb{Q}}\left[b^{\text{neg}}G_n^{\text{neg}}(T)\right]. \end{aligned}$$

The difference with the reference study lies in the fact that we now allow a regime-switching models for the asset prices and in the remaining of this section, we will study the explicit expressions for $L_K^{AG}(0)$, $U_K^{AG}(0)$ and $C_K^{AG}(0)$ in this framework.

Theorem 5.2. *The lower bound $L_K^{AG}(0)$ is given by the formula:*

$$\begin{aligned} L_K^{AG}(0) = \left(\frac{e^{-\delta k}}{\pi}\int_0^\infty e^{-i\gamma k}\Psi_T^{Geo}(\gamma;\delta,\alpha,K)d\gamma\right)^+ + b^{neg}\left(G_n^{neg}(0)\boldsymbol{p}e^{(Q-D^{neg})T}\mathbf{1}\right) \\ - \frac{b^{neg}}{\sum_{k\in J^{neg}}|w_k|}\left(\sum_{k\in J^{neg}}|w_k|s_k\boldsymbol{p}e^{(Q-L_k)T}\mathbf{1}\right) \end{aligned}$$

where δ is the damping factor,

$$\begin{aligned} \alpha &= \frac{\mathbb{E}^{\mathbb{Q}}\left[b^{neg}G_n^{neg}(T)\right]}{\mathbb{E}^{\mathbb{Q}}\left[b^{neg}G_n^{neg}(T)\right] + K}, \\ k &= \ln\left(\mathbb{E}^{\mathbb{Q}}\left[b^{neg}G_n^{neg}(T)\right] + K\right), \end{aligned} \tag{5.8}$$

$$\Psi_T^{Geo}(\gamma;\delta,\alpha,K) = \frac{\exp\left\{(\delta+i\gamma)\ln\left((b^{neg}G_n^{neg}(0))^{\alpha}\mathbf{p}e^{(Q-B^{Geo})T}\mathbf{1}\right)\right\}}{\delta+i\gamma}\times$$
$$\Bigg(\exp\left\{(1+\delta+i\gamma)h_{pos}-\alpha(\delta+i\gamma)h_{neg}\right\}\mathbf{p}e^{(Q-C_1)T}\mathbf{1}$$
$$-\exp\left\{(\delta+i\gamma)h_{pos}+(1-\alpha(\delta+i\gamma))h_{neg}\right\}\mathbf{p}e^{(Q-C_2)T}\mathbf{1}$$
$$-K\exp\left\{(\delta+i\gamma)h_{pos}-\alpha(\delta+i\gamma)h_{neg}\right\}\mathbf{p}e^{(Q-C_3)T}\mathbf{1}\Bigg),$$

with $h_{pos} = \ln(b^{pos}) + \sum_{k\in J^{pos}} w_k^{pos}\ln(s_k)$, $h_{neg} = \ln(b^{neg}) + \sum_{k\in J^{neg}} w_k^{neg}\ln(s_k)$, *and with the vectors* $\boldsymbol{a}_1 = (1+\delta+i\gamma)\boldsymbol{w}^{pos} - \alpha(\delta+i\gamma)\boldsymbol{w}^{neg}$, $\boldsymbol{a}_2 = (\delta+i\gamma)\boldsymbol{w}^{pos} + (1-\alpha(\delta+i\gamma))\boldsymbol{w}^{neg}$, $\boldsymbol{a}_3 = (\delta+i\gamma)\boldsymbol{w}^{pos} - \alpha(\delta+i\gamma)\boldsymbol{w}^{neg}$,

$$D^{neg} = diag\left(\boldsymbol{\Phi}(-i\boldsymbol{w}^{neg}) + \boldsymbol{r} - \sum_{k\in J^{neg}} w_k^{neg}\boldsymbol{\mu}_k\right),$$
$$L_k = diag\left(\boldsymbol{\Phi}(-i\boldsymbol{e}_k) + \boldsymbol{r} - \boldsymbol{\mu}_k\right),$$
$$B^{Geo} = diag\left(\boldsymbol{\Phi}(-i\alpha\boldsymbol{w}^{neg}) - \alpha\sum_{k\in J^{neg}} w_k^{neg}\boldsymbol{\mu}_k\right),$$

and

$$C_1 = diag\left(\boldsymbol{\Phi}(-i\boldsymbol{a}_1) + \boldsymbol{r} - (1+\delta+i\gamma))\sum_{k\in J^{pos}} w_k^{pos}\boldsymbol{\mu}_k + \alpha(\delta+i\gamma)\sum_{k\in J^{neg}} w_k^{neg}\boldsymbol{\mu}_k\right),$$
$$C_2 = diag\left(\boldsymbol{\Phi}(-i\boldsymbol{a}_2) + \boldsymbol{r} - (\delta+i\gamma))\sum_{k\in J^{pos}} w_k^{pos}\boldsymbol{\mu}_k - (1-\alpha(\delta+i\gamma))\sum_{k\in J^{neg}} w_k^{neg}\boldsymbol{\mu}_k\right),$$
$$C_3 = diag\left(\boldsymbol{\Phi}(-i\boldsymbol{a}_3) + \boldsymbol{r} - (\delta+i\gamma))\sum_{k\in J^{pos}} w_k^{pos}\boldsymbol{\mu}_k + \alpha(\delta+i\gamma)\sum_{k\in J^{neg}} w_k^{neg}\boldsymbol{\mu}_k\right).$$

Proof. Noting that

$$G_n^{\text{neg}}(T) = \Pi_{k\in J^{\text{neg}}}\left(e^{\ln(s_k)+\Lambda_k(T)+X_k(T)}\right)^{w_k^{\text{neg}}},$$

the expectation $\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}G_n^{\text{neg}}(T)\right]$ can be calculated by using Lemma 2.2 with $\mathbf{a} = \boldsymbol{w}^{\text{neg}}$ and $C(t) = -U(t) + \sum_{k\in J^{\text{neg}}} w_k^{\text{neg}}\Lambda_k(t)$. Hence, analogously to the calculations performed in the previous sections, we obtain

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}G_n^{\text{neg}}(T)\right] = \Pi_{k\in J^{\text{neg}}} s_k^{w_k^{\text{neg}}}\mathbf{p}e^{\left(Q-D^{\text{neg}}\right)T}\mathbf{1}. \tag{5.9}$$

In order to determine the expectation $\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}A_n^{\text{neg}}(T)\right]$, we firstly point out that

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}A_n^{\text{neg}}(T)\right] = \frac{1}{\sum_{k\in J^{\text{neg}}}|w_k|}\sum_{k\in J^{\text{neg}}}|w_k|s_k\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)+\Lambda_k(T)+X_k(T)}\right].$$

The expectation $\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)+\Lambda_k(T)+X_k(T)}\right]$ is easily computed by Lemma 2.2 with $C(t) = -U(t) + \Lambda_k(t)$ and $\mathbf{a} = \boldsymbol{e}_k$:

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)+\Lambda_k(T)+X_k(T)}\right] = \mathbf{p}e^{(Q-L_k)T}\mathbf{1}.$$

Using this result, the expectation $\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}A_n^{\text{neg}}(T)\right]$ is then explicitly rewritten as

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}A_n^{\text{neg}}(T)\right] = \frac{1}{\sum_{k\in J^{\text{neg}}}|w_k|}\sum_{k\in J^{\text{neg}}}|w_k|s_k\mathbf{p}e^{(Q-L_k)T}\mathbf{1}.$$

Finally, we observe that the calculation of the expectation

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}(b^{\text{pos}}G_n^{\text{pos}}(T) - b^{\text{neg}}G_n^{\text{neg}}(T) - K)^+\right]$$

can be reduced to the problem of pricing a spread option with underlyings $b^{\text{pos}}G_n^{\text{pos}}(T)$ and $b^{\text{neg}}G_n^{\text{neg}}(T)$. Therefore, using the methodology given in Section 4.2, we can explicitly calculate the expectation $\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}(b^{\text{pos}}G_n^{\text{pos}}(T) - b^{\text{neg}}G_n^{\text{neg}}(T) - K)^+\right]$. To be more precise, we firstly determine a set Ξ^{Geo} yielding a lower bound $V_K^{\text{Geo}}(0)$

$$\begin{aligned}V_K^{\text{Geo}}(0) &:= \mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}(b^{\text{pos}}G_n^{\text{pos}}(T) - b^{\text{neg}}G_n^{\text{neg}}(T) - K)\mathbb{1}_{(\Xi^{\text{Geo}})}\right] \\ &\leq \mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}(b^{\text{pos}}G_n^{\text{pos}}(T) - b^{\text{neg}}G_n^{\text{neg}}(T) - K)^+\right]\end{aligned}$$

to the exact option price:

$$\Xi^{\text{Geo}} = \left\{\omega\in\Omega : \frac{b^{\text{pos}}G_n^{\text{pos}}(T)}{(b^{\text{neg}}G_n^{\text{neg}}(T))^{\alpha}} > \frac{e^k}{\mathbb{E}^{\mathbb{Q}}\left[(b^{\text{neg}}G_n^{\text{neg}}(T))^{\alpha}\right]}\right\}$$

by using α and k given by equation (5.8).

Notice that $\mathbb{E}^{\mathbb{Q}}\left[(b^{\text{neg}}G_n^{\text{neg}}(T))^{\alpha}\right]$ can be explicitly expressed by applying Lemma 2.2 with $C(t) = \alpha\sum_{k\in J^{\text{neg}}} w_k^{\text{neg}}\Lambda_k(t)$ and $\mathbf{a} = \alpha\mathbf{w}^{\text{neg}}$:

$$\mathbb{E}^{\mathbb{Q}}\left[(b^{\text{neg}}G_n^{\text{neg}}(T))^{\alpha}\right] = \left(b^{\text{neg}}\Pi_{k\in J^{\text{neg}}}s_k^{w_k^{\text{neg}}}\right)^{\alpha}\mathbf{p}e^{\left(Q-B^{\text{Geo}}\right)T}\mathbf{1}. \tag{5.10}$$

In light of this result, the set Ξ^{Geo} can be expressed as:

$$\Xi^{\text{Geo}} = \left\{\omega\in\Omega : \tilde{X}_1(T) - \alpha\tilde{X}_2(T) > k - \ln\left((b^{\text{neg}}G_n^{\text{neg}}(0))^{\alpha}\mathbf{p}e^{\left(Q-B^{\text{Geo}}\right)T}\mathbf{1}\right)\right\},$$

where

$$\begin{aligned}\tilde{X}_1(T) &= \ln\left(b^{\text{pos}}G_n^{\text{pos}}(T)\right) = \ln b^{\text{pos}} + \sum_{k\in J^{\text{pos}}} w_k^{\text{pos}}(\ln(s_k) + \Lambda_k(T) + X_k(T)),\\ \tilde{X}_2(T) &= \ln\left(b^{\text{neg}}G_n^{\text{neg}}(T)\right) = \ln b^{\text{neg}} + \sum_{k\in J^{\text{neg}}} w_k^{\text{neg}}(\ln(s_k) + \Lambda_k(T) + X_k(T)).\end{aligned}$$

Let $f(\tilde{x}_1, \tilde{x}_2, u)$ be the density function of $(\ln(b^{\text{pos}} G_n^{\text{pos}}(T)), \ln(b^{\text{neg}} G_n^{\text{neg}}(T)), U(T))$. Then, introducing a damping factor δ, as in the study of Carr and Madan [11], and applying the Fourier transform to the lower bound $V_K^{\text{Geo}}(0)$, it is clear that

$$V_K^{\text{Geo}}(0) = \frac{e^{-\delta k}}{\pi} \int_0^{\infty} e^{-i\gamma k} \Psi_T^{\text{Geo}}(\gamma; \delta, \alpha, K) \mathrm{d}\gamma, \tag{5.11}$$

where

$$\begin{aligned}\Psi_T^{\text{Geo}}(\gamma; \delta, \alpha, K) &= \int_{-\infty}^{\infty} e^{(\delta + i\gamma)k} \mathbb{E}^{\mathbb{Q}} \left[e^{-U(T)} (b^{\text{pos}} G_n^{\text{pos}}(T) - b^{\text{neg}} G_n^{\text{neg}}(T) - K) \mathbb{1}_{(\Xi^{\text{Geo}})} \right] \mathrm{d}k \\ &= \int_{-\infty}^{\infty} e^{(\delta + i\gamma)k} \int_{\mathbb{R}^3} e^{-u} (e^{\tilde{x}_1} - e^{\tilde{x}_2} - K) \mathbb{1}_{(\Xi^{\text{Geo}})} f(\tilde{x}_1, \tilde{x}_2, u) \mathrm{d}\tilde{x}_1 \mathrm{d}\tilde{x}_2 \mathrm{d}u \mathrm{d}k.\end{aligned}$$

Then, by using analogous calculations as in the previous sections, the following expression is obtained:

$$\begin{aligned}\Psi_T^{\text{Geo}}(\gamma; \delta, \alpha, K) &= \frac{\exp\left\{ (\delta + i\gamma) \ln\left((b^{\text{neg}} G_n^{\text{neg}}(0))^{\alpha} \mathbf{p} e^{(Q - B^{\text{Geo}})T} \mathbf{1} \right) \right\}}{\delta + i\gamma} \\ \times \Bigg(\mathbb{E}^{\mathbb{Q}} &\left[e^{\left(-U(T) + (1 + \delta + i\gamma) \tilde{X}_1(T) - \alpha(\delta + i\gamma) \tilde{X}_2(T) \right)} \right] - \mathbb{E}^{\mathbb{Q}} \left[e^{\left(-U(T) + (\delta + i\gamma) \tilde{X}_1(T) + (1 - \alpha(\delta + i\gamma)) \tilde{X}_2(T) \right)} \right] \\ &- K \mathbb{E}^{\mathbb{Q}} \left[e^{\left(-U(T) + (\delta + i\gamma) \tilde{X}_1(T) - \alpha(\delta + i\gamma) \tilde{X}_2(T) \right)} \right] \Bigg).\end{aligned}$$

Using Lemma 2.2, the expectations in the last equality may be rewritten as

$$\begin{aligned}&\mathbb{E}^{\mathbb{Q}} \left[e^{\left(-U(T) + (1 + \delta + i\gamma) \tilde{X}_1(T) - \alpha(\delta + i\gamma) \tilde{X}_2(T) \right)} \right] \\ &= \exp\left\{ (1 + \delta + i\gamma) h_{\text{pos}} - \alpha(\delta + i\gamma) h_{\text{neg}} \right\} \mathbf{p} e^{(Q - C_1)T} \mathbf{1}, \\ &\mathbb{E}^{\mathbb{Q}} \left[e^{\left(-U(T) + (\delta + i\gamma) \tilde{X}_1(T) + (1 - \alpha(\delta + i\gamma)) \tilde{X}_2(T) \right)} \right] \\ &= \exp\left\{ (\delta + i\gamma) h_{\text{pos}} + (1 - \alpha(\delta + i\gamma)) h_{\text{neg}} \right\} \mathbf{p} e^{(Q - C_2)T} \mathbf{1}, \\ &\mathbb{E}^{\mathbb{Q}} \left[e^{\left(-U(T) + (\delta + i\gamma) \tilde{X}_1(T) - \alpha(\delta + i\gamma) \tilde{X}_2(T) \right)} \right] \\ &= \exp\left\{ (\delta + i\gamma) h_{\text{pos}} - \alpha(\delta + i\gamma) h_{\text{neg}} \right\} \mathbf{p} e^{(Q - C_3)T} \mathbf{1}.\end{aligned}$$

These calculations imply the desired result. □

Now, the following proposition states the explicit expressions for the upper bound $U_K^{AG}(0)$ and the approximate price $C_K^{AG}(0)$. As mentioned before, the corresponding calculations are analogous to the ones given for the lower bound $L_K^{AG}(0)$.

Theorem 5.3. *The upper bound $U_K^{AG}(0)$ and the approximate price $C_K^{AG}(0)$ are given by the formulas:*

$$\begin{aligned} U_K^{AG}(0) &= \left(\frac{e^{-\delta k}}{\pi}\int_0^\infty e^{-i\gamma k}\Psi_T^{Geo}(\gamma;\delta,\alpha,K)d\gamma\right)^+ - b^{pos}\left(G_n^{pos}(0)\boldsymbol{p}e^{(Q-D^{pos})T}\mathbf{1}\right) \\ &+ \frac{b^{pos}}{\sum_{k\in J^{pos}} w_k}\left(\sum_{k\in J^{pos}} w_k s_k \boldsymbol{p}e^{(Q-L_k)T}\mathbf{1}\right), \\ C_K^{AG}(0) &= \left(\frac{e^{-\delta k^*}}{\pi}\int_0^\infty e^{-i\gamma k^*}\Psi_T^{Geo}(\gamma;\delta,\alpha^*,K^*)d\gamma\right)^+, \end{aligned}$$

where L_k, $\Psi_T^{Geo}(\gamma;\delta,\alpha,K)$, α and k are given in Theorem 5.2, α^* and k^* are obtained by replacing the strike price K in α and k by K^*, and

$$D^{pos} = diag\left(\boldsymbol{\Phi}(-i\boldsymbol{w}^{pos}) + \boldsymbol{r} - \sum_{k\in J^{pos}} w_k^{pos}\boldsymbol{\mu}_k\right).$$

Proof. Analogously to the calculation of $\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}A_n^{\text{neg}}(T)\right]$ and $\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}G_n^{\text{neg}}(T)\right]$, one easily shows that

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}A_n^{\text{pos}}(T)\right] = \frac{1}{\sum_{k\in J^{\text{pos}}} w_k}\sum_{k\in J^{\text{pos}}} w_k s_k \mathbf{p}e^{(Q-L_k)T}\mathbf{1},$$

and

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}G_n^{\text{pos}}(T)\right] = \Pi_{k\in J^{\text{pos}}} s_k^{w_k^{\text{pos}}}\mathbf{p}e^{\left(Q-D^{\text{pos}}\right)T}\mathbf{1}$$

with $D^{\text{pos}} = \text{diag}\left(\boldsymbol{\Phi}(-i\boldsymbol{w}^{\text{pos}}) + \boldsymbol{r} - \sum_{k\in J^{\text{pos}}} w_k^{\text{pos}}\boldsymbol{\mu}_k\right)$. Indeed, we only replace the terms corresponding to negative weights with the ones of positive weigths. These yield the desired upper bound $U_K^{AG}(0)$.

For the approximation

$$C_K^{AG}(0) = \mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}(b^{\text{pos}}G_n^{\text{pos}}(T) - b^{\text{neg}}G_n^{\text{neg}}(T) - K^*)^+\right],$$

we follow the approach of spread option pricing as in the derivation of the bounds $L_K^{AG}(0)$ and $U_K^{AG}(0)$, but with a different strike price K^*. Indeed, the result follows by replacing the strike price K in the expressions $\Psi_T^{\text{Geo}}(\gamma;\delta,\alpha,K)$, α and k by K^*. □

Remark 5.1. It is worth mentioning that in the case of a basket option with positive weights, the expectation

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}(b^{\text{pos}}G_n^{\text{pos}}(T) - b^{\text{neg}}G_n^{\text{neg}}(T) - K)^+\right]$$

is converted into

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}(G_n(T)-K)^+\right],$$

which denotes the price of a European call option whose underlying is the geometric average of risky assets $G_n(t)$. In this regard, the approximations $L_K^{AG}(0)$, $U_K^{AG}(0)$ and $C_K^{AG}(0)$ turn to

$$\begin{aligned}
L_K^{AG}(0) =& \mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}(G_n(T)-K)^+\right], \\
U_K^{AG}(0) =& \mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}(G_n(T)-K)^+\right] + \mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}A_n(T)\right] - \mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}G_n(T)\right], \\
C_K^{AG}(0) =& \mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}(G_n(T)-K^*)^+\right],
\end{aligned}$$

where the price of the corresponding European call option is stated in Theorem A.1-1 and the expectations $\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}A_n(T)\right]$ and $\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}G_n(T)\right]$ are easily calculated by considering a similar argument given in Theorem 5.2 or Theorem 5.3.

CHAPTER 6

PRICING SPREAD OPTIONS UNDER A MARKOV-MODULATED LÉVY MODEL WITH SYNCHRONOUS JUMPS

This chapter extends the evaluation of spread options given in Chapter 4 to a more general Lévy-driven regime-switching framework, allowing asset prices to jump synchronously due to a transition in the Markov process. To be more precise, in the case of a regime change we do not only consider a switch in the model parameters, but also take into account the *possibility* of synchronous jumps in the asset prices. These synchronous jumps, therefore, address an interesting correlation between the asset price processes, resulting from the underlying Markov process.

For the motivation to synchronous jumps that arise in the asset dynamics, we refer to Chourdakis [14] who shows that these Lévy-based regime-switching models can successfully capture asymmetric volatility skews and to Hainaut and Colwell [22] who relate the so-called synchronous jumps to the events leading an immediate change in the model parameters, such as certain economic events, natural catastrophes or terrorist attacks. Hainaut and Colwell [22] also point out that these processes fit well to the time series of asset prices. In both papers, the asset prices always jump synchronously whenever a transition occurs in the Markov process. Although these studies inspire to our modeling framework, our regime-switching set-up can be considered more general than the ones they proposed in the fact that we also take into account the possibility of occuring no synchronous jump after a phase change. Our study also differs from those studies in the sense that they do not consider the evaluation of spread options under this newly proposed framework.

6.1 The Market Model

In this section, we will closely follow the modeling framework of Chapter 4. The important distinction is that we now take into account the possibility of synchronous jumps in the underlying prices when the state of the Markov process changes.

To incorporate synchronous jumps, we introduce the following set-up:

(i) We consider the Markov process M with a finite state space $\mathcal{S} = \{1, 2, \ldots, N\}$ and initial vector $\boldsymbol{p}$. Moreover, let $\boldsymbol{X} = (X_1, X_2)$ be a 2-dimensional MMLP whose behaviour when $M = k$ is governed by the 2-dimensional Lévy process $\boldsymbol{Y}_k$ with the characteristic exponent $\Phi_k(\boldsymbol{u})$:

$$\mathbb{E}^{\mathbb{Q}}\left[e^{i\langle \boldsymbol{u}, \boldsymbol{Y}_k(t)\rangle}\right] = e^{-\Phi_k(\boldsymbol{u})t},$$

where $\boldsymbol{u} = (u_1, u_2) \in \mathbb{R}^2$ and $\mathbb{Q}$ is the risk-neutral proabbility measure.

(ii) We assume that when $M = k$, a transition to phase $l \neq k$ happens at rate q_{kl} and synchronous jumps may occur at rate γ_k.

(iii) After a jump, a new state of M is chosen to modulate $\boldsymbol{X} = (X_1, X_2)$ such that the distribution of jumps and the state after jumps are governed by the matrix $G = (G_{kl})_{N\times N}$, where

$$G_{kl}(\boldsymbol{x}) \\ = \mathbb{P}\left(\text{jump of } X_1 \leq x_1, \text{jump of } X_2 \leq x_2, M = l \text{ after jump} \,|\, \text{jump in phase k}\right)$$

for $\boldsymbol{x} = (x_1, x_2)$. Notice that this construction implies $G_{kk}(\boldsymbol{x}) = 0$ for all k. We denote by $\hat{G}(\boldsymbol{u})$ the Fourier transform of G:

$$\hat{G}(\boldsymbol{u}) = \int_{\mathbb{R}^2} e^{i\langle \boldsymbol{u}, \boldsymbol{x}\rangle}\, dG(\boldsymbol{x}).$$

(iv) Finally, we denote $\Gamma = \operatorname{diag}(\gamma_k)$ and define the matrix $Q = (q_{kl})_{N\times N}$ by assuming that all diagonal elements q_{kk} are negative, and

$$\mathbb{P}\left(M(t+dt)=k, \text{no synchronous jump on } [t,t+dt] \mid M(t)=k\right)$$
$$= 1 + q_{kk}dt + o(dt),$$
$$\mathbb{P}\left(M(t+dt)=l \ \ (l \neq k), \text{no synchronous jump on } [t,t+dt] \mid M(t)=k\right)$$
$$= q_{kl}dt + o(dt),$$
$$\mathbb{P}\left(\text{synchronous jump on } [t,t+dt] \mid M(t)=k\right) = \gamma_k dt + o(dt).$$

Since the sum of probabilities above over $M(t)=k$ must be 1, we also have

$$(Q+\Gamma)\mathbf{1} = \mathbf{0},$$

where $\mathbf{1}$ is a column vector with each component equal to 1 and $\mathbf{0}$ is a column vector with each component equal to 0.

Notice that Q is now the sub-generator of the instantaneous transitions in M when there is no synchronous jump. As mentioned before, we also remark that we take into account the possibility of occurring no synchronous jumps after a phase change and this differs from the framework of Chourdakis [14] and Hainaut and Colwell [22], in which a transition in the Markov process certainly yields a synchronous jump.

In the following we state a very useful result, whose proof for the n-dimensional setting can be found in Lemma A.1. We note that this result is the extension of Lemma 2.2 [16] to the synchronous jumps with $n=2$.

Lemma 6.1. *Consider a Markov-modulated drift process* $C(t) = \int_0^t c(M(s))\,ds$ *where*

$$c(M(t)) = \sum_{k=1}^{N} c_k \mathbb{1}_{\{M(t)=k\}},$$

with $c_k,\ k=1,\ldots,N,$ *being constants.*

Then, for every $\boldsymbol{a} \in \mathbb{C}^2$ *and* $\forall t \geq 0,$ *we have*

$$\mathbb{E}\left[e^{C(t)+\langle \boldsymbol{a}, \boldsymbol{X}(t)\rangle}\right] = \boldsymbol{p} e^{\left(-A+Q+\Gamma\hat{G}(-i\boldsymbol{a})\right)t}\mathbf{1},$$

where A *is the diagonal matrix such that* $A_{kk} = \Phi_k(-i\boldsymbol{a}) - c_k$*, under the assumption that* $\hat{G}(-i\mathbf{a})$ *exists and* $\Phi_k(-i\mathbf{a})$ *is known analytically.*

We further consider the 2-dimensional price vector $\boldsymbol{S}(t) = (S_1(t), S_2(t))$ given by:

$$S_j(t) = s_j e^{\Lambda_j(t)+X_j(t)}, \quad j = 1, 2,$$

where $s_j = S_j(0)$ and $\Lambda_j(t)$ denotes the drift process:

$$\Lambda_j(t) = \int_0^t \mu_j(M(s))\, ds, \;\; \mu_j(M(t)) = \sum_{k=1}^{N} \mu_{jk} \mathbb{1}_{\{M(t)=k\}},$$

with constant coefficients μ_{jk}.

We also denote the Markov-modulated interest rate process by $r(M(t))$ and assume that

$$r(M(t)) = \sum_{k=1}^{N} r_k \mathbb{1}_{\{M(t)=k\}},$$

where the coefficients r_k are constant. Then, the integrated interest rate process $U(t)$ is given by

$$U(t) = \int_0^t r(M(s))\, ds.$$

Noting that $\boldsymbol{u}^\top$ denotes the transpose of a vector $\boldsymbol{u}$, we define:

$$\begin{aligned} \boldsymbol{r} &= (r_1, r_2, \cdots, r_N)^\top, \\ \boldsymbol{\mu}_j &= (\mu_{j1}, \mu_{j2}, \ldots, \mu_{jN})^\top, \quad j = 1, 2, \\ \boldsymbol{\Phi}(\boldsymbol{u}) &= (\Phi_1(\boldsymbol{u}), \Phi_2(\boldsymbol{u}), \cdots, \Phi_N(\boldsymbol{u}))^\top. \end{aligned}$$

Herewith, $\boldsymbol{e}_j$ denotes the jth standard basis vector of $\mathbb{R}^2$.

In the sequel, we present the sufficient constraints on the vectors $\boldsymbol{\mu}_j$ in order to have a correct model under the risk-neutral measure $\mathbb{Q}$. For the proof of the n-dimensional setting, see Lemma A.2.

Lemma 6.2. *If the vectors $\boldsymbol{\mu}_j$ are chosen as*

$$\boldsymbol{\mu}_j = \boldsymbol{r} + \boldsymbol{\Phi}(-i\boldsymbol{e}_j) + \Gamma\left(I - \hat{G}(-i\boldsymbol{e}_j)\right)\mathbf{1} \tag{6.1}$$

for $j = 1, 2$, then the processes $\left(e^{-U(t)} S_j(t)\right)_t$ are martingales under $\mathbb{Q}$, where I denotes the $N \times N$ identitty matrix.

6.2 Spread Option Pricing by the Use of a Lower Bound

In this section, we consider the valuation of spread options in a Markov-modulated Lévy framework, but now including synchronous jumps. We will follow the same considerations given in Chapter 4.

Recall that the fair price of a spread option can be approximated by the following lower bound:

$$V_K^{k,\alpha}(0) = \mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}(S_1(T) - S_2(T) - K)\mathbb{1}_{(H)}\right], \tag{6.2}$$

where

$$H = \left\{\omega \in \Omega : \frac{S_1(T)}{S_2^\alpha(T)} > \frac{e^k}{\mathbb{E}^{\mathbb{Q}}\left[S_2^\alpha(T)\right]}\right\},$$
$$\alpha = \frac{F_2(0,T)}{F_2(0,T)+K},$$
$$k = \ln\left(F_2(0,T)+K\right),$$

with $F_2(0,T) = \mathbb{E}^{\mathbb{Q}}\left[S_2(T)\right]$.

Following the arguments given in Chapter 4, we will begin with introducing a more explicit formulation for the set H. Below are the remarks to achieve this.

1. By applying Lemma 6.1 with $\mathbf{a} = (0,1)$ and $C(t) = \Lambda_2(t)$, it is easy to verify that the forward price of the second asset equals

$$\begin{aligned} F_2(0,T) = \mathbb{E}^{\mathbb{Q}}\left[S_2(T)\right] &= s_2\mathbb{E}^{\mathbb{Q}}\left[e^{\Lambda_2(T)+X_2(T)}\right] \\ &= s_2\mathbf{p}e^{(Q-A+\Gamma\hat{G}(-i\boldsymbol{a}))T}\mathbf{1}, \end{aligned} \tag{6.3}$$

   where $A = \text{diag}(\mathbf{\Phi}(0,-i) - \boldsymbol{\mu}_2)$. As a result of (6.3),

$$\alpha = \frac{s_2\mathbf{p}e^{(Q-A+\Gamma\hat{G}(0,-i))T}\mathbf{1}}{s_2\mathbf{p}e^{(Q-A+\Gamma\hat{G}(0,-i))T}\mathbf{1}+K} \tag{6.4}$$

   and

$$k = \ln\left(s_2\mathbf{p}e^{(Q-A+\Gamma\hat{G}(0,-i))T}\mathbf{1}+K\right). \tag{6.5}$$

2. Based on Lemma 6.1 with $\mathbf{a} = (0,\alpha)$ and $C(t) = \alpha\Lambda_2(t)$, we have

$$\mathbb{E}^{\mathbb{Q}}\left[S_2^\alpha(T)\right] = s_2^\alpha\mathbb{E}^{\mathbb{Q}}\left[e^{\alpha\Lambda_2(T)+\alpha X_2(T)}\right] = s_2^\alpha\mathbf{p}e^{(Q-B+\Gamma\hat{G}(0,-i\alpha))T}\mathbf{1},$$

   where $B = \text{diag}(\mathbf{\Phi}(0,-i\alpha) - \alpha\boldsymbol{\mu}_2)$.

Considering these remarks, we obtain

$$\begin{aligned} H &= \left\{\omega \in \Omega : \ln\left(S_1(T)\right) - \alpha \ln\left(S_2(T)\right) > k - \ln\left(s_2^{\alpha} \mathbf{p} e^{(Q-B+\Gamma \hat{G}(0,-i\alpha))T} \mathbf{1}\right)\right\} \\ &= \left\{\omega \in \Omega : \tilde{X}_1(T) - \alpha \tilde{X}_2(T) > k - \ln\left(s_2^{\alpha} \mathbf{p} e^{(Q-B+\Gamma \hat{G}(0,-i\alpha))T} \mathbf{1}\right)\right\}, \end{aligned}$$

where $\tilde{X}_1(T) = \ln(s_1) + \Lambda_1(T) + X_1(T)$ and $\tilde{X}_2(T) = \ln(s_2) + \Lambda_2(T) + X_2(T)$.

In the following theorem, we present an explicit expression for the lower bound $V_K^{k,\alpha}(0)$ assuming that asset prices may jump synchronously in the case of a regime change.

Theorem 6.1. *The lower bound $V_K^{k,\alpha}(0)$ equals the following expression:*

$$V_K^{k,\alpha}(0) = \left(\frac{e^{-\delta k}}{\pi} \int_0^{\infty} e^{-i\gamma k} \Psi_T(\gamma; \delta, \alpha, K) \mathrm{d}\gamma\right)^{+},$$

where δ is the damping factor, α and k are given by (6.4) and (6.5), respectively, and

$$\begin{aligned} \Psi_T(\gamma; \delta, \alpha, K) &= \frac{\exp\left\{(\delta + i\gamma) \ln\left(s_2^{\alpha} \mathbf{p} e^{(Q-B+\Gamma \hat{G}(0,-i\alpha))T} \mathbf{1}\right)\right\}}{\delta + i\gamma} \\ &\times \Bigg(\exp\left\{(1+\delta+i\gamma)\ln(s_1) - \alpha(\delta+i\gamma)\ln(s_2)\right\} \mathbf{p} e^{\left(Q-L_1+\Gamma \hat{G}(-i\boldsymbol{a}_1)\right)T} \mathbf{1} \\ &- \exp\left\{(\delta+i\gamma)\ln(s_1) + (1-\alpha(\delta+i\gamma))\ln(s_2)\right\} \mathbf{p} e^{\left(Q-L_2+\Gamma \hat{G}(-i\boldsymbol{a}_2)\right)T} \mathbf{1} \\ &- K \exp\left\{(\delta+i\gamma)\ln(s_1) - \alpha(\delta+i\gamma)\ln(s_2)\right\} \mathbf{p} e^{\left(Q-L_3+\Gamma \hat{G}(-i\boldsymbol{a}_3)\right)T} \mathbf{1} \Bigg), \end{aligned}$$

with $\boldsymbol{a}_1 = (1+\delta+i\gamma, -\alpha(\delta+i\gamma))$, $\boldsymbol{a}_2 = (\delta+i\gamma, 1-\alpha(\delta+i\gamma))$, $\boldsymbol{a}_3 = (\delta + i\gamma, -\alpha(\delta+i\gamma))$ and

$$\begin{aligned} B &= \mathit{diag}\left(\boldsymbol{\Phi}(0,-i\alpha) - \alpha \boldsymbol{\mu}_2\right), \\ L_1 &= \mathit{diag}\Big(\boldsymbol{\Phi}(-i\boldsymbol{a}_1) + \boldsymbol{r} - (1+\delta+i\gamma)\boldsymbol{\mu}_1 + \alpha(\delta+i\gamma)\boldsymbol{\mu}_2\Big), \\ L_2 &= \mathit{diag}\Big(\boldsymbol{\Phi}(-i\boldsymbol{a}_2) + \boldsymbol{r} - (\delta+i\gamma)\boldsymbol{\mu}_1 - (1-\alpha(\delta+i\gamma))\boldsymbol{\mu}_2\Big), \\ L_3 &= \mathit{diag}\Big(\boldsymbol{\Phi}(-i\boldsymbol{a}_3) + \boldsymbol{r} - (\delta+i\gamma)\boldsymbol{\mu}_1 + \alpha(\delta+i\gamma)\boldsymbol{\mu}_2\Big). \end{aligned}$$

Proof. Carr and Madan [11] formula implies that

$$V_K^{k,\alpha}(0) = \frac{e^{-\delta k}}{\pi} \int_0^{\infty} e^{-i\gamma k} \Psi_T(\gamma; \delta, \alpha, K) \mathrm{d}\gamma,$$

where δ is the damping factor, the parameters α and k are defined in (6.4) and (6.5), respectively, and

$$\begin{aligned}\Psi_T(\gamma;\delta,\alpha,K) &= \int_{-\infty}^{\infty} e^{(\delta+i\gamma)k}\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}(S_1(T)-S_2(T)-K)\mathbb{1}_{(H)}\right]\mathrm{d}k \\ &= \int_{-\infty}^{\infty} e^{(\delta+i\gamma)k}\int_{\mathbb{R}^3} e^{-u}(e^{\tilde{x}_1}-e^{\tilde{x}_2}-K)\mathbb{1}_{(H)}f(\tilde{x}_1,\tilde{x}_2,u)\mathrm{d}\tilde{x}_1\mathrm{d}\tilde{x}_2\mathrm{d}u\mathrm{d}k.\end{aligned}$$

By taking the set H into account in the boundaries of integrals and by Fubini arguments, we obtain

$$\begin{aligned}&\Psi_T(\gamma;\delta,\alpha,K) \\ &= \int_{\mathbb{R}^3}\int_{\alpha\tilde{x}_2+k-\ln(s_2^{\alpha}\mathbf{p}e^{(Q-B+\Gamma\hat{G}(0,-i\alpha))T}\mathbf{1})}^{\infty} e^{-u+(\delta+i\gamma)k}(e^{\tilde{x}_1}-e^{\tilde{x}_2}-K)f(\tilde{x}_1,\tilde{x}_2,u)\mathrm{d}\tilde{x}_1\mathrm{d}\tilde{x}_2\mathrm{d}u\mathrm{d}k \\ &= \int_{\mathbb{R}^3}\int_{-\infty}^{\tilde{x}_1-\alpha\tilde{x}_2+\ln(s_2^{\alpha}\mathbf{p}e^{(Q-B+\Gamma\hat{G}(0,-i\alpha))T}\mathbf{1})} e^{-u+(\delta+i\gamma)k}(e^{\tilde{x}_1}-e^{\tilde{x}_2}-K)f(\tilde{x}_1,\tilde{x}_2,u)\mathrm{d}k\mathrm{d}\tilde{x}_1\mathrm{d}\tilde{x}_2\mathrm{d}u.\end{aligned}$$

Elaborating the product and recognizing that the resulting integrals are expectations, we lead to the following expression for $\Psi_T(\gamma;\delta,\alpha,K)$:

$$\begin{aligned}&\Psi_T(\gamma;\delta,\alpha,K) \\ &= \int_{\mathbb{R}^3}\frac{e^{-u+(\delta+i\gamma)(\tilde{x}_1-\alpha\tilde{x}_2+\ln(s_2^{\alpha}\mathbf{p}e^{(Q-B+\Gamma\hat{G}(0,-i\alpha))T}\mathbf{1}))}}{\delta+i\gamma}(e^{\tilde{x}_1}-e^{\tilde{x}_2}-K)f(\tilde{x}_1,\tilde{x}_2,u)\mathrm{d}\tilde{x}_1\mathrm{d}\tilde{x}_2\mathrm{d}u. \\ &= \frac{e^{(\delta+i\gamma)\ln(s_2^{\alpha}\mathbf{p}e^{(Q-B+\Gamma\hat{G}(0,-i\alpha))T}\mathbf{1})}}{\delta+i\gamma}\Bigg(\int_{\mathbb{R}^3} e^{-u+\tilde{x}_1(1+\delta+i\gamma)-\alpha(\delta+i\gamma)\tilde{x}_2}f(\tilde{x}_1,\tilde{x}_2,u)d\tilde{x}_1d\tilde{x}_2du \\ &\qquad -\int_{\mathbb{R}^3} e^{-u+\tilde{x}_1(\delta+i\gamma)+(1-\alpha(\delta+i\gamma))\tilde{x}_2}f(\tilde{x}_1,\tilde{x}_2,u)d\tilde{x}_1d\tilde{x}_2du \\ &\qquad -K\int_{\mathbb{R}^3} e^{-u+\tilde{x}_1(\delta+i\gamma)-\alpha(\delta+i\gamma)\tilde{x}_2}f(\tilde{x}_1,\tilde{x}_2,u)d\tilde{x}_1d\tilde{x}_2du\Bigg) \\ &= \frac{e^{(\delta+i\gamma)\ln(s_2^{\alpha}\mathbf{p}e^{(Q-B+\Gamma\hat{G}(0,-i\alpha))T}\mathbf{1})}}{\delta+i\gamma}\Bigg(\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)+\tilde{X}_1(T)(1+\delta+i\gamma)-\alpha(\delta+i\gamma)\tilde{X}_2(T)}\right] \\ &-\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)+\tilde{X}_1(T)(\delta+i\gamma)+(1-\alpha(\delta+i\gamma))\tilde{X}_2(T)}\right]-K\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)+\tilde{X}_1(T)(\delta+i\gamma)-\alpha(\delta+i\gamma)\tilde{X}_2(T)}\right]\Bigg).\end{aligned}$$

Note that the first expectation is easily computed by using Lemma 6.1 with $C(t) = (1+\delta+i\gamma)\Lambda_1(t)-\alpha(\delta+i\gamma)\Lambda_2(t)-U(t)$ and $\mathbf{a}_1 = (1+\delta+i\gamma,-\alpha(\delta+i\gamma))$. Indeed,

$$\begin{aligned}&\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)+\tilde{X}_1(T)(1+\delta+i\gamma)-\alpha(\delta+i\gamma)\tilde{X}_2(T)}\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)+(1+\delta+i\gamma)(\ln(s_1)+\Lambda_1(T)+X_1(T))-\alpha(\delta+i\gamma)(\ln(s_2)+\Lambda_2(T)+X_2(T))}\right] \\ &= e^{(1+\delta+i\gamma)\ln(s_1)-\alpha(\delta+i\gamma)\ln(s_2)}\mathbf{p}e^{(Q-L_1+\Gamma\hat{G}(-i\mathbf{a}_1))T}\mathbf{1}.\end{aligned}$$

Analogously, by an application of Lemma 6.1 with $C(t) = (\delta + i\gamma)\Lambda_1(t) + (1 - \alpha(\delta + i\gamma))\Lambda_2(t) - U(t)$ and $\mathbf{a}_2 = (\delta + i\gamma, (1 - \alpha(\delta + i\gamma)))$, the second expectation turns out to equal

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)+(\delta+i\gamma)\tilde{X}_1(T)+(1-\alpha(\delta+i\gamma))\tilde{X}_2(T)}\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)+(\delta+i\gamma)(\ln(s_1)+\Lambda_1(T)+X_1(T))+(1-\alpha(\delta+i\gamma))(\ln(s_2)+\Lambda_2(T)+X_2(T))}\right] \\
&= e^{(\delta+i\gamma)\ln(s_1)+(1-\alpha(\delta+i\gamma))\ln(s_2)}\mathbf{p}e^{(Q-L_2+\Gamma\hat{G}(-i\mathbf{a}_2))T}\mathbf{1}.
\end{aligned}$$

The third expectation is also followed by Lemma 6.1, but now with $C(t) = (\delta + i\gamma)\Lambda_1(t) - \alpha(\delta + i\gamma)\Lambda_2(t) - U(t)$ and $\mathbf{a}_3 = (\delta + i\gamma, -\alpha(\delta + i\gamma))$:

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)+(\delta+i\gamma)\tilde{X}_1(T)-\alpha(\delta+i\gamma)\tilde{X}_2(T)}\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)+(\delta+i\gamma)(\ln(s_1)+\Lambda_1(T)+X_1(T))-\alpha(\delta+i\gamma)(\ln(s_2)+\Lambda_2(T)+X_2(T))}\right] \\
&= e^{(\delta+i\gamma)\ln(s_1)-\alpha(\delta+i\gamma)\ln(s_2)}\mathbf{p}e^{(Q-L_3+\Gamma\hat{G}(-i\mathbf{a}_3))T}\mathbf{1}.
\end{aligned}$$

When we combine these results and change the negative prices to zero, we obtain the announced pricing formula $V_K^{k,\alpha}(0)$. □

CHAPTER 7

PRICING BASKET OPTIONS UNDER A MARKOV-MODULATED LÉVY MODEL WITH SYNCHRONOUS JUMPS

In Chapter 5, we have studied at length pricing of basket options written on underlying assets whose price dynamics evolve according to MMLPs without synchronous jumps. Now, we will extend our previous results obtained for basket options to the Lévy-based regime-switching models with synchronous jumps.

7.1 The Market Model

The evaluation of basket options first requires a straightforward extension of the setting given in Chapter 6 to the n-asset case.

More precisely,

(i) We assume an n-dimensional MMLP $\boldsymbol{X} = (X_1, \ldots, X_n)$ which behaves like the n-dimensional Lévy process $\boldsymbol{Y}_k$ when $M = k$:

$$\mathbb{E}^{\mathbb{Q}}\left[e^{i\langle \boldsymbol{u}, \boldsymbol{Y}_k(t)\rangle}\right] = e^{-\Phi_k(\boldsymbol{u})t},$$

where $\Phi_k(\boldsymbol{u})$ denotes the characteristic exponent with $\boldsymbol{u} = (u_1, \ldots, u_n) \in \mathbb{R}^n$ and $\mathbb{Q}$ is the risk-neutral proabbility measure.

(ii) M is a Markov process with a finite state space $\mathcal{S} = \{1, 2, \ldots, N\}$ and initial vector $\boldsymbol{p}$. Moreover, a transition to phase $l \neq k$ happens at rate q_{kl} and synchronous jumps may occur at rate γ_k.

(iii) Matrix $G = (G_{kl})_{N\times N}$ now defines the distribution of jumps and the state after jumps for the n-dimensional process $\boldsymbol{X} = (X_1, X_2, \dots, X_n)$. Namely, we have

$$G_{kl}(\boldsymbol{x})$$
$$= \mathbb{P}\left(\text{jump of } X_1 \le x_1, \dots, \text{jump of } X_n \le x_n, M = l \text{ after jump} \,|\, M(t) = k\right),$$

where $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$ and all diagonal elements equate to zero with $G_{kk}(\boldsymbol{x}) = 0$.

In the following, we also use the Fourier transform of G, which is denoted as $\hat{G}(\boldsymbol{u})$ with

$$\hat{G}(\boldsymbol{u}) = \int_{\mathbb{R}^n} e^{i\langle \boldsymbol{u}, \boldsymbol{x}\rangle}\, dG(\boldsymbol{x}).$$

where $\boldsymbol{u} = (u_1, u_2, \dots, u_n)$.

(iv) Finally, we define the matrices $\Gamma = \text{diag}(\gamma_k)$ and $Q = (q_{kl})_{N\times N}$ by assuming that $q_{kk} < 0$ for all k and

$$\mathbb{P}\left(M(t+dt) = k, \text{no synchronous jump on } [t, t+dt] \,|\, M(t) = k\right)$$
$$= 1 + q_{kk}dt + o(dt),$$
$$\mathbb{P}\left(M(t+dt) = l \;\; (l \neq k), \text{no synchronous jump on } [t, t+dt] \,|\, M(t) = k\right)$$
$$= q_{kl}dt + o(dt),$$
$$\mathbb{P}\left(\text{synchronous jump on } [t, t+dt] \,|\, M(t) = k\right) = \gamma_k dt + o(dt).$$

Therefore, we also have

$$(Q + \Gamma)\mathbf{1} = \mathbf{0},$$

where $\mathbf{1}$ is a column vector with each component equal to 1 and $\mathbf{0}$ is a column vector with each component equal to 0.

In the sequel, price vector $\boldsymbol{S}(t) = (S_1(t), \dots, S_n(t))$ of risky assets is defined by

$$S_j(t) = s_j e^{\Lambda_j(t) + X_j(t)}, \quad j = 1, \dots, n,$$

where $s_j = S_j(0)$ and $\Lambda_j(t)$ denotes the drift process:

$$\Lambda_j(t) = \int_0^t \mu_j(M(s))\, ds, \;\; \mu_j(M(t)) = \sum_{k=1}^{N} \mu_{jk} \mathbb{1}_{\{M(t)=k\}},$$

with constant coefficients μ_{jk}. We further denote the column vector $\boldsymbol{\mu}_j = (\mu_{j1}, \mu_{j2}, \ldots, \mu_{jN})^\top$ for each j.

Moreover, we give the Markov-modulated interest rate process $r(M(t))$:

$$r(M(t)) = \sum_{k=1}^{N} r_k \mathbb{1}_{\{M(t)=k\}},$$

where the coefficients r_k are constant within each regime k. We also consider the vector $\boldsymbol{r} = (r_1, r_2, \cdots, r_N)^\top$ and integrated interest rate process $U(t)$ with

$$U(t) = \int_0^t r(M(s))\, ds.$$

Defining the column vector of characteristic exponents

$$\boldsymbol{\Phi}(\boldsymbol{u}) = (\Phi_1(\boldsymbol{u}), \Phi_2(\boldsymbol{u}), \cdots, \Phi_N(\boldsymbol{u}))^\top,$$

we finally examine the drift condition of the basket underlyings. As shown in Lemma A.2, when the vectors $\boldsymbol{\mu}_j$ are chosen as

$$\boldsymbol{\mu}_j = \boldsymbol{r} + \boldsymbol{\Phi}(-i\boldsymbol{e}_j) + \Gamma\left(I - \hat{G}(-i\boldsymbol{e}_j)\right)\mathbf{1} \tag{7.1}$$

for $j = 1, 2, \ldots, n$, then the discounted asset prices $\left(e^{-U(t)} S_j(t)\right)_t$ become martingales under $\mathbb{Q}$. Note that we denote by $\boldsymbol{e}_j$ the jth standard basis vector of $\mathbb{R}^n$.

In the following sections, we focus on pricing basket options based on the framework given above.

7.2 Basket Option Pricing by the Use of a Lower Bound

In this section, basket options are priced via a lower bound based on the geometric average of underlyings in the context of synchronous jumps. Recall that this study is inspired from the paper of Caldana et al. [9] which explores the evaluation of basket options under several models without regime-switching.

Consider $\mathbf{w} = (w_1, w_2, \ldots, w_n) \in \mathbb{R}^n$ denoting the vector of weights and the process

$$A_n(T) = \sum_{j=1}^{n} w_j S_j(T)$$

showing the value of an underlying basket.

It is well known from Chapter 5 that with $\mathcal{G}(x)$ representing the set constructed on the geometric average of underlyings, the process $V_K^{\mathcal{G}}(0)$ given by

$$V_K^{\mathcal{G}}(0) = \max_{x \in \mathbb{R}} V_K^{\mathcal{G}}(0, x) =: V_K^{\mathcal{G}}(0, x^*) \tag{7.2}$$

serves as a lower bound to the fair price of the basket option. Here, $H_n(T) = \ln G_n(T)$ with $G_n(T) = \prod_{j=1}^{n} S_j(T)^{w_j}$ and

$$V_K^{\mathcal{G}}(0, x) = \mathbb{E}^{\mathbb{Q}} \left[e^{-U(T)} (A_n(T) - K) \mathbb{1}_{\mathcal{G}(x)} \right]^+,$$

where $\mathcal{G}(x)$ is the set defined by $\mathcal{G}(x) = \{\omega \in \Omega : H_n(T) > x\}$.

Noting that $H_n(T)$ is also expressed as

$$H_n(T) = \sum_{j=1}^{n} w_j R_j(T) + H_n(0),$$

with

$$R_j(T) = \ln \left(\frac{S_j(T)}{S_j(0)} \right) = \Lambda_j(T) + X_j(T),$$
$$H_n(0) = \sum_{j=1}^{n} w_j \ln S_j(0),$$

we introduce an explicit representation of the lower bound $V_K^{\mathcal{G}}(0)$ regarding synchronous jumps:

Theorem 7.1. *The lower bound price $V_K^{\mathcal{G}}(0)$ is obtained by*

$$V_K^{\mathcal{G}}(0) = \max_{x \in \mathbb{R}} V_K^{\mathcal{G}}(0, x),$$

with

$$V_K^{\mathcal{G}}(0, x) = \left(\frac{e^{-\delta x}}{\pi} \int_0^{\infty} e^{-i\gamma x} \Psi_T(\gamma; \delta, K) d\gamma \right)^+,$$

where δ is the damping factor, $\mathbf{a}_j = (\delta + i\gamma)\mathbf{w} + \boldsymbol{e}_j$ for $j = 1, 2, \ldots, n$, $\boldsymbol{b} = (\delta + i\gamma)\mathbf{w}$ and $\Psi_T(\gamma; \delta, K) = \Psi_T^1(\gamma; \delta, K) - \Psi_T^2(\gamma; \delta, K)$ with

$$\Psi_T^1(\gamma; \delta, K) = \frac{e^{(\delta + i\gamma) H_n(0)}}{\delta + i\gamma} \sum_{j=1}^{n} w_j S_j(0) \boldsymbol{p} e^{\left(Q + \Gamma \hat{G}(-i\mathbf{a}_j) - D_j\right) T} \mathbf{1},$$

$$\Psi_T^2(\gamma; \delta, K) = K \frac{e^{(\delta + i\gamma) H_n(0)}}{\delta + i\gamma} \boldsymbol{p} e^{\left(Q + \Gamma \hat{G}(-i\boldsymbol{b}) - L\right) T} \mathbf{1},$$

where $H_n(0) = \sum_{j=1}^{n} w_j \ln S_j(0)$ *and*

$$D_j = diag\left(\Phi(-i\mathbf{a}_j) + \boldsymbol{r} - \boldsymbol{\mu}_j - (\delta + i\gamma)\sum_{l=1}^{n} w_l \boldsymbol{\mu}_l\right), \quad j = 1, \ldots, n,$$
$$L = diag\left(\Phi(-i\boldsymbol{b}) + \boldsymbol{r} - (\delta + i\gamma)\sum_{l=1}^{n} w_l \boldsymbol{\mu}_l\right).$$

Proof. As shown in Chapter 5, if we apply the well-known Carr-Madan formula [11], the lower bound $V_K^{\mathcal{G}}(0, x)$ is reduced to the computation of terms $\Psi_T^1(\gamma; \delta, K)$ and $\Psi_T^2(\gamma; \delta, K)$ in the sense that

$$V_K^{\mathcal{G}}(0, x) = \left(\frac{e^{-\delta x}}{\pi}\int_0^\infty e^{-i\gamma x}\Psi_T(\gamma; \delta, K) d\gamma\right)^+,$$

where $\Psi_T(\gamma; \delta, K) = \Psi_T^1(\gamma; \delta, K) - \Psi_T^2(\gamma; \delta, K)$,

$$\Psi_T^1(\gamma; \delta, K) = \frac{1}{\delta + i\gamma}\sum_{j=1}^{n} w_j S_j(0)\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)+R_j(T)+(\delta+i\gamma)H_n(T)}\right],$$
$$\Psi_T^2(\gamma; \delta, K) = \frac{K}{\delta + i\gamma}\mathbb{E}^{\mathbb{Q}}\left[e^{(\delta+i\gamma)H_n(T)-U(T)}\right].$$

Note that through the computation of expectations given above, the matrices Γ and $\hat{G}$, which adress the possibility of synchronous jumps, now enter into play. Indeed, Lemma A.1 with $C(t) = -U(t) + \Lambda_j(t) + (\delta + i\gamma)\sum_{l=1}^{n} w_l \Lambda_l(t)$ and $\mathbf{a}_j = (\delta + i\gamma)\mathbf{w} + \boldsymbol{e}_j$ implies that

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)+R_j(T)+(\delta+i\gamma)H_n(T)}\right] = e^{(\delta+i\gamma)H_n(0)}\boldsymbol{p}e^{\left(Q+\Gamma\hat{G}(-i\mathbf{a}_j)-D_j\right)T}\mathbf{1}.$$

Similarly, the use of Lemma A.1 with $C(t) = -U(t) + (\delta + i\gamma)\sum_{l=1}^{n} w_l \Lambda_l(t)$ and $\boldsymbol{b} = (\delta + i\gamma)\mathbf{w}$ leads to an explicit representation of the expectation in the second term $\Psi_T^2(\gamma; \delta, K)$:

$$\mathbb{E}^{\mathbb{Q}}\left[e^{(\delta+i\gamma)H_n(T)-U(T)}\right] = e^{(\delta+i\gamma)H_n(0)}\boldsymbol{p}e^{\left(Q+\Gamma\hat{G}(-i\boldsymbol{b})-L\right)T}\mathbf{1}.$$

When we combine these results and maximize $V_K^{\mathcal{G}}(0, x)$ over the values of x, we obtain the desired result. □

Since the maximization procedure we considered can make the pricing problem time consuming, we also provide faster approximations followed from the arithmetic-geometric mean inequality.

7.3 Basket Option Pricing by the Arithmetic-Geometric Mean Inequality

Using the same considerations given in Section 5.3, we denote the sets of indices corresponding to positive and negative weights by J^{pos} and J^{neg}, respectively. Then, we define

$$A_n(T) = \sum_{k \in J^{\text{pos}}} w_k S_k(T) - \sum_{k \in J^{\text{neg}}} |w_k| S_k(T) = b^{\text{pos}} A_n^{\text{pos}}(T) - b^{\text{neg}} A_n^{\text{neg}}(T),$$

where

$$A_n^{\text{pos}}(T) = \frac{\sum_{k \in J^{\text{pos}}} w_k S_k(T)}{\sum_{k \in J^{\text{pos}}} w_k}, \quad b^{pos} = \sum_{k \in J^{\text{pos}}} w_k,$$

and

$$A_n^{\text{neg}}(T) = \frac{\sum_{k \in J^{\text{neg}}} |w_k| S_k(T)}{\sum_{k \in J^{\text{neg}}} |w_k|}, \quad b^{\text{neg}} = \sum_{k \in J^{\text{neg}}} |w_k|.$$

Furthermore, we consider the vector $\boldsymbol{w}^{\text{pos}}$ whose kth component w_k^{pos} is given by

$$w_k^{\text{pos}} = \begin{cases} w_k / \sum_{k \in J^{\text{pos}}} w_k & \text{if} \quad k \in J^{\text{pos}}, \\ 0 & \text{if} \quad k \in J^{\text{neg}}. \end{cases}$$

Similarly, let the vector $\boldsymbol{w}^{\text{neg}}$ be the vector whose kth component w_k^{neg} is:

$$w_k^{\text{neg}} = \begin{cases} |w_k| / \sum_{k \in J^{\text{neg}}} |w_k| & \text{if} \quad k \in J^{\text{neg}}, \\ 0 & \text{if} \quad k \in J^{\text{pos}}. \end{cases}$$

Finally, we assume that

$$G_n^{\text{pos}}(T) = \Pi_{k \in J^{\text{pos}}} S_k(T)^{w_k^{\text{pos}}}, \quad G_n^{\text{neg}}(T) = \Pi_{k \in J^{\text{neg}}} S_k(T)^{w_k^{\text{neg}}},$$

and define

$$H_n^{\text{pos}}(T) = \ln G_n^{\text{pos}}(T), \quad H_n^{\text{neg}}(T) = \ln G_n^{\text{neg}}(T).$$

Based on these notions, we analyze the approximations

$$\begin{aligned} L_K^{AG}(0) =& \mathbb{E}^{\mathbb{Q}} \left[e^{-U(T)} (b^{\text{pos}} G_n^{\text{pos}}(T) - b^{\text{neg}} G_n^{\text{neg}}(T) - K)^+ \right] \\ &+ b^{\text{neg}} \left(\mathbb{E}^{\mathbb{Q}} \left[e^{-U(T)} G_n^{\text{neg}}(T) \right] - \mathbb{E}^{\mathbb{Q}} \left[e^{-U(T)} A_n^{\text{neg}}(T) \right] \right), \\ U_K^{AG}(0) =& \mathbb{E}^{\mathbb{Q}} \left[e^{-U(T)} (b^{\text{pos}} G_n^{\text{pos}}(T) - b^{\text{neg}} G_n^{\text{neg}}(T) - K)^+ \right] \\ &+ b^{\text{pos}} \left(\mathbb{E}^{\mathbb{Q}} \left[e^{-U(T)} A_n^{\text{pos}}(T) \right] - \mathbb{E}^{\mathbb{Q}} \left[e^{-U(T)} G_n^{\text{pos}}(T) \right] \right), \\ C_K^{AG}(0) =& \mathbb{E}^{\mathbb{Q}} \left[e^{-U(T)} (b^{\text{pos}} G_n^{\text{pos}}(T) - b^{\text{neg}} G_n^{\text{neg}}(T) - K^*)^+ \right], \end{aligned}$$

so as to price basket options in a faster way. Recall that all these approximations take the arithmetic-geometric inequality, $G_n(T) \leq A_n(T)$, into account with the relation $L_K^{AG}(0) \leq C_K^{AG}(0) \leq U_K^{AG}(0)$; and further, K^* is the strike price defined by

$$K^* = K - \mathbb{E}^{\mathbb{Q}}\left[b^{\text{pos}} A_n^{\text{pos}}(T)\right] + \mathbb{E}^{\mathbb{Q}}\left[b^{\text{pos}} G_n^{\text{pos}}(T)\right] + \mathbb{E}^{\mathbb{Q}}\left[b^{\text{neg}} A_n^{\text{neg}}(T)\right] - \mathbb{E}^{\mathbb{Q}}\left[b^{\text{neg}} G_n^{\text{neg}}(T)\right].$$

In order to interpret $L_K^{AG}(0)$, $U_K^{AG}(0)$ and $C_K^{AG}(0)$ in a more explicit way, considering the framework with synchronus jumps, we introduce the following theorems. First, we present the derivation of the lower bound $L_K^{AG}(0)$ under a MMLP setting with synchronous jumps.

Theorem 7.2. *The lower bound $L_K^{AG}(0)$ has the following explicit form:*

$$\begin{aligned} L_K^{AG}(0) = & \left(\frac{e^{-\delta k}}{\pi} \int_0^{\infty} e^{-i\gamma k} \Psi_T^{Geo}(\gamma; \delta, \alpha, K) d\gamma \right)^+ \\ & + b^{neg} \left(G_n^{neg}(0) \boldsymbol{p} e^{\left(Q + \Gamma \hat{G}(-i\boldsymbol{w}^{neg}) - D^{neg}\right)T} \mathbf{1} \right) \\ & - \frac{b^{neg}}{\sum_{k \in J^{neg}} |w_k|} \left(\sum_{k \in J^{neg}} |w_k| S_k(0) \boldsymbol{p} e^{\left(Q + \Gamma \hat{G}(-i\boldsymbol{e}_k) - L_k\right)T} \mathbf{1} \right), \end{aligned}$$

where δ is the damping factor,

$$\alpha = \frac{\mathbb{E}^{\mathbb{Q}}\left[b^{neg} G_n^{neg}(T)\right]}{\mathbb{E}^{\mathbb{Q}}\left[b^{neg} G_n^{neg}(T)\right] + K} \quad \text{and} \quad k = \ln\left(\mathbb{E}^{\mathbb{Q}}\left[b^{neg} G_n^{neg}(T)\right] + K\right), \tag{7.3}$$

and

$$\begin{aligned} \Psi_T^{Geo}(\gamma; \delta, \alpha, K) = & \frac{\exp\left\{(\delta + i\gamma) \ln\left((b^{neg} G_n^{neg}(0))^{\alpha} \mathbf{p} e^{(Q - B^{Geo} + \Gamma \hat{G}(-i\alpha \boldsymbol{w}^{neg}))T} \mathbf{1} \right) \right\}}{\delta + i\gamma} \times \\ & \Bigg(\exp\left\{(1 + \delta + i\gamma) h_{pos} - \alpha(\delta + i\gamma) h_{neg}\right\} \mathbf{p} e^{\left(Q - C_1 + \Gamma \hat{G}(-i\mathbf{a}_1)\right)T} \mathbf{1} \\ & - \exp\left\{(\delta + i\gamma) h_{pos} + (1 - \alpha(\delta + i\gamma)) h_{neg}\right\} \mathbf{p} e^{\left(Q - C_2 + \Gamma \hat{G}(-i\mathbf{a}_2)\right)T} \mathbf{1} \\ & - K \exp\left\{(\delta + i\gamma) h_{pos} - \alpha(\delta + i\gamma) h_{neg}\right\} \mathbf{p} e^{\left(Q - C_3 + \Gamma \hat{G}(-i\mathbf{a}_3)\right)T} \mathbf{1} \Bigg), \end{aligned}$$

with $h_{pos} = \ln(b^{pos}) + \sum_{k \in J^{pos}} w_k^{pos} \ln(S_k(0))$, $h_{neg} = \ln(b^{neg}) + \sum_{k \in J^{neg}} w_k^{neg} \ln(S_k(0))$, and with the vectors $\mathbf{a}_1 = (1 + \delta + i\gamma)\boldsymbol{w}^{pos} - \alpha(\delta + i\gamma)\boldsymbol{w}^{neg}$, $\mathbf{a}_2 = (\delta + i\gamma)\boldsymbol{w}^{pos} +$

$(1-\alpha(\delta+i\gamma))\boldsymbol{w}^{neg}$, $\mathbf{a}_3=(\delta+i\gamma)\boldsymbol{w}^{pos}-\alpha(\delta+i\gamma)\boldsymbol{w}^{neg}$,

$$D^{neg}=diag\left(\boldsymbol{\Phi}(-i\boldsymbol{w}^{neg})+\boldsymbol{r}-\sum_{k\in J^{neg}}w_k^{neg}\boldsymbol{\mu}_k\right),$$
$$L_k=diag\left(\boldsymbol{\Phi}(-i\boldsymbol{e}_k)+\boldsymbol{r}-\boldsymbol{\mu}_k\right),$$
$$B^{Geo}=diag\left(\boldsymbol{\Phi}(-i\alpha\boldsymbol{w}^{neg})-\alpha\sum_{k\in J^{neg}}w_k^{neg}\boldsymbol{\mu}_k\right),$$

and

$$C_1=diag\left(\boldsymbol{\Phi}(-i\mathbf{a}_1)+\boldsymbol{r}-(1+\delta+i\gamma))\sum_{k\in J^{pos}}w_k^{pos}\boldsymbol{\mu}_k+\alpha(\delta+i\gamma)\sum_{k\in J^{neg}}w_k^{neg}\boldsymbol{\mu}_k\right),$$
$$C_2=diag\left(\boldsymbol{\Phi}(-i\mathbf{a}_2)+\boldsymbol{r}-(\delta+i\gamma))\sum_{k\in J^{pos}}w_k^{pos}\boldsymbol{\mu}_k-(1-\alpha(\delta+i\gamma))\sum_{k\in J^{neg}}w_k^{neg}\boldsymbol{\mu}_k\right),$$
$$C_3=diag\left(\boldsymbol{\Phi}(-i\mathbf{a}_3)+\boldsymbol{r}-(\delta+i\gamma))\sum_{k\in J^{pos}}w_k^{pos}\boldsymbol{\mu}_k+\alpha(\delta+i\gamma)\sum_{k\in J^{neg}}w_k^{neg}\boldsymbol{\mu}_k\right).$$

Proof. We will first focus on the explicit representation of $\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}G_n^{\text{neg}}(T)\right]$. Since

$$G_n^{\text{neg}}(T)=\Pi_{k\in J^{\text{neg}}}\left(e^{\ln(S_k(0))+\Lambda_k(T)+X_k(T)}\right)^{w_k^{\text{neg}}},$$

the expectation $\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}G_n^{\text{neg}}(T)\right]$ is given in a more explicit form if we apply Lemma A.1 with $\mathbf{a}=\boldsymbol{w}^{\text{neg}}$ and $C(t)=-U(t)+\sum_{k\in J^{\text{neg}}}w_k^{\text{neg}}\Lambda_k(t)$. As a result, we have

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}G_n^{\text{neg}}(T)\right]=\Pi_{k\in J^{\text{neg}}}S_k(0)^{w_k^{\text{neg}}}\mathbf{p}e^{\left(Q-D^{\text{neg}}+\Gamma\hat{G}(-i\boldsymbol{w}^{\text{neg}})\right)T}\mathbf{1}. \tag{7.4}$$

Secondly, we compute the expectation $\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}A_n^{\text{neg}}(T)\right]$ regarding that

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}A_n^{\text{neg}}(T)\right]=\frac{1}{\sum_{k\in J^{\text{neg}}}|w_k|}\sum_{k\in J^{\text{neg}}}|w_k|S_k(0)\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)+\Lambda_k(T)+X_k(T)}\right].$$

By using Lemma A.1 with $C(t)=-U(t)+\Lambda_k(t)$ and $\mathbf{a}=\boldsymbol{e}_k$, we obtain:

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)+\Lambda_k(T)+X_k(T)}\right]=\mathbf{p}e^{(Q-L_k+\Gamma\hat{G}(-i\boldsymbol{e}_k))T}\mathbf{1},$$

and hence,

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}A_n^{\text{neg}}(T)\right]=\frac{1}{\sum_{k\in J^{\text{neg}}}|w_k|}\sum_{k\in J^{\text{neg}}}|w_k|S_k(0)\mathbf{p}e^{(Q-L_k+\Gamma\hat{G}(-i\boldsymbol{e}_k))T}\mathbf{1}.$$

The last expectation $\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}(b^{\text{pos}}G_n^{\text{pos}}(T) - b^{\text{neg}}G_n^{\text{neg}}(T) - K)^+\right]$, as in the case of no synchronous jumps, shows the value of a spread option written on the underlyings $b^{\text{pos}}G_n^{\text{pos}}(T)$ and $b^{\text{neg}}G_n^{\text{neg}}(T)$. In the line of the approach developed for spread options, we construct a set Ξ^{Geo} :

$$\Xi^{\text{Geo}} = \left\{\omega \in \Omega : \frac{b^{\text{pos}}G_n^{\text{pos}}(T)}{(b^{\text{neg}}G_n^{\text{neg}}(T))^{\alpha}} > \frac{e^k}{\mathbb{E}^{\mathbb{Q}}\left[(b^{\text{neg}}G_n^{\text{neg}}(T))^{\alpha}\right]}\right\},$$

leading the lower bound $V_K^{\text{Geo}}(0)$

$$V_K^{\text{Geo}}(0) = \mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}(b^{\text{pos}}G_n^{\text{pos}}(T) - b^{\text{neg}}G_n^{\text{neg}}(T) - K)\mathbb{1}_{(\Xi^{\text{Geo}})}\right],$$

where α and k are defined in (7.3).

As a result of Lemma A.1 with $C(t) = \alpha \sum_{k\in J^{\text{neg}}} w_k^{\text{neg}}\Lambda_k(t)$ and $\mathbf{a} = \alpha\mathbf{w}^{\text{neg}}$, we find that

$$\mathbb{E}^{\mathbb{Q}}\left[(b^{\text{neg}}G_n^{\text{neg}}(T))^{\alpha}\right] = \left(b^{\text{neg}}\Pi_{k\in J^{\text{neg}}}S_k(0)^{w_k^{\text{neg}}}\right)^{\alpha}\mathbf{p}e^{\left(Q - B^{\text{Geo}} + \Gamma\hat{G}(-i\alpha\boldsymbol{w}^{\text{neg}})\right)T}\mathbf{1}, \quad (7.5)$$

and therefore,

$$\Xi^{\text{Geo}}$$

$$= \left\{\omega \in \Omega : \tilde{X}_1(T) - \alpha\tilde{X}_2(T) > k - \ln\left((b^{\text{neg}}G_n^{\text{neg}}(0))^{\alpha}\mathbf{p}e^{\left(Q - B^{\text{Geo}} + \Gamma\hat{G}(-i\alpha\boldsymbol{w}^{\text{neg}})\right)T}\mathbf{1}\right)\right\},$$

where

$$\begin{aligned}\tilde{X}_1(T) &= \ln\left(b^{\text{pos}}G_n^{\text{pos}}(T)\right)\\ &= \ln b^{\text{pos}} + \sum_{k\in J^{\text{pos}}} w_k^{\text{pos}}(\ln\left(S_k(0)\right) + \Lambda_k(T) + X_k(T)), \qquad (7.6)\\ \tilde{X}_2(T) &= \ln\left(b^{\text{neg}}G_n^{\text{neg}}(T)\right)\\ &= \ln b^{\text{neg}} + \sum_{k\in J^{\text{neg}}} w_k^{\text{neg}}(\ln\left(S_k(0)\right) + \Lambda_k(T) + X_k(T)). \qquad (7.7)\end{aligned}$$

After stating the set Ξ^{Geo} more explicitly, we apply Carr and Madan formula [11] in order to derive the spread option price. That is, we lead to

$$V_K^{\text{Geo}}(0) = \frac{e^{-\delta k}}{\pi}\int_0^{\infty} e^{-i\gamma k}\Psi_T^{\text{Geo}}(\gamma; \delta, \alpha, K)\mathrm{d}\gamma, \quad (7.8)$$

with

$$\begin{aligned}\Psi_T^{\text{Geo}}(\gamma; \delta, \alpha, K) &= \int_{-\infty}^{\infty} e^{(\delta + i\gamma)k}\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}(b^{\text{pos}}G_n^{\text{pos}}(T) - b^{\text{neg}}G_n^{\text{neg}}(T) - K)\mathbb{1}_{(\Xi^{\text{Geo}})}\right]\mathrm{d}k\\ &= \int_{-\infty}^{\infty} e^{(\delta + i\gamma)k}\int_{\mathbb{R}^3} e^{-u}(e^{\tilde{x}_1} - e^{\tilde{x}_2} - K)\mathbb{1}_{(\Xi^{\text{Geo}})}f(\tilde{x}_1, \tilde{x}_2, u)\mathrm{d}\tilde{x}_1\mathrm{d}\tilde{x}_2\mathrm{d}u\mathrm{d}k,\end{aligned}$$

where δ is the damping factor and $f(\tilde{x}_1, \tilde{x}_2, u)$ denotes the joint density function of $(\ln(b^{\text{pos}} G_n^{\text{pos}}(T)), \ln(b^{\text{neg}} G_n^{\text{neg}}(T)), U(T))$.

As in the proofs given in the previous sections, a Fubini argument and elaboration of the product reduce the term $\Psi_T^{\text{Geo}}(\gamma; \delta, \alpha, K)$ to the computation of three expectations:

$$\Psi_T^{\text{Geo}}(\gamma; \delta, \alpha, K) = \frac{\exp\left\{(\delta + i\gamma) \ln\left((b^{\text{neg}} G_n^{\text{neg}}(0))^{\alpha} \mathbf{p} e^{(Q - B^{\text{Geo}} + \Gamma \hat{G}(-i\alpha \boldsymbol{w}^{\text{neg}}))T} \mathbf{1}\right)\right\}}{\delta + i\gamma} \times$$

$$\left(\mathbb{E}^{\mathbb{Q}}\left[e^{\left(-U(T) + (1+\delta+i\gamma)\tilde{X}_1(T) - \alpha(\delta+i\gamma)\tilde{X}_2(T)\right)}\right] - \mathbb{E}^{\mathbb{Q}}\left[e^{\left(-U(T) + (\delta+i\gamma)\tilde{X}_1(T) + (1-\alpha(\delta+i\gamma))\tilde{X}_2(T)\right)}\right]\right.$$

$$\left. - K\mathbb{E}^{\mathbb{Q}}\left[e^{\left(-U(T) + (\delta+i\gamma)\tilde{X}_1(T) - \alpha(\delta+i\gamma)\tilde{X}_2(T)\right)}\right]\right).$$

Finally, substituting $\tilde{X}_1(T)$ and $\tilde{X}_2(T)$ as expressed in (7.6) and (7.7) and using Lemma A.1, we have

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[e^{\left(-U(T) + (1+\delta+i\gamma)\tilde{X}_1(T) - \alpha(\delta+i\gamma)\tilde{X}_2(T)\right)}\right] \\
&= \exp\{(1 + \delta + i\gamma)h_{\text{pos}} - \alpha(\delta + i\gamma)h_{\text{neg}}\}\, \mathbf{p} e^{\left(Q - C_1 + \Gamma \hat{G}(-i\mathbf{a}_1)\right)T} \mathbf{1}, \\
&\mathbb{E}^{\mathbb{Q}}\left[e^{\left(-U(T) + (\delta+i\gamma)\tilde{X}_1(T) + (1-\alpha(\delta+i\gamma))\tilde{X}_2(T)\right)}\right] \\
&= \exp\{(\delta + i\gamma)h_{\text{pos}} + (1 - \alpha(\delta + i\gamma))h_{\text{neg}}\}\, \mathbf{p} e^{\left(Q - C_2 + \Gamma \hat{G}(-i\mathbf{a}_2)\right)T} \mathbf{1}, \\
&\mathbb{E}^{\mathbb{Q}}\left[e^{\left(-U(T) + (\delta+i\gamma)\tilde{X}_1(T) - \alpha(\delta+i\gamma)\tilde{X}_2(T)\right)}\right] \\
&= \exp\{(\delta + i\gamma)h_{\text{pos}} - \alpha(\delta + i\gamma)h_{\text{neg}}\}\, \mathbf{p} e^{\left(Q - C_3 + \Gamma \hat{G}(-i\mathbf{a}_3)\right)T} \mathbf{1}.
\end{aligned}$$

Combining all the results we complete the proof. □

Next theorem states the upper bound $U_K^{AG}(0)$ and the approximate price $C_K^{AG}(0)$ in the context of synchronous jumps. As mentioned before, we follow very similar arguments considered for the lower bound $L_K^{AG}(0)$.

Theorem 7.3. *The upper bound $U_K^{AG}(0)$ and the approximate price $C_K^{AG}(0)$ are given by the formulas:*

$$
\begin{aligned}
U_K^{AG}(0) = & \left(\frac{e^{-\delta k}}{\pi}\int_0^\infty e^{-i\gamma k}\Psi_T^{Geo}(\gamma;\delta,\alpha,K)d\gamma\right)^+ \\
& - b^{pos}\left(G_n^{pos}(0)\boldsymbol{p}e^{\left(Q+\Gamma\hat{G}(-i\boldsymbol{w}^{pos})-D^{pos}\right)T}\mathbf{1}\right) \\
& + \frac{b^{pos}}{\sum_{k\in J^{pos}} w_k}\left(\sum_{k\in J^{pos}} w_k S_k(0)\boldsymbol{p}e^{\left(Q+\Gamma\hat{G}(-i\boldsymbol{e}_k)-L_k\right)T}\mathbf{1}\right), \\
C_K^{AG}(0) = & \left(\frac{e^{-\delta k^*}}{\pi}\int_0^\infty e^{-i\gamma k^*}\Psi_T^{Geo}(\gamma;\delta,\alpha^*,K^*)d\gamma\right)^+,
\end{aligned}
$$

where L_k, $\Psi_T^{Geo}(\gamma;\delta,\alpha,K)$, α and k are given in Theorem 7.2, α^* and k^* are obtained by replacing the strike price K in α and k by K^* in (7.3)*, and*

$$
D^{pos} = diag\left(\boldsymbol{\Phi}(-i\boldsymbol{w}^{pos}) + \boldsymbol{r} - \sum_{k\in J^{pos}} w_k^{pos}\boldsymbol{\mu}_k\right).
$$

Proof. Similar to the computation of $\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}A_n^{\text{neg}}(T)\right]$ and $\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}G_n^{\text{neg}}(T)\right]$, which are explored in the previous theorem, we obtain that

$$
\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}A_n^{\text{pos}}(T)\right] = \frac{1}{\sum_{k\in J^{\text{pos}}} w_k}\sum_{k\in J^{\text{pos}}} w_k S_k(0)\mathbf{p}e^{(Q-L_k+\Gamma\hat{G}(-i\boldsymbol{e}_k))T}\mathbf{1},
$$

and

$$
\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}G_n^{\text{pos}}(T)\right] = \Pi_{k\in J^{\text{pos}}} S_k(0)^{w_k^{\text{pos}}}\mathbf{p}e^{\left(Q-D^{\text{pos}}+\Gamma\hat{G}(-i\boldsymbol{w}^{\text{pos}})\right)T}\mathbf{1}
$$

with $D^{\text{pos}} = \text{diag}\left(\boldsymbol{\Phi}(-i\boldsymbol{w}^{\text{pos}}) + \boldsymbol{r} - \sum_{k\in J^{\text{pos}}} w_k^{\text{pos}}\boldsymbol{\mu}_k\right)$.

Furthermore, the expectation

$$
C_K^{AG}(0) = \mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}(b^{\text{pos}}G_n^{\text{pos}}(T) - b^{\text{neg}}G_n^{\text{neg}}(T) - K^*)^+\right]
$$

represents the price of a spread option written on the underlyings of the bounds $L_K^{AG}(0)$ and $U_K^{AG}(0)$, but with a different strike price K^*. So as to find a more explicit expression of this approximation, and in order to complete the proof, we substitute the strike K^* in the expressions $\Psi_T^{\text{Geo}}(\gamma;\delta,\alpha,K)$, α and k. □

Remark 7.1. As in the case of no synchronous jumps, the approximations $L_K^{AG}(0)$,

$U_K^{AG}(0)$ and $C_K^{AG}(0)$ become

$$
\begin{aligned}
L_K^{AG}(0) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}(G_n(T)-K)^+\right], \\
U_K^{AG}(0) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}(G_n(T)-K)^+\right] + \mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}A_n(T)\right] - \mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}G_n(T)\right], \\
C_K^{AG}(0) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}(G_n(T)-K^*)^+\right],
\end{aligned}
$$

when we consider a basket option with only positive weights. Differently, the price $\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}(G_n(T)-K)^+\right]$ of the European call option is now determined as in Theorem A.1-2 and the expectations $\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}A_n(T)\right]$ and $\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}G_n(T)\right]$ are computed by using the arguments given in Theorem 7.2 or Theorem 7.3.

CHAPTER 8

NUMERICAL RESULTS

In this chapter, we study the performance of the approximate pricing formulas derived in the previous chapters by exploring different MMLP models and by using several data sets.

All numerical experiments with synchronous jumps are governed by a two-state Markov process $M(t)$ with

$$Q = \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix}, \ \Gamma = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}, \ \boldsymbol{p} = \begin{bmatrix} 1 & 0 \end{bmatrix}. \tag{8.1}$$

When we consider the case without synchronous jumps, we assume the generator

$$Q = \begin{bmatrix} -3 & 3 \\ 1 & -1 \end{bmatrix}.$$

The matrix G of the synchronous jumps distribution is assumed to be given by

$$G(\boldsymbol{x}) = \begin{bmatrix} 0 & G_{12}(\boldsymbol{x}) \\ G_{21}(\boldsymbol{x}) & 0 \end{bmatrix},$$

where the component $G_{12}(\boldsymbol{x})$ (respectively $G_{21}(\boldsymbol{x})$) specifies the distribution of jumps that may occur when there is a transition from phase 1 to 2 (respectively from phase 2 to 1). Note that the assumptions in this numerical section imply that every phase change incurs a synchronous jump, as in Hainaut and Colwell [22] (due to $q_{12} = q_{21} = 0$). We will assume that phase 1 represents a (rather) bad economic situation with high volatility (and more jumps if applicable), whereas phase 2 is assumed to be a (rather) good economic environment with low volatility.

We mainly concentrate on positive and negative exponential synchronous jumps, as in Hainaut and Colwell [22]. For the two-asset cases, when a jump occurs at the occasion of a transition from phase 1 to 2, the two components of $\boldsymbol{X}$ jump upwards. Here, the size of the jump of X_1 is exponentially distributed with parameter λ_1, and is independent of the size of the jump of X_2, which is exponentially distributed with parameter λ_2. Similarly, when there is a jump at a transition from phase 2 to 1, the two components of $\boldsymbol{X}$ jump downwards (modeling that after these downward jumps, the rather bad economic phase starts). In this case, the size of the jump of X_1 is exponentially distributed with parameter ξ_1, and is independent of the size of the jump of X_2, which is exponentially distributed with parameter ξ_2. The synchronous jumps induce dependence in the sense that both components jump at the same time and in the same direction. These specifications correspond with the following Fourier transform matrix $\hat{G}(u_1, u_2)$:

$$\hat{G}(u_1, u_2) = \begin{bmatrix} 0 & \prod_{k=1}^{2} \frac{\lambda_k}{(\lambda_k - iu_k)} \\ \prod_{k=1}^{2} \frac{\xi_k}{(\xi_k + iu_k)} & 0 \end{bmatrix}.$$

Analogous assumptions and notations lead to the matrix $\hat{G}(u_1, \ldots, u_n)$ in the n-asset case

$$\hat{G}(u_1, \ldots, u_n) = \begin{bmatrix} 0 & \prod_{k=1}^{n} \frac{\lambda_k}{(\lambda_k - iu_k)} \\ \prod_{k=1}^{n} \frac{\xi_k}{(\xi_k + iu_k)} & 0 \end{bmatrix}.$$

We remark that in all examples with 2 assets, we consider the same parameters for the exponential synchronous jumps, namely $\lambda_1 = 4.5$, $\lambda_2 = 4$, $\xi_1 = 2.7$ and $\xi_2 = 2.5$.

As an alternative, we also examine the case of normally distributed jumps (although we will only report these results in the second example). Considering two assets in this case, we assume that the means of the jump sizes of the two components of $\boldsymbol{X}$ are both positive when a jump from phase 1 to phase 2 occurs. For this transition, the size of the jump of X_1 is assumed to be normally distributed with positive mean β_{11} and variance τ_{11}^2, and is independent of the size of the jump of X_2, which is normally distributed with positive parameters β_{21} and τ_{21}^2. In the case of a transition from phase 2 to phase 1, the two components of $\boldsymbol{X}$ have on average a negative jump size. Here, the size of the jump of X_1 is assumed to be normally distributed with negative mean β_{12} and variance τ_{12}^2, and is independent of the size of the jump of X_2, which is normally distributed with negative mean β_{22} and variance τ_{22}^2. Differently from the arguments given for exponentially distributed jumps, both components do not always jump in the same direction, but the average jump sizes of both components have the same sign. In these settings, the matrix $\hat{G}(u_1, u_2)$ is given by

$$\hat{G}(u_1, u_2) = \begin{bmatrix} 0 & \prod_{k=1}^{2} e^{\left(i\beta_{k1}u_k - (1/2)\tau_{k1}^2 u_k^2\right)} \\ \prod_{k=1}^{2} e^{\left(i\beta_{k2}u_k - (1/2)\tau_{k2}^2 u_k^2\right)} & 0 \end{bmatrix}.$$

8.1 Implementation Details

This section gives a brief description about how to apply the numerical methods for the valuation of spread and basket options.

We study the accuracy of the formulas $V_K^{k,\alpha}(0)$ and $V_K^{\mathcal{G}}(0)$, derived for spread and basket options respectively, by comparing the approximations with estimates obtained by a control variate Monte Carlo (MC) technique for different exercise prices. For the spread option evaluation, the lower bound $V_K^{k,\alpha}(0)$ is used as a control variate in the sense that the true option price can be rewritten as

$$V(0) = V_K^{k,\alpha}(0) - \mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}(S_1(T) - S_2(T) - K)\mathbb{1}_{(\Xi)}\right]^+ + \mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}(S_1(T) - S_2(T) - K)^+\right]. \tag{8.2}$$

Similarly, the true basket option price can be evaluated by the formula

$$V^{\text{Basket}}(0) = V_K^{\mathcal{G}}(0) - \mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}(A_n(T) - K)\mathbb{1}_{\mathcal{G}(x^*)}\right]^+ + \mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}(A_n(T) - K)^+\right], \tag{8.3}$$

where $V_K^{\mathcal{G}}(0)$ is the lower bound to the exact basket option price, corresponding to the optimal value x^*. Calculation of the spread bound $V_K^{k,\alpha}(0)$ is carried out by Theorem 4.1 for the case without synchronous jumps and by Theorem 6.1 for the case with synchronous jumps. For the basket bound $V_K^{\mathcal{G}}(0)$, we consider Theorem 5.1 when there is no synchronous jumps and Theorem 7.1 when synchronous jumps are allowed. Note that the two expected values given in equations (8.2) and (8.3), which are for each equation highly correlated, are computed via a crude MC method.

For the implementation of the MC technique, we first determine all transition times $0 < \tau_1^j < \tau_2^j < \ldots < \tau_{I^j}^j \leq T$ and the corresponding phases at each realization

$j = 1, 2, \ldots, J$. Then, between each transition time we obtain a simulated path of the underlying process by using an Euler-Maruyama scheme. Note that since between the transitions the parameters are assumed to be constant, we can carry out the computations as in the case of no regime shifts. We also want to remark that once we obtained a path in the time interval $[\tau_k^j, \tau_{k+1}^j]$, its final value is assigned as the initial value of the next path which is simulated in the time interval $[\tau_{k+1}^j, \tau_{k+2}^j]$. All simulation results are performed with 100 time steps and 10^5 simulations, unless mentioned otherwise. Furthermore, all tables with pricing results include columns entitled *CI-length*. These columns report the length of the 95% confidence intervals.

In the case of a GBM framework without synchronous jumps, we also consider the Markov-modulated Kirk's formula for which we work with stochastic differential equations leading a MC-based numerical implementation. More precisely, as for the MC approach given above, we evaluate the transition times $0 < \tau_1^j < \tau_2^j < \ldots < \tau_{I^j}^j \leq T$ and the corresponding phases at each realization $j = 1, 2, \ldots, J$. Then, we calculate the integrated processes $\int_0^T (\tilde{r}(M(t)) - r(M(t))dt$ and $\int_0^T \sigma^2(M(t))dt$ by splitting them into $I^j + 1$ integrals:

$$\int_0^T (\tilde{r}(M(t)) - r(M(t)))\,dt = \int_0^{\tau_1^j} (\tilde{r}(M(t)) - r(M(t)))\,dt + \int_{\tau_1^j}^{\tau_2^j} (\tilde{r}(M(t)) - r(M(t)))\,dt + \ldots + \int_{\tau_{I^j}^j}^{T} (\tilde{r}(M(t)) - r(M(t)))\,dt,$$

$$\int_0^T \sigma^2(M(t))dt = \int_0^{\tau_1^j} \sigma^2(M(t))dt + \int_{\tau_1^j}^{\tau_2^j} \sigma^2(M(t))dt + \ldots + \int_{\tau_{I^j}^j}^{T} \sigma^2(M(t))dt.$$

With this splitting, the integral terms over the transition times are performed as if no regime-switching, similarly observed in the implementation of MC simulations. Regarding these integrated processes, we derive the Black-Scholes type formula (3.6) for the corresponding realization. At the final step, we take the average of the Black-Scholes prices computed at each realization, for obtaining the price given in (3.7).

The calculation of the lower bounds $V_K^{k,\alpha}(0)$ and $V_K^{\mathcal{G}}(0)$ are performed with the FFT algorithm, based upon Carr and Madan [11], with the damping factor chosen to be equal to $\delta = 0.75$. Noting that the FFT implementation of these two bounds $V_K^{k,\alpha}(0)$ and $V_K^{\mathcal{G}}(0)$ are similar, we summarize the corresponding implementation steps for $V_K^{k,\alpha}(0)$:

- Consider a discretized grid over the values of γ such that $\gamma_m = (m-1)\Delta_\gamma$ with $\Delta_\gamma > 0$ being the size of grid and $m = 1, \dots, \mathcal{M}$. Over this grid size, we apply the Simpson's rule in the sense that

$$V_K^{k,\alpha}(0) \approx \left(\frac{e^{-\delta k}}{\pi} \sum_{m=1}^{\mathcal{M}} e^{-i\gamma_m k} \Psi_T(\gamma_m; \delta, \alpha, K) \frac{\Delta_\gamma}{3} w_m \right)^+,$$

where $w_m = 4$ when m is odd, 2 when m is even and 1 when $m = 0$.

- Construct a grid with respect to k such that $k_y = -b + \Delta_k(y-1)$ where $b = \mathcal{M}\Delta_k/2$ and $\Delta_k > 0$ is the size of the grid and $y = 1, \dots, \mathcal{M}$.

- Moreover, determine the grid sizes Δ_γ and Δ_k considering $\Delta_\gamma = \frac{2\pi}{\mathcal{M}\Delta_k}$.

- As a result, the option price for values of k_y can be approximated by the following expression:

$$\begin{aligned} V_K^{k_y,\alpha}(0) &\approx \left(\frac{e^{-\delta k_y}}{\pi} \sum_{m=1}^{\mathcal{M}} e^{-i\gamma_m k_y} \Psi_T(\gamma_m; \delta, \alpha, K_y) \frac{\Delta_\gamma}{3} w_m \right)^+ \\ &= \left(\frac{e^{-\delta k_y}}{\pi} \sum_{m=1}^{\mathcal{M}} e^{i(-2\pi(m-1)(y-1)/N + m\pi)} \Psi_T(\gamma_m; \delta, \alpha, K_y) \frac{\Delta_\gamma}{3} w_m \right)^+. \end{aligned}$$

In the implementations, we set $\mathcal{M} = 4096$, $\Delta_k = 600/4096$ and hence $\Delta_\gamma = 2\pi/600$.

The optimization procedure arising in the computation of $V_K^{\mathcal{G}}(0)$ in this regime-switching framework is performed via a two-step procedure, as in Caldana et al. [9], and is implemented via the built-in-function `fminbnd` of MATLAB. For the sake of completeness, we briefly mention how to carry out this maximization procedure in two-steps. We start with computing $V_K^{\mathcal{G}}(0, x)$ over the equidistant values $\{x_1, \dots, x_k\}$ and then find the point $x_i \in \{x_1, \dots, x_k\}$ for which $V_K^{\mathcal{G}}(0, x)$ is maximized. In the second step, we regard an optimization over the all values x in the interval $[x_i, x_{i+1}]$, by defining this x_i found in the first step as the starting value. For a more detailed overview, see the paper of [9].

As an example for the basket option pricing when the asset price dynamics are ruled by a Markov-modulated Merton jump-diffusion model, we will provide results for the approximations $U_K^{AG}(0)$, $V_K^{AG}(0)$ and $C_K^{AG}(0)$. Recall that the computation of these approximations does not require an optimization procedure.

8.2 Examples

In the following, we will present and discuss different regime-switching models and examples.

Example 8.1 (Bivariate GBM)**.** This example examines the valuation of spread options when the underyling price dynamics are ruled by a Markov-modulated GBM framework without synchronous jumps (see Chapter 3). To this end, we focus on Markov-modulated Kirk's formula and the lower bound $V_K^{k,\alpha}(0)$ whose performances are established by MC simulations. Note that for the lower bound we consider a 2-dimensional process $\boldsymbol{X}(t) = (X_1(t), X_2(t))$ with two possible phases such that when $M = j,\ \boldsymbol{X}$ is characterized by the GBM process $\boldsymbol{Y_j}(t) = (\sigma_{1j}W_1(t), \sigma_{2j}W_2(t))$.

Here, the joint characteristic exponent $\boldsymbol{\Phi}(u_1, u_2) = [(\Phi_1(u_1, u_2), \Phi_2(u_1, u_2)]$ is given as

$$\Phi_j(u_1, u_2) = \frac{1}{2}\left(\sigma_{1j}^2 u_1^2 + \sigma_{2j}^2 u_2^2 + 2\rho_j \sigma_{1j}\sigma_{2j} u_1 u_2\right), \ \ j = 1, 2.$$

For the parameter set, we choose $S_1(0) = 110$, $S_2(0) = 100$, $T = 1$, $r_1 = r_2 = 0.05$ and $\rho_1 = \rho_2 = 0.5$. Moreover, when the economy is bad, volatilites are relatively high with $\sigma_{11} = 0.5$ and $\sigma_{21} = 0.4$. For the good economic environment, we choose lower volatilities $\sigma_{12} = 0.1$ and $\sigma_{22} = 0.05$.

Table 8.1 illustrates the performance of the lower bound $V_K^{k,\alpha}(0)$ as well as Kirk's formula by comparing with MC simulations. First notice that when lower bound $V_K^{k,\alpha}(0)$ is used as a control variate, the performance of the crude MC method is improved by reducing the length of CI to a large extent. Therefore, it is reasonable to consider the control variate MC results as benchmark prices. Based on these benchmarks, we see that the lower bound $V_K^{k,\alpha}(0)$ is a good candidate for the fair price of the option. Especially, when $K = 0$ (namely, when we evaluate exchange options), the lower bound seems to be indifferent from the true price. Note that this result will also be observed in the other examples. Furthermore, Kirk's approximation seems to perform well for small strike prices, especially for the exchange option case. However, an increase in the strike price reduces the performance of this approximation method, as expected.

Table 8.1: Spread option prices in the GBM model studied in Example 8.1. Number of MC simulations is 10^5.

	$V_K^{k,\alpha}(0)$	MC$^{\text{Crude}}$	CI-length (Crude)	Kirk	MC$^{\text{Control}}$	CI-length (Control)
$K = 0$	17.9472	17.8968	3.2422e-01	17.9454	17.9472	3.5527e-14
$K = 0.8$	17.4809	17.4304	3.2174e-01	17.5010	17.4809	9.1418e-06
$K = 1.6$	17.0233	16.9728	3.1924e-01	17.0645	17.0233	3.2849e-05
$K = 2.4$	16.5744	16.5236	3.1672e-01	16.6360	16.5746	7.9650e-05
$K = 3.2$	16.1344	16.0831	3.1417e-01	16.2155	16.1346	9.6609e-05
$K = 4$	15.7033	15.6514	3.1161e-01	15.8032	15.7035	9.4859e-05

Example 8.2 (Bivariate Variance Gamma model, with only dependence due to regime-switching)**.** This example is based on Variance Gamma processes, which is first proposed by Madan and Seneta [27] to explain the properties of log-returns such as leptokurtosis and skewness. More precisely, a Variance Gamma process behaves as a time-changed Brownian motion with drift subordinated by the Gamma process.

In this example, we consider a bivariate MMLP $\boldsymbol{X}(t) = (X_1(t), X_2(t))$ such that when $M = j$ (with $j = 1, 2$), the process $\boldsymbol{X}$ is driven by a bivariate Lévy process $\boldsymbol{Y_j} = (Y_{1j}, Y_{2j})$ whose components evolve like independent Variance Gamma processes. In particular, for $j = 1, 2$,

$$Y_{1j}(t) = \theta_{1j} J_{1j}(t) + \sigma_{1j} W_{1j}(J_{1j}(t)),$$
$$Y_{2j}(t) = \theta_{2j} J_{2j}(t) + \sigma_{2j} W_{2j}(J_{2j}(t)),$$

where $W_{1j}(t)$ are $W_{2j}(t)$ are Brownian motions, $J_{1j}(t)$ and $J_{2j}(t)$ are Gamma subordinators with $\Gamma(t, \kappa_{1j} t)$ and $\Gamma(t, \kappa_{2j} t)$. Here, $\Gamma(., .)$ denotes the Gamma distribution, being different from the matrix of synchronous jumps Γ.

Here, the characteristic exponent of $Y_{lj}(t)$ is given by

$$\varphi_{lj}(u) = \frac{1}{\kappa_{lj}} \log \left(1 - i\theta_{lj}\kappa_{lj} u + \frac{\sigma_{lj}^2}{2} \kappa_{lj} u^2 \right), \quad l, j = 1, 2.$$

Since the Lévy components are independent within each state, the joint characteristic exponent $\boldsymbol{\Phi}(u_1, u_2) = [(\Phi_1(u_1, u_2), \Phi_2(u_1, u_2)]$ can be expressed as

$$\Phi_j(u_1, u_2) = \varphi_{1j}(u_1) + \varphi_{2j}(u_2), \quad j = 1, 2.$$

Table 8.2: Parameter values for the Variance Gamma model studied in Example 8.2.

	Phase 1			Phase 2		
	κ	θ	σ	κ	θ	σ
X_1	0.0236	-0.1421	0.4460	0.0011	0.0196	0.1234
X_2	0.0374	-0.1135	0.2459	0.0015	0.0043	0.1534

We notice that since the underlying price processes are independent in each phase, the dependence between the asset prices is only implied by the hidden Markov process and in particular the synchronous jumps.

For the numerical experiments in this example, we choose the parameters of the Variance Gamma processes as in the study of Hainaut and Colwell [22] (see Table 8.2), although we use another transition matrix and synchronous jump modelling. In this interesting paper, the authors performed a calibration with a slightly modified version of the Hamilton's filter which allowed them to justify the use of switching Lévy processes with synchronous jumps. In particular, they fit the Markov-modulated Variance Gamma processes with synchronous jumps to time series of asset prices of some French firms and conclude that the Variance Gamma model with two states and exponential jumps seems to outperform some other models. In our study, the parameter values of the first process X_1 are the ones inferred for the firm *Axa*, which is an insurance company affected severely by the credit crunch and by the sovereign debts crisis of 2008. On the other side, the parameters of X_2 are the ones calibrated for *STMicroelectonics*, which is a non-financial firm not showing a visible impact of this crisis. We further assume that $\boldsymbol{r} = (0.01, 0.005)$, $S_1(0) = 100$, $S_2(0) = 100$ and $T = 1$.

Table 8.3 contains, for different exercise prices, spread option prices obtained via the lower bound approximation and via MC simulation techniques, both in the case of no synchronous jumps and in the case where the synchronous jumps are exponentially distributed with parameters $\lambda_1 = 4.5$, $\lambda_2 = 4$, $\xi_1 = 2.7$, and $\xi_2 = 2.5$. Table 8.4 presents the spread option prices in the case of normally distributed jumps with parameters of mean $\beta_{k1} = 0.1$, $\beta_{k2} = -0.4$ and of variance $\tau_{k1}^2 = 0.05$, $\tau_{k2}^2 = 0.05$ for $k = 1, 2$. In both situations, the synchronous jumps yield higher option prices. When comparing the length of the confidence intervals of the crude MC method and of the

Table 8.3: Spread option prices in the Variance Gamma model studied in Example 8.2 with and without exponential synchronous jumps. The parameters of the synchronous jumps are $\lambda_1 = 4.5$, $\lambda_2 = 4$, $\xi_1 = 2.7$, and $\xi_2 = 2.5$. Number of MC simulations is 10^5.

	Jumps	$V_K^{k,\alpha}(0)$	MC^{Crude}	CI-length (Crude)	MC^{Control}	CI-length (Control)
$K = 0$	with	23.4043	23.4309	5.5114e-01	23.4043	3.5527e-14
	without	14.0983	14.0257	3.0613e-01	14.0983	2.1316e-14
$K = 0.8$	with	23.0078	23.0344	5.4856e-01	23.0079	3.4959e-04
	without	13.7261	13.6553	3.0317e-01	13.7261	1.5097e-06
$K = 1.6$	with	22.6171	22.6440	5.4597e-01	22.6175	1.0737e-04
	without	13.3617	13.2927	3.0021e-01	13.3617	3.2332e-06
$K = 2.4$	with	22.2322	22.2594	5.4377e-01	22.2331	1.9164e-04
	without	13.0051	12.9378	3.0021e-01	13.0051	8.3254e-06
$K = 3.2$	with	21.8530	21.8807	5.4077e-01	21.8545	3.0328e-04
	without	12.6562	12.5904	2.9429e-01	12.6562	2.3116e-05
$K = 4$	with	21.4796	21.5077	5.3817e-01	21.4821	4.4009e-04
	without	12.3150	12.2507	2.9132e-01	12.3150	2.3963e-05

ones based on a control variate, both based on 10^5 simulations, it is clear that the use of the lower bound $V_K^{k,\alpha}(0)$ as a control variate provides a significant improvement. Therefore, we will only report the control variate MC results in the following. Using the control variate MC prices as a benchmark, the lower bound clearly seems to have a high precision. In general, the lower bounds are a little closer to the control variate MC results when there are no synchronous jump; but even in the presence of synchronous jumps, they still show nice approximations.

Example 8.3 (Bivariate Variance Gamma model with dependence due to a systematic part)**.** In this example, we assume dependent Variance Gamma processes inspired by the method in Ballotta and Bonfiglioli [3], namely we generate dependence between the different assets by a common systematic component. The advantages of this model construction can be summarized as follows: a flexible correlation structure, being parsimonious in the sense that a linear increase in the overall number of parameters is observed with the inclusion of new assets, and readily computable characteristic functions which facilitate the calibration procedure.

More precisely, when the phase does not change, $\boldsymbol{X}(t) = (X_1(t), X_2(t))$ is assumed

Table 8.4: Spread option prices in the Variance Gamma model studied in Example 8.2 with normal synchronous jumps. The parameters of the synchronous jumps are $\beta_{k1} = 0.1$, $\beta_{k2} = -0.4$, $\tau_{k1}^2 = 0.05$, and $\tau_{k2}^2 = 0.05$ for $k = 1, 2$. Number of MC simulations is 10^5.

	$V_K^{k,\alpha}(0)$	MC^{Crude}	CI-length (Crude)	$\text{MC}^{\text{Control}}$	CI-length (Control)
$K = 0$	20.6679	20.6888	4.3529e-01	20.6679	5.6843e-14
$K = 0.8$	20.2884	20.3104	4.3227e-01	20.2884	2.6229e-05
$K = 1.6$	19.9143	19.9375	4.2925e-01	19.9144	5.7169e-05
$K = 2.4$	19.5456	19.5701	4.2622e-01	19.5460	1.4245e-04
$K = 3.2$	19.1823	19.2079	4.2318e-01	19.1830	1.8799e-04
$K = 4$	18.8244	18.8511	4.2015e-01	18.8255	2.7407e-04

to be driven by a bivariate Variance Gamma model in which there are two idiosyncratic parts Z_1 and Z_2, specific for each underlying, and a common systematic component Z_C, which implies the dependency between X_1 and X_2. Indeed, the dynamics of the processes $X_1(t)$ and $X_2(t)$ are formulated as:

$$X_1(t) = Z_1(t) + d_1(t)Z_C(t),$$
$$X_2(t) = Z_2(t) + d_2(t)Z_C(t),$$

where the processes Z_1, Z_2 and Z_C are independent one-dimensional Markov-modulated Variance Gamma processes and

$$d_l(t) = \sum_{j=1}^{N} d_{lj}(t)\mathbb{1}_{M(t)=j}, \quad l = 1, 2.$$

For the sake of clarity, we suppose that when $M = j$, the dynamics of Z_1, Z_2 and Z_C are ruled by the Variance-Gamma processes Z_{1j}, Z_{2j} and Z_{Cj}:

$$Z_{1j}(t) = \theta_{1j}J_{1j}(t) + \sigma_{1j}W_{1j}(J_{1j}(t)),$$
$$Z_{2j}(t) = \theta_{2j}J_{2j}(t) + \sigma_{2j}W_{2j}(J_{2j}(t)), \qquad (j = 1, 2),$$
$$Z_{Cj}(t) = \theta_{Cj}J_{Cj}(t) + \sigma_{Cj}W_{Cj}(J_{Cj}(t)),$$

where $W_{1j}(t)$, $W_{2j}(t)$ and $W_{Cj}(t)$ are Brownian motions, $J_{lj}(t)$, $J_{2j}(t)$ and $J_{Cj}(t)$ are Gamma subordinators with $\Gamma(t, \kappa_{1j}t)$, $\Gamma(t, \kappa_{2j}t)$ and $\Gamma(t, \kappa_{Cj}t)$ [27]. Note that $\Gamma(.,.)$ denotes Gamma distribution.

Hence, when $M(t) = j$, the joint characteristic exponent $\mathbf{\Phi}(u_1, u_2)$ easily follows:

$$\Phi_j(u_1, u_2) = \varphi_{1j}(u_1) + \varphi_{2j}(u_2) + \varphi_{Cj}(d_{1j}u_1 + d_{2j}u_2), \tag{8.4}$$

Table 8.5: Parameter values for the Variance Gamma model studied in Example 8.3

	Phase 1			Phase 2		
d_1		0.2			0.05	
d_2		0.5			0.3	
	k	θ	σ	k	θ	σ
Z_1	0.0236	-0.1421	0.4460	0.0011	0.0196	0.1234
Z_2	0.0374	-0.1135	0.2459	0.0015	0.0043	0.1534
Z_C	0.05	-0.1	0.3	0.001	0.008	0.1

Table 8.6: Spread option prices in the Variance Gamma model studied in Example 8.3 with and without exponential synchronous jumps. The parameters of the synchronous jumps are $\lambda_1 = 4.5$, $\lambda_2 = 4$, $\xi_1 = 2.7$, and $\xi_2 = 2.5$. Number of MC simulations is 10^5.

	With synchronous jumps			**Without synchronous jumps**		
	$V_K^{k,\alpha}(0)$	$\text{MC}^{\text{Control}}$	CI-length	$V_K^{k,\alpha}(0)$	$\text{MC}^{\text{Control}}$	CI-length
$K = 0$	23.5082	23.5082	1.5632e-13	14.2948	14.2948	4.2633e-14
$K = 0.8$	23.1094	23.1095	3.8169e-05	13.9188	13.9188	5.0960e-06
$K = 1.6$	22.7165	22.7169	1.1546e-04	13.5506	13.5506	8.3719e-06
$K = 2.4$	22.3293	22.3302	2.1195e-04	13.1900	13.1901	1.8162e-05
$K = 3.2$	21.9478	21.9495	3.4138e-04	12.8372	12.8373	3.2827e-05
$K = 4$	21.5721	21.5746	4.4520e-04	12.4920	12.4921	3.1884e-05

where

$$\varphi_{lj}(u) = \frac{1}{k_{lj}} \ln\left(1 - iuk_{lj}\theta_{lj} + \frac{1}{2}u^2 k_{lj}\sigma_{lj}^2\right), \quad l = 1,2;\ j = 1,2,$$
$$\varphi_{Cj}(u) = \frac{1}{k_{Cj}} \ln\left(1 - iuk_{Cj}\theta_{Cj} + \frac{1}{2}u^2 k_{Cj}\sigma_{Cj}^2\right), \quad j = 1,2.$$

Table 8.5 summarizes the parameters of this Variance Gamma based model for the two phases. Notice that we choose the parameter set given in Example 8.2 for the processes Z_1 and Z_2. We also set $\boldsymbol{r} = (0.01, 0.005)$, $S_1(0) = 100$, $S_2(0) = 100$, and $T = 1$.

In Table 8.6, we report prices of spread options obtained via the lower bound approximation $V_K^{k,\alpha}(0)$ and control variate Monte Carlo simulations (based on 10^5 simulations) for both cases without and with synchronous jumps. The numerical results clearly show that, as in Example 8.2, i) higher prices are observed when we allow exponentially distributed synchronous jumps with parameters $\lambda_1 = 4.5$, $\lambda_2 = 4$, $\xi_1 = 2.7$, and $\xi_2 = 2.5$, ii) the lower bound $V_K^{k,\alpha}(0)$ has a very good performance in the sense that the bounds are very close to the control variate MC prices.

Example 8.4 (Multivariate jump-diffusion model with dependence due to a systematic part)**.** In this example, we focus on Merton jump-diffusion model, which is first introduced by Merton [30] to incorporate the abnormal movements of asset prices by using compund Poisson processes. This example now illustrates the method of Ballotta and Bonfiglioli [3] for modeling n risky assets, $S_l(t)$, $l = 1, \ldots, n$, by using the regime-switching version of Merton jump-diffusion model. Namely, the Markov-modulated process $X_l(t)$ has the form

$$X_l(t) = Z_l(t) + d_l(t) Z_C(t), \quad l = 1, \ldots, n, \tag{8.5}$$

where for all l, Z_l and Z_C are independent, one-dimensional Markov-modulated Merton jump-diffusion processes with jumps having zero mean. For $l = 1, \ldots, n$, Z_l refers to the idiosyncratic part of $X_l(t)$, Z_C denotes the common systematic component which implies the dependence between the processes $X_l(t)$ and

$$d_l(t) = \sum_{j=1}^{N} d_{lj}(t) \mathbb{1}_{M(t)=j}, \quad l = 1, \ldots, n.$$

To be more precise about the implementation of this model, it is assumed that when $M = j$, the dynamics of Z_l nad Z_C are governed by the following Merton jump-diffusion processes Z_{lj} and Z_{Cj}:

$$\begin{aligned} Z_{lj}(t) &= \sigma_{lj} W_{lj}(t) + \sum_{i=1}^{N_{lj}(t)} J_{lj}^i, && j = 1, 2; \quad l = 1, \ldots, n, \\ Z_{Cj}(t) &= \sigma_{Cj} W_{Cj}(t) + \sum_{i=1}^{N_{Cj}(t)} J_{Cj}^i, && j = 1, 2, \end{aligned}$$

where σ_{lj} and σ_{Cj} are the volatilities, $W_{lj}(t)$ and $W_{Cj}(t)$ are Brownian motions and $\sum_{i=1}^{N_{lj}(t)} J_{lj}^i$ and $\sum_{i=1}^{N_{Cj}(t)} J_{Cj}^i$ are compound Poisson processes. Herewith, $N_{lj}(t)$ and $N_{Cj}(t)$ are Poisson processes with intensities θ_{lj} and θ_{Cj}, respectively, and $\{J_{lj}^i\}$ and $\{J_{Cj}^i\}$ are normally distributed jumps sizes with $J_{lj}^i \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, \tau_{lj}^2)$ and $J_{Cj}^i \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, \tau_{Cj}^2)$.

Table 8.7: Parameter values for the Markov-modulated Merton jump-diffusion model studied in Example 8.4

	Phase 1			Phase 2		
d_1		0.2			0.05	
d_2		0.5			0.3	
d_3		0.4			0.4	
	τ	θ	σ	τ	θ	σ
Z_1	0.1	1	0.2	0.05	0.2	0.05
Z_2	0.1	1	0.2	0.05	0.2	0.05
Z_3	0.1	1	0.2	0.05	0.2	0.05
Z_C	0	0	0.25	0	0	0.1

Then, analogously to equation (8.4), the characteristic exponent $\mathbf{\Phi}(u_1, \ldots, u_n)$ is determined by

$$\Phi_j(u_1, \ldots, u_n) = \sum_{l=1}^{n} \varphi_{lj}(u_l) + \varphi_{Cj}\left(\sum_{l=1}^{n} d_{lj} u_l\right),$$

where

$$\varphi_{lj}(u) = \frac{1}{2}u^2\sigma_{lj}^2 + \theta_{lj}\left(1 - e^{-\frac{1}{2}u^2\tau_{lj}^2}\right), \qquad j = 1, 2; \quad l = 1, \ldots, n,$$
$$\varphi_{Cj}(u) = \frac{1}{2}u^2\sigma_{Cj}^2 + \theta_{Cj}\left(1 - e^{-\frac{1}{2}u^2\tau_{Cj}^2}\right), \qquad j = 1, 2.$$

where $\varphi_{lj}(u)$ are $\varphi_{Cj}(u)$ are the characteristic components of Z_{lj} and Z_{Cj}.

For the numerical illustrations, we consider the cases $n = 2$ and $n = 3$. Table 8.7 presents the parameter set of the Markov-modulated processes in these cases. We further choose as in the previous examples $\boldsymbol{r} = (0.01, 0.005)$, $S_1(0) = 100$, $S_2(0) = 100$, $T = 1$, and additionally set $S_3(0) = 100$.

Table 8.8 illustrates the performance of the lower bounds $V_K^{k,\alpha}(0)$ and $V_K^{\mathcal{G}}(0)$ for spread options, both without and with synchronous jumps. We study the accuracy of these bounds $V_K^{k,\alpha}(0)$ and $V_K^{\mathcal{G}}(0)$ by using the control variate MC prices as benchmark. To start with, both in the case of no synchronous jumps (in paranthesis) and in the case where synchronus jumps are regarded, the value of basket bound $V_K^{\mathcal{G}}(0)$, which needs an optimization procedure, is close to the value of spread bound $V_K^{k,\alpha}(0)$. Especially, when $K = 0$, the two bounds seem to give the same values. However, in general, the spread option bound $V_K^{k,\alpha}(0)$ turns out to perform better than the basket bound $V_K^{\mathcal{G}}(0)$. Notice that the simulation errors resulting from the control variate MC

Table 8.8: Spread option prices in the jump-diffusion model studied in Example 8.4. The first row for each strike K shows the prices with exponential synchronous jumps whereas the second row (in paranthesis) are those without synchronous jumps. The parameters of the synchronous jumps are $\lambda_1 = 4.5$, $\lambda_2 = 4$, $\xi_1 = 2.7$, and $\xi_2 = 2.5$. Number of MC simulations is 10^5.

	$V_K^{k,\alpha}(0)$	MC$^{\text{Control}}$	CI-length	$V_K^{\mathcal{G}}(0)$	MC$^{\text{Control}}$	CI-length
$K = 0$	20.6292	20.6292	3.5527e-14	20.6292	20.6292	1.4211e-13
	(8.4423)	(8.4423)	(4.6185e-14)	(8.4423)	(8.4423)	(2.1316e-14)
$K = 0.8$	20.2110	20.2111	4.1607e-05	20.2104	20.2111	1.5265e-04
	(8.0477)	(8.0477)	(1.5129e-06)	(8.0476)	(8.0477)	(3.2495e-05)
$K = 1.6$	19.7999	19.8003	1.1399e-04	19.7975	19.8004	4.4719e-04
	(7.6668)	(7.6668)	(3.1694e-06)	(7.6665)	(7.6668)	(8.9685e-05)
$K = 2.4$	19.3959	19.3969	2.0170e-04	19.3906	19.3969	8.0990e-04
	(7.2996)	(7.2996)	(1.0107e-05)	(7.2989)	(7.2996)	(1.5238e-04)
$K = 3.2$	18.9992	19.0009	2.8998e-04	18.9896	19.0008	1.2130e-03
	(6.9459)	(6.9459)	(1.2834e-05)	(6.9447)	(6.9460)	(2.4552e-04)
$K = 4$	18.6097	18.6124	4.2021e-04	18.5946	18.6125	1.7726e-03
	(6.6057)	(6.6057)	(1.2007e-05)	(6.6038)	(6.6057)	(3.2290e-04)

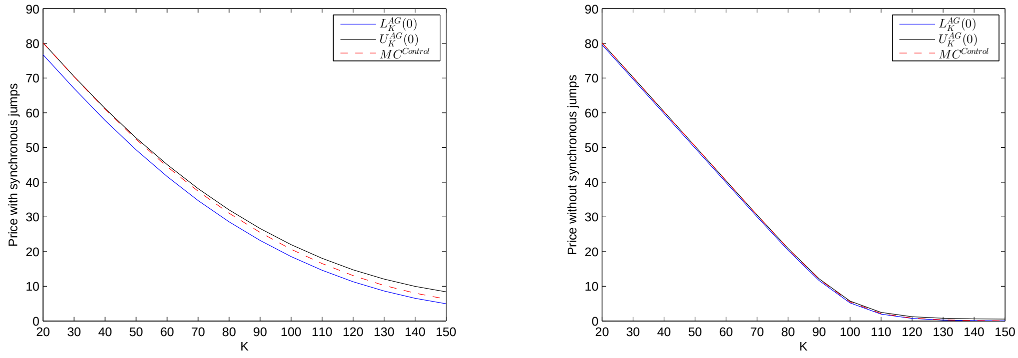
are smaller when we use the spread option bound $V_K^{k,\alpha}(0)$ as a control variate. Finally, we compare the prices without synchronus jumps and those with synchronous jumps. Mainly, we observe that in the case of no synchronous jumps option prices $V_K^{k,\alpha}(0)$ and $V_K^{\mathcal{G}}(0)$ are lower and generally closer to the MC prices, showing a nicer precision. Even so, we emphasize that the bounds have a good performance regardless of the presence of synchronous jumps.

Table 8.9 refers to the case of a 2-asset basket option with positive weights, which are chosen as $\mathbf{w} = (0.3, 0.7)$. We begin with discussing the results obtained without synchronous jumps (in paranthesis). Noting that we also reported values of $L_K^{AG}(0)$, $U_K^{AG}(0)$ and $C_K^{AG}(0)$, which were derived considering the arithmetic-geometric mean inequality, $V_K^{\mathcal{G}}(0)$ provides the most accurate prices among all the bounds we obtained. In particular, benchmarking with MC prices, this bound seems to be exact for the strikes less than $K = 60$. Regarding the performance of $L_K^{AG}(0)$, $U_K^{AG}(0)$ and $C_K^{AG}(0)$ within this range of strikes, we see that the upper bound $U_K^{AG}(0)$ gives the best results, having the same precision with the bound $V_K^{\mathcal{G}}(0)$. But, for higher strikes it is outperformed by the approximation $C_K^{AG}(0)$. Although the bound $V_K^{\mathcal{G}}(0)$ is much

Table 8.9: Basket option prices in the jump-diffusion model studied in Example 8.4. The first row for eack strike K shows the prices with exponential synchronous jumps whereas the second row (in paranthesis) are those without synchronous jumps. The weights of the basket are $\mathbf{w} = (0.3, 0.7)$. The parameters of the synchronous jumps are $\lambda_1 = 4.5$, $\lambda_2 = 4$, $\xi_1 = 2.7$, and $\xi_2 = 2.5$. Number of MC simulations is 10^5.

	$V_K^{\mathcal{G}}(0)$	MC$^{\text{Control}}$	CI-length	$L_K^{AG}(0)$	$U_K^{AG}(0)$	$C_K^{AG}(0)$
$K = 20$	80.1496	80.1613	3.3376e-03	76.7436	80.1876	80.1620
	(80.1429)	(80.1429)	(2.8422e-13)	(79.6235)	(80.1429)	(80.1433)
$K = 30$	70.3319	70.3693	7.2089e-03	67.0265	70.4704	70.3614
	(70.2143)	(70.2143)	(2.5580e-13)	(69.6949)	(70.2143)	(70.2147)
$K = 40$	60.9553	61.0153	9.1903e-03	57.8026	61.2466	60.9215
	(60.2857)	(60.2857)	(1.5632e-13)	(59.7663)	(60.2857)	(60.2862)
$K = 50$	52.3102	52.3886	1.3971e-02	49.3321	52.7760	52.1812
	(50.3572)	(50.3572)	(8.5265e-14)	(49.8379)	(50.3573)	(50.3577)
$K = 60$	44.4242	44.5159	1.6029e-02	41.6369	45.0808	44.2174
	(40.4309)	(40.4311)	(2.9788e-04)	(39.9130)	(40.4324)	(40.4323)
$K = 70$	37.3117	37.4155	1.6595e-02	34.7226	38.1666	37.0314
	(30.5361)	(30.5376)	(7.4771e-04)	(30.0285)	(30.5479)	(30.5431)
$K = 80$	30.9801	31.0877	1.6384e-02	28.5889	32.0328	30.6284
	(20.8564)	(20.8605)	(1.1106e-03)	(20.3856)	(20.9050)	(20.8767)
$K = 90$	25.4138	25.5278	1.7391e-02	23.2153	26.6592	24.9945
	(12.0405)	(12.0478)	(1.4520e-03)	(11.6494)	(12.1688)	(12.0646)
$K = 100$	20.5826	20.6933	1.7126e-02	18.5688	22.0127	20.0999
	(5.4876)	(5.4945)	(1.2753e-03)	(5.2098)	(5.7292)	(5.4655)
Average CPU (seconds)	13.2014					0.6174

closer to the true option price, we can conclude that $L_K^{AG}(0)$, $U_K^{AG}(0)$ and $C_K^{AG}(0)$ based on the arithmetic-geometric mean inequality may also be very useful in practice. In the case of synchronous jumps, we observe a little different pattern in the performance of the bounds. More precisely, although the lower bound $V_K^{\mathcal{G}}(0)$ again leads to very good results, taking the control variate MC results as benchmark, the approximate price $C_K^{AG}(0)$ has the highest precision for strikes $K = 20$ and 30 (very deep-in-the-money option). But for other strikes given in this table, the true option price can best be approximated by $V_K^{\mathcal{G}}(0)$. Also the upper bound $U_K^{AG}(0)$ is generally more accurate than the lower bound $L_K^{AG}(0)$, but not as accurate as $V_K^{\mathcal{G}}(0)$ or $C_K^{AG}(0)$. All in all, taking the synchronous jumps into account, we again achieve very



(a) (b)

Figure 8.1: Price of a two-asset basket option carried out by the lower bound $L_K^{AG}(0)$, upper bound $U_K^{AG}(0)$ and control variate Monte Carlo $\text{MC}^{\text{Control}}$. The weights of the basket are $\mathbf{w} = (0.3, 0.7)$. (a) with synchronous jumps (b) without synchronous jumps.

good approximations. We finally remark that the prices obtained with synchronous jumps are higher than those without jumps. Notice that in the lowermost row we additionally report the average CPU times for $V_K^{\mathcal{G}}(0)$ and $C_K^{AG}(0)$. Since the bounds $U_K^{AG}(0)$ and $L_K^{AG}(0)$ have a similar CPU performance, only the results of $C_K^{AG}(0)$ are reported. We observe that although the lower bound $V_K^{\mathcal{G}}(0)$ in general shows a better performance than $L_K^{AG}(0)$, $U_K^{AG}(0)$, and $C_K^{AG}(0)$ for both cases, it is slower than these approximations due to the optimization procedure we considered.

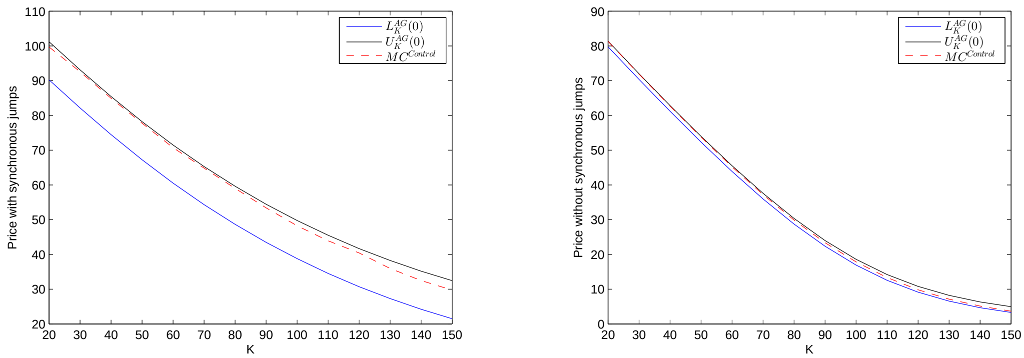
Figure 8.1 shows the lower bound $L_K^{AG}(0)$, upper bound $U_K^{AG}(0)$ and control variate Monte Cralo prices for strikes ranging from 20 to 150. The difference $U_K^{AG}(0) - L_K^{AG}(0)$ seems to be null for the case without synchronous jumps. When we take into account synchronous jumps, the difference turns out to be wider. Additionally, we see that the upper bound $U_K^{AG}(0)$ with synchronous jumps is very close to the Monte Carlo prices for deep-in-the-money options. For increasing values of strike price, its precision is slightly reduced.

Table 8.10 illustrates the valuation of a 3-asset basket option in which the third asset has a negative weight. Note that in order to have sufficiently small MC confidence intervals, we need more simulations and we therefore used here 5.10^5 simulations. Moreover, it is worth noticing that the lower bound $V_K^{\mathcal{G}}(0)$ is still accurate but not as

Table 8.10: Basket option prices in the jump-diffusion model studied in Example 8.4. The first row for each strike K shows the prices with exponential synchronous jumps whereas the second row (in paranthesis) are those without synchronous jumps. The weights of the basket are $\mathbf{w} = (2, 1, -2)$. The parameters of the synchronous jumps are $\lambda_1 = 4.5$, $\lambda_2 = 4$, $\lambda_3 = 3.8$, $\xi_1 = 2.7$, $\xi_2 = 2.5$ and $\xi_3 = 2.4$. Number of MC simulations is 5.10^5.

	$V_K^{\mathcal{G}}(0)$	$\text{MC}^{\text{Control}}$	CI-length	$L_K^{AG}(0)$	$U_K^{AG}(0)$	$C_K^{AG}(0)$
$K = 20$	98.2063	100.1263	4.7553e-02	90.2579	101.2252	99.5981
	(81.2050)	(81.3030)	(5.3651e-03)	(79.7515)	(81.4001)	(81.3285)
$K = 30$	90.7255	92.3049	4.1907e-02	82.1615	93.1288	91.1228
	(71.7801)	(71.8792)	(5.0999e-03)	(70.3630)	(72.0115)	(71.9097)
$K = 40$	83.5495	84.8272	4.0837e-02	74.4730	85.4402	82.9875
	(62.5704)	(62.6646)	(4.5146e-03)	(61.1953)	(62.8438)	(62.7002)
$K = 50$	76.6998	77.7113	3.7038e-02	67.2488	78.2161	75.2536
	(53.6555)	(53.7413)	(4.0123e-03)	(52.3298)	(53.9784)	(53.7780)
$K = 60$	70.1935	71.0171	3.5338e-02	60.5302	71.4975	67.9788
	(45.1383)	(45.2102)	(3.4254e-03)	(43.8717)	(45.5203)	(45.2438)
$K = 70$	64.0433	64.7536	3.4458e-02	54.3357	65.3029	61.2062
	(37.1468)	(37.2038)	(2.8050e-03)	(35.9520)	(37.6006)	(37.2245)
$K = 80$	58.2577	58.9241	3.3629e-02	48.6621	59.6294	54.9568
	(29.8316)	(29.8732)	(2.3671e-03)	(28.7238)	(30.3723)	(29.8702)
$K = 90$	52.8413	53.5405	3.4016e-02	43.4910	54.4582	49.2296
	(23.3487)	(23.3802)	(2.1864e-03)	(22.3449)	(23.9934)	(23.3400)
$K = 100$	47.7950	48.5741	3.5415e-02	38.7961	49.7634	44.0073
	(17.8242)	(17.8523)	(2.0434e-03)	(16.9407)	(18.5893)	(17.7680)
Average CPU (seconds)	18.1433					1.8358

promising as in the case with 2-asset option. Therefore, we also report the values of the basket bounds $L_K^{AG}(0)$, $U_K^{AG}(0)$ and the approximate price $C_K^{AG}(0)$ for both framework (without synchronous jumps and with synchronous jumps). Firstly, visiting the results of no synchronus jumps, we see that the bound $V_K^{\mathcal{G}}(0)$ is not sharp as the approximation $C_K^{AG}(0)$ for the strikes less than $K = 90$. Even so, $V_K^{\mathcal{G}}(0)$ seems to give tighter results than the bounds $U_K^{AG}(0)$ and $L_K^{AG}(0)$. In the case of synchronous jumps, numerical results show that when the strike equals $K = 20$, the lower bound $V_K^{\mathcal{G}}(0)$ is again outperformed by the approximation $C_K^{AG}(0)$, which provides the tightest result in these settings. When K varies between 30 and 70, the performance of the lower



(a) (b)

Figure 8.2: Price of a three-asset basket option carried out by the lower bound $L_K^{AG}(0)$, upper bound $U_K^{AG}(0)$ and control variate Monte Carlo simulations $\text{MC}^{\text{Control}}$. The weights of the basket are $\mathbf{w} = (2, 1, -2)$. (a) with synchronous jumps (b) without synchronous jumps

bound $V_K^{\mathcal{G}}(0)$ is better than the approximation $C_K^{AG}(0)$, but not as good as the upper bound $U_K^{AG}(0)$, which is the closest to the MC results. When we consider strikes larger than 70, the lower bound $V_K^{\mathcal{G}}(0)$ turns out to achieve the best results. Besides the approximate prices, we examine the average CPU times of the lower bound $V_K^{\mathcal{G}}(0)$ and approximation $C_K^{AG}(0)$. Recall that $U_K^{AG}(0)$ and $L_K^{AG}(0)$ have a similar CPU as for the approximation $C_K^{AG}(0)$. As in the case of 2-asset basket option, we see that on the one hand lower bound $V_K^{\mathcal{G}}(0)$ is close to the fair price of the option, but on the other hand it is slower than the approximations based on the arithmetic-geometric mean inequality.

Figure 8.2 displays the difference $U_K^{AG}(0) - L_K^{AG}(0)$ for different strike prices. It seems that this difference is very small for the case without synchronous jumps. For the case with synchronous jumps, we see that the difference is wider. Furthermore, the lower bound $L_K^{AG}(0)$ is not so tight and is outperformed by the upper bound $U_K^{AG}(0)$. As for the two-asset basket options, the upper bound $U_K^{AG}(0)$ is more promising for very-deep-in-the-money option.

Example 8.5 (Multivariate jump-diffusion model with dependence due to a stochastic time change)**.** This example focuses on a regime-switching version of a multivariate Kou model proposed by Mai et al. [28]. This approach, which will be explained in more detail below, aims to model the marginal distribution of each asset by a one-dimensional Markov-modulated Kou process, and, moreover, to construct dependency among the jump-components of different assets by a stochastic time change. This stochastic time change is defined by using the parameters of the marginal distributions as well as supplementary dependency parameters $\kappa_k^+ \in (0,1)$ and $\kappa_k^- \in (0,1)$, corresponding to respectively positive and negative jumps. The calibration of this multivariate model without regime-switching is reported to be very practical since the parameters of the univariate Kou models can be estimated first, and then the dependence parameters can be treated separately.

We start by discussing the modeling framework when the Markov process M is in phase k. Let us assume a two-dimensional MMLP $\mathbf{X}(t) = (X_1(t), X_2(t))$ whose dynamics within phase k are characterized by a 2-dimensional Lévy process $\mathbf{Y}_k = (Y_{1k}, Y_{2k})$, with components following a univariate Kou model. In particular, $Y_{jk}(t)$, $j = 1, 2$, is governed by a one dimensional standard Brownian motion $W_{jk}(t)$ and two independent compound Poisson processes $\Xi_{jk}^+(t)$ and $\Xi_{jk}^-(t)$, representing the positive and negative jumps. In this state k, the dependency between the Lévy processes $Y_{1k}(t)$ and $Y_{2k}(t)$ follows from: (1) correlated Brownian motions $W_{1k}(t)$ and $W_{2k}(t)$; and (2) jump components which are dependent by using a stochastic time change. More precisely, each Lévy process component $Y_{jk}(t), j = 1, 2$, in phase k is formulated as:

$$Y_{jk}(t) = \sigma_{jk} W_{jk}(t) + \Xi_{jk}^+(t) - \Xi_{jk}^-(t),$$

$$\Xi_{jk}^+(t) = \sum_{i=1}^{N_{jk}^+(\Theta_k^+(t))} J_{jk}^{i,+},$$

$$\Xi_{jk}^-(t) = \sum_{i=1}^{N_{jk}^-(\Theta_k^-(t))} J_{jk}^{i,-},$$

where:

(i) $\Xi_{jk}^+(t)$ is a compound Poisson process with intensity ϑ_{jk}^+ and exponentially distributed jumps with parameter d_{jk}^+.

(ii) $\Xi_{jk}^-(t)$ is a compound Poisson process with intensity ϑ_{jk}^- and exponentially distributed jumps with parameter d_{jk}^-. Here, $\Xi_{jk}^-(t)$ is independent of the compound Poisson $\Xi_{jk}^+(t)$ for each $j = 1, 2$.

Table 8.11: Parameter values for the jump-diffusion model in Example 8.5.

Phase 1											
ϑ_1^+	ϑ_2^+	ϑ_1^-	ϑ_2^-	d_1^+	d_2^+	d_1^-	d_2^-	σ_1	σ_2	κ^+	κ^-
0	0	0.6	0.4	0	0	3.895	2.838	0.3	0.4	0.1	0.7
Phase 2											
ϑ_1^+	ϑ_2^+	ϑ_1^-	ϑ_2^-	d_1^+	d_2^+	d_1^-	d_2^-	σ_1	σ_2	κ^+	κ^-
0.05	0.1	0.273	0.164	7	5	6	4	0.167	0.182	0.2	0.5

(iii) $\Theta_k^+(t)$ and $\Theta_k^-(t)$ are two compound Poisson processes with intensities $\vartheta_k^{0,+}$ and $\vartheta_k^{0,-}$ and exponentially distributed jumps Exp$(d^{0,+})$ and Exp$(d^{0,-})$ where

$$\vartheta_k^{0,+} = \max_{1\le j\le n} (\vartheta_{jk}^+/\kappa_k^+), \quad \vartheta_k^{0,-} = \max_{1\le j\le n} (\vartheta_{jk}^-/\kappa_k^-)$$

with $\kappa_k^+ \in (0,1)$ and $\kappa_k^- \in (0,1)$ for each regime k.

(iv) $N_{jk}^+(t)$ and $N_{jk}^-(t)$ are independent Poisson processes for each $j = 1, 2$:

$$N_{jk}^+(t) \sim \text{Poisson}\left(\frac{\vartheta_{jk}^+ d^{0,+}}{\vartheta_k^{0,+} - \vartheta_{jk}^+}\right),$$

$$N_{jk}^-(t) \sim \text{Poisson}\left(\frac{\vartheta_{jk}^- d^{0,-}}{\vartheta_k^{0,-} - \vartheta_{jk}^-}\right),$$

where all Poisson processes given above are independent of $\Theta_k^+(t)$ and $\Theta_k^-(t)$.

(v) $\{J_{jk}^{i,+}\}_{i\in\mathbb{N}}$ and $\{J_{jk}^{i,-}\}_{i\in\mathbb{N}}$ are i.i.d. random variables for each $j = 1, 2$:

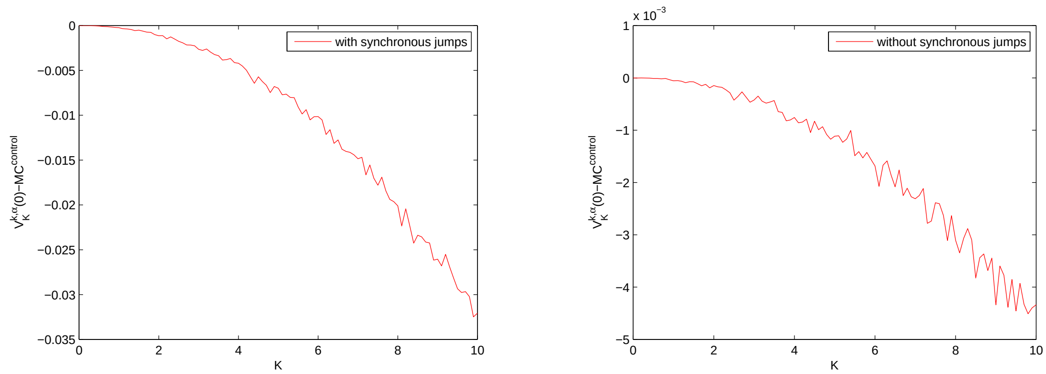
$$J_{jk}^{i,+} \sim \text{Exp}\left(\frac{\vartheta_k^{0,+} d_{jk}^+}{\vartheta_k^{0,+} - \vartheta_{jk}^+}\right), \quad i \in \mathbb{N},$$
$$J_{jk}^{i,-} \sim \text{Exp}\left(\frac{\vartheta_k^{0,-} d_{jk}^-}{\vartheta_k^{0,-} - \vartheta_{jk}^-}\right), \quad i \in \mathbb{N}.$$

By assumption, these random variables are independent of $\Theta_k^+(t)$, $\Theta_k^-(t)$ and the Poisson processes $N_{jk}^+(t)$, $N_{jk}^-(t)$ for $j = 1, 2$.

(vi) $W_{1k}(t)$ and $W_{2k}(t)$ are Brownian motions with correlation coefficient ρ_k within state k. These Brownian motions are independent from all processes introduced above.

Table 8.12: Spread option prices in the jump-diffusion model studied in Example 8.5 both without and with exponential synchronous jumps. The parameters of the synchronous jumps are $\lambda_1 = 4.5$, $\lambda_2 = 4$, $\xi_1 = 2.7$, and $\xi_2 = 2.5$. Number of MC simulations is 10^5.

	With synchronous jumps			**Without synchronous jumps**		
	$V_K^{k,\alpha}(0)$	$\text{MC}^{\text{Control}}$	CI-length	$V_K^{k,\alpha}(0)$	$\text{MC}^{\text{Control}}$	CI-length
$K = 0$	31.1912	31.1912	2.1316e-14	28.0053	28.0053	1.4211e-14
$K = 0.8$	30.5244	30.5246	7.8651e-05	27.2542	27.2543	4.9114e-05
$K = 1.6$	29.8636	29.8642	1.6927e-04	26.5069	26.5071	1.0451e-04
$K = 2.4$	29.2089	29.2104	3.1233e-04	25.7637	25.7640	1.4077e-04
$K = 3.2$	28.5607	28.5635	5.3566e-04	25.0249	25.0254	2.2449e-04
$K = 4$	27.9192	27.9236	8.8259e-04	24.2909	24.2916	3.0281e-04



(a) (b)

Figure 8.3: Difference between the lower bound $V_K^{k,\alpha}(0)$ and control variate Monte Carlo: (a) with synchronous jumps (b) without synchronous jumps

This model construction implies the following characteristic exponent within state k, see Mai et al. [28]:

$$\Phi_k(\mathbf{u}) = \frac{1}{2}\mathbf{u}^T \hat{\Sigma}\mathbf{u} - \frac{\vartheta_k^{0,+}\beta_k^+(\mathbf{u})}{1-\beta_k^+(\mathbf{u})} + \frac{\vartheta_k^{0,-}\beta_k^-(\mathbf{u})}{1+\beta_k^-(\mathbf{u})},$$

where $\hat{\Sigma}_k$ is the covariance matrix of $(\sigma_{1k}W_{1k}(t), \sigma_{2k}W_{2k}(t))$ in state k,

$$\beta_k^+(\mathbf{u}) = \sum_{m=1}^{2} \frac{\vartheta_{mk}^+ iu_m}{\vartheta_k^{0,+}d_{mk}^+ - iu_k(\vartheta_k^{0,+} - \vartheta_{mk}^+)},$$

$$\beta_k^-(\mathbf{u}) = \sum_{m=1}^{2} \frac{\vartheta_{mk}^- iu_m}{\vartheta_k^{0,-}d_{mk}^- + iu_k(\vartheta_k^{0,-} - \vartheta_{mk}^-)}.$$

Table 8.11 presents the parameter set chosen in this example, which is inspired by Mai et al. [28]. We further choose $\boldsymbol{r} = (0, 0)$, $S_1(0) = 69.468$, $S_2 = 42.10$, $\boldsymbol{\rho} = (0.5, 0.5)$ and $T = 1$. Table 8.12 illustrates the valuation of spread options via the lower bound $V_K^{k,\alpha}(0)$, benchmarked with the control variate MC method by using 10^5 simulations. For the simulation of the asset price processes, we use Algorithm 5.12 in Schulz [34]. As it is apparent from Table 8.12, despite the complexity of the model, the lower bound provides very tight prices for both cases without and with synchronous jumps. Furthermore, we observe higher prices with a slightly less precision when asset prices are modeled with synchronous jumps. Additionally, Figure 8.3 shows the differences between the lower bound $V_K^{k,\alpha}(0)$ and control variate Monte Carlo simulations $\text{MC}^{\text{Control}}$ for strikes ranging from 0 to 10. It is confirmed that although under a modeling framework with synchronous jumps the price differences are larger, the lower bound prices $V_K^{k,\alpha}(0)$ continue to agree well with the Monte Carlo simulations.

CHAPTER 9

CONCLUSION

Being an important research area due to their use in many different markets, this thesis concentrates upon pricing spread and basket options under the different regime-switching frameworks.

Mainly, we provide approximations to the exact option prices based upon ideas from the literature without regime-switching. We begin with examining the valuation of spread options in a MMGBM setting without synchronous jumps. To this end, we derive a Markov-modulated Kirk's formula by using the regime-switching version of Black-Scholes formula. After considering the GBM setting, we proceed to price spread and basket options when risky asset prices are ruled by MMLPs. We provide several bounds for the corresponding options. In particular, we derive a lower bound for spread options, which is obtained via a univariate Fourier inversion. This bound is applicable to a wide variety of models for which the joint characteristic function of MMLPs is known explicitly. For the basket options, we provide several approximations to the true option price. By defining a set based on the geometric average of the weighted assets, we first introduce a lower bound that can be obtained via a univariate Fourier inversion and an optimization procedure. Since this optimization procedure increases the computational cost of the bound, we then study the analogous approximations followed by the arithmetic-geometric mean inequality. As in the spread option case, all bounds introduced for basket options are manageable whenever the joint characteristic function of MMLPs are known analytically. Finally, we study evaluation of spread and basket options under a more generalized MMLP framework, in which a regime change yields not only a switch in the model parameters, but also *may* lead synchronous jumps in the asset prices. In this context, we generalize the aforementioned pricing formulas to the case of synchronous jumps.

After theoretical arguments, the accuracy of all aproximations included in this thesis has been verified by several numerical examples. The first example is introduced for showing the precision of Markov-modulated Kirk's formula; namely we consider a MMGBM setting without synchronous jumps. Then, we consider a Markov-modulated Variance Gamma model and we assume that the underlying price processes are independent in each phase. In this framework, we illustrate the performance of the spread option bound, considering both cases with and without synchronous jumps. With the purpose of discussing the case of dependent asset prices (in each phase), we then focus upon a regime-switching model, using a common systematic component to explain the dependency structure between the underlying asset prices. In this example, we assume that both the idiosyncratic parts as well as the common systematic part of the underlying assets are modeled by Variance Gamma processes. Particularly, we study the prices of spread options for the cases without and with synchronous jumps. Next, we observe the valuation of spread and basket options under the methodology of the previous example, but now the asset components are driven according to a Markov-modulated Merton jump-diffusion model. In this framework, we first discuss approximate prices for spread options and afterwards for basket options with two and three underlying assets. As in the previous examples, we show the precison of the approximations in the case of no synchronous jump and in the case when synchronous jumps occur. Lastly, we consider the regime-switching generalization of a multivariate Kou model in which the dependency between the jump components is constructed by means of a stochastic time change. For this example, we concentrate upon the approximate prices of spread options both without and with synchronous jumps.

This thesis provides contributions in different aspects:

- We derive pricing formulas under frameworks without and with sycnhronous jumps.
- Benchmarking with control variate MC simulations, it is shown that especially for small strikes the Markov-modulated Kirk's formula has a nice precision.

Here, the control variate MC simulations are conducted with the spread bound $V_K^{k,\alpha}(0)$, which indeed seems to be more promising than the Kirk's formula, especially for larger strikes.

- We ensure the accuracy of the spread bound $V_K^{k,\alpha}(0)$ by numerical illustrations given above. This accuracy surely depends on the presence of synchronous jumps. Considering the effect of synchronous jumps, it is clear that the performance of the bound is slightly better when asset prices are modeled without synchronous jumps. However, even including synchronus jumps we achieve very promising results. Additionally, we observe that when $K = 0$ the bound becomes exact regardless of the chosen model. As a result, our approximation gives very good results for our two modelling frameworks, namely both without and with sychronous jumps.

- As an example, namely when asset prices are modeled by a Markov-modulated Variance Gamma model, we also provide spread option prices with normally distributed synchronous jumps. It is shown that the bound remains accurate also for this kind of synchronous jumps.

- For a modeling framework with Merton jump-diffusion processes, we compare the precision of the spread $V_K^{k,\alpha}(0)$ and the basket bound $V_K^{\mathcal{G}}(0)$. In general, the spread bound turns out to be more sharp than the basket bound, for both without and with synchronous jump. However, the basket bound $V_K^{\mathcal{G}}(0)$ may also be considered as reasonably accurate.

- When we regard basket options, we observe that the precision of the basket bounds, $V_K^{\mathcal{G}}(0)$, $L_K^{AG}(0)$, $U_K^{AG}(0)$ and $C_K^{AG}(0)$ are also affected by the presence of synchronous jumps. In particular, the precision of all approximations are reduced due to inclusion of synchronous jumps. But even for these cases we achieve to obtain prices very close to those of MC simulations.

- For the 2-asset basket option with positive weights, the lower bound $V_K^{\mathcal{G}}(0)$ in general shows a better performance than the other approximations. For the 3-asset basket option with negative weights, the lower bound $V_K^{\mathcal{G}}(0)$ is still accurate but not as promising as in the case of 2-asset basket option with positive weights. In particular, when there is no synchronous jump, the true option price seems to be best approximated by $C_K^{AG}(0)$ for strikes less than $K = 90$. For other strikes, such as very deep out of the money options, the lower bound $V_K^{\mathcal{G}}(0)$ becomes sharper. In the case of synchronous jumps, the upper bound $U_K^{AG}(0)$ is in general the most precise bound for in the money options whereas for strikes larger than 70, the lower bound $V_K^{\mathcal{G}}(0)$ achieves the best results.

Although the most promising bound changes according to the baskets or strike values we considered, we always provide very accurate approximations to the true option price.

As an outlook, the Markov-modulated Kirk's formula can also be derived for a MMGBM setting with synchronous jumps. Moreover, we can carry out a calibration procedure for a MMLP framework both without and with synchronous jumps. However, we should take into account the fact that the calibration of the parameters may be challenging if we consider a large basket or more complex underlying models.

REFERENCES

[1] C. Alexander and A. Venkatramanan, Analytic approximations for multi-asset option pricing, Mathematical Finance, 22(4), pp. 667–689, 2012.

[2] S. Asmussen, *Applied probability and queues*, volume 51, Springer Science & Business Media, 2008.

[3] L. Ballotta and E. Bonfiglioli, Multivariate asset models using Lévy processes and applications, The European Journal of Finance, 22(13), pp. 1320–1350, 2016.

[4] F. Benth, G. Di Nunno, A. Khedher, and M. Schmeck, Pricing of spread options on a bivariate jump market and stability to model risk, Applied Mathematical Finance, 22(1), pp. 28–62, 2015.

[5] P. Bjerksund and G. Stensland, Closed form spread option valuation, Quantitative Finance, 14(10), pp. 1785–1794, 2014.

[6] P. Boyle and T. Draviam, Pricing exotic options under regime switching, Insurance: Mathematics and Economics, 40(2), pp. 267 – 282, 2007.

[7] J. Buffington and R. J. Elliott, American options with regime switching, International Journal of Theoretical and Applied Finance, 5(05), pp. 497–514, 2002.

[8] R. Caldana and G. Fusai, A general closed-form spread option pricing formula, Journal of Banking & Finance, 37(12), pp. 4893–4906, 2013.

[9] R. Caldana, G. Fusai, A. Gnoatto, and M. Grasselli, General closed-form basket option pricing bounds, Quantitative Finance, 16(4), pp. 535–554, 2016.

[10] R. Carmona and V. Durrleman, Pricing and hedging spread options, SIAM Review, 45(4), pp. 627–685, 2003.

[11] P. Carr and D. Madan, Option valuation using the fast Fourier transform, Journal of computational finance, 2(4), pp. 61–73, 1999.

[12] G. Cheang and C. Chiarella, Exchange options under jump-diffusion dynamics, Applied Mathematical Finance, 18(3), pp. 245–276, 2011.

[13] S.-N. Chen, M.-H. Chiang, P.-P. Hsu, and C.-Y. Li, Valuation of quanto options in a Markovian regime-switching market: A Markov-modulated Gaussian HJM model, Finance Research Letters, 11(2), pp. 161–172, 2014.

[14] K. Chourdakis, Switching Lévy models in continuous time: Finite distributions and option pricing, Technical report, University of Essex, Centre for Computational Finance and Economic Agents (CCFEA), 2005.

[15] G. Deelstra, A. Petkovic, and M. Vanmaele, Pricing and hedging Asian basket spread options, Journal of Computational and Applied Mathematics, 233(11), pp. 2814–2830, 2010.

[16] G. Deelstra and M. Simon, Multivariate European option pricing in a Markov-modulated Lévy framework, Journal of Computational and Applied Mathematics, 317, pp. 171–187, 2017.

[17] M. Dempster and S. Hong, Spread option valuation and the fast Fourier transform, in H. Geman, D. Madan, S. Pliska, and T. Vorst, editors, *Mathematical Finance — Bachelier Congress 2000: Selected Papers from the First World Congress of the Bachelier Finance Society, Paris, June 29–July 1, 2000*, pp. 203–220, Springer Berlin Heidelberg, 2002.

[18] R. Elliott and C.-J. Osakwe, Option pricing for pure jump processes with Markov switching compensators, Finance and Stochastics, 10(2), pp. 250–275, 2006.

[19] R. J. Elliott, L. Chan, and T. K. Siu, Option pricing and Esscher transform under regime switching, Annals of Finance, 1(4), pp. 423–432, 2005.

[20] K. Fan and R. Wang, Valuation of correlation options under a stochastic interest rate model with regime switching, Frontiers of Mathematics in China, 12(5), pp. 1113–1130, 2017.

[21] G. Grimmett and D. Stirzaker, *Probability and random processes*, Oxford university press, 2001.

[22] D. Hainaut and D. B. Colwell, A structural model for credit risk with switching processes and synchronous jumps, The European Journal of Finance, 22(11), pp. 1040–1062, 2016.

[23] T. Hurd and Z. Zhou, A Fourier transform method for spread option pricing, SIAM Journal of Financial Mathematics, 1(1), pp. 142–157, 2010.

[24] E. Kirk, Correlation in the energy markets, Managing energy price risk, 1, pp. 71–78, 1995.

[25] M. Konikov and D. Madan, Option pricing using variance-gamma Markov chains, Review of Derivatives Research, 5(1), pp. 81–115, 2002.

[26] D. Linders and B. Stassen, The multivariate Variance Gamma model: Basket option pricing and calibration, Quantitative Finance, 16(4), pp. 555–572, 2016.

[27] D. B. Madan and E. Seneta, The variance gamma (VG) model for share market returns, Journal of Business, pp. 511–524, 1990.

[28] J.-F. Mai, M. Scherer, and T. Schulz, Sequential modeling of dependent jump processes, Wilmott, 2014(70), pp. 54–63, 2014.

[29] W. Margrabe, The value of an option to exchange one asset for another, The Journal of Finance, 33(1), pp. 177–186, 1978.

[30] R. C. Merton, Option pricing when underlying stock returns are discontinuous, Journal of Financial Economics, 3(1-2), pp. 125–144, 1976.

[31] J. R. Norris, *Markov Chains*, 2, Cambridge University Press, 1998.

[32] A. Ramponi, Fourier transform methods for regime-switching jump-diffusions and the pricing of forward starting options, International Journal of Theoretical and Applied Finance, 15(05), pp. 1–26, 2012.

[33] S. Ross, *Stochastic Processes*, Wiley series in probability and statistics: Probability and Statistics, Wiley, 1996.

[34] T. Schulz, *Stochastic Dependencies in Derivative Pricing: Decoupled BNS-volatility, Sequential Modeling of Jumps, and Extremal WWR*, Ph.D. thesis, Technische Universität München, 2017.

[35] G. Tour, N. Thakoor, A. Khaliq, and D. Tangman, COS method for option pricing under a regime-switching model with time-changed Lévy processes, Quantitative Finance, 18(4), pp. 673–692, 2018.

[36] A. Venkatramanan and C. Alexander, Closed form approximations for spread options, Applied Mathematical Finance, 18(5), pp. 447–472, 2011.

[37] T. Vorst, Prices and hedge ratios of average exchange rate options, International Review of Financial Analysis, 1(3), pp. 179–193, 1992.

[38] G. Xu and H. Zheng, Lower bound approximation to basket option values for local volatility jump-diffusion models, International Journal of Theoretical and Applied Finance, 17(01), pp. 1–15, 2014.

[39] D. D. Yao, Q. Zhang, and X. Y. Zhou, A regime-switching model for European options, in *Stochastic processes, optimization, and control theory: applications in financial engineering, queueing networks, and manufacturing systems*, pp. 281–300, Springer, 2006.

[40] J. Yoon, B.-G. Jang, and K.-H. Roh, An analytic valuation method for multivariate contingent claims with regime-switching volatilities, Operations Research Letters, 39(3), pp. 180–187, 2011.

[41] J. Zhou and X. Wang, Accurate closed-form approximation for pricing Asian and basket options, Applied Stochastic Models in Business and Industry, 24(4), pp. 343–358, 2008.

[42] S.-P. Zhu, A. Badran, and X. Lu, A new exact solution for pricing European options in a two-state regime-switching economy, Computers & Mathematics with Applications, 64(8), pp. 2744 – 2755, 2012.

APPENDIX A

PROOFS OF SOME THEOREMS AND LEMMAS

By assuming the notions given in Chapter 6 and Chapter 7, we give the proofs of some important results below.

With the following lemma, we obtain the joint characteristic function of MMLPs with synchronous jumps.

Lemma A.1. *Consider a Markov-modulated drift process* $C(t) = \int_0^t c(M(s))\,ds$ *where*

$$c(M(t)) = \sum_{j=1}^{N} c_j \mathbb{1}_{M(t)=j},$$

with $c_j,\ j = 1, \ldots, N$*, being some constants.*

Then, under a MMLP framework with synchronous jumps, for all $\boldsymbol{a} \in \mathbb{C}^n$ *and* $t \geq 0$*:*

$$\mathbb{E}\left[e^{C(t)+\langle \boldsymbol{a}, \boldsymbol{X}(t)\rangle}\right] = \boldsymbol{p} e^{\left(-A+Q+\Gamma\hat{G}(-i\boldsymbol{a})\right)t}\mathbf{1},$$

where A is the diagonal matrix with $A_{jj} = \Phi_j(-i\boldsymbol{a}) - c_j$, under the assumption that $\hat{G}(-i\mathbf{a})$ exists and $\Phi_j(-i\mathbf{a})$ is known analytically.

Proof. We will closely follow the arguments given in Deelstra and Simon [16] by taking the sycnhronous jumps into account and by using XI-Proposition 2.2 from Asmussen [2], as motivated by this reference paper.

Let us define the conditional expectation $O_{lj}(t)$:

$$O_{lj}(t) = \mathbb{E}\left[e^{C(t)+\langle \boldsymbol{a}, \boldsymbol{X}(t)\rangle}\mathbb{1}_{M(t)=j} \mid M(0) = l\right],$$

and consider the fact that the number of transitions during the time interval $[t, t + y]$ is more than one with a probability of $o(y)$.

Our first aim is to compute $O_{lj}(t+y)$ by means of $O_{lj}(t)$. For this purpose, we will consider the following arguments:

Notice that the conditinal expectation $O_{lj}(t+y)$ can be rewritten as:

$$\begin{aligned}
O_{lj}(t+y) = {} & \mathbb{E}\left[e^{C(t)+c_j y+\langle \boldsymbol{a},\boldsymbol{X}(t)\rangle+\langle \boldsymbol{a},\boldsymbol{Y_j}(t+y)-\boldsymbol{Y_j}(t)\rangle}\mathbb{1}_{M(t)=j} \mid M(0)=l\right] \times \\
& \mathbb{P}\left(M(t+y)=j, \text{no synchronous jump on } [t,t+y] \mid M(t)=j\right) \\
& + \sum_{k\neq j}^{N} \mathbb{E}\left[e^{C(t)+c_k y+\langle \boldsymbol{a},\boldsymbol{X}(t)\rangle+\langle \boldsymbol{a},\boldsymbol{Y_k}(t+y)-\boldsymbol{Y_k}(t)\rangle}\mathbb{1}_{M(t)=k} \mid M(0)=l\right] \times \\
& \mathbb{P}\left(M(t+y)=j, \text{no synchronous jump on } [t,t+y] \mid M(t)=k\right) \\
& + \sum_{k\neq j}^{N} \mathbb{E}\left[e^{C(t)+c_k y+\langle \boldsymbol{a},\boldsymbol{X}(t)\rangle+\langle \boldsymbol{a},\boldsymbol{Y_k}(t+y)-\boldsymbol{Y_k}(t)\rangle}\mathbb{1}_{M(t)=k} \mid M(0)=l\right] \times \\
& \sum_{k\neq j}^{N} \mathbb{E}\left[e^{\langle \boldsymbol{a},\boldsymbol{J_k}\rangle}\mathbb{1}_{M(t+y)=j} \mid M(t)=k\right] \mathbb{P}\left(\text{synchronous jump on } [t,t+y] \mid M(t)=k\right),
\end{aligned}$$

which is followed from conditioning on $M(t)$ and possibility of synchronous jumps. Also note that synchronous jumps $\boldsymbol{J_k}$ are independent of the increments $\boldsymbol{Y_j}(t+y)-\boldsymbol{Y_j}(t)$ and $M(0)$.

Substituting the values of the above probabilities, we obtain

$$\begin{aligned}
O_{lj}(t+y) = {} & \mathbb{E}\left[e^{C(t)+c_j y+\langle \boldsymbol{a},\boldsymbol{X}(t)\rangle+\langle \boldsymbol{a},\boldsymbol{Y_j}(t+y)-\boldsymbol{Y_j}(t)\rangle}\mathbb{1}_{M(t)=j} \mid M(0)=l\right] \times \\
& (1+q_{jj}y+o(y)) + \sum_{k\neq j}^{N} \mathbb{E}\left[e^{C(t)+c_k y+\langle \boldsymbol{a},\boldsymbol{X}(t)\rangle+\langle \boldsymbol{a},\boldsymbol{Y_k}(t+y)-\boldsymbol{Y_k}(t)\rangle}\mathbb{1}_{M(t)=k} \mid M(0)=l\right] \times \\
& (q_{kj}y+o(y)) + \sum_{k\neq j}^{N} \mathbb{E}\left[e^{C(t)+c_k y+\langle \boldsymbol{a},\boldsymbol{X}(t)\rangle+\langle \boldsymbol{a},\boldsymbol{Y_k}(t+y)-\boldsymbol{Y_k}(t)\rangle}\mathbb{1}_{M(t)=k} \mid M(0)=l\right] \times \\
& \sum_{k\neq j}^{N} \mathbb{E}\left[e^{\langle \boldsymbol{a},\boldsymbol{J_k}\rangle}\mathbb{1}_{M(t+y)=j} \mid M(t)=k\right] (\gamma_k y+o(y)).
\end{aligned}$$

By regarding the independent and stationary increments property of Lévy processes, we notice that $\boldsymbol{Y_j}(t+y)-\boldsymbol{Y_j}(t)$ is independent of $\boldsymbol{X}(t)$ and equal in disribution to $\boldsymbol{Y_j}(y)$, for all $j=1,2,\ldots,n$. Since these increments $\boldsymbol{Y_j}(t+y)-\boldsymbol{Y_j}(t)$ are also independent of M, the above equation turns out to be:

$$O_{lj}(t+y) = e^{c_j y}\mathbb{E}\left[e^{C(t)+\langle \boldsymbol{a},\boldsymbol{X}(t)\rangle}\mathbb{1}_{M(t)=j} \mid M(0)=l\right]\mathbb{E}\left[e^{\langle \boldsymbol{a},\boldsymbol{Y_j}(y)\rangle}\right](1+q_{jj}y+o(y))$$
$$+\sum_{k\neq j}^{N} e^{c_k y}\mathbb{E}\left[e^{C(t)+\langle \boldsymbol{a},\boldsymbol{X}(t)\rangle}\mathbb{1}_{M(t)=k} \mid M(0)=l\right]\mathbb{E}\left[e^{\langle \boldsymbol{a},\boldsymbol{Y_k}(y)\rangle}\right](q_{kj}y+o(y))$$
$$+\sum_{k\neq j}^{N} e^{c_k y}\mathbb{E}\left[e^{C(t)+\langle \boldsymbol{a},\boldsymbol{X}(t)\rangle}\mathbb{1}_{M(t)=k} \mid M(0)=l\right]\mathbb{E}\left[e^{\langle \boldsymbol{a},\boldsymbol{Y_j}(y)\rangle}\right]\times$$
$$\mathbb{E}\left[e^{\langle \boldsymbol{a},\boldsymbol{J_k}\rangle}\mathbb{1}_{M(t+y)=j} \mid M(t)=k\right](\gamma_k y+o(y)).$$

Remembering the definiton of $O_{lj}(t)$ as well as characteristic functions of $\boldsymbol{Y_j}$ and $\boldsymbol{J_k}$, we then have:

$$O_{lj}(t+y) = O_{lj}(t)e^{y(c_j-\Phi_j(-i\boldsymbol{a}))}(1+q_{jj}y+o(y))+\sum_{k\neq j}^{N} O_{lk}(t)e^{y(c_k-\Phi_k(-i\boldsymbol{a}))}(q_{kj}y+o(y))$$
$$+\sum_{k\neq j}^{N} O_{lk}(t)e^{y(c_k-\Phi_k(-i\boldsymbol{a}))}\hat{G}_{kj}(-i\boldsymbol{a})(\gamma_k y+o(y))$$
$$= O_{lj}(t)(1+y(c_j-\Phi_j(-i\boldsymbol{a}))+o(y))(1+q_{jj}y+o(y))$$
$$+\sum_{k\neq j}^{N} O_{lk}(t)(1+y(c_k-\Phi_k(-i\boldsymbol{a}))+o(y))(q_{kj}y+o(y))$$
$$+\sum_{k\neq j}^{N} O_{lk}(t)(1+y(c_k-\Phi_k(-i\boldsymbol{a}))+o(y))\hat{G}_{kj}(-i\boldsymbol{a})(\gamma_k y+o(y)),$$

where in the last equality we replace the term $e^{y(c_j-\Phi_j(-i\boldsymbol{a}))}$ with its Taylor expansion $1+y(c_j-\Phi_j(-i\boldsymbol{a}))+o(y)$.

By straightforward calculations, $O_{lj}(t+y)$ is finally rewritten as:

$$O_{lj}(t+y) = O_{lj}(t) + y\Bigg(O_{lj}(t)(c_j-\Phi_j(-i\boldsymbol{a})) + \sum_{k=1}^{N} O_{lk}(t)q_{kj}$$
$$+\sum_{k\neq j}^{N} O_{lk}(t)\hat{G}_{kj}(-i\boldsymbol{a})\gamma_k\Bigg),$$

adressing the following equality:

$$\frac{1}{y}\left(O_{lj}(t+y)-O_{lj}(t)\right) = O_{lj}(t)(c_j-\Phi_j(-i\boldsymbol{a})) + \sum_{k=1}^{N} O_{lk}(t)q_{kj}$$
$$+\sum_{k\neq j}^{N} O_{lk}(t)\hat{G}_{kj}(-i\boldsymbol{a})\gamma_k.$$

In what follows, we define a matrix $O(t) := (O_{lj}(t))_{N\times N}$ and a diagonal matrix A with $A_{jj} = \Phi_j(-i\boldsymbol{a}) - c_j$ so that the above equality becomes

$$\frac{1}{y}\left(O(t+y) - O(t)\right) = O(t)(Q - A + \Gamma\hat{G}(-i\boldsymbol{a})).$$

Next, we take the limit as y approaches to zero and obtain

$$\frac{dO(t)}{dt} = O(t)(Q - A + \Gamma\hat{G}(-i\boldsymbol{a})),$$

or, equivalently, $O(t) = e^{(Q-A+\Gamma\hat{G}(-i\boldsymbol{a}))t}$ with $O(0) = I$. Here, I denotes the $N\times N$ identity matrix.

By recognizing that the expectation $\mathbb{E}\left[e^{C(t)+\langle \boldsymbol{a},\boldsymbol{X}(t)\rangle}\right]$ can be expressed as

$$\begin{aligned}\mathbb{E}\left[e^{C(t)+\langle \boldsymbol{a},\boldsymbol{X}(t)\rangle}\right] &= \sum_{j=1}^{N}\sum_{l=1}^{N}\mathbb{E}\left[e^{C(t)+\langle \boldsymbol{a},\boldsymbol{X}(t)\rangle}\mathbb{1}_{M(t)=j}\,|\,M(0)=l\right]\mathbb{P}\left(M(0)=l\right)\\ &= \sum_{j=1}^{N}\sum_{l=1}^{N} e^{(Q-A+\Gamma\hat{G}(-i\boldsymbol{a}))t}p_l,\end{aligned}$$

we conclude the proof. □

The next result examines the drift condition that makes the discounted asset prices martingale when their dynamics are driven by MMLPs with synchronous jumps.

Lemma A.2. *Suppose that $M(t)$ is a homogeneous continuous-time Markov process defined by a finite state space $\mathcal{S} = \{1,2,\ldots,N\}$, generator Q and initial probability vector $\mathbf{p} = [p_1,p_2,\ldots,p_N]$ with $p_j = \mathbb{P}\left(M(0)=j\right)$. Assume that $\mathbf{X}(t) = (X_1(t)\ldots,X_n(t))$ is a n-dimensional MMLP such that when $M=k$, $\mathbf{X}$ is identified by the n-dimensional Lévy process $\mathbf{Y}_k = (Y_{1k},\ldots,Y_{nk})$ that has the characteristic exponent $\Phi_k(\mathbf{u})$:*

$$\mathbb{E}^{\mathbb{Q}}\left[e^{i\langle \mathbf{u},\mathbf{Y}_k(t)\rangle}\right] = e^{-\Phi_k(\mathbf{u})t}.$$

Then, under a MMLP framework with synchronous jumps, if the vectors $\boldsymbol{\mu}_l$ are chosen as

$$\boldsymbol{\mu}_l = \boldsymbol{r} + \boldsymbol{\Phi}(-i\boldsymbol{e}_l) + \Gamma\left(I - \hat{G}(-i\boldsymbol{e}_l)\right)\mathbf{1}$$

for $l = 1,2,\ldots,n$, then the processes $\left(e^{-U(t)}S_l(t)\right)_t$ are martingales under $\mathbb{Q}$, where I is the $N\times N$ identity matrix.

Proof. In order to check the $\mathbb{Q}$-martingality of $e^{-U(t)}S_l(t)$, we verify whether

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-U(t)+\Lambda_l(t)+X_l(t)}\right] = 1 \quad \forall t.$$

The left hand side of this last equality follows easily from Lemma A.1 with $\mathbf{a} = \boldsymbol{e}_l$ and $C(t) = \Lambda_l(t) - U(t)$:

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-U(t)+\Lambda_l(t)+X_l(t)}\right] = \mathbf{p}e^{B_l t}\mathbf{1}$$

with $B_l = Q - A_l + \Gamma\hat{G}(-i\boldsymbol{e}_l)$ and $A_l = \text{diag}\,(\boldsymbol{r} - \boldsymbol{\mu}_l + \boldsymbol{\Phi}(-i\boldsymbol{e}_l))$. It therefore suffices to prove that $B_l\mathbf{1} = 0$, which is equivalent to the equality:

$$Q\mathbf{1} + \boldsymbol{\mu}_l - \boldsymbol{r} - \boldsymbol{\Phi}(-i\boldsymbol{e}_l) + \Gamma\hat{G}(-i\boldsymbol{e}_l)\mathbf{1} = \mathbf{0}.$$

Since $Q\mathbf{1} = -\Gamma\mathbf{1}$, the announced relation follows. □

Considering also the notations given for basket options, the following theorem will state the price $V^{\text{Eu-Geo}}(0)$ of a European call option written on the geometric average $G_n(T)$ of underlying assets:

$$V^{\text{Eu-Geo}}(0) = \mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}(G_n(T) - K)^+\right],$$

where K is the strike price of the option and T is the maturity time.

Theorem A.1. *1. Let the market dynamics be driven by MMLP processes without synchronous jumps, as given in Section 5.1. Then, the European option price $V^{\text{Eu-Geo}}(0)$ written on $G_n(T)$ is given explicitly in the following form:*

$$V^{\text{Eu-Geo}}(0) = \left(\frac{e^{-\delta k}}{\pi}\int_0^\infty e^{-i\gamma k}\Psi_T^{\text{Eu-Geo}}(\gamma;\delta,K)d\gamma\right)^+,$$

where δ is the damping factor, $H_n(0) = \sum_{j=1}^n w_j \ln s_j$,

$$\Psi_T^{\text{Eu-Geo}}(\gamma;\delta,K) = \frac{e^{(1+\delta+i\gamma)H_n(0)}}{(\delta+i\gamma)(1+\delta+i\gamma)}\mathbf{p}e^{(Q-A)T}\mathbf{1}$$

with

$$A = \text{diag}\left(\boldsymbol{\Phi}(-i\mathbf{a}) + \mathbf{r} - (1+\delta+i\gamma)\sum_{j=1}^n w_j\boldsymbol{\mu}_j\right),$$

$\mathbf{a} = (1+\delta+i\gamma)\mathbf{w}$ and $\boldsymbol{\mu}_j$, $j = 1,2,\ldots,n$, being the martingale condition given in Equation (5.2).

2. *In the case of a MMLP framework with synchronous jumps, the corresponding European option price* $V^{\textit{Eu-Geo}}(0)$ *then turns out to be:*

$$V^{\textit{Eu-Geo}}(0) = \left(\frac{e^{-\delta k}}{\pi} \int_0^\infty e^{-i\gamma k} \Psi_T^{\textit{Eu-Geo}}(\gamma; \delta, K) d\gamma \right)^+,$$

where δ *is the damping factor,* $H_n(0) = \sum_{j=1}^n w_j \ln s_j$, $\mathbf{a} = (1+\delta+i\gamma)\mathbf{w}$,

$$\Psi_T^{\textit{Eu-Geo}}(\gamma; \delta, K) = \frac{\exp{(1+\delta+i\gamma)H_n(0)}}{(\delta+i\gamma)(1+\delta+i\gamma)} \mathbf{p} e^{(Q-A+\Gamma\hat{G}(-i\mathbf{a}))T}\mathbf{1}$$

with

$$A = \textit{diag}\left(\mathbf{\Phi}(-i\mathbf{a}) + \mathbf{r} - (1+\delta+i\gamma) \sum_{j=1}^n w_j \boldsymbol{\mu}_j \right),$$

and $\boldsymbol{\mu}_j$, $j = 1, 2, \ldots, n$, *being the martingale condition given in Equation* (7.1).

Proof. In order to compute the option price $V^{\text{Eu-Geo}}(0)$, we will follow Carr and Madan arguments [11], as done in the previous sections. Indeed, these results will be very similar to the one obtained for the European call option written on the risky asset S, see e.g. Deelstra and Simon [16] for the case without synchronous jumps.

Recalling that $G_n(T) = e^{H_n(T)}$, option price $V^{\text{Eu-Geo}}(0)$ can be expressed in the following integral form:

$$\begin{aligned} V^{\text{Eu-Geo}}(0) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}(G_n(T) - K^+\right] \\ &= \int_{\mathbb{R}^2} e^{-u}(e^{h_n} - e^k)\mathbf{1}_{(h_n \geq k)} f(h_n, u) \mathrm{d}h_n \mathrm{d}u \\ &= \int_{-\infty}^{\infty} \int_k^\infty e^{-u}(e^{h_n} - e^k) f(h_n, u) \mathrm{d}h_n \mathrm{d}u \end{aligned}$$

where $k = \ln(K)$ and $f(h_n, u)$ denotes the density function of $(H_n(T), U(T))$.

The well-known Carr and Madan argument [11] implies that

$$V_K^{k,\alpha}(0) = \frac{e^{-\delta k}}{\pi} \int_0^\infty e^{-i\gamma k} \Psi_T^{\text{Eu-Geo}}(\gamma; \delta, K) d\gamma,$$

where δ is the damping factor, and

$$\begin{aligned} \Psi_T^{\text{Eu-Geo}}(\gamma; \delta, K) &= \int_{-\infty}^{\infty} e^{(\delta+i\gamma)k} \mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)}(G_n(T) - K)\mathbf{1}_{(h_n \geq k)}\right] \mathrm{d}k \\ &= \int_{-\infty}^{\infty} e^{(\delta+i\gamma)k} \int_{-\infty}^{\infty} \int_k^\infty e^{-u}(e^{h_n} - e^k) f(h_n, u) \mathrm{d}h_n \mathrm{d}u \mathrm{d}k. \end{aligned}$$

When we switch the order of integrals by using Fubini theorem and elaborate the product, $\Psi_T^{\text{Eu-Geo}}(\gamma;\delta,K)$ turns out to be

$$\begin{aligned}\Psi_T^{\text{Eu-Geo}}(\gamma;\delta,K) &= \int_{\mathbb{R}^2}\int_{-\infty}^{h_n} e^{-u+(\delta+i\gamma)k}(e^{h_n}-e^k)f(h_n,u)\mathrm{d}k\mathrm{d}h_n\mathrm{d}u \\ &= \int_{\mathbb{R}^2}\int_{-\infty}^{h_n} e^{-u+(\delta+i\gamma)k+h_n}f(h_n,u)\mathrm{d}k\mathrm{d}h_n\mathrm{d}u \\ &\quad - \int_{\mathbb{R}^2}\int_{-\infty}^{h_n} e^{-u+(1+\delta+i\gamma)k}f(h_n,u)\mathrm{d}k\mathrm{d}h_n\mathrm{d}u.\end{aligned}$$

As a result of straightforward calculations, we then have

$$\begin{aligned}\Psi_T^{\text{Eu-Geo}}(\gamma;\delta,K) &= \frac{1}{\delta+i\gamma}\int_{\mathbb{R}^2} e^{-u+(1+\delta+i\gamma)h_n}f(h_n,u)\mathrm{d}h_n\mathrm{d}u \\ &\quad - \frac{1}{1+\delta+i\gamma}\int_{\mathbb{R}^2}\int_{-\infty}^{h_n} e^{-u+(1+\delta+i\gamma)h_n}f(h_n,u)\mathrm{d}h_n\mathrm{d}u \\ &= \frac{1}{(1+\delta+i\gamma)(\delta+i\gamma)}\int_{\mathbb{R}^2} e^{-u+(1+\delta+i\gamma)h_n}f(h_n,u)\mathrm{d}h_n\mathrm{d}u.\end{aligned}$$

Since the above integral is an expectation, we finally have

$$\begin{aligned}\Psi_T^{\text{Eu-Geo}}(\gamma;\delta,K) &= \frac{1}{(1+\delta+i\gamma)(\delta+i\gamma)}\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)+H_n(T)(1+\delta+i\gamma)}\right] \\ &= \frac{e^{(1+\delta+i\gamma)H_n(0)}}{(1+\delta+i\gamma)(\delta+i\gamma)}\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)+(1+\delta+i\gamma)\sum_{j=1}^n w_j(\Lambda_j(T)+X_j(T))}\right].\end{aligned}$$

In the last equality, we used the relation $H_n(T)=\sum_{j=1}^n w_j(\Lambda_j(T)+X_j(T))+H_n(0)$. Notice that when the market dynamics are ruled by MMLPs without synchronous jumps, the resulting term is expressed in the form of

$$\begin{aligned}\Psi_T^{\text{Eu-Geo}}(\gamma;\delta,K) &= \frac{e^{(1+\delta+i\gamma)H_n(0)}}{(1+\delta+i\gamma)(\delta+i\gamma)}\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)+(1+\delta+i\gamma)\sum_{j=1}^n w_j(\Lambda_j(T)+X_j(T))}\right] \\ &= \frac{e^{(1+\delta+i\gamma)H_n(0)}}{(1+\delta+i\gamma)(\delta+i\gamma)}\mathbf{p}e^{(Q-A)T}\mathbf{1},\end{aligned}$$

by using Lemma 2.2 with $C(t)=-U(t)+(1+\delta+i\gamma)\sum_{j=1}^n w_j\Lambda_j(T)$ and $\mathbf{a}=(1+\delta+i\gamma)\mathbf{w}$.

Instead, in the case of a framework with synchronous jumps, we have the following interpretation:

$$\begin{aligned}\Psi_T^{\text{Eu-Geo}}(\gamma;\delta,K) &= \frac{e^{(1+\delta+i\gamma)H_n(0)}}{(1+\delta+i\gamma)(\delta+i\gamma)}\mathbb{E}^{\mathbb{Q}}\left[e^{-U(T)+(1+\delta+i\gamma)\sum_{j=1}^n w_j(\Lambda_j(T)+X_j(T))}\right] \\ &= \frac{e^{(1+\delta+i\gamma)H_n(0)}}{(1+\delta+i\gamma)(\delta+i\gamma)}\mathbf{p}e^{(Q-A+\Gamma\hat{G}(-i\mathbf{a}))T}\mathbf{1},\end{aligned}$$

where we now consider Lemma A.1 for the drift process $C(t) = -U(t) + (1 + \delta + i\gamma) \sum_{j=1}^{n} w_j \Lambda_j(T)$ and $\mathbf{a} = (1 + \delta + i\gamma)\mathbf{w}$.

It is important to mention that although the drift process $C(t)$ seems to be same under these two different framework, $\Lambda_j(T)$'s are ruled by different martingale conditions.

$\square$

CURRICULUM VITAE

PERSONAL INFORMATION

Surname, Name: Kozpınar, Sinem

Nationality: Turkish (TC)

Date and Place of Birth: 26.12.1986, Eskişehir

Marital Status: Single

Phone: +90(312)2102987

Fax: +90(312)2102985

EDUCATION

Degree	Institution	Year of Graduation
Ph.D.	Middle East Technical University	2018
M.S.	Middle East Technical University	2013
B.S.	Hacettepe University	2011
High School	Bülent Ecevit Anatolian High School	2004

PROFESSIONAL EXPERIENCE

Year	Place	Enrolment
2012-Present	Middle East Technical University	Research Assistant

PUBLICATIONS

- Deelstra, G., Kozpınar, S., Simon, M., Spread and Basket Option Pricing in a Markov-Modulated Lévy Framework with Synchronous Jumps, Applied Stochastic Models in Business and Industry, pp.1-21, 2018.

- Kozpınar, S., Yolcu Okur, Y., Evcin, C. and Uğur, Ö., Pricing Stochastic Barrier Equity Options under a Double-Exponential Jump-Diffusion Process, submitted to Mathematics and Computers in Simulation, 2018.

- Kozpınar, S., Uzunca, M., Yolcu Okur, Y. and Karasözen, B., Pricing European and American Options under Heston Model using Discontinuous Galerkin Finite Elements, submitted to Computational Economics, 2018.

Presentations:

- Kozpınar, S., Yolcu Okur, Y., Uğur, Ö. and Tekin, Ö., Pricing Stochastic Barrier Options in Presence of Jumps, 55th Meeting of the EWGCFM, Ankara, Turkey, 14-16 May 2015.

- Kozpınar, S. and Tekin, Ö., Pricing Equity Options with Stochastic Barrier in Presence of Jumps, Advanced Modelling in Mathematical Finance, Kiel, Almanya, 20-22 May 2015. (Poster Presentation, supervised by Yolcu Okur, Y.).

- Kozpınar, S., Pricing Equity Options with Stochastic Barrier in Presence of Jumps, 2nd Ankara-İstanbul Workshop on Stochastic Processes, Istanbul, Turkey, 2015, (Poster Presentation, supervised by Yolcu Okur, Y. and Uğur, Ö.).

- Kozpınar, S., Uzunca, M. and Karasözen, B., Model Order Reduction for Paramet rized Option Pricing Models, Reduced Basis Summer School 2016, Hedersleben, Germany, 3-7 October 2016.

- Kozpınar, S., Yolcu Okur, Y. and Eksi-Altay, Z., Markov-modulated Spread Option Pricing, 8th General AMaMeF Conference, Amsterdam, Holland, 19-23 June 2017.

- Kozpınar, S., Deelstra, G., Simon, M., Pricing Basket and Spread Options under a Markov-Modulated Lévy Framework with Synchronous Jumps, Brussels, Belgium, 8-9 February 2018 (Poster Presentation).