ANALYSIS OF A PROJECTION-BASED VARIATIONAL MULTISCALE METHOD FOR A LINEARLY EXTRAPOLATED BDF2 TIME DISCRETIZATION OF THE NAVIER-STOKES EQUATIONS

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ABSTRACT

ANALYSIS OF A PROJECTION-BASED VARIATIONAL MULTISCALE METHOD FOR A LINEARLY EXTRAPOLATED BDF2 TIME DISCRETIZATION OF THE NAVIER-STOKES EQUATIONS

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This thesis studies a projection-based variational multiscale (VMS) method based on a linearly extrapolated second order backward difference formula (BDF2) to simulate the incompressible time-dependent Navier-Stokes equations (NSE). The method concerns adding stabilization based on projection acting only on the small scales. To give a basic notion of the projection-based VMS method, a three-scale VMS method is explained. Also, the principles of the projection-based VMS stabilization are provided. By using this stabilization scheme for spatial discretization and the linearly extrapolated BDF2 for time discretization of NSE, the fully discrete approximation of them is obtained. The existence, uniqueness, unconditional stability and convergence of the approximate solutions are proven. Also, to verify the theoretical findings, numerical experiments which indicate the efficiency of the proposed scheme are presented.

Keywords: Navier-Stokes equations; projection-based variational multiscale method; BDF2; error analysis

ÖZ

NAVIER-STOKES DENKLEMLERİNİN LİNEER EKSTRAPOLASYONLU BDF2 ZAMAN AYRIKLAŞTIRILMASI İÇİN PROJEKSİYON ESASLI VARYASYONEL ÇOKLU ÖLÇEK METODUNUN ANALİZİ

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Bu tez sıkıştırılamaz zamana bağlı Navier-Stokes denklemlerinin ikinci mertebeden geriye doğru fark formülüne (BDF2) dayalı projeksiyon (izdüşüm) esaslı varyasyonel çoklu ölçek (VMS) metodu ile çözümlerini incelemektedir. Bu metot sadece küçük ölçeklere etki eden projeksiyona dayalı bir kararlaştırma ekleme işlemi ile alakalıdır. Projeksiyon esaslı VMS metodunun temek fikrini verebilmek için, üç ölçekli VMS metot açıklanmıştır. Sonrasında, projeksiyon esaslı VMS metodunun prensipleri verilmiştir. Bu kararlaştırmalı algoritmayı uzay ayrıklaştırılması için ve lineer ekstrapolasyonlu (dışkestirim) BDF2 yöntemini zaman ayrıklaştırılması için kullanarak, Navier-Stokes denklemlerinin tamamen ayrıştırılmış yaklaşık çözümü elde edilmiştir. Elde edilen çözümün varlığı, tekliği, koşulsuz kararlılığı ve yakınsaklığı ispatlanmıştır. Ayrıca, teorik sonuçları doğrulamak için, önerilen algoritmanın etkinliğini gösteren sayısal testler sunulmuştur.

Anahtar Kelimeler: Navier-Stokes Denklemeleri; projeksiyon esaslı varyasyonel çoklu ölçek metodu; BDF2; hata analizi

To my family

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CHAPTER 1

INTRODUCTION

Turbulence is the concept used when describing the behavior of many flows in nature. In daily life, people have to deal with these kinds of flows. For example, when traveling with airplane, we are exposed to turbulence in air flow or when investigating the motion of the stream, we confront the turbulent flows of water. In short, most of the gas and water flows are turbulent.

Turbulent flows are highly dissipative, irregular, diffusive and rotational flows. Because of these characteristics features of the turbulence, coping with turbulent flows is very compelling. Despite of the difficulty of turbulence problems, down through the ages, many scientists have tried to find solutions for them. However, there are still incomprehensible notions about turbulence phenomena and this is tempting many scientists, especially engineers, physicians and mathematicians, since the turbulence problem not only have a scientific challenge, but also practical challenges. Hence, solving them serves several advantages for easier human life.

Turbulent flows are modelled with Navier-Stokes equations (NSE). In NSE, there is a control parameter which is called Re, and as this nondimensional parameters' value is increasing, flows' motion changes from stationary to turbulent.

The NSE are so complex equations with regard to existence, uniqueness and stability of their solutions. In practice, they are too difficult to solve analytically. When we go back to previous studies on mathematical theory of NSE, we see that they started with Leray's paper [43]. Today, we still consider his description of NSE's solutions, namely weak solutions. In this work, he proved the existence and uniqueness of the weak solutions for NSE for all cases in two dimension. However, in three dimension, the existence and uniqueness of the weak solutions of the NSE for turbulence problems can not be shown, although they have been proven for smaller Re (laminar and stationary flows) [42]. This dark side of the mathematical theory of NSE have allured many scientists all over the world, since it was presented by Leray [43] in 1934. Clay Mathematics Institute want to celebrate this efforts. Hence, it is presented existence and uniqueness of the weak solutions of NSE in 3D as one million dolar clay prize problem. For detailed description of the problem, we refer to [14].

Since we can not find analytical solution of NSE with direct computation, we must solve them with some numerical methods. In this process, richness of the scale in turbulent flows is the major obstacle to computation. As long as range of scales expands (with respect to Re), cost of computation increases. Hence, we benefit from some turbulence models.

Large eddy simulation (LES) is one of the well-known method to solve turbulence problem. This method considers only large scales of turbulent flows between variety of scales in it, and try to simulate them by taking average of solutions in space. At the first glance, LES seems very promising method to simulate multiscale flows. However, there are some drawbacks of LES. These are explained in [36] as following. The first drawback arises when flows are given in bounded domain, and most of time we consider these cases in applications. The main cause of this problem is regarding averaging NSE in LES. When deriving equation for large scales, some additional terms are introduced. In [13], it is called commutation error. Although we neglect this term in applications, according to [13], it does not vanish asymptotically in all cases. Also, second drawback is how the appropriate boundary conditions can be defined in bounded domain. There is no cure for this drawback. However, this difficulty of LES constructs a basis for variational multiscale (VMS) methods in which large scales are defined with projection technique into suitable space.

VMS method was established in [24] and [25] as a general technique to model the subgrid scales in the numerical solution of partial differential equations. In these papers, the VMS method was presented to deal with multiscale phenomena. In parallel, in [19], J.-L. Guermond introduced an alternative technique for multiscale subgrid modeling. In [26], Hughes and his coworkers constructed a bridge between general ideas on VMS and turbulence problem. After this seminal paper, first numerical results were examined in [27, 28].

Core concept of the VMS method is based on the variational formulation of the partial differential equation and separation of the scales as resolved and unresolved scales. However, we can decompose resolved scales into many other scales. For example, three-scale VMS method relies on a decomposition of the flow field into large, small resolved scales and unresolved scales [36]. After the scale decomposition, another key feature of VMS method is that the projection of large scales into appropriate function spaces. In this thesis, we focus on a projection-based VMS method.

This thesis is organized as follows:

Chapter 2: Some definitions, theorems, lemmas and remarks frequently used throughout the thesis are presented.

Chapter 3: The Navier-Stokes equations and their weak formulation are presented. Also, in this chapter, the Galerkin finite element formulation of NSE to set the stage for our numerical method, which is the main concern of this thesis, is proposed.

Chapter 4: The knowledge required for our method is provided by presenting three-scale VMS method. By means of this method, a basis to understand how our method work is constructed.

Chapter 5: The main method used in the thesis is presented, which is the projection-based VMS method with the light of information from previous chapter.

Chapter 6: This chapter considers a numerical and mathematical analysis of a projection-based VMS method. The algorithm is presented by using linearly extrapolated two-step backward Euler formula (BDF2). Existence and uniqueness of the numerical solution is presented. Then, the unconditional stability and convergence of the method is proven. Lastly, in this chapter, efficiency of the method is verified by testing with three well-known flow problems in computer.

CHAPTER 2

MATHEMATICAL PRELIMINARIES

In this chapter, we will give a summary of the basic notations, terminologies, theorems and techniques which will be used throughout thesis. For the sake of formality, we use bold face letter for a vector-valued functions. In addition, C denotes generic constants which are independent from all flow parameters.

We use the standard notations for function spaces with their definitions which was mentioned in [1] by R. Adams. We also state some important definitions and theorems from [8, 32].

Definition 2.0.1. *(Lebesgue Spaces $L^p(\Omega)$) Let $\Omega \in \mathbb{R}^d$ and let p be a positive real number. The class of all measurable functions is defined by*

$$L^p(\Omega) := \{\mathbf{u} : \int_\Omega |\mathbf{u}(x)|^p dx < \infty\},$$

where $\mathbf{u}$ is also a measurable function defined on Ω and $1 \leq p < \infty$. The norm in $L^p(\Omega)$ space is

$$\|\mathbf{u}\|_{L^p(\Omega)} := \Big(\int_\Omega |\mathbf{u}(x)|^p dx\Big)^{1/p}.$$

Remark 2.0.1. *In this thesis, we use frequently L^2-space which is defined as*

$$L^2(\Omega) := \{\mathbf{u} : \int_\Omega |\mathbf{u}(x)|^2 dx < \infty\}.$$

Then, we denote the L^2-norm by

$$\|\mathbf{u}\| := \Big(\int_\Omega |\mathbf{u}(x)|^2 dx\Big)^{1/2}$$

and inner product in L^2-space by

$$(\mathbf{u}, \mathbf{v}) = \int_\Omega \mathbf{u}(x)\mathbf{v}(x)dx.$$

Definition 2.0.2. *(The Space $L^\infty(\Omega)$) The space $L^\infty(\Omega)$ is the space of all functions which are bounded for almost all $x \in \Omega$ and defined by*

$$L^\infty := \{\mathbf{u} : |\mathbf{u}(x)| < \infty \text{ for almost all } x \in \Omega\}.$$

Then, the norm on $L^\infty(\Omega)$ space is

$$\|\mathbf{u}\|_\infty := ess \sup_{x\in\Omega} |\mathbf{u}(x)|.$$

Definition 2.0.3. *(Sobolev Spaces $W^{k,p}(\Omega)$) Let k is a positive integer and $1 \leq p \leq \infty$. Sobolev space $W^{k,p}(\Omega)$ consists of all integrable functions $\mathbf{u} : \Omega \to \mathbb{R}$ such that for each multi-index α with $|\alpha| \leq k$, the derivative $D^\alpha \mathbf{u}$ exists in the weak sense and it belongs to $L^p(\Omega)$.*

$$W^{k,p}(\Omega) := \{\mathbf{u} \in L^p(\Omega) : D^\alpha \mathbf{u} \in L^p(\Omega) \text{ for } |\alpha| \leq k\}.$$

The norm on $W^{k,p}(\Omega)$ is

$$\|\mathbf{u}\|_{W^{k,p}} = \begin{cases} \Big(\sum_{|\alpha|\leq k} \|D^\alpha \mathbf{u}\|^p_{L^p(\Omega)} \Big)^{1/p} & \text{if } p \in [1, \infty), \\ \sum_{|\alpha|\leq k} ess \sup_{x\in\Omega} |\mathbf{u}(x)| & \text{if } p = \infty, \end{cases}.$$

Remarks 2.0.1.

- *If $k = 0$ in Sobolev space $W^{k,p}$, then $L^p(\Omega) = W^{0,p}(\Omega)$.*
- *The Sobolev spaces for $p = 2$ are Hilbert spaces. This is the most commonly used space in finite element analysis.*

$$W^{k,2}(\Omega) = H^k(\Omega).$$

Then, the norm on Hilbert Space $H^k(\Omega)$ is defined by

$$\|\mathbf{u}\|_{H^k} = \|\mathbf{u}\|_k = \Big(\sum_{|\alpha|\leq k} \|D^\alpha \mathbf{u}\|^2 \Big)^{1/2} = \Big[\int_\Omega \mathbf{u}(x)^2 + \sum_{\substack{j=1 \\ |\alpha|=j}}^{k} \Big(\frac{\partial^\alpha \mathbf{u}(x)}{\partial x^\alpha} \Big)^2 \Big]^{1/2}.$$

Also, seminorm in $H^k(\Omega)$ is defined by

$$|\mathbf{u}|_{H^k} = |\mathbf{u}|_k = \Big(\sum_{|\alpha|\leq k} |D^\alpha \mathbf{u}|^2 \Big)^{1/2} = \Big[\int_\Omega \sum_{\substack{j=1 \\ |\alpha|=j}}^{k} \Big(\frac{\partial^\alpha \mathbf{u}(x)}{\partial x^\alpha} \Big)^2 \Big]^{1/2}.$$

- *In the case $k = 1$, the Sobolev spaces are important for the study of Navier-Stokes equations.*

$$W^{1,p}(\Omega) = \{\mathbf{u} : \int_\Omega |\mathbf{u}(x)|^p + |\nabla \mathbf{u}(x)|^p dx \leq \infty\}, \quad p \in [1, \infty).$$

Also, the spaces $W^{1,p}(\Omega)$ are equipped with the norm

$$\|\mathbf{u}\|_{W^{1,p}(\Omega)} := \Big(\int_\Omega |\mathbf{u}(x)|^p + |\nabla \mathbf{u}(x)|^p dx \Big)^{1/p}, \quad p \in [1, \infty).$$

Definition 2.0.4. *(Spaces of Functions with Compact Support) A function $\mathbf{u}$ is said to have a compact support if $supp(\mathbf{u}) = \overline{\{x : \mathbf{u}(x) \neq 0\}} \subset \Omega$. This implies that $\mathbf{u}$ vanishes on the boundary of Ω. The spaces of functions which have a compact support are denoted by a subscript 0.*
The space $C_0^\infty(\Omega)$ is given by

$$C_0^\infty(\Omega) = \{\mathbf{u} : \mathbf{u} \in C^\infty(\Omega) \text{ and } \mathbf{u} \text{ has compact support}\}.$$

The Sobolev spaces $W_0^{k,p}(\Omega)$ are defined by the closure of $C_0^\infty(\Omega)$ in the norm of $W^{k,p}(\Omega)$. The most important one we use in this thesis is

$$W_0^{1,2}(\Omega) = H_0^1(\Omega) = \{\mathbf{u} : \mathbf{u} \in H^1(\Omega) \text{ and } \mathbf{u} = 0 \text{ on } \partial\Omega\}.$$

Definition 2.0.5. *(Spaces of Space-time Functions) For functions $\mathbf{u}(x, t)$ defined on* *time interval $(0,T)$, the space of space-time function is defined as*

$$L^p(0,T;H^k) = \Big\{\mathbf{u}(x,t) : \int_0^T \|\mathbf{u}(\cdot,t)\|_k^p \, dt < \infty\Big\},\ p \in [1,\infty),$$

with the norm

$$\|\mathbf{u}\|_{p,k} = \Big(\int_0^T \|\mathbf{u}(\cdot,t)\|_k^p \, dt\Big)^{1/p},\ p \in [1,\infty).$$

Also, when p is infinity, the norm is denoted as

$$\|\mathbf{u}\|_{\infty,k} = ess \sup_{t\in[0,T]} \|\mathbf{u}(\cdot,t)\|_k \,.$$

Theorem 2.0.1. *(The Sobolev Imbedding Theorem [1, p. 85]) Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a locally Lipschitz boundary. Let j and k are non-negative integers and let p satisfies $1 \leq p < \infty$. For $kp = d$,*

$$W^{k,p}(\Omega) \hookrightarrow L^q(\Omega), \quad 1 \leq q < \infty.$$

Remark 2.0.2. *In this thesis, we use Theorem 2.0.1 for $k = 1$, $p = 2$ and $j = 0$ such that*

$$H^1(\Omega) \hookrightarrow L^2(\Omega).$$

This means that

$$\|\mathbf{u}\| \leq \|\mathbf{u}\|_{H^1} \,.$$

Theorem 2.0.2. *(Divergence Theorem) If $\partial\Omega$ is smooth enough to be the graph of a Lipschitz function, $\widehat{n}$ is its outward unit normal and $\mathbf{u}$ is smooth enough, then*

$$\int_\Omega \nabla \cdot \mathbf{u} d\mathbf{x} = \int_{\partial\Omega} \mathbf{u} \cdot \widehat{n} ds.$$

Definition 2.0.6. *(Dual Norm) For $f \in L^2(\Omega)$, the H^{-1} norm defined as*

$$\|f\|_{-1} := \sup_{\mathbf{v} \in H_0^1(\Omega)} \frac{(f, \mathbf{v})}{\|\nabla \mathbf{v}\|},$$

Note that, the function space $H^{-1}(\Omega)$ is the closure of $L^2(\Omega)$ with the norm $\|\cdot\|_{-1}$.

Theorem 2.0.3. *(Hölder's Inequality) Let $\mathbf{u} \in L^p(\Omega)$ and $\mathbf{v} \in L^q(\Omega)$ with $p, q \in [1, \infty]$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then $\mathbf{uv} \in L^1(\Omega)$ and the Hölder's inequality holds*

$$\int_\Omega |\mathbf{u}(x)\mathbf{v}(x)| \, dx \leq \Big[\int_\Omega |\mathbf{u}(x)|^p \, dx \Big]^{1/p} \Big[\int_\Omega |\mathbf{v}(x)|^q \, dx \Big]^{1/q}.$$

The case $p = q = 2$ in Theorem 2.0.3, leads to the following Cauchy-Schwarz inequality.

Theorem 2.0.4. *(Cauchy-Schwarz Inequality): Let $\mathbf{u}, \mathbf{v} \in L^2(\Omega)$. Then the Cauchy-Schwarz Inequality holds*

$$|(\mathbf{u}, \mathbf{v})| \leq \|\mathbf{u}\| \, \|\mathbf{v}\| \, .$$

Theorem 2.0.5. *(Poincaré-Friedrichs' Inequality) For $\mathbf{u} \in H_0^1(\Omega)$,*

$$\|\mathbf{u}\| \leq C_{PF} \, \|\nabla \mathbf{u}\| \, ,$$

where $C_{PF} = C_{PF}(\Omega)$ is a positive constant.

Theorem 2.0.6. *(Korn's Inequality) For all $\mathbf{u} \in H_0^1(\Omega)$ it holds*

$$\|\nabla \mathbf{u}\| \leq C \, \|\mathbb{D}(\mathbf{u})\| \, ,$$

where $\mathbb{D}(\mathbf{u}) = \frac{\nabla \mathbf{u} + \nabla \mathbf{u}^T}{2}$ is the velocity deformation tensor.

Theorem 2.0.7. *(Inverse Estimate) For $\mathbf{v} \in H_0^1(\Omega)$*

$$\|\nabla \mathbf{v}\| \leq C h^{-1} \, \|\mathbf{v}\| \, .$$

Theorem 2.0.8. *(Young's Inequality) For* $a, b \geq 0,\ \ p, q \leq 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Also, the most commonly used case of Young's inequality is

$$ab \leq \frac{1}{2\varepsilon}a^2 + \frac{\varepsilon}{2}b^2, \quad \forall \varepsilon > 0.$$

Theorem 2.0.9. *(The Projection Theorem): Let G be a closed subspace of a Hilbert space H.*

(i) Given any $\mathbf{u} \in H$*, there is a unique* $\mathbf{g} \in G$ *so that*

$$\|\mathbf{u} - \mathbf{g}\| = \inf_{\chi \in G} \|\mathbf{u} - \chi\|.$$

(ii) $\mathbf{g}$ *is determined by the system of equations*

$$(\mathbf{u} - \mathbf{g}, \chi) = 0, \quad \forall \chi \in G.$$

(iii) $\|\mathbf{u}\|^2 = \|\mathbf{g}\|^2 + \|\mathbf{u} - \mathbf{g}\|^2.$

Here $\mathbf{g}$ *is called the projection of* $\mathbf{u}$ *onto* G *and written* $\mathbf{g} = P_G\mathbf{u}$.

Definition 2.0.7. *(L^2-projection) The* L^2 *projection operator* $P : L^2(\Omega) \to H_0^1(\Omega)$ *is defined by*

$$(\mathbf{u} - P\mathbf{u}, \boldsymbol{\psi}) = 0, \quad \forall \boldsymbol{\psi} \in H_0^1(\Omega).$$

Lemma 2.0.1. *(Discrete Gronwall Inequality) Let* $\Delta t,\ B,\ a_n,\ b_n,\ c_n,\ d_n$ *for integers* $n \geq 0$ *be nonnegative numbers such that for all* $N \geq 1$. *If*

$$a_N + \Delta t \sum_{n=0}^{N} b_n \leq \Delta t \sum_{n=0}^{N-1} d_n\, a_n + \Delta t \sum_{n=0}^{N} c_n + B \quad \forall N \geq 0,$$

then for all $\Delta t > 0$,

$$a_N + \Delta t \sum_{n=0}^{N} b_n \leq \exp\left(\Delta t \sum_{n=0}^{N-1} d_n\right)\left(\Delta t \sum_{n=0}^{N} c_n + B\right) \quad \forall N \geq 0.$$

Theorem 2.0.10. *(The Lax-Milgram Theorem) Let* $a(\cdot,\cdot) : \mathbf{X} \times \mathbf{X} \to \mathbb{R}$ *where* $\mathbf{X} = H_0^1(\Omega)$ *be a bilinear form which satisfies*

$$\begin{aligned} \text{continuity } a(\mathbf{u},\mathbf{v}) &\leq c_1 \|\mathbf{u}\|_{\mathbf{X}} \|\mathbf{v}\|_{\mathbf{X}}, \ \forall \mathbf{u},\mathbf{v} \in \mathbf{X}. && (2.1) \\ \text{coercivity } a(\mathbf{u},\mathbf{u}) &\geq c_2 \|\mathbf{u}\|_{\mathbf{X}}^2, \ \forall \mathbf{u} \in \mathbf{X}. && (2.2) \end{aligned}$$

for some positive constants c_1, c_2*. Let* $F : \mathbf{X} \to \mathbb{R}$ *be a linear functional satisfying*

$$\text{continuity } F(\mathbf{v}) \leq C \|\mathbf{v}\|_{\mathbf{X}}, \ \forall \mathbf{v} \in \mathbf{X}. \tag{2.3}$$

Then, there exists a unique $\mathbf{u} \in \mathbf{X}$ *satisfying*

$$a(\mathbf{u},\mathbf{v}) = F(\mathbf{v}), \ \forall \mathbf{v} \in \mathbf{X}.$$

Remark 2.0.3. *(Remainder Term of the Second Order Backward Difference Formula (BDF2) with Taylor's Theorem) To find remainder term of BDF2, we write the Taylor's expansions of all terms in BDF2 around* t^{n+1} *with integral remainders:*

$$\mathbf{u}_t(t^{n+1}) = \mathbf{u}_t(t^{n+1}) \tag{2.4}$$

$$\mathbf{u}^{n+1} = \mathbf{u}^{n+1} \tag{2.5}$$

$$\mathbf{u}^n = \mathbf{u}^{n+1} - \mathbf{u}_t(t^{n+1})\Delta t + \mathbf{u}_{tt}(t^{n+1})\frac{\Delta t^2}{2} + \int_{t^{n+1}}^{t^n} \mathbf{u}_{ttt}(t)\frac{(t^n - t)^2}{2}\, dt \tag{2.6}$$

$$\mathbf{u}^{n-1} = \mathbf{u}^{n+1} - 2\mathbf{u}_t(t^{n+1})\Delta t + 2\mathbf{u}_{tt}(t^{n+1})\Delta t^2 + \int_{t^{n+1}}^{t^{n-1}} \mathbf{u}_{ttt}(t)\frac{(t^{n-1} - t)^2}{2}\, dt \tag{2.7}$$

Then, by multiplying (2.4) with -1*, (2.5) with* $3/2\Delta t$*, (2.6) with* $-4/2\Delta t$ *and (2.7) with* $1/2\Delta t$ *and summing these 4 equations, we obtain the remainder term of BDF2:*

$$\begin{aligned} &\frac{3\mathbf{u}^{n+1} - 4\mathbf{u}^n + \mathbf{u}^{n-1}}{2\Delta t} - \mathbf{u}_t(t^{n+1}) \\ &\quad = \frac{1}{4\Delta t}\left[\int_{t^{n+1}}^{t^{n-1}} \mathbf{u}_{ttt}(t)(t^{n-1} - t)^2\, dt - 4\int_{t^{n+1}}^{t^n} \mathbf{u}_{ttt}(t)(t^n - t)^2\, dt\right] \end{aligned} \tag{2.8}$$

Remark 2.0.4. *(Remainder Term of the Linear Extrapolation of* $\mathbf{u}^{n+1}$ *with Taylor's*

Theorem) To find truncation error of the linear extrapolation applied to $\mathbf{u}^{n+1}$*, let us do Taylor's expansion of the terms in the linear extrapolation around* t^{n+1} *with integral remainders:*

$$\mathbf{u}(t^{n+1}) = \mathbf{u}^{n+1} \tag{2.9}$$

$$\mathbf{u}^{n} = \mathbf{u}^{n+1} + \mathbf{u}_t(t^{n+1})(-\Delta t) + \int_{t^{n+1}}^{t^n} \mathbf{u}_{tt}(t)(t^n - t)\, dt \tag{2.10}$$

$$\mathbf{u}^{n-1} = \mathbf{u}^{n+1} + \mathbf{u}_t(t^{n+1})(-2\Delta t) + \int_{t^{n+1}}^{t^{n-1}} \mathbf{u}_{tt}(t)(t^{n-1} - t)\, dt \tag{2.11}$$

Then, by multiplying (2.9) and (2.11) with -1*, (2.10) with* 2*, and then taking a summation of all these three equations, we get the truncation error:*

$$\begin{aligned}
&(2\mathbf{u}^{n} - \mathbf{u}^{n+1}) - \mathbf{u}(t^{n+1}) \\
&\quad = 2\int_{t^{n+1}}^{t^n} \mathbf{u}_{tt}(t)(t^n - t)\, dt - \int_{t^{n+1}}^{t^{n-1}} \mathbf{u}_{tt}(t)(t^{n-1} - t)\, dt
\end{aligned} \tag{2.12}$$

CHAPTER 3

NAVIER-STOKES EQUATIONS

The Navier-Stokes equations (NSE) are partial differential equations which describe the motion of viscous fluid substances such as water, oil, air etc. They represent the general laws of continuum mechanics which are conservation of mass and conservation of linear momentum (i.e. Newton's second law) [15].

In this thesis, we are interested in mathematical analysis and numerical simulations of the NSE with finite element methods.

The dimensionless incompressible NSE given by

$$\begin{aligned} \mathbf{u}_t - Re^{-1}\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } (0,T]\times\Omega, \\ \nabla\cdot\mathbf{u} &= 0 \quad \text{in } [0,T]\times\Omega, \\ \mathbf{u}(0,\mathbf{x}) &= \mathbf{u}_0 \quad \text{in } \Omega, \\ \int_\Omega p\, d\mathbf{x} &= 0 \quad \text{in } (0,T], \end{aligned} \tag{3.1}$$

where $\Omega \subset \mathbb{R}^d, d \in \{2,3\}$ is bounded domain of fluid with boundary $\partial\Omega$. If domain Ω is proper subset of $\mathbb{R}^d$, the NSE have to be equipped with boundary conditions on boundary $\partial\Omega$ of Ω. If the boundary conditions are not mentioned, to satisfy simplicity when finding a solution of NSE, we can take boundary conditions as homogeneous Dirichlet boundary conditions;

$$\mathbf{u} = 0 \quad \text{on} \quad [0,T]\times\partial\Omega.$$

The functions and symbols in the NSE (3.1) are

$\mathbf{u}(x,t)$: velocity,	$p(x,t)$: pressure,
$\mathbf{f}(x,t)$: body force,	$\mathbf{u}_0$: initial velocity,
Re : Reynolds number,	T : final time.

Our goal is to compute the value of velocity $\mathbf{u}$ and the pressure p by using given other data listed above. In NSE, the first equation in (3.1) represents the conservation of momentum. The descriptions of the terms in (3.1) as follows:

(i) The term $\mathbf{u}_t$ describes the rate of change in velocity.

(ii) The term $(\mathbf{u} \cdot \nabla)\mathbf{u}$ is the convective term which governs the inertial effects. The summation of these two terms $\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u}$ describes a convective transport.

(iii) The term $-Re^{-1}\Delta\mathbf{u}$ describes the diffusive transport.

(iv) With the term ∇p, the pressure describes the forces acting on the surface of each fluid volume.

(v) The control parameter in (3.1) is Reynolds number, Re, given as

$$Re = \frac{UL}{\nu}$$

where L and U are characteristic length and velocity scales of the flow respectively, and ν represents kinematic viscosity of the flow.

Flow regimes change with respect to the size of Re. Typical sizes of Re are given below.

If Re is small (approximately less than 1), the flow field is stationary (the term $\mathbf{u}_t$ vanishes and indicating initial condition $\mathbf{u}_0$ is unnecessary) and fluid flow is dominated by viscosity term $\nu\Delta\mathbf{u}$. Also, in this case, the effect of convection term $(\mathbf{u} \cdot \nabla)\mathbf{u}$ is negligible. This kind of flow is called creeping flows.

If Re is moderate (less than the its critical value 2000), the flow field is laminar which is the fluid flow in which the fluid travels smoothly or in regular paths. However, in this case flow field is time dependent, [44].

If Re is very large (greater than 2000), the flow field will be turbulent flow in which the fluid undergoes irregular fluctuations. In this case, the convection term $(\mathbf{u} \cdot \nabla)\mathbf{u}$ become dominant.

The second equation in (3.1) is incompressibility constraint representing the conservation of mass. Furthermore, the last equation in (3.1) is the usual normalization condition on pressure such as requiring the pressure to have mean value zero.

3.1 Weak Formulation of NSE

In order to derive the weak formulation of (3.1), we introduce function spaces **X** and Q for the velocity $\mathbf{u}$ and the pressure p, respectively.

$$\begin{aligned} \mathbf{X} &:= (H_0^1(\Omega))^d = \{\mathbf{v} \in L^2(\Omega) : \nabla \mathbf{v} \in L^2(\Omega) \text{ and } \mathbf{v} = 0 \text{ on } \partial\Omega\} \\ Q &:= L_0^2(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q \, d\mathbf{x} = 0\} \end{aligned}$$

We also define divergence free functions' space

$$\mathbf{V} := \{\mathbf{v} \in \mathbf{X} : (\nabla \cdot \mathbf{v}, q) = 0 \; \forall q \in Q\}.$$

Now, multiply the momentum equation of (3.1) by test function $\mathbf{v} \in \mathbf{X}$ and the incompressibility constraint by test function $q \in Q$, and integrate them on Ω. Then for velocity $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)^T$ and the pressure $p = (p_1, p_2, p_3)^T$, we obtain the following terms.

Let us start with viscous term. The viscous term will be

$$-Re^{-1}(\Delta\mathbf{u}, \mathbf{v}) = \int_\Omega -Re^{-1}\Delta\mathbf{u}\,\mathbf{v}\,d\mathbf{x} = \int_\Omega -Re^{-1}\nabla \cdot \nabla\mathbf{u}\,\mathbf{v}\,d\mathbf{x}. \tag{3.2}$$

In this thesis, we will use the symmetric part of velocity gradient which is called velocity deformation tensor, and it is defined by

$$\mathbb{D}(\mathbf{u}) = \frac{\nabla\mathbf{u} + \nabla\mathbf{u}^T}{2}.$$

$$\mathbb{D}(\mathbf{u}) = \frac{1}{2}\begin{bmatrix} \frac{\partial \mathbf{u}_1}{\partial x} & \frac{\partial \mathbf{u}_1}{\partial y} & \frac{\partial \mathbf{u}_1}{\partial z} \\ \frac{\partial \mathbf{u}_2}{\partial x} & \frac{\partial \mathbf{u}_2}{\partial y} & \frac{\partial \mathbf{u}_2}{\partial z} \\ \frac{\partial \mathbf{u}_3}{\partial x} & \frac{\partial \mathbf{u}_3}{\partial y} & \frac{\partial \mathbf{u}_3}{\partial z} \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \frac{\partial \mathbf{u}_1}{\partial x} & \frac{\partial \mathbf{u}_2}{\partial x} & \frac{\partial \mathbf{u}_3}{\partial x} \\ \frac{\partial \mathbf{u}_1}{\partial y} & \frac{\partial \mathbf{u}_2}{\partial y} & \frac{\partial \mathbf{u}_3}{\partial y} \\ \frac{\partial \mathbf{u}_1}{\partial z} & \frac{\partial \mathbf{u}_2}{\partial z} & \frac{\partial \mathbf{u}_3}{\partial z} \end{bmatrix}$$

$$
= \frac{1}{2} \begin{bmatrix} 2\frac{\partial \mathbf{u}_1}{\partial x} & \frac{\partial \mathbf{u}_1}{\partial y} + \frac{\partial \mathbf{u}_2}{\partial x} & \frac{\partial \mathbf{u}_1}{\partial z} + \frac{\partial \mathbf{u}_3}{\partial x} \\ \frac{\partial \mathbf{u}_2}{\partial x} + \frac{\partial \mathbf{u}_1}{\partial y} & 2\frac{\partial \mathbf{u}_2}{\partial y} & \frac{\partial \mathbf{u}_2}{\partial z} + \frac{\partial \mathbf{u}_3}{\partial y} \\ \frac{\partial \mathbf{u}_3}{\partial x} + \frac{\partial \mathbf{u}_1}{\partial z} & \frac{\partial \mathbf{u}_3}{\partial y} + \frac{\partial \mathbf{u}_2}{\partial z} & 2\frac{\partial \mathbf{u}_3}{\partial z} \end{bmatrix}.
$$

Now, if we take the divergence of velocity deformation tensor, we obtain

$$
\begin{aligned}
\nabla \cdot \mathbb{D}(\mathbf{u}) = \frac{\nabla \cdot \nabla \mathbf{u} + \nabla \cdot \nabla \mathbf{u}^T}{2} &= \frac{1}{2} \begin{bmatrix} \frac{\partial^2 \mathbf{u}_1}{\partial x^2} + \frac{\partial^2 \mathbf{u}_1}{\partial y^2} + \frac{\partial^2 \mathbf{u}_1}{\partial z^2} \\ \frac{\partial^2 \mathbf{u}_2}{\partial x^2} + \frac{\partial^2 \mathbf{u}_2}{\partial y^2} + \frac{\partial^2 \mathbf{u}_2}{\partial z^2} \\ \frac{\partial^2 \mathbf{u}_3}{\partial x^2} + \frac{\partial^2 \mathbf{u}_3}{\partial y^2} + \frac{\partial^2 \mathbf{u}_3}{\partial z^2} \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \frac{\partial^2 \mathbf{u}_1}{\partial x^2} + \frac{\partial^2 \mathbf{u}_2}{\partial x \partial y} + \frac{\partial^2 \mathbf{u}_3}{\partial x \partial z} \\ \frac{\partial^2 \mathbf{u}_1}{\partial y \partial x} + \frac{\partial^2 \mathbf{u}_2}{\partial y^2} + \frac{\partial^2 \mathbf{u}_3}{\partial y \partial z} \\ \frac{\partial^2 \mathbf{u}_1}{\partial z \partial x} + \frac{\partial^2 \mathbf{u}_2}{\partial x \partial y} + \frac{\partial^2 \mathbf{u}_3}{\partial z^2} \end{bmatrix} \\
&= \frac{1}{2} \Delta \mathbf{u} + \frac{1}{2} \begin{bmatrix} \frac{\partial}{\partial x}\left(\frac{\partial \mathbf{u}_1}{\partial x} + \frac{\partial \mathbf{u}_2}{\partial y} + \frac{\partial \mathbf{u}_3}{\partial z}\right) \\ \frac{\partial}{\partial y}\left(\frac{\partial \mathbf{u}_1}{\partial x} + \frac{\partial \mathbf{u}_2}{\partial y} + \frac{\partial \mathbf{u}_3}{\partial z}\right) \\ \frac{\partial}{\partial z}\left(\frac{\partial \mathbf{u}_1}{\partial x} + \frac{\partial \mathbf{u}_2}{\partial y} + \frac{\partial \mathbf{u}_3}{\partial z}\right) \end{bmatrix} \\
&= \frac{1}{2} \Delta \mathbf{u} + \frac{1}{2} \begin{bmatrix} \frac{\partial}{\partial x}(\nabla \cdot \mathbf{u}) \\ \frac{\partial}{\partial y}(\nabla \cdot \mathbf{u}) \\ \frac{\partial}{\partial z}(\nabla \cdot \mathbf{u}) \end{bmatrix} \\
&= \frac{1}{2} \Delta \mathbf{u} \qquad (3.3)
\end{aligned}
$$

If the velocity field is divergence free, the second part of summation in the last step vanishes.

The equality (3.3) implies

$$
\Delta \mathbf{u} = 2 \nabla \cdot \mathbb{D}(\mathbf{u}). \qquad (3.4)
$$

So by using (3.4) in (3.2) and divergence theorem, we obtain

$$
\begin{aligned}
&- Re^{-1}(\Delta \mathbf{u}, \mathbf{v}) = \int_\Omega -Re^{-1} \Delta \mathbf{u} \, \mathbf{v} \, d\mathbf{x} = \int_\Omega -Re^{-1} \nabla \cdot \nabla \mathbf{u} \, \mathbf{v} \, d\mathbf{x} \\
&= \int_\Omega -2Re^{-1} \nabla \cdot \mathbb{D}(\mathbf{u}) \, \mathbf{v} \, d\mathbf{x} = -2Re^{-1} \int_{\partial\Omega} \mathbf{v} \mathbb{D}(\mathbf{u}) \cdot \mathbf{n} \, ds + 2Re^{-1} \int_\Omega \mathbb{D}(\mathbf{u}) \, \nabla \mathbf{v} \, d\mathbf{x} \\
&= 2Re^{-1}(\mathbb{D}(\mathbf{u}), \nabla \mathbf{v}).
\end{aligned}
$$

Since the velocity deformation tensor is symmetric, one gets

$$
\begin{aligned}
2Re^{-1}(\mathbb{D}(\mathbf{u}), \nabla \mathbf{v}) &= Re^{-1}(\mathbb{D}(\mathbf{u}), \nabla \mathbf{v}) + Re^{-1}(\mathbb{D}(\mathbf{u})^T, \nabla \mathbf{v}^T) \\
&= Re^{-1}(\mathbb{D}(\mathbf{u}), \nabla \mathbf{v}) + Re^{-1}(\mathbb{D}(\mathbf{u}), \nabla \mathbf{v}^T) \\
&= 2Re^{-1}(\mathbb{D}(\mathbf{u}), \mathbb{D}(\mathbf{v})). \qquad (3.5)
\end{aligned}
$$

Secondly, the convective term is

$$((\mathbf{u}\cdot\nabla)\mathbf{u},\mathbf{v}) = b^*(\mathbf{u},\mathbf{u},\mathbf{v}). \tag{3.6}$$

where $b^*(\mathbf{u},\mathbf{u},\mathbf{v})$ is the skew-symmetric trilinear form and defined by

$$b^*(\mathbf{u},\mathbf{u},\mathbf{v}) = \frac{1}{2}((\mathbf{u}\cdot\nabla)\mathbf{u},\mathbf{v}) - \frac{1}{2}((\mathbf{u}\cdot\nabla)\mathbf{v},\mathbf{u}). \tag{3.7}$$

The key property of skew-symmetric trilinear form is

$$((\mathbf{u}\cdot\nabla)\mathbf{v},\mathbf{v}) = 0, \tag{3.8}$$

or more generally

$$((\mathbf{u}\cdot\nabla)\mathbf{u},\mathbf{v}) = -((\mathbf{u}\cdot\nabla)\mathbf{v},\mathbf{u}).$$

Lemma 3.1.1. *(Estimation of the Convective Term for Skew-symmetric Form, [40]): For all* $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbf{X}$

$$b^*(\mathbf{u},\mathbf{v},\mathbf{w}) \leq M(\Omega)\left\|\nabla\mathbf{u}\right\|\left\|\nabla\mathbf{v}\right\|\left\|\nabla\mathbf{w}\right\|, \tag{3.9}$$

for $M(\Omega)$ *is a finite constant. Also, for* $\mathbf{u}\in L^2(\Omega)$*,* $\mathbf{w}\in\mathbf{X}$ *and* $\mathbf{v},\nabla\mathbf{v}\in L^\infty(\Omega)$

$$b^*(\mathbf{u},\mathbf{v},\mathbf{w}) \leq \frac{1}{2}\big(\left\|\mathbf{u}\right\|\left\|\nabla\mathbf{v}\right\|_\infty\left\|\mathbf{w}\right\| + \left\|\mathbf{u}\right\|\left\|\mathbf{v}\right\|_\infty\left\|\nabla\mathbf{w}\right\|\big). \tag{3.10}$$

Proof. If we take the absolute value of the skew-symmetric trilinear form of convective term, then by Cauchy-Schwarz inequality, one gets

$$\begin{aligned}|b^*(\mathbf{u},\mathbf{v},\mathbf{w})| =& |\frac{1}{2}((\mathbf{u}\cdot\nabla)\mathbf{v},\mathbf{w}) - \frac{1}{2}((\mathbf{u}\cdot\nabla)\mathbf{w},\mathbf{v})| \\ \leq& \frac{1}{2}\big(|((\mathbf{u}\cdot\nabla)\mathbf{v},\mathbf{w})| + |((\mathbf{u}\cdot\nabla)\mathbf{w},\mathbf{v})|\big) \\ =& \frac{1}{2}\Big(\Big|\int_\Omega \nabla\mathbf{v}\;\mathbf{u}\;\mathbf{w}\;d\mathbf{x}\Big| + \Big|\int_\Omega \nabla\mathbf{w}\;\mathbf{u}\;\mathbf{v}\;d\mathbf{x}\Big|\Big)\end{aligned} \tag{3.11}$$

When applying Hölder's inequality, if we choose $q = 2, p = r = 4$ to satisfy condition $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} = 1$ and Poincaré-Friedrichs' (in general case, for $p = 4$) inequality and Sobolev imbedding theorem, we obtain

$$
\begin{aligned}
|b^*(\mathbf{u},\mathbf{v},\mathbf{w})| \leq & \frac{1}{2}\Big[\big((\int_\Omega |\nabla \mathbf{v}|^q \, d\mathbf{x})^{1/q}(\int_\Omega |\mathbf{u}|^p \, d\mathbf{x})^{1/p}(\int_\Omega |\mathbf{w}|^r \, d\mathbf{x})^{1/r}\big) \\
& + \big((\int_\Omega |\nabla \mathbf{w}|^q \, d\mathbf{x})^{1/q}(\int_\Omega |\mathbf{u}|^p \, d\mathbf{x})^{1/p}(\int_\Omega |\mathbf{w}|^r \, d\mathbf{x})^{1/r}\big)\Big] \\
= & \frac{1}{2}\Big[\, \|\nabla \mathbf{v}\| \, \|\mathbf{u}\|_{L^4} \, \|\mathbf{w}\|_{L^4} + \|\nabla \mathbf{w}\| \, \|\mathbf{u}\|_{L^4} \, \|\mathbf{v}\|_{L^4} \Big] \\
\leq & C\Big[\, \|\nabla \mathbf{v}\| \, \|\nabla \mathbf{u}\|_{L^4} \, \|\nabla \mathbf{w}\|_{L^4} + \|\nabla \mathbf{w}\| \, \|\nabla \mathbf{u}\|_{L^4} \, \|\nabla \mathbf{v}\|_{L^4} \Big] \\
\leq & C \, \|\nabla \mathbf{v}\| \, \|\nabla \mathbf{u}\| \, \|\nabla \mathbf{w}\|
\end{aligned}
$$

On the other hand, after the step (3.11), when applying Hölder's inequality, if we choose $q = \infty, p = r = 2$ and $m = \infty, k = l = 2$ to satisfy $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$ and $\frac{1}{k} + \frac{1}{l} + \frac{1}{m} = 1$, we obtain

$$
\begin{aligned}
|b^*(\mathbf{u},\mathbf{v},\mathbf{w})| \leq & \frac{1}{2}\Big[\big((\int_\Omega |\nabla \mathbf{v}|^q \, d\mathbf{x})^{1/q}(\int_\Omega |\mathbf{u}|^p \, d\mathbf{x})^{1/p}(\int_\Omega |\mathbf{w}|^r \, d\mathbf{x})^{1/r}\big) \\
& + \big((\int_\Omega |\nabla \mathbf{w}|^k \, d\mathbf{x})^{1/k}(\int_\Omega |\mathbf{u}|^l \, d\mathbf{x})^{1/l}(\int_\Omega |\mathbf{w}|^m \, d\mathbf{x})^{1/m}\big)\Big] \\
= & \frac{1}{2}\Big[\, \|\nabla \mathbf{v}\|_\infty \, \|\mathbf{u}\| \, \|\mathbf{w}\| + \|\nabla \mathbf{w}\| \, \|\mathbf{u}\| \, \|\mathbf{v}\|_\infty \Big]
\end{aligned}
$$

□

In the vector form, by using divergence theorem, the pressure term become

$$
\begin{aligned}
(\nabla p, \mathbf{v}) = & \int_\Omega \nabla p \; \mathbf{v} \; d\mathbf{x} \\
= & \int_\Omega \begin{bmatrix} \frac{\partial p_1}{\partial x} & \frac{\partial p_1}{\partial y} & \frac{\partial p_1}{\partial z} \\ \frac{\partial p_2}{\partial x} & \frac{\partial p_2}{\partial y} & \frac{\partial p_2}{\partial z} \\ \frac{\partial p_3}{\partial x} & \frac{\partial p_3}{\partial y} & \frac{\partial p_3}{\partial z} \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{bmatrix} d\mathbf{x} \\
= & \begin{bmatrix} \int_\Omega \frac{\partial p_1}{\partial x}\mathbf{v}_1 + \frac{\partial p_1}{\partial y}\mathbf{v}_1 + \frac{\partial p_1}{\partial z}\mathbf{v}_1 \; d\mathbf{x} \\ \int_\Omega \frac{\partial p_2}{\partial x}\mathbf{v}_2 + \frac{\partial p_2}{\partial y}\mathbf{v}_2 + \frac{\partial p_2}{\partial z}\mathbf{v}_2 \; d\mathbf{x} \\ \int_\Omega \frac{\partial p_3}{\partial x}\mathbf{v}_3 + \frac{\partial p_3}{\partial y}\mathbf{v}_3 + \frac{\partial p_3}{\partial z}\mathbf{v}_3 \; d\mathbf{x} \end{bmatrix} \\
= & \begin{bmatrix} \int_\Omega \big(\frac{\partial}{\partial x}(p_1 \, \mathbf{v}_1) - p_1 \, \frac{\partial \mathbf{v}_1}{\partial x} + \frac{\partial}{\partial y}(p_1 \, \mathbf{v}_1) - p_1 \, \frac{\partial \mathbf{v}_1}{\partial y} + \frac{\partial}{\partial z}(p_1 \, \mathbf{v}_1) - p_1 \, \frac{\partial \mathbf{v}_1}{\partial z}\big) d\mathbf{x} \\ \int_\Omega \big(\frac{\partial}{\partial x}(p_2 \, \mathbf{v}_2) - p_2 \, \frac{\partial \mathbf{v}_2}{\partial x} + \frac{\partial}{\partial y}(p_2 \, \mathbf{v}_2) - p_2 \, \frac{\partial \mathbf{v}_2}{\partial y} + \frac{\partial}{\partial z}(p_2 \, \mathbf{v}_2) - p_2 \, \frac{\partial \mathbf{v}_2}{\partial z}\big) d\mathbf{x} \\ \int_\Omega \big(\frac{\partial}{\partial x}(p_1 \, \mathbf{v}_3) - p_3 \, \frac{\partial \mathbf{v}_3}{\partial x} + \frac{\partial}{\partial y}(p_1 \, \mathbf{v}_3) - p_3 \, \frac{\partial \mathbf{v}_3}{\partial y} + \frac{\partial}{\partial z}(p_1 \, \mathbf{v}_3) - p_3 \, \frac{\partial \mathbf{v}_3}{\partial z}\big) d\mathbf{x} \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&= \begin{bmatrix} \int_{\partial\Omega} p_1 \mathbf{v}_1 \cdot \mathbf{n}\, ds - \int_\Omega \left(p_1 \frac{\partial \mathbf{v}_1}{\partial x} + p_1 \frac{\partial \mathbf{v}_1}{\partial y} + p_1 \frac{\partial \mathbf{v}_1}{\partial z} \right) d\mathbf{x} \\ \int_{\partial\Omega} p_2 \mathbf{v}_2 \cdot \mathbf{n}\, ds - \int_\Omega \left(p_2 \frac{\partial \mathbf{v}_2}{\partial x} + p_2 \frac{\partial \mathbf{v}_2}{\partial y} + p_2 \frac{\partial \mathbf{v}_2}{\partial z} \right) d\mathbf{x} \\ \int_{\partial\Omega} p_3 \mathbf{v}_3 \cdot \mathbf{n}\, ds - \int_\Omega \left(p_3 \frac{\partial \mathbf{v}_3}{\partial x} + p_3 \frac{\partial \mathbf{v}_3}{\partial y} + p_3 \frac{\partial \mathbf{v}_3}{\partial z} \right) d\mathbf{x} \end{bmatrix} \\
&= -\int_\Omega \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \begin{bmatrix} \frac{\partial \mathbf{v}_1}{\partial x} + \frac{\partial \mathbf{v}_1}{\partial y} + \frac{\partial \mathbf{v}_1}{\partial z} \\ \frac{\partial \mathbf{v}_2}{\partial x} + \frac{\partial \mathbf{v}_2}{\partial y} + \frac{\partial \mathbf{v}_2}{\partial z} \\ \frac{\partial \mathbf{v}_3}{\partial x} + \frac{\partial \mathbf{v}_3}{\partial y} + \frac{\partial \mathbf{v}_3}{\partial z} \end{bmatrix} d\mathbf{x} \\
&= -\int_\Omega p \, \nabla \cdot \mathbf{v} d\mathbf{x} \\
&= -(p, \nabla \cdot \mathbf{v}) \qquad (3.12)
\end{aligned}
$$

Hence, with (3.2), (3.6) and (3.12), the variational form of NSE (3.1): Find $(\mathbf{u}, p) \in (\mathbf{X}, Q)$ satisfying

$$
\begin{aligned}
(\mathbf{u}_t, \mathbf{v}) + 2Re^{-1}(\mathbb{D}(\mathbf{u}), \mathbb{D}(\mathbf{v})) + b^*(\mathbf{u}, \mathbf{u}, \mathbf{v}) - (p, \nabla \cdot \mathbf{v}) &= (\mathbf{f}, \mathbf{v}), \\
(\nabla \cdot \mathbf{u}, q) &= 0,
\end{aligned} \qquad (3.13)
$$

for all $\mathbf{v} \in \mathbf{X}$ and $q \in Q$.

3.2 Galerkin Finite Element Formulation of NSE

First step of Galerkin finite element formulation of NSE (3.1) is to choose suitable finite element spaces. We need to take $\mathbf{X}_h \subset \mathbf{X}$ for the velocity and $Q_h \subset Q$ for pressure which have to satisfy the inf-sup condition [4], [9]. This condition or with other saying Ladyzhenskaya-Babuska-Brezzi condition or inf-sup condition is given by

$$
\inf_{q_h \in Q_h} \sup_{\mathbf{v}_h \in \mathbf{X}_h} \frac{(q_h, \nabla \cdot \mathbf{v}_h)}{\|\nabla \mathbf{v}_h\| \, \|q_h\|} \geq \beta_h > 0. \qquad (3.14)
$$

Then, Galerkin finite element discretization of NSE (3.1) with respect to the variational formulation (3.13) as follows: Find $(\mathbf{u}_h, p_h) \in (\mathbf{X}_h, Q_h)$ satisfying

$$
\begin{aligned}
(\partial_t \mathbf{u}_h, \mathbf{v}_h) + 2Re^{-1}(\mathbb{D}(\mathbf{u}_h), \mathbb{D}(\mathbf{v}_h)) + b^*(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) - (p_h, \nabla \cdot \mathbf{v}_h) &= (\mathbf{f}, \mathbf{v}_h) \\
(\nabla \cdot \mathbf{u}_h, q_h) &= 0
\end{aligned} \qquad (3.15)
$$

for all $\mathbf{v}_h \in \mathbf{X}_h$ and $q_h \in Q_h$.

Also, if under the inf-sup condition (3.14) we consider the discretely divergence free functions' space

$$\mathbf{V}_h = \{\mathbf{v}_h \in \mathbf{X}_h : (\nabla \cdot \mathbf{v}_h, q_h) = 0, \ \forall q_h \in Q_h\}.$$

Thus, variational formulation (3.15) can be rewritten in the space $\mathbf{V}_h$ as: Find $\mathbf{u}_h \in \mathbf{V}_h$ satisfying

$$(\partial_t \mathbf{u}_h, \mathbf{v}_h) + Re^{-1}(\mathbb{D}(\mathbf{u}_h), \mathbb{D}(\mathbf{v}_h)) + b^*(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) \tag{3.16}$$

for all $\mathbf{v}_h \in \mathbf{V}_h$.

Note that, the variational formulation of NSE in $(\mathbf{X}_h, Q_h)$ and $(\mathbf{V}_h, Q_h)$ is equivalent when spaces satisfy inf-sup condition (3.14), see [17].

However, Galerkin finite element discretization remains incapable when trying to solve turbulence problems. For a large Re, since the convective term become dominant, the simulation of turbulent flow will be difficult, sometimes not possible. To overcome this difficulty, we need some stabilization techniques. Several methods have been presented for this purpose. In this thesis, we use the variational multiscale (VMS) methods to model turbulence. There are different realizations of it. For example, in [24, 25], residual based VMS method is introduced. In [35] John and Kindl deal turbulence flow with three-scale bubble VMS method. Also, in [37], the error obtained when solving convection-dominated convection-diffusion equations by two-level VMS method is studied.

In addition, in literature, for some multiphysics problems VMS method has been considered. To illustrate, in [45], the method is used to simulate non-isothermal flows. Also, in [5], the method is implemented as post-processing step for incompressible, non-isothermal fluid flows. In [10], analysis of the method is done for the Darcy–Brinkman equations in double-diffusive convection. In this thesis, we consider the projection-based VMS method based on linearly extrapolated BDF2.

Now, before starting detailed review of projection-based VMS method, we present brief information about three-scale VMS method which constructs basis for our method.

CHAPTER 4

AN OVERVIEW ON THREE-SCALE VARIATIONAL MULTISCALE METHOD

The basic idea of the VMS method is based on separation of scales presented in [19, 24]. There are several ways to do this separation. Before introducing the projection-based VMS method, to explain ideas underlying this method, we present three-scale VMS method mentioned in [2].

In the first step of the method, all scales of turbulent flow are decomposed with large and small resolved scales and unresolved scales. To denote these scales, we use $\overline{(\cdot)}$, $\widehat{(\cdot)}$ and $(\cdot)'$ notations respectively.

After applying this separation to spaces for velocity and pressure, respectively, we obtain

$$\mathbf{X} = \overline{\mathbf{X}} \oplus \widehat{\mathbf{X}} \oplus \mathbf{X}', \quad Q = \overline{Q} \oplus \widehat{Q} \oplus Q'$$

where $\oplus$ denotes the direct sum.

Thus, the scale decompositions of the solution become

$$\mathbf{u} = \overline{\mathbf{u}} + \widehat{\mathbf{u}} + \mathbf{u}', \quad p = \overline{p} + \widehat{p} + p'$$

and the test functions become

$$\mathbf{v} = \overline{\mathbf{v}} + \widehat{\mathbf{v}} + \mathbf{v}', \quad q = \overline{q} + \widehat{q} + q'.$$

The variational formulation of NSE is presented as (3.13). Let us denote it by the linear forms $A(\cdot;\cdot,\cdot)$ for left hand side of (3.13) and $\mathbf{f}(\cdot)$ for the right hand side of

(3.13), then obtain

$$A(\mathbf{u};(\mathbf{u},p),(\mathbf{v},q)) = \mathbf{f}(\mathbf{v}) \tag{4.1}$$

Inserting three-scales decomposition in (4.1), and choosing different test functions from spaces of these three scales leads to three subproblems:

- Large scale problem

$$A(\mathbf{u};(\overline{\mathbf{u}},\overline{p}),(\overline{\mathbf{v}},\overline{q})) + A(\mathbf{u};(\widehat{\mathbf{u}},\widehat{p}),(\overline{\mathbf{v}},\overline{q})) + A(\mathbf{u};(\mathbf{u}',p'),(\overline{\mathbf{v}},\overline{q})) = \mathbf{f}(\overline{\mathbf{v}}) \tag{4.2}$$

- Small resolved scale problem

$$A(\mathbf{u};(\overline{\mathbf{u}},\overline{p}),(\widehat{\mathbf{v}},\widehat{q})) + A(\mathbf{u};(\widehat{\mathbf{u}},\widehat{p}),(\widehat{\mathbf{v}},\widehat{q})) + A(\mathbf{u};(\mathbf{u}',p'),(\widehat{\mathbf{v}},\widehat{q})) = \mathbf{f}(\widehat{\mathbf{v}}) \tag{4.3}$$

- Unresolved scale problem

$$A(\mathbf{u};(\overline{\mathbf{u}},\overline{p}),(\mathbf{v}',q')) + A(\mathbf{u};(\widehat{\mathbf{u}},\widehat{p}),(\mathbf{v}',q')) + A(\mathbf{u};(\mathbf{u}',p'),(\mathbf{v}',q')) = \mathbf{f}(\mathbf{v}') \tag{4.4}$$

Now, we have to solve these three problems. However, It is a very hard process. So, to simplify it, we consider some assumptions.

Assumption 1. *We do not expect to find explicit solution of (4.4). Hence, we drop it.*

Assumption 2. *Direct influence of unresolved scales on large scales is negligible. So, we take*

$$A(\mathbf{u};(\mathbf{u}',p'),(\overline{\mathbf{v}},\overline{q}) = 0$$

in (4.2).

Assumption 3. *Influence of unresolved scales on to small resolved scales needs to be modeled such that;*

$$A(\mathbf{u};(\mathbf{u}',p'),(\widehat{\mathbf{v}},\widehat{q})) \approx c(\mathbf{u};(\overline{\mathbf{u}},\overline{p}),(\widehat{\mathbf{u}},\widehat{p}),(\widehat{\mathbf{v}},\widehat{q}))$$

in (4.3).

With these assumptions, we obtain the following variational formulation:

Find $(\overline{\mathbf{u}}, \overline{p}) \times (\widehat{\mathbf{u}}, \widehat{p}) : (0,T) \to (\overline{\mathbf{X}}, \overline{Q}) \times (\widehat{\mathbf{X}}, \widehat{Q})$ satisfying

$$A(\overline{\mathbf{u}} + \widehat{\mathbf{u}}; (\overline{\mathbf{u}}, \overline{p}), (\overline{\mathbf{v}}, \overline{q})) + A(\overline{\mathbf{u}} + \widehat{\mathbf{u}}; (\widehat{\mathbf{u}}, \widehat{p}), (\overline{\mathbf{v}}, \overline{q})) = \mathbf{f}(\overline{\mathbf{v}}) \tag{4.5}$$

$$\begin{aligned} A(\overline{\mathbf{u}} + \widehat{\mathbf{u}}; (\overline{\mathbf{u}}, \overline{p}), (\widehat{\mathbf{v}}, \widehat{q})) + A(\overline{\mathbf{u}} + \widehat{\mathbf{u}}; (\widehat{\mathbf{u}}, \widehat{p}), (\widehat{\mathbf{v}}, \widehat{q})) \\ + c(\overline{\mathbf{u}} + \widehat{\mathbf{u}}; (\overline{\mathbf{u}}, \overline{p}), (\widehat{\mathbf{u}}, \widehat{p}), (\widehat{\mathbf{v}}, \widehat{q})) = \mathbf{f}(\widehat{\mathbf{v}}) \end{aligned} \tag{4.6}$$

for all $(\overline{\mathbf{v}}, \overline{q}) \times (\widehat{\mathbf{v}}, \widehat{q}) \in (\overline{X}, \overline{Q}) \times (\widehat{\mathbf{X}}, \widehat{Q})$.

We note that the form $c(\mathbf{u}; (\overline{\mathbf{u}}, \overline{p}), (\widehat{\mathbf{u}}, \widehat{p}), (\widehat{\mathbf{v}}, \widehat{q}))$ introducing the artificial viscosity works as a stabilization.

Now, our problem become the coupled system (4.5)-(4.6) in resolved small and large scale spaces. Let us represent all resolved scales with $\mathbf{X}_h = \overline{\mathbf{X}} \oplus \widehat{\mathbf{X}}$ and $Q_h = \overline{Q} \oplus \widehat{Q}$ to yield more simple form of the problem.

Then, by using the restriction operators, we define the large scales:

$$\overline{P_{vel}} : \mathbf{X}_h \mapsto \overline{\mathbf{X}} \quad \text{such that} \quad \overline{P_{vel}} \mathbf{u}_h = \overline{\mathbf{u}}$$

$$\overline{P_{pre}} : Q_h \mapsto \overline{Q} \quad \text{such that} \quad \overline{P_{pre}} p_h = \overline{p}$$

where $\mathbf{u}_h = \overline{\mathbf{u}} + \widehat{\mathbf{u}}$ from space $\mathbf{X}_h = \overline{\mathbf{X}} \oplus \widehat{\mathbf{X}}$ and $p_h = \overline{p} + \widehat{p}$ from space $Q_h = \overline{Q} \oplus \widehat{Q}$.

Then, along with the restriction operators, the summation of coupled equations (4.5) and (4.6) gives the variational formulation of the problem: Find $(\mathbf{u}_h, p_h) \in (\mathbf{X}_h, Q_h)$ satisfying

$$\begin{aligned} A(\mathbf{u}_h; (\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)) \\ + c(\mathbf{u}_h; \overline{P}(\mathbf{u}_h, p_h), (I - \overline{P})(\mathbf{u}_h, p_h), (I - \overline{P})(\mathbf{v}_h, q_h)) = \mathbf{f}(\mathbf{v}_h) \end{aligned} \tag{4.7}$$

where I is identity operator and we define $P = (\overline{P_{vel}}, \overline{P_{pre}})$ which can be projection or interpolation operators.

Turbulent model $c(\cdot; \cdot, \cdot)$ acts only on small resolved scales and it influences indirectly large scales. With the (4.7), we present three-scale VMS method.

Now, with the light of these ideas, we present the projection-based VMS method which is the main concern of this thesis. VMS method that we consider uses the spaces for all resolved scales as standard finite element spaces $(\mathbf{X}_h, Q_h)$. In addition, the restriction operator on the large resolved scales is determined as L^2 orthogonal projection and turbulence model $c(\cdot;\cdot,\cdot)$ is chosen as a positive and bounded constant.

CHAPTER 5

PROJECTION-BASED VARIATIONAL MULTISCALE METHOD

We now present the projection-based VMS method. As it is mentioned in Chapter 4, this method is the special case of three-scale VMS method and it is based on, like three scale VMS method, scale decomposition and variational formulation of the NSE. In [36], the main idea of the projection-based VMS was introduced in every aspect.

In the projection-based VMS method, the separation of scales are done by using two scales; large and small resolved scales. The effect of unresolved scales will be neglected. Then, large and small resolved scales are defined with velocity deformation tensor. With this definition of the small resolved scales, obtained turbulence model influences directly only small resolved scales. Furthermore, it influences indirectly the large scales.

Now, to present the projection-based VMS method, first of all we have to define spaces for velocity and pressure. $\mathbf{X}_h$ and Q_h are the standard finite element spaces which satisfy the discrete inf-sup condition (3.14) for velocity and pressure respectively. They contain all resolved scales. Secondly, we need to define $\mathbf{L}_H$ which is the coarse or large scale space which consists of symmetric $d \times d$ tensor valued functions on Ω and stated as

$$\mathbf{L}_H \subset \mathbf{L} = \{\mathbb{L} \in (L^2(\Omega))^{d\times d},\ \mathbb{L} = \mathbb{L}^T\}.$$

The variational formulation of the NSE has the form: Find $\mathbf{u}_h : [0,T] \to \mathbf{X}_h$, $p_h : (0,T] \to Q_h$, and $\mathbb{G}_H : [0,T] \to \mathbf{L}_H$ satisfying

$$
\begin{aligned}
(\partial_t \mathbf{u}_h, \mathbf{v}_h) + ((2\nu + \nu_T)\mathbb{D}(\mathbf{u}_h), \mathbb{D}(\mathbf{v}_h)) + b(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) & \\
-(p_h, \nabla \cdot \mathbf{v}_h) - (\nu_T \mathbb{G}_H, \mathbb{D}(\mathbf{v}_h)) &= (\mathbf{f}, \mathbf{v}_h) \\
(q_h, \nabla \cdot \mathbf{u}_h) &= 0 \\
(\mathbb{G}_H - \mathbb{D}(\mathbf{u}_h), \mathbb{L}_H) &= 0
\end{aligned}
\tag{5.1}
$$

for all $\mathbf{v}_h \in \mathbf{X}_h$, $q_h \in Q_h$ and $\mathbb{L}_H \in \mathbf{L}_H$.

This method (5.1) was presented in [36] based on the ideas in [26]. In the variational formulation (5.1),

- $\mathbb{G}_H$ represents the large scales,
- $\mathbb{D}(\mathbf{u}_h)$ represents all resolved scales,
- $\mathbb{D}(\mathbf{u}_h) - \mathbb{G}_H$ represents the small resolved scales.

Also, as it is seen in the last equation of (5.1), $\mathbb{G}_H$ is stated as an L^2-projection of $\mathbb{D}(\mathbf{u}_h)$ into $\mathbf{L}_H$ which is the space of symmetric tensors for large scales.

The L^2-projection $P_{\mathbf{L}_H} : \mathbf{L} \to \mathbf{L}_H$ is defined by

$$
(P_{\mathbf{L}_H}\mathbb{D}(\mathbf{v}) - \mathbb{D}(\mathbf{v}), \mathbb{L}_H) = 0, \quad \forall \mathbb{L}_H \in \mathbf{L}_H. \tag{5.2}
$$

Hence, (5.2) implies $\mathbb{G}_H = P_{\mathbf{L}_H}\mathbb{D}(\mathbf{u}_h)$.

Then, another parameter in the turbulence model is ν_T which is called an additional eddy viscosity parameter. It is used for describing the direct influence of unresolved scales onto small resolved scales. It is a nonnegative function that might depend on the finite element velocity, pressure and the mesh width [2].

Remark 5.0.1. *The additional eddy viscosity parameter ν_T has different choices. The Smagorinsky type choices are given in [26] as:*

$$
\begin{aligned}
\nu_T &= C_s \delta^2 \left\| \mathbb{D}(\mathbf{u}_h) \right\|_F, \\
\nu_T &= C_s \delta^2 \left\| \mathbb{D}(\mathbf{u}_h) - \mathbb{G}_H \right\|_F, \\
\nu_T &= C_s \frac{\delta^2}{|K|^{\frac{1}{2}}} \left\| \mathbb{D}(\mathbf{u}_h) \right\|_{L^2(K)}.
\end{aligned}
$$

However, throughout the thesis, we consider ν_T as a positive constant.

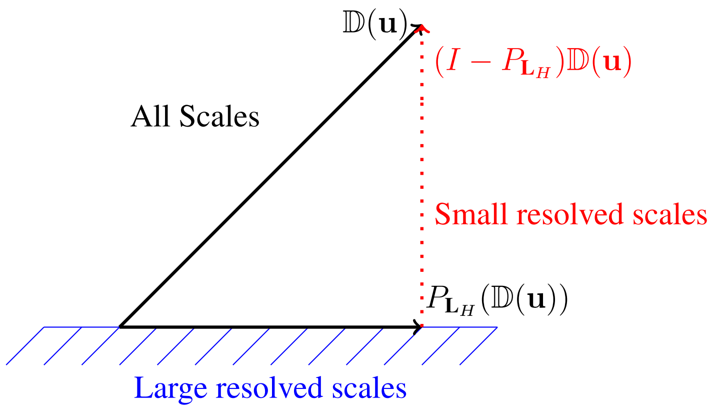


Figure 5.1: L^2-projection of $\mathbb{D}(\mathbf{u})$.

If we write the formulation of model (5.1) in the short form: Find $\mathbf{u}_h : [0,T] \to \mathbf{X}_h,\ p_h : (0,T] \to Q_h$

$$\begin{aligned} A(\mathbf{u}_h; (\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)) + (\nu_T (I - P_{\mathbf{L}_H})\mathbb{D}(\mathbf{u}_h), \mathbb{D}(\mathbf{v}_h)) &= (\mathbf{f}, \mathbf{v}_h) \\ (q_h, \nabla \cdot \mathbf{u}_h) &= 0 \\ (\mathbb{G}_H - \mathbb{D}(\mathbf{u}_h), \mathbb{L}_H) &= 0 \end{aligned} \tag{5.3}$$

for all $(\mathbf{v}_h, q_h) \in (\mathbf{X}_h, Q_h)$ along with the projection (5.2).

Consequently, we obtain the finite element formulation of NSE (3.1) with an extra term. This extra term effects only on the small resolved scales, because of the definition of L^2-projection.

Note that, in [36, 38], the connection between our method and the idea presented in [26] is established. By utilizing the commutation of differentiation and projection operators, and adding defined L^2-projection which is given in the third equation of (5.1), the turbulence model can be rewritten in that form:

$$\begin{aligned} &(\nu_T (I - P_{\mathbf{L}_H})\mathbb{D}(\mathbf{u}_h), \mathbb{D}(\mathbf{v}_h)) + \nu_T(\mathbb{G}_H - \mathbb{D}(\mathbf{u}_h), \mathbb{L}_H) \\ &\quad = (\nu_T (I - P_{\mathbf{L}_H})\mathbb{D}(\mathbf{u}_h), \mathbb{D}(\mathbf{v}_h)) + \nu_T (P_{\mathbf{L}_H}\mathbb{D}(\mathbf{u}_h) - \mathbb{D}(\mathbf{u}_h), P_{\mathbf{L}_H}\mathbb{D}(\mathbf{v}_h)) \\ &\quad = (\nu_T \mathbb{D}(\mathbf{u}_h), \mathbb{D}(\mathbf{v}_h)) - (\nu_T P_{\mathbf{L}_H}\mathbb{D}(\mathbf{u}_h), \mathbb{D}(\mathbf{v}_h)) \\ &\qquad + (\nu_T P_{\mathbf{L}_H}\mathbb{D}(\mathbf{u}_h), P_{\mathbf{L}_H}\mathbb{D}(\mathbf{v}_h)) - (\nu_T \mathbb{D}(\mathbf{u}_h), P_{\mathbf{L}_H}\mathbb{D}(\mathbf{v}_h)) \end{aligned}$$

$$= (\nu_T \mathbb{D}(\mathbf{u}_h), (I - P_{\mathbf{L}_H})\mathbb{D}(\mathbf{v}_h)) - (\nu_T P_{\mathbf{L}_H}\mathbb{D}(\mathbf{u}_h), (I - P_{\mathbf{L}_H})\mathbb{D}(\mathbf{v}_h))$$
$$= \nu_T((I - P_{\mathbf{L}_H})\mathbb{D}(\mathbf{u}_h), (I - P_{\mathbf{L}_H})\mathbb{D}(\mathbf{v}_h)). \tag{5.4}$$

With this way, by using (5.4) in (5.3), an equivalent formulation of (5.3) can be given in the following way: Find $\mathbf{u}_h : [0, T] \to \mathbf{X}_h,\ p_h : (0, T] \to Q_h$ satisfying

$$\begin{aligned} A(\mathbf{u}_h; (\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)) + (\nu_T(I - P_{\mathbf{L}_H})\mathbb{D}(\mathbf{u}_h)), (I - P_{\mathbf{L}_H})\mathbb{D}(\mathbf{v}_h))) &= (\mathbf{f}, \mathbf{v}_h) \\ (q_h, \nabla \cdot \mathbf{u}_h) &= 0 \end{aligned} \tag{5.5}$$

for all $(\mathbf{v}_h, q_h) \in (\mathbf{X}_h, Q_h)$.

However, throughout this thesis, we use the (5.1) projection-based formulation (its short form is (5.3)). In the next chapter, we analyze our method with this formulation along with linearly extrapolated BDF2 time-stepping scheme in every aspect.

CHAPTER 6

A PROJECTION-BASED VMS ON LINEARLY EXTRAPOLATED BDF2 TIME-STEPPING SCHEME FOR NAVIER-STOKES EQUATIONS

In this chapter, we present a finite element approximation of NSE which are discretized by a projection-based VMS method in space and linearly extrapolated 2-step backward difference formula (BDF2) in time. First of all, we discuss the stability of the fully discretized form of NSE and then existence and uniqueness of it. Finally, we give a priori error estimation of a fully discrete problem.

Let us start with the discretization of (3.1) in space with the projection-based VMS method. This discretization procedure is discussed in Chapter 5, and the semi-discrete approximation of NSE by using the projection-based VMS method is presented as (5.1).

Now, to obtain fully discrete approximation, along with the spatial discretization, we discretize with respect to time by using linearly extrapolated BDF2. By applying this extrapolation, we solve linear problem at each time step. But, if we used Newton or fixed-point method for linearization, we would have to implement same linearization procedure for each time step.

Let $t^n = n\Delta t, n = 0, 1, 2, ..., N$ and $T = N\Delta t$. Then, the linearly extrapolated BDF2 discretization in time and the projection-based VMS method in space for NSE (3.1) is: Given $\mathbf{u}_h^n,\ \mathbf{u}_h^{n-1},\ p_h^n,\ p_h^{n-1}$ and find $\mathbf{u}_h^{n+1} \in \mathbf{X}_h,\ p_h^{n+1} \in Q_h$ satisfying

$$\Big(\frac{3\mathbf{u}_h^{n+1} - 4\mathbf{u}_h^n + \mathbf{u}_h^{n-1}}{2\Delta t}, \mathbf{v}_h\Big) + b^*(2\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, \mathbf{u}_h^{n+1}, \mathbf{v}_h) + 2\nu(\mathbb{D}(\mathbf{u}_h^{n+1}), \mathbb{D}(\mathbf{v}_h))$$

$$+\nu_T(\mathbb{D}(\mathbf{u}_h^{n+1}), \mathbb{D}(\mathbf{v}_h)) - (p_h^{n+1}, \nabla \cdot \mathbf{v}_h) - \nu_T(\mathbb{G}_H^n, \mathbb{D}(\mathbf{v}_h)) = (\mathbf{f}^{\,n+1}, \mathbf{v}_h), \quad (6.1)$$

$$(\nabla \cdot \mathbf{u}_h^{n+1}, q_h) = 0, \quad (6.2)$$

$$(\mathbb{G}_H^n - \mathbb{D}(\mathbf{u}_h^n), \mathbb{L}_H) = 0, \quad (6.3)$$

for all $\mathbf{v}_h \in \mathbf{X}_h$, $q_h \in Q_h$, $\mathbb{L}_H \in \mathbf{L}_H$.

Recall that, under inf-sup condition (3.14), the formulation in $\mathbf{X}_h$ and $\mathbf{V}_h$ are equivalent. Then, the variational formulation (6.1)-(6.3) can be written as

$$\Big(\frac{3\mathbf{u}_h^{n+1} - 4\mathbf{u}_h^n + \mathbf{u}_h^{n-1}}{2\Delta t}, \mathbf{v}_h\Big) + b^*(2\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, \mathbf{u}_h^{n+1}, \mathbf{v}_h) + 2\nu(\mathbb{D}(\mathbf{u}_h^{n+1}), \mathbb{D}(\mathbf{v}_h))$$

$$+\nu_T(\mathbb{D}(\mathbf{u}_h^{n+1}), \mathbb{D}(\mathbf{v}_h)) - \nu_T(\mathbb{G}_H^n, \mathbb{D}(\mathbf{v}_h)) = (\mathbf{f}^{\,n+1}, \mathbf{v}_h), \quad (6.4)$$

$$(\mathbb{G}_H^n - \mathbb{D}(\mathbf{u}_h^n), \mathbb{L}_H) = 0, \quad (6.5)$$

for all $\mathbf{v}_h \in \mathbf{V}_h$, $\mathbb{L}_H \in \mathbf{L}_H$.

For time discretization of NSE, we use linearly extrapolated 2-step backward differentiation formula (BDF2) based on (6.4)-(6.5). The analysis of BDF2 requires G-stability. Now, we explain G-stability for this purpose.

6.1 G-Stability of BDF2

Definition 6.1.1. *(Dahlquist 1976) [11]: The multistep (k-step) method is defined by*

$$\sum_{i=0}^{k} \alpha^i v^{m+i} = \Delta t\, f\Big(\sum_{i=0}^{k} \beta^i t^{m+i}, \sum_{i=0}^{k} \beta^i v^{m+i}\Big)$$

in order to solve ordinary differential equation $v_t = v' = f(t, v)$.
G-norm is defined by

$$\|\chi^m\|_G^2 = \big(\chi^m, G\chi^m\big)$$

where

$$\chi^m = \begin{bmatrix} v^{m+k-1} \\ \vdots \\ v^m \end{bmatrix}, \quad \chi \in \mathbb{R}^{d\cdot k}.$$

Note that, $(\cdot,\cdot)$ is the inner product in $\mathbb{R}^n$.
The multistep method is called G-stable, if there exists a real, symmetric, and positive definite matrix G which satisfies the inequality

$$\left\|\chi^{m+1} - \hat{\chi}^{m+1}\right\|_G \leq \|\chi^m - \hat{\chi}^m\|_G \,, \; \forall \Delta t > 0 \tag{6.6}$$

where $\{v^m\}$ and $\{\hat{v}^m\}$ are two different numerical solutions of our ODE.
Also, for all ODE's satisfy the Lipschitz condition

$$(f(t^m, v^m) - f(t^m, \hat{v}^m), v^m - \hat{v}^m) \leq k \,\|v^m - \hat{v}^m\|^2$$

in case of $k = 0$.

The following results from [11] states the relation between G-stability and A-stability.

Theorem 6.1.1. *G-stability implies A-stability.*

Additionally, in 1978, Dahlquist explains equivalence of these two types stabilities in [12].

Now, we consider the G-stability of BDF2 method, [21]. We let

$$\frac{3v^{m+1} - 4v^m + v^{m-1}}{2} = \Delta t \; f(t^{m+1}, v^{m+1}), \tag{6.7}$$

to solve the problem $v_t = f(t, v)$. To show G-stability of the method, we have to show that the condition (6.6) is satisfied by the two different numerical solutions of the problem. Let $\{v_m\}$ and $\{\hat{v}_m\}$ be the solutions of (6.7).

Now, insert the BDF2 method into the Lipschitz condition with $k = 0$

$$(f(t^{m+1}, v^{m+1}) - f(t^{m+1}, \hat{v}^{m+1}), v^{m+1} - \hat{v}^{m+1}) \leq 0$$

$$(\frac{3v^{m+1} - 4v^m + v^{m-1}}{2} - \frac{3\hat{v}^{m+1} - 4\hat{v}^m + \hat{v}^m}{2}, v^{m+1} - \hat{v}^{m+1}) \leq 0$$

By denoting the difference of two numerical solutions with $\Delta\mathbf{v}^m = \mathbf{v}^m - \hat{\mathbf{v}}^m$, we obtain

$$D = (\frac{3\Delta v^{m+1} - 4\Delta v^m + \Delta v^{m-1}}{2}, \Delta v^{m+1}) \leq 0 \tag{6.8}$$

To show (6.6) is satisfied, we arrange D as

$$D = \|\Delta\chi^m\|_G^2 - \left\|\Delta\chi^{m-1}\right\|_G^2 + \left\|a_2\Delta v^{m+1} + a_1\Delta v^m + a_0\Delta v^{m-1}\right\|^2 \tag{6.9}$$

where G is real, symmetric, positive definite matrix defined by

$$G = \begin{bmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{bmatrix}.$$

Expanding the terms in (6.9) gives

$$\begin{aligned} \|\Delta\chi^m\|^2 &= \left(\begin{bmatrix} \Delta v^{m+1} \\ \Delta v^m \end{bmatrix}, G \begin{bmatrix} \Delta v^{m+1} \\ \Delta v^m \end{bmatrix} \right) \\ &= g_{11}(\Delta v^{m+1})^2 + 2g_{12}\Delta v^m \Delta v^{m+1} + g_{22}(\Delta v^m)^2, \end{aligned} \tag{6.10}$$

$$\begin{aligned} \left\|\Delta\chi^{m-1}\right\|^2 &= \left(\begin{bmatrix} \Delta v^m \\ \Delta v^{m-1} \end{bmatrix}, G \begin{bmatrix} \Delta v^m \\ \Delta v^{m-1} \end{bmatrix} \right) \\ &= g_{11}(\Delta v^m)^2 + 2g_{12}\Delta v^{m-1}\Delta v^m + g_{22}(\Delta v^{m-1})^2, \end{aligned} \tag{6.11}$$

$$\begin{aligned} \left\|a_2\Delta v^{m+1} + a_1\Delta v^m + a_0\Delta v^{m-1}\right\|^2 &= a_2^2(\Delta v^{m+1})^2 + a_1^2(\Delta v^m)^2 + a_0^2(\Delta v^{m-1})^2 \\ &+ 2a_1a_2\Delta v^{m+1}\Delta v^m + 2a_0a_2\Delta v^{m+1}\Delta v^{m-1} + 2a_0a_1\Delta v^m\Delta v^{m-1}. \end{aligned} \tag{6.12}$$

After this expansion, by the equivalence of (6.8) and (6.9), we obtain

$$\frac{3}{2} = g_{11} + a_2^2, \qquad 0 = g_{22} - g_{11} + a_1^2, \qquad 0 = -g_{22} + a_0^2, \tag{6.13}$$

$$-2 = 2g_{12} + 2a_2a_1, \qquad \frac{1}{2} = 2a_0a_2, \qquad 0 = -2g_{12} + 2a_1a_0. \tag{6.14}$$

By adding all six equations, we find relation between $a_0,\ a_1,\ a_2$. Then, by adding first and third equations in (6.14), we find a_1. Lastly, by adding equations in (6.13) and using the second equation in (6.14), we find a_0 and a_2. By using these all calculated values, we finally obtain

$$g_{11} = \frac{5}{4} \quad g_{12} = \frac{-1}{2} \quad g_{22} = \frac{1}{4}.$$

That is,

$$G = \frac{1}{4}\begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}.$$

Hence the Lipschitz condition become

$$D = \|\Delta\chi^m\|_G^2 - \left\|\Delta\chi^{m-1}\right\|_G^2 + \left\|\pm\frac{1}{2}\Delta v^{m+1} \mp 1\Delta v^m \pm \frac{1}{2}\Delta v^{m-1}\right\|^2. \tag{6.15}$$

We know that the third term of D in (6.15) is always positive and $D \leq 0$ from (6.8). So, we get

$$\begin{aligned} \|\Delta\chi^m\|_G^2 - \left\|\Delta\chi^{m-1}\right\|_G^2 &\leq 0 \\ \|\Delta\chi^m\|_G^2 &\leq \left\|\Delta\chi^{m-1}\right\|_G^2 \\ \|v^m - \hat{v}^m\|_G^2 &\leq \left\|v^{m-1} - \hat{v}^{m-1}\right\|_G^2. \end{aligned}$$

Thus, the BDF2 method satisfies the condition (6.6). Hence, the G-stability of method is established.

In the analysis, we use the following two important inequalities which are presented in [31].

Lemma 6.1.1. *For any vector* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$*, we have*

$$\begin{aligned} \left(\begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix}, G\begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix}\right) &= \frac{3}{4}\|\mathbf{u}\|^2 - \frac{1}{4}\|\mathbf{v}\|^2 + \frac{1}{2}\|\mathbf{u}-\mathbf{v}\|^2 \\ &\geq \frac{3}{4}\|\mathbf{u}\|^2 - \frac{1}{4}\|\mathbf{v}\|^2 \end{aligned} \tag{6.16}$$

and

$$\begin{aligned} \left(\begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix}, G\begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix}\right) &\leq \frac{3}{4}\|\mathbf{u}\|^2 + \frac{1}{2}\|\mathbf{u}-\mathbf{v}\|^2 \\ &\leq \frac{7}{4}\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 \end{aligned} \tag{6.17}$$

Proof. We presented the G matrix above. By using it, we obtain

$$\left(\begin{bmatrix}\mathbf{u}\\ \mathbf{v}\end{bmatrix}, G\begin{bmatrix}\mathbf{u}\\ \mathbf{v}\end{bmatrix}\right) = \left(\begin{bmatrix}\mathbf{u}\\ \mathbf{v}\end{bmatrix}, \begin{bmatrix}\frac{5\mathbf{u}}{4}-\frac{2\mathbf{v}}{4}\\ \frac{-2\mathbf{u}}{4}+\frac{\mathbf{v}}{4}\end{bmatrix}\right) \tag{6.18}$$

$$= \frac{5}{4}\mathbf{u}^2 - \frac{2}{4}\mathbf{uv} - \frac{2}{4}\mathbf{uv} + \frac{1}{4}\mathbf{v}^2 \tag{6.19}$$

$$= \frac{3}{4}\|\mathbf{u}\|^2 - \frac{1}{4}\|\mathbf{v}\|^2 + \frac{1}{2}\|\mathbf{u}-\mathbf{v}\|^2 \tag{6.20}$$

$$\geq \frac{3}{4}\|\mathbf{u}\|^2 - \frac{1}{4}\|\mathbf{v}\|^2 \tag{6.21}$$

Since, we can eliminate the non-negative term in (6.20). Hence, we produce the inequality (6.16).

Secondly, if we eliminate the non-positive term in (6.20) and use the inequality $\|\mathbf{u}-\mathbf{v}\| \leq 2(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2)$, we obtain

$$\left(\begin{bmatrix}\mathbf{u}\\ \mathbf{v}\end{bmatrix}, G\begin{bmatrix}\mathbf{u}\\ \mathbf{v}\end{bmatrix}\right) \leq \frac{3}{4}\|\mathbf{u}\|^2 + \frac{1}{2}\|\mathbf{u}-\mathbf{v}\|^2 \tag{6.22}$$

$$\leq \frac{3}{4}\|\mathbf{u}\|^2 + \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 \tag{6.23}$$

By this way, we produce the inequality (6.17). □

In error analysis, we use the inequalities

$$\|\chi_{\phi}^N\|_G^2 = \left(\begin{bmatrix}\phi_h^{N+1}\\ \phi_h^N\end{bmatrix}, G\begin{bmatrix}\phi_h^{N+1}\\ \phi_h^N\end{bmatrix}\right) \geq \frac{3}{4}\|\phi_h^{N+1}\|^2 - \frac{1}{4}\|\phi_h^N\|^2 \tag{6.24}$$

and

$$\|\chi_{\phi}^0\|_G^2 = \left(\begin{bmatrix}\phi_h^1\\ \phi_h^0\end{bmatrix}, G\begin{bmatrix}\phi_h^1\\ \phi_h^0\end{bmatrix}\right) \leq \frac{7}{4}\|\phi_h^1\|^2 + \|\phi_h^0\|^2 \tag{6.25}$$

obtained by Lemma 6.1.1.

6.2 Stability Analysis of (6.4)-(6.5)

In this section, the stability analysis of (3.1) based on the finite element formulation (6.4)-(6.5) is considered. By using standard energy arguments, we prove the stability of the method.

Define $\chi_{\mathbf{v}}^{n} := \begin{bmatrix} \mathbf{v}^{n+1} \\ \mathbf{v}^{n} \end{bmatrix}$ and $\chi_{\mathbf{v}}^{n-1} := \begin{bmatrix} \mathbf{v}^{n} \\ \mathbf{v}^{n-1} \end{bmatrix}$.

Then, from (6.15), one can write

$$\Big(\frac{3\mathbf{v}^{n+1} - 4\mathbf{v}^{n} + \mathbf{v}^{n-1}}{2}, \mathbf{v}^{n+1}\Big) = \|\chi_{\mathbf{v}}^{n}\|_G^2 - \left\|\chi_{\mathbf{v}}^{n-1}\right\|_G^2 + \frac{\|\mathbf{v}^{n+1} - 2\mathbf{v}^{n} + \mathbf{v}^{n-1}\|^2}{4} \tag{6.26}$$

for all $\mathbf{v}^i \in L^2(\Omega)$.

Lemma 6.2.1. *The algorithm (6.4)-(6.5) is unconditionally stable in the following sense, for all $N \geq 1$*

$$\left\|\mathbf{u}_h^N\right\|^2 + \frac{2\Delta t\, \nu}{3} \sum_{n=1}^{N-1} \left\|\mathbb{D}(\mathbf{u}_h^{n+1})\right\|^2 + \frac{2\Delta t\, \nu_T}{3} \left\|\mathbb{D}(\mathbf{u}_h^{N})\right\|^2$$
$$\leq \Big(\frac{1}{3}\Big)^N \left\|\mathbf{u}_h^0\right\|^2 + \frac{4N}{3} \left\|\chi_{\mathbf{u}}^0\right\|_G^2 + \frac{2N\Delta t\, \nu_T}{3} \left\|\mathbb{D}(\mathbf{u}_h^1)\right\|^2 + \frac{2N\Delta t}{9\nu} \sum_{n=1}^{N-1} \left\|\boldsymbol{f}^{\,n+1}\right\|_{-1}^2.$$

Proof. To show stability, choose $\mathbf{v}_h = \Delta t\ \mathbf{u}_h^{n+1} \in \mathbf{V}_h$ in (6.4),

$$\Delta t\, \Big(\frac{3\mathbf{u}_h^{n+1} - 4\mathbf{u}_h^{n} + \mathbf{u}_h^{n-1}}{2\Delta t}, \mathbf{u}_h^{n+1}\Big) + \Delta t\, b^*(2\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, \mathbf{u}_h^{n+1}, \mathbf{u}_h^{n+1})$$
$$+ \Delta t\, 2\nu(\mathbb{D}(\mathbf{u}_h^{n+1}), \mathbb{D}(\mathbf{u}_h^{n+1})) + \Delta t\, \nu_T(\mathbb{D}(\mathbf{u}_h^{n+1}), \mathbb{D}(\mathbf{u}_h^{n+1}))$$
$$- \Delta t\, \nu_T(\mathbb{G}_H^n, \mathbb{D}(\mathbf{u}_h^{n+1})) = \Delta t\, (\mathbf{f}^{\,n+1}, \mathbf{u}_h^{n+1}). \tag{6.27}$$

Now, the trilinear term vanishes, because of (3.8). Then, by using the relation (6.26), we rearrange (6.27)

$$\|\chi_{\mathbf{u}}^n\|_G^2 - \left\|\chi_{\mathbf{u}}^{n-1}\right\|_G^2 + \frac{\left\|\mathbf{u}_h^{n+1} - 2\mathbf{u}_h^{n} + \mathbf{u}_h^{n-1}\right\|^2}{4} + \Delta t\, 2\nu \left\|\mathbb{D}(\mathbf{u}_h^{n+1})\right\|^2$$
$$+ \Delta t\, \nu_T \left\|\mathbb{D}(\mathbf{u}_h^{n+1})\right\|^2 = \Delta t\, \nu_T(\mathbb{G}_H^n, \mathbb{D}(\mathbf{u}_h^{n+1})) + \Delta t\, (\mathbf{f}^{\,n+1}, \mathbf{u}_h^{n+1}). \tag{6.28}$$

By applying Cauchy-Schwarz inequality, Young's inequality and using dual norm for the right hand side terms in (6.28), we obtain

$$\begin{aligned}
\|\chi_{\mathbf{u}}^n\|_G^2 - \left\|\chi_{\mathbf{u}}^{n-1}\right\|_G^2 + \frac{\left\|\mathbf{u}_h^{n+1} - 2\mathbf{u}_h^n + \mathbf{u}_h^{n-1}\right\|^2}{4} + \Delta t\, 2\nu \left\|\mathbb{D}(\mathbf{u}_h^{n+1})\right\|^2 \\
+ \Delta t\, \nu_T \left\|\mathbb{D}(\mathbf{u}_h^{n+1})\right\|^2 \leq \frac{\Delta t\, \nu_T^2}{2\varepsilon_1} \|\mathbb{G}_H^n\|^2 + \frac{\Delta t \varepsilon_1}{2} \left\|\mathbb{D}(\mathbf{u}_h^{n+1})\right\|^2 + \frac{\Delta t}{2\varepsilon_2} \left\|\mathbf{f}^{\,n+1}\right\|_{-1}^2 \\
+ \frac{\Delta t \varepsilon_2}{2} \left\|\mathbb{D}(\mathbf{u}_h^{n+1})\right\|^2 . \qquad (6.29)
\end{aligned}$$

To bound the first term in the right hand side of (6.29), we let $\mathbb{L}_H = \mathbb{G}_H^n$ in (6.5). Then, by Cauchy-Schwarz inequaliy, we have

$$\begin{aligned}
\|\mathbb{G}_H^n\|^2 =& (\mathbb{D}(\mathbf{u}_h^n), \mathbb{G}_H^n) \\
\|\mathbb{G}_H^n\| \leq& \|\mathbb{D}(\mathbf{u}_h^n)\| . \qquad (6.30)
\end{aligned}$$

Now, we obtain

$$\begin{aligned}
\|\chi_{\mathbf{u}}^n\|_G^2 - \left\|\chi_{\mathbf{u}}^{n-1}\right\|_G^2 + \frac{\left\|\mathbf{u}_h^{n+1} - 2\mathbf{u}_h^n + \mathbf{u}_h^{n-1}\right\|^2}{4} + \Delta t\, 2\nu \left\|\mathbb{D}(\mathbf{u}_h^{n+1})\right\|^2 \\
+ \Delta t\, \nu_T \left\|\mathbb{D}(\mathbf{u}_h^{n+1})\right\|^2 \leq \frac{\Delta t\, \nu_T^2}{2\varepsilon_1} \|\mathbb{D}(\mathbf{u}_h^n)\|^2 + \frac{\Delta t\, \varepsilon_1}{2} \left\|\mathbb{D}(\mathbf{u}_h^{n+1})\right\|^2 \\
+ \frac{\Delta t}{2\varepsilon_2} \left\|\mathbf{f}^{\,n+1}\right\|_{-1}^2 + \frac{\Delta t\, \varepsilon_2}{2} \left\|\mathbb{D}(\mathbf{u}_h^{n+1})\right\|^2 . \qquad (6.31)
\end{aligned}$$

By dropping nonnegative term $\dfrac{\left\|\mathbf{u}_h^{n+1} - 2\mathbf{u}_h^n + \mathbf{u}_h^{n-1}\right\|^2}{4}$, we can choose $\varepsilon_1 = \nu_T$ and $\varepsilon_2 = 3\nu$ to yield

$$\begin{aligned}
\|\chi_{\mathbf{u}}^n\|_G^2 - \left\|\chi_{\mathbf{u}}^{n-1}\right\|_G^2 + \frac{\Delta t\, \nu}{2} \left\|\mathbb{D}(\mathbf{u}_h^{n+1})\right\|^2 + \frac{\Delta t\, \nu_T}{2} \left\|\mathbb{D}(\mathbf{u}_h^{n+1})\right\|^2 \\
\leq \frac{\Delta t\, \nu_T}{2} \|\mathbb{D}(\mathbf{u}_h^n)\|^2 + \frac{\Delta t}{6\nu} \left\|\mathbf{f}^{\,n+1}\right\|_{-1}^2 . \qquad (6.32)
\end{aligned}$$

Now, take the sum from $n = 1$ to $n = N - 1$

$$\left\|\chi_{\mathbf{u}}^{N-1}\right\|_G^2 - \left\|\chi_{\mathbf{u}}^{0}\right\|_G^2 + \frac{\Delta t\ \nu}{2}\sum_{n=1}^{N-1}\left\|\mathbb{D}(\mathbf{u}_h^{n+1})\right\|^2 + \frac{\Delta t\ \nu_T}{2}\left\|\mathbb{D}(\mathbf{u}_h^{N})\right\|^2$$

$$\leq \frac{\Delta t\ \nu_T}{2}\left\|\mathbb{D}(\mathbf{u}_h^{1})\right\|^2 + \frac{\Delta t}{6\nu}\sum_{n=1}^{N-1}\left\|\mathbf{f}^{\,n+1}\right\|_{-1}^2. \tag{6.33}$$

By using inequality (6.16) in Lemma 6.1.1, we get

$$\left\|\mathbf{u}_h^{N}\right\|^2 + \frac{2\Delta t\ \nu}{3}\sum_{n=1}^{N-1}\left\|\mathbb{D}(\mathbf{u}_h^{n+1})\right\|^2 + \frac{2\Delta t\ \nu_T}{3}\left\|\mathbb{D}(\mathbf{u}_h^{N})\right\|^2$$

$$\leq \frac{1}{3}\left\|\mathbf{u}_h^{N-1}\right\|^2 + \frac{4}{3}\left\|\chi_{\mathbf{u}}^{0}\right\|_G^2 + \frac{2\Delta t\ \nu_T}{3}\left\|\mathbb{D}(\mathbf{u}_h^{1})\right\|^2 + \frac{2\Delta t}{9\nu}\sum_{n=1}^{N-1}\left\|\mathbf{f}^{\,n+1}\right\|_{-1}^2. \tag{6.34}$$

Lastly, by induction on n, we find the required stability bound. □

6.3 Existence and Uniqueness of a Solution Obtained by Projection-based VMS Method with Linearly Extrapolated BDF2

After time discretization with linearly extrapolated BDF2, we obtain a steady problem at each time step. Thus, a fully discrete problem (6.4)-(6.5) is very similar to steady-state NSE problem. In our scheme, along with the BDF2 for discretization in time, we apply linear extrapolation to convective term. If we had not applied linear extrapolation, we would have computed solution of nonlinear problem at each time step.

The proof of existence and uniqueness of the solution obtained by projection-based VMS method with linearly extrapolated BDF2 follows from ideas in the book by John [32].

Theorem 6.3.1. *(Existence and uniqueness of the solution) Let $(\boldsymbol{V}_h, Q_h)$ be conforming finite element spaces which satisfy the discrete inf-sup condition (3.14). If $\mathbf{u}_h^n \in \boldsymbol{V}_h$, then (6.4)-(6.5) has a unique solution.*

Proof. The existence-uniqueness of the velocity solution is proved with Lax-Milgram theorem (2.0.10). The weak formulation (6.4) can be arranged as

$$\begin{aligned}&\big(\frac{3\mathbf{u}_h^{n+1}}{2\Delta t},\mathbf{v}_h\big)+b^*(2\mathbf{u}_h^n-\mathbf{u}_h^{n-1},\mathbf{u}_h^{n+1},\mathbf{v}_h)+2\nu(\mathbb{D}(\mathbf{u}_h^{n+1}),\mathbb{D}(\mathbf{v}_h))\\&+\nu_T(\mathbb{D}(\mathbf{u}_h^{n+1}),\mathbb{D}(\mathbf{v}_h))=(\mathbf{f}^{\,n+1},\mathbf{v}_h)+\big(\frac{4\mathbf{u}_h^n-\mathbf{u}_h^{n-1}}{2\Delta t},\mathbf{v}_h\big)+\nu_T(\mathbb{G}_H^n,\mathbb{D}(\mathbf{v}_h))\end{aligned} \tag{6.35}$$

$$(\mathbb{G}_H^n-\mathbb{D}(\mathbf{u}_h^n),\mathbb{L}_H)=0$$

for all $\mathbf{v}_h\in\mathbf{V}_h$ and $\mathbb{L}_H\in\mathbf{L}_H$.

The equation (6.35) can be rewritten in the form

$$a(\mathbf{u}_h^{n+1},\mathbf{v}_h)=F(\mathbf{v}_h)$$

where $a(\cdot,\cdot)$ denotes the left hand side of (6.35) and $F(\cdot)$ denotes the right hand side of (6.35). First of all, we show $a(\mathbf{u}_h^{n+1},\mathbf{v}_h)$ satisfies the coercivity condition. To do this, we choose $\mathbf{v}_h=\mathbf{u}_h^{n+1}\in\mathbf{V}_h$

$$a(\mathbf{u}_h^{n+1},\mathbf{u}_h^{n+1})=\frac{3}{2\Delta t}\left\|\mathbf{u}_h^{n+1}\right\|^2+2\nu\left\|\mathbb{D}(\mathbf{u}_h^{n+1})\right\|^2+\nu_T\left\|\mathbb{D}(\mathbf{u}_h^{n+1})\right\|^2 \tag{6.36}$$

where trilinear form $b^*(2\mathbf{u}_h^n-\mathbf{u}_h^{n-1},\mathbf{u}_h^{n+1},\mathbf{u}_h^{n+1})=0$, because of (3.8).
Then, by Korn's inequality to (6.36), we obtain

$$\begin{aligned}a(\mathbf{u}_h^{n+1},\mathbf{u}_h^{n+1})\geq&\frac{3}{2\Delta t}\left\|\mathbf{u}_h^{n+1}\right\|^2+\frac{2\nu+\nu_T}{\sqrt{2}}\left\|\nabla\mathbf{u}_h^{n+1}\right\|^2\\\geq&C\big(\left\|\mathbf{u}_h^{n+1}\right\|^2+\left\|\nabla\mathbf{u}_h^{n+1}\right\|^2\big)\\=&C\left\|\mathbf{u}_h^{n+1}\right\|_{H^1}^2\end{aligned}$$

where $C=min\{\frac{3}{2\Delta t},\frac{2\nu+\nu_T}{\sqrt{2}}\}$.

Secondly, to prove the continuity of $a(\mathbf{u}_h^{n+1},\mathbf{v}_h)$, use Cauchy-Schwarz inequality, the estimation (3.1.1), Poincaré-Friedrichs', Korn's inequalities and Sobolev imbedding theorem :

$$\begin{aligned}
a(\mathbf{u}_h^{n+1}, \mathbf{v}_h) \leq & \frac{3}{2\Delta t} \left\|\mathbf{u}_h^{n+1}\right\| \|\mathbf{v}_h\| + M \left\|\nabla(2\mathbf{u}_h^n - \mathbf{u}_h^{n-1})\right\| \left\|\nabla(\mathbf{u}_h^{n+1})\right\| \|\nabla(\mathbf{v}_h)\| \\
& + 2\nu \left\|\mathbb{D}(\mathbf{u}_h^{n+1})\right\| \|\mathbb{D}(\mathbf{v}_h)\| + \nu_T \left\|\mathbb{D}(\mathbf{u}_h^{n+1})\right\| \|\mathbb{D}(\mathbf{v}_h)\| \\
\leq & C \left\|\nabla(2\mathbf{u}_h^n - \mathbf{u}_h^{n-1})\right\| \left\|\mathbb{D}(\mathbf{u}_h^{n+1})\right\| \|\mathbb{D}(\mathbf{v}_h)\| \\
\leq & C \left\|\mathbb{D}(\mathbf{u}_h^{n+1})\right\| \|\mathbb{D}(\mathbf{v}_h)\| \\
\leq & C \left\|\mathbf{u}_h^{n+1}\right\|_{H^1} \|\mathbf{v}_h\|_{H^1}
\end{aligned}$$

where C is the constant determined as minimum among constants from Poincaré-Friedrichs' and Korn's inequalities and finite value of $\left\|2\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\right\|$ (from stability result).

Lastly, let's show the continuity condition for $F(\mathbf{v}_h)$. By using the dual norm (2.0.6), Cauchy-Schwarz inequality, the bound (6.30), Korn's and Poincaré-Friedrichs' inequalities, we get

$$\begin{aligned}
F(\mathbf{v}_h) = & (\mathbf{f}^{\,n+1}, \mathbf{v}_h) + \big(\frac{4\mathbf{u}_h^n - \mathbf{u}_h^{n-1}}{2\Delta t}, \mathbf{v}_h\big) + \nu_T(\mathbb{G}_H^n, \mathbb{D}(\mathbf{v}_h)) \\
\leq & \left\|\mathbf{f}^{\,n+1}\right\|_{-1} \|\nabla \mathbf{v}_h\| + \frac{1}{2\Delta t} \left\|4\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\right\| \|\mathbf{v}_h\| + \nu_T \|\mathbb{D}(\mathbf{u}_h^n)\| \|\mathbb{D}(\mathbf{v}_h)\| \\
\leq & C \|\mathbb{D}(\mathbf{v}_h)\| \Big[\left\|\mathbf{f}^{\,n+1}\right\|_{-1} + \frac{C_2}{2\Delta t} \left\|4\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\right\| + \nu_T \|\mathbb{D}(\mathbf{u}_h^n)\| \Big] \\
\leq & C \|\mathbb{D}(\mathbf{v}_h)\| \\
\leq & C \|\mathbf{v}_h\|_{H^1}
\end{aligned}$$

which is the continuity bound for the $F(\mathbf{v}_h)$.

Hence, the fully discrete problem (6.4) satisfies all conditions of the Lax-Milgram theorem (2.0.10) such that the existence and uniqueness of the velocity solution follows. Since the finite element spaces of the problem (6.4)-(6.5) satisfy the discrete inf-sup condition (3.14), existence and uniqueness of the pressure is guaranteed, [32]. □

6.4 Error Analysis

In this section, we present an error analysis of our method. We show that solution of the fully discrete approximation obtained by projection-based VMS method with linearly extrapolated BDF2 time discretization is convergent.

We introduce the discrete in time version of the norms used in the continuous time case given by Definition 2.0.5. The discrete norms are given as

$$|||v|||_{\infty,k} := \max_{0\leq n\leq N_T} \|v^n\|_k \, , \qquad |||v|||_{p,k} := \left(\sum_{n=0}^{N_T} \|v^n\|_k^p \, \Delta t \right)^{1/p} .$$

The following approximation properties are used through the analysis. These properties are presented in the book by Brenner and Scott [8]. If piecewise polynomials of degree k for the velocity, and piecewise polynomial of degree $(s-1)$ for the pressure are chosen, the finite element spaces for velocity and pressure satisfy

$$\begin{aligned} \inf_{\mathbf{v}_h\in X_h} \|\mathbf{u}-\mathbf{v}_h\| &\leq Ch^{k+1}|\mathbf{u}|_{k+1}, \ \mathbf{u}\in H^{k+1}(\Omega), \\ \inf_{\mathbf{v}_h\in X_h} \|\nabla(\mathbf{u}-\mathbf{v}_h)\| &\leq Ch^{k}|\mathbf{u}|_{k+1}, \ \mathbf{u}\in H^{k+1}(\Omega), \\ \inf_{q^h\in Q^h} \|p-q_h\| &\leq Ch^{s+1}|p|_{s+1}, \ p\in H^{s+1}(\Omega). \end{aligned} \tag{6.37}$$

Additionally, approximation on coarse space $\mathbf{L}_H$ is given by:

$$\|\mathbb{L}-P_{\mathbf{L}_H}\mathbb{L}\| \leq CH^k \|\mathbf{v}\|_{k+1} \ \forall \mathbb{L}\in \mathbf{L}\cap H^{k+1}(\Omega). \tag{6.38}$$

We also assume that the exact solution satisfies the following regularity assumptions:

$$\begin{aligned} &\mathbf{u}\in H^1(0,T;H^{k+1}(\Omega))\cap H^2(0,T;H^1(\Omega))\cap L^\infty(0,T;H^1(\Omega))\cap H^3(0,T;L^2(\Omega)), \\ &p\in L^2(0,T;H^s(\Omega)), \\ &\mathbf{f}\in L^2(0,T;L^2(\Omega)). \end{aligned} \tag{6.39}$$

Theorem 6.4.1. *Let $(\mathbf{u}, p)$ be the solution pair of NSE (3.1) such that the regularity assumptions (6.39) are hold. Then, the error $\mathbf{e}^n = \mathbf{u}^n - \mathbf{u}_h^n$ satisfies*

$$\left\|\mathbf{e}^N\right\|^2 + \frac{1}{3}\sum_{n=1}^{N-1}\left\|\mathbf{e}^{n+1} - 2\mathbf{e}^n + \mathbf{e}^{n-1}\right\|^2 + \Delta t\frac{2\nu + 2\nu_T}{3}\sum_{n=1}^{N-1}\left\|\mathbb{D}(\mathbf{e}^{n+1})\right\|^2$$

$$\leq exp\Big(C\nu^{-1}T\Big)\Bigg[\Big(\frac{1}{3}\Big)^N\left\|\mathbf{e}^0\right\|^2 + C\Big(1 - \big(\frac{1}{3}\big)^N\Big)(\left\|\mathbf{e}^1\right\| + \left\|\mathbf{e}^0\right\|) + C\Big(1 - \big(\frac{1}{3}\big)^N\Big)$$

$$\Big(\nu^{-1}h^{2k+2}\left|||\mathbf{u}_t|\right||^2_{2,k+1} + \nu^{-1}h^{2k}\left|||\mathbf{u}|\right||^2_{2,k+1}\left|||\nabla\mathbf{u}|\right||^2_{\infty,0} + (2\nu + \nu_T)h^{2k}\left|||\mathbf{u}|\right||^2_{2,k+1}$$

$$+ \nu^{-1}h^{2s+2}\left|||p|\right||^2_{2,s+1} + \nu^{-1}\Delta t^4\left|||\mathbf{u}_{ttt}|\right||^2_{2,0} + \nu^{-1}\Delta t^4\left|||\nabla\mathbf{u}|\right||^2_{\infty,0}\left|||\nabla\mathbf{u}_{tt}|\right||^2_{2,0}$$

$$+ \nu^{-1}\nu_T^2h^{2k}\left|||\mathbf{u}|\right||_{2,k+1} + \nu^{-1}\nu_T^2H^{2k}\left|||\mathbf{u}|\right||_{2,k+1} + \nu^{-1}\nu_T^2\Delta t^2\left\|\mathbf{u}_t|\right||_{\infty,0}\Big)\Bigg]. \quad (6.40)$$

Remark 6.4.1. *The pair of Taylor-Hood element spaces is one of the most suitable choices among finite elements spaces satisfying discrete inf-sup condition (3.14). They consist of k^{th} degree piecewise continuous functions for velocity and $(k-1)^{th}$ degree piecewise continuous functions for pressure where $k \geq 2$, [23]. In our analysis, we prefer to use Taylor-Hood elements for $k = 2$, i.e, (P_2, P_1). This serves us good approximation solutions for both velocity and pressure with relatively low computation cost compared with other finite element choices.*

Corollary 6.4.1. *Under the assumptions of (6.39), let the finite element spaces $(\boldsymbol{X}_h, Q_h)$ be (P_2, P_1) Taylor-Hood element, the coarse mesh size and additional viscosity are chosen as $H \leq O(\sqrt{h})$ and $\nu_T = h^2$ or $\nu_T = h$ respectively, then the error in velocity satisfies*

$$\left\|\mathbf{e}^N\right\|^2 + \frac{1}{3}\sum_{n=1}^{N-1}\left\|\mathbf{e}^{n+1} - 2\mathbf{e}^n + \mathbf{e}^{n-1}\right\|^2 + \Delta t\frac{2\nu + 2\nu_T}{3}\sum_{n=1}^{N-1}\left\|\mathbb{D}(\mathbf{e}^{n+1})\right\|^2$$

$$\leq C\big[\left\|\mathbf{e}^1\right\| + \left\|\mathbf{e}^0\right\| + h^4 + \Delta t^4 + h^2\Delta t^2\big],$$

for all $\Delta t > 0$.

Proof. If one uses the approximation assumptions (6.37) in (6.40), then the result follows. □

Remark 6.4.2. *It is clear that if we choose $\mathbf{u}_h^0$ and $\mathbf{u}_h^1$ such that $\|\mathbf{e}^0\|$ and $\|\mathbf{e}^1\|$ are optimal, then we get second order accuracy.*

Proof. We now present the proof of Theorem 6.4.1.

At time level t^{n+1}, by denoting $\mathbf{u}^n = \mathbf{u}(t^n)$, the true solution of the NSE (3.1) can be written as

$$\Big(\frac{3\mathbf{u}^{n+1} - 4\mathbf{u}^n + \mathbf{u}^{n-1}}{2\Delta t}, \mathbf{v}_h\Big) + b^*(2\mathbf{u}^n - \mathbf{u}^{n-1}, \mathbf{u}^{n+1}, \mathbf{v}_h) + 2\nu(\mathbb{D}(\mathbf{u}^{n+1}), \mathbb{D}(\mathbf{v}_h))$$
$$+ \nu_T(\mathbb{D}(\mathbf{u}^{n+1}), \mathbb{D}(\mathbf{v}_h)) - \nu_T(\mathbb{D}(\mathbf{u}^{n+1}), \mathbb{D}(\mathbf{v}_h)) - (p^{n+1}, \nabla \cdot \mathbf{v}_h)$$
$$= (\mathbf{f}^{\,n+1}, \mathbf{v}_h) + Intp(\mathbf{u}^{n+1}; \mathbf{v}_h) \tag{6.41}$$

where

$$Intp(\mathbf{u}^{n+1}; \mathbf{v}_h) = \Big(\frac{3\mathbf{u}^{n+1} - 4\mathbf{u}^n + \mathbf{u}^{n-1}}{2\Delta t} - \partial_t \mathbf{u}(t^{n+1}), \mathbf{v}_h\Big)$$
$$+ b^*(2\mathbf{u}^n - \mathbf{u}^{n-1}, \mathbf{u}^{n+1}, \mathbf{v}_h) - b^*(\mathbf{u}^{n+1}, \mathbf{u}^{n+1}, \mathbf{v}_h). \tag{6.42}$$

is the local truncation error.

We split the error:

$$\mathbf{e}^n := \mathbf{u}^n - \mathbf{u}_h^n = (\mathbf{u}^n - \tilde{\mathbf{u}}^n) - (\mathbf{u}_h^n - \tilde{\mathbf{u}}^n) := \boldsymbol{\eta}^n - \boldsymbol{\phi}_h^n. \tag{6.43}$$

where $\tilde{\mathbf{u}}$ approximates $\mathbf{u}$ in $\mathbf{V}_h$, $\eta = \mathbf{u} - \tilde{\mathbf{u}}$ and $\boldsymbol{\phi}_h = \mathbf{u}_h - \tilde{\mathbf{u}} \in \mathbf{V}_h$.

By subtracting (6.4) from (6.41), we obtain

$$\Big(\frac{3\mathbf{e}^{n+1} - 4\mathbf{e}^n + \mathbf{e}^{n-1}}{2\Delta t}, \mathbf{v}_h\Big) + b^*(2\mathbf{u}^n - \mathbf{u}^{n-1}, \mathbf{u}^{n+1}, \mathbf{v}_h) - b^*(2\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, \mathbf{u}_h^{n+1}, \mathbf{v}_h)$$
$$+ (2\nu + \nu_T)(\mathbb{D}(\mathbf{e}^{n+1}), \mathbb{D}(\mathbf{v}_h)) = (p^{n+1} - q_h, \nabla \cdot \mathbf{v}_h) + \nu_T(\mathbb{D}(\mathbf{u}^{n+1}), \mathbb{D}(\mathbf{v}_h))$$
$$- \nu_T(\mathbb{G}_H^n, \mathbb{D}(\mathbf{v}_h)) + Intp(\mathbf{u}^{n+1}; \mathbf{v}_h). \tag{6.44}$$

We first rewrite the convective term in (6.44):

$$\begin{aligned}
&b^*(2\mathbf{u}^n - \mathbf{u}^{n-1}, \mathbf{u}^{n+1}, \mathbf{v}_h) - b^*(2\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, \mathbf{u}_h^{n+1}, \mathbf{v}_h) \\
&\quad = b^*(2\mathbf{u}^n - \mathbf{u}^{n-1}, \mathbf{u}^{n+1}, \mathbf{v}_h) - b^*(2\mathbf{u}^n - \mathbf{u}^{n-1}, \mathbf{u}_h^{n+1}, \mathbf{v}_h) \\
&\quad + b^*(2\mathbf{u}^n - \mathbf{u}^{n-1}, \mathbf{u}_h^{n+1}, \mathbf{v}_h) - b^*(2\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, \mathbf{u}_h^{n+1}, \mathbf{v}_h) \\
&\quad = b^*(2\mathbf{u}^n - \mathbf{u}^{n-1}, \mathbf{e}^{n+1}, \mathbf{v}_h) + b^*(2\mathbf{e}^n - \mathbf{e}^{n-1}, \mathbf{u}_h^{n+1}, \mathbf{v}_h). \qquad (6.45)
\end{aligned}$$

Then, by using (6.45) and error decomposition (6.43), we obtain

$$\begin{aligned}
&(\frac{3\boldsymbol{\phi}_h^{n+1} - 4\boldsymbol{\phi}_h^n + \boldsymbol{\phi}_h^{n-1}}{2\Delta t}, \mathbf{v}_h) + b^*(2\mathbf{u}^n - \mathbf{u}^{n-1}, \boldsymbol{\phi}_h^{n+1}, \mathbf{v}_h) \\
&\quad + b^*(2\boldsymbol{\phi}_h^n - \boldsymbol{\phi}_h^{n-1}, \mathbf{u}_h^{n+1}, \mathbf{v}_h) + (2\nu + \nu_T)(\mathbb{D}(\boldsymbol{\phi}_h^{n+1}), \mathbb{D}(\mathbf{v}_h)) \\
&\quad = (\frac{3\boldsymbol{\eta}^{n+1} - 4\boldsymbol{\eta}^n + \boldsymbol{\eta}^{n-1}}{2\Delta t}, \mathbf{v}_h) - (p^{n+1} - q_h, \nabla \cdot \mathbf{v}_h) \\
&\qquad + b^*(2\mathbf{u}^n - \mathbf{u}^{n-1}, \boldsymbol{\eta}^{n+1}, \mathbf{v}_h) + b^*(2\boldsymbol{\eta}^n - \boldsymbol{\eta}^{n-1}, \mathbf{u}_h^{n+1}, \mathbf{v}_h) \\
&\qquad + \nu_T(\mathbb{G}_H^n - \mathbb{D}(\mathbf{u}^{n+1}), \mathbb{D}(\mathbf{v}_h)) + (2\nu + \nu_T)(\mathbb{D}(\boldsymbol{\eta}^{n+1}), \mathbb{D}(\mathbf{v}_h)) - Intp(\mathbf{u}^{n+1}; \mathbf{v}_h). \\
&\qquad\qquad (6.46)
\end{aligned}$$

Setting $\mathbf{v}_h = \boldsymbol{\phi}_h^{n+1} \in \mathbf{V}_h$ and using G-stability (6.26) in (6.46), we yield

$$\begin{aligned}
&\frac{1}{\Delta t}\left(\left\|\chi_{\phi}^n\right\|_G^2 - \left\|\chi_{\phi}^{n-1}\right\|_G^2 \right) + \frac{1}{4\Delta t}\left\|\boldsymbol{\phi}_h^{n+1} - 2\boldsymbol{\phi}_h^n + \boldsymbol{\phi}_h^{n-1}\right\|^2 + (2\nu + \nu_T)\left\|\mathbb{D}(\boldsymbol{\phi}_h^{n+1})\right\|^2 \\
&= (\frac{3\boldsymbol{\eta}^{n+1} - 4\boldsymbol{\eta}^n + \boldsymbol{\eta}^{n-1}}{2\Delta t}, \boldsymbol{\phi}_h^{n+1}) + b^*(2\mathbf{u}^n - \mathbf{u}^{n-1}, \boldsymbol{\eta}^{n+1}, \boldsymbol{\phi}_h^{n+1}) \\
&\quad + b^*(2\boldsymbol{\eta}^n - \boldsymbol{\eta}^{n-1}, \mathbf{u}_h^{n+1}, \boldsymbol{\phi}_h^{n+1}) - b^*(2\boldsymbol{\phi}_h^n - \boldsymbol{\phi}_h^{n-1}, \mathbf{u}_h^{n+1}, \boldsymbol{\phi}_h^{n+1}) \\
&\qquad + (2\nu + \nu_T)(\mathbb{D}(\boldsymbol{\eta}^{n+1}), \mathbb{D}(\boldsymbol{\phi}_h^{n+1})) - (p^{n+1}, \nabla \cdot \boldsymbol{\phi}_h^{n+1}) \\
&\qquad + \nu_T(\mathbb{G}_H^n - \mathbb{D}(\mathbf{u}^{n+1}), \mathbb{D}(\boldsymbol{\phi}_h^{n+1})) - Intp(\mathbf{u}^{n+1}; \boldsymbol{\phi}_h^{n+1}). \qquad (6.47)
\end{aligned}$$

To bound the first term in the right hand side of (6.47), we use Cauchy-Schwarz, Poincaré-Friedrichs', Korn's inequalities, fundamental theorem of Calculus and Young's inequality, then we obtain

$$
\begin{aligned}
&\left(\frac{3\boldsymbol{\eta}^{n+1}-4\boldsymbol{\eta}^{n}+\boldsymbol{\eta}^{n-1}}{2\Delta t},\boldsymbol{\phi}_h^{n+1}\right)\\
&\quad\leq\left\|\frac{3\boldsymbol{\eta}^{n+1}-4\boldsymbol{\eta}^{n}+\boldsymbol{\eta}^{n-1}}{2\Delta t}\right\|\left\|\boldsymbol{\phi}_h^{n+1}\right\|\\
&\quad\leq C\left\|\frac{(\boldsymbol{\eta}^{n+1}-\boldsymbol{\eta}^{n-1})+2(\boldsymbol{\eta}^{n+1}-\boldsymbol{\eta}^{n})-2(\boldsymbol{\eta}^{n}-\boldsymbol{\eta}^{n-1})}{2\Delta t}\right\|\left\|\mathbb{D}(\boldsymbol{\phi}_h^{n+1})\right\|\\
&\quad\leq\frac{C}{\Delta t}\left\|\frac{1}{2}\int_{t^{n-1}}^{t^{n+1}}\boldsymbol{\eta}_t\mathrm{d}t+\int_{t^{n}}^{t^{n+1}}\boldsymbol{\eta}_t\mathrm{d}t-\int_{t^{n-1}}^{t^{n}}\boldsymbol{\eta}_t\mathrm{d}t\right\|\left\|\mathbb{D}(\boldsymbol{\phi}_h^{n+1})\right\|\\
&\quad\leq\frac{C}{\Delta t}\left\|\frac{1}{2}\int_{t^{n-1}}^{t^{n+1}}|\boldsymbol{\eta}_t|\mathrm{d}t+\int_{t^{n}}^{t^{n+1}}|\boldsymbol{\eta}_t|\mathrm{d}t+\int_{t^{n-1}}^{t^{n}}|\boldsymbol{\eta}_t|\mathrm{d}t\right\|\left\|\mathbb{D}(\boldsymbol{\phi}_h^{n+1})\right\|\\
&\quad\leq\frac{C}{(\Delta t)^2\nu}\left\|\int_{t^{n-1}}^{t^{n+1}}|\boldsymbol{\eta}_t|\mathrm{d}t\right\|^2+\frac{\nu}{16}\left\|\mathbb{D}(\boldsymbol{\phi}_h^{n+1})\right\|^2\\
&\quad\leq\frac{C}{\nu\Delta t}\int_{t^{n-1}}^{t^{n+1}}\left\|\boldsymbol{\eta}_t\right\|^2dt+\frac{\nu}{16}\left\|\mathbb{D}(\boldsymbol{\phi}_h^{n+1})\right\|^2.
\end{aligned}
\tag{6.48}
$$

Bounding the convective terms in (6.47) requires (3.9) in Lemma 3.1.1, Korn's and Young's inequalities as

$$
\begin{aligned}
&b^*(2\mathbf{u}^n-\mathbf{u}^{n-1},\boldsymbol{\eta}^{n+1},\boldsymbol{\phi}_h^{n+1})\\
&\quad\leq C\left\|\nabla(2\mathbf{u}^n-\mathbf{u}^{n-1})\right\|\left\|\nabla\boldsymbol{\eta}^{n+1}\right\|\left\|\nabla\boldsymbol{\phi}_h^{n+1}\right\|\\
&\quad\leq C(\left\|\nabla\mathbf{u}^n\right\|+\left\|\nabla\mathbf{u}^{n-1}\right\|)\left\|\nabla\boldsymbol{\eta}^{n+1}\right\|\left\|\mathbb{D}(\boldsymbol{\phi}_h^{n+1})\right\|\\
&\quad\leq\frac{C}{\nu}\left\|\nabla\boldsymbol{\eta}^{n+1}\right\|^2(\left\|\nabla\mathbf{u}^n\right\|^2+\left\|\nabla\mathbf{u}^{n-1}\right\|^2)+\frac{\nu}{16}\left\|\mathbb{D}(\boldsymbol{\phi}_h^{n+1})\right\|^2
\end{aligned}
\tag{6.49}
$$

and

$$
\begin{aligned}
&b^*(2\boldsymbol{\eta}^n - \boldsymbol{\eta}^{n-1}, \mathbf{u}_h^{n+1}, \boldsymbol{\phi}_h^{n+1}) \\
&\quad \le C \left\|\nabla(2\boldsymbol{\eta}^n - \boldsymbol{\eta}^{n-1})\right\| \left\|\nabla \mathbf{u}_h^{n+1}\right\| \left\|\nabla \boldsymbol{\phi}_h^{n+1}\right\| \\
&\quad \le C(\|\nabla \boldsymbol{\eta}^n\| + \left\|\nabla \boldsymbol{\eta}^{n-1}\right\|) \left\|\nabla \mathbf{u}_h^{n+1}\right\| \left\|\mathbb{D}(\boldsymbol{\phi}_h^{n+1})\right\| \\
&\quad \le \frac{C}{\nu} \left\|\nabla \mathbf{u}_h^{n+1}\right\|^2 (\|\nabla \boldsymbol{\eta}^n\|^2 + \left\|\nabla \boldsymbol{\eta}^{n-1}\right\|^2) + \frac{\nu}{16} \left\|\mathbb{D}(\boldsymbol{\phi}_h^{n+1})\right\|^2 .
\end{aligned}
\tag{6.50}
$$

With the bound (3.10) in Lemma 3.1.1 and Korn's and Young's inequalities, we bound the next term as

$$
\begin{aligned}
&b^*(2\boldsymbol{\phi}_h^n - \boldsymbol{\phi}_h^{n-1}, \mathbf{u}_h^{n+1}, \boldsymbol{\phi}_h^{n+1}) \\
&\quad \le \frac{1}{2}\big(\left\|2\boldsymbol{\phi}_h^n - \boldsymbol{\phi}_h^{n-1}\right\| \left\|\nabla \mathbf{u}_h^{n+1}\right\|_\infty \left\|\boldsymbol{\phi}_h^{n+1}\right\| + \left\|2\boldsymbol{\phi}_h^n - \boldsymbol{\phi}_h^{n-1}\right\| \left\|\mathbf{u}_h^{n+1}\right\|_\infty \left\|\nabla \boldsymbol{\phi}_h^{n+1}\right\| \big) \\
&\quad \le C\big(\|\boldsymbol{\phi}_h^n\| + \left\|\boldsymbol{\phi}_h^{n-1}\right\| \big) \left\|\mathbb{D}(\boldsymbol{\phi}_h^{n+1})\right\| \big(\left\|\nabla \mathbf{u}_h^{n+1}\right\|_\infty + \left\|\mathbf{u}_h^{n+1}\right\|_\infty \big) \\
&\quad \le \frac{C}{\nu}\big(\|\boldsymbol{\phi}_h^n\|^2 + \left\|\boldsymbol{\phi}_h^{n-1}\right\|^2 \big)\big(\left\|\nabla \mathbf{u}_h^{n+1}\right\|_\infty^2 + \left\|\mathbf{u}_h^{n+1}\right\|_\infty^2 \big) + \frac{\nu}{16} \left\|\mathbb{D}(\boldsymbol{\phi}_h^{n+1})\right\|^2 .
\end{aligned}
\tag{6.51}
$$

For viscous term, we apply Cauchy-Schwarz and Young's inequalities, then obtain

$$
\begin{aligned}
&(2\nu + \nu_T)(\mathbb{D}(\boldsymbol{\eta}^{n+1}), \mathbb{D}(\boldsymbol{\phi}_h^{n+1})) \\
&\quad \le C(2\nu + \nu_T) \left\|\mathbb{D}(\boldsymbol{\eta}^{n+1})\right\|^2 + \frac{2\nu + \nu_T}{2} \left\|\mathbb{D}(\boldsymbol{\phi}_h^{n+1})\right\|^2 .
\end{aligned}
\tag{6.52}
$$

Note that since $\boldsymbol{\phi}_h \in \mathbf{V}_h, (q_h, \nabla.\boldsymbol{\phi}_h) = 0$. The pressure term is bounded by applying Cauchy-Schwarz, Korn's and Young's inequalities:

$$
\begin{aligned}
(p^{n+1}, \nabla \cdot \boldsymbol{\phi}_h^{n+1}) &= (p^{n+1} - q_h, \nabla \cdot \boldsymbol{\phi}_h^{n+1}) \qquad \forall q_h \in Q_h \\
&\le \frac{\nu}{16} \left\|\mathbb{D}(\boldsymbol{\phi}_h^{n+1})\right\|^2 + \frac{C}{\nu} \left\|p^{n+1} - q_h\right\|^2 .
\end{aligned}
\tag{6.53}
$$

We now bound the additional viscous term in (6.47). Recall that we define $P_{\mathbf{L}_H}$ as an L^2-projection operator (to project the all solutions into large scales). Hence, the last equation in (6.4) implies that we can take $\mathbb{G}_H^n = P_{\mathbf{L}_H}\mathbb{D}(\mathbf{u}_h^n)$. After that, by adding and subtracting $P_{\mathbf{L}_H}\mathbb{D}(\mathbf{u}^n)$ and $\mathbb{D}(\mathbf{u}^n)$ to additional viscous term respectively, Cauchy-Schwarz and Young's inequalities, yields the following estimation along with the inverse estimation in Theorem 2.0.7 and the bound (6.30):

$$
\begin{aligned}
&|\nu_T(\mathbb{G}_H^n - \mathbb{D}(\mathbf{u}^{n+1}), \mathbb{D}(\boldsymbol{\phi}_h^{n+1}))| \\
&= \nu_T(P_{\mathbf{L}_H}\mathbb{D}(\mathbf{e}^n) + (I - P_{\mathbf{L}_H})\mathbb{D}(\mathbf{u}^n) + (\mathbb{D}(\mathbf{u}^{n+1}) - \mathbb{D}(\mathbf{u}^n)), \mathbb{D}(\boldsymbol{\phi}_h^{n+1})) \\
&= \nu_T(P_{\mathbf{L}_H}\mathbb{D}(\boldsymbol{\eta}^n), \mathbb{D}(\boldsymbol{\phi}_h^{n+1})) - \nu_T(P_{\mathbf{L}_H}\mathbb{D}(\boldsymbol{\phi}_h^n), \mathbb{D}(\boldsymbol{\phi}_h^{n+1})) \\
&\quad + \nu_T((I - P_{\mathbf{L}_H})\mathbb{D}(\mathbf{u}^n), \mathbb{D}(\boldsymbol{\phi}_h^{n+1})) + \nu_T(\mathbb{D}(\mathbf{u}^{n+1}) - \mathbb{D}(\mathbf{u}^n), \mathbb{D}(\boldsymbol{\phi}_h^{n+1})) \\
&\le \frac{C\nu_T^2}{\nu}\big[\, \|P_{\mathbf{L}_H}\mathbb{D}(\boldsymbol{\eta}^n)\|^2 + \|P_{\mathbf{L}_H}\mathbb{D}(\boldsymbol{\phi}_h^n)\|^2 + \|(I - P_{\mathbf{L}_H})\mathbb{D}(\mathbf{u}^n)\|^2 \\
&\quad + \left\|\mathbb{D}(\mathbf{u}^{n+1}) - \mathbb{D}(\mathbf{u}^n)\right\|^2 \big] + \frac{\nu}{16}\left\|\mathbb{D}(\boldsymbol{\phi}_h^{n+1})\right\|^2 \\
&\le \frac{C\nu_T^2}{\nu}\big[\, \|\mathbb{D}(\boldsymbol{\eta}^n)\|^2 + h^{-2}\|\boldsymbol{\phi}_h^n\|^2 + \|(I - P_{\mathbf{L}_H})\mathbb{D}(\mathbf{u}^n)\|^2 \\
&\quad + \left\|\mathbb{D}(\mathbf{u}^{n+1} - \mathbf{u}^n)\right\|^2 \big] + \frac{\nu}{16}\left\|\mathbb{D}(\boldsymbol{\phi}_h^{n+1})\right\|^2 .
\end{aligned}
\tag{6.54}
$$

Now, we bound the local truncation error $Intp(\mathbf{u}^{n+1}; \boldsymbol{\phi}_h^{n+1})$.

Firstly, we estimate the first term of $Intp(\mathbf{u}^{n+1}; \boldsymbol{\phi}_h^{n+1})$ by using Cauchy-Schwarz, Poincaré-Friedrichs', Korn's, Young's and Hölder's inequalities along with truncation error of BDF2 (2.8) presented in Remark 2.0.3.

$$
\begin{aligned}
&Intp(\mathbf{u}^{n+1}; \boldsymbol{\phi}_h^{n+1}) \\
&= \Big(\frac{3\mathbf{u}^{n+1} - 4\mathbf{u}^n + \mathbf{u}^{n-1}}{2\Delta t} - \mathbf{u}_t(t^{n+1}), \boldsymbol{\phi}_h^{n+1}\Big) \\
&\le \frac{\nu}{16}\left\|\boldsymbol{\phi}_h^{n+1}\right\|^2 + \frac{C}{2\nu}\left\|\frac{3\mathbf{u}^{n+1} - 4\mathbf{u}^n + \mathbf{u}^{n-1}}{2\Delta t} - \mathbf{u}_t(t^{n+1})\right\|^2 \\
&\le \frac{\nu}{16}\left\|\mathbb{D}(\boldsymbol{\phi}_h^{n+1})\right\|^2 + \frac{C}{\Delta t^2 \nu}\left\|\left(\int_{t^{n+1}}^{t^n}(t^n - t)^4\, dt\right)^{\frac{1}{2}}\left(\int_{t^{n+1}}^{t^n}\mathbf{u}_{ttt}^2(t)\, dt\right)^{\frac{1}{2}}\right. \\
&\quad \left. - \left(\int_{t^{n-1}}^{t^{n+1}}(t^{n-1} - t)^4\, dt\right)^{\frac{1}{2}}\left(\int_{t^{n-1}}^{t^{n+1}}\mathbf{u}_{ttt}^2(t)\, dt\right)^{\frac{1}{2}}\right\|^2
\end{aligned}
$$

$$\leq \frac{\nu}{16}\left\|\mathbb{D}(\boldsymbol{\phi}_h^{n+1})\right\|^2 + \frac{C}{\Delta t^2 \nu}\Big[\Big(\int_{t^{n+1}}^{t^n}(t^n - t)^4\,dt\Big)\Big(\int_\Omega \int_{t^{n+1}}^{t^n} \mathbf{u}_{ttt}^2(t)\,dtd\mathbf{x}\Big)$$
$$- \Big(\int_{t^{n-1}}^{t^{n+1}}(t^{n-1} - t)^4\,dt\Big)\Big(\int_\Omega \int_{t^{n-1}}^{t^{n+1}} \mathbf{u}_{ttt}^2(t)\,dtd\mathbf{x}\Big)\Big]$$
$$\leq \frac{\nu}{16}\left\|\mathbb{D}(\boldsymbol{\phi}_h^{n+1})\right\|^2 + \frac{C\Delta t^3}{\nu}\Big(\int_\Omega \int_{t^{n-1}}^{t^{n+1}} \mathbf{u}_{ttt}^2(t)\,dtd\mathbf{x}\Big)$$
$$= \frac{\nu}{16}\left\|\mathbb{D}(\boldsymbol{\phi}_h^{n+1})\right\|^2 + \frac{C\Delta t^3}{\nu}\int_{t^{n-1}}^{t^{n+1}} \|\mathbf{u}_{ttt}\|^2\,dt. \tag{6.55}$$

Now, let's continue with the remaining terms in $Intp(\mathbf{u}^{n+1};\boldsymbol{\phi}_h^{n+1})$.

We start with the rearrangement of the terms. Then, by using the bound (3.9) in Lemma 3.1.1, Young's, Korn's and Hölder's inequalities with truncation error of linear extrapolation (2.12) presented in Remark 2.0.4, we get

$$b^*(2\mathbf{u}^n - \mathbf{u}^n, \mathbf{u}^{n+1}, \boldsymbol{\phi}_h^{n+1}) - b^*(\mathbf{u}^{n+1}, \mathbf{u}^{n+1}, \boldsymbol{\phi}_h^{n+1})$$
$$= b^*(2\mathbf{u}^n - \mathbf{u}^n - \mathbf{u}^{n+1}, \mathbf{u}^{n+1}, \boldsymbol{\phi}_h^{n+1}) - b^*(\mathbf{u}^{n+1}, \mathbf{u}^{n+1} - \mathbf{u}^{n+1}, \boldsymbol{\phi}_h^{n+1})$$
$$= b^*(2\mathbf{u}^n - \mathbf{u}^n - \mathbf{u}^{n+1}, \mathbf{u}^{n+1}, \boldsymbol{\phi}_h^{n+1})$$
$$\leq C\left\|\nabla(2\mathbf{u}^n - \mathbf{u}^n - \mathbf{u}^{n+1})\right\|\left\|\nabla \mathbf{u}^{n+1}\right\|\left\|\nabla\boldsymbol{\phi}_h^{n+1}\right\|$$
$$\leq \frac{\nu}{16}\left\|\mathbb{D}(\boldsymbol{\phi}_h^{n+1})\right\|^2 + \frac{C}{\nu}\left\|\int_{t^{n+1}}^{t^n}(t^n - t)\nabla\mathbf{u}_{tt}\,dt - \int_{t^{n+1}}^{t^n-1}(t^{n-1} - t)\nabla\mathbf{u}_{tt}\,dt\right\|^2\left\|\nabla\mathbf{u}^{n+1}\right\|^2$$
$$\leq \frac{\nu}{16}\left\|\mathbb{D}(\boldsymbol{\phi}_h^{n+1})\right\|^2 + \frac{C}{\nu}\Bigg\|\Big(\int_{t^{n+1}}^{t^n}(t^n - t)^2\,dt\Big)^{1/2}\Big(\int_{t^{n+1}}^{t^n}\nabla\mathbf{u}_{tt}^2\,dt\Big)^{1/2}$$
$$- \Big(\int_{t^{n+1}}^{t^n-1}(t^{n-1} - t)^2\,dt\Big)^{1/2}\Big(\int_{t^{n+1}}^{t^n-1}\nabla\mathbf{u}_{tt}^2\,dt\Big)^{1/2}\Bigg\|^2\left\|\nabla\mathbf{u}^{n+1}\right\|^2$$
$$\leq \frac{\nu}{16}\left\|\mathbb{D}(\boldsymbol{\phi}_h^{n+1})\right\|^2 + \frac{C\Delta t^3}{\nu}\Big(\int_\Omega\int_{t^{n-1}}^{t^{n+1}}\nabla\mathbf{u}_{tt}^2\,dt\,d\mathbf{x}\Big)\left\|\nabla\mathbf{u}^{n+1}\right\|^2$$
$$= \frac{\nu}{16}\left\|\mathbb{D}(\boldsymbol{\phi}_h^{n+1})\right\|^2 + \frac{C\Delta t^3}{\nu}\left\|\nabla\mathbf{u}^{n+1}\right\|^2\int_{t^{n-1}}^{t^{n+1}}\left\|\nabla\mathbf{u}_{tt}\right\|^2\,dt. \tag{6.56}$$

We bound all the terms on the right hand side of (6.47).

$$
\begin{aligned}
&\frac{1}{\Delta t}\left\|\chi_{\boldsymbol{\phi}}^{n}\right\|_{G}^{2}-\frac{1}{\Delta t}\left\|\chi_{\boldsymbol{\phi}}^{n-1}\right\|_{G}^{2}+\frac{1}{4\Delta t}\left\|\boldsymbol{\phi}_h^{n+1}-2\boldsymbol{\phi}_h^{n}+\boldsymbol{\phi}_h^{n-1}\right\|^{2}+\frac{\nu+\nu_T}{2}\left\|\mathbb{D}(\boldsymbol{\phi}_h^{n+1})\right\|^{2}\\
&\quad\le \frac{C}{\nu\Delta t}\int_{t^{n-1}}^{t^{n+1}}\|\boldsymbol{\eta}_t\|^2\,\mathrm{d}t+\frac{C}{\nu}\left\|\nabla\boldsymbol{\eta}^{n+1}\right\|^{2}(\|\nabla\mathbf{u}^n\|^2+\left\|\nabla\mathbf{u}^{n-1}\right\|^2)\\
&\qquad+\frac{C}{\nu}\left\|\nabla\mathbf{u}_h^{n+1}\right\|^{2}(\|\nabla\boldsymbol{\eta}^n\|^2+\left\|\nabla\boldsymbol{\eta}^{n-1}\right\|^2)+\frac{C}{\nu}\big(\|\boldsymbol{\phi}_h^n\|^2+\left\|\boldsymbol{\phi}_h^{n-1}\right\|^2\big)\\
&\qquad\big(\left\|\nabla\mathbf{u}_h^{n+1}\right\|_\infty^{2}+\left\|\mathbf{u}_h^{n+1}\right\|_\infty^{2}\big)+C(2\nu+\nu_T)\left\|\mathbb{D}(\boldsymbol{\eta}^{n+1})\right\|^2\\
&\qquad+\frac{C}{\nu}\left\|p^{n+1}-q_h\right\|^2+\frac{C\nu_T^2}{\nu}\big[\,\|\mathbb{D}(\boldsymbol{\eta}^n)\|^2+h^{-2}\|\boldsymbol{\phi}_h^n\|^2+\|(I-P_{\mathbf{L}_H})\mathbb{D}(\mathbf{u}^n)\|^2\\
&\qquad+\left\|\mathbb{D}(\mathbf{u}^{n+1}-\mathbf{u}^n)\right\|^2\big]+\frac{C\Delta t^3}{\nu}\int_{t^{n-1}}^{t^{n+1}}\|\mathbf{u}_{ttt}\|^2\,dt\\
&\qquad+\frac{C\Delta t^3}{\nu}\left\|\nabla\mathbf{u}^{n+1}\right\|^2\int_{t^{n-1}}^{t^{n+1}}\|\nabla\mathbf{u}_{tt}\|^2\,dt. \qquad (6.57)
\end{aligned}
$$

Now, sum from $n=1$ and $n=N-1$, and multiply both sides by Δt

$$
\begin{aligned}
&\left\|\chi_{\boldsymbol{\phi}}^{N-1}\right\|_{G}^{2}-\left\|\chi_{\boldsymbol{\phi}}^{0}\right\|_{G}^{2}+\frac{1}{4}\sum_{n=1}^{N-1}\left\|\boldsymbol{\phi}_h^{n+1}-2\boldsymbol{\phi}_h^{n}+\boldsymbol{\phi}_h^{n-1}\right\|^{2}+\Delta t\frac{\nu+\nu_T}{2}\sum_{n=1}^{N-1}\left\|\mathbb{D}(\boldsymbol{\phi}_h^{n+1})\right\|^{2}\\
&\quad\le C\Delta t\sum_{n=1}^{N-1}\Bigg[\frac{1}{\nu\Delta t}\int_{t^{n-1}}^{t^{n+1}}\|\boldsymbol{\eta}_t\|^2\,\mathrm{d}t+\frac{1}{\nu}\left\|\nabla\boldsymbol{\eta}^{n+1}\right\|^{2}(\|\nabla\mathbf{u}^n\|^2+\left\|\nabla\mathbf{u}^{n-1}\right\|^2)\\
&\qquad+\frac{1}{\nu}\left\|\nabla\mathbf{u}_h^{n+1}\right\|^{2}(\|\nabla\boldsymbol{\eta}^n\|^2+\left\|\nabla\boldsymbol{\eta}^{n-1}\right\|^2)+\frac{1}{\nu}\big(\|\boldsymbol{\phi}_h^n\|^2+\left\|\boldsymbol{\phi}_h^{n-1}\right\|^2\big)\\
&\qquad\big(\left\|\nabla\mathbf{u}_h^{n+1}\right\|_\infty^{2}+\left\|\mathbf{u}_h^{n+1}\right\|_\infty^{2}\big)+(2\nu+\nu_T)\left\|\mathbb{D}(\boldsymbol{\eta}^{n+1})\right\|^2+\frac{1}{\nu}\left\|p^{n+1}-q_h\right\|^2\\
&\qquad+\frac{\nu_T^2}{\nu}\big[\,\|\mathbb{D}(\boldsymbol{\eta}^n)\|^2+h^{-2}\|\boldsymbol{\phi}_h^n\|^2+\|(I-P_{\mathbf{L}_H})\mathbb{D}(\mathbf{u}^n)\|^2+\left\|\mathbb{D}(\mathbf{u}^{n+1}-\mathbf{u}^n)\right\|^2\big]\\
&\qquad+\frac{\Delta t^3}{\nu}\int_{t^{n-1}}^{t^{n+1}}\|\mathbf{u}_{ttt}\|^2\,dt+\frac{\Delta t^3}{\nu}\left\|\nabla\mathbf{u}^{n+1}\right\|^2\int_{t^{n-1}}^{t^{n+1}}\|\nabla\mathbf{u}_{tt}\|^2\,dt\Bigg]. \qquad (6.58)
\end{aligned}
$$

Using the bound (6.24) for G-norm obtained from Lemma 6.1.1

$$
\begin{aligned}
&\|\phi_h^N\|^2 + \frac{1}{3}\sum_{n=1}^{N-1}\|\phi_h^{n+1} - 2\phi_h^n + \phi_h^{n-1}\|^2 + \Delta t\frac{2\nu + 2\nu_T}{3}\sum_{n=1}^{N-1}\|\nabla\phi_h^{n+1}\|^2 \\
&\quad\leq \frac{1}{3}\|\phi_h^{N-1}\|^2 + \frac{4}{3}\|\chi_\phi^0\|_G^2 + C\Delta t\sum_{n=1}^{N-1}\Bigg[\frac{1}{\nu\Delta t}\int_{t^{n-1}}^{t^{n+1}}\|\eta_t\|^2\,\mathrm{d}t \\
&\qquad + \frac{1}{\nu}\|\nabla\eta^{n+1}\|^2(\|\nabla\mathbf{u}^n\|^2 + \|\nabla\mathbf{u}^{n-1}\|^2) + \frac{1}{\nu}\|\nabla\mathbf{u}_h^{n+1}\|^2(\|\nabla\eta^n\|^2 + \|\nabla\eta^{n-1}\|^2) \\
&\qquad + \frac{1}{\nu}\big(\|\phi_h^n\|^2 + \|\phi_h^{n-1}\|^2\big)\big(\|\nabla\mathbf{u}_h^{n+1}\|_\infty^2 + \|\mathbf{u}_h^{n+1}\|_\infty^2\big) + (2\nu + \nu_T)\|\mathbb{D}(\eta^{n+1})\|^2 \\
&\qquad + \frac{1}{\nu}\|p^{n+1} - q_h\|^2 + \frac{\nu_T^2}{\nu}\big[\|\mathbb{D}(\eta^n)\|^2 + h^{-2}\|\phi_h^n\|^2 + \|(I - P_{\mathbf{L}_H})\mathbb{D}(\mathbf{u}^n)\|^2 \\
&\qquad + \|\mathbb{D}(\mathbf{u}^{n+1} - \mathbf{u}^n)\|^2\big] + \frac{\Delta t^3}{\nu}\int_{t^{n-1}}^{t^{n+1}}\|\mathbf{u}_{ttt}\|^2\,dt + \frac{\Delta t^3}{\nu}\|\nabla\mathbf{u}^{n+1}\|^2\int_{t^{n-1}}^{t^{n+1}}\|\nabla\mathbf{u}_{tt}\|^2\,dt\Bigg].
\end{aligned}
\tag{6.59}
$$

With approximation properties (6.37) and (6.38), we get the following estimation by induction on N:

$$
\begin{aligned}
&\|\phi_h^N\|^2 + \frac{1}{3}\sum_{n=1}^{N-1}\|\phi_h^{n+1} - 2\phi_h^n + \phi_h^{n-1}\|^2 + \Delta t\frac{2\nu + 2\nu_T}{3}\sum_{n=1}^{N-1}\|\nabla\phi_h^{n+1}\|^2 \\
&\leq \Big(\frac{1}{3}\Big)^N\|\phi_h^0\|^2 + 2\Big(1 - \big(\tfrac{1}{3}\big)^N\Big)\Bigg[\|\chi_\phi^0\|_G^2 + C\Big(\nu^{-1}h^{2k+2}|||\mathbf{u}_t|||_{2,k+1}^2 \\
&+ \nu^{-1}h^{2k}|||\mathbf{u}|||_{2,k+1}^2|||\nabla\mathbf{u}|||_{\infty,0}^2 + (2\nu + \nu_T)h^{2k}|||\mathbf{u}|||_{2,k+1}^2 + \nu^{-1}h^{2s+2}|||p|||_{2,s+1}^2 \\
&+ \nu^{-1}\Delta t^4|||\mathbf{u}_{ttt}|||_{2,0}^2 + \nu^{-1}\Delta t^4|||\nabla\mathbf{u}|||_{\infty,0}^2|||\nabla\mathbf{u}_{tt}|||_{2,0}^2 + \nu^{-1}\nu_T^2h^{2k}|||\mathbf{u}|||_{2,k+1} \\
&+ \nu^{-1}\nu_T^2H^{2k}|||\mathbf{u}|||_{2,k+1} + \nu^{-1}\nu_T^2\Delta t^2|||\mathbf{u}_t|||_{\infty,0}\Big) + C\Delta t\nu^{-1}(1 + \nu_T^2h^{-2})\sum_{n=0}^{N-1}\|\phi_h^n\|^2\Bigg].
\end{aligned}
\tag{6.60}
$$

Note that, with the typical choice $\nu_T = h$, the multiplier of last summation in (6.60) is constant. We use this choice for only that term. After obtaining the error bound of our method, as in the Corollary 6.4.1 we use this choice for all ν_T.

After then by Lemma 2.0.1,

$$\|\phi_h^N\|^2 + \frac{1}{3}\sum_{n=1}^{N-1}\|\phi_h^{n+1} - 2\phi_h^n + \phi_h^{n-1}\|^2 + \Delta t\frac{2\nu + 2\nu_T}{3}\sum_{n=1}^{N-1}\|\nabla\phi_h^{n+1}\|^2$$

$$\leq exp\Big(C\nu^{-1}T\Big)\Bigg[\Big(\frac{1}{3}\Big)^N\|\phi_h^0\|^2 + 2\Big(1-\big(\frac{1}{3}\big)^N\Big)\|\chi_\phi^0\|_G^2 + C\Big(1-\big(\frac{1}{3}\big)^N\Big)$$

$$\Bigg(\nu^{-1}h^{2k+2}\,|||\mathbf{u}_t|||_{2,k+1}^2 + \nu^{-1}h^{2k}\,|||\mathbf{u}|||_{2,k+1}^2\,|||\nabla\mathbf{u}|||_{\infty,0}^2 + (2\nu+\nu_T)h^{2k}\,|||\mathbf{u}|||_{2,k+1}^2$$

$$+ \nu^{-1}h^{2s+2}\,|||p|||_{2,s+1}^2 + \nu^{-1}\Delta t^4\,|||\mathbf{u}_{ttt}|||_{2,0}^2 + \nu^{-1}\Delta t^4\,|||\nabla\mathbf{u}|||_{\infty,0}^2\,|||\nabla\mathbf{u}_{tt}|||_{2,0}^2$$

$$+ \nu^{-1}\nu_T^2h^{2k}\,|||\mathbf{u}|||_{2,k+1} + \nu^{-1}\nu_T^2H^{2k}\,|||\mathbf{u}|||_{2,k+1} + \nu^{-1}\nu_T^2\Delta t^2\,|||\mathbf{u}_t|||_{\infty,0}\Bigg)\Bigg] \quad (6.61)$$

Note that, the triangle inequality implies that

$$\|\mathbf{e}^n\| = \|\boldsymbol{\eta}^n - \phi_h^n\| \leq \|\boldsymbol{\eta}^n\| + \|\phi_h^n\| \quad (6.62)$$

Thus, by using the bound (6.25) for G-norm obtained from Lemma 6.1.1, and the triangle inequality (6.62), we obtain the required result (6.40). □

6.5 Numerical Experiments

In this part, three examples will be considered in the numerical experiments to test the scheme (6.1)-(6.3). The first experiment will verify the predicted order of convergence rates which were proven in the previous section. The second and third experiments are well-known driven cavity flow and flow around cylinder problems, respectively. They will be tested to demonstrate the efficiency of the method. All numerical experiments are done by using the software Freefem++ [22] by considering Taylor-Hood finite element spaces (P_2, P_1).

6.5.1 Convergence rate verification

To verify convergence rates for our numerical scheme (6.1)-(6.3), we consider NSE problem in unit square domain $\Omega = [0, 1]^2$ with analytical solution such that

$$\mathbf{u} = \begin{bmatrix} (1+0.01t)sin(2\pi y) \\ (1+0.01t)cos(2\pi x) \end{bmatrix}, \qquad p = x + y$$

given in [7]. Simulations are performed with the kinematic viscosity $\nu = 1$. By substituting analytic solution to momentum equation of NSE (3.1) with these data, we obtain the external force. Also, all boundary conditions are determined by analytical solutions. By using (P_2, P_1) on uniform meshes, we compute errors and rates by changing mesh width h and time step Δt simultaneously, when the final time $T = 0.01$. Uniform triangular grids are used and the coarse mesh size is determined with $H = \sqrt{h}$ and $\nu_T = h^2$. All computations are listed in Table 6.1. All values of rates are around 2. Hence, we can observe optimal convergence rates which are mentioned in Corollary 6.4.1.

Table 6.1: Errors and convergence rates for the projection-based VMS method on linearly extrapolated BDF2 scheme

h	Δt	$\|\mathbf{u} - \mathbf{u}_h\|_{2,1}$	Rate
$1/4$	T	$5.2366e-2$	-
$1/8$	$T/2$	$1.3363e-2$	1.9704
$1/16$	$T/4$	$3.3530e-3$	1.9947
$1/32$	$T/8$	$8.3846e-4$	1.9997
$1/64$	$T/16$	$2.0968e-4$	1.9995
$1/128$	$T/32$	$5.2390e-5$	2.0008

6.5.2 Driven Cavity Problem

Now, we will test our numerical scheme, the projection-based VMS method along with linearly extrapolated BDF2, to solve the driven cavity problem. The driven cavity problem is very useful to test how our scheme works when simulating fluid flows. Moreover, it is highly preferable, because of the practicality when implementing.

We consider two dimensional driven cavity flow. It is modeled on a square domain. The top of the domain moves with velocity $\mathbf{u} = (1, 0)$ along the length, and other sides of the domain are stationary. Hence, two top corners of the boundary are singular points. Also, the other two corners are weakly singular. These singularities are important, because in nature, two dimensional fluid flows in cornered structured is very common [3]. Hence, the driven cavity problem serves a good understanding for these kind of fluid flows.

We consider the unit square domain $[0,1] \times [0,1]$. As it is seen in Figure 6.1, at the top boundary $0 < x < 1$ and $y = 1$, the velocity of the flow is $\mathbf{u} = (1,0)$, and at other boundaries, there are no-slip boundary condition.

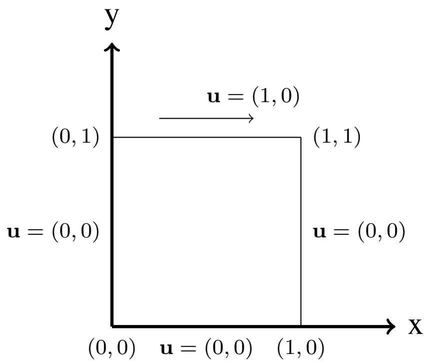


Figure 6.1: Driven Cavity Flow

First of all, we investigate the behavior of cavity flow for different Reynolds numbers which are $Re = 1, 100, 400, 1000, 5000$. The streamlines and velocity vectors are plotted for each Re in Figures 6.2-6.3 by taking 40×40 grid points on the domain. As long as the value of Re increases, the center of velocity vector circulation moves away from the driven wall and nearly approaches to the geometric center of cavity. At the same time, we observe reverse circulations in lower corners, when Re increases. Similarly, streamlines' centers come close to the geometric center of cavity by becoming circular. All of these results in Figures 6.2 - 6.3 are compared with the Akin's results presented in [3], and good matches are observed with them.

Secondly, we examine the values of velocity along the vertical and horizontal lines passing through the geometric center of cavity. For three different Reynolds numbers ($Re = 100, 400, 1000$), we draw profiles of x component of the velocity along the line $x = 0.5$, and y component of the velocity along the line $y = 0.5$ by comparing them with the data of Ghia et al. [16].

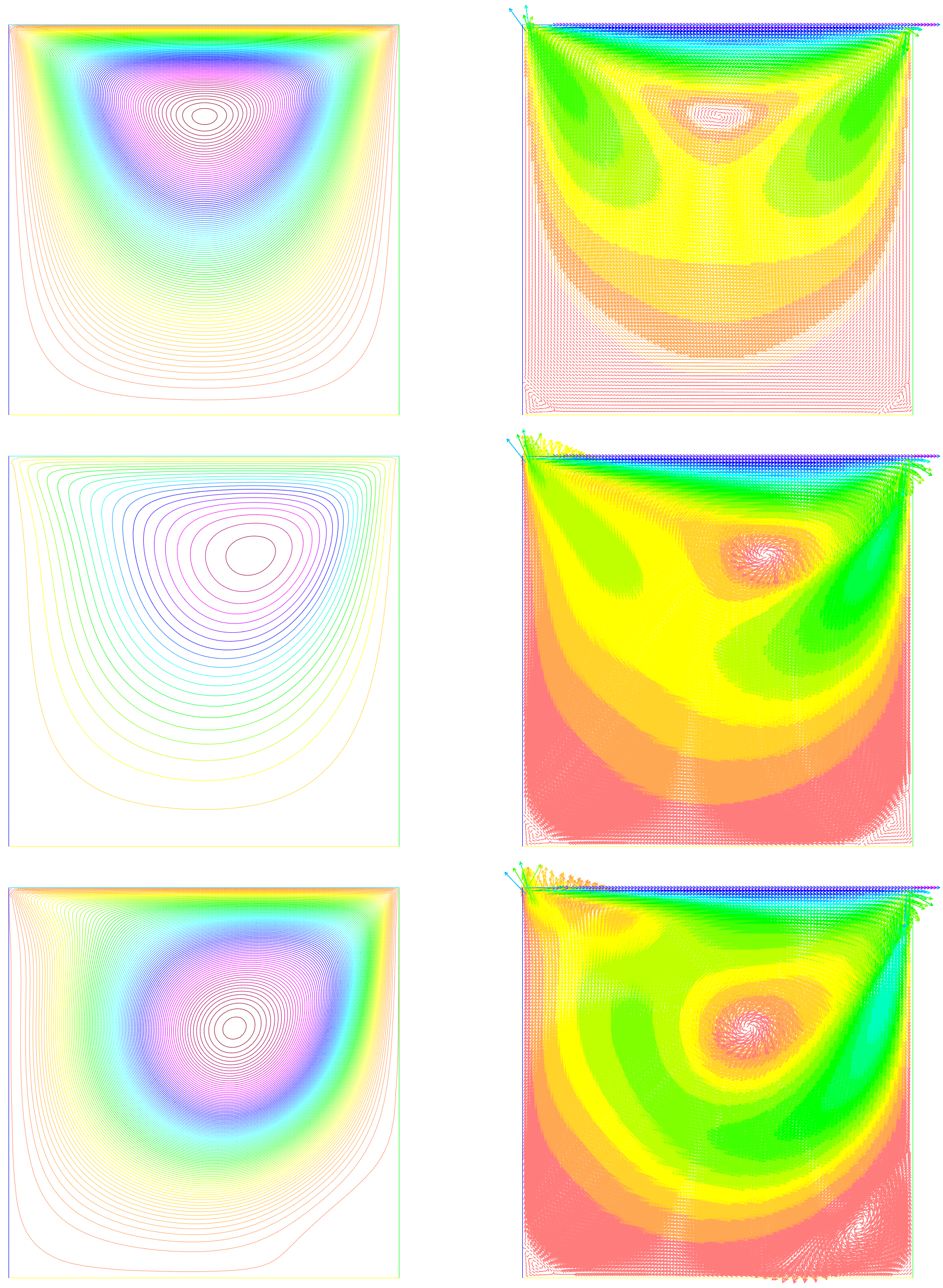


Figure 6.2: Streamlines (left) and velocity vectors (right) for each $Re = 1, 100, 400$ (form up to down).

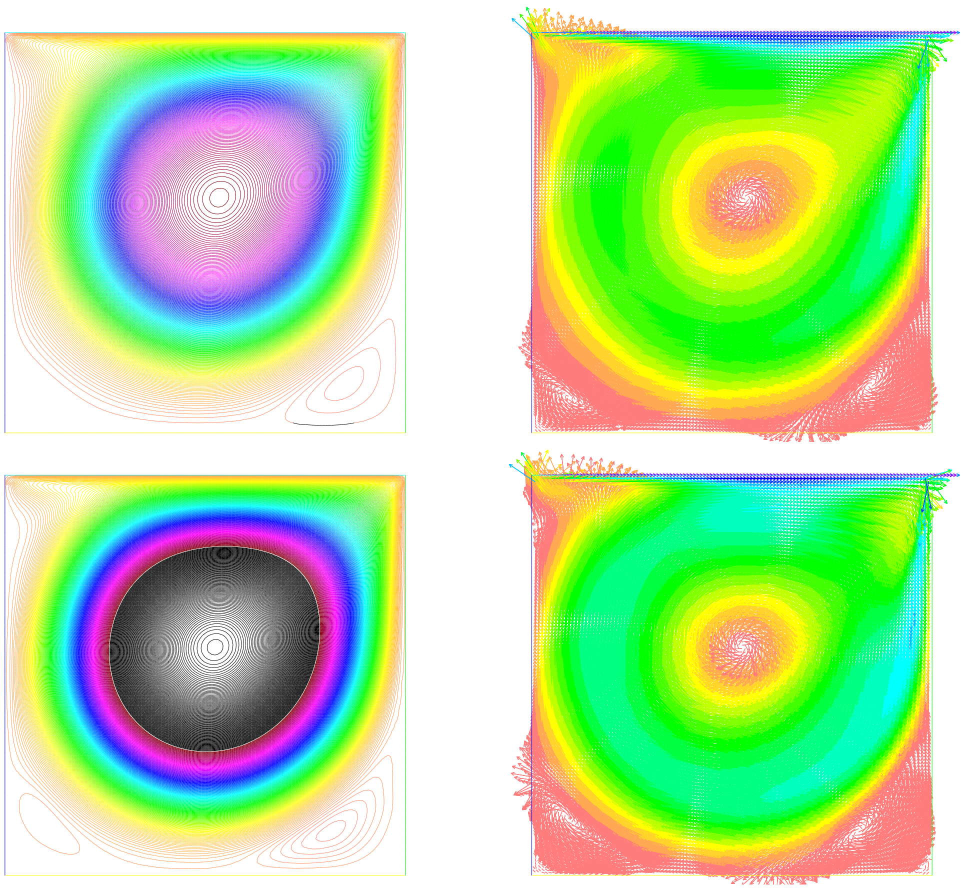


Figure 6.3: Streamlines (left) and velocity vectors (right) for each $Re = 1000, 5000$ (form up to down).

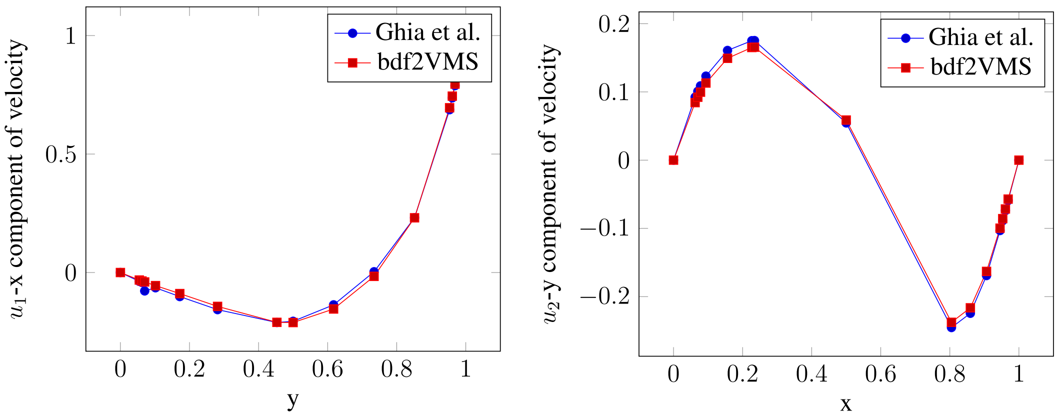


Figure 6.4: Change in velocity along vertical and horizontal midlines for $Re = 100$.

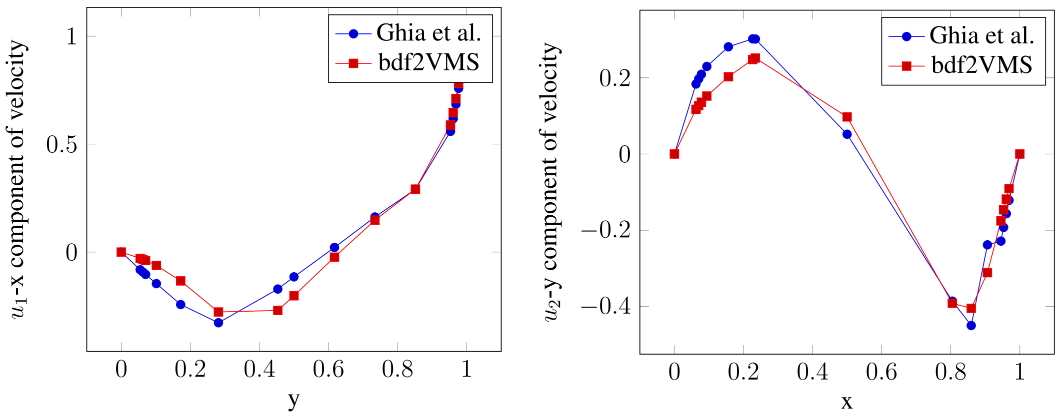


Figure 6.5: Change in velocity along vertical and horizontal midlines for $Re = 400$.

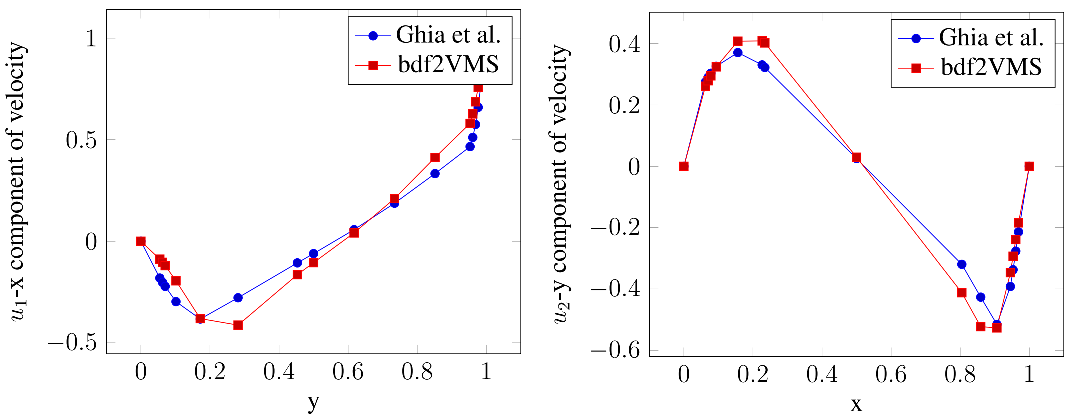


Figure 6.6: Change in velocity along vertical and horizontal midlines for $Re = 1000$.

Even if we use coarser mesh than Ghia et al.'s mesh (obtained with 129×129 grid points), the good agreement is satisfied between Ghia's data in [16] and our data obtained by implementing the scheme (6.1)-(6.3) as it is seen in Figures 6.4, 6.5, 6.6.

6.5.3 Flow Around a Cylinder

The third example is considered to test efficiency of the scheme (6.1)-(6.3) is the flow passing through the channel with a cylinder depicted in Figure 6.7. This well-known problem is taken from the papers [34, 46]. Taylor-Hood element (P_2, P_1) is used with 31295 total degrees of freedom. The time dependent inflow and outflow profile presented as

$$\begin{aligned}\mathbf{u}_1(0,y,t) =& \mathbf{u}_1(2.2,y,t) = \frac{6}{0.41^2}\sin(\frac{\pi t}{8})y(0.41-y),\\ \mathbf{u}_2(0,y,t) =& \mathbf{u}_2(2.2,y,t) = 0. \end{aligned} \tag{6.63}$$

At rest of all boundaries, there are no-slip condition. Additionally, we take kinematic viscosity $\nu = 10^{-3}$ and external force $\mathbf{f} = 0$. Also, an additional viscosity is determined by $\nu_T = h^2$ and coarse mesh size $H = \sqrt{h}$.

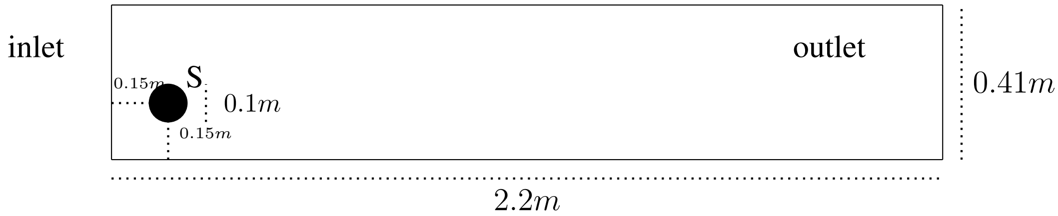


Figure 6.7: Domain Ω of the test problem

With this problem, we observe the change of velocity vectors at time $t = 2,\ 4,\ 5,\ 6,\ 7$ and 8 for the time step $\Delta t = 0.005$ and final time $T = 8$ with the Figure 6.8. As time goes on, two vortices start to develop behind the cylinder seen in the velocity profile at $t = 2$ and 4. After $t = 4$, as in the profile at $t = 5$, development of the vortex street is observed. Also we still see these vortices at the final time $T = 8$. These results agree with the results of [34, 46].

We deal with the drag $c_d(t)$ and lift $c_l(t)$ values at cylinder, and pressure difference $\Delta p(t) = p(t; 0.15, 0.2) - p(t; 0.25, 0.2)$ at a final time $T = 8$. Definitions of these values presented in [46] as follow:

$$\begin{aligned} c_d(t) =& \frac{2}{\rho L U_{max}^2}\int_S \big(\rho\nu\frac{\partial \mathbf{u}_{ts}}{\partial n}n_y - p(t)n_x\big)dS,\\ c_l(t) =& -\frac{2}{\rho L U_{max}^2}\int_S \big(\rho\nu\frac{\partial \mathbf{u}_{ts}}{\partial n}n_x + p(t)n_y\big)dS, \end{aligned}$$

where U_{max} is the maximum mean flow, L is the diameter of the cylinder, $n = (n_x, n_y)^T$ is the normal vector on S and $\mathbf{u}_{ts}$ is the tangential velocity.

We obtain the evolution profile of drag $c_d(t)$, lift $c_l(t)$ and pressure difference $\Delta p(t)$ values in Figure 6.9, when we implement the scheme (6.1)-(6.3). These graphs clearly coincide with the results presented in [34].

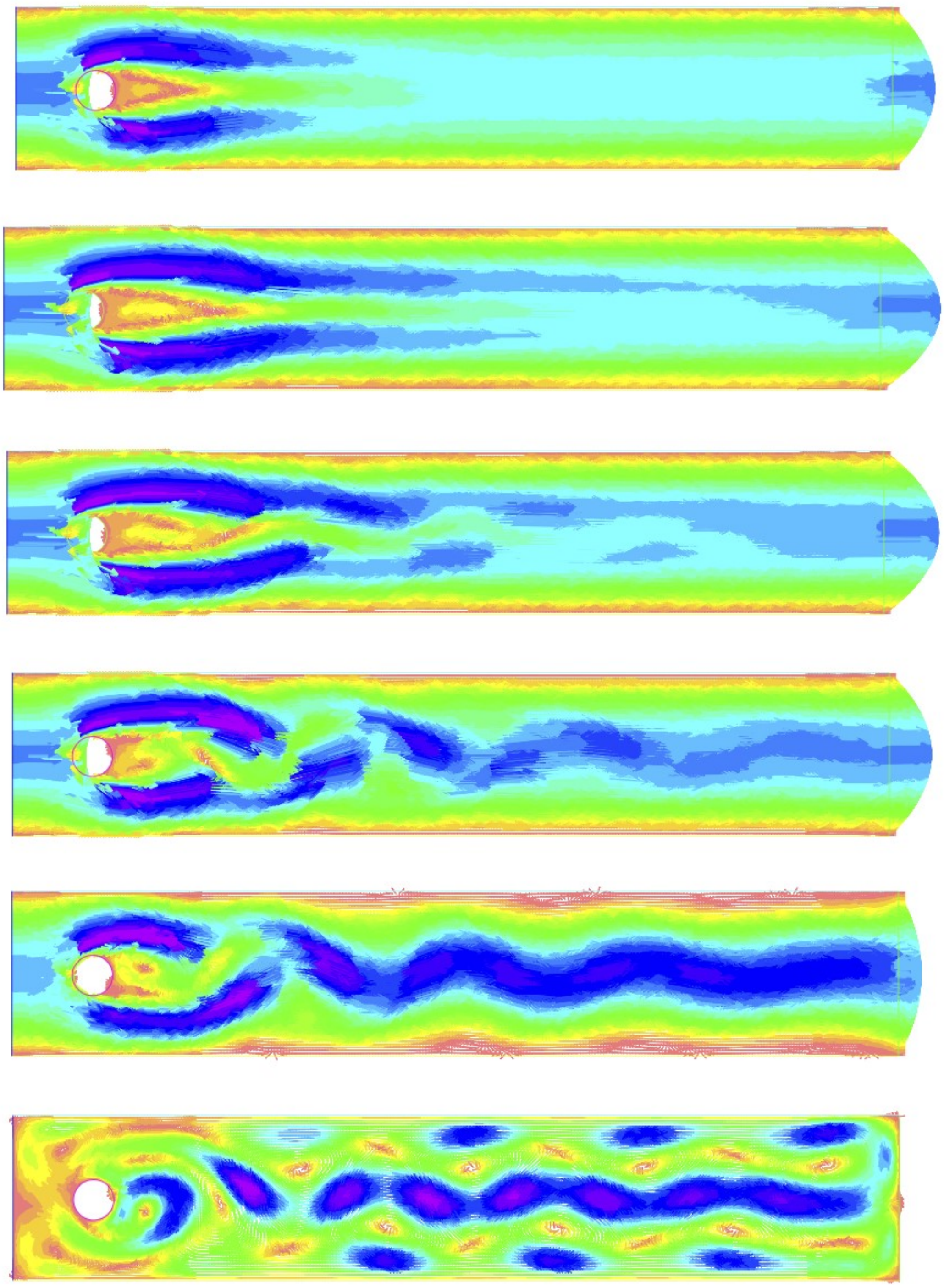
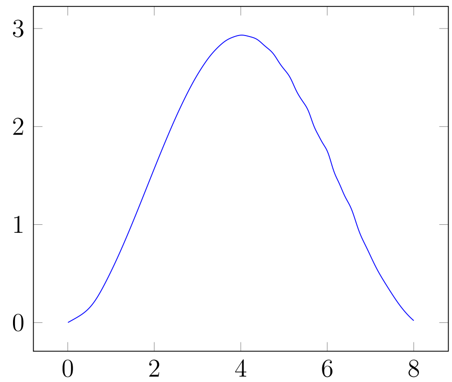
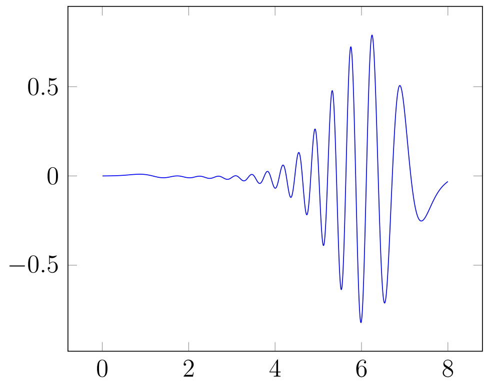


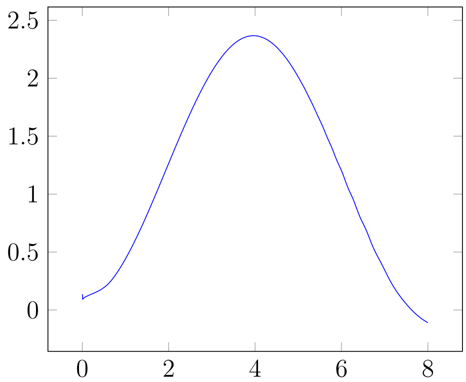
Figure 6.8: The velocity at $t = 2,\ 4,\ 5,\ 6,\ 7,\ 8$ by the scheme (6.1)-(6.3) (from up to down).



(a) Evolution of $c_{d,max}$



(b) Evolution of $c_{l,max}$



(c) Evolution of Δp

Figure 6.9: Evolution of maximum value of drag values, lift values and pressure differences obtained when using the scheme (6.1)-(6.3) with $\Delta t = 0.005$.

CHAPTER 7

SUMMARY AND CONCLUSION

In this thesis, the projection-based Variational Multiscale (VMS) method is considered based on linearly extrapolated BDF2 time discretization for the Navier-Stokes equations.

First, we presented the basic theorems, lemmas and definitions. After that, we started our review with explaining basic notion of the projection based VMS method, by giving the three-scale VMS method. Then, we explained the idea of projection-based VMS method in every respect. Afterward, we used the linearly extrapolated second order backward difference formula BDF2 to build the fully time discretization. After obtaining the fully discrete problem, we studied on stability and error analysis of it. When working on stability analysis, we utilized the G-stability of BDF2. Furthermore, by using Lax- Milgram theorem, we established the existence and uniqueness of the fully discrete problem's solution. Lastly, we performed Numerical tests to verify the stability and accuracy of the method.

With the light of these studies, we yielded that following remarks: Fully discrete algorithm obtained with projection-based VMS method with linearly extrapolated BDF2,

(i) is unconditionally stable,

(ii) provides a unique solution,

(iii) with some regularity assumptions on exact solution, the error obtained when simulating algorithm is bounded. In other words, the method is convergent.

(iv) When the finite element spaces satisfy the inf-sup condition (3.14), the optimal convergence rates are obtained.

(v) The use of projection-based VMS method gives expected accuracy of the solution and physically correct behavior for benchmark problems namely driven cavity flow and flow around cylinder.

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