EQUIVARIANT REDUCTION OF MATRIX GAUGE THEORIES AND EMERGENT CHAOTIC DYNAMICS

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ABSTRACT

EQUIVARIANT REDUCTION OF MATRIX GAUGE THEORIES AND EMERGENT CHAOTIC DYNAMICS

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In this thesis we focus on a massive deformation of a Yang-Mills matrix gauge theory. We first layout the essential features of this model including fuzzy 4- sphere extremum of the mass deformed potential as well as its relation with string theoretic matrix models such as the BFSS model. Starting with such a model with $U(4N)$ gauge symmetry, we determine the $SU(4)$ equivariant fluctuations modes. We trace over the fuzzy 4-spheres at the matrix levels $N = \frac{1}{6}(n+1)(n+2)(n+3)$, $(n : 1, 2 \ldots 5)$ and obtain the corresponding low energy effective actions(LEA).This reduction over fuzzy 4-sphere breaks the $U(4)$ gauge symmetry down to $U(1) \times U(1)$, which is further broken to $\mathbb{Z}_2 \times \mathbb{Z}_2$ by the Gauss Law constraint on the gauge fields. We solve numerically the Hamilton's equations of motions for the corresponding phase space variables and using the latter obtain the Lyapunov exponents, from which we conclude the presence of chaotic dynamics in the LEA. Finally in the Euclidean time, we also find that the reduced LEA's have kink solutions with topological charges in $\mathbb{Z}_2 \times \mathbb{Z}_2$

Keywords: Matrix Models, Fuzzy Spaces, Yang Mills Models, Mass deformed matrix models

ÖZ

MATRİS AYAR TEORİLERİNİN SİMETRİK İNDİRGENMESİ VE KAOTİK DİNAMİĞİ

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Bu tezde, ilk olarak kütle deformasyonu taşıyan Yang Mills matris ayar teorilerine odaklanıldı. Bu modelin kütle bozunumlu potansiyelinin ekstremumu olan fuzzy 4-küre konfigürasyonları ile sicim teorisi kaynaklı matris teorileri, örneğin BFSS modeli, ile ilişkisi ana hatlarıyla ortaya konuldu. $U(4N)$ ayar simetrisini taşıyan bir modelden başlıyarak, $SU(4)$ simetrik salınım modlarını elde ettik. $N = \frac{1}{6}(n+1)(n+2)(n+3)$, $(n = 1, \ldots, 5)$ matris mertebelerindeki fuzzy 4-küreler üzerinde iz işlemi yapılarak, bu mertebelere denk düşen düşük enerjili etkin eylemleri hesapladık. Faz uzayı değişkenlerinin Hamilton hareket denklemlerini nümerik metotlar ile çözüp Lyapunov üstlerini de elde ettik. Bu bilgilerin ışığında ilgili düşük enerjili etkin eylemlerin kaotik dinamiği olduğu sonucuna vardık. Son olarak, Öklidyen zaman iminde düşük enerjili etkin eylemlerin $1+0$ boyutta bulunan tipik yapıda instanton çözümleri taşıdığını da gösterdik.

Anahtar Kelimeler: Matris Modelleri, Fuzzy Uzaylar, Yang Mills Modelleri

To my family

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IRR	Irreducible Representation
LEA	Low Energy Action
LLE	Largest Lyapunov Exponent
KSE	Kolmogorov Sinai Entropy
$\mu, \nu, \rho \ldots$	In Chapter 2 takes the following values $= (1 \ldots 3)$
$a, b, c, \ldots$	In Chapter 3 takes the following values $= (1, \ldots 5)$
$A, B, C, \ldots$	In Chapter 3 takes the following values $= (1, \ldots 6)$

CHAPTER 1

INTRODUCTION

Matrix gauge theories occupy an important place in current research in theoretical physics, due to their various connections with M-theory and String theories. Among these models, it may be useful to mention Banks-Fischler-Shenker-Suskind (BFSS)[1] model, since the matrix model studied in this thesis is strongly tied to it as will be explained in detail in Chapter 2. BFSS model is a supersymmetric quantum mechanics matrix model, whose bosonic part contains $N \times N$ matrices transforming under the adjoint representation of a local $U(N)$ gauge symmetry. These nine matrices coupled with a single gauge field, through a covariant derivation and they couple each other with a fourth order potential term. The model is invariant under a $U(N)$ gauge symmetry and also under a global $SO(9)$ symmetry, which is a rigid rotation of the nine matrices.

The matrix entries depend on time only. We will only focus on the bosoinc part of this model, therefore we do not discuss its fermionic content. This model arises discrete light-cone quantization of M-theory on flat backgrounds(DLCQ). The massive deformation of this model is known as the BMN matrix model[2], and it arises as the DLCQ of M-theory on certain curved backgrounds. BMN model has fuzzy 2 spheres and their direct sum as vacuum solutions. These matrix models provide a description of the dynamics of N coincident $D0$-branes, which has the dual description of a black hole in the large N and strong coupling limit[3]. Recently, BFSS & BMN matrix models have been subject to various studies exploring their chaotic dynamics at the large temperature i.e. near classical level, with the motivation of gaining further insights on the black hole description in the gravity dual [4, 3, 5, 6]

In this thesis, we focus on a massive deformation of Yang-Mills matrix gauge theory with 5 matrices. This may be seen as a massive deformation of a subsector of the BFSS model as we will explain in Chapter 2. In contrast with fuzzy 2-spheres appearing as classical solutions in the BMN theory this model have classical solutions which are fuzzy four spheres[7, 8, 9]. Our main objective in the thesis is to examine certain fluctuations satisfying symmetry constraints about this classical configuration and utilizing their explicit form to obtain reduced effective actions. Subsequently, we reveal the chaotic dynamics emerging from these effective actions by calculating their Lyapunov exponents using numerical solutions to their Hamilton's equations of motion.

Next, we provide a detail outline of the thesis and a brief summary of our results. In Chapter 2, we provide a general discussion of the fuzzy spaces[10, 11, 12], first focusing on the construction and properties of the simplest of all fuzzy spaces, namely the fuzzy sphere. After giving a brief description on how this construction generalizes to complex projective spaces $\mathbb{C}P^N$ and in particular, $\mathbb{C}P^2$, we focus on the central subject of this thesis which is an extensive review of the construction of the fuzzy 4-sphere[13, 14, 15, 16]. This chapter also includes a detailed description of Yang-Mills matrix model with 5-matrices and massive and/or Chern-Simmons(CS) type deformation terms. In particular, various features of the work of [13] on this matrix model with CS term is reviewed here. Chapter 3 contains all the original results obtained in the study in collaboration with Ü. H. Coşkun, S. Kürkçüoğlu & G. Ünal [17]. Here we determine the equivariant parameterizations of the gauge fields and the fluctuations of a mass-deformed $U(4N)$ Yang-Mills matrix model about the four concentric fuzzy 4-sphere configurations. The latter are solutions of the matrix model only for negative value ($\mu^2 = -8$) of the mass squared which may be an indication of instabilities. Nevertheless the low energy effective actions(LEA) that we obtain by performing traces over the equivariantly symmetric configurations at several different matrix levels(sizes) all have potentials which are bounded from below, which implies that the negativity of μ^2 do not lead to instabilities under equivariant fluctuations. After dimensional reduction $U(4)$ gauge symmetry of the concentric S_F^4 configuration is reduced to $U(1) \times U(1)$. Since the gauge fields in the LEAs are not dynamical, their equations of motion yield not differential but algebraic equations, which are constraint equations. They are known as the Gauss law constraints in the literature[4]. Solving these equations turns out to be equal to enforcing the two complex fields in the LEAs to have zero charge, that is they are real under the abelian gauge fields. This breaks $U(1) \times U(1)$ further down to $\mathbb{Z}_2 \times \mathbb{Z}_2$. This chapter contains our findings providing ample evidence for the emergent Chaotic dynamics of the LEAs. In particular, we find the Lyapunov spectrum at several different energies, and give a number of plots demonstrating the time development of the Lyapunov exponents which all converge to fixed values well before the computation time used in numerical calculations. (3.1) summarizes our numerical findings for the Lyapunov exponents. In the last section of this chapter we consider the structure of the LEA's in the Euclidean signature, and exhibit that they have kink solutions i.e. instantons in $1 + 0$ dimensions[18, 19, 20]. Chapter 4 gives a summary of the results obtained & conclusion reached in this thesis.

CHAPTER 2

INTRODUCTION TO FUZZY SPACES

In this chapter we introduce fuzzy spaces and explain our interest in them. In this thesis we will focus on on Yang-Mills theories with fuzzy four sphere configurations. In order to understand the Fuzzy S^4, first we have to familiarize ourselves with the fuzzy S^2 and $\mathbb{C}P^3$ which are more elementary examples of fuzzy spaces and they will be of use in the construction of fuzzy S^4 and understanding its detailed structure. Lets start with stating some of our motivations to study fuzzy and non-commutative geometries (First of all use of non commutative theory emerges in the context of the String theory from the D brane configurations. In the theory of N coincident D- branes described by the $U(n)$ Yang-Mills theory. Coordinates of this theory are represented by $U(n)$ matrices and expectedly they are non-commutative). A more intuitive motivation for studying fuzzy spaces is that as we dwell into smaller scales, space time itself behaves quantized which can be understood in the context of non-commutative geometry. Since we will construct S_F^4 as a S^2 bundle over $\mathbb{C}P^3$ lets first investigate the construction of S_F^2 and $\mathbb{C}P^n$ before introducing the Fuzzy S^4. Here we will follow the articles of Kimura[13],Steinacker[7], notes of Ydri[8] and the book of Balachandran et. al[10]

2.1 Construction of S_F^2 and $\mathbb{C}P_F^N$

We start our discussion with one of the most basic example of the fuzzy spaces S_F^2[21, 10]. Our plan is to review the construction and properties of the Fuzzy S^2 and then Fuzzy $\mathbb{C}P^N$ in order to familiarize the reader with the ideas of fuzzy spaces and prepare for further developments.

2.1.1 Geometry of S^2

We can define S^2 as the compact manifold embedded in $\mathbb{R}^3$ fulfilling the relation[22]

$$x_1^2 + x_2^2 + x_3^2 = r^2, \qquad (x_1, x_2, x_3) \in \mathbb{R}^3. \tag{2.1}$$

The algebra of smooth bounded functions on S^2 denoted as $\mathbb{C}P^\infty(S^2)$. Letting $f(x) \in \mathbb{C}P^\infty(S^2)$ with the coordiantes x_i fulfilling the relation (2.1) we can expand these functions as follows

$$f(x) = \sum_{a_1, a_2, \ldots a_n = 1}^{3} f_{a_1 \ldots a_n} x_{a_1} \ldots x_{a_n}. \tag{2.2}$$

Equivalently functions on S^2 can be expanded in terms of Spherical Harmonics as

$$f(\theta, \phi) = \sum_{lm} c_{lm} Y_{lm}(\theta, \phi) \tag{2.3}$$

With functions on S^2 in hand, we are in position to define the derivatives on S^2. This is a map from $C^\infty(S^2)$ onto itself which is given by the generators of the rotation group of S^2 and satisfies the Leibniz rule. We can write it as

$$L_\mu = -i\epsilon_{\mu\nu\rho} x_\nu \partial_\rho, \tag{2.4}$$

where the Greek indices μ, ν, ρ runs from 1 to 3. This relation can also be given in vector notation as

$$\vec{\boldsymbol{L}} = -i\boldsymbol{x} \times \boldsymbol{\nabla} \tag{2.5}$$

L_μ's fulfill the $SU(2)$ commutation relations as expected.

$$[L_\mu, L_\nu] = i\epsilon_{\mu\nu\rho} L_\rho \tag{2.6}$$

as expected it readily follows from (2.4) that $x_\mu L_\mu = 0$ so that $\vec{x} \perp \vec{L}$ which lead us to realize that $\vec{L}$ is perpendicular to the radial direction on S^2 which means that it is the tangential to the S^2

Laplacian on S^2 is given by

$$\boldsymbol{L}^2 = L_\mu L_\mu \tag{2.7}$$

From group theory and also from quantum mechanics we know that the eigenvalues of this operator is $l(l+1)$ since it corresponds to the Casimir operator of the group $SU(2)$ from the former and they are essentially the angular momentum operators from the latter, perspective. Also, note that we can define a scalar product on functions on S^2 via

$$(f,g) = \frac{1}{2\pi R}\int d^3x\, \delta(x_\mu^2 - r^2) f^*(x) g(x) \tag{2.8}$$

For both $f(x)$ and $g(x) \in C^\infty(S^2)$.

We can also define S^2 by making use of the first Hopf fibration. To this end consider the Projection map P. Let $g \in SU(2)$ generated by the usual Pauli matrices τ. We may write,

$$\begin{aligned} g\sigma_3 g^{-1} &= \hat{n}.\vec{\tau}\,. \\ g\dot{\hat{n}}\vec{\tau} g^{-1} &= \hat{n}.\vec{\tau}, \end{aligned} \tag{2.9}$$

where $\hat{n}_0 = (0,0,1)$,$\hat{n}$, are unit vectors in $\mathbb{R}^3$. One can understand this map in the following way. Action of g rotates the unit vector $\hat{n}_0$ point along the positive x_3 direction to a general radially outward vector $\hat{n}$. Indeed squaring both sides in (2.10) we get

$$\begin{aligned} g\tau_3 g^{-1} g\tau_3 g^{-1} &= \hat{n}_\mu \hat{n}_\nu \tau_\mu \tau_\nu, \\ g\tau_3^2 g^{-1} &= \hat{n}_\mu \hat{n}_\nu (\delta_{\mu\nu} + i\epsilon_{\mu\nu\rho}\tau_\rho), \\ 1 &= \hat{n}_\mu \hat{n}_\nu \delta_{\mu\nu} + i\epsilon_{\mu\nu\rho}\hat{n}_\mu \hat{n}_\nu \tau_\rho, \\ 1 &= \hat{n}_\mu \hat{n}_\mu. \end{aligned} \tag{2.10}$$

Let us note that if we take $g \to gh$, with $h \in U(1) \subset SU(2)$ given as $h = e^{\frac{i}{2}\theta\sigma_3}$. Using the (2.10) we find

$$g\sigma_3 g^{-1} \to gh\tau_3 h^{-1} g^{-1} = g e^{\frac{i}{2}\theta\sigma_3}\tau_3 e^{\frac{i}{2}\theta\tau_3} g^{-1} = g\sigma_3 g^{-1}, \tag{2.11}$$

which shows us that (2.10) is left invariant under the transformation $g \to gh$. This means that all element of $gh = ge^{\frac{i}{2}\theta\sigma_3}$ of $S^3 \equiv SU(2)$ are projected on to the same point on S^2 We can express this equivalence class as

$$\hat{n} \in S^2 \longleftrightarrow [ge^{\frac{i}{2}\theta\tau_3}] \in \frac{SU(2)}{U(1)}, \tag{2.12}$$

This amount to the construction of S^2 as the adjoint orbit of $SU(2)$ through σ_3. In other words, the equivalence class of parts $g \equiv gh$ on S^3 are mapped to a single point of S^2. Since (2.10) is equivalent to rotation of $\hat{n}$ by an orthogonal matrix $R \in SO(3) \equiv SU(2)$ as

$$\hat{n}' = R\hat{n}, \tag{2.13}$$

There is also another way to understand this Hopf Fibration. Starting from the 2 dimensional flat complex space $\mathbb{C}^2$ spanned by $Z = \{z_1, z_2\}$ and removing the origin we can write coordinates of the complex plane as $\xi = \frac{z}{|z|}$. Since we may write

$$\xi = (\xi_1, \xi_2) = (\alpha_1 + i\alpha_2, \beta_1 + i\beta_2), \qquad |\xi|^2 = \alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2 = 1 \tag{2.14}$$

Observe that ξ define the manifold S^3. Now we can consider the projection map $\Pi : S^3 \to S^2$ which can be given as

$$x_\mu(\xi) = \xi^* \tau_\mu \xi \tag{2.15}$$

Obviously (2.15) is invariant under the $U(1)$ transformations $\xi \to \xi e^{i\theta}$. We can check that square of the $x_\mu \in S^2$, since

$$\begin{aligned} x_\mu x_\mu = (\xi^\dagger \tau_\mu \xi)^2 = &\quad (\xi^\dagger_\alpha (\tau_\mu)_{\alpha\beta} \xi_\beta)(\xi^\dagger_\gamma (\tau_\mu)_{\gamma\delta} \xi_\delta), \\ = &\quad \xi^\dagger_\alpha \xi_\beta \xi^\dagger_\gamma \xi_\delta (\tau_\mu)_{\alpha\beta} (\tau_\mu)_{\gamma\delta}, \\ = &\quad \xi^\dagger_\alpha \xi_\beta \xi^\dagger_\gamma \xi_\delta (\delta_{\alpha\beta}\delta_{\gamma\delta} - 2\epsilon_{\alpha\gamma}\epsilon_{\beta\delta}), \\ = &\quad \xi^\dagger_\alpha \xi_\alpha \xi^\dagger_\gamma \xi_\gamma - 2\xi^\dagger_\alpha \xi_\beta \xi^\dagger_\gamma \epsilon_{\alpha\gamma}\epsilon_{\beta\delta}, \\ = &\quad 1 - 2\xi^\dagger_\alpha \xi_\beta \xi^\dagger_\gamma \xi_\delta (\delta_{\alpha\beta}\delta_{\alpha\delta}\delta_{\gamma\beta}), \\ = &\quad 1 - 2 + 2, \\ &= 1. \end{aligned} \tag{2.16}$$

Where we have used the Fierz identity for Pauli matrices in passing from the 2^{nd} to 3^{rd} line of (2.16) in the previous calculation. Which can be given more explicitly as.

$$\begin{aligned} (\tau_\mu)_{\alpha\beta}(\tau_\mu)_{\gamma\sigma} = &\quad \delta_{\alpha\beta}\delta_{\gamma\sigma} - 2\epsilon_{\alpha\gamma}\epsilon_{\beta\sigma}, \\ = &\quad \delta_{\alpha\beta}\delta_{\gamma\sigma} - 2\delta_{\alpha\beta}\delta_{\gamma\sigma} + 2\delta_{\alpha\sigma}\delta_{\beta\gamma}, \\ = &\quad -\delta_{\alpha\beta}\delta_{\gamma\sigma} + 2\delta_{\alpha\sigma}\delta_{\beta\gamma}. \end{aligned} \tag{2.17}$$

Thus we have showed that indeed $x_\mu \in S^2$ with $(\alpha, \beta, \gamma, \sigma) = (1, 2)$. And we have constructed the descent chain of manifolds as $C^2 \to S^3 \to S^2$ where the second arrow in this chain corresponds to the 1^{st} Hopf Map, which we have previously discussed from another perspective. $SU(2) \to S^2$.

2.1.2 Construction of Fuzzy S^2

In order to get the fuzzy versions of these spaces we can quantize these manifolds in the following manner. Consider replacing the complex coordinates z_α and $\bar{z}_\alpha$ by the operators a_α and $a^\dagger_\beta$

$$\left[a_\alpha, a^\dagger_\beta\right] = 1 \tag{2.18}$$

Where α, β runs from 1 to 2, in other words, we have two pairs of annihilation and creation operators. We can also define the number operator N as

$$N = a^\dagger_i a_i \tag{2.19}$$

Where sum over repeated indices is implied. Now we need to parametrize the ξ_α. As an intermediate step we may write.

$$\xi_\alpha \equiv a_i \frac{1}{\sqrt{N}} \qquad \xi^\dagger_\alpha \equiv a^\dagger_i \frac{1}{\sqrt{N}} \qquad N \neq 0 \tag{2.20}$$

The condition $N \neq 0$ tells us that we have excluded the vacuum from the Fock space. But clearly such condition can not be satisfied as successive application of the annihilation operator $a_\alpha \frac{1}{\sqrt{\hat{N}}}$ will naturally create it from any Fock state $|m\rangle$. This problem, will not perisist as we move on the quantization of S^2.

To get the quantized version of S^2 we replace (2.15) with

$$\begin{aligned} x_\mu = & \quad \xi^\dagger \tau_\mu \xi, \\ = & \quad \frac{1}{\sqrt{\hat{N}}} a^\dagger \tau_\mu a \frac{1}{\sqrt{\hat{N}}}, \\ = & \quad \frac{1}{\hat{N}} a^\dagger \tau_\mu a. \end{aligned} \tag{2.21}$$

Observe that $[x_\mu, N] = 0$, so we can restrict x_μ to act on a subspace with eigenvalue of N equal to $n \neq 0$. More specifically we may restrict to the $(n+1)$-dimensional Hilbert space spanned by the vectors.

$$|n_1, n_2\rangle = \frac{(a^\dagger_1)^{n_1}}{\sqrt{n_1!}} \frac{(a^\dagger_2)^{n_2}}{\sqrt{n_2!}} |0, 0\rangle, \qquad n_1 + n_2 = n. \tag{2.22}$$

Thus x_μ are $(n+1) \times (n+1)$ Hermitian matrices acting on $|n_1, n_2\rangle$. By the irreducible action of x_μ on these states the full matrix algebra of $(n+1) \times (n+1)$ is generated.

Note that we could relate the x_i to the Schwinger construction of $SU(2)$ generators in the following way. For such L_μ that is generating $SU(2)$ we have.

$$L_\mu = \frac{1}{2} a^\dagger \tau_\mu a, \tag{2.23}$$

with the usual commutation relations

$$[L_\mu, L_\nu] = i\epsilon_{\mu\nu\rho} L_\rho. \tag{2.24}$$

Letting $N = n$ in this subspace we get

$$L_\mu = \frac{n}{2} x_\mu. \tag{2.25}$$

Observe that the adjoint action of L_μ on $a_\alpha^\dagger$

$$\begin{aligned}\left[L_\mu, a_\alpha^\dagger\right] &= \frac{1}{2}(\tau_\mu)_{\delta\gamma}\left[a_\delta^\dagger a_\gamma, a_\alpha^\dagger\right], \\ &= \frac{1}{2}(\tau_\mu)_{\gamma\delta} a_\gamma^\dagger \delta_{\delta\alpha}, \\ &= \frac{1}{2}(\tau_\mu)_{\gamma\alpha} a_\gamma^\dagger,\end{aligned} \tag{2.26}$$

And likewise we get

$$[L_\mu, a_\alpha] = \frac{1}{2}(\tau_\mu)_{\alpha\mu} a_\mu \tag{2.27}$$

which shows that the operators a_α and $a_\alpha^\dagger$ transform as spin $\frac{1}{2}$ under $SU(2)$. Hence the Hilbert space spanned by the state vectors created $a_\alpha^\dagger$ span the n-fold symmetric tensor product representation of $SU(2)$.

$$\left(\frac{1}{2} \otimes \frac{1}{2} \cdots \otimes \frac{1}{2}\right)_{sym} \equiv \frac{n}{2}. \tag{2.28}$$

From standard quantum mechanics, we know that Casimir operator for $SU(2)$ is given by

$$L_\mu^2 = \frac{n}{2}(\frac{n}{2} + 1)\mathbb{1}_{n+1}, \tag{2.29}$$

in the spin $\frac{n}{2}$ irreducible representation. Now using the fact that $x_\mu = \frac{2}{n} L_i$, as given in (2.25) We can obtain the commutation relation between the coordinates x_μ as

$$[x_\mu, x_\nu] = \frac{2}{n}\epsilon_{\mu\nu\rho} x_\rho, \tag{2.30}$$

and these fulfill

$$x_\mu^2 = \left(1 + \frac{2}{n}\right)\mathbb{1}_{n+1}. \tag{2.31}$$

We can refine the relation (2.31) by adjusting the scaling as $x_\mu = \frac{1}{\sqrt{l(l+1)}} L_\mu$ rather than (2.25), which yields.

$$[x_\mu, x_\nu] = \frac{i\epsilon_{\mu\nu\rho}x_\rho}{\sqrt{l(l+1)}}, \tag{2.32}$$

which is more commonly encountered in the literature. Relations (2.1) and (2.32) describe a fuzzy sphere $S_F^2(l)$, at the level $n = 2\ell$. Any element ,μ, of the matrix algebra $Mat(2l+1)$ is an element of this fuzzy sphere and it can be expanded in terms of generators of S_F^2 as

$$M = \sum C_{i_1 \dots i_k} x_{i_1} \dots x_{i_k}. \tag{2.33}$$

Now that we have obtained the S_F^2 and functions on it we are in position to define the scalar product on S_F^2, which is given by the matrix scalar product, i.e. trace

$$(M_1, M_2) = TrM_1^\dagger M_2 = \frac{1}{2\ell+1} M_1^\dagger M_2 \tag{2.34}$$

Which in the fuzzy setting replaces the integral given in (2.8). Note that from now on Tr will stand for the normalized trace i.e. $Tr\mathbb{1}_{n+1} = 1$. We can also define the left and right acting linear operators on $Mat(2\ell+1)$. Consider α^l and α^R as linear operators on $Mat(2\ell+1)$. Since they are are also $(n+1)\times(n+1)$ matrices. they are naturally in the algebra $Mat(2\ell+1)$ too. To be more explicit we have

$$\alpha^L M = \alpha M, \qquad \alpha^R M = M\alpha, \tag{2.35}$$

where $\alpha^L, \alpha^R, M \in Mat(2\ell+1)$. It is clear that these two operators commute.

$$\left[\alpha^L, \beta^R\right] = 0 \tag{2.36}$$

and

$$(\alpha\beta)^L = \alpha^L\beta^L \qquad (\alpha\beta)^R = \beta^R\alpha^R \tag{2.37}$$

A set of these right and left acting operators can be the angular momentum operators L_i^L and L_i^R. We can write the action of the respective Casimirs

$$(L_i^L)^2 |n_1, n_2\rangle = L_i^2 |n_1, n_2\rangle = \ell(\ell+1) |n_1, n_2\rangle, \tag{2.38}$$

$$(L_i^R)^2 \langle n_1, n_2| = \langle n_1, n_2| L_i^2 = \ell(\ell+1) \langle n_1, n_2|, \tag{2.39}$$

As expected both L_μ^L and L_μ^R carries the spin $-\ \ell\ IRR$ of $SU(2)$. With left and right acting operators in hand we can define the adjoint action

$$\mathcal{L}_\mu M = \text{ad}L_\mu M = (L_\mu^L - L_\mu^R)M = [L_\mu, M]. \tag{2.40}$$

Let's sum up our progress up to here. We have defined the fuzzy space S_F^2, showed how to write the functions on it and also showed that how the operator acts on these functions. In order to progress further we need a certain kind of operators that allows us to define the derivatives on S_F^2. And for that purpose we can use the angular momentum operators we have just defined. Since they obey the Leibniz rule $\mathcal{L}_i$'s are derivatives over the matrix algebra $Mat(2l+1)$ and hence on S_F^2. In the commutative limit $l \to \infty$, we get

$$\mathcal{L}_\mu \to -i\epsilon_{\mu\nu\rho}x_\nu\partial_\rho \tag{2.41}$$

as can be compared with equation (2.6). This consolidates our interpretation of the operators $\mathcal{L}_\mu$ as derivative operators. But, what is the spectrum of these operators? Since we have $\mathcal{L}_\mu = L_\mu^L - L_\mu^R$ It carries the tensor product representation

$$\frac{n}{2} \otimes \frac{n}{2} = l \otimes \ell = 0 \oplus 1 \oplus 2 \cdots \oplus 2\ell \tag{2.42}$$

Thus $\mathcal{L}^2$ can take on the eigenvalues $l(l+1)$ where $l = 0, 1, \ldots 2\ell$. We see that the angular momentum operator is truncated at a maximum value 2ℓ, which is a characteristic property of the fuzzy spaces. Next natural step is to investigate the eigenvectors of $\mathcal{L}^2$ and $\mathcal{L}_3$. Eigenvectors of these operators are given in terms of the polarization tensors $T_{lm}(n)$ where l and m runs as follows $j : 0 \ldots 2\ell$, $m = -j \ldots j$ which are elements of $Mat(2l+1)$. These are $(2\ell+1)^2$ linearly independent matrices and they form a basis for the algebra $Mat(2\ell+1)$ and we have

$$\mathcal{L}^2 T_{lm} = l(l+1)T_{lm}, \tag{2.43}$$

$$\mathcal{L}_3 T_{lm} = mT_{lm}, \tag{2.44}$$

with the following inner product

$$(T_{l'm'}, T_{lm}) = \delta_{ll'}\delta_{mm'}. \tag{2.45}$$

Finally we can show the transformation properties of the T_{lm}'s under $SU(2)$ as

$$\begin{aligned} T'_{lm'} &= D(g)T_{lm'}D(g)^{-1}, \\ &= \sum_m D(g)^l_{mm'}T_{lm}. \end{aligned} \tag{2.46}$$

Observe that this is the same transformation of the $Y_{lm}(\theta, \phi)$ under finite rotations

$$Y_{lm'}(\theta, \phi) = \sum_m D(g)^l_{mm'}Y_{lm}(\theta, \phi), \tag{2.47}$$

where (2.45) is what replaces the orthogonality relation $\frac{1}{4\pi}\int Y^*_{lm}(\theta,\phi)Y_{l'm}(\theta,\phi)d\omega = \delta_{ll'}\delta_{mm'}$ of the spherical harmonics. Hence we can deduce that T_{lm} operators carries the spin l IRR of $SU(2)$. We can conclude this section with one last remark on the expansion of functions on the S^2_F in terms of polarization tensors, which is given as $M = \sum C_{lm}T_{lm}$, which can be compared with (2.33) and also the expansions given in (2.8) for the ordinary sphere.

2.1.3 Fuzzy Complex Projective Spaces

In the previous section we have stated that the S^2_F can be obtained by the quantization of the first Hopf Map. A Hopf like map can also be used to obtain the fuzzy versions of the complex projective spaces $\mathbb{C}P^n$'s. So let us briefly explain the structure and construction of these spaces since they will be an important part of the future discussions. Just as we represent $S^2 \equiv \mathbb{C}P^1 = \frac{SU(2)}{U(1)}$ we may represent $\mathbb{C}P^n$ as the coset space

$$\begin{aligned} \mathbb{C}P^k &\equiv SU(k+1)/U(k) \\ &\simeq SU(k+1)/SU(k)\times U(1) \end{aligned} \tag{2.48}$$

Now we will focus on the specific case of fuzzy $\mathbb{C}P^2$, but note that following discussion can be generalized to $\mathbb{C}P^n$. Here we will follow the references [23] We start our construction with denoting the generators of the $SU(3)$ group T_a $(a = 1, \ldots, 8)$ which carries the $(n, 0)$ IRR of $SU(3)$. Dimensions of this IRR of $SU(3)$ is given as

$$N := dim(n,0) = \frac{1}{2}(n+1)(n+2) \tag{2.49}$$

For instance the usual 3-dimensional fundamental IRR is denoted by $(1, 0) \equiv (\mathbf{m}, 3)$ while the anti-fundamental IRR is given by $(0, 1) \equiv \bar{3}$ and the adjoint IRR is $(1, 1) \equiv 8$ They satisfy the usual $SU(3)$ relations

$$[T_a, T_b] = if_{abc}T_c, \tag{2.50}$$

$$T_a^2 = \tfrac{1}{3}n(n+3)\mathbb{1},$$

$$d_{abc}T_aT_b = \tfrac{2n+3}{6}T_c. \tag{2.51}$$

where d_{abc} is the totally symmetric tensor from

$$T_aT_b = \frac{2}{n}\delta_{ab} + (d_{abc} + f_{abc})T_c \tag{2.52}$$

We can introduce the Gell-Mann matrices λ_a of $SU(3)$ in the fundamental representation $(1,0)$. Taking the n-fold symmetric tensor product we construct the generators T_a

$$T_a = (\frac{\lambda_a}{2} \otimes 1 \otimes \ldots 1 + \ldots 1 \otimes 1 \otimes \ldots \frac{\lambda_a}{2})_{sym} \tag{2.53}$$

Now we will show that indeed these T_a generates the fuzzy $\mathbb{C}P^2$. To do that first we need to go back to the commutative $\mathbb{C}P^2$. We know that vectors of $\mathbb{C}P^2$ have the $U(1)$ symmetry, meaning that $e^{i\theta}|\psi\rangle$ for all $\theta \in [0, 2\pi]$ denotes the same point. Since $\mathbb{C}P^2$ is the space of all rank one projection operators on $\mathbb{C}^3$ the vector $e^{i\theta}|\psi\rangle$ also corresponds to the same projector $P = |\psi\rangle\langle\psi|$. Let H_n and H_3 be the Hilbert spaces of the $SU(3)$ IRR's $(n,0)$ and $(1,0)$. Starting from a vector in R^8 which we denote as $\vec{n}$ we attempt to define the projector.

$$P_3 = \frac{1}{3}\mathbb{1} + n_a t_a, \tag{2.54}$$

where $t_a = \frac{\lambda_a}{2}$ are the generators of $SU(3)$ in the fundamental representation $(1,0)$ that satisfy the following

$$\begin{aligned} 2t_a t_b &= \frac{1}{3}\delta_{ab} + (d_{abc} + i f_{abc}) t_c, \\ Tr(t_a t_b) &= \frac{1}{2}\delta_{ab}, \\ Tr(t_a t_b t_c) &= \frac{1}{4}(d_{abc} + i f_{abc}). \end{aligned} \tag{2.55}$$

And the usual requirement of projection operators that they square to themselves, i.e. $P_3^2 = P_3$ leads us to $\vec{n}$ being a point on $\mathbb{C}P^2$ satisfying the following conditions

$$\begin{aligned} n_a^2 &= \tfrac{4}{3}, \\ d_{abc} n_a n_b &= \tfrac{2}{3} n_c, \end{aligned} \tag{2.56}$$

With the coherent states of H_3 denoted as $|\vec{n},3\rangle$ we can write the projection operator

$$P_3 = |\vec{n},3\rangle\langle 3,\vec{n}|\,. \tag{2.57}$$

Using P_3 we may write down it's generalization to P_n on H_N

$$P_N = |\vec{n},N\rangle\langle N,\vec{n}| = (P_3 \otimes P_3 \otimes P_3 \ldots P_3)_{sym} \tag{2.58}$$

Computation of trace relations

$$Tr(t_a P_3) = \langle \vec{n},3| t_a |\vec{n},3\rangle = \frac{1}{2} n_a \qquad Tr(T_a P_n) = \langle \vec{n},N| T_a |\vec{n},N\rangle = \frac{n}{2} n_a \tag{2.59}$$

Using these trace identities we can identify fuzzy $\mathbb{C}P^2$ at level $N = \frac{1}{2}(n+1)(n+2)$ by the coordinate operators.

$$x_a = \frac{2}{n}T_a, \tag{2.60}$$

satisfying the following

$$\begin{aligned} [x_a, x_b] &= \tfrac{2i}{n} f_{abc} x_c, \\ x_a^2 &= \tfrac{4}{3}(1 + \tfrac{3}{n}), \\ d_{abc} x_a x_b &= \tfrac{2}{3}(1 + \tfrac{3}{2n}) x_c. \end{aligned} \tag{2.61}$$

from these relations we can see that in the limit $N \to \infty$ we retrieve the commutative $\mathbb{C}P^2$ as $x_a \to n_a$. The algebra of function of fuzzy $\mathbb{C}P^2$ can be identified with the matrix algebra of $N \times N$ matrices which can be denoted by Mat_N. Similar to the case of S_F^2 now the $SU(3)$ group has two distinct actions on these algebra. The left action of $SU(3)$ is generated by $(n, 0)$ on the other hand right action is generated by $(0, n)$. Thus we have the following decomposition of Mat_N under the adjoint $SU(3)$ action.

$$(n, 0) \otimes (0, n) = \oplus_{p=0}^{n} (p, p) \tag{2.62}$$

In the same manner as discussed for fuzzy S^2, we can expand a general function on fuzzy $\mathbb{C}P_N^2$ in terms of the polarization tensor of $SU(3)$ IRR as

$$G = \sum_{p=0}^{n} G^{(p)}_{I^2, I_3, Y} T^{(p,p)}_{I^2, I_3, Y}, \tag{2.63}$$

Here $T^{(p,p)}_{I^2,I_3,Y}$ are $N \times N$ matrices and the set (I^2, I_3, Y) are the eigenvalues of the casimir of $SU(2)$ subgroup respectively its third component and the hypercharge of the isotropy subgroup in (2.48). Now that we have defined the Hilbert space and matrix algebra of fuzzy $\mathbb{C}P^2$ what remains is the Laplacian on $\mathbb{C}P_F^2$. To find it we may define the derivations on $\mathbb{C}P_F^2$. We know from our previous construction of S_F^2 that, the adjoint action of the generators of the symmetry group $SU(3)$ on $\mathbb{C}P_F^2$. Since, we can naturally see that the derivations on $\mathbb{C}P_F^2$ are generated by

$$[T_a, .] \tag{2.64}$$

and the Laplacian can be given as

$$\triangle_N := [T_a, [T_a, .]], \tag{2.65}$$

This concludes our introduction to the basic examples fuzzy spaces. We will now proceed to discuss the construction of fuzzy S^4 and its basic properties.

2.2 Introduction to Fuzzy Four Sphere S_F^4

2.2.1 Construction of Fuzzy S^4

Now we are in a position to start constructing the fuzzy S^4. In this section we will be following the previous works of [7, 8, 13]

First of all let us start with the ordinary definition of 4-dimensional sphere, S^4. To do so, our first job is to define and embed it in $\mathbb{R}^5$. Then we have

$$S^4 \equiv \langle \vec{X} = (X_1, X_2, \cdots, X_5) \in \mathbb{R}^5 \big| \vec{X} \cdot \vec{X} = R^2 \rangle. \tag{2.66}$$

It is known that S^4 can be described as a coset space of

$$S^4 \equiv SO(5)/SO(4) \tag{2.67}$$

so either from (2.66) or (2.67) we see that coordinates of S^4, given as $\vec{X}$, transforms as a vector under the $SO(5)$ rotations.

Now we can start to quantize the S^4 in the pursuit of fuzzy S^4. A fuzzy four sphere can be constructed using the following relations

$$\epsilon^{abcde} x_a x_b x_c x_d = C x_e, \tag{2.68}$$

$$x_a x_a = \rho^2, \tag{2.69}$$

where x_a,$(a : 1, \ldots, 5)$ represent the non-commuting coordinates of S_F^4 , ρ is the radius and $C = (8n + 16)$. Equations (2.68) and (2.69) requires some explanation. We need to understand, in what way we can introduce a matrix algebra, to describe the fuzzy 4-sphere, and connect this to the relations (2.68) and (2.69). To this end let us introduce the Γ-matrices associated to the $SO(5)$ group.

Γ-matrices are the 4×4 matrices satisfying the clifford algebra $\{\Gamma_a, \Gamma_b\} = 2\delta_{ab}$, $(a, b : 1, \ldots, 5)$ An explicit form may be given as

$$\Gamma_a = \begin{pmatrix} 0 & -i\sigma_a \\ i\sigma_a & 0 \end{pmatrix}, \quad \Gamma_4 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}, \quad \Gamma_5 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \tag{2.70}$$

Generators of the $SO(5)$ are defined by $\Sigma_{ab} = \frac{1}{2}[\Gamma_a, \Gamma_b]$. Σ_{ab}'s generate the 4-dimensional spinor representation of $SO(5)$, which is given as $(0, 1)$ in terms of the Dynkin labels. Γ_a also act on the 4-dimensional spinor space and therefor we may say that they carry the IRR $(0, 1)$. Now we consider the n-fold symetric tensor product of Γ_a's given as

$$X_a^{(n)} = (\Gamma_a \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes 1 \cdots \otimes \Gamma_a)_{\text{Sym}} \qquad (2.71)$$

X_a carry the $(0, n)$ IRR of $SO(5)$ and they have the dimension

$$N = dim(0, n) = \frac{1}{6}(n+1)(n+2)(n+3). \qquad (2.72)$$

The matrices X_a satisfy the relations

$$X_a X_a \quad = \quad n(n+4)\mathbb{1}_N, \qquad (2.73)$$

$$\epsilon^{abcde} X_a X_b X_c X_d = \epsilon^{abcde} X_{ab} X_{cd} \quad = \quad (8n+16)X_e, \qquad (2.74)$$

In fact, it is not hard to prove (2.73) and we will do so a little later on. A somewhat more detailed calculation is necesseary to show that (2.74) holds. A detailed calculation is given in [24], but we will not attempt to reproduce it here. Comparing (2.73) and (2.74) with (2.68) we see that x_a and X_a are related by a constant of dimension length i.e. $x_a = \alpha X_a$ and we have $C = (8n+16)\alpha^3$,$\rho^2 = n(n+4)\alpha^2$ based on the dimensional analysis.

Using the commutator of these X_a we can construct the generators of the $SO(5)$ algebra

$$X_{ab} = \frac{1}{2}[X_a, X_b]. \qquad (2.75)$$

These X_{ab} satisfy the usual $SO(5)$ algebra. With the commutation relations

$$[X_{ab}, X_{cd}] = 2(\delta_{bc} X_{ad} + \delta_{ad} X_{bc} - \delta_{ac} X_{bd} - \delta_{bd} X_{ac}). \qquad (2.76)$$

From (2.75) we see that there is something unusual about the fuzzy S^4 construction, compared to the fuzzy S^2 and $\mathbb{C}P^2$. Namely, we see that commutators of X_a are no longer expressed in terms of linear combinations of X_a (2.24). In other words the algebra of X_a do not close. This means that algebra of fuzzy S^4 is larger than the matrix algebra generated by X_a, It also contains the matrices generated by X_{ab} too.

Quadratic Casimir of $SO(5)$ given in the $(0, n)$ IRR reads

$$C_2(SO(5)) = \frac{1}{2} X_{ab} X_{ba} = 2n(n+4)\mathbb{1}_N, \qquad (2.77)$$

Using the (2.68) it is straightforward to get

$$X_{ab}X_{bc} = n(n+4)\delta_{ac} + X_a X_c - 2X_c X_a, \tag{2.78}$$

Observe that (2.74) can also be written as.

$$X_{ab} = -\frac{1}{2(n+2)}\epsilon^{abcde}X_{cd}X_e = -\frac{1}{2(n+2)}X_c X_d X_e. \tag{2.79}$$

Using (2.68) we also find that X_a transforms as vectors. under the adjoint action of $SO(5)$, indeed we have

$$[X_{ab}, X_c] = 2(\delta_{ac}X_b - \delta_{bc}X_a) \tag{2.80}$$

Alternatively we can en up with the same result by employing a generalized and adapted form of Schwinger construction suitable for the present problem. Introducing four pairs of annihilation-creation operators we can write.

$$X_a = a^\dagger_\alpha (\Gamma_a)_{\alpha\beta} a_\beta \qquad \alpha, \beta : 1, 2 \tag{2.81}$$

where a and $a^\dagger$ are the creation and annihilation operators corresponding to the 4 bosonic oscillators. Acting on the Fock space $\mathcal{H}_N = a^\dagger_{i_1} \ldots a^\dagger_{i_N} |0\rangle$. Note that $\mathcal{H}_N$ is same as the space that X_μ's act on $\mathcal{H}_N = (C^4 \otimes \ldots C^4)_{sym}$. Which comes from the symmetric product of C^4 where gamma matrices acts on naturally. Using this X_μ we can obtain the following commutation relations. Introducing $\mathcal{X}_{ab} = a^\dagger \Sigma_{ab} a$ we have that all the relations given previously for X_a, X_{ab} are satisfied with in this equivalent formulation. We know that functions on four sphere, S^4, can be expanded in terms of the spherical harmonics

$$a(x) = \sum_{l=0}^{\infty} \sum_{m_i} a_{lm_i} Y_{lm_i}(x), \tag{2.82}$$

where the latter are defined as

$$Y_{lm_i}(x) = \frac{1}{\rho^l} \sum_a f^{(lm_i)}_{a_1,\ldots,a_l} x^{a_1} \ldots x^{a_l}. \tag{2.83}$$

with $f^{(lm_i)}_{a_1,\ldots,a_l}$ being a symmetric traceless tensor(because of the requirement $X_i X_i = \rho^2$). and m_i denoting the necessary labels in spherical harmonics on S^2. We may have expected a similarity with developments given for S^2_F and $\mathbb{C}P^N_F$ that for the fuzzy S^4 ther matrices replacing the functions $a(x)$ are

$$M = \sum_{l=0}^{\infty} \sum_{m_i} c_{lm_i} T_{lm_i} \tag{2.84}$$

T_{lm_i} being the the spherical polarization tensors replacing Y_{lm_i}.
Nevertheless our discussion so far have shown us that, the matrix algebra generated by X_a's only do not close(see (2.75)) and therefore we can not simply represent matrices for S_F^4 by (2.84). In the next section we discuss this point in more detail.

2.2.2 S_F^2 fiber over S_F^4

We will now provide an argument revealing that there is indeed a fuzzy two sphere attached to every point on the fuzzy S^4. Attachment of these fuzzy 2 spheres on S_F^4 may be seen as the presence of an internal spin degree of freedom for S_F^4. We may always choose to diagonalize one of the matrices X_a, say we diagonalize X_5.

$$X_{5_{\text{diag}}} = (n, n-2, \ldots, -n+2, -n) \tag{2.85}$$

with eigenvalue m having the degeneracy $((n+2)^2 - m^2)/4$. Now out of the 10 generators of $SO(5)$ algebra, we take a set of 6 generators that $SO(4) \equiv SU(2) \oplus SU(2)$ subalgebra. Let N_μ and M_μ be the generators of $SU(2) \oplus SU(2)$ algebras. We may take them as

$$\begin{aligned} N_1 &= -\frac{i}{4}(X_{23} - X_{14}), & M_1 &= -\frac{i}{4}(X_{23} + X_{14}), \\ N_2 &= -\frac{i}{4}(X_{13} - X_{24}), & M_2 &= -\frac{i}{4}(X_{13} - X_{24}), \\ N_3 &= -\frac{i}{4}(X_{12} - X_{34}), & M_3 &= -\frac{i}{4}(X_{12} - X_{34}), \end{aligned} \tag{2.86}$$

We find that N_μ and M_μ indeed satisfy the $SU(2) \oplus SU(2)$ Lie algebra commutation relaions.

$$\begin{aligned} [N_\mu, N_\nu] &= i\epsilon_{\mu\nu\rho} N_\rho, \\ [M_\mu, M_\nu] &= i\epsilon_{\mu\nu\rho} M_\rho, \\ [M_\mu, N_\nu] &= 0, \end{aligned} \tag{2.87}$$

We can reverse this relation in order to get the X_{ab}'s in terms of N_μ's and M_μ's We can calculate the Casimir operator for each of these $SU(2)$ generators as. We find using (2.68),(2.77),(2.78)

$$\begin{aligned} N_\mu N_\mu &= \frac{1}{16}(n + X_5)(n + 4 + X_5), \\ M_\mu M_\mu &= \frac{1}{16}(n - X_5)(n + 4 - X_5). \end{aligned} \tag{2.88}$$

For a given eigenvalue of X_5 from (2.85) we see from (2.88) that N_μ and M_μ carry the $(n + X_5 + 2)/2$ and $(n - X_5 + 2)/2$ dimensional representations of the $SU(2)$ they generate. The dimension of the $SU(2) \oplus SU(2)$ IRR (J_1, J_2) is given by $(2J_1 + 1)(2J_2 + 1)$. Thus, in order to check dimension of the matrices N_μ and M_μ we sum over the dimensions of possible IRRs, which we find to be

$$\begin{aligned} & \textstyle\sum_{X_5=-n}^{n} \left(\frac{n+X_5+2}{2}\right)\left(\frac{n-X_5+2}{2}\right), \\ = & \quad \tfrac{1}{6}(n+1)(n+2)(n+3), \\ = & \quad N \end{aligned} \tag{2.89}$$

which is equal to the size of our matrices as expected. At the north pole on S_F^4, we have X_5 taking its maximal eigenvalues $X_5 = n$ This gives from (2.88) that

$$\begin{aligned} N_\mu N_\mu = & \quad \tfrac{n(n+2)}{4}, \\ M_\mu M_\mu = & \quad 0. \end{aligned} \tag{2.90}$$

From (2.90) we observe that the N_μ operators Casimir eigenvalue at the north pole is equal to that of a spin $j = \frac{n}{2}$ IRR of $SU(2)$, while it is clearly the spin 0, trivial representation for M_μ's. a So we can argue that starting from the $SO(5)$ generators we have obtained a fuzzy S^2 attached at the north pole of the fuzzy S^4 with the radius given in (2.73) which only a factor of $\frac{1}{4}$ less then the radius of the original fuzzy S^4 given in (2.73). Since the fuzzy 4-sphere has $SO(5)$ symmetry we can conclude that there is a S^2 attached to every point of the fuzzy S^4. This extra degrees of freedom coming as a fuzzy S^2 can be interpreted as an internal spin degree of freedom. .

In the commutative limit, S_F^4 is not only S^4 but in fact given by a S^2 fiber on S^4. The following discussion is based on [11].To see this limit it is suitable to scale the S_F^4 defining the following relations.

$$Y_a = \frac{X_a}{\sqrt{n(n+4)}}, \quad Y_{ab} = \frac{X_{ab}}{\sqrt{n(n+4)}}, \tag{2.91}$$

(2.92)

$$\epsilon^{abcde} Y_a Y_b Y_c Y_d = \frac{8(n+2)}{(n(n+4))^{3/2}} Y_e. \tag{2.93}$$

In the commutative limit we have $Y_a \to x_a$, $Y_{ab} \to \omega_{ab}$. Furthermore, it can proved using Schur's Lemma that (2.68) leads to

$$\frac{1}{4}\{Y_a, Y_b\} + \sum_b \{Y_{ab}, Y_{cd}\} = \frac{1}{2}\delta_{ab}.. \tag{2.94}$$

(2.93) can expressed using (2.68) as

$$\epsilon^{abcde} Y_{ab} Y_{cd} = \frac{8(n+2)}{\sqrt{n(n+4)}} Y_e, \tag{2.95}$$

the equation (2.95) is not independent from (2.93) but in the large n limit Y_{ab}, Y_a decouple and the left hand side of (2.95) becomes a constraint.

$$\epsilon^{abcde} \omega_{ab} \omega_{cd} = 0. \tag{2.96}$$

Now let us introduce a 5 component vector which is an element of $\mathbb{R}^5$

$$\mathbf{V}_a = (2i\omega_{a1}, 2i\omega_{a2}, 2i\omega_{a3}, 2i\omega_{a4}, 2i\omega_{a5}). \tag{2.97}$$

This vector seems to be 5 dimensional but if one observers that it will always have at least one zero component it can be stated that it is effectively 4 dimensional. We may look at the properties of this vector. We have

$$\begin{aligned} \mathbf{V}_a.\mathbf{V}_c &= -4 \sum_{b=1}^{5} Y_{ab} Y_{cb}, \\ &= \delta_{ac} - x_a x_c, \end{aligned} \tag{2.98}$$

where we have used the relation

$$\frac{1}{4} x_a x_c + \omega_{ab}\omega_{bc} = \frac{1}{4}\delta_{ac}. \tag{2.99}$$

$\mathbf{V}_a$ also satisfies

$$\mathbf{V}_a.\vec{x} = 2i\omega_{ab} x_c \tag{2.100}$$

Taking the square of both sides

$$\begin{aligned} &= 4\omega_{ba}\omega_{ad} x_b x_d, \\ &= 4 - (\tfrac{1}{4}\delta_{bd} - \tfrac{1}{4} x_b x_d) x_b x_d, \\ &= x_d x_d - x_b x_b x_d x_d, \\ &= 0. \end{aligned} \tag{2.101}$$

In the last line we used the fact that coordinates squares to 1. Together these two relations describe a 4 dimensional manifold. We can choose $\vec{x} = (0, 0, 0, 0, 1)$ to be the north pole of S^4 by use of the rotational symmetry of the sphere. Then our relations become

$$\begin{aligned} \mathbf{V}_a.\mathbf{V}_b &= \delta_{ab}, \\ \mathbf{V}_a.\vec{x} &= 0, \\ \mathbf{V}_5 &= 0 \end{aligned} \tag{2.102}$$

where a, b runs from 1 to 4. V_a spans an orthonormal basis of four vectors which are all tangent to S^4. Finally using the constraint (2.96) further eliminates one more degree of freedom, leaving only 3 independent degrees of freedom in ω_{an} which still fulfills $\omega_{ab}\omega_{ab} = 1$, which is nothing but a S^2.

2.3 Basic Features of Gauge Theory on Fuzzy S_F^4

In this subsection we will focus on the matrix gauge theories related to fuzzy four sphere. Starting from the BFSS model action[1] and adding suitable deformations like a mass term and/or Chern-Simmons like terms we can construct actions with fuzzy S^4 extremums. Each term has its benefits and disadvantages[1],[25],[2].Let's start this section by briefly reviewing the matrix models to set the stage for further developments. Starting from the Yang-Mills 5-matrix model in Minkowski signature and with $U(N)$ gauge symmetry, whose action may be given as

$$\mathcal{S}_{YM} = \int dt\, \mathcal{L}_{YM} = \frac{1}{g^2}\int dt\, Tr\left(\frac{1}{2}(\mathcal{D}_t A_a)^2 + \frac{1}{4}[A_a, A_b]^2\right) \tag{2.103}$$

where A_a $(a : 1, \ldots 5)$ are $N \times N$ Hermitian matrices transforming under the adjoint representation of $U(N)$ as

$$A_a \to U^\dagger A_a U\,, \quad U \in U(N)\,, \tag{2.104}$$

$\mathcal{D}_t A_a = \partial_t A_a - i[\mathcal{A}_0, A_a]$ are the covariant derivatives, $\mathcal{A}$ is a $U(N)$ gauge field transforming as

$$\mathcal{A} \to U^\dagger \mathcal{A} U - iU^\dagger \partial_t U\,, \tag{2.105}$$

and Tr stands for the normalized trace. For future reference we write out the potential part of L_{YM} separately as

$$\mathcal{V}_{YM} = -\frac{1}{4g^2} Tr[A_a, A_b]^2\,. \tag{2.106}$$

Clearly, $\mathcal{S}_{YM}$ is invariant under the $U(N)$ gauge transformations given by (2.104) and (2.105). $\mathcal{S}_{YM}$ is also invariant under the global $SO(5)$ rotations of A_a i.e $A_a \rightarrow A'_a = A_{ab}A_b$, $R \in SO(5)$ rigid rotations (Note that matrix elements of R are not time dependent). It can be obtained from the dimensional reduction of the $U(4N)$ gauge theory in $5+1$-dimensions to $0+1$-dimensions, where the $SO(5,1)$ Lorentz symmetry of the latter yields to the global $SO(5)$ of the reduced theory. There are two distinct deformations of $\mathcal{S}_{YM}$ preserving its $U(N)$ gauge and the $SO(5)$ global symmetries. One of these is obtained by adding a fifth rank Chern-Simons term to $\mathcal{S}_{YM}$ (i.e. a Myers like term) which is given as

$$\mathcal{S}_{CS} = \frac{1}{g^2}\int dt\, Tr\, \frac{\lambda}{5}\epsilon^{abcde}A_aA_bA_cA_dA_e\,, \tag{2.107}$$

while the other is a massive deformation term of the form

$$\mathcal{S}_{mass} = -\frac{1}{g^2}\int dt\, Tr\, \mu^2 A_a^2\,. \tag{2.108}$$

Clearly both $\mathcal{S}_{CS}$ and $\mathcal{S}_{mass}$ are gauge invariant and invariant under rigid $SO(5)$ rotations.

For future purposes it is convenient to write out the potential terms for $\mathcal{S}_1$ and $\mathcal{S}_2$ explicitly:

$$\mathcal{V}_1 = \tfrac{1}{g^2}Tr\left(-\tfrac{1}{4}[A_a, A_b]^2 + \mu^2 A_a^2\right)\,, \tag{2.109}$$

$$\mathcal{V}_2 = -\tfrac{1}{g^2}Tr\left(\tfrac{1}{4}[A_a, A_b]^2 + \tfrac{\lambda}{5}\epsilon^{abcde}A_aA_bA_cA_dA_e\right)\,. \tag{2.110}$$

Note that $\mathcal{S}_1$ and $\mathcal{S}_2$ can be thought as deformations of the bosonic part of the BFSS [1] matrix quantum mechanics. Whose action has the same form as in (2.103) except that there are $N \times N$ matrices in other words the index a takes values from 1 to 9 in that case. We also know that BFSS model can be obtained from the dimensional reduction of the YM theory in $9+1$ dimensions to $0+1$ dimensions [8] with the $SO(9,1)$ symmetry of the YM theory yielding to the global $SO(9)$ symmetry of the BFSS. By the use of deformation terms $\mathcal{S}_{mass}$ and/or $\mathcal{S}_{CS}$ this symmetry can be broken down to $SO(5) \otimes SO(4)$ and naturally splits the A_a to a $SO(5)$ and a $SO(4)$ vector.

In the rest of this section we will be interested in the pure matrix models, i.e. matrices without any time-dependence[13]. We can give our actions as,

$$\tilde{S}_1 = \quad -\tfrac{1}{g^2}Tr([A_a, A_b][A_a, A_b] - \mu^2 A_a A_a). \tag{2.111}$$

$$\tilde{S}_2 = \quad -\tfrac{1}{g^2}Tr(\tfrac{1}{4}[A_a, A_b][A_a, A_b] + \tfrac{\lambda}{5}\epsilon^{abcd} A_a A_b A_c A_d A_e), \tag{2.112}$$

$\tilde{S}_2$ and $\tilde{S}_1$ are invariant under the global $SO(5)$ symmetry as well as the $U(N)$ gauge symmetry, which simply takes the form

$$A_a \to U A_a U^\dagger, \tag{2.113}$$

$$A_a \to A_a + c_a \mathbf{1}, \tag{2.114}$$

here. Equation of motion for $\tilde{S}_2$ follows from the variation $A_a \to A_a + \delta A_a$ which gives.

$$[A_b, [A_a, A_b]] + \lambda \epsilon^{abcde} A_a A_b A_c A_d A_e = 0, \tag{2.115}$$

One of the two classical solutions for the equation of motion (2.115) is simply given by the diagonal matrices.

$$A_a = \text{diag}\left(x_a^{(N)}, \ldots, x_a^{(1)}\right), \tag{2.116}$$

$\tilde{S}_2$ evaluated for this solution immediately yields $\tilde{S}_2 = 0$ Note that the diagonal matrix solution is more stable at the classical level than the fuzzy four sphere solution since corresponding values of the action at classical level. Another solution is provided by a fuzzy 4-sphere i.e. for $A_a = X_a$, provided that we take $\lambda = \frac{2}{n+2}$ The action then takes the value

$$\tilde{S}_2 = -\frac{4\rho^4}{5g^2}\frac{N}{n(n+4)} \tag{2.117}$$

Which is negative and therefore less then $\tilde{S}_2 = 0$. So it appears that fuzzy 4-sphere is a more stable solution to (2.115) then diagonal, commuting matrices. Let's start with expanding the matrices A_μ around the classical solution. In order to explore the fluctuations about this classical solution we write

$$A_a = l\rho\left(\frac{1}{\rho}X_a + a_a\right) \tag{2.118}$$

For convenience we can write the matrices which has dimensions of length

$$\omega_{ab} \equiv \alpha X_{ab} \tag{2.119}$$

Obviously α_{ab} satisfy the same relations as X_{ab}. Commutative limit of this construction can be obtained in the limit $l \to 0$ with fixed radius. In this limit coordinates and α_{ab} becomes commutative.

We have already seen that on S^4 that scalar fields can be expanded in terms of spherical harmonics. This was given in (2.83). Discussions of the previous sections indicate that, for fuzzy four sphere one has to also take into account that the fuzzy 2 spheres attached to the fuzzy four sphere at every point. This expansion for the fuzzy 4-sphere can achieved by expanding the $N \times N$ matrices in terms of the matrix spherical harmonics(i.e. polarization tensor) $y_{n_1 n_2 \tilde{m}_i}(x, \omega)$ as

$$M(x, \omega) = \sum_{n_2=0}^{n} \sum_{n_1, \tilde{m}_i} M_{n_1 n_2 \tilde{m}_i} y_{n_1 n_2 \tilde{m}_i}(x, \omega), \tag{2.120}$$

Here (n_1, n_2) label the $SO(5)$ IRR's (in the Dynkin labelling scheme) and with $1 \geq n_1 \geq n_2$. Only $n_1 = 0$ in this expansion overlap expansion of functions given in (2.83) on S^4 other terms indicate and characterize the internal spin structure of S_F^4 the internal spin is labeled by n_1 and is cut off at a maximal value n_2 for a given $n_2 \leq n$. We have $\sum_{n_1 \geq n_2}^{n} \dim(n_1, n_2) = N^2$. Corresponding functions may be written as

$$M(x, \omega) = \sum \sum M_{n_1 n_2 \tilde{m}_1} y_{n_1 n_2 \tilde{m}_i}(x, \omega) \tag{2.121}$$

The algebra generated by these functions is noncommutative but associative. Here the noncommutativity is generated by ω_{ab} Which also generate the fuzzy two spheres attached at each point on the fuzzy four sphere. A product which is commutative but not associative. We know how to act with X_a and X_{ab} on the M given in (2.120). They act adjointly as

$$AdX_a M = [X_a, M], \qquad AdX_{ab} M = [X_{ab}, M] \tag{2.122}$$

On the corresponding functions (2.121), and in the commutative limit $\alpha \to 0$, $n \to \infty$ with $\rho \to$ fixed adjoint actions of X_a & X_{ab} the differential forms

$$Ad(X_a) \to 2i \left(\omega_{ab} \partial_{x_b} - x_b \partial_{\omega_{ab}} \right), \tag{2.123}$$

$$Ad(X_{ab}) \to 2 \left(x_a \partial_{x_b} - x_b \partial_{x_a} - \omega_{ac} \partial_{\omega cb} + \omega_{bc} \partial_{\omega_{ca}} \right). \tag{2.124}$$

Where the derivative with respect to ω's are defined as

$$\frac{\partial \omega_{cd}}{\partial \omega_{ab}} = \delta_{ac} \delta_{bd} - \delta_{ad} \delta_{cb} \tag{2.125}$$

In the commutative limit the adjoint action of X_a and X_{ab} become the differential operators [13]

$$\mathrm{ad}\, X_a \quad \rightarrow \quad \nabla_a := 2i\left(\omega_{ab}\partial_{x_b} - x_b\partial_{\omega_{ab}}\right), \tag{2.126a}$$

$$\mathrm{ad}\, G_{ab} \quad \rightarrow \quad \nabla_{ab} := 2\left(x_a\partial_{x_b} - x_b\partial_a - \omega_{ac}\partial_{\omega_{cb}} + \omega_{bc}\partial_{\omega_{ca}}\right), \tag{2.126b}$$

We may note for future use that $x_a\nabla_a = 2i\left(x_a\omega_{ab}\partial_{x_b} - x_b x_b\partial_{\omega_{ab}}\right) = 0$, since the first term in the r.h.s is already noted to vanish and the second term vanishes due to the antisymmetry of ω_{ab}. We can explore these parts even more concretely by inspecting the transformation of $M(x,\omega)$ under $SO(5)$ rotations. $SO(5)$ acts on $M(x,\omega)$ by adjoint action as

$$AdX_{ab}M(x,\omega) = e^{iX_{ab}\omega_{ab}}M(x,\omega)e^{-iX_{ab}\omega_{ab}}. \tag{2.127}$$

Expanding the exponential terms to first order in ω_{ab} and using (2.124) we have

$$M(x,\omega) + i\omega_{ab}Ad(X)_{ab}M(x,\omega) \tag{2.128}$$

$$= a(x,\alpha) + 2i\alpha_{ab}\left(x_a\partial_{x_b} - x_b\partial_{x_a} - \alpha_{ac}\partial_{\alpha_{cb}} + \alpha_{cb}\partial_{\alpha_{ac}}\right)a(x,\alpha) \tag{2.129}$$

Note that when we are working with the differential forms of the operators we use the functions corresponding to matrices. Now we can determine the form of (2.129) at the north pole of the fuzzy 4-sphere. In this case last two terms in (2.129) can be expressed using $\theta_\mu = (\omega_{23},\omega_{31},\omega_{12})$, $\beta_\mu = (\omega_{41},\omega_{42},\omega_{43})$ amd N_μ representing the coordinates corresponding to N_μ in (2.86). More precisely writing $N_\mu = -\frac{i}{4}(\frac{1}{2}\epsilon_{\mu\nu\rho}\omega_{\nu\rho} - \omega_{\mu 4})$ and $\partial_{N_\mu} = 2i(\frac{1}{2}\epsilon_{\mu\nu\rho}\partial_{\omega_{\nu\rho}} - \partial_{\omega_{\mu 4}})$ We can find that at the North pole (2.129) takes the form

$$M(x,\omega) + 2i\omega_{ab}(x_a\partial_{x_b} - x_b\partial_{x_a})M(x,\omega) - 4i(\theta_\mu + \beta_\mu)\epsilon_{\mu\nu\rho}N_\nu\partial_{N_\rho}M(x,\omega), \tag{2.130}$$

The greek indices μ,ν runs from 1 to 3 as we have already stated earlier. We can easily see that the second term of (2.130) corresponds to the usual orbital angular momentum. While the last term governs the infinitesimal transformation under the internal spin angular momentum. Fields $a(x,\alpha)$ can also be Taylor Expanded in terms of coordinates N_μ at the North pole,

$$M(x,\omega) = a(x,0) + N_{\mu_1}\partial_{N_{\mu_1}}M\big|_{N=0} + \cdots + \frac{1}{n}N_{\mu_1}\ldots N_{\mu_n}\partial^n_{N_{\mu_1}\ldots N_{\mu_n}}M\big|_{N=0} \tag{2.131}$$

The first term in (2.131) can be interpreted as a scalar field with zero spin while the $(m+1)$th term can be carries spin-m i.e. $\tilde{M}_{\mu_1\mu_2\ldots\mu_m} := \partial_{N_{\mu_1}\ldots N_{\mu_m}}\big|_{N=0}$ is a spin m field..

2.3.1 Action $\tilde{S}_2$ as a Gauge Theory on S_F^4

In the last part of this chapter we follow [13] to discuss how $\tilde{S}_2$ in (2.112) can be given the structure of a gauge theory on S_F^4. A gauge covariant field strength tensor may be introduced as,

$$F_{ab} := \frac{1}{(\alpha\rho)^2}\left([A_a, A_b] + \frac{1}{2}\lambda\epsilon^{abcde}[A_c, A_d]A_e\right), \tag{2.132}$$

Using the gauge transformation of A_a, we immediately see that F_{ab} transforms covariantly under $SU(N)$ gauge group as $F_{ab} \to U^\dagger F_{ab} U$. Inserting (2.118) in (2.132) We find

$$= \frac{1}{\rho}[X_a, a_b] - \frac{1}{\rho}[X_b, a_a] + [a_a, a_b] + \alpha\rho\lambda\epsilon^{abcde}\left(\frac{1}{\rho^2}X_{cd}a_e + \frac{1}{\rho}[X_c, a_d]\left(\frac{1}{\rho}X_e d + a_e\right)\right). \tag{2.133}$$

Clearly $TrF_{ab}F_{ab}$ is a gauge invariant term, which what we intend to introduce in rewriting $\tilde{S}_2$. We observe, from (2.132) that $F_{ab}F_{ab}$ involves 5^{th} & 4^{th} order terms in A_a which are not present in $\tilde{S}_2$, so these terms must be subtracted from $TrF_{ab}F_{ab}$. A long but a straightforward calculation gives that

$$\begin{aligned}\tilde{S}_2 = -\frac{(l\rho)^4}{g^2}Tr(\frac{1}{4}F_{ab}F_{ab} - \frac{9\lambda}{40(l\rho)^2}\epsilon^{abcde}[A_a, A_b][A_c, A_d]A_e \\ -\frac{\lambda^2}{16(l\rho)^2}f^{abcdef}[A_a, A_b]A_c[A_d, A_e]A_f),\end{aligned} \tag{2.134}$$

where f^{abcdef} follows from the contraction of two epsilon tensors $\epsilon^{ghabc}\epsilon^{ghdef}$ which we have explicitly showed in the appendix (B.5). For the infinitesimal gauge transformations

$$U = e^{i\lambda} \cong 1 + i\lambda + O(\lambda^2). \tag{2.135}$$

We have $A'_a \to A_a + i[\lambda, A_a] = A + \delta A$ and therefore inserting (2.118) we get.

$$\begin{aligned}\delta A_a \quad &= i\alpha\rho\Big[\lambda, \tfrac{1}{\rho}X_a + a_a\Big], \qquad (2.136)\\ &= i\alpha[\lambda, X_a] + i\alpha\rho[\lambda, a_a], \\ &= \rho\alpha\left(\tfrac{i}{\rho}[X_a, \lambda] + i[\lambda, a_a]\right),\end{aligned}$$

Thus we get δa_a as

$$\delta a_a(x, \alpha) = \frac{i}{\rho}[X_a, \lambda(x, \alpha)] + i[\lambda(x, \alpha), a_a(x, \alpha)], \tag{2.137}$$

Thus a_a are the gauge fields on the fuzzy S^4 background geometry while the (2.137) represents the infinitesimal gauge transformations of a_a Finally, let us make a few comments about the content of the gauge theory action. We will be brief here, as this is not going to be the direction that we pursue to develop in the next chapter. A natural laplacian operator on S_F^4 is given by $ad(X_{ab})^2 = [X_{ab}, [X_{ab}, .]]$ since it is the quadratic Casimir operator of the symmetry group of S_F^4 which is $SO(5)$. On the other hand we can also choose the operator $ad(X_a)^2 = [X_a, [X_a, .]]$. This is also an invariant of $SO(5)$ and can indeed be written as difference of Casimir operators of $SO(6)$ and $SO(5)$ We see from (2.133) and the first term of (2.134) that the term involving the double commutators of X_a is given by

$$S_{\text{Laplacian}} = \frac{(l\rho)^4}{2g^2} Tr\left(a_b\left[\frac{X_a}{\rho}, \left[\frac{X_a}{\rho}, a_b\right]\right]\right), \tag{2.138}$$

and we therefore infer that the second option for the Laplacian is naturally appears in the action (2.134). The corresponding spectrum for $(adX_a)^2$ may be calculated using group theory. On the spherical polarization tensors it takes its diagonal form and given by

$$adX_a^2 Y_{r_1 r_2} = [X_a, [X_a, Y_{r_1 r_2}]] = 4(r_1(r_1+3) - r_2(r_2+1))Y_{r_1 r_2}. \tag{2.139}$$

On the other hand the spectrum of $(ad(X_{ab}))^2$ on $Y_{r_1 r_2}$ is given by

$$adX_{ab}^2 Y_{r_1 r_2} = [X_{ab}, [X_{ab}, Y_{r_1 r_2}]] = 8(r_1(r_1+3) + r_2(r_2+1))Y_{r_1 r_2} \tag{2.140}$$

Up to this point we have worked with the $U(1)$ gauge group. One can easily generalize this to a general $U(n)$ gauge group in the following way.

$$x_a \rightarrow x_a \otimes \mathbb{1}_n \tag{2.141}$$

and the replacement for the fluctuations are

$$a = \sum_{a=1}^{n^2} a^k \otimes T^k \tag{2.142}$$

Where T^k are denoting the generators of the $U(n)$ algebra.

CHAPTER 3

EQUIVARIANT FIELDS ON S_F^4

3.1 Mass Deformed Yang Mills Matrix Model

3.1.1 Matrix Models & the Fuzzy S^4 Configurations

From now on we will focus on the Yang Mills Model with the mass deformation and with the gauge group $U(4N)$. The action is given as

$$\mathcal{S}_1 = \frac{1}{g^2} \int dx Tr \left(\frac{1}{2} (\mathcal{D}_t A_a)^2 + \frac{1}{4} [A_a, A_b][A_a, A_b] - \mu^2 A_a A_a \right), \tag{3.1}$$

where A_a are $4N \times 4N$ matrices. The potential part can be written out separately as

$$\mathcal{V}_1 = \frac{1}{g^2} Tr \left(-\frac{1}{4} [\mathcal{X}_a, \mathcal{X}_b]^2 + \mu^2 \mathcal{X}_a^2 \right) \tag{3.2}$$

$\mathcal{V}_1$ is extremized by the matrices fulfilling the equation

$$[A_b, [A_a, A_b]] - 2\mu^2 A_a = 0 \tag{3.3}$$

Equation (3.3) is solved by a configuration given by four concentric S_F^4 as

$$A_a = X_a \otimes \mathbb{1}_4 \tag{3.4}$$

where X_a are $N \times N$ matrices that form S_F^4 as discussed in the previous chapter. Dimension of N of these matrices is determined by the level n of the fuzzy 4-spheres as(a detailed calculation from group theory is given in the appendix (A.1.4).)

$$N = \frac{1}{6}(n+1)(n+2)(n+3), \tag{3.5}$$

Fuzzy four spheres S_F^4 and their direct sums (even from different matrix levels) are solutions of this equation for $\mu^2 = -8$. In a recent article Steinacker [7] showed that the superficial instability implied by the negativity of μ^2 is actually cured by quantum corrections in pure YM matrix model (i.e. in matrix models with no time dependence). Let us also note that, we will see that superficial instability implied by negativity of μ^2 does not actually lead to a problem when we consider the equivariant fluctuations of the $\mathcal{S}_1$ action about the S_F^4 backgrounds. The reason for this is essentially that the potential of the emergent equivariantly reduced action is bounded from below at any finite matrix level, as we will see and discuss in more detail later on. We may as well interpret this outcome as being due to the fact that the equivariant parametrization of the fluctuations introduces fluxes through the S_F^4 stabilizing its radius. As we will see later on non-trivial fluxes leaves its imprints as kink type solutions in the reduced action in Euclidean signature. This vacuum configuration breaks the $U(4N)$ symmetry of the action to $U(N) \times U(4)$ and after setting A_a as coordinates of S_F^4's indicated by (3.4), we have only a $U(4)$ gauge symmetry left. Fluctuations about (3.4) may be written in general as.

$$A_a = X_a \otimes \mathbb{1}_4 + F_a \tag{3.6}$$

3.1.2 Equivariant Fluctuation & Their Parametrization

Our aim is to find the fluctuations which are left invariant under the $SO(5)$ rotations of S_F^4 up to $SU(4)$ gauge transformations. To do so, we introduce the symmetry generators.

$$W_{ab} = X_{ab} \otimes \mathbb{1}_4 + \mathbb{1}_N \otimes \Sigma_{ab}\,, \tag{3.7}$$

Where

$$\Sigma_{ab} = \frac{1}{2}[\Gamma_a, \Gamma_b] \tag{3.8}$$

are the generators of $SO(5)$ in the 4-dimensional fundamental spinor representation labeled by $(0,1)$. They can be embedded into the generators of $SO(6)$ as

$$\Sigma_{AB} \equiv (\Sigma_{ab}, \Sigma_{a6}) \equiv (\Sigma_{ab}, i\Gamma_a) \tag{3.9}$$

where A, B takes the values $1 \ldots 6$. Σ_{AB} generate of the $SO(6) \equiv \frac{SU(4)}{\mathbb{Z}_2}$ in the fundamental spinor representation labeled by $(1,0,0)$. Coming back to (3.7) we clearly see that W_{ab} satisfies the $SO(5)$ commutation relations. However, it carries reducible representation of $SO(5)$, since X_{ab} in the first terms carries the $(0,n)$ IRR, while Σ_{ab}

carries the $(0,1)$ IRR of $SO(5)$. The irreducible representation of W_{ab} can be determined from the decomposition of the tensor product of these two representations. This gives [26]

$$(0,n)\otimes(0,1)\equiv(0,n+1)\oplus(1,n-1)\oplus(0,n-1) \tag{3.10}$$

This discussion can be easily lifted to $SO(6)$ group by writing $W_{AB}\equiv(W_{ab},W_{a6})$ with has the decomposition under $SO(6)$ irreducibles as

$$(n,0,0)\otimes(1,0,0)\equiv(n+1,0,0)\oplus(n-1,1,0) \tag{3.11}$$

Branching of (3.11) under $SO(5)$ IRRs yields precisely (3.10) (We show this explicitly in the Appendix(A.1.1). Let us now investigate the adjoint action of W_{ab}, which is given as

$$adW_{ab}=[W_{ab},.]=[X_{ab},.]+[\Sigma_{ab},.] \tag{3.12}$$

Suppressing the tensor product with the identity matrices for ease in notation in (3.12)

First term generates the infinitesimal $SO(5)$ rotations of the S_F^4, while the second term generates the $SO(6)\equiv\frac{SU(4)}{\mathbb{Z}_2}$ gauge transformations in $SO(5)$. The irreducible representation content of adW_{ab} is given by the tensor product of r.h.s. of (3.10) with itself, That is

$$[(0,n+1)\oplus(1,n-1)\oplus(0,n-1)]^{\otimes 2} \tag{3.13}$$

Using the LieArt package of Mathematica we infer that (3.12) has the IRR content

$$\mathbf{3}(0,0)\oplus\mathbf{7}(1,0)\oplus\text{Higher dimensional IRRs} \tag{3.14}$$

where the bold terms represents the multiplicities of the respective IRRs. The part of the direct sum given in (3.14) will be sufficient for our purposes. Using (3.12) we define and impose the equivariant symmetry constraint on the gauge field $\mathcal{A}$ and the fluctuations F_c around the (3.4) as

$$adW_{ab}\mathcal{A}_0=[W_{ab},\mathcal{A}_0] \;=\; 0\,, \tag{3.15a}$$

$$adW_{ab}F_c=[W_{ab},F_c] \;=\; -2(\delta_{ac}F_b-\delta_{bc}F_a)\,, \tag{3.15b}$$

The first requirement means that the gauge fields $\mathcal{A}_0$ are transforming as scalars of $SO(5)$ under the adjoint action of W_{ab}, which is naturally expected since they do not carry a $SO(5)$ index. The second requirement implies that the fluctuations F_a around the (3.4) transform as a vector of $SO(5)$. So we can deduce from the decomposition (3.14) given above that the space of rotational invariants that may be constructed from $(0,n)$ and $(0,1)$ IRRs of $SO(5)$ is 3-dimensional and the space of vectors that may be constructed in terms of the intertwiners $(0,n)\&(0,1)$ IRRs and X_a is of dimension 7. In order to explicitly obtain the mentioned intertwiners of IRRs we can introduce the projection operator.

$$P_I := \prod_{J\neq I} \frac{-(X_{ab}+\Sigma_{ab})^2 - 2\,C_2(\lambda_J)}{2\,C_2(\lambda_I) - 2\,C_2(\lambda_J)}\,, \quad P_I^2 = P_I\,, \quad P_I^\dagger = P_I\,, \quad I:1,2,3\,, \tag{3.16}$$

where the factors of two in front of Casimirs are due to the unrestricted sum over a's and b's. P_I are projections to the IRRs of $SO(5)$ in the order given in the r.h.s of (3.10) and $C_2(\lambda_I)$ stand for the quadratic Casimirs of $SO(5)$ in the IRRs labeled by $\lambda_I \equiv ((0,n+1),(1,n-1),(0,n-1))$ Using (3.16) and the fact that idempotents can be given as $Q_I = \mathbb{1}_{4N} - 2P_I$ we may compute the intertwiners $(0,n)\&(0,1)$ IRRs as the idempotents

$$Q_1 \;=\; \frac{(X\cdot\Sigma - 4)(X\cdot\Sigma - 4n - 16) - 16(n+1)(n+2)}{16(n+1)(n+2)}\,, \tag{3.17a}$$

$$Q_2 \;=\; \frac{(X\cdot\Sigma + 4n)(X\cdot\Sigma - 4n - 16) + 2(2n+2)(2n+6)}{-2(2n+2)(2n+6)}\,, \tag{3.17b}$$

$$Q_3 \;=\; \frac{(X\cdot\Sigma - 4)(X\cdot\Sigma + 4n) - 16(n+3)(n+2)}{16(n+3)(n+2)}\,. \tag{3.17c}$$

By construction it is obvious that we do have $Q_I^2 = \mathbb{1}_{4N}$ and $Q_I^\dagger = Q_I$ (As shown in appendix (B.1)). Let us also note that Q_I are not all independent from each other as we have $\sum_I Q_I = -\mathbb{1}_{4N}$.

Observe that $(X.\Sigma)$ appears in formulae (3.17a). A straightforward, but long calculation, whose details are given in the Appendix (B.15), gives

$$(X\cdot\Sigma)^2 = 12\,\Gamma_a\Gamma_b G_{ab} + 8n(n+2)X_a\Gamma_a + 8n(n+4)\mathbb{1}_{4N}\,. \tag{3.18}$$

Adjoint representation of $SO(6)\approx SU(4)$ branches under $SO(5)$ as $\mathbf{15}\to\mathbf{5}\oplus\mathbf{10}$, or in Dynkin notation[26]:

$$(1,0,1)\equiv(1,0)\oplus(0,2)\,. \tag{3.19}$$

Thus, further insight on how $SO(6)\approx SU(4)$ generators sits in these intertwiners is gained by observing that Q_I contain, 10 of these generators as Σ_{ab}, and the remaining

5 as Γ_a as seen from (3.18), transforming in $\mathbf{2}(0,1)$ and $(1,0)$ of $SO(5)$, respectively. Alluding to our remarks in the previous subsection after (3.5) we may say that the equivariant parametrization of the fluctuations introduces topological fluxes through the concentric S_F^4's, preventing the latter to shrink to zero radius and thereby stabilizes the configuration. Using

$$\Omega_{AB} \equiv (\omega_{ab}, x_a) := \lim_{n\to\infty} \frac{X_{AB}}{n} \,, \tag{3.20}$$

we find the commutative limit of (3.18) takes the form

$$\lim_{n\to\infty} \frac{(X\cdot\Sigma)^2}{n^2} = 8(x_a\Gamma_a + \mathbb{1}_4) \,. \tag{3.21}$$

Consequently, we find for $q_I := \lim\limits_{n\to\infty} Q_I$:

$$q_1 = \frac{1}{2}\left(x_a\Gamma_a - \sum_{a<b}\omega_{ab}\Sigma_{ab} - \mathbb{1}_4\right) \,, \tag{3.22a}$$

$$q_2 = -x_a\Gamma_a \,, \tag{3.22b}$$

$$q_3 = \frac{1}{2}\left(x_a\Gamma_a + \sum_{a<b}\omega_{ab}\Sigma_{ab} - \mathbb{1}_4\right) \,. \tag{3.22c}$$

Without going into any technicalities, regarding the S^2 effective fiber coordinates ω_{ab} over S^4, our previous remark supported by the observation that the commutative limit of this topological flux may be seen to be characterized via the second Chern number on S^4

$$c_2(S^4) = \frac{1}{8\pi^2}\int_{S^4} p_2\,(d\,p_2)\,(d\,p_2) = 1 \,, \tag{3.23}$$

for the rank 4 projectors $p_2 = \frac{1}{2}(1-q_2)$ [22].

Using these Q's we can solve the constraints given in (3.15a) and (3.15b) as follows. To satisfy (3.15a), we may choose to parameterize the gauge field A_0 as

$$\mathcal{A}_0 = \frac{1}{2}\alpha_1 Q_1 + \frac{1}{2}\alpha_2\mathbb{1}_{4N} + \frac{1}{2}\alpha_3 Q_3 \,, \tag{3.24}$$

where $\alpha_i = \alpha_i(t)$ $(i:1,2,3)$ are functions of time only, and Q_2 is eliminated in favor of $\mathbb{1}_{4N}$ using $\sum_I Q_I = -\mathbb{1}_{4N}$. From this form of the gauge field it can be easily observed that the $SU(4)$ gauge symmetry is broken down to $U(1)\times U(1)\times U(1)$. However, later on we will see that the term proportional to identity matrix in (3.24) does not survive the dimensional reduction and the gauge symmetry of the reduced action is essentially $U(1)\times U(1)$.

Now let us investigate the fluctuations F_a that satisfy the requirement (3.15b) To parametrize such vector we will again use the idempotents Q_I and the matrices X_a we have calculated previously. We have found the most useful parametrization for future purposes is

$$
\begin{aligned}
F_a = i\frac{\phi_1}{2}[X_a, Q_1] + i\frac{\chi_1}{2}[X_a, Q_3] + \frac{\phi_2+1}{2}Q_1[X_a, Q_1] + \frac{\chi_2+1}{2}Q_3[X_a, Q_3] \\
+ \phi_3\left(\left\{\hat{X}_a, Q_1\right\} - Q_3[\hat{X}_a, Q_3]\right) + \chi_3\left(\left\{\hat{X}_a, Q_3\right\} - Q_1[\hat{X}_a, Q_1]\right) \\
+ \phi_4\left(\hat{X}_a + \hat{\Gamma}_a + Q_3[\hat{X}_a, Q_3]\right) .
\end{aligned}
\tag{3.25}
$$

where curly brackets stand for anti-commutators and we have introduced $\phi_\mu = \phi_\mu(t)$ and $\chi_\nu = \chi_\nu(t)$ $(\mu, \nu) : (1, 2, 3, 4)$ as real functions of time only and the notation

$$
\hat{X}_a = \frac{X_a \otimes \mathbb{1}_4}{n} \equiv \frac{X_a}{n} , \quad \hat{\Gamma}_a = \frac{\mathbb{1}_N \otimes \Gamma_a}{n} \equiv \frac{\Gamma_a}{n} . \tag{3.26}
$$

The $\frac{1}{n}$ factors appearing in the last three terms of F_a via, $\hat{X}_a$ and $\hat{\Gamma}_a$ are naturally expected to obtain a finite F_a in the commutative limit $n \to \infty$. Similar analysis, on previous work on equivariant parameterizations of fluctuations over S_F^2 and $S_F^2 \times S_F^2$ [27, 12] also carries the same features. Indeed, as $n \to \infty$, we find

$$
\begin{aligned}
F_a \to f_a := i\frac{\phi_1}{2}\nabla_a q_1 + i\frac{\chi_1}{2}\nabla_a q_3 + \frac{\phi_2+1}{2}q_1\nabla_a q_1 + \frac{\chi_2+1}{2}q_3\nabla_a q_3 \\
+ \phi_3 2x_a q_1 + \chi_3 2x_a q_3 + \phi_4 x_a .
\end{aligned}
\tag{3.27}
$$

Where q_I are given in (3.22a). Demanding the fluctuations f_a to be tangential to S^4 means that we have to take $x_a f_a = 0$. Since $x_a \nabla_a = 0$, as noted after (2.125), the latter condition is satisfied if and only if $\phi_3(t)$, $\chi_3(t)$ and $\phi_4(t)$ all vanish in this limit.

3.2 Dimensional Reduction of the Action $\mathcal{S}_1$

3.2.1 Structure of the Kinetic Term

Inserting the parametrizations (3.24) and (3.25) into the covariant derivative $D_t A_a = \partial_t A_a - i[\mathcal{A}_0, A_a]$ we have

$$\begin{aligned}
D_t X_a =& i\frac{\partial_t \phi_1}{2}[X_a, Q_1] + i\frac{\partial_t \chi_1}{2}[X_a, Q_3] \\
&+ \frac{\partial_t \phi_2}{2} Q_1 [X_a, Q_1] + \frac{\partial_t \chi_2}{2} Q_3 [X_a, Q_3] \\
&+ \partial_t \phi_3 (\{X_a, Q_1\} - Q_3[X_a, Q_3]) + \partial_t \chi_3 (\{X_a, Q_3\} - Q_1 [X_a, Q_1]) \\
&+ \partial_t \phi_4 (X_a + \gamma_a + Q_3 [X_a, Q_3]) + i\frac{\alpha_1 \phi_2}{4}[Q_1, [X_a, Q_1]] \\
&+ i\frac{\alpha_1 \chi_1}{4}[Q_1, [X_a, Q_3]] + \frac{\alpha_1(\phi_2 + 1)}{4}[Q_1, Q_1[X_a, Q_1]] \\
&+ \frac{\alpha_1(\chi_2 + 1)}{4}[Q_1, Q_3[X_a, Q_3]] + \frac{\alpha_1 \phi_3}{2}[Q_1, \{X_a, Q_1\} - Q_3[X_a, Q_3]] \\
&+ \frac{\alpha_1 \chi_3}{2}[Q_1, \{X_a, Q_3\} - Q_1[X_a, Q_1]] + \frac{\alpha_1 \phi_4}{2}[Q_1, X_a + \gamma_a + Q_3[X_a, Q_3]] \\
&+ i\frac{\alpha_2 \phi_1}{2}[Q_3, [X_a, Q_1]] + i\frac{\alpha_2 \chi_2}{4}[Q_3, [X_a, Q_3]] + \frac{\alpha_2(\phi_1 + 1)}{4}[Q_3, Q_1[X_a, Q_1]] \\
&+ \frac{\alpha_2(\chi_2 + 1)}{4}[Q_3, Q_3[X_a, Q_3]] + \frac{\alpha_2 \phi_3}{2}[Q_3, \{X_a, Q_1\} - Q_3[X_a, Q_3]] \\
&+ \frac{\alpha_2 \chi_3}{2}[Q_3, \{X_a, Q_3\} - Q_1[X_a, Q_1]] + \frac{\alpha_2 \phi_4}{2}[Q_3, X_a + \Gamma_a + Q_3[X_a, Q_3]] + \\
&+ \frac{\alpha_1}{2}[Q_1, X_a] + \frac{\alpha_2}{2}[Q_3, X_a].
\end{aligned} \tag{3.28}$$

Using the identities we provided in appendix (B.31) and some more commutation relations we can simplify the (3.28) to

$$\begin{aligned}
D_t X_a =& \frac{i}{2}(\partial_t \phi_1 - i\alpha_1 \phi_2)[X_a, Q_1] + \frac{i}{2}(\partial_t \chi_1 - i\alpha_2 \chi_2)[X_a, Q_3] \\
&+ \frac{1}{2}(\partial_t \phi_2 + i\alpha_1 \phi_1) Q_1 [X_a, Q_1] + \frac{1}{2}(\partial_t \chi_2 + i\alpha_1 \chi_2) Q_3 [X_a, Q_3] \\
&+ \partial_t \phi_3 (\{X_a, Q_1\} - Q_3 [X_a, Q_3]) + \partial_t \chi_3 (\{X_a, Q_3\} - Q_1 [X_a, Q_1]) \\
&+ \partial_t \phi_4 (X_a + \Gamma_a + Q_3 [X_a, Q_3]).
\end{aligned} \tag{3.29}$$

Now we can introduce covariant derivative for the fields ϕ_i and χ_i where i runs from 1 to 2 as

$$\begin{aligned} D_t\phi_i &= \partial_t\phi_i + i\epsilon_{ji}\alpha_1\phi_j, \\ D_t\chi_1 &= \partial_t\chi_i + i\epsilon_{ji}\alpha_2\chi_j. \end{aligned} \tag{3.30}$$

Inserting these two into (3.29) we get

$$\begin{aligned} D_tX_a =& \frac{i}{2}(D_t\phi_1 - iQ_1D_t\phi_2)[X_a, Q_1] + \frac{i}{2}(D_t\chi_1 - iQ_3D_t\chi_2)[X_a, Q_3] \\ &+ \partial_t\phi_3(\{X_a, Q_1\} - Q_3[X_a, Q_3]) + \partial_t\chi_3(\{X_a, Q_3\} - Q_1[X_a, Q_1]) \\ &+ \partial_t\phi_4(X_a + \gamma_a + Q_3[X_a, Q_3]). \end{aligned} \tag{3.31}$$

We are in position to calculate trace in the kinetic term of $\mathcal{S}_1$. Using both analytic techniques & Mathematica we found the following results for the kinetic term. After performing the traces over the matrices.:

$$\begin{aligned} \frac{1}{2}Tr(D_0X_a)^2 =& \frac{n(n+4)}{(n+1)^2}|D_0\phi|^2 + \frac{n(n+4)}{(n+3)^2}|D_0\chi|^2 \\ &+ \frac{2(n+4)(n^5+8n^4+18n^3+8n^2-11n)}{n^2(n+1)^2(n+3)^2}(\partial_0\phi_3)^2 \\ &- \frac{12n(n+4)}{n^2(n+1)(n+3)}(\partial_0\phi_3\partial_0\chi_3) \\ &+ \frac{n(n+4)(-n^3-3n^2+17n+35)}{n^2(n+1)(n+3)^2}(\partial_0\phi_3\partial_t\phi_4) \\ &- \frac{n(n+4)(n+5)}{n^2(n+3)}(\partial_0\phi_4\partial_0\chi_3) \\ &+ \frac{(n^4+10n^3+30n^2+34n+45)}{2n^2(n+3)^2}(\partial_0\phi_4)^2 \\ &+ \frac{2n(n+4)(n^4+8n^3+18n^2+8n-11)}{n^2(n+3)^2(n+1)^2}(\partial_0\chi_3)^2 \end{aligned} \tag{3.32}$$

As it stands (3.32) does not seem manifestly to be positively definite but with a quick Mathematica check we can confirm that this is indeed, as it should be by construction. In fact, we can make a linear field redefinition in the sector spanned by ϕ_3, ϕ_4, χ_3 such that the kinetic term (3.32) becomes diagonalized. The generic form of the diagonalized kinetic term at a given value of n may be calculated, but it appears to be a rather cluttered formula, with not practical value. So we don't give it here. In the ensuing section, we will work with actions with such redefined fields for the span of values $n = 2, 3, 4, 5$. In the large n limit the trace (3.32) becomes the following

$$\lim_{n\to\infty} \frac{1}{2} Tr(D_0 X_a)^2 = |D_0\phi|^2 + |D_0\chi|^2 + 2(\partial_t\phi_3)^2 \\ - \partial_t\phi_3\partial_0\phi_4 - \partial_0\phi_4\partial_0\chi_3 + \frac{1}{2}(\partial_0\phi_4)^2 + 2(\partial_0\chi_3)^2 . \tag{3.33}$$

3.2.2 Structure of the mass term

Now that we have calculated kinetic term we can continue with the calculation of the mass term which is

$$Tr(A_a A_a) = Tr(X_a X_a + X_a F_a + F_a X_a + F_a F_a) \tag{3.34}$$

where X_a are the vacuum solutions that describe S_F^4. As in the previous case performing analytic calculations and exploiting Mathematica we find the trace of the mass term as.

$$\begin{aligned} &-\mu^2 Tr(A_a A_a) = -\mu^2 Tr(X_a X_a + 2X_a F_a + F_a F_a) \\ &= -\mu^2 \Bigg(\frac{2n(n+4)}{(n+1)^2}|\phi|^2 + \frac{2n(n+4)}{(n+3)^2}|\chi|^2 + \frac{4(n+4)(n^4+8n^3+18n^2+8n-11))}{n(n+1)^2(n+3)^2}(\phi_3^2+\chi_3^2) \\ &\quad + \frac{n^4+10n^3+30n^2+34n+45}{n^2(n+3)^2}\phi_4^2 + \frac{2(n+4)(-n^3-3n^2-17n+35)}{n(n+1)(n+3)^2}\phi_3\phi_4 \\ &\quad - \frac{24(n+4)}{n(n+1)(n+3)}\phi_3\chi_3 - \frac{2(n+4)(n+5)}{n(n+3)}\chi_3\phi_4 + \frac{(n+4)(-n^3-4n^2+7n+22)}{(n+3)^2(n+1)}\phi_3 \\ &\quad + \frac{(n+4)(-n^3-8n^2-9n+6)}{(n+1)^2(n+3)}\chi_3 + \frac{(n+4)(n^2+6n+5)}{(n+3)^2}\phi_4 + C(n) \Bigg) . \end{aligned} \tag{3.35}$$

where $C(n)$ is an irrelevant constant term. Later, when we consider dynamics of reduced actions we will adjust the overall constant factors in the action so that the minimum of the potentials take the value zero.

Let us make a few remarks on (3.35) it contains terms that are linear in the fields ϕ_3, ϕ_4 and χ_3. These terms cause no harm for any finite values of n which is what we will be interested in the following sections. However, to obtain a finite limit for the $n \to \infty$ case. it is required to assume that ϕ_3, ϕ_4 and χ_3 vanish faster than $\frac{1}{n}$. Then(3.35) converges to $-2\mu^2(|\phi|^2 + |\chi|^2)$ in this limit. Let us also note that we have $\mu^2 = -8$ since we are inspecting this term around the S_F^4 extremum satisfying (3.3)

3.2.3 Structure of the quartic term in $\mathcal{S}_1$

Now we are in position to discuss the quartic interaction term in $\mathcal{S}_1$ which is

$$\frac{1}{4}Tr[A_a, A_b][A_a, A_b], \tag{3.36}$$

whit $A_a = X_a + F_a$ and F_a given in (3.25). From (3.25) and (3.4) it is readily observed that analytic calculations of these terms appears to be a formidable task so instead we calculate (3.36) for values $n = 1, 2, 3, 4, 5$ using Mathematica. Since these values already corresponds to large span of matrix sizes $4N = 16, 40, 80, 140, 224$ respectively they will give us sufficient information to explore the dynamics of the low energy reduced action.

3.3 Dynamics of the Reduced Action

3.3.1 Gauge symmetry and the Gauss Law Constraint

From (3.32) we see that the gauge fields $\alpha_2(t)$ decouples completely after dimensional reduction. Therefore the reduced actions obtained from $\mathcal{S}_1$ are invariant under the remaining $U(1) \times U(1)$ gauge group. The gauge transformations are given as

$$\begin{aligned} \phi' &= e^{-i\Lambda_1(t)}\phi\,, \quad \alpha_1'(t) = \alpha_1(t) + \partial_t\Lambda_1(t)\,, \\ \chi' &= e^{-i\Lambda_3(t)}\chi\,, \quad \alpha_3'(t) = \alpha_3(t) + \partial_t\Lambda_3(t) \end{aligned} \tag{3.37}$$

The ϕ_3,ϕ_4 and χ_3 are real and thus uncharged under this $U(1)\times U(1)$ symmetry. Since time derivatives of the fields $\alpha_1(t)$ and $\alpha_2(t)$ does not appear in our action they have no dynamics on their own. Thus their equations of motion will be algebraic in other words they will be constraints that need to be fulfilled by the complex fields ϕ and χ These are called the the Gauss Law constraints and from the equation of motions of $\alpha_1(t)$ & $\alpha_2(t)$ that can br calculated using (3.32) we find

$$\begin{aligned} \frac{1}{2i}\frac{1}{|\phi|^2}(\phi(\partial_t\phi)^* - (\partial_t\phi)\phi^*) &= \alpha_1(t)\,, \\ \frac{1}{2i}\frac{1}{|\chi|^2}(\chi(\partial_t\chi)^* - (\partial_t\chi)\chi^*) &= \alpha_3(t)\,. \end{aligned} \tag{3.38}$$

We can choose to work in the gauge $\alpha_1(t) = 0 = \alpha_3(t)$. This means that even if these gauge fields are not zero we can make them vanish by the $U(1) \times U(1)$ gauge transformation given as

$$\alpha_i(t) \to \alpha_i'(t) = \alpha_i(t) + \partial_0 F_i(t), \tag{3.39}$$

where F_i are gauge functions and we can choose them such that $\alpha_i'(t) = 0$. Thus we assume that such a gauge transformation is already made and we set $\alpha_i(t) = 0$. This gauge choice can also be realized as the reality conditions $\phi^* = \phi$ and $\chi^* = \chi$. Further investigating the gauge choice we observe that the Gauss law constraints (3.40) does not break the $U(1) \times U(1)$ gauge symmetry completely, but a residual $\mathbb{Z}_2 \times \mathbb{Z}_2$ remains. To be more explicit, we can write $\phi \equiv (\phi_1, \phi_2) = |\phi|(\cos\theta, \sin\theta)$ and $\chi \equiv (\chi_1, \chi_2) = |\chi|(\cos\sigma, \sin\sigma)$, to express the constraints (3.38) in the form

$$\partial_t\theta = \frac{1}{|\phi|^2}\varepsilon_{ij}\phi_i\partial_t\phi_j = \partial_t\Lambda_1 = 0\,, \quad \partial_t\sigma = \frac{1}{|\chi|^2}\varepsilon_{ij}\chi_i\partial_t\chi_j = \partial_t\Lambda_3 = 0\,. \tag{3.40}$$

Therefore, the remaining $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry is encoded in the gauge functions as $\Lambda_1(t) = \Lambda_1^0 + \pi k_1$ and $\Lambda_3(t) = \Lambda_3^0 + \pi k_3$, where Λ_1^0 and Λ_3^0 are constants and $k_1, k_3 \in \mathbb{Z}_2$. This indicates that, for either of the gauge functions, Λ_1 or Λ_3, we have more generally

$$\int_{-\infty}^{\infty} dt\, \partial_t\Lambda = \Lambda(\infty) - \Lambda(-\infty) = \pi k \tag{3.41}$$

Due to (3.40), we have $\theta(t) = \theta^0 + \pi k_1$ and $\sigma(t) = \sigma^0 + \pi k_3$, and (3.41) holds for both $\theta(t)$ and $\sigma(t)$, as well. Having noted these points, we set $\phi_2(t)$ and $\chi_2(t)$ to zero (i.e., we have both θ^0 and σ^0 set to zero). Then, the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry is implemented by $(\phi_1\,, \chi_1) \to (\pm\phi_1\,, \pm\chi_1)$ & $(\phi_1\,, \chi_1) \to (\pm\phi_1\,, \mp\chi_1)$. In section (3.5) we will consider the structure of the LEAs in the Euclidean time τ. Due to the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry, we will be able to explore possible kink type solutions of the LEAs by choosing the appropriate boundary conditions on $\phi_1(\tau)$ and $\chi_1(\tau)$ as $\tau \to \pm\infty$. Presence of topologically non-trivial boundary conditions on the latter can then be attributed to the property (3.41) of the restricted gauge functions, which holds the same in the Euclidean signature.

3.4 Structure of the Reduced Actions

Diagonalizing the kinetic term and setting ϕ_2 and χ_2 to the zero as discussed in the previous section. LEA's take a relatively simple form. For $n = 2, 3, 4, 5$ we have that the Lagrangian is a functional of the five fields $\phi_1, \chi_1, \phi_3, \chi_3, \phi_4$ and their generalized velocities $\dot{\phi}_1, \dot{\chi}_1, \dot{\phi}_3, \dot{\chi}_3, \dot{\phi}_4$. For instance, at $n = 2$, we have

$$\begin{aligned}
L_{(n=2)} =& \frac{1}{2}\left(0.96\dot{\chi}_1^2 + 2.7\dot{\phi}_1^2 + 12.94\dot{\phi}_3^2 + 6.32\dot{\phi}_4^2 + 0.88\dot{\chi}_3^2\right) - 1.09\chi_1^4 - 0.252\chi_3^4 \\
& - 2.03\chi_3^3 + 6.99\chi_1^2 - 0.26\chi_1^2\chi_3^2 - 4.80\chi_3^2 + 2.69\chi_1^2\chi_3 \\
& + 0.11\chi_3 - 4.8\chi_1^2\phi_1^2 - 0.10\chi_3^2\phi_1^2 + 3.77\chi_3\phi_3\phi_1^2 - 0.77\chi_3\phi_4\phi_1^2 \\
& - 2.79\chi_3\phi_1^2 - 1.46\chi_3^3\phi_3 + 0.44\chi_3^3\phi_4 - 1.62\chi_1^2\phi_3^2 - 2.71\chi_1^2\phi_4^2 \\
& + 5.02\chi_1^2\phi_3 - 5.11\chi_1^2\phi_3\phi_4 + 3.81\chi_1^2\phi_4 - 3.36\chi_3^2\phi_3^2 \\
& - 0.33\chi_3^2\phi_4^2 - 8.51\chi_3^2\phi_3 + 1.92\chi_3^2\phi_3\phi_4 + 2.75\chi_3^2\phi_4 - 0.64\chi_3\phi_3^3 \\
& - 0.67\chi_3\phi_4^3 - 19.2\chi_3\phi_3^2 - 1.45\chi_3\phi_3\phi_4^2 + 1.80\chi_3\phi_4^2 \\
& - 1.36\chi_1^2\chi_3\phi_3 - 2.51\chi_1^2\chi_3\phi_4 - 13.05\chi_3\phi_3 + 2.16\chi_3\phi_3^2\phi_4 \\
& + 10.25\chi_3\phi_3\phi_4 + 1.07\chi_3\phi_4 - 3.70\phi_1^4 - 32.51\phi_3^2\phi_1^2 \\
& + 0.90\phi_4^2\phi_1^2 + 41.66\phi_3\phi_1^2 + 19.59\phi_3\phi_4\phi_1^2 - 20.62\phi_4\phi_1^2 \\
& + 12.20\phi_1^2 - 14.33\phi_3^4 - 5.46\phi_4^4 + 41.31\phi_3^3 - 5.89\phi_3\phi_4^3 \\
& + 28.77\phi_4^3 - 28.88\phi_3^2 - 3.423\phi_3^2\phi_4^2 + 22.60\phi_3\phi_4^2 - 43.37\phi_4^2 \\
& - 46.70\phi_3 + 4.18\phi_3^3\phi_4 + 3.42\phi_3^2\phi_4 - 15.50\phi_3\phi_4 + 16.80\phi_4 - 29.6\,.
\end{aligned} \tag{3.42}$$

The equivariantly reduced Lagrangians at the levels $n = 3, 4, 5$ are given in the appendix. Let us summarize the steps do taken, notations and conventions in obtaining the LEA's $L_{(n)}$ *i)* performed the linear transformation among the fields $\phi_3 \to \phi_3'$, $\phi_4 \to \phi_4'$, $\chi_3 \to \chi_3'$ which diagonalizes the kinetic term , and dropped the $'$'s in the final form, *ii)* have set $\mu^2 = -8$, *iii)* have imposed the Gauss law constraints as discussed in the previous section by setting $\phi_2 = 0$ and $\chi_2 = 0$, *iv)* adjusted the constant in the final form of each $L_{(n)}$, so that the potentials $V_{(n)}$, take the value zero at their minima and *v)* introduced an over-dot $(\dot{...})$ to denote the time derivatives and *vi)* have set the coupling constant g to one, as it has no effect on the classical physics save for determining a global normalization in the energy unit.

A very important property of the reduced Lagrangian $L_{(n)}$ is that their potentials are all bounded from below. Due to this reason we can conclude that at any level n the equivariant fluctuations around the S_F^4 vacuum solutions does not cause any instability. Reduced action for $n = 1$ appears as a special case since $n = 1$ case contains only the combinations of the real fields ϕ_3, ϕ_4, χ_3 with a new parametrization $\Phi = \phi_3 + \phi_4 - \chi_3$ we can redefine the result for $n = 1$ as

$$\begin{aligned} L_{(n=1)} =& \frac{5}{16}\dot{\chi}_1^2 + \frac{5}{4}\dot{\phi}_1^2 + \frac{15}{4}\dot{\Phi}^2 - \frac{5}{8}\chi_1^4 + 5\chi_1^2 - 30\Phi^4 - 60\phi^3 - 45\Phi^2\phi_1^2 \\ & + 15\Phi^2 - 45\Phi\phi_1^2 + 45\Phi - \frac{15}{4}\chi_1^2\phi_1^2 - \frac{5}{2}\phi_1^4 + \frac{35}{4}\phi_1^2 - \frac{215}{8} \,. \end{aligned} \tag{3.43}$$

This can be expressed in a more elegant form as

$$\begin{aligned} L_{(n=1)} = \frac{1}{8}\dot{\chi}_1^2 + \frac{1}{2}\dot{\phi}_1^2 + \frac{3}{2}\dot{\Phi}^2 - \frac{1}{4}\left(\phi_1^2 + \chi_1^2 - 4\right)^2 - \frac{3}{4}\left(\phi_1^2 + 4\Phi(1+\Phi) - 3\right)^2 \\ - \phi_1^2\chi_1^2 - 3(1+2\Phi)^2\phi_1^2 \,. \end{aligned} \tag{3.44}$$

3.4.1 Lyapunov Spectrum for LEAs and Chaotic Dynamics

Reduced actions have Chaotic Dynamics. To reveal this we calculate the Lyapunov spectrum. One of the basic tools to probe the presence of chaos in a dynamical system is to compute the Lyapunov exponents, which measures the exponential growth in perturbations. If, say, $x(t)$ is a phase space coordinate, in a chaotic system the perturbation in $x(t)$, denoted by $\delta x(t)$, deviates exponentially from its initial value at $t = 0$; $|\delta x(t)| = |\delta x(0)| e^{\lambda_L t}$, λ_L being the corresponding Lyapunov exponent corresponding to the phase space variable $x(t)$. We outline a well-known procedure for calculating the Lyapunov spectrum in Appendix(D)

The phase space corresponding the LEA are 10-dimensional, except for the $n = 1$ case, and spanned by

$$(\phi_3, p_{\phi_3}, \phi_4, p_{\phi_4}, \chi_3, p_{\chi_3}, \phi_1, p_{\phi_1}, \chi_1, p_{\chi_1}) \,, \tag{3.45}$$

where p_i are the corresponding conjugate momenta and the Hamiltonians, $H_{(n)}$, are obtained from $L_{(n)}$ in the usual manner using $H = p_i\dot{q}_i - L$. We have obtained the Lyapunov spectrum for $n = 1, 2, 3, 4, 5$ at various energies (as determined by the initial conditions) using numerical solutions for the Hamilton's equations of motion. For $n = 1$, dimension of the phase space phase space reduces to 6 as easily observed from $L_{(1)}$ in (3.44). The table below summarizes our numerical findings for the largest Lyapunov exponent, λ_{max}, and the sum of the positive Lyapunov exponents, $\sum_{\lambda>0}\lambda$, for $n = 1, 2, 3, 4, 5$ at several different values of the energy E. We have shaded the $n = 1$ column in this table to indicate the noted differences between this case and the rest. Using the algorithm discussed in the appendix(D). We have created a MatLab code to obtain the Lyapunov spectrum where we input the initial conditions that satisfy a certain energy to obtain the Lyapunov spectrum.

Table 3.1: LLE and KS Values

Energy	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	
$E = 20$	0.33	0.39	0.35	0.07	0.09	λ_{max}
	0.51	0.79	0.41	0.18	0.20	$\sum_{\lambda>0} \lambda$
$E = 30$	0.58	0.84	0.84	0.56	0.32	λ_{max}
	0.85	1.67	1.78	1.09	0.69	$\sum_{\lambda>0} \lambda$
$E = 100$	0.96	1.94	1.87	1.52	1.37	λ_{max}
	1.34	4.60	4.15	2.94	2.91	$\sum_{\lambda>0} \lambda$
$E = 250$	1.12	2.27	2.16	1.99	1.78	λ_{max}
	1.65	5.65	5.48	4.76	3.93	$\sum_{\lambda>0} \lambda$

In the table (3.1) we give values for $\sum_{\lambda>0} \lambda$ for different values of energy E. The reason for this is that these values are equal to the Kolmogorov-Sinai (KS) Entropy[28, 29] which is also known as the metric entropy. First of all KS Entropy can be thought of as a single number say κ that depends only on the chaotic dynamical system considered which measures the time rate of creation of information as the chaos evolves. Secondly it is not really an physical entropy rather it provides a connection with the physical entropy $S(t)$. Naively this connection can be given as

$$\left| \frac{dS}{dt} \right| \leq \kappa \tag{3.46}$$

A more detailed explanation of the relationship between KS Entropy and Physical Entropy can be found in Latora et.al. [29]. Uses of KS Entropy includes it's relations with Entanglement Entropy [30] and Bekenstein-Hawking Entropy [31] in various contexts. Although we only state the values of KS Entropy and not explore those directions that has been discussed above they remain interesting endeavors for future studies.

Now we will present the time evolution of the Lyapunov exponents $\lambda_i, (i, 1, \ldots 10)$ for $n = 1, 2, 3, 4, 5$ at the energies $E = 20, 30, 100, 250$. We observe that for all of these cases the Lyapunov exponents rapidly converges to constant values. We also look at how the Lyapunov spectrum changes for given initial condition as n and as the energy increases. The plots Fig.(3.21-3.24) are given for the initial condition

$$(1.25, 1.2, 1.35, 1.2, 1.06, 1.2, 1.5, 1.4, 1.9, 1.3)\,. \tag{3.47}$$

Plots (3.25-3.26) shows the evolution of λ_{max} for $n = 2, 3, 4, 5$. We observe a decrease in the values of λ_{max} as n increases further but it still is significantly larger than zero. If one keeps on increasing n while keeping the energy fixed it is expected that λ_{max} to get smaller. This result is not surprising since, to keep the energy fixed we would need to choose the initial conditions closer and closer to zero. Which eventually becomes not useful to probe the chaotic dynamics of our system.
From the figures (3.25)-(3.30) we see that the rate of decrease in λ_{max} at different energies as n takes on the values $n = 3, 4, 5$ appears to be almost the same. Hence we may argue that λ_{max} will remain significantly larger than zero for increasing values of n, provided that the system has sufficiently large energy. In the figure (3.31) we give the values that LLE converges for energies $E = 20, 30, 100, 250$. Here we need to keep in mind that phase space dimensions for $n = 1$ and other cases are different so their detached values from the characteristic behavior does not raise any problem.

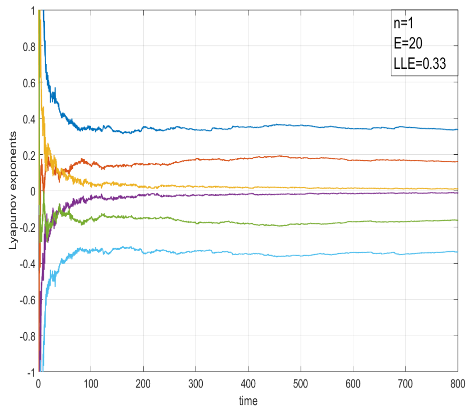

Figure 3.1: $n = 1$, $E = 20$

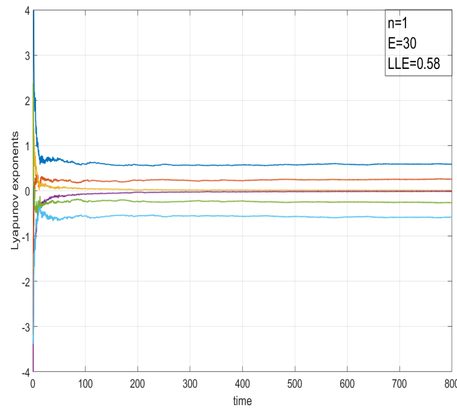

Figure 3.2: $n = 1$, $E = 30$

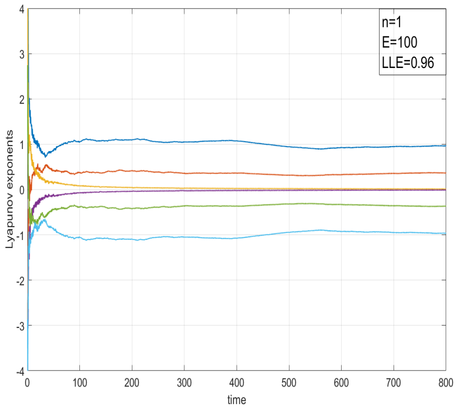

Figure 3.3: $n = 1, E = 100$

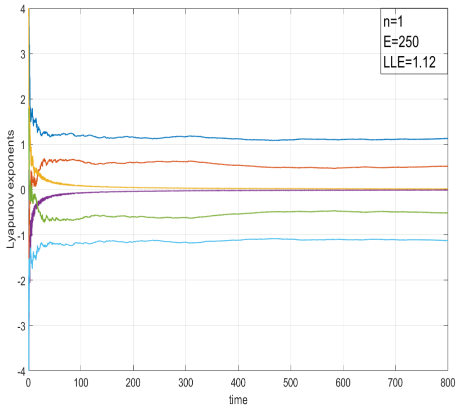

Figure 3.4: $n = 1$, $E = 250$

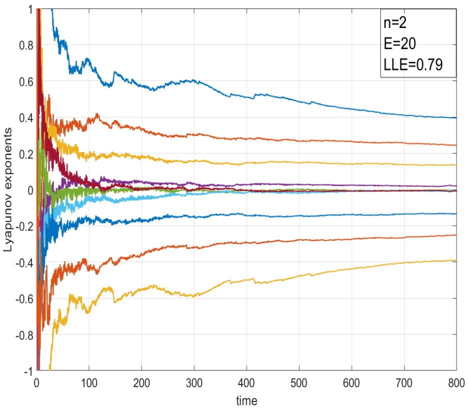

Figure 3.5: $n = 2$ $E = 20$

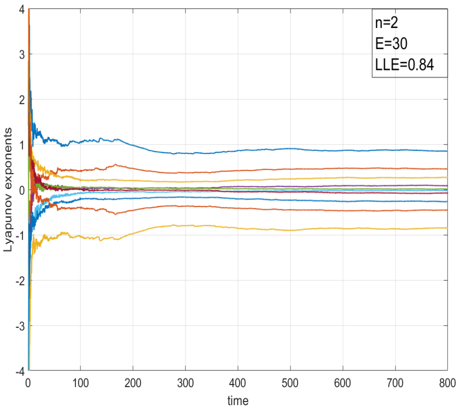

Figure 3.6: $n = 2$ $E = 30$

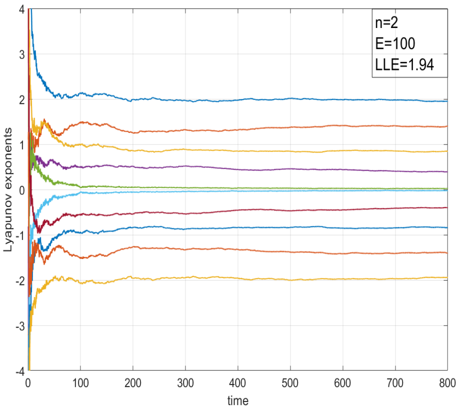

Figure 3.7: $n = 2$ $E = 100$

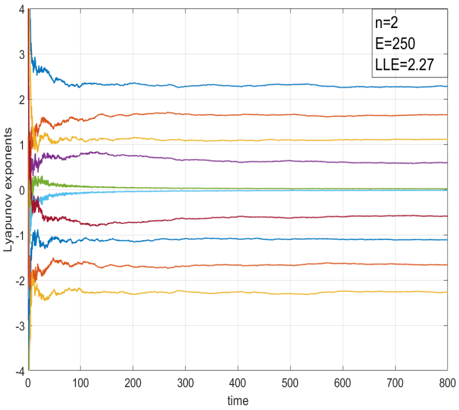

Figure 3.8: $n = 2$ $E = 250$

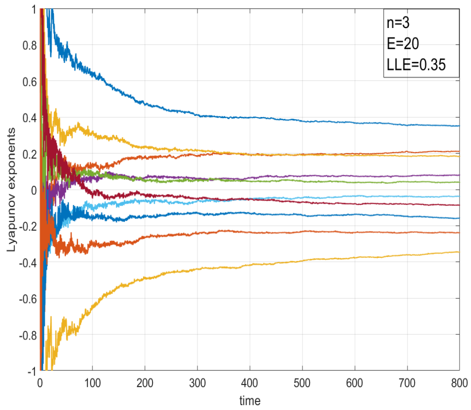

Figure 3.9: $n = 3$ $E = 20$

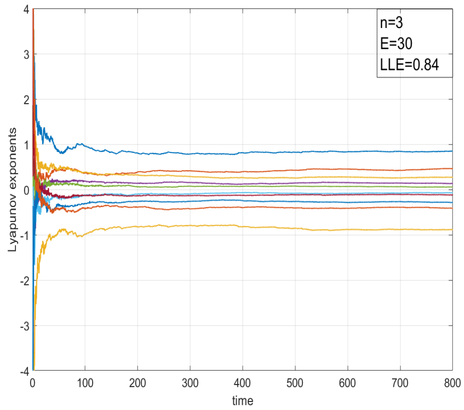

Figure 3.10: $n = 3$ $E = 30$

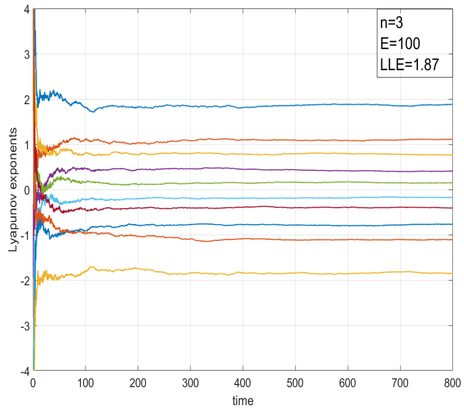

Figure 3.11: $n = 3$ $E = 100$

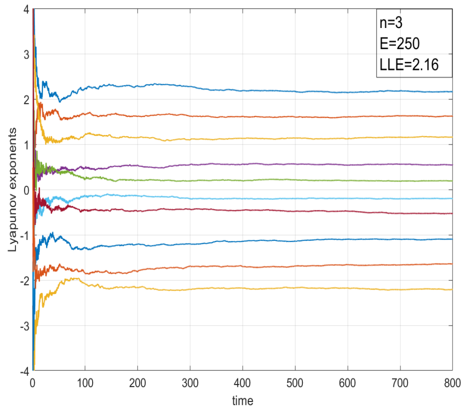

Figure 3.12: $n = 3$ $E = 250$

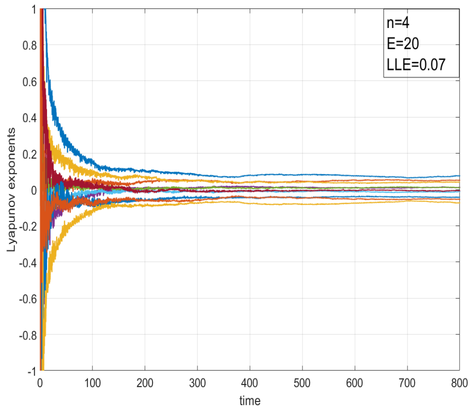

Figure 3.13: $n = 4$ $E = 20$

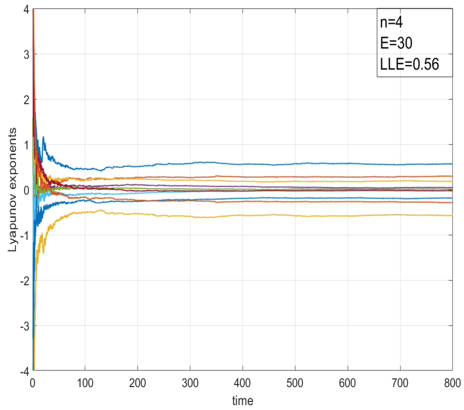

Figure 3.14: $n = 4$ $E = 30$

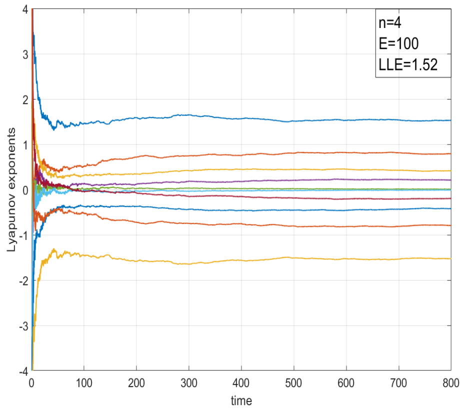

Figure 3.15: $n = 4$ $E = 100$

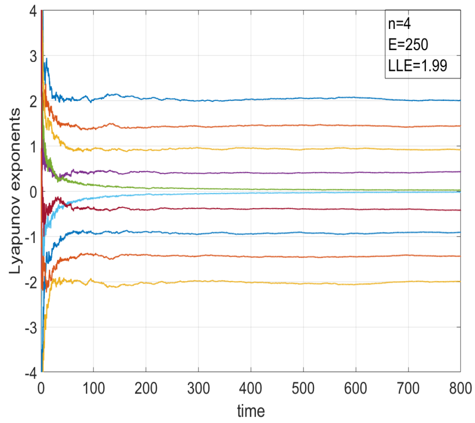

Figure 3.16: $n = 4$ $E = 250$

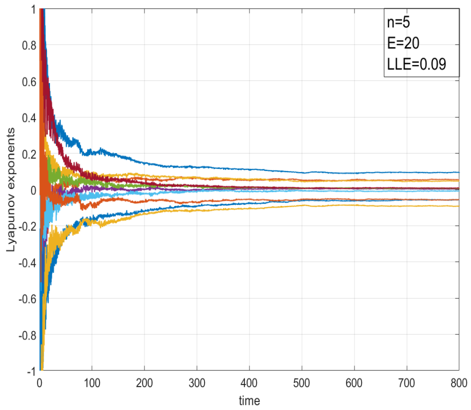

Figure 3.17: $n = 5$ $E = 20$

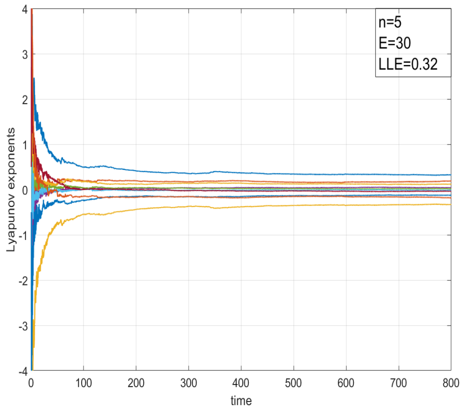

Figure 3.18: $n = 5$ $E = 30$

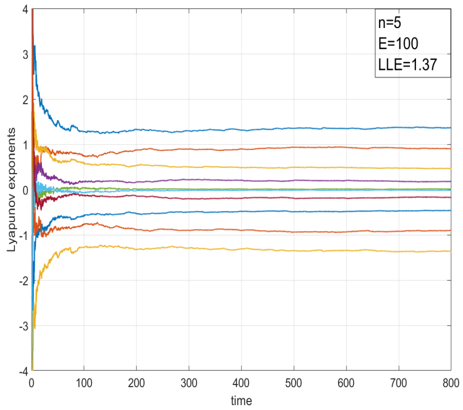

Figure 3.19: $n = 5$ $E = 100$

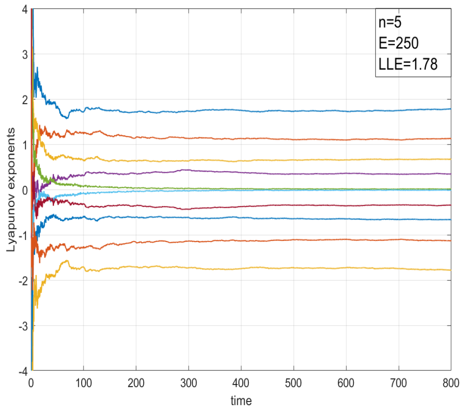

Figure 3.20: $n = 5$ $E = 250$

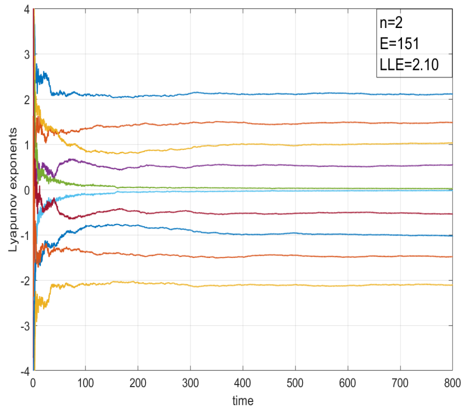

Figure 3.21: $n = 2$ $E = 151$

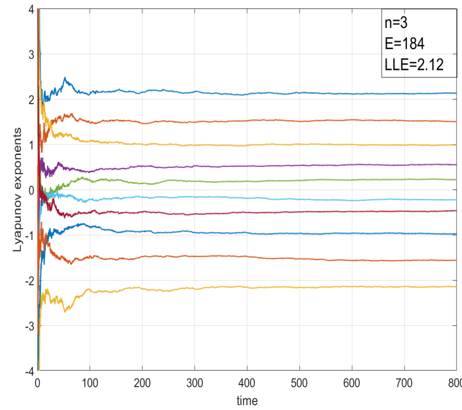

Figure 3.22: $n = 3$ $E = 185$

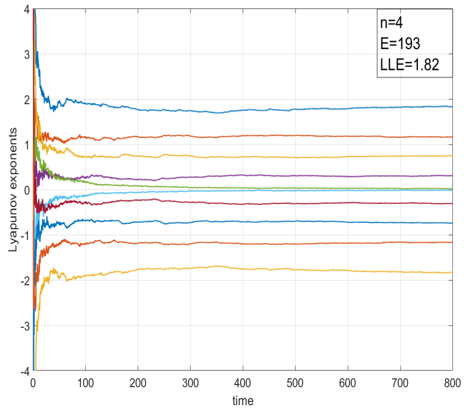

Figure 3.23: $n = 4$ $E = 194$

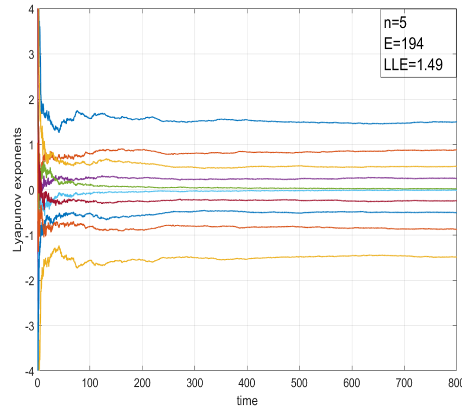

Figure 3.24: $n = 5$ $E = 194$

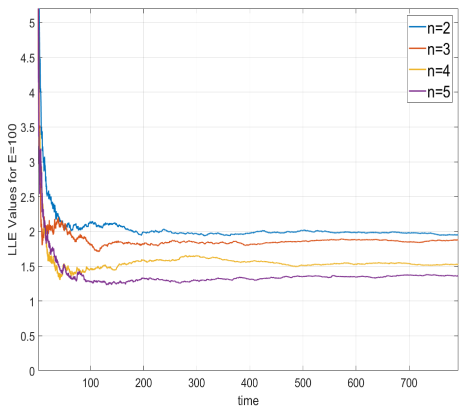

Figure 3.25: LLE Values for $E = 100$

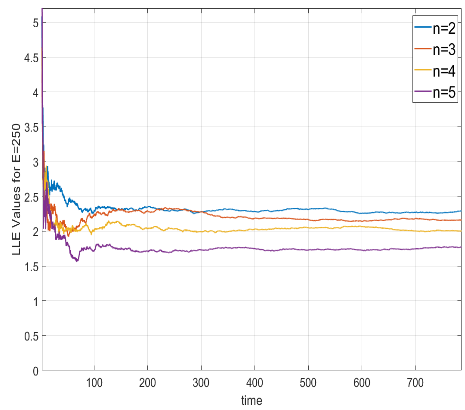

Figure 3.26: LLE Values for $E = 250$

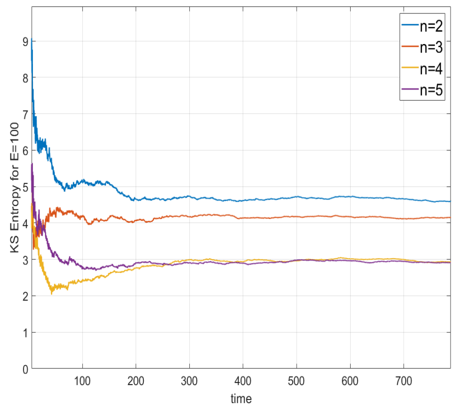

Figure 3.27: KSE Values for $E = 100$

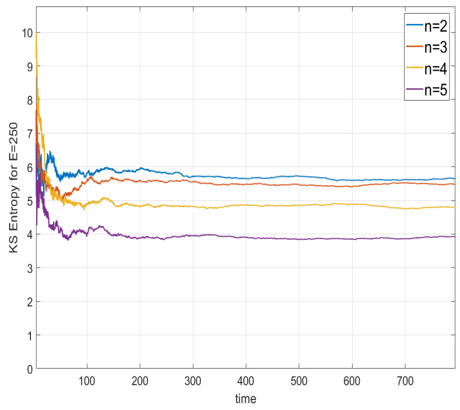

Figure 3.28: KSE Values for $E = 250$

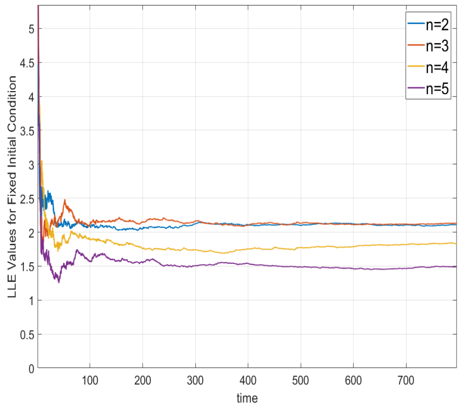

Figure 3.29: LLE Values for Fixed Initial Condition

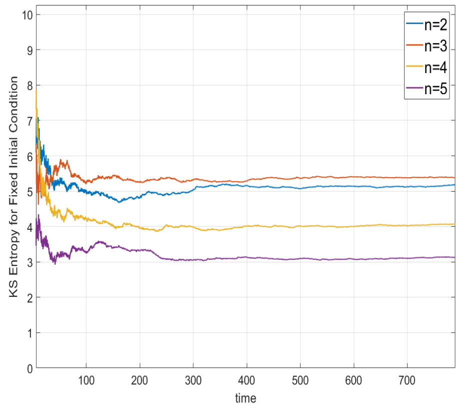

Figure 3.30: KS Entropy for fixed Initial Condition

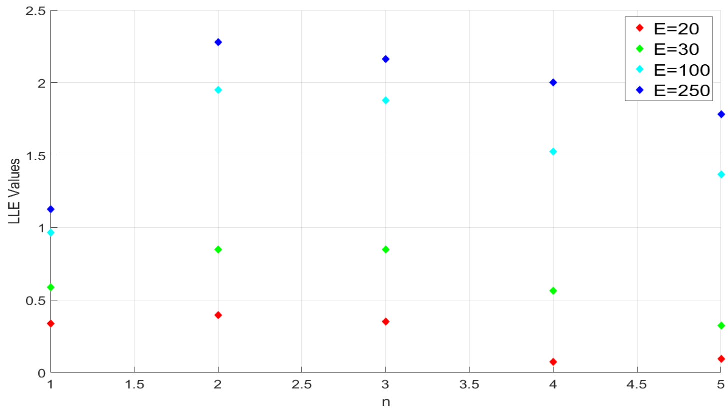

Figure 3.31: LLE Values vs n

3.5 Kink Solutions of LEA's

In this section we will consider the matrix model in the Euclidean signature. The Lagrangian for $n = 1$ can be given as

$$L_{(n=1)} = \frac{1}{8}\chi_1'^2 + \frac{1}{2}\phi_1'^2 + \frac{3}{2}\Phi'^2 + \frac{1}{4}\left(\phi_1^2 + \chi_1^2 - 4\right)^2 + \frac{3}{4}\left(\phi_1^2 + 4\Phi(1+\Phi) - 3\right)^2 + \phi_1^2\chi_1^2 + 3(1+2\Phi)^2\phi_1^2 . \tag{3.48}$$

where $'$ stands for derivatives with respect to the Euclidean time τ. We can easily see from (3.48) that there are three different pairs of vacua, which are given by the configurations

$$\begin{aligned} \phi_1 = \pm 2\,, \quad \chi_1 = 0\,, \quad \Phi &= -\frac{1}{2}\,, \\ \phi_1 = 0\,, \quad \chi_1 = \pm 2\,, \quad \Phi &= \frac{1}{2} \text{ or } -\frac{3}{2}\,, \end{aligned} \tag{3.49}$$

Since either ϕ_1 or χ_1 vanish in these vacua, we infer that, the kink solutions could be of the type with topological indices $(\pm 1, 0)$ or $(0, \pm 1) \in \mathbb{Z}_2 \oplus \mathbb{Z}_2$. These are the familiar kink solutions. Indeed, we find that the equations of motion are of the form

$$\phi_1'' - \left(4\phi_1^3 + 3\chi_1^2\phi_1 + \phi_1(7+6\Phi)(6\Phi - 1)\right) = 0\,, \tag{3.50a}$$

$$\chi_1'' - 4\left(\chi_1^3 + 3\phi_1^2\chi_1 - 4\chi_1\right) = 0\,, \tag{3.50b}$$

$$\Phi'' - \left(2(1+2\Phi)\left(3\phi_1^2 + 4\Phi(1+\Phi) - 3\right)\right) = 0\,. \tag{3.50c}$$

which have the following solutions

$$\phi_1(\tau) = 2\tanh\left(2\sqrt{2}\,\tau\right), \quad \chi_1(\tau) = 0\,, \quad \Phi(\tau) = -\frac{1}{2}\,, \tag{3.51}$$

$$\text{with} \quad \phi_1(\pm\infty) = \pm 2\,,$$

$$\phi_1(\tau) = 0\,, \quad \chi_1(\tau) = 2\tanh\left(2\sqrt{2}\,\tau\right), \quad \Phi(\tau) = \frac{1}{2} \text{ or } -\frac{3}{2}, \tag{3.52}$$

$$\text{with} \quad \chi_1(\pm\infty) = \pm 2\,.$$

It is instructive to see how the equations of motions (3.50) yield solutions given in (3.51),(3.52). Substituting $\chi_1 = 0$ and $\Phi = -\frac{1}{2}$ we see that (3.50b) & (3.50c) are trivially satisfied and from (3.50a) we are left with

$$\phi_1'' - (4\phi_1^3 - 16\phi_1) = 0, \tag{3.53}$$

which can be simplified to the first order equation

$$\phi_1'^2 = 2(\phi_1^2 - 4)^2. \tag{3.54}$$

Where the constant of integration is fixed from the form of the potential in (3.48), with $\chi_1 = 0$ and $\Phi = -\frac{1}{2}$. Choosing the positive square root for the kink solution we have

$$\phi_1' = \sqrt{2}(\phi_1^2 - 4), \tag{3.55}$$

which integrates as

$$\int \frac{d\phi_1}{\sqrt{2}(\phi_1^2 - 4)} = \int d\tau, \tag{3.56}$$

$$\frac{1}{2\sqrt{2}}\text{arctanh}(\frac{\phi_1}{2}) = \tau \tag{3.57}$$

or we can represent it as

$$\phi_1(\tau) = 2\tanh\left(2\sqrt{2}\tau\right) \tag{3.58}$$

As we wanted to show. For further visualization we add a graphic of this kink solution

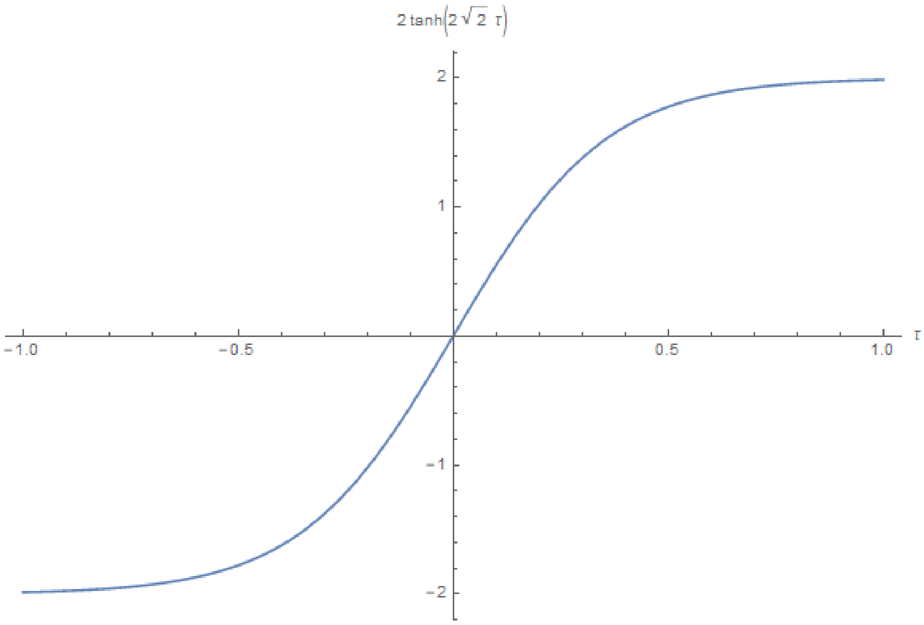

Figure 3.32: τ vs $2\tanh\left(2\sqrt{2}\tau\right)$

3.5.1 Kinks at levels $n \geq 2$

For $n = 2, 3, 4, 5$, the number of degenerate vacua increases. This may be expected due to the larger number degrees of freedom in the LEAs. A similar structure in vacuum configurations to that of $n = 1$ is observed, and allow for the kink solutions. At $n = 3$, for instance, we have eight pairs of degenerate vacua, which are given as

$$\begin{aligned}
&\{\phi_1 \to 0., \phi_3 \to 2.56, \phi_4 \to 3.42, \chi_1 \to \pm 2., \chi_3 \to -11.5\}\,, \\
&\{\phi_1 \to 0., \phi_3 \to -0.28, \phi_4 \to 0.55, \chi_1 \to \pm 2., \chi_3 \to 1.46\}\,, \\
&\{\phi_1 \to 0., \phi_3 \to 2.55, \phi_4 \to -0.60, \chi_1 \to \pm 2., \chi_3 \to -5.26\}\,, \\
&\{\phi_1 \to 0., \phi_3 \to -0.27, \phi_4 \to 4.60, \chi_1 \to \pm -2., \chi_3 \to -4.79\}\,, \\
&\{\phi_1 \to \pm 2., \phi_3 \to 2.30, \phi_4 \to 4.13, \chi_1 \to 0., \chi_3 \to -3.01\}\,, \\
&\{\phi_1 \to \pm 2., \phi_3 \to -0.02, \phi_4 \to -0.16, \chi_1 \to 0., \chi_3 \to -7.04\}\,, \\
&\{\phi_1 \to \pm 2., \phi_3 \to 0.28, \phi_4 \to -0.55, \chi_1 \to 0., \chi_3 \to -1.46\}\,, \\
&\{\phi_1 \to \pm 2., \phi_3 \to 2.00, \phi_4 \to 4.51, \chi_1 \to 0., \chi_3 \to \pm 8.59\}\,,
\end{aligned} \tag{3.59}$$

The equations of motion for $L_{(n=3)}$ are coupled non-linear differential equations, which are not easily solved. We may look at the linearized system of equations around one of the minima. For notational simplicity, let us write $(\phi_1, \chi_1, \phi_3, \phi_4, \chi_3) \equiv (S_1, S_2, S_3, S_4, S_5) := \boldsymbol{S}$ and also write $\boldsymbol{S} = \boldsymbol{S}^0 + \boldsymbol{s}$, where $\boldsymbol{S}^0$ is one of the vacuum configurations and $\boldsymbol{s}$ are the fluctuations. The linearized system of equations is given by

$$s_i'' = \left.\frac{\partial^2 V_{(3)}}{\partial s_i \partial s_j}\right|_{\boldsymbol{S}^0} s_j \,, \tag{3.60}$$

and for, say, $\boldsymbol{S}^0 \equiv \{\phi_1 \to \pm 2., \phi_3 \to 2.00, \phi_4 \to 4.51, \chi_1 \to 0., \chi_3 \to \pm 8.59\}$, these take the form

$$\begin{aligned}
2.6 s_1'' - 125.3 s_1 - 30.1 s_3 + 51.9 s_4 + 5.41 s_5 &= 0\,, \\
0.52 s_2'' - 38.91 s_2, &= 0\,, \\
9.8 s_3'' - 30.1 s_1 - 216.3 s_3 + 29.7 s_4 + 3.7 s_5, &= 0\,, \\
6.9 s_4'' + 51.9 s_1 + 29.7 s_3 - 110.8 s_4 - 6.4 s_5, &= 0 \\
0.92 s_5'' + 5.4 s_1 + 3.7 s_3 - 6.4 s_4 - 5.8 s_5 &= 0\,.
\end{aligned} \tag{3.61}$$

The leading order profiles of the solutions of these equations which are regular as

$\tau \to \infty$ are given below, while the complete solutions are given in the Appendix.

$$\begin{aligned}
s_1(\tau) &\approx (-0.98c_1 - 0.18c_2 + 4.98c_3 + 1.11c_4)\, e^{-2.38\tau}\,, \quad s_2(\tau) = c_5 e^{-8.65068\tau}\,, \\
s_3(\tau) &\approx (-0.65c_1 - 0.12c_2 + 3.28c_3 + 0.73c_4)\, e^{-2.38\tau}\,, \\
s_4(\tau) &\approx (7.75c_1 + 1.49c_2 - 39.25c_3 - 8.73c_4)\, e^{-2.38\tau}\,, \\
s_5(\tau) &\approx (-73.40c_1 - 13.99c_2 + 404.89c_3 + 89.86c_4)\, e^{-2.38\tau}\,, \qquad (3.62)
\end{aligned}$$

where c_i $(i : 1 \cdots 5)$ are arbitrary constants. These results give the profile of the kink solution as for large τ.

CHAPTER 4

CONCLUSIONS

In this thesis, We have started with an introduction of fuzzy spaces starting from the basic examples of S_F^2 and $\mathbb{C}P^2$ and continue to a richer space S_F^4 we have demonstrated how to obtain the S_F^4 through several ways. Then we have showed how to introduce functions living on S_F^4 and how to interpret the action $\tilde{S}_2$ as a gauge theory on S_F^4. In Chapter 2 and 3 we have investigated two different gauge theories, One with a fifth order Chern-Simmons like and one with a mass deformation. We have seen that choosing a fifth order deformation forces us on a fixed S_F^4 while with mass deformation one can choose any stack of S_F^4 for the purposes of our equivariant parametrization we choose stack of 4 fuzzy 4 spheres. However we need to take a negative mass term in order to satisfy the vacuum solutions for the mass deformed action. Which at first seems as a source of instability that is cured by the equivariant dimensional reduction. We introduced the $SU(4)$ equivariant gauge fields on the S_F^4 and expand the S_F^4 extremum solutions around the vacuum with fluctuations that are elements of this $SU(4)$ equivariant fields. We have analytically calculated the two terms in our action which are the mass term and the Dynamical term. But calculating the remaining part of the potential term turns out to be a formidable problem so instead we get the result for $n = 1, 2, 3, 4, 5$ numerically using Mathematica. Using the numerical results we have obtained the Lagrangians for the $n = 1, 2, 3, 4, 5$ and we observed that they facilitate chaotic behavior. In order to verify our observation we have calculated the Lyapunov spectrum for various Energies such as $E = 20, 30, 100, 200$ and for a fixed initial condition $(1.25, 1.2, 1.35, 1.2, 1.06, 1.2, 1.5, 1.4, 1.9, 1.3)$ we observed that the Largest Lyapunov Exponents(LLE) converges rapidly to constant values which are significantly larger than zero. Such values of LLE indicates highly chaotic behavior as we expected. We also plotted the n vs LLE and n vs KSE plots to investigate the characteristics of this chaotic behavior. Finally we investigated our actions in the Euclidean time τ and observed that they facilitates kink type solutions with topological charges $\mathbb{Z}_2 \times \mathbb{Z}_2$.

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APPENDIX A

USEFUL RESULTS FROM GROUP THEORY

A.1 Brief Review of Group Theoretical Identities

A.1.1 Branching Rules

For this part of appendix we will briefly review some result from group theory from our previous work.[32]. We know that irreducible representations of $SO(2k)$ and $SO(2k{-}1)$can be given in terms of the highest weight labels $[\lambda] \equiv (\lambda_1\,, \lambda_2\,, \cdots\,, \lambda_{k-1}\,, \lambda_k)$ and $[\mu] \equiv (\mu_1\,, \mu_2\,, \cdots\,,\ \mu_{k-1})$ respectively. Branching of the IRR $[\lambda]$ of $SO(2k)$ under $SO(2k-1)$ IRRs follows from the rule [26]

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \mu_{k-1} \geq |\lambda_k|\,, \tag{A.1}$$

A.1.2 Quadratic Casimir operators of $SO(2k)$ and $SO(2k-1)$ Lie algebras

Eigenvalues for the quadratic Casimir operators of $SO(2k)$ and $SO(2k-1)$ in the IRRs $[\lambda] \equiv (\lambda_1, \lambda_2 \cdots \lambda_k)$, $[\mu] \equiv (\mu_1, \mu_2 \cdots \mu_{k-1})$, respectively are given as [26]:

$$C_2^{SO(2k)}[\lambda] = \sum_{i=1}^{k} \lambda_i(\lambda_i + 2k - 2i) \tag{A.2}$$

$$C_2^{SO(2k-1)}[\mu] = \sum_{i=1}^{k-1} \mu_i(\mu_i + 2k - 1 - 2i)\,. \tag{A.3}$$

Eigenvalues of quadratic Casimir operators of some specific IRRs are given as

$$C_2^{SO(4)}\left(n+\frac{I}{2},s\right)=\frac{I^2}{4}+In+I+n^2+2n+s^2 \tag{A.4}$$

$$C_2^{SO(3)}\left(\frac{I}{2}\right)=\frac{I^2}{4}+\frac{I}{2} \tag{A.5}$$

$$C_2^{SO(6)}\left(n+\frac{I}{2},\frac{I}{2},s\right)=\frac{I^2}{2}+In+3I+n^2+4n+s^2 \tag{A.6}$$

$$C_2^{SO(5)}\left(\frac{I}{2},\frac{I}{2}\right)=\frac{I^2}{2}+2I \tag{A.7}$$

A.1.3 Relationship between Dynkin and Highest weight labels

Throughout this thesis we have used highest weight labels (HW) and Dykin labels to label the irreducible representations of Lie algebras. To be more precise other than in section (2.3) we have used Dykin labeling. Although the difference between them is just algebraic. We give the relationship between Dykin labels and highest weight labels to be more clear as follows For a $SO(5)$ IRR , the labels are given as

$$(p,q)_{Dynkin}\equiv(\lambda_1,\lambda_2)_{HW}$$

and the relation between these labels are given by

$$\lambda_1=\frac{p+q}{2} \qquad \lambda_2=\frac{q}{2}$$

To illustrate, $(I/2,I/2)_{HW}$ corresponds to $(0,I)_{Dynkin}$.

For a $SO(6)$ IRR , the labels are given as

$$(p,q,r)_{Dynkin}\equiv(\lambda_1,\lambda_2,\lambda_3)_{HW}$$

and the relation between these labels are given by

$$\lambda_1=q+\frac{p+r}{2} \qquad \lambda_2=\frac{p+r}{2} \qquad \lambda_3=\frac{p-r}{2}$$

For a $SO(4)$ IRR the labels are

$$(p,q)_{Dynkin}\equiv(\lambda_1,\lambda_2)_{HW}$$

and the relation between these labels are given by

$$\lambda_1=\frac{p+q}{2} \qquad \lambda_2=\frac{p-q}{2}$$

To illustrate, $(n+I/2,s)_{HW}$ corresponds to $(\frac{1}{2}(n+I/2+s),\frac{1}{2}(n+I/2-s))_{Dynkin}$.

A.1.4 Dimensional Relations

Using these labellings we can obtain the dimensional relations we have used in the following way. As stated in previous sections dimension of X_μ can be obtained from the dimension relations of the $SO(5)$ algebra since they are constructed from the symmetric tensor product of the Γ matrices they also respect the $SO(5)$ relations

$$dim(N, M) = \frac{1}{6}(N+1)(M+1)(N+M+2)(2N+M+3), \tag{A.8}$$

Setting $N = 0$ and $M = n$ we get the dimensional relation (2.72)

APPENDIX B

CALCULATIONS ON EQUIVARAINT REDUCTION

B.1 Details on the Dimensional Reduction

Let us show that the square of these operators are indeed 1. To do that we need to find $(G \cdot \Sigma)^2$

$$G_{ij}G_{kl}\Sigma_{ij}\Sigma_{kl} = \gamma_i\gamma_j\gamma_k\gamma_l G_{ij}G_{kl} \tag{B.1}$$

Starting from relation $\epsilon^{ijklm}\gamma_i\gamma_j\gamma_k\gamma_l = 24\gamma_m$ and multiplying both sides with ϵ^{abcdm} One can obtain

$$\epsilon^{abcdm}\epsilon^{ijklm}\gamma_i\gamma_j\gamma_k\gamma_l G_{ab}G_{cd} = 24\epsilon^{abcdm}\gamma_m G_{ab}G_{cd} \tag{B.2}$$

Left hand side can be calculated by expanding the product of epsilons as a determinant

$$\epsilon^{abcdm}\epsilon^{ijklm} = \begin{vmatrix} \delta^{ai} & \delta^{aj} & \delta^{ak} & \delta^{al} \\ \delta^{bi} & \delta^{bj} & \delta^{bk} & \delta^{bl} \\ \delta^{ci} & \delta^{cj} & \delta^{ck} & \delta^{cl} \\ \delta^{di} & \delta^{dj} & \delta^{dk} & \delta^{dl} \end{vmatrix} \tag{B.3}$$

When calculated gives the result

$$\begin{aligned}\epsilon^{abcdm}\epsilon^{ijklm} = {} & \delta^{ai}\left(\delta^{bj}(\delta^{ck}\delta^{dl} - \delta^{dk}\delta^{cl}) - \delta^{bk}(\delta^{cj}\delta^{dl} - \delta^{cl}\delta^{dj}) + \delta^{bl}(\delta^{cj}\delta^{dk} - \delta^{ck}\delta^{dj})\right) \\ & -\delta^{aj}\left(\delta^{bi}(\delta^{ck}\delta^{dl} - \delta^{cl}\delta^{dk}) - \delta^{bk}(\delta^{ci}\delta^{dl} - \delta^{cl}\delta^{di}) + \delta^{bl}(\delta^{ci}\delta^{dk} - \delta^{di}\delta^{ck})\right) \\ & +\delta^{ak}\left(\delta^{bi}(\delta^{cj}\delta^{dl} - \delta^{cl}\delta^{dj}) - \delta^{bj}(\delta^{ci}\delta^{dl} - \delta^{di}\delta^{cl}) + \delta^{bl}(\delta^{ci}\delta^{dj} - \delta^{cj}\delta^{di})\right) \\ & -\delta^{al}\left(\delta^{bi}(\delta^{cj}\delta^{dk} - \delta^{ck}\delta^{dj}) - \delta^{bj}(\delta^{ci}\delta^{dk} - \delta^{ck}\delta^{di}) + \delta^{bk}(\delta^{ci}\delta^{dj} - \delta^{di}\delta^{cj})\right)\end{aligned} \tag{B.4}$$

Which is the most general form of the operator we have previously defined as f^{abcdef} in (2.134). Multiplying (B.4) with $\gamma_i\gamma_j\gamma_k\gamma_l G_{ab}G_{cd}$ we get

$$\begin{aligned} &4\gamma_a\gamma_b\gamma_c\gamma_d G_{ab}G_{cd} + 4\gamma_c\gamma_d\gamma_a\gamma_b G_{ab}G_{cd} + 4\gamma_a\gamma_c\gamma_d\gamma_b G_{ab}G_{cd} \\ &+4\gamma_a\gamma_d\gamma_b\gamma_c G_{ab}G_{cd} + 4\gamma_c\gamma_a\gamma_b\gamma_d G_{ab}G_{cd} + 4\gamma_d\gamma_a\gamma_c\gamma_b G_{ab}G_{cd} \end{aligned} \tag{B.5}$$

Using anti commutation relation of gamma matrices we can arrange these terms as

$$4(6\gamma_a\gamma_b\gamma_c\gamma_d G_{ab}G_{cd} + 12G_{ab}G_{ab} + 24\gamma_a\gamma_c G_{ab}G_{cb}) \tag{B.6}$$

so

$$\epsilon^{abcdm}\gamma_m G_{ab}G_{cd} = \gamma_a\gamma_b\gamma_c\gamma_d G_{ab}G_{cd} - 2G_{ab}G_{ab} - 4\gamma_a\gamma_c G_{ab}G_{cb} \tag{B.7}$$

we know that $\gamma_a\gamma_b\gamma_c\gamma_d G_{ab}G_{cd}$ is equal to $(G\cdot\Sigma)^2$ so

$$\gamma_a\gamma_b\gamma_c\gamma_d G_{ab}G_{cd} = \epsilon^{abcdm}\gamma_m G_{ab}G_{cd} + 2G_{ab}G_{ab} - 4\gamma_a\gamma_c G_{ab}G_{cb} \tag{B.8}$$

Lets focus on the first term using $G_{ab} = \frac{1}{2}[X_a, X_b]$ we can expand the term $\epsilon^{abcdm}\gamma_m G_{ab}G_{cd}$ as

$$\epsilon^{abcdm}\gamma_m G_{ab}G_{cd} = \frac{1}{4}\epsilon^{abcdm}\gamma_m\left(X_aX_bX_cX_d - X_aX_bX_dX_c - X_bX_aX_cX_d + X_bX_aX_dX_c\right) \tag{B.9}$$

Which is equal to

$$\epsilon^{abcdm}\gamma_m X_aX_bX_dX_c \tag{B.10}$$

We know that $\epsilon^{abcdm}X_aX_bX_cX_d = 8(n+2)X_m$, and using this relation we find the first term as

$$\epsilon^{abcdm}\gamma_m G_{ab}G_{cd} = 8(n+2)X_m\gamma_m \tag{B.11}$$

Now that we have found the first term we can move on to the second term which is the Casimir of $(0, n)$ IRR of $SO(5)$ which can be written as

$$G_{ab}G_{ab} = -4n(n+4) \tag{B.12}$$

Lastly the third term can be calculated in the following way

$$\begin{aligned} \gamma_a\gamma_c G_{ab}G_{cb} &= \frac{1}{2}\gamma_a\gamma_c G_{ab}G_{cb} + \frac{1}{2}\gamma_a\gamma_c G_{ab}G_{cb} \\ &= \frac{1}{2}\gamma_a\gamma_c G_{ab}G_{cb} + G_{ab}G_{ab} - \frac{1}{2}\gamma_a\gamma_c G_{cb}G_{ab} \\ &= \frac{1}{2}\gamma_a\gamma_c[G_{ab}, G_{cb}] + G_{ab}G_{ab} \end{aligned} \tag{B.13}$$

Using the commutation relation of $SO(5)$ generators we found the last term as

$$12\gamma_a\gamma_c G_{ac} - 4G_{ab}G_{ab} \tag{B.14}$$

Combining all the terms together we get

$$(G\cdot\Sigma)^2 = 8n(n+2)X_a\gamma_a + 12\gamma_a\gamma_c G_{ac} + 8n(n+4)\mathbb{1}_{4N} \tag{B.15}$$

Lets investigate $\frac{(G\cdot\Sigma)^2}{n^2}$ as n goes to ∞

$$\lim_{n\to\infty}\left(\frac{8(n+2)X_m\gamma_m + 8n(n+4) + 12\Sigma_{ac}G_{ac}}{n^2}\right) = 8(x_m\gamma_m + \mathbb{1}_4) \tag{B.16}$$

Where x_m represents the coordinates of R^5. Now that we have found the limit of $\frac{(G\cdot\Sigma)^2}{n^2}$ we can calculate the limit of Q's Lets start with Q_2

$$\begin{aligned}\lim_{n\to\infty} Q_2 &= \frac{(G\cdot\Sigma)^2}{-8n^2} + \frac{16n^2}{8n^2} - 1 \\ &= -x_m\gamma_m - 1 + 2 + 1 = -x_m\gamma_m = q_2\end{aligned} \tag{B.17}$$

Taking the square

$$\begin{aligned} q_2^2 &= x_m x_n \gamma_m \gamma_n \\ &= \frac{1}{2}x_m x_n\gamma_m\gamma_n + \frac{1}{2}x_m x_n\gamma_m\gamma_n \\ &= \frac{1}{2}x_m x_n\{\gamma_m,\gamma_n\} \\ &= \frac{1}{2}x_m x_n 2\delta_{mn} \\ &= 1\end{aligned} \tag{B.18}$$

Now focusing on Q_1

$$\lim_{n\to\infty} = q_1 = \frac{(G\cdot\Sigma)^2}{16n^2} - \frac{4nG\Sigma}{16n^2} - 1 = \frac{x_m\gamma_m}{2} - \frac{1}{4}\alpha_{cd}\Sigma_{cd} - 1 \tag{B.19}$$

Taking the square

$$q_1^2 = \frac{x_m x_n\gamma_m\gamma_n}{4} + \frac{\alpha_{cd}\Sigma_{cd}\alpha_{cd}\Sigma_{cd}}{16} - \frac{x_a\gamma_a\alpha_{cd}\Sigma_{cd}}{8} - \frac{\alpha_{cd}\Sigma_{cd}x_a\gamma_a}{8} - \frac{x_m\gamma_m}{2} + \frac{\alpha_{cd}\Sigma_{cd}}{4} + \frac{1}{4} \tag{B.20}$$

Where α_{ab} represents coordinates of R^{15}. We know that $G_{ab}G_{cd}\Sigma_{ab}\Sigma_{cd}$ is equal to

$$= 8(n+2)X_m\gamma_m + 8n(n+4) + 12\Sigma_{ac}G_{ac} \tag{B.21}$$

So divide both sides with n^2 and find

$$\alpha_{ab}\alpha_{cd}\Sigma_{ab}\Sigma_{cd} = 8x_m\gamma_m + 8 \tag{B.22}$$

Putting this in (B.20) and organizing terms a little bit more leads us to

$$q_1^2 = 1 - \frac{x_a\gamma_a\alpha_{cd}\Sigma_{cd}}{8} - \frac{\alpha_{cd}\Sigma_{cd}x_a\gamma_a}{8} + \frac{\alpha_{cd}\Sigma_{cd}}{4} \tag{B.23}$$

For q_1^2 be equal to 1 we need last three terms of (B.23) to be equal to 0 which leads us to the equality

$$x_a\alpha_{cd}\gamma_a\Sigma_{cd} + \alpha_{cd}x_a\Sigma_{cd}\gamma_a = 2\alpha_{cd}\Sigma_{cd} \tag{B.24}$$

Lets show that RHS is indeed equal to LHS. We can put LHS in the following form

$$x_a\alpha_{cd}\{\gamma_a, \Sigma_{cd}\} = \frac{1}{2}x_a\alpha_{cd}\{\gamma_a, [\gamma_c, \gamma_d]\} \tag{B.25}$$

Using the relation $\epsilon^{abcde}\gamma_b\gamma_c\gamma_d = 3[\gamma_a, \gamma_e]$ and multiplying both sides of this equation with γ_m gives us

$$\epsilon^{abcde}\gamma_m\gamma_b\gamma_c\gamma_d = 3\gamma_m[\gamma_a, \gamma_e] \tag{B.26}$$

Using this equation we can conclude that

$$\{\gamma_m, \{\gamma_a, \gamma_c\}\} = 2\epsilon^{abcde}\gamma_c\gamma_d \tag{B.27}$$

Combining this with $\alpha_{ab} = -1/2\epsilon^{abcde}\alpha_{cd}x_e$ (B.25) becomes

$$\begin{aligned} &-x_g\alpha_{cd}x_e(\epsilon^{abcde})(\epsilon^{abgfh})\gamma_f\gamma_h \\ &= x_g\alpha_{fh}x_g\gamma_f\gamma_h - x_g\alpha_{hf}\gamma_f\gamma_h \\ &= \alpha_{fh}[\gamma_f, \gamma_h] = 2\alpha_{fh}\Sigma_{fh} \end{aligned} \tag{B.28}$$

As we wanted. Now we can focus on Q_3. Observe that it has the same structure as Q_1.

$$\lim_{n\to\infty} \frac{(G\cdot\Sigma)^2}{16n^2} + \frac{4nG\Sigma}{16n^2} - 1 = \frac{x_m\gamma_m}{2} + \frac{1}{4}\alpha_{cd}\Sigma_{cd} - \frac{1}{2} \tag{B.29}$$

$$q_3^2 = 1 + \frac{x_a\gamma_a\alpha_{cd}\Sigma_{cd}}{8} + \frac{\alpha_{cd}\Sigma_{cd}x_a\gamma_a}{8} + \frac{\alpha_{cd}\Sigma_{cd}}{4} \tag{B.30}$$

So same condition holds and we have q_3^2 equal to 1.

Some useful identities among Q_1, Q_3 and X_a that greatly simplify the analytic calculations are listed below[33].

$$
\begin{aligned}
[Q_1, [X_a, Q_3]] = 0\,, &\quad [Q_3, [X_a, Q_1]] = 0 \\
[Q_1, Q_3[X_a, Q_3]] = 0\,, &\quad [Q_3, Q_1[X_a, Q_1]] = 0 \\
[Q_3, Q_1[X_a, Q_1]] = 0\,, &\quad [Q_1, \{X_a, Q_2\}] = 0 \\
[Q_1, \{X_a, Q_1\} - Q_3[X_a, Q_3]] = 0\,, &\quad [Q_3, \{X_a, Q_1\} - Q_3[X_a, Q_3]] = 0 \\
[Q_1, \{X_a, Q_3\} - Q_1[X_a, Q_1]] = 0\,, &\quad [Q_3, \{X_a, Q_3\} - Q_1[X_a, Q_1]] = 0
\end{aligned}
\tag{B.31}
$$

APPENDIX C

EXPLICIT FORMULA FOR LOW ENERGY REDUCED ACTIONS AND THEIR MINIMUMS

C.1 Explicit Formula for LEA

Here we list the equivariantly reduced Lagrangians for $n = 4, 5, 3$:

$$
\begin{aligned}
L_{(n=4)} = & \frac{1}{2}\left(1.3\dot{\chi}_1^2 + 2.6\dot{\phi}_1^2 + 8.38\dot{\phi}_3^2 + 6.77\dot{\phi}_4^2 + 0.88\dot{\chi}_3^2\right) - 1.66\chi_1^4 \\
& - 1.66\chi_1^4 - 0.015\chi_3^4 - 0.39\chi_3^3 + 8.81\chi_1^2 - 0.287\chi_1^2\chi_3^2 - 2.75\chi_3^2 \\
& + 3.47\chi_1^2\chi_3 + 0.252\chi_3 - 5.48\chi_1^2\phi_1^2 - 0.182\chi_3^2\phi_1^2 + 3.55\chi_3\phi_3\phi_1^2 \\
& - 1.32\chi_3\phi_4\phi_1^2 - 3.72\chi_3\phi_1^2 - 0.164\chi_3^3\phi_3 + 0.044\chi_3^3\phi_4 - 7.09\chi_1^2\phi_3^2 \\
& - 0.796\chi_1^2\phi_4^2 + 14.2\chi_1^2\phi_3 - 5.37\chi_1^2\phi_3\phi_4 + 3.18\chi_1^2\phi_4 - 0.852\chi_3^2\phi_3^2 \\
& - 0.0543\chi_3^2\phi_4^2 - 2.59\chi_3^2\phi_3 + 0.40\chi_3^2\phi_3\phi_4 + 0.785\chi_3^2\phi_4 + 0.010\chi_3\phi_3^3 \\
& - 0.020\chi_3\phi_4^3 - 14.3\chi_3\phi_3^2 - 0.33\chi_3\phi_3\phi_4^2 - 0.18\chi_3\phi_4^2 - 2.92\chi_1^2\chi_3\phi_3 \\
& - 1.22\chi_1^2\chi_3\phi_4 - 10.38\chi_3\phi_3 + 0.402\chi_3\phi_3^2\phi_4 + 7.01\chi_3\phi_3\phi_4 + 1.59\chi_3\phi_4 \\
& - 3.94\phi_1^4 - 16.04\phi_3^2\phi_1^2 - 2.03\phi_4^2\phi_1^2 + 13.4\phi_1^2 + 28.6\phi_3\phi_1^2 + 14.2\phi_3\phi_4\phi_1^2 \\
& - 17.7\phi_4\phi_1^2 - 5.49\phi_3^4 - 0.21\phi_4^4 + 24.2\phi_3^3 - 0.73\phi_3\phi_4^3 + 3.47\phi_4^3 \\
& - 66.4\phi_3^2 - 1.99\phi_3^2\phi_4^2 + 9.79\phi_3\phi_4^2 - 16.53\phi_4^2 - 42.75\phi_3 \\
& + 2.24\phi_3^3\phi_4 + 5.81\phi_3^2\phi_4 - 11.5\phi_3\phi_4 + 14.5\phi_4 - 31.0\,.
\end{aligned}
\tag{C.1}
$$

$$
\begin{aligned}
L_{(n=5)} = & \frac{1}{2}\left(1.5\dot{\chi}_1^2 + 2.5\dot{\phi}_1^2 + 7.549\dot{\phi}_3^2 + 6.537\dot{\phi}_4^2 + 0.83\dot{\chi}_3^2\right) - 1.85\chi_1^4 \\
& - 1.85\chi_1^4 - 0.006\chi_3^4 - 0.22\chi_3^3 + 9.31\chi_1^2 - 0.26\chi_1^2\chi_3^2 - 2.20\chi_3^2 + 3.45\chi_1^2\chi_3 \\
& + 0.245\chi_3 - 5.625\chi_1^2\phi_1^2 - 0.181\chi_3^2\phi_1^2 + 2.8\chi_3\phi_3\phi_1^2 - 0.967\chi_3\phi_4\phi_1^2 - 3.69\chi_3\phi_1^2 \\
& - 0.07\chi_3^3\phi_3 + 0.015\chi_3^3\phi_4 - 10.4\chi_1^2\phi_3^2 - 0.190\chi_1^2\phi_4^2 + 18.01\chi_1^2\phi_3 \\
& - 3.23\chi_1^2\phi_3\phi_4 + 1.60\chi_1^2\phi_4 - 0.45\chi_3^2\phi_3^2 - 0.018\chi_3^2\phi_4^2 \\
& - 1.45\chi_3^2\phi_3 + 0.169\chi_3^2\phi_3\phi_4 + 0.372\chi_3^2\phi_4 + 0.104\chi_3\phi_3^3 + 0.0003\chi_3\phi_4^3 \\
& - 10.40\chi_3\phi_3^2 - 0.1003\chi_3\phi_3\phi_4^2 - 0.205\chi_3\phi_4^2 - 3.37\chi_1^2\chi_3\phi_3 - 0.577\chi_1^2\chi_3\phi_4 \\
& - 8.29\chi_3\phi_3 + 0.071\chi_3\phi_3^2\phi_4 + 4.34\chi_3\phi_3\phi_4 + 1.16\chi_3\phi_4 - 3.88\phi_1^4 \\
& - 9.66\phi_3^2\phi_1^2 - 1.197\phi_4^2\phi_1^2 + 13.4\phi_1^2 + 20.9\phi_3\phi_1^2 + 8.14\phi_3\phi_4\phi_1^2 \\
& - 11.95\phi_4\phi_1^2 - 3.66\phi_3^4 - 0.019\phi_4^4 + 22.15\phi_3^3 - 0.12\phi_3\phi_4^3 + 0.61\phi_4^3 - 77.93\phi_3^2 \\
& - 0.897\phi_3^2\phi_4^2 + 3.98\phi_3\phi_4^2 - 6.40\phi_4^2 - 38.93\phi_3 + 0.883\phi_3^3\phi_4 + 5.42\phi_3^2\phi_4 \\
& - 7.43\phi_3\phi_4 + 10.35\phi_4 - 31\,.
\end{aligned}
\tag{C.2}
$$

$$
\begin{aligned}
L_{(n=3)} = & \frac{1}{2}\left(0.52\dot{\chi}_1^2 + 0.92\dot{\chi}_3^2 + 6.903\dot{\phi}_4^3 + 2.6\dot{\phi}_1^2 + 9.8\dot{\phi}_3^2\right) - 1.43\chi_1^4 \\
& - 0.053\chi_3^4 - 0.814\chi_3^3 + 8.10\chi_1^2 - 0.305\chi_1^2\chi_3^2 - 3.63\chi_3^2 + 3.339\chi_1^2\chi_3 \\
& + 0.226\chi_3 - 5.25\chi_1^2\phi_1^2 - 0.165\chi_3^2\phi_1^2 + 4.04\chi_3\phi_3\phi_1^2 - 1.34\chi_3\phi_4\phi_1^2 \\
& - 3.56\chi_3\phi_1^2 - 0.447\chi_3^3\phi_3 + 0.128\chi_3^3\phi_4 - 4.09\chi_1^2\phi_3^2 - 1.79\chi_1^2\phi_4^2 \\
& + 9.83\chi_1^2\phi_3 - 6.24\chi_1^2\phi_3\phi_4 + 4.20\chi_1^2\phi_4 - 1.66\chi_3^2\phi_3^2 \\
& - 0.133\chi_3^2\phi_4^2 - 4.70\chi_3^2\phi_3 + 0.86\chi_3^2\phi_3\phi_4 + 1.49\chi_3^2\phi_4 \\
& - 0.23\chi_3\phi_3^3 - 0.142\chi_3\phi_4^3 - 17.80\chi_3\phi_3^2 - 0.75\chi_3\phi_3\phi_4^2 \\
& + 0.34\chi_3\phi_4^2 - 2.30\chi_1^2\chi_3\phi_3 - 1.96\chi_1^2\chi_3\phi_4 - 12.48\chi_3\phi_3 \\
& + 1.08\chi_3\phi_3^2\phi_4 + 9.19\chi_3\phi_3\phi_4 + 1.64\chi_3\phi_4 - 3.94\phi_1^4 - 23.22\phi_3^2\phi_1^2 \\
& - 1.65\phi_4^2\phi_1^2 + +13.14\phi_1^2 + 35.3\phi_3\phi_1^2 + 18.84\phi_3\phi_4\phi_1^2 - 21.27\phi_4\phi_1^2 \\
& - 8.46\phi_3^4 - 1.19\phi_4^4 + 29.74\phi_3^3 - 2.33\phi_3\phi_4^3 + 11.24\phi_4^3 - 51.36\phi_3^2 \\
& - 2.91\phi_3^2\phi_4^2 + 16.41\phi_3\phi_4^2 - 29.82\phi_4^2 - 45.17\phi_3 + 3.54\phi_3^3\phi_4 \\
& + 4.90\phi_3^2\phi_4 - 14.55\phi_3\phi_4 + 17.07\phi_4 - 30.61\,.
\end{aligned}
\tag{C.3}
$$

C.2 Minimum Values of the Potential

For $n = 3$

$$\begin{aligned}
&\{\phi_1 \to 0., \phi_3 \to 2.55953, \phi_4 \to 3.41521, \chi_1 \to \pm 2., \chi_3 \to -11.5045\}\,, \\
&\{\phi_1 \to 0., \phi_3 \to -0.277688, \phi_4 \to 0.548444, \chi_1 \to \pm 2., \chi_3 \to 1.45505\}\,, \\
&\{\phi_1 \to 0., \phi_3 \to 2.55117, \phi_4 \to -0.596206, \chi_1 \to \pm 2., \chi_3 \to -5.25775\}\,, \\
&\{\phi_1 \to 0., \phi_3 \to -0.269331, \phi_4 \to 4.55986, \chi_1 \to \pm - 2., \chi_3 \to -4.79172\}\,, \\
&\{\phi_1 \to \pm 2., \phi_3 \to 2.30365, \phi_4 \to 4.12784, \chi_1 \to 0., \chi_3 \to -3.00725\}\,, \\
&\{\phi_1 \to \pm 2., \phi_3 \to -0.0218119, \phi_4 \to -0.164188, \chi_1 \to 0., \chi_3 \to -7.04222\}\,, \\
&\{\phi_1 \to \pm 2., \phi_3 \to 0.277688, \phi_4 \to -0.548444, \chi_1 \to 0., \chi_3 \to -1.45505\}\,, \\
&\{\phi_1 \to \pm 2., \phi_3 \to 2.00415, \phi_4 \to 4.5121, \chi_1 \to 0., \chi_3 \to \pm 8.59443\}\,,
\end{aligned} \tag{C.4}$$

For $n = 4$

$$\begin{aligned}
&\{\phi_1 \to 0, \phi_3 \to 2.4648, \phi_4 \to 2.89574, \chi_1 \to \pm 1., \chi_3 \to -12.403\}\,, \\
&\{\phi_1 \to 0, \phi_3 \to 2.27251, \phi_4 \to -0.169024, \chi_1 \to \pm 1., \chi_3 \to -7.91332\}\,, \\
&\{\phi_1 \to 0, \phi_3 \to 0.87219, \phi_4 \to 4.6719, \chi_1 \to \pm 1., \chi_3 \to -7.43008\}\,, \\
&\{\phi_1 \to 0, \phi_3 \to 0.679905, \phi_4 \to 1.60714, \chi_1 \to \pm 1., \chi_3 \to -2.94038\}\,, \\
&\{\phi_1 \to \pm 1., \phi_3 \to 0.892447, \phi_4 \to 0.644303, \chi_1 \to 0, \chi_3 \to -4.73132\}\,, \\
&\{\phi_1 \to \pm 1., \phi_3 \to 2.25226, \phi_4 \to 3.85858, \chi_1 \to 0, \chi_3 \to -10.6121\}\,, \\
&\{\phi_1 \to \pm 1., \phi_3 \to 2.42106, \phi_4 \to 3.8452, \chi_1 \to 0, \chi_3 \to -6.37635\}\,, \\
&\{\phi_1 \to \pm 1., \phi_3 \to 0.723642, \phi_4 \to 0.65768, \chi_1 \to 0, \chi_3 \to -8.96706\}\,,
\end{aligned} \tag{C.5}$$

For $n = 5$

$$\begin{aligned}
&\{\phi_1 \to 0, \phi_3 \to 3.03956, \phi_4 \to 2.63677, \chi_1 \to \pm 1., \chi_3 \to -16.9732\}\,,\\
&\{\phi_1 \to 0, \phi_3 \to 0.921837, \phi_4 \to 2.07711, \chi_1 \to \pm 1., \chi_3 \to -4.2708\}\,,\\
&\{\phi_1 \to 0, \phi_3 \to 2.59883, \phi_4 \to -2.66941, \chi_1 \to \pm 1., \chi_3 \to -11.0127\}\,,\\
&\{\phi_1 \to 0, \phi_3 \to 1.36257, \phi_4 \to 7.38328, \chi_1 \to \pm 1., \chi_3 \to -10.2313\}\,,\\
&\{\phi_1 \to \pm 1., \phi_3 \to 1.05886, \phi_4 \to 0.279828, \chi_1 \to 0, \chi_3 \to -6.35122\}\,,\\
&\{\phi_1 \to \pm 1., \phi_3 \to 2.90254, \phi_4 \to 4.43405, \chi_1 \to 0, \chi_3 \to -14.8928\}\,,\\
&\{\phi_1 \to \pm 1., \phi_3 \to 3.10959, \phi_4 \to 4.79955, \chi_1 \to 0, \chi_3 \to -9.04226\}\,,\\
&\{\phi_1 \to \pm 1., \phi_3 \to 0.851816, \phi_4 \to -0.0856701, \chi_1 \to 0, \chi_3 \to -12.2018\}\,,
\end{aligned} \tag{C.6}$$

C.3 Asymptotic Profiles of the Kink Solution for $L_{(n=3)}$

Solutions of (3.62), which are regular as $\tau \to \infty$ are given below

$$s_1(\tau) = (3.1c_1 + 0.49c_2 - 6.52c_3 - 1.52c_4)\, e^{-3.45\tau} + (0.07c_1 - 0.005c_2 + 0.25c_3 + 0.16c_4)\, e^{-4.56\tau}$$
$$+(-1.17c_1 - 0.3c_2 + 1.29c_3 + 0.26c_4)\, e^{-7.36\tau} + (-0.98c_1 - 0.18c_2 + 4.98c_3 + 1.11c_4)\, e^{-2.38\tau}\,, \tag{C.7}$$

$$s_3(\tau) = (0.92c_1 + 0.15c_2 - 1.95c_3 - 0.46c_4)\, e^{-3.44\tau} + (-0.139c_1 + 0.01c_2 - 0.48c_3 - 0.30c_4)\, e^{-4.56\tau}$$
$$+(-0.14c_1 - 0.035c_2 + 0.15c_3 + 0.03c_4)\, e^{-7.36\tau} + (-0.65c_1 - 0.12c_2 + 3.28c_3 + 0.73c_4)\, e^{-2.38\tau}$$

$$s_4(\tau) = (7.74c_1 + 1.45c_2 - 39.25c_3 - 8.73c_4)\, e^{-2.38\tau} + (5.79c_1 + 0.93c_2 - 12.24c_3 - 2.86c_4)\, e^{-3.45\tau}$$
$$+(0.25c_1 + 0.065c_2 - 0.28c_3 - 0.055c_4)\, e^{-7.36\tau} + (0.016c_1 - 0.001c_2 + 0.055c_3 + 0.035c_4)\, e^{-4.56\tau}$$

$$s_5(\tau) = (3.09c_1 + 0.49c_2 - 6.26c_3 - 1.47c_4)\, e^{-3.45\tau} + (0.20c_1 + 0.051c_2 - 0.24c_3 - 0.048c_4)\, e^{-7.36\tau}$$
$$+(0.025c_1 + 0.00025c_2 + 0.03c_3 + 0.03c_4)\, e^{-4.56\tau} + (-73.40c_1 - 13.99c_2 + 404.89c_3 + 89.86c_4)\, e^{-2.38\tau}$$

APPENDIX D

BASICS OF CALCULATION OF LYAPUNOV EXPONENTS

Looking at the results we obtained one can easily see that they can have chaotic behavior. To investigate if it really is the case we will compute the Lyapunov exponents corresponding to our Lagrangians for various energies. Let us start this section with a brief review of how to compute the Lyapunov exponents [4, 3, 5]. Suppose we have a Hamiltonian H with a solution $x(t)$ which demonstrates chaotic behavior. If we consider a nearby point $x(t) + \delta(t)$. We expect this deviation to grow exponentially in the regime of chaotic behavior.

$$|\delta(t)| = |\delta(0)|e^{\lambda t} \tag{D.1}$$

Where λ is a positive constant called a Lyapunov exponent. This exponential growth we see in (D.1) is the realization of heavy sensitivity on initial conditions in chaotic system. So we can use these constant to decide if a system is chaotic or not. To compute the Lyapunov constant for a general system we will do the following. Consider a system of First order differential equations which represents the Hamilton's equation of motion of a Hamiltonian system.

$$\dot{x} = F(x) \tag{D.2}$$

where x is N dimensional vector. Taking a variation around $\dot{x}$ we obtain the following

$$\delta\dot{x}_i = \frac{\partial F^i}{\partial x^j}\delta x^j \tag{D.3}$$

Hence we see that we can express a deviation vector $\delta x(t)$ can be expressed as

$$\delta x(t) = U(t)\delta x(0) \tag{D.4}$$

Where $U(t)$ is the time translation operator. And by the use of linearity property of the time translation operator we can obtain a general deviation vector in the form.

$$\delta x(t+t') = U(t)U(t')\delta(0) \tag{D.5}$$

And the Lyapunov exponent for this deviation vector is defined as

$$\lambda \equiv \lim_{T\to\infty} \frac{1}{T} \log \frac{|\delta x(T)|}{|\delta x(0)|} \tag{D.6}$$

And using the linearity property

$$\lambda \equiv \lim_{n\to\infty} \log \frac{|U_{\Delta t}.....U_{\Delta t}\delta x(0)|}{|\delta x(0)|} \tag{D.7}$$

Now we are in position to generalize this procedure for n Lyapunov exponents.Firstly we need to construct an orthonormal set of basis that spans the vector which is tangent to the trajectory. Let this set of basis be labeled as $\{k_0^1, k_0^2, \ldots, k_0^n\}$. Then by applying the time translation operator $U_{\Delta t}$ we evolve each basis vector by Δt. Resulting in a new set of basis vectors $\{w_1^1, w_1^2, \ldots, w_1^n\}$. Note that after the transformation new basis vector are no longer orthonormal. To take care of this we follow the usual Gram-Schmidt orthogonalization process. Doing so we obtain a new set of orthogonal vectors $\{\tilde{k}_1^1, \tilde{k}_1^2, \ldots, \tilde{k}_1^n\}$.Now we will compute the expansion rate of each vector which will be used in the definition of Lyapunov exponent can be given as

$$a_1^k \equiv \frac{\left|\tilde{k}_1^k\right|}{p_0^k} = \left|\tilde{k}_1^k\right| \tag{D.8}$$

Finally we normalize the orthogonal set of vectors $\{\tilde{k}_1^1, \tilde{k}_1^2, \ldots, \tilde{k}_1^n\}$ which completes a single cycle. Then this process should be repeated for N times and the spectrum Lyapunov exponents can be obtained as

$$\lambda_k \equiv \lim_{N\to\infty} \frac{1}{n\Delta t} \sum_{i=1}^{N} \log\left(a_i^k\right) \tag{D.9}$$

By construction v_i^1's direction is the one with the highest sensitivity to initial conditions so it's corresponding expansion rate has the largest value in this region.