FINITE RIGID SETS IN CURVE COMPLEXES OF NON-ORIENTABLE SURFACES

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ABSTRACT

FINITE RIGID SETS IN CURVE COMPLEXES OF NON-ORIENTABLE SURFACES

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A finite rigid set in a curve complex of a surface is a subcomplex such that every locally injective simplicial map defined on this subcomplex into the curve complex is induced from an automorphism of curve complex. In this thesis, we find finite rigid sets in the curve complexes of connected, non-orientable surfaces of genus g with n holes, where $g+n \neq 4$.

Keywords: Finite Rigid Sets, Curve Complex, Nonorientable Surfaces, Locally Injective Simplicial Maps

ÖZ

YÖNLENDİRİLEMEYEN YÜZEYLERİN EĞRİ KOMPLEKSİNDEKİ SONLU KATI KÜMELER

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Bir yüzeyin eğri kompleksindeki bir sonlu katı küme, bu kümeden eğri kompleksine tanımlı her yerel birebir simpleksel gönderimin, eğri kompleksinin bir otomorfizminden indirgendiği altkompleksdir. Bu tezde, $g + n \neq 4$ için g cins ve n delikli bağlantılı, yönlendirilemeyen yüzeylerin eğri komplekslerindeki sonlu katı kümeleri bulduk.

Anahtar Kelimeler: Sonlu Katı Kümeler, Eğri Kompleksi, Yönlendirilemeyen Yüzeyler, Yerel Birebir Simpleksel Gönderimler

To the tears of my mother and my niece Beyza

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CHAPTER 1

INTRODUCTION

In [7], Harvey introduced the complex of curves $\mathcal{C}(S)$ on an orientable surface S as the abstract simplicial complex whose n simplices are the sets of $n+1$ pairwise disjoint, distinct, nontrivial isotopy classes of simple closed curves. Harvey also stated that the extended mapping class group $\mathrm{Mod}^*(S)$, the group of the isotopy classes of self-homeomorphisms of S, acts simplicially on $\mathcal{C}(S)$. This action gives rise to the natural question: Is the natural homomorphism $\Phi : \mathrm{Mod}^*(S) \to \mathrm{Aut}(\mathcal{C}(S))$ induced from the aforementioned action an isomorphism? Ivanov showed that Φ is an isomorphism for a surface with genus at least 2 in [13]. For a sphere with at least 5 punctures or a torus with at least 3 punctures , Korkmaz [14] and Luo [15] completed the result $\mathrm{Mod}^*(S) \simeq \mathrm{Aut}(\mathcal{C}(S)$, except twice punctured torus case.

Recall that a superinjective simplicial map $\lambda : \mathcal{C}(S) \to \mathcal{C}(S)$ is a simplicial map with the following property: The vertices α and β span an edge in $\mathcal{C}(S)$ if and only if the vertices $\lambda(\alpha)$ and $\lambda(\beta)$ span an edge in $\mathcal{C}(S)$. The result about an automorphism of $\mathcal{C}(S)$ was first generalized by Irmak [9] to superinjective simplicial map: For closed orientable surface with genus at least 3, every superinjective simplicial map of $\mathcal{C}(S)$ is induced from a homeomorphism of S. Behrstock-Margalit [4] obtained the result for a genus 1 orientable surface S with at least 3 punctures. For an orientable surface S, the simplicial embeddings of the curve complex $\mathcal{C}(S)$ were studied by Shackleton in [19] and it was shown that any simplicial embedding of the curve complex $\mathcal{C}(S)$ is obtained from a homeomorphism of S. Behrstock-Margalit and Shackleton independently introduced the adjacency graph of pants decomposition as a new tool to distinguish curve types on S in their work.

Aramayona-Leininger [1] described a finite subcomplex $\mathfrak{X}$ of the curve complex $\mathcal{C}(S)$ such that for any locally injective simplicial map $\phi : \mathfrak{X} \to \mathcal{C}(S)$ (see Definition 2.1.9), there exists an automorphism $\Phi : \mathcal{C}(S) \to \mathcal{C}(S)$ satisfying $\Phi|_{\mathfrak{X}} = \phi$, and Φ is unique up to the point-wise stabilizer of $\mathfrak{X}$ in $\mathrm{Aut}(\mathcal{C}(S))$. Such a subcomplex $\mathfrak{X}$ is called a *finite rigid set*. The main idea of the work is to show that a locally injective simplicial map $\mathfrak{X} \to \mathcal{C}(S)$ is induced from a homeomorphism $S \to S$ and by the results in [13], [14], [15], it is concluded that a locally injective simplicial map is a restriction of an automorphism.

After Aramayona-Leininger introduced the finite rigid sets in $\mathcal{C}(S)$, Maungchang used the finite rigid sets to find the finite rigid subgraphs of the pants graphs of punctured spheres in his dissertation [16]. Hernández Hernández [8] and Aramayona-Leininger [2] used the finite rigid sets to *exhaust* the curve graph and the curve complex of an orientable surface of finite topological type, respectively: They show that for an orientable surface S, there exists a sequence of the finite rigid sets each has trivial point-wise stabilizer in $\mathrm{Mod}^*(S)$ so that the union of these sets is the curve graph of S and the curve complex of S, respectively.

On the other hand for the non-orientable surfaces, the curve complex was studied by Scharlemann [18]. Atalan-Korkmaz [3] proved that the mapping class group $\mathrm{Mod}(N_g^n)$ of a non-orientable surface N_g^n of genus g with n holes is isomorphic to the automorphism group $\mathrm{Aut}(\mathcal{C}(N_g^n))$ of the curve complex of the surface N_g^n for $g + n \geq 5$. Irmak generalized this results by showing a superinjective map and an edge-preserving map $\mathcal{C}(N_g^n) \to \mathcal{C}(N_g^n)$ is induced from a homeomorphism $N_g^n \to N_g^n$ in [11] and [12], respectively. Irmak showed that the adjacency and nonadjacency with respect to the top dimensional maximal simplex in the curve complex of a non-orientable surface are preserved under a superinjective simplicial map in [11]. Irmak used this result to distinguish the topological types of the curves on a non-orientable surface under a superinjective simplicial map.

In this dissertation, we establish a finite rigid set in the curve complex of a non-orientable surface. Here is an outline of the dissertation: In Chapter 2, we give the basic notions about the curve complex of a non-orientable surface, recall the definitions of the locally injective simplicial map and of the adjacency graph of a pants decomposition, and introduce the notation used throughout this dissertation.

In Chapter 3, we introduce the finite subcomplex $\mathfrak{X}_g^n$ in the curve complex $\mathcal{C}(N_g^n)$, where $g + n \geq 5$ and present our results about the topological types of vertices in $\mathfrak{X}_g^n$ under a locally injective simplicial map.

In Chapter 4, we state and prove the main result of the dissertation: Let N_g^n be a compact connected non-orientable surface of genus g with n holes, $g + n \geq 5$ and $\mathfrak{X}_g^n$ be the finite subcomplex in $\mathcal{C}(N_g^n)$ introduced in Chapter 3.

Theorem 4.2.1. *Any locally injective simplicial map $\phi : \mathfrak{X}_g^n \to \mathcal{C}(N_g^n)$ is induced from a homeomorphism $F : N_g^n \to N_g^n$.*

Hence, we deduce that the subcomplex $\mathfrak{X}_g^n$ is a finite rigid set in $\mathcal{C}(N_g^n)$.

In Chapter 5, we investigate the finite rigid sets for $g + n < 5$ and give a conclusion of the dissertation.

CHAPTER 2

PRELIMINARIES

The background chapter consists of three sections. In the first section, we give the basic definitions and some facts about the curve complex of a non-orientable surface. In the second section, we present the notation to indicate a non-orientable surface and the simple closed curves on a non-orientable surface that will be used throughout this dissertation. In the third section, we cite an adjacency graph of a pants decomposition which will frequently be used in the proofs of the results about topological types of vertices in the chapter three.

2.1 The Curve Complex $\mathcal{C}(N_g^n)$

In this section, we recall the basic definitions about the curve complex of a non-orientable surface. We state a result about the topological type of the simple closed curves in a top dimensional pants decomposition of a non-orientable surface. We cite the result in [3] which relates the mapping class group of a non-orientable surface and the automorphism group of a curve complex of a non-orientable surface. Finally we recall the definition of a locally injective simplicial map.

Let $N = N_g^n$ be a compact connected non-orientable surface of genus g with n boundary components and let S_g^n denote a compact connected orientable surface of genus g with n boundary components. The *genus* of a non-orientable surface is defined as the maximum number of real projective planes in a connected sum decomposition. By convention, we assume that a sphere is a non-orientable surface of genus 0. We will use the term "hole" instead of boundary component.

2.1.1 Basic Notions

Let ∂N denote the set of holes of N. A *simple closed curve* is an embedding $a : \mathbb{S}^1 \to N \setminus \partial N$, where $\mathbb{S}^1$ is the unit circle. For a simple closed curve a, we denote by N_a the surface obtained by cutting N along a. A simple closed curve a is *non-separating* if N_a is connected and *separating* if N_a has two connected components. A simple closed curve on N is *nontrivial* if it bounds neither a disc, nor a disc with one hole, nor a Möbius band in N. A simple closed curve on N is called *two-sided* (respectively *one-sided*) if its regular neighbourhood is an annulus (respectively a Möbius band). A one-sided simple closed curve a is called *essential* if either $g = 1$ or $g \geq 2$ and N_a is non-orientable. We say a simple closed curve a is of *type* (p, q) if N_a is homeomorphic to the disjoint union of N_p^{q+1} and N_{g-p}^{n-q+1}, where p, q are nonnegative integers. In particular if N_a is disjoint union of N_0^{q+1} and N_g^{n-q+1}, that is if a bounds a disk with q-holes, then we say that a is of type $(0, q)$. Note that a simple closed curve of type (p, q) is also of type $(g - p, n - q)$.

Let a and b be two simple closed curves on N. We say a and b are *isotopic* if there is a continuous map $F : \mathbb{S}^1 \times [0, 1] \to N$ such that $F(x, 0) = a(x),\ F(x, 1) = b(x)$ and $F(x, t)$ is an embedding for each $t \in [0, 1]$. The isotopy class of simple closed curve a is denoted by α.

Let α and β be two isotopy classes of the simple closed curves a and b, respectively. The *geometric intersection number* $i(\alpha, \beta)$ is defined as

$$i(\alpha, \beta) = min\{\mathrm{Card}(a \cap b) \mid a \in \alpha, b \in \beta\},$$

where $\mathrm{Card}(a \cap b)$ denotes the cardinality of $a \cap b$. We assume that representatives of isotopy classes intersect minimally, that is transversely and in the minimal number. We say two isotopy classes α and β are *disjoint* if $i(\alpha, \beta) = 0$.

Definition 2.1.1. *[17] Let V be a finite set. An abstract simplicial complex K is a family of nonempty subsets of V, called simplices of K, such that*

- *if $v \in V$, then $\{v\} \in K$, and*
- *if $\sigma \in K$ and $\tau \subset \sigma$ is nonempty, then $\tau \in K$.*

The set V is called the vertex set of K. A simplex $\sigma \in K$ having $n+1$ distinct vertices is called n-simplex.

Definition 2.1.2. *The complex of curves $\mathcal{C}(N)$ is an abstract simplicial complex with the set of vertices*

$$\mathcal{V} = \{\alpha \mid \alpha \text{ is an isotopy class of a nontrivial simple closed curve on } N\}$$

and a k-simplex of $\mathcal{C}(N)$ is the set of $(k+1)$ distinct vertices $\{\alpha_0, \alpha_1, \ldots, \alpha_k\}$ on N having pairwise disjoint representatives.

For simplicity, we do not distinguish between a vertex of $\mathcal{C}(N)$, the corresponding isotopy class of a nontrivial simple closed curve and its representative curve. In this dissertation the term "simple closed curve" corresponds to any of the aforementioned three terms.

2.1.2 Top Dimensional Maximal Simplices

There is a bijection between the set of the maximal simplices of $\mathcal{C}(N)$ and the set of the pants decompositions of N. In particular we are interested in the top dimensional pants decompositions on N, and hence the top dimensional maximal simplices in $\mathcal{C}(N)$.

Definition 2.1.3. *Let P be a set of pairwise disjoint, non-isotopic, nontrivial simple closed curves on N. The set P is called a pair of pants decomposition of N, or briefly pants decomposition of N, if the complement of the elements of P is disjoint union of three-holed spheres.*

We shall prove a lemma about the topological types of vertices in a top dimensional maximal simplex of $\mathcal{C}(N)$. In the proof we will use the following results about the dimension of a maximal simplex of $\mathcal{C}(N)$ in [3] and the number of simple closed curves in a pants decomposition of an orientable surface S_g^n.

Lemma 2.1.4. *[3, Proposition 2.3] Let N_g^n be a connected non-orientable surface of genus $g \geq 2$ with n holes. Suppose that $(g, n) \neq (2, 0)$. Let $a_r = 3r+n-2$ and $b_r = 4r + n - 2$ if $g = 2r + 1$, and $a_r = 3r + n - 4$ and $b_r = 4r + n - 4$ if $g = 2r$. Then, there is a maximal simplex of dimension q in $\mathcal{C}(N)$ if and only if $a_r \leq q \leq b_r$. In particular, there are precisely $\lceil g/2 \rceil$ values which occur as the dimension of a maximal simplex, where $\lceil g/2 \rceil$ denotes the smallest integer greater than $g/2$.*

A pants decomposition P is called *top dimensional* if it contains $b_r + 1$ elements. By the Lemma 2.1.4, we have the following corollary.

Corollary 2.1.5. *Let P be a top dimensional pants decomposition of N_g^n. Then, P contains $2g + n - 3$ simple closed curves.*

Lemma 2.1.6. *Let S_g^n be an orientable surface of genus g with n holes. The number of simple closed curves in a pair of pants decomposition is $3g - 3 + n$.*

Proof. See pages 248-249 in [5]. □

Lemma 2.1.7. *Let P be a top dimensional pants decomposition of N. Topological type of an element of P can be:*

1. *essential one-sided,*
2. *of type (p, q) where $0 \leq p \leq g, 1 \leq q \leq n - 1$ (except $(0, 1)$).*

Proof. We will prove the lemma by showing that a top dimensional pants decomposition P of N does not contain a simple closed curve of type that is not mentioned in the lemma. The topological types of simple closed curves that are not mentioned in the lemma are the followings:

1. A two-sided non-separating simple closed curve whose complement is orientable (in the case g is even).
2. A one-sided simple closed curve whose complement is orientable (in the case g is odd and $g \geq 3$).

3. A separating simple closed curve whose complement is disjoint union of S_k^{l+1} and N_{g-2k}^{n-l+1} $(k \geq 1)$.

4. A two-sided non-separating simple closed curve whose complement is non-orientable $(g \geq 3)$.

Case 1: Let β_1 be a two-sided non-separating simple closed curve with the complement N_{β_1} is orientable and Q_1 be a pants decomposition of N containing β_1. In this case g is even, say $g = 2r$, $r \geq 1$. Since N_{β_1} is homeomorphic to S_{r-1}^{n+2}, by Lemma 2.1.6, the pants decomposition $Q_1 \setminus \{\beta_1\}$ of N_{β_1} contains

$$3(r-1) - 3 + (n+2) = 3r + n - 4$$

elements. Thus, Q_1 contains $3r + n - 3$ elements. Since

$$3r + n - 3 < 2g + n - 3,$$

Q_1 is not a top dimensional pants decomposition.

Case 2: Let β_2 be a one-sided simple closed curve such that N_{β_2} is orientable and Q_2 be a pants decomposition of N containing β_2. Now g is odd, say $g = 2r + 1$. Since N_{β_2} is homeomorphic to S_r^{n+1}, by Lemma 2.1.6, the pants decomposition $Q_2 \setminus \{\beta_2\}$ of N_{β_2} contains

$$3r - 3 + (n+1) = 3r + n - 2$$

elements, hence Q_2 contains $3r + n - 1$ elements. Since

$$3r + n - 1 < 2g + n - 3,$$

Q_2 is not a top dimensional pants decomposition.

Case 3: Let β_3 be a separating simple closed curve such that N_{β_3} is disjoint union of an orientable surface S_k^{l+1} and a non-orientable surface N_{g-2k}^{n-l+1} where $k \geq 1$. Let Q_3 be a pants decomposition of N containing β_3. Then, there is a pants decomposition Q_1^3 of S_k^{l+1} and a pants decomposition Q_2^3 of N_{g-2k}^{n-l+1} such that $Q_3 = Q_1^3 \cup Q_2^3 \cup \{\beta_2\}$. By Lemma 2.1.6, the pants decomposition Q_1^3 contains

$$3k - 3 + l + 1 = 3k + l - 2$$

elements. There are two cases about the number of simple closed curves in Q_2^3:

If g is even, say $g = 2r$, by Lemma 2.1.4, Q_2^3 contains at most

$$4(r-k) + (n-l+1) - 3 = 4(r-k) + n - l - 2$$

elements. Then, Q_3 contains at most

$$(3k + l - 2) + (4(r-k) + n - l - 2) + 1 = 4r - k + n - 3$$

elements. Since this number is less than $2g + n - 3$, the pants decomposition Q_3 can not be a top dimensional pants decomposition.

If g is odd, say $g = 2r + 1$, by Lemma 2.1.4, Q_2^3 contains at most

$$4(r-k) + (n-l+1) - 1 = 4(r-k) + n - l$$

elements. Thus, Q_3 contains at most

$$(3k + l - 2) + (4(r-k) + n - l) + 1 = 4r - k + n - 1$$

elements. Since this number is less than $2g + n - 3$, Q_3 can not be a top dimensional pants decomposition.

Case 4: Let β_4 be a two-sided non-separating simple closed curve such that N_{β_4} is non-orientable, Q_4 be a pants decomposition of N containing β_4. Since N_{β_4} is homeomorphic to N_{g-2}^{n+2}, by Lemma 2.1.4, the pants decomposition $Q_4 \setminus \{\beta_4\}$ contains at most

$$4(r-1) + n + 2 - 3 = 4r + n - 5$$

elements if g is even, say $g = 2r$; the pants decomposition $Q_4 \setminus \{\beta_4\}$ contains at most

$$4(r-1) + n + 2 - 1 = 4r + n - 3$$

elements if g is odd, say $g = 2r + 1$.

Thus, Q_4 contains at most

$$4r + n - 5 + 1 = 4r + n - 4,$$

elements if $g = 2r$,

$$4r + n - 3 + 1 = 4r + n - 2,$$

elements if $g = 2r + 1$. In both cases, these numbers are less than $2g + n - 3$. Hence, Q_4 can not be a top dimensional pants decomposition. $\square$

2.1.3 Mapping Class Groups

Let f and g be two homeomorphisms of N. We say f and g are *homotopic* if there exists a continuous function $F : N \times [0,1] \to N$ such that

$$F(x,0) = f(x) \text{ and } F(x,1) = g(x) \text{ for all } x \in N.$$

If $F(x,t)$ is a homeomorphism for each $t \in [0,1]$, then F is called an *isotopy*.

Let $Homeo(N)$ denote the group of the self homeomorphisms of N, with compact-open topology and $Homeo_0(N)$ be the subgroup of $Homeo(N)$ consisting of elements which are isotopic to the identity $Id : N \to N$.

The *mapping class group* of N is defined as

$$\mathrm{Mod}(N) = Homeo(N)/Homeo_0(N)$$

Recall that an *automorphism* of $\mathcal{C}(N)$ is a bijective simplicial map. Let $\mathrm{Aut}(\mathcal{C}(N))$ denote the group of the automorphisms of $\mathcal{C}(N)$. The group $\mathrm{Mod}(N)$ acts simplicially on $\mathcal{C}(N)$. Hence, there exists the natural homomorphism $\mathrm{Mod}(N) \to \mathrm{Aut}(\mathcal{C}(N))$. The following result is proven by Atalan and Korkmaz in [3] showing the natural homomorphism is an isomorphism:

Theorem 2.1.8. *[3, Theorem 1] Let $g+n \geq 5$ and let N_g^n be a compact connected non-orientable surface of genus g with n holes. Then, $\mathrm{Mod}(N_g^n)$ is isomorphic to $\mathrm{Aut}(\mathcal{C}(N_g^n))$.*

2.1.4 Locally Injective Simplicial Map

Let K be an abstract simplicial complex and γ be a vertex in K. The *star* of a vertex $\gamma \in K$ is the subcomplex of K consisting of all simplices of K containing γ and all faces of such simplices.

Definition 2.1.9. *Let K be an abstract simplicial complex and L be a subcomplex of K. A simplicial map $\phi : L \to K$ is locally injective if it is injective on the restriction to the star of every vertex of L.*

Lemma 2.1.10. *Let L be a subcomplex of $\mathcal{C}(N)$, th map $\phi : L \to \mathcal{C}(N)$ be a locally injective simplicial map and $P \subset L$ be a top dimensional pants decomposition of N. Then, the set $\phi(P)$ is a top dimensional pants decomposition of N.*

Proof. Let $P = \{\gamma_i \mid i = 1, 2, \ldots, 2g+n-3\}$ be a top dimensional pants decomposition and Δ be its corresponding top dimensional maximal simplex. Since ϕ is injective on the star of each vertex of Δ, each $\phi(\gamma_i)$ is distinct. Furthermore ϕ is simplicial. Hence, $\phi(\Delta)$ is a simplex spanned by $2g + n - 3$ vertices, that is $\phi(\Delta)$ is a top dimensional maximal simplex in $\mathcal{C}(N)$. Hence, $\phi(P)$ is a top dimensional pants decomposition of N. $\square$

2.2 Notation

In this section, we introduce the notation used throughout this dissertation. We explain how we use it to illustrate a non-orientable surface and the simple closed curves on this surface.

Let N_g^n be a non-orientable surface of genus g with n holes. We now describe the model to represent N_g^n. Let us take a regular $g+n$-gon R. Let $\{e_1, e_2, \ldots, e_{g+n}\}$ be the set of edges of R. We index the edges of R with indices $i = 1, 2, \ldots, g+n$ in a counter-clockwise order. We remove the all vertices between the edges and insert the semicircle s_i between the edges e_{i-1} and e_i, $i = 1, 2, \ldots, g$ and indices in modulo $g + n$ (e.g, see Figure2.1).

We obtain N_g^n by doubling the model described above as follows: The points on the edges with the same indices are glued together identically and the antipodal points on the circle coming from duplicating the semicircles with same indices are identified. If the number n of the holes is at least one, the vertices in the model denote the holes of N_g^n.

A non-orientable surface N_g^n is obtained from a sphere as follows: Remove $g+n$ disjoint, distinct, open discs where g-many of them have a cross inside, n many of them are denoted by dots and identify the antipodal points of the each resulting

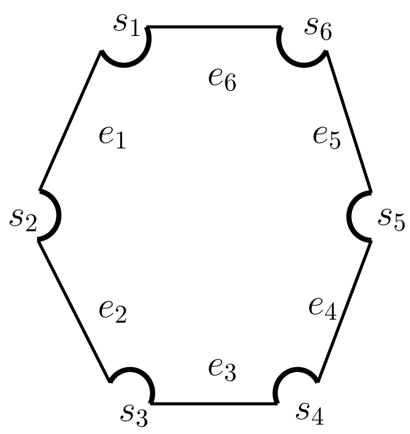

Figure 2.1: Models for N_6^0 and N_3^4.

boundary components coming from removing g crossed discs (see Figure 2.2).

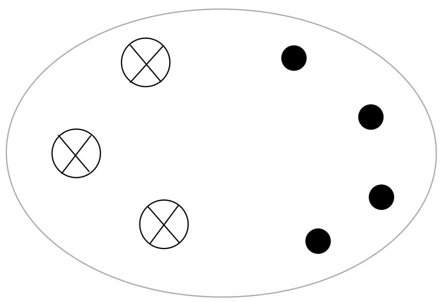

Figure 2.2: Representative of N_3^4.

The arcs on the model introduced above represents the simple closed curves on N_g^n as follows: The arc, parallel to the semicircle s_i represents the one-sided simple closed curve denoted by α_i. The arc connecting the semicircle s_i to the edge e_j indicates the one-sided simple closed curve denoted by α_i^j. The arc connecting two non-consecutive edges e_i and e_j, represents the two-sided simple closed curve denoted by β_i^j (e.g., see Figure 2.3).

2.3 Adjacency Graph of a Pants Decomposition

In this section, we define the adjacency graph of a pants decomposition and some terms related to it.

Definition 2.3.1. *[4] Let P be a pants decomposition of N, α and β be two elements of P. The simple closed curve α is adjacent to β with respect to P if*

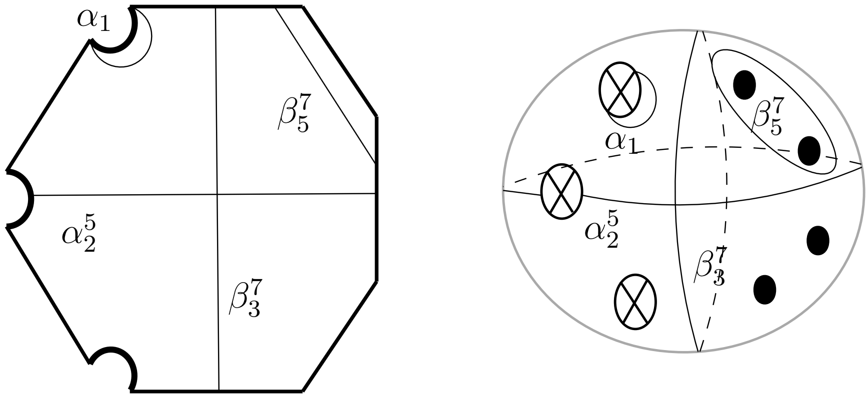

Figure 2.3: Arcs represent simple closed curves on N_3^4.

there exists a component of N_P containing α and β in its closure in N. The adjacency graph Γ_P of P is a graph whose set of vertices consists of the simple closed curves in P and two vertices α and β are connected by an edge if the simple closed curves α and β are adjacent with respect to P.

We say P is *linear* if Γ_P is linear. (e.g., see Figure2.4)

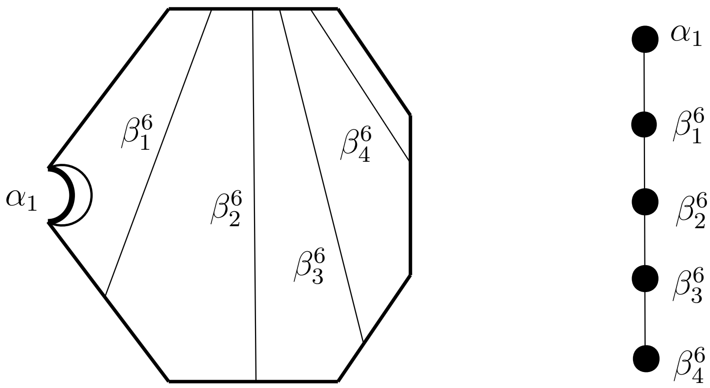

Figure 2.4: A linear pants decomposition P of N_1^6 and its adjacency graph Γ_P.

For an example of an adjacency graph of a pants decomposition of N_3^4, see Figure 2.5.

The *valency of a vertex* in a graph is the number of edges incident to the vertex. For example in Figure 2.5, α_1 has valency one and β_1^3 has valency three in the adjacency graph Γ_P of the pants decomposition $P = \{\alpha_1, \alpha_2, \alpha_3, \beta_1^3, \beta_1^4, \beta_1^5\}$.

By the *type* of a vertex in an adjacency graph Γ_P, we mean the topological type

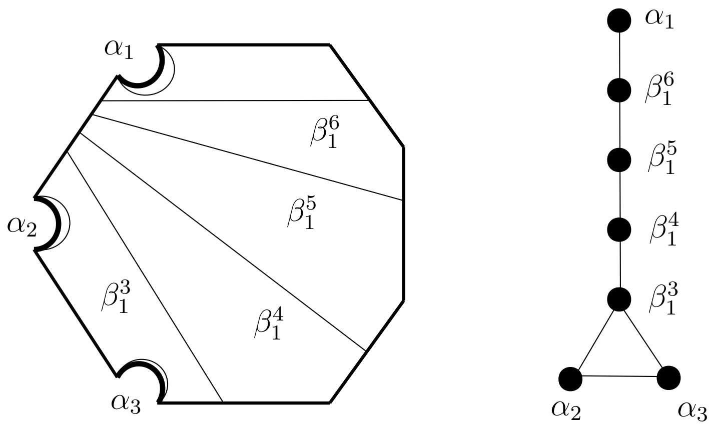

Figure 2.5: The simple closed curves in P and its adjacency graph Γ_P.

of the simple closed curve in P corresponding to the vertex in Γ_P.

By a *path* in an adjacency graph Γ_P, we mean an edge path determined by a sequence of distinct adjacent vertices $v_1, v_2, \ldots, v_k$ of Γ_P. A *cycle* in Γ_P is a subgraph homeomorphic to a circle. A cycle is called a *triangle* if it has three vertices.

Let $\alpha \in \Gamma_P$ be a vertex of valency one. If Γ_P is linear, then every vertex of Γ_P is a linear successor of α. If Γ_P is not linear, there is a unique vertex $\beta \in \Gamma_P$ closest to α and of valency at least three such that every vertex of a path from α to β has valency two except for α and β. The vertices in this path are called *linear successors* of α. In Figure 2.5, the vertices $\beta_1^3, \beta_1^4, \beta_1^5, \beta_1^6$ are linear successors of α_1.

CHAPTER 3

FINITE SUBCOMPLEX $\mathfrak{X}_g^n$ OF CURVE COMPLEX

In this chapter, we introduce the finite subcomplex $\mathfrak{X}_g^n \subset \mathcal{C}(N_g^n)$ for $g+n \geq 5$ that will be shown to be a finite rigid set in $\mathcal{C}(N_g^n)$ in Chapter 4. By the term "finite rigid set" in $\mathcal{C}(N_g^n)$, we mean a finite subcomplex in $\mathcal{C}(N_g^n)$ with the following property: Any locally injective simplicial map $\mathfrak{X}_g^n \to \mathcal{C}(N_g^n)$ is induced from a mapping class in $\mathrm{Mod}(N_g^n)$. We shall investigate the topological types of the image of the simple closed curves in $\mathfrak{X}_g^n \subset \mathcal{C}(N_g^n)$ under a locally injective simplicial map $\mathfrak{X}_g^n \to \mathcal{C}(N_g^n)$.

3.1 Finite Subcomplex

Let N_g^n be a compact connected non-orientable surface of genus g with n holes, where $g+n \geq 5$. We use the notation given in the Section 2.2 to introduce the finite subcomplex $\mathfrak{X}_g^n$. We consider the set $\mathfrak{X}_g^n \subset \mathcal{C}(N_g^n)$ as a disjoint union of two sets: $\mathfrak{X}_1$, the set of one-sided simple closed curves in $\mathfrak{X}_g^n$ and $\mathfrak{X}_2$, the set of two-sided simple closed curves in $\mathfrak{X}_g^n$. We notice that we use the indices, the subscripts and the superscripts in modulo $g+n$ in this dissertation.

$$\mathfrak{X}_1 = \{\alpha_i, \alpha_i^j \ \mid \ 1 \leq i \leq g,\ 1 \leq j \leq g+n \text{ and } j \neq i-1, j \neq i\}.$$

Note that each of the one-sided simple closed curves in $\mathfrak{X}_1$ is essential.

$$\mathfrak{X}_2 = \{\beta_i^j \mid 1 \leq i, j \leq g+n,\ 2 \leq |i-j| \leq g+n-2\}.$$

Note that each of the simple closed curves in $\mathfrak{X}_2$ is separating and of type (p, q) for some p, q.

For the arcs corresponding to the simple closed curves in the subcomplexes $\mathfrak{X}_1$ and $\mathfrak{X}_2$ in $\mathcal{C}(N_0^6)$, $\mathcal{C}(N_1^5)$, $\mathcal{C}(N_3^3)$ and $\mathcal{C}(N_6^0)$ see Figure3.1, Figure3.2, Figure3.3 and Figure3.4, respectively.

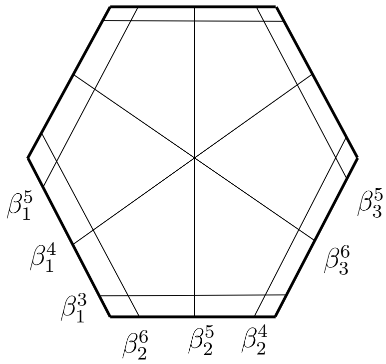

Figure 3.1: $\mathfrak{X}_0^6$ in $\mathcal{C}(N_1^5)$.

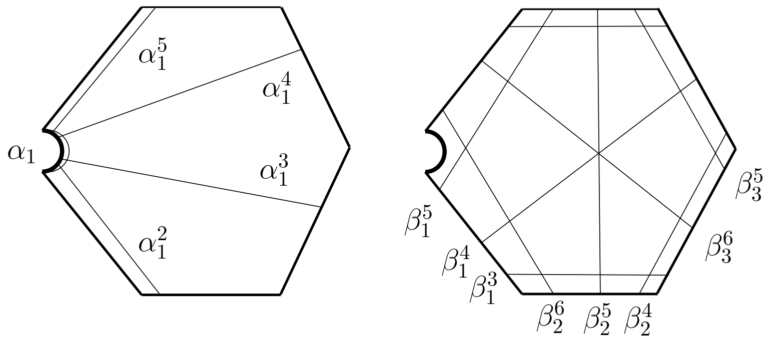

Figure 3.2: $\mathfrak{X}_1$ and $\mathfrak{X}_2$ in $\mathcal{C}(N_1^5)$.

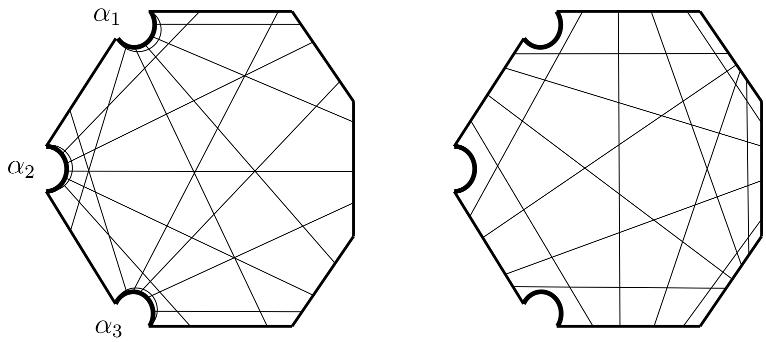

Figure 3.3: $\mathfrak{X}_1$ and $\mathfrak{X}_2$ in $\mathcal{C}(N_3^4)$.

If we cut a non-orientable surface N_g^n of genus g with n holes along the essential one-sided simple closed curve α_g, we obtain a surface homeomorphic to a non-

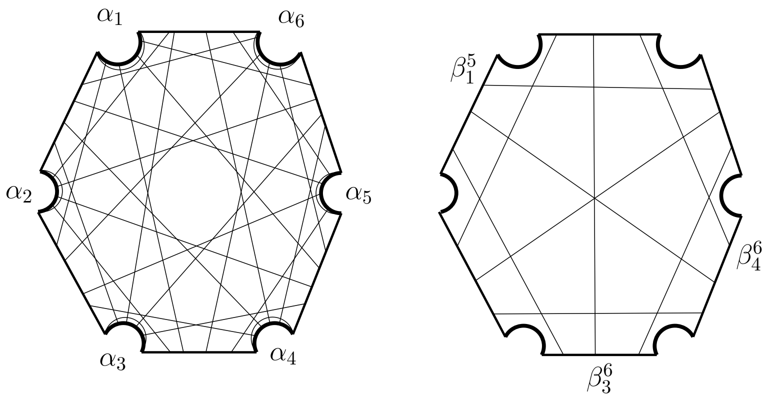

Figure 3.4: $\mathfrak{X}_1$ and $\mathfrak{X}_2$ in $\mathcal{C}(N_6^0)$.

orientable surface N_{g-1}^{n+1} of genus $g-1$ with $n+1$ holes. One of the holes of N_{g-1}^{n+1}, name δ_g, comes from α_g. We define the quotient map $q : N_{g-1}^{n+1} \to N_g^n$ which identifies the antipodal points of δ_g. This map gives rise to an injective simplicial map $q_* : \mathcal{C}(N_{g-1}^{n+1}) \to \mathcal{C}(N_g^n)$ such that $q_*(\mathfrak{X}_{g-1}^{n+1}) \subset \mathfrak{X}_g^n$.

Let S denote an orientable surface and let L be a subcomplex of $\mathcal{C}(N_g^n)$. It was shown in [19, Lemma 7] that any locally injective simplicial map $\mathcal{C}(S) \to \mathcal{C}(S)$ preserves nonadjacency in the adjacency graph of a pants decomposition of S. Now we shall show that a locally injective simplicial map $L \to \mathcal{C}(N_g^n)$ preserves nonadjacency in the adjacency graph of a pants decomposition $P \subset L$ of a non-orientable surface N_g^n with some modifications of [19, Lemma 7]. In the proof of this lemma we need the following fact in [6].

Lemma 3.1.1. *[6, Corollary 2.11] Let P^2 be a pair of pants, i.e. the 2-sphere minus the interior of three disjoint open discs. The set $A(P^2)$ of isotopy classes properly embedded arcs consists of exactly six elements, classified by the connected components of ∂P^2 in which the endpoints of respective arcs fall. See Figure 3.5 for the representatives of the isotopy classes of properly embedded arcs.*

Lemma 3.1.2. *Let L be a subcomplex of $\mathcal{C}(N_g^n)$, let the set $P \subset L$ be a top dimensional pants decomposition of N_g^n and let Γ_P be its adjacency graph. Suppose that for every pair $\alpha \in P$ and $\beta \in P$ that is nonadjacent in Γ_P, there exist two disjoint simple closed curves δ_α and δ_β in L such that*

$$\delta_\alpha \cap \alpha \neq \emptyset, \delta_\beta \cap \beta \neq \emptyset \textit{ and } \delta_\alpha \cap \beta = \emptyset, \delta_\beta \cap \alpha = \emptyset$$

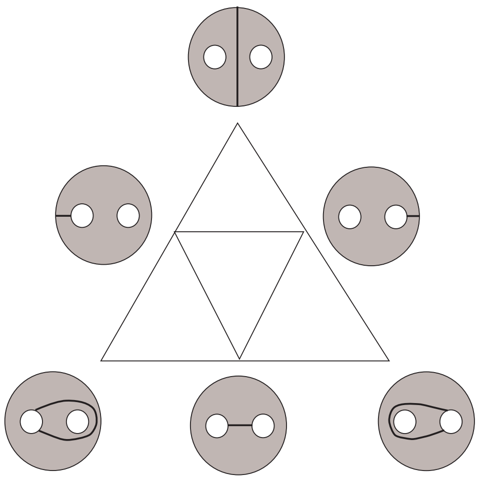

Figure 3.5: Isotopy classes of properly embedded arcs in P^2 and their simplex in arc complex.

and each of them is disjoint from the simple closed curves in $P \setminus \{\alpha, \beta\}$. Then, nonadjacency in Γ_P is preserved under a locally injective simplicial map $\phi : L \to \mathcal{C}(N_g^n)$.

Proof. Let α and β be two simple closed curves in $P \subset L$ such that they are nonadjacent in Γ_P. By the assumption, there exist two disjoint simple closed curves δ_α and δ_β on N_g^b such that $\delta_\alpha \cap \alpha \neq \emptyset, \delta_\beta \cap \beta \neq \emptyset$ and $\delta_\alpha \cap \beta = \emptyset, \delta_\beta \cap \alpha = \emptyset$ and each of them are disjoint from the simple closed curves in $P \setminus \{\alpha, \beta\}$.

Assume $\phi(\alpha)$ and $\phi(\beta)$ are adjacent in $\Gamma_{\phi(P)}$. The complement of the simple closed curves in $\phi(P)$ on N_g^n is a bunch of a pair of pants, one of them, say X, has boundary components come from $\phi(\alpha)$ and $\phi(\beta)$. Since δ_α and δ_β are disjoint from all simple closed curves in $P \setminus \{\alpha, \beta\}$, $\phi(\delta_\alpha)$ and $\phi(\delta_\beta)$ become properly embedded arcs on X with endpoints on the boundary components arising from by cutting $\phi(\alpha)$ and $\phi(\beta)$, respectively. By Lemma 3.1.1, these arcs must intersect . This gives a contradiction since δ_α and δ_β are disjoint. $\square$

Remark 3.1.3. *We will introduce the pants decomposition in $\mathfrak{X}_g^n$; the set of simple closed curves in $\mathfrak{X}_g^n$ satisfies the assumptions of Lemma 3.1.2 when we use nonadjacency in the adjacency graph of the pants decomposition.*

3.2 Topological Types of Vertices when $g = 1$

Let N_1^n denote the non-orientable surface of genus 1 with $n \geq 4$ holes. Throughout this section, we use N instead of N_1^n. In this section, we shall show that the topological types of $\alpha_1 \in \mathfrak{X}_1^n$ and two-sided simple closed curves in $\mathfrak{X}_1^n \subset \mathcal{C}(N)$ under a locally injective simplicial map $\mathfrak{X}_1^n \to \mathcal{C}(N)$. Before this, we state some results about an adjacency graph of a linear pants decomposition of N.

3.2.1 Linear Pants Decompositions

We shall obtain some results about the type of vertices in an adjacency graph of a linear pants decomposition of N. Note that there exist $n-1$ simple closed curves in a pants decomposition of N.

Lemma 3.2.1. *Let P be a linear pants decomposition of N, Γ_P be its adjacency graph and $\gamma \in \Gamma_P$ be a vertex. The vertex γ is of type $(0,k)$ for some $k \geq 3$ if and only if γ has valency two in Γ_P.*

Proof. Let γ be of type $(0,k)$, for some $k \geq 3$. The complement of γ in N has two connected components, none of which is a pair of pants. Thus, each connected component contains an element of P, so that γ is adjacent to at least two, hence exactly two, vertices in Γ_P since P is linear.

Now let γ have valency two in Γ_P. Assume γ is one-sided. If we cut N along the simple closed curves in P, we get $n-1$ pair of pants. There is a unique pants X such that its boundary components come from γ and two other simple closed curves β_1, β_2. But in that case γ, β_1, β_2 form a triangle in the adjacency graph, contradicting the linearity of Γ_P.

Assume that γ is of type $(0,2)$. If we cut N along the simple closed curves in P, there exist two pair of pants X_1 and X_2 such that one of the boundary components of X_i comes from γ for $i = 1, 2$. Since γ is of type $(0,2)$, one of them, say X_1, is two-holed disc. Then, boundary components of X_2 arise from γ and two other simple closed curves β_1, β_2, since the valency of γ is two. This gives contradiction by the same reason above. $\square$

Remark 3.2.2. *Note that a vertex in the adjacency graph of a linear pants decomposition which have valency one must be either one-sided or of type* $(0,2)$*. In fact, one of such vertex is one-sided, the other is of type* $(0,2)$*.*

Lemma 3.2.3. *Let* P *be a linear pants decomposition of* N*,* Γ_P *be the adjacency graph of* P *with vertices* $\gamma_2, \gamma_3, \ldots, \gamma_n$ *such that* γ_i *is connected to* γ_{i+1} *for* $i = 2, \ldots, n-1$ *in* Γ_P*. Then, by changing the label* i *with* $n+2-i$ *if necessary,* γ_n *is one-sided and* γ_i *is of type* $(0,i)$ *for* $2 \le i \le n-1$*.*

Proof. By Remark 3.2.2, either γ_2 or γ_n is one-sided. By changing the labeling if necessary, we may assume that γ_n is one-sided.

Claim 1. *For* $2 \le k \le n-1$*,* γ_k *is of type* $(0,k)$*,*

The proof is by induction on k. For $k = 2$, γ_2 has valency one and it is disjoint from one-sided γ_n. So γ_2 is of type $(0,2)$. Assume that γ_i is of type $(0,i)$ for $i \le k-1$. Now if we cut N along the simple closed curves in P, there is a unique pair of pants X such that two of the boundary components come from γ_{k-1} and γ_k. If the third hole in X comes from a circle $\beta \in P \setminus \{\gamma_{k-1}, \gamma_k\}$, then Γ_P contains a triangle with vertices $\{\beta, \gamma_{k-1}, \gamma_k\}$. This contradicts with the linearity of P. Hence, the third hole is a hole of N. Since γ_{k-1} bounds a disc with $k-1$ holes, γ_k bounds a disc with k holes, i.e. it is of type $(0,k)$. □

The following lemma is based on Lemma 5.2 in [4]. In this lemma, Behrstock-Margalit prove that if the adjacency graph of a pants decomposition of a torus with $n \ge 4$ holes does not contain a triangle, then this pants decomposition is either linear or a cyclic pants decomposition.

Lemma 3.2.4. *Let* N *be a compact connected non-orientable surface of genus* 1 *with* $n \ge 4$ *holes,* P *be a pants decomposition of* N *and let* Γ_P *be its adjacency graph. If* Γ_P *does not contain any triangle, then* P *is linear.*

Proof. Suppose that Γ_P does not contain any triangle. The surface obtained by cutting N along P is a disjoint union of pair of pants. Since P does not contain any triangle, at least one hole of each such pair pants is a hole of N. Let X be such a pair of pants. Then,

1. exactly two holes of X are simple closed curves in P one of which is one-sided, or

2. exactly two holes of X are two-sided simple closed curves in P, or

3. only one hole of X is a two-sided curve in P.

Since the genus of N is 1, there is only one pair of pants X_1 of type 1. The pair of pant X_2 glued to X_1 must be of type 2; otherwise (if type 3), the surface would be N_1^3. The pair of pant X_3 glued to $X_1 \cup X_2$ can be of type 2 of 3. If type 3, then $N = N_1^4$. By arguing in this way we conclude that $N = X_1 \cup X_2 \cup X_3 \cup \cdots \cup X_{n-1}$, where X_1 is of type 1, X_{n-1} is of type 3 and all others are of type 2. It follows that Γ_P is linear. $\square$

Lemma 3.2.5. *Let $P \subset \mathfrak{X}_1^n$ be a linear pants decomposition of N satisfying the assumptions of Lemma 3.1.2 and $\phi : \mathfrak{X}_1^n \to \mathcal{C}(N)$ be a locally injective simplicial map. Then, $\phi(P)$ is a linear pants decomposition of N.*

Proof. Let $P \subset \mathfrak{X}_1^n$ be a linear pants decomposition of N and Γ_P be its the adjacency graph.

If $n = 4$, Γ_P contains three vertices and two of them are nonadjacent to each other. If $\Gamma_{\phi(P)}$ is a triangle, each vertex has valency two, that is each vertices are pairwise adjacent to each other in $\Gamma_{\phi(P)}$. Since ϕ preserves nonadjacency with respect to P, this is not possible. Hence, $\phi(P)$ is linear by Lemma 3.2.4.

If $n > 4$, P contains at least four simple closed curves. Assume that $\Gamma_{\phi(P)}$ contains a triangle. Since $\Gamma_{\phi(P)}$ contains at least four vertices, there exists a vertex $\phi(\gamma)$ in this triangle which has valency at least three. Thus, there exist at most $n-5$ vertices in $\Gamma_{\phi(P)}$ nonadjacent to $\phi(\gamma)$. On the other hand the number of vertices in Γ_P nonadjacent to γ is either $n-3$ or $n-4$. This implies that there exists at least one vertex δ in Γ_P nonadjacent to γ but $\phi(\delta)$ is adjacent to $\phi(\gamma)$ in $\Gamma_{\phi(P)}$. Since ϕ preserves nonadjacency, this is a contradiction. Hence, $\phi(P)$ is linear, by Lemma 3.2.4. $\square$

3.2.2 Topological Types of Vertices

We shall show that the topological types of simple closed curves in a linear pants decomposition in $\mathfrak{X}_1^n$ containing α_1 are preserved under a locally injective simplicial map $\mathfrak{X}_1^n \to \mathcal{C}(N_1^n)$.

Lemma 3.2.6. *A locally injective simplicial map $\phi : \mathfrak{X}_1^n \to \mathcal{C}(N)$ preserves the topological types of simple closed curves in a linear pants decomposition of N containing α_1.*

Proof. Let $P \subset \mathfrak{X}_1^n$ be a linear pants decomposition of N containing α_1 and satisfying assumptions of Lemma 3.1.2, and let Γ_P be its adjacency graph with vertices $\gamma_2, \gamma_3, \ldots, \gamma_n = \alpha_1$ such that γ_i is connected to γ_{i+1} in Γ_P and γ_i is of type $(0, i)$ for $i = 2, \ldots, n-1$. By Lemma 3.2.5, $\phi(P)$ is a linear pants decomposition. Note that γ_2 and γ_n has valency one in Γ_P. This implies $\phi(\gamma_2)$ and $\phi(\gamma_n)$ have valency one. By Remark 3.2.2, one of $\phi(\gamma_2)$ and $\phi(\gamma_n)$ is one-sided and the other is of type $(0, 2)$.

We claim that $\phi(\gamma_n)$ is one-sided and $\phi(\gamma_2)$ is of type $(0, 2)$. Assume the converse; so that $\phi(\gamma_n)$ is of type $(0, 2)$ and $\phi(\gamma_2)$ is one-sided. Since $n \geq 4$, there exist at least two simple closed curves β_1 and β_2 in $\mathfrak{X}_1^n$ of type $(0, 2)$ that are disjoint from each other. For $i = 1, 2$, choose a linear pants decomposition P_i containing γ_n and β_i, and not containing β_{3-i}. By looking at the adjacency graph Γ_{P_i} of P_i, we conclude from the previous paragraph that both $\phi(\beta_1)$ and $\phi(\beta_2)$ must be one-sided. Since β_1 and β_2 are distinct and disjoint, this is impossible. By this contradiction, $\phi(\gamma_n)$ is one-sided and $\phi(\gamma_2)$ is of type $(0, 2)$. Since $\phi(P)$ is linear, by Lemma 3.2.3, $\phi(\gamma_i)$ is of type $(0, i)$ for $i = 2, \ldots, n-1$. $\square$

Now we introduce the pants decompositions $P \subset \mathfrak{X}_g^n$ containing $\alpha_1 \in \mathfrak{X}_1^n$ and the set of the simple closed curves satisfying assumptions of Lemma 3.1.2. We introduce these sets in three cases with respect to the two-sided curve $\beta_i^j \in \mathfrak{X}_g^n$:

Case 1. Let the subscript i of β_i^j be 1. In other words, for any two-sided $\beta_1^k \in \mathfrak{X}_1^n$ where $3 \leq k \leq n$, we take the pants decomposition

$$P = \{\alpha_1, \beta_1^n, \beta_1^{n-1}, \dots, \beta_1^k, \dots, \beta_1^3\} \subset \mathfrak{X}_g^n$$

and the set of simple closed curves

$$I = \{\alpha_1^n, \beta_{n-1}^{n+1}, \beta_{n-2}^n, \dots, \beta_2^4\}$$

see Figure 3.6.

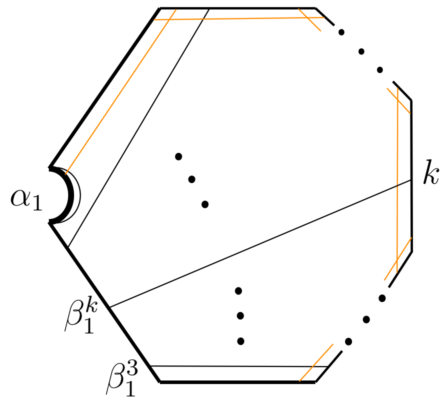

Figure 3.6: The top dimensional pants decomposition P (in black) containing β_1^k and the simple closed curves (in orange) in I on N_1^n.

Case 2. Let the superscript j of β_i^j be $n+1$. In other words for any two-sided $\beta_k^{n+1} \in \mathfrak{X}_g^n$ where $2 \leq k \leq n-1$, we take the pants decomposition

$$P = \{\alpha_1, \beta_2^{n+1}, \beta_3^{n+1}, \dots, \beta_k^{n+1}, \dots, \beta_{n-1}^{n+1}\} \subset \mathfrak{X}_1^n$$

and the set of simple closed curves

$$I = \{\alpha_1^2, \beta_1^3, \beta_2^4, \dots, \beta_{n-2}^n\}$$

see Figure 3.7.

Case 3. Let $i \neq 1$ and $j \neq n+1$ of β_i^j where $j - i \geq 2$. We take the pants decomposition

$$P = \{\alpha_1, \beta_2^{n+1}, \beta_3^{n+1}, \dots, \beta_{i-1}^{n+1}, \beta_i^{n+1}, \beta_i^n, \beta_i^{n-1}, \dots, \beta_i^j, \beta_i^{j-1}, \dots, \beta_i^{i+2}\} \subset \mathfrak{X}_1^n$$

and the set of simple closed curves

$$I = \{\alpha_1^2, \beta_1^3, \beta_2^4, \dots, \beta_{i-2}^i, \beta_{i-1}^n, \beta_{n-1}^{n+1}, \beta_{n-2}^n, \dots, \beta_{i+1}^{i+3}\}$$

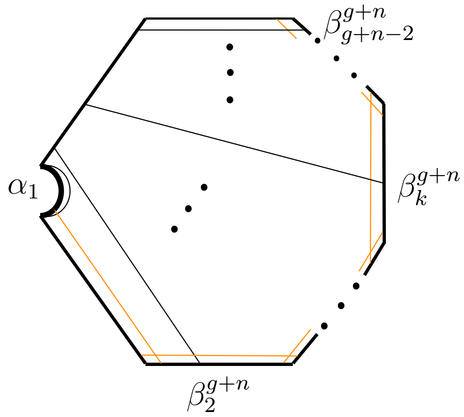

Figure 3.7: The top dimensional pants decomposition P (in black) containing β^{g+n}_k and the simple closed curves (in orange) in I on N^n_1.

see Figure 3.8.

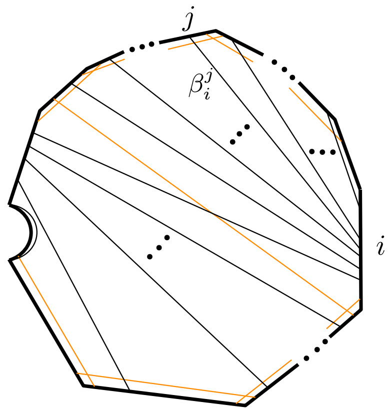

Figure 3.8: The top dimensional pants decomposition P (in black) containing β^j_i and the simple closed curves (in orange) in I on N^n_1.

For any two-sided curve in $\mathfrak{X}^n_1$ one can consider one of the linear pants decomposition P that is introduced above. Hence, by Lemma 3.2.3, we get

Corollary 3.2.7. *A locally injective simplicial map $\phi : \mathfrak{X}^n_1 \to \mathcal{C}(N)$ preserves topological types of α_1 and of two-sided simple closed curves in $\mathfrak{X}^n_1$.*

3.3 Topological Types of Vertices when $g \geq 2, n \geq 2$.

In this section, we let $g + n \geq 5$, and N denotes the non-orientable surface N^n_g of genus $g \geq 2$ with $n \geq 2$ holes given in Section 2.2. Recall that by

Corollary 2.1.5, there exist $2g+n-3$ simple closed curves in a top dimensional pants decomposition of N, hence there exist $2g+n-3$ vertices in an adjacency graph of a top dimensional pants decomposition of N. Hereafter $\phi : \mathfrak{X}_g^n \to \mathcal{C}(N_g^n)$ is a locally injective simplicial map. We show that $\phi(\alpha_1)$ and $\phi(\alpha_g)$ are essential one-sided.

Lemma 3.3.1. *Let P be a top dimensional pants decomposition of N, Γ_P be its adjacency graph and let γ be a vertex of valency one in Γ_P. Then, γ is either essential one-sided or of type $(0,2)$.*

Proof. Assume that γ is neither essential one-sided nor of type $(0,2)$. Since γ is contained in the top dimensional pants decomposition P, by the Lemma 2.1.7, γ is separating and, the complement N_γ of γ is disjoint union of two subsurfaces of N, none of which is a pair of pants. On each of these components, there is a vertex of P adjacent to γ, so that the valency of γ in Γ_P is at least two. By this contradiction, γ is either essential one-sided or is of type $(0,2)$. □

Lemma 3.3.2. *Let $P \subset \mathfrak{X}_g^n$ be a top dimensional pants decomposition which contains a linear path $(\gamma_0, \gamma_1, \ldots, \gamma_k)$ such that γ_0 has k linear successors in Γ_P and satisfies the assumptions of Lemma 3.1.2. Then, $(\phi(\gamma_0), \phi(\gamma_1), \ldots, \phi(\gamma_k))$ is a linear path and $\phi(\gamma_0)$ has at least k linear successors in $\Gamma_{\phi(P)}$.*

Proof. Note that the vertex γ_0 is adjacent only to the vertex γ_1 in Γ_P. Since ϕ preserves nonadjacency, $\phi(\gamma_0)$ is adjacent to only $\phi(\gamma_1)$. Hence, $\phi(\gamma_0)$ has valency one in $\Gamma_{\phi(P)}$. The vertex γ_i is adjacent to γ_{i-1} and γ_{i+1} for $i = 1, 2, \ldots, k-1$. Since ϕ preserves nonadjacency in Γ_P, the valency of $\phi(\gamma_i)$ is either one or two in $\Gamma_{\phi(P)}$. On the other hand since $\Gamma_{\phi(P)}$ is connected, $\phi(\gamma_i)$ has valency two in $\Gamma_{\phi(P)}$. Hence, the vertex $\phi(\gamma_i)$ is adjacent to the vertices $\phi(\gamma_{i-1})$ and $\phi(\gamma_{i+1})$. Thus, $(\phi(\gamma_0), \phi(\gamma_1), \ldots, \phi(\gamma_k))$ is a linear path and $\phi(\gamma_0)$ has at least k linear successors. □

Lemma 3.3.3. *The topological type of the simple closed curves α_1 and α_g in $\mathfrak{X}_g^n$ is preserved under ϕ.*

Proof. Consider the top dimensional pants decomposition

$$P = \{\alpha_1, \alpha_2, \ldots, \alpha_{g-1}, \alpha_g, \beta_1^{g+n-1}, \beta_1^{g+n-2}, \ldots, \beta_1^{g+1}, \beta_1^g, \beta_2^g, \beta_3^g, \ldots, \beta_{g-2}^g\} \subset \mathfrak{X}_g^n,$$

so that the vertex β_1^{g+n-1} is the unique vertex adjacent to α_1 in the adjacency graph Γ_P of P (see Figure 3.9).

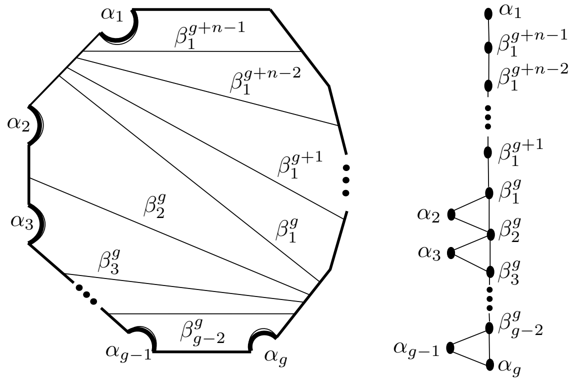

Figure 3.9: The top dimensional pants decomposition P and its adjacency graph Γ_P for the non-orientable surface N_g^n.

Note that $(\alpha_1, \beta_1^{g+n-1}, \beta_1^{g+n-2}, \ldots, \beta_1^{g+1}, \beta_1^g)$ is a linear path in Γ_P with $n+1$ vertices, and α_1 has n linear successors.

The top dimensional pants decomposition P satisfies the conditions of Lemma 3.1.2 if we consider the following set of the simple closed curves

$$I = \{\alpha_1^{g+n-1}, \alpha_2^g, \ldots, \alpha_{g-1}^g, \alpha_g^{g-2}, \beta_{g+n-2}^{g+n}, \beta_{g+n-3}^{g+n-1}, \ldots, \beta_g^{g+2}, \beta_2^{g+1}, \beta_1^3, \beta_2^4, \ldots, \beta_{g-3}^{g-1}\}$$

which is the subset of $\mathfrak{X}_g^n$. the (see Figure 3.10). Hence, the map ϕ preserves the nonadjacency in the adjacency graph Γ_P of P. Since α_1 has unique adjacent in Γ_P, there exist $2g+n-5$ vertices in Γ_P that are nonadjacent to γ. Since ϕ preserves nonadjacency in Γ_P, there exist $2g+n-5$ vertices in $\Gamma_{\phi(P)}$, nonadjacent to $\phi(\gamma)$. Thus, $\phi(\gamma)$ has valency one in $\Gamma_{\phi(P)}$. By Lemma 3.3.1, $\phi(\gamma)$ is either essential one-sided or of type $(0,2)$.

Suppose that $\phi(\alpha_1)$ is of type $(0,2)$ in the adjacency graph $\Gamma_{\phi(P)}$ of $\phi(P)$. By Lemma 3.3.2, the vertex $\phi(\alpha_1)$ has at least n linear successors in the adjacency graph of $\Gamma_{\phi(P)}$. Since $\phi(\alpha_1)$ is of type $(0,2)$, the vertex $\phi(\beta_1^{g+1})$ is of type $(0,n+1)$ and the vertex $\phi(\beta_1^g)$ is of type $(0,n+2)$ in $\Gamma_{\phi(P)}$. But there are no simple closed

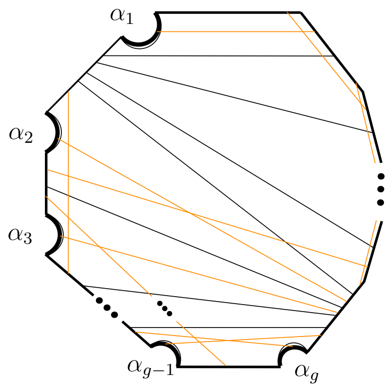

Figure 3.10: The pants decomposition P (in black) containing α_1 and the simple closed curves (in orange) in I on N_g^n.

curves of type $(0, n+1)$ and $(0, n+2)$ on N. This gives a contradiction. Hence, $\phi(\alpha_1)$ is essential one-sided.

Exactly the same conclusion can be drawn for $\phi(\alpha_g)$ by considering the top dimensional pants decomposition

$$P' = \{\alpha_1, \alpha_2, \ldots, \alpha_g, \beta_{g-1}^{g+1}, \beta_{g-1}^{g+2}, \ldots, \beta_{g-1}^{g+n}, \beta_2^{g+n}, \beta_3^{g+n} \ldots, \beta_{g-1}^{g+n}\} \subset \mathfrak{X}_g^n$$

where β_{g-1}^{g+1} is the unique vertex adjacent to α_g in Γ_P and the set

$$\{\alpha_g, \beta_{g-1}^{g+1}, \beta_{g-1}^{g+2}, \ldots, \beta_{g-1}^{g+n}\}$$

in Γ_P is the linear path with $n+1$ vertices including α_g. □

3.4 Topological Types of Vertices for Surfaces with at Most One Hole

In this section, we shall show that the topological type of the simple closed curve α_i in the finite subcomplex in the curve complex of a non-orientable surface with at most one hole is preserved under a locally injective simplicial map $\phi : \mathfrak{X}_g^n \to \mathcal{C}(N_g^n)$. Recall that N_g^0 denotes the non-orientable surface of genus g and N_{g-1}^1 denotes the non-orientable surface of genus $g-1$ with one hole.

Lemma 3.4.1. *Let P be a top dimensional pants decomposition of N_g^0, Γ_P be its adjacency graph and α be a vertex in Γ_P. The vertex α is essential one-sided if and only if α has valency two in Γ_P.*

Proof. Let P be a top dimensional pants decomposition of N_g^0 containing α. Since N_g^0 is closed, the set P contains either essential one-sided simple closed curves or simple closed curves of type $(k, 0)$ for some k where $2 \leq k \leq g-2$, by Lemma 2.1.7.

Suppose that α is a essential one-sided simple closed curve. The complement of the simple closed curves in $P \setminus \{\alpha\}$ on N_g^0 is the disjoint union of $g-3$ pair of pants and the subsurface X homeomorphic to the real projective plane with two holes and α lies on X. The holes of $X \subset N_g^0$ come from two distinct simple closed curves in $P \setminus \{\alpha\}$. Hence, α has valency two in Γ_P.

Let γ be a simple closed curve in P which is not one-sided. Hence, the simple closed curve γ is of type $(k, 0)$ on N_g^0 for some k where $2 \leq k \leq g-2$. The complement of γ in N_g^0 has two connected components X_1 and X_2, one of them is homeomorphic to N_k^1, the other is homeomorphic to N_{g-k}^1. Each of X_1 and X_2 contains a pair of elements in P such that γ is adjacent to these vertices in Γ_P. Hence, valency of γ is four in Γ_P. This implies if α has valency two in Γ_P, then α is essential one-sided. $\square$

By the arguments similar to the proof of the Lemma 3.4.1, one can prove the following lemma:

Lemma 3.4.2. *Let P be a top dimensional pants decomposition of N_{g-1}^1, Γ_P be its adjacency graph and α be a vertex in Γ_P. The vertex α is essential one-sided if and only if α has valency at most two in Γ_P.*

Lemma 3.4.3. *The topological type of the essential one-sided simple closed curve $\alpha_i \in \mathfrak{X}_g^0$ is preserved under ϕ, for every $i = 1, 2, \ldots, g$.*

Proof. Let α_i be a one-sided simple closed curve in $\mathfrak{X}_g^0$ and let $P \subset \mathfrak{X}_g^0$ be the top dimensional pants decomposition of N_g^0 containing α and satisfying the assumptions of Lemma 3.1.2. Note that by Corollary 2.1.5, Γ_P contain $2g-3$ vertices. By Lemma 3.4.1, the valency of α_i is two in Γ_P. In other words, there exist $2g-6$ vertices in Γ_P nonadjacent to α_i. By Lemma 3.1.2, there exist at least $2g-6$ vertices nonadjacent to $\phi(\alpha_i)$. Hence, the valency of $\phi(\alpha_i)$ is one or two in $\Gamma_{\phi(P)}$. By Lemma 2.1.10, the set $\phi(P)$ is a top dimensional pants decomposition of N_g^0. Hence, by Lemma 2.1.7, the set $\phi(P)$ contains either essential one-sided simple closed curves or simple closed curves of type $(k, 0)$ for some k. Since N_g^0 is closed, the holes of the pair of pants containing $\phi(\alpha_i)$ come from two simple closed curves in $\phi(P)$, that is the valency of $\phi(\alpha_i)$ is two in $\Gamma_{\phi(P)}$. By Lemma 3.4.1, the simple closed curve $\phi(\alpha_i)$ is essential one-sided. $\square$

Indeed for any $\alpha_i \in \mathfrak{X}_g^0$, one can consider the pants decomposition

$$P = \{\alpha_1, \alpha_2, \ldots, \alpha_{i-1}, \alpha_i, \ldots, \alpha_g, \beta_1^i, \beta_2^i, \ldots, \beta_{i-2}^i, \beta_i^{i+2}, \ldots, \beta_i^g\}$$

and the set of simple closed curves

$$I = \{\alpha_1^i, \alpha_2^i, \ldots, \alpha_{i-1}^i, \alpha_i^{i-2}, \beta_2^g, \beta_1^3, \ldots, \beta_{i-3}^{i-1}, \beta_{i+1}^{i+3}, \ldots, \beta_1^{g-1}\},$$

Hence, the pants decomposition P satisfies the assumptions of Lemma 3.1.2 (see Figure 3.11).

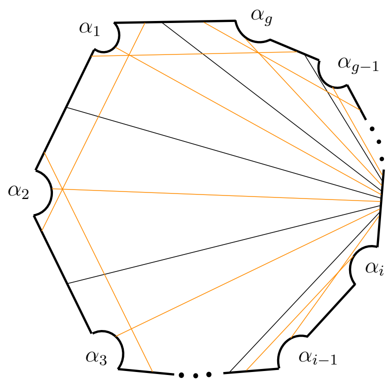

Figure 3.11: The pants decomposition P (in black) containing α_i and the simple closed curves (in orange) in I on N_g^0.

The following lemma can be proven by the arguments similar to those in the proof of Lemma 3.4.3.

Lemma 3.4.4. *The topological type of the essential one-sided simple closed curve $\alpha_i \in \mathfrak{X}_{g-1}^1$ is preserved under a locally injective simplicial map ϕ, for every $i = 1, 2, \ldots, g-1$.*

CHAPTER 4

THE MAIN RESULT

Let $\mathfrak{X}_g^n$ be the finite subcomplex introduced in Chapter 3 and let $g + n \geq 5$. In this chapter, we shall show that any locally injective simplicial map $\mathfrak{X}_g^n \to \mathcal{C}(N_g^n)$ is induced from a homeomorphism of N_g^n.

4.1 g=0 Case

Recall that by convention, we assume that a sphere is a non-orientable surface of genus 0. We give the result of Aramayona and Leininger in [1] which is the base step of the proof of the main result.

Theorem 4.1.1. *[1, Theorem 3.1] Let N_0^n be a sphere with $n \geq 5$ holes and $\mathfrak{X}_0^n \subset \mathcal{C}(N_0^n)$ be the finite subcomplex of the curve complex of $\mathcal{C}(N_0^n)$ given in Section 3.1. For any locally injective simplicial map $\phi : \mathfrak{X}_0^n \to \mathcal{C}(N_0^n)$, there is an element $h \in \mathrm{Mod}(N_0^n)$ such that ϕ is induced from h. Moreover h is unique up to pointwise stabilizer of $\mathfrak{X}_0^n$ in $\mathrm{Mod}(N_0^n)$.*

4.2 The Main Result

In this section, we state and prove the main result of this dissertation.

Theorem 4.2.1. *Let $g+n \geq 5$ and let N_g^n be a compact connected non-orientable surface of genus g with n holes. Any locally injective simplicial map $\phi : \mathfrak{X}_g^n \to \mathcal{C}(N_g^n)$ is induced from an element $F \in \mathrm{Mod}(N_g^n)$. The mapping class F by which*

ϕ is induced is unique up to composition with the involution that interchanges the faces of the model of N_g^n.

Proof. We prove the theorem by induction on g. Recall that we consider S_0^n as a non-orientable surface of genus 0. For $g = 0$ the base step is proven in Theorem 4.1.1. Now assume that $g \geq 1$ and any locally injective simplicial map $\mathfrak{X}_{g-1}^m \to \mathcal{C}(N_{g-1}^m)$ for $g - 1 + m \geq 5$, is induced from a homeomorphism $N_{g-1}^m \to N_{g-1}^m$. Let $\phi : \mathfrak{X}_g^n \to \mathcal{C}(N_g^n)$ be a locally injective simplicial map, $g + n \geq 5$. By Chapter 3, we know that for the essential one-sided simple closed curve $\alpha_g \in \mathfrak{X}_g^n$, $\phi(\alpha_g)$ is an essential one-sided simple closed curve. So there exists a homeomorphism $h_0 : N_g^n \to N_g^n$ such that $(h_0(\phi))(\alpha_g) = \alpha_g$. Hereafter we may take ϕ as $h_0 \circ \phi$, so that $\phi(\alpha_g) = \alpha_g$. Hence, the surface $N_{\phi(\alpha_g)}$ obtained by cutting N along α_g is a non-orientable surface of genus $g - 1$ with $n + 1$ holes which is homeomorphic to N_{g-1}^{n+1}. One of the hole, say δ_g, on N_{g-1}^{n+1} comes from α_g.

Recall that we introduced in Section 3.1 the injective simplicial map $q_* : \mathcal{C}(N_{g-1}^{n+1}) \to \mathcal{C}(N_g^n)$ induced from the quotient map $q : N_{g-1}^{n+1} \to N_g^n$ which identifies the antipodal points of the hole δ_g. Since ϕ fixes α_g, ϕ induces a locally injective simplicial map $\bar{\phi} : \mathfrak{X}_{g-1}^{n+1} \to \mathcal{C}(N_{g-1}^{n+1})$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{X}_g^n & \xrightarrow{\phi} & \mathcal{C}(N_g^n) \\ {\scriptstyle q_*}\uparrow & & {\scriptstyle q_*}\uparrow \\ \mathfrak{X}_{g-1}^{n+1} & \xrightarrow{\bar{\phi}} & \mathcal{C}(N_{g-1}^{n+1}) \end{array}$$

More precisely, $\bar{\phi}(\bar{\gamma}) = \overline{\phi(\gamma)}$ where $\overline{\phi(\gamma)}$ is the preimage of $\phi(\gamma)$ under q_* for any $\bar{\gamma} \in \mathfrak{X}_{g-1}^{n+1}$.

By induction step $\bar{\phi}$ is induced from a homeomorphism $\bar{f} : N_{g-1}^{n+1} \to N_{g-1}^{n+1}$, that is $\bar{f}(\bar{\gamma}) = \bar{\phi}(\bar{\gamma})$ for every $\bar{\gamma} \in \mathfrak{X}_{g-1}^{n+1}$. This implies that $\bar{\phi}(\bar{\alpha_i})$ is essential one-sided for $i = 1, 2, \ldots, g - 1$. Hence,

$$q_*\bar{\phi}(\bar{\alpha_i}) = \phi(q_*(\bar{\alpha_i})) = \phi(\alpha_i)$$

essential one-sided simple closed curve.

Claim:$\bar{f}$ fixes the hole δ_g coming from α_g.

Proof of Claim: Now we separate the proof in two cases:

Case 1: $g = 1$. Suppose that $\bar{f}$ does not fix the hole δ_1 coming from α_1. There exists a simple closed curve $\bar{\beta} \in \mathfrak{X}_0^{n+1}$ such that $\bar{\beta}$ bounds δ_1 and another hole δ' of N_0^{n+1} such that image under $\bar{f}(\delta') \neq \delta_1$. Hence, $q_*(\bar{\beta})$ is of type $(1,1)$ in $\mathfrak{X}_1^n$. By Corollary 3.2.7, the locally injective simplicial map $\phi : \mathfrak{X}_1^n \to \mathcal{C}(N_1^n)$ preserves topological types of two-sided simple closed curves. Thus, $\phi(q_*(\bar{\beta}))$ is of type $(1,1)$. On the other hand, since $\bar{f}(\delta_1) \neq \delta_1$ and $\bar{f}(\delta_1) \neq \delta_1$, the simple closed curve $\bar{\phi}(\bar{\beta}) = \bar{f}(\bar{\beta})$ is of type $(0,2)$. Thus, $q_*(\bar{\phi}(\bar{\beta})) = \phi(q_*(\bar{\beta}))$ is of type $(0,2)$ in $\mathcal{C}(N_1^n)$. Note that $g+n \geq 5$, hence any curve of type $(1,1)$ is not isotopic to a curve of type $(0,2)$. Since diagram commutes, this is a contradiction.

Case 2: $g \geq 2$. If $n = 0$, then $\bar{f}(\delta_g) = \delta_g$.

Let $n \geq 1$ and β_{i-1}^{i+1} be the simple closed curve of type $(2,0)$ in $\mathfrak{X}_g^n$, that is β_{i-1}^{i+1} bounds a subsurface in N_g^n homeomorphic to Klein Bottle with one hole where α_i and α_{i+1} lie, $i = 1, 2, \ldots, g-1$. We want to show that ϕ preserves the topological type of β_{i-1}^{i+1}.

Consider the top dimensional pants decomposition

$$P = \{\alpha_1, \ldots, \alpha_i, \alpha_{i+1}, \ldots, \alpha_g, \beta_1^{i-1}, \beta_2^{i-1} \ldots, \beta_{i-3}^{i-1}, \beta_{i-1}^{i+1}, \ldots, \beta_{i-1}^{g+n}\}$$

and the set of the simple closed curves in $\mathfrak{X}_g^n$

$$I = \{\alpha_1^{i-1}, \ldots, \alpha_i^{i+1}, \alpha_{i+1}^{i-1}, \ldots, \alpha_g^{i-1}, \beta_2^{g+n}, \beta_1^3, \ldots, \beta_{i-4}^{i-2}, \beta_i^{i+2}, \ldots, \beta_1^{g+n-1}\}$$

(see Figure 4.1).

With the set of simple closed curves I, the pants decomposition P satisfies the assumptions of Lemma 3.1.2. Hence, ϕ preserves the nonadjacency in the adjacency graph Γ_P of P.

Note that the vertices α_i, α_{i+1} and β_{i-1}^{i+1} form a triangle in Γ_P. We notice that the subscripts and superscripts in the elements of P are taken mod $g+n$. Since $\phi(\alpha_j)$ is essential one-sided for each $j = 1, 2, \ldots, g$ and the simple closed curve β_{i-1}^{i+1} is distinct from α_i, $\phi(\beta_{i-1}^{i+1})$ is a two-sided curve. Note that β_{i-1}^{i+1} is the

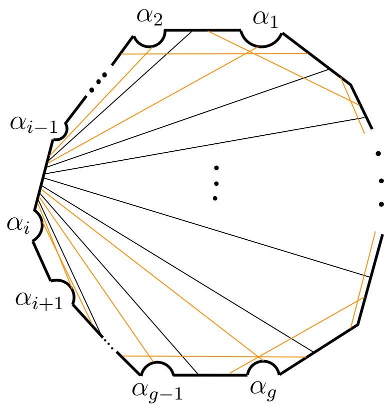

Figure 4.1: The pants decomposition P (in black) containing α_i, α_{i+1} and β_{i-1}^{i+1} and the simple closed curves (in orange) in I on N_g^n.

unique two-sided simple closed curve that is adjacent to α_i and α_{i+1} in Γ_P. Since ϕ preserves nonadjacency in Γ_P, $\phi(\beta_{i-1}^{i+1})$ is the unique vertex in $\Gamma_{\phi(P)}$ that can be adjacent to both $\phi(\alpha_i)$ and $\phi(\alpha_{i+1})$. If it can be shown that the vertices $\phi(\alpha_i)$, $\phi(\alpha_{i+1})$ and $\phi(\beta_{i-1}^{i+1})$ form a triangle in $\Gamma_{\phi(P)}$, we conclude that $\phi(\beta_{i-1}^{i+1})$ is of type $(2,0)$.

Since ϕ is a top dimensional pants decomposition, $\Gamma_{\phi(P)}$ is connected graph, the vertices $\phi(\alpha_i)$, $\phi(\alpha_{i+1})$ and $\phi(\beta_{i-1}^{i+1})$ can form the following subgraphs in $\Gamma_{\phi(P)}$ (see. Figure 4.2):

We know that $\phi(\alpha_i)$ and $\phi(\alpha_{i+1})$ are essential one-sided simple closed curves. As in the first and second case in Figure 4.2, if $\phi(\alpha_i)$ and $\phi(\alpha_{i+1})$ are adjacent in $\phi(P)$, then they lies on Klein Bottle with one hole. If this hole comes from $\phi(\beta_{i-1}^{i+1})$, then $\phi(\alpha_i)$, $\phi(\alpha_{i+1})$ and $\phi(\beta_{i-1}^{i+1})$ form a triangle in $\Gamma_{\phi(P)}$. If this hole is a hole of the surface N_g^n, then N_g^n is homeomorphic to Klein Bottle with one hole. For the case (3) in Figure 4.2, this configuration is just possible on a surface X homeomorphic to Klein Bottle with two holes where $\phi(\beta_{i-1}^{i+1})$ separates X into two subsurfaces X_1, X_2 of X each of homeomorphic to the real projective plane with two holes such that $\phi(\alpha_i)$ lies on X_1 and $\phi(\alpha_{i+1})$ lies on X_2 (see. Figure 4.3). Since $g+n \geq 5$, the cases (1), (2) and (3) in Figure 4.2 are not possible. Hence, ϕ preserves the topological type of simple closed curves of type $(2,0)$ in $\mathfrak{X}_g^n$.

Now suppose that $\bar{f}$ does not fix the boundary component δ_g. Let $\overline{\beta_{g-2}^{g}} \in \mathfrak{X}_{g-1}^{n+1}$

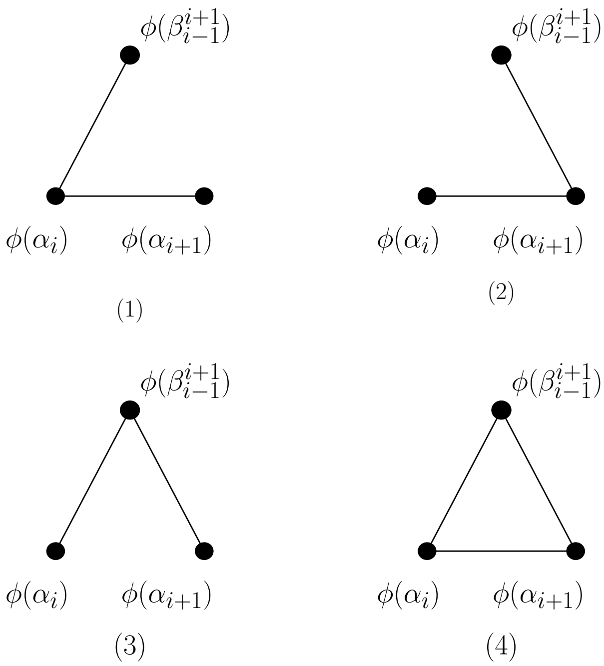

Figure 4.2: Possible configuration for $\phi(\alpha_i)$, $\phi(\alpha_{i+1})$ and $\phi(\beta_{i-1}^{i+1})$ in $\Gamma_{\phi(P)}$.

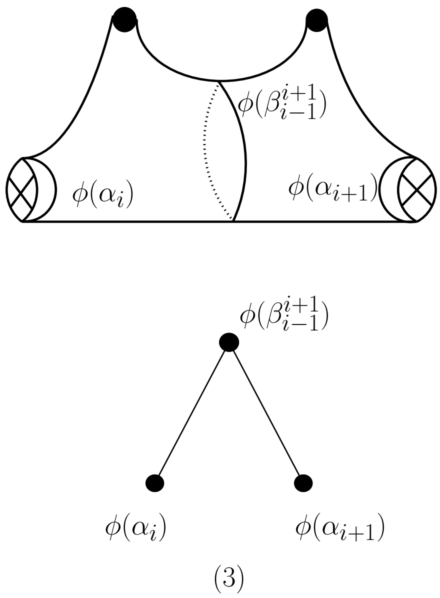

Figure 4.3: $\phi(\alpha_i)$, $\phi(\alpha_{i+1})$ and $\phi(\beta_{i-1}^{i+1})$ lie on Klein Bottle with two holes when they form the graph in Figure4.2 (3).

be the curve which bounds a subsurface in N_{g-1}^{n+1} which is homeomorphic to the real projective plane with two holes where δ_g is one of the boundary component and α_{g-1} lies. The simple closed curve $q_*(\overline{\beta_{g-2}^{g}})$ is of type $(2,0)$ in $\mathfrak{X}_g^n$. Hence, $\phi(q_*(\overline{\beta_{g-2}^{g}}))$ is of type $(2,0)$ since ϕ preserves the topological type of a simple closed curve of type $(2,0)$ in $\mathfrak{X}_g^n$. On the other hand $\bar{\phi}(\overline{\beta_{g-2}^g}) = \bar{f}(\overline{\beta_{g-2}^g})$ is of type $(1,1)$ in $\mathcal{C}(N_{g-1}^{n+1})$ and it does not bound δ_g since $\bar{f}$ does not fix δ_g. Hence, $q_*(\bar{\phi}(\overline{\beta_{g-2}^g})) = \phi(q_*(\overline{\beta_{g-2}^g}))$ is of type $(1,1)$ in $\mathcal{C}(N_g^n)$. Note that since $g+n \geq 5$, hence a curve of type $(1,1)$ is not isotopic to a curve of type $(2,0)$. Since diagram commutes, this gives a contradiction.

By applying a suitable isotopy on the regular neighborhood of boundary component δ_g coming from α_g, one can assume $\bar{f}$ sends antipodal points to antipodal points of δ_g. Hence, $\bar{f}$ descends a homeomorphism $F : N_g^n \to N_g^n$ so that $F(\gamma) = \phi(\gamma)$ for every $\gamma \in \mathfrak{X}_g^n \setminus A$ where

$$A = \{\alpha_g^j \mid 1 \leq j \leq g+n, j \neq g-1, \; j \neq g\}.$$

Claim:$F(\alpha_g^j) = \phi(\alpha_g^j)$, where $\alpha_g^j \in A$.
Proof of Claim: Consider the sets $\mathcal{A} = \{\alpha_1, \alpha_2, \ldots, \alpha_{g-1}\}$ and $\mathcal{B} = \{\beta_i^j \mid j - i = \pm 2 \mod g+n\}$, each of them is a subset of $\mathfrak{X}_g^n$.

The simple closed curves α_g^{g-2} and α_g are the only simple closed curves in $\mathcal{C}(N_g^n)$ which are distinct and disjoint from the simple closed curves in $\mathcal{D}_{g-2} = \mathcal{A} \cup \mathcal{B} \setminus \{\beta_{g-3}^{g-1}, \beta_{g-1}^{g+1}\}$ (e.g., see Figure 4.4 (1)). The simple closed curves α_g^{g+1} and α_g are the only simple closed curves in $\mathcal{C}(N_g^n)$ which are distinct and disjoint from the simple closed curves in $D_{g+1} = \mathcal{A} \cup \mathcal{B} \setminus \{\beta_{g-2}^g, \beta_g^{g+2}\}$ (e.g., see Figure 4.4 (2)). The simple closed curve $\alpha_g^j \in A \setminus \{\alpha_g^{g-2}, \alpha_g^{g+1}\}$ and α_g are the only simple closed curves in $\mathcal{C}(N_g^n)$ which are distinct and disjoint from the simple closed curves in $\mathcal{D}_j = \mathcal{A} \cup \mathcal{B} \cup \{\beta_{g-1}^j, \beta_g^j\} \setminus \{\beta_{g-2}^g, \beta_{g-1}^{g+1}, \beta_{j-1}^{j+1}\}$ where $j = 1, 2, \ldots, g-3, g+2, \ldots, g+n$ (e.g., see Figure 4.4 (3)).

Note that the complement of the simple closed curves in D_j are union of disks, annuli and the subsurface which is homeomorphic to the real projective plane with two holes and α_g and α_g^j lie on. Since $F(\gamma) = \phi(\gamma)$ for every $\gamma \in \mathcal{D}_j \subset \mathfrak{X}_g^n \setminus A$, the simple closed curves $\phi(\alpha_g)$ and $\phi(\alpha_g^j)$ lies in the complement of $\phi_g(\mathcal{D}_j) = F(\mathcal{D}_j)$, which is real projective plane with two holes. By [18], there exist only two simple closed curves in the curve complex of the real projective plane with two holes. Since ϕ is locally injective, $\phi(\alpha_g^j) = F(\alpha_g^j)$. Therefore, $F \mid_{\mathfrak{X}_g^n} = \phi$.

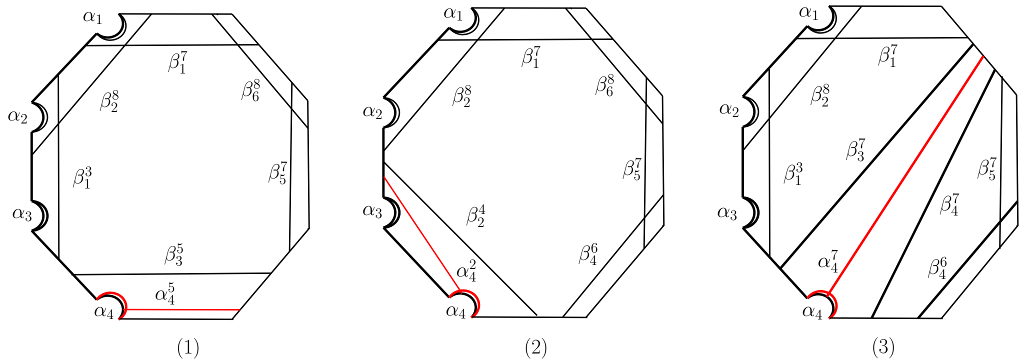

Figure 4.4: The simple closed curves in D_j, α_g and α_g^j for $j = 2, 5, 7$ on N_4^4.

Let $r : N_g^n \to N_g^n$ be the reflection of N_g^n which interchanges the faces of the model of N_g^n. We now prove the uniqueness of the homeomorphism F, which ϕ is induced from, up to composition with r. Let $G : N_g^n \to N_g^n$ be another homeomorphism of N_g^n such that $G(\gamma) = \phi(\gamma)$, for $\gamma \in \mathfrak{X}_g^n$. Then, $(F \circ G^{-1})(\gamma) = \gamma$ for $\gamma \in \mathfrak{X}_g^n$. By composing F with r, if necessary, one can assume the orientation of the simple closed curve $(F^{-1}((G))(\gamma)$ is same with the orientation of the simple closed curve $Id(\gamma)$, where $Id : N_g^n \to N_g^n$ is the identity homeomorphism of N_g^n. In particular $(F^{-1}((G))(\sigma) = \sigma$, for $\sigma \in \mathfrak{C}$ where

$$\mathfrak{C} = \{\alpha_1, \alpha_2, \dots \alpha_g, \beta_i^j \mid j - i = \pm 2 \text{ mod } g + n\} \subset \mathfrak{X}_g^n$$

(see Figure 4.5) Note that the complement of the simple closed curves in $\mathfrak{C}$ on

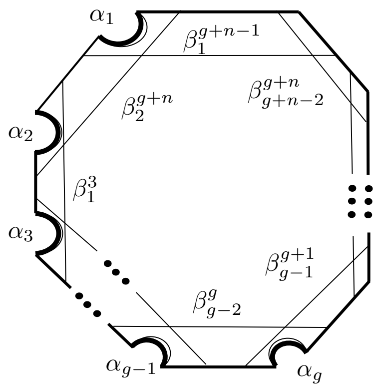

Figure 4.5: The simple closed curves in $\mathfrak{C}$.

N_g^n is disjoint union of disks and annuli. The homeomorphism $F^{-1} \circ G$ fixes each of the boundaries of these components with their orientation. Hence, $F^{-1} \circ G$ can be extended to each components identically. So $F^{-1} \circ G$ is isotopic to the identity homeomorphism of N_g^n. So F is unique up to composition with the reflection r. $\square$

Hence, by Theorem 2.1.8, we have the following corollary:

Corollary 4.2.2. *Let N_g^n be a compact connected non-orientable surface of genus g with n holes, $g+n \geq 5$. Any automorphism $\Psi : \mathcal{C}(N_g^n) \to \mathcal{C}(N_g^n)$ induces a locally injective simplicial map $\Psi|_{\mathfrak{X}_g^n} : \mathfrak{X}_g^n \to \mathcal{C}(N_g^n)$. Since any locally injective map $\mathfrak{X}_g^n \to \mathcal{C}(N_g^n)$ is induced from a mapping class by Theorem 4.2.1, the natural homomorphism $\Phi : \mathrm{Mod}(N_g^n) \to \mathrm{Aut}(\mathcal{C}(N_g^n))$, induced from the action of $\mathrm{Mod}(N_g^n)$ on $\mathcal{C}(N_g^n)$, is onto.*

CHAPTER 5

EXCEPTIONAL CASES

This chapter consists of two sections. In the first section we investigate finite rigid sets in $\mathcal{C}(N_g^n)$ where $g + n < 5$ which we call as the exceptional cases. Second section is a conclusion part of this dissertation.

We now cite the definition of a Dehn Twist along a two-sided non-trivial simple closed curve which is an important example of a mapping class. Before the definition, we need to recall *twist map* which is used in the definition of Dehn Twist.

Let $C = \mathbb{S}^1 \times [0, 1]$ be the cylinder. The *twist map* is the homeomorphism $T : C \to C$ given by

$$T(z, t) = (e^{2i\pi t}z, t)$$

Definition 5.0.1. *Let β be a two-sided nontrivial simple closed curve on N and A be the regular neighbourhood of β in N which is homeomorphic to the cylinder $C = \mathbb{S}^1 \times [0, 1]$. Hence, there exists an orientation preserving homeomorphism $\psi : A \to C$. The homeomorphism $t_\beta : N \to N$ defined by*

$$t_\beta(x) = \begin{cases} (\psi^{-1} \circ T \circ \psi)(x) & \textit{if } x \in A \\ x & \textit{if } x \notin A \end{cases}$$

is called Dehn twist along β.

5.1 Finite Rigid Sets for $g + n < 5$

Let N be projective plane with at most one hole. The curve complex $\mathcal{C}(N)$ consists of the unique essential one-sided simple closed curve α. (see Figure 5.1)

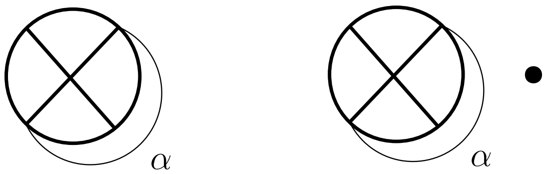

Figure 5.1: The simple closed curve α in $\mathcal{C}(N_1^0)$ and $\mathcal{C}(N_1^1)$, respectively.

So there is a unique locally injective simplicial map identity $Id : \mathfrak{X} \to \mathcal{C}(N)$ which is induced from the identity homeomorphism of N.

Let N_1^2 be two-holed projective plane. By Scharlemann in [18], there exist two vertices α_1 and α_2 in $\mathcal{C}(N_1^2)$ (see Figure 5.2).

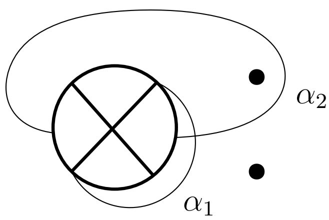

Figure 5.2: The simple closed curves in $\mathcal{C}(N_1^2)$.

$\mathfrak{X}_1^2$ can be chosen as $\{\alpha_1\}$ or $\{\alpha_2\}$. Since α_1 and α_2 have same topological type that is essential one-sided simple closed curve, $\mathfrak{X}_1^2$ is a finite rigid set.

The set of vertices of the curve complex $\mathcal{C}(N_2^0)$ consists of two essential one-sided α_1, α_2 simple closed curves and one two-sided simple closed curve β (see Figure 5.3). In $\mathcal{C}(N_2^0)$, there exists an edge between α_1 and α_2 since they are disjoint.

If we choose finite subcomplex $\mathfrak{X}_2^0 = \{\alpha_1, \alpha_2\}$, the image of $\mathfrak{X}_2^0$ is a one simplex in $\mathcal{C}(N_2^0)$ since ϕ is locally injective simplicial map. Hence, $\phi(\alpha_1)$ and $\phi(\alpha_2)$ are disjoint essential one-sided simple closed curves. $\mathfrak{X}_2^0$ is a finite rigid set in $\mathcal{C}(N_2^0)$.

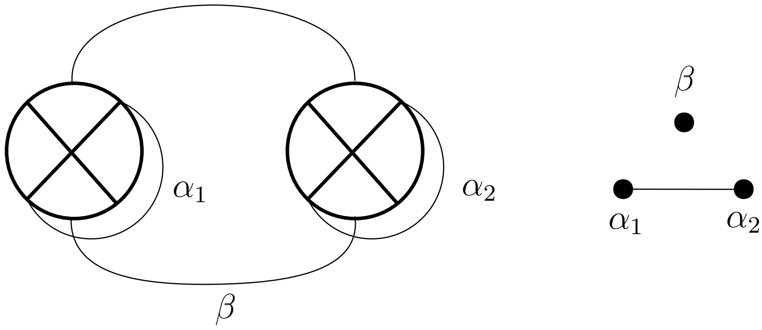

Figure 5.3: The simple closed curves in $\mathcal{C}(N_2^0)$ and the curve complex $\mathcal{C}(N_2^0)$.

The set of vertices of the curve complex $\mathcal{C}(N_2^1)$ is $\{\beta, t_\beta^k(\alpha) \mid k \in \mathbb{Z}\}$ where t_β is the Dehn twist about the two-sided simple closed curve β on N_2^1. The curve complex is defined in [3] as follows: "$\mathcal{C}(N_1^2)$ is the disjoint union of a vertex which represents two-sided simple closed curve β and the real line on which a vertex is placed at each integer point (cf. [18])." If we choose $\mathfrak{X}_2^1 = \{\alpha, \gamma = t_\beta(\alpha)\}$, since ϕ is locally injective simplicial map, the simple closed curves $\phi(\alpha)$ and $\phi(\gamma)$ span an edge in $\mathcal{C}(N_2^1)$. Hence, ϕ is induced from $t_\beta^{\pm n}$ for some $n \in \mathbb{Z}^+$, that is $\phi(\alpha) = t_\beta^{\pm n}(\alpha)$ and $\phi(\gamma) = t_\beta^{\pm n}(\gamma)$.

If $(g, n) = (3, 0)$, we introduce finite subcomplex $\mathfrak{X}_3^0 = \{\alpha_1, \alpha_2, \alpha_3\} \subset \mathcal{C}(N_3^0)$ (see Figure 5.4).

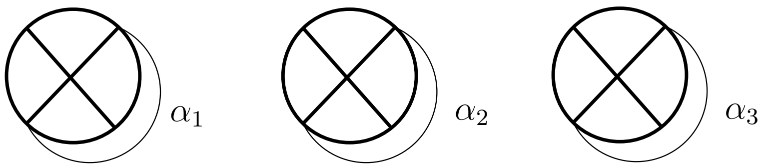

Figure 5.4: The simple closed curves in $\mathfrak{X}_3^0 \subset \mathcal{C}(N_3^0)$.

$P = \{\alpha_1, \alpha_2, \alpha_3\}$ is a top dimensional pants decomposition of N_3^0. Hence, by Lemma 2.1.10, $\phi(P)$ is a top dimensional pants decomposition. By Lemma 2.1.7, in $\phi(P)$, there exist either essential one-sided simple closed curves or separating simple closed curves. Since there does not exist any non-trivial separating simple closed curve on N_3^0, each of $\phi(\alpha_i)$ is essential one-sided simple closed curve, $i = 1, 2, 3$. Since ϕ is locally injective, each $\phi(\alpha_i)$ is distinct. Note that $\{\phi(\alpha_1), \phi(\alpha_2), \phi(\alpha_3)\}$ forms a two-simplex in $\mathcal{C}(N_3^0)$ since ϕ is simplicial. Hence, they are pairwise disjoint. This implies ϕ is induced from a homeomorphism F

of N_3^0.

We do not know whether there exist a finite rigid set in $\mathcal{C}(N_g^n)$ for the cases $g + n = 4$. Now we finish dissertation with a conclusion.

5.2 Conclusion

In [1], Aramayona and Leininger introduce finite rigid sets in the curve complex of a orientable surface. In this dissertation, we contribute for the non-orientable cases. We firstly investigate the topological types of simple closed curves in curve complex of non-orientable surface under a locally injective simplicial map. Different than an orientable surface, there is more options of topological types of simple closed curves on a non-orientable surface. Hence, this topic needs more specific techniques. We use the top dimensional pants decompositions which is firstly used by Irmak in [10] and we observe the topological types of simple closed curves in a top dimensional pants decomposition. We use the adjacency graph of a pants decomposition to distinguish the topological types of simple closed curves. Different than existing works, we limit ourselves to use nonadjacency and linearity in the adjacency graph of a pants decomposition to minimize the finite rigid set. In other words, if we use adjacency in the adjacency graph of a pants decomposition, then the set would contain more element.

As in the [3], we can not find a solution for the cases $g + n = 4$. In [1] by modifying the complex of curve for four-holed sphere, Aramayona and Leininger introduce a finite rigid set. For $(g, n) = (1, 3)$ we can not distinguish topological types of simple closed curves. For $(g, n) = (2, 2)$ the geometric intersection number of the simple closed curves under a locally injective simplicial map can not be controlled.

Recently, Hernández and Aramayona-Leininger use the rigid sets introduced in [1] to find the sequence of rigid sets whose union is curve complex respectively in [8] and [2] with different techniques. This is called as exhaustion of curve complex by rigid sets. Non-orientable case of this work may be a future work.

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