APPLICATION OF STOCHASTIC VOLATILITY MODELS WITH JUMPS TO BIST OPTIONS

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ABSTRACT

APPLICATION OF STOCHASTIC VOLATILITY MODELS WITH JUMPS TO BIST OPTIONS

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This thesis gives a derivation of call and put option pricing formulas under stochastic volatility models with jumps; the precise model is a combination of Merton and Heston models. The derivation is based on the computation of the characteristic function of the underlying process. We use the derived formulas to fit the model to options written on two stocks in the BIST30 index covering the first two months of 2017. The fit is done by minimizing a weighted L_2 distance between the observed prices and the model prices.

Keywords: Merton Model, Heston Model, Heston-Merton Model, Option pricing, Characteristic function, VIOP, BIST

ÖZ

SICRAMALI STOKASTIK VOLATILITE MODELLERININ BIST OPSIYONLARINA UYGULAMASI

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Bu tez Heston ve Merton modellerinin birleşimi olan hem fiyatta sıçramaya izin veren, hem stokastik volatilite içeren bir model için Avrupa opsiyon fiyatlaması formüllerinin çıkarımını vermektedir. Formül, ilgili fiyat sürecinin karateristik fonksiyonu için açık bir formül bularak ve bu formülün Fourier tersini alarak çıkarılmaktadır. Çıkarımı verilen formüller kullanılarak model, BIST30 indeksinde yer alan iki hisse senedi fiyati üzerinde yazılı ve VIOP'de alınıp satılan opsiyon fiyatlarına fit edilmiştir; fit 2017'nin ilk iki ayını kapsamaktadır ve piyasada gözlemlenen opsiyon fiyatlarıyla model fiyatları arasındaki L_2 uzaklık en az yapılarak gerçekleştirilmiştir.

Anahtar Kelimeler : Merton modeli, Heston modeli, Heston-Merton modeli, Opsiyon fiyatlama, Karakteristik fonksiyon, VIOP, BIST

To My Mom
and
To, The Best Uncle Ever,
Kazem Rahiminejat

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CHAPTER 1

Introduction

Options (derivatives) are financial products that derive their value from the value of an underlying security. An option is European if its price (payoff) at a future deterministic time T is given as an explicit function of the price S_T of an underlying security at time T. The goal of this thesis is to study stochastic processes used to model prices that allow both stochastic volatility and jumps in the dynamics of the price process and an application of these models to European options traded in Borsa Istanbul. In the following paragraphs we briefly introduce derivatives and the evolution of price models in the literature leading to models that we will focus on this thesis.

European type call and put options are the most basic options. A variety of European option contracts are available in Turkish financial markets and are traded in the VIOP (Vadeli Islemler ve Opsiyon Piyasasi) of Borsa Istanbul [6]. Let S_t denote the price of the underlying at time t. European call and put options have the following payoffs:

$$C_T = (S_T - K)^+, \;\; P_T = (S_T - K)^- = (K - S_T)^+,$$

where T is the maturity of the option, S_T is the price of the underlying at maturity T and $K > 0$ is called the strike. Pricing formulas for these products as a function of T, K, S_0, the volatility σ and the interest rate r were derived in [1] by Black and Scholes. The primary assumption of the Black-Scholes (BS) model is that the underlying price process S_t has dynamics

$$S_t = S_0 + \int_0^t \mu S_s ds + \int_0^t S_s \sigma dW_s,$$

where $\mu > 0$, $\sigma > 0$ are assumed to be constant, and W is a standard Brownian motion (BM). The interest rate $r > 0$ is assumed constant. In modern notation [8, page 92], the pricing formula of [1] is the following:

$$C_t = \mathbb{E}^*[e^{-r(T-t)}(S_T - K)^+ | \mathscr{F}_t], \tag{1.1}$$

where $\{\mathscr{F}_t\}$ denotes the filtration generated by W and $\mathbb{E}^*$ denotes expectation with respect to a measure under $\mathbb{P}^*$ under which the discounted price process $\tilde{S}_t = e^{-rt} S_t$ is a martingale; the existance of $\mathbb{P}^*$ is guaranteed by Girsanov's theorem [8, Chapter 4]. The formula (1.1) gives the price of the call option, the price of the put option can be obtained from this using the put-call parity:

$$\begin{aligned}C_t - P_t &= \mathbb{E}^*[e^{-r(T-t)}(S_T - K)|\mathscr{F}_t] \\ &= e^{rt}\mathbb{E}^*[e^{-rT}S_T|\mathscr{F}_t] - e^{-r(T-t)}K \\ &= e^{rt}\bar{S}_t - e^{-r(T-t)}K \\ &= S_t - e^{-r(T-t)}K,\end{aligned}$$

i.e.,

$$P_t = C_t - S_t + e^{-r(T-t)}K. \tag{1.2}$$

For the BS model, the dynamics of (S_t) under $\mathbb{P}^*$ is explicitly given as

$$S_t = S_0 e^{(r-\sigma^2/2)t+\sigma B_t},$$

where B is the Brownian motion under $\mathbb{P}^*$; the iid increments and Markov propertis of the BM reduce (1.1) to

$$\begin{aligned}C_t &= \mathbb{E}^*\left[\left(S_0 e^{(r-\sigma^2/2)T+\sigma B_T} - K)\right)^+ |\mathscr{F}_t\right] \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(S_t e^{(r-\sigma^2/2)(T-t)+\sigma x\sqrt{T-t}} - K\right)^+ e^{-x^2/2}dx;\end{aligned} \tag{1.3}$$

this and the put-call parity (1.2) gives also an explicit formula for P_t, the price of a European put. Let us denote the right side of the above formula by $BSC(r,\sigma,S_t,T-t,K)$, i.e., the price of a call option with strike K, maturity T at time t, when the constant interest rate is r and the volatility of the underlying is σ. One can show by differentation that C_t is monotone increasing in σ. This implies that for any observed price C_t there is a unique $\sigma^* > 0$ for which $C_t = BS(r,\sigma^*,S_t,T-t,K)$, this unique value is called the implied volatility of the stock (by the option price C_t); let us denote implied volatility by $IV(r,C_t,S_t,T-t,K)$; by definition IV satisfies

$$C_t = BS(r, IV(r,C_t,S_t,T-t,K), S_t, T-t, K).$$

The set of all implied volatilities observed as the maturity T and strike K range in $(0,\infty)$ is called the volatility surface. Because the volatility $\sigma > 0$ is a constant in the BS model, the BS model implies a flat volatility surface. This is one of the weakest sides of the BS model because the volatility surfaces observed in practice are not flat, see, e.g., [4, page 72, Figure 5.9]. We give another example from VIOP. The volatility surface for options written on the stock price of Garanti Bankasi observed on February 24 2017 is given in Table 1.1.

This surface consists of points computed from 18 prices at VIOP written on Garanti Bankasi with maturities ranging from 6 to 65 days and strikes ranging from 7 to 9.25, the closing price of GARAN that day was 8.69. The implied volatilities for these 18 options range from 0.12 to 1.0025. Given the phenomenon of nonconstant volatility surfaces it is natural to seek new models that allow more complex shapes for the volatility surfaces. The first step in this direction was taken by Merton in [9] by adding a jump process to the dynamics of the price process. Merton's model is reviewed in

price	K	T (in days)	$Type$
0.57	8.25	34	C
0.38	8.5	34	C
0.13	9	34	C
0.07	9.25	34	C
0.35	8.75	65	C
0.02	8	34	P
0.07	8.5	34	P
0.15	8.75	34	P
0.03	8	65	P
0.06	8.25	65	P

Table 1.1: Options available at VIOP on Garanti Bankasi with maturities longer than 5 days, February 24, 2017 (those options with prices less than 0.01 and those maturities less than 5 days are excluded)

section 3.1. In the same chapter we give a derivation of the characteristic function of compound Poisson processes which will be used again in Chapter 5. A second approach is to allow the volatility to be a process as well. These types of models are known as stochastic volatility models, the most well-known is called the Heston model, developed by Heston in 1993 [5]. The Heston model is reviewed in Chapter 4.1. The advantage of the Heston framework is that, it gives explicit formulas for prices of European options via the computation of characteristic functions. The derivation of the characteristic function associated with the Heston model is given in Chapter 4.1, following [4]. How does one compute the price of a Eureopean option starting from the characteristic function of $\log(S_t)$? One basic way is reviewed in Chapter 2.1, based on the Fourier inversion theorem. It is observed in [4] neither the Merton model nor the Heston model are versatile enough to fully match volatility surfaces observed in actual markets. One way to overcome this shortcoming is to combine these models, i.e., to use a model for the price process that has Heston stochastic volatility as well as adding a jump process to its dynamics. [4] observes that this general model can be fit perfectly in the examples studied in that work. Chapter 4 reviews the Heston model with jumps and derives the characteristic function of $\log(S_T)$ for that model. One can then use the ideas in Chapter 2.1 to use this characteristic function to compute prices of European call options in this framework. The resulting formulas can be computed efficiently on a computer. This allows one to fit this model very easily to observed prices. This fitting to prices observed in VIOP is carried out in Chapter 6. The fit is performed on two stocks in the BIST30 Index: Garanti Bankasi and Koc Holding between December 1, 2015, and February 24, 2017. In the same chapter we comment on the results of these fits. Our most important result is that the observation made in [4] continue to hold for the prices studied in this thesis: it is possible to almost perfectly fit the Heston volatility model with jumps to the option prices studied in this thesis. The Conclusion summarizes the thesis and comments on future work.

CHAPTER 2

Pricing formulas based on the characteristic function

2.1 Pricing formulas based on the characteristic function

In the risk neutral pricing framework of mathematical finance theory the price of a European option is given by the conditional expectation

$$C_t = e^{-r(T-t)}\mathbb{E}^*[h(S_T)|\mathscr{F}_t],$$

where $\mathbb{E}^*$ is the risk neutral measure under which $\tilde{S}_t = e^{-rt}S_t$ is a martingale. Assuming S_t is a Markov process, to compute C_t it suffices to know the conditional density $f(\cdot|s)$ of S_T given $S_t = s$ under $\mathbb{P}^*$, in terms of which the above expectation can be written as

$$C_t = e^{-r(T-t)}\int_{-\infty}^{\infty} h(x)f(x|s)ds.$$

In the Black-Scholes model this distribution is log-normal, which leads to the explicit formula (1.3). In many models, this conditional density has no explicit formula but it is possible to compute its characteristic function explicitly. Then, to compute an expectation one must know how to derive the density of a random variable from its characteristic function. By definition the characteristic function of a random variable X is defined by

$$\phi_X(t) = \mathbb{E}[e^{itX}].$$

The theorem that computes the distribution of X from ϕ_X is called the inversion theorem; one can use this theorem to compute the conditional density of S_T from its characteristic function. The goal of this section is to review this computation. In this regard, we will follow [3, Chapter 3]. Let μ denote the distribution of X, i.e., for a Borel set $A \subset \mathbb{R}$,

$$\mu(A) = P(X \in A);$$

then, by Definition,

$$\phi_X(t) = \int_{-\infty}^{\infty} e^{itx}\mu(dx).$$

The inversion theorem ([3, Theorem 3.3.4]) is the following formula connecting ϕ_X to μ:

Theorem 2.1. *For $a < b$*

$$\mu((a,b)) + \frac{1}{2}\mu(\{a,b\}) = \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it} \phi_X(t)dt.$$

Proof.

$$I_T = \int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it} \phi_X(t)dt = \int_{-T}^{T} \int \frac{e^{-ita} - e^{-itb}}{it} e^{itx} \mu(dx)dt, \tag{2.1}$$

the integrand is properly defined near zero since

$$\frac{e^{-ita} - e^{-itb}}{it} = \int_a^b e^{ity} dy$$

it is obvious that the integrand is bounded by $b - a$. As μ is a probability measure, $[-T, T]$ is finite, $\cos(-x) = \cos(x)$, and $\sin(-x) = -\sin(x)$, based on Fubini's theorem we have

$$\begin{aligned} I_T &= \int \int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it} e^{itx} dt\mu(dx) \qquad (2.2) \\ &= \int \int_{-T}^{T} \frac{e^{-it(a-x)} - e^{-it(b-x)}}{d} t\mu(dx) \\ &= \int \left\{ \int_{-T}^{T} \frac{\sin t(x-a)}{t} dt - \int_{-T}^{T} \frac{\sin t(x-b)}{t} dt \right\} \mu(dx), \end{aligned}$$

noting that

$$\frac{e^{-itx}}{it} = \frac{-\sin(tx) - i\cos(tx)}{t}.$$

Defining $R(\theta, T) = \int_{-T}^{T} \frac{\sin \theta t}{t} dt$, the last equation will be

$$I_T = \int R(x-a, T) - R(x-b, T)\mu(dx). \tag{2.3}$$

If $S(T) = \int_0^T \frac{\sin x}{x} dx$, then, we change variables $t = \frac{x}{\theta}$. We have

$$R(\theta, T) = 2 \int_0^{T\theta} \frac{sinx}{x} = 2S(T\theta),$$

when $\theta < 0$, $R(\theta, T) = -R(|\theta|, T)$. By using the function $sgn(x)$, we can combine two formulas as

$$R(\theta, T) = 2(sgn\theta)S(T|\theta|).$$

As $T \to \infty$, $S(T) \to \pi/2$; so, $R(\theta, T) \to \pi sgn\theta$, and

$$R(x-a, T) - R(x-b, T) \to \begin{cases} 2\pi, & a < x < b, \\ \pi, & x = a or x = b, \\ 0, & x < a or x > b. \end{cases}$$

We note that $|R(\theta, T)| \leq 2\sup_y S(y) < \infty$, so that bounded convergence theorem and 2.1 implies

$$\frac{2}{\pi} I_T \to \mu(a,b) + \frac{1}{2}\mu(\{a,b\})$$

□

The inversion formula gives a formula for the distribution μ of X given its characteristic function. The next theorem provides a condition in terms of ϕ_X under which X has a density and a formula for the density function (this is offered as [3, Theorem 3.3.5]).

Theorem 2.2. *If* $\int_{-\infty}^{\infty} |\phi_X(t)| dt < \infty$ *then* μ *has a bounded continuous density*

$$f(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ity} \phi_X(t) dt.$$

Proof. In the proof of last theorem, we observed

$$\left| \frac{e^{-ita} - e^{-itb}}{it} \right| = \left| \int_a^b e^{-ity} dy \right| \leq |b-a|. \tag{2.4}$$

This integral is the same as the integral in last theorem, but the convergence here is absolute. So

$$\mu((a,b)) + \frac{1}{2}\mu(\{a,b\}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-ita} - e^{-itb}}{it} \phi_X(t) dt \leq \frac{(b-a)}{2\pi} \int_{-\infty}^{\infty} |\phi_X(t)| dt$$

From the last equation, it is obvious that μ has got no point masses. Then,

$$\begin{aligned} \mu(x, x+h) &= \frac{1}{2\pi} \int \frac{e^{-itx} - e^{-it(x+h)}}{it} \phi_X(t) dt \\ &= \frac{1}{2\pi} \int \left(\int_x^{x+h} e^{-ity} dy \right) \phi_X(t) dt \\ &= \int_x^{x+h} \left(\frac{1}{2\pi} \int e^{-ity} \phi_X(t) dt \right) dy, \end{aligned}$$

where the last equation is stated by Fubini's theorem. Thus, the density function of distribution μ is

$$f(y) = \frac{1}{2\pi} \int e^{-ity} \phi_X(t) dt.$$

By dominated convergence theorem, we conclude that f is continuous, and the proof is done.

□

We will apply the last theorem to the characteristic function computed in Chapter 5 to get the risk neutral density of S_T under the Heston stochastic volatility model with jumps, which in turn will allow us to compute prices of European call options under that model.

The approach outlined above is not the only way to connect the characteristic function to prices of options. A further study of this question can be found in [11, Chapter 3].

CHAPTER 3

Merton Model

3.1 Merton Model

In the rest of this thesis we assume that all of the random variables are defined on a probability space $(\Omega, \mathscr{F}, P)$ equipped with a filtration $\{\mathscr{F}_t\}$, satisfying the usual hypotheses (see [10, Definition, page 3]); all processes below are assumed as adapted to $\{\mathscr{F}_t\}$.

Furthermore, unless otherwise noted, we will take the constant interest rate to be $r = 0$; the constant nonzero interest rate case can be reduced to this case by working with discounted prices. Let us give an example: a call option with strike K and maturity T, corresponds to the option with strike $\tilde{K} = e^{-rT}K$ with the same maturity. Once the discounted price $\tilde{C}_t$ is computed, it can be converted to undiscounted price via $C_t = e^{rt}\tilde{C}_t$.

The first option pricing that allows jumps in the price of the underlying process is the one introduced by Merton [9], which we now review. In addition to the Brownian motion W let us introduce a Poisson process N independent of W with rate λ and an independent and identically distributed (iid) sequence of random variables X_i. The compound Poisson Process Y is defined as follows:

$$Y_t \doteq \sum_{i=1}^{N_t} \left(e^{X_i} - 1\right).$$

Throughout this thesis we will assume X_i to be $N(\alpha, \delta^2)$, i.e., normal distributed with mean α and standard deviation δ. Merton's model assumes the following dynamics for the price process:

$$dS_t = S_t dX_t, \tag{3.1}$$

where

$$X_t = \mu t + \sigma W_t + Y_t, \tag{3.2}$$

and $\mu, \sigma > 0$ are assumed constant. Using Ito's formula one can find an explicit formula for (3.1). Because X has jumps, we need an Ito formula that allows jumps. Such a formula is given in [10, Theorem 32, page 78]; we review it in the following subsection.

3.1.1 Ito's Formula for semimartingales

For the general version of Ito's formula allowing jumps we need the concept of semimartingales and finite variation (FV) processes:

Definition 3.1. A process V is said to be of finite variation if for any t and a sequence of partitions $0 = t_0^n < t_1^n < t_2^n < \cdots < t_{k_n}^n$ of $[0, t]$ satisfying $\sup_j |t_{j+1}^n - t_j^n| \to 0$ one has

$$\lim_{n\to\infty} \sum_{j=0}^{k_n-1} |V_{t_{j+1}^n} - V_{t_j^n}| < \infty.$$

Definition 3.2. An adapted cadlag (right continuous, left limits) process Z is called a semimartingale if it has the form

$$Z_t = Y_0 + M_t + V_t,$$

where M is a local martingale and V is a finite variation process.

By its definition, a Poisson process can jump at most finitely many times in an interval $[0, t]$ and Y is constant in between these jumps, these imply that the Y process is FV. Furthermore, $t \mapsto \mu t$ is monotone increasing and therefore FV. Finally, the sum of two FV processes is again FV. It follows from these that $\mu t + Y_t$ is a FV process. On the other hand, the Brownian motion W is a martingale. These imply that X process of (3.2) is a semimartingale.

For a semimartingale Z, its quadratic variation is given by (see [10, Section II.6])

$$[Z, Z]_t \doteq Z_t^2 - 2\int_0^t Z_{s-} dZ_s.$$

For a cadlag process Z, let Z^c denote its continuous part:

$$Z_t^c = Z_t - \sum_{s\le T} \Delta Z_s, \quad \Delta Z_s = Z_s - Z_{s-}.$$

The process X of (3.2) is already given as a sum of its continuous and discontinuous parts:

$$X_t^c = \mu t + \sigma W_t.$$

Because X has jumps, $[X, X]_t$ will also jumps. The relation between $[X, X]^c$ and X^c is given in [10, page 70]:

$$[X, X]^c = [X^c, X^c] = [\mu t + \sigma W_t, \mu t + \sigma W_t] = \sigma^2 t,$$

where we used the well known quadratic variation of the Brownian motion (see [7, Chapter 2]).

With these let us now state the Ito formula for semimartingales:

Theorem 3.1. *Let X be a semimartingale an let f be a C^2 real function. Then $f(X)$ is again a semimartingale, and the following formula holds:*

$$
\begin{aligned}
f(X_t) = f(X_0) &+ \int_{0+}^{t} f'(X_{s-})dX_s + \frac{1}{2}\int_{0+}^{t} f''(X_{s-})d[X,X]_s^c \\
&+ \sum_{0<s\leq t} \left(f(X_s) - f(X_{s-}) - f'(X_{s-})\Delta X_s\right).
\end{aligned}
$$

The above theorem is given as [10, Theorem 32]; its proof can be found in the same work.

3.1.2 Risk neutral measure

Before we use Ito's formula to solve (3.1) let us address an important issue. Prices of options are computed under a risk neutral measure; the question is "what risk neutral measure should be used in the Merton framework?" As opposed to the BS model, a market consisting of S_t and the riskless bond is no longer complete (see [2, Chapter 10]) when we add the jump process Y to (3.2). It follows that there are a family of measures on $(\Omega, \mathscr{F})$ under which S is a martingale. In BS, one replaces the Brownian motion W_t with $B_t = \frac{\mu - r}{\sigma}t + W_t$, and the unique measure $\mathbb{P}^*$ under which B is a Brownian motion is the unique measure making S a martingale; this change of measure modifies only the constant drift of W. In the present case, the changes of measures can also change the dynamics of Y by modifying the jump rate λ and the parameters α and δ of the jump sizes. Different ways of doing this are given in [2, Chapter 9] and in the references cited in this work. A simplistic solution is to proceed as in the Black-Scholes model and not to change the dynamics of the Y process and only modify the constant drift of X, as is done in the BS framework. This is the approach taken in [9]. Arguments for and against it can be found in [2, Chapter 10]. We will also use the same approach. Therefore, under our risk neutral measure $\mathbb{P}^*$ only the drift μ in (3.2) will be modified so that the solution S of (3.1) becomes a Martingale. Under $\mathbb{P}^*$, Y, N and X_i will continue have the same distributions as under the original measure. By (3.1), S is a martingale under $\mathbb{P}^*$, if and only if X is. Let us write X as follows:

$$X_t = \mu t + \mathbb{E}^*[Y_t] + W_t + \bar{Y}_t,$$

where

$$\bar{Y}_t = Y_t - \mathbb{E}[Y_t].$$

Here, $\bar{Y}_t$ is the compensated version of Y. By [10, Theorem 41, page 30], $\bar{Y}_t$ is a martingale. The expectation of a compound Poisson process is well known:

$$\mathbb{E}^*[Y_t] = \mathbb{E}[Y_t] = \lambda t \mathbb{E}\left[e^{X_i} - 1\right] = \lambda \omega t, \quad \omega \doteq e^{\alpha + \frac{\delta^2}{2}} - 1, \tag{3.3}$$

where $e^{\alpha + \frac{\delta^2}{2}}$ comes from the mean of the log-normal distribution. Then X can be written as

$$X_t = (\mu + \omega)t + \sigma W_t + \bar{Y}_t.$$

We would like this process to be a martingale under $\mathbb{P}^*$. For this it suffices to include the drift term inside the Brownian motion, i.e., we define

$$B_t = \frac{\mu + \omega}{\sigma} t + W_t.$$

Then, we write X as

$$X_t = \sigma B_t + \bar{Y}_t = \sigma B_t + Y_t - \omega t.$$

We will choose $\mathbb{P}^*$ so that B_t is a martingale under $\mathbb{P}^*$. The explicit change of measure that gives $\mathbb{P}^*$ is once again given by Girsanov's theorem. In the section below, we will use this $\mathbb{P}^*$ measure as our risk neutral measure.

3.1.3 An Explicit Formula for S_t

Define the following compound Poisson process:

$$J_t = \sum_{i=1}^{N_t} X_i;$$

let us that Y and J have the same jump times; J has jumps X_i, whereas Y has jumps $e^{X_i} - 1$. The process J will be useful in our explicit formula for S.

By the existence and uniqueness theorem [10, Theorem 6, Section V.3] the SDE (3.1) has a unique solution S_t. Ito's formula, Theorem 3.1, gives the following explicit formula for this solution:

Proposition 3.2. *The solution of* (3.1) *is given by*

$$S_t = S_0 e^{L_t},$$

where

$$L_t = -(\omega + \sigma^2/2)t + \sigma B_t + J_t,$$

and ω is as in (3.3).

Proof. We know that (3.1) has a unique solution. Therefore, it suffices to show that $E_t = e^{L_t}$ satisfies (3.1). By Ito's formula Theorem 3.1 we have

$$E_t = E_0 + \int_{0+}^{t} E_{s-} dL_s + \frac{1}{2} \int_{0}^{t} E_{s-} [L, L]_s^c + \sum_{s \leq t} \left(f(L_s) - f(L_{s-}) - E_{s-} \Delta X_s \right). \tag{3.4}$$

We note

$$\int_{0+}^{t} E_{s-} dL_s = \int_{0+}^{t} E_{s-} \left(- \left(\omega + \sigma^2/2 \right) ds + \sigma B_s \right) + \int_{0+}^{t} E_{s-} dJ_s,$$

and

$$\int_{0+}^{t} E_{s-} dJ_s = \sum_{s \leq t} E_{s-} \Delta X_s.$$

Cancelling these two terms in (3.4) gives

$$E_t = E_0 + \int_{0+}^{t} E_{s-}\left(-(\omega + \sigma^2/2)dt + \sigma dB_s\right)$$
$$+ \frac{1}{2}\int_0^t E_{s-}[L, L]_s^c + \sum_{s\le t}(e^{L_s} - e^{L_{s-}}).$$

Recall from Section 3.1.1 that $[L, L]_t^c = [L^c, L^c]_t = \sigma^2 t$; substituting this in the last display reduces it to

$$E_t = E_0 + \int_{0+}^{t} E_{s-}\left(-\omega dt + \sigma dB_s\right) + \sum_{s\le t}(e^{L_s} - e^{L_{s-}}). \tag{3.5}$$

Finally, $e^{L_s} - e^{L_{s-}}$ is nonzero only when $s = T_j$, where $\{T_j\}$ are the jump times of the Poisson process N N. Let T_j be a jump time of N; by definition:

$$\begin{aligned} e^{L_{T_j}} - e^{L_{T_j -}} &= e^{L_{T_j -} + X_j} - e^{L_{T_j -}} \\ &= e^{L_{T_j}-}(e^{X_j} - 1) \\ &= E_{T_j -}\Delta Y_{T_j}. \end{aligned}$$

Therefore,

$$\sum_{s\le t}(e^{L_s} - e^{L_{s-}}) = \int_{0+}^{t} E_s dY_s.$$

Substituting this in (3.5) gives

$$\begin{aligned} E_t &= E_0 + \int_{0+}^{t} E_{s-}\left(-\omega ds + \sigma dB_s + dY_s\right) \\ &= E_0 + \int_{0+}^{t} E_{s-} dX_s, \end{aligned}$$

i.e., E indeed is the solution of (3.1). □

We can use this explicit formula for S_t to derive a formula for prices of European options under the Merton model

Proposition 3.3. *Under the Merton model, the price at time t of a European option with payoff $h(S_T)$ is given by*

$$C_t = \frac{1}{2\pi}\sum_{k=0}^{\infty}\frac{1}{k!}(\lambda(T-t))^k e^{-\lambda(T-t)} \tag{3.6}$$
$$\times \int_{-\infty}^{\infty} h\left(e^{-(\omega+\sigma^2)(T-t)+k\alpha+\sqrt{k\delta^2+\sigma^2(T-t)}x}\right)e^{-x^2/2}dx.$$

Proof. Note that

$$S_T = S_t e^{L_T - L_t}$$

where L is a Markov Process with independent and identically distributed increments (i.e., it is a Levy process). Therefore, the conditional density of $L_T - L_t$ equals the density of L_{T-t}. This implies

$$\mathbb{E}^*[h(S_T)|\mathscr{F}_t] = g(S_t),$$

where

$$g(s) = \mathbb{E}^* \left[h \left(s e^{L_{T-t}} \right) \right].$$

To compute g we proceed as follows: partitioning the last expectation on the different values that N_{T-t} can take, we have

$$g(s) = \mathbb{E}^*[h \left(s e^{-(\omega+\sigma^2/2)(T-t)+\sigma B_{T-t}+\sum_{i=1}^{k} X_i} 1_{\{N_{T-t}=k\}} \right)].$$

The independence of B, N and the sequence $\{X_i\}$ and the Poisson distribution of N_{T-t} give

$$= \sum_{k=0}^{\infty} \frac{1}{k!} e^{-\lambda(T-t)} (\lambda(T-t))^k \left(\mathbb{E}^*[h \left(s e^{-(\omega+\sigma^2/2)(T-t)+\sigma B_{T-t}+\sum_{i=1}^{k} X_i} \right) \right].$$

The only remaining random variable in this display is $\sigma B_{T-t} + \sum_{i=1}^{k} X_i$, which is normal distributed with mean $k\alpha$ and variance $\sigma^2(T-t) + k\delta^2$. Computing the last expectation using this density gives (3.6). $\square$

3.1.4 Implied volatility surface for the Merton Model

Let us now use the explicit pricing formula derived in the previous section to look at the implied volatility surface under the Merton framework. An example is given in Figure 3.1; this is the volatility surface of the Merton model for the following parameter values:

$$\sigma = 0.2, \lambda = 0.1, \alpha = 0.02, \delta = 0.3, s = 1.$$

As is clear from Figure 3.1, the Merton model is able to produce nonflat implied volatility surfaces as opposed to the BS model which always implies a flat volatility surface. The next chapter will review the Heston model, which is another model with nonflat implied volatility surfaces.

3.1.5 The Characteristic Function of J_t

As seen in Proposition 3.3, under the Merton model, one can directly compute the prices of European securities using the densities of the underlying random variables; therefore, for the Merton model one does not need the characteristic function of S_T to

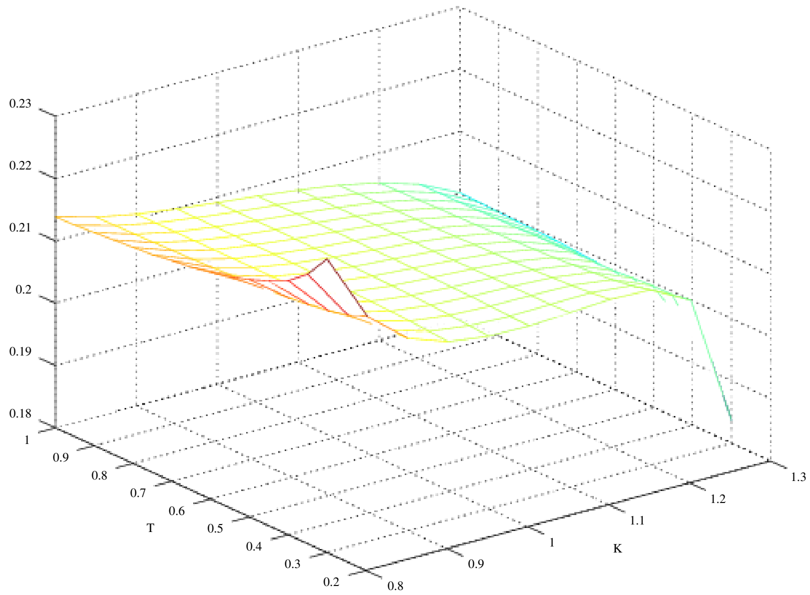

Figure 3.1: The implied volatility surface of the Merton model for $\lambda = 0.1$, $\sigma = 0.2$, $\alpha = 0.02$ and $\delta = 0.3$

compute option prices. Nonetheless, in Chapter 5 when we will combine stochastic volatility models with jumps, we will no longer have explicit formulas for densities and we will need to use the characteristic function approach. For this it will be useful to know the characteristic function of J_t. Here, J_t is a compound Poisson process, so its characteristic function is well known. Given its importance in our calculations, we repeat this computation below.

Proposition 3.4. *The characteristic function of J_T equals*

$$\phi_J(\theta) = \mathbb{E}^* \left[e^{i\theta J_T}\right] = \exp\left(\lambda t \left(\phi_X(\theta) - 1\right)\right),$$

where

$$\phi_X(\theta) = e^{i\alpha\theta - \frac{1}{2}\delta^2\theta^2},$$

is the characteristic function of X_i.

Proof. We begin by conditioning on N_T:

$$\begin{aligned}
\phi_J(\theta) &= \mathbb{E}^* \left[\exp(i\theta \sum_{n=k}^{N_T} X_n)\right] \\
&= \mathbb{E}^* \left[\mathbb{E}^* \left[\exp\left((i\theta \sum_{n=k}^{N_T} X_n\right) |N_T\right]\right]
\end{aligned}$$

the fact that N_T takes countably many values implies the representaion

$$= \mathbb{P}^*(N_T = 0) + \sum_{k=1}^{\infty} \mathbb{E}^* \left[i\theta \sum_{n=1}^{N_T} X_n | N_T = k \right] \mathbb{P}^*(N_T = k).$$

The independence of N and X_n leades to the form

$$\begin{aligned}
&= e^{-\lambda t} + \sum_{k=1}^{\infty} \mathbb{E}^*(e^{i\theta X_1})\mathbb{E}^*(e^{i\theta X_2})...\mathbb{E}^*(e^{i\theta X_k}) \frac{(\lambda t)^k}{k!} e^{-\lambda t} \\
&= e^{-\lambda t} + e^{-\lambda t} \sum_{k=1}^{\infty} \frac{(\varphi_X(\theta)\lambda t)^k}{k!} \\
&= e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\varphi_X(\theta)\lambda t)^k}{k!} \\
&= \exp \lambda t(\varphi_X(\theta) - 1); .
\end{aligned}$$

this proves the proposition. $\square$

CHAPTER 4

Heston Model

4.1 Heston Model

A very popular framework for option pricing is the "stochastic volatility" models. These models assume that the volatility parameter σ itself is a stochastic process. The first and most popular stochastic volatility model is the Heston model, first introduced in [5]. Two book length treatments of this model are [4, 11]. This chapter follows [4]. The Heston model assumes the following price dynamics:

$$dS_t = S_t dX_t, \tag{4.1}$$

where

$$\begin{aligned} dX_t &= \mu dt + \sqrt{v_t} dW_t \\ dv_t &= -\lambda_h \left(v_t - \bar{\nu}\right) dt + \eta \sqrt{v_t} dW_t^h, \end{aligned} \tag{4.2}$$

and (W_t, W_t^h) is a two-dimensional Brownian motion with cross variation

$$\langle W, W^h \rangle = \rho t.$$

The parameters of the Heston model are: μ, λ_h, $\bar{\nu} > 0$, $\lambda_h > 0$, $\eta > 0$ and $\rho \in [-1, 1]$. $\bar{\nu}$ is the mean-variance level to which v reverts, λ_h is the speed of reversion. The presence of a two Brownian motion implies that a market consisting only of S and the riskless bond is incomplete and, therefore, there are a family of risk neutral measures under which S is a martingale. Similar to the Merton model, this brings the question of which risk-neutral pricing measure to use. The most commonly used solution to this problem is to proceed as we did in the Merton model and to set $\mathbb{P}^*$ to the unique measure that only modifies the drift of the Brownian motion appearing in X, so that X becomes a martingale (see [4, 11]):

$$dX_t = \sqrt{v_t} \left(\frac{\mu}{\sqrt{v_t}} dt + dW_t \right) = \sqrt{v_t} dB_t,$$

where

$$B_t^h = \frac{\mu}{\sqrt{v_t}} t + W_t.$$

We choose $\mathbb{P}^*$ so that B is a standard Brownian motion under $\mathbb{P}^*$. Once again, Girsanov's theorem provides the explicit change of measure that defines $\mathbb{P}^*$ in terms of $\mathbb{P}$. Going from $\mathbb{P}$ to $\mathbb{P}^*$ only the Brownian motion changes from W to B; the rest of the processes remain the same, in particular, the parameters ρ, η, λ_h and $\bar{\nu}$ remain unchanged.

Let us define

$$L_t^h \doteq -1/2 \int_0^t v_s ds + \int_0^t \sqrt{v_s} B_s^h.$$

An argument parallel to the proof of Proposition 3.2 gives the following formula for S_t:

Proposition 4.1. *The solution of* (4.1) *is*

$$S_t = S_0 e^{L_t^h}. \tag{4.3}$$

As opposed to the situation in the Merton framework, there is no simple formula for the density of L_t^h; therefore, one cannot simply use (4.3) to find explicit formulas for prices of call options, similar to the derivation of (3.6). Nonetheless, it is still possible to find fairly explicit formulas for prices of European call options under the Heston model using PDE methods. The reference [4] gives a derivation of this formula; in the rest of this chapter we follow the derivation in [4], which is based on the partial differential equation (PDE) satisfied by the price of the option. This PDE is given as follows in [4]:

$$-\frac{\partial C}{\partial \tau} + \frac{1}{2} v C_{11} - \frac{1}{2} v C_1 + \frac{1}{2} \eta^2 v C_{22} + \rho \eta v C_{12} - \lambda (v - \bar{v}) C_2 = 0,$$

where C is the price function and subscripts 1 and 2 refer to differentiation with respect to x and v respectively. suppose the solution of the equation has the form

$$C(x, v, \tau) = K\{e^x P_1(x, v, \tau) - P_0(x, v, \tau)\}.$$

Having calculated all partial derivatives for proposed solution, we will insert them into the PDE; the partial derivatives are:

$$\begin{aligned}
\frac{\partial C}{\partial \tau} &= K \left[e^x \frac{\partial P_1}{\partial \tau} - \frac{\partial P_0}{\partial \tau} \right], \\
C_1 = \frac{\partial C}{\partial x} &= K \left[e^x P_1 + e^x \frac{\partial P_1}{\partial x} - \frac{\partial P_0}{\partial x} \right], \\
C_{11} = \frac{\partial^2 C}{\partial x^2} &= K \left[e^x P_1 + e^x \frac{\partial P_1}{\partial x} + e^x \frac{\partial P_1}{\partial x} + e^x \frac{\partial^2 P_1}{\partial x^2} - \frac{\partial^2 P_0}{\partial x^2} \right], \\
C_2 = \frac{\partial C}{\partial v} &= K \left[e^x \frac{\partial P_1}{\partial v} - \frac{\partial P_0}{\partial v} \right], \\
C_{22} = \frac{\partial C}{\partial v} &= K \left[e^x \frac{\partial^2 P_1}{\partial v^2} - \frac{\partial^2 P_0}{\partial v^2} \right], \\
C_{12} = \frac{\partial}{\partial v} (\frac{\partial C}{\partial x}) &= K \left[e^x \frac{\partial P_1}{\partial v} + e^x \frac{\partial^2 P_1}{\partial v \partial x} - \frac{\partial^2 P_0}{\partial v \partial x} \right].
\end{aligned}$$

By substituting these derivatives in the PDE and simplifying the result, we get

$$\begin{aligned}
&- e^x\frac{\partial P_1}{\partial \tau}+\frac{1}{2}ve^x\frac{\partial^2 P_1}{\partial x^2}+\frac{1}{2}ve^x\frac{\partial P_1}{\partial x}+\frac{1}{2}\eta^2 ve^x\frac{\partial^2 P_1}{\partial v^2}+\rho\eta ve^x\frac{\partial^2 P_1}{\partial v\partial x}\\
&+(\rho\eta ve^x-\lambda(v-\bar{v}))\frac{\partial P_1}{\partial v}+\frac{\partial P_0}{\partial \tau}-\frac{1}{2}v\frac{\partial^2 P_0}{\partial x^2}+\frac{1}{2}v\frac{\partial P_0}{\partial x}-\frac{1}{2}\eta^2 v\frac{\partial^2 P_0}{\partial v^2}-\rho\eta v\frac{\partial^2 P_0}{\partial x\partial v}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad-(\lambda\bar{v}-\lambda v)\frac{\partial P_0}{\partial v}=0.
\end{aligned}$$

This is the same as

$$\begin{aligned}
&-(e^x\frac{\partial P_1}{\partial \tau}-\frac{\partial P_0}{\partial \tau})+\frac{1}{2}v(e^x\frac{\partial^2 P_1}{\partial x^2}-\frac{\partial^2 P_0}{\partial x^2})+\frac{1}{2}v(e^x\frac{\partial P_1}{\partial x}+\frac{\partial P_0}{\partial x})+\frac{1}{2}\eta^2 v(e^x\frac{\partial^2 P_1}{\partial v^2}-\frac{\partial^2 P_0}{\partial v^2})\\
&\qquad+\rho\eta v(e^x\frac{\partial^2 P_1}{\partial v\partial x}+\frac{\partial^2 P_0}{\partial x\partial v})+\lambda(\bar{v}-v)(e^x\frac{\partial P_1}{\partial v}-\frac{\partial P_0}{\partial v})+\rho\eta ve^x\frac{\partial P_1}{\partial v}=0;
\end{aligned}$$

Since the solution stands for C; so, it must stand for both P_1, and P_2. Then:

$$-\frac{\partial P_j}{\partial \tau}+\frac{1}{2}v\frac{\partial^2 P_j}{\partial x^2}-(\frac{1}{2}-j)v\frac{\partial P_j}{\partial x}+\frac{1}{2}\eta^2 v\frac{\partial^2 P_j}{\partial v^2}+\rho\eta v\frac{\partial^2 P_j}{\partial x\partial v}+(a-b_j v)\frac{\partial P_j}{\partial v}=0 \tag{4.4}$$

when $j=0{,}1$ where

$$a=\lambda\bar{v},\quad b_j=\lambda-j\rho\eta,$$

$$\theta(x):=\lim_{\tau\to 0}P_j(x,v,\tau)=\begin{cases}1, & if x>0,\\ 0, & if x\le 0,\end{cases}$$

where $\theta(x)$ is the terminal value. By the definition of equation $\theta(x)$ and via Fourier transform technique, we will solve equation (4.4). Consider the Fourier transform of P_j as

$$\tilde{P}(x,v,\tau)=\int_{-\infty}^{\infty}dxe^{-iux}P(x,v,\tau).$$

Then the forward transform is

$$\tilde{P}(x,v,0)=\int_{-\infty}^{\infty}dxe^{-iux}\theta(x)=\frac{1}{iu},$$

and for inverse transform we have

$$P(x,v,\tau)=\int_{-\infty}^{\infty}\frac{du}{2\pi}e^{iux}\tilde{P}(x,v,\tau). \tag{4.5}$$

Having calculated all those derivatives in (4.4) with respect to τ, x^2, x, v^2, v, for (4.5) we have

$$\frac{\partial P_j}{\partial x}=\frac{iu}{2\pi}e^{iux}\tilde{P}_j,$$
$$\frac{\partial^2 P_j}{\partial x^2}=\frac{-u^2}{2\pi}e^{iux}\tilde{P}_j,$$

$$\frac{\partial^2 P_j}{\partial x \partial v} = \frac{iu}{2\pi} e^{iux} \frac{\partial \tilde{P}_j}{\partial v}.$$

We substitute the last equation into (4.4). Since $\frac{e^{iux}}{2\pi}$ is not zero, we can easily ignore it. So, we have

$$- \frac{\partial \tilde{P}_j}{\partial \tau} - \frac{1}{2}u^2 v \tilde{P}_j - \frac{1}{2}u^2 v \tilde{P}_j - (\frac{1}{2} - j) iuv \tilde{P}_j + \frac{1}{2}\eta^2 v \frac{\partial^2 \tilde{P}_j}{\partial v^2} + \rho\eta iuv \frac{\partial \tilde{P}_j}{\partial v} + (a - \lambda + \rho j \eta) = 0.$$

By rearranging this equation, we have

$$- v(\frac{1}{2}u^2 - \frac{1}{2}iuv + jiuv)\partial \tilde{P}_j - v(\lambda - \rho\eta j - \rho\eta iu)\frac{\partial \tilde{P}_j}{\partial v} + \frac{1}{2}\eta^2 v \frac{\partial^2 \tilde{P}_j}{\partial v^2} + a\frac{\partial \tilde{P}_j}{\partial v} - \frac{\partial \tilde{P}_j}{\partial \tau} = 0. \quad (4.6)$$

Now, we define

$$\begin{aligned}
\alpha &= -\frac{u^2}{2} - \frac{iu}{2} + iju, \\
\beta &= \lambda - \rho\eta j - \rho\eta iu, \\
\gamma &= \frac{\eta^2}{2}.
\end{aligned}$$

Then, (4.6) turns out to be

$$v \left[\alpha \tilde{P}_j - \beta \frac{\partial \tilde{P}_j}{\partial v} + \gamma \frac{\partial^2 \tilde{P}_j}{\partial v^2} \right] + a \frac{\partial \tilde{P}_j}{\partial v} - \frac{\partial \tilde{P}_j}{\partial \tau} = 0. \quad (4.7)$$

Now, consider the answer in below form and substitute it in (4.7):

$$\begin{aligned}
\tilde{P}_j(u, v, \tau) &= \exp\left[C_j(u,\tau)\bar{v} + D_j(u,\tau)v\right] \\
&= \frac{1}{iu} exp\left[C_j(u,\tau)\bar{v} + D_j(u,\tau)v\right]
\end{aligned}$$

$$v \left[\alpha \tilde{P}_j - \beta \frac{\partial \tilde{P}_j}{\partial v} + \gamma \frac{\partial^2 \tilde{P}_j}{\partial v^2} \right] + \lambda \bar{v} \frac{\partial \tilde{P}_j}{\partial v} = \left[\bar{v} \frac{\partial C_j}{\partial \tau} + v \frac{\partial D_j}{\partial \tau} \right] P_j. \quad (4.8)$$

Therefore, we have

$$\begin{aligned}
\frac{\partial \tilde{P}_j}{\partial \tau} &= \left[\bar{v} \frac{\partial C_j}{\partial \tau} + v \frac{\partial D_j}{\partial \tau} \right] P_j, \\
\frac{\partial \tilde{P}_j}{\partial v} &= D_j \tilde{P}_j, \\
\frac{\partial^2 \tilde{P}_j}{\partial v^2} &= D_j^2 \tilde{P}_j.
\end{aligned}$$

So,(4.8) holds if

$$\begin{aligned}\frac{\partial C_j}{\partial \tau} &= \lambda D_j \\ \frac{\partial D_j}{\partial \tau} &= \alpha - \beta D_j + \gamma D_j^2 = \gamma(D_j - r_+)(D_j - r_-)\end{aligned} \tag{4.9}$$

define

$$r_{j,\pm} = \frac{\beta \pm \sqrt{B^2 - 4\alpha\gamma}}{2\gamma} := \frac{\beta \pm d}{\eta^2}$$

By integrating (4.9), with respect to the terminal condition $C_j(u, 0) = 0$ and $D_j(u, 0) = 0$, we have

$$\begin{aligned}\frac{\partial D_j}{(D_j - r_+)(D_j - r_-)} &= \gamma \partial \tau \\ (\frac{1}{D_j - r_-} - \frac{1}{D_j - r_+})\partial D &= (r_- - r_+)\gamma\partial\tau \\ \ln \frac{D_j - r_-}{D_j - r_+} &= \frac{\gamma\partial\tau}{r_- - r_+} \\ (D_j - r_+) \exp\left[\frac{\gamma\tau}{r_- - r_+}\right] &= D_j - r_-.\end{aligned}$$

Let us consider $d := \frac{\gamma}{r_- - r_+}$ and $g := \frac{r_-}{r_+}$. So,

$$\begin{aligned}D_j(u, \tau) &= \frac{r_- - r_+ e^{d\tau}}{1 - e^{d\tau}} \\ C_j(u, \tau) &= \lambda \left[r_-\tau - \frac{2}{\eta^2} \log(\frac{1 - ge^{-d\tau}}{1 - g}) \right]\end{aligned}$$

Complex integration in (4.5) gives the final form of the P_j:

$$P_j(x, v, \tau) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty du Re \left[\frac{exp\{C_j(u, \tau)\bar{v} + D_j(u, \tau)v + iux\}}{iu} \right].$$

Let us summarize the foregoing computations in the following proposition.

Proposition 4.2. *The price C_t of a European call option under the Heston model at time t is given by*

$$C(x, v, t) = K(e^x P_1(x, v, T - t) - P_0(x, v, T - t)). \tag{4.10}$$

4.1.1 Characteristic function of L_T^h

In Chapter 5 we will combine the Heston and the Merton models; to derive prices of European options under that framework we will use the characteristic function approach outlined in Chapter 2.1. To compute the characteristic function of $\log(S_T)$ in Chapter 5 it will be useful to have a formula for the characteristic function of L_T^h; [4] derives the following formula for this characteristic function starting from (4.10).

Proposition 4.3. *Characteristic function of Heston model is*

$$\phi_T^h(\theta) = \mathbb{E}^*\left[e^{i\theta L_T^h}\right] = \exp[C(\theta,\tau)\bar{v} + D(\theta,\tau)v].$$

Proof. The final log-stock price x_T is greater than the strike price. This probability is

$$\begin{aligned} Pr(L_T^h > x) &= P_0(x,v,\tau) \\ &= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty duRe\left[\frac{exp\{C_j(\theta,\tau)\bar{v} + D_j(\theta,\tau)v + i\theta x\}}{i\theta}\right] \end{aligned}$$

with $x = log(S_t/K)$, $\tau = T - t$, and log-strike price is defined by $k = log(K/S_t) = -x$. Then, the probability density function $p(k)$ is

$$\begin{aligned} Pr(k) &= -\frac{\partial P_0}{\partial k} \\ &= \frac{1}{k}\int_{-\infty}^\infty \theta' \exp[C(\theta',\tau)\bar{v} + D(\theta',\tau)v - i\theta' k]. \end{aligned}$$

Here, with

$$\begin{aligned} \phi_T^h(\theta) &= \int_{-\infty}^\infty dkp(k)e^{i\theta k} \\ &= \frac{1}{2\pi}\int_{-\infty}^\infty d\theta' \exp[C(\theta',\tau)\bar{v} + D(\theta',\tau)v]\int_{-\infty}^\infty due^{i(\theta-\theta')k} \\ &= \int_{-\infty}^\infty d\theta' \exp[C(\theta',\tau)\bar{v} + D(\theta',\tau)v]\delta(\theta-\theta') \\ &= \exp[C(\theta,\tau)\bar{v} + D(\theta,\tau)v]. \end{aligned}$$

□

CHAPTER 5

Combination of Heston and Merton Models

One of the observations made in [4] is that the Heston model itself is not flexible enough to fully fit implied volatility surfaces given in practice. To overcome this difficulty, [4] combines the Merton jump and stochastic volatility models. The resulting model uses the following price dynamics:

$$dS_t = S_t dX_t, \tag{5.1}$$

where

$$\begin{aligned} Y_t &= \sum_{i=1}^{N_t} \left(e^{X_i} - 1\right), \\ dX_t &= \mu dt + \sqrt{v_t} dW_t + dY_t \\ dv_t &= -\lambda_h \left(v_t - \bar{\nu}\right) dt + \eta \sqrt{v_t} dW_t^h, \end{aligned}$$

(W_t, W_t^h) is a two dimensional Brownian motion with cross variation

$$\langle W, W^h \rangle = \rho t,$$

N_t is a Poisson process with jump rate λ and X_i is an iid sequence with distribution $N(\alpha, \delta^2)$. The components of the Y process (the X_i variables and the Poisson process N) are *independent of each other*. We note that this is the combination of (3.1), (3.2) and (4.2). The equation (5.1) is a linear equation and X is a semimartingale. Therefore, (5.1) has a unique solution S_t.

As it is always the case in risk neutral pricing of options, the μ parameter has no role and can be ignored. Therefore, the Merton-Heston model has the following seven parameters: λ_h,, $\bar{\nu}$, η, ρ, α, δ^2 and λ.

5.1 Risk neutral measure

Given that the Heston-Merton model is even more general than the Heston and Merton models, like them, it gives an incomplete market when the market consists of only the riskless security and S. Therefore, the Heston-Merton model does not have a unique martingale measure either. Parallel to the earlier chapters and to [4] we will use the simplest possible change of measure and only change the drift of X to 0 to make the resulting process a martingale. The X process has two sources of drift: the μ term and the drift, i.e., the compensator of Y. The latter was computed in Chapter 3, see (3.3):

$$\mathbb{E}[Y_t] = \omega t = \left(e^{\alpha+\frac{\delta^2}{2}} - 1\right) t \tag{5.2}$$

and

$$dX_t = (\mu + \omega)dt + \sqrt{v_t}dW_t - d\bar{Y}_t,$$

where

$$\bar{Y}_t = Y_t - \omega t.$$

Now, we rewrite dX_t as

$$dX_t = \sqrt{v_t}\left(\frac{\mu+\omega}{\sqrt{v_t}}dt + dW_t\right) - d\bar{Y}_t.$$

For X to be a martingale it suffices to change the measure to $\mathbb{P}^*$ under which

$$W_t^* = \int_0^t \frac{\mu+\sigma}{\sqrt{v_s}}ds + W_t$$

is a martingale. Such a change of measure exists, once again, by the Girsanov theorem. Parallel to earlier chapters, this is the measure that we will use in our price computations.

5.2 Formula for the price and the characteristic function

Similar to the Merton model, the price model (5.1) has jumps. Parallel to Chapter 3 one can use the Ito formula for semimartingale with jumps to derive the following representation for $\log(X_t)$; the proof is entirely parallel to that of Proposition 3.2 and is omitted.

Proposition 5.1. *The solution of* (5.1) *is given by*

$$S_t = S_0 e^{L_t^{hm}},$$

where

$$L_t^{hm} = -\omega t - \frac{1}{2}\int_0^t v_s ds + \int_0^t \sqrt{v_s}dW_s^* + J_t$$

$$J_t = \sum_{i=1}^{N_t} X_i,$$

and ω is as in (5.2).

As in the previous chapter we will use the characteristic function of $\log(S_t)$ to compute prices of European options. Without loss of generality, let us take $S_0 = 1$; then, by the previous proposition we have

$$\log(S_t) = L_t^{hm}.$$

Therefore, to compute prices of European options we only have to compute the characteristic function of L_t; this is done in the next proposition:

Proposition 5.2. *The characteristic function of L_t equals*

$$\phi_T^{hm}(\theta) = \mathbb{E}^*[e^{i\theta L_T^{hm}}] = \phi_T^h(\theta)\phi_T^J(\theta). \tag{5.3}$$

Proof. Note that by definition the processes J_t and

$$H_t = -\omega t - \frac{1}{2}\int_0^t v_s ds + \int_0^t \sqrt{v_s} dW_s$$

are independent of each other. Therefore,

$$\mathbb{E}^*[e^{i\theta L_T^{hm}}] = \mathbb{E}^*[e^{i\theta(H_T+J_T)}] = \mathbb{E}^*[e^{i\theta H_T}]\mathbb{E}^*[e^{i\theta J_T}] = \phi_T^h(\theta)\phi_T^J(\theta),$$

which proves (5.3). □

We notice that ϕ_T^{hm} was given in Proposition 4.3 of Chapter 4 and ϕ_T^j was computed in Proposition 3.4 of Chapter3. Now one can use the Fourier inversion theorem to get the density of L_T, which in turn can be employed to compute European option prices for the Heston-Merton model.

CHAPTER 6

Fitting to VIOP

The goal of this section is to fit the Heston-Merton model, reviewed in the previous chapter, to European option prices observed in VIOP. Our study looks at the first two months of 2017. Perhaps the most popular way of fitting volatility models to option data is via the loss function approach. In this approach, one defines a loss function (or a distance function) that measures the distance between observed option prices and those implied by the values of the model parameters. The fitting process minimizes this distance over the range of model parameters. The distance function we use for this chapter is of the following form:

$$L(\Theta) = \sum_{i=1}^{N} w(T_i, K_i)(C(T_i, K_i, \Theta) - C_M(T_i, K_i))^p, \tag{6.1}$$

where $p > 0$, N is the number of prices observed in the market, Θ is the vector of model parameters (in the case of the Heston-Merton model Θ consists of eight components, the seven are the model parameters λ_h,, $\bar{\nu}$, η, ρ, α, δ^2 and λ; the position of the unobserved v process is also taken as a parameter. Here, N is the number of prices observed in the market, $C_M(T_i, K_i)$ is the price observed in the market for the call option with strike K_i and maturity T_i, $C(T_i, K_i, \Theta)$ is the prices of the call option given by the model (put option prices are converted to call option prices via the put call parity) and $w(T_i, K_i) > 0$ is a weight that can be chosen to give different distance functions. A list of possible weight functions is given in [11, Chapter 6]; for the purposes of this thesis we will use $p = 2$ and w is implied by the vega of the option under the BS model.

6.1 Results of the fit

We will fit the Heston model to two sets of option prices: options of Garanti Bankasi and those of Koc Holding. The price data is available on BIST's website (`http://www.borsaistanbul.com/en/`) and for the interest rate we use the overnight rate of the Turkish Central Bank (available at `http://www.tcmb.gov.tr/`). The price movements of these two assets between January 2, 2017, and February 24, 2017 (two months, 40 trading days) are given in Figure 6.1. For fitting the model, we keep those options (1) for which there is an open interest, (2) which have at least 3 days

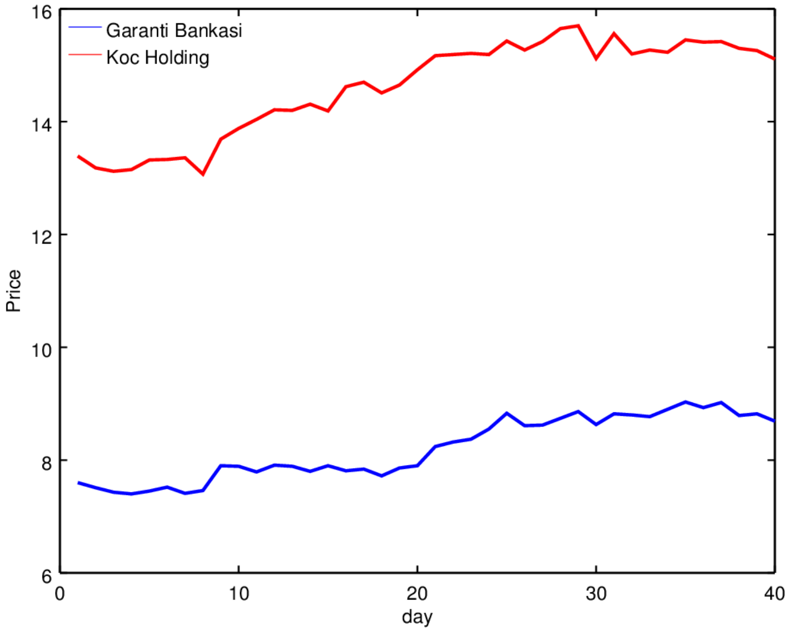

Figure 6.1: The prices of Garanti Bankasi and Koc Holding between January 2, 2017, and February 24, 2017

to their maturity, and (3) which have a valid implied volatility value. The number of options on each days satisfying these conditions are shown in Figure 6.2. Note that the resulting parameter fits for the eight model parameters are given in Figures 6.3 to 6.10. We have performed the fit using the `fminsearch` function of octave using at most 20 iterations per optimization.

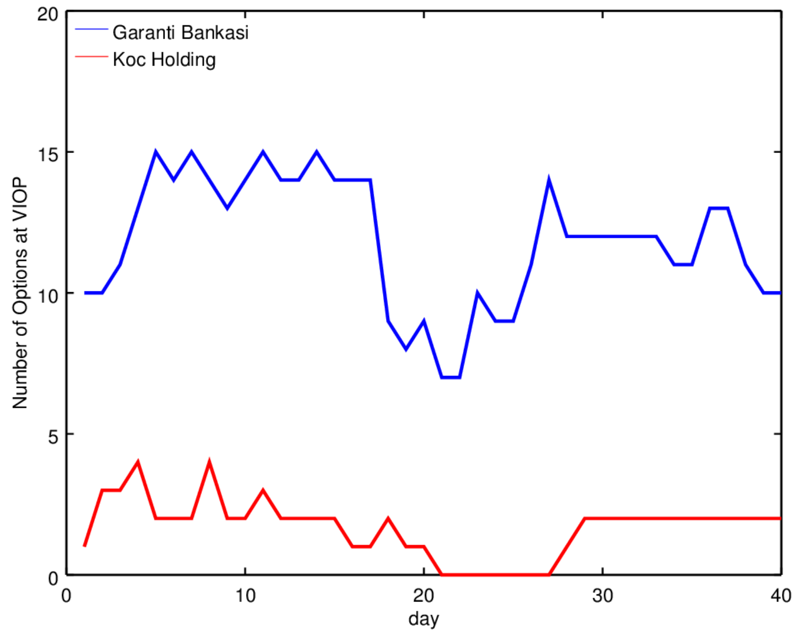

Figure 6.2: Number of European options written on Garanti and Koc traded on VIOP between January 2 2017, February 24, 2017

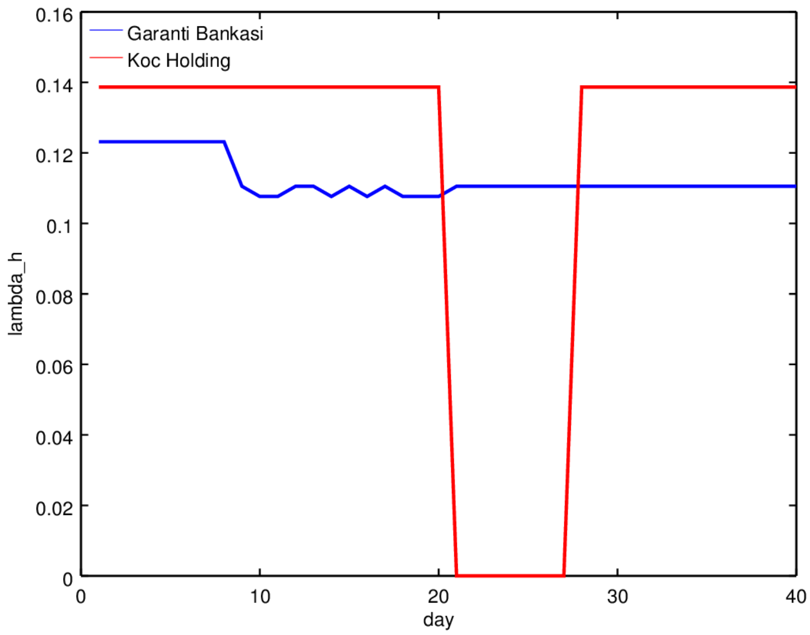

Figure 6.3: The estimated value of the λ_h parameter

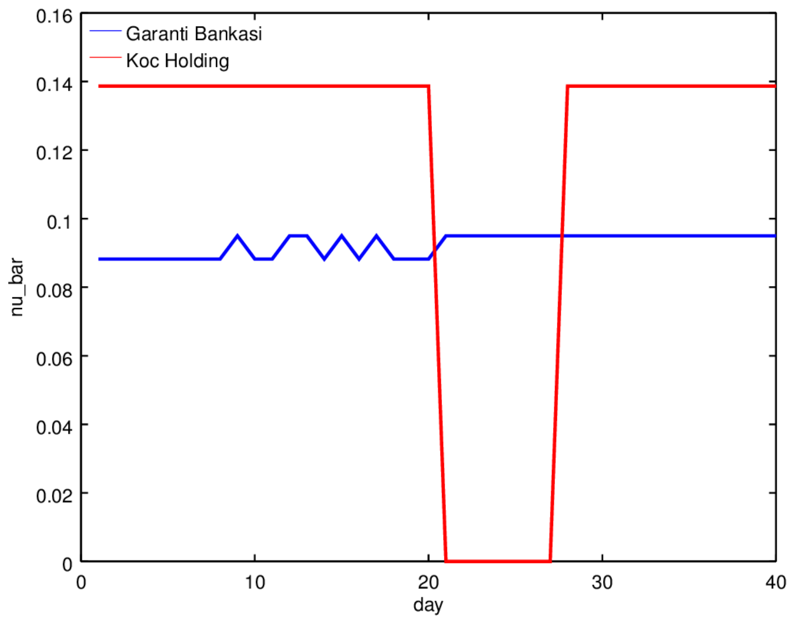

Figure 6.4: The estimated value of the $\bar{\nu}$ parameter

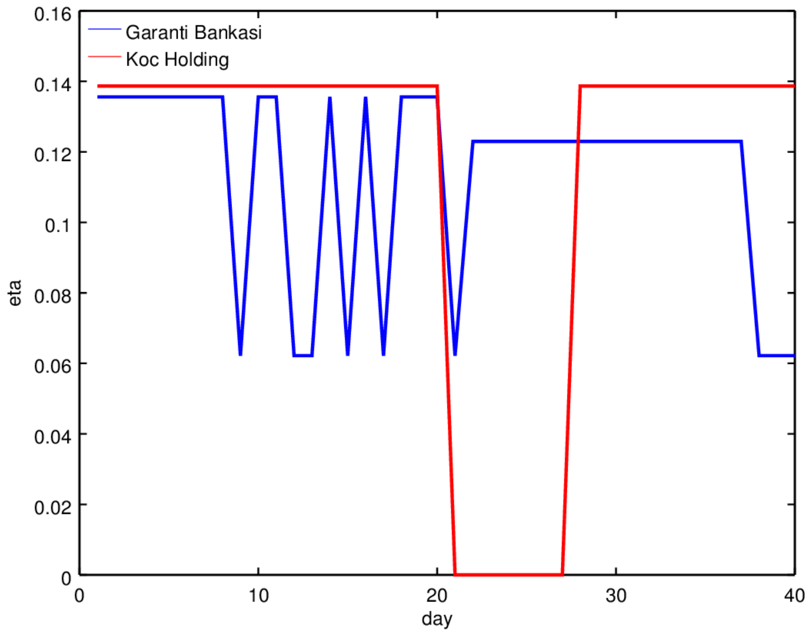

Figure 6.5: The estimated value of the η parameter

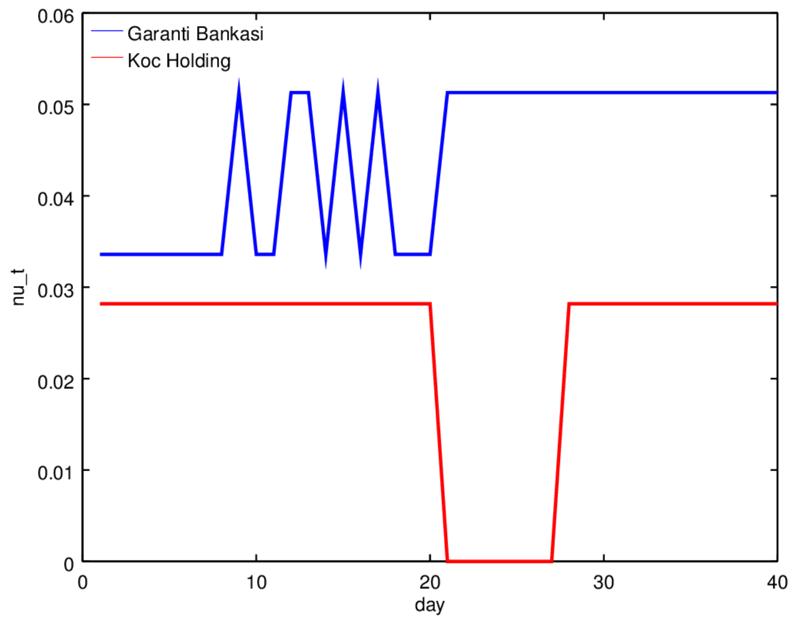

Figure 6.6: The estimated value of v_t

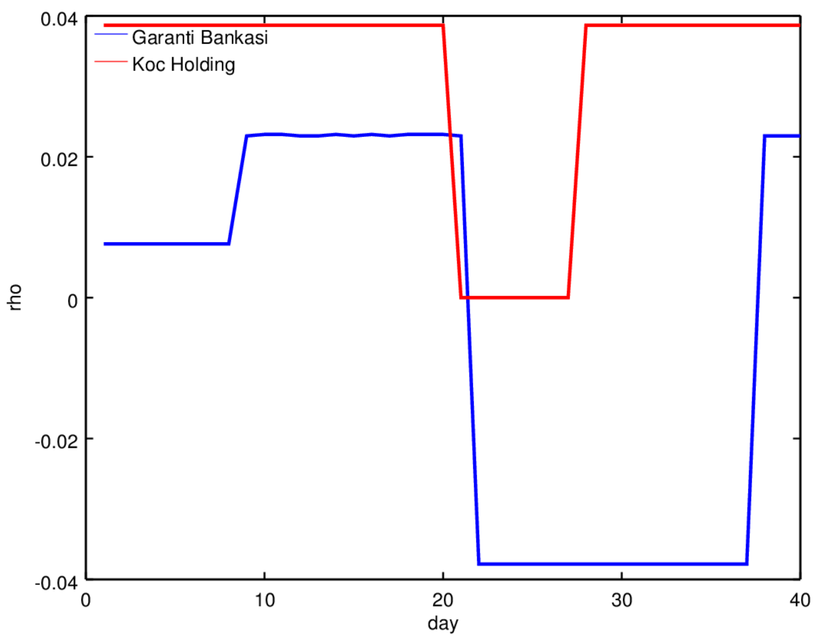

Figure 6.7: The estimated value of the ρ parameter

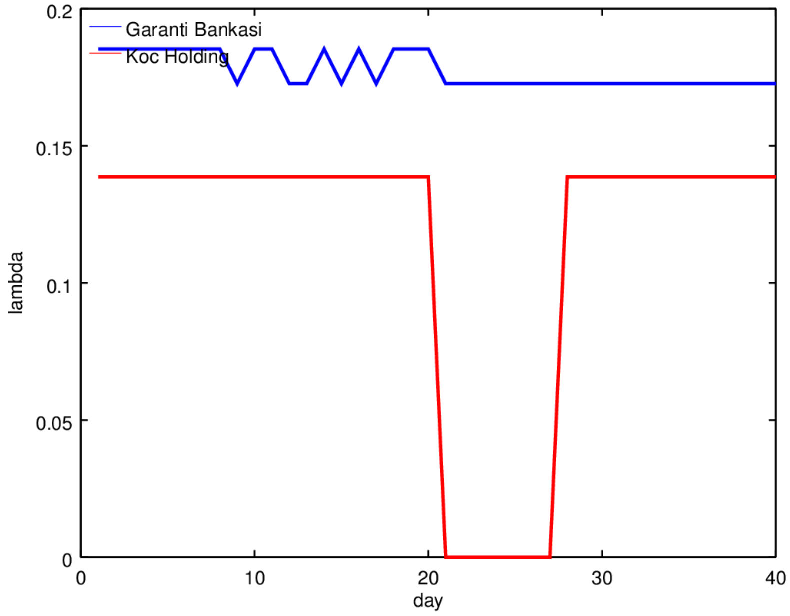

Figure 6.8: The estimated value of the λ parameter

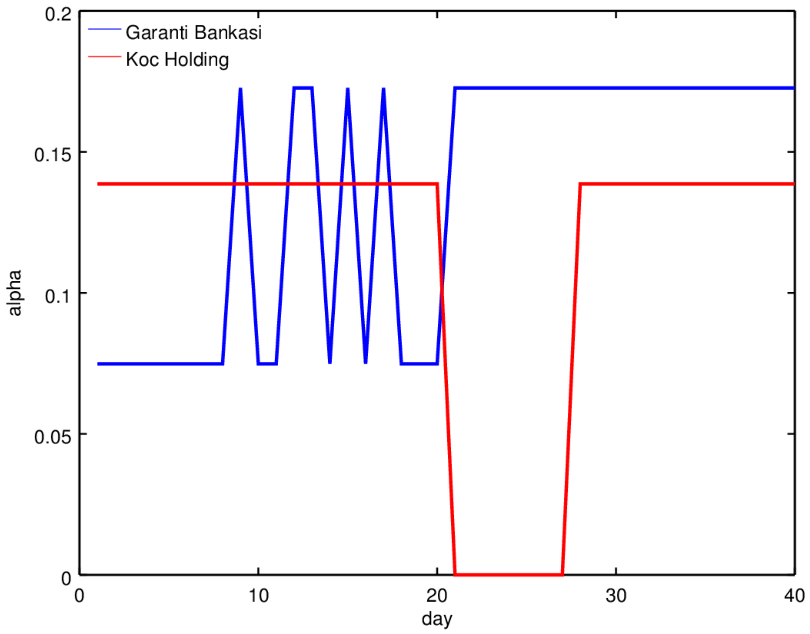

Figure 6.9: The estimated value of the α parameter

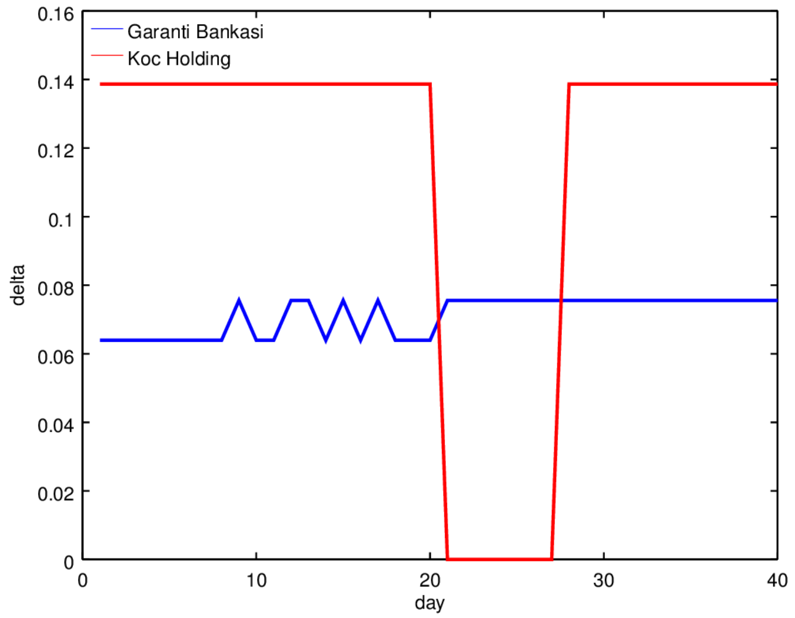

Figure 6.10: The estimated value of the δ parameter

Our observations regarding these results are as follows:

1. Garanti has more options available than Koc Holding. Number of options on Koc goes down to 0 between days 20 and 30, during which time interval the model could not be fit to Koc Holding. This small number of options is problematic, because the model has 8 parameters; fitting 8 parameters to 4 data points leads to over-fitting.

2. The paramater values for these two companies are different from each other. For example, the $\bar{\nu}$ parameter for Garanti is typically around 0.09, whereas it is near 0.14 for Koc.

3. The λ_h, $\bar{\nu}$, λ, δ and ρ paramaters for Garanti fluctuate little. The α and η parameters show a greater variation.

4. The parameter values for Koc holding fluctuate a little, perhaps because of the small number of option prices available for this asset.

5. The ρ parameter for both of the assets is very small across the 40 days covered in this study.

6. Both models have significant α, λ values indicating a nontrivial jump feature.

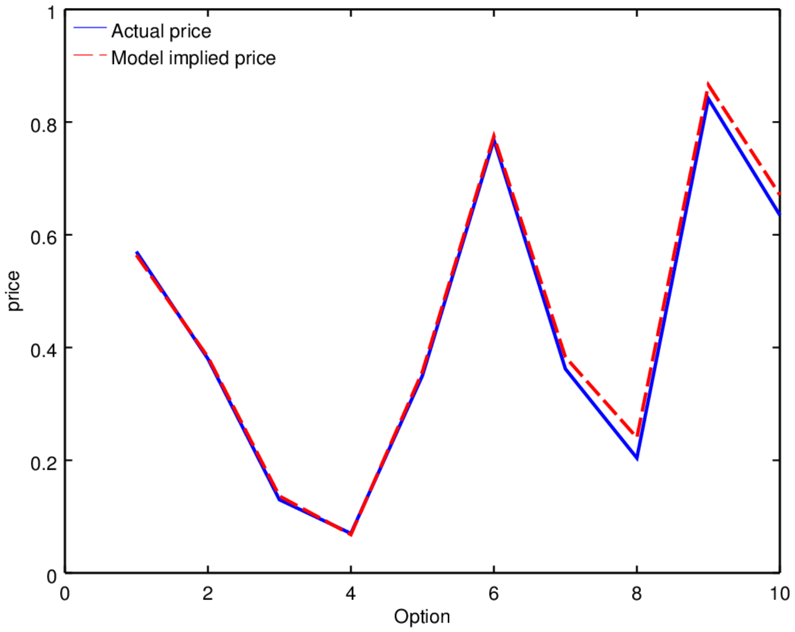

Figure 6.11: Actual and Model prices on February 24

6.2 Approximation performance of the fitted model

The loss function L of (6.1) used in the model-fitting measures the difference between model prices and the observed prices. Therefore, we begin by comparing these two quantities.

Figures 6.11 to 6.14 show the option prices observed in the market in three different days. In these figures, the maturity and strikes of the options are not explicitly given. This information for February 24. was already given in Chapter 1. The rest of the days have similar maturity and strike values. We see from these figures that, at the level of absolute prices, the fit performs rather well.

However, more important measures of closeness are relative error and the difference between implied volatilities. In terms of these measures, we see that the fits we have found perform not so well. As an example, we give the relative errors and the implied volatilities for 10 February 2017 are given in Figures 6.15 and 6.16.

The relative error for February 10 is around 5%; however, we see that for one of the options it goes up to 25%.

We note that the model IV curve is flatter, when compared to the actual IV curve; nonetheless, in terms of their absolute values, these two curves lie near one together.

Results for Koc are less interesting, because for most of the 40 days which study,

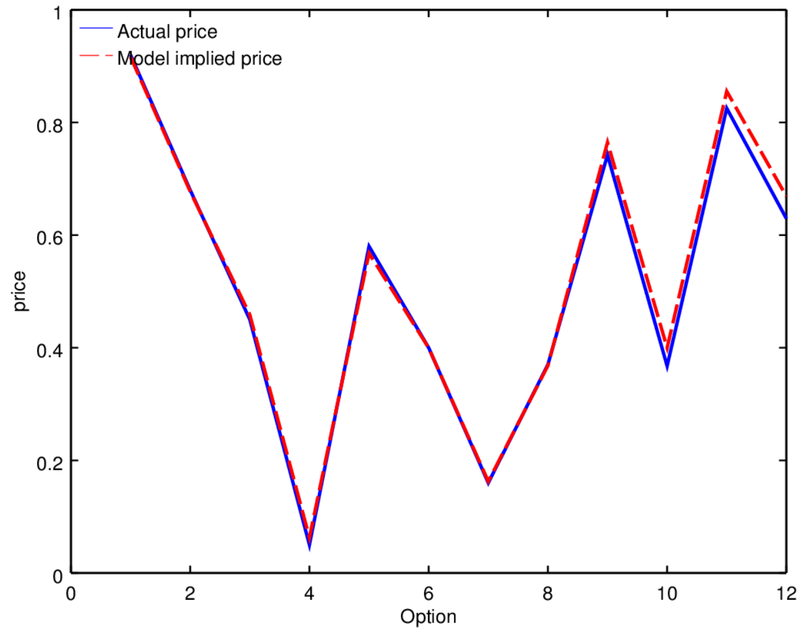

Figure 6.12: Actual and Model prices on February 10

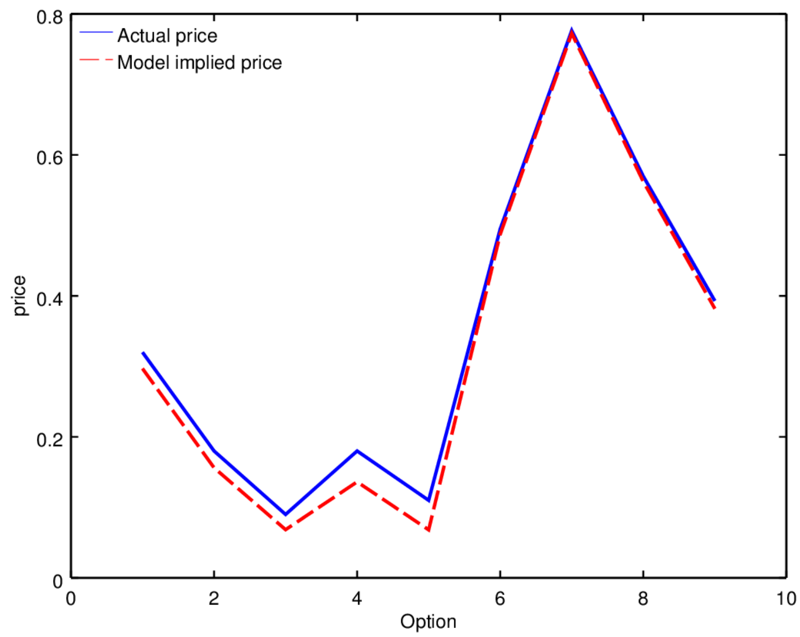

Figure 6.13: Actual and Model prices on January 27

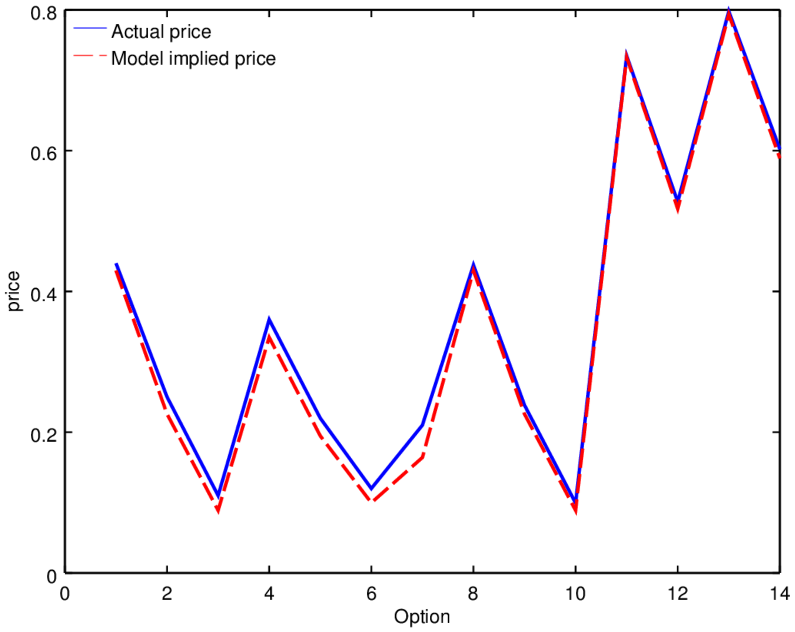

Figure 6.14: Actual and Model prices on January 13

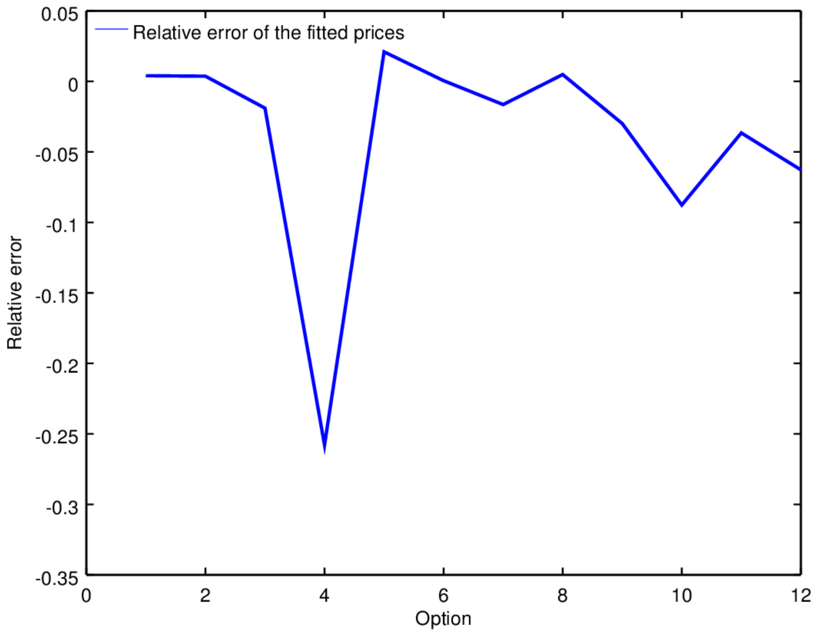

Figure 6.15: Relative error between model prices and actual prices, February 10

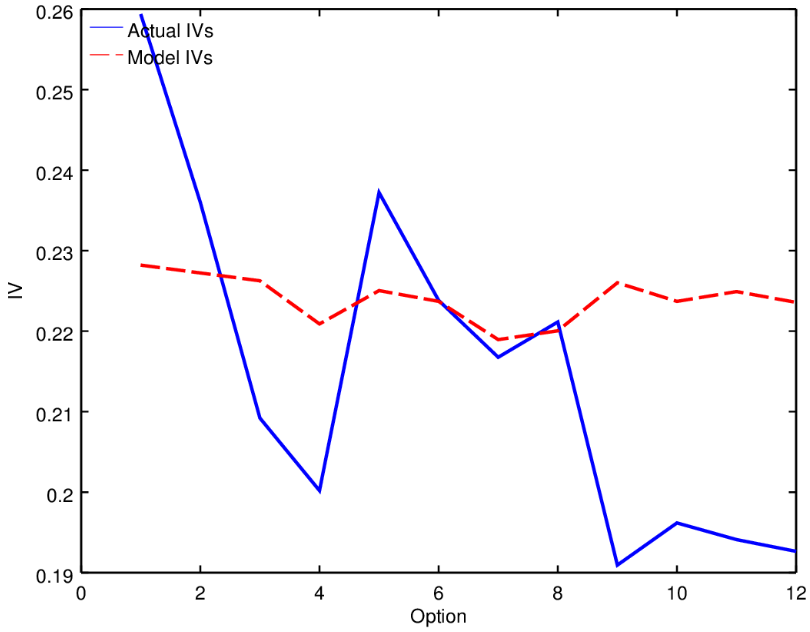

Figure 6.16: The market and model implied volatilities of Garanti Options, February 10

the number of options available for the fit is very small. Therefore, often, the prices, relative prices and the implied volatilities can match very well. An example is given in Figure 6.17, again for February 10. We observe that the implied volatilities observed in the market and those given by the model are close. But this not very interesting, because the model is fit only to two options.

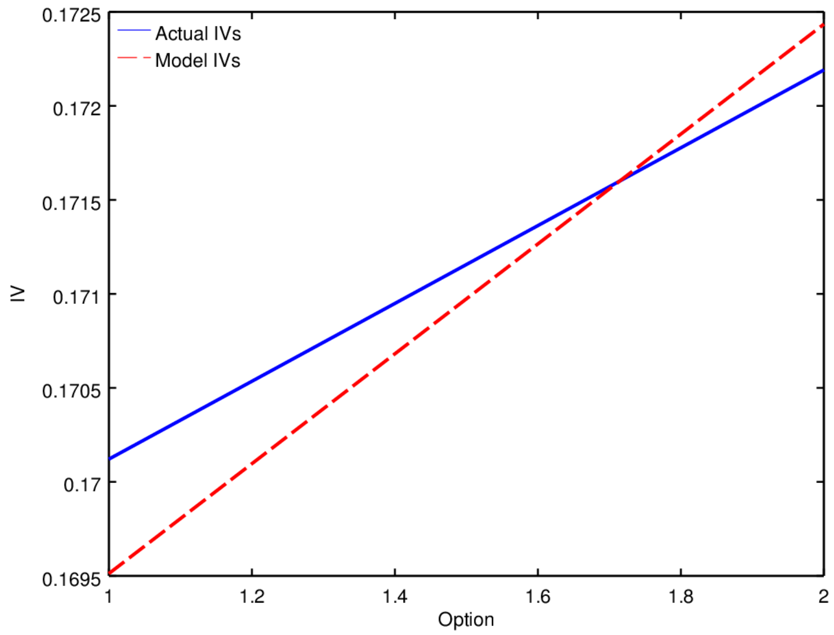

Figure 6.17: The market and model implied volatilities of Koc Holding Options, February 10

CHAPTER 7

Conclusion

In this thesis we have reviewed the Merton and Heston models and their combination for the pricing of European options. The Heston-Merton model does not have an explicit formula for its density, but its characteristic function is explicitly computable. We reviewed the derivation of this characteristic function. Once the characteristic function is available, one can use Fourier inversion to get the density. We based our pricing computations on this approach.

We have fitted the combined Heston-Merton model to several option prices observed in VIOP / BIST. We focused on the options of two companies: Garanti Bankasi and Koc Holding and on the first two months of 2017. For the fit, have minimized a loss function based on the L_2-distance between market prices and model prices. The fitting processes gave stable parameter values across the 40 day period we focused on. The fit was accurate in terms of absolute prices, but it was mediocre in terms of relative error and implied volatilities.

In future research we hope to do the following:

1. Using a loss function based on implied volatilities; the difficulty with this is that computation of implied volatilities requires a lot of computation. For this reason, this may need the implementation of our code in C or some other language with a compiler producing machine level code.
2. Studying the hedging performance of the Heston-Merton model.
3. Extending our study to longer periods and more options traded in VIOP.

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