NEW MODELING AND ANALYSIS METHODS FOR MICRO-PLATES

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Submitted by **REZA AGHAZADEH** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy in Mechanical Engineering Department, Middle East Technical University** by,

Prof. Dr. Gülbin Dural Ünver _____
Dean, Graduate School of **Natural and Applied Sciences**

Prof. Dr. Raif Tuna Balkan _____
Head of Department, **Mechanical Engineering**

Prof. Dr. Serkan Dağ _____
Supervisor, **Mechanical Engineering Dept., METU**

Assoc. Prof. Dr. Ender Ciğeroğlu _____
Co-Supervisor, **Mechanical Engineering Dept., METU**

Examining Committee Members:

Prof. Dr. Hakan I. Tarman _____
Mechanical Engineering Dept., METU

Prof. Dr. Serkan Dağ _____
Mechanical Engineering Dept., METU

Assist. Prof. Ulaş Yaman _____
Mechanical Engineering Dept., METU

Assist. Prof. Dr. Can U. Doğruer _____
Mechanical Engineering Dept., Hacettepe University

Assoc. Prof. Dr. Recep Güneş _____
Mechanical Engineering Dept., Erciyes University

Date: _____

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Last name: REZA AGHAZADEH

Signature:

ABSTRACT

NEW MODELING AND ANALYSIS METHODS FOR MICRO-PLATES

Aghazadeh, Reza
Ph.D., Department of Mechanical Engineering
Supervisor: Prof. Dr. Serkan Dağ
Co-Supervisor: Assoc. Prof. Dr. Ender Ciğeroğlu

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This study presents strain gradient elasticity based procedures for static bending, free vibration and buckling analyses of functionally graded rectangular micro-plates subjected to mechanical and thermal loadings. Mathematically the non-classical modified couple stress and classical elasticity theories are the two special cases of the new model. The methods developed allow taking into account spatial variations of length scale parameters of strain gradient elasticity and modified couple stress theory. Governing partial differential equations and boundary conditions are derived by following variational approaches and applying Hamilton's principle. Displacement field is expressed in a unified way to produce numerical results in accordance with Kirchhoff, Mindlin, and third order shear deformation theories. All material properties, including the length scale parameters, are assumed to be functions of the plate thickness coordinate in the derivations. Developed equations are solved numerically by means of differential quadrature method. Proposed procedures are verified through comparisons made to the results available in the literature for certain limiting cases. Further numerical results are provided to illustrate the effects of material and geometric parameters upon static deflection, vibration frequency, and critical buckling load. Presented numerical results clearly illustrate size effect at micro-scale, impact of

length scale parameter variations and influence of initial thermal stresses upon mechanical behavior of functionally graded rectangular micro-plates.

Keywords: functionally graded micro-plates; strain gradient elasticity; modified couple stress theory; length scale parameters; thermal stresses; bending; free vibrations; buckling

ÖZ

MİKRO-PLAKLAR İÇİN YENİ MODELLEME VE ANALİZ TEKNİKLERİ

Aghazadeh, Reza
Doktora, Makina Mühendisliği Bölümü
Tez Yöneticisi: Prof. Dr Serkan Dağ
Eş Tez Yöneticisi: Doç. Dr. Ender Ciğeroğlu

Eylül 2017, 99 sayfa

Bu araştırma mekanik veya termal yükleme altındaki fonksiyonel derecelendirilmiş mikro-plakların statik eğilme, serbest titreşim ve burkulma analizleri için gerinim gradyanı elastisitesine dayalı yeni yöntemler ortaya koymaktadır. Matematiksel olarak, klasik olmayan modifiye edilmiş kuvvet çifti gerilmesi ve klasik elastisiste teorileri, yeni modelin özel halleridir. Geliştirilen yöntemler, gerinim gradyanı ve modifiye edilmiş kuvve çifti gerilmesi teorilerinin uzunluk ölçeği parametrelerindeki uzaysal değişimlerinin hesaba katılmasını mümkün kılmaktadır. Yönetici kısmı diferansiyel denklemler ve sınır koşulları varyasyonel yöntemler ve Hamilton prensibi uygulanarak türetilmiştir. Yerdeğiştirme alanı, Kirchhoff, Mindlin ve üçüncü derece kayma deformasyon teorilerine göre sayısal sonuçlar üretmek için birleşik bir şekilde ifade edilmiştir. Türetmede uzunluk ölçeği parametreleri de dahil olmak üzere tüm malzeme özellikleri plağın kalınlık koordinatının fonksiyonları olarak varsayılmıştır. Geliştirilen denklemler sayısal olarak, diferansiyel kare yapma (quadrature) metodu ile çözülmüştür. Önerilen yöntemler, bazı kısıtlayıcı durumlar için literatürde mevcut sonuçlar ile karşılaştırmalar yapılarak doğrulanmıştır. Malzeme ve geometrik parametrelerin statik yerdeğiştirme, titreşim frekansı, ve kritik burkulma yükü üzerinde etkilerini göstermek için ayrıntılı sayısal sonuçlar

verilmiştir. Sunulan sayısal sonuçlar, mikro ölçekte boyutun, uzunluk ölçeği parametre değişiminin ve başlangıç termal gerilmelerin fonksiyonel derecelendirilmiş dikdörtgen mikro-plakların mekanik davranışına olan etkilerini açıkça göstermektedir.

Anahtar kelimeler: fonksiyonel derecelendirilmiş mikro-plaklar; gerinim gradyanı elastisitesi; modifiye edilmiş kuvve çifti gerilmesi teorisi; uzunluk ölçeği parametreleri; termal gerilmeler; eğilme; serbest titreşim; burkulma

To My Family

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A	Area of mid-plane of micro-plate
a	Length of micro-plate
b	Width of micro-plate
c	Ceramic phase index
E	Young's modulus
e_{ijk}	Alternating tensor
f	Shape function for plate theories
h	Thickness of micro-plate
K	Kinetic energy
k_s	Shear correction factor
l_0, l_1, l_2	Material length scale parameters
m	Metallic phase index
m_{ij}^s	Higher order stress, work-conjugate to χ_{ij}^s
n	Volume fraction exponent
n_{x_1}, n_{x_2}	Direction cosines of unit normal of the boundary
N_{x_1}, N_{x_2}	Number of grid points in x_1, x_2 directions
$M_{pq}^{\ i}$, P_p^i	Stress resultants associated with σ_{ij}, p_i
P_{x_1}, P_{x_2}	In-plane buckling loads
$P_{x_1}^0$, $P_{x_2}^0$, $P_{x_1x_2}^0$	Thermally induced initial in-plane forces
P	Critical buckling load
p_i	Higher order stress, work-conjugate to γ_i
q	Distributed load
$T_{pqr}^{\ i}$	Stress resultants associated with $\tau_{ijk}^{(1)}$
T_0	Stress-free state temperature

U	Strain energy
$u_1,\ u_2,\ u_3$	Displacements along $x_1,\ x_2,\ x_3$ directions
u	Displacement of mid-plane along x_1 direction
V	Volume fraction
v	Displacement of mid-plane along x_2 direction
W	Work done by external forces
w	Displacement of mid-plane along x_3 direction
Y^{i}_{pq}	Stress resultant associated with m^{s}_{ij}
α	Coefficient of thermal expansion
β	Length scale parameter ratio
Γ	Boundary curve enclosing mid-plane of micro-plate
γ_i	Dilatation gradient vector
ΔT	Temperature change from T_0
ΔT_{cr}	Critical buckling temperature difference
δ_{ij}	Kronecker delta
ε_{ij}	Strain tensor
$\eta^{(1)}_{ijk}$	Deviatoric stretch gradient tensor
$\theta_1,\ \theta_2$	Transverse shear strains of any point on the mid-plane
μ	Shear modulus
ν	Poisson's ratio
ρ	Mass density
σ_{ij}	Cauchy stress tensor
$\tau^{(1)}_{ijk}$	Higher order stress tensor, work-conjugate to $\eta^{(1)}_{ijk}$
$\phi_1,\ \phi_2$	Rotations of the transverse normal about $x_2,\ x_1$
χ^{s}_{ij}	Symmetric curvature tensor
Ω	Volume
ω	Natural frequency

LIST OF ABBREVIATIONS

CCCC	All edges clamped
CT	Classical (elasticity) theory
DQM	Differential quadrature method
FEM	Finite element method
FGM	Functionally graded material
KPT	Kirchhoff plate theory
MCST	Modified couple stress theory
MEMS	Micro-electro-mechanical system
MPT	Mindlin plate theory
SFSM	Spline finite strip method
SGT	Strain gradient theory
SSSS	All edges simply supported
TSDT	Third order shear deformation theory

CHAPTER 1

INTRODUCTION

1.1 Introduction

Small scale structures such as micron-sized beams and plates are commonly used in micro-electro-mechanical system (MEMS) devices such as micro-sensors, micro-actuators, and micro-resonators. For an accurate and comprehensive design of MEMS devices, the mechanical features of micro-structures should be examined. For example, in a micro-mechanical gyroscope to avoid bias there is need to study dynamic behavior and estimate the resonant frequency of sensitive element [1, 2]. As other examples of technological applications requiring thorough understanding of design considerations in small-scale structures, one can mention nano-plate resonator [3] and micro-valves in micro-fluidic applications [4]. It is experimentally observed that, micro-scale structures exhibit size-dependent mechanical behavior [5-7]. Traditional continuum theories fail to predict the size effect in small-scale structures due to lack of a length scale parameter. Various higher-order continuum theories have been proposed to address the size-dependency. These theories employ one or more intrinsic length scale parameters. Among examples of such theoretical frameworks, we can mention nonlocal elasticity [8], surface elasticity [9], strain gradient theories [6, 10] and couple stress theories [11-13]. Strain gradient theory (SGT) introduced by Lam et al. [6] and modified couple stress theory (MCST) proposed by Yang et al. [13] are the two most commonly used higher order continuum theories in investigations involving small-scale structures. Strain gradient theory is derived by taking into account the second order deformation gradient

beside the classical first order deformation gradient, resulting in three material length scale parameters in constitutive relations. Yang et al. [13] incorporated the concept of moment of couples into classical couple stress theory and put forward modified couple stress theory, which employs a single length scale parameter.

The main objective of this study is to put forward new methods for the analysis of functionally graded micro-plates that are under the effect of mechanical or thermal loading. As the higher order continuum theory, strain gradient theory is used in the derivation of system of governing equations and boundary conditions. Shear deformation plate theories are employed, so that, it is feasible to take into account the distribution of shear stress through the thickness of plates. The study is undertaken to be able to develop new analysis methods that can take into account thermal effects and the spatial variations in the length scale parameters of functionally graded materials.

1.2 Literature Survey

Using modified couple stress and strain gradient theories, researchers have developed various models to investigate behavior of homogeneous micro-plates undergoing static bending, free vibrations, and buckling. Akgöz and Civalek [14], Asghari [15], Jomehzadeh et al. [16], Tsiatas [17], Yin et al. [18], Farokhi and Ghayesh [19], Şimşek et al. [20], Zhong et al. [21], Wang et al. [22] adopted Kirchhoff plate model and analyzed mechanical behavior of homogeneous micro-plates in accordance with modified couple stress theory. Note that in Kirchhoff plate model the effects of transverse shear deformation is neglected. In a number of studies, Mindlin plate model, i.e. first order shear deformation theory, and modified couple stress theory are utilized to examine structural mechanics problems of small-scale plates [23, 24]. Farokhi and Ghayesh [25] used third-order shear deformation theory in conjunction with modified couple stress theory. Lazopoulos [26], Ramezani [27], Wang et al. [28], Akgöz and Civalek [29], Ansari et al. [30], Ramezani [31] adopted strain gradient theory to capture the size effect in homogeneous micro-plates.

It should be noted that the afore-cited studies are carried out for micro-plates made of homogeneous materials. Functionally graded materials (FGMs) are inhomogeneous composites which are processed by combining the best properties of two distinct phases. FGMs possess smooth spatial variation in the volume fractions of constituents. The gradual variation of composition across the volume of FGMs prevents high stress concentrations and makes FGMs ideal to be used in harsh working conditions such as high-temperature environment. Although FGMs were initially developed as thermal barrier materials in aerospace structures [32, 33], nowadays they have found widespread applications from their use in high temperature environment [34] to electronics [35] and biomedical industry [36, 37]. In recent years, incorporation of functionally graded materials (FGMs) into small-scale structures has become feasible with advances in manufacturing technologies such as magnetron sputtering [38], chemical vapor deposition [39], and modified soft lithography [40]. As a result, structural problems involving functionally graded micro-beams and micro-plates have attracted researchers' attention. In a number of studies modified couple stress theory in accordance with Kirchhoff plate model is used to examine behavior of functionally graded small-scale structures [41-44]. In research work conducted by Ke et al. [45], Thai and Choi [42], Lou and He [44], Noori and Jomehzadeh [43] and Mahmoud and Shaat [46], modified couple stress theory is used in conjunction with Mindlin plate model to examine the problems regarding FGM micro-plates. Examples of studies based on modified couple stress theory and third order shear deformation plate model include the articles by Kim and Reddy [47], Reddy and Kim [48], Kim and Reddy [49] and Thai and Kim [50]. Li and Pan [51] and Thai and Vo [52] put forward functionally graded micro-plate models based on modified couple stress theory and sinusoidal shear deformation plate model. In a study by Mohseni et al. [53] modified couple stress theory along with higher-order shear and normal deformable plate theory is used to assess static bending behavior of functionally graded rectangular micro-plates. Salehipour et al. [54] developed a model based on modified couple stress and three-dimensional elasticity theories to analyze free vibrations of functionally graded micro-plates. Strain gradient theory is also utilized to examine behavior of functionally graded

small-scale structures. Farahmand et al. [55] employed strain gradient theory and Kirchhoff plate model for free vibration analysis of functionally graded micro-plates. In research work conducted by Sahmani and Ansari [56] and Mohammadimehr et al. [57], strain gradient theory is used in conjunction with third order shear deformation plate model to solve problems regarding FGM micro-plates. Examples of studies based on strain gradient theory and first order shear deformation theory include the articles by Ansari et al. [58], Gholami and Ansari [59], and Shenas and Malekzadeh [60].

In all studies mentioned in the foregoing paragraph, length scale parameters of functionally graded micro-plates are assumed to be constants. However, this is a strictly simplifying assumption since the length scale parameter is itself a material property [61-64]; and similar to the other material properties of a functionally graded medium it should vary as a function of spatial coordinates. For examlpe, , in strain gradient theory, the three length scale parameters are defined in terms of shear modulus and material parameters associated with higher-order deformation measures. In modified couple stress theory, the length scale parameter is defined as the square root of the ratio of modulus of curvature to shear modulus [61, 64]. Both modulus of curvature and shear modulus are material properties indicating that the length scale parameter is itself a material property. As an another example implying the length scale parameter to be material constant, we can mention polymers for which the material length scale parameter depends on chain stiffness, chain interactions and cross-link density which are micro-structural properties of constituent [63]. Thus, for a functionally graded micro-structure, all of the length scale parameters are themselves material properties, whose spatial variations need to be represented by suitable functions that depend on the coordinates.

There are several studies in the literature that account for the spatial variation of the length scale parameter. Kahrobaiyan et al. [65] and Aghazadeh et al. [66] incorporated through-the-thickness variation of the length scale parameter into the analysis of functionally graded micro-beams. Eshraghi et al. [67], Eshraghi et al. [68] solved problems involving micro-scale FGM annular plates by considering the variation of length scale parameter. Alipour Ghassabi et al. [69] applied nonlocal

elasticity to examine free vibrations of rectangular nano-plates having a spatially variable nonlocal parameter. However, in the technical literature, there are no strain gradient theory based studies that take into account smooth spatial variations of the three length scale parameters of micro-scale functionally graded rectangular plates. Note that developments presented in Aghazadeh et al. [66] and Kahrobaiyan et al. [65] are applicable for beams, those given in Eshraghi et al. [67] and Eshraghi et al. [68] are valid for annular plates and those described in Alipour Ghassabi et al. [69] are derived in accordance with nonlocal elasticity. Analysis of rectangular FGM micro-plates by means of strain gradient theory requires derivation and solution of completely different partial differential equations compared to those considered in these articles. One of the main objectives in the present study is to put forward *strain gradient theory* based bending, free vibrations and buckling solutions for functionally graded rectangular micro-plates, that possess *spatially variable length scale parameters*.

The micro-structural elements usually operate under thermomechanical conditions, leading to thermal stresses developed in these structures. The bending, vibrational and buckling characteristics of micro-structures are very sensitive to induced thermal stresses. FGM micro-structures can be idealized to have a high performance in thermal environments and, therefore, they have found increasing applications in MEMS as cooling unit, thermal barrier and other heat transfer devices. Therefore, it is essential to account for thermal effects in modeling and analyzing functionally graded micro-beams and micro-plates. Although there are studies regarding the thermal analysis of FGM plates by employing classical elasticity theory in the literature [70-73], there is not sufficient effort to address mechanical problems of micro-plates undergoing thermal loads in technical literature. Mirsalehi et al. [74] developed a modified couple stress based model to investigate stability of functionally graded micro-plate. Based on modified couple stress theory, Reddy and Kim [48] put forward a third order shear deformation thermally loaded micro-plate model. In their work no numerical results are presented. Eshraghi et al. [68] used modified couple stress theory in conjunction with unified plate model to treat mechanical problems of functionally graded annular and

circular micro-plates in thermal environment. Ansari et al. [58] investigated the effects of boundary conditions, size, and volume fraction exponent of functionally graded micro-plates on the temperature difference required for buckling. Shenas and Malekzadeh [60] and Ghorbani Shenas and Malekzadeh [75] presented a strain gradient based formulation for free vibration analysis of micro-scale functionally graded plates in thermal environment possessing quadrilateral and isosceles triangular shapes, respectively.

1.3 Motivation and Scope of Study

The main objective in this study on one hand is to put forward a general plate model capable of capturing size effect in small scale FGM plates with variable length scale parameters; on the other hand is to investigate the effects of temperature change on mechanical behavior of micro-plates. The study is organized as follows:

In CHAPTER 2, governing partial differential equations and associated boundary conditions for bending, free vibrations, and buckling of rectangular FGM micro-plates are derived in accordance with strain gradient theory. Hamilton's principle is utilized in derivations. All material properties, including the three length scale parameters of strain gradient elasticity, are assumed to be functions of the thickness coordinate. Displacement field of the rectangular micro-plate is expressed in a unified way to be able to produce numerical results corresponding to three different plate theories, which are Kirchhoff, Mindlin, and third order shear deformation theories.

In CHAPTER 3, on the basis of differential quadrature method (DQM), a solution procedure is developed to solve equation system comprising partial differential equations and boundary conditions. In order to produce numerical results, MATLAB software is utilized to implement developed numerical technique.

In CHAPTER 4, parametric analyses and related numerical results are presented. Developed procedures are verified through comparisons made with the results available for limiting cases in the literature. In analysis of free vibrations under thermal conditions, thermally induced initial displacements and thermal stresses are

also computed. Presented numerical results illustrate influences of length scale parameter variation, geometric and material parameters, and temperature change upon static deflections, vibration frequencies, and critical buckling loads of functionally graded rectangular micro-plates.

Finally, in CHAPTER 5, a conclusion is given and future work is discussed

CHAPTER 2

FORMULATION

2.1 Shear Deformation Plate Theories

Figure 1 depicts a functionally graded rectangular micro-plate having a thickness h. Mid-plane of the undeformed plate is coincident with $x_1 - x_2$ plane. Deformed shape of the mid-plane in $x_1 - x_3$ plane is also shown in Figure 1. Displacements of any point at time t along x_1, x_2 and x_3 directions are denoted by u_1, u_2 and u_3, respectively; and can be expressed in a unified form as given below:

$$u_1(x_1, x_2, x_3, t) = u(x_1, x_2, t) - x_3 w_{,x_1} + f(x_3)\theta_1(x_1, x_2, t), \tag{1.1}$$

$$u_2(x_1, x_2, x_3, t) = v(x_1, x_2, t) - x_3 w_{,x_2} + f(x_3)\theta_2(x_1, x_2, t), \tag{1.2}$$

$$u_3(x_1, x_2, x_3, t) = w(x_1, x_2, t), \tag{1.3}$$

Where u, v and w are displacements of the mid-plane along x_1, x_2 and x_3, respectively; θ_1 and θ_2 are transverse shear strains of any point on the mid-plane due to bending in $x_1 - x_3$ and $x_2 - x_3$ planes; and a comma stands for differentiation. Transverse shear strains θ_1 and θ_2 are written in terms of rotations ϕ_1 and ϕ_2 of the transverse normal at $x_3 = 0$ about x_2 and x_1 axes as follows:

$$\theta_1(x_1, x_2, t) = w_{,x_1}(x_1, x_2, t) + \phi_1(x_1, x_2, t), \tag{2.1}$$

$$\theta_2(x_1, x_2, t) = w_{,x_2}(x_1, x_2, t) + \phi_2(x_1, x_2, t). \tag{2.2}$$

Shape function f in Eq. (1) controls through-the-thickness distributions of transverse shear strain and stress. In the present study, we produce numerical results for three different plate theories, namely Kirchhoff plate theory (KPT), Mindlin plate theory (MPT), and third order shear deformation theory (TSDT). f-functions corresponding to these theories are given by

$$f\left(x_3\right)=\begin{cases} 0, & \text{for KPT,} \\ x_3, & \text{for MPT,} \\ x_3\left(1-\dfrac{4x_3^2}{3h^2}\right), & \text{for TSDT.} \end{cases} \tag{3}$$

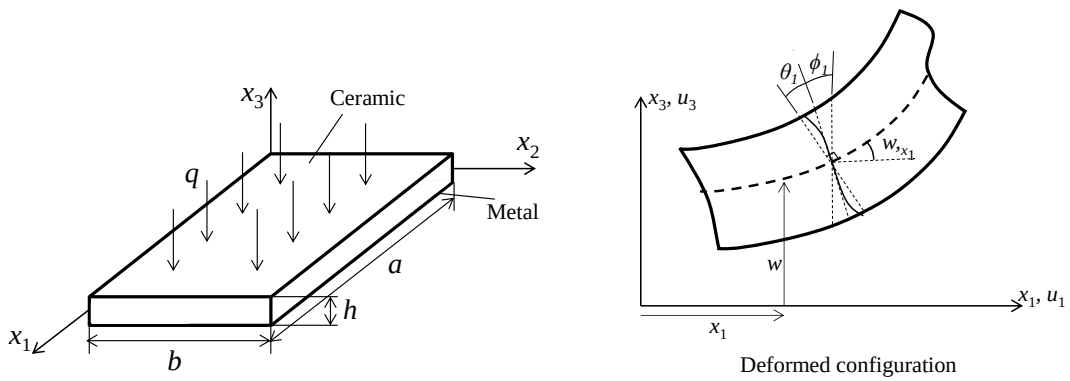

Figure 1. Functionally graded rectangular micro-plate configuration and deformed shape.

Note that in Kirchhoff plate theory transverse shear strain is assumed to be zero. Mindlin plate theory presumes constant transverse shear on the cross section. In third order shear deformation theory, transverse shear has a parabolic distribution.

2.2 Strain Gradient Theory

According to strain gradient theory (SGT), strain energy of the micro-plate is written as:

$$U = \frac{1}{2}\int_{\Omega}\left(\sigma_{ij}\left(\varepsilon_{ij} - \alpha\Delta T\delta_{ij}\right) + p_i\gamma_i + \tau_{ijk}^{(1)}\eta_{ijk}^{(1)} + m_{ij}^{s}\chi_{ij}^{s}\right)dV, \tag{4}$$

δ_{ij} in Eq. (4) is the Kronecker delta; σ_{ij} is Cauchy stress; ε_{ij} is strain; p_i, $\tau_{ijk}^{(1)}$, m_{ij}^{s} are higher order stress tensors; γ_i denotes dilatation gradient vector; $\eta_{ijk}^{(1)}$ represents deviatoric stretch gradient tensor; χ_{ij}^{s} is symmetric curvature tensor; ΔT is temperature change from stress-free state temperature T_0, α is the coefficient of thermal expansion; and Ω is volume. Deformation measures in Eq. (4) are defined by

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \tag{5.1}$$

$$\gamma_i = \varepsilon_{mm,i}, \tag{5.2}$$

$$\eta_{ijk}^{(1)} = \frac{1}{3}\left(\varepsilon_{jk,i} + \varepsilon_{ki,j} + \varepsilon_{ij,k}\right) - \frac{1}{15}\delta_{ij}\left(\varepsilon_{mm,k} + 2\varepsilon_{mk,m}\right) - \frac{1}{15}\left\{\delta_{jk}\left(\varepsilon_{mm,i} + 2\varepsilon_{mi,m}\right) + \delta_{ki}\left(\varepsilon_{mm,j} + 2\varepsilon_{mj,m}\right)\right\}, \tag{5.3}$$

$$\chi_{ij}^{s} = \frac{1}{2}\left(e_{ipq}\varepsilon_{qj,p} + e_{jpq}\varepsilon_{qi,p}\right). \tag{5.4}$$

e_{ijk} here designates alternating tensor. By substituting displacement field given in Eq. (1) into Eq. (5), ε_{ij}, γ_i, $\eta_{ijk}^{(1)}$ and χ_{ij}^{s} are derived as:

$$\varepsilon_{11} = \frac{\partial u}{\partial x_1} - x_3\frac{\partial^2 w}{\partial x_1^2} + f\frac{\partial\theta_1}{\partial x_1}, \tag{6.1}$$

$$\varepsilon_{22} = \frac{\partial v}{\partial x_2} - x_3\frac{\partial^2 w}{\partial x_2^2} + f\frac{\partial\theta_2}{\partial x_2}, \tag{6.2}$$

$$\varepsilon_{12} = \varepsilon_{21} = \frac{1}{2}\frac{\partial u}{\partial x_2} + \frac{1}{2}\frac{\partial v}{\partial x_1} - x_3\frac{\partial^2 w}{\partial x_1\partial x_2} + \frac{1}{2}f\frac{\partial\theta_1}{\partial x_2} + \frac{1}{2}f\frac{\partial\theta_2}{\partial x_1}, \tag{6.3}$$

$$\varepsilon_{13} = \varepsilon_{31} = \frac{1}{2} f'\theta_1, \tag{6.4}$$

$$\varepsilon_{23} = \varepsilon_{32} = \frac{1}{2} f'\theta_2, \tag{6.5}$$

$$\gamma_1 = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 v}{\partial x_1 \partial x_2} - x_3 \frac{\partial^3 w}{\partial x_1^3} - x_3 \frac{\partial^3 w}{\partial x_1 \partial x_2^2} + f \frac{\partial^2 \theta_1}{\partial x_1^2} + f \frac{\partial^2 \theta_2}{\partial x_1 \partial x_2}, \tag{6.6}$$

$$\gamma_2 = \frac{\partial^2 u}{\partial x_1 \partial x_2} + \frac{\partial^2 v}{\partial x_2^2} - x_3 \frac{\partial^3 w}{\partial x_2^3} - x_3 \frac{\partial^3 w}{\partial x_1^2 \partial x_2} + f \frac{\partial^2 \theta_1}{\partial x_1 \partial x_2} + f \frac{\partial^2 \theta_2}{\partial x_2^2}, \tag{6.7}$$

$$\gamma_3 = -\frac{\partial^2 w}{\partial x_1^2} - \frac{\partial^2 w}{\partial x_2^2} + f' \frac{\partial \theta_1}{\partial x_1} + f' \frac{\partial \theta_2}{\partial x_2}, \tag{6.8}$$

$$\eta_{111}^{(1)} = \frac{2}{5}\left(\frac{\partial^2 u}{\partial x_1^2} - \frac{1}{2}\frac{\partial^2 u}{\partial x_2^2} - \frac{\partial^2 v}{\partial x_1 \partial x_2} - x_3 \frac{\partial^3 w}{\partial x_1^3} + \frac{3}{2} x_3 \frac{\partial^3 w}{\partial x_1 \partial x_2^2} + f \frac{\partial^2 \theta_1}{\partial x_1^2} - \frac{1}{2} f \frac{\partial^2 \theta_1}{\partial x_2^2} \right.$$
$$\left. -\frac{1}{2} f''\theta_1 - f \frac{\partial^2 \theta_2}{\partial x_1 \partial x_2} \right), \tag{6.9}$$

$$\eta_{222}^{(1)} = \frac{2}{5}\left(-\frac{\partial^2 u}{\partial x_1 \partial x_2} - \frac{1}{2}\frac{\partial^2 v}{\partial x_1^2} + \frac{\partial^2 v}{\partial x_2^2} - x_3 \frac{\partial^3 w}{\partial x_2^3} + \frac{3}{2} x_3 \frac{\partial^3 w}{\partial x_1^2 \partial x_2} - f \frac{\partial^2 \theta_1}{\partial x_1 \partial x_2} \right.$$
$$\left. -\frac{1}{2} f \frac{\partial^2 \theta_2}{\partial x_1^2} + f \frac{\partial^2 \theta_2}{\partial x_2^2} - \frac{1}{2} f''\theta_2 \right), \tag{6.10}$$

$$\eta_{333}^{(1)} = \frac{1}{5}\left(\frac{\partial^2 w}{\partial x_1^2} + \frac{\partial^2 w}{\partial x_2^2} - 2 f' \frac{\partial \theta_1}{\partial x_1} - 2 f' \frac{\partial \theta_2}{\partial x_2} \right), \tag{6.11}$$

$$\eta_{112}^{(1)} = \eta_{211}^{(1)} = \eta_{121}^{(1)} = \frac{1}{15}\left(8 \frac{\partial^2 u}{\partial x_1 \partial x_2} + 4 \frac{\partial^2 v}{\partial x_1^2} - 3 \frac{\partial^2 v}{\partial x_2^2} + 3 x_3 \frac{\partial^3 w}{\partial x_2^3} - 12 x_3 \frac{\partial^3 w}{\partial x_1^2 \partial x_2} + 8 f \frac{\partial^2 \theta_1}{\partial x_1 \partial x_2} \right.$$
$$\left. +4 f \frac{\partial^2 \theta_2}{\partial x_1^2} - 3 f \frac{\partial^2 \theta_2}{\partial x_2^2} - f''\theta_2 \right), \tag{6.12}$$

$$\eta_{113}^{(1)} = \eta_{311}^{(1)} = \eta_{131}^{(1)} = -\frac{1}{15}\left(4 \frac{\partial^2 w}{\partial x_1^2} - \frac{\partial^2 w}{\partial x_2^2} - 8 f' \frac{\partial \theta_1}{\partial x_1} + 2 f' \frac{\partial \theta_2}{\partial x_2} \right), \tag{6.13}$$

$$\eta_{221}^{(1)}=\eta_{122}^{(1)}=\eta_{212}^{(1)}=\frac{1}{15}\left(-3\frac{\partial^2 u}{\partial x_1^2}+4\frac{\partial^2 u}{\partial x_2^2}+8\frac{\partial^2 v}{\partial x_1\partial x_2}+3x_3\frac{\partial^3 w}{\partial x_1^3}-12x_3\frac{\partial^3 w}{\partial x_1\partial x_2^2}\right.$$
$$\left.-3f\frac{\partial^2\theta_1}{\partial x_1^2}+4f\frac{\partial^2\theta_1}{\partial x_2^2}-f''\theta_1+8f\frac{\partial^2\theta_2}{\partial x_1\partial x_2}\right), \tag{6.14}$$

$$\eta_{223}^{(1)}=\eta_{322}^{(1)}=\eta_{232}^{(1)}=-\frac{1}{15}\left(-\frac{\partial^2 w}{\partial x_1^2}+4\frac{\partial^2 w}{\partial x_2^2}+2f'\frac{\partial\theta_1}{\partial x_1}-8f'\frac{\partial\theta_2}{\partial x_2}\right), \tag{6.15}$$

$$\eta_{331}^{(1)}=\eta_{133}^{(1)}=\eta_{313}^{(1)}=-\frac{1}{15}\left(3\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}+2\frac{\partial^2 v}{\partial x_1\partial x_2}-3x_3\frac{\partial^3 w}{\partial x_1^3}-3x_3\frac{\partial^3 w}{\partial x_1\partial x_2^2}\right.$$
$$\left.+3f\frac{\partial^2\theta_1}{\partial x_1^2}+f\frac{\partial^2\theta_1}{\partial x_2^2}-4f''\theta_1+2f\frac{\partial^2\theta_2}{\partial x_1\partial x_2}\right), \tag{6.16}$$

$$\eta_{332}^{(1)}=\eta_{233}^{(1)}=\eta_{323}^{(1)}=-\frac{1}{15}\left(2\frac{\partial^2 u}{\partial x_1\partial x_2}+\frac{\partial^2 v}{\partial x_1^2}+3\frac{\partial^2 v}{\partial x_2^2}-3x_3\frac{\partial^3 w}{\partial x_2^3}-3x_3\frac{\partial^3 w}{\partial x_1^2\partial x_2}\right.$$
$$\left.+2f\frac{\partial^2\theta_1}{\partial x_1\partial x_2}+f\frac{\partial^2\theta_2}{\partial x_1^2}+3f\frac{\partial^2\theta_2}{\partial x_2^2}-4f''\theta_2\right), \tag{6.17}$$

$$\eta_{123}^{(1)}=\eta_{312}^{(1)}=\eta_{231}^{(1)}=\eta_{132}^{(1)}=\eta_{213}^{(1)}=\eta_{321}^{(1)}=\frac{1}{3}\left(-\frac{\partial^2 w}{\partial x_1\partial x_2}+f'\frac{\partial\theta_1}{\partial x_2}+f'\frac{\partial\theta_2}{\partial x_1}\right), \tag{6.18}$$

$$\chi_{11}^{s}=\frac{1}{2}\left(2\frac{\partial^2 w}{\partial x_1\partial x_2}-f'\frac{\partial\theta_2}{\partial x_1}\right), \tag{6.19}$$

$$\chi_{22}^{s}=-\frac{1}{2}\left(2\frac{\partial^2 w}{\partial x_1\partial x_2}-f'\frac{\partial\theta_1}{\partial x_2}\right), \tag{6.20}$$

$$\chi_{33}^{s}=-\frac{1}{2}\left(f'\frac{\partial\theta_1}{\partial x_2}-f'\frac{\partial\theta_2}{\partial x_1}\right), \tag{6.21}$$

$$\chi_{12}^{s}=\chi_{21}^{s}=\frac{1}{4}\left(-2\frac{\partial^2 w}{\partial x_1^2}+2\frac{\partial^2 w}{\partial x_2^2}+f'\frac{\partial\theta_1}{\partial x_1}-f'\frac{\partial\theta_2}{\partial x_2}\right), \tag{6.22}$$

$$\chi_{13}^{s}=\chi_{31}^{s}=\frac{1}{4}\left(-\frac{\partial^2 u}{\partial x_1\partial x_2}+\frac{\partial^2 v}{\partial x_1^2}-f\frac{\partial^2\theta_1}{\partial x_1\partial x_2}+f\frac{\partial^2\theta_2}{\partial x_1^2}-f''\theta_2\right), \tag{6.23}$$

$$\chi_{23}^{s} = \chi_{32}^{s} = \frac{1}{4}\left(-\frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 v}{\partial x_1 \partial x_2} - f\frac{\partial^2 \theta_1}{\partial x_2^2} + f''\theta_1 + f\frac{\partial^2 \theta_2}{\partial x_1 \partial x_2} \right). \tag{6.24}$$

Constitutive relations of strain gradient theory are expressed as

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{Bmatrix} = \frac{E}{1-\upsilon^2} \begin{bmatrix} 1 & \upsilon & 0 & 0 & 0 \\ \upsilon & 1 & 0 & 0 & 0 \\ 0 & 0 & 1-\upsilon & 0 & 0 \\ 0 & 0 & 0 & k_s(1-\upsilon) & 0 \\ 0 & 0 & 0 & 0 & k_s(1-\upsilon) \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} - \alpha\Delta T \\ \varepsilon_{22} - \alpha\Delta T \\ \varepsilon_{12} \\ \varepsilon_{13} \\ \varepsilon_{23} \end{Bmatrix} \tag{7.1}$$

$$p_i = 2\mu l_0^2 \gamma_i, \tag{7.2}$$

$$\tau_{ijk}^{(1)} = 2\mu l_1^2 \eta_{ijk}^{(1)}, \tag{7.3}$$

$$m_{ij}^{s} = 2\mu l_2^2 \chi_{ij}^{s}. \tag{7.4}$$

where E is modulus of elasticity, ν is Poisson's ratio, μ is shear modulus, k_s is shear correction factor which is equal to unity in KPT and TSDT; and 5/6 in MPT, and l_i, $i = 0,1,2,$ are length scale parameters. All of the material properties, including the length scale parameters l_i, $i = 0,1,2,$ are assumed to be functions of the thickness coordinate x_3. Note that in modified couple stress theory (MCST) proposed by Yang and Shen [70] l_2 exists as the only nonzero length scale parameter and classical elasticity theory lacks length scale parameter to capture size effect.

2.3 Derivation of Governing Equations and Boundary Conditions Using Hamilton's principle

Partial differential equations of motion and boundary conditions are derived by using Hamilton's principle, which postulates that

$$\delta \int_{t_1}^{t_2} \left(K - \left(U - W \right) \right) dt = 0, \tag{8}$$

where K, U, and W are kinetic energy, total strain energy, and work done by external forces, respectively.

The kinetic energy of the micro-plate is obtained by

$$K = \frac{1}{2}\int_{\Omega} \rho \left\{ \left(\frac{\partial u_1}{\partial t} \right)^2 + \left(\frac{\partial u_2}{\partial t} \right)^2 + \left(\frac{\partial u_3}{\partial t} \right)^2 \right\} dV, \tag{9}$$

where ρ is mass density; Using integration by parts and Green's theorem and assuming that the initial and final configurations of the plate respectively at $t = t_1$ and $t = t_2$ are prescribed, the first variation of kinetic energy on the time interval $[t_1, t_2]$ can be reached

$$\begin{aligned}
\delta \int_{t_1}^{t_2} K dt = \int_{t_1}^{t_2} \int_{A} \Bigg\{ & -\left(I_0 \frac{\partial^2 u}{\partial t^2} - I_1 \frac{\partial^3 w}{\partial x_1 \partial t^2} + I_3 \frac{\partial^2 \theta_1}{\partial t^2} \right) \delta u \\
& - \left(I_1 \frac{\partial^3 u}{\partial x_1 \partial t^2} - I_2 \frac{\partial^4 w}{\partial x_1^2 \partial t^2} + I_4 \frac{\partial^3 \theta_1}{\partial x_1 \partial t^2} \right) \delta w - \left(I_3 \frac{\partial u^2}{\partial t^2} - I_4 \frac{\partial^3 w}{\partial x_1 \partial t^2} + I_5 \frac{\partial^2 \theta_1}{\partial t^2} \right) \delta \theta_1 \\
& - \left(I_0 \frac{\partial^2 v}{\partial t^2} - I_1 \frac{\partial^3 w}{\partial x_2 \partial t^2} + I_3 \frac{\partial^2 \theta_2}{\partial t^2} \right) \delta v - \left(I_1 \frac{\partial^3 v}{\partial x_2 \partial t^2} - I_2 \frac{\partial^4 w}{\partial x_2^2 \partial t^2} + I_4 \frac{\partial^3 \theta_2}{\partial x_2 \partial t^2} \right) \delta w \\
& - \left(I_3 \frac{\partial^2 v}{\partial t^2} - I_4 \frac{\partial^3 w}{\partial x_2 \partial t^2} + I_5 \frac{\partial^2 \theta_2}{\partial t^2} \right) \delta \theta_2 - \left(I_0 \frac{\partial^2 w}{\partial t^2} \right) \delta w \Bigg\} dA dt \\
& + \oint_{\Gamma} \Bigg(\left(I_1 \frac{\partial^2 u}{\partial t^2} - I_2 \frac{\partial^3 w}{\partial x_1 \partial t^2} + I_4 \frac{\partial^2 \theta_1}{\partial t^2} \right) n_{x_1} \delta w \\
& + \left(I_1 \frac{\partial^2 v}{\partial t^2} - I_2 \frac{\partial^3 w}{\partial x_2 \partial t^2} + I_4 \frac{\partial^2 \theta_2}{\partial t^2} \right) n_{x_2} \delta w \Bigg) d\Gamma
\end{aligned} \tag{10}$$

where A is the area occupied by the mid-plane of the micro-plate; and Γ is the boundary curve enclosing the area A. The inertia terms are given by

$$\{I_0, I_1, I_2, I_3, I_4, I_5\} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho(x_3) \{1, x_3, x_3^2, f, x_3 f, f^2\} dx_3 \tag{11}$$

In the present study as the external forces the effects of distributed loading $q(x_1, x_2)$ and the in-plane buckling loads P_{x_1} and P_{x_2} will be considered. Note that $q(x_1, x_2)$ is applied transversely and P_{x_1} and P_{x_2} are applied axially along x_1 and x_2 directions, respectively. Consequently, the work done by the external forces is of the form:

$$W = \int_A qwdA + \frac{1}{2}\int_A \left(P_{x_1}\left(\frac{\partial w}{\partial x_1}\right)^2 + P_{x_2}\left(\frac{\partial w}{\partial x_2}\right)^2 \right) dA. \tag{12}$$

Substituting the variations of strain energy, kinetic energy and work into Eq. (8), employing Eq. (2) and considering the arbitrariness of δu, δv, δw, $\delta\phi_1$ and $\delta\phi_2$ the following governing equations of motion are derived, with the help of integration by parts and Green's theorem,

δu:

$$\begin{aligned}
&\frac{\partial M_{11}^0}{\partial x_1} + \frac{\partial M_{12}^0}{\partial x_2} - \frac{\partial^2 P_1^0}{\partial x_1^2} - \frac{\partial^2 P_2^0}{\partial x_1 \partial x_2} - \frac{2}{5}\frac{\partial^2 T_{111}^0}{\partial x_1^2} + \frac{1}{5}\frac{\partial^2 T_{111}^0}{\partial x_2^2} + \frac{2}{5}\frac{\partial^2 T_{222}^0}{\partial x_1 \partial x_2} \\
&-\frac{8}{5}\frac{\partial^2 T_{112}^0}{\partial x_1 \partial x_2} + \frac{3}{5}\frac{\partial^2 T_{221}^0}{\partial x_1^2} - \frac{4}{5}\frac{\partial^2 T_{221}^0}{\partial x_2^2} + \frac{3}{5}\frac{\partial^2 T_{331}^0}{\partial x_1^2} + \frac{1}{5}\frac{\partial^2 T_{331}^0}{\partial x_2^2} + \frac{2}{5}\frac{\partial^2 T_{332}^0}{\partial x_1 \partial x_2} \\
&+\frac{1}{2}\frac{\partial^2 Y_{13}^0}{\partial x_1 \partial x_2} + \frac{1}{2}\frac{\partial^2 Y_{23}^0}{\partial x_2^2} = I_0 \frac{\partial^2 u}{\partial t^2} + (I_3 - I_1)\frac{\partial^3 w}{\partial x_1 \partial t^2} + I_3 \frac{\partial^2 \phi_1}{\partial t^2}
\end{aligned} \tag{13.1}$$

δv:

$$\begin{aligned}
&\frac{\partial M_{22}^0}{\partial x_2} + \frac{\partial M_{12}^0}{\partial x_1} - \frac{\partial^2 P_1^0}{\partial x_1 \partial x_2} - \frac{\partial^2 P_2^0}{\partial x_2^2} + \frac{2}{5}\frac{\partial^2 T_{111}^0}{\partial x_1 \partial x_2} + \frac{1}{5}\frac{\partial^2 T_{222}^0}{\partial x_1^2} - \frac{2}{5}\frac{\partial^2 T_{222}^0}{\partial x_2^2} \\
&-\frac{4}{5}\frac{\partial^2 T_{112}^0}{\partial x_1^2} + \frac{3}{5}\frac{\partial^2 T_{112}^0}{\partial x_2^2} - \frac{8}{5}\frac{\partial^2 T_{221}^0}{\partial x_1 \partial x_2} + \frac{2}{5}\frac{\partial^2 T_{331}^0}{\partial x_1 \partial x_2} + \frac{1}{5}\frac{\partial^2 T_{332}^0}{\partial x_1^2} + \frac{3}{5}\frac{\partial^2 T_{332}^0}{\partial x_2^2} \\
&-\frac{1}{2}\frac{\partial^2 Y_{13}^0}{\partial x_1^2} - \frac{1}{2}\frac{\partial^2 Y_{23}^0}{\partial x_1 \partial x_2} = I_0 \frac{\partial^2 v}{\partial t^2} + (I_3 - I_1)\frac{\partial^3 w}{\partial x_2 \partial t^2} + I_3 \frac{\partial^2 \phi_2}{\partial t^2}
\end{aligned} \tag{13.2}$$

δw:

$$
\begin{aligned}
&\frac{\partial^2 M_{11}^1}{\partial x_1^2}-\frac{\partial^2 M_{11}^2}{\partial x_1^2}+\frac{\partial^2 M_{22}^1}{\partial x_2^2}-\frac{\partial^2 M_{22}^2}{\partial x_2^2}+2\frac{\partial^2 M_{12}^1}{\partial x_1\partial x_2}-2\frac{\partial^2 M_{12}^2}{\partial x_1\partial x_2}+\frac{\partial M_{13}^3}{\partial x_1}+\frac{\partial M_{23}^3}{\partial x_2}\\
&-\frac{\partial^3 P_1^1}{\partial x_1^3}+\frac{\partial^3 P_1^2}{\partial x_1^3}-\frac{\partial^3 P_1^1}{\partial x_1\partial x_2^2}+\frac{\partial^3 P_1^2}{\partial x_1\partial x_2^2}-\frac{\partial^3 P_2^1}{\partial x_2^3}+\frac{\partial^3 P_2^2}{\partial x_2^3}-\frac{\partial^3 P_2^1}{\partial x_1^2\partial x_2}+\frac{\partial^3 P_2^2}{\partial x_1^2\partial x_2}+\frac{\partial^2 P_3^0}{\partial x_1^2}-\frac{\partial^2 P_3^3}{\partial x_1^2}\\
&+\frac{\partial^2 P_3^0}{\partial x_2^2}-\frac{\partial^2 P_3^3}{\partial x_2^2}-\frac{2}{5}\frac{\partial^3 T_{111}^1}{\partial x_1^3}+\frac{2}{5}\frac{\partial^3 T_{111}^2}{\partial x_1^3}+\frac{3}{5}\frac{\partial^3 T_{111}^1}{\partial x_1\partial x_2^2}-\frac{2}{5}\frac{\partial^3 T_{111}^2}{\partial x_1\partial x_2^2}-\frac{1}{5}\frac{\partial^3 T_{111}^2}{\partial x_1\partial x_2^2}-\frac{1}{5}\frac{\partial T_{111}^4}{\partial x_1}\\
&-\frac{2}{5}\frac{\partial^3 T_{222}^1}{\partial x_2^3}+\frac{2}{5}\frac{\partial^3 T_{222}^2}{\partial x_2^3}+\frac{3}{5}\frac{\partial^3 T_{222}^1}{\partial x_1^2\partial x_2}-\frac{1}{5}\frac{\partial^3 T_{222}^2}{\partial x_1^2\partial x_2}-\frac{2}{5}\frac{\partial^3 T_{222}^2}{\partial x_1^2\partial x_2}-\frac{1}{5}\frac{\partial T_{222}^4}{\partial x_2}\\
&-\frac{1}{5}\frac{\partial^2 T_{333}^0}{\partial x_1^2}+\frac{2}{5}\frac{\partial^2 T_{333}^3}{\partial x_1^2}-\frac{1}{5}\frac{\partial^2 T_{333}^0}{\partial x_2^2}+\frac{2}{5}\frac{\partial^2 T_{333}^3}{\partial x_2^2}+\frac{3}{5}\frac{\partial^3 T_{112}^1}{\partial x_2^3}-\frac{3}{5}\frac{\partial^3 T_{112}^2}{\partial x_2^3}-\frac{12}{5}\frac{\partial^3 T_{112}^1}{\partial x_1^2\partial x_2}\\
&+\frac{4}{5}\frac{\partial^3 T_{112}^2}{\partial x_1^2\partial x_2}+\frac{8}{5}\frac{\partial^3 T_{112}^2}{\partial x_1^2\partial x_2}-\frac{1}{5}\frac{\partial T_{112}^4}{\partial x_2}+\frac{4}{5}\frac{\partial^2 T_{113}^0}{\partial x_1^2}-\frac{8}{5}\frac{\partial^2 T_{113}^3}{\partial x_1^2}-\frac{1}{5}\frac{\partial^2 T_{113}^0}{\partial x_2^2}+\frac{2}{5}\frac{\partial^2 T_{113}^3}{\partial x_2^2}\\
&+\frac{3}{5}\frac{\partial^3 T_{221}^1}{\partial x_1^3}-\frac{3}{5}\frac{\partial^3 T_{221}^2}{\partial x_1^3}-\frac{12}{5}\frac{\partial^3 T_{221}^1}{\partial x_1\partial x_2^2}+\frac{8}{5}\frac{\partial^3 T_{221}^2}{\partial x_1\partial x_2^2}+\frac{4}{5}\frac{\partial^3 T_{221}^2}{\partial x_1\partial x_2^2}-\frac{1}{5}\frac{\partial T_{221}^4}{\partial x_1}\\
&-\frac{1}{5}\frac{\partial^2 T_{223}^0}{\partial x_1^2}+\frac{2}{5}\frac{\partial^2 T_{223}^3}{\partial x_1^2}+\frac{4}{5}\frac{\partial^2 T_{223}^0}{\partial x_2^2}-\frac{8}{5}\frac{\partial^2 T_{223}^3}{\partial x_2^2}+\frac{3}{5}\frac{\partial^3 T_{331}^1}{\partial x_1^3}-\frac{3}{5}\frac{\partial^3 T_{331}^2}{\partial x_1^3}+\frac{3}{5}\frac{\partial^3 T_{331}^1}{\partial x_1\partial x_2^2}\\
&-\frac{2}{5}\frac{\partial^3 T_{331}^2}{\partial x_1\partial x_2^2}-\frac{1}{5}\frac{\partial^3 T_{331}^2}{\partial x_1\partial x_2^2}+\frac{4}{5}\frac{\partial T_{331}^4}{\partial x_1}+\frac{3}{5}\frac{\partial^3 T_{332}^1}{\partial x_2^3}-\frac{3}{5}\frac{\partial^3 T_{332}^2}{\partial x_2^3}+\frac{3}{5}\frac{\partial^3 T_{332}^1}{\partial x_1^2\partial x_2}\\
&-\frac{1}{5}\frac{\partial^3 T_{332}^2}{\partial x_1^2\partial x_2}-\frac{2}{5}\frac{\partial^3 T_{332}^2}{\partial x_1^2\partial x_2}+\frac{4}{5}\frac{\partial T_{332}^4}{\partial x_2}+2\frac{\partial^2 T_{123}^0}{\partial x_1\partial x_2}-4\frac{\partial^2 T_{123}^3}{\partial x_1\partial x_2}-\frac{\partial^2 Y_{11}^0}{\partial x_1\partial x_2}+\frac{1}{2}\frac{\partial^2 Y_{11}^3}{\partial x_1\partial x_2}\\
&+\frac{\partial^2 Y_{22}^0}{\partial x_1\partial x_2}-\frac{1}{2}\frac{\partial^2 Y_{22}^3}{\partial x_1\partial x_2}+\frac{\partial^2 Y_{12}^0}{\partial x_1^2}-\frac{1}{2}\frac{\partial^2 Y_{12}^3}{\partial x_1^2}-\frac{\partial^2 Y_{12}^0}{\partial x_2^2}+\frac{1}{2}\frac{\partial^2 Y_{12}^3}{\partial x_2^2}-\frac{1}{2}\frac{\partial Y_{13}^4}{\partial x_2}+\frac{1}{2}\frac{\partial Y_{23}^4}{\partial x_1}\\
&+q-P_{x_1}\frac{\partial^2 w}{\partial x_1^2}-P_{x_2}\frac{\partial^2 w}{\partial x_2^2}+P_{x_1}^0\frac{\partial^2 w}{\partial x_1^2}+P_{x_2}^0\frac{\partial^2 w}{\partial x_2^2}+2P_{x_1x_2}^0\frac{\partial^2 w}{\partial x_1\partial x_2}\\
&=\left(I_1-I_3\right)\frac{\partial^3 u}{\partial x_1\partial t^2}+\left(I_1-I_3\right)\frac{\partial^3 v}{\partial x_2\partial t^2}+\left(2I_4-I_2-I_5\right)\frac{\partial^4 w}{\partial x_1^2\partial t^2}\\
&+\left(2I_4-I_2-I_5\right)\frac{\partial^4 w}{\partial x_2^2\partial t^2}+I_0\frac{\partial^2 w}{\partial t^2}+\left(I_4-I_5\right)\frac{\partial^3 \phi_1}{\partial x_1\partial t^2}+\left(I_4-I_5\right)\frac{\partial^3 \phi_2}{\partial x_2\partial t^2}
\end{aligned}
\tag{13.3}
$$

$\delta\phi_1$:

$$\frac{\partial M_{11}^2}{\partial x_1}+\frac{\partial M_{12}^2}{\partial x_2}-M_{13}^3-\frac{\partial^2 P_1^2}{\partial x_1^2}-\frac{\partial^2 P_2^2}{\partial x_1\partial x_2}+\frac{\partial P_3^3}{\partial x_1}-\frac{2}{5}\frac{\partial^2 T_{111}^2}{\partial x_1^2}+\frac{1}{5}\frac{\partial^2 T_{111}^2}{\partial x_2^2}+\frac{1}{5}T_{111}^4$$
$$+\frac{2}{5}\frac{\partial^2 T_{222}^2}{\partial x_1\partial x_2}-\frac{2}{5}\frac{\partial T_{333}^3}{\partial x_1}-\frac{8}{5}\frac{\partial^2 T_{112}^2}{\partial x_1\partial x_2}+\frac{8}{5}\frac{\partial T_{113}^3}{\partial x_1}+\frac{3}{5}\frac{\partial^2 T_{221}^2}{\partial x_1^2}-\frac{4}{5}\frac{\partial^2 T_{221}^2}{\partial x_2^2}+\frac{1}{5}T_{221}^4-\frac{2}{5}\frac{\partial T_{223}^3}{\partial x_1}$$
$$+\frac{3}{5}\frac{\partial^2 T_{331}^2}{\partial x_1^2}+\frac{1}{5}\frac{\partial^2 T_{331}^2}{\partial x_2^2}-\frac{4}{5}T_{331}^4+\frac{2}{5}\frac{\partial^2 T_{332}^2}{\partial x_1\partial x_2}+2\frac{\partial T_{123}^3}{\partial x_2}$$
$$+\frac{1}{2}\frac{\partial Y_{22}^3}{\partial x_2}-\frac{1}{2}\frac{\partial Y_{33}^3}{\partial x_2}+\frac{1}{2}\frac{\partial Y_{12}^3}{\partial x_1}+\frac{1}{2}\frac{\partial^2 Y_{13}^2}{\partial x_1\partial x_2}+\frac{1}{2}\frac{\partial^2 Y_{23}^2}{\partial x_2^2}-\frac{1}{2}Y_{23}^4$$
$$=I_3\frac{\partial^2 u}{\partial t^2}+\left(I_5-I_4\right)\frac{\partial^3 w}{\partial x_1\partial t^2}+I_5\frac{\partial^2\phi_1}{\partial t^2} \tag{13.4}$$

$\delta\phi_2$:

$$\frac{\partial M_{22}^2}{\partial x_2}+\frac{\partial M_{12}^2}{\partial x_1}-M_{23}^3-\frac{\partial^2 P_1^2}{\partial x_1\partial x_2}-\frac{\partial^2 P_2^2}{\partial x_2^2}+\frac{\partial P_3^3}{\partial x_2}+\frac{2}{5}\frac{\partial^2 T_{111}^2}{\partial x_1\partial x_2}$$
$$+\frac{1}{5}\frac{\partial^2 T_{222}^2}{\partial x_1^2}-\frac{2}{5}\frac{\partial^2 T_{222}^2}{\partial x_2^2}+\frac{1}{5}T_{222}^4-\frac{2}{5}\frac{\partial T_{333}^3}{\partial x_2}-\frac{4}{5}\frac{\partial^2 T_{112}^2}{\partial x_1^2}+\frac{3}{5}\frac{\partial^2 T_{112}^2}{\partial x_2^2}+\frac{1}{5}T_{112}^4-\frac{2}{5}\frac{\partial T_{113}^3}{\partial x_2}$$
$$-\frac{8}{5}\frac{\partial^2 T_{221}^2}{\partial x_1\partial x_2}+\frac{8}{5}\frac{\partial T_{223}^3}{\partial x_2}+\frac{2}{5}\frac{\partial^2 T_{331}^2}{\partial x_1\partial x_2}+\frac{1}{5}\frac{\partial^2 T_{332}^2}{\partial x_1^2}+\frac{3}{5}\frac{\partial^2 T_{332}^2}{\partial x_2^2}-\frac{4}{5}T_{332}^4+2\frac{\partial T_{123}^3}{\partial x_1}$$
$$-\frac{1}{2}\frac{\partial Y_{11}^3}{\partial x_1}+\frac{1}{2}\frac{\partial Y_{33}^3}{\partial x_1}-\frac{1}{2}\frac{\partial Y_{12}^3}{\partial x_2}-\frac{1}{2}\frac{\partial^2 Y_{13}^2}{\partial x_1^2}+\frac{1}{2}Y_{13}^4-\frac{1}{2}\frac{\partial^2 Y_{23}^2}{\partial x_1\partial x_2}$$
$$=I_3\frac{\partial^2 v}{\partial t^2}+\left(I_5-I_4\right)\frac{\partial^3 w}{\partial x_2\partial t^2}+I_5\frac{\partial^2\phi_2}{\partial t^2} \tag{13.5}$$

and the boundary conditions read

$\delta u=0$ or

$$\left(M_{11}^0-\frac{\partial P_1^0}{\partial x_1}-\frac{1}{2}\frac{\partial P_2^0}{\partial x_2}-\frac{2}{5}\frac{\partial T_{111}^0}{\partial x_1}+\frac{1}{5}\frac{\partial T_{222}^0}{\partial x_2}-\frac{4}{5}\frac{\partial T_{112}^0}{\partial x_2}+\frac{3}{5}\frac{\partial T_{221}^0}{\partial x_1}+\frac{3}{5}\frac{\partial T_{331}^0}{\partial x_1}+\frac{1}{5}\frac{\partial T_{332}^0}{\partial x_2}\right.$$
$$\left.+\frac{1}{4}\frac{\partial Y_{13}^0}{\partial x_2}\right)n_{x_1}+\left(M_{12}^0-\frac{1}{2}\frac{\partial P_2^0}{\partial x_1}+\frac{1}{5}\frac{\partial T_{111}^0}{\partial x_2}+\frac{1}{5}\frac{\partial T_{222}^0}{\partial x_1}-\frac{4}{5}\frac{\partial T_{112}^0}{\partial x_1}-\frac{4}{5}\frac{\partial T_{221}^0}{\partial x_2}\right.$$
$$\left.+\frac{1}{5}\frac{\partial T_{331}^0}{\partial x_2}+\frac{1}{5}\frac{\partial T_{332}^0}{\partial x_1}+\frac{1}{4}\frac{\partial Y_{13}^0}{\partial x_1}+\frac{1}{2}\frac{\partial Y_{23}^0}{\partial x_2}\right)n_{x_2}=0 \tag{14.1}$$

$$\delta \frac{\partial u}{\partial x_1} = 0 \quad \text{or}$$

$$\left(P_1^0 + \frac{2}{5} T_{111}^0 - \frac{3}{5} T_{221}^0 - \frac{3}{5} T_{331}^0 \right) n_{x_1} + \left(\frac{P_2^0}{2} - \frac{1}{5} T_{222}^0 + \frac{4}{5} T_{112}^0 - \frac{T_{332}^0}{5} - \frac{Y_{13}^0}{4} \right) n_{x_2} = 0 \qquad (14.2)$$

$$\delta \frac{\partial u}{\partial x_2} = 0 \quad \text{or}$$

$$\left(\frac{P_2^0}{2} - \frac{T_{222}^0}{5} + \frac{4}{5} T_{112}^0 - \frac{T_{332}^0}{5} - \frac{Y_{13}^0}{4} \right) n_{x_1} + \left(-\frac{T_{111}^0}{5} + \frac{4}{5} T_{221}^0 - \frac{T_{331}^0}{5} - \frac{Y_{23}^0}{2} \right) n_{x_2} = 0 \qquad (14.3)$$

$$\delta v = 0 \quad \text{or}$$

$$\left(M_{12}^0 - \frac{1}{2} \frac{\partial P_1^0}{\partial x_2} + \frac{1}{5} \frac{\partial T_{111}^0}{\partial x_2} + \frac{1}{5} \frac{\partial T_{222}^0}{\partial x_1} - \frac{4}{5} \frac{\partial T_{112}^0}{\partial x_1} - \frac{4}{5} \frac{\partial T_{221}^0}{\partial x_2} + \frac{1}{5} \frac{\partial T_{331}^0}{\partial x_2} + \frac{1}{5} \frac{\partial T_{332}^0}{\partial x_1} \right.$$
$$\left. - \frac{1}{2} \frac{\partial Y_{13}^0}{\partial x_1} - \frac{1}{4} \frac{\partial Y_{23}^0}{\partial x_2} \right) n_{x_1} + \left(M_{22}^0 - \frac{1}{2} \frac{\partial P_1^0}{\partial x_1} - \frac{\partial P_2^0}{\partial x_2} + \frac{1}{5} \frac{\partial T_{111}^0}{\partial x_1} - \frac{2}{5} \frac{\partial T_{222}^0}{\partial x_2} \right.$$
$$\left. + \frac{3}{5} \frac{\partial T_{112}^0}{\partial x_2} - \frac{4}{5} \frac{\partial T_{221}^0}{\partial x_1} + \frac{1}{5} \frac{\partial T_{331}^0}{\partial x_1} + \frac{3}{5} \frac{\partial T_{332}^0}{\partial x_2} - \frac{1}{4} \frac{\partial Y_{23}^0}{\partial x_1} \right) n_{x_2} = 0 \qquad (14.4)$$

$$\delta \frac{\partial v}{\partial x_1} = 0 \quad \text{or}$$

$$\left(-\frac{T_{222}^0}{5} + \frac{4}{5} T_{112}^0 - \frac{T_{332}^0}{5} + \frac{Y_{13}^0}{2} \right) n_{x_1} + \left(\frac{P_1^0}{2} - \frac{T_{111}^0}{5} + \frac{4}{5} T_{221}^0 - \frac{T_{331}^0}{5} + \frac{Y_{23}^0}{4} \right) n_{x_2} = 0 \qquad (14.5)$$

$$\delta \frac{\partial v}{\partial x_2} = 0 \quad \text{or}$$

$$\left(\frac{P_1^0}{2} - \frac{T_{111}^0}{5} + \frac{4}{5} T_{221}^0 - \frac{T_{331}^0}{5} + \frac{Y_{23}^0}{4} \right) n_{x_1} + \left(P_2^0 + \frac{2}{5} T_{222}^0 - \frac{3}{5} T_{112}^0 - \frac{3}{5} T_{332}^0 \right) n_{x_2} = 0 \qquad (14.6)$$

$\delta w = 0$ or

$$
\begin{aligned}
&\left(\frac{\partial M_{11}^{1}}{\partial x_1}-\frac{\partial M_{11}^{2}}{\partial x_1}+\frac{\partial M_{12}^{1}}{\partial x_2}-\frac{\partial M_{12}^{2}}{\partial x_2}+M_{13}^{3}-\frac{\partial^2 P_1^1}{\partial x_1^2}+\frac{\partial^2 P_1^2}{\partial x_1^2}-\frac{\partial^2 P_2^1}{\partial x_1\partial x_2}+\frac{\partial^2 P_2^2}{\partial x_1\partial x_2}\right.\\
&+\frac{\partial P_3^0}{\partial x_1}-\frac{\partial P_3^3}{\partial x_1}-\frac{2}{5}\frac{\partial^2 T_{111}^{1}}{\partial x_1^2}+\frac{2}{5}\frac{\partial^2 T_{111}^{2}}{\partial x_1^2}-\frac{1}{5}\frac{\partial^2 T_{111}^{2}}{\partial x_2^2}-\frac{1}{5}T_{111}^{4}+\frac{3}{5}\frac{\partial T_{222}^{1}}{\partial x_1\partial x_2}-\frac{2}{5}\frac{\partial^2 T_{222}^{2}}{\partial x_1\partial x_2}\\
&-\frac{1}{5}\frac{\partial T_{333}^{0}}{\partial x_1}+\frac{2}{5}\frac{\partial T_{333}^{3}}{\partial x_1}-\frac{12}{5}\frac{\partial^2 T_{112}^{1}}{\partial x_1\partial x_2}+\frac{8}{5}\frac{\partial^2 T_{112}^{2}}{\partial x_1\partial x_2}+\frac{4}{5}\frac{\partial T_{113}^{0}}{\partial x_1}-\frac{8}{5}\frac{\partial T_{113}^{3}}{\partial x_1}\\
&+\frac{3}{5}\frac{\partial^2 T_{221}^{1}}{\partial x_1^2}-\frac{3}{5}\frac{\partial^2 T_{221}^{2}}{\partial x_1^2}+\frac{4}{5}\frac{\partial^2 T_{221}^{2}}{\partial x_2^2}-\frac{1}{5}T_{221}^{4}-\frac{1}{5}\frac{\partial T_{223}^{0}}{\partial x_1}+\frac{2}{5}\frac{\partial T_{223}^{3}}{\partial x_1}\\
&+\frac{3}{5}\frac{\partial^2 T_{331}^{1}}{\partial x_1^2}-\frac{3}{5}\frac{\partial^2 T_{331}^{2}}{\partial x_1^2}-\frac{1}{5}\frac{\partial^2 T_{331}^{2}}{\partial x_2^2}+\frac{4}{5}T_{331}^{4}+\frac{3}{5}\frac{\partial^2 T_{332}^{1}}{\partial x_1\partial x_2}-\frac{2}{5}\frac{\partial^2 T_{332}^{2}}{\partial x_1\partial x_2}\\
&\left.+\frac{\partial T_{123}^{0}}{\partial x_2}-2\frac{\partial T_{123}^{3}}{\partial x_2}-\frac{1}{2}\frac{\partial Y_{11}^{0}}{\partial x_2}+\frac{1}{2}\frac{\partial Y_{22}^{0}}{\partial x_2}-\frac{1}{2}\frac{\partial Y_{22}^{3}}{\partial x_2}+\frac{\partial Y_{12}^{0}}{\partial x_1}-\frac{1}{2}\frac{\partial Y_{12}^{3}}{\partial x_1}+\frac{1}{2}Y_{23}^{4}-\frac{\partial w}{\partial x_1}P_{x_1}\right)n_{x_1}\\
&+\left(\frac{\partial M_{22}^{1}}{\partial x_2}-\frac{\partial M_{22}^{2}}{\partial x_2}+\frac{\partial M_{12}^{1}}{\partial x_1}-\frac{\partial M_{12}^{2}}{\partial x_1}+M_{23}^{3}-\frac{\partial^2 P_1^1}{\partial x_1\partial x_2}+\frac{\partial^2 P_1^2}{\partial x_1\partial x_2}-\frac{\partial^2 P_2^1}{\partial x_2^2}+\frac{\partial^2 P_2^2}{\partial x_2^2}\right.\\
&+\frac{\partial P_3^0}{\partial x_2}-\frac{\partial P_3^3}{\partial x_2}+\frac{3}{5}\frac{\partial^2 T_{111}^{1}}{\partial x_1\partial x_2}-\frac{2}{5}\frac{\partial^2 T_{111}^{2}}{\partial x_1\partial x_2}-\frac{2}{5}\frac{\partial^2 T_{222}^{1}}{\partial x_2^2}-\frac{1}{5}\frac{\partial^2 T_{222}^{2}}{\partial x_1^2}-\frac{1}{5}T_{222}^{4}+\frac{2}{5}\frac{\partial^2 T_{222}^{2}}{\partial x_2^2}\\
&-\frac{1}{5}\frac{\partial T_{333}^{0}}{\partial x_2}+\frac{2}{5}\frac{\partial T_{333}^{3}}{\partial x_2}+\frac{3}{5}\frac{\partial^2 T_{112}^{1}}{\partial x_2^2}+\frac{4}{5}\frac{\partial^2 T_{112}^{2}}{\partial x_1^2}-\frac{3}{5}\frac{\partial^2 T_{112}^{2}}{\partial x_2^2}-\frac{1}{5}T_{112}^{4}-\frac{1}{5}\frac{\partial T_{113}^{0}}{\partial x_2}+\frac{2}{5}\frac{\partial T_{113}^{3}}{\partial x_2}\\
&-\frac{12}{5}\frac{\partial^2 T_{221}^{1}}{\partial x_1\partial x_2}+\frac{8}{5}\frac{\partial^2 T_{221}^{2}}{\partial x_1\partial x_2}+\frac{4}{5}\frac{\partial T_{223}^{0}}{\partial x_2}-\frac{8}{5}\frac{\partial T_{223}^{3}}{\partial x_2}\\
&+\frac{3}{5}\frac{\partial^2 T_{331}^{1}}{\partial x_1\partial x_2}-\frac{2}{5}\frac{\partial^2 T_{331}^{2}}{\partial x_1\partial x_2}+\frac{3}{5}\frac{\partial^2 T_{332}^{1}}{\partial x_2^2}-\frac{1}{5}\frac{\partial^2 T_{332}^{2}}{\partial x_1^2}-\frac{3}{5}\frac{\partial^2 T_{332}^{2}}{\partial x_2^2}+\frac{4}{5}T_{332}^{4}\\
&\left.+\frac{\partial T_{123}^{0}}{\partial x_1}-2\frac{\partial T_{123}^{3}}{\partial x_1}-\frac{1}{2}\frac{\partial Y_{11}^{0}}{\partial x_1}+\frac{1}{2}\frac{\partial Y_{11}^{3}}{\partial x_1}+\frac{1}{2}\frac{\partial Y_{22}^{0}}{\partial x_1}-\frac{\partial Y_{12}^{0}}{\partial x_2}+\frac{1}{2}\frac{\partial Y_{12}^{3}}{\partial x_2}-\frac{1}{2}Y_{13}^{4}-\frac{\partial w}{\partial x_2}P_{x_2}\right)n_{x_2}\\
&=\left(\left(I_1-I_3\right)\frac{\partial^2 u}{\partial t^2}+\left(2I_4-I_2-I_5\right)\frac{\partial^3 w}{\partial x_1\partial t^2}+\left(I_4-I_5\right)\frac{\partial^2 \phi_1}{\partial t^2}\right)n_{x_1}\\
&+\left(\left(I_1-I_3\right)\frac{\partial^2 v}{\partial t^2}+\left(2I_4-I_2-I_5\right)\frac{\partial^3 w}{\partial x_2\partial t^2}+\left(I_4-I_5\right)\frac{\partial^2 \phi_2}{\partial t^2}\right)n_{x_2}
\end{aligned}
\tag{14.7}
$$

$$\delta \frac{\partial w}{\partial x_1} = 0 \quad \text{or}$$

$$\left(-M_{11}^1 + M_{11}^2 + \frac{\partial P_1^1}{\partial x_1} - \frac{\partial P_1^2}{\partial x_1} + \frac{\partial P_2^1}{\partial x_2} - \frac{\partial P_2^2}{\partial x_2} - P_3^0 + P_3^3 + \frac{2}{5}\frac{\partial T_{111}^1}{\partial x_1} - \frac{2}{5}\frac{\partial T_{111}^2}{\partial x_1} - \frac{3}{5}\frac{\partial T_{222}^1}{\partial x_2} \right.$$

$$+ \frac{3}{5}\frac{\partial T_{222}^2}{\partial x_2} + \frac{1}{5}T_{333}^0 - \frac{2}{5}T_{333}^3 + \frac{12}{5}\frac{\partial T_{112}^1}{\partial x_2} - \frac{12}{5}\frac{\partial T_{112}^2}{\partial x_2} - \frac{4}{5}T_{113}^0 + \frac{8}{5}T_{113}^3 - \frac{3}{5}\frac{\partial T_{221}^1}{\partial x_1} + \frac{3}{5}\frac{\partial T_{221}^2}{\partial x_1}$$

$$\left. + \frac{1}{5}T_{223}^0 - \frac{2}{5}T_{223}^3 - \frac{3}{5}\frac{\partial T_{331}^1}{\partial x_1} + \frac{3}{5}\frac{\partial T_{331}^2}{\partial x_1} - \frac{3}{5}\frac{\partial T_{332}^1}{\partial x_2} + \frac{3}{5}\frac{\partial T_{332}^2}{\partial x_2} - Y_{12}^0 + \frac{1}{2}Y_{12}^3 \right) n_{x_1}$$

$$+ \left(-M_{12}^1 + M_{12}^2 - T_{123}^0 + 2T_{123}^3 + \frac{1}{2}Y_{11}^0 - \frac{1}{2}Y_{22}^0 + \frac{1}{2}Y_{22}^3 \right) n_{x_2} = 0 \qquad (14.8)$$

$$\delta \frac{\partial w}{\partial x_2} = 0 \quad \text{or}$$

$$\left(-M_{12}^1 + M_{12}^2 - T_{123}^0 + 2T_{123}^3 + \frac{1}{2}Y_{11}^0 - \frac{1}{2}Y_{11}^3 - \frac{1}{2}Y_{22}^0 \right) n_{x_1} + \left(-M_{22}^1 + M_{22}^2 \right.$$

$$+ \frac{\partial P_1^1}{\partial x_1} - \frac{\partial P_1^2}{\partial x_1} + \frac{\partial P_2^1}{\partial x_2} - \frac{\partial P_2^2}{\partial x_2} - P_3^0 + P_3^3 - \frac{3}{5}\frac{\partial T_{111}^1}{\partial x_1} + \frac{3}{5}\frac{\partial T_{111}^2}{\partial x_1} + \frac{2}{5}\frac{\partial T_{222}^1}{\partial x_2} - \frac{2}{5}\frac{\partial T_{222}^2}{\partial x_2}$$

$$+ \frac{1}{5}T_{333}^0 - \frac{2}{5}T_{333}^3 - \frac{3}{5}\frac{\partial T_{112}^1}{\partial x_2} + \frac{3}{5}\frac{\partial T_{112}^2}{\partial x_2} + \frac{1}{5}T_{113}^0 - \frac{2}{5}T_{113}^3 + \frac{12}{5}\frac{\partial T_{221}^1}{\partial x_1} - \frac{12}{5}\frac{\partial T_{221}^2}{\partial x_1}$$

$$\left. - \frac{4}{5}T_{223}^0 + \frac{8}{5}T_{223}^3 - \frac{3}{5}\frac{\partial T_{331}^1}{\partial x_1} + \frac{3}{5}\frac{\partial T_{331}^2}{\partial x_1} - \frac{3}{5}\frac{\partial T_{332}^1}{\partial x_2} + \frac{3}{5}\frac{\partial T_{332}^2}{\partial x_2} + Y_{12}^0 - \frac{1}{2}Y_{12}^3 \right) n_{x_2} = 0 \qquad (14.9)$$

$$\delta \frac{\partial^2 w}{\partial x_1^2} = 0 \quad \text{or}$$

$$\left(-P_1^1 + P_1^2 - \frac{2}{5}T_{111}^1 + \frac{2}{5}T_{111}^2 + \frac{3}{5}T_{221}^1 - \frac{3}{5}T_{221}^2 + \frac{3}{5}T_{331}^1 - \frac{3}{5}T_{331}^2 \right) n_{x_1}$$

$$+ \left(-P_2^1 + P_2^2 + \frac{3}{5}T_{222}^1 - \frac{3}{5}T_{222}^2 - \frac{12}{5}T_{112}^1 + \frac{12}{5}T_{112}^2 + \frac{3}{5}T_{332}^1 - \frac{3}{5}T_{332}^2 \right) n_{x_2} = 0 \qquad (14.10)$$

$$\delta \frac{\partial^2 w}{\partial x_2^2} = 0 \quad \text{or}$$

$$\left(-P_1^1 + P_1^2 + \frac{3}{5}T_{111}^1 - \frac{3}{5}T_{111}^2 - \frac{12}{5}T_{221}^1 + \frac{12}{5}T_{221}^2 + \frac{3}{5}T_{331}^1 - \frac{3}{5}T_{331}^2 \right) n_{x_1}$$

$$+ \left(-P_2^1 + P_2^2 - \frac{2}{5}T_{222}^1 + \frac{2}{5}T_{222}^2 + \frac{3}{5}T_{112}^1 - \frac{3}{5}T_{112}^2 + \frac{3}{5}T_{332}^1 - \frac{3}{5}T_{332}^2 \right) n_{x_2} = 0 \qquad (14.11)$$

$\delta\phi_1 = 0$ or

$$\left(M_{11}^2 - \frac{\partial P_1^2}{\partial x_1} - \frac{1}{2}\frac{\partial P_2^2}{\partial x_2} + P_3^3 - \frac{2}{5}\frac{\partial T_{111}^2}{\partial x_1} + \frac{1}{5}\frac{\partial T_{222}^2}{\partial x_2} - \frac{1}{5}2T_{333}^3 \right.$$

$$\left. -\frac{4}{5}\frac{\partial T_{112}^2}{\partial x_2} + \frac{8}{5}T_{113}^3 + \frac{3}{5}\frac{\partial T_{221}^2}{\partial x_1} - \frac{2}{5}T_{223}^3 + \frac{3}{5}\frac{\partial T_{331}^2}{\partial x_1} + \frac{1}{5}\frac{\partial T_{332}^2}{\partial x_2} + \frac{1}{2}Y_{12}^3 + \frac{1}{4}\frac{\partial Y_{13}^2}{\partial x_2} \right) n_{x_1}$$

$$+\left(M_{12}^2 - \frac{1}{2}\frac{\partial P_2^2}{\partial x_1} + \frac{1}{5}\frac{\partial T_{111}^2}{\partial x_2} + \frac{1}{5}\frac{\partial T_{222}^2}{\partial x_1} - \frac{4}{5}\frac{\partial T_{112}^2}{\partial x_1} - \frac{4}{5}\frac{\partial T_{221}^2}{\partial x_2} + \frac{1}{5}\frac{\partial T_{331}^2}{\partial x_2} + \frac{1}{5}\frac{\partial T_{332}^2}{\partial x_1} \right.$$

$$\left. +2T_{123}^3 + \frac{Y_{22}^3}{2} - \frac{Y_{33}^3}{2} + \frac{1}{4}\frac{\partial Y_{13}^2}{\partial x_1} + \frac{1}{2}\frac{\partial Y_{23}^2}{\partial x_2} \right) n_{x_2} = 0 \qquad (14.12)$$

$\delta\dfrac{\partial\phi_1}{\partial x_1} = 0$ or

$$\left(P_1^2 + \frac{2}{5}T_{111}^2 - \frac{3}{5}T_{221}^2 - \frac{3}{5}T_{331}^2 \right) n_{x_1} + \left(\frac{P_2^2}{2} - \frac{T_{222}^2}{5} + \frac{4}{5}T_{112}^2 - \frac{T_{332}^2}{5} - \frac{Y_{13}^2}{4} \right) n_{x_2} = 0 \qquad (14.13)$$

$\delta\dfrac{\partial\phi_1}{\partial x_2} = 0$ or

$$\left(\frac{P_2^2}{2} - \frac{T_{222}^2}{5} + \frac{4}{5}T_{112}^2 - \frac{T_{332}^2}{5} - \frac{Y_{13}^2}{4} \right) n_{x_1} + \left(-\frac{T_{111}^2}{5} + \frac{4}{5}T_{221}^2 - \frac{T_{331}^2}{5} - \frac{Y_{23}^2}{2} \right) n_{x_2} = 0 \qquad (14.14)$$

$\delta\phi_2 = 0$ or

$$\left(M_{12}^2 - \frac{1}{2}\frac{\partial P_1^2}{\partial x_2} + \frac{1}{5}\frac{\partial T_{111}^2}{\partial x_2} + \frac{1}{5}\frac{\partial T_{222}^2}{\partial x_1} - \frac{4}{5}\frac{\partial T_{112}^2}{\partial x_1} - \frac{4}{5}\frac{\partial T_{221}^2}{\partial x_2} + \frac{1}{5}\frac{\partial T_{331}^2}{\partial x_2} + \frac{1}{5}\frac{\partial T_{332}^2}{\partial x_1} \right.$$

$$\left. +2T_{123}^3 - \frac{1}{2}Y_{11}^3 + \frac{1}{2}Y_{33}^3 - \frac{1}{2}\frac{\partial Y_{13}^2}{\partial x_1} - \frac{1}{4}\frac{\partial Y_{23}^2}{\partial x_2} \right) n_{x_1}$$

$$+\left(M_{22}^2 - \frac{1}{2}\frac{\partial P_1^2}{\partial x_1} - \frac{\partial P_2^2}{\partial x_2} + P_3^3 + \frac{1}{5}\frac{\partial T_{111}^2}{\partial x_1} - \frac{2}{5}\frac{\partial T_{222}^2}{\partial x_2} - \frac{2}{5}T_{333}^3 + \frac{3}{5}\frac{\partial T_{112}^2}{\partial x_2} - \frac{2}{5}T_{113}^3 \right.$$

$$\left. -\frac{4}{5}\frac{\partial T_{221}^2}{\partial x_1} + \frac{8}{5}T_{223}^3 + \frac{1}{5}\frac{\partial T_{331}^2}{\partial x_1} + \frac{3}{5}\frac{\partial T_{332}^2}{\partial x_2} - \frac{1}{2}Y_{12}^3 - \frac{1}{4}\frac{\partial Y_{23}^2}{\partial x_1} \right) n_{x_2} = 0 \qquad (14.15)$$

$\delta\dfrac{\partial\phi_2}{\partial x_1} = 0$ or

$$\left(-\frac{T_{222}^2}{5} + \frac{4}{5}T_{112}^2 - \frac{T_{332}^2}{5} + \frac{Y_{13}^2}{2} \right) n_{x_1} + \left(\frac{P_1^2}{2} - \frac{T_{111}^2}{5} + \frac{4}{5}T_{221}^2 - \frac{T_{331}^2}{5} + \frac{Y_{23}^2}{4} \right) n_{x_2} = 0 \qquad (14.16)$$

$$\delta \frac{\partial \phi_2}{\partial x_2} = 0 \quad \text{or}$$

$$\left(\frac{P_1^2}{2} - \frac{T_{111}^2}{5} + \frac{4}{5} T_{221}^2 - \frac{T_{331}^2}{5} + \frac{Y_{23}^2}{4} \right) n_{x_1} + \left(P_2^2 + \frac{2}{5} T_{222}^2 - \frac{3}{5} T_{112}^2 - \frac{3}{5} T_{332}^2 \right) n_{x_2} = 0 \qquad (14.17)$$

n_{x_1} and n_{x_2} are the direction cosines of unit normal of the boundary of mid-plane; $P_{x_1}^0$, $P_{x_2}^0$ and $P_{x_1 x_2}^0$ are thermally induced initial in-plane forces determined from a static thermal bending analysis; M, P, T and Y are utilized to denote the stress resultants through the plate thickness, respectively. These stress resultants are expressed as

$$M_{11}^i = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{11} \zeta_i dx_3 \ \left(i = 0,\ 1,\ 2\right), \quad M_{22}^i = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{22} \zeta_i dx_3 \ \left(i = 0,\ 1,\ 2\right),$$

$$M_{12}^i = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{12} \zeta_i dx_3 \ \left(i = 0,\ 1,\ 2\right), \quad M_{13}^i = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{13} \zeta_i dx_3 \ \left(i = 3\right),$$

$$M_{23}^i = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{23} \zeta_i dx_3 \ \left(i = 3\right) \qquad (15.1)$$

$$P_1^i = \int_{-\frac{h}{2}}^{\frac{h}{2}} p_1 \zeta_i dx_3 \ \left(i = 0,\ 1,\ 2\right), \quad P_2^i = \int_{-\frac{h}{2}}^{\frac{h}{2}} p_2 \zeta_i dx_3 \ \left(i = 0,\ 1,\ 2\right),$$

$$P_3^i = \int_{-\frac{h}{2}}^{\frac{h}{2}} p_3 \zeta_i dx_3 \ \left(i = 0,\ 3\right) \qquad (15.2)$$

$$T_{111}^i = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{111}^{(1)} \zeta_i dx_3 \ \left(i = 0,\ 1,\ 2,\ 4\right), \quad T_{222}^i = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{222}^{(1)} \zeta_i dx_3 \ \left(i = 0,\ 1,\ 2,\ 4\right),$$

$$T_{333}^i = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{333}^{(1)} \zeta_i dx_3 \ \left(i = 0,\ 3\right), \quad T_{112}^i = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{112}^{(1)} \zeta_i dx_3 \ \left(i = 0,\ 1,\ 2,\ 4\right),$$

$$T_{113}^i = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{113}^{(1)} \zeta_i dx_3 \ \left(i = 0,\ 3\right), \quad T_{221}^i = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{221}^{(1)} \zeta_i dx_3 \ \left(i = 0,\ 1,\ 2,\ 4\right),$$

$$T_{223}^i = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{223}^{(1)} \zeta_i dx_3 \ \left(i = 0,\ 3\right), \quad T_{331}^i = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{331}^{(1)} \zeta_i dx_3 \ \left(i = 0,\ 1,\ 2,\ 4\right),$$

$$T_{332}^i = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{332}^{(1)} \zeta_i dx_3 \ \left(i = 0,\ 1,\ 2,\ 4\right), \quad T_{123}^i = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{123}^{(1)} \zeta_i dx_3 \ \left(i = 0,\ 3\right), \qquad (15.3)$$

$$Y_{11}^{i}=\int_{-\frac{h}{2}}^{\frac{h}{2}} m_{11}^{s}\zeta_i dx_3 \ \left(i=0,\ 3\right), \quad Y_{22}^{i}=\int_{-\frac{h}{2}}^{\frac{h}{2}} m_{22}^{s}\zeta_i dx_3 \ \left(i=0,\ 3\right),$$

$$Y_{33}^{i}=\int_{-\frac{h}{2}}^{\frac{h}{2}} m_{33}^{s}\zeta_i dx_3 \ \left(i=3\right),$$

$$Y_{12}^{i}=\int_{-\frac{h}{2}}^{\frac{h}{2}} m_{12}^{s}\zeta_i dx_3 \ \left(i=0,\ 3\right), \quad Y_{13}^{i}=\int_{-\frac{h}{2}}^{\frac{h}{2}} m_{13}^{s}\zeta_i dx_3 \ \left(i=0,\ 2,\ 4\right),$$

$$Y_{23}^{i}=\int_{-\frac{h}{2}}^{\frac{h}{2}} m_{23}^{s}\zeta_i dx_3 \ \left(i=0,\ 2,\ 4\right) \tag{15.4}$$

where $\zeta_0=1,\ \zeta_1=x_3,\ \zeta_2=f,\ \zeta_3=f'$ and $\zeta_4=f''$. A prime mark is used for differentiation with respect to x_3.

By inserting the expressions for stress resultants given in Eq. (15) into governing equations and boundary conditions, and introducing the following parameters

$$\left\{A_{11},B_{11},D_{11},F_{11},F_{22},F_{33}\right\}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\frac{E\left(x_3\right)}{1-\nu\left(x_3\right)^2}\left\{1,x_3,x_3^2,f,x_3f,f^2\right\}dx_3, \tag{16.1}$$

$$\left\{A_{L11},B_{L11},D_{L11},F_{L11},F_{L22},F_{L33}\right\}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\frac{E\left(x_3\right)\nu\left(x_3\right)}{1-\nu\left(x_3\right)^2}\left\{1,x_3,x_3^2,f,x_3f,f^2\right\}dx_3, \tag{16.2}$$

$$\begin{aligned}&\left\{A_{55},B_{55},D_{55},F_{44},F_{46},F_{47},F_{48},F_{55},F_{57},F_{66},F_{67},F_{68}\right\}\\&=\int_{-\frac{h}{2}}^{\frac{h}{2}}\frac{E\left(x_3\right)}{2\left(1+\nu\left(x_3\right)\right)}\left\{1,x_3,x_3^2,f^2,ff'',f,x_3f,f'^2,f',f''^2,f'',x_3f''\right\}dx_3,\end{aligned} \tag{16.3}$$

$$\begin{aligned}&\left\{A_{550},B_{550},D_{550},F_{440},F_{460},F_{470},F_{480},F_{550},F_{570},F_{660},F_{670},F_{680}\right\}\\&=\int_{-\frac{h}{2}}^{\frac{h}{2}}\frac{E\left(x_3\right)l_0\left(x_3\right)^2}{2\left(1+\nu\left(x_3\right)\right)}\left\{1,x_3,x_3^2,f^2,ff'',f,x_3f,f'^2,f',f''^2,f'',x_3f''\right\}dx_3,\end{aligned} \tag{16.4}$$

$$\begin{aligned}&\left\{A_{551},B_{551},D_{551},F_{441},F_{461},F_{471},F_{481},F_{551},F_{571},F_{661},F_{671},F_{681}\right\}\\&=\int_{-\frac{h}{2}}^{\frac{h}{2}}\frac{E\left(x_3\right)l_1\left(x_3\right)^2}{2\left(1+\nu\left(x_3\right)\right)}\left\{1,x_3,x_3^2,f^2,ff'',f,x_3f,f'^2,f',f''^2,f'',x_3f''\right\}dx_3,\end{aligned} \tag{16.5}$$

$$\left\{A_{552},B_{552},D_{552},F_{442},F_{462},F_{472},F_{482},F_{552},F_{572},F_{662},F_{672},F_{682}\right\}$$
$$=\int_{-\frac{h}{2}}^{\frac{h}{2}}\frac{E\left(x_3\right)l_2\left(x_3\right)^2}{2\left(1+\nu\left(x_3\right)\right)}\left\{1,x_3,x_3^2,f^2,ff'',f,x_3f,f'^2,f',f''^2,f'',x_3f''\right\}dx_3, \quad (16.6)$$

$$\left\{A_{T11},B_{T11},F_{T11}\right\}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\frac{E\left(x_3\right)}{\left(1-\upsilon\left(x_3\right)\right)}\alpha\left(x_3\right)\left\{1,x_3,f\right\}\Delta T\left(x_3\right)\mathrm{d}x_3 \quad (16.7)$$

equations of motion are obtained in terms of displacements as the following form

δu :

$$\left(-2A_{550}-\frac{4}{5}A_{551}\right)\frac{\partial^4 u}{\partial x_1^4}+\left(-\frac{8}{15}A_{551}-\frac{1}{4}A_{552}\right)\frac{\partial^4 u}{\partial x_2^4}$$
$$+\left(-2A_{550}-\frac{4}{3}A_{551}-\frac{1}{4}A_{552}\right)\frac{\partial^4 u}{\partial x_1^2\partial x_2^2}+A_{11}\frac{\partial^2 u}{\partial x_1^2}+A_{55}\frac{\partial^2 u}{\partial x_2^2}$$
$$+\left(-2A_{550}-\frac{4}{15}A_{551}+\frac{1}{4}A_{552}\right)\frac{\partial^4 v}{\partial x_1^3\partial x_2}+\left(-2A_{550}-\frac{4}{15}A_{551}+\frac{1}{4}A_{552}\right)\frac{\partial^4 v}{\partial x_1\partial x_2^3}$$
$$+\left(A_{55}+A_{L11}\right)\frac{\partial^2 v}{\partial x_1\partial x_2}+\left(-2F_{470}+2B_{550}+\frac{4}{5}B_{551}-\frac{4}{5}F_{471}\right)\frac{\partial^5 w}{\partial x_1^5}$$
$$+\left(4B_{550}-4F_{470}+\frac{8}{5}B_{551}-\frac{8}{5}F_{471}\right)\frac{\partial^5 w}{\partial x_1^3\partial x_2^2}$$
$$+\left(-2F_{470}+2B_{550}+\frac{4}{5}B_{551}-\frac{4}{5}F_{471}\right)\frac{\partial^5 w}{\partial x_1\partial x_2^4}+\left(F_{11}-B_{11}+\frac{2}{5}F_{671}\right)\frac{\partial^3 w}{\partial x_1^3}$$
$$+\left(F_{11}-B_{11}+\frac{2}{5}F_{671}\right)\frac{\partial^3 w}{\partial x_1\partial x_2^2}+\left(-2F_{470}-\frac{4}{5}F_{471}\right)\frac{\partial^4\phi_1}{\partial x_1^4}+\left(-\frac{8}{15}F_{471}-\frac{1}{4}F_{472}\right)\frac{\partial^4\phi_1}{\partial x_2^4}$$
$$+\left(-2F_{470}-\frac{4}{3}F_{471}-\frac{1}{4}F_{472}\right)\frac{\partial^4\phi_1}{\partial x_1^2\partial x_2^2}+\left(F_{11}+\frac{2}{5}F_{671}\right)\frac{\partial^2\phi_1}{\partial x_1^2}$$
$$+\left(F_{47}+\frac{2}{15}F_{671}+\frac{1}{4}F_{672}\right)\frac{\partial^2\phi_1}{\partial x_2^2}+\left(-2F_{470}-\frac{4}{15}F_{471}+\frac{1}{4}F_{472}\right)\frac{\partial^4\phi_2}{\partial x_1^3\partial x_2}$$
$$+\left(-2F_{470}-\frac{4}{15}F_{471}+\frac{1}{4}F_{472}\right)\frac{\partial^4\phi_2}{\partial x_1\partial x_2^3}+\left(F_{L11}+F_{47}+\frac{4}{15}F_{671}-\frac{1}{4}F_{672}\right)\frac{\partial^2\phi_2}{\partial x_1\partial x_2}$$
$$-\frac{\partial\left(A_{T11}\right)}{\partial x_1}=I_0\frac{\partial^2 u}{\partial t^2}+\left(I_3-I_1\right)\frac{\partial^3 w}{\partial x_1\partial t^2}+I_3\frac{\partial^2\phi_1}{\partial t^2}, \quad (17.1)$$

δv :

$$\left(-2A_{550}-\frac{4}{15}A_{551}+\frac{1}{4}A_{552}\right)\frac{\partial^4 u}{\partial x_1^3\partial x_2}+\left(-2A_{550}-\frac{4}{15}A_{551}+\frac{1}{4}A_{552}\right)\frac{\partial^4 u}{\partial x_1\partial x_2^3}$$
$$+\left(A_{55}+A_{L11}\right)\frac{\partial^2 u}{\partial x_1\partial x_2}+\left(-\frac{8}{15}A_{551}-\frac{1}{4}A_{552}\right)\frac{\partial^4 v}{\partial x_1^4}+\left(-2A_{550}-\frac{4}{5}A_{551}\right)\frac{\partial^4 v}{\partial x_2^4}$$
$$+\left(-2A_{550}-\frac{4}{3}A_{551}-\frac{1}{4}A_{552}\right)\frac{\partial^4 v}{\partial x_1^2\partial x_2^2}+A_{55}\frac{\partial^2 v}{\partial x_1^2}+A_{11}\frac{\partial^2 v}{\partial x_2^2}$$
$$+\left(-2F_{470}+2B_{550}+\frac{4}{5}B_{551}-\frac{4}{5}F_{471}\right)\frac{\partial^5 w}{\partial x_2^5}+\left(2B_{550}+\frac{4}{5}B_{551}-2F_{470}-\frac{4}{5}F_{471}\right)\frac{\partial^5 w}{\partial x_1^4\partial x_2}$$
$$+\left(4B_{550}+\frac{8}{5}B_{551}-4F_{470}-\frac{8}{5}F_{471}\right)\frac{\partial^5 w}{\partial x_1^2\partial x_2^3}+\left(F_{11}-B_{11}+\frac{2}{5}F_{671}\right)\frac{\partial^3 w}{\partial x_2^3}$$
$$+\left(F_{11}-B_{11}+\frac{2}{5}F_{671}\right)\frac{\partial^3 w}{\partial x_1^2\partial x_2}+\left(-2F_{470}-\frac{4}{15}F_{471}+\frac{1}{4}F_{472}\right)\frac{\partial^4 \phi_1}{\partial x_1^3\partial x_2}$$
$$+\left(-2F_{470}-\frac{4}{15}F_{471}+\frac{1}{4}F_{472}\right)\frac{\partial^4 \phi_1}{\partial x_1\partial x_2^3}+\left(F_{L11}+F_{47}+\frac{4}{15}F_{671}-\frac{1}{4}F_{672}\right)\frac{\partial^2 \phi_1}{\partial x_1\partial x_2}$$
$$+\left(-\frac{8}{15}F_{471}-\frac{1}{4}F_{472}\right)\frac{\partial^4 \phi_2}{\partial x_1^4}+\left(-2F_{470}-\frac{4}{5}F_{471}\right)\frac{\partial^4 \phi_2}{\partial x_2^4}$$
$$+\left(-2F_{470}-\frac{4}{3}F_{471}-\frac{1}{4}F_{472}\right)\frac{\partial^4 \phi_2}{\partial x_1^2\partial x_2^2}+\left(F_{47}+\frac{2}{15}F_{671}+\frac{1}{4}F_{672}\right)\frac{\partial^2 \phi_2}{\partial x_1^2}$$
$$+\left(F_{11}+\frac{2}{5}F_{671}\right)\frac{\partial^2 \phi_2}{\partial x_2^2}-\frac{\partial\left(A_{T11}\right)}{\partial x_2}=I_0\frac{\partial^2 v}{\partial t^2}+\left(I_3-I_1\right)\frac{\partial^3 w}{\partial x_2\partial t^2}+I_3\frac{\partial^2 \phi_2}{\partial t^2}, \qquad (17.2)$$

$\delta w:$

$$\left(-2B_{550}+2F_{470}-\frac{4}{5}B_{551}+\frac{4}{5}F_{471}\right)\frac{\partial^5 u}{\partial x_1^5}+\left(-4B_{550}+4F_{470}-\frac{8}{5}B_{551}+\frac{8}{5}F_{471}\right)\frac{\partial^5 u}{\partial x_1^3\partial x_2^2}$$

$$+\left(-2B_{550}+2F_{470}-\frac{4}{5}B_{551}+\frac{4}{5}F_{471}\right)\frac{\partial^5 u}{\partial x_1\partial x_2^4}+\left(B_{11}-F_{11}-\frac{2}{5}F_{671}\right)\frac{\partial^3 u}{\partial x_1^3}$$

$$+\left(B_{11}-F_{11}-\frac{2}{5}F_{671}\right)\frac{\partial^3 u}{\partial x_1\partial x_2^2}+\left(-2B_{550}+2F_{470}-\frac{4}{5}B_{551}+\frac{4}{5}F_{471}\right)\frac{\partial^5 v}{\partial x_2^5}$$

$$+\left(-2B_{550}+2F_{470}-\frac{4}{5}B_{551}+\frac{4}{5}F_{471}\right)\frac{\partial^5 v}{\partial x_1^4\partial x_2}$$

$$+\left(-4B_{550}+4F_{470}-\frac{8}{5}B_{551}+\frac{8}{5}F_{471}\right)\frac{\partial^5 v}{\partial x_1^2\partial x_2^3}+\left(B_{11}-F_{11}-\frac{2}{5}F_{671}\right)\frac{\partial^3 v}{\partial x_2^3}$$

$$+\left(B_{11}-F_{11}-\frac{2}{5}F_{671}\right)\frac{\partial^3 v}{\partial x_1^2\partial x_2}+\left(2D_{550}+2F_{440}-4F_{480}+\frac{4}{5}D_{551}+\frac{4}{5}F_{441}-\frac{8}{5}F_{481}\right)\frac{\partial^6 w}{\partial x_1^6}$$

$$+\left(2D_{550}+2F_{440}-4F_{480}+\frac{4}{5}D_{551}+\frac{4}{5}F_{441}-\frac{8}{5}F_{481}\right)\frac{\partial^6 w}{\partial x_2^6}$$

$$+\left(6D_{550}+6F_{440}-12F_{480}+\frac{12}{5}D_{551}+\frac{12}{5}F_{441}-\frac{24}{5}F_{481}\right)\frac{\partial^6 w}{\partial x_1^4\partial x_2^2}$$

$$+\left(6D_{550}+6F_{440}-12F_{480}+\frac{12}{5}D_{551}+\frac{12}{5}F_{441}-\frac{24}{5}F_{481}\right)\frac{\partial^6 w}{\partial x_1^2\partial x_2^4}$$

$$+\left(-D_{11}+2F_{22}-F_{33}-2A_{550}-2F_{550}+4F_{570}-\frac{8}{15}A_{551}-\frac{4}{5}F_{461}-\frac{32}{15}F_{551}+\frac{32}{15}F_{571}\right.$$

$$\left.+\frac{4}{5}F_{681}-A_{552}-\frac{1}{4}F_{552}+F_{572}\right)\frac{\partial^4 w}{\partial x_1^4}+\left(-D_{11}+2F_{22}-F_{33}-2A_{550}-2F_{550}+4F_{570}\right.$$

$$\left.-\frac{8}{15}A_{551}-\frac{4}{5}F_{461}-\frac{32}{15}F_{551}+\frac{32}{15}F_{571}+\frac{4}{5}F_{681}-A_{552}-\frac{1}{4}F_{552}+F_{572}\right)\frac{\partial^4 w}{\partial x_2^4}$$

$$+\left(-2D_{11}+4F_{22}-2F_{33}-4A_{550}-4F_{550}+8F_{570}-\frac{16}{15}A_{551}-\frac{8}{5}F_{461}-\frac{64}{15}F_{551}+\frac{64}{15}F_{571}\right.$$

$$\left.+\frac{8}{5}F_{681}-2A_{552}-\frac{1}{2}F_{552}+2F_{572}\right)\frac{\partial^4 w}{\partial x_1^2\partial x_2^2}+\left(k_sF_{55}+\frac{8}{15}F_{661}+\frac{1}{4}F_{662}\right)\frac{\partial^2 w}{\partial x_1^2}$$

$$+\left(k_sF_{55}+\frac{8}{15}F_{661}+\frac{1}{4}F_{662}\right)\frac{\partial^2 w}{\partial x_2^2}+\left(2F_{440}-2F_{480}+\frac{4}{5}F_{441}-\frac{4}{5}F_{481}\right)\frac{\partial^5 \phi_1}{\partial x_1^5}$$

$$+\left(4F_{440}-4F_{480}+\frac{8}{5}F_{441}-\frac{8}{5}F_{481}\right)\frac{\partial^5 \phi_1}{\partial x_1^3\partial x_2^2}$$

$$+\left(2F_{440}-2F_{480}+\frac{4}{5}F_{441}-\frac{4}{5}F_{481}\right)\frac{\partial^5 \phi_1}{\partial x_1\partial x_2^4}+\left(F_{22}-F_{33}-2F_{550}+2F_{570}-\frac{4}{5}F_{461}\right.$$

$$-\frac{32}{15}F_{551}+\frac{16}{15}F_{571}+\frac{2}{5}F_{681}-\frac{1}{4}F_{552}+\frac{1}{2}F_{572}\Bigg)\frac{\partial^3\phi_1}{\partial x_1^3}+\Bigg(F_{22}-F_{33}-2F_{550}+2F_{570}-\frac{4}{5}F_{461}$$

$$-\frac{32}{15}F_{551}+\frac{16}{15}F_{571}+\frac{2}{5}F_{681}-\frac{1}{4}F_{552}+\frac{1}{2}F_{572}\Bigg)\frac{\partial^3\phi_1}{\partial x_1\partial x_2^2}+\Bigg(k_sF_{55}+\frac{8}{15}F_{661}+\frac{1}{4}F_{662}\Bigg)\frac{\partial\phi_1}{\partial x_1}$$

$$+\Bigg(2F_{440}-2F_{480}+\frac{4}{5}F_{441}-\frac{4}{5}F_{481}\Bigg)\frac{\partial^5\phi_2}{\partial x_2^5}+\Bigg(2F_{440}-2F_{480}+\frac{4}{5}F_{441}-\frac{4}{5}F_{481}\Bigg)\frac{\partial^5\phi_2}{\partial x_1^4\partial x_2}$$

$$+\Bigg(4F_{440}-4F_{480}+\frac{8}{5}F_{441}-\frac{8}{5}F_{481}\Bigg)\frac{\partial^5\phi_2}{\partial x_1^2\partial x_2^3}+\Bigg(F_{22}-F_{33}-2F_{550}+2F_{570}-\frac{4}{5}F_{461}$$

$$-\frac{32}{15}F_{551}+\frac{16}{15}F_{571}+\frac{2}{5}F_{681}-\frac{1}{4}F_{552}+\frac{1}{2}F_{572}\Bigg)\frac{\partial^3\phi_2}{\partial x_2^3}+\Bigg(F_{22}-F_{33}-2F_{550}+2F_{570}-\frac{4}{5}F_{461}$$

$$-\frac{32}{15}F_{551}+\frac{16}{15}F_{571}+\frac{2}{5}F_{681}-\frac{1}{4}F_{552}+\frac{1}{2}F_{572}\Bigg)\frac{\partial^3\phi_2}{\partial x_1^2\partial x_2}+\Bigg(k_sF_{55}+\frac{8}{15}F_{661}+\frac{1}{4}F_{662}\Bigg)\frac{\partial\phi_2}{\partial x_2}$$

$$+q-P_{x_1}\frac{\partial^2 w}{\partial x_1^2}-P_{x_2}\frac{\partial^2 w}{\partial x_2^2}+P_{x_1}^0\frac{\partial^2 w}{\partial x_1^2}+P_{x_2}^0\frac{\partial^2 w}{\partial x_2^2}+2P_{x_1x_2}^0\frac{\partial^2 w}{\partial x_1\partial x_2}-\frac{\partial^2\left(B_{T11}-F_{T11}\right)}{\partial x_1^2}$$

$$-\frac{\partial^2\left(B_{T11}-F_{T11}\right)}{\partial x_2^2}=\left(I_1-I_3\right)\frac{\partial^3 u}{\partial x_1\partial t^2}+\left(I_1-I_3\right)\frac{\partial^3 v}{\partial x_2\partial t^2}+\left(2I_4-I_2-I_5\right)\frac{\partial^4 w}{\partial x_1^2\partial t^2}$$

$$+\left(2I_4-I_2-I_5\right)\frac{\partial^4 w}{\partial x_2^2\partial t^2}+I_0\frac{\partial^2 w}{\partial t^2}+\left(I_4-I_5\right)\frac{\partial^3\phi_1}{\partial x_1\partial t^2}+\left(I_4-I_5\right)\frac{\partial^3\phi_2}{\partial x_2\partial t^2}, \qquad (17.3)$$

$\delta\phi_1$:

$$\left(-2F_{470}-\frac{4}{5}F_{471}\right)\frac{\partial^4 u}{\partial x_1^4}+\left(-\frac{8}{15}F_{471}-\frac{1}{4}F_{472}\right)\frac{\partial^4 u}{\partial x_2^4}+\left(-2F_{470}-\frac{4}{3}F_{471}-\frac{1}{4}F_{472}\right)\frac{\partial^4 u}{\partial x_1^2\partial x_2^2}$$
$$+\left(F_{11}+\frac{2}{5}F_{671}\right)\frac{\partial^2 u}{\partial x_1^2}+\left(F_{47}+\frac{2}{15}F_{671}+\frac{1}{4}F_{672}\right)\frac{\partial^2 u}{\partial x_2^2}$$
$$+\left(-2F_{470}-\frac{4}{15}F_{471}+\frac{1}{4}F_{472}\right)\frac{\partial^4 v}{\partial x_1^3\partial x_2}+\left(-2F_{470}-\frac{4}{15}F_{471}+\frac{1}{4}F_{472}\right)\frac{\partial^4 v}{\partial x_1\partial x_2^3}$$
$$+\left(F_{L11}+F_{47}+\frac{4}{15}F_{671}-\frac{1}{4}F_{672}\right)\frac{\partial^2 v}{\partial x_1\partial x_2}+\left(-2F_{440}+2F_{480}-\frac{4}{5}F_{441}+\frac{4}{5}F_{481}\right)\frac{\partial^5 w}{\partial x_1^5}$$
$$+\left(-4F_{440}+4F_{480}-\frac{8}{5}F_{441}+\frac{8}{5}F_{481}\right)\frac{\partial^5 w}{\partial x_1^3\partial x_2^2}+\left(-2F_{440}+2F_{480}-\frac{4}{5}F_{441}\right.$$
$$\left.+\frac{4}{5}F_{481}\right)\frac{\partial^5 w}{\partial x_1\partial x_2^4}+\left(-F_{22}+F_{33}+2F_{550}-2F_{570}+\frac{4}{5}F_{461}+\frac{32}{15}F_{551}-\frac{16}{15}F_{571}-\frac{2}{5}F_{681}\right.$$
$$\left.+\frac{1}{4}F_{552}-\frac{1}{2}F_{572}\right)\frac{\partial^3 w}{\partial x_1^3}+\left(-F_{22}+F_{33}+2F_{550}-2F_{570}+\frac{4}{5}F_{461}+\frac{32}{15}F_{551}-\frac{16}{15}F_{571}\right.$$
$$\left.-\frac{2}{5}F_{681}+\frac{1}{4}F_{552}-\frac{1}{2}F_{572}\right)\frac{\partial^3 w}{\partial x_1\partial x_2^2}+\left(-k_sF_{55}-\frac{8}{15}F_{661}-\frac{1}{4}F_{662}\right)\frac{\partial w}{\partial x_1}$$
$$+\left(-2F_{440}-\frac{4}{5}F_{441}\right)\frac{\partial^4\phi_1}{\partial x_1^4}+\left(-\frac{8}{15}F_{441}-\frac{1}{4}F_{442}\right)\frac{\partial^4\phi_1}{\partial x_2^4}$$
$$+\left(-2F_{440}-\frac{4}{3}F_{441}-\frac{1}{4}F_{442}\right)\frac{\partial^4\phi_1}{\partial x_1^2\partial x_2^2}+\left(F_{33}+2F_{550}+\frac{4}{5}F_{461}+\frac{32}{15}F_{551}+\frac{1}{4}F_{552}\right)\frac{\partial^2\phi_1}{\partial x_1^2}$$
$$+\left(F_{44}+\frac{4}{15}F_{461}+\frac{4}{3}F_{551}+\frac{1}{2}F_{462}+F_{552}\right)\frac{\partial^2\phi_1}{\partial x_2^2}+\left(-k_sF_{55}-\frac{8}{15}F_{661}-\frac{1}{4}F_{662}\right)\phi_1$$
$$+\left(-2F_{440}-\frac{4}{15}F_{441}+\frac{1}{4}F_{442}\right)\frac{\partial^4\phi_2}{\partial x_1^3\partial x_2}+\left(-2F_{440}-\frac{4}{15}F_{441}+\frac{1}{4}F_{442}\right)\frac{\partial^4\phi_2}{\partial x_1\partial x_2^3}$$
$$+\left(F_{L33}+F_{44}+2F_{550}+\frac{8}{15}F_{461}+\frac{4}{5}F_{551}-\frac{1}{2}F_{462}-\frac{3}{4}F_{552}\right)\frac{\partial^2\phi_2}{\partial x_1\partial x_2}$$
$$-\frac{\partial\left(F_{T11}\right)}{\partial x_1}=I_3\frac{\partial^2 u}{\partial t^2}+\left(I_5-I_4\right)\frac{\partial^3 w}{\partial x_1\partial t^2}+I_5\frac{\partial^2\phi_1}{\partial t^2}, \tag{17.4}$$

$\delta\phi_2$:

$$\left(-2F_{470}-\frac{4}{15}F_{471}+\frac{1}{4}F_{472}\right)\frac{\partial^4 u}{\partial x_1^3\partial x_2}+\left(-2F_{470}-\frac{4}{15}F_{471}+\frac{1}{4}F_{472}\right)\frac{\partial^4 u}{\partial x_1\partial x_2^3}$$
$$+\left(F_{L11}+F_{47}+\frac{4}{15}F_{671}-\frac{1}{4}F_{672}\right)\frac{\partial^2 u}{\partial x_1\partial x_2}$$
$$+\left(-\frac{8}{15}F_{471}-\frac{1}{4}F_{472}\right)\frac{\partial^4 v}{\partial x_1^4}+\left(-2F_{470}-\frac{4}{5}F_{471}\right)\frac{\partial^4 v}{\partial x_2^4}$$
$$+\left(-2F_{470}-\frac{4}{3}F_{471}-\frac{1}{4}F_{472}\right)\frac{\partial^4 v}{\partial x_1^2\partial x_2^2}+\left(F_{47}+\frac{2}{15}F_{671}+\frac{1}{4}F_{672}\right)\frac{\partial^2 v}{\partial x_1^2}$$
$$+\left(F_{11}+\frac{2}{5}F_{671}\right)\frac{\partial^2 v}{\partial x_2^2}+\left(-2F_{440}+2F_{480}-\frac{4}{5}F_{441}+\frac{4}{5}F_{481}\right)\frac{\partial^5 w}{\partial x_2^5}$$
$$+\left(-4F_{440}+4F_{480}-\frac{8}{5}F_{441}+\frac{8}{5}F_{481}\right)\frac{\partial^5 w}{\partial x_1^2\partial x_2^3}+\left(-2F_{440}+2F_{480}-\frac{4}{5}F_{441}\right.$$
$$\left.+\frac{4}{5}F_{481}\right)\frac{\partial^5 w}{\partial x_1^4\partial x_2}+\left(-F_{22}+F_{33}+2F_{550}-2F_{570}+\frac{4}{5}F_{461}+\frac{32}{15}F_{551}-\frac{16}{15}F_{571}-\frac{2}{5}F_{681}\right.$$
$$\left.+\frac{1}{4}F_{552}-\frac{1}{2}F_{572}\right)\frac{\partial^3 w}{\partial x_2^3}+\left(-F_{22}+F_{33}+2F_{550}-2F_{570}+\frac{4}{5}F_{461}+\frac{32}{15}F_{551}-\frac{16}{15}F_{571}\right.$$
$$\left.-\frac{2}{5}F_{681}+\frac{1}{4}F_{552}-\frac{1}{2}F_{572}\right)\frac{\partial^3 w}{\partial x_1^2\partial x_2}+\left(-k_sF_{55}-\frac{8}{15}F_{661}-\frac{1}{4}F_{662}\right)\frac{\partial w}{\partial x_2}$$
$$+\left(-2F_{440}-\frac{4}{15}F_{441}+\frac{1}{4}F_{442}\right)\frac{\partial^4\phi_1}{\partial x_1^3\partial x_2}+\left(-2F_{440}-\frac{4}{15}F_{441}+\frac{1}{4}F_{442}\right)\frac{\partial^4\phi_1}{\partial x_1\partial x_2^3}$$
$$+\left(F_{L33}+F_{44}+2F_{550}+\frac{8}{15}F_{461}+\frac{4}{5}F_{551}-\frac{1}{2}F_{462}-\frac{3}{4}F_{552}\right)\frac{\partial^2\phi_1}{\partial x_1\partial x_2}$$
$$+\left(-\frac{8}{15}F_{441}-\frac{1}{4}F_{442}\right)\frac{\partial^4\phi_2}{\partial x_1^4}+\left(-2F_{440}-\frac{4}{5}F_{441}\right)\frac{\partial^4\phi_2}{\partial x_2^4}$$
$$+\left(-2F_{440}-\frac{4}{3}F_{441}-\frac{1}{4}F_{442}\right)\frac{\partial^4\phi_2}{\partial x_1^2\partial x_2^2}+\left(F_{44}+\frac{4}{15}F_{461}+\frac{4}{3}F_{551}+\frac{1}{2}F_{462}+F_{552}\right)\frac{\partial^2\phi_2}{\partial x_1^2}$$
$$+\left(F_{33}+2F_{550}+\frac{4}{5}F_{461}+\frac{32}{15}F_{551}+\frac{1}{4}F_{552}\right)\frac{\partial^2\phi_2}{\partial x_2^2}+\left(-k_sF_{55}-\frac{8}{15}F_{661}-\frac{1}{4}F_{662}\right)\phi_2$$
$$-\frac{\partial\left(F_{T11}\right)}{\partial x_2}=I_3\frac{\partial^2 v}{\partial t^2}+\left(I_5-I_4\right)\frac{\partial^3 w}{\partial x_2\partial t^2}+I_5\frac{\partial^2\phi_2}{\partial t^2}. \tag{17.5}$$

and the corresponding boundary conditions are determined as

$\delta u = 0$ or

$$
\begin{aligned}
&\Bigg(\left(-2A_{550}-\frac{4}{5}A_{551}\right)\frac{\partial^3 u}{\partial x_1^3}+\left(-A_{550}-\frac{2}{3}A_{551}-\frac{1}{8}A_{552}\right)\frac{\partial^3 u}{\partial x_1\partial x_2^2}+A_{11}\frac{\partial u}{\partial x_1}\\
&+\left(-A_{550}+\frac{2}{15}A_{551}\right)\frac{\partial^3 v}{\partial x_2^3}+\left(-2A_{550}+\frac{4}{15}A_{551}+\frac{1}{8}A_{552}\right)\frac{\partial^3 v}{\partial x_1^2\partial x_2}+A_{L11}\frac{\partial v}{\partial x_2}\\
&+\left(-2F_{470}+2B_{550}+\frac{4}{5}B_{551}-\frac{4}{5}F_{471}\right)\frac{\partial^4 w}{\partial x_1^4}+\left(-F_{470}+B_{550}-\frac{2}{5}B_{551}+\frac{2}{5}F_{471}\right)\frac{\partial^4 w}{\partial x_2^4}\\
&+\left(3B_{550}-3F_{470}+\frac{2}{5}B_{551}-\frac{2}{5}F_{471}\right)\frac{\partial^4 w}{\partial x_1^2\partial x_2^2}+\left(F_{11}-B_{11}+\frac{2}{5}F_{671}\right)\frac{\partial^2 w}{\partial x_1^2}\\
&+\left(F_{L11}-B_{L11}+\frac{2}{15}F_{671}-\frac{1}{8}F_{672}\right)\frac{\partial^2 w}{\partial x_2^2}+\left(-2F_{470}-\frac{4}{5}F_{471}\right)\frac{\partial^3\phi_1}{\partial x_1^3}\\
&+\left(-F_{470}-\frac{2}{3}F_{471}-\frac{1}{8}F_{472}\right)\frac{\partial^3\phi_1}{\partial x_1\partial x_2^2}+\left(F_{11}+\frac{2}{5}F_{671}\right)\frac{\partial\phi_1}{\partial x_1}+\left(-F_{470}+\frac{2}{5}F_{471}\right)\frac{\partial^3\phi_2}{\partial x_2^3}\\
&+\left(-2F_{470}+\frac{4}{15}F_{471}+\frac{1}{8}F_{472}\right)\frac{\partial^3\phi_2}{\partial x_1^2\partial x_2}+\left(F_{L11}+\frac{2}{15}F_{671}-\frac{1}{8}F_{672}\right)\frac{\partial\phi_2}{\partial x_2}-A_{T11}\Bigg)n_{x_1}\\
&+\Bigg(\left(-\frac{8}{15}A_{551}-\frac{1}{4}A_{552}\right)\frac{\partial^3 u}{\partial x_2^3}+\left(-A_{550}-\frac{2}{3}A_{551}-\frac{1}{8}A_{552}\right)\frac{\partial^3 u}{\partial x_1^2\partial x_2}+A_{55}\frac{\partial u}{\partial x_2}\\
&+\left(-\frac{8}{15}A_{551}+\frac{1}{8}A_{552}\right)\frac{\partial^3 v}{\partial x_1^3}+\left(-A_{550}-\frac{2}{3}A_{551}+\frac{1}{4}A_{552}\right)\frac{\partial^3 v}{\partial x_1\partial x_2^2}+A_{55}\frac{\partial v}{\partial x_1}\\
&+\left(B_{550}-F_{470}+\frac{6}{5}B_{551}-\frac{6}{5}F_{471}\right)\frac{\partial^4 w}{\partial x_1^3\partial x_2}+\left(-F_{470}+B_{550}+\frac{6}{5}B_{551}-\frac{6}{5}F_{471}\right)\frac{\partial^4 w}{\partial x_1\partial x_2^3}\\
&+\left(2F_{47}-2B_{55}+\frac{4}{15}F_{671}+\frac{1}{8}F_{672}\right)\frac{\partial^2 w}{\partial x_1\partial x_2}+\left(-\frac{8}{15}F_{471}-\frac{1}{4}F_{472}\right)\frac{\partial^3\phi_1}{\partial x_2^3}\\
&+\left(-F_{470}-\frac{2}{3}F_{471}-\frac{1}{8}F_{472}\right)\frac{\partial^3\phi_1}{\partial x_1^2\partial x_2}+\left(F_{47}+\frac{2}{15}F_{671}+\frac{1}{4}F_{672}\right)\frac{\partial\phi_1}{\partial x_2}\\
&+\left(-\frac{8}{15}F_{471}+\frac{1}{8}F_{472}\right)\frac{\partial^3\phi_2}{\partial x_1^3}+\left(-F_{470}-\frac{2}{3}F_{471}+\frac{1}{4}F_{472}\right)\frac{\partial^3\phi_2}{\partial x_1\partial x_2^2}\\
&+\left(F_{47}+\frac{2}{15}F_{671}-\frac{1}{8}F_{672}\right)\frac{\partial\phi_2}{\partial x_1}\Bigg)n_{x_2}=0,
\end{aligned}
\tag{18.1}
$$

$$\delta \frac{\partial u}{\partial x_1} = 0 \quad \text{or}$$

$$\Bigg(\left(2A_{550} + \frac{4}{5}A_{551}\right)\frac{\partial^2 u}{\partial x_1^2} - \frac{2}{5}A_{551}\frac{\partial^2 u}{\partial x_2^2} + \left(2A_{550} - \frac{4}{5}A_{551}\right)\frac{\partial^2 v}{\partial x_1 \partial x_2}$$

$$+\left(2F_{470} - 2B_{550} - \frac{4}{5}B_{551} + \frac{4}{5}F_{471}\right)\frac{\partial^3 w}{\partial x_1^3}$$

$$+\left(-2B_{550} + 2F_{470} + \frac{6}{5}B_{551} - \frac{6}{5}F_{471}\right)\frac{\partial^3 w}{\partial x_1 \partial x_2^2} - \frac{2}{5}F_{671}\frac{\partial w}{\partial x_1} + \left(2F_{470} + \frac{4}{5}F_{471}\right)\frac{\partial^2 \phi_1}{\partial x_1^2}$$

$$-\frac{2}{5}F_{471}\frac{\partial^2 \phi_1}{\partial x_2^2} - \frac{2}{5}F_{671}\phi_1 + \left(2F_{470} - \frac{4}{5}F_{471}\right)\frac{\partial^2 \phi_2}{\partial x_1 \partial x_2}\Bigg)n_{x_1}$$

$$+\Bigg(\left(A_{550} + \frac{16}{15}A_{551} + \frac{1}{8}A_{552}\right)\frac{\partial^2 u}{\partial x_1 \partial x_2} + \left(\frac{8}{15}A_{551} - \frac{1}{8}A_{552}\right)\frac{\partial^2 v}{\partial x_1^2} + \left(A_{550} - \frac{2}{5}A_{551}\right)\frac{\partial^2 v}{\partial x_2^2}$$

$$+\left(F_{470} - B_{550} + \frac{2}{5}B_{551} - \frac{2}{5}F_{471}\right)\frac{\partial^3 w}{\partial x_2^3} + \left(-B_{550} + F_{470} - \frac{8}{5}B_{551} + \frac{8}{5}F_{471}\right)\frac{\partial^3 w}{\partial x_1^2 \partial x_2}$$

$$+\left(-\frac{2}{15}F_{671} + \frac{1}{8}F_{672}\right)\frac{\partial w}{\partial x_2} + \left(F_{470} + \frac{16}{15}F_{471} + \frac{1}{8}F_{472}\right)\frac{\partial^2 \phi_1}{\partial x_1 \partial x_2}$$

$$+\left(\frac{8}{15}F_{471} - \frac{1}{8}F_{472}\right)\frac{\partial^2 \phi_2}{\partial x_1^2} + \left(F_{470} - \frac{2}{5}F_{471}\right)\frac{\partial^2 \phi_2}{\partial x_2^2}$$

$$+\left(-\frac{2}{15}F_{671} + \frac{1}{8}F_{672}\right)\phi_2\Bigg)n_{x_2} = 0, \qquad (18.2)$$

$$\delta \frac{\partial u}{\partial x_2} = 0 \quad \text{or}$$

$$\Bigg(\Bigg(A_{550} + \frac{16}{15}A_{551} + \frac{1}{8}A_{552}\Bigg)\frac{\partial^2 u}{\partial x_1 \partial x_2} + \Bigg(\frac{8}{15}A_{551} - \frac{1}{8}A_{552}\Bigg)\frac{\partial^2 v}{\partial x_1^2} + \Bigg(A_{550} - \frac{2}{5}A_{551}\Bigg)\frac{\partial^2 v}{\partial x_2^2}$$

$$+\Bigg(F_{470} - B_{550} + \frac{2}{5}B_{551} - \frac{2}{5}F_{471}\Bigg)\frac{\partial^3 w}{\partial x_2^3} + \Bigg(-B_{550} + F_{470} - \frac{8}{5}B_{551} + \frac{8}{5}F_{471}\Bigg)\frac{\partial^3 w}{\partial x_1^2 \partial x_2}$$

$$+\Bigg(-\frac{2}{15}F_{671} + \frac{1}{8}F_{672}\Bigg)\frac{\partial w}{\partial x_2} + \Bigg(F_{470} + \frac{16}{15}F_{471} + \frac{1}{8}F_{472}\Bigg)\frac{\partial^2 \phi_1}{\partial x_1 \partial x_2}$$

$$+\Bigg(\frac{8}{15}F_{471} - \frac{1}{8}F_{472}\Bigg)\frac{\partial^2 \phi_2}{\partial x_1^2} + \Bigg(F_{470} - \frac{2}{5}F_{471}\Bigg)\frac{\partial^2 \phi_2}{\partial x_2^2} + \Bigg(-\frac{2}{15}F_{671} + \frac{1}{8}F_{672}\Bigg)\phi_2\Bigg)n_{x_1}$$

$$+\Bigg(-\frac{2}{5}A_{551}\frac{\partial^2 u}{\partial x_1^2} + \Bigg(\frac{8}{15}A_{551} + \frac{1}{4}A_{552}\Bigg)\frac{\partial^2 u}{\partial x_2^2} + \Bigg(\frac{16}{15}A_{551} - \frac{1}{4}A_{552}\Bigg)\frac{\partial^2 v}{\partial x_1 \partial x_2}$$

$$+\Bigg(\frac{2}{5}B_{551} - \frac{2}{5}F_{471}\Bigg)\frac{\partial^3 w}{\partial x_1^3} + \Bigg(-\frac{8}{5}B_{551} + \frac{8}{5}F_{471}\Bigg)\frac{\partial^3 w}{\partial x_1 \partial x_2^2} + \Bigg(-\frac{2}{15}F_{671} - \frac{1}{4}F_{672}\Bigg)\frac{\partial w}{\partial x_1}$$

$$-\frac{2}{5}F_{471}\frac{\partial^2 \phi_1}{\partial x_1^2} + \Bigg(\frac{8}{15}F_{471} + \frac{1}{4}F_{472}\Bigg)\frac{\partial^2 \phi_1}{\partial x_2^2}$$

$$+\Bigg(-\frac{2}{15}F_{671} - \frac{1}{4}F_{672}\Bigg)\phi_1 + \Bigg(\frac{16}{15}F_{471} - \frac{1}{4}F_{472}\Bigg)\frac{\partial^2 \phi_2}{\partial x_1 \partial x_2}\Bigg)n_{x_2} = 0, \qquad (18.3)$$

$\delta v = 0$ or

$$
\begin{aligned}
&\Bigg(\Bigg(-\frac{8}{15}A_{551}+\frac{1}{8}A_{552}\Bigg)\frac{\partial^3 u}{\partial x_2^3}+\Bigg(-A_{550}-\frac{2}{3}A_{551}+\frac{1}{4}A_{552}\Bigg)\frac{\partial^3 u}{\partial x_1^2\partial x_2}+A_{55}\frac{\partial u}{\partial x_2}\\
&+\Bigg(-A_{550}-\frac{2}{3}A_{551}-\frac{1}{8}A_{552}\Bigg)\frac{\partial^3 v}{\partial x_1\partial x_2^2}+\Bigg(-\frac{8}{15}A_{551}-\frac{1}{4}A_{552}\Bigg)\frac{\partial^3 v}{\partial x_1^3}+A_{55}\frac{\partial v}{\partial x_1}\\
&+\Bigg(B_{550}-F_{470}+\frac{6}{5}B_{551}-\frac{6}{5}F_{471}\Bigg)\frac{\partial^4 w}{\partial x_1^3\partial x_2}+\Bigg(B_{550}+\frac{6}{5}B_{551}-F_{470}-\frac{6}{5}F_{471}\Bigg)\frac{\partial^4 w}{\partial x_1\partial x_2^3}\\
&+\Bigg(2F_{47}-2B_{55}+\frac{4}{15}F_{671}+\frac{1}{8}F_{672}\Bigg)\frac{\partial^2 w}{\partial x_1\partial x_2}+\Bigg(-\frac{8}{15}F_{471}+\frac{1}{8}F_{472}\Bigg)\frac{\partial^3 \phi_1}{\partial x_2^3}\\
&+\Bigg(-F_{470}-\frac{2}{3}F_{471}+\frac{1}{4}F_{472}\Bigg)\frac{\partial^3 \phi_1}{\partial x_1^2\partial x_2}+\Bigg(F_{47}+\frac{2}{15}F_{671}-\frac{1}{8}F_{672}\Bigg)\frac{\partial \phi_1}{\partial x_2}\\
&+\Bigg(-\frac{8}{15}F_{471}-\frac{1}{4}F_{472}\Bigg)\frac{\partial^3 \phi_2}{\partial x_1^3}+\Bigg(-F_{470}-\frac{2}{3}F_{471}-\frac{1}{8}F_{472}\Bigg)\frac{\partial^3 \phi_2}{\partial x_1\partial x_2^2}\\
&+\Bigg(F_{47}+\frac{2}{15}F_{671}+\frac{1}{4}F_{672}\Bigg)\frac{\partial \phi_2}{\partial x_1}\Bigg)n_{x_1}+\Bigg(\Bigg(-A_{550}+\frac{2}{5}A_{551}\Bigg)\frac{\partial^3 u}{\partial x_1^3}\\
&+\Bigg(-2A_{550}+\frac{4}{15}A_{551}+\frac{1}{8}A_{552}\Bigg)\frac{\partial^3 u}{\partial x_1\partial x_2^2}+A_{L11}\frac{\partial u}{\partial x_1}+\Bigg(-2A_{550}-\frac{4}{5}A_{551}\Bigg)\frac{\partial^3 v}{\partial x_2^3}\\
&+\Bigg(-A_{550}-\frac{2}{3}A_{551}-\frac{1}{8}A_{552}\Bigg)\frac{\partial^3 v}{\partial x_1^2\partial x_2}+A_{11}\frac{\partial v}{\partial x_2}+\Bigg(B_{550}-F_{470}-\frac{2}{5}B_{551}+\frac{2}{5}F_{471}\Bigg)\frac{\partial^4 w}{\partial x_1^4}\\
&+\Bigg(-2F_{470}+2B_{550}+\frac{4}{5}B_{551}-\frac{4}{5}F_{471}\Bigg)\frac{\partial^4 w}{\partial x_2^4}+\Bigg(3B_{550}+\frac{2}{5}B_{551}-3F_{470}-\frac{2}{5}F_{471}\Bigg)\frac{\partial^4 w}{\partial x_1^2\partial x_2^2}\\
&+\Bigg(F_{L11}-B_{L11}+\frac{2}{15}F_{671}-\frac{1}{8}F_{672}\Bigg)\frac{\partial^2 w}{\partial x_1^2}+\Bigg(F_{11}-B_{11}+\frac{2}{5}F_{671}\Bigg)\frac{\partial^2 w}{\partial x_2^2}\\
&+\Bigg(-F_{470}+\frac{2}{5}F_{471}\Bigg)\frac{\partial^3 \phi_1}{\partial x_1^3}+\Bigg(-2F_{470}+\frac{4}{15}F_{471}+\frac{1}{8}F_{472}\Bigg)\frac{\partial^3 \phi_1}{\partial x_1\partial x_2^2}\\
&+\Bigg(F_{L11}+\frac{2}{15}F_{671}-\frac{1}{8}F_{672}\Bigg)\frac{\partial \phi_1}{\partial x_1}+\Bigg(-2F_{470}-\frac{4}{5}F_{471}\Bigg)\frac{\partial^3 \phi_2}{\partial x_2^3}\\
&+\Bigg(-F_{470}-\frac{2}{3}F_{471}-\frac{1}{8}F_{472}\Bigg)\frac{\partial^3 \phi_2}{\partial x_1^2\partial x_2}+\Bigg(F_{11}+\frac{2}{5}F_{671}\Bigg)\frac{\partial \phi_2}{\partial x_2}-A_{T11}\Bigg)n_{x_2}=0, \qquad (18.4)
\end{aligned}
$$

$$\delta \frac{\partial v}{\partial x_1} = 0 \quad \text{or}$$

$$\Bigg(\left(\frac{16}{15} A_{551} - \frac{1}{4} A_{552} \right) \frac{\partial^2 u}{\partial x_1 \partial x_2} + \left(\frac{8}{15} A_{551} + \frac{1}{4} A_{552} \right) \frac{\partial^2 v}{\partial x_1^2} - \frac{2}{5} A_{551} \frac{\partial^2 v}{\partial x_2^2}$$

$$+ \left(\frac{2}{5} B_{551} - \frac{2}{5} F_{471} \right) \frac{\partial^3 w}{\partial x_2^3} + \left(-\frac{8}{5} B_{551} + \frac{8}{5} F_{471} \right) \frac{\partial^3 w}{\partial x_1^2 \partial x_2} + \left(-\frac{2}{15} F_{671} - \frac{1}{4} F_{672} \right) \frac{\partial w}{\partial x_2}$$

$$+ \left(\frac{16}{15} F_{471} - \frac{1}{4} F_{472} \right) \frac{\partial^2 \phi_1}{\partial x_1 \partial x_2} + \left(\frac{8}{15} F_{471} + \frac{1}{4} F_{472} \right) \frac{\partial^2 \phi_2}{\partial x_1^2} - \frac{2}{5} F_{471} \frac{\partial^2 \phi_2}{\partial x_2^2}$$

$$+ \left(-\frac{2}{15} F_{671} - \frac{1}{4} F_{672} \right) \phi_2 \Bigg) n_{x_1} + \Bigg(\left(A_{550} - \frac{2}{5} A_{551} \right) \frac{\partial^2 u}{\partial x_1^2} + \left(\frac{8}{15} A_{551} - \frac{1}{8} A_{552} \right) \frac{\partial^2 u}{\partial x_2^2}$$

$$+ \left(A_{550} + \frac{16}{15} A_{551} + \frac{1}{8} A_{552} \right) \frac{\partial^2 v}{\partial x_1 \partial x_2} + \left(-B_{550} + F_{470} + \frac{2}{5} B_{551} - \frac{2}{5} F_{471} \right) \frac{\partial^3 w}{\partial x_1^3}$$

$$+ \left(-B_{550} + F_{470} - \frac{8}{5} B_{551} + \frac{8}{5} F_{471} \right) \frac{\partial^3 w}{\partial x_1 \partial x_2^2}$$

$$+ \left(-\frac{2}{15} F_{671} + \frac{1}{8} F_{672} \right) \frac{\partial w}{\partial x_1} + \left(F_{470} - \frac{2}{5} F_{471} \right) \frac{\partial^2 \phi_1}{\partial x_1^2} + \left(\frac{8}{15} F_{471} - \frac{1}{8} F_{472} \right) \frac{\partial^2 \phi_1}{\partial x_2^2}$$

$$+ \left(-\frac{2}{15} F_{671} + \frac{1}{8} F_{672} \right) \phi_1 + \left(F_{470} + \frac{16}{15} F_{471} + \frac{1}{8} F_{472} \right) \frac{\partial^2 \phi_2}{\partial x_1 \partial x_2} \Bigg) n_{x_2} = 0, \qquad (18.5)$$

$$\delta \frac{\partial v}{\partial x_2} = 0 \quad \text{or}$$

$$\Bigg(\Bigg(A_{550} - \frac{2}{5}A_{551}\Bigg)\frac{\partial^2 u}{\partial x_1^2} + \Bigg(\frac{8}{15}A_{551} - \frac{1}{8}A_{552}\Bigg)\frac{\partial^2 u}{\partial x_2^2} + \Bigg(A_{550} + \frac{16}{15}A_{551} + \frac{1}{8}A_{552}\Bigg)\frac{\partial^2 v}{\partial x_1 \partial x_2}$$

$$+\Bigg(-B_{550} + F_{470} + \frac{2}{5}B_{551} - \frac{2}{5}F_{471}\Bigg)\frac{\partial^3 w}{\partial x_1^3} + \Bigg(-B_{550} + F_{470} - \frac{8}{5}B_{551} + \frac{8}{5}F_{471}\Bigg)\frac{\partial^3 w}{\partial x_1 \partial x_2^2}$$

$$+\Bigg(-\frac{2}{15}F_{671} + \frac{1}{8}F_{672}\Bigg)\frac{\partial w}{\partial x_1} + \Bigg(F_{470} - \frac{2}{5}F_{471}\Bigg)\frac{\partial^2 \phi_1}{\partial x_1^2} + \Bigg(\frac{8}{15}F_{471} - \frac{1}{8}F_{472}\Bigg)\frac{\partial^2 \phi_1}{\partial x_2^2}$$

$$+\Bigg(-\frac{2}{15}F_{671} + \frac{1}{8}F_{672}\Bigg)\phi_1 + \Bigg(F_{470} + \frac{16}{15}F_{471} + \frac{1}{8}F_{472}\Bigg)\frac{\partial^2 \phi_2}{\partial x_1 \partial x_2}\Bigg)n_{x_1}$$

$$+\Bigg(\Bigg(2A_{550} - \frac{4}{5}A_{551}\Bigg)\frac{\partial^2 u}{\partial x_1 \partial x_2} - \frac{2}{5}A_{551}\frac{\partial^2 v}{\partial x_1^2} + \Bigg(2A_{550} + \frac{4}{5}A_{551}\Bigg)\frac{\partial^2 v}{\partial x_2^2}$$

$$+\Bigg(-2B_{550} + 2F_{470} + \frac{6}{5}B_{551} - \frac{6}{5}F_{471}\Bigg)\frac{\partial^3 w}{\partial x_1^2 \partial x_2}$$

$$+\Bigg(2F_{470} - 2B_{550} - \frac{4}{5}B_{551} + \frac{4}{5}F_{471}\Bigg)\frac{\partial^3 w}{\partial x_2^3} - \frac{2}{5}F_{671}\frac{\partial w}{\partial x_2} + \Bigg(2F_{470} - \frac{4}{5}F_{471}\Bigg)\frac{\partial^2 \phi_1}{\partial x_1 \partial x_2}$$

$$-\frac{2}{5}F_{471}\frac{\partial^2 \phi_2}{\partial x_1^2} + \Bigg(2F_{470} + \frac{4}{5}F_{471}\Bigg)\frac{\partial^2 \phi_2}{\partial x_2^2} - \frac{2}{5}F_{671}\phi_2\Bigg)n_{x_2} = 0, \qquad (18.6)$$

$\delta w = 0$ or

$$\Bigg(\Bigg(-2B_{550}+2F_{470}-\frac{4}{5}B_{551}+\frac{4}{5}F_{471}\Bigg)\frac{\partial^4 u}{\partial x_1^4}+\frac{8}{15}F_{471}\frac{\partial^4 u}{\partial x_2^4}+\Bigg(-2B_{550}+2F_{470}-\frac{14}{5}B_{551}$$

$$+\frac{4}{3}F_{471}\Bigg)\frac{\partial^4 u}{\partial x_1^2\partial x_2^2}+\Bigg(B_{11}-F_{11}-\frac{2}{5}F_{671}\Bigg)\frac{\partial^2 u}{\partial x_1^2}+\Bigg(B_{55}-F_{47}-\frac{2}{15}F_{671}-\frac{1}{4}F_{672}\Bigg)\frac{\partial^2 u}{\partial x_2^2}$$

$$+\Bigg(-2B_{550}+2F_{470}-\frac{4}{5}B_{551}+\frac{4}{15}F_{471}\Bigg)\frac{\partial^4 v}{\partial x_1^3\partial x_2}+\Bigg(-2B_{550}+2F_{470}+\frac{6}{5}B_{551}$$

$$+\frac{4}{15}F_{471}\Bigg)\frac{\partial^4 v}{\partial x_1\partial x_2^3}+\Bigg(B_{55}+B_{L11}-F_{47}-F_{L11}-\frac{4}{15}F_{671}+\frac{1}{4}F_{672}\Bigg)\frac{\partial^2 v}{\partial x_1\partial x_2}$$

$$+\Bigg(2D_{550}+2F_{440}-4F_{480}+\frac{4}{5}D_{551}+\frac{4}{5}F_{441}-\frac{8}{5}F_{481}\Bigg)\frac{\partial^5 w}{\partial x_1^5}$$

$$+\Bigg(4D_{550}+4F_{440}-8F_{480}+\frac{18}{5}D_{551}+\frac{8}{5}F_{441}-\frac{26}{5}F_{481}\Bigg)\frac{\partial^5 w}{\partial x_1^3\partial x_2^2}$$

$$+\Bigg(2D_{550}+2F_{440}-4F_{480}-\frac{6}{5}D_{551}+\frac{4}{5}F_{441}+\frac{2}{5}F_{481}\Bigg)\frac{\partial^5 w}{\partial x_1\partial x_2^4}$$

$$+\Bigg(-D_{11}+2F_{22}-F_{33}-2A_{550}-2F_{550}+4F_{570}-\frac{8}{15}A_{551}-\frac{4}{5}F_{461}-\frac{32}{15}F_{551}+\frac{32}{15}F_{571}$$

$$+\frac{4}{5}F_{681}-A_{552}-\frac{1}{4}F_{552}+F_{572}\Bigg)\frac{\partial^3 w}{\partial x_1^3}+\Big(-D_{11}+2F_{22}-F_{33}-2A_{550}-2F_{550}+4F_{570}$$

$$-\frac{8}{15}A_{551}-\frac{4}{5}F_{461}-\frac{32}{15}F_{551}+\frac{32}{15}F_{571}+\frac{4}{5}F_{681}-A_{552}-\frac{1}{4}F_{552}+F_{572}\Bigg)\frac{\partial^3 w}{\partial x_1\partial x_2^2}$$

$$+\Bigg(k_sF_{55}+\frac{8}{15}F_{661}+\frac{1}{4}F_{662}\Bigg)\frac{\partial w}{\partial x_1}+\Bigg(2F_{440}-2F_{480}+\frac{4}{5}F_{441}-\frac{4}{5}F_{481}\Bigg)\frac{\partial^4\phi_1}{\partial x_1^4}+\frac{8}{15}F_{441}\frac{\partial^4\phi_1}{\partial x_2^4}$$

$$+\Bigg(2F_{440}-2F_{480}+\frac{4}{3}F_{441}-\frac{14}{5}F_{481}\Bigg)\frac{\partial^4\phi_1}{\partial x_1^2\partial x_2^2}+\Bigg(F_{22}-F_{33}-2F_{550}+2F_{570}-\frac{4}{5}F_{461}$$

$$-\frac{32}{15}F_{551}+\frac{16}{15}F_{571}+\frac{2}{5}F_{681}-\frac{1}{4}F_{552}+\frac{1}{2}F_{572}\Bigg)\frac{\partial^2\phi_1}{\partial x_1^2}+\Bigg(-F_{44}+F_{48}-\frac{4}{15}F_{461}$$

$$-\frac{4}{3}F_{551}+\frac{2}{3}F_{571}-\frac{1}{4}F_{462}-\frac{1}{2}F_{552}+\frac{1}{2}F_{572}\Bigg)\frac{\partial^2\phi_1}{\partial x_2^2}+\Bigg(k_sF_{55}+\frac{8}{15}F_{661}+\frac{1}{4}F_{662}\Bigg)\phi_1$$

$$+\Bigg(2F_{440}-2F_{480}+\frac{4}{15}F_{441}-\frac{4}{5}F_{481}\Bigg)\frac{\partial^4\phi_2}{\partial x_1^3\partial x_2}+\Bigg(2F_{440}-2F_{480}+\frac{4}{15}F_{441}+\frac{6}{5}F_{481}\Bigg)\frac{\partial^4\phi_2}{\partial x_1\partial x_2^3}$$

$$+\Bigg(F_{48}+F_{L22}-F_{44}-F_{L33}-2F_{550}+2F_{570}-\frac{8}{15}F_{461}-\frac{4}{5}F_{551}+\frac{2}{5}F_{571}+\frac{2}{5}F_{681}$$

$$+\frac{1}{4}F_{462}+\frac{1}{4}F_{552}\Bigg)\frac{\partial^2\phi_2}{\partial x_1\partial x_2}-\frac{\partial w}{\partial x_1}P_{x_1}+\frac{\partial\left(F_{T11}-B_{T11}\right)}{\partial x_1}\Bigg)n_{x_1}$$

$$+\left(\left(-2B_{550}+2F_{470}+\frac{6}{5}B_{551}+\frac{4}{15}F_{471}\right)\frac{\partial^4 u}{\partial x_1^3\partial x_2}+\left(-2B_{550}+2F_{470}-\frac{4}{5}B_{551}\right.\right.$$

$$\left.+\frac{4}{15}F_{471}\right)\frac{\partial^4 u}{\partial x_1\partial x_2^3}+\left(B_{55}+B_{L11}-F_{47}-F_{L11}-\frac{4}{15}F_{671}+\frac{1}{4}F_{672}\right)\frac{\partial^2 u}{\partial x_1\partial x_2}+\frac{8}{15}F_{471}\frac{\partial^4 v}{\partial x_1^4}$$

$$+\left(-2B_{550}+2F_{470}-\frac{4}{5}B_{551}+\frac{4}{5}F_{471}\right)\frac{\partial^4 v}{\partial x_2^4}+\left(-2B_{550}+2F_{470}-\frac{14}{5}B_{551}+\frac{4}{3}F_{471}\right)\frac{\partial^4 v}{\partial x_1^2\partial x_2^2}$$

$$+\left(B_{11}-F_{11}-\frac{2}{5}F_{671}\right)\frac{\partial^2 v}{\partial x_2^2}+\left(B_{55}-F_{47}-\frac{2}{15}F_{671}-\frac{1}{4}F_{672}\right)\frac{\partial^2 v}{\partial x_1^2}$$

$$+\left(2D_{550}+2F_{440}-4F_{480}+\frac{4}{5}D_{551}+\frac{4}{5}F_{441}-\frac{8}{5}F_{481}\right)\frac{\partial^5 w}{\partial x_2^5}$$

$$+\left(2D_{550}+2F_{440}-4F_{480}-\frac{6}{5}D_{551}+\frac{4}{5}F_{441}+\frac{2}{5}F_{481}\right)\frac{\partial^5 w}{\partial x_1^4\partial x_2}$$

$$+\left(4D_{550}+4F_{440}-8F_{480}+\frac{18}{5}D_{551}+\frac{8}{5}F_{441}-\frac{26}{5}F_{481}\right)\frac{\partial^5 w}{\partial x_1^2\partial x_2^3}$$

$$+\left(-D_{11}+2F_{22}-F_{33}-2A_{550}-2F_{550}+4F_{570}-\frac{8}{15}A_{551}-\frac{4}{5}F_{461}-\frac{32}{15}F_{551}+\frac{32}{15}F_{571}\right.$$

$$\left.+\frac{4}{5}F_{681}-A_{552}-\frac{1}{4}F_{552}+F_{572}\right)\frac{\partial^3 w}{\partial x_2^3}+\left(-D_{11}+2F_{22}-F_{33}-2A_{550}-2F_{550}+4F_{570}\right.$$

$$\left.-\frac{8}{15}A_{551}-\frac{4}{5}F_{461}-\frac{32}{15}F_{551}+\frac{32}{15}F_{571}+\frac{4}{5}F_{681}-A_{552}-\frac{1}{4}F_{552}+F_{572}\right)\frac{\partial^3 w}{\partial x_1^2\partial x_2}$$

$$+\left(k_sF_{55}+\frac{8}{15}F_{661}+\frac{1}{4}F_{662}\right)\frac{\partial w}{\partial x_2}+\left(2F_{440}-2F_{480}+\frac{4}{15}F_{441}+\frac{6}{5}F_{481}\right)\frac{\partial^4\phi_1}{\partial x_1^3\partial x_2}$$

$$+\left(2F_{440}-2F_{480}+\frac{4}{15}F_{441}-\frac{4}{5}F_{481}\right)\frac{\partial^4\phi_1}{\partial x_1\partial x_2^3}$$

$$+\left(F_{48}+F_{L22}-F_{44}-F_{L33}-2F_{550}+2F_{570}-\frac{8}{15}F_{461}-\frac{4}{5}F_{551}+\frac{2}{5}F_{571}+\frac{2}{5}F_{681}+\frac{1}{4}F_{462}\right.$$

$$\left.+\frac{1}{4}F_{552}\right)\frac{\partial^2\phi_1}{\partial x_1\partial x_2}+\frac{8}{15}F_{441}\frac{\partial^4\phi_2}{\partial x_1^4}+\left(2F_{440}-2F_{480}+\frac{4}{5}F_{441}-\frac{4}{5}F_{481}\right)\frac{\partial^4\phi_2}{\partial x_2^4}$$

$$+\left(2F_{440}-2F_{480}+\frac{4}{3}F_{441}-\frac{14}{5}F_{481}\right)\frac{\partial^4\phi_2}{\partial x_1^2\partial x_2^2}$$

$$+\left(-F_{44}+F_{48}-\frac{4}{15}F_{461}-\frac{4}{3}F_{551}+\frac{2}{3}F_{571}-\frac{1}{4}F_{462}-\frac{1}{2}F_{552}+\frac{1}{2}F_{572}\right)\frac{\partial^2\phi_2}{\partial x_1^2}$$

$$+\left(F_{22}-F_{33}-2F_{550}+2F_{570}-\frac{4}{5}F_{461}-\frac{32}{15}F_{551}+\frac{16}{15}F_{571}+\frac{2}{5}F_{681}-\frac{1}{4}F_{552}+\frac{1}{2}F_{572}\right)\frac{\partial^2\phi_2}{\partial x_2^2}$$

$$+\left(k_s F_{55}+\frac{8}{15}F_{661}+\frac{1}{4}F_{662}\right)\phi_2-\frac{\partial w}{\partial x_2}P_{x_2}+\frac{\partial\left(F_{T11}-B_{T11}\right)}{\partial x_2}\Bigg)n_{x_2}$$

$$=\left(\left(I_1-I_3\right)\frac{\partial^2 u}{\partial t^2}+\left(2I_4-I_2-I_5\right)\frac{\partial^3 w}{\partial x_1\partial t^2}+\left(I_4-I_5\right)\frac{\partial^2\phi_1}{\partial t^2}\right)n_{x_1}$$

$$+\left(\left(I_1-I_3\right)\frac{\partial^2 v}{\partial t^2}+\left(2I_4-I_2-I_5\right)\frac{\partial^3 w}{\partial x_2\partial t^2}+\left(I_4-I_5\right)\frac{\partial^2\phi_2}{\partial t^2}\right)n_{x_2}, \tag{18.7}$$

$$\delta\frac{\partial w}{\partial x_1}=0 \quad \text{or}$$

$$\Bigg(\left(2B_{550}-2F_{470}+\frac{4}{5}B_{551}-\frac{4}{5}F_{471}\right)\frac{\partial^3 u}{\partial x_1^3}+\left(2B_{550}-2F_{470}+\frac{14}{5}B_{551}-\frac{14}{5}F_{471}\right)\frac{\partial^3 u}{\partial x_1\partial x_2^2}$$

$$+\left(-B_{11}+F_{11}\right)\frac{\partial u}{\partial x_1}+\left(2B_{550}-2F_{470}-\frac{6}{5}B_{551}+\frac{6}{5}F_{471}\right)\frac{\partial^3 v}{\partial x_2^3}+\Bigg(2B_{550}-2F_{470}+\frac{4}{5}B_{551}$$

$$-\frac{4}{5}F_{471}\Bigg)\frac{\partial^3 v}{\partial x_1^2\partial x_2}+\left(-B_{L11}+F_{L11}\right)\frac{\partial v}{\partial x_2}+\Bigg(-2D_{550}-2F_{440}+4F_{480}-\frac{4}{5}D_{551}-\frac{4}{5}F_{441}$$

$$+\frac{8}{5}F_{481}\Bigg)\frac{\partial^4 w}{\partial x_1^4}+\left(-2D_{550}+4F_{480}-2F_{440}+\frac{6}{5}D_{551}+\frac{6}{5}F_{441}-\frac{12}{5}F_{481}\right)\frac{\partial^4 w}{\partial x_2^4}$$

$$+\left(-4D_{550}+8F_{480}-4F_{440}-\frac{18}{5}D_{551}-\frac{18}{5}F_{441}+\frac{36}{5}F_{481}\right)\frac{\partial^4 w}{\partial x_1^2\partial x_2^2}$$

$$+\Bigg(D_{11}+F_{33}-2F_{22}+2A_{550}+2F_{550}-4F_{570}+\frac{8}{15}A_{551}+\frac{2}{5}F_{461}+\frac{32}{15}F_{551}-\frac{32}{15}F_{571}$$

$$-\frac{2}{5}F_{681}+A_{552}+\frac{1}{4}F_{552}-F_{572}\Bigg)\frac{\partial^2 w}{\partial x_1^2}+\Big(D_{L11}-2F_{L22}+F_{L33}+2A_{550}+2F_{550}-4F_{570}$$

$$-\frac{2}{15}A_{551}+\frac{2}{5}F_{461}-\frac{8}{15}F_{551}+\frac{8}{15}F_{571}-\frac{2}{5}F_{681}-A_{552}-\frac{1}{4}F_{552}+F_{572}\Bigg)\frac{\partial^2 w}{\partial x_2^2}$$

$$+\left(-2F_{440}+2F_{480}-\frac{4}{5}F_{441}+\frac{4}{5}F_{481}\right)\frac{\partial^3\phi_1}{\partial x_1^3}+\Bigg(-2F_{440}+2F_{480}-\frac{14}{5}F_{441}$$

$$+\frac{14}{5}F_{481}\Bigg)\frac{\partial^3\phi_1}{\partial x_1\partial x_2^2}+\Bigg(-F_{22}+F_{33}+2F_{550}-2F_{570}+\frac{2}{5}F_{461}+\frac{32}{15}F_{551}-\frac{16}{15}F_{571}$$

$$-\frac{2}{5}F_{681}+\frac{1}{4}F_{552}-\frac{1}{2}F_{572}\Bigg)\frac{\partial\phi_1}{\partial x_1}+\left(-2F_{440}+2F_{480}+\frac{6}{5}F_{441}-\frac{6}{5}F_{481}\right)\frac{\partial^3\phi_2}{\partial x_2^3}$$

$$+\left(-2F_{440}+2F_{480}-\frac{4}{5}F_{441}+\frac{4}{5}F_{481}\right)\frac{\partial^3\phi_2}{\partial x_1^2\partial x_2}+\Bigg(-F_{L22}+F_{L33}+2F_{550}-2F_{570}+\frac{2}{5}F_{461}$$

$$-\frac{8}{15}F_{551}+\frac{4}{15}F_{571}-\frac{2}{5}F_{681}-\frac{1}{4}F_{552}+\frac{1}{2}F_{572}\Bigg)\frac{\partial\phi_2}{\partial x_2}+\left(B_{T11}-F_{T11}\right)\Bigg)n_{x_1}$$

$$+\left(\left(-B_{55}+F_{47}\right)\frac{\partial u}{\partial x_2}+\left(-B_{55}+F_{47}\right)\frac{\partial v}{\partial x_1}\right.$$

$$+\left(2D_{55}+2F_{44}-4F_{48}+\frac{2}{3}A_{551}+\frac{8}{3}F_{551}-\frac{8}{3}F_{571}+2A_{552}+\frac{1}{2}F_{552}-2F_{572}\right)\frac{\partial^2 w}{\partial x_1\partial x_2}$$

$$+\left(F_{44}-F_{48}+\frac{4}{3}F_{551}-\frac{2}{3}F_{571}+\frac{1}{4}F_{552}-\frac{1}{2}F_{572}\right)\frac{\partial \phi_1}{\partial x_2}$$

$$\left.+\left(F_{44}-F_{48}+\frac{4}{3}F_{551}-\frac{2}{3}F_{571}+\frac{1}{4}F_{552}-\frac{1}{2}F_{572}\right)\frac{\partial \phi_2}{\partial x_1}\right)n_{x_2}=0, \qquad (18.8)$$

$$\delta\frac{\partial w}{\partial x_2}=0 \quad \text{or}$$

$$\left(\left(-B_{55}+F_{47}\right)\frac{\partial u}{\partial x_2}+\left(-B_{55}+F_{47}\right)\frac{\partial v}{\partial x_1}+\left(2D_{55}+2F_{44}-4F_{48}+\frac{2}{3}A_{551}+\frac{8}{3}F_{551}\right.\right.$$

$$\left.-\frac{8}{3}F_{571}+2A_{552}+\frac{1}{2}F_{552}-2F_{572}\right)\frac{\partial^2 w}{\partial x_1\partial x_2}+\left(F_{44}-F_{48}+\frac{4}{3}F_{551}-\frac{2}{3}F_{571}+\frac{1}{4}F_{552}\right.$$

$$\left.\left.-\frac{1}{2}F_{572}\right)\frac{\partial \phi_1}{\partial x_2}+\left(F_{44}-F_{48}+\frac{4}{3}F_{551}-\frac{2}{3}F_{571}+\frac{1}{4}F_{552}-\frac{1}{2}F_{572}\right)\frac{\partial \phi_2}{\partial x_1}\right)n_{x_1}$$

$$+\left(\left(2B_{550}-2F_{470}-\frac{6}{5}B_{551}+\frac{6}{5}F_{471}\right)\frac{\partial^3 u}{\partial x_1^3}+\left(2B_{550}-2F_{470}+\frac{4}{5}B_{551}-\frac{4}{5}F_{471}\right)\frac{\partial^3 u}{\partial x_1\partial x_2^2}\right.$$

$$+\left(-B_{L11}+F_{L11}\right)\frac{\partial u}{\partial x_1}+\left(2B_{550}-2F_{470}+\frac{4}{5}B_{551}-\frac{4}{5}F_{471}\right)\frac{\partial^3 v}{\partial x_2^3}$$

$$+\left(2B_{550}-2F_{470}+\frac{14}{5}B_{551}-\frac{14}{5}F_{471}\right)\frac{\partial^3 v}{\partial x_1^2\partial x_2}+\left(-B_{11}+F_{11}\right)\frac{\partial v}{\partial x_2}$$

$$+\left(-2D_{550}+4F_{480}-2F_{440}+\frac{6}{5}D_{551}+\frac{6}{5}F_{441}-\frac{12}{5}F_{481}\right)\frac{\partial^4 w}{\partial x_1^4}$$

$$+\left(-2D_{550}-2F_{440}+4F_{480}-\frac{4}{5}D_{551}-\frac{4}{5}F_{441}+\frac{8}{5}F_{481}\right)\frac{\partial^4 w}{\partial x_2^4}$$

$$+\left(-4D_{550}+8F_{480}-4F_{440}-\frac{18}{5}D_{551}-\frac{18}{5}F_{441}+\frac{36}{5}F_{481}\right)\frac{\partial^4 w}{\partial x_1^2\partial x_2^2}$$

$$+\Bigg(D_{L11}-2F_{L22}+F_{L33}+2A_{550}+2F_{550}-4F_{570}-\frac{2}{15}A_{551}+\frac{2}{5}F_{461}-\frac{8}{15}F_{551}+\frac{8}{15}F_{571}$$

$$-\frac{2}{5}F_{681}-A_{552}-\frac{1}{4}F_{552}+F_{572}\Bigg)\frac{\partial^2 w}{\partial x_1^2}+\Bigg(D_{11}+F_{33}-2F_{22}+2A_{550}+2F_{550}-4F_{570}+\frac{8}{15}A_{551}$$

$$+\frac{2}{5}F_{461}+\frac{32}{15}F_{551}-\frac{32}{15}F_{571}-\frac{2}{5}F_{681}+A_{552}+\frac{1}{4}F_{552}-F_{572}\Bigg)\frac{\partial^2 w}{\partial x_2^2}$$

$$+\Bigg(-2F_{440}+2F_{480}+\frac{6}{5}F_{441}-\frac{6}{5}F_{481}\Bigg)\frac{\partial^3\phi_1}{\partial x_1^3}+\Bigg(-2F_{440}+2F_{480}-\frac{4}{5}F_{441}+\frac{4}{5}F_{481}\Bigg)\frac{\partial^3\phi_1}{\partial x_1\partial x_2^2}$$

$$+\Bigg(-F_{L22}+F_{L33}+2F_{550}-2F_{570}+\frac{2}{5}F_{461}-\frac{8}{15}F_{551}$$

$$+\frac{4}{15}F_{571}-\frac{2}{5}F_{681}-\frac{1}{4}F_{552}+\frac{1}{2}F_{572}\Bigg)\frac{\partial\phi_1}{\partial x_1}+\Bigg(-2F_{440}+2F_{480}-\frac{4}{5}F_{441}+\frac{4}{5}F_{481}\Bigg)\frac{\partial^3\phi_2}{\partial x_2^3}$$

$$+\Bigg(-2F_{440}+2F_{480}-\frac{14}{5}F_{441}+\frac{14}{5}F_{481}\Bigg)\frac{\partial^3\phi_2}{\partial x_1^2\partial x_2}+\Bigg(-F_{22}+F_{33}+2F_{550}-2F_{570}+\frac{2}{5}F_{461}$$

$$+\frac{32}{15}F_{551}-\frac{16}{15}F_{571}-\frac{2}{5}F_{681}+\frac{1}{4}F_{552}-\frac{1}{2}F_{572}\Bigg)\frac{\partial\phi_2}{\partial x_2}+\left(B_{T11}-F_{T11}\right)\Bigg)n_{x_2}=0, \qquad (18.9)$$

$$\delta \frac{\partial^2 w}{\partial x_1^2} = 0 \quad \text{or}$$

$$\Bigg(\Bigg(-2B_{550} + 2F_{470} - \frac{4}{5}B_{551} + \frac{4}{5}F_{471}\Bigg)\frac{\partial^2 u}{\partial x_1^2} + \Bigg(+\frac{2}{5}B_{551} - \frac{2}{5}F_{471}\Bigg)\frac{\partial^2 u}{\partial x_2^2}$$

$$+\Bigg(-2B_{550} + 2F_{470} + \frac{4}{5}B_{551} - \frac{4}{5}F_{471}\Bigg)\frac{\partial^2 v}{\partial x_1 \partial x_2} + \Bigg(2D_{550} + 2F_{440} - 4F_{480} + \frac{4}{5}D_{551}$$

$$+\frac{4}{5}F_{441} - \frac{8}{5}F_{481}\Bigg)\frac{\partial^3 w}{\partial x_1^3} + \Bigg(2D_{550} + 2F_{440} - 4F_{480} - \frac{6}{5}D_{551} - \frac{6}{5}F_{441} + \frac{12}{5}F_{481}\Bigg)\frac{\partial^3 w}{\partial x_1 \partial x_2^2}$$

$$+\Bigg(-\frac{2}{5}F_{461} + \frac{2}{5}F_{681}\Bigg)\frac{\partial w}{\partial x_1} + \Bigg(2F_{440} - 2F_{480} + \frac{4}{5}F_{441} - \frac{4}{5}F_{481}\Bigg)\frac{\partial^2 \phi_1}{\partial x_1^2}$$

$$+\Bigg(-\frac{2}{5}F_{441} + \frac{2}{5}F_{481}\Bigg)\frac{\partial^2 \phi_1}{\partial x_2^2} + \Bigg(-\frac{2}{5}F_{461} + \frac{2}{5}F_{681}\Bigg)\phi_1$$

$$+\Bigg(2F_{440} - 2F_{480} - \frac{4}{5}F_{441} + \frac{4}{5}F_{481}\Bigg)\frac{\partial^2 \phi_2}{\partial x_1 \partial x_2}\Bigg)n_{x_1}$$

$$+\Bigg(\Bigg(-2B_{550} + 2F_{470} - \frac{16}{5}B_{551} + \frac{16}{5}F_{471}\Bigg)\frac{\partial^2 u}{\partial x_1 \partial x_2}$$

$$+\Bigg(-\frac{8}{5}B_{551} + \frac{8}{5}F_{471}\Bigg)\frac{\partial^2 v}{\partial x_1^2} + \Bigg(-2B_{550} + 2F_{470} + \frac{6}{5}B_{551} - \frac{6}{5}F_{471}\Bigg)\frac{\partial^2 v}{\partial x_2^2}$$

$$+\Bigg(2D_{550} + 2F_{440} - 4F_{480} - \frac{6}{5}D_{551} - \frac{6}{5}F_{441} + \frac{12}{5}F_{481}\Bigg)\frac{\partial^3 w}{\partial x_2^3}$$

$$+\Bigg(2D_{550} + 2F_{440} - 4F_{480} + \frac{24}{5}D_{551} + \frac{24}{5}F_{441} - \frac{48}{5}F_{481}\Bigg)\frac{\partial^3 w}{\partial x_1^2 \partial x_2} + \Bigg(-\frac{2}{5}F_{461}$$

$$+\frac{2}{5}F_{681}\Bigg)\frac{\partial w}{\partial x_2} + \Bigg(2F_{440} - 2F_{480} + \frac{16}{5}F_{441} - \frac{16}{5}F_{481}\Bigg)\frac{\partial^2 \phi_1}{\partial x_1 \partial x_2} + \Bigg(\frac{8}{5}F_{441} - \frac{8}{5}F_{481}\Bigg)\frac{\partial^2 \phi_2}{\partial x_1^2}$$

$$+\Bigg(2F_{440} - 2F_{480} - \frac{6}{5}F_{441} + \frac{6}{5}F_{481}\Bigg)\frac{\partial^2 \phi_2}{\partial x_2^2} + \Bigg(-\frac{2}{5}F_{461} + \frac{2}{5}F_{681}\Bigg)\phi_2\Bigg)n_{x_2} = 0, \qquad (18.10)$$

$$\delta \frac{\partial^2 w}{\partial x_2^2} = 0 \quad \text{or}$$

$$\Bigg(\Bigg(-2B_{550} + 2F_{470} + \frac{6}{5}B_{551} - \frac{6}{5}F_{471}\Bigg)\frac{\partial^2 u}{\partial x_1^2} + \Bigg(-\frac{8}{5}B_{551} + \frac{8}{5}F_{471}\Bigg)\frac{\partial^2 u}{\partial x_2^2}$$

$$+\Bigg(-2B_{550} + 2F_{470} - \frac{16}{5}B_{551} + \frac{16}{5}F_{471}\Bigg)\frac{\partial^2 v}{\partial x_1 \partial x_2} + \Bigg(2D_{550} + 2F_{440} - 4F_{480} - \frac{6}{5}D_{551}$$

$$-\frac{6}{5}F_{441} + \frac{12}{5}F_{481}\Bigg)\frac{\partial^3 w}{\partial x_1^3} + \Bigg(2D_{550} + 2F_{440} - 4F_{480} + \frac{24}{5}D_{551} + \frac{24}{5}F_{441} - \frac{48}{5}F_{481}\Bigg)\frac{\partial^3 w}{\partial x_1 \partial x_2^2}$$

$$+\Bigg(-\frac{2}{5}F_{461} + \frac{2}{5}F_{681}\Bigg)\frac{\partial w}{\partial x_1} + \Bigg(2F_{440} - 2F_{480} - \frac{6}{5}F_{441} + \frac{6}{5}F_{481}\Bigg)\frac{\partial^2 \phi_1}{\partial x_1^2}$$

$$+\Bigg(\frac{8}{5}F_{441} - \frac{8}{5}F_{481}\Bigg)\frac{\partial^2 \phi_1}{\partial x_2^2} + \Bigg(-\frac{2}{5}F_{461} + \frac{2}{5}F_{681}\Bigg)\phi_1 + \Bigg(2F_{440} - 2F_{480} + \frac{16}{5}F_{441}$$

$$-\frac{16}{5}F_{481}\Bigg)\frac{\partial^2 \phi_2}{\partial x_1 \partial x_2}\Bigg)n_{x_1} + \Bigg(\Bigg(-2B_{550} + 2F_{470} + \frac{4}{5}B_{551} - \frac{4}{5}F_{471}\Bigg)\frac{\partial^2 u}{\partial x_1 \partial x_2}$$

$$+\Bigg(\frac{2}{5}B_{551} - \frac{2}{5}F_{471}\Bigg)\frac{\partial^2 v}{\partial x_1^2} + \Bigg(-2B_{550} + 2F_{470} - \frac{4}{5}B_{551} + \frac{4}{5}F_{471}\Bigg)\frac{\partial^2 v}{\partial x_2^2}$$

$$+\Bigg(2D_{550} + 2F_{440} - 4F_{480} + \frac{4}{5}D_{551} + \frac{4}{5}F_{441} - \frac{8}{5}F_{481}\Bigg)\frac{\partial^3 w}{\partial x_2^3}$$

$$+\Bigg(2D_{550} + 2F_{440} - 4F_{480} - \frac{6}{5}D_{551} - \frac{6}{5}F_{441} + \frac{12}{5}F_{481}\Bigg)\frac{\partial^3 w}{\partial x_1^2 \partial x_2} + \Bigg(-\frac{2}{5}F_{461} + \frac{2}{5}F_{681}\Bigg)\frac{\partial w}{\partial x_2}$$

$$+\Bigg(2F_{440} - 2F_{480} - \frac{4}{5}F_{441} + \frac{4}{5}F_{481}\Bigg)\frac{\partial^2 \phi_1}{\partial x_1 \partial x_2} + \Bigg(-\frac{2}{5}F_{441} + \frac{2}{5}F_{481}\Bigg)\frac{\partial^2 \phi_2}{\partial x_1^2}$$

$$+\Bigg(2F_{440} - 2F_{480} + \frac{4}{5}F_{441} - \frac{4}{5}F_{481}\Bigg)\frac{\partial^2 \phi_2}{\partial x_2^2} + \Bigg(-\frac{2}{5}F_{461} + \frac{2}{5}F_{681}\Bigg)\phi_2\Bigg)n_{x_2} = 0, \qquad (18.11)$$

$\delta\phi_1 = 0$ or

$$\Bigg(\left(-2F_{470} - \frac{4}{5}F_{471}\right)\frac{\partial^3 u}{\partial x_1^3} + \left(-F_{470} - \frac{2}{3}F_{471} - \frac{1}{8}F_{472}\right)\frac{\partial^3 u}{\partial x_1 \partial x_2^2} + F_{11}\frac{\partial u}{\partial x_1}$$

$$+\left(-2F_{470} + \frac{4}{15}F_{471} + \frac{1}{8}F_{472}\right)\frac{\partial^3 v}{\partial x_1^2 \partial x_2} + \left(-F_{470} + \frac{2}{5}F_{471}\right)\frac{\partial^3 v}{\partial x_2^3} + F_{L11}\frac{\partial v}{\partial x_2}$$

$$+\left(-2F_{440} + 2F_{480} - \frac{4}{5}F_{441} + \frac{4}{5}F_{481}\right)\frac{\partial^4 w}{\partial x_1^4} + \left(-F_{440} + F_{480} + \frac{2}{5}F_{441} - \frac{2}{5}F_{481}\right)\frac{\partial^4 w}{\partial x_2^4}$$

$$+\left(-3F_{440} + 3F_{480} - \frac{2}{5}F_{441} + \frac{2}{5}F_{481}\right)\frac{\partial^4 w}{\partial x_1^2 \partial x_2^2}$$

$$+\left(-F_{22} + F_{33} + 2F_{550} - 2F_{570} + \frac{2}{5}F_{461} + \frac{32}{15}F_{551} - \frac{16}{15}F_{571} + \frac{1}{4}F_{552} - \frac{1}{2}F_{572}\right)\frac{\partial^2 w}{\partial x_1^2},$$

$$+\Bigg(-F_{L22} + F_{L33} + 2F_{550} - 2F_{570} + \frac{2}{15}F_{461} - \frac{8}{15}F_{551} + \frac{4}{15}F_{571} - \frac{1}{8}F_{462} - \frac{1}{4}F_{552}$$

$$+\frac{1}{2}F_{572}\Bigg)\frac{\partial^2 w}{\partial x_2^2} + \left(-2F_{440} - \frac{4}{5}F_{441}\right)\frac{\partial^3 \phi_1}{\partial x_1^3} + \left(-F_{440} - \frac{2}{3}F_{441} - \frac{1}{8}F_{442}\right)\frac{\partial^3 \phi_1}{\partial x_1 \partial x_2^2}$$

$$+\left(F_{33} + 2F_{550} + \frac{2}{5}F_{461} + \frac{32}{15}F_{551} + \frac{1}{4}F_{552}\right)\frac{\partial \phi_1}{\partial x_1} + \left(-F_{440} + \frac{2}{5}F_{441}\right)\frac{\partial^3 \phi_2}{\partial x_2^3}$$

$$+\left(-2F_{440} + \frac{4}{15}F_{441} + \frac{1}{8}F_{442}\right)\frac{\partial^3 \phi_2}{\partial x_1^2 \partial x_2}$$

$$+\left(F_{L33} + 2F_{550} + \frac{2}{15}F_{461} - \frac{8}{15}F_{551} - \frac{1}{8}F_{462} - \frac{1}{4}F_{552}\right)\frac{\partial \phi_2}{\partial x_2} - F_{T11}\Bigg)n_{x_1}$$

$$+\Bigg(\left(-\frac{8}{15}F_{471} - \frac{1}{4}F_{472}\right)\frac{\partial^3 u}{\partial x_2^3} + \left(-F_{470} - \frac{2}{3}F_{471} - \frac{1}{8}F_{472}\right)\frac{\partial^3 u}{\partial x_1^2 \partial x_2} + F_{47}\frac{\partial u}{\partial x_2}$$

$$+\left(-\frac{8}{15}F_{471} + \frac{1}{8}F_{472}\right)\frac{\partial^3 v}{\partial x_1^3} + \left(-F_{470} - \frac{2}{3}F_{471} + \frac{1}{4}F_{472}\right)\frac{\partial^3 v}{\partial x_1 \partial x_2^2} + F_{47}\frac{\partial v}{\partial x_1}$$

$$+\left(-F_{440} + F_{480} - \frac{6}{5}F_{441} + \frac{6}{5}F_{481}\right)\frac{\partial^4 w}{\partial x_1^3 \partial x_2} + \left(-F_{440} + F_{480} - \frac{6}{5}F_{441} + \frac{6}{5}F_{481}\right)\frac{\partial^4 w}{\partial x_1 \partial x_2^3}$$

$$+\left(2F_{44} - 2F_{48} + \frac{4}{15}F_{461} + \frac{8}{3}F_{551} - \frac{4}{3}F_{571} + \frac{1}{8}F_{462} + \frac{1}{2}F_{552} - F_{572}\right)\frac{\partial^2 w}{\partial x_1 \partial x_2} + \Bigg(-\frac{8}{15}F_{441}$$

$$-\frac{1}{4}F_{442}\Bigg)\frac{\partial^3 \phi_1}{\partial x_2^3} + \left(-F_{440} - \frac{2}{3}F_{441} - \frac{1}{8}F_{442}\right)\frac{\partial^3 \phi_1}{\partial x_1^2 \partial x_2} + \Bigg(F_{44} + \frac{2}{15}F_{461} + \frac{4}{3}F_{551}$$

$$+\frac{1}{4}F_{462} + F_{552}\Bigg)\frac{\partial \phi_1}{\partial x_2} + \left(-\frac{8}{15}F_{441} + \frac{1}{8}F_{442}\right)\frac{\partial^3 \phi_2}{\partial x_1^3} + \left(-F_{440} - \frac{2}{3}F_{441} + \frac{1}{4}F_{442}\right)\frac{\partial^3 \phi_2}{\partial x_1 \partial x_2^2}$$

$$+\left(F_{44}+\frac{2}{15}F_{461}+\frac{4}{3}F_{551}-\frac{1}{8}F_{462}-\frac{1}{2}F_{552}\right)\frac{\partial\phi_2}{\partial x_1}\Bigg)n_{x_2}=0, \tag{18.12}$$

$$\delta\frac{\partial\phi_1}{\partial x_1}=0 \quad \text{or}$$

$$\Bigg(\left(2F_{470}+\frac{4}{5}F_{471}\right)\frac{\partial^2 u}{\partial x_1^2}-\frac{2}{5}F_{471}\frac{\partial^2 u}{\partial x_2^2}+\left(2F_{470}-\frac{4}{5}F_{471}\right)\frac{\partial^2 v}{\partial x_1\partial x_2}$$

$$+\left(2F_{440}-2F_{480}+\frac{4}{5}F_{441}-\frac{4}{5}F_{481}\right)\frac{\partial^3 w}{\partial x_1^3}+\left(2F_{440}-2F_{480}-\frac{6}{5}F_{441}+\frac{6}{5}F_{481}\right)\frac{\partial^3 w}{\partial x_1\partial x_2^2}$$

$$-\frac{2}{5}F_{461}\frac{\partial w}{\partial x_1}+\left(2F_{440}+\frac{4}{5}F_{441}\right)\frac{\partial^2\phi_1}{\partial x_1^2}-\frac{2}{5}F_{441}\frac{\partial^2\phi_1}{\partial x_2^2}-\frac{2}{5}F_{461}\phi_1$$

$$+\left(2F_{440}-\frac{4}{5}F_{441}\right)\frac{\partial^2\phi_2}{\partial x_1\partial x_2}\Bigg)n_{x_1}+\Bigg(\left(F_{470}+\frac{16}{15}F_{471}+\frac{1}{8}F_{472}\right)\frac{\partial^2 u}{\partial x_1\partial x_2}$$

$$+\left(\frac{8}{15}F_{471}-\frac{1}{8}F_{472}\right)\frac{\partial^2 v}{\partial x_1^2}+\left(F_{470}-\frac{2}{5}F_{471}\right)\frac{\partial^2 v}{\partial x_2^2}+\left(F_{440}-F_{480}-\frac{2}{5}F_{441}+\frac{2}{5}F_{481}\right)\frac{\partial^3 w}{\partial x_2^3}$$

$$+\left(F_{440}-F_{480}+\frac{8}{5}F_{441}-\frac{8}{5}F_{481}\right)\frac{\partial^3 w}{\partial x_1^2\partial x_2}+\left(-\frac{2}{15}F_{461}+\frac{1}{8}F_{462}\right)\frac{\partial w}{\partial x_2}$$

$$+\left(F_{440}+\frac{16}{15}F_{441}+\frac{1}{8}F_{442}\right)\frac{\partial^2\phi_1}{\partial x_1\partial x_2}+\left(\frac{8}{15}F_{441}-\frac{1}{8}F_{442}\right)\frac{\partial^2\phi_2}{\partial x_1^2}$$

$$+\left(F_{440}-\frac{2}{5}F_{441}\right)\frac{\partial^2\phi_2}{\partial x_2^2}+\left(-\frac{2}{15}F_{461}+\frac{1}{8}F_{462}\right)\phi_2\Bigg)n_{x_2}=0, \tag{18.13}$$

$$\delta \frac{\partial \phi_1}{\partial x_2} = 0 \quad \text{or}$$

$$\Bigg(\Bigg(F_{470} + \frac{16}{15}F_{471} + \frac{1}{8}F_{472}\Bigg)\frac{\partial^2 u}{\partial x_1 \partial x_2} + \Bigg(\frac{8}{15}F_{471} - \frac{1}{8}F_{472}\Bigg)\frac{\partial^2 v}{\partial x_1^2} + \Bigg(F_{470} - \frac{2}{5}F_{471}\Bigg)\frac{\partial^2 v}{\partial x_2^2}$$

$$+\Bigg(F_{440} - F_{480} + \frac{8}{5}F_{441} - \frac{8}{5}F_{481}\Bigg)\frac{\partial^3 w}{\partial x_1^2 \partial x_2} + \Bigg(F_{440} - F_{480} - \frac{2}{5}F_{441} + \frac{2}{5}F_{481}\Bigg)\frac{\partial^3 w}{\partial x_2^3}$$

$$+\Bigg(-\frac{2}{15}F_{461} + \frac{1}{8}F_{462}\Bigg)\frac{\partial w}{\partial x_2} + \Bigg(F_{440} + \frac{16}{15}F_{441} + \frac{1}{8}F_{442}\Bigg)\frac{\partial^2 \phi_1}{\partial x_1 \partial x_2}$$

$$+\Bigg(\frac{8}{15}F_{441} - \frac{1}{8}F_{442}\Bigg)\frac{\partial^2 \phi_2}{\partial x_1^2} + \Bigg(F_{440} - \frac{2}{5}F_{441}\Bigg)\frac{\partial^2 \phi_2}{\partial x_2^2} + \Bigg(-\frac{2}{15}F_{461} + \frac{1}{8}F_{462}\Bigg)\phi_2\Bigg)n_{x_1}$$

$$+\Bigg(-\frac{2}{5}F_{471}\frac{\partial^2 u}{\partial x_1^2} + \Bigg(\frac{8}{15}F_{471} + \frac{1}{4}F_{472}\Bigg)\frac{\partial^2 u}{\partial x_2^2} + \Bigg(\frac{16}{15}F_{471} - \frac{1}{4}F_{472}\Bigg)\frac{\partial^2 v}{\partial x_1 \partial x_2}$$

$$+\Bigg(-\frac{2}{5}F_{441} + \frac{2}{5}F_{481}\Bigg)\frac{\partial^3 w}{\partial x_1^3} + \Bigg(\frac{8}{5}F_{441} - \frac{8}{5}F_{481}\Bigg)\frac{\partial^3 w}{\partial x_1 \partial x_2^2} + \Bigg(-\frac{2}{15}F_{461} - \frac{1}{4}F_{462}\Bigg)\frac{\partial w}{\partial x_1}$$

$$-\frac{2}{5}F_{441}\frac{\partial^2 \phi_1}{\partial x_1^2} + \Bigg(\frac{8}{15}F_{441} + \frac{1}{4}F_{442}\Bigg)\frac{\partial^2 \phi_1}{\partial x_2^2}$$

$$+\Bigg(-\frac{2}{15}F_{461} - \frac{1}{4}F_{462}\Bigg)\phi_1 + \Bigg(\frac{16}{15}F_{441} - \frac{1}{4}F_{442}\Bigg)\frac{\partial^2 \phi_2}{\partial x_1 \partial x_2}\Bigg)n_{x_2} = 0, \qquad (18.14)$$

$\delta\phi_2 = 0$ or

$$\Bigg(\left(-\frac{8}{15}F_{471}+\frac{1}{8}F_{472}\right)\frac{\partial^3 u}{\partial x_2^3}+\left(-F_{470}-\frac{2}{3}F_{471}+\frac{1}{4}F_{472}\right)\frac{\partial^3 u}{\partial x_1^2\partial x_2}+F_{47}\frac{\partial u}{\partial x_2}$$

$$+\left(-\frac{8}{15}F_{471}-\frac{1}{4}F_{472}\right)\frac{\partial^3 v}{\partial x_1^3}+\left(-F_{470}-\frac{2}{3}F_{471}-\frac{1}{8}F_{472}\right)\frac{\partial^3 v}{\partial x_1\partial x_2^2}+F_{47}\frac{\partial v}{\partial x_1}$$

$$+\left(-F_{440}+F_{480}-\frac{6}{5}F_{441}+\frac{6}{5}F_{481}\right)\frac{\partial^4 w}{\partial x_1\partial x_2^3}+\left(-F_{440}+F_{480}-\frac{6}{5}F_{441}+\frac{6}{5}F_{481}\right)\frac{\partial^4 w}{\partial x_1^3\partial x_2}$$

$$+\left(2F_{44}-2F_{48}+\frac{4}{15}F_{461}+\frac{8}{3}F_{551}-\frac{4}{3}F_{571}+\frac{1}{8}F_{462}+\frac{1}{2}F_{552}-F_{572}\right)\frac{\partial^2 w}{\partial x_1\partial x_2}$$

$$+\left(-\frac{8}{15}F_{441}+\frac{1}{8}F_{442}\right)\frac{\partial^3\phi_1}{\partial x_2^3}+\left(-F_{440}-\frac{2}{3}F_{441}+\frac{1}{4}F_{442}\right)\frac{\partial^3\phi_1}{\partial x_1^2\partial x_2}$$

$$+\left(F_{44}+\frac{2}{15}F_{461}+\frac{4}{3}F_{551}-\frac{1}{8}F_{462}-\frac{1}{2}F_{552}\right)\frac{\partial\phi_1}{\partial x_2}+\left(-\frac{8}{15}F_{441}-\frac{1}{4}F_{442}\right)\frac{\partial^3\phi_2}{\partial x_1^3}$$

$$+\left(-F_{440}-\frac{2}{3}F_{441}-\frac{1}{8}F_{442}\right)\frac{\partial^3\phi_2}{\partial x_1\partial x_2^2}+\left(F_{44}+\frac{2}{15}F_{461}+\frac{4}{3}F_{551}+\frac{1}{4}F_{462}+F_{552}\right)\frac{\partial\phi_2}{\partial x_1}\Bigg)n_{x_1}$$

$$+\Bigg(\left(-2F_{470}+\frac{4}{15}F_{471}+\frac{1}{8}F_{472}\right)\frac{\partial^3 u}{\partial x_1\partial x_2^2}+\left(-F_{470}+\frac{2}{5}F_{471}\right)\frac{\partial^3 u}{\partial x_1^3}+F_{L11}\frac{\partial u}{\partial x_1}$$

$$+\left(-2F_{470}-\frac{4}{5}F_{471}\right)\frac{\partial^3 v}{\partial x_2^3}+\left(-F_{470}-\frac{2}{3}F_{471}-\frac{1}{8}F_{472}\right)\frac{\partial^3 v}{\partial x_1^2\partial x_2}+F_{11}\frac{\partial v}{\partial x_2}$$

$$+\left(-2F_{440}+2F_{480}-\frac{4}{5}F_{441}+\frac{4}{5}F_{481}\right)\frac{\partial^4 w}{\partial x_2^4}$$

$$+\left(-3F_{440}+3F_{480}-\frac{2}{5}F_{441}+\frac{2}{5}F_{481}\right)\frac{\partial^4 w}{\partial x_1^2\partial x_2^2}+\left(-F_{440}+F_{480}+\frac{2}{5}F_{441}-\frac{2}{5}F_{481}\right)\frac{\partial^4 w}{\partial x_1^4}$$

$$+\left(-F_{22}+F_{33}+2F_{550}-2F_{570}+\frac{2}{5}F_{461}+\frac{32}{15}F_{551}-\frac{16}{15}F_{571}+\frac{1}{4}F_{552}-\frac{1}{2}F_{572}\right)\frac{\partial^2 w}{\partial x_2^2}$$

$$+\Bigg(-F_{L22}+F_{L33}+2F_{550}-2F_{570}+\frac{2}{15}F_{461}-\frac{8}{15}F_{551}+\frac{4}{15}F_{571}-\frac{1}{8}F_{462}-\frac{1}{4}F_{552}$$

$$+\frac{1}{2}F_{572}\Bigg)\frac{\partial^2 w}{\partial x_1^2}+\left(-F_{440}+\frac{2}{5}F_{441}\right)\frac{\partial^3\phi_1}{\partial x_1^3}+\left(-2F_{440}+\frac{4}{15}F_{441}+\frac{1}{8}F_{442}\right)\frac{\partial^3\phi_1}{\partial x_1\partial x_2^2}$$

$$+\left(F_{L33}+2F_{550}+\frac{2}{15}F_{461}-\frac{8}{15}F_{551}-\frac{1}{8}F_{462}-\frac{1}{4}F_{552}\right)\frac{\partial\phi_1}{\partial x_1}+\left(-2F_{440}-\frac{4}{5}F_{441}\right)\frac{\partial^3\phi_2}{\partial x_2^3}$$

$$+\left(-F_{440}-\frac{2}{3}F_{441}-\frac{1}{8}F_{442}\right)\frac{\partial^3\phi_2}{\partial x_1^2\partial x_2}$$

$$+\left(F_{33}+2F_{550}+\frac{2}{5}F_{461}+\frac{32}{15}F_{551}+\frac{1}{4}F_{552}\right)\frac{\partial\phi_2}{\partial x_2}-F_{T11}\Bigg)n_{x_2}=0, \tag{18.15}$$

$$\delta\frac{\partial\phi_2}{\partial x_1}=0 \quad \text{or}$$

$$\Bigg(\left(\frac{16}{15}F_{471}-\frac{1}{4}F_{472}\right)\frac{\partial^2 u}{\partial x_1\partial x_2}+\left(\frac{8}{15}F_{471}+\frac{1}{4}F_{472}\right)\frac{\partial^2 v}{\partial x_1^2}-\frac{2}{5}F_{471}\frac{\partial^2 v}{\partial x_2^2}$$

$$+\left(-\frac{2}{5}F_{441}+\frac{2}{5}F_{481}\right)\frac{\partial^3 w}{\partial x_2^3}+\left(\frac{8}{5}F_{441}-\frac{8}{5}F_{481}\right)\frac{\partial^3 w}{\partial x_1^2\partial x_2}+\left(-\frac{2}{15}F_{461}-\frac{1}{4}F_{462}\right)\frac{\partial w}{\partial x_2}$$

$$+\left(\frac{16}{15}F_{441}-\frac{1}{4}F_{442}\right)\frac{\partial^2\phi_1}{\partial x_1\partial x_2}+\left(\frac{8}{15}F_{441}+\frac{1}{4}F_{442}\right)\frac{\partial^2\phi_2}{\partial x_1^2}-\frac{2}{5}F_{441}\frac{\partial^2\phi_2}{\partial x_2^2}$$

$$+\left(-\frac{2}{15}F_{461}-\frac{1}{4}F_{462}\right)\phi_2\Bigg)n_{x_1}+\Bigg(\left(F_{470}-\frac{2}{5}F_{471}\right)\frac{\partial^2 u}{\partial x_1^2}+\left(\frac{8}{15}F_{471}-\frac{1}{8}F_{472}\right)\frac{\partial^2 u}{\partial x_2^2}$$

$$+\left(F_{470}+\frac{16}{15}F_{471}+\frac{1}{8}F_{472}\right)\frac{\partial^2 v}{\partial x_1\partial x_2}+\left(F_{440}-F_{480}-\frac{2}{5}F_{441}+\frac{2}{5}F_{481}\right)\frac{\partial^3 w}{\partial x_1^3}$$

$$+\left(F_{440}-F_{480}+\frac{8}{5}F_{441}-\frac{8}{5}F_{481}\right)\frac{\partial^3 w}{\partial x_1\partial x_2^2}+\left(-\frac{2}{15}F_{461}+\frac{1}{8}F_{462}\right)\frac{\partial w}{\partial x_1}$$

$$+\left(F_{440}-\frac{2}{5}F_{441}\right)\frac{\partial^2\phi_1}{\partial x_1^2}+\left(\frac{8}{15}F_{441}-\frac{1}{8}F_{442}\right)\frac{\partial^2\phi_1}{\partial x_2^2}+\left(-\frac{2}{15}F_{461}+\frac{1}{8}F_{462}\right)\phi_1$$

$$+\left(F_{440}+\frac{16}{15}F_{441}+\frac{1}{8}F_{442}\right)\frac{\partial^2\phi_2}{\partial x_1\partial x_2}\Bigg)n_{x_2}=0, \tag{18.16}$$

$$\delta\frac{\partial\phi_2}{\partial x_2}=0 \quad \text{or}$$

$$\Bigg(\left(F_{470}-\frac{2}{5}F_{471}\right)\frac{\partial^2 u}{\partial x_1^2}+\left(\frac{8}{15}F_{471}-\frac{1}{8}F_{472}\right)\frac{\partial^2 u}{\partial x_2^2}+\left(F_{470}+\frac{16}{15}F_{471}+\frac{1}{8}F_{472}\right)\frac{\partial^2 v}{\partial x_1\partial x_2}$$
$$+\left(F_{440}-F_{480}-\frac{2}{5}F_{441}+\frac{2}{5}F_{481}\right)\frac{\partial^3 w}{\partial x_1^3}+\left(F_{440}-F_{480}+\frac{8}{5}F_{441}-\frac{8}{5}F_{481}\right)\frac{\partial^3 w}{\partial x_1\partial x_2^2}$$
$$+\left(-\frac{2}{15}F_{461}+\frac{1}{8}F_{462}\right)\frac{\partial w}{\partial x_1}+\left(F_{440}-\frac{2}{5}F_{441}\right)\frac{\partial^2\phi_1}{\partial x_1^2}+\left(\frac{8}{15}F_{441}-\frac{1}{8}F_{442}\right)\frac{\partial^2\phi_1}{\partial x_2^2}+\Bigg(-\frac{2}{15}F_{461}$$
$$+\frac{1}{8}F_{462}\Bigg)\phi_1+\left(F_{440}+\frac{16}{15}F_{441}+\frac{1}{8}F_{442}\right)\frac{\partial^2\phi_2}{\partial x_1\partial x_2}\Bigg)n_{x_1}+\Bigg(\left(2F_{470}-\frac{4}{15}F_{471}\right)\frac{\partial^2 u}{\partial x_1\partial x_2}$$
$$-\frac{2}{5}F_{471}\frac{\partial^2 v}{\partial x_1^2}+\left(2F_{470}+\frac{4}{5}F_{471}\right)\frac{\partial^2 v}{\partial x_2^2}+\left(2F_{440}-2F_{480}+\frac{4}{5}F_{441}-\frac{4}{5}F_{481}\right)\frac{\partial^3 w}{\partial x_2^3}$$
$$+\left(2F_{440}-2F_{480}-\frac{6}{5}F_{441}+\frac{6}{5}F_{481}\right)\frac{\partial^3 w}{\partial x_1^2\partial x_2}-\frac{2}{5}F_{461}\frac{\partial w}{\partial x_2}+\left(2F_{440}-\frac{4}{5}F_{441}\right)\frac{\partial^2\phi_1}{\partial x_1\partial x_2}$$
$$-\frac{2}{5}F_{441}\frac{\partial^2\phi_2}{\partial x_1^2}+\left(2F_{440}+\frac{4}{5}F_{441}\right)\frac{\partial^2\phi_2}{\partial x_2^2}-\frac{2}{5}F_{461}\phi_2\Bigg)n_{x_2}=0. \tag{18.17}$$

Although all types of boundary conditions including combinations of clamped (C), simply supported (S) and free (F) can be achieved by using Eq. (18), in the present study, all edges clamped (CCCC) and all edges simply supported (SSSS) micro-plates are considered to generate numerical results. The boundary conditions for all edges clamped (CCCC) micro-plate are given by

At edges $x_1=0,a$

$$u=\frac{\partial u}{\partial x_1}=v=\frac{\partial v}{\partial x_1}=w=\frac{\partial w}{\partial x_1}=\frac{\partial^2 w}{\partial x_1^2}=\phi_1=\frac{\partial\phi_1}{\partial x_1}=\phi_2=\frac{\partial\phi_2}{\partial x_1}=0, \tag{19.1}$$

At edges $x_2=0,b$

$$u=\frac{\partial u}{\partial x_2}=v=\frac{\partial v}{\partial x_2}=w=\frac{\partial w}{\partial x_2}=\frac{\partial^2 w}{\partial x_2^2}=\phi_1=\frac{\partial\phi_1}{\partial x_2}=\phi_2=\frac{\partial\phi_2}{\partial x_2}=0. \tag{19.2}$$

Also, the boundary conditions associated with all edges simply supported are in the form

At edges $x_1 = 0, a$

$$\frac{\partial u}{\partial x_1} = v = w = \frac{\partial^2 w}{\partial x_1^2} = \frac{\partial \phi_1}{\partial x_1} = \phi_2 = 0,$$
$$u \neq 0, \quad \frac{\partial v}{\partial x_1} \neq 0, \quad \frac{\partial w}{\partial x_1} \neq 0, \quad \phi_1 \neq 0, \quad \frac{\partial \phi_2}{\partial x_1} \neq 0, \tag{20.1}$$

At edges $x_2 = 0, b$

$$u = \frac{\partial v}{\partial x_2} = w = \frac{\partial^2 w}{\partial x_2^2} = \phi_1 = \frac{\partial \phi_2}{\partial x_2} = 0,$$
$$v \neq 0, \quad \frac{\partial u}{\partial x_2} \neq 0, \quad \frac{\partial w}{\partial x_2} \neq 0, \quad \frac{\partial \phi_1}{\partial x_2} \neq 0, \quad \phi_2 \neq 0, \tag{20.2}$$

where a and b are the length and width of functionally graded micro-plate, respectively, as depicted in Figure 1.

CHAPTER 3

NUMERICAL SOLUTION

3.1 Differential Quadrature Method

Differential quadrature method is employed to solve the system comprising governing partial differential equations and boundary conditions. The technique, which was originally proposed by Bellman and Casti [76], is based on approximating derivative of a function by a weighted sum of functional values. m^{th} derivative of a function $z(x,t)$ with respect to x at a point x_i is represented as:

$$\frac{\partial^m z(x,t)}{\partial x^m}\Big|_{x=x_i} = \sum_{j=1}^{N} c_{ij}^{(m)} z(x_j,t), \quad i = 1, 2, ..., N \tag{21}$$

N here is the number of nodes, and $c_{ij}^{(m)}$ are the weighting coefficients for the mth derivative. The coefficients are available in the book by Shu [77]. Figure 2 depicts discretization of the mid-plane of the FGM micro-plate. N_{x_1} and N_{x_2} in the figure are numbers of grid points in x_1 and x_2 directions, respectively. Using DQM, derivatives of a function $z(x,y,t)$ at a point (i, j), $i = 1, ..., N_{x_1}$ and $j = 1, ..., N_{x_2}$, are expressed as:

$$\frac{\partial^n z(x_1,x_2,t)}{\partial x_1^n}\Big|_{(i,j)} = \sum_{k=1}^{N_{x_1}} \alpha_{ik}^{(n)} \left(z(x_1,x_2,t)\big|_{(k,j)} \right), \tag{22.1}$$

$$\frac{\partial^n z\left(x_1, x_2, t\right)}{\partial x_2^n}\Big|_{(i,j)} = \sum_{k=1}^{N_{x_2}} \beta_{jk}^{(n)} \left(z\left(x_1, x_2, t\right)\big|_{(i,k)} \right), \tag{22.2}$$

$$\frac{\partial^{(n+m)} z\left(x_1, x_2, t\right)}{\partial x_1^n \partial x_2^m}\Big|_{(i,j)} = \sum_{l=1}^{N_{x_2}} \beta_{jl}^{(m)} \sum_{k=1}^{N_{x_1}} \alpha_{ik}^{(n)} \left(z\left(x_1, x_2, t\right)\big|_{(k,l)} \right). \tag{22.3}$$

where $\alpha_{ij}^{(m)}$ and $\beta_{ij}^{(m)}$ are the weighting coefficients for the mth derivative in x_1 and x_2 directions. An important factor influencing stability of DQM is the distribution of grid points. In the present study, we used Chebyshev-Gauss-Lobatto points, which result in a more stable procedure compared to uniform grid points [78]. Chebyshev-Gauss-Lobatto points for a two dimensional setting $0 \le x_1 \le 1$ and $0 \le x_2 \le 1$, are given by

$$x_{1i} = \frac{1}{2}\left\{1 - \cos\left(\frac{\pi\left(i-1\right)}{N_{x_1} - 1}\right)\right\}, \qquad i = 1, 2, ..., N_{x_1}, \tag{23.1}$$

$$x_{2j} = \frac{1}{2}\left\{1 - \cos\left(\frac{\pi\left(j-1\right)}{N_{x_2} - 1}\right)\right\}, \qquad j = 1, 2, ..., N_{x_2}. \tag{23.2}$$

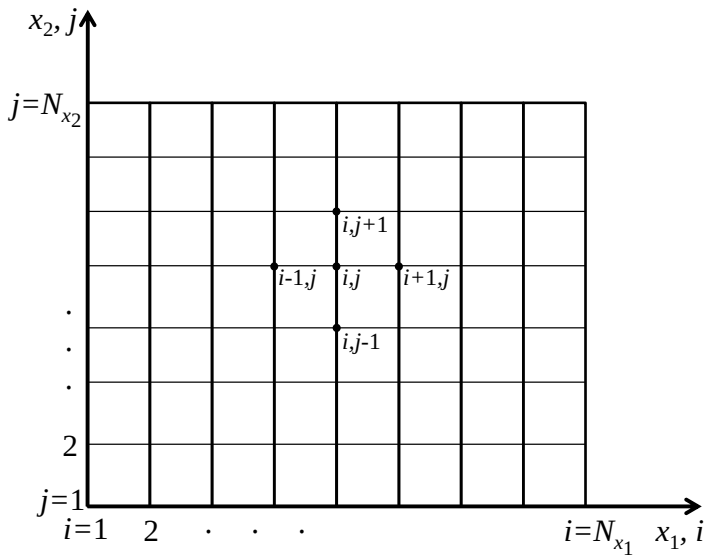

Figure 2. Discretization of a plate mid-plane.

3.2 Static Thermal Bending Analysis

Before proceeding to the free vibration analysis of functionally graded micro-plate in thermal environment, the initial stress state and hence initial in-plane forces $P_{x_1}^0$, $P_{x_2}^0$ and $P_{x_1x_2}^0$ must be determined from static thermal bending problem. The static problem whose solution gives the thermally induced initial displacements is obtained by dropping inertia related terms, initial in-plane forces including thermally induced and mechanical in-plane loads and distributed load in Eqs. (17) and (18). Once the static thermal displacements are worked out, $P_{x_1}^0$, $P_{x_2}^0$ and $P_{x_1x_2}^0$ are evaluated by inserting these displacements respectively in M_{11}^0, M_{22}^0 and M_{12}^0 given by Eq. (15). For all edges clamped, since thermally induced initial displacements are zero, in-plane forces are obtained as follows:

$$P_{x_1}^0 = -A_{T11}, \quad P_{x_2}^0 = -A_{T11}, \quad P_{x_1x_2}^0 = 0 \tag{24}$$

3.3 Static Bending

For a rectangular FGM micro-plate under static loading, unknown generalized displacement vector **d** is defined as follows:

$$\mathbf{d} = \left\{ \{u_p\}^{\mathrm{T}},\ \{v_p\}^{\mathrm{T}},\ \{w_p\}^{\mathrm{T}},\ \{\phi_{1p}\}^{\mathrm{T}},\ \{\phi_{2p}\}^{\mathrm{T}} \right\}^{\mathrm{T}}, \quad \text{for } p = 1,\ 2,\ \ldots,\ N_{x_1} \times N_{x_2}, \tag{25}$$

where $\{u_p\}$, $\{v_p\}$, $\{w_p\}$, $\{\phi_{1p}\}$ and $\{\phi_{2p}\}$ are unknown vectors with $N_{x_1} \times N_{x_2}$ elements. When MPT or TSDT is employed in numerical analysis, the displacement vector **d** comprises of $5 \times N_{x_1} \times N_{x_2}$ components. However, in the computations based on KPT, **d** contains $3 \times N_{x_1} \times N_{x_2}$ unknown displacements because governing equations associated with ϕ_1 and ϕ_2 are not included.

Eliminating in-plane forces and the terms including time derivatives; and utilizing DQM, governing equations and boundary conditions are recast into the following matrix form:

$$\mathbf{Dd} + \mathbf{Q} = \mathbf{0}, \tag{26}$$

in which $\mathbf{D}$ is coefficient matrix associated with grid points, and $\mathbf{Q}$ is forcing vector resulting from distributed loading. For grid points located on the boundaries, i.e. $i = 1$, $j = 1$, $i = N_{x_1}$ and $j = N_{x_2}$ in Figure 2, boundary conditions are implemented. If there are more than one conditions on a boundary, the nodes next to the boundary grid points, $i = 2$, $j = 2$, $i = N_{x_1} - 1$ and $j = N_{x_2} - 1$, may be used to apply boundary equations.

3.4 Free Vibrations

The total displacement is the sum of static thermal displacement vector $\mathbf{d}^0$ and dynamic displacement vector $\mathbf{d}$. By inserting the total displacement vector $\mathbf{d} + \mathbf{d}^0$ in governing equations and boundary conditions and knowing that $\mathbf{d}^0$ satisfies the thermal static equilibrium, the governing equations in terms of dynamic displacements are obtained in which the nonhomogeneous terms are dropped.

To conduct the free vibration analysis the dynamic displacement vector is defined in the following form

$$\mathbf{d} = \mathbf{d}^* \mathrm{e}^{\mathrm{i}\omega t}, \tag{27}$$

where ω represents the natural frequency and $\mathbf{d}^*$ is the vibration mode shape vector consisting of dynamic displacement vectors of field variables

$$\mathbf{d}^* = \left\{ \left\{ u_p^* \right\}^{\mathrm{T}}, \left\{ v_p^* \right\}^{\mathrm{T}}, \left\{ w_p^* \right\}^{\mathrm{T}}, \left\{ \phi_{1p}^* \right\}^{\mathrm{T}}, \left\{ \phi_{2p}^* \right\}^{\mathrm{T}} \right\}^{\mathrm{T}}, \quad \text{for } p = 1,\ 2,\ \ldots,\ N_{x_1} \times N_{x_2} \tag{28}$$

Substitution of Eq. (27) into governing equations leads to

$$\mathbf{D_b d^{*e}} + \mathbf{D_d d^{*i}} - \omega^2 \mathbf{M d^{*i}} = \mathbf{0}. \tag{29}$$

where $\mathbf{d^{*e}}$ and $\mathbf{d^{*i}}$ are dynamic displacement vectors for boundary and internal points, respectively; $\mathbf{D_b}$ and $\mathbf{D_d}$ are coefficient matrices associated with boundary and internal points; and $\mathbf{M}$ is mass matrix. Using Eq. (27) and boundary conditions one finds

$$\mathbf{B_b d^{*e}} + \mathbf{B_d d^{*i}} = \mathbf{0}. \tag{30}$$

$\mathbf{B_b}$ and $\mathbf{B_d}$ are coefficient matrices obtained from boundary conditions associated with boundary and internal points, respectively. Combining Eqs. (29) and (30), we derive the eigenvalue problem:

$$\left\{\mathbf{K} - \omega^2 \mathbf{M}\right\} \mathbf{d^{*i}} = \mathbf{0}. \tag{31}$$

$\mathbf{K}$ here is stiffness matrix given by

$$\mathbf{K} = -\mathbf{D_b B_b^{-1} B_d} + \mathbf{D_d}. \tag{32}$$

3.5 Buckling

In buckling analysis, the in-plane loading is comprised of externally applied biaxial mechanical loading with $P_{x_1} = P_{x_2} = P$ and initial forces $P_{x_1}^0$, $P_{x_2}^0$ and $P_{x_1 x_2}^0$ due to static thermal bending problem. Similar to the vibration formulation, buckling solution procedure leads to an eigenvalue problem. However, instead of mass matrix $\mathbf{M}$, a coefficient matrix $\mathbf{X}$ is generated from the derivative terms $\partial^2 w / \partial x_1^2$ and $\partial^2 w / \partial x_2^2$. The eigenvalue problem is derived in the following form:

$$\left\{\mathbf{K} - P\mathbf{X}\right\} \mathbf{d_b^*} = \mathbf{0}. \tag{33}$$

The smallest eigenvalue P computed from Eq. (33) is the critical buckling load of the rectangular FGM micro-plate, and $\mathbf{d}_{\mathrm{b}}^{*}$ is buckling mode shape vector.

CHATER 4

NUMERICAL RESULTS

On the base of numerical method described in previous section static bending, free vibration and buckling analyses of micro-plates with all edges simply-supported (SSSS) and all edges clamped (CCCC) have been established.

4.1 Functionally Graded Material

The geometry of a small-scale functionally graded rectangular plate is depicted in Figure 1. The mechanical properties of the functionally graded micro-plate are assumed to be functions of the thickness coordinate x_3. It is assumed that the material of micro-plate varies from metal phase at $x_3 = -h/2$ to ceramic at $x_3 = h/2$. The volume fraction V of each phase material is determined by the following power-law

$$V_c(x_3) = (0.5 + x_3/h)^n, \quad V_c + V_m = 1 \tag{34}$$

where the subscripts c and m represent ceramic and metallic constituents, respectively and n is the volume fraction exponent. A typical effective material property Z of a functionally graded micro-plate such as Young's modulus $E(x_3)$, Poisson's ratio $\nu(x_3)$, density $\rho(x_3)$, material length scale parameters $l_i(x_3)$, $i = 0,1,2,$ coefficient of thermal expansion $\alpha(x_3)$ and thermal conductivity $k(x_3)$ vary continuously along the thickness according to the rule of mixture

$$Z(x_3) = Z_c V_c(x_3) + Z_m V_m(x_3), \tag{35}$$

The particular metal-ceramic functionally graded micro-plate considered in parametric analyses, unless otherwise stated, is made of aluminum (Al) and silicon carbide (SiC) for which, according to Eshraghi et al. [68], material properties are given as

$$E_c = 427 \text{ GPa}, \quad E_m = 70 \text{ GPa}, \tag{36.1}$$

$$\nu_c = 0.17, \quad \nu_m = 0.3, \tag{36.2}$$

$$\rho_c = 3100 \text{ kg/m}^3, \quad \rho_m = 2702 \text{ kg/m}^3, \tag{36.3}$$

$$\alpha_c = 4.3(10)^{-6} \text{ 1/K}, \quad \alpha_m = 23.0(10)^{-6} \text{ 1/K}, \tag{36.4}$$

$$k_c = 65 \text{ W/(mK)}, \quad k_m = 204 \text{ W/(mK)}. \tag{36.5}$$

Since there is not sufficient characterization data in the literature on length scale parameters, approximate values are used for functionally graded small-scale structures [56, 59]. In the present study, length scale parameters of strain gradient theory of the metallic phase are taken as $l_{0_m} = l_{1_m} = l_{2_m} = l = 15\mu\text{m}$; and those of the ceramic component are defined as $l_{0_c} = l_{1_c} = l_{2_c} = \beta l$, where β is length scale parameter ratio. Similarly, the length scale parameter associated with modified couple stress theory of metallic component is assumed to be $l_m = l = 15\mu\text{m}$; and that of ceramic phase is characterized by using length scale parameter ratio β as $l_c = \beta l$. Note that when $\beta = 1$, length scale parameters are constant within the micro-plate. Any positive β value other than unity implies spatial length scale parameter variations.

4.2 Convergence Study

In order to study the convergence of DQM, first dimensionless transverse natural frequencies of simply-supported micro-plate are provided in Table 1. Dimensionless natural frequency is defined as

$$\bar{\omega} = \omega \frac{a^2}{h} \sqrt{\rho_m / E_m}. \tag{37}$$

The results for three different elasticity theories, namely, classical theory (CT), MCST and SGT are generated, by using third order shear deformation plate model. It can be seen that for better accuracy and convergence of results the number of grid points can be chosen as $N_{x_1} = N_{x_2} = 11$.

Table 1. Convergence study on first dimensionless transverse natural frequencies of SSSS micro-plate, *l* = 15 μm, *l*/*h* = 0.4, *a*/*h* = 10, *b*/*a* = 1.0, *n* = 2.0, *β* = 2.0.

$N_{x_1} \times N_{x_2}$	CT	MCST	SGT
9×9	8.3671	14.9849	24.6728
11×11	8.3672	14.9851	24.6706
13×13	8.3672	14.9851	24.6706
15×15	8.3672	14.9851	24.6706

4.3 Numerical Results in Absence of Thermal Effets

4.3.1 Static bending

In order to be able to verify the developed procedures, we first present some comparisons to the static bending results provided by Ansari et al. [30]. Table 1 tabulates comparisons regarding normalized maximum deflection $w_{\max}/h$ of a simply-supported homogeneous micro-plate under uniform loading *q*. Maximum deflection occurs at the mid-point of the simply-supported plate. Material properties are given by:

$$E = 1.44 \text{ GPa}, \quad \rho = 1220 \text{ kg/m}^3, \quad l_0 = l_1 = l_2 = l = 17.6 \mu\text{m}. \tag{38}$$

The results are generated in accordance with Mindlin plate model and strain gradient theory. The excellent agreement between the deflections is indicative of the high degree of accuracy attained by the application of the developed procedures. Note that Ansari et al. [30] used finite element method (FEM) and proposed a new size-dependent triangular plate element to capture size effect in micro-plates.

Table 2. Comparisons of the dimensionless deflection (w_{max}/h) for a homogeneous micro-plate, ν = 0.3, b/a = 1.0, q = 1000 N/m^2.

a/h		***h/l***		
		1.0	2.0	5.0
10	Present	3.3427E-5	8.0393E-5	2.0811E-4
	Reference [30]	3.3211E-5	8.0250E-5	2.0834E-4
50	Present	0.0125	0.0412	0.1209
	Reference [30]	0.0125	0.0413	0.1212

Figure 3 shows dimensionless maximum deflection $\bar{w}$ for a functionally graded simply-supported micro-plate subjected to uniform loading q = 1 N/m^2. Normalized deflection is defined by

$$\bar{w} = w_{\max} \frac{100 E_m h^3}{q a^4}. \tag{39}$$

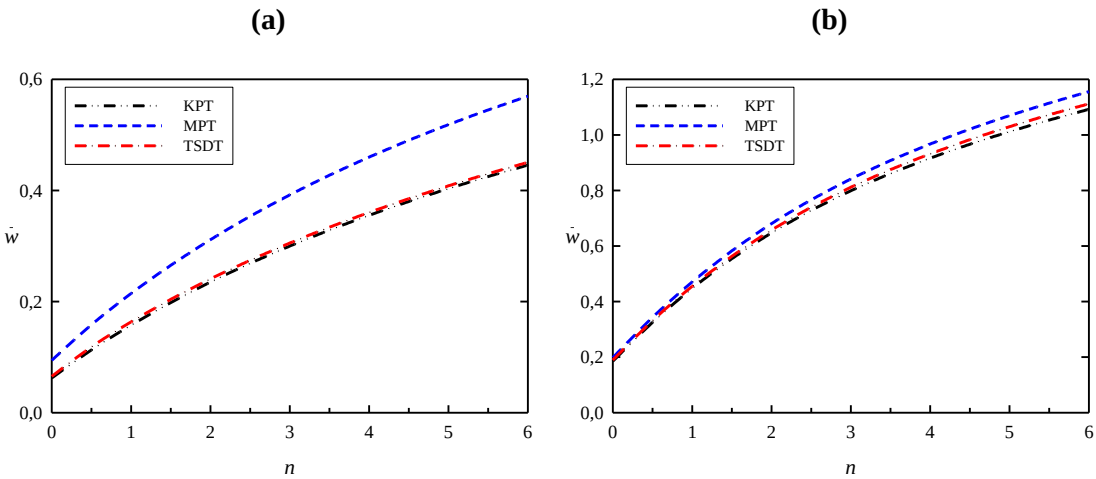

Figure 3. Dimensionless maximum deflection of SSSS micro-plate, l = 15 μm, l/h = 0.4, a/h = 10, b/a = 1.0, β = 2.0, q = 1 N/m^2, predicted by (a) SGT and (b) MCST.

$\bar{w}$ is plotted with respect to power function exponent *n* for three different plate theories. When *n* = 1, all material variation profiles are linear. The plate is metal-rich for *n* > 1, and ceramic-rich if *n* < 1. Mid-point deflection is found to be an increasing function of the exponent *n*. This is the expected result since for larger *n* the plate is metal-rich, and elastic modulus of the metallic component is much smaller than that of ceramic component. Deflection profiles found in accordance with Kirchhoff plate theory and third-order shear deformation theory are close to each other, whereas MPT overestimates the micro-plate deflection. Further results generated for static bending presented in Figures 4-7 are calculated by considering TSDT.

Figure 4 illustrates influence of the length scale parameter variation upon the static bending behavior. Normalized maximum plate deflection is plotted as a function of the exponent *n* for four different values of the length scale parameter ratio β. Note that when $\beta = 1$, all length scale parameters are constant, while any value other than unity implies through-the-thickness variations for these parameters. Maximum micro-plate deflection $\bar{w}$ decreases as the ratio β is varied from 1/2 to 4. Impact of β upon static bending behavior underlines the importance of inclusion of length scale parameter variations in the formulation of micro-scale structural problems. Figure 5 depicts variations of the maximum deflection with respect to the normalized length scale parameter l/h, where *l* is length scale parameter of the metallic component. The curves are plotted for four different values of the exponent *n*. Maximum normalized deflection decreases as l/h is increased from 0 to 1.2. Reduction in the deflection is due to size effect, which is more prominent when *h* is close to *l*. As l/h approaches zero, size effect weakens and this causes considerably larger deflections.

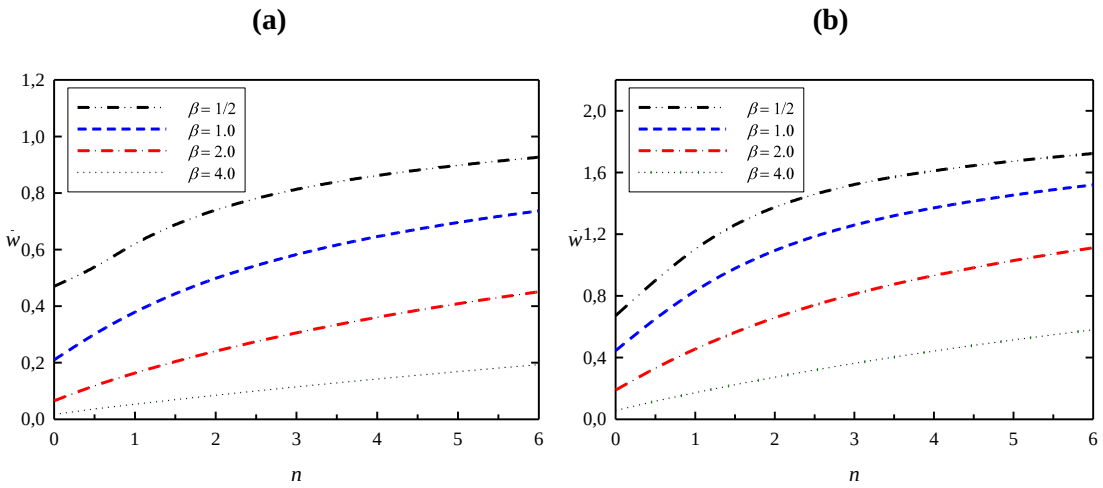

Figure 4. Dimensionless maximum deflection of SSSS micro-plate, l = 15 μm, l/h = 0.4, a/h = 10, b/a = 1.0, q = 1 N/m^2, predicted by (a) SGT and (b) MCST.

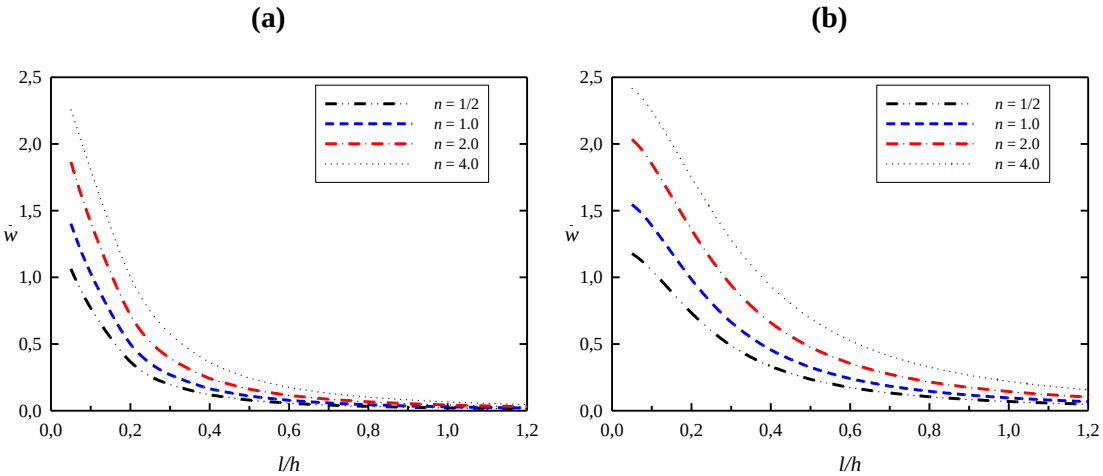

Figure 5. Dimensionless maximum deflection of SSSS micro-plate, h = 25 μm, a/h = 10, b/a = 1.0, β = 2.0, q = 1 N/m^2 predicted by (a) SGT and (b) MCST.

Provided in Figure 6 are the dimensionless maximum deflections of simply-supported micro-plate predicted by three different elasticity theories, namely, classical theory (CT), modified couple stress theory (MCST) and strain gradient theory (SGT). Classical theory disregards size effects; and dimensionless maximum deflections of plates with the same aspect ratios a/h and b/a are not affected by the change in the thickness when computed by classical elasticity theory. Smaller deflections are found in accordance with MCST and SGT specially as l/h gets larger which is indication of significance of higher order continuum theories in small-

scales. As $l/h \to 0$, all three approaches converge to the same result as that of CT. The stiffest plate is predicted based on SGT.

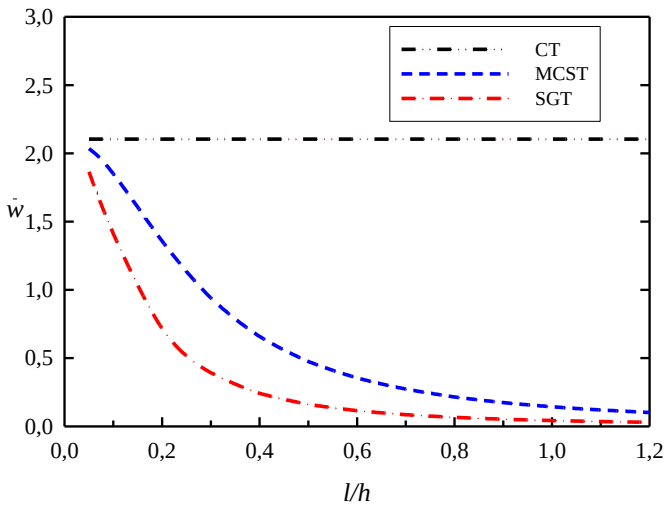

Figure 6. Dimensionless maximum deflection of SSSS micro-plate, *h* = 25 μm, *a*/*h* = 10, *b*/*a* = 1.0, *β* = 2.0, *n* = 2.0, *q* = 1 N/m^2.

Since all edges clamped micro-plates show similar behaviors to all edges simply-supported micro-plates, all the forgoing results, for brevity, are generated for all edges simply-supported micro-plates. As an example, based on MCST, the dimensionless maximum deflections of CCCC micro-plates as functions of the exponent *n* and l/h are plotted in Figure 7. It is observed that an increase in l/h leads to reduction in maximum normalized deflection. The curves are descending functions of the exponent *n*. Comparing Figure 7 with Figure 5 (b) it is revealed that, under a similar loading condition, in a CCCC microplate the deflections are smaller than a SSSS micro-plate of the same size; This can be justified by the more rigid boundaries of an all edges clamped micro-plate.

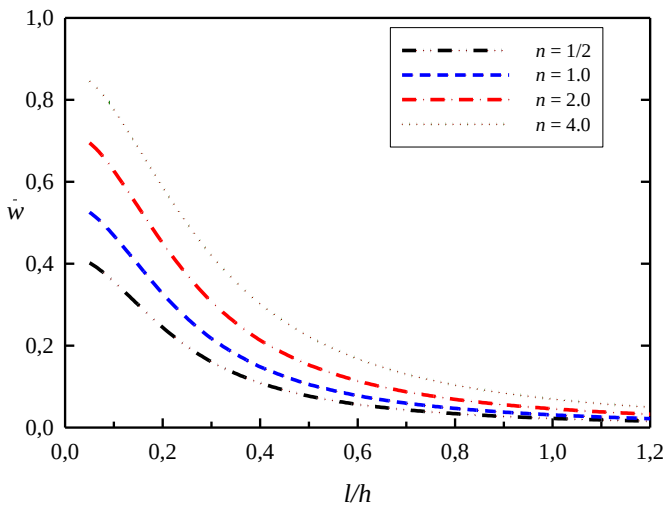

Figure 7. Dimensionless maximum deflection of CCCC micro-plate, h = 25 μm, a/h = 10, b/a = 1.0, β = 2.0, q = 1 N/m^2, predicted by MCST.

4.3.2 Free vibrations

In order to verify free vibration analysis techniques, in Table 2 we compare first natural frequencies of a simply-supported homogeneous micro-plate produced by DQM to the frequencies given by Ansari et al. [30] which are computed by FEM. Material properties are the same as those given by Eq. (38), and MPT along with SGT are used in analyses. Natural frequencies computed are found to be in excellent agreement.

Table 3. Comparisons of the first natural frequency ω_1 (in MHz) for a homogeneous micro-plate, ν = 0.38, a/h = 10, b/a = 1.0.

	h/l				
	1.0	1.5	2.0	5.0	10
Present	1.7107	0.8889	0.5564	0.1421	0.0618
Reference [30]	1.7094	0.8887	0.5564	0.1420	0.0617

In Table 4 the first dimensionless natural frequencies $\bar{\omega}_1$ of functionally graded micro-plates with all edges simply-supported are compared to those presented by Thai and Choi [42]. Thai and Choi [42] used Navier approach to solve the bending, buckling, and vibration problems regarding functionally graded rectangular micro-plates. The results in Table 4 are generated based on modified couple stress theory

and two different plate models, i.e. KPT and MPT. Material properties used in this set of results are given by

$$E_c = 14.4 \text{ GPa}, \quad E_m = 1.44 \text{ GPa}, \tag{40.1}$$

$$\rho_c = 12.2(10)^3 \text{ kg/m}^3, \quad \rho_m = 1.22(10)^3 \text{ kg/m}^3, \tag{40.2}$$

Dimensionless natural frequency in this set of results is given by Eq. (37). It can be observed that results of the current study are identical with those reported by Thai and Choi [42].

Table 4. Comparisons of first dimensionless natural frequency $\bar{\omega}_1$ of SSSS micro-plate, *ν* = 0.38, *h* = 17.6 μm, *a*/*h* = 10, *b*/*a* = 1.0.

Plate model	***l*/*h***	***n*=0**		***n*=10**	
		Present	Reference [42]	Present	Reference [42]
KPT	0	6.1103	6.1103	6.3958	6.3958
	0.2	6.5491	6.5491	6.8156	6.8156
	0.4	7.7174	7.7174	7.9431	7.9431
	0.6	9.3453	9.3453	9.5303	9.5303
	0.8	11.2349	11.2349	11.3866	11.3866
	1	13.2749	13.2749	13.4006	13.4006
MPT	0	5.9301	5.9301	6.1903	6.1903
	0.2	6.3559	6.3559	6.5967	6.5967
	0.4	7.4807	7.4807	7.6797	7.6797
	0.6	9.0261	9.0261	9.1829	9.1829
	0.8	10.7848	10.7848	10.9066	10.9066
	1	12.6360	12.6360	12.7303	12.7303

Figures 8-15 and Table 5 present our results regarding free vibrations of functionally graded rectangular micro-plates. The dimensionless natural frequency is defined the same as Eq. (37).

Figure 8 shows the first dimensionless transverse natural frequency as a function of the exponent *n* for three different plate theories. Dimensionless frequency is found to be a decreasing function of *n*. Therefore, ceramic-rich micro-plates possess higher natural frequencies compared to metal-rich plates. Frequencies found in accordance with KPT and TSDT are in close agreement. MPT slightly underestimates the natural frequency. Remaining parametric analyses on free vibrations are thus carried out in accordance with TSDT.

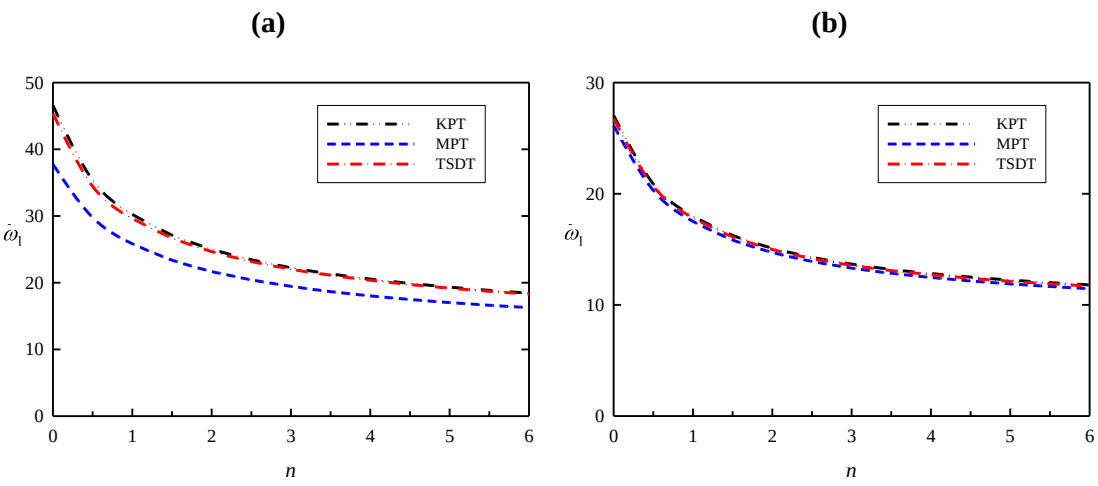

Figure 8. First dimensionless transverse natural frequencies of SSSS micro-plate, *l* = 15 μm, *l*/*h* = 0.4, *a*/*h* = 10, *b*/*a* = 1.0, *β* = 2.0, predicted by (a) SGT and (b) MCST.

Table 5 tabulates first ten dimensionless natural frequencies computed for length scale parameter ratios β = 1/2 and β = 4. The table also lists dominant mode of vibration at each frequency. The dominant mode is determined by comparing axial (u and v), transverse (w) and rotational (ϕ_1 and ϕ_2) mode shapes. Axial and transverse vibration mode shapes of a typical simply-supported FGM micro-plate are illustrated in Figures 9 and 10. The frequencies at which the rotational vibration is dominant are generally the higher frequencies. Note that the weighting coefficients used in DQM are determined by choosing Lagrange interpolated polynomial as the set of test functions. The mode shapes given in Figures 9 and 10, initially are obtained by utilizing $N_{x_1} = N_{x_2} = 11$ in numerical calculations, then by means of multivariate Lagrange interpolation, high resolution plots are generated. Examining Table 3, it is seen that dominant mode of vibration strongly depends upon the length scale parameter ratio β.

Table 5. Dominant modes and corresponding frequencies of SSSS micro-plate, l = 15 μm, l/h = 0.4, a/h = 10, b/a = 1.0, n = 2.0, predicted by SGT.

β=1/2		β=4.0	
$\bar{\omega}$	Dominant Mode	$\bar{\omega}$	Dominant Mode
14,0703	Transverse *w*, mode 1	33,7745	Axial, mode 1
32,6332	Axial, mode 1	33,7745	Axial, mode 1
32,6332	Axial, mode 1	41,4124	Transverse *w*, mode 1
33,7162	Transverse *w*, mode 2	48,8784	Axial, mode 2
33,7162	Transverse *w*, mode 2	74,6516	Axial, mode 3
45,6546	Axial, mode 2	74,6516	Axial, mode 3
52,0493	Transverse *w*, mode 3	82,6270	Axial, mode 4
63,6646	Transverse *w*, mode 4	85,4552	Axial, mode 5
63,6646	Transverse *w*, mode 4	85,4552	Axial, mode 5
65,8387	Axial, mode 3	99,8940	Transverse *w*, mode 2

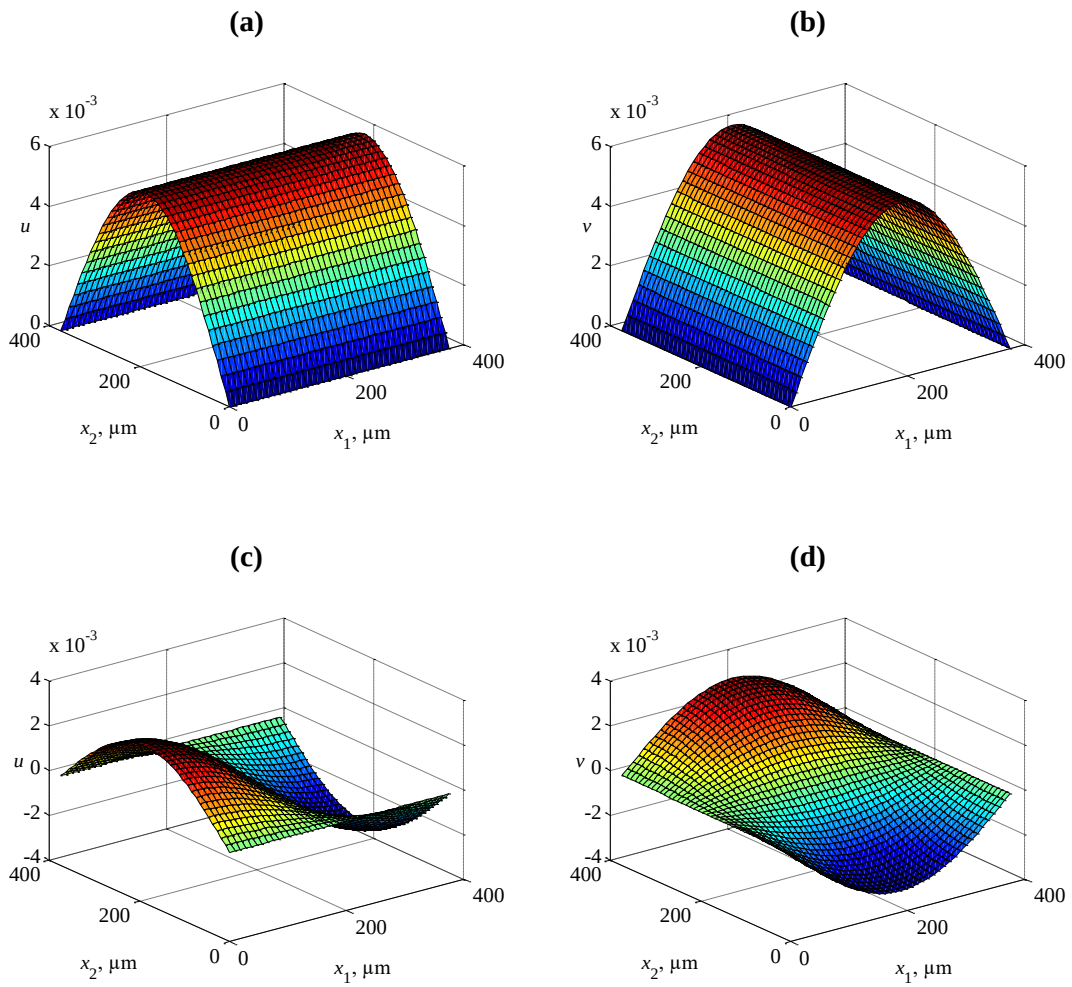

Figure 9. Dominant axial mode shapes of SSSS micro-plate: (a) Axial mode 1 (u); (b) axial mode 1 (v); (c) axial mode 2 (u); (d) axial mode 2 (v). l = 15 μm, l/h = 0.4, a/h = 10, b/a = 1.0, n = 2.0, β = 2.0, predicted by SGT.

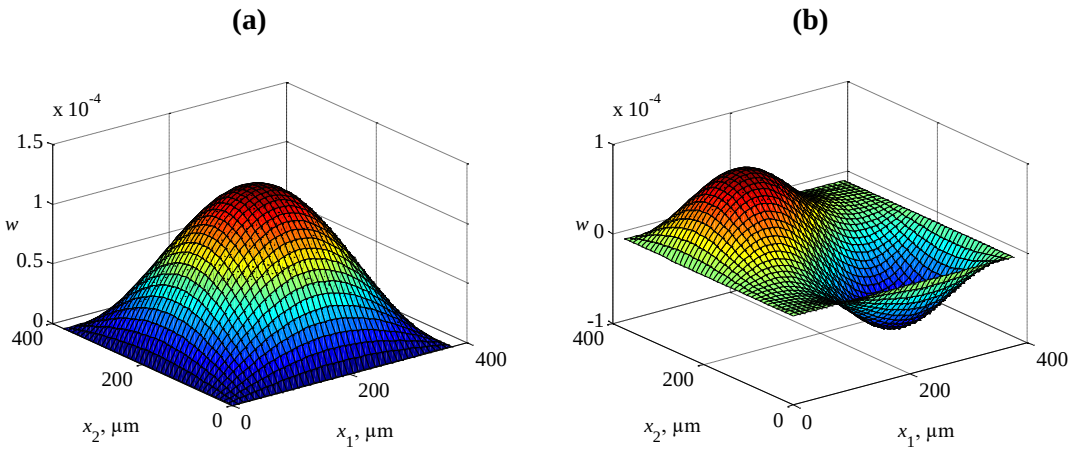

Figure 10. Dominant transverse mode shapes of SSSS micro-plate: (a) Transverse mode 1 (*w*); (b) transverse mode 2 (*w*). *l* = 15 μm, *l*/*h* = 0.4, *a*/*h* = 10, *b*/*a* = 1.0, *n* = 2.0, *β* = 2.0, predicted by SGT.

Figure 11 depicts variation of the first dimensionless transverse natural frequency $\bar{\omega}_1$ with respect to *n*, for four different values of the length scale parameter ratio β. Increase in β causes a notable rise in the first dimensionless frequency. Thus, the micro-plate behaves in a stiffer manner for larger values of the length scale parameter ratio. Influence of β on vibration behavior is another finding illustrating the need to account for length scale parameter variation in structural analysis of micro-scale components. The first two dimensionless frequencies of transverse vibrations are plotted in Figures 12 and 13 as functions of *n* and l/h. Both of the natural frequencies increase notably as l/h is increased from 0 to 1.2. Note that for a macro-scale plate $l/h \to 0$, and vibration frequency is considerably smaller due to less stiff behavior.

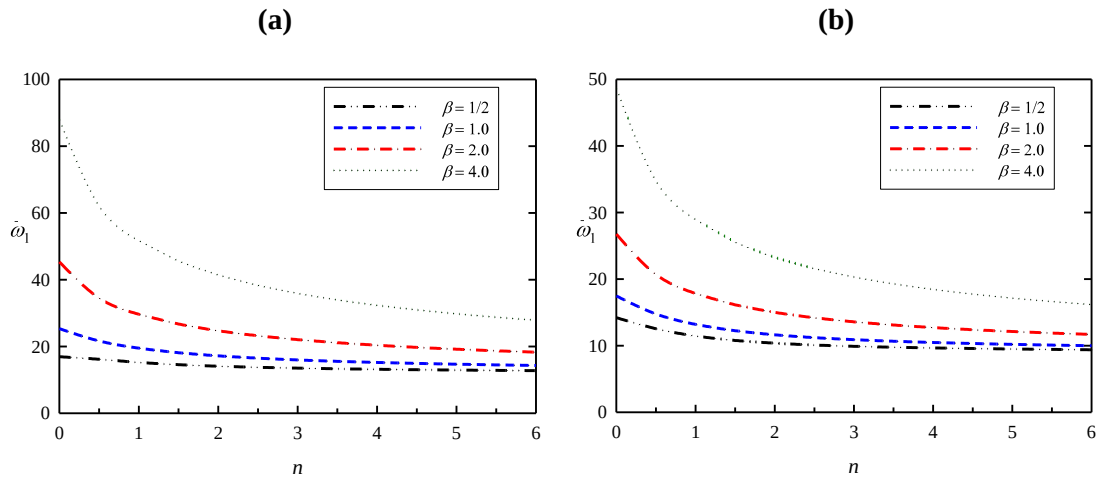

Figure 11. First dimensionless transverse natural frequencies of SSSS micro-plate, *l* = 15 µm, *l*/*h* = 0.4, *a*/*h* = 10, *b*/*a* = 1.0, predicted by (a) SGT and (b) MCST.

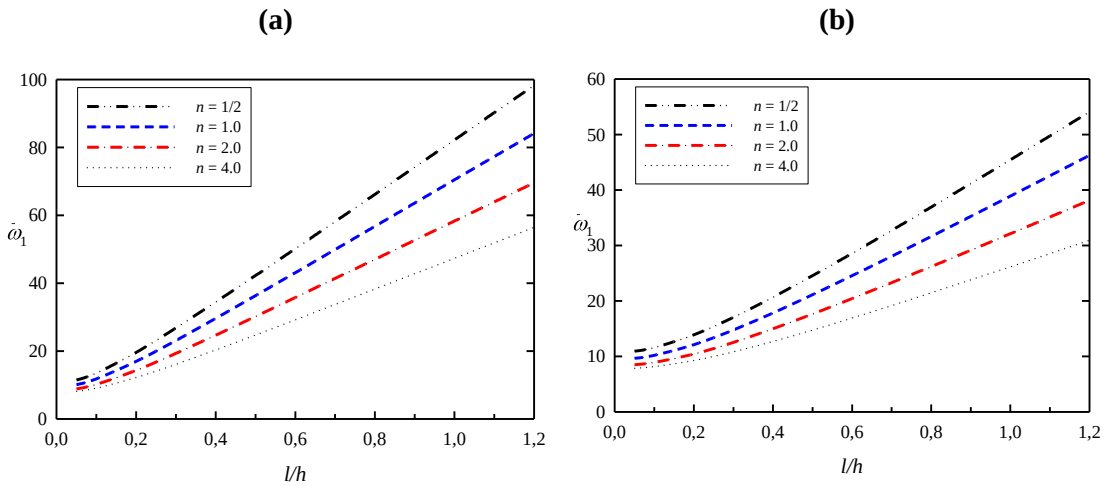

Figure 12. First dimensionless transverse natural frequencies of SSSS micro-plate, *h* = 25 µm, *a*/*h* = 10, *b*/*a* = 1.0, *β* = 2.0, predicted by (a) SGT and (b) MCST.

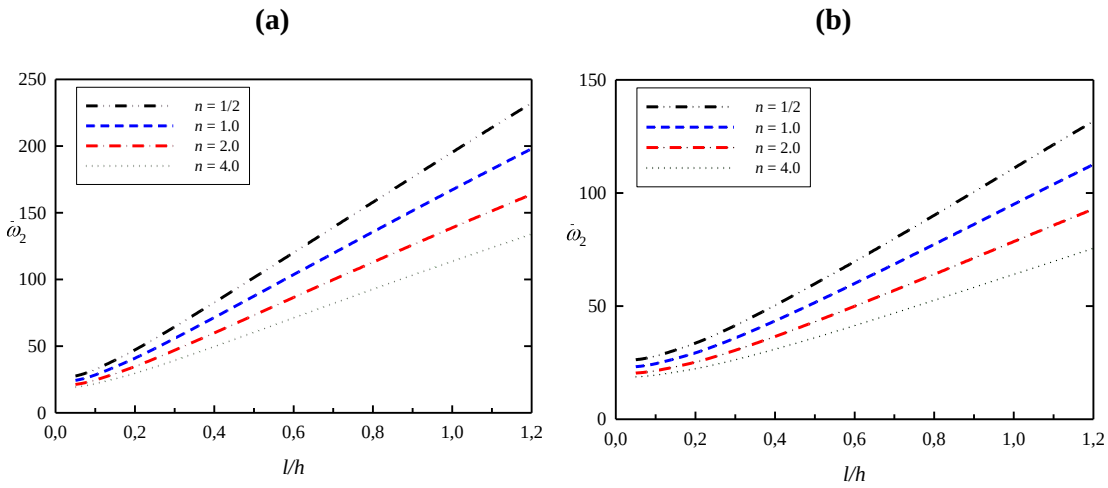

Figure 13. Second dimensionless transverse natural frequencies of SSSS micro-plate, *h* = 25 μm, *a*/*h* = 10, *b*/*a* = 1.0, *β* = 2.0, predicted by (a) SGT and (b) MCST.

Figure 14 depicts first dimensionless natural frequencies of simply-supported micro-plates as a function of the dimensionless length scale parameter l/h. The results are calculated for the classical and the two higher order continuum theories. Frequency plots obtained by these theories are convergent to a same result as l/h decreases. Variation of l/h has no effect on the results predicted by classical theory. As l approaches to value of the thickness of micro-plate, frequencies computed by MCST and SGT increase. The increase in natural frequencies seems to be larger for SGT compared to those found by MCST.

The first and second dimensionless natural frequencies of all edges clamped micro-plates are provided in Figure 15. The results are generated based on MCST. Comprehensive examination of Figure 15 and the previous results regarding frequencies of all edges simply-supported micro-plate reveals the fact that similar trends can be observed in micro-plates with the both types of boundary conditions. The first two natural frequencies of a CCCC micro-plate are higher than those of a SSSS micro-plate.

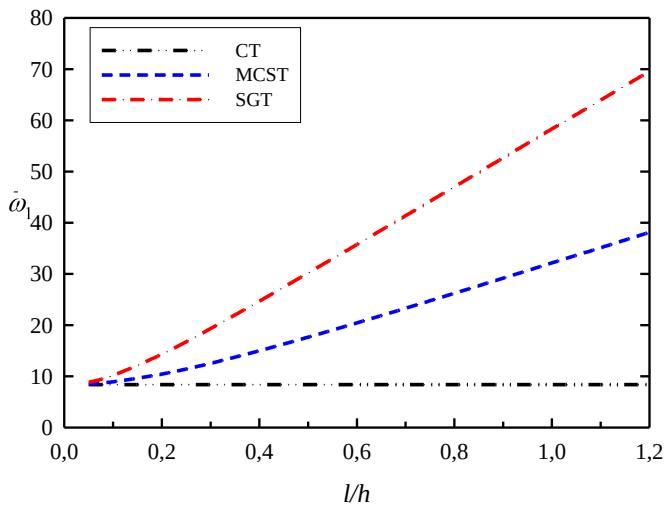

Figure 14. First dimensionless transverse natural frequencies of SSSS micro-plate, *h* = 25 µm, *a*/*h* = 10, *b*/*a* = 1.0, *β* = 2.0.

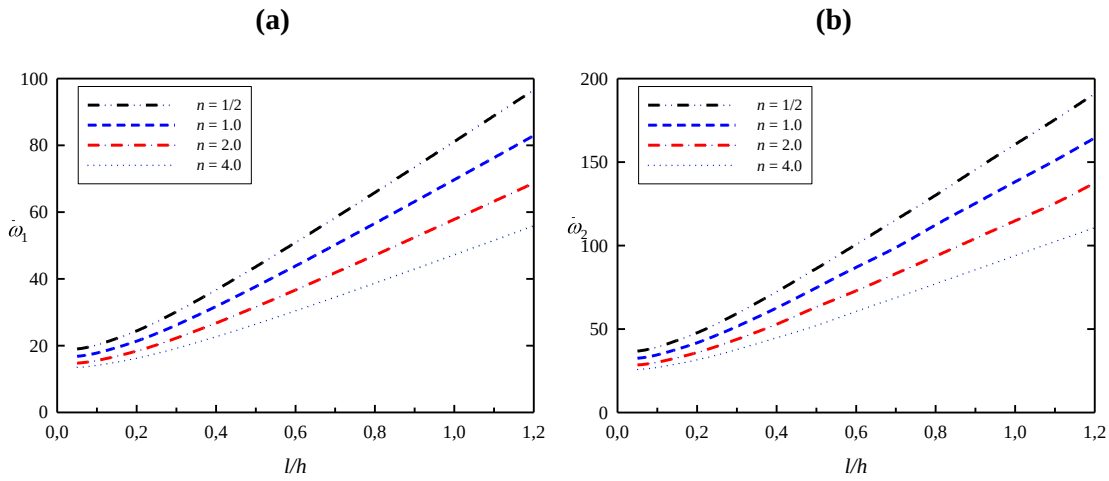

Figure 15. First two dimensionless transverse natural frequencies of CCCC micro-plate: (a) First dimensionless transverse natural frequency; (b) second dimensionless transverse natural frequency. *h* = 25 µm, *a*/*h* = 10, *b*/*a* = 1.0, *β* = 2.0, predicted by MCST.

4.3.3 Buckling

To be able to verify computational developments provided in the current study, comparison results are given for dimensionless critical buckling loads of all edges simply supported micro-plate, according to KPT and MPT. The results are generated based on modified couple stress theory. Material properties are the same as those given by Eq. (36). Dimensionless critical buckling load is expressed in the form:

$$\bar{P} = \frac{Pa^2}{E_m h^3}. \tag{41}$$

It can be seen that the DQM results of this study are in good agreement with those provided by Thai and Choi [42] by utilizing Navier method.

Table 6. Comparisons of dimensionless critical buckling loads $\bar{P}$ of SSSS micro-plate, ν = 0.38, h = 17.6 μm, a/h = 10, b/a = 1.0.

Plate model	***l/h***	***n*=0**		***n*=10**	
		Present	Reference [42]	Present	Reference [42]
KPT	0	19.2255	19.2255	3.8359	3.8359
	0.2	22.0863	22.0863	4.3560	4.3560
	0.4	30.6686	30.6685	5.9164	5.9164
	0.6	44.9723	44.9723	8.5171	8.5171
	0.8	64.9976	64.9976	12.1581	12.1581
	1	90.7444	90.7444	16.8394	16.8393
MPT	0	18.0746	18.0746	3.5854	3.5854
	0.2	20.7607	20.7607	4.0710	4.0710
	0.4	28.7478	28.7478	5.5152	5.5151
	0.6	41.8272	41.8271	7.8803	7.8802
	0.8	59.6657	59.6657	11.1065	11.1065
	1	81.8270	81.8269	15.1153	15.1152

Numerical results regarding critical buckling loads of functionally graded rectangular micro-plates are provided in Figures 16-20.

In Figure 16, we present critical buckling load as a function of the exponent n for three different plate theories. Drop in $\bar{P}$ as the exponent n is increased from 0 to 6 indicates that ceramic-rich FGM micro-plates possess larger critical buckling loads compared to metal-rich ones. Buckling loads computed in accordance with KPT and TSDT are close to each other. MPT seems to slightly underestimate the critical buckling load. Further results provided in Figures 17-20 are generated by TSDT. Figure 17 shows n-variation of the buckling load for four different values of the length scale parameter ratio β. Buckling load is highly sensitive to the changes in the length scale parameter ratio. It rises significantly as the length scale parameter ratio is increased from 1/2 to 4. Figure 18 presents variation of critical buckling load with respect to l/h for four different values of n. This figure is illustrative of the size

effect in that it shows that critical load computed for larger l/h is considerably larger than that evaluated for macro-scale plates for which l/h tends to zero.

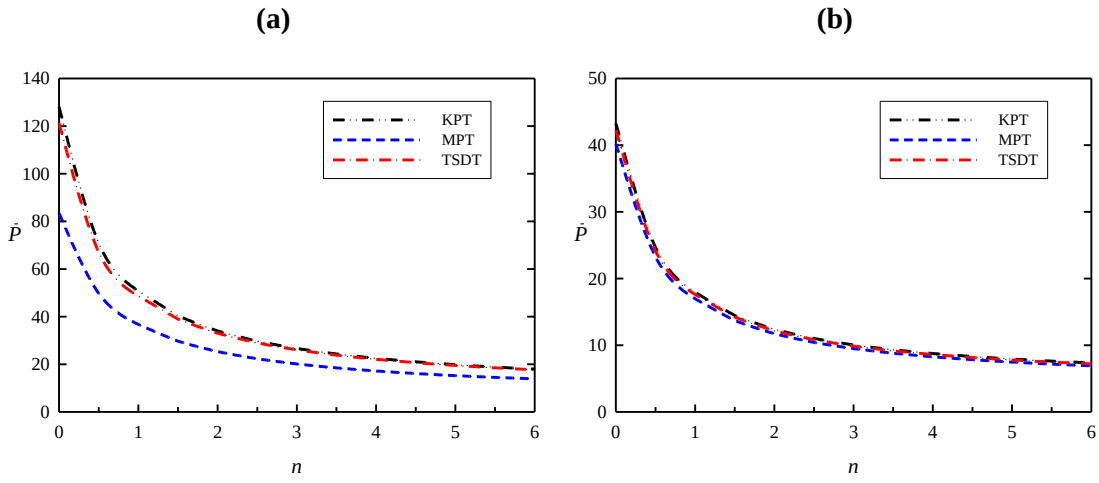

Figure 16. Dimensionless critical buckling load of SSSS micro-plate, *l* = 15 μm, *l*/*h* = 0.4, *a*/*h* = 10, *b*/*a* = 1.0, *β* = 2.0, predicted by (a) SGT and (b) MCST.

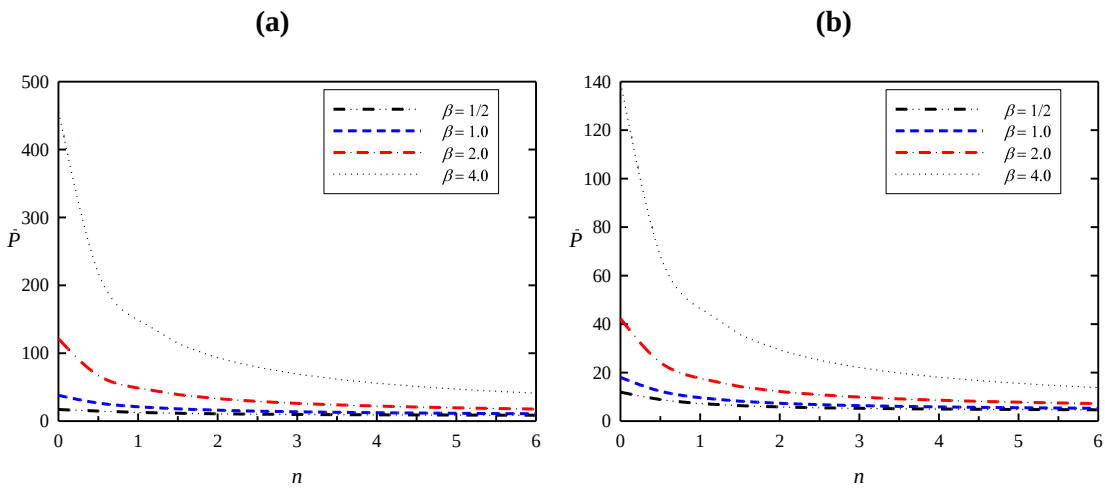

Figure 17. Dimensionless critical buckling load of SSSS micro-plate, *l* = 15 μm, *l*/*h* = 0.4, *a*/*h* = 10, *b*/*a* = 1.0, predicted by (a) SGT and (b) MCST.

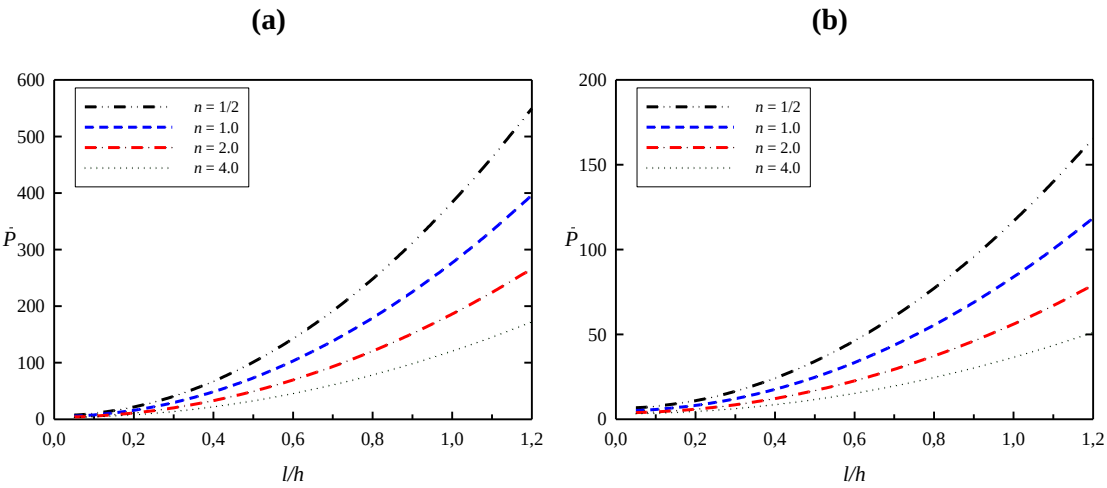

Figure 18. Dimensionless critical buckling load of SSSS micro-plate, *h* = 25 μm, *a*/*h* = 10, *b*/*a* = 1.0, *β* = 2.0, predicted by (a) SGT and (b) MCST.

In Figure 19 $\bar{P}$ is plotted with respect to l/h for three different elasticity theories. The curve of classical elasticity theory is independent of length scale parameter l and remains as a straight line, by taking aspect ratios *a*/*h* and *b*/*a* as constants. SGT and MCST give dimensionless critical buckling loads close to classical theory when $l \to 0$. $\bar{P}$-plots generated by SGT and MCST are increasing functions of l/h. The more rapid increase in critical buckling loads occurs in results of SGT rather than those of MCST.

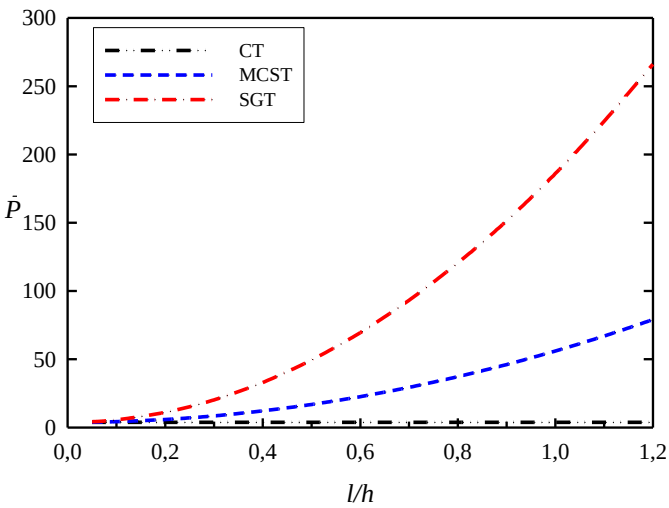

Figure 19. Dimensionless critical buckling load of SSSS micro-plate, *h* = 25 μm, *a*/*h* = 10, *b*/*a* = 1.0, *β* = 2.0.

To examine the effects of type of boundary conditions, in Figure 20 dimensionless critical buckling load of CCCC micro-plate is illustrated. Similar to all edges simply-supported micro-plate, critical buckling load is seen to be a decreasing function of the exponent *n*. Also, $\bar{P}$ increases with a corresponding increase in l/h. All edges clamped micro-plate shows more stiff behavior with respect to all edges simply-supported micro-plate.

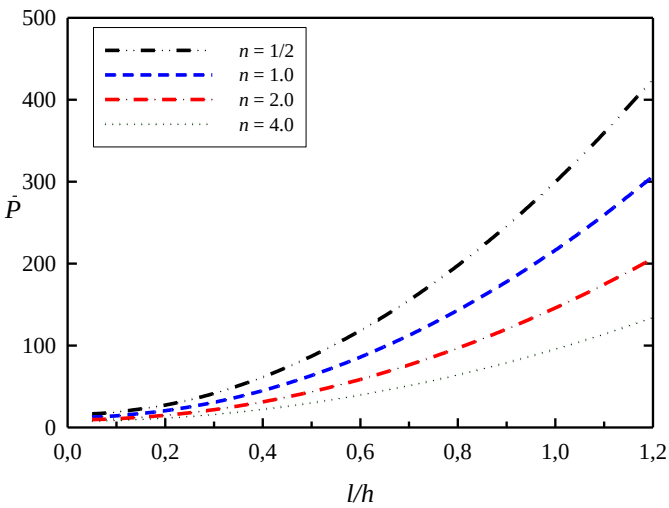

Figure 20. Dimensionless critical buckling load of CCCC micro-plate, *h* = 25 μm, *a*/*h* = 10, *b*/*a* = 1.0, *β* = 2.0, predicted by MCST.

4.4 Numerical Results in Thermal Environment

Due to lack of studies on static thermal bending analysis and, also, free vibrations of functionally graded small-scale rectangular plates in thermal environment, the results verification is only made on thermal buckling results.

For a plate initially at temperature T_0, the temperature can be uniformly raised to T_f such that the plate buckles. This temperature change is denoted by ΔT_{cr} and is called critical buckling temperature difference. In Table 7 we compare critical buckling temperature difference of a functionally graded micro-plate to those given by Mirsalehi et al. [74]. In their study, the spline finite strip method (SFSM) is adopted for analyzing the stability of FGM micro plates. Modified couple stress theory is used in conjunction with KPT to produce the results provided in Table 7. FGM micro-plate in this particular comparison is assumed to be fabricated of alumina and aluminum. Material properties of aluminum are the same as those given by Eq. (33). Alumina possesses the following properties

$$E_c = 380 \text{ GPa}, \quad \alpha_c = 7.4(10)^{-6} \text{ 1/K}, \tag{41}$$

The results are in excellent conformity with those of Mirsalehi et al. [74]. Note that to generate these results the prebuckling static thermal displacements are not worked out for simply-supported micro-plate, and thermally induced initial in-plane forces are considered to be the same as those given by Eq. (24) which is a simplifying assumption for SSSS micro-plates. In order to justify significance of need to account for the initial thermal stress analysis, in Table 8 we have provided maximum static thermal deflections $w_{\max}$ and first dimensionless natural frequencies $\bar{\omega}_1$ of a simply-supported micro-plate under different temperature rises and β-values. To generate these results TSDT is used. Material properties are the same as those given in Eq. (33). It is observed that the results show differences, especially in high temperature rises and small values of β.

Table 7. Comparisons of critical buckling temperature difference ΔT_{cr} (K) of SSSS and CCCC micro-plate under uniform temperature rise, ν = 0.3, h = 17.6 μm, a/h = 100, b/a = 1.0.

Boundary conditions	l/h	n = 0		n = 5	
		Present	Reference [74]	Present	Reference [74]
SSSS	0.0	17.0991	17.0992	7.2658	7.2657
	1.0	88.9154	88.9157	36.9707	36.9707
	2.0	304.3641	304.3652	126.0856	126.0858
CCCC	0.0	45.3437	45.3471	19.2674	19.2688
	1.0	235.7870	235.8052	98.0394	98.0467
	2.0	807.1167	807.1792	334.3553	334.3806

Table 8. Maximum deflection w_{max} and dimensionless natural frequencies under uniform temperature rise, l = 15 μm, l/h = 0.4, a/h = 10, b/a = 1.0, n = 2, predicted by MCST.

ΔT (K)	β	w_{max} (μm)		$\bar{\omega}_1$	
		SSSS*	SSSS**	SSSS*	SSSS**
100	0.5	0.4911	0	10.2262	9.9084
	1.0	0.3931	0	11.4968	11.2154
	2.0	0.2354	0	14.8943	14.6786
	4.0	0.0904	0	23.1977	23.0602
500	0.5	2.4556	0	9.6815	7.8603
	1.0	1.9657	0	11.0147	9.4552
	2.0	1.1771	0	14.5247	13.3826
	4.0	0.4521	0	22.9617	22.2584

* Static thermal bending analysis are carried out based on procedure developed in this work
** In-plane forces are assumed to be the same as those given in Eq. (24)

4.4.1 Static thermal bending

In this section deflections resulted from static thermal bending analysis under uniform temperature rises are provided. Since MCST and SGT exhibit similar trends regarding the mechanical responses of micro-plate in thermal environment, unless otherwise mentioned, the results are mostly computed by MCST and at the end of each section sample results obtained by SGT are presented. Illustrated in Figure 21 are the deflections of SSSS and CCCC micro-plates under 100 K temperature rise. As it is observed thermally induced initial displacements are zero for all edges clamped micro-plate; because the geometric boundary conditions and governing equations for static thermal bending problem of CCCC micro-plate form a homogeneous system of differential equations with trivial solution. The nonzero initial displacements due to temperature change in all edges simply-supported micro-

plate necessitates consideration of these displacements in calculation of thermally induced initial in-plane forces.

Maximum deflections of SSSS micro-plate, undergoing a temperature rise of 100 K, as a function of the exponent n and predicted by three different plate theories, i.e. KPT, MPT and TSDT, are given in Figure 22. Maximum deflection occurs at the mid-point of plate. All of the maximum deflections increase as the exponent n is increased. Thus, ceramic rich functionally graded micro-plates show smaller deflections. This is the expected result since the ceramic phase of the FGM micro-plate has larger modulus of elasticity and coefficient of thermal expansion compared to the metallic phase. Note that, thermal bending occurs in FGM plates due to the differences in material properties of upper and lower surfaces. When n = 0, plate is homogeneous, and is not deflected under temperature change. Further inspections on Figure 22 show that the three plate theories are seen to lead to almost identical maximum thermally induced deflection curves. In computation of the results given in Figures 23-25 third-order shear deformation theory is employed because it produces more accurate results due to its quadratic transverse shear strain distribution.

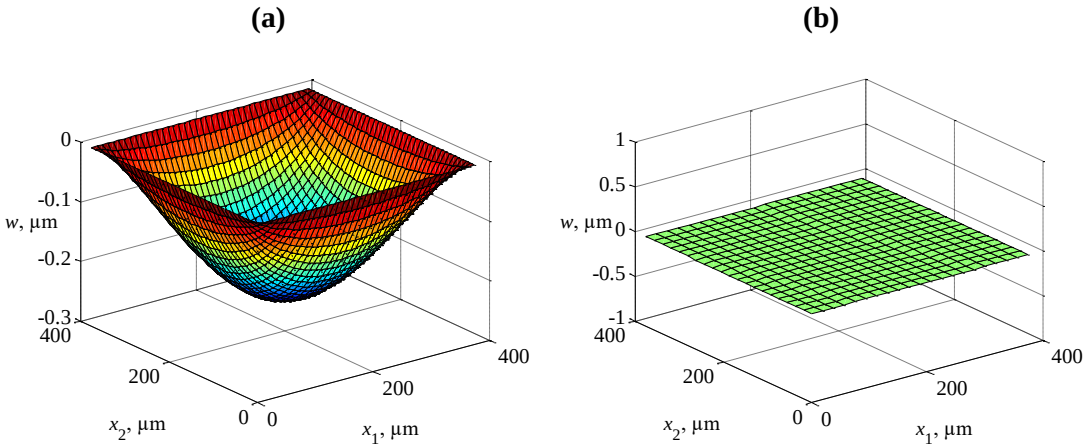

Figure 21. Static deflection under uniform temperature rise of (a) SSSS micro-plate and (b) CCCC micro-plate, l = 15 μm, l/h = 0.4, a/h = 10, b/a = 1.0, β = 2.0, n = 2, ΔT = 100 K, predicted by MCST.

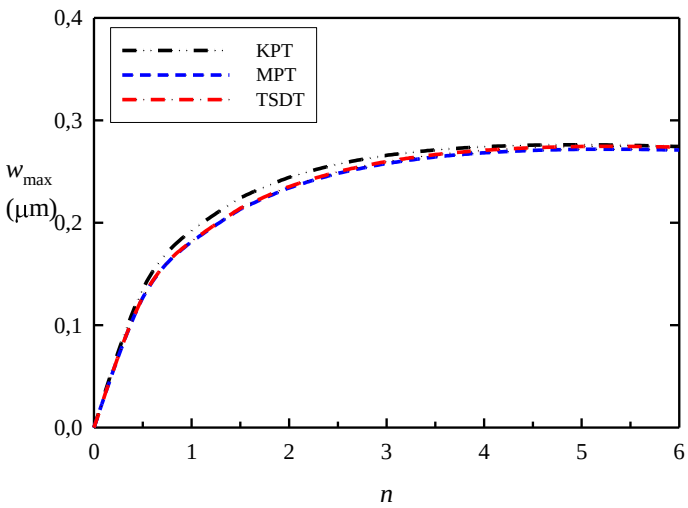

Figure 22. Maximum deflection of SSSS micro-plate, l = 15 μm, l/h = 0.4, a/h = 10, b/a = 1.0, β = 2.0, ΔT = 100 K, predicted by MCST.

Figure 23 depicts influence of the length scale parameter ratio β on maximum thermal deflections of all edges simply-supported micro-plates. It can be seen that β has a significant impact on the maximum thermal deflection. Thus, it can be infered that variation of the length scale parameter needs to be taken into account to be able to produce more accurate numerical results regarding the structural behavior of FGM micro-plates. Thermal deflections get smaller as β is increased. As n is increased, different β-curves tend to converge to a single value because the equivalent length scale parameters of metal rich functionally graded micro-plates approach to that of pure metal phase, i.e. l = 15 μm.

The results provided in Figure 24 demonstrate the impact of temperature change ΔT upon static thermal deformations. It is seen that maximum deflections increase as temperature increases from 0 K to 500 K. It is also observed that as the exponent n gets larger than a specfic value and the micro-plate approaches to a homogeneous metal configuration, slopes of w_{max}-curves drop.

In order to be able to compare static thermal results of three elasticity theories considered in this study, maximum deflections of SSSS micro-plates computed by CT, MCST and SGT are given in Figure 25. Due to lack of length scale parameter, classical theory predicts the less rigid micro-plate with the highest values of

thermally induced maximum deflections. Employing SGT leads to the smallest thermal displacements. The curves of ΔT = 0 K are coincident with horizontal axis.

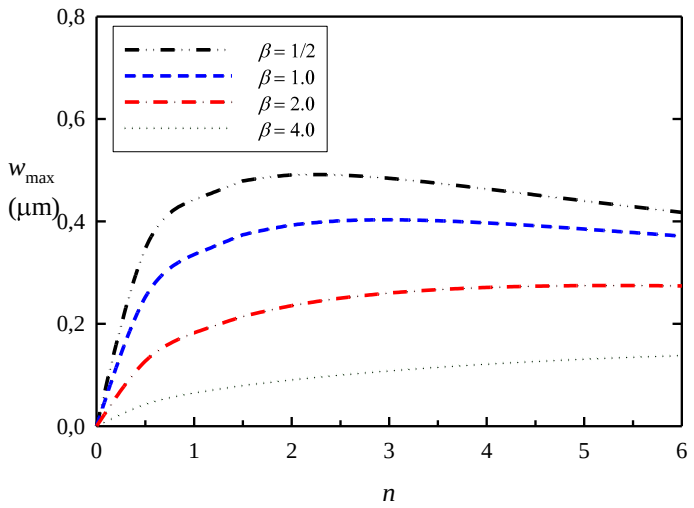

Figure 23. Maximum deflection of SSSS micro-plate, *l* = 15 μm, *l*/*h* = 0.4, *a*/*h* = 10, *b*/*a* = 1.0, Δ*T* = 100 K, predicted by MCST.

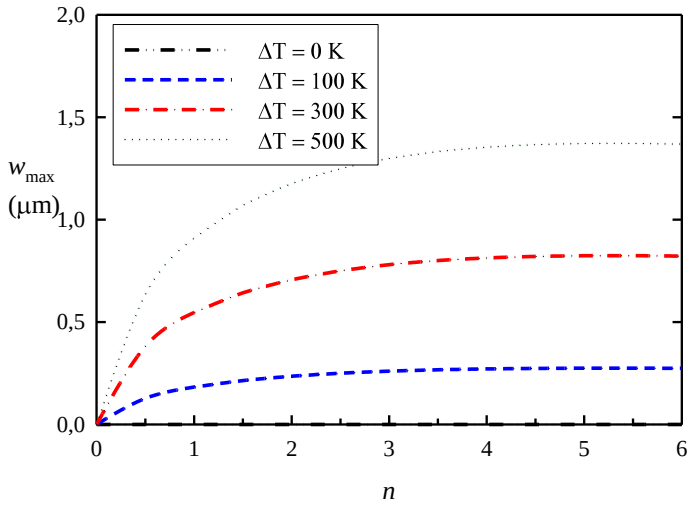

Figure 24. Maximum deflection of SSSS micro-plate, *l* = 15 μm, *l*/*h* = 0.4, *a*/*h* = 10, *b*/*a* = 1.0, *β* = 2.0, predicted by MCST.

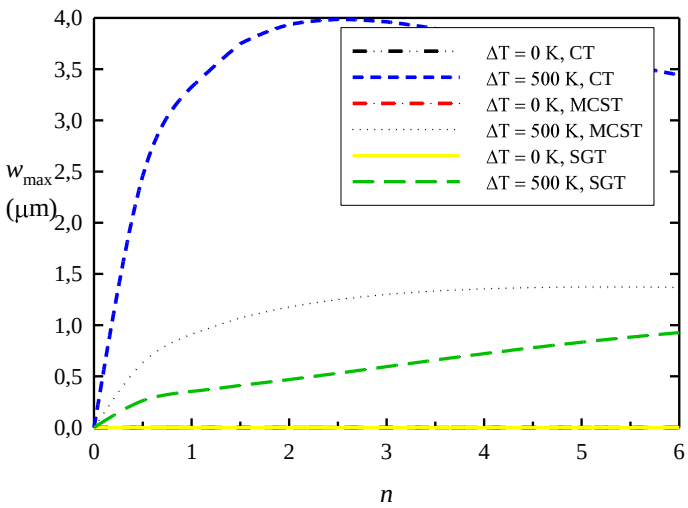

Figure 25. Maximum deflection of SSSS micro-plate, *l* = 15 μm, *l*/*h* = 0.4, *a*/*h* = 10, *b*/*a* = 1.0, *β* = 2.0.

4.4.2 Free vibrations

The results regarding free vibrations of functionally graded small-scale plates are provided in Figures 26-31. To produce these results pre-vibration thermal bending analysis has been also carried out. In Figure 26 we present first diensionless natural frequencies of micro-plate computed using Kirchhoff, Mindlin and third-order shear deformation plate theories. Differences of the results of different plate models seem to be larger for all edges clamped micro-plate compared to those of simply-supported one. This fact can be justified by examining the shear effects in both of the boundary configurations. All edges clamped micro-plate possesses higher degree of constraint in boundaries which leads to larger shear effect with respect to all edges simply-supported one. Almost identical natural frequencies are predicted by KPT, MPT and TSDT for SSSS micro-plate. Natural frequencies of a plate with all edges clamped configuration are seen to be higher than natural frequencies obtained for all edges simply-supported plate.

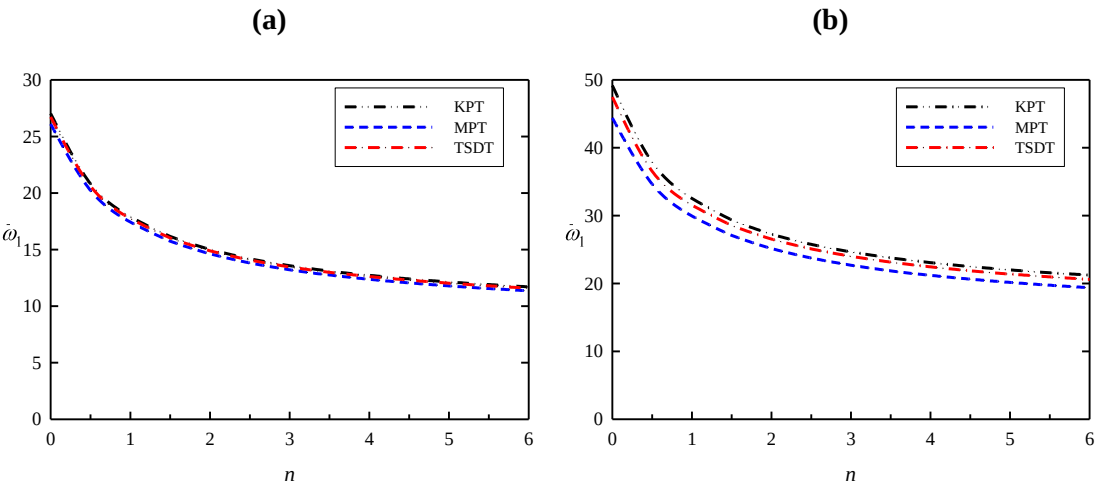

Figure 26. First dimensionless transverse natural frequencies of (a) SSSS micro-plate and (b) CCCC micro-plate, l = 15 μm, l/h = 0.4, a/h = 10, b/a = 1.0, β = 2.0, ΔT = 100 K, predicted by MCST.

Dependences of the first two natural frequencies of a micro-plate in a thermal environment with ΔT = 100 K on the length scale parameter ratio β are examined in Figures 27 and 28. As is the case for analysis in absence of thermal loading, a micro-plate with higher β displays higher natural frequencies. The first two dimensionless natural frequencies approach to specific values as length scale parameter ratio β increases. These constant values are the first two dimensionless natural frequencies obtained for a micro-plate made of fully metal.

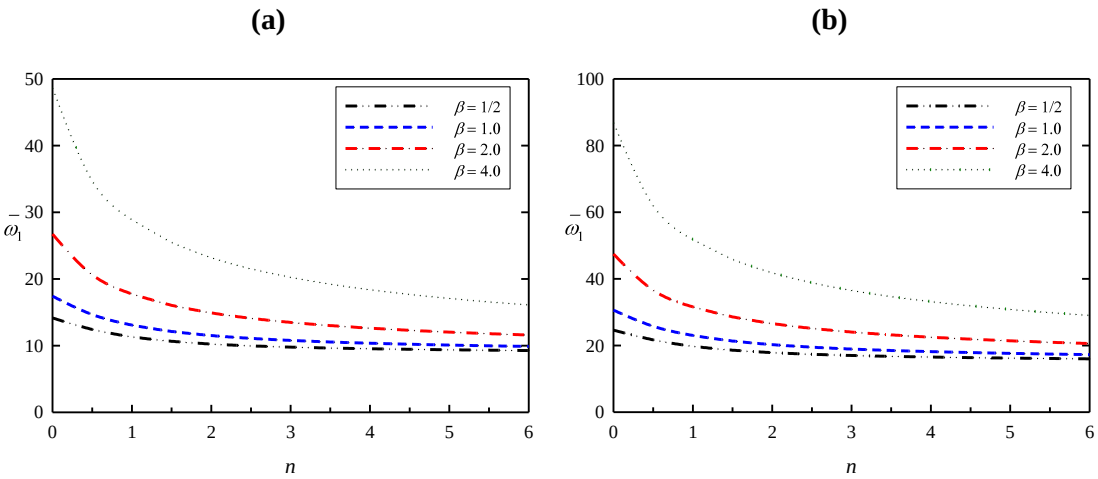

Figure 27. First dimensionless transverse natural frequencies of (a) SSSS micro-plate and (b) CCCC micro-plate, l = 15 μm, l/h = 0.4, a/h = 10, b/a = 1.0, ΔT = 100 K, predicted by MCST.

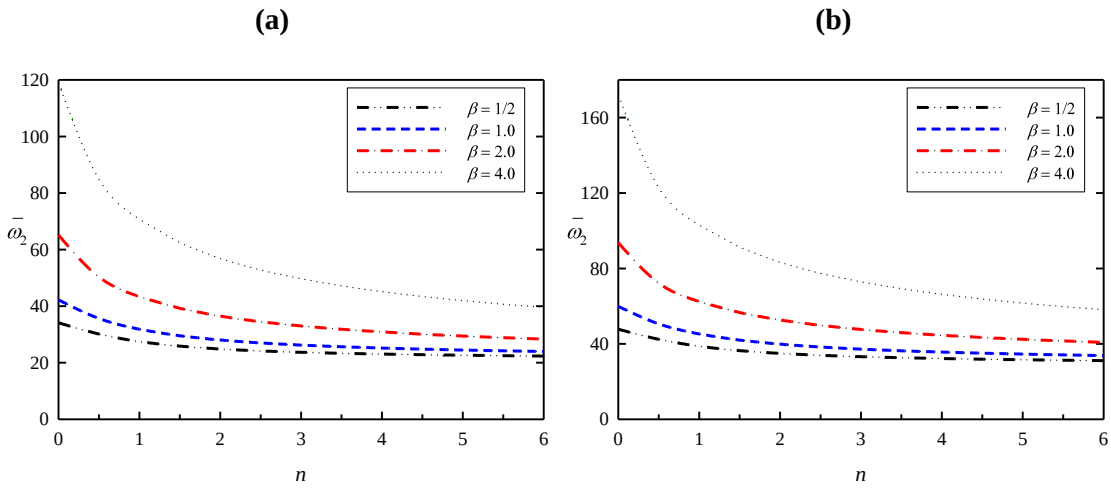

Figure 28. Second dimensionless transverse natural frequencies of (a) SSSS micro-plate and (b) CCCC micro-plate, *l* = 15 μm, *l*/*h* = 0.4, *a*/*h* = 10, *b*/*a* = 1.0, Δ*T* = 100 K, predicted by MCST.

Figures 29 and 30 show, respectively, variations of $\bar{\omega}_1$ and $\bar{\omega}_2$ with respect to the exponent *n* for four different values of Δ*T*. Results reveal that the first two natural frequencies are not that sensitive to the change in temperature. Increasing temperature of micro-plate leads to a slight drop in frequencies. The decrease in natural frequencies are more sensible in all edges clamped micro-plates because of higher values of initial thermal stresses due to temperature rise.

The first ten dimensionless natural frequencies at uniform temperature rises Δ*T* = 0 K and Δ*T* = 500 K are tabulated in Tables 9 and 10 for SSSS and CCCC micro-plates, respectively. As it is observed, thermal loading has no influence on the order of dominant modes of vibration. Although increase in temperature leads to drop in transverse natural frequencies, its effect on axial vibration frequencies is negligible. To compute the first three natural frequencies in absence of thermal loading, $N_{x_1} = N_{x_2} = 11$ is used as the number of grid points; However, the number of nodes is increased to $N_{x_1} = N_{x_2} = 17$ for convergence of results in thermal environment and also higher natural frequencies.

Depicted in Figure 31 is $\bar{\omega}_1$ at Δ*T* = 0 K and Δ*T* = 500 K, obtained by utilizing classical elasticity theory and the two non-classical models developed in the current study, namely MCST and SGT. It is clearly shown that natural frequencies predicted

by classical model are smaller than those generated by non-classical approaches. Using SGT results in the highest values of frequencies. Increase in temperature leads to drop in natural frequencies. However, thermal effects seem to be negligible in results produced by SGT. The decrease in natural frequencies are more pronounced when classical theory is employed.

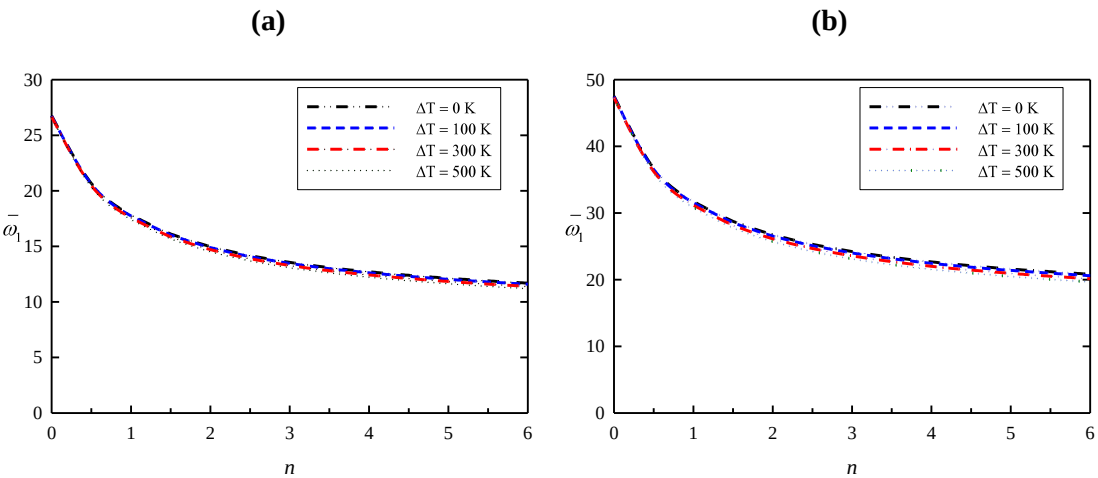

Figure 29. First dimensionless transverse natural frequencies of (a) SSSS micro-plate and (b) CCCC micro-plate, *l* = 15 μm, *l*/*h* = 0.4, *a*/*h* = 10, *b*/*a* = 1.0, *β* = 2.0, predicted by MCST.

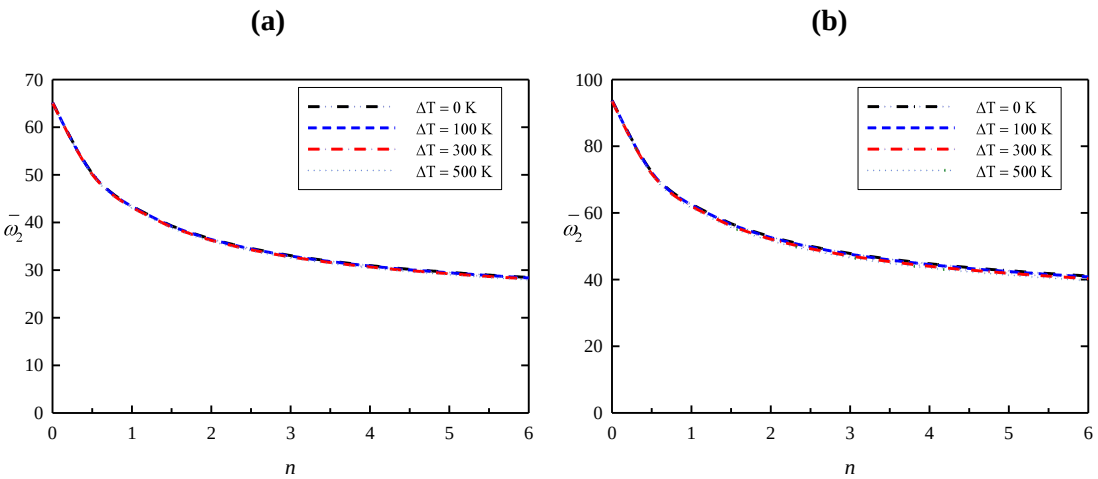

Figure 30. Second dimensionless transverse natural frequencies of (a) SSSS micro-plate and (b) CCCC micro-plate, *l* = 15 μm, *l*/*h* = 0.4, *a*/*h* = 10, *b*/*a* = 1.0, *β* = 2.0, predicted by MCST.

Table 9. Dominant modes and corresponding frequencies of SSSS micro-plate, l = 15 μm, l/h = 0.4, a/h = 10, b/a = 1.0, β = 2.0, n = 2.0, predicted by MCST.

ΔT = 0 K		ΔT = 500 K	
$\bar{\omega}$	Dominant Mode	$\bar{\omega}$	Dominant Mode
14.9851	Transverse *w*, mode 1	14.5247	Transverse *w*, mode 1
32.6824	Axial, mode 1	32.6824	Axial, mode 1
32.6824	Axial, mode 1	32.6824	Axial, mode 1
36.5330	Transverse *w*, mode 2	36.0600	Transverse *w*, mode 2
36.5330	Transverse *w*, mode 2	36.0600	Transverse *w*, mode 2
45.7911	Axial, mode 2	45.7911	Axial, mode 2
57.1706	Transverse *w*, mode 3	56.6300	Transverse *w*, mode 3
66.2338	Axial, mode 3	66.2338	Axial, mode 3
66.2338	Axial, mode 3	66.2338	Axial, mode 3
70.4532	Transverse *w*, mode 4	69.9352	Transverse *w*, mode 4

Table 10. Dominant modes and corresponding frequencies of CCCC micro-plate, l = 15 μm, l/h = 0.4, a/h = 10, b/a = 1.0, β = 2.0, n = 2.0, predicted by MCST.

ΔT = 0 K		ΔT = 500 K	
$\bar{\omega}$	Dominant Mode	$\bar{\omega}$	Dominant Mode
26.7523	Transverse *w*, mode 1	25.7090	Transverse *w*, mode 1
52.8506	Transverse *w*, mode 2	51.6489	Transverse *w*, mode 2
52.8506	Transverse *w*, mode 2	51.6489	Transverse *w*, mode 2
61.3364	Axial, mode 1	61.3258	Axial, mode 1
61.3364	Axial, mode 1	61.3258	Axial, mode 1
76.0491	Transverse *w*, mode 3	74.7717	Transverse *w*, mode 3
79.5756	Axial, mode 2	79.5756	Axial, mode 2
90.7455	Transverse *w*, mode 4	89.4982	Transverse *w*, mode 4
91.5897	Transverse *w*, mode 5	90.3193	Transverse *w*, mode 5
92.1342	Axial, mode 3	92.1240	Axial, mode 3

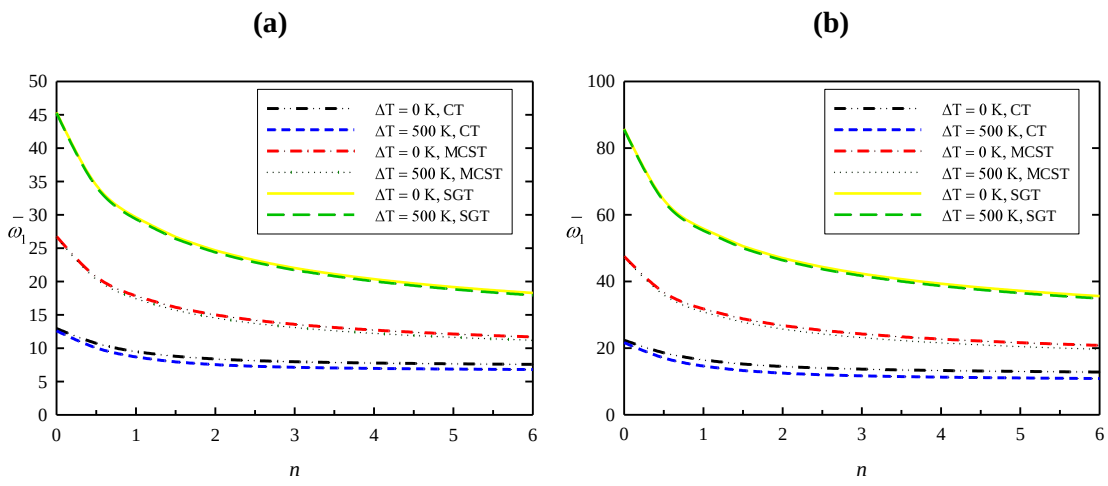

Figure 31. First dimensionless transverse natural frequencies of (a) SSSS micro-plate and (b) CCCC micro-plate, l = 15 μm, l/h = 0.4, a/h = 10, b/a = 1.0, β = 2.0.

4.4.3 Buckling

Numerical results regarding dimensionless critical buckling load $\bar{P}$, defined by Eq. (41), are provide in Figures 32-35. Figure 32 displays the variation of $\bar{P}$ with the power function exponent n predicted by Kirchhoff, Mindlin, and third order shear deformation theories. Despite differences, the results produced by utilizing different plate models are very close. Due to shear effects, the differences between predictions of the three plate theories are more visible in the plot for CCCC micro-plate.

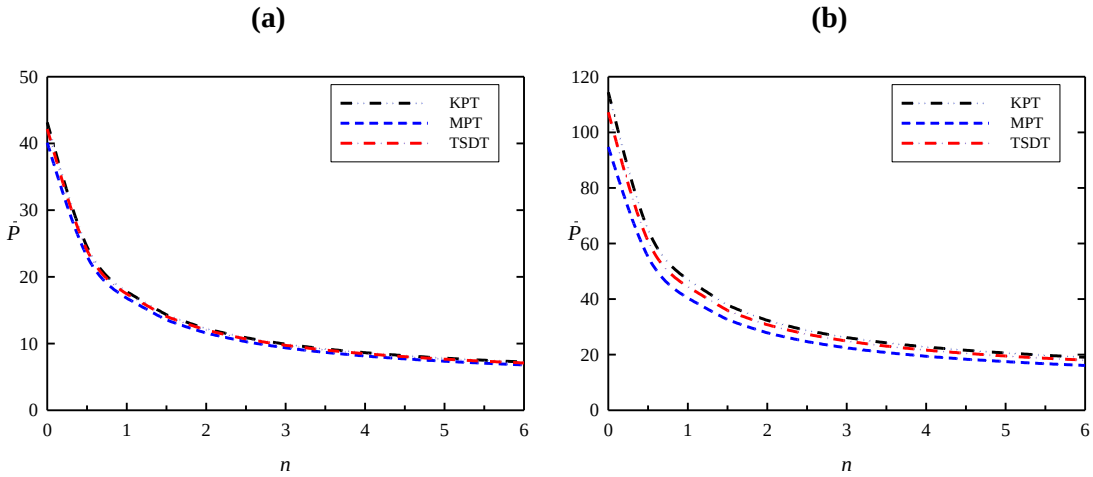

Figure 32. Dimensionless critical buckling load of (a) SSSS micro-plate and (b) CCCC micro-plate, l = 15 μm, l/h = 0.4, a/h = 10, b/a = 1.0, β = 2.0, ΔT = 100 K, predicted by MCST.

Figure 33 illustrates the influence of β upon critical buckling load of micro-plate whose temperature is uniformly raised as 100 K from room temperature T_0. The effects of length scale paramete ratio β and the index n at ΔT = 100 K are similar to those discussed on the results produced in absence of thermal effects in Section 4.3.3. Higher values of β result in larger values of $\bar{P}$, especially in smaller values of n in which ceramic phase is dominant constituent of FGM micro-plate.

Plotted in Figure 34 is the variation of critical buckling load with n for four different values of temperature change ΔT. Increases in temperature from 0 K to 500 K slightly lead to decrease in values of $\bar{P}$. Generally, micro-plates behave less stiff as temperature is raised.

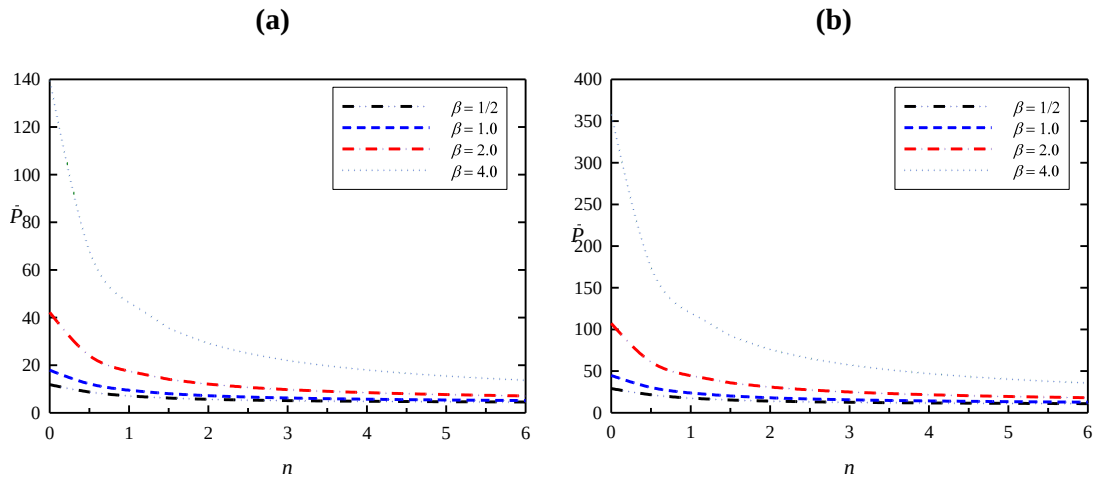

Figure 33. Dimensionless critical buckling load of (a) SSSS micro-plate and (b) CCCC micro-plate, , *l* = 15 μm, *l*/*h* = 0.4, *a*/*h* = 10, *b*/*a* = 1.0, Δ*T* = 100 K, predicted by MCST.

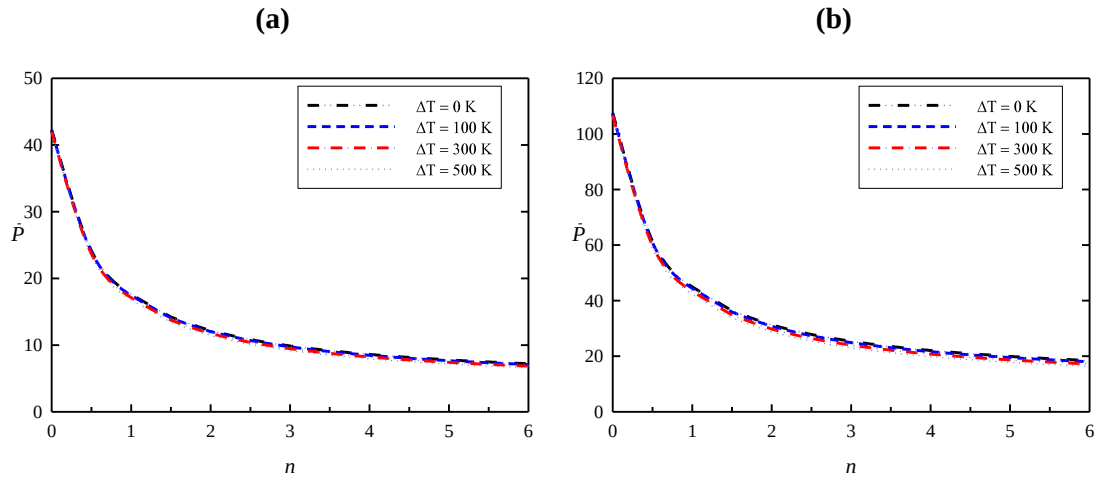

Figure 34. Dimensionless critical buckling load of (a) SSSS micro-plate and (b) CCCC micro-plate, *l* = 15 μm, *l*/*h* = 0.4, *a*/*h* = 10, *b*/*a* = 1.0, *β* = 2.0, predicted by MCST.

Figure 35 shows how the dimnsionless critical buckling load varies with the volume fraction exponent *n* and temperature change, where results of three elasticity theories are presented. Although $\bar{P}$ is a decreasing function of Δ*T*, the sensitivity of buckling load to temperature rise is poor, especially in the results obtained by stain gradient theory.

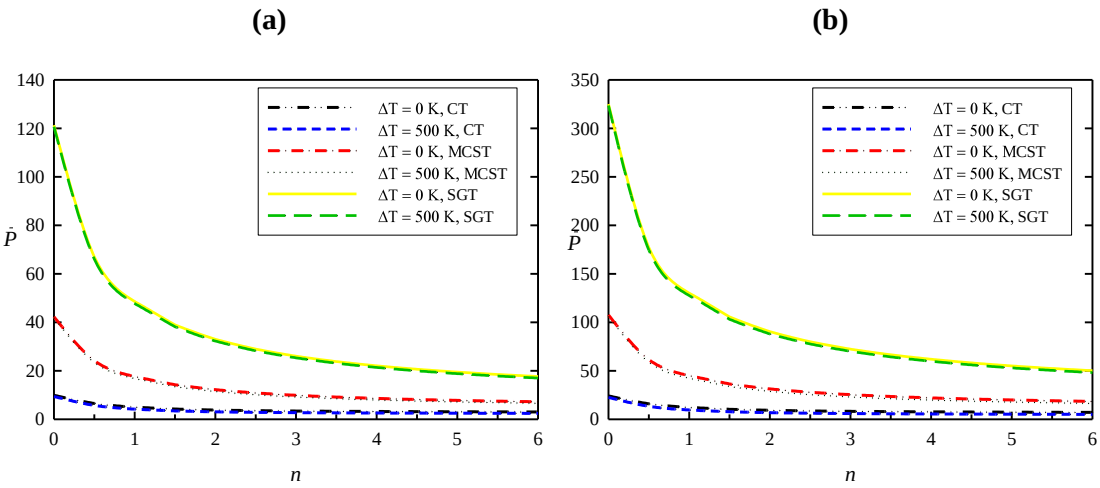

Figure 35. Dimensionless critical buckling load of (a) SSSS micro-plate and (b) CCCC micro-plate, l = 15 μm, l/h = 0.4, a/h = 10, b/a = 1.0, β = 2.0.

CHAPTER 5

CONCLUDING REMARKS AND FUTURE WORKS

In this work, a new strain gradient elasticity based analysis procedures for static bending, free vibrations, and buckling of functionally graded rectangular micro-plates undergoing mechanical and thermal loadings are presented. Proposed methods allow taking into account spatial variations of the length scale parameters. Governing partial differential equations and associated boundary conditions are derived by applying Hamilton's principle. All material properties, including the length scale parameters, are assumed to be functions of the thickness coordinate in these derivations. A unified expression for displacement field is utilized which allows producing results for both classical and shear deformation plate theories, namely, Kirchhoff, Mindlin, and third order shear deformation theories. The equations are solved numerically by means of the differential quadrature method. Comparisons to the findings available in the literature for certain limiting cases do verify the developed techniques. Presented numerical results include static deflections, natural frequencies, mode shapes, and critical buckling loads of simply-supported and all edges clamped functionally graded rectangular micro-plates.

Length scale parameter ratio β identifies the degree of spatial variations of the length scale parameters. The ratio β is shown to have a significant impact on static deflection, vibration frequency, and buckling load of a rectangular FGM micro-plate. A rise in β leads to a drop in dimensionless maximum deflection, and increases in dimensionless vibration frequency and buckling load. Hence, sufficiently accurate results can be generated only if spatial variations of the length scale parameters are taken into account in the formulation of the relevant problem.

The influence of transverse shear deformation upon the stiffness of the micro-plate is revealed through the numerical analyses according to three different plate theories. Mindlin plate theory underestimates natural frequencies and critical buckling loads. Since third order shear deformation theory allows a nonlinear distribution of shear strains across the thickness, the results produced by employing this theory seem to be more accurate.

In conducting numerical analyses for simply supported micro-plates undergoing thermal loading, initially induced static thermal bending and thermal stresses must be worked out. It is shown that natural frequencies and critical buckling loads of simply-supported plates computed by taking into account the initial thermal displacements show differences compared to those obtained by assuming thermal deflections to be zero.

Further numerical results provided in this study exhibit the effects of different material and geometrical properties and various loading conditions on mechanical responses of micro-plates. The methods presented in this article could be useful in analysis, design, and optimization studies involving functionally graded micro-scale plates.

The models developed based on higher order continuum theories are highly dependent on the values of length scale parameters. Except for epoxy, there are no sufficient data in the literature regarding the values of small-scale parameters used in modified couple stress and strain gradient theories. To obtain more accurate results for development of functionally graded micro-structures, there is need for experimental efforts to characterize micro-structural properties of materials.

The current model can also be developed for wider appalicability in dynamical systems by considering damping effects and time dependency of the responses. In some vibrational applications in which energy loss due to mechanical damping, such as Coulomb and viscous dampings, is high, damping effects must be appropriately modeled in derivations. Damping elements changes the response of the system and its resonance frequencies. Transient responses of micro-structures under daynamic loads is another problem encountered in design and development of micro-electro-mechanical systems.

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CURRICULUM VITAE

PERSONAL INFORMATION

Surname, Name: Aghazadeh, Reza
Nationality: Iranian (IR)
Date and Place of Birth: 21 March 1985, Oroumieh, Iran
Marital Status: Married
Phone: +90 534 454 1965
email: reza.aghazadeh@metu.edu.tr

EDUCATION

Degree	Institution	Year of Graduation
MS	METU Mechanical Engineering	2013
BS	IUST Mechanical Engineering, Iran	2008
High School	Imam Khomeini High School, Oroumieh, Iran	2002

WORK EXPERIENCE

Year	Place	Enrollment
2015-Present	University of Turkish Aeronautical Association	Lecturer
2008-2010	Iran-Persit	Technical Designer
2007-2008	L.S.U.	Quality Control Chief

FOREIGN LANGUAGES

Fluent Turkish, Fluent English, Native Azerbaijani, Native Persian

PUBLICATIONS

1. R Aghazadeh, E Cigeroglu, and S Dag. Static and free vibration analyses of small-scale functionally graded beams possessing a variable length scale parameter using different beam theories. *European Journal of Mechanics - A/Solids*, 46: 1-11, 2014.

2. U Tosun, R Aghazadeh, C Sert, and M B Özer. Tracking free surface and estimating sloshing force using image processing. *Experimental Thermal and Fluid Science*, 88:423-433, 2017.

RESEARCH INTERESTS

Mechanics of micro-structures, Structural Analysis, Vibrations, Computational Mechanics